Roles and characteristics of problem solving in the mathematics curriculum

Since problem solving became one of the foci of mathematics education, numerous researches have been carried out to improve its teaching, developing students' higher skills or evaluating its learning. Usually these researches were devoted to studying particular aspects of problem solving. However, more investigation is still needed to understand how to address problem solving from a more comprehensive perspective, especially in the curricular field. So we ask, what characteristics should problem solving have in each stage of the curriculum enactment process for a successful implementation? What role should problem solving have from a global curriculum perspective? To answer these questions we have made a literature review to identify the roles and characteristics of problem solving that facilitate its successful implementation. After a qualitative analysis of data, we organize the results into the categories: conditions of the educational system, curriculum structure, characteristics of the planned curriculum and instructional design, types of problems in the intended curriculum and instructional materials, characteristics of the implemented tasks, class management, evaluation, considerations about students and teacher roles during problem solving. In addition, we identified five principles to implement problem solving in teaching mathematics: understanding, reasoning, autonomy, collaboration, and affective factors.

Keywords: mathematics curriculum, curriculum implementation, problem solving, literature review

Introduction

Since problem solving became one of the foci of mathematics education, numerous investigations have been carried out to improve its teaching, developing students' higher skills, emphasizing its formative character or evaluating its learning (Castro, 2008; Lester & Cai, 2016). Problem solving has become a major objective of mathematics curriculum in much of the world. However, despite sharing this common goal, each educational system seems to follow different ways to achieve it, which is reflected in diverse evaluation results (Burkhardt, 2014). So, one can ask; are the curricula considering the advances that have been made since the research in problem solving? What characteristics should problem solving have in each stage of the curriculum enactment process for a successful implementation? What role should problem solving have from a global curriculum perspective? To answer this question, we need to know what research says about how to incorporate problem solving in the curriculum, and what aspects are the most relevant in each stage of curricular enactment process. In this work, we make a literature review with a focus on recommendations to implement problem solving.

Problem solving in mathematics teaching and research

There is not only one approach to define what a problem is. An element of consensus seems to be the non-existence of a previous path, experience or method that allows finding the solution when we face a problem, which implies that the presence of a problem depends on who faces it (Chapman, 2015; Schoenfeld, 1985). A similar situation happens when we try to figure out what problem solving means. Ayllón (2012), making a review about this question, identifies three positions:

(a) A mental activity that a solver puts into action since a problem arises, assumes it as such, wants to solve it and considers the task finished;
(b) A task in which the subject who faces it can understand it due to previous learning, but does not have a method to solve it, which causes confusion. Problem solving would be the process by which the subject gets rid of his disappointment;
(c) The process where previous knowledge is applied in new unfamiliar situations.

What is meant by problem solving influences their role in the teaching of mathematics. Schroeder and Lester (1989) indicate that problem solving can be considered in three ways: teaching for problem solving (using problems as application exercises), teaching about problem solving (in the use of heuristics and solving steps) and teaching through problem solving (as a teaching method). According to these authors, none of the three should have an absolute predominance, although the third role deserves to be considered more since it allows students to develop superior skills and mathematical concepts in the context of inquiry-oriented experiences.

Regarding the research activity, problem solving has been a topic of interest for a long time. Since the publication of Polya's classic book, researchers has been interested in different aspects of this field. Castro (2008) makes a synthesis of the main issues that have been addressed, including the isolation of key determinants in the difficulty of problems, identification of characteristics of good problem solvers, training in heuristics, metacognition, affective factors and beliefs, social interactions, problem posing, assessment, representations, and the use of technology to solve problems.

Researchers still show interest in problem solving in recent years. Currently, a group of works have special relevance. These works study how to incorporate problem solving into the mathematics curriculum, in either curricular regulations, textbooks, instructional design, or lessons. In this group, stands out the work carried out by Stacey (2005) on the analysis of curricular regulations. Anderson's research on curricular development in Australia is also interesting (2014), as well as the methodology to introduce problem solving in the curriculum, of the Study on Math Teachers National Institute of Education in Singapore (Kin et al., 2019; Toh et al., 2019). All these investigations require applying the knowledge accumulated over years of research in problem solving, so it is important having an overview of the priorities and recommendations about different aspects involved in the curriculum enactment process.

The mathematics curriculum and its design and enactment process

Regarding the mathematics curriculum, we adopted the proposal of Remillard and Heck (2014), who define curriculum as

> a plan for the experiences that learners will encounter, as well as the actual experiences they do encounter, that are designed to help them reach specified mathematics objectives. We use the term ‘‘experiences’’ to signal that curriculum is more than the specification of topics to be covered or objectives to be met; it includes what students are intended to or actually experience to support their learning. (p. 707)

These authors propose a model of the curricular system, placing the focus on two domains: the official curriculum, and the operational curriculum. The official curriculum is the one designed by governments, and it includes aims and objectives, usually prescribed in curricular regulations. It also includes materials designed to facilitate the implementation of the objectives, and the content of important evaluations.

The operational curriculum refers to what happens when the official curriculum is implemented. Within the operational curriculum, Remillard and Heck place the intended curriculum, the enacted curriculum, and the students' outcomes. The intended curriculum includes the interpretations and decisions taken to anticipate and plan the teaching. In contrast, the enacted curriculum is made up of the interactions between students and teachers around a task in each class throughout a unit. Finally, instructional materials are defined as resources designed to support teaching, including textbooks, curricular guides and educational software.

In this study we consider as a reference the elements of the curriculum system model of Remillard and Heck. We will use them to guide the pursuit of recommendations from research about the characteristics of problem solving at each stage of the curriculum design and enactment processes.

Method

To identify the roles and characteristics of problem solving that facilitate its successful implementation in the curriculum, we made an exploratory systematic review. A systematic exploratory review differs from the traditional review due to its potential to obtain the essence of a diverse body of evidence (Davis, Drey, & Gould, 2009). This allows knowing what has been done, including all types of documents that are selected for their relevance (Manchado et al., 2009). Our review began with the inquiry of research articles and relevant texts in the databases of Springer Link[1], ScienceDirect[2], and Eric[3]. A first search with keywords such as 'mathematical problem solving' or 'problem solving math curriculum' resulted in 12073 documents. From this set, we made the first selection considering the reading of titles, abstracts, and keywords, using the following inclusion criteria:

- Documents about characteristics of problem solving in some of the stages of curricular enactment.
- Studies in peer-reviewed journals, book chapters, or conference proceedings.
- Relevant authors in the field of mathematics education research.

[1] https://link.springer.com/

[2] https://www.sciencedirect.com/

[3] https://eric.ed.gov/

- School educational level.
- English or Spanish language.

As excluding criteria, we applied the following:

- Characteristics that could only be applied to one area of mathematics.
- Works on curricular reforms of more than 10 years.
- Works on the use of technology of more than 5 years.

As a result, we obtained a set of 285 documents. Having them, we started a general reading of the complete texts, starting on those that seemed most relevant to our research. This reading also allowed us to identify the first possible categories of analysis, using the grounded theory method. After this review, we were left with 120 documents. The initial review also led us to a first open coding, understood as ‘the analytic process through which concepts are identified and their properties and dimensions are discovered in data' (Strauss & Corbin, 1990, p. 101). Based on these first readings, we carried out a theoretical sampling, which aimed to collect data guided by the concepts that were found to enrich the categories in terms of their properties and dimensions (Strauss & Corbin, 1990). The theoretical sampling led us to find another group of documents that emerged from the reading of the first ones. Adding this manual search, we obtained a total sample of 131 documents. We made a second, more detailed reading. The reading and categorization process ended when we reached the point of theoretical saturation, that is ‘the point in the research where collecting additional data seems counterproductive; the “new” that is uncovered does not add that much more to the explanation at this time’ (Strauss & Corbin, 1990, p. 136). At the end of the process, we were left with a total of 78 documents that we analyzed in depth and served as the basis for the construction of a system of categories. After the analysis of these documents we carried out a new codification to verify the consistency. We concluded the process with an audit of an external expert, which allowed us to safeguard the rigor of the process and refine some of the categories. A synthesis of the process is observed in figure 1.

Figure 1. Synthesis of the review process: selection and qualitative analysis of documents.

Results

Because of the analysis, we obtained a series of characteristics and roles of problem solving that the research recommends putting into practice when the curriculum is enacted. We synthesized them in categories, organized according to the levels of Remillard and Heck model (2014). Accordingly, for the level of official curriculum we obtained the categories 'Curricular structure' and 'Conditions of the educational system'. For the level of the intended curriculum, we obtained the categories 'Characteristics of the planned curriculum and instructional design' and 'Types of problems in intended curriculum and instructional materials'. For the level of the enacted curriculum, the categories we considered were 'Characteristics of the implemented tasks', 'Class management' and 'Evaluation'. In addition, we got two categories related with the agents of the process: 'Considerations about students' and 'Teacher roles during problem solving'. Next, we will describe the results found according to each category.

Curriculum structure

The category 'curriculum structure' is like a framework that gives context and influences the rest of categories. In the analyzed documents, all agree that in any mathematics program, problem solving must have a fundamental place (Burkhardt, 2014; Leong et al., 2016; Lester & Cai, 2016). There is a consensus that it should be an integral part of the curriculum, imbricated with the rest of the contents (Anderson, 2014; Lester, 2013).

However, just giving a central role to problem solving is not enough. As an example, Singapore is a country that establishes problem solving as a central part of the curriculum. Since 1990 it has been organized into five components located around mathematical problem solving (Fan & Zhu, 2007). Nevertheless, in recent studies it is recognized that teachers still apply a limited view of problem solving (Kin et al., 2019; Toh et al., 2019). According to these studies, it is observed a pattern that consists of teaching contents and then training to solve the types of problems that appear in national and international exams. Beyond the ability to solve this type of tasks, successful curricula are characterized by giving problem solving a specific role. Possible roles include teaching 'for', 'about' or 'through' problem solving (Schroeder & Lester, 1989). The highest degree that can be integrated into the curriculum is 'teaching through problem solving' (English & Sriraman, 2010; Safrudiannur & Rott, 2018; Schroeder & Lester, 1989). The current recommendation is to teach through this approach since it allows students to develop new knowledge, improve performance and have a deeper conceptual knowledge (Lester & Cai, 2016). Curricula that have adopted this approach promote a more open learning of mathematics with tasks that allow connections between different ideas (Stacey, 2005). One of these is the Japanese curriculum, supported by the Lesson Study method (Schoenfeld & Kilpatrick, 2008). This approach it is characterized by lessons organized in four moments (problem posed by the teacher, independent solving, comparison and group discussion, and synthesis). These lessons are carefully planned for the students to discover valuable mathematical ideas (Fujii, 2018; Isoda, 2015).

The role assigned in curricular regulations, especially if we talk about 'teaching through problem solving', should be aligned with the rest of the curriculum (Burkhardt, 2014). On this account, the role of problem solving must be considered in each level of the enactment process, in the context of a reasoning-oriented curriculum. Cai (2015), in a longitudinal and quasi-experimental study with statistical controls conducted in the United States, provides empirical evidence on how a reformed curriculum, reasoning-oriented and with predominance of teaching 'through problem solving', has a positive effect on the development of skills, reasoning and even basic skills in students.

Conditions of the educational system

In addition to the curriculum structure, other characteristics of the education system should be considered. For example, for teachers to be able to handle the curriculum, they must have sufficient skills and knowledge. To achieve this, the opportunities that the system promotes for their autonomy and professional development are important. Successful problem solving will not be reached if these opportunities are not available (Kin et al., 2019). The conditions for a teaching autonomy are reported in different ways in literature. According to (Schoenfeld, 2014a), cultural aspects determine teachers' autonomy. The author exemplifies the case of Finland, where teachers have a high level of support and trust from the community, which allows them to maintain control of the curriculum. In Japan, teachers write guides and manuals based on the results of their Lesson Studies (Isoda, 2010). On the other hand, Kin et al. (2019) argue that for teachers to develop their ability to teach using problem solving there must be greater investment. Teacher autonomy must be supported by conditions that ensure their professional development, such as adequate salaries, planning time, time for collaborative work and opportunities for professional growth (Schoenfeld, 2014a). The curriculum can also support teachers, providing clear guidance on what is expected about problem solving and how to work it (Anderson, 2014).

Another important aspect refers to the care that should be put to the influence of external evaluations, especially in countries where they permeate what is taught and valued in the classroom (Burkhardt, 2014). For Doorman et al. (2007) this is a great difficulty due to the scarce presence of non-routine problems in these tests. Di Martino and Signorini (2019) suggest some causes for the influence of standardized evaluations on the political decisions that determine the curriculum. Among these are the numerical results that provide immediate and comparable information, which, however, ends up having a high impact on what is considered relevant in didactic terms.

Finally, in literature it is emphasized that the system should promote conditions for implementing the curriculum with flexibility. Curricular documents must provide sufficient opportunities for interpretation and redesign according to the characteristics of the context and the students, without losing the depth of the initial objectives (Leong et al., 2011; Quebec & Ma, 2018).

Characteristics of the planned curriculum and instructional design

After establishing the goals of mathematics education and having a curricular structure, the next level of enactment corresponds to the intended curriculum. A first approach to the intended curriculum is manifested through the planning of units and instructional design. Regarding this aspect, elements present in teaching guides, lesson plans from textbooks, or lesson designs made by teachers should be considered. One of these elements is the expected role of problem solving, explained above. Another element is the regular use of problems. This implies, first, that problem solving should be used for teaching and learning regular content (Lester & Cai, 2016). Secondly, its use should serve even to develop specific or basic skills (Cai, 2015; NCTM, 2000). Third, students should be able to develop skills that can only be achieved in the long term (van Zanten

& van den Heuvel-Panhuizen, 2018).

For students to learn regular content through problem solving, learning experiences must be carefully designed (Anderson, 2014; Lester, 1994). Curriculum planning has generally been ignored as a factor in the teaching of problem solving, but it is an important aspect in order to improve learning (Guven, Aydin-Guc, & Ozmen, 2016; Lester, 2013). Furthermore, the sequence of planned content must be coherent across units and school years. A clear and coherent sequence allows teachers to know what prior knowledge students would have at the moment of facing a problem and anticipate their reasoning (Fujii, 2016, 2018). It also allows them to know what other concepts a problem might connect to (Fujii, 2016, Lester & Cai, 2016). In addition, teachers need to see how problems logically fit into the treatment of units and appreciate student progress across levels (Fujii, 2016, Leong et al., 2016).

Researches focused on the approach of teaching through problem solving show a common aspect that is the careful selection of tasks according to their convenience to achieve the learning objectives. This aspect usually appears as the responsibility of teachers and relates to opportunities for professional development (Fujii, 2018; Lester & Cai, 2016). However, curricular materials must also offer examples of well-designed lessons, where tasks are consistent with the objectives and centered on one main idea (Fujii, 2018; NCTM, 2000).

The curriculum should also anticipate how to carry out work with the class as a whole. Beyond the individual solving processes, teachers need to visualize how to work on problem solving with the whole class, whether students work in small groups, or individually (Godin, 2018; Lester, 2013). Tasks should generate opportunities for all students to participate, regardless of the complexity of the mathematics they work with (Schoenfeld, 2014b).

Literature also refer to the importance to anticipate students' thinking (Quebec & Ma, 2018). One of the roles of the teacher is to predict what will be the students' reasoning about different aspects: typical solutions, incorrect solutions, types of strategies, among others (Fujii, 2018; Isoda, 2015). Instructional materials should provide aids to anticipate what would be the reasoning of students with certain tasks (Remillard, Harris, & Agodini, 2014). By anticipating possible ways of thinking, aids can be designed to allow students to advance in understanding and to commit themselves to the development of the task (Peter Liljedahl, 2019; Schoenfeld, 2014a).

Another characteristic of instructional design widely mentioned is the use of steps proposed by Polya. Polya (1945) suggested a four-step model: understanding the problem, designing a plan, carrying it out and looking back. These steps, especially the first two, could be useful to help students start up their solving processes (Lee, 2014). The fourth step can be used for students to reflect on the mathematical content (Toh et al., 2019). However, there are also more critical points of view. Lester (1994), for example, indicates that until that date there was no conclusive evidence of the contribution of the use of the four steps. Nowadays a great number of researches continue showing enthusiasm for their use, although the results that they report do not focus specifically on their effectiveness (Chanudet, 2019; Kin et al., 2019).

Finally, in this category, multiple studies agree on the contribution of technology use (Chapman, 2015). Spreadsheets and programming languages provide an environment for exploring solutions (Burkhardt, 2014). New technologies also allow new ways of interpreting problems and transforming the meanings of mathematical concepts, due to their potential for representation and possibilities for manipulation (Santos-Trigo & Moreno-Armella, 2016).

Types of problems in the intended curriculum and instruccional materials

In literature, we can find diverse kinds of problems, according to different criteria and to each branch of mathematics. Zhu and Fan (2006), in their studies on problem solving in textbooks, propose a general classification that includes non-routines vs. routine problems, non-traditional vs. traditional problems, open-ended vs. closed-ended problems, problems applicable to the real world vs. problems without application, and problems with sufficient, insufficient or extraneous data to be solved. Through literature, some of these are promoted more intensely, especially those that seem more apt for the 'teaching through problem solving' approach. To this approach, the most important consideration is the use of non-routine problems, since it is an indispensable requirement to solve real problems. If students have to remember a method previously taught to solve a question, then it becomes a simple exercise (Burkhardt, 2014; Safrudiannur & Rott, 2018).

There is less agreement in the type of situations that problems should address. Some authors appeal to the use of real-life situations, the social and natural world (Bostic, Pape, & Jacobbe, 2016; Pang, 2014), the immediate environment of students (Isoda & Olfos, 2009), and contexts from other scientific disciplines (English & Sriraman, 2010). Others suggest paying more attention to how problematic the tasks are instead of the type of situation they address (Mason, 2016).

General characteristics of the implemented tasks

In literature, there is a distinction between the concepts of task, problem, and activity. A task constitutes what is demanded to the students, that mobilizes their knowledge about a mathematical topic and stipulates specific objectives in terms of actions, while an activity would be the student' response to this demand (Lupiáñez, 2009). A task becomes a problem when for the student it results 'problematic' (Mason, 2016).

For students to face problematic tasks, it is important that these can be addressed in multiple ways (Lester & Cai, 2016; Santos-Trigo, 2019). To get all students to learn, a problem must allow them to test several approaches, and evaluate the advantages and disadvantages of each one (Fujii, 2016). A task of this type can be addressed using several representations, different mathematical relations, or various solving arguments (Leikin, 2011).

A good math lesson should also allow students to discuss and establish connections between concepts and procedures, as well as provide opportunities to learn, apply, and connect important mathematical ideas (Lester & Cai, 2016; Schoenfeld, 2014b). Thus, teachers must lead classes taking advantage of the potential of the problems to generate valuable mathematical knowledge (Burkhardt, 2014; Fujii, 2016).

This kind of lessons needs to have adequate levels of cognitive demand (Lester & Cai, 2016). Schoenfeld (2014b, p. 407) defines the cognitive demand of a classroom as 'the extent to which classroom interactions create and maintain an environment of productive intellectual challenge that is conducive to students' mathematical development'. The higher the demand, the greater the gain in learning (Lester & Cai, 2016). However, researchers recommend maintaining a balance between tasks of high levels of cognitive demand and the students' current skills (Fujii, 2018; Schoenfeld, 2014b).

Finally, problem posing is highlighted as an inseparable part of problem solving (Leung, 2016). Problem posing can have a positive impact on students learning, in terms of their ability to solve problems, the development of thinking, flexibility and their attitudes and confidence (Chapman, 2015). Some studies describe a relationship between the inclusion of problem posing tasks and high levels of cognitive demand

(Cai, Jiang, Hwang, Nie, & Hu, 2016). In addition, some research has shown that it can be used as a strategy to detect and work with students with mathematical talent (Espinoza, Lupiáñez, & Segovia, 2016).

Class management

As for the enacted curriculum, literature agrees on a set of conditions that facilitate implementing problem solving successfully in lessons. First, it is important the role of problem solving, which is closely linked to the teaching method. When teaching focuses only on mastering concepts and procedures, students do not develop higher order skills that are necessary to use mathematics (Schoenfeld, 1985). According to Liljedahl's research (2019), in traditional lessons, focused on the direct transmission of content, activities of the type 'now you try' are usual. In them, teachers first show a procedure and then ask the students to apply it on their own. The purpose would be to verify if the students have understood. On the contrary, through a discovery method, students are encouraged to explain their reasoning, privileging the students' argumentation before the teacher's explanations (Toh et al., 2019).

The learning environment is one of the characteristics that distinguishes traditional teaching from problem-based teaching (Schoenfeld & Kilpatrick, 2008). In a problem-based lesson, the teacher along with the students are responsible for maintaining a favorable environment for exploration, open to sharing ideas and taking risks (Lester, 2013). In addition, conditions must be created so that students feel safe sharing their conjectures, and feel that it is not wrong to make mistakes (Mason, 2016).

Time management is another key factor. To solve challenging and mathematically rich problems, a considerable amount of time is needed (Anderson, 2014; Lester, 2013). The time available for problem solving should consider the discussion of student strategies, giving the same relevance in terms of time and interest, both to the solutions and processes (Lester & Cai, 2016; Pang, 2014).

Evaluation

Compared to the previous categories, the literature refers little to aspects of the evaluated curriculum that favor problem solving. One of the accepted points is that problem solving must be part of the students' assessment (Chapman, 2015; Schoenfeld, 2014a). Leong et al. (2016) indicate that the lack of success in the problem solving approach is because it is not usually evaluated. For this reason, the solving process loses importance and students prefer to pay attention to contents that are actually evaluated. Regarding the way in which the evaluation is implemented, it cannot be reduced to the application of tests or summative works, but the teacher must evaluate at all times the performance of the students in order to support them (Chanudet, 2019).

The second point related to the evaluation is the support that the official curriculum can give to teachers to evaluate problem solving. The curriculum must provide sufficient guidance to focus evaluation on thinking processes, beyond solutions, to adapt teaching accordingly to students' learning (Di Martino & Signorini, 2019; O'Shea & Leavy, 2013; Quebec & Ma, 2018). Students' reasoning can be accessed through interviews, questions during lessons, use of worksheets, incorporation of problem posing tasks, among others (Cai & Hwang, 2019; Godin, 2018).

Considerations about students

This and the next category, instead of levels of curricular implementation, describe some aspects that literature recommends to consider when enacting the curriculum. For the students, in the first place, it is convenient to take into account their particular characteristics. It is fundamental to know their conceptions and previous learning, since these determine if for them the proposed tasks could be problematic or not (Toh, Leong, Dindyal, & Quek, 2010). Schoenfeld (1985) calls ‘resources’ the body of knowledge that a person uses when solving a problem. It is also important to pay attention to the diversity of students, whether they are students with learning difficulties or with special talents (Castro, 2008).

Students' attitudes and beliefs toward problem solving are equally relevant. The belief system is a central part of the framework proposed by Schoenfeld (1985), although he later calls it ‘orientations’ to include values, preferences, and tastes (Schoenfeld, 2013). One of the most widespread beliefs reported is that a problem has only one correct answer. This generates a vision of mathematics as something that does not require creativity or greater intellectual activity, but only memorization (Isoda & Olfos, 2009). As a counterpoint, there are studies that show that this type of beliefs can be modified, for example, by changing traditional teaching to the approach of ‘teaching through problem solving’ (Lester & Cai, 2016).

In addition, teachers need access to information on metacognitive processes, since implementing the problem solving approach requires students to put these processes into practice (Chapman, 2015; Quebec & Ma, 2018). According to Lester (1994), in relation to metacognition, during problem solving, at least three elements have been accepted in the community of researchers:

- An effective metacognitive activity requires the solver to know what, when and how to monitor learning;
- The development of metacognition skills is more effective during the learning of specific concepts than in the context of their teaching in general;
- The development of metacognition skills is hard and requires unlearning inappropriate behaviors.

Metacognition is considered a fundamental part of problem solving performance, although it has been largely ignored in favor of other aspects such as the learning of heuristic skills (Kilpatrick, 2016).

The strategies and heuristics used to solve problems also depend on students. In literature, this is a controversial aspect. Chapman (2015) mentions that direct teaching of heuristics has not been shown as to have a significant impact on learning. However, it also highlights the need for teachers to have knowledge about heuristics to understand, thus, the reasoning processes of their students. Other authors reach the same conclusion (Lester, 1994, Lester & Cai, 2016). Schoenfeld (1985) is also careful in pointing out that students must access the use of heuristics during their learning years, but that it is wrong to think that they can be taught to elementary students in the sense given in Polya's text. Van Zanten and van den Heuvel-Panhuizen (2018) even indicate that the direct teaching of heuristics makes them become more rules to follow, causing problem solving to lose its potential for discovery. Other authors are more moderate in suggesting that the success of teaching heuristics depends on how they are used (O'Shea & Leavy, 2013). English and Sriraman (2010) state that the failure can be due to the little prescriptive and descriptive power of the lists of heuristics to which teachers have access. In summary, the treatment given to the use of heuristics in the curriculum should be careful. It is better to provide opportunities for the use of strategies naturally through the contents of the curriculum than to teach them as a content in itself (NCTM, 2000).

Teacher's roles during problem solving

Through the curriculum in any of its stages of enactment, a key element is to make explicit and clear the role of teachers within problem solving activities (Lester, 2013). During lessons, the teacher must decide when and how to intervene to ensure that the students succeed in their strategies, but without directly giving the answers (Chapman, 2015). The teacher must also motivate students to find complete solutions and give them credit for their contributions. To achieve this it will be necessary to develop their own listening and observation skills (Lester, 2013; Lester & Cai, 2016).

Teachers' attitudes and beliefs about problem solving are a relevant issue because of the impact they have on a student’s performance (Chapman, 2015; Lester & Cai, 2016). For students to have the opportunity to become successful solvers, they must feel that the teacher believes that problem solving is important (Lester, 1994). Among the elements highlighted by Lester (2013) in this aspect are emotions, beliefs, attitudes, metacognition that affects behavior during lessons, among others. Schoenfeld and Kilpatrick (2008) also highlight that teachers' awareness of their own theories about teaching affects activities and interaction with students during lessons.

Finally, the teacher should be aware of the nature and purpose of the proposed problems to guide the students in their solving process, so the curriculum should help in this regard (Chapman, 2015). Some aspects to consider are: the type of problem, the language used, the number of steps to solve them and their mathematical structure (Castro, 2008; Chapman, 2015).

Discussion

The categories generated by the qualitative review correspond to characteristics of the problem solving in the curriculum, which have been more prominent in literature. They have been organized in table 1.

Table 1. *Characteristics of problem solving in the curriculum: categories and subcategories.*

Stage	Categories	Subcategories
Official Curriculum	Curriculum structure	Centrality of problem solving in the curriculum Role of the problem solving in curricular regulations Reasoning-oriented curriculum
	Conditions of the educational system	Conditions for the autonomy and professional development of teachers Attention to the influence of external assessments Conditions for curriculum implementation: place for flexibility
Intended curriculum and instruccional materials	Characteristics of the planned curriculum and instructional design	Role of the problem solving in planned curriculum and lessons design Regular use of problem solving Convenience of the tasks for the achievement of the curriculum objectives. Representation of the problem solving process in the context of working with the whole class Anticipation of student thinking. Contributions of the use of the 4 Polya's steps Consideration of the use of ICT
	Types of problems in intended curriculum and instructional materials	Promoting non-routine problems Promoting non-traditional problems Promoting open-ended problems Addressing different types of situations Promoting sufficient data problems as well as extraneous data problems, and insufficient data problems
Enacted Curriculum	Characteristics of the implemented tasks	Multiple ways of solving or representing Generating and connecting mathematics Adequate level of cognitive demand Inclusion of problem posing tasks
	Class management	Role of problem solving during the implementation of lessons Teaching method: lessons focused on reasoning Learning environment Time to the discussion of strategies
	Evaluation	PS as a component of the evaluation Attention to thinking processes
Agents of the process	Considerations about students	Characteristics of students Students' attitudes and beliefs Metacognition Awareness of 'problematicity' Strategies and heuristics used by students
	Teacher roles during problem solving	Forms of action during problem solving Observation and listening skills Teacher attitudes and beliefs Consideration of the nature of problems

The analysis of documents also revealed a set of principles that support implementing problem solving. We understand it as a series of characteristics attributed to the curriculum and that without their presence would be difficult to implement problem solving successfully. Since problem solving has a complex nature (Lester & Cai, 2016), these precepts should not be taken as rules to be applied, but rather as a basis to work with. The proposed principles emerged from our analysis are the following: understanding, reasoning, autonomy, collaboration, and affective aspects.

Understanding

Firstly, it is incorrect or at least insufficient thinking that the primary goal of the mathematics curriculum is solving problems. We saw that even in problem solving-centered curricula, a narrow view still applies. Seen in this way, even the teaching of strategies and heuristics can be understood as a set of procedures that must be memorized (van Zanten & van den Heuvel-Panhuizen, 2018). Instead, the main goal of the mathematics curriculum seems to be understanding. This is the only method to change the vision to one where mathematics is acknowledged as a way of thinking and organizing the world (Schroeder & Lester, 1989).

Reasoning

Once understanding is assumed as the main objective of the mathematics curriculum, the means to reach understanding is reasoning (Quebec & Ma, 2018). The curriculum must be oriented to reasoning instead of memorization. Here is where problem solving finds its main role (if not the only one): as a means to develop higher mental processes. In the official curriculum, guidelines should be clear about what is expected of mathematics and problems. The intended curriculum and instructional materials should be aligned in this regard (Burkhardt, 2014). Concerning the enacted curriculum, teachers must have the skills and flexibility to understand the reasoning of their students, as well as the knowledge to be able to interpret the curriculum and adapt it to the context.

Autonomy and collaboration

Implementing problem solving requires high levels of autonomy and collaboration. Students need autonomy to seek solutions and methods, supported by the careful guidance of teachers, without restraining their reasoning (O'Shea & Leavy, 2013). Collaboration is essential if we accept that we learn from interactions with others (Fujii, 2018; Mason, 2016). As for teachers, the curriculum should encourage instances of autonomy, an indispensable requirement to design lessons according to their particular students (Quebec & Ma, 2018). Collaboration turns necessary to create communities that facilitate professional development, discussion, observation and reflection (Doorman et al., 2007). Autonomy and collaboration are two inseparable aspects, which can be seen in experiences where problem solving is successfully addressed.

Affective aspects

Finally, affective aspects are one of the weightiest, and most difficult to change (Chapman, 2015; Schoenfeld, 1985). Both students and teachers, and even curriculum designers bring with them beliefs, attitudes, and values, which influence the implementation of problem solving, even more than other purely cognitive factors (Schoenfeld, 2013). Then, the curriculum should facilitate that the proposed tasks become desirable problems to face, that awaken students' and teachers' emotions to impulse the cognitive activity and enjoy experiences of delight and amazement (Isoda, 2010; Mason, 2016).

Conclusion

After decades of studies on problem solving, it is possible to say that now we have a body of knowledge that reflects important advances on the subject. The most outstanding advances are related with particular aspects of problem solving. Some of these focus on the characteristics of the agents of the educational process, such as the cognitive and metacognitive processes in which solvers are involved, how affective factors influence students and teachers, and what characterizes successful solvers.

Others are related to the process of solving problems, such as the use of ideal solving steps, how students use strategies and heuristics, the type of tasks implemented or their level of cognitive demand. In recent times, there have been progress in elements of the context that must be considered in order to lead the teaching of problem solving with successful results. For instance, the roles that problem solving has in the curriculum, the result of different class configurations, and the influence of certain evaluations.

All of these aspects have an impact on the teaching of problem solving, and therefore, on the curriculum. However, as knowledge advances, other pending issues are now emerging. Among these is the study of the presence of problem solving in the curriculum in a more global and in-depth way. It is needed more research on how different elements interact at each level of curriculum implementation. It is also necessary to visualize how these elements and relationships are expressed in natural contexts and in a non-isolated way. If it were possible to understand how each part takes place while curricular designers, publishers, administrators, and teachers do their work, it could be easy to take advantage of all the research that has been done so far, applied to the educational field. These are points that still need to be addressed.

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