The Daugavet equation for polynomials on C*-algebras and JB*-triples

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A B S T R A C T

We prove that every JB*-triple E (in particular, every C*-algebra) satisfying the Daugavet property also satisfies the stronger polynomial Daugavet property, that is, every weakly compact polynomial $P\colon E \longrightarrow E$ satisfies the Daugavet equation $\|\operatorname{Id}_X + P\| = 1 + \|P\|$. The analogous conclusion also holds for the alternative Daugavet property.

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1. Introduction

A Banach space X satisfies the *Daugavet property* (*DPr* in short) if the norm equality

$$\| \operatorname{Id}_X +T\| = 1 + \|T\| \tag{DE}$$

holds for all rank-one bounded linear operators $T\colon X \longrightarrow X$ (equivalently, for all weakly compact linear operators on X, [37]). Basic examples of Banach spaces with the Daugavet property include $C(K)$ spaces when K is perfect, $L_1(\mu)$ spaces when μ is atomless, uniform algebras whose Choquet boundary is perfect, and isometric preduals of $L_1(\mu)$ spaces for which the set of extreme points of the dual ball is weak-star perfect up to rotations, among many others. We refer the interested reader to the seminal paper [37] and to the more recent contributions [32,44,48,59] and references therein for more information and background.

A very related property to the DPr is the following: A Banach space X satisfies the *alternative Daugavet property* (*ADP* in short) if the norm equality

$$\max_{|w|=1} \| \operatorname{Id}_X +wT\| = 1 + \|T\| \tag{aDE}$$

holds for all rank-one bounded linear operators $T\colon X \longrightarrow X$ (equivalently, for all weakly compact linear operators on X, [47, Theorem 2.2]). Here, the basic examples are $C(K)$ and $L_1(\mu)$ for all compact spaces K and positive measures μ, all uniform algebras, and all isometric preduals of $L_1(\mu)$, among many others. This property was formally introduced and deeply studied in [47] (some related ideas had appeared before). We also refer the reader to [4,36,45] for instance, for more information, background, and for the relationship between the ADP and the study of the numerical index of Banach spaces.

As this manuscript will deal with C*-algebras and JB*-triples, it makes sense to present the characterizations of the DPr and the ADP for these spaces given in [11,45, 47,49]. We refer to Section 3 for the definition of the involved concepts.

Theorem 1.1 ([11,45,47,49]). (a). *Let X be a C*-algebra. Then the following statements hold:*

(a1) *X has the DPr if and only if X is diffuse (i.e., it contains no atomic projections).*
(a2) *X has the ADP if and only if every atomic projection is central.*

(b). *Let X be a JB*-triple. Then the following assertions hold:*

(b1) *X has the DPr if and only if X contains no minimal tripotent.*
(b2) *X has the ADP if and only if every minimal tripotent is diagonalizing.*

This manuscript is devoted to study the natural extensions of the DPr and the ADP for polynomials. Let us introduce some notation. Even though both the DPr and the ADP have sense for both real and complex spaces, we will only deal with complex Banach spaces in this paper, as our main interest will be (complex) C*-algebras and JB*-triples. The closed unit ball of a Banach space X will be denoted by $\mathcal{B}_X$. Let X and Y be Banach spaces. A 0-*homogeneous polynomial* from X to Y is just a constant mapping. For $N \geq 1$, a (continuous) N-*homogeneous polynomial* P from X to Y is a mapping $P \colon X \longrightarrow Y$ for which we can find an operator $T \colon \overbrace{X \times \ldots \times X}^{N} \longrightarrow Y$ (continuous) multilinear and symmetric (i.e., $T(x_1, \ldots, x_N) = T(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ for every permutation σ of the set $\{1, \ldots, N\}$) satisfying $P(x) = T(x, \ldots, x)$ for every $x \in X$. According to the usual notation, the symbol $\mathcal{P}(^N X, Y)$ will stand for the space of continuous N-homogeneous polynomials from X to Y. Given $P \in \mathcal{P}(^N X, Y)$, we shall write $\widehat{P}$ for the unique continuous symmetric N-linear mapping associated with P. The scalar N-homogeneous polynomials on X (i.e., $\mathcal{P}(^N X, \mathbb{C})$) will be simply denoted by $\mathcal{P}(^N X)$. A (general) polynomial from X to Y is a mapping $P \colon X \longrightarrow Y$ which can be written as a finite sum of homogeneous polynomials. We shall write $\mathcal{P}(X, Y)$ for the space of all polynomials from X to Y, and $\mathcal{P}(X)$ for $\mathcal{P}(X, \mathbb{C})$. We say that $P \in \mathcal{P}(X, Y)$ is weakly compact when the closure of $P(\mathcal{B}_X)$ is weakly compact. We will consider $\mathcal{P}(X, Y)$ endowed with the usual supremum norm

$$\|P\| = \sup\{\|P(x)\| \colon \|x\| \leq 1\}.$$

Using this norm, it makes sense to consider the equations (DE) and (aDE) in the space $\mathcal{P}(X, X)$ and hence to consider, as it is done in [16,17,62], the following two properties: We say that X has the *polynomial Daugavet property* [16,17] if every weakly compact $P \in \mathcal{P}(X, X)$ satisfies (DE). X has the *alternative polynomial Daugavet property* if every weakly compact $P \in \mathcal{P}(X, X)$ satisfies (aDE). It is immediate that the polynomial versions of the Daugavet and the alternative Daugavet properties imply the usual ones, respectively.

It is an open problem whether the Daugavet property implies the polynomial Daugavet property, but it is known that this is the case in some families of Banach spaces as $C(K)$ spaces and some generalizations [16,17], for $L_1(\mu)$ spaces and for vector-valued L_1 spaces [46], and also for isometric preduals of $L_1(\mu)$, for uniform algebras and for spaces of Lipschitz functions [48], among others. For the case of the ADP, the situation is different as, for instance, the complex space ℓ_1 fails the alternative polynomial Daugavet property [16] (despite of the fact that ℓ_1 has the ADP). It is however known that every complex $C(K)$ space has the alternative polynomial Daugavet property [16]. The real case is even worse: real c_0 and real ℓ_1 fail the alternative polynomial Daugavet property [16]. E. Santos established in [61] that every continuous polynomial of finite type or approximable on a JB*-triple having the DPr (respectively, the ADP) satisfies the Daugavet equation (DE) (respectively, the alternative Daugavet equation (aDE)).

The main aim of this paper is to show that for C*-algebras and JB*-triples, the DPr implies the polynomial Daugavet property and the ADP implies the alternative polynomial Daugavet property.

Theorem 1.2 *(Main result). Let X be a JB*-triple (in particular, a C*-algebra). Then the following statements hold:*

(a) *If X has the DPr, then it has the polynomial Daugavet property.*
(b) *If X has the ADP, then it has the alternative polynomial Daugavet property.*

It should be noticed that this result provides a complete positive solution to the problems posed by E. Santos in [61, §4].

The proofs of these two statements will appear in Theorem 4.1 and Theorem 4.2, respectively, where we actually rediscover a new proof of Theorem 1.1.

Let us comment that the main idea to get the polynomial Daugavet property in $C(K)$ spaces and in some related classes of Banach spaces is to produce a number of "good" c_0-sequences in the space (or in the bidual), and then use that continuous scalar polynomials in c_0 are weakly continuous on bounded sets. This idea does not seem to work for noncommutative C*-algebras nor for general JB*-triples. Therefore, we need to find a suitable substitute of it, and the alternative tool is the sequential continuity of scalar polynomials on JB*-triples (and so on C*-algebras) for the strong*-topology (see Corollary 3.7), a result which is of interest by itself. The remaining strategy to show that JB*-triples having the DPr and the ADP actually satisfy their polynomial versions consists in extending and improving ideas from the original proofs given in [47,49] for the linear properties. There is however a substantial turn here, we do not rely on the arguments devoted to control the norm of a determined finite set of linear functionals when restricted to the Peirce-2 and Peirce-1 subspaces of an appropriate tripotent in the bidual, instead of that we shall show that the joint strong*-continuity of the triple product in the bidual spaces simplifies the computation of the norms for scalar polynomials via pointwise convergence.

The outline of the paper is as follows: Section 2 is devoted to revisit some known tools and results to study the Daugavet equation for polynomials on a Banach space X which are weakly continuous on bounded sets. We prove that, assuming that X satisfies the DPr, every weakly compact polynomial on X which is weakly continuous on bounded sets satisfies the Daugavet equation (see Theorem 2.3). Section 3 contains a very brief introduction to JB*-triples together with some basic tools and results required in our arguments (like the strong*-topology and its main properties). We shall also revisit the results guaranteeing that every scalar polynomial on a general JB*-triple is sequentially strong*-continuous. Finally, section 4 is devoted to present the proofs of the main results.

2. Revisiting the Daugavet equation for polynomials

Our goal here is to recall some basic facts on the polynomial Daugavet property and the alternative polynomial Daugavet property which will be useful in the sequel. We start with characterizations of both properties in terms of rank-one polynomials.

We begin with a result for the polynomial Daugavet property.

Proposition 2.1 *([17, Theorem 1.1 or Corollary 2.2]). Let X be a Banach space. Then, the following are equivalent:*

(a) *X has the polynomial Daugavet property (i.e., every weakly compact polynomial on X satisfies (DE)).*

(b) *Every $P \in \mathcal{P}(X, X)$ of the form $x \longmapsto p(x)a$ for suitable $p \in \mathcal{P}(X)$ and $a \in X$, satisfies (DE).*

(c) *Given $p \in \mathcal{P}(X)$ with $\|p\| = 1$, $a \in X$ with $\|a\| = 1$, and $\varepsilon > 0$, there are $x \in \mathcal{B}_X$ and $w \in \mathbb{C}$ with $|w| = 1$ such that*

$$\operatorname{Re} wp(y) > 1 - \varepsilon \quad and \quad \|a + wx\| > 2 - \varepsilon.$$

The analogous result for the alternative polynomial Daugavet property also holds.

Proposition 2.2 *([17, Corollary 1.2]). Let X be a Banach space. Then, the following are equivalent:*

(a) *X has the alternative polynomial Daugavet property (i.e., every weakly compact polynomial on X satisfies (aDE)).*

(b) *Every $P \in \mathcal{P}(X, X)$ of the form $x \longmapsto p(x)a$ for suitable $p \in \mathcal{P}(X)$ and $a \in X$, satisfies (aDE).*

(c) *Given $p \in \mathcal{P}(X)$ with $\|p\| = 1$, $a \in X$ with $\|a\| = 1$, and $\varepsilon > 0$, there are $x \in \mathcal{B}_X$ and $w \in \mathbb{C}$ with $|w| = 1$ such that*

$$|p(x)| > 1 - \varepsilon \quad and \quad \|a + wx\| > 2 - \varepsilon.$$

As we already mentioned in the introduction, a strategy to prove the polynomial Daugavet property in some Banach spaces like $C(K)$ spaces, uniform algebras, or isometric preduals of L_1-spaces is to show that these spaces (or their biduals) are full of "good copies" of c_0, and then use the fact that scalar polynomials on c_0 are weakly continuous on bounded sets (see [17, Proposition 6.3] and [48, Proposition 4.3]). Of course this strategy cannot be used in all Banach spaces with the DPr (for instance, $L_1[0, 1]$ contains no copies of c_0). Here we can get a partial result showing that in a Banach space with the DPr, weakly compact polynomials which are weakly continuous on bounded sets satisfy (DE). As far as we know, this result is new.

Theorem 2.3. *Let X be a Banach space with the DPr. Then, given $p \in \mathcal{P}(X)$ which is weakly continuous on bounded sets and satisfies $\|p\| = 1$, $a \in X$ with $\|a\| = 1$, and $\varepsilon > 0$, there are $x \in \mathcal{B}_X$ and $w \in \mathbb{C}$ with $|w| = 1$ such that*

$$\operatorname{Re} wp(x) > 1 - \varepsilon \quad \text{and} \quad \|a + wx\| > 2 - \varepsilon.$$

As a consequence, every weakly compact $P \in \mathcal{P}(X, X)$ which is weakly continuous on bounded sets satisfies (DE).

We will make use of [63, Lemma 3] for which only the following weaker version will be needed.

Lemma 2.4 *([63, Lemma 3]). Let X be a Banach space with the DPr. Then, for every $y \in \mathcal{B}_X$ with $\|y\| = 1$ and every $\varepsilon > 0$, the set*

$$\{x \in \mathcal{B}_X \colon \|y + x\| > 2 - \varepsilon\}$$

is weakly dense in $\mathcal{B}_X$.

Let us observe that the lemma actually provides a characterization of Banach spaces with the Daugavet property, since the reciprocal implication is immediate by the well known characterization of the DPr using slices (see [37, Lemma 2.1]).

Proof of Theorem 2.3. Take $w \in \mathbb{C}$ with $|w| = 1$ such that $\sup_{x \in \mathcal{B}_X} \operatorname{Re} wp(x) > 1 - \varepsilon$, and observe that p being weakly continuous on $\mathcal{B}_X$ implies that the set

$$U := \{x \in \mathcal{B}_X \colon \operatorname{Re} wp(x) > 1 - \varepsilon\}$$

is non-empty and weakly open relative to $\mathcal{B}_X$. Now, Lemma 2.4, applied to $\bar{w}a$, assures that there is $x_0 \in U$ such that $\|\bar{w}a + x_0\| > 2 - \varepsilon$. It clearly follows that

$$\operatorname{Re} wp(x_0) > 1 - \varepsilon \quad \text{and} \quad \|a + wx_0\| > 2 - \varepsilon.$$

To prove the last statement in the theorem, we just have to follow the lines of the proof of $(c) \Rightarrow (a)$ in Proposition 2.1 given in [17, Theorem 1.1], and observe that in order to prove that every weakly compact and weakly continuous on bounded sets $P \in \mathcal{P}(X, X)$ satisfies (DE), we only need to check that (c) holds for scalar polynomials which are weakly continuous on bounded sets, that is, we only need what we already proved in the first part of this proposition. $\square$

Some remarks are worth mentioning.

Remarks 2.5.

- Theorem 2.3 cannot be applied to show that C*-algebras or JB*-triples with the DPr have the polynomial Daugavet property. Actually, it cannot be used to get the polynomial Daugavet property from the DPr in any Banach space. Indeed, every Banach space with the DPr contains an isomorphic copy of ℓ_1 [37, Theorem 2.9] and it is known that every Banach space containing ℓ_1 admits a continuous scalar polynomial which is not weakly continuous on bounded sets (cf. [23, Proposition 2.36 and its proof]).

- In case that every continuous scalar polynomial on X is weakly sequentially continuous (as happens in Banach spaces with the Dunford-Pettis property, see [23, Proposition 2.34]), then a "sequential" version of Lemma 2.4 for such an X (i.e., weak sequential denseness instead of weak denseness) would be enough to get the polynomial Daugavet property from the DPr on X. We do not know if such a sequential version of Shvidkoy lemma is true for C*-algebras or JB*-triples, but it is certainly not true for all Banach spaces with the DPr: there is a Banach space satisfying both the DPr and the Schur property [38].

- It was proved in [61, Theorem 3.5] that when X is a C*-algebra or a JB*-triple and X has the DPr, then the polynomials of finite-type on X satisfy (DE). Recall that a polynomial $P \in \mathcal{P}(^N X, Y)$ is said to be of *finite type* if there exists a finite subset $\{\varphi_i\}_{i=1}^m \subset X^*$ and elements y_i's in Y such that $P(x) = \sum_{i=1}^m y_i \varphi_i(x)^N$ $(x \in X)$. It is clear that polynomials of finite type are weakly continuous (in particular, weakly continuous on bounded sets), so the cited result [61, Theorem 3.5] actually follows from our Theorem 2.3 and it is indeed true for all Banach spaces with the DPr, not only for C*-algebras and JB*-triples.

- The analogous result to Theorem 2.3 for the ADP and the polynomial alternative Daugavet property is not true. Indeed, the (real or complex) two-dimensional space ℓ_1^2 has the ADP [47], but fails the polynomial alternative Daugavet property (see Example 3.12 and the paragraph below Remark 3.13 in [16]). Being finite-dimensional, it is immediate that all polynomials on ℓ_1^2 are weakly continuous (on bounded sets). By the way, this shows that a result analogous to Shvidkoy's Lemma 2.4 for the ADP cannot be true.

3. A closer look at the theory of C*-algebras and JB*-triples

This section is not only devoted to summarize the basic definitions of JB*-triples, but to revisit and adapt the geometric and topological tools required in our arguments. We shall also rediscover and extend some geometric tools employed in previous references from a quite different point of view.

There are several worthy arguments to consider and study the complex Banach spaces in the category of JB*-triples. First, the class of JB*-triples is strictly wider than the class

of C*-algebras, and in many cases the geometric properties are easier affordable from this more general point of view. Perhaps the most important motivation to introduce JB*-triples arises in holomorphic theory and the classification of bounded symmetric domains in arbitrary complex Banach spaces, domains that play the role of simply connected domains in the celebrated Riemann mapping theorem (cf. [41]). Let us briefly recall the definition. A complex Banach space E is called a *JB*-triple* if it admits a continuous triple product $\{\cdot, \cdot, \cdot\} : E \times E \times E \longrightarrow E$, which is symmetric and bilinear in the first and third variables, conjugate-linear in the middle one, and satisfies the following axioms:

(a) (Jordan identity)

$$L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$$

for a, b, x, y in E, where $L(a, b)$ is the operator on E given by $x \longmapsto \{a, b, x\}$;
(b) $L(a, a)$ is a hermitian operator with non-negative spectrum for all $a \in E$;
(c) $\|\{a, a, a\}\| = \|a\|^3$ for each $a \in E$.

The triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \tag{1}$$

provides a structure of JB*-triple for every C*-algebra and every closed subspace of $B(H)$ which is closed for the triple product just commented – in particular the space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces and all complex Hilbert spaces are JB*-triples. There exist JB*-triples which cannot be embedded as JB*-subtriples of $B(H)$, they are related to the so-called exceptional Cartan factors (cf. [28]). A JBW*-triple is a JB*-triple which is also a dual Banach space. In analogy with Sakai's theorem, every JBW*-triple admits a unique (isometric) predual and its triple product is separately weak* continuous (cf. [7]).

Let A be a C*-algebra. It is known that the fixed points of the triple product (1) are precisely the partial isometries in A (i.e., those $e \in A$ such $ee^*e = e$, equivalently, ee^* or e^*e is a projection). If we fix a partial isometry $e \in A$, we can easily decompose A in the form

$$A = ee^*Ae^*e + ((1 - ee^*)Ae^*e + ee^*A(1 - e^*e)) + (1 - ee^*)A(1 - e^*e),$$

decomposition which is known under the name of Peirce decomposition of A associated to e. Clearly, projections in A are nothing but those positive partial isometries in A.

More generally, if we have a JB*-triple E, the elements $e \in E$ satisfying $\{e, e, e\} = e$ are called *tripotents*. Each tripotent e in E produces a *Peirce decomposition* of the space E in terms of the eigenspaces of the operator $L(e, e)$:

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e), \tag{2}$$

where $E_k(e) := \{x \in E : L(e, e)x = \frac{k}{2}x\}$ is a subtriple of E called the *Peirce-k subspace* ($k = 0, 1, 2$). The natural projection of E onto $E_k(e)$ is known as the *Peirce-k projection*, and it is usually denoted by $P_k(e)$. We shall later employ that Peirce projections are all contractive (cf. [27, Corollary 1.2]).

The concrete algebraic expressions of Peirce projections in terms of triple products read as follows: for each $a, b \in E$ we write $Q(a, b)$ for the conjugate linear operator on E given by $Q(a, b)(x) := \{a, x, b\}$ ($x \in E$). The operator $Q(a, a)$ is simply denoted by $Q(a)$. According to this notation, $P_2(e) = Q(e)^2$, $P_1(e) = 2L(e, e) - 2Q(e)^2$, and $P_0(e) = \mathrm{Id}_E - 2L(e, e) + Q(e)^2$.

A tripotent e in a JB*-triple E is called *minimal* if e is non-zero and $E_2(e) = \mathbb{C}e$.

Triple products among elements in different Peirce subspaces obey certain laws known as *Peirce arithmetic*. Concretely, the inclusion $\{E_k(e), E_l(e), E_m(e)\} \subseteq E_{k-l+m}(e)$, and the identity $\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = \{0\}$, hold for all $k, l, m \in \{0, 1, 2\}$, where $E_{k-l+m}(e) = \{0\}$ whenever $k - l + m$ is not in $\{0, 1, 2\}$. The Peirce-2 subspace $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ_e y = \{x, e, y\}$ and $x^{*_e} = \{e, x, e\}$, respectively (cf. [15, Corollary 4.2.30]).

A tripotent e in a JB*-triple E is said to be *diagonalizing* if $E_1(e) = \{0\}$. Let us observe that a projection p in a C*-algebra A is diagonalizing as tripotent if and only if it is a central projection.

Projections p, q in a C*-algebra A are called orthogonal if $pq = 0$. If we consider general elements $a, b \in A$, according to the usual notation, we shall say that a and b are *orthogonal* ($a \perp b$ in short) if $ab^* = b^*a = 0$. It is known that $a \perp b$ if and only if $L(a, b) = 0$ (cf. [14, Lemma 1]). In a general JB*-triple E, elements a, b are called *orthogonal* if $L(a, b) = 0$. In case that a and b are tripotents, it is easy to see that $a \perp b$ if and only if $a \in E_0(b)$, if and only if $b \in E_0(a)$. The second identity in the so-called Peirce arithmetic precisely tells that $a \perp b$ whenever $a \in E_0(e)$ and $b \in E_2(e)$ for a tripotent $e \in E$. Another geometric property of orthogonal elements in JB*-triples assures that every two of orthogonal elements a, b in a JB*-triple are M-orthogonal, that is,

$$\|a \pm b\| = \max\{\|a\|, \|b\|\} \text{ (cf. [27, Lemma 1.3(a)]).} \tag{3}$$

There is a natural partial ordering among tripotents in a JB*-triple E given by $e \leq u$ if $u - e$ is a tripotent and $(u - e) \perp e$.

The Gelfand-Naimark-Segal construction is one of the best known results in representation theory of C*-algebras. One of the many ideas in this construction provides a tool to define Hilbertian structures associated with positive functionals in the dual of a C*-algebra. Concretely, if ϕ is a positive functional in the dual of a C*-algebra A, the mapping $(a, b) \longmapsto \phi(b^*a)$ defines a semi-positive sesquilinear form on A. The symmetric version of these sesquilinear form (i.e., $(a, b) \longmapsto \phi(\frac{b^*a + a^*b}{2})$) defines the pre-Hilbertian seminorm of the form $\|a\|_\phi^2 = \phi(\frac{a^*a + aa^*}{2})$ ($a \in A$) appearing in the non-commutative version of Grothendieck's inequalities by G. Pisier and U. Haagerup (cf. [56,30,57]).

Despite of the lacking of a positive cone in general JB*-triples, J. T. Barton and Y. Friedman established in [5, Proposition 1.2] the following procedure to define a pre-Hilbertian seminorm associated with a functional φ in the dual, E^*, of a JB*-triple E: for each $z \in E^{**}$ with $\|z\| = 1$ and $\varphi(z) = \|\varphi\|$, the mapping

$$E \times E \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto \varphi\{x, y, z\}$$

is a semi-positive sesquilinear form on E, which does not depend on the choice of the element $z \in E^{**}$. The corresponding prehilbertian seminorm on E is denoted by $\|x\|_\varphi^2 := \varphi\{x, x, z\}$ ($x \in E$). All pre-Hilbertian seminorms associated with positive functionals in the dual space of a C*-algebra A arise in this way, since for each positive functional ϕ we have $\phi(\mathbf{1}) = \|\phi\|$ for the unit element $\mathbf{1} \in A^{**}$, and hence $\|x\|_\phi^2 = \phi\{x, x, \mathbf{1}\} = \phi(\frac{x^*x + xx^*}{2})$. These pre-Hilbertian seminorms play a fundamental role in the results known as Grothedieck's inequalities for JB*-triples and in the definition of the strong*-topology which will be recalled later (cf. [6,31]).

It is known that

$$|\varphi(x)| \le \|\varphi\| \, \|x\|_\varphi$$

for all $\varphi \in E^*$, $x \in E$ (see [6, comments before Definition 3.1]).

Among the results we shall revisit here, we shall present a quantitative version of results stated by M. Martín and T. Oikhberg [47, Lemma 4.16] and E. Santos [61, Lemma 3.3]. We begin by recalling a result borrowed from [26], which offers a technical control of a functional on Peirce-1 and -2 subspaces associated with a tripotent at which the pre-Hilbertian seminorms of the functionals are "small".

Lemma 3.1. [26, Lemma 3.2] *Let e be a tripotent in a JB*-triple E, and let φ be a norm-one element in E^* such that $\|e\|_\varphi^2 < \delta$. Then, the inequalities*

$$|\varphi P_2(e)(x)| < 3\sqrt{\delta} \, \|P_2(e)(x)\|, \text{ and } |\varphi P_1(e)(x)| < 6\sqrt{\delta} \, \|P_1(e)(x)\|,$$

hold for all $x \in E$.

It is known, and easily deducible from the definition of orthogonality, that the square of each pre-Hilbertian seminorm behaves additively on orthogonal elements, that is, if $a_1, \ldots, a_n$ are mutually orthogonal elements in a JB*-triple E and $\varphi \in E^*$, we have $\left\| \sum_{j=1}^n a_j \right\|_\varphi^2 = \sum_{j=1}^n \|a_j\|_\varphi^2$. Our next goal is a simplified proof of [47, Lemma 4.16] and [61, Lemma 3.3], which additionally provides a quantitative conclusion. The desired statement is analyzed in the following remark.

Remark 3.2. For each $\varepsilon > 0$ and $k \in \mathbb{N}$ take a natural number n_0 satisfying $n_0 > \frac{12^2 k}{\varepsilon^2}$. Then, for every finite family $\{\varphi_1, \ldots, \varphi_k\}$ of non-zero functionals in the closed unit ball of the dual space of a JB*-triple E, and each finite family $\{e_1, \ldots, e_{m_0}\}$ of non-zero mutually orthogonal tripotents in E with $m_0 \geq n_0$, there is $m \in \{1, \ldots, m_0\}$ satisfying $\|\varphi_i|_{E_2(e_m) \oplus E_1(e_m)}\| = \|\varphi_i(P_2(e_m) + P_1(e_m))\| < \varepsilon$, for all $1 \leq i \leq k$.

Namely, let us fix $\varepsilon > 0$, $k \in \mathbb{N}$. The chosen $n_0 \in \mathbb{N}$ satisfies $\frac{k}{n_0} < \frac{\varepsilon^2}{12^2}$. Consider functionals and mutually orthogonal tripotents as in the statement with $m_0 \geq n_0$, and define the following pre-Hilbertian seminorm:

$$\|x\|_{\varphi_1, \ldots, \varphi_k}^2 := \sum_{j=1}^{k} \|x\|_{\varphi_j}^2, \quad (x \in E).$$

Since the square of every $\|\cdot\|_{\varphi_j}$ is additive on orthogonal elements, the square of the seminorm $\|\cdot\|_{\varphi_1, \ldots, \varphi_k}$ is also additive on orthogonal elements. Having in mind that the element $\sum_{i=1}^{m_0} e_i$ is a tripotent and the fact that $e_1, \ldots, e_{m_0}$ are mutually orthogonal, we deduce that

$$\sum_{i=1}^{m_0} \|e_i\|_{\varphi_1, \ldots, \varphi_k}^2 = \left\| \sum_{i=1}^{m_0} e_i \right\|_{\varphi_1, \ldots, \varphi_k}^2 = \sum_{j=1}^{k} \left\| \sum_{i=1}^{m_0} e_i \right\|_{\varphi_j}^2 \leq \sum_{j=1}^{k} \|\varphi_j\| \left\| \sum_{i=1}^{m_0} e_i \right\|^2 \leq k.$$

Therefore, there exists $m \in \{1, \ldots, m_0\}$ such that $\|e_m\|_{\varphi_1, \ldots, \varphi_k}^2 \leq \frac{k}{m_0} \leq \frac{k}{n_0} < \frac{\varepsilon^2}{12^2}$. Since for each $i \in \{1, \ldots, k\}$ we have $\|e_m\|_{\varphi_i}^2 \leq \|e_m\|_{\varphi_1, \ldots, \varphi_k}^2 < \frac{\varepsilon^2}{12^2}$, Lemma 3.1 assures that

$$|\varphi_i \left(P_2(e_m) + P_1(e_m) \right) (x)| < 3 \frac{\varepsilon}{12} \ \|P_2(e_m)(x)\| + 6 \frac{\varepsilon}{12} \ \|P_1(e_m)(x)\|$$

$$\leq 9 \frac{\varepsilon}{12} \ \|P_2(e_m)(x) + P_1(e_m)(x)\|$$

for all $1 \leq i \leq k$ and all $x \in E$, where in the last inequality we applied that Peirce projections are contractive and $P_2(e_m)P_1(e_m) = P_1(e_m)P_2(e_m) = 0$.

The arguments in previous studies, like [47,61], rely on controlling the norms of the restriction to the Peirce-2 and Peirce-1 subspaces associated with a tripotent in the bidual space of all elements of a finite family of functionals in the dual space of a JB*-triple E, as in the previous remark. The same conclusion does not seem easily achievable for an arbitrary scalar polynomial. For the proof in this paper we shall rely on a weaker pointwise convergence.

One of the novelties in this note, compared with previous forerunners, consists in replacing finite families of mutually orthogonal elements by sequences of mutually orthogonal elements. In order to sum countable families of mutually orthogonal tripotents, we need to employ an appropriate topology to assure the convergence. Actually, the following stronger property holds: every arbitrary family $\{e_i\}_{i \in \Lambda}$ of mutually orthogonal

tripotents in a JBW*-triple M is summable with respect to the weak* topology of M, and its limit, denoted by $\sum_{i \in \Lambda} e_i = \text{w}^*\text{-}\sum_{i \in \Lambda} e_i$, is another tripotent in M (cf. [34, Corollary 3.13] or [8, Proposition 3.8]).

We shall also make use of the strong*-topology of JB*-triples. For this reason, it is worth to recall some basic facts on this topology. According to [6, §3], the *strong*-topology* of a JB*-triple E is the topology determined by all the seminorms of the form $\| \cdot \|_\varphi$, with φ running in the dual space of E (or equivalently, in the unit sphere of E^*). If M is a JBW*-triple, the strong*-topology of M is the topology determined by all the seminorms of the form $\| \cdot \|_\varphi$ with φ running in the predual space of M. Let us observe that the strong*-topology of E is precisely the restriction to E of the strong*-topology of E^{**}. This topology enjoys some useful properties, it is compatible with the duality (E, E^*) and stronger than the weak topology of E.

When a C*-algebra A is regarded as a JB*-triple, the strong*-topology on A in the triple sense coincides with the strong* topology in the usual C*- sense (cf. [60, 1.8.7] and [6, p. 258-259]). In what concerns this paper, we remark that the triple product is jointly strong*-continuous on bounded sets (cf. [58], [53, §4]). It is well known that every sequence of mutually orthogonal projections in a C*-algebra is strong*-null, and the same property holds for every sequence of mutually orthogonal tripotents in a JB*-triple (cf. [52, Comments in page 86]).

A (closed) subtriple I of a JB*-triple E is said to be an *ideal* (respectively, an *inner ideal*) of E if $\{E, E, I\} + \{E, I, E\} \subseteq I$ (respectively, $\{I, E, I\} \subseteq I$). For example, if p, q are two projections in a C*-algebra A, the subtriple pAq is an inner ideal of A, in particular, the Peirce-2 subspace associated with a partial isometry is an inner ideal.

It is shown in [61, Lemma 3.2] that given two tripotents e, e_1 in a JB*-triple E the conditions $e \geq e_1$ (i.e., $e = (e - e_1) + e_1$ with $e - e_1$ a tripotent orthogonal to e_1) and e_1 minimal in the Peirce-2 subspace $E_2(e)$, imply that e_1 is minimal in E. This is actually a consequence of the following more general property, where we are not assuming any order relationship among the tripotents.

Lemma 3.3. *Let I be an inner ideal of a JB*-triple E. Suppose that e_1 is a minimal tripotent in I. Then e_1 is minimal in E. In particular, if e_1 is a minimal tripotent in $E_2(e)$ or in $E_0(e)$ for a fixed tripotent $e \in E$, then e_1 is a minimal tripotent in E.*

Proof. By assumptions $I_2(e_1) = \mathbb{C}e_1$. Each element x in the unital JB*-algebra $E_2(e_1)$ decomposes in the form $x = h + ik$ with h, k self-adjoint elements in $E_2(e_1)$. In particular, $k = k^{*e_1} = \{e_1, k, e_1\}$ and $h = h^{*e_1} = \{e_1, h, e_1\}$. Having in mind that I is an inner ideal with $e_1 \in I$, we get $k = \{e_1, k, e_1\}$, $h = \{e_1, h, e_1\} \in I$. Thus h and k are self-adjoint elements in the JB*-algebra $I_2(e_1) = \mathbb{C}e_1$, and consequently $x = h + ik \in \mathbb{C}e_1$, which gives the desired statement. The rest is clear because, for each tripotent $e \in E$, the Peirce subspaces $E_2(e)$ and $E_0(e)$ are inner ideals. $\square$

The C*-subalgebra of a C*-algebra generated by a single non-normal element is not always representable as a C*-algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space. However, the JB*-subtriple, E_a, generated by a single element a in a JB*-triple E is JB*-triple isomorphic to $C_0(Sp(a))$ for some unique compact set $Sp(a) \subseteq [0, \|a\|]$ with $\|a\| \in Sp(a)$, where $C_0(Sp(a))$ denotes the commutative C*-algebra of all continuous complex-valued functions on $Sp(a)$ vanishing at zero if $0 \in Sp(a)$. It is further known that we can actually find a triple isomorphism $\Psi_a : E_a \longrightarrow C_0(Sp(a))$ mapping a to the natural inclusion of $Sp(a)$ into $\mathbb{C}$ (cf. [41, Corollary 1.15] and [42, Proposition 3.5(iii)]). The set $Sp(a)$ is called the *triple spectrum of* a. It is worth to note that for $a \neq 0$, the triple spectrum $Sp(a)$ is precisely the set of all positive square roots of the elements in the spectrum of the operator $L(a, a)|_{E_a}$ in the unital complex Banach algebra $B(E_a)$, that is,

$$
\begin{aligned}
Sp(a) &= \{t \in \mathbb{R}_0^+ : t^2 \in \sigma_{B(E_a)}(L(a,a)|_{E_a})\} \\
&= \{t \in \mathbb{R}_0^+ : L(a,a)|_{E_a} - t^2 Id_{E_a} \text{ not invertible in } B(E_a)\} \\
&= \{t \in \mathbb{R}_0^+ : a \notin (L(a,a) - t^2 Id)(E)\}
\end{aligned}
$$

(cf. [42, Corollary 3.4]). For $a = 0$ we set $Sp(a) = \{0\}$ –in such a case $B(E_a)$ is not a unital Banach algebra.

Orthogonality is the key notion to define the rank of a JB*-triple E. A subset $\mathcal{S} \subset E$ is *orthogonal* if $0 \notin \mathcal{S}$ and $a \perp b$ for all $a, b \in \mathcal{S}$. The minimal cardinal number r satisfying card$(\mathcal{S}) \leq r$ for every orthogonal subset $\mathcal{S} \subseteq E$ is called the *rank* of E ($r(E)$ in short). The rank of a tripotent e in E, $r(e)$, is defined as the rank of the Peirce-2 space $E_2(e)$. A JB*-triple has finite rank if and only if it is reflexive (cf. [13, Proposition 4.5] and [18, Theorem 6] or [10,9,25]).

We would like to remark some known facts on tripotents of finite rank.

Remark 3.4. We observe that every minimal tripotent in a JBW*-triple M lies in the atomic part of M, actually the atomic part of M is the weak*-closed ideal of M generated by all minimal tripotents in M (cf. [27]). The atomic part of M coincides with an ℓ_∞-sum of Cartan factors (see [28, Proposition 2] and [35, Corollary 1.8]). By working on each of the summands appearing in the atomic part of M, we deduce that if a tripotent e writes as the orthogonal sum of a finite family of n minimal tripotents, then it has finite rank n, and reciprocally, if a tripotent has finite rank n, it can be written as an orthogonal sum of n minimal tripotents (cf. [43, Comments after Lemma 3.3 in page 200] and the previous Lemma 3.3 to deal with minimality). Let E be a JB*-triple of finite rank $n \in \mathbb{N}$. Then each tripotent in E can be written as a finite sum of at most n mutually orthogonal minimal tripotents in E (see [13, Proposition 4.5] and [18, Theorem 6] or [9]).

It is well known that a C*-algebra A is reflexive as a Banach space if and only if it is finite dimensional. Actually, dimension and rank are intrinsically related in the setting of C*-algebras. More concretely, every finite dimensional C*-algebra must have finite rank

by basic structure theory (cf. [64, Theorem I.11.2]). On the other hand, every infinite-dimensional C*-algebra must have infinite rank (see, for example, [39, Exercise 4.6.13]). If A is an infinite-dimensional von Neumann algebra, we can clearly find an infinite sequence of mutually orthogonal non-zero projections. Contrary to what is known in the case of C*-algebras and projections, the rank and not the dimension of the Peirce-2 subspace associated with a tripotent e in a JB*-triple E determines the number of mutually orthogonal elements in $E_2(e)$. We can find tripotents e whose Peirce-2 subspace is infinite dimensional with finite rank. For example, every spin factor C has rank 2 (cf. [43], or e.g. the comments after Lemma 3.1 of [40]). If C is infinite-dimensional, it contains many unitary tripotents, and for each one of them the Peirce-2 subspace is the whole C, and thus it is infinite-dimensional, reflexive, and does not contain more than 2 mutually orthogonal tripotents. This particularity of JB*-triples produces a subtle gap in the proofs of [61, Theorems 3.5 and 3.6] and [47, Theorem 4.5], where it is assumed that for a tripotent e in a JB*-triple E, the hypothesis $\dim(E_2(e)) = \infty$ implies that its bidual contains as many mutually orthogonal tripotents as desired whose sum is e. We shall see in the proof of Theorem 4.2 an argument to fix this small inconvenient.

The next technical lemma is thought to clarify the application of the notion of rank in JB*-triples.

Lemma 3.5. *Let e be a tripotent in a JBW*-triple M. Suppose that $M_2(e)$ is a JBW*-triple having infinite rank. Then, there exists an infinite sequence (e_n) of mutually orthogonal non-zero tripotents in $E_2(e)$ with $e = w^*\text{-}\sum_{n=1}^{\infty} e_n$.*

Proof. We shall first prove the following property: for each tripotent v in M such that $M_2(v)$ has infinite rank, there exists a tripotent v_1 such that $v_1 \leq v$, $v_1 \neq v$ and $M_2(v_1)$ has infinite rank.

We begin by observing that v cannot be minimal in M, otherwise, $M_2(v) = \mathbb{C}e$ which is impossible because $\mathbb{C}e$ has rank one.

Since minimality in the set of tripotents of a JBW*-triple is precisely minimality with respect to the natural partial order (cf. [24, Corollary 4.8] and [8, Lemma 4.7]), we can write v as the orthogonal sum of two non-zero tripotents w_1 and w_2. We claim that $M_2(w_1)$ or $M_2(w_2)$ has infinite rank. Otherwise, both of them are finite rank JBW*-triples, and hence each w_j can be expressed as an orthogonal finite sum of minimal tripotents in $M_2(w_j)$ $(j = 1, 2)$. Lemma 3.3 assures that minimal tripotents in $M_2(w_j)$ are minimal in M. Therefore, $v = w_1 + w_2$ writes as an orthogonal finite sum of minimal tripotents in M, and hence $M_2(v)$ has finite rank (see Remark 3.4), which is a contradiction.

Let us now prove the desired conclusion. By applying what we proved in the first part to e, there exists a tripotent v_2 such that $v_2 \lneqq e$ and $M_2(v_2)$ has infinite rank. Set $e_2 := e - v_2 \neq 0$. Now, by repeating the same argument to v_2, we find another tripotent v_3 such that $v_3 \lneqq v_2$ and $M_2(v_3)$ has infinite rank. We set $e_3 := v_2 - v_3$. Clearly, $0 \neq e_3 \perp e_2$. We inductively define a sequence of tripotents $(v_n)_n$ satisfying

$v_{n+1} \lneqq v_n$ and $M_2(v_n)$ has infinite rank for all natural $n \geq 2$. Setting $e_{n+1} := v_n - v_{n+1}$ ($n \geq 2$) and $e_1 = e - \text{w*-}\sum_{n\geq 2}^{\infty} e_n$ we obtain the desired sequence. $\quad \square$

Pełczyński's Property (V) is a powerful tool to study weak compactness. Let us recall that a series $\sum_{n\geq 1} x_n$ in a Banach space X is called weakly unconditionally Cauchy (wuC) if $\sum_{n\geq 1} |\varphi(x_n)| < \infty$ for each $\varphi \in X^*$. As shown by Bessaga and Pełczyński, wuC series are essentially linked to the canonical basis of c_0 [22, §VI]. According to [50], a Banach space X satisfies property (V) if for any (bounded) non relatively weakly compact set $K \subset X^*$ there exists a wuC series $\sum_{n\geq 1} x_n$ in X such that $\sup_{\varphi\in K} |\varphi(x_n)|$ does not converge to 0. If X is a Banach space satisfying property (V), then a bounded linear operator T from X into any other Banach space Y is either weakly compact or fixes an isomorphic copy of c_0 (cf. [50] or [33, §III.3.3]). Consequently, each bounded linear operator from X into a Banach space not containing c_0 is weakly compact. It is further known that the dual space of a Banach space having property (V) is weakly sequentially complete (see [50]). It follows that the dual space of a Banach space satisfying property (V) cannot contain a copy of c_0, and if X and Y satisfy property (V), every bounded linear operator $T: X \longrightarrow Y^*$ is weakly compact.

The list of Banach spaces satisfying Pełczyński's Property (V) includes all C*-algebras [55, Corollary 6], and the strictly wider class of JB*-triples [20]. Consequently, every bounded linear operator from a JB*-triple E into the dual space, F^*, of any other JB*-triple, F, is weakly compact. The latter conclusion can be strengthened to the fact that every bounded linear operator $T: E \longrightarrow F^*$ actually factors through a Hilbert space (cf. [19, Lemma 5]).

The procedure initiated by R. Arens in [2] for bilinear operators was later extended by R. M. Aron and P. D. Berner in [3], materializing in the most employed tool to produce norm-preserving extensions of multilinear operators to the bidual spaces. Let $T: X_1 \times X_2 \times \cdots \times X_N \longrightarrow Y$ be a bounded multilinear map (where $X_1, X_2, \ldots, X_N, Y$ are Banach spaces). The Aron-Berner procedure guarantees the existence of $N!$ norm-preserving "natural" extensions of T to multilinear maps from $X_1^{**} \times X_2^{**} \times \ldots \times X_N^{**}$ to Y^{**}, each one enjoying certain weak* continuity properties. These Aron-Berner extensions are separately weak*-to-weak* continuous if and only if they all coincide. A result by F. Bombal and I. Villanueva proves that, assuming that for each $i \neq j$, every bounded linear operator from X_i into X_j^* is weakly compact, every bounded multilinear operator $T: X_1 \times X_2 \times \cdots \times X_N \longrightarrow Y$ admits a unique norm-preserving separately weak*-to-weak* continuous Aron-Berner extension $\widetilde{T}: X_1^{**} \times X_2^{**} \times \ldots \times X_N^{**} \longrightarrow Y^{**}$ [12, Theorem 1].

Another basic principle of functional analysis asserts that a bounded linear operator $T: X \longrightarrow Y$ (where X and Y are Banach spaces) is weakly compact if and only if $T^{**}(X^{**}) \subseteq Y$ (cf. [21, Theorem 5.5]). The setting of multilinear operators differs from the linear case. One implication remains true. Namely, in order to guarantee the uniqueness of the Aron-Berner extension, let us assume that for each $i \neq j$ every bounded linear operator from X_i into X_j^* is weakly compact. Then it is known that each weakly

compact multilinear mapping $T\colon X_1 \times X_2 \times \cdots \times X_N \longrightarrow Y$ satisfies that its Aron-Berner extension is Y-valued (cf. [12, Corollary 2]). However, for $N > 1$ there exist non-weakly compact N-linear operators whose Aron-Berner extension remains valued in the same codomain space (see, for example, [51, page 385] or [29, Lemma 7] and [65, Theorem 5]).

In the case of multilinear maps from the Cartesian product of a collection of JB*-triples $E_1, \ldots, E_N$ into a complex Banach space X, the precise characterization of those satisfying that their Aron-Berner extension remains valued in the same codomain space was established by A. M. Peralta, I. Villanueva, J. D. M. Wright, and K. Ylinen in [54]. We recall that a multilinear operator $T\colon E_1 \times \cdots \times E_N \longrightarrow X$ is *quasi completely continuous* if whenever we choose strong* Cauchy sequences $(x_n^i)_n \subset E_i$ $(1 \leq i \leq N)$, it follows that the sequence $(T(x_n^1, \ldots, x_n^N))_n$ is norm convergent, equivalently, given sequences $(x_n^i) \subset E_i$ which are strong* convergent to $x^i \in E_i$ $(1 \leq i \leq k)$, we have

$$\lim_n \|T(x_n^1, \ldots, x_n^N) - T(x^1, \ldots, x^k)\| = 0.$$

Theorem 3.6 *([54, Theorem 3.9]). Let $E_1, \ldots, E_N$ be JB*-triples, X a complex Banach space, and $T\colon E_1 \times \cdots \times E_N \longrightarrow X$ a bounded multilinear operator. Then, the following assertions are equivalent:*

(1) *T is quasi completely continuous.*
(2) *The unique Aron-Berner extension of T is X-valued.*

A former version of the above theorem for C*-algebras was obtained by J. D. M. Wright and K. Ylinen (see [66, Corollary 3.6]), while the case of abelian C*-algebras was treated by I. Villanueva in [65] (see also [66, Corollary 3.7]).

Let E be a JB*-triple. Each scalar polynomial $p\colon E \longrightarrow \mathbb{C}$ writes as a finite sum $p = p_0 + p_1 + \cdots + p_N$, where each p_k is a scalar k-homogeneous polynomial. Clearly, Theorem 3.6 above implies that the symmetric multilinear form $\widehat{p_k}$ is quasi completely continuous, and hence $p_k\colon E \longrightarrow \mathbb{C}$ is sequentially strong*(-to-norm) continuous for all $0 \leq k \leq N$. The conclusion in the next corollary just follows by gluing the different k-homogeneous summands expressing p.

Corollary 3.7. *Every scalar polynomial on a JB*-triple is sequentially strong* continuous.*

4. Main results

We can now state and prove that every JB*-triple satisfying the Daugavet property also satisfies the stronger polynomial Daugavet property.

Theorem 4.1. *Let E be a JB*-triple satisfying the Daugavet property. Then E satisfies the polynomial Daugavet property, that is, every weakly compact polynomial $P\colon E \longrightarrow E$ satisfies the Daugavet equation* (DE).

Proof. Let $P\colon E \longrightarrow E$ be a weakly compact polynomial. We can assume, via [16, Theorem 1.1] (see also Proposition 2.1), that P is of the form $P(x) = p(x)a$, where $a \in E$, $p\colon E \longrightarrow \mathbb{C}$ is a scalar polynomial with $\|a\| = 1 = \|p\|$. For each $\varepsilon > 0$, we can find x_0 in the closed unit ball of E and $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $\operatorname{Re}\omega p(x_0) > 1 - \varepsilon$. Let $\widehat{p}\colon E^{**} \longrightarrow \mathbb{C}$ denote the Aron-Berner extension of the polynomial p, obtained by extending each homogeneous summand of p.

As in the argument in the proof of [61, Theorem 3.5] (or [11,47]), the key information is in the triple spectrum of a. Let $Sp(a) \subseteq [0, 1]$ denote the triple spectrum of a in E, where $1 \in Sp(a)$.

Our next goal will consist in proving that for each $\varepsilon > 0$ there exists a sequence $(e_n)_n$ of mutually orthogonal tripotents in E^{**} such that

$$P_2(e_n)(a) = \lambda_n e_n \text{ with } \lambda_n > 1 - \varepsilon, \text{ for all } n \in \mathbb{N}. \tag{4}$$

If 1 is isolated in $Sp(a)$, the element $e = \chi_{\{1\}}$ –i.e., the characteristic function corresponding to the set $\{1\}$– is a continuous function in $C_0(Sp(a)) \equiv E_a$ and lies in the JB*-subtriple E_a generated by a. Furthermore, $a = e + a_0$ with $a_0 \perp e$. We shall next show that $E_2(e)$ has infinite rank. Otherwise it must be reflexive (cf. [13, Proposition 4.5] and [18, Theorem 6] or [10,9]). It is known that every reflexive JB*-triple is generated by its minimal tripotents (cf. [13, Proposition 4.5 and Remark 4.6]). However, every minimal tripotent in $E_2(e)$ is a minimal tripotent in E (cf. Lemma 3.3) but the latter contains no minimal tripotents by hypothesis (cf. Theorem 1.1). This concludes the proof of the fact that $E_2(e)$ has infinite rank. By observing that $E_2(e)$ is a weak*-dense JB*-subtriple of $E_2^{**}(e)$, we conclude that the latter has finite rank (equivalently, is reflexive) if and only if $E_2(e)$ has finite rank. Therefore $E_2^{**}(e)$ must have infinite rank.

Now, by applying Lemma 3.5 to the JBW*-triple E^{**} and the tripotent e, we deduce the existence of an infinite sequence $(e_n)_n$ of mutually orthogonal non-zero tripotents in $E_2^{**}(e)$ with $e = \text{w*-}\sum_{n=1}^{\infty} e_n$. Let us recall that $a = e + a_0$ with $a_0 \perp e$, therefore, by orthogonality, we get

$$P_2(e_n)(a) = P_2(e_n)(e) = e_n = \lambda_n e_n, \text{ with } \lambda_n = 1 > 1 - \varepsilon.$$

This proves the statement in (4) in the case in which 1 is isolated in $Sp(a)$.

If 1 is non-isolated in $Sp(a)$, by [1, Proposition 3.6] we can find an infinite collection of mutually orthogonal non-zero tripotents $(e_n)_n$ in E^{**} satisfying (4).

We have already proved (4). Now, having in mind that $e_n \perp P_0(e_n)(x_0)$ and (3), we define the following sequence in the unit sphere of E^{**} by the rule

$$x_n := \overline{\omega}e_n + P_0(e_n)(x_0).$$

We note that

$$x_0 - x_n = P_1(e_n)(x_0) + P_2(e_n)(x_0 - \overline{\omega}e_n) = P_1(e_n)(x_0) + P_2(e_n)(x_0) - \overline{\omega}e_n.$$

The sequence $(e_n)_n$ converges to zero in the strong*-topology of E^{**} (see comments in page 12), and the same occurs to $(\overline{\omega}e_n)_n$. We deduce from the joint strong* continuity of the triple product on bounded sets of E^{**} that the sequences $(L(e_n, e_n)(x_0))_n = (P_2(e_n)(x_0) + \frac{1}{2}P_1(e_n)(x_0))_n$, $(Q(e_n)(x_0))_n$ and $(P_2(e_n)(x_0))_n = (Q(e_n)^2(x_0))_n$ converge to zero in the strong*-topology of E^{**}. In particular, $(P_2(e_n)(x_0))_n$, $(P_1(e_n)(x_0))_n \longrightarrow 0$ in the strong*-topology of E^{**}. This guarantees that $(x_n - x_0) \longrightarrow 0$ and $(x_n) \longrightarrow x_0$ in the strong*-topology of E^{**}.

By Corollary 3.7 the polynomials $p \colon E \longrightarrow \mathbb{C}$ and $\hat{p} \colon E^{**} \longrightarrow \mathbb{C}$ are sequentially strong* continuous. Therefore $\hat{p}(x_n) \longrightarrow \hat{p}(x_0) = p(x_0)$. It is now time to pick n_0 in $\mathbb{N}$ such that $\operatorname{Re} \omega \hat{p}(x_{n_0}) > 1 - \varepsilon$. Our choice of n_0 implies the following facts: $P_2(e_{n_0})(x_{n_0}) = \overline{\omega}e_{n_0}$ and

$$\begin{aligned}
\|x_{n_0} + \hat{p}(x_{n_0})a\| &\geq \|P_2(e_{n_0})(x_{n_0} + \hat{p}(x_{n_0})a)\| = \|\overline{\omega}e_{n_0} + \hat{p}(x_{n_0})\lambda_{n_0}e_{n_0}\| \\
&= |\overline{\omega} + \hat{p}(x_{n_0})\lambda_{n_0}| = |1 + \omega\hat{p}(x_{n_0})\lambda_{n_0}| \\
&\geq \operatorname{Re}\left(1 + \omega\hat{p}(x_{n_0})\lambda_{n_0}\right) > 2 - 2\varepsilon,
\end{aligned}$$

witnessing that $\|\operatorname{Id}_E + P\| = \|\operatorname{Id}_E + pa\| = \|\operatorname{Id}_{E^{**}} + \widehat{pa}\| > 2 - 2\varepsilon$, and the arbitrariness of ε gives the desired conclusion. $\square$

Let us now deal with the alternative Daugavet Property.

Theorem 4.2. *Let E be a JB*-triple satisfying the alternative Daugavet property. Then, E satisfies the alternative polynomial Daugavet property, that is, every weakly compact polynomial $P \colon E \longrightarrow E$ satisfies the alternative Daugavet equation* (aDE).

Proof. Since E satisfies the ADP every minimal tripotent in E is diagonalizing (see Theorem 1.1). Suppose $P \colon E \longrightarrow E$ is a weakly compact polynomial. We can assume, via [16, Corollary 1.2] (see Proposition 2.2), that $P(x) = p(x)a$, where $p \colon E \longrightarrow \mathbb{C}$ is a polynomial and $a \in X$. There is no loss of generality in assuming that $\|p\| = 1 = \|a\|$.

Let us refine a bit our previous arguments in the proof of Theorem 4.1. If 1 is isolated in the triple spectrum of a, the characteristic function $\chi_{\{1\}}$ is a tripotent e in $C_0(Sp(a)) \equiv E_a$, where the latter denotes the JB*-subtriple of E generated by a.

The dichotomy is the following: $E_2(e)$ has finite or infinite rank. In the first case $E_2(e)$ must be reflexive (cf. [13, Proposition 4.5] or [18, Theorem 6] or [10,9]), and the whole $E_2(e)$ is generated by its minimal tripotents ([13, Proposition 4.5 and Remark 4.6]). Therefore, $E_2(e)$ contains a minimal tripotent e_1 which is also minimal in E by Lemma 3.3, and diagonalizing by hypotheses. Therefore E decomposes in the form $E = \mathbb{C}e_1 \oplus_\infty E_0(e_1)$ $(E_2(e_1) = \mathbb{C}e_1)$. We note that $e = \gamma e_1 + P_0(e_1)(e)$ for a suitable unitary $\gamma \in \mathbb{C}$.

Since $\|p\| = 1$, for each $\varepsilon > 0$, there exists $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and x_0 in the closed unit ball of $E_0(e_1)$ such that $|p(\alpha e_1 + x_0)| > 1 - \varepsilon$. Having in mind that p is a scalar polynomial, the mapping $\zeta \longmapsto p(\zeta \gamma e_1 + x_0)$ is a polynomial in $\zeta \in \mathbb{C}$, and thus an entire function. By the maximum modulus principle, there exists a unitary scalar λ_1 such that

$$|p(\lambda_1 \gamma e_1 + x_0)| = \max\{|p(\beta e_1 + x_0)| \colon \beta \in \mathbb{C}, |\beta| \leq 1\} \geq |p(\alpha e_1 + x_0)| > 1 - \varepsilon. \quad (5)$$

Let us observe that $a = e + P_0(e)(a) = \gamma e_1 + P_0(e_1)(a)$ because $\gamma e_1 \leq e$. Setting $x_1 := \lambda_1 \gamma e_1 + x_0$, we define a norm-one element in E satisfying the following:

$$\|(\mathrm{Id}_E + \omega P)(x_1)\| = \|x_1 + \omega p(x_1)a\| \geq \|P_2(e_1)(x_1 + \omega p(x_1)a)\|$$
$$= \|\lambda_1 \gamma e_1 + \omega p(x_1)\gamma e_1\| = |\lambda_1 + \omega p(x_1)| = |1 + \omega \overline{\lambda_1} p(x_1)|,$$

for every unitary $\omega \in \mathbb{C}$. It is straightforward to check, from the previous inequality, that $\max_{|\omega|=1} \|(\mathrm{Id}_E + \omega P)(x_1)\| = 1 + |p(x_1)| > 2 - \varepsilon$ (see (5)), witnessing that $\max_{|\omega|=1} \|\mathrm{Id}_E + \omega P\| = 2$ as desired.

The above argument shows that every polynomial of the form $P(x) = p(x)a$ with $p \in \mathcal{P}(E)$, $a \in E$, and $\|p\| = \|a\| = 1$, satisfies the alternative Daugavet equation (aDE) whenever 1 is isolated in the triple spectrum of a and there exists a minimal tripotent e_1 in E with $e_1 \in E_2(e)$ (in particular, when $E_2(e)$ has finite rank).

In the remaining cases one of the next statements holds:

(a) 1 is isolated in the triple spectrum of a and, for $e = \chi_{\{1\}} \in E$, the Peirce-2 subspace $E_2(e)$ has infinite rank.

(b) 1 is non-isolated in the triple spectrum of a.

In both cases, we can literally repeat the arguments in the proof of Theorem 4.1 to show that P satisfies the Daugavet equation and thus the alternative Daugavet equation. $\quad \square$

Obviously, the conclusions in Theorems 4.1 and 4.2 actually characterize those JB*-triples satisfying the Daugavet property and the alternative Daugavet property, respectively. We have indeed rediscovered the conclusions in Theorem 1.1 because we have proved that E has the DPr (respectively, the ADP) provided that E contains no minimal tripotents (respectively, every minimal tripotent in E is diagonalizing).

We conclude this paper with two open problems.

Problem 4.3. Let E be a JB*-triple satisfying the Daugavet property (respectively, the alternative Daugavet property). In view of Theorem 3.6 ([54, Theorem 3.9]), it seems natural to ask whether every polynomial $P \colon E \longrightarrow E$ whose Aron-Berner extension remains E-valued satisfies the (DE) (respectively, the (aDE)).

Problem 4.4. It is known from previous studies that a JBW*-triple has the Daugavet Property or the alternative Daugavet Property if and only if its predual enjoys the same

property (cf. [47, Theorem 4.7] and [45, Theorem 2.3]). In the case of the polynomial versions, it is known that the alternative polynomial Daugavet property does not pass from a JBW*-triple to its predual, nor the other way around (the complex ℓ_∞ gives a counterexample). For the polynomial Daugavet property, we do not know whether the equivalence holds (actually, no implication is known to be true in this case).

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