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Sobolev orthogonal polynomials and spectral methods in boundary value problems

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A B S T R A C T

In the variational formulation of a boundary value problem for the harmonic oscillator, Sobolev inner products appear in a natural way. First, we study the sequences of Sobolev orthogonal polynomials with respect to such an inner product. Second, their representations in terms of a sequence of Gegenbauer polynomials are deduced as well as an algorithm to generate them in a recursive way is stated. The outer relative asymptotics between the Sobolev orthogonal polynomials and classical Legendre polynomials is obtained. Next we analyze the solution of the boundary value problem in terms of a Fourier-Sobolev projector. Finally, we provide numerical tests concerning the reliability and accuracy of the Sobolev spectral method.

1. Introduction and motivation

Orthogonal polynomials with respect to Sobolev inner products associated with a vector of positive measures $(\mu_0, \mu_1, ..., \mu_N)$ supported on the real line have attracted the interest of many researchers. They are interesting from several points of view: smooth approximations by polynomials, Fourier expansions in terms of those polynomials as an alternative to the standard ones (see [7]), spectral methods for boundary value problems for differential equations, where the Sobolev orthogonal polynomials play an efficient role with respect to the classical ones (see [2], [3], [4]). Indeed, they have been recently studied in the framework of the so called diagonalized spectral methods for boundary value problems for some elliptic differential operators, see [1], [19].

Aside from the classical Gram-Schmidt method, a key problem is the generation of sequences of Sobolev orthogonal polynomials. Let us consider the Sobolev inner product

$$\langle f, g\rangle_S = \sum_{k=0}^{N} \int_{E_k} f^{(k)}(x) g^{(k)}(x) d\mu_k(x),$$

for f, g functions on the appropriate weighted Sobolev space, where E_k is the support of the measure $\mu_k, k = 0, 1, ..., N$, respectively. As it is well known, the multiplication by x is not a symmetric operator with respect to this inner product and,

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as a consequence, the three term recurrence relation that standard orthogonal polynomials satisfy (see [5]) does not hold any more. This relation constitutes, from a computational point of view, a very useful tool to construct such sequences of standard orthogonal polynomials (see [6]).

Just a few examples of Sobolev orthogonal polynomials were known before the last decade of the 20th century, (see [11] for a survey on the state of the art). Nevertheless, the concept of coherent pairs of measures was introduced in the pioneering work ([8]). Furthermore, Sobolev orthogonal polynomials for vectors of measures associated with such measures have been extensively studied (see [13] as an updated survey).

A pair of measures (μ_0, μ_1) is said to be a coherent pair if the corresponding sequences of monic orthogonal polynomials $\{P_n(x;\mu_k)\}_{n\geqslant 0}, k=0,1$, are related by a simple relation as

$$P_n(x;\mu_1)=\frac{1}{n+1}[P'_{n+1}(x;\mu_0)-\rho_n P'_n(x;\mu_0)], \quad n\geqslant 1.$$

In such a case, if $\{S_n(x;\mu_0,\mu_1)\}_{n\geqslant 0}$ denotes the sequence of monic Sobolev orthogonal polynomials with respect to the vector of measures (μ_0,μ_1), then they can be generated by the following recursive relations in terms of the standard orthogonal polynomials

$$\begin{aligned} S_{n+1}(x;\mu_0,\mu_1)-\gamma_n S_n(x;\mu_0,\mu_1)&=P_{n+1}(x;\mu_0)-\rho_n P_n(x;\mu_0),\\ S'_{n+1}(x;\mu_0,\mu_1)-\gamma_n S'_n(x;\mu_0,\mu_1)&=(n+1)P_n(x;\mu_1). \end{aligned} \tag{1.1}$$

Coherent pairs of measures are described in [17]. Essentially, one of the measures must be a classical one (Jacobi, Laguerre) and the other one is a rational perturbation of it. The above connection formulas proved to be very useful in the study of analytic properties of the Sobolev orthogonal polynomials associated with a coherent pair of measures. As an illustrative sample, outer relative asymptotics have been deeply analyzed in the literature (see [14], [15], [16] as well as the recent survey [13], where an updated list of references concerning this topic is presented).

A natural extension of coherent pairs has been presented in [17] where μ_0 and μ_1 are symmetric measures supported on symmetric intervals of the real line. In such a case

$$P_n(\mu_1;x)=\frac{1}{n+1}\left[P'_{n+1}(\mu_0;x)-\rho_{n-1}P'_{n-1}(\mu_0;x)\right], \quad \rho_{n-1}\neq 0, \quad n\geq 2.$$

The symmetrically coherent pairs of measures are described in [17]. Essentially, one of the measures must be a classical one (Hermite, Gegenbauer) and the other one is a rational perturbation of it. Other extensions of coherent pairs of measures have been studied in the literature, either involving more terms in the right hand side of (1.1) (see [10]) or higher order derivatives in both sides ([9]).

Next, we will show how Sobolev orthogonal polynomials appear as a useful tool in the framework of spectral methods for boundary value problems.

The solution of the boundary value problem (BVP, in short) for the ordinary differential equation associated with the harmonic oscillator, a stationary Schrödinger equation with potential $V(x)=x^2$,

$$\begin{aligned} -u''+\lambda\, x^2\, u&=f(x),\\ u(-1)=u(1)&=0, \end{aligned} \tag{1.2}$$

where $\lambda>0$, can be studied from a variational perspective taking into account the Sobolev inner product

$$\langle u,v\rangle_\lambda=\lambda\int_{-1}^{1}u(x)\,v(x)\,x^2\,dx+\int_{-1}^{1}u'(x)\,v'(x)\,dx, \tag{1.3}$$

associated with the variational formulation of (1.2).

Let $\mathbb{P}$ denote the linear space of real polynomials. The test functions for (1.2) should be chosen in the linear space $(x^2-1)\,\mathbb{P}$. Then we have to deal with an orthogonal basis in $(x^2-1)\,\mathbb{P}$ of polynomials vanishing at the ends of the interval $[-1,1]$ associated with the above Sobolev inner product.

The structure of the manuscript is as follows. In Section 1 we have presented a motivation and basic background about sequences of orthogonal polynomials with respect to weighted Sobolev inner products. Next, in such a section we will provide some basic results about Jacobi orthogonal polynomials which will useful in the sequel. In Section 2, Sobolev orthogonal polynomials with respect to the pair of measures (x^2dx, dx) supported on the interval $[-1,1]$ are introduced. The connection formulas with generalized Jacobi polynomials $P_n^{(-1,-1)}(x)$ are stated and, as a consequence, estimates for the Sobolev norms as well as outer relative asymptotics with respect to the Legendre polynomials are deduced. In Section 3, a family of polynomials orthogonal with respect to the above Sobolev inner product satisfying the boundary conditions at ± 1 is studied. Notice that such polynomials constitute a basis of the linear space $(x^2-1)\,\mathbb{P}$. From them, we generate a new Sobolev inner product with an orthogonal polynomial basis. The connection formula between such Sobolev orthogonal polynomials and Jacobi polynomials $P_n^{(1,1)}(x)$ is stated. As a consequence, in Section 4, the Sobolev Fourier coefficients of a function satisfying the BVP conditions (1.2) are deduced in a recursive way. Finally, some numerical experiments are shown.

1.1. Jacobi orthogonal polynomials

Along this paper, we deal with the sequence of monic polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geqslant 0}$, $\alpha,\beta > -1$, orthogonal with respect to the classical Jacobi inner product defined by the beta distribution, i.e.,

$$\langle f, g\rangle_{\alpha,\beta} = \int_{-1}^{1} f(t)g(t)w^{(\alpha,\beta)}(t)dt, \tag{1.4}$$

where $w^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$. Expressions for monic Jacobi polynomials are obtained from the corresponding formulas in [18].

The square of the norms $\|P_n^{(\alpha,\beta)}\|^2 = \langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\rangle_{\alpha,\beta}$ is given by

$$\|P_n^{(\alpha,\beta)}\|^2 = 2^{2n+\alpha+\beta+1} n! \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+2)\Gamma(2n+\alpha+\beta+1)}, \tag{1.5}$$

(see [18, (4.3.3)]).

The explicit representation of monic orthogonal Jacobi polynomials ([18, (4.3.2)]) can be given in terms of Pochhammer symbols as

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) =& \frac{n!}{(n+\alpha+\beta+1)_n} \\ &\times \sum_{m=0}^{n} \frac{(\alpha+m+1)_{n-m}}{(n-m)!} \frac{(\beta+n-m+1)_m}{m!} (x+1)^{n-m}(x-1)^m, \end{aligned} \tag{1.6}$$

where for $a\in\mathbb{R}$, $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1), n\geqslant 1$, denotes the usual Pochhammer symbol.

By using the explicit (hypergeometric) expression (1.6), we can define monic Jacobi polynomials for $\alpha,\beta\in\mathbb{R}$ such that

$$(n+\alpha+\beta+1)_n \neq 0, \quad n\in\mathbb{N}.$$

Monic Jacobi orthogonal polynomials $\{P_n^{(\alpha,\beta)}\}_{n\geqslant 0}$ satisfy the three-term recurrence relation ([18])

$$\begin{aligned} &x\,P_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta)}(x) + \lambda_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(x) + \gamma_n^{(\alpha,\beta)}\,P_{n-1}^{(\alpha,\beta)}(x), \quad n\geqslant 0, \\ &P_{-1}^{(\alpha,\beta)}(x) = 0, \quad P_0^{(\alpha,\beta)}(x) = 1, \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \lambda_n^{(\alpha,\beta)} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}, \quad n\geqslant 0, \\ \gamma_n^{(\alpha,\beta)} &= \frac{4n\,(n+\alpha)\,(n+\beta)\,(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)}, \quad n\geqslant 1. \end{aligned}$$

In the symmetric case, that is, when $\alpha=\beta$,

$$\lambda_n^{(\alpha,\alpha)} = 0, \quad n\geqslant 0, \quad \gamma_n^{(\alpha,\alpha)} = \frac{n\,(n+2\alpha)}{(2n+2\alpha+1)(2n+2\alpha-1)}, \quad n\geqslant 1,$$

and the three term recurrence relation takes the form

$$x\,P_n^{(\alpha,\alpha)}(x) = P_{n+1}^{(\alpha,\alpha)}(x) + \frac{n\,(n+2\alpha)}{(2n+2\alpha+1)(2n+2\alpha-1)}\,P_{n-1}^{(\alpha,\alpha)}(x). \tag{1.8}$$

As it is well known, for $\alpha,\beta > -1$, Jacobi polynomials are orthogonal with respect to the inner product (1.4). Following Favard's theorem, for $\alpha,\beta\in\mathbb{R}$ such that $-\alpha, -\beta, -\alpha-\beta\notin\mathbb{N}$, Jacobi polynomials are orthogonal with respect to a quasi-definite moment functional ([5]).

Singular cases appear when $\alpha=-l$ or $\beta=-l$, $l\in\mathbb{N}$ ([18, p. 64]). If $\alpha=-l$, then $x=1$ is a zero of order l of $P_n^{(-l,\beta)}(x)$. Similarly, if $\beta=-l$, then $x=-1$ is a zero of order l of $P_n^{(\alpha,-l)}(x)$.

Monic Jacobi polynomials satisfy the derivation formula

$$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x) = n\,P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n\geqslant 1. \tag{1.9}$$

Moreover, Jacobi polynomials satisfy two structure relations

$$(x^2-1)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = n\left[P_{n+1}^{(\alpha,\beta)}(x) + a_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(x) + b_n^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x)\right], \tag{1.10}$$

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\alpha+1,\beta+1)}(x) + c_n^{(\alpha,\beta)}P_{n-1}^{(\alpha+1,\beta+1)}(x) + d_n^{(\alpha,\beta)}P_{n-2}^{(\alpha+1,\beta+1)}(x), \tag{1.11}$$

for $n \geqslant 1$, where

$$\begin{aligned}
a_n^{(\alpha,\beta)} &= \frac{2(\beta-\alpha)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)},\\
b_n^{(\alpha,\beta)} &= -\frac{4(n+\alpha)(n+\beta)(n+\alpha+\beta+1)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)},\\
c_n^{(\alpha,\beta)} &= \frac{2n(\beta-\alpha)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)},\\
d_n^{(\alpha,\beta)} &= -\frac{4n(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)}.
\end{aligned}$$

1.2. Connection formulas for Legendre polynomials

We work with the generalized monic Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n\geqslant 2}$ defined as the natural extension of monic Jacobi polynomials when $\alpha, \beta \to -1$.

From (1.6), we can define the polynomials for $n \geqslant 2$, and we can complete the basis by defining $P_0^{(-1,-1)}(x) = 1$, $P_1^{(-1,-1)}(x) = x$.

This monic singular Jacobi polynomials also satisfy the three-term recurrence relation (1.8) in the form

$$\begin{aligned}
&P_0^{(-1,-1)}(x) = 1, \quad P_1^{(-1,-1)}(x) = x,\\
&P_n^{(-1,-1)}(x) = x\,P_{n-1}^{(-1,-1)}(x) - \frac{(n-1)\,(n-3)}{(2n-3)(2n-5)}\,P_{n-2}^{(-1,-1)}(x), \quad n \geqslant 2.
\end{aligned}$$

It well known (see [18]) that

$$P_n^{(-1,-1)}(x) = (x^2-1)\,P_{n-2}^{(1,1)}(x), \quad n \geqslant 2. \tag{1.12}$$

On the one hand, using (1.9), we get

$$\frac{d}{dx}P_n^{(-1,-1)}(x) = n\,P_{n-1}^{(0,0)}(x) = n\,P_{n-1}(x), \quad n \geqslant 2, \tag{1.13}$$

where $\{P_n\}_{n\geqslant 0}$ denotes the monic Legendre polynomials, orthogonal with respect to the inner product (1.4) for $\alpha = \beta = 0$. In addition, we will denote by

$$\|P_n\|^2 = \langle P_n, P_n\rangle_{0,0}$$

the norm of the monic Legendre polynomials. By (1.5), we can see that, for $k \geqslant 0$,

$$\lim_{n\to+\infty}\frac{\|P_n\|^2}{\|P_{n-k}\|^2} = \left(\frac{1}{4}\right)^k. \tag{1.14}$$

In addition, both families of polynomials are related by (1.11)

$$P_n^{(-1,-1)}(x) = P_n(x) + d_n\,P_{n-2}(x), \quad n \geqslant 2, \tag{1.15}$$

where

$$d_n = d_n^{(-1,-1)} = -\frac{n\,(n-1)}{(2n-1)(2n-3)} = -\frac{1}{4} - \frac{1}{4n} - \frac{5}{16n^2} + O\,(n^{-3}). \tag{1.16}$$

The three-term recurrence relation for Legendre polynomials reads as

$$x\,P_n(x) = P_{n+1}(x) + \gamma_n\,P_{n-1}(x), \quad n \geqslant 0, \tag{1.17}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, $\gamma_n = \gamma_n^{(0,0)}$, with

$$\gamma_n = \frac{n^2}{4n^2-1} = \frac{1}{4} + \frac{1}{8}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{64n^4} + O\,(n^{-6}). \tag{1.18}$$

Iterating (1.17) we get

$$x^2 P_n(x) = P_{n+2}(x) + (\gamma_{n+1} + \gamma_n) P_n(x) + \gamma_n \gamma_{n-1} P_{n-2}(x), \quad n \geqslant 1. \tag{1.19}$$

2. Sobolev orthogonal polynomials from the variational formulation

In this section we study the sequence of orthogonal polynomials with respect to the Sobolev inner product

$$\langle u, v\rangle_\lambda = \lambda \int_{-1}^{1} u(x)\, v(x)\, x^2\, dx + \int_{-1}^{1} u'(x)\, v'(x)\, dx, \tag{2.1}$$

for $\lambda > 0$.

Let us denote by $\{\phi_n\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials with respect to this Sobolev inner product. It is easy to check that

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2 - \frac{3}{5}, \quad \phi_3(x) = x^3 - \frac{5(\lambda+7)}{7(\lambda+5)}x. \tag{2.2}$$

The connection between these polynomials and $P_n^{(-1,-1)}$ is given in the next Proposition.

Proposition 2.1. *For $n \geqslant 2$,*

$$P_n^{(-1,-1)}(x) = \phi_n(x) + a_n\, \phi_{n-2}(x) + b_n\, \phi_{n-4}(x), \tag{2.3}$$

where

$$a_n = \lambda[\gamma_n \gamma_{n-1} + d_n(\gamma_{n-1} + \gamma_{n-2} + d_{n-2} - a_{n-2})]\frac{\|P_{n-2}\|^2}{\|\phi_{n-2}\|_\lambda^2}, \quad n \geqslant 2, \tag{2.4}$$

$$b_n = \lambda\, d_n \frac{\|P_{n-2}\|^2}{\|\phi_{n-4}\|_\lambda^2} \neq 0, \quad n \geqslant 4, \quad b_2 = b_3 = 0, \tag{2.5}$$

and d_n, γ_n are given in (1.16) and (1.18), respectively.

Moreover,

$$P_n(x) + d_n P_{n-2}(x) = \phi_n(x) + a_n \phi_{n-2}(x) + b_n \phi_{n-4}(x). \tag{2.6}$$

Proof. Expanding

$$P_n^{(-1,-1)}(x) = \phi_n(x) + \sum_{i=0}^{n-1} \alpha_i^n\, \phi_i(x),$$

and applying orthogonality,

$$\alpha_i^n = \frac{\langle P_n^{(-1,-1)}, \phi_i\rangle_\lambda}{\langle \phi_i, \phi_i\rangle_\lambda},$$

we get $\alpha_i^n = 0$, for $0 \leqslant i \leqslant n-5$ and, using the symmetry, (2.3) holds. Using (1.13) and (1.15), we observe that

$$\lambda \int_{-1}^{1} (P_n(x) + d_n P_{n-2}(x))\phi_{n-4}(x)\, x^2\, dx = \lambda\, d_n\, \|P_{n-2}\|^2, \quad n \geqslant 4,$$

so (2.5) holds. To compute a_n, we study the integral

$$\begin{aligned} I_1 &= \int_{-1}^{1} (P_n(x) + d_n P_{n-2}(x))\phi_{n-2}(x)x^2 dx \\ &= \|P_n\|^2 + d_n \int_{-1}^{1} x^2 P_{n-2}(x)\phi_{n-2}(x) dx. \end{aligned}$$

Using (1.19) we obtain

$$I_2 = \int_{-1}^{1} x^2 P_{n-2}(x)\phi_{n-2}(x)dx$$
$$= \int_{-1}^{1} \big(P_n(x) + (\gamma_{n-1}+\gamma_{n-2})P_{n-2}(x) + \gamma_{n-2}\,\gamma_{n-3}P_{n-4}(x)\big)\phi_{n-2}(x)dx$$
$$= (\gamma_{n-1}+\gamma_{n-2})\|P_{n-2}\|^2 + \gamma_{n-2}\,\gamma_{n-3}\int_{-1}^{1} P_{n-4}(x)\phi_{n-2}(x)dx.$$

Now, using (2.3) and (1.15) we get

$$\phi_{n-2}(x) = P_{n-2}^{(-1,-1)}(x) - a_{n-2}\phi_{n-4}(x) - b_{n-2}\phi_{n-6}(x)$$
$$= P_{n-2}(x) + d_{n-2}P_{n-4}(x) - a_{n-2}\phi_{n-4}(x) - b_{n-2}\phi_{n-6}(x),$$

then

$$\int_{-1}^{1} P_{n-4}(x)\phi_{n-2}(x)dx = (d_{n-2}-a_{n-2})\|P_{n-4}\|^2,$$

and, as a consequence,

$$I_1 = \|P_n\|^2 + d_n[(\gamma_{n-1}+\gamma_{n-2})\|P_{n-2}\|^2 + \gamma_{n-2}\,\gamma_{n-3}(d_{n-2}-a_{n-2})\|P_{n-4}\|^2].$$

By (1.19), $\gamma_{n-2}\gamma_{n-3}\|P_{n-4}\|^2 = \|P_{n-2}\|^2$, so

$$I_1 = \|P_n\|^2 + d_n\big(\gamma_{n-1}+\gamma_{n-2}+d_{n-2}-a_{n-2}\big)\|P_{n-2}\|^2,$$

and again using that $\|P_n\|^2 = \gamma_n\gamma_{n-1}\|P_{n-2}\|^2$ the result follows.

Finally, we get (2.6) from (1.15) and (2.3). □

Remark 2.2. Notice that the initial conditions read as $a_2 = -\frac{2}{5}$, $a_3 = -\frac{2\lambda}{7(\lambda+5)}$.

Proposition 2.3. *The norms of the Sobolev orthogonal polynomials, for $n \geqslant 4$, satisfy*

$$\|\phi_n\|_\lambda^2 = \lambda(\gamma_n+\gamma_{n+1}+2d_n-a_n)\|P_n\|^2$$
$$+\lambda d_n\big((\gamma_{n-1}+\gamma_{n-2})(d_n-a_n) - a_n(d_{n-2}-a_{n-2}-b_n)\big)\|P_{n-2}\|^2$$
$$+n^2\|P_{n-1}\|^2.$$

In addition,

$$\|\phi_0\|_\lambda^2 = \frac{2\lambda}{3}, \quad \|\phi_1\|_\lambda^2 = 2+\frac{2\lambda}{5}, \quad \|\phi_2\|_\lambda^2 = \frac{8}{3}+\frac{8\lambda}{175},$$
$$\|\phi_3\|_\lambda^2 = \frac{8}{45}\frac{5\lambda^2+511\lambda+2205}{49\lambda+245}.$$

Proof. The norms of ϕ_0, ϕ_1, ϕ_2 and ϕ_3 are obtained directly from their explicit expressions (2.2). For $n \geqslant 3$, using (1.15) and (1.13),

$$\|\phi_n\|_\lambda^2 = \langle \phi_n, P_n^{(-1,-1)}\rangle_\lambda$$
$$= \lambda\int_{-1}^{1} \phi_n(x)(P_n(x)+d_nP_{n-2}(x))x^2dx + \int_{-1}^{1} \phi_n'(x)nP_{n-1}(x)dx$$
$$= \lambda\int_{-1}^{1} \phi_n(x)(x^2P_n(x)+d_nx^2P_{n-2}(x))dx + n^2\|P_{n-1}\|^2.$$

Applying (1.19),

$$\|\phi_n\|_\lambda^2 = \lambda \int_{-1}^{1} \phi_n(x)\Big(P_{n+2}(x) + (\gamma_{n+1}+\gamma_n)P_n(x) + \gamma_n\gamma_{n-1}P_{n-2}(x) \\ + d_n(P_n(x) + (\gamma_{n-1}+\gamma_{n-2})P_{n-2}(x) + \gamma_{n-2}\gamma_{n-3}P_{n-4}(x)\Big)dx \\ + n^2\|P_{n-1}\|^2,$$

and using the orthogonality of the Legendre polynomials, we get

$$\|\phi_n\|_\lambda^2 = \lambda(\gamma_{n+1}+\gamma_n+d_n)\|P_n\|^2 \\ + \lambda\Big(\gamma_n\gamma_{n-1} + d_n(\gamma_{n-1}+\gamma_{n-2})\Big)\int_{-1}^{1}\phi_n(x)P_{n-2}(x)dx \\ + \lambda\, d_n\gamma_{n-2}\gamma_{n-3}\int_{-1}^{1}\phi_n(x)P_{n-4}(x)dx + n^2\|P_{n-1}\|^2.$$

But, from Proposition 2.1 and (1.15)

$$\int_{-1}^{1}\phi_n(x)P_{n-2}(x)dx \\ = \int_{-1}^{1}\Big(P_n^{(-1,-1)}(x) - a_n\phi_{n-2}(x) - b_n\phi_{n-4}(x)\Big)P_{n-2}(x)dx \\ = \int_{-1}^{1}\Big(P_n(x) + d_nP_{n-2}(x) - a_n\phi_{n-2}(x) - b_n\phi_{n-4}(x)\Big)P_{n-2}(x)dx \\ = (d_n - a_n)\|P_{n-2}\|^2.$$

Using the same argument as above and the previous integral for $n-2$, we get, for $n \geqslant 4$,

$$\int_{-1}^{1}\phi_n(x)P_{n-4}(x)dx \\ = \int_{-1}^{1}\Big(P_n(x) + d_nP_{n-2}(x) - a_n\phi_{n-2}(x) - b_n\phi_{n-4}(x)\Big)P_{n-4}(x)dx \\ = -a_n\int_{-1}^{1}\phi_{n-2}(x)P_{n-4}(x)dx - b_n\|P_{n-4}\|^2 \\ = -a_n(d_{n-2} - a_{n-2})\|P_{n-4}\|^2 - b_n\|P_{n-4}\|^2.$$

Then

$$\|\phi_n\|_\lambda^2 = \lambda(\gamma_{n+1}+\gamma_n+d_n)\|P_n\|^2 \\ + \lambda\Big(\gamma_n\gamma_{n-1} + d_n(\gamma_{n-1}+\gamma_{n-2})\Big)(d_n - a_n)\|P_{n-2}\|^2 \\ - \lambda\, d_n\gamma_{n-2}\gamma_{n-3}\Big(a_n(d_{n-2} - a_{n-2}) + b_n\Big)\|P_{n-4}\|^2 + n^2\|P_{n-1}\|^2.$$

Taking into account that $\gamma_n\gamma_{n-1}\|P_{n-2}\|^2 = \|P_n\|^2$ we obtain

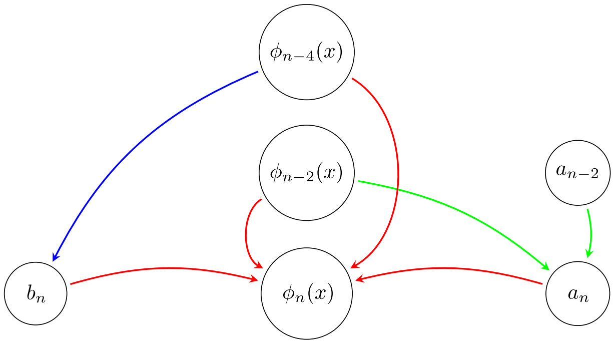

Fig. 1. Relation between norms and coefficients for $n \geqslant 4$. Color code for the arrows: blue arrows, step 1: we use (2.5); green arrows, step 2: we use (2.4); red arrows, step 3: we use (2.3). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\begin{aligned}\|\phi_n\|_\lambda^2 &= \lambda(\gamma_{n+1}+\gamma_n+2d_n-a_n)\|P_n\|^2\\&\quad+\lambda d_n\Big((\gamma_{n-1}+\gamma_{n-2})(d_n-a_n)-a_n(d_{n-2}-a_{n-2})-b_n\Big)\|P_{n-2}\|^2\\&\quad+n^2\|P_{n-1}\|^2. \quad\square\end{aligned}$$

Remark 2.4. From Proposition 2.1 we get an algorithm to generate the family of Sobolev orthogonal polynomials $\{\phi_n(x)\}_{n\geqslant 0}$ and the coefficients a_n and b_n in terms of the previous ones. We compute the even index and the odd index separately. First of all, from the explicit expressions of $\phi_n(x)$ and $P_n^{(-1,-1)}(x)$, for $n=0,1,2,3$, given in (2.2) and (1.12), we initialize our algorithm with $a_2=-\frac{2}{5}$, $a_3=-\frac{2\lambda}{7(\lambda+5)}$.

Therefore, given $n\geqslant 4$, we suppose that we have computed all the Sobolev orthogonal polynomials $\{\phi_k(x)\}_{k=0}^{n-1}$, and the coefficients $\{a_k\}_{k=2}^{n-1}$, $\{b_k\}_{k=4}^{n-1}$. The algorithm is given by the following steps

1. $\phi_{n-4}(x)\longrightarrow b_n$, by using expression (2.5), blue arrow,
2. $(\phi_{n-2}(x), a_{n-2})\longrightarrow a_n$, using (2.4), green arrows,
3. $(\phi_{n-4}(x), \phi_{n-2}(x), b_n, a_n)\longrightarrow \phi_n$, using (2.3), red arrows.

In a graphic mode, we show the algorithm in Fig. 1.

Now we study some asymptotic relations of the norms of the Sobolev orthogonal polynomials. The first one deals with the behavior of the Sobolev norm of the monic polynomials $\phi_n(x)$ in terms of the L^2 norm of the monic Legendre orthogonal polynomials, while the second one focus the attention on the ratio of the Sobolev norms of two consecutive monic polynomials $\phi_n(x)$ and $\phi_{n-1}(x)$. The behavior of such norms will be a useful tool in order to get the outer relative asymptotics of Sobolev polynomials $\phi_n(x)$ in terms of Legendre polynomials to be considered later on.

Proposition 2.5. *The following asymptotic relations hold.*

(i) $\displaystyle\lim_{n\to+\infty}\frac{\|\phi_n\|_\lambda^2}{n^2\|P_{n-1}\|^2}=1.$

(ii) $\displaystyle\lim_{n\to+\infty}\frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-1}\|_\lambda^2}=\frac{1}{4}.$

Notice that the limits are independent of λ.

Proof. According to the extremal property of Legendre polynomials we get

$$\|\phi_n\|_\lambda^2=\lambda\int_{-1}^{1}x^2\phi_n^2(x)dx+\int_{-1}^{1}(\phi_n'(x))^2dx\geqslant\lambda\|P_{n+1}\|^2+n^2\|P_{n-1}\|^2$$

On the other hand, by (1.13) and (1.15) and taking into account the extremal property of monic Sobolev orthogonal polynomials, we deduce

$$\|\phi_n\|_\lambda^2 \leqslant \|P_n^{(-1,-1)}\|_\lambda^2 = \lambda \int_{-1}^{1} x^2 (P_n^{(-1,-1)}(x))^2 dx + n^2 \int_{-1}^{1} (P_{n-1}(x))^2 dx$$
$$= \lambda \int_{-1}^{1} \Big(x(P_n(x) + d_n P_{n-2}(x)) \Big)^2 dx + n^2 \int_{-1}^{1} (P_{n-1}(x))^2 dx.$$

Using the three-term recurrence relation for Legendre polynomials (1.17),

$$\int_{-1}^{1} \Big(x(P_n(x) + d_n P_{n-2}(x)) \Big)^2 dx$$
$$= \int_{-1}^{1} \Big(P_{n+1}(x) + (\gamma_n + d_n) P_{n-1}(x) + d_n \gamma_{n-2} P_{n-3} \Big)^2 dx$$
$$= \|P_{n+1}\|^2 + (\gamma_n + d_n)^2 \|P_{n-1}\|^2 + d_n^2 \gamma_{n-2}^2 \|P_{n-3}(x)\|^2.$$

Therefore,

$$\|\phi_n\|_\lambda^2 \leqslant \lambda \Big(\|P_{n+1}\|^2 + (\gamma_n + d_n)^2 \|P_{n-1}\|^2 + d_n^2 \gamma_{n-2}^2 \|P_{n-3}\|^2 \Big) + n^2 \|P_{n-1}\|^2.$$

Thus we have proved that

$$\lambda \|P_{n+1}\|^2 + n^2 \|P_{n-1}\|^2 \leqslant \|\phi_n\|_\lambda^2$$
$$\leqslant \lambda \Big(\|P_{n+1}\|^2 + (\gamma_n + d_n)^2 \|P_{n-1}\|^2 + d_n^2 \gamma_{n-2}^2 \|P_{n-3}\|^2 \Big) + n^2 \|P_{n-1}\|^2.$$

Dividing by $\|P_{n-1}\|^2$ and using (1.14), as a consequence we have

$$1 \leqslant \lim_{n\to+\infty} \frac{\|\phi_n\|_\lambda^2}{n^2 \|P_{n-1}\|^2} \leqslant 1,$$

so (i) follows.

Now, let us compare the norms of two consecutive Sobolev orthogonal polynomials. We get

$$\lim_{n\to+\infty} \frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-1}\|_\lambda^2} = \lim_{n\to+\infty} \frac{\|\phi_n\|_\lambda^2}{n^2 \|P_{n-1}\|^2} \frac{n^2 \|P_{n-1}\|^2}{\|\phi_{n-1}\|_\lambda^2}$$
$$= \lim_{n\to+\infty} \frac{\|\phi_n\|_\lambda^2}{n^2 \|P_{n-1}\|^2} \frac{n^2 \|P_{n-2}\|^2}{\|\phi_{n-1}\|_\lambda^2} \frac{\|P_{n-1}\|^2}{\|P_{n-2}\|^2}$$
$$= \lim_{n\to+\infty} \frac{\|P_{n-1}\|^2}{\|P_{n-2}\|^2} = \frac{1}{4},$$

where (i) and the expression (1.14) of the norms of monic Legendre polynomials has been used. □

We can also give the asymptotics for the coefficients.

Lemma 2.6. *The coefficients a_n and b_n satisfy*

$$\lim_{n\to+\infty} a_n = 0, \tag{2.7}$$
$$\lim_{n\to+\infty} b_n = 0. \tag{2.8}$$

More precisely, $a_n = o(1/n^2)$ and $b_n = o(1/n^2)$.

Proof. From the expression of the connection coefficients b_n (2.5),

$$b_n = \lambda\, d_n \frac{\|P_{n-2}\|^2}{\|\phi_{n-4}\|_\lambda^2} = \frac{n^2 \|P_{n-2}\|^2}{\|\phi_{n-4}\|_\lambda^2} \frac{\lambda d_n}{n^2},$$

as well as the expression of the coefficients d_n (1.16), we deduce (2.8), and $b_n = o(1/n^2)$.

Moreover, by Proposition 2.1,

$$a_n^2\|\phi_{n-2}\|_\lambda^2 = \|P_n^{(-1,-1)}\|_\lambda^2 - \|\phi_n\|_\lambda^2 - b_n^2\|\phi_{n-4}\|_\lambda^2$$

and, dividing by $\|\phi_{n-2}\|_\lambda^2$, we get

$$a_n^2 = \frac{\|P_n^{(-1,-1)}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} - \frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} - b_n^2\frac{\|\phi_{n-4}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2}. \tag{2.9}$$

But, using (1.15) and (1.13),

$$\frac{\|P_n^{(-1,-1)}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} = \frac{\lambda\int_{-1}^1 x^2(P_n(x)+d_nP_{n-2}(x))^2dx + n^2\|P_{n-1}\|^2}{\|\phi_{n-2}\|_\lambda^2}$$

and from the three-term recurrence relation for Legendre polynomials (1.17)

$$\int_{-1}^{1} x^2(P_n(x)+d_nP_{n-2}(x))^2dx = \|P_{n+1}\|^2 + (\gamma_n+d_n)\|P_{n-1}\|^2 + d_n^2\gamma_{n-2}^2\|P_{n-3}\|^2.$$

Then,

$$\frac{\|P_n^{(-1,-1)}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} = \frac{\lambda}{n^2}\frac{n^2(\|P_{n+1}\|^2+(\gamma_n+d_n)\|P_{n-1}\|^2+d_n^2\gamma_{n-2}^2\|P_{n-3}\|^2)}{\|\phi_{n-2}\|_\lambda^2} + \frac{n^2\|P_{n-1}\|^2}{\|\phi_{n-2}\|_\lambda^2},$$

and by Proposition 2.5

$$\lim_{n\to+\infty}\frac{\|P_n^{(-1,-1)}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} = \lim_{n\to+\infty}\frac{n^2\|P_{n-1}\|^2}{\|\phi_{n-2}\|_\lambda^2} = \lim_{n\to+\infty}\frac{n^2\|P_{n-1}\|^2}{\|\phi_n\|_\lambda^2}\frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2}$$
$$= \lim_{n\to+\infty}\frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} = \lim_{n\to+\infty}\frac{\|\phi_n\|_\lambda^2}{\|\phi_{n-1}\|_\lambda^2}\frac{\|\phi_{n-1}\|_\lambda^2}{\|\phi_{n-2}\|_\lambda^2} = \frac{1}{16}.$$

Replacing in (2.9), we prove (2.7), and $a_n = o(1/n^2)$. □

Using identity (2.6) we get the outer relative asymptotics of $\{\phi_n\}_{n\geqslant 0}$ with respect to the sequence of Legendre polynomials $\{P_n\}_{n\geqslant 0}$.

Theorem 2.7. *Let $\{\phi_n\}_{n\geqslant 0}$ be the Sobolev MOPS associated with (2.1), and let $\{P_n\}_{n\geqslant 0}$ be the Legendre MOPS. Then,*

$$\lim_{n\to+\infty}\frac{\phi_n(x)}{P_n(x)} = \frac{1}{\Phi'(x)},$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$, where

$$\Phi(x) = \frac{x+\sqrt{x^2-1}}{2}.$$

Proof. It is well known (see [18, Th. 8.21.1]) that

$$\lim_{n\to+\infty}\frac{P_{n-1}(x)}{P_n(x)} = \frac{1}{\Phi(x)},$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$. Obviously this implies

$$\lim_{n\to+\infty}\frac{P_{n-k}(x)}{P_n(x)} = \frac{1}{\Phi(x)^k}, \tag{2.10}$$

for $k\geqslant 0$.

Using the notation

$$Y_n(x) := \frac{\phi_n(x)}{P_n(x)}$$
$$\alpha_n(x) := a_n \frac{P_{n-2}(x)}{P_n(x)}$$
$$\beta_n(x) := b_n \frac{P_{n-4}(x)}{P_n(x)}$$
$$\delta_n(x) := 1 + d_n \frac{P_{n-2}(x)}{P_n(x)}$$

equation (2.6) can be rewritten as

$$Y_n(x) + \alpha_n(x) Y_{n-2}(x) + \beta_n(x) Y_{n-4}(x) = \delta_n(x), \tag{2.11}$$

which uniquely defines the sequence $\{Y_n\}_{n\geqslant 0}$ of analytic functions in $\mathbb{C}\setminus[-1,1]$, with their corresponding initial values. It is clear that

$$|Y_n(x)| \leqslant |\alpha_n(x)||Y_{n-2}(x)| + |\beta_n(x)||Y_{n-4}(x)| + |\delta_n(x)|.$$

From (2.7), (2.8), and (2.10) we deduce

$$\lim_{n\to+\infty} \alpha_n(x) = \lim_{n\to+\infty} \beta_n(x) = 0, \tag{2.12}$$

thus, for a fixed $\gamma \in \mathbb{R}$ with $0 < \gamma < \frac{1}{2}$ and a given compact subset of $\mathbb{C}\setminus[-1,1]$ there exists $n_0 \in \mathbb{N}$ such that

$$|\alpha_n(x)| < \gamma, \quad |\beta_n(x)| < \gamma, \text{ for } n \geqslant n_0.$$

In the same way, from

$$\lim_{n\to+\infty} \delta_n(x) = 1 - \frac{1}{4}\frac{1}{\Phi(x)^2}, \tag{2.13}$$

and the inequality $|\Phi(x)| > \frac{1}{2}$ for $x \notin [-1,1]$, we deduce that there exist $B > 0$ and $n_1 \in \mathbb{N}$ such that

$$|\delta_n(x)| < B, \quad n \geqslant n_1.$$

Then, for $n \geqslant \max(n_0, n_1)$, we have

$$|Y_n(x)| < \gamma|Y_{n-2}(x)| + \gamma|Y_{n-4}(x)| + B. \tag{2.14}$$

Now, if we iterate inequality (2.14) for $0 < k < \frac{n}{2}$ we get

$$|Y_n(x)| < p_k|Y_{n-2k}(x)| + q_k|Y_{n-2k-2}(x)| + r_k,$$

where the sequences $\{p_k\}_{k\geqslant 0}$, $\{q_k\}_{k\geqslant 0}$, and $\{r_k\}_{k\geqslant 0}$ satisfy

$$p_{k+1} = \gamma p_k + q_k,$$
$$q_{k+1} = \gamma p_k,$$
$$r_{k+1} = r_k + B p_k,$$

which gives

$$p_{k+1} = \gamma p_k + \gamma p_{k-1},$$
$$r_{k+1} = B\sum_{j=0}^{k} p_k,$$

with initial conditions $p_0 = 1$, $p_1 = \gamma$. Solving the difference equation, we can write

$$p_k = \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^{k+1} - \lambda_2^{k+1}\right),$$
$$q_k = \gamma\frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^{k} - \lambda_2^{k}\right),$$
$$r_k = B\frac{1}{\lambda_1 - \lambda_2}\left(\frac{1-\lambda_1^{k+1}}{1-\lambda_1} - \frac{1-\lambda_2^{k+1}}{1-\lambda_2}\right),$$

where λ_1 and λ_2 are the two different real zeros of the polynomial $p(\lambda) = \lambda^2 - \gamma\lambda - \gamma$. Condition $0 < \gamma < \frac{1}{2}$ implies $|\lambda_1| < 1$ and $|\lambda_2| < 1$, which gives

$$\lim_{n\to+\infty} p_n = \lim_{n\to+\infty} q_n = 0, \quad \lim_{n\to+\infty} r_n = \frac{1}{1-2\gamma}.$$

So, we have

$$|Y_{2n}(x)| < p_{n-1}|Y_2(x)| + q_{n-1}|Y_0(x)| + r_{n-1},$$
$$|Y_{2n+1}(x)| < p_{n-1}|Y_3(x)| + q_{n-1}|Y_1(x)| + r_{n-1}.$$

Taking limits when $n \to +\infty$ on the right hand side, we easily deduce that $|Y_n(x)|$ is uniformly bounded. Finally, taking limits in (2.11), from (2.12) and (2.13) we have

$$\lim_{n\to+\infty} Y_n(x) = \lim_{n\to+\infty} \delta_n(x) = 1 - \frac{1}{4\Phi(x)^2} = \frac{\sqrt{x^2-1}}{\Phi(x)} = \frac{1}{\Phi'(x)},$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$. □

Remark 2.8. Notice that in the previous theorem we have extended the outer relative asymptotic results for Sobolev orthogonal polynomials defined by coherent and symmetrically coherent pairs of measures in [12] and [15], respectively, by using a higher order difference equation. Outer relative asymptotics for general Sobolev orthogonal polynomials with respect to more general pairs of measures supported in bounded intervals have been given in [14] and [16].

Remark 2.9. As a direct consequence and taking into account the Hurwitz theorem, we get that for n large enough the zeros of $\phi_n(x)$ are located in $(-1,1)$.

3. The test functions

The test functions for (1.2) should be chosen in the linear space $(x^2-1)\,\mathbb{P}$ of polynomials vanishing at the ends of the interval $[-1,1]$, and they should be orthogonal with respect to the Sobolev inner product (1.3). A basis of such polynomials could be computed applying the Gram-Schmidt process to the family $\{(x^2-1)\,x^k\}_{k\geqslant 0}$. However, in this section we generate a basis in a recursive way and study its properties.

Let us denote by $\{(x^2-1)\psi_n\}_{n\geqslant 0}$ the monic orthogonal polynomials with respect to the Sobolev inner product $\langle\cdot,\cdot\rangle_\lambda$ defined in (1.3).

Observe that

$$\langle (x^2-1)\psi_n, (x^2-1)\psi_m\rangle_\lambda = \lambda\int_{-1}^{1} \psi_n(x)\psi_m(x)x^2(x^2-1)^2 dx$$
$$+ \int_{-1}^{1} [2x\psi_n(x) + (x^2-1)\psi_n'(x)][2x\psi_m(x) + (x^2-1)\psi_m'(x)]dx.$$

We study the term of crossing derivatives and apply integration by parts, obtaining

$$\int_{-1}^{1} x(x^2-1)(\psi_n(x)\psi_m'(x) + \psi_n'(x)\psi_m(x))dx = \int_{-1}^{1} x(x^2-1)(\psi_n(x)\psi_m(x))'dx$$
$$= -\int_{-1}^{1} (3x^2-1)\psi_n(x)\psi_m(x)dx.$$

Therefore, the sequence of monic polynomials $\{\psi_n\}_{n\geqslant 0}$ is orthogonal with respect to the Sobolev inner product

$$\langle p,q\rangle_S = \int_{-1}^{1} (\lambda x^2(1-x^2)^2 + 2(1-x^2))p(x)q(x)dx + \int_{-1}^{1} p'(x)q'(x)(1-x^2)^2dx. \tag{3.1}$$

A direct computation shows that

$$\psi_0(x)=1,\ \ \psi_1(x)=x,\ \ \psi_2(x)=x^2-\frac{2\lambda+21}{6\lambda+105},\ \ \psi_3(x)=x^3-\frac{10\lambda+297}{22\lambda+693}x,$$

and

$$\|\psi_0\|_S^2=\frac{2(8\lambda+140)}{105},\qquad \|\psi_1\|_S^2=\frac{2(8\lambda+252)}{315},$$
$$\|\psi_2\|_S^2=\frac{2^7(\lambda^2+144\lambda+2079)}{3^3\,5(154\lambda+2695)}.$$

Remark 3.1. The measures involved in the above inner product constitute a generalized symmetrically coherent pair. If we denote

$$w_0(x)=\lambda x^2(1-x^2)^2+2(1-x^2),\quad w_1(x)=(1-x^2)^2=w^{(2,2)}(x),$$

the respective weight functions, and $\{Q_n(x;w_0)\}_{n\geqslant 0}$ and $\{Q_n(x;w_1)\}_{n\geqslant 0}\equiv\{P_n^{(2,2)}(x)\}_{n\geqslant 0}$ the corresponding sequences of orthogonal polynomials, then

$$Q_n(x;w_1)=\frac{1}{n+1}\frac{d}{dx}Q_{n+1}(x;w_0)+\widetilde{\alpha}_n\frac{d}{dx}Q_{n-1}(x;w_0)+\widetilde{\beta}_n\frac{d}{dx}Q_{n-3}(x;w_1),$$

for $n\geqslant 1$, where $\widetilde{\beta}_n\neq 0$, $n\geqslant 3$, and $\widetilde{\beta}_1=\widetilde{\beta}_2=0$.

For k-coherent pairs of measures and the corresponding sequences of Sobolev orthogonal polynomials, see [9] and [10].

We get the following connection formula between test functions and classical Jacobi polynomials.

Proposition 3.2. *The following relation holds*

$$P_n^{(1,1)}(x)=\psi_n(x)+\widehat{a}_n\psi_{n-2}(x)+\widehat{b}_n\psi_{n-4}(x),\quad n\geqslant 2, \tag{3.2}$$

where

$$\widehat{a}_n=-\lambda\big(\gamma_{n+1}^{(1,1)}+\gamma_n^{(1,1)}+\gamma_{n-1}^{(1,1)}+\gamma_{n-2}^{(1,1)}-\widehat{a}_{n-2}\big)\frac{\|P_n^{(1,1)}\|^2}{\|\psi_{n-2}\|_S^2},\quad n\geqslant 4, \tag{3.3}$$

$$\widehat{b}_n=-\lambda\frac{\|P_n^{(1,1)}\|^2}{\|\psi_{n-4}\|_S^2}\neq 0,\quad n\geqslant 4, \tag{3.4}$$

and

$$\widehat{a}_2=\frac{4\lambda}{15(3\lambda+35)},\qquad \widehat{a}_3=\frac{4\lambda}{77(2\lambda+63)}.$$

Proof. The Fourier expansion of $P_n^{(1,1)}$ in terms of the Sobolev orthogonal polynomials $\{\psi_n\}_{n\geqslant 0}$ yields

$$P_n^{(1,1)}(x)=\psi_n(x)+\sum_{k=0}^{n-1}\mu_{n,k}\psi_k(x),\quad n\geqslant 2,$$

where

$$\mu_{n,k}=\frac{\langle P_n^{(1,1)},\psi_k\rangle_S}{\|\psi_k\|_S^2}.$$

Taking into account (3.1) as well as the orthogonality relations of the Gegenbauer polynomials $P_n^{(1,1)}$ ((1.4) with $\alpha=\beta=1$) we get, for $k\leqslant n-1$,

$$\langle P_n^{(1,1)}, \psi_k\rangle_S = \int_{-1}^{1} P_n^{(1,1)}(x)\psi_k(x)(\lambda x^2(1-x^2)+2)(1-x^2)dx$$
$$+ n\int_{-1}^{1} P_{n-1}^{(2,2)}(x)\psi_k'(x)(1-x^2)^2dx$$
$$= \int_{-1}^{1} P_n^{(1,1)}(x)\psi_k(x)(\lambda x^2(1-x^2)+2)(1-x^2)dx.$$

Therefore $\mu_{n,k}=0, k<n-4$. In addition, due to the symmetry of both $P_n^{(1,1)}$ and ψ_n, it is clear that $\mu_{n,n-1}=\mu_{n,n-3}=0$, and then (3.2) holds.

On the other hand, the iteration of the three term recurrence relation (1.7) of the sequence $\{P_n^{(1,1)}\}_{n\geqslant 0}$ yields

$$\begin{aligned}(\lambda x^2(1-x^2)+2)P_n^{(1,1)}(x) = &-\lambda P_{n+4}^{(1,1)}(x) + r_{n,0}P_{n+2}^{(1,1)}(x) + r_{n,1}P_n^{(1,1)}(x)\\ &+ r_{n,2}P_{n-2}^{(1,1)}(x) + r_{n,3}P_{n-4}^{(1,1)}(x),\end{aligned} \tag{3.5}$$

obtaining

$$r_{n,1} = -\lambda\left[\gamma_{n+1}^{(1,1)}(\gamma_{n+2}^{(1,1)}+\gamma_{n+1}^{(1,1)}+\gamma_n^{(1,1)}-1)+\gamma_n^{(1,1)}(\gamma_{n+1}^{(1,1)}+\gamma_n^{(1,1)}+\gamma_{n-1}^{(1,1)}-1)\right]+2, \tag{3.6}$$
$$r_{n,2} = -\lambda\gamma_n^{(1,1)}\gamma_{n-1}^{(1,1)}(\gamma_{n+1}^{(1,1)}+\gamma_n^{(1,1)}+\gamma_{n-1}^{(1,1)}+\gamma_{n-2}^{(1,1)}), \tag{3.7}$$
$$r_{n,3} = -\lambda\gamma_n^{(1,1)}\gamma_{n-1}^{(1,1)}\gamma_{n-2}^{(1,1)}\gamma_{n-3}^{(1,1)}. \tag{3.8}$$

Therefore, taking into account that

$$\|P_n^{(1,1)}\|^2 = \gamma_n^{(1,1)}\|P_n^{(1,1)}\|^2, \tag{3.9}$$

we get for $n\geqslant 4$

$$\begin{aligned}\widehat{b}_n = &\mu_{n,n-4} = \frac{\int_{-1}^{1} P_n^{(1,1)}(x)\psi_{n-4}(x)(\lambda x^2(1-x^2)+2)(1-x^2)dx}{\|\psi_{n-4}\|_S^2}\\ = &\frac{r_{n,3}\|P_{n-4}^{(1,1)}\|^2}{\|\psi_{n-4}\|_S^2} = -\lambda\frac{\|P_n^{(1,1)}\|^2}{\|\psi_{n-4}\|_S^2}.\end{aligned}$$

Moreover,

$$\widehat{a}_n = \mu_{n,n-2} = \frac{\int_{-1}^{1} P_n^{(1,1)}(x)\psi_{n-2}(x)(\lambda x^2(1-x^2)+2)(1-x^2)dx}{\|\psi_{n-2}\|_S^2}.$$

But the numerator reads as

$$r_{n,2}\|P_{n-2}^{(1,1)}\|^2 + r_{n,3}\int_{-1}^{1} P_{n-4}^{(1,1)}(x)\psi_{n-2}(x)(1-x^2)dx.$$

Taking into account

$$\psi_{n-2}(x) = P_{n-2}^{(1,1)}(x) - \widehat{a}_{n-2}\psi_{n-4}(x) - \widehat{b}_{n-2}\psi_{n-6}(x),$$

we get

$$\widehat{a}_n = \frac{r_{n,2}\|P_{n-2}^{(1,1)}\|^2 - r_{n,3}\widehat{a}_{n-2}\|P_{n-4}^{(1,1)}\|^2}{\|\psi_{n-2}\|_S^2}, \quad n\geqslant 4.$$

Then, (3.3) follows using again (3.9).

The expressions of $\widehat{a}_2, \widehat{a}_3$ follow in a straightforward way from the fact that $P_2^{(1,1)}(x) = \psi_2(x) + \widehat{a}_2\psi_0(x)$ and $P_3^{(1,1)}(x) = \psi_3(x) + \widehat{a}_3\psi_1(x)$ and taking into account the polynomials involved therein have been explicitly given after (3.1). □

The norms of the Sobolev orthogonal polynomials for $n\geqslant 4$ satisfy the following property.

Proposition 3.3.

$$\begin{aligned}\|\psi_n\|_S^2 =& \Big[n(n+3) + r_{n,1} + \lambda \widehat{a}_n (\gamma_{n+1}^{(1,1)} + \gamma_n^{(1,1)} + \gamma_{n-1}^{(1,1)} + \gamma_{n-2}^{(1,1)}) \\ & - \lambda(\widehat{a}_n \widehat{a}_{n-2} - \widehat{b}_n)\Big] \|P_n^{(1,1)}\|^2.\end{aligned}$$

Proof. Using the definition of the Sobolev inner product, (1.9) and the expression (3.5), we get

$$\begin{aligned}\|\psi_n\|_S^2 =& \langle \psi_n, P_n^{(1,1)}\rangle_S = n^2 \|P_{n-1}^{(2,2)}\|^2 + r_{n,1}\|P_n^{(1,1)}\|^2 \\ & + r_{n,2} \int_{-1}^{1} P_{n-2}^{(1,1)}(x)\psi_n(x)(1-x^2)dx \\ & + r_{n,3} \int_{-1}^{1} P_{n-4}^{(1,1)}(x)\psi_n(x)(1-x^2)dx \\ =& n^2 \|P_{n-1}^{(2,2)}\|^2 + r_{n,1}\|P_n^{(1,1)}\|^2 - r_{n,2}\widehat{a}_n \|P_{n-2}^{(1,1)}\|^2 \\ & + r_{n,3}(\widehat{a}_n \widehat{a}_{n-2} - \widehat{b}_n)\|P_{n-4}^{(1,1)}\|^2,\end{aligned}$$

where the constants are given by (3.6), (3.7), and (3.8). Using $r_{n,3}\|P_{n-4}^{(1,1)}\|^2 = -\lambda \|P_n^{(1,1)}\|^2$ and $r_{n,2}\|P_{n-2}^{(1,1)}\|^2 = -\lambda(\gamma_{n+1}^{(1,1)} + \gamma_n^{(1,1)} + \gamma_{n-1}^{(1,1)} + \gamma_{n-2}^{(1,1)})\|P_n^{(1,1)}\|^2$ the result follows. □

Remark 3.4. An alternative expression for the norm of ψ_n follows from (3.2).

$$\|P_n^{(1,1)}\|_S = \|\psi_n\|_S^2 + \widehat{a}_n^2 \|\psi_{n-2}\|_S^2 + \widehat{b}_n^2 \|\psi_{n-4}\|_S^2.$$

But

$$\|P_n^{(1,1)}\|_S = n^2 \|P_{n-1}^{(2,2)}\|^2 + r_{n,1}\|P_n^{(1,1)}\|^2.$$

As a consequence,

$$\|\psi_n\|_S^2 = n^2 \|P_{n-1}^{(2,2)}\|^2 + r_{n,1}\|P_n^{(1,1)}\|^2 - \widehat{a}_n^2 \|\psi_{n-2}\|_S^2 - \widehat{b}_n^2 \|\psi_{n-4}\|_S^2.$$

Remark 3.5. As in the previous Section, using Proposition 3.2, the algorithm to generate the second family of Sobolev orthogonal polynomials $\{\psi_n(x)\}_{n\geqslant 0}$ and the coefficients $\widehat{a}_n$ and $\widehat{b}_n$ in terms of the previous ones has the scheme given in Fig. 1, substituting the ϕ's by ψ's, the a's by $\widehat{a}$'s, the b's by $\widehat{b}$'s, and initializing the algorithm this time with $\widehat{a}_2 = \frac{4\lambda}{15(3\lambda+35)}$, $\widehat{a}_3 = \frac{4\lambda}{77(2\lambda+63)}$, and $\widehat{b}_4 = -\frac{2\lambda}{2079(2\lambda+35)}$.

4. Fourier analysis

Let denote by

$$\sum_{n=0}^{+\infty} \widehat{u}_n (x^2-1)\psi_n(x),$$

the Fourier expansion of the solution of the BVP (1.2) in terms of the orthogonal sequence $\{(x^2-1)\psi_n(x)\}_{n\geqslant 0}$, where

$$\widehat{u}_n = \frac{\langle u, (x^2-1)\psi_n\rangle_\lambda}{\langle (x^2-1)\psi_n, (x^2-1)\psi_n\rangle_\lambda} = \frac{\langle u, (x^2-1)\psi_n\rangle_\lambda}{\|(x^2-1)\psi_n\|_\lambda^2}.$$

On the one hand, since $u(-1) = u(1) = 0$, there exists a function $v(x)$ such that $u(x) = (x^2-1)v(x)$. Then, using (3.1), we get

$$\begin{aligned}\widehat{u}_n =& \frac{\langle u, (x^2-1)\psi_n\rangle_\lambda}{\langle (x^2-1)\psi_n, (x^2-1)\psi_n\rangle_\lambda} = \frac{\langle (x^2-1)v, (x^2-1)\psi_n\rangle_\lambda}{\|(x^2-1)\psi_n\|_\lambda^2} \\ =& \frac{\langle v, \psi_n\rangle_S}{\|\psi_n\|_S^2} = \widehat{v}_n,\end{aligned}$$

where $\widehat{v}_n$ is the coefficient of the Fourier expansion of v in terms of the Sobolev orthogonal basis $\{\psi_n(x)\}_{n\geqslant 0}$, as

$$\sum_{n=0}^{+\infty}\widehat{v}_n\psi_n(x).$$

On the other hand,

$$\begin{aligned}\widehat{u}_n\|\psi_n\|_S^2 =&\lambda\int_{-1}^{1} x^2u(x)(x^2-1)\psi_n(x)dx+\int_{-1}^{1} u'(x)[(x^2-1)\psi_n(x)]'dx\\ &=\int_{-1}^{1}[-u''(x)+\lambda x^2u(x)](x^2-1)\psi_n(x)dx\\ &=\int_{-1}^{1} f(x)(x^2-1)\psi_n(x)dx =: \widetilde{f}(n).\end{aligned}$$

This is the so called diagonalized spectral method for the BVP.

Taking into account (3.2), we get

$$\int_{-1}^{1} f(x)P_n^{(1,1)}(x)(x^2-1)dx=\widetilde{f}(n)+\widehat{a}_n\widetilde{f}(n-2)+\widehat{b}_n\widetilde{f}(n-4).$$

Moreover, if we denote by $\overline{f}_n^{(1,1)}$ the nth Fourier coefficient of the function $f(x)$ with respect to the basis of Jacobi polynomials, $\{P_n^{(1,1)}\}_{n\geqslant 0}$, the above expression can be written as

$$-\overline{f}_n^{(1,1)}\|P_n^{(1,1)}\|^2=\widetilde{f}(n)+\widehat{a}_n\widetilde{f}(n-2)+\widehat{b}_n\widetilde{f}(n-4).$$

Thus, we have a method to generate recursively the sequence $\{\widetilde{f}(n)\}_{n\geqslant 0}$ assuming the initial conditions

$$\begin{aligned}\widetilde{f}(0)&=\int_{-1}^{1} f(x)(x^2-1)dx, & \widetilde{f}(1)&=\int_{-1}^{1} f(x)x(x^2-1)dx,\\ \widetilde{f}(2)&=\int_{-1}^{1} f(x)\psi_2(x)(x^2-1)dx, & \widetilde{f}(3)&=\int_{-1}^{1} f(x)\psi_3(x)(x^2-1)dx.\end{aligned}$$

Notice that an alternative way to use (3.2) is the following

$$\widehat{u}_n\|\psi_n\|_S^2+\widehat{a}_n\widehat{u}_{n-2}\|\psi_{n-2}\|_S^2+\widehat{b}_n\widehat{u}_{n-4}\|\psi_{n-4}\|_S^2=\int_{-1}^{1} f(x)P_n^{(1,1)}(x)(x^2-1)dx. \tag{4.1}$$

Substituting expressions (3.3) and (3.4), we get

$$\begin{aligned}\widehat{u}_n\|\psi_n\|_S^2-&\lambda\left[\sum_{i=n-2}^{n+1}\gamma_i^{(1,1)}-\widehat{a}_{n-2}\right]\frac{\|P_n^{(1,1)}\|^2}{\|\psi_{n-2}\|_S^2}\widehat{u}_{n-2}\|\psi_{n-2}\|_S^2\\ &-\lambda\frac{\|P_n^{(1,1)}\|^2}{\|\psi_{n-4}\|_S^2}\widehat{u}_{n-4}\|\psi_{n-4}\|_S^2\\ &=-\overline{f}_n^{(1,1)}\|P_n^{(1,1)}\|^2,\end{aligned}$$

and finally

$$\widehat{u}_n=\frac{\|P_n^{(1,1)}\|^2}{\|\psi_n\|_S^2}\left\{\lambda\left[\sum_{i=n-2}^{n+1}\gamma_i^{(1,1)}-\widehat{a}_{n-2}\right]\widehat{u}_{n-2}+\lambda\widehat{u}_{n-4}-\overline{f}_n^{(1,1)}\right\}.$$

Remark 4.1. According to (1.10) and (1.9)

$$(x^2-1)P_n^{(1,1)}(x) = P_{n+2}(x) - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_n(x),$$

we have the expression of the right hand side of (4.1) in terms of Fourier coefficients of f in terms of the sequence of Legendre polynomials $\{P_n(x)\}_{n\geqslant 0}$.

5. Numerical experiments

In this section, we explore the reliability and accuracy of the Sobolev spectral method for solving elliptic boundary problems on the interval $[-1,1]$.

We examine the second order Dirichlet boundary value problem associated to the non-homogeneous Schrödinger equation with an harmonic potential

$$\begin{aligned} -u''(x) + \lambda x^2 u(x) &= f(x), \\ u(-1) = u(1) &= 0. \end{aligned}$$

For $\lambda = 1$, the sequence of monic polynomials $\{(x^2-1)\psi_n(x)\}_{n\geqslant 0}$ orthogonal with respect to the Sobolev inner product

$$\langle u, v\rangle_\lambda = \int_{-1}^{1} u(x)\, v(x)\, x^2\, dx + \int_{-1}^{1} u'(x)\, v'(x)\, dx,$$

can be easily computed. For instance, the first five monic polynomials are given by

$$\begin{aligned} \psi_0(x) &= 1, \\ \psi_1(x) &= x, \\ \psi_2(x) &= x^2 - \frac{23}{111}, \\ \psi_3(x) &= x^3 - \frac{307}{715}x, \\ \psi_4(x) &= x^4 - \frac{9641}{14456}x^2 + \frac{7779}{159016}. \end{aligned}$$

In our first example, we analyze the case where $f(x)$ is a $\mathcal{C}^\infty$ function, namely $f(x) = e^x(-1-4x-2x^2+x^4)$. Notice that the solution of the boundary value problem is $u(x) = (x^2-1)e^x$.

Let us denote by

$$\sum_{n=0}^{+\infty} \hat{u}_n (x^2-1)\psi_n(x),$$

the Fourier–Sobolev expansion of $u(x)$. As we know, the coefficients $\hat{u}_n$ satisfy

$$\begin{aligned} \hat{u}_n \|(x^2-1)\psi_n\|_\lambda^2 &= \int_{-1}^{1} x^2 u(x)(x^2-1)\psi_n(x)dx + \int_{-1}^{1} u'(x)[(x^2-1)\psi_n(x)]'dx \\ &= \int_{-1}^{1} [-u''(x) + x^2 u(x)](x^2-1)\psi_n(x)dx \\ &= \int_{-1}^{1} f(x)(x^2-1)\psi_n(x)dx. \end{aligned}$$

In Fig. 2, we plot the solution $u(x)$ and the successive approximants

$$u_N(x) = \sum_{n=0}^{N} \hat{u}_n (x^2-1)\psi_n(x),$$

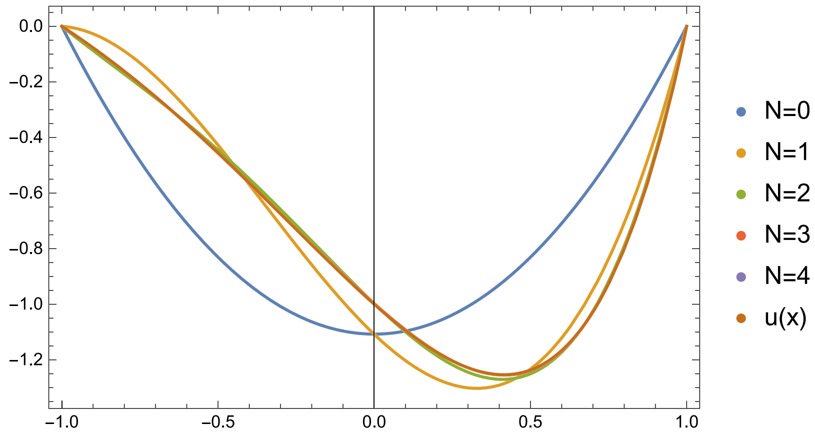

Fig. 2. The solution $u(x)$ and the successive approximants $u_N(x)$ for $N = 0, 1, \ldots, 4$.

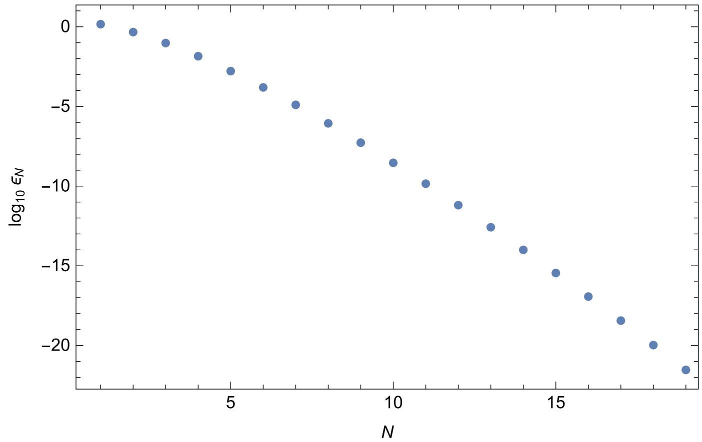

Fig. 3. A logarithmic plot of the errors for $N = 0, 1, \ldots, 18$.

for $N = 0, 1, \ldots, 4$.

In Fig. 3, we show a logarithmic plot of the errors in the Sobolev norm

$$\epsilon_N = \|u(x) - u_N(x)\|_\lambda,$$

for $N = 0, 1, \ldots, 18$. Clearly, the near straight aligned points indicate an exponential convergence rate. Observe that this happens not only for the approximants but also for their derivatives.

In our last example, we consider a BVP where both functions $f(x)$ and $u(x)$ are non-differentiable at $x = -1$. In fact, we take

$$f(x) = \frac{x\left(4(x-1)x(x+1)^2 - 15\right) - 9}{4\sqrt{x+1}},$$

and the solution is $u(x) = (x^2 - 1)\sqrt{x+1}$. Of course, as a consequence of the non-differentiability at $x = -1$, we can not expect an exponential convergence rate of the approximants.

In Fig. 4, we show a simultaneous logarithmic plot of the errors in the Sobolev norm

$$\epsilon_N = \|u(x) - u_N(x)\|_\lambda$$

and the errors in the L_2 norm

$$\tilde{\epsilon}_N = \|u(x) - u_N(x)\|_{L_2} = \left(\int_{-1}^{1} (u(x) - u_N(x))^2 dx\right)^{\frac{1}{2}},$$

for $N = 0, 1, \ldots, 18$. The graph indicates an algebraic convergence rate for the approximants in the L_2 norm.

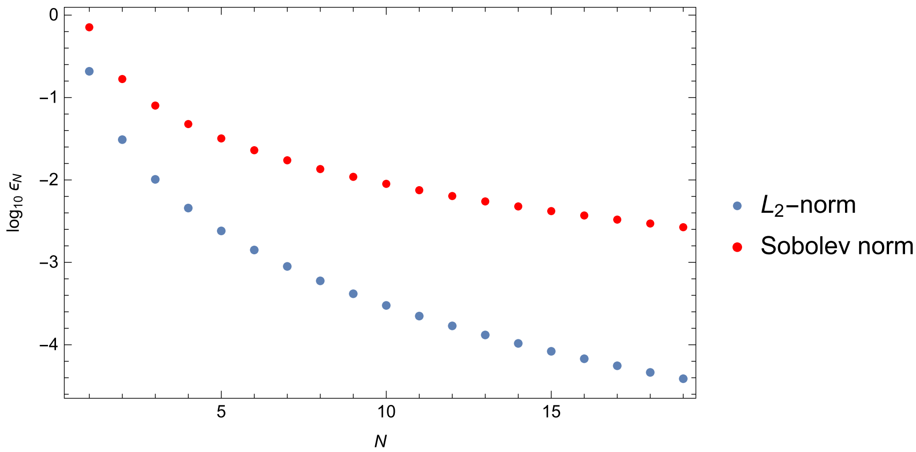

Fig. 4. A logarithmic plot of the Sobolev and L_2 errors for $N = 0, 1, \ldots, 18$.

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