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Tesis Doctoral

Avances en aproximación en el disco unidad. El caso Zernike

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PhD Thesis

Advances in approximation in the unit disk. The Zernike case

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Dedicatoria

Este trabajo está dedicado a la memoria de mi padre Regis Nery Recarte (1954 – 2022). Él no pudo verlo terminado pero sé lo orgulloso que estaba con cada logro de sus hijos.

Resumen

El objetivo de esta Tesis Doctoral es el estudio de aproximantes para funciones definidas en la bola unidad. Estos aproximantes se consideran utilizando dos enfoques diferentes: aproximación por mínimos cuadrados y aproximación uniforme. Como es bien conocido, la aproximación por mínimos cuadrados se basa en considerar productos escalares definidos sobre la bola unidad, y la aproximación uniforme se basa en considerar la norma uniforme, en este caso en el disco unidad. Damos especial énfasis a la aproximación por mínimos cuadrados basada en los polinomios ortogonales de Zernike, es decir, polinomios bivariados que son ortogonales con respecto a la medida de Lebesgue en el disco unidad, debido a las aplicaciones en Óptica y Optometría.

El primer enfoque se basa en aproximar funciones definidas en la bola d-dimensional mediante el estudio de modificaciones del producto escalar clásico (que incluye los polinomios de Zernike como caso particular cuando la función peso es una constante) mediante operadores diferenciales multivariados como gradientes o Laplacianos, los llamados productos escalares de Sobolev, de dos maneras diferentes. Primero, tratamos la bola unidad d-dimensional dotada con un producto escalar construido agregando un punto de masa en el origen al producto escalar clásico de la bola aplicado a los gradientes de las funciones. Determinamos una base explícita de polinomios ortogonales, y estudiamos las propiedades de aproximación de los desarrollos

de Fourier en términos de esta base. Deducimos relaciones entre las sumas parciales de Fourier en términos de los nuevos polinomios ortogonales y las sumas parciales de Fourier en términos de los polinomios clásicos sobre la bola. También damos una estimación del error de aproximación por polinomios de grado no mayor a n en el espacio de Sobolev correspondiente, demostrando que podemos aproximar una función usando su gradiente.

El siguiente capítulo se dedica al estudio de la estructura ortogonal inducida por un producto escalar que involucra los Laplacianos de las funciones, una extensión del producto estudiado estudiado por Xu en 2008 intentando resolver el problema planteado por Atkinson y Hansen de encontrar la solución numérica de la ecuación de Poisson no lineal con condiciones de contorno nulas en la bola unitaria de d dimensiones. Analizamos los polinomios ortogonales asociados a este nuevo producto escalar, demostrando que satisfacen una ecuación diferencial parcial de cuarto orden. También estudiamos las propiedades de aproximación de las sumas de Fourier con respecto a estos polinomios ortogonales y estimamos el error de aproximación simultánea de una función, sus derivadas parciales y su Laplaciano. En ambos casos, se presentan ejemplos numéricos para ilustrar el comportamiento de la aproximación en la base de Sobolev.

El segundo enfoque consiste en la construcción y estudio de sucesiones de operadores tipo Bernstein que actúan sobre funciones bivariadas definidas en el disco unitario. Para ello, se estudian los operadores tipo Bernstein bajo una transformación de dominio, se analizan los operadores bivariados de Bernstein-Stancu y se introducen los operadores tipo Bernstein en los cuadrantes del disco mediante transformaciones continuamente diferenciables de la función. Se establecen resultados de convergencia para funciones continuas y se estima la velocidad de convergencia. Se presentan varios ejemplos numéricos interesantes que comparan las aproximaciones utilizando los operadores de Bernstein-Stancu desplazados y los operadores tipo Bernstein sobre los cuadrantes del disco.

Abstract

The objective of this Doctoral Thesis is the study of approximants for functions defined in the unit ball. These approximants are considered using two different approaches: least-squares approximation and uniform approximation. As is well known, least-squares approximation is based on considering inner products defined on the unit ball, and uniform approximation is based on considering the uniform norm, in this case on the unit disk . We give special emphasis to the least-squares approximation based on Zernike orthogonal polynomials, that is, bivariate polynomials which are orthogonal with respect to the Lebesgue measure on the unit disk, due to applications in Optics and Optometry.

The first approach is based on approximating functions defined on the d-dimensional ball by studying modifications of the classical inner product (that includes the Zernike polynomials as particular case when the weight function is a constant function) by means of multivariate differential operators such as gradients or Laplacians, the so-called Sobolev inner products in two different ways. First, we deal with the d-dimensional unit ball equipped with an inner product constructed by adding a mass point at the origin to the classical ball inner product applied to the gradients of the functions. We determine an explicit orthogonal polynomial basis, and we study approximation properties of Fourier expansions in terms of this basis. We deduce relations between the partial Fourier sums in terms of the new orthogonal

polynomials and the partial Fourier sums in terms of the classical ball polynomials. We also give an estimate of the approximation error by polynomials of degree at most n in the corresponding Sobolev space, proving that we can approximate a function by using its gradient.

The next chapter is devoted to study the orthogonal structure induced by an inner product involving the Laplacians of the functions, an extension of the inner product studied by Xu in 2008 trying to solve the problem posed by Atkinson and Hansen of finding the numerical solution of the nonlinear Poisson equation with zero boundary conditions on the d-dimensional unit ball. We analyze the orthogonal polynomials associated with this new inner product, proving that they satisfy a fourth-order partial differential equation. We also study the approximation properties of the Fourier sums with respect to these orthogonal polynomials and we estimate the error of simultaneous approximation of a function, its partial derivatives, and its Laplacian. In both cases, numerical examples are given to illustrate the approximation behavior of the Sobolev basis.

For the second approach, we construct and study sequences of operators of Bernstein type acting on bivariate functions defined on the unit disk. To this end, we study Bernstein-type operators under a domain transformation, we analyze the bivariate Bernstein-Stancu operators, and we introduce Bernstein-type operators on disk quadrants by means of continuously differentiable transformations of the function. We state convergence results for continuous functions and we estimate the rate of convergence. Several interesting numerical examples are given, comparing approximations using the shifted Bernstein-Stancu and the Bernstein-type operator on disk quadrants.

Contents

Introduction

Any sufficiently regular phase function defined on the unit disk can be represented by its Fourier expansion in terms of the Zernike polynomials with certain coefficients. The alteration of these coefficients allows detection of the possible aberrations of the studied optical system. Zernike polynomials show many applications in the manufacture of precision optical devices, because they allow the characterization of higher order errors observed in the interferometric analysis to achieve the desired performance of the system. They are also used to describe the aberrations of the cornea or lens from an ideal spherical shape in optometry and ophthalmology. Finally, they can be effectively used in adaptive optics to cancel atmospheric distortion, allowing images to be improved in infrared (IR) or visual astronomy and satellite images. However, in practice, Zernike polynomials present convergence problems when working on the edges of the disk, producing distortions that could be eliminated by dealing with a modification of the associated inner product ([24], [43], among others). This modification could be, for instance, of Sobolev type.

Sobolev orthogonal polynomials, that is, families of polynomials that are orthogonal with respect to inner product involving both values of functions as well as derivative operators such as partial derivatives, gradients, normal derivatives, Laplacians, and others, have been recently studied. A recent survey on this topic can be found in [30]. A clear range of application of orthogonal polynomials is the field of approximation of functions, with multiple technological applications within the multivariate case. Recently, there has been a renovated interest for approximants based on multivariate Sobolev orthogonal polynomials, showing that it is not necessary to use the derivatives of the function. Examples of this kind of studies can be found in [53], [38], [31], among others.

Sobolev orthogonal polynomials in several variables have already been applied in the analysis of polishing tools in the manufacture of optical surfaces [52]. In the case of applications of orthogonal polynomials to the clinical problems related to human vision, we consider interesting, for example, the study of the efficiency of bivariate Zernike-Sobolev orthogonal polynomials within this context. The approximation of functions in the unit disk is a research topic with multiple applications in the reconstruction of functions defined within that domain. This topic has found a wide range of scientific and technological applications, particularly in fields such as Optics and Ophthalmology, where the reconstruction of surfaces defined in circular domains is essential. The reconstruction of three-dimensional surfaces is a current and complex topic that can be approached from various perspectives. There are several approaches that have been proposed to solve this problem, and in this doctoral dissertation, we will address two of them: least-squares approximation and uniform approximation.

Our main contributions are based on the objectives detailed below.

- First, we are interested in finding approximating functions defined on the d-dimesional ball by studying modifications of the classical inner product (that includes the Zernike polynomials as particular case when the weight function is a constant function) by means of multivariate differential operators such as gradients or Laplacians, in two different ways.

- Second, we are interested in constructing an extension of the Bernstein operator to approximate functions defined in the unit disk. Thus, we will mainly study two types of modifications: by transforming the argument of the function to be approximated, and by defining a suitable basis of functions. We will study two types of Bernstein approximants and compare them through several examples.

This work is organized as follows:

- In Chapter 1, we establish the fundamental background and notation necessary for this doctoral dissertation. It covers the general properties of orthogonal polynomials in several variables, including basic definitions of polynomials and orthogonal polynomials over the unit ball. The current state of research on Sobolev orthogonal polynomials on the unit ball is also presented. Furthermore, the properties of univariate bivariate Bernstein operators are discussed. Overall, this Chapter provides a comprehensive overview of the background information needed to understand the rest of the dissertation.

- In Chapter 2, we focus on the Sobolev inner product with a mass point at the origin in a d-dimensional unit ball, which is constructed by adding a mass point at zero to the classical ball inner product applied to the gradients of the functions. This inner product is defined as follows:

  $$\langle f, g\rangle_{\nabla,\mu} = f(0)g(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla g(x)\,(1-\|x\|^2)^\mu\,dx,$$

  where $\lambda > 0$ and $\mu \geqslant 0$. Our objective is to study the analytic properties of approximation by means of the corresponding Fourier sums, which can be computed without using derivatives. In contrast to the classical orthogonal expansions, we show that if we only know the gradients of the functions, we can compute approximants using the Sobolev Fourier orthogonal expansions, and the approximation is similar or even better.

  Our focus is on extending and completing the results obtained by Xu in [52] to the more general case $\mu \geqslant 0$. Finally, we illustrate our results with numerical examples.

  These results are published in

  - [31] M. E. Marriaga, T. E. Pérez, M. A. Piñar, and M. J. Recarte. Approximation via gradients on the ball. the Zernike case. *Journal of Computational and Applied Mathematics*, 430:115258, 2023.

- In Chapter 3, we analyze the impact of an additional term in the Sobolev inner product introduced by Y. Xu in [51] used to find the numerical solution of the Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions (Atkinson and Hansen [2]). We modify the inner product by introducing a term on the spherical border of the ball taking into account the possible information that may be observed from some modification of the nonlinear Poisson equation:

$$\langle f, g \rangle_\Delta = \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi)\, g(\xi)\, d\sigma(\xi) \\ + \frac{1}{8\,\sigma_{d-1}} \int_{\mathbf{B}^d} \Delta[(1-\|x\|^2) f(x)]\, \Delta[(1-\|x\|^2) g(x)]\, dx.$$

  This term is introduced by means of a real positive constant $\lambda > 0$ to modulate the influence of that term in the problem, and the normalization constants are chosen to simplify expressions in the sequel.

  Appart from the exhaustive description of the orthogonal structure, giving explicit expressions for the basis, we provide the Fourier coefficients of the approximations and related them with the Fourtier coefficients for the original case. We must remark that the spherical term in the Sobolev inner product has influence over the angular part of the orthogonal expansion. Moreover, the error of approximation for the orthogonal expansions with respect to the Sobolev inner products appearing in this work (in particular, the one introduced in [51]) have not been previously studied in the literature.

  The contents of this chapter appear in

  – [32] M. E. Marriaga, T. E. Pérez, and M.J. Recarte. Simultaneous approximation via Laplacians on the unit ball. Submitted.

- In Chapter 4, we find the extension of the Bernstein operator for approximating functions defined on the unit disk. Specifically, we present and analyze two types of Bernstein type approximants and compare them using various examples. To achieve this, we explore two modifications: transforming the argument of the function to be approximated and defining an appropriate basis of functions. We also review

Stancu's method for obtaining Bernstein-type operators in two variables [44] and describe an extension of linear combinations of univariate Bernstein operators that provide better order of approximation. Finally, we present several numerical examples to compare the approximation results obtained using both Bernstein-type operators on the disk and the linear combinations introduced in our analysis.

Overall, our findings provide valuable insights into the extension of the Bernstein operator for approximating functions defined on the unit disk.

The contents of this chapter are collected in

- [40] M. E. Marriaga, T. E. Pérez, and M.J. Recarte. A class of Bernstein-type operators on the unit disk. Submitted.

Finally, in the Appendix, we present all the Mathematica® used to generate the graphics and perform error calculations presented in this work. These codes can be a useful resource for further research and experimentation. For writing the codes, an important source of reference was [50]. In the Future Work section, open problems that could be subject to future research are discussed, highlighting the importance of continuing this work in future investigation. Additionally, the bibliography contains the references used throughout this work

Chapter 1

Preliminaries

In this Chapter, we establish the notation and basic background that we will need in this doctoral dissertation.

From the first to the fourth section, the general properties of orthogonal polynomials are summarized in several variables; the basic definitions of polynomials in several variables and orthogonal polynomials over the unit ball are presented mainly following [18]. In the fifth section, the state of the art concerning Sobolev orthogonal polynomials on the unit ball is presented. Finally, the last section in this Chapter is devoted to collect the properties of univariate Bernstein operators that will be necessary throughout this doctoral dissertation. Moreover we recall the method introduced by Stancu [44] for obtaining Bernstein-type operators in two variables by the successive application of Bernstein operators in one variable.

Throughout this Chapter, several results will be stated without proofs; they can be found in the references to be indicated.

1.1 Multivariate orthogonal polynomials

We denote by $\mathbb{N}_0$ the set of nonnegative integers. A multi–index is usually denoted by α,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d.$$

Let us define $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. For $\alpha \in \mathbb{N}_0^d$ and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, a monomial in the variables $x_1, x_2, \ldots, x_d$ is a product:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d},$$

where $|\alpha|$ is the total degree of x^α. A real polynomial P in d variables is a finite linear combination of monomials,

$$P(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where the coefficients c_α are real numbers. The total degree of the polynomial is defined as the highest total degree of its monomials. We denote by Π^d the linear space of all polynomials in x with coefficients in $\mathbb{R}$. In addition, for $n \geqslant 0$, we denote by Π_n^d the polynomials with total degree less than or equal to n. Note that $\Pi^d = \bigcup_{n=0}^{\infty} \Pi_n^d$. It is also known that ([18])

$$\dim \Pi_n^d = \binom{n+d}{n}.$$

A polynomial is said to be homogeneous if all its monomials have the same total degree n. We denote the linear space of homogeneous polynomials in d variables of degree n by $\mathcal{P}_n^d$; that is,

$$\mathcal{P}_n^d = \left\{ P \in \Pi^d : P(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha \right\}.$$

Denote by r_n^d the dimension of $\mathcal{P}_n^d$, which is equal to ([18])

$$r_n^d = \binom{n+d-1}{n}.$$

For every homogeneous polynomial P of total degree n, we have that

$$\sum_{i=1}^{d} x_i \frac{\partial P}{\partial x_i} = nP, \tag{1.1}$$

this is known as Euler's equation for homegeneous polynomials.

Let $\langle \cdot, \cdot \rangle$ be an inner product defined on Π^d, that is $\langle \cdot, \cdot \rangle : \Pi^d \times \Pi^d \to \mathbb{R}$.

Definition 1.1.1. *Two polynomials $P, Q \in \Pi^d$ are said to be orthogonal with respect to the inner product if:*

$$\langle P, Q \rangle = 0.$$

Definition 1.1.2. *A polynomial P is called an orthogonal polynomial if P is orthogonal to all polynomial of lower degree; that is*

$$\langle P, Q \rangle = 0, \quad \forall Q \in \Pi^d, \quad \textit{with} \quad \deg(Q) < \deg(P).$$

Denote by $\mathcal{V}_n^d$ the linear space of orthogonal polynomials of degree exactly n; that is

$$\mathcal{V}_n^d = \left\{ P \in \Pi_n^d : \langle P, Q \rangle = 0, \quad \deg(P) = n, \quad \forall Q \in \Pi_{n-1}^d \right\}.$$

The dimension of $\mathcal{V}_n^d$ is r_n^d, the same dimension as $\mathcal{P}_n^d$ ([18]).

For $n \geqslant 0$, let $\{P_\nu^n(x) : 1 \leqslant \nu \leqslant r_n^d\}$ be a basis of $\mathcal{V}_n^d$. Notice that every element of $\mathcal{V}_n^d$ is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, $\langle P_\nu^n, P_\eta^n \rangle = 0$ whenever $\nu \neq \eta$, we call the basis *mututally orthogonal*. If, in addition, $\langle P_\nu^n, P_\nu^n \rangle = 1$, we say that the basis is *orthonormal*.

If the inner product is given in terms of a weight function W in the form

$$\langle P, Q \rangle_W = \int_\Omega P(x) Q(x) W(x) dx,$$

where Ω is a domain in $\mathbb{R}^d$, we say that orthogonal polynomials are orthogonal with respect to the weight function W. Denote by $\mathcal{V}_n^d(W)$ the linear space of orthogonal polinomials with respect to $\langle \cdot, \cdot \rangle_W$.

1.2 Spherical harmonics

For $x \in \mathbb{R}^d$, we denote by $\|\cdot\|$ the usual Euclidean norm, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$. The unit ball and the unit sphere in $\mathbb{R}^d$ are denoted by $\mathbf{B}^d = \left\{ x \in \mathbb{R}^d : \|x\| \leqslant 1 \right\}$

and $\mathbf{S}^{d-1} = \left\{\xi \in \mathbb{R}^d : \|\xi\| = 1\right\}$, respectively.

The integral over the unit ball represents the volume of the unit d–ball and is expressed in terms of the Gamma function $\Gamma(\cdot)$,

$$V_d = \int_{\mathbf{B}^d} dx = \int_{\mathbf{B}^{d-1}} \prod_{j=1}^{d} dx_j = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}. \tag{1.2}$$

Let $d\sigma$ be the measure on the sphere and let σ_{d-1} be the area of the surface of $\mathbf{S}^{d-1}$, then

$$\sigma_{d-1} = \int_{\mathbf{S}^{d-1}} d\sigma(\xi) = 2\int_{\mathbf{B}^{d-1}} \sqrt{1 + \sum_{i=1}^{d-1} \left(\frac{dx_n}{dx_i}\right)^2} \prod_{j=1}^{d-1} dx_j = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{1.3}$$

Notice than $V_d = \frac{2}{d/2}\sigma_{d-1} = \frac{4}{d}\sigma_{d-1}$.

The spherical-polar coordinates or generalized spherical coordinates are defined by ([14])

$$\begin{aligned}
x_1 &= r\cos(\theta_{d-1}),\\
x_2 &= r\sin(\theta_{d-1})\cos(\theta_{d-2}),\\
&\vdots\\
x_{d-1} &= r\sin(\theta_{d-1})\ldots\sin(\theta_2)\cos(\theta_1),\\
x_d &= r\sin(\theta_{d-1})\ldots\sin(\theta_2)\sin(\theta_1),
\end{aligned}$$

where $r \geqslant 0$, $0 \leqslant \theta_1 \leqslant 2\pi$, $0 \leqslant \theta_i \leqslant \pi$, for $2 \leq i \leq d-1$.

It can be easily verified that

$$V_d = \int_0^1 \int_{\mathbf{S}^{d-1}} r^{d-1} dr d\sigma(\xi),$$

and

$$d\sigma(\xi) = \prod_{j=1}^{d-2} \left(\sin\theta_{d-j}\right)^{d-j-1} d\theta_1 d\theta_2 \cdots d\theta_{d-1}.$$

Let $\mathcal{H}_n^d$ denote the linear space of harmonic polynomials in d variables of total degree exactly n, that is, homogeneous polynomials of degree n satisfying the Laplace equation $\Delta P = 0$, that is,

$$\mathcal{H}_n^d = \left\{P \in \mathcal{P}_n^d : \Delta P = 0\right\},$$

where $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator. It is well known that ([14])

$$a_n^d := \dim\ \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n}.$$

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}_n^d$, then in spherical-polar coordinates $x = r\xi$ where $r \geqslant 0$ and $\xi \in \mathbf{S}^{d-1}$, we get

$$Y(x) = r^n\, Y(\xi),$$

so that Y is uniquely determined by its restriction to the sphere. We will also use $\mathcal{H}_n^d$ to denote the linear space of spherical harmonics of degree n.

It is a consequence of Green's theorem that the homogeneous harmonic polynomials of different degrees are orthogonal with respect to the surface measure $d\sigma$. Suppose that f, g are polynomials on $\mathbb{R}^d$; then ([18])

$$\int_{\mathbf{S}^{d-1}} \left(\frac{\partial f}{\partial n} g - \frac{\partial g}{\partial n} f \right) d\sigma(\xi) = \int_{\mathbf{B}^d} (g\Delta f - f\Delta g)\, dx,$$

where $\dfrac{\partial}{\partial n}$ denotes the normal derivative. If f, g are homogeneous then

$$(\deg f - \deg g) \int_{\mathbf{S}^{d-1}} fg d\sigma(\xi) = \int_{\mathbf{B}^d} (g\Delta f - f\Delta g)\, dx = 0.$$

Using the previous fact, spherical harmonics are orthogonal on $\mathbf{S}^{d-1}$ with respect to the inner product

$$\langle f, g \rangle_{\mathbf{S}^{d-1}} = \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi)\, g(\xi)\, d\sigma(\xi).$$

Let ∇ denote the gradient operator, $\nabla f = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \cdots \dfrac{\partial f}{\partial x_d} \right)^\top$, where $A^\top$ denotes the transpose of A. If we define the operator

$$x \cdot \nabla = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i},$$

then, by Euler's equation for homogeneous polynomials, we deduce

$$(x \cdot \nabla) Y(x) = n\, Y(x), \quad \forall Y \in \mathcal{H}_n^d.$$

The differential operators ∇, Δ and $x\cdot\nabla$ can be expressed in spherical-polar coordinates $x = r\xi$, $r \geqslant 0$, $\xi \in \mathbf{S}^{d-1}$ as ([14])

$$\nabla = \frac{1}{r}\nabla_0 + \xi\frac{\partial}{\partial r}, \tag{1.4}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{1.5}$$

$$x\cdot\nabla = r\frac{\partial}{\partial r}, \tag{1.6}$$

where ∇_0 and $\Delta_0 = \nabla_0 \cdot \nabla_0$ are the spherical part of the gradient and the Laplacian respectively; Δ_0 is called the Laplace–Beltrami operator. The operator Δ_0 has spherical harmonics as eigenfunctions. More precisely, it holds that ([14])

$$\Delta_0\, Y(\xi) = -n\,(n+d-2)\,Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbf{S}^{d-1}. \tag{1.7}$$

We will also need the following family of differential operators, $D_{i,j}$, defined by

$$D_{i,j} = x_i\,\partial_j - x_j\,\partial_i, \quad 1 \leqslant i < j \leqslant d.$$

These are angular derivatives since $D_{i,j} = \partial_{\theta_{i,j}}$ in the polar coordinates of the (x_i, x_j)–plane, where $(x_i, x_j) = r_{i,j}\,(\cos\theta_{i,j}, \sin\theta_{i,j})$. Furthermore, the angular derivatives $D_{i,j}$ and the Laplace-Beltrami operator Δ_0 are related by

$$\Delta_0 = \sum_{1\leqslant i<j\leqslant d} D_{i,j}^2.$$

1.3 Classical orthogonal polynomials on $\mathbf{B}^d$

The classical orthogonal polynomials on the unit ball $\mathbf{B}^d$ are defined with respect to the inner product

$$\langle f, g\rangle_\mu = b_\mu \int_{\mathbf{B}^d} f(x)g(x)W_\mu(x)dx, \tag{1.8}$$

with the weight function

$$W_\mu(x) = (1-\|x\|^2)^\mu\,,\ \mu > -1,\ x \in \mathbf{B}^d.$$

In the above expression, b_μ is a normalization constant such that $\langle 1,1\rangle_\mu = 1$. From this:

$$1 = b_\mu \int_{\mathbf{B}^d} W_\mu(x)dx.$$

The normalization constant is given by

$$b_\mu = \left(\int_{\mathbf{B}^d} W_\mu(x)dx\right)^{-1}.$$

The value of the above integral can be obtained by making a change of variable using the spherical-polar coordinates $x = r\xi,\ r \geqslant 0,\ \xi \in \mathbf{S}^{d-1}$. Then:

$$\int_{\mathbf{B}^d} W_\mu(x)dx = \int_0^1 r^{d-1}(1-r^2)^\mu dr \int_{\mathbf{S}^{d-1}} d\sigma(\xi) = \sigma_{d-1}\int_0^1 r^{d-1}(1-r^2)^\mu dr.$$

If we use the value of σ_{d-1} and some properties of the beta function ($\mathcal{B}(\cdot,\cdot)$), (see for example [1]) we obtain:

$$\int_{B^d} W_\mu(x)dx = \frac{1}{2}\mathcal{B}\left(\frac{d}{2},\mu+1\right)\frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(\mu+1)\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\mu+\frac{d+2}{2}\right)}.$$

Therefore, b_μ is given by

$$b_\mu = \frac{\Gamma\left(\mu+\frac{d+2}{2}\right)}{\Gamma(\mu+1)\pi^{d/2}}. \tag{1.9}$$

Let $s = \{s_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ a multi-sequence of real numbers and let $\mathbf{u}$ be the linear functional defined by

$$(\mathbf{u}, x^\alpha) = s_\alpha, \quad \alpha \in \mathbb{N}_0^d,$$

and extended on Π^d by linearity. We call $\mathbf{u}$ the moment functional defined by the sequence s. The moment functional associated with the weight function on the ball is defined by:

$$(\mathbf{u}_\mu, x^\alpha) = b_\mu \int_{\mathbf{B^d}} x^\alpha\ W_\mu(x)dx,\ \ \alpha \in \mathbb{N}_0^d\ .$$

Let us denote by $\mathcal{V}_n^d(W_\mu)$ the linear space of orthogonal polynomials of total degree exactly n with respect to $\langle\cdot,\cdot\rangle_\mu$ and by $P_n^{(\alpha,\beta)}(t)$ the classical univariate Jacobi polynomial of degree n on $[-1,1]$. For $\alpha, \beta > -1$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ of degree n is given by (see for example page 41 of [46])

$$P_n^{(\alpha,\beta)}(t) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(k+\alpha+1)_{n-k}(k+\alpha+\beta+1)_k\left(\frac{t-1}{2}\right)^k,$$

where $(a)_n = a\,(a+1)\cdots(a+n-1)$, $(a)_0 = 1$, denotes the Pochhammer symbol. They are orthogonal with respect to the Jacobi weight function $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ on $[-1,1]$.

A family of mutually orthogonal polynomials with respect to (1.8) can be expressed in terms of the Jacobi polynomials and spherical harmonics.

Theorem 1.3.1 ([18]). For $n \geqslant 0$ and $0 \leqslant j \leqslant \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ denote an orthonormal basis of $\mathcal{H}_{n-2j}^d$. For $\mu > -1$, define the polynomials

$$P_{j,\nu}^{n,\mu}(x) = P_j^{\left(\mu, n-2j+\frac{d-2}{2}\right)}\left(2\,\|x\|^2 - 1\right)\, Y_\nu^{n-2j}(x). \tag{1.10}$$

Then the set $\{P_{j,\nu}^{n,\mu} : 0 \leqslant j \leqslant \frac{n}{2}, 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$. Moreover,

$$\langle P_{j,\nu}^{n,\mu}, P_{k,\eta}^{m,\mu}\rangle_\mu \;=\; H_{j,n}^\mu\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where

$$H_{j,n}^\mu = \frac{(\mu+1)_j\;(d/2)_{n-j}\;(n-j+\mu+d/2)}{j!\;(\mu+d/2+1)_{n-j}\;(n+\mu+d/2)}. \tag{1.11}$$

The square of the norm of the Jacobi polynomial $P_j^{(\alpha,\beta)}(x)$, denoted by $h_j^{(\alpha,\beta)}$, is related with $H_{j,n}^\mu$ as follows:

$$H_{j,n}^\mu = \frac{\gamma_{\mu,d}}{2^{n-2j}}\,h_j^{(\mu,n-2j+\frac{d-2}{2})}, \tag{1.12}$$

where $\gamma_{\mu,d} = \dfrac{b_\mu\,\sigma_{d-1}}{2^{\mu+\frac{d}{2}+1}}$.

Other orthogonal basis of $\mathcal{V}_n^d(W_\mu)$ are known, for a more detailed description of them see [18].

With respect to the basis (1.10), the Fourier orthogonal expansion of $f \in L^2(W_\mu;\mathbf{B}^d)$ is defined as

$$f(x) = \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}^d}\widehat{f}_{j,\nu}^{\,n,\mu}P_{j,\nu}^{n,\mu}(x) \quad \text{with} \quad \widehat{f}_{j,\nu}^{\,n,\mu} := \frac{1}{H_{j,n}^\mu}\left\langle f, P_{j,\nu}^{n,\mu}\right\rangle_\mu. \tag{1.13}$$

Since $\|f\|_\mu$ is finite, the Parseval identity holds: for $\mu > -1$,

$$\|f\|_\mu^2 = \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \left| \widehat{f}_{j,\nu}^{\,n,\mu} \right|^2 H_{j,n}^\mu. \tag{1.14}$$

Let $\mathrm{proj}_n^\mu : L^2(W_\mu; \mathbf{B}^d) \to \mathcal{V}_n^d(W_\mu)$ and $S_n^\mu : L^2(W_\mu; \mathbf{B}^d) \to \Pi_n^d$ denote the projection operator and the partial sum operator, respectively. Then,

$$\mathrm{proj}_m^\mu f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{\,m,\mu} P_{j,\nu}^{m,\mu}(x) \quad \text{and} \quad S_n^\mu f(x) = \sum_{m=0}^{n} \mathrm{proj}_m^\mu f(x).$$

By definition, $\langle f - S_n^\mu f, Q \rangle_\mu = 0$ for all $Q \in \Pi_n^d$, and $S_n^\mu f = f$ if $f \in \Pi_n^d$.

We consider the error, $\mathcal{E}_n(f)_\mu$, of best approximation by polynomials in Π_n^d in the space $L^2(W_\mu; \mathbf{B}^d)$, defined by

$$\mathcal{E}_n(f)_\mu = \inf_{p_n \in \Pi_n^d} \|f - p_n\|_\mu,$$

and notice that the infimum is achieved by $S_n^\mu f$.

1.4 Zernike Polynomials

The natural field of application of orthogonal polynomials is that of the approximation of functions, which can be found in multiple technological applications. The reconstruction and representation of surfaces is a basic tool of graphical computing, medical imaging, and other branches. For example, in ophthalmological practice, the Hartmann-Shack sensor (or wavefront sensor) is used to determine the refractive errors of the human optical system, measuring slopes or normals of the wavefront at different points starting from the displacement of some luminous points in a target.

A systematic method of classifying forms of aberrations is to express the corresponding function in an appropriate basis. The so-called Zernike polynomials, originally described by Frits Zernike in 1934 ([54]) to describe the diffraction of the wavefront in the phase contrast image microscope, are recognized as the standard basis of wavefront developments by the Optical Society of America, (OSA). In addition, they are implemented in the standard measuring devices used in optics, see [5, 22, 28, 36, 47, 48].

From our point of view, Zernike polynomials are polynomials in two variables which are orthogonal on the unit disk with respect to the Lebesgue measure. They are represented in polar coordinates as a product of a radial polynomial part times a trigonometric function. The even polynomials are multiples of the cosine, and the odd polynomials are multiples of the sine. Any sufficiently regular phase function defined on the unit disk can be represented by its Fourier expansion in terms of the Zernike polynomials with certain coefficients. The alteration of these coefficients allows the detection of the possible aberrations of the studied optical system.

Zernike polynomials are defined in terms of a double index (n, m) as follows.

Definition 1.4.1. *Zernike polynomials are defined as*

$$Z_n^m(\rho,\theta) = \begin{cases} N_n^m R_n^{|m|}(\rho)\cos(m\theta), & m \geqslant 0, \\ N_n^m R_n^{|m|}(\rho)\sin(|m|\theta), & m < 0, \end{cases} \tag{1.15}$$

where $0 \leqslant \rho \leqslant 1$ *and* $0 \leqslant \theta \leqslant 2\pi$ *are the polar coordinates, and the double index* (n, m) *satisfy restrictions:* $n \geqslant 0$ *,* $|m| \leqslant n$*, and* $n-m$ *an even integer.*

The radial part $R_n^{|m|}(\rho)$ is a Jacobi polynomial, in the form

$$R_n^{|m|}(\rho) = (-1)^{(n-m)/2}\rho^m P_{(n-m)/2}^{(m,0)}(1-2\rho^2).$$

Using that $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ (see page 59 of [46]), the radial part can be expressed as

$$R_n^{|m|}(\rho) = \rho^m P_{(n-m)/2}^{(0,m)}(2\rho^2-1).$$

The orthogonality of the radial polynomials is:

$$\int_0^1 R_n^{|m|}(\rho)R_{n'}^{|m|}(\rho)\rho d\rho \;\; = \;\; \frac{1}{2(n+1)}\delta_{n,n'}.$$

The normalization term N_n^m is sometimes taken to be 1 for simplicity, thus giving only orthogonality but not orthonormality. The value of this normalization constant can be defined as:

$$N_n^m = \sqrt{(2-\delta_{0,m})(n+1)},$$

to grant orthonormality.

We will consider the last option, so the orthonormality of the Zernike polynomials on the unit disk $\mathbf{B}^2$ can be expressed as:

$$\int\int_{\mathbf{B}^2} Z_n^m(x_1,x_2) Z_r^s(x_1,x_2) dx_1 dx_2 = \delta_{n,r}\delta_{m,s}.$$

Writing it in polar coordinates we get:

$$\int_0^1 \int_0^{2\pi} Z_n^m(\rho,\theta) Z_r^s(\rho,\theta)\rho d\rho d\theta = \delta_{n,r}\delta_{m,s}.$$

In practice, the use of a double index is sometimes troublesome, and there are at least two different ways of converting the double index into a single index, known as the Noll index ([36]) and the OSA index ([48]). In the Noll system, polynomials with cosine and sine terms alternate, starting the index at 1; whereas in OSA numbering, all polynomials with the same radial order n are consecutive and the index starts at 0.

The conversion from one system to another is given by the expressions

$$\begin{aligned} j &= \frac{n(n+2)+m}{2}, \\ n &= \left\lceil \frac{-3+\sqrt{9+8j}}{2} \right\rceil, \\ m &= 2j - n(n+2). \end{aligned}$$

In Table 1.1, we show the explicit expressions for Zernike polynomials up to degree 4 and in Figure (1.1) some of their graphs are shown.

j	n	m	Polar Coordinates	Cartesian Coordinates
0	0	0	1	1
1	1	-1	$2\rho\sin(\theta)$	$2y$
2	1	1	$2\rho\cos(\theta)$	$2x$
3	2	-2	$\sqrt{6}\rho^2\sin(2\theta)$	$2\sqrt{6}xy$
4	2	0	$\sqrt{3}(2\rho^2-1)$	$\sqrt{3}(2x^2+2y^2-1)$
5	2	2	$\sqrt{6}\rho^2\cos(2\theta)$	$\sqrt{6}(x^2-y^2)$
6	3	-3	$\sqrt{8}\rho^3\sin(3\theta)$	$\sqrt{8}(3x^2y-y^3)$
7	3	-1	$\sqrt{8}(3\rho^3-2\rho)\sin(\theta)$	$\sqrt{8}(3x^2y+3y^3-2y)$
8	3	1	$\sqrt{8}(3\rho^3-2\rho)\cos(\theta)$	$\sqrt{8}(3x^2y+3y^3-2x)$
9	3	3	$\sqrt{8}\rho^3\cos(3\theta)$	$\sqrt{8}(x^3-2xy^2)$
10	4	-4	$\sqrt{10}\rho^4\sin(4\theta)$	$\sqrt{10}(4x^3y-4xy^3)$
11	4	-2	$\sqrt{10}(4\rho^4-3\rho^2)\sin(2\theta)$	$\sqrt{10}(8x^3y+8xy^3-6xy)$
12	4	0	$\sqrt{5}(6\rho^4-6\rho^2+1)$	$\sqrt{5}(6x^4+12x^2y^2+6y^4-6x^2-6y^2+1)$
13	4	2	$\sqrt{10}(4\rho^4-3\rho^2)\cos(2\theta)$	$\sqrt{10}(4x^4-4y^4-3x^2+3y^2)$
14	4	4	$\sqrt{10}\rho^4\cos(4\theta)$	$\sqrt{10}(x^4-6x^2y^2+y^4)$

Table 1.1: Zernike Polynomials, with $N_n^m = \sqrt{(2-\delta_{0,m})(n+1)}$.

1.5 Sobolev orthogonal polynomials on the unit ball

One type of modification that can be made to an inner product defined on Π^d is the so-called Sobolev inner product. In the theory of orthogonal polynomials in one variable, the Sobolev name is associated with polynomials that are orthogonal with respect to an inner product involving both functions and their derivatives or other differential operators.

This type of polynomials has been widely studied and is the main subject of a large portion of the literature (see, for example, [20, 25, 29, 30, 34] and the references cited therein). However, as far as we know, orthogonal Sobolev polynomials in several variables have been studied only in some particular cases. At this time, some references on the subject related to the orthogonal polynomials on the unit ball can be found in [7, 15, 33, 37, 38, 39, 51, 52].

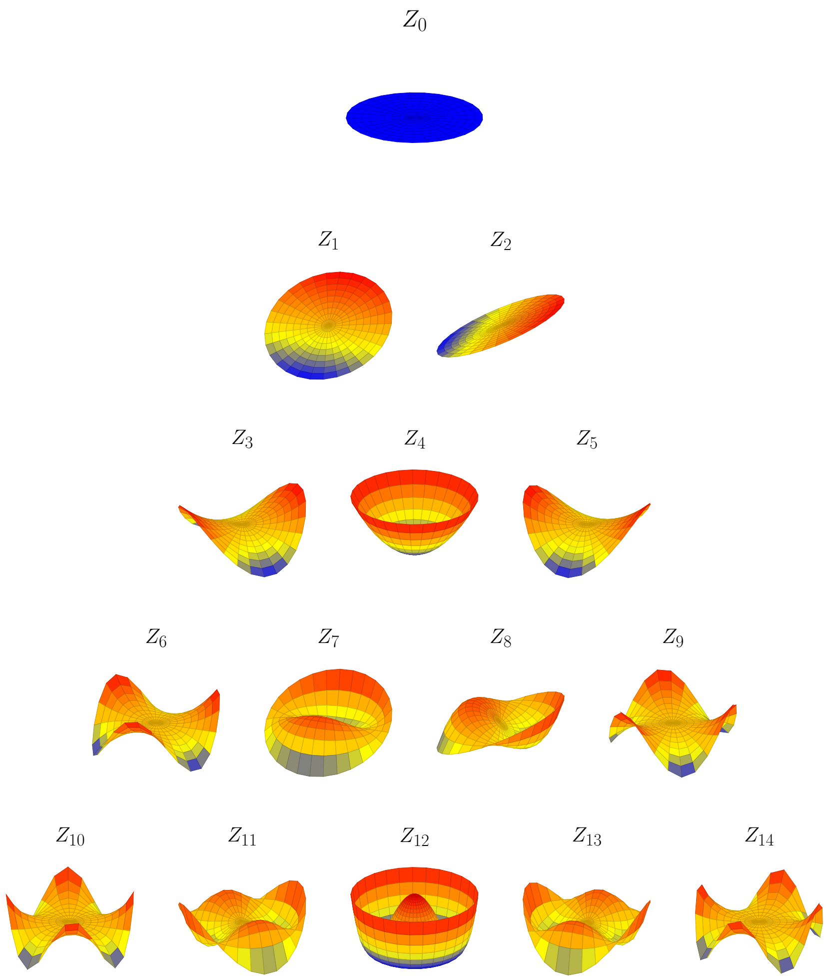

Figure 1.1: Some Zernike Polynomials with OSA index. Graph made using the tikz package in LaTeX.

In [51], the author considers an inner product motivated by an application in the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions (see [2]). This inner product is defined by

$$\langle f, g\rangle_\Delta = \alpha_d \int_{\mathbf{B}^d} \Delta[(1-\|x\|^2)f(x)]\Delta[(1-\|x\|^2)g(x)]dx, \tag{1.16}$$

where Δ is the usual Laplace operator and $\alpha_d = 1/(4d^2V_d)$ is a normalization constant so that $\langle 1, 1\rangle_\Delta = 1$.

We denote by $\mathcal{V}_n^d(\Delta)$ the linear space of orthogonal polynomials of degree

n with respect to $\langle\cdot,\cdot\rangle_\Delta$. The symmetry of the inner product allows the construction of a basis of mutually orthogonal polynomials, which can be expressed in terms of the harmonic polynomials. Hence, emulating the construction of the basis in (1.10), Xu searched for a basis of the form ([51])

$$Q^n_{j,\nu}(x) = q_j(2\|x\|^2-1)Y^{n-2j}_\nu(x), \quad 0 \leqslant j \leqslant n/2, \tag{1.17}$$

where q_j is a polynomial of degree j in one variable and $\{Y^{n-2j}_\nu : 1 \leq \nu \leq \sigma_{n-2j}\}$ is an orthonormal basis for $\mathcal{H}^d_{n-2j}$.

In [51], it was also proved that the q_j in (1.17) is orthogonal with respect to a univariate Sobolev inner product, which depends on the degree of the polynomials. In fact, the polynomial q_j is orthogonal with respect to the inner product

$$(f,g)_{\beta_j} := \int_{-1}^1 (\mathcal{J}_{\beta_j}f)(s)(\mathcal{J}_{\beta_j}g)(s)(1+s)^{\beta_j}ds,$$

where $\beta_j = n-2j+(d-2)/2$, and

$$(\mathcal{J}_{\beta_j}q_j)(s) = (1-s^2)q_j''(s) + (\beta_j - 1 - (\beta_j+3)s)q_j'(s) - (\beta_j+1)q_j(s).$$

Using this property, an explicit representation of the orthogonal Sobolev polynomials can be obtained.

Theorem 1.5.1. ([51]). A mutually orthogonal basis for $\mathcal{V}^d_n(\Delta)$ is given by

$$\begin{aligned} Q^n_{0,\nu}(x) &= Y^n_\nu(x), \\ Q^n_{j,\nu}(x) &= (1-\|x\|^2)P^{(2,n-2j+\frac{d-2}{2})}_{j-1}(2\|x\|^2-1)Y^{n-2j}_\nu(x), \quad 1 \leqslant j \leqslant n/2. \end{aligned}$$

The explicit formula of the basis given in Theorem 1.5.1 leads to an interesting result, which involves the orthogonal polynomials with respect to $W_2(x) = (1-\|x\|^2)^2$

$$\mathcal{V}^d_n(\Delta) = \mathcal{H}^d_n \oplus (1-\|x\|^2)\mathcal{V}^d_{n-2}(W_2). \tag{1.18}$$

In addition, the explicit formula can be used to study other properties of the orthogonal basis. In particular, it turns out that the orthogonal expansion of a function f in the basis presented in Theorem 1.5.1 can be computed without involving of the derivatives of f.

In [52], the author analyzes orthogonal polynomials with respect to other types of inner products defined on the ball. There, inner products involving the usual gradient operator ∇ are considered. In particular, two Sobolev inner products are defined as follows

$$\langle f,g\rangle_I := \frac{\lambda}{\sigma_{d-1}}\int_{\mathbf{B}^d}\nabla f(x)\cdot\nabla g(x)dx + \frac{1}{\sigma_{d-1}}\int_{\mathbf{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi),\ \xi\in\mathbf{S}^{d-1},$$

$\lambda > 0$ and

$$\langle f,g\rangle_{II} := \frac{\lambda}{\sigma_{d-1}}\int_{\mathbf{B}^d}\nabla f(x)\cdot\nabla g(x)dx + f(0)g(0), \quad \lambda>0,$$

where the normalization constants have been chosen in such a way that $\langle 1,1\rangle_I = \langle 1,1\rangle_{II} = 1$. Let denote by $\mathcal{V}_n^d(I)$ and $\mathcal{V}_n^d(II)$ the spaces of orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_I$ and $\langle\cdot,\cdot\rangle_{II}$, respectively. Using the same construction as in the previous Theorem, an orthonormal basis for the spaces is constructed explicitly. The basis in [51] and the one constructed in [52] involve the Jacobi polynomials. It is interesting to note that the basis of $\langle\cdot,\cdot\rangle_I$ and that of $\langle\cdot,\cdot\rangle_\Delta$ have the same structure as (1.17) which consists of an univariate polynomial multiplied by a harmonic polynomial, but differ only in the parameters of the Jacobi polynomials.

The explicit representation of the orthogonal Sobolev polynomials with respect to $\langle\cdot,\cdot\rangle_I$, is given in the following theorem.

Theorem 1.5.2. ([52]). A mutually orthogonal basis for $\mathcal{V}_n^d(I)$ is given by

$$\begin{aligned} U_{0,\nu}^n(x) &= Y_\nu^n(x),\\ U_{j,\nu}^n(x) &= (1-\|x\|^2)P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)Y_\nu^{n-2j}(x), \quad 1\leqslant j\leqslant n/2. \end{aligned}$$

It follows that

$$\mathcal{V}_n^d(I) = \mathcal{H}_n^d \oplus (1-\|x\|^2)\mathcal{V}_{n-2}^d(W_1). \tag{1.19}$$

For the inner product $\langle\cdot,\cdot\rangle_{II}$, the main result also provides an explicit of mutually orthogonal basis. The proposed basis for $\mathcal{V}_n^d(II)$ turns out to be similar to the basis of $\mathcal{V}_n^d(I)$ given in Theorem 1.5.2. In fact, if n is odd, the two basis are identical, while for n even, the two basis differ only by one element.

Theorem 1.5.3. ([52]) A mutually orthogonal basis for $\mathcal{V}_n^d(II)$ is given by

$$\begin{aligned} V_{j,\nu}^n(x) &= U_{j,\nu}^n(x), \quad 1 \leqslant j \leqslant \frac{n-1}{2}, \\ V_{\frac{n}{2}}^n(x) &= \frac{1}{n+\frac{d-2}{2}} \left(P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\|x\|^2-1) - (-1)^{\frac{n}{2}} \frac{(d/2)_{\frac{n}{2}}}{(n/2)!} \right), \end{aligned}$$

where $V_{\frac{n}{2}}^n(x) := V_{\frac{n}{2},\nu}^n(x)$ is valid only when n is even.

An interesting consequence of these explicit formulas is that the Fourier expansion of a function f with respect to these orthogonal basis can be computed without the use of the derivatives of f.

Another context in which Sobolev orthogonal polynomial appear are in partial differential equations.

As it is well known (see [18]), for $\mu > -1$, the orthogonal polynomials of degree n with respect to the weight function $W_\mu(x) = (1-\|x\|^2)^\mu$, $x \in \mathbf{B}^d$, satisfy a partial differential equation, which in compact form can be written as follows

$$\left[\Delta - \langle x, \nabla\rangle^2 - (2\mu+d)\langle x, \nabla\rangle\right] P = -n(n+2\mu+d)P. \tag{1.20}$$

In [39], the singular case corresponding to the values $\mu = -1, -2, \ldots$, is studied. In particular, explicit polynomial solutions are studied and it is proved that the equation for $\mu = -2, -3, \ldots$, has a complete system of polynomial solutions if the dimension d is odd. One of the most interesting results obtained by the authors is that the orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_I = \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla g(x) dx + \int_{\mathbf{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi), \qquad \lambda > 0, \tag{1.21}$$

which were studied in [52], satisfy (1.20) for $\mu = -1$.

In [7], the authors considered a bivariate orthogonal polynomials with respect to the following inner product:

$$\langle f, g\rangle_{S_1} = (\mathbf{u}, fg) + \left(\mathbf{v}, (\nabla f)^t \Theta \nabla g\right), \tag{1.22}$$

where $\mathbf{u}$, $\mathbf{v}$ are linear funtionals and Θ is a diagonal matrix whose entries are polynomials in two variables of degree at most 2. It should be noted that the polynomials studied in [52] are a particular case of (1.22) when Θ is the identity matrix. The connection between the coefficients of the second order partial differential operator and the moment functionals defining the Sobolev inner product is also explored.

In [15], Sobolev orthogonal polynomials with respect to an inner product on $\mathbf{B}^d$ are studied; the new inner product is defined as a modification of (1.8) by adding the evaluation of derivatives at several points, that is

$$\langle f, g\rangle_{S_2} = \langle f, g\rangle_\mu + \lambda \sum_{k=0}^{N} \frac{\partial f(s_k)}{\partial n}\frac{\partial g(s_k)}{\partial n}, \tag{1.23}$$

where $s_k \in \mathbf{S}^{d-1}$. As main results, relations between the polynomials associated to $\langle \cdot, \cdot\rangle_\mu$ and the modified inner product are established. Asymptotics for the Christoffel function of the Sobolev polynomials are also deduced.

Extensions of (1.21) are studied in [37] by considering the inner products

$$\langle f, g\rangle_{\nabla, W_\mu, I} = \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla g(x) W_{\mu+1}(x)dx + \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi), \tag{1.24}$$

$$\langle f, g\rangle_{\nabla, W_\mu, II} = b_\mu \left[\int_{\mathbf{B}^d} f(x)g(x)W_\mu(x)dx + \lambda \int_{B^d} \nabla f(x) \cdot \nabla g(x) W_\mu(x)dx\right], \tag{1.25}$$

for $\lambda > 0$. Orthogonal bases are constructed, having the same form of (1.17) where the corresponding q_j is orthogonal with respect to an interesting Sobolev inner product in one variable.

1.6 Polynomial approximation on the unit ball

We define the Sobolev space: For $\mathbf{m} \in \mathbb{N}_0^d$, let $\partial^{\mathbf{m}} = \partial_1^{m_1} \cdots \partial_d^{m_d}$. For $\mu > -1$ and $s \geqslant 1$, we denote by $\mathcal{W}_2^s(W_\mu; \mathbf{B}^d)$ the Sobolev space

$$\mathcal{W}_2^s(W_\mu; \mathbf{B}^d) = \left\{ f \in L^2(W_\mu; \mathbf{B}^d);\ \partial^{\mathbf{m}} f \in L^2(W_{\mu+|\mathbf{m}|}; \mathbf{B}^d),\ |\mathbf{m}| \leqslant s,\ \mathbf{m} \in \mathbb{N}_0^d \right\}.$$

The following estimate was proved in [38]: for $n \geqslant 2s$ and $f \in \mathcal{W}_2^{2s}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{n^{2s}} \Big[\mathcal{E}_{n-2s}(\Delta^s f)_{\mu+2s} + \mathcal{E}_n(\Delta_0^s f)_\mu \Big], \tag{1.26}$$

and for $n \geqslant 2s+1$ and $f \in \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{n^{2s+1}} \left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-1}(\partial_i \, \Delta^s f)_{\mu+2s+1} + \sum_{1 \leqslant i < j \leqslant d} \mathcal{E}_n(D_{i,j} \, \Delta_0^s f)_\mu \right], \quad (1.27)$$

where c is a generic constant independent of n and f but may depend on μ and d, and its value may be different from one instance to the next. As pointed out in [38], each term involving Δ and Δ_0 on the right hand side of the above inequalities is necessary since the first term deals with the radial component of f and the second one deals with the harmonic component of f defined on the ball.

It is known that the elements of the basis of $\mathcal{V}_n^d(W_\mu)$ defined in (1.10) are eigenfunctions of another second order linear partial differential operator $\mathcal{L}_\mu$ which is clearly different from (1.20). More precisely, we have ([38])

$$\mathcal{L}_\mu[P_{j,\nu}^{n,\mu}(x)] = \lambda_{n,j}^\mu \, P_{j,\nu}^{n,\mu}(x) \,, \quad (1.28)$$

where

$$\mathcal{L}_\mu = (1 - \|x\|^2) \, \Delta - 2(\mu+1) \, (x \cdot \nabla), \quad (1.29)$$

and

$$\lambda_{n,j}^\mu = -[4\, j \, (n - j + \mu + d/2) + 2 \, (n - 2\, j) \, (\mu+1)]. \quad (1.30)$$

Unlike in (1.20), where the eigenvalues depend solely on n, in this case they depend on both n and j.

In [38], the authors also show that the basis defined in (1.10) satisfy a relation involving the Laplacians and also show that the gradients of this basis satisfy an orthogonality property on the unit ball. Let $\mu > -1$ and let $P_{j,\nu}^{n,\mu}(x)$ be the mutually orthogonal polynomials in $\mathcal{V}_n^d(W_\mu)$ defined in (1.10). Then,

$$b_\mu \int_{\mathbf{B}^d} \nabla P_{j,\nu}^{n,\mu}(x) \cdot \nabla P_{k,\eta}^{m,\mu}(x) \, W_{\mu+1}(x) \, dx = H_{j,n}^\mu(\nabla) \, \delta_{n,m} \, \delta_{j,k} \, \delta_{\nu,\eta}, \quad (1.31)$$

where

$$H_{j,n}^\mu(\nabla) = [4\, j \, (n - j + \mu + d/2) + 2 \, (n - 2\, j) \, (\mu+1)] \, H_{j,n}^\mu,$$

and

$$\Delta \, P_{j,\nu}^{n,\mu}(x) = \kappa_{n-j}^\mu \, P_{j-1,\nu}^{n-2,\mu+2}(x) \quad \text{and} \quad \Delta_0 \, P_{j,\nu}^{n,\mu}(x) = \varrho_{n-2j} \, P_{j,\nu}^{n,\mu}(x), \quad (1.32)$$

where

$$\kappa_n^\mu = 4\left(n+\mu+\frac{d}{2}\right)\left(n+\frac{d-2}{2}\right) \quad \text{and} \quad \varrho_n = -n\,(n+d-2).$$

It turns out that the partial derivatives commute with the partial sum operator, as shown in the following Theorem.

Theorem 1.6.1 ([38])**.** Let $\mu > -1$. Then,

$$\partial_i\, S_n^\mu f \;=\; S_{n-1}^{\mu+1}(\partial_i\, f), \quad 1 \leqslant i \leqslant d,$$

and

$$D_{i,j}\, S_n^\mu f \;=\; S_n^\mu(D_{i,j}\, f), \quad 1 \leqslant i < j \leqslant d.$$

The relations in the above Theorem pass down to the Fourier coefficients.

Theorem 1.6.2 ([38])**.** Let $f \in \mathcal{W}_2^2(W_\mu;\mathbf{B}^d)$, $\mu > -1$. Then,

$$\widehat{\Delta\, f}_{j,\nu}^{\,n-2,\mu+2} \;=\; \kappa_{n-j-1}^\mu\, \widehat{f}_{j+1,\nu}^{\,n,\mu}, \quad 0 \leqslant j \leqslant \frac{n-2}{2},$$

and

$$\widehat{\Delta_0\, f}_{j,\nu}^{\,n,\mu} \;=\; \varrho_{n-2j}\, \widehat{f}_{j,\nu}^{\,n,\mu}, \quad 0 \leqslant j \leqslant \frac{n}{2}.$$

1.7 Bernstein polynomials

In 1912, S. Bernstein [4] published a constructive proof of the Weierstrass approximation theorem that affirms that every continuous function $f(x)$ defined on a closed interval can be uniformly approximated by polynomials. For a given function $f \in C[0,1]$, Bernstein constructed a sequence of polynomials (lately called Bernstein polynomials) in the form

$$B_n f(x) \equiv (B_n\, f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \tag{1.33}$$

for $0 \leqslant x \leqslant 1$, and $n \geqslant 0$.

Clearly, $B_n\, f$ is a polynomial in the variable x of degree less than or equal to n, and (1.33) can be seen as a linear operator that transforms functions defined on $[0,1]$ to polynomials of degree at most n.

Hence, in the sequel, we will refer to B_n as the n-th classical univariate *Bernstein operator*.

If we define

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant k \leqslant n, \tag{1.34}$$

then, the set $\{p_{n,k}(x) : 0 \leqslant k \leqslant n\}$ is a basis of the linear space of polynomials with real coefficients of degree at most n, that we will denote by Π_n, called *Bernstein basis*. Then, the n-th Bernstein polynomial associated with $f(x)$ is usually written as

$$B_n\, f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x).$$

Among others, classical Bernstein operators satisfy the following properties ([27]):

- They are linear and positive operators acting on the function f, and preserve the constant functions as well as polynomials of degree 1, that is,
$$B_n\, 1 = 1, \quad B_n\, x = x, \quad n \geqslant 0.$$
- If f is continuous at a point x, then $B_n\, f(x)$ converges to $f(x)$, and $B_n\, f$ converges uniformly if f is continuous on the whole interval $[0,1]$. Moreover, the order of approximation is $\omega_f(n^{-1/2})$, where ω_f denotes the modulus of continuity of f. Because of this property, Bernstein operators are called *Bernstein Approximants*.
- Bernstein operators satisfy a Voronowskaya type theorem, that is, if f is twice differentiable at x, then $B_n\, f(x) - f(x) = \mathcal{O}(1/n)$.

Bernstein operators admit a complete system of polynomial eigenfunctions; however, this system depends on n and, therefore, is associated with each operator B_n. Another inconvenience of Bernstein operators associated to an adequate function f is its slow rate of convergence towards f.

For years, several modifications and extensions of Bernstein operators have been studied. The modifications have been introduced in several directions, and we only recall a few interesting cases and cite some papers. For instance, it is possible to substitute the values of the function on equally spaced points by other mean values such as integrals, as was stated in the pioneering papers of Durrmeyer ([19]) and Derriennic ([16], [17]). In [9], the operator is modified in order to preserve some properties of the original function. Another group of modifications given by the transformation of the function by means of convenient continuous and differentiable functions is analyzed in [10]; and, of course, the extension of the Bernstein operators to the multivariate case. The most common extension of the Bernstein operator is defined on the unit simplex in higher dimensions ([27], [3], [42], [44], among others), since the basic polynomials (1.34) can be easily extended to the simplex.

1.7.1 Univariate Bernstein operators

In this section, we recall the modified univariate Bernstein-type operators that we will need later. We start by shifting the univariate Bernstein operator.

Using the change of variable

$$x = (\beta - \alpha)\, s + \alpha, \quad \alpha < \beta, \quad 0 \leqslant s \leqslant 1, \tag{1.35}$$

the univariate Bernstein basis can be defined on the interval $[\alpha, \beta]$. Indeed, if we let

$$\widetilde{p}_{n,k}(x; [\alpha, \beta]) \;=\; p_{n,k}\left(\frac{x-\alpha}{\beta-\alpha}\right) = \frac{1}{(\beta-\alpha)^n}\binom{n}{k}(x-\alpha)^k(\beta-x)^{n-k}, \quad \alpha \leqslant x \leqslant \beta,$$

then the set $\{\widetilde{p}_{n,k}(x; [\alpha, \beta]) : n \geqslant 0,\, 0 \leqslant k \leqslant n,\, \alpha \leqslant x \leqslant \beta\}$ is a basis of Π_n on the interval $[\alpha, \beta]$ satisfying

$$\begin{aligned}\sum_{k=0}^{n} \widetilde{p}_{n,k}(x; [\alpha, \beta]) = &\sum_{k=0}^{n} p_{n,k}\left(\frac{x-\alpha}{\beta-\alpha}\right) = \frac{1}{(\beta-\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}(x-\alpha)^k(\beta-x)^{n-k}\\ =&\frac{1}{(\beta-\alpha)^n}(x-\alpha+\beta-x)^n = 1.\end{aligned}$$

Moreover, since

$$\widetilde{p}_{n,k}(x;[\alpha,\beta]) \;=\; p_{n,k}(s), \quad 0 \leqslant s \leqslant 1, \quad 0 \leqslant n, \quad 0 \leqslant k \leqslant n,$$

we have that Bernstein basis on $[\alpha,\beta]$ (see Figure 1.2) satisfies the following properties:

- $\widetilde{p}_{n,k}(x;[\alpha,\beta]) \geqslant 0$ for $\alpha \leqslant x \leqslant \beta$,
- $\widetilde{p}_{n,k}(\alpha) = \delta_{0,k}$ and $\widetilde{p}_{n,k}(\beta) = \delta_{k,n}$, where, as usual, $\delta_{\nu,\eta}$ denotes the Kronecker delta,
- $(\beta-\alpha)\,\widetilde{p}\,'_{n,k}(x;[\alpha,\beta]) = n\,(\widetilde{p}_{n-1,k-1}(x;[\alpha,\beta]) - p_{n-1,k}(x;[\alpha,\beta]))$,
- If $n \neq 0$, then $\widetilde{p}_{n,k}(x;[\alpha,\beta])$ has a unique local maximum on $[\alpha,\beta]$ at $x = (\beta-\alpha)\,\frac{k}{n} + \alpha$. This maximum takes the value

$$\widetilde{p}_{n,k}\left((\beta-\alpha)\,\frac{k}{n} + \alpha;[\alpha,\beta]\right) \;=\; p_{n,k}\left(\frac{k}{n}\right) = \binom{n}{k}\frac{k^k}{n^n}(n-k)^{n-k}.$$

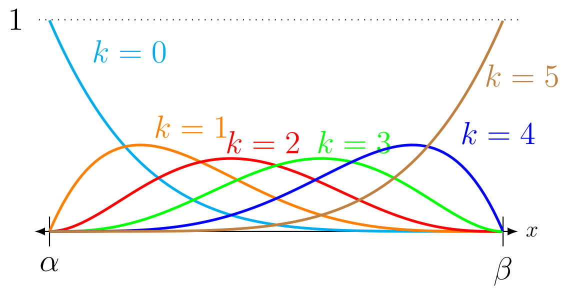

Figure 1.2: Bernstein on basis $[\alpha,\beta]$ for $n=5$.

For every function f defined on $I = [\alpha,\beta]$, we can define the shifted univariate n-th Bernstein operator as

$$\widetilde{B}_n\,[f(x),I] = \sum_{k=0}^{n} f\left((\beta-\alpha)\frac{k}{n} + \alpha\right)\,\widetilde{p}_{n,k}(x;I).$$

Note that $\widetilde{B}_n\,[f(x),I]$ is a polynomial of degree at most n. In this way,

$$\widetilde{B}_n\,[f(x),I] = B_n\,F(s), \quad 0 \leqslant s \leqslant 1,$$

where $F(s) = f((\beta - \alpha)\, s + \alpha)$ is a function defined on $[0,1]$ associated with f. From this, and since the change of variable (1.35) is linear, it is clear that $\widetilde{B}_n$ has analogous properties to those satisfied by the classical Bernstein operator.

1.7.2 Bivariate Bernstein operators

In 1963, Stancu ([44]) studied a method for deducing polynomials of Bernstein type of two variables. This method is based on obtaining an operator in two variables from the successive application of Bernstein operators of one variable.

Let $\phi_1 \equiv \phi_1(x)$ and $\phi_2 \equiv \phi_2(x)$ be two continuous functions such that $\phi_1 < \phi_2$ on $[0,1]$. Let $\Omega \subseteq \mathbb{R}^2$ be the domain bounded by the curves $y = \phi_1(x)$, $y = \phi_2(x)$, and the straight lines $x = 0$, $x = 1$. For every function $f(x,y)$ defined on Ω, taking into view

$$y = (\phi_2(x) - \phi_1(x))\, t + \phi_1(x), \tag{1.36}$$

let us define the function

$$F(x,t) \;=\; f(x, (\phi_2(x) - \phi_1(x))\, t + \phi_1(x)), \tag{1.37}$$

where $0 \leqslant t \leqslant 1$.

The *n-th Bernstein-Stancu operator* is defined as

$$\mathscr{B}_n[f(x,y),\Omega] = \sum_{k=0}^{n} \sum_{j=0}^{n_k} F\left(\frac{k}{n}, \frac{j}{n_k}\right) p_{n,k}(x)\, p_{n_k,j}(t), \tag{1.38}$$

where each n_k is a non negative integer associated with the k-th node $x_k = k/n$, and t is given by (1.36).

If we denote by $B_n^{(t)}$ the univariate Bernstein operator acting on the variable t, then the Bernstein-Stancu operator can be written as

$$\mathscr{B}_n[f(x,y),\Omega] = \sum_{k=0}^{n} \left[B_{n_k}^{(t)} F\left(\frac{k}{n}, t\right)\right] p_{n,k}(x).$$

We have the following representation of $\mathscr{B}_n$ in terms of a matrix determinant.

Proposition 1.7.1. Let $f(x, y)$ be a function defined on the domain Ω, and let F be the function defined on (1.37). Denote by $B_n^{(t)}$ the univariate Bernstein operator acting on the variable t. Then, the n-th Bernstein-Stancu operator is given by the determinant

$$\mathscr{B}_n[f(x,y),\Omega] = - \left| \begin{array}{cccc|c} 1 & & & \bigcirc & B_{n_0}^{(t)} F(0,t) \\ & 1 & & & B_{n_1}^{(t)} F(1/n,t) \\ & & \ddots & & \vdots \\ \bigcirc & & & 1 & B_{n_n}^{(t)} F(1,t) \\ \hline p_{n,0}(x) & p_{n,1}(x) & \dots & p_{n,n}(x) & 0 \end{array} \right| .$$

Remark 1.7.2. Observe that the step size of the partition of the x axis is $1/n$ and, for a fixed node $x_k = k/n$, the step size of the partition of the t axis is $1/n_k$. Therefore, the step size of the partition of the y axis is $1/m_k$, where

$$m_k = \frac{n_k}{\phi_2\left(\frac{k}{n}\right) - \phi_1\left(\frac{k}{n}\right)},$$

and, thus,

$$F\left(\frac{k}{n}, \frac{j}{n_k}\right) = f\left(\frac{k}{n}, \frac{j}{m_k} + \phi_1\left(\frac{k}{n}\right)\right).$$

We point out that, in general, $\mathscr{B}_n[f(x,y),\Omega]$ is not a polynomial. However, it is possible to obtain polynomials by an appropriate choice of ϕ_1, ϕ_2, and n_k. For instance:

1. *The Bernstein-Stancu operator on the unit square* $\mathbf{Q} = [0,1] \times [0,1]$ (see for instance [27], [44]) are obtained by letting $\phi_1(x) = 0$ and $\phi_2(x) = 1$. Hence, for a function f defined on $\mathbf{Q}$, we get

$$F\left(\frac{k}{n}, \frac{j}{n_k}\right) = f\left(\frac{k}{n}, \frac{j}{n_k}\right),$$

and

$$\mathscr{B}_n[f(x,y),\mathbf{Q}] = \sum_{k=0}^{n} \sum_{j=0}^{n_k} f\left(\frac{k}{n}, \frac{j}{n_k}\right) p_{n,k}(x) p_{n_k,j}(y).$$

Note that when n_k is independent of k (e.g., $n_k = m$ for some positive integer m), $\mathscr{B}_n$ is the tensor product of univariate Bernstein operators on $\mathbf{Q}$.

2. *The Bernstein-Stancu operators can be defined on the simplex* $\mathbf{T}^2 = \{(x,y) \in \mathbb{R}^2 : x, y \geqslant 0, 1-x-y \geqslant 0\}$ (see for instance [3] and [44]). In this case, we set $\phi_1(x) = 0$, $\phi_2(x) = 1-x$, and $n_k = n-k$, $0 \leqslant k \leqslant n$. In this way, $m_k = n$ and, since

$$F\left(\frac{k}{n}, \frac{j}{n-k}\right) = f\left(\frac{k}{n}, \frac{j}{n}\right),$$

we have

$$\begin{aligned}\mathscr{B}_n[f(x,y), \mathbf{T}^2] &= \sum_{k=0}^{n}\sum_{j=0}^{n-k} f\left(\frac{k}{n}, \frac{j}{n}\right) p_{n,k}(x)\, p_{n-k,j}\left(\frac{y}{1-x}\right) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n-k} f\left(\frac{k}{n}, \frac{j}{n}\right) \begin{pmatrix} & n & \\ k & & j\end{pmatrix} x^k\, y^j\, (1-x-y)^{n-k-j},\end{aligned}$$

for $(x,y) \in \mathbf{T}^2$, where

$$\begin{pmatrix} & n & \\ k & & j\end{pmatrix} = \frac{n!}{k!\, j!\, (n-k-j)!}, \quad 0 \leqslant k+j \leqslant n.$$

In [44], Stancu proved the following convergence result on $\mathbf{T}^2$.

Theorem 1.7.3 ([44])**.** Let f be a continuous function on $\mathbf{T}^2$. Then $\mathscr{B}_n[f(x,y), \mathbf{T}^2]$ converges uniformly to $f(x,y)$ as $n \to +\infty$.

Stancu only gave a detailed proof of the approximation properties of $\mathscr{B}_n$ on triangles. In Chapter 2, we consider a slightly general operator and prove the uniform convergence on any bounded domain Ω, and we recover Stancu's result when $\Omega = \mathbf{T}^2$.

Chapter 2

Approximation via gradients on the ball. The Zernike case

We deal with a d dimensional unit ball equipped with an inner product constructed by adding a mass point at zero to the classical ball inner product applied to the gradients of the functions. Apart from determining an explicit orthogonal polynomial basis, we study approximation properties of Fourier expansions in terms of this basis. In particular, we deduce relations between the partial Fourier sums in terms of the new orthogonal polynomials and the partial Fourier sums in terms of the classical ball polynomials. We also give an estimate of the approximation error by polynomials of degree at most n in the corresponding Sobolev space, proving that we can approximate a function by using its gradient. Numerical examples to illustrate the approximation behavior of the Sobolev basis are given. The results presented in this chapter have recently been published in [31].

2.1 Introduction

In this Chapter we start dealing with a d dimensional unit ball equipped with the inner

$$\langle f, g\rangle_{\nabla,\mu} = f(0)g(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla g(x)\,(1-\|x\|^2)^{\mu}\,dx,$$

for $\lambda > 0$ and $\mu > -1$. While these conditions are necessary for the positive definiteness of the inner product, our contribution involves only the case when $\mu \geqslant 0$. We show that if we only know the gradients of the functions, we can compute approximants by using the Sobolev Fourier orthogonal expansions, and the approximation is similar or even better than the classical one. In the introduction of [26], Li and Xu explain why and how the approximation based on the Sobolev orthogonal expansions could be better than the classical orthogonal expansions.

The particular inner product involving gradients that we will study in our paper was introduced and studied in the particular case $\mu = 0$ by Xu in [52], where the author finds a complete system of orthonormal polynomials with respect to these inner products and explores their properties.

Our objective, apart from the extension and completion of the results obtained by Xu in [52] to the more general case $\mu \geqslant 0$, is to study the analytic properties of approximation by means of the corresponding Fourier sums. We remark that, using our results, we can compute the coefficients of the Sobolev-Fourier sums without using derivatives. The Chapter is organized as follows. The Sobolev inner product with a mass point at the origin and associated bases of orthogonal polynomials are discussed in Section 2.2. Section 3.3 is devoted to the study of Sobolev Fourier orthogonal expansions and their approximation behavior. Finally, in Section 2.5, we illustrate our results with numerical examples.

2.2 Sobolev orthogonal polynomials with a mass point at zero

This section is devoted to the study of the orthogonal structure on the unit ball with respect to the *Sobolev inner product*

$$\langle f, g\rangle_{\nabla,\mu} = f(0)g(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla g(x) W_\mu(x) dx, \quad \lambda > 0. \tag{2.1}$$

Orthogonal polynomials with respect to inner products involving derivatives are called Sobolev orthogonal polynomials. Let us denote by $\mathcal{V}_n^d(\nabla, W_\mu)$ the linear space of Sobolev orthogonal polynomials of degree n with respect (3.2).

Let $\mathcal{U}(W_\mu;\mathbf{B}^d)$ denote the Sobolev space

$$\mathcal{U}(W_\mu;\mathbf{B}^d) = \{f \in \mathcal{W}_2^1(W_\mu;\mathbf{B}^d) : |f(0)| < +\infty\},$$

and let $\|\cdot\|_{\nabla,\mu}$ denote the norm of $\mathcal{U}(W_\mu;\mathbf{B}^d)$ defined by

$$\|f\|_{\nabla,\mu} = \left(f(0)^2 + \frac{\lambda}{b_\mu}\sum_{i=1}^{d}\|\partial_i f\|_\mu^2\right)^{1/2}. \tag{2.2}$$

In the following proposition, we construct a mutually orthogonal polynomial basis with respect to the inner product (3.2).

Proposition 2.2.1. For $\mu > 0$, define the polynomials

$$\begin{aligned} Q_{j,\nu}^{n,\mu}(x) &:= P_{j,\nu}^{n,\mu-1}(x) - P_{j,\nu}^{n,\mu-1}(0), \quad n \geqslant 1, \\ Q_{0,0}^{0,\mu}(x) &:= 1. \end{aligned} \tag{2.3}$$

Then $\{Q_{j,\nu}^{n,\mu} :\ 0 \leqslant j \leqslant \frac{n}{2},\ 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ constitutes a mutually orthogonal basis of $\mathcal{V}_n^d(\nabla, W_\mu)$. Moreover,

$$\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = H_{j,n}^{\nabla,\mu}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where

$$H_{j,n}^{\nabla,\mu} = \frac{\lambda}{b_{\mu-1}}[4\,j\,(n-j+\mu+d/2-1) + 2\,(n-2\,j)\,\mu]\,H_{j,n}^{\mu-1}. \tag{2.4}$$

Proof. We note that since $Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d$ are homogeneous polynomials, $Y_\nu^{n-2j}(0) = 0$ when $n-2j \geqslant 1$, and, therefore, $P_{j,\nu}^{n,\mu-1}(0) = 0$ for $n-2j \geqslant 1$. Moreover, $Q_{j,\nu}^{n,\mu}(0) = 0$ for $n \geqslant 1$.

On one hand, it is clear that

$$\left\langle Q_{0,0}^{0,\mu}, Q_{0,0}^{0,\mu}\right\rangle_{\nabla,\mu} = 1 \quad \text{and} \quad \left\langle Q_{0,0}^{0,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = 0, \quad m \geqslant 1.$$

On the other hand, we compute

$$\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} = \lambda \int_{\mathbf{B}^d} \nabla P_{j,\nu}^{n,\mu-1}(x)\cdot\nabla P_{k,\eta}^{m,\mu-1}(x)\,W_\mu(x)\,dx.$$

From (1.31) we get

$$\begin{aligned} &\left\langle Q_{j,\nu}^{n,\mu}, Q_{k,\eta}^{m,\mu}\right\rangle_{\nabla,\mu} \\ &\qquad = \frac{\lambda}{b_{\mu-1}}[4\,j\,(n-j+\mu+d/2-1) + 2\,(n-2\,j)\,\mu]\,H_{j,n}^{\mu-1}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta}. \end{aligned}$$

□

Observe that we can write the basis (3.2.2) as follows:

$$Q^{n,\mu}_{0,\nu}(x) := Y^n_\nu(x),$$
$$Q^{n,\mu}_{j,\nu}(x) := P_j^{\left(\mu-1,n-2j+\frac{d-2}{2}\right)}\left(2\,\|x\|^2-1\right)\,Y^{n-2j}_\nu(x), \quad 1\leqslant j\leqslant \frac{n-1}{2},$$
$$Q^{n,\mu}_{\frac{n}{2},1}(x) := P_{\frac{n}{2}}^{(\mu-1,\frac{d-2}{2})}(2\,\|x\|^2-1)-(-1)^{\frac{n}{2}}\frac{(d/2)_{\frac{n}{2}}}{(n/2)!},$$

where $Q^{n,\mu}_{\frac{n}{2},1}(x)$ holds only when n is even. Here, we have used the fact that $P_k^{(\alpha,\beta)}(-1)=(-1)^k(\beta+1)_k/k!$

The case when $\mu=0$ was previously studied in [52]. Here, we recall the explicit expression for the basis in this case. Let us denote by $q_k(x)$ the univariate polynomial defined by

$$q_0(x)=1, \quad q_k(x)=\int_{-1}^{x}P_{k-1}^{(0,\frac{d}{2})}(t)\,dt, \quad k\geqslant 1.$$

The Jacobi polynomials $P_k^{(-1,\frac{d-2}{2})}(x)$ are well defined for $k\geqslant 1$, satisfying [46, (4.5.5), p.72]

$$\frac{d}{dx}P_k^{(-1,\frac{d-2}{2})}(x)=\frac{1}{2}\left(k+\frac{d-2}{2}\right)\,P_{k-1}^{(0,\frac{d}{2})}(x). \tag{2.5}$$

Hence, we have

$$q_k(x)=\frac{4}{2k+d-2}\left(P_k^{(-1,\frac{d-2}{2})}(x)-(-1)^k\frac{(d/2)_k}{k!}\right).$$

Proposition 2.2.2. A mutually orthogonal basis for $\mathcal{V}^d_n(\nabla,W_0)$ is given by

$$\begin{aligned} Q^{n,0}_{0,\nu}(x) &= Y^n_\nu(x),\\ Q^{n,0}_{j,\nu}(x) &= (1-\|x\|^2)\,P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\,Y^{n-2j}_\nu(x), \quad 1\leqslant j\leqslant \tfrac{n-1}{2},\\ Q^{n,0}_{\frac{n}{2},1}(x) &= \frac{4}{n+d-2}\left(P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\|x\|^2-1)-(-1)^{\frac{n}{2}}\frac{(d/2)_{\frac{n}{2}}}{(n/2)!}\right), \end{aligned} \tag{2.6}$$

where $Q^{n,0}_{\frac{n}{2},1}(x)$ holds only when n is even. Furthermore,

$$\begin{aligned} \left\langle Q^{n,0}_{0,\nu},Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0} &= n\,\lambda\,\sigma_{d-1},\\ \left\langle Q^{n,0}_{j,\nu},Q^{n,0}_{j,\nu}\right\rangle_{\nabla,0} &= \frac{2\,j^2}{n+\frac{d-2}{2}}\lambda\,\sigma_{d-1}, \quad 1\leqslant j\leqslant \tfrac{n-1}{2},\\ \left\langle Q^{n,0}_{\frac{n}{2},1},Q^{n,0}_{\frac{n}{2},1}\right\rangle_{\nabla,0} &= \frac{8}{n+\frac{d-2}{2}}\,\lambda\,\sigma_{d-1}. \end{aligned} \tag{2.7}$$

2.3 Sobolev Fourier orthogonal expansions and approximation

For $\mu \geqslant 0$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$, let us denote by $\widehat{f}_{j,\nu}^{n,\mu}(\nabla)$ the Fourier coefficients with respect to the basis of $\mathcal{V}_n^d(\nabla, W_\mu)$ defined in (3.2.2), that is,

$$\widehat{f}_{j,\nu}^{n,\mu}(\nabla) = \frac{1}{H_{j,n}^{\nabla,\mu}} \left\langle f, Q_{j,\nu}^{n,\mu} \right\rangle_{\nabla,\mu}.$$

Let $\mathrm{proj}_m^{\nabla,\mu} : \mathcal{U}(W_\mu; \mathbf{B}^d) \to \mathcal{V}_n^d(\nabla,\, W_\mu)$ and $S_n^{\nabla,\mu} : \mathcal{U}(W_\mu; \mathbf{B}^d) \to \Pi_n^d$ denote the projection operator and partial sum operators

$$\mathrm{proj}_m^{\nabla,\mu} f(x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{m,\mu}(\nabla)\, Q_{j,\nu}^{m,\mu}(x) \quad \text{and} \quad S_n^{\nabla,\mu} f(x) = \sum_{m=0}^{n} \mathrm{proj}_m^{\nabla,\mu} f(x).$$

2.3.1 The case $\mu > 0$

The Fourier coefficients with respect to the basis (1.10) of $\mathcal{V}_n^d(W_\mu)$ are related to the Fourier coefficients $\widehat{f}_{j,\nu}^{n,\mu}(\nabla)$.

Proposition 2.3.1. Let $\mu > 0$. Then, for $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$,

$$\begin{aligned} \widehat{f}_{j,\nu}^{n,\mu}(\nabla) &= \widehat{f}_{j,\nu}^{\,n,\mu-1}, \quad n \geqslant 1, \\ \widehat{f}_{0,1}^{0,\mu}(\nabla) &= f(0). \end{aligned}$$

Proof. Since $Q_{0,0}^{0,\mu}(x) = 1$ and $H_{0,0}^{\nabla,\mu} = 1$ for $\mu \geqslant 0$, $\widehat{f}_{0,1}^{0,\mu}(\nabla) = \langle f, Q_{0,1}^{0,\mu} \rangle_{\nabla,\mu} = f(0)$.

Let $\mu > 0$. For $n \geqslant 1$,

$$\begin{aligned} \left\langle f, Q_{j,\nu}^{n,\mu} \right\rangle_{\nabla,\mu} &= f(0)\, Q_{j,\nu}^{n,\mu}(0) + \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla Q_{j,\nu}^{n,\mu}(x)\, W_\mu(x)\, dx \\ &= \lambda \int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla P_{j,\nu}^{n,\mu-1}(x)\, W_\mu(x)\, dx. \end{aligned}$$

Using Green's formula and (1.28), we get

$$\left\langle f, Q_{j,\nu}^{n,\mu} \right\rangle_{\nabla,\mu} = \lambda \left[4\, j\, (n - j + \mu + d/2 - 1) + 2\, (n - 2\, j)\, \mu\right] \int_{\mathbf{B}^d} f(x)\, P_{j,\nu}^{n,\mu-1}(x)\, W_{\mu-1}(x)\, dx$$

$$= \frac{\lambda}{b_{\mu-1}} [4\,j\,(n-j+\mu+d/2-1) + 2\,(n-2\,j)\,\mu] \left\langle f, P_{j,\nu}^{n,\mu-1} \right\rangle_{\mu-1}.$$

From (3.10), we get

$$\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla) = \frac{1}{H_{j,n}^{\nabla,\mu}} \left\langle f, Q_{j,\nu}^{n,\mu} \right\rangle_{\nabla,\mu} = \frac{1}{H_{j,n}^{\mu-1}} \left\langle f, P_{j,\nu}^{n,\mu-1} \right\rangle_{\mu-1} = \widehat{f}_{j,\nu}^{\,n,\mu-1}.$$

□

It is important to note that the Fourier coefficients can be computed without involving the derivatives of f.

For $\mu > 0$, we can deduce the relationship between the projection operators with respect to the classical and Sobolev bases.

Proposition 2.3.2. For $\mu > 0$,

$$\begin{aligned} &\operatorname{proj}_0^{\nabla,\mu} f(x) = f(0), \\ &\operatorname{proj}_m^{\nabla,\mu} f(x) = \operatorname{proj}_m^{\mu-1} f(x) - \operatorname{proj}_m^{\mu-1} f(0), \quad m \geqslant 1, \end{aligned}$$

and

$$\begin{aligned} &S_0^{\nabla,\mu} f(x) = f(0), \\ &S_n^{\nabla,\mu} f(x) = f(0) + S_n^{\mu-1} f(x) - S_n^{\mu-1} f(0), \quad n \geqslant 0. \end{aligned}$$

Proof. Clearly, $\operatorname{proj}_0^{\nabla,\mu} f(x) = f(0)$. For $m \geqslant 1$,

$$\begin{aligned} \operatorname{proj}_m^{\nabla,\mu} f(x) &= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{\,m,\mu}(\nabla)\, Q_{j,\nu}^{m,\mu}(x) \\ &= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu=1}^{a_{m-2j}^d} \left[\widehat{f}_{j,\nu}^{\,m,\mu-1} P_{j,\nu}^{m,\mu-1}(x) - \widehat{f}_{j,\nu}^{\,m,\mu-1} P_{j,\nu}^{m,\mu-1}(0) \right]. \end{aligned}$$

Then,

$$\operatorname{proj}_m^{\nabla,\mu} f(x) = \operatorname{proj}_m^{\mu-1} f(x) - \operatorname{proj}_m^{\mu-1} f(0), \quad m \geqslant 1,$$

where we have used Proposition 2.3.1 to write $\widehat{f}_{j,\nu}^{\,n,\mu}(\nabla) = \widehat{f}_{j,\nu}^{\,n,\mu-1}$ for $n \geqslant 1$.

Moreover, since $\operatorname{proj}_0^{\mu-1} f(x) - \operatorname{proj}_0^{\mu-1} f(0) = 0$

$$\begin{aligned} S_n^{\nabla,\mu} f(x) &= \operatorname{proj}_0^{\nabla,\mu} f(x) + \sum_{m=1}^{n} \operatorname{proj}_m^{\nabla,\mu} f(x) \\ &= f(0) + \sum_{m=0}^{n} \left[\operatorname{proj}_m^{\mu-1} f(x) - \operatorname{proj}_m^{\mu-1} f(0)\right]. \end{aligned}$$

Therefore,

$$S_n^{\nabla,\mu} f(x) = S_n^{\mu-1} f(x) + f(0) - S_n^{\mu-1} f(0), \quad \mu > 0, \tag{2.8}$$

and, consequently, $S_n^{\nabla,\mu} f(0) = f(0)$. □

We have the following proposition stating the interaction between differentiation and the partial sum operator $S_n^{\nabla,\mu}$ for $\mu > 0$.

Proposition 2.3.3. Let $\mu > 0$ and $n \geqslant 1$. Then,

$$\partial_i\, S_n^{\nabla,\mu} f(x) = S_{n-1}^{\mu}(\partial_i\, f)(x), \quad 1 \leqslant i \leqslant d,$$

or, equivalently,

$$\partial_i\, S_n^{\nabla,\mu} f(x) = S_{n-1}^{\nabla,\mu+1}(\partial_i\, f)(x) + S_{n-1}^{\mu}(\partial_i f)(0) - (\partial_i\, f)(0).$$

Proof. Differentiating (2.8) and using Theorem 1.6.1, we obtain

$$\partial_i\, S_n^{\nabla,\mu} f(x) = \partial_i\, S_n^{\mu-1} f(x) = S_{n-1}^{\mu}(\partial_i\, f)(x).$$

Using (2.8) again, we get

$$S_{n-1}^{\mu}(\partial_i\, f)(x) = S_{n-1}^{\nabla,\mu+1}(\partial_i\, f)(x) + S_{n-1}^{\mu}(\partial_i\, f)(0) - (\partial_i\, f)(0),$$

and the result follows. □

We have the following expression for the Parseval identity.

Corollary 2.3.4. For $\mu > 0$ and $f \in \mathcal{U}(W_\mu; \mathbf{B}^d)$,

$$\|f\|_{\nabla,\mu}^2 = f(0)^2 + \frac{\lambda}{b_{\mu-1}} \sum_{n=1}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} \left|\lambda_{n,j}^{\mu-1}\right| \left|\widehat{f}_{j,n}^{\,n,\mu-1}\right|^2 H_{j,n}^{\mu-1},$$

where $\lambda_{n,j}^{\mu-1}$ are defined in (1.30).

Consequently,

$$b_{\mu-1}\int_{B^d}\|\nabla f(x)\|^2\,W_\mu(x)\,dx=\sum_{n=1}^{\infty}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}}\left|\lambda_{n,j}^{\mu-1}\right|\left|\widehat{f}_{j,n}^{\,n,\mu-1}\right|^2\,H_{j,n}^{\mu-1}.$$

Proof. Parseval's identity for $f\in\mathcal{U}(W_\mu;\mathbf{B}^d)$ with respect to the orthogonal basis defined in (3.2.2) is written as

$$\|f\|_{\nabla,\mu}^2=f(0)^2+\sum_{n=1}^{\infty}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{\nu=1}^{a_{n-2j}}\left|\widehat{f}_{j,n}^{\,n,\mu}(\nabla)\right|^2\,H_{j,n}^{\nabla,\mu}.$$

The result follows from Proposition 2.3.1 and (3.10).

The last equation follows from

$$\int_{B^d}\nabla f(x)\cdot\nabla f(x)\,W_\mu(x)\,dx=\lim_{\lambda\to+\infty}\frac{\|f\|_{\nabla,\mu}^2}{\lambda}.$$

□

Let $\mathcal{E}_n(f)_{\nabla,\mu}$ denote the error of best approximation by polynomials in Π_n^d in the space $\mathcal{U}(W_\mu;\mathbf{B}^d)$:

$$\mathcal{E}_n(f)_{\nabla,\mu}=\|f-S_n^{\nabla,\mu}f\|_{\nabla,\mu}.$$

We have the following estimate.

Theorem 2.3.5. Let $\mu>0$. Then, for $n\geqslant 2s+1$ and $f\in\mathcal{U}(W_\mu;\mathbf{B}^d)\cap\mathcal{W}_2^{2s+1}(W_\mu;\mathbf{B}^d)$,

$$\mathcal{E}_n(f)_{\nabla,\mu}\leqslant\frac{c}{(n-1)^{2s}}\sum_{i=1}^{d}\left[\mathcal{E}_{n-2s-1}(\Delta^s\partial_i f)_{\mu+2s}+\mathcal{E}_{n-1}(\Delta_0^s\partial_i f)_\mu\right],$$

and for $n\geqslant 2s+2$ and $f\in\mathcal{U}(W_\mu;\mathbf{B}^d)\cap\mathcal{W}_2^{2s+2}(W_\mu;\mathbf{B}^d)$,

$$\mathcal{E}_n(f)_{\nabla,\mu}$$
$$\leqslant\frac{c}{(n-1)^{2s+1}}\sum_{i=1}^{d}\left[\sum_{k=1}^{d}\mathcal{E}_{n-2s-2}(\partial_k\,\Delta^s\partial_i f)_{\mu+2s+1}+\sum_{1\leqslant k<\ell\leqslant d}\mathcal{E}_{n-1}(D_{k,\ell}\Delta_0^s\partial_i f)_\mu\right].$$

Proof. For $n\geqslant 1$, we have

$$\mathcal{E}_n(f)_{\nabla,\mu}^2=\|f-S_n^{\nabla,\mu}f\|_{\nabla,\mu}^2=\frac{\lambda}{b_\mu}\sum_{i=1}^{d}\|\partial_i f-\partial_i S_n^{\nabla,\mu}f\|_\mu^2.$$

Using Proposition 2.3.3, we get

$$\mathcal{E}_n(f)^2_{\nabla,\mu} = \frac{\lambda}{b_\mu}\sum_{i=1}^{d} \|\partial_i f - S^\mu_{n-1}(\partial_i f)\|^2_\mu = \frac{\lambda}{b_\mu}\sum_{i=1}^{d} \mathcal{E}_{n-1}(\partial_i f)^2_\mu,$$

and the result follows from (1.26) and (1.27). □

Lemma 2.3.6. Let $\mu \geqslant 0$. For $f \in \mathcal{U}(W_{\mu+1}; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\mu \leqslant \frac{c}{\sqrt{n}}\,\mathcal{E}_n(f)_{\nabla,\mu+1}.$$

Proof. From Corollary 2.3.4 and the Parseval identity (1.14), we get

$$\begin{aligned}\|f - S^\mu_n f\|^2_\mu &= \sum_{m=n+1}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu=1}^{a_{m-2j}} \left|\widehat{f}^{\,m,\mu}_{j,\nu}\right|^2 H^\mu_{j,m}\\ &= \sum_{m=n+1}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{1}{\left|\lambda^\mu_{m,j}\right|}\sum_{\nu=1}^{a_{m-2j}} \left|\lambda^\mu_{m,j}\right| \left|\widehat{f}^{\,n,\mu}_{j,\nu}\right|^2 H^\mu_{j,n}\\ &\leqslant \frac{c}{2\,(\mu+1)\,n}\mathcal{E}_n(f)^2_{\nabla,\mu+1},\end{aligned}$$

where c is a constant. The inequality follows from $\left|\lambda^\mu_{n,j}\right|^{-1} \leqslant \dfrac{1}{2\,(\mu+1)\,n}$ for $n \geqslant 1$ and $0 \leqslant j \leqslant \frac{n}{2}$. □

Moreover, the following proposition shows that the rate of convergence of $S^{\nabla,\mu}_n f$ towards f is faster than the rate stated in Theorem 2.3.5.

Theorem 2.3.7. Let $\mu > 0$. Then, for $n \geqslant 2s+1$ and $f \in \mathcal{U}(W_{\mu+2s+1}; \mathbf{B}^d) \cap \mathcal{W}^{2s+1}_2(W_\mu; \mathbf{B}^d)$,

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}}\sum_{i=1}^{d}\left[\mathcal{E}_{n-2s-1}(\Delta^s\partial_i f)_{\nabla,\mu+2s+1} + \mathcal{E}_{n-1}(\Delta^s_0\partial_i f)_{\nabla,\mu+1}\right],$$

and for $n \geqslant 2s+2$ and $f \in \mathcal{U}(W_{\mu+2s+2}; \mathbf{B}^d) \cap \mathcal{W}^{2s+2}_2(W_\mu; \mathbf{B}^d)$,

$$\begin{aligned}\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+3/2}}\sum_{i=1}^{d}\Bigg[&\sum_{k=1}^{d}\mathcal{E}_{n-2s-2}(\partial_k\,\Delta^s\partial_i f)_{\nabla,\mu+2s+2}\\ &+ \sum_{1\leqslant k<\ell\leqslant d}\mathcal{E}_{n-1}(D_{k,\ell}\Delta^s_0\partial_i f)_{\nabla,\mu+1}\Bigg].\end{aligned}$$

Consequently,

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1} + \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right],$$

and

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+3/2}} \sum_{i=1}^{d} \left[\sum_{k=1}^{d} \|\partial_k \, \Delta^s \partial_i f\|_{\nabla,\mu+2s+2} + \sum_{1\leqslant k<\ell\leqslant d} \|D_{k,\ell} \Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right],$$

respectively.

Proof. For $n \geqslant 2s+1$ and $f \in \mathcal{U}(W_{\mu+2s+1}; \mathbf{B}^d) \cap \mathcal{W}_2^{2s+1}(W_\mu; \mathbf{B}^d)$, Theorem 2.3.5, together with Lemma 2.3.6, implies

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\nabla,\mu+2s+1} + \mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_{\nabla,\mu+1} \right].$$

Moreover, from the fact that, for $i = 1, \ldots, d$,

$$\mathcal{E}_{n-2s-1}(\Delta^s \partial_i f)_{\nabla,\mu+2s+1} \leqslant c \, \|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1},$$

and

$$\mathcal{E}_{n-1}(\Delta_0^s \partial_i f)_{\nabla,\mu+1} \leqslant c \, \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1},$$

we get

$$\mathcal{E}_n(f)_{\nabla,\mu} \leqslant \frac{c}{(n-1)^{2s+1/2}} \sum_{i=1}^{d} \left[\|\Delta^s \partial_i f\|_{\nabla,\mu+2s+1} + \|\Delta_0^s \partial_i f\|_{\nabla,\mu+1} \right].$$

The remaining part of the theorem follows similarly. □

2.4 The Zernike case $\mu = 0$

This case is more complicated than the previous one. Hence, we consider it separately here.

Proposition 2.4.1. For $f \in \mathcal{U}(W_0; \mathbf{B}^d)$,

$$\begin{aligned}
\widehat{f}^{\,n,0}_{0,\nu}(\nabla) &= \langle f, Y^n_\nu \rangle_{\mathbf{S}^{d-1}}\,, \\
\widehat{f}^{\,n,0}_{j,\nu}(\nabla) &= \frac{n+\frac{d-2}{2}}{j}\left[\frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{j-1,\nu}(x)\,dx \right. \\
&\qquad\qquad \left. - \left\langle f, Y^{n-2j}_\nu \right\rangle_{\mathbf{S}^{d-1}}\right], \quad 1 \leqslant j \leqslant \frac{n-1}{2}, \\
\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla) &= \frac{n+\frac{d-2}{2}}{2}\left[\langle f, 1\rangle_{\mathbf{S}^{d-1}} - \frac{(n+d-2)}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{\frac{n}{2}-1,1}(x)\,dx\right], \\
\widehat{f}^{\,0,0}_{0,1}(\nabla) &= f(0),
\end{aligned}$$

where $\widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla)$ holds when n is even.

Proof. Here, we use spherical-polar coordinates $x = r\,\xi$ where $r > 0$ and $\xi \in \mathbf{S}^{d-1}$.

Let $n \geqslant 1$. Using Green's identity and the fact that $Q^{n,0}_{0,\nu}(0) = Y^n_\nu(0) = 0$, we get

$$\left\langle f, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0} = \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\,\frac{\partial\, Q^{n,0}_{0,\nu}}{\partial r}(\xi)\,d\sigma(\xi) - \lambda \int_{\mathbf{B}^d} f(x)\,\Delta Q^{n,0}_{0,\nu}(x)\,dx.$$

Observe that $\Delta\, Q^{n,0}_{0,\nu}(x) = \Delta\, Y^n_\nu(x) = 0$. Moreover, by Euler's equation for homogeneous polynomials (1.1), we have that

$$\frac{\partial}{\partial r} Y^n_\nu(x)\bigg|_{r=1} = n\,Y^n_\nu(\xi).$$

Then, from (2.7), we get

$$\widehat{f}^{\,n,0}_{0,\nu}(\nabla) = \frac{\left\langle f, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0}}{\left\langle Q^{n,0}_{0,\nu}, Q^{n,0}_{0,\nu}\right\rangle_{\nabla,0}} = \langle f, Y^n_\nu\rangle_{\mathbf{S}^{d-1}}\,.$$

Similarly, for $j \geqslant 1$, we have

$$\left\langle f, Q^{n,0}_{j,\nu}\right\rangle = \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\frac{\partial\, Q^{n,0}_{j,\nu}}{\partial r}(\xi)\,d\sigma(\xi) - \lambda \int_{\mathbf{B}^d} f(x)\,\Delta\, Q^{n,0}_{j,\nu}(x)\,dx.$$

Using the following facts ([52]):

$$\begin{aligned}
\frac{\partial}{\partial r} Q_{j,\nu}^{n,0}(x)\Big|_{r=1} &= -2\, P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(1)\, Y_\nu^{n-2j}(\xi), \\
P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(1) &= j, \\
\Delta\, Q_{j,\nu}^{n,0}(x) &= -4\, j \left(n - j + \frac{d-2}{2}\right) P_{j-1,\nu}^{n-2,1}(x),
\end{aligned}$$

we get

$$\begin{aligned}
\left\langle f, Q_{j,\nu}^{n,0} \right\rangle = -2\, j\, \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\, Y_\nu^{n-2j}(\xi)\, d\sigma(\xi) \\
+ 4\, j \left(n - j + \frac{d-2}{2}\right) \lambda \int_{\mathbf{B}^d} f(x)\, P_{j-1,\nu}^{n-2,1}(x)\, dx,
\end{aligned}$$

and, therefore, from (2.7), we obtain

$$\begin{aligned}
\widehat{f}_{j,\nu}^{\,n,0}(\nabla) = -\frac{n + \frac{d-2}{2}}{j} \left\langle f, Y_\nu^{n-2j} \right\rangle_{\mathbf{S}^{d-1}} \\
+ \frac{2\left(n + \frac{d-2}{2}\right)\left(n - j + \frac{d-2}{2}\right)}{j\, \sigma_{d-1}} \int_{\mathbf{B}^d} f(x)\, P_{j-1\,\nu}^{n-2,1}(x)\, dx.
\end{aligned}$$

Similarly, $\widehat{f}_{\frac{n}{2},1}^{\,n,0}(\nabla)$ is deduced by using the fact that $P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\,\|x\|^2 - 1)$ is a radial function and that

$$\begin{aligned}
\frac{\partial}{\partial r} P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2r^2 - 1)\Big|_{r=1} &= (n + d - 2), \\
\Delta P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\,\|x\|^2 - 1) &= (n + d - 2)^2\, P_{\frac{n}{2}-1}^{(1,\frac{d-2}{2})}(2\,\|x\|^2 - 1), \qquad (2.9)
\end{aligned}$$

where we have used (1.5) and the identities ([46] and [37], respectively)

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$\beta\, P_n^{(\alpha,\beta)}(t) + (1 + t)\, \frac{d}{dt} P_n^{(\alpha,\beta)}(t) = (\beta + n)\, P_n^{(\alpha+1,\beta-1)}(t),$$

to compute $\Delta P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\,\|x\|^2 - 1)$. □

For the case when $\mu = 0$, the Parseval identity reads

$$\|f\|^2_{\nabla,0} = f(0)^2 + \lambda\,\sigma_{d-1} \sum_{n=1}^{\infty} \Bigg[\sum_{\nu=1}^{a_n^d} n\ |\langle f, Y_\nu^n \rangle_{\mathbf{S}^{d-1}}|^2$$

$$+ \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} (2n+d-2)$$

$$\times \left| \frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}} \int_{\mathbf{B}^d} f(x)\, P_{j-1\,\nu}^{n-2,1}(x)\,dx - \left\langle f, Y_\nu^{n-2j} \right\rangle_{\mathbf{S}^{d-1}} \right|^2 \Bigg]$$

$$+ \lambda\,\sigma_{d-1} \sum_{k=1}^{\infty} (4k+d-2) \left| \langle f, 1 \rangle_{\mathbf{S}^{d-1}} - \frac{(2k+d-1)}{\sigma_{d-1}} \int_{\mathbf{B}^d} f(x)\, P_{k-1,1}^{2k-2,1}(x)\,dx \right|^2.$$

Since

$$\int_{\mathbf{B}^d} \nabla f(x) \cdot \nabla f(x)\,dx = \lim_{\lambda\to+\infty} \frac{\|f\|^2_{\nabla,0}}{\lambda},$$

we have the following corollary.

Corollary 2.4.2. For $f \in \mathcal{U}(W_0; \mathbf{B}^d)$,

$$\frac{1}{\sigma_{d-1}} \int_{\mathbf{B}^d} \|\nabla f(x)\|^2\ dx = \sum_{n=1}^{\infty} \Bigg[\sum_{\nu=1}^{a_n} n\ |\langle f, Y_\nu^n \rangle_{\mathbf{S}^{d-1}}|^2$$

$$+ \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} (2n+d-2)$$

$$\times \left| \frac{2\left(n-j+\frac{d-2}{2}\right)}{\sigma_{d-1}} \int_{B^d} f(x)\, P_{j-1\,\nu}^{n-2,1}(x)\,dx - \left\langle f, Y_\nu^{n-2j} \right\rangle_{\mathbf{S}^{d-1}} \right|^2 \Bigg]$$

$$+ \sum_{k=1}^{\infty} (4k+d-2) \left| \langle f, 1 \rangle_{\mathbf{S}^{d-1}} - \frac{(2k+d-1)}{\sigma_{d-1}} \int_{\mathbf{B}^d} f(x)\, P_{k-1,1}^{2k-2,1}(x)\,dx \right|^2.$$

Therefore, if $f(x) = (1-\|x\|^2)\, g(x) \in \mathcal{U}(W_0; \mathbf{B}^d)$,

$$\sigma_{d-1}\, b_1^2 \int_{\mathbf{B}^d} \|\nabla f(x)\|^2\ dx$$

$$= 4 \sum_{n=1}^{\infty} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}} (2n+d-2) \left(n-j+\frac{d-2}{2}\right)^2 \left| H^1_{j-1,n-2}\, \widehat{g}\,{}_{j-1\,\nu}^{n-2,1} \right|^2$$

$$+ \sum_{k=1}^{\infty} (4k+d-2)\ (2k+d-1)^2 \left| H^1_{k-1,2k-2}\, \widehat{g}\,{}_{k-1,1}^{2k-2,1} \right|^2.$$

We will denote the projection operator on $\mathcal{H}_n^d$ by $\mathrm{proj}_{\mathcal{H}_n^d}$. It is well known that

$$\mathrm{proj}_{\mathcal{H}_n^d} f(x) = \|x\|^2 \frac{n+\frac{d-2}{2}}{\frac{d-2}{2}} \frac{1}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} f(y)\, C_n^{(\frac{d-2}{2})}(x'\cdot y)\, d\sigma(y),$$

for $x \in \mathbf{B}^d$ and $x' = x/\|x\| \in \mathbf{S}^{d-1}$, where $C_n^{(\lambda)}(t)$ denotes the Gegenbauer polynomial of degree n and $x\cdot y$ is the usual dot product in $\mathbb{R}^d$. Moreover, we will denote by $\mathbb{P}_n^\mu(\cdot,\cdot)$ the reproducing kernel on $\mathcal{V}_n^d(W_\mu)$ given by

$$\mathbb{P}_n^\mu(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\nu=1}^{a_{n-2j}^d} \frac{P_{j,\nu}^{n,\mu}(x)\, P_{j,\nu}^{n,\mu}(y)}{H_{j,n}^\mu}.$$

Proposition 2.4.3. For $f \in \mathcal{U}(W_0;\mathbf{B}^d)$ and $n \geqslant 1$,

$$\begin{aligned}\mathrm{proj}_n^{\nabla,0} f(x) = \mathrm{proj}_{\mathcal{H}_n^d} f(x) + (1-\|x\|^2)\Bigg[\frac{d\,(d/2+1)}{\sigma_{d-1}} \int_{\mathbf{B}^d} f(y)\mathbb{P}_{n-2}^1(x,y)\,dy \\ - \left(n+\frac{d-2}{2}\right) \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{j} P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\, \mathrm{proj}_{\mathcal{H}_{n-2j}^d} f(x)\Bigg] \\ + \widehat{f}^{\,n,0}_{\frac{n}{2},1}(\nabla)\, Q^{n,0}_{\frac{n}{2},1}(x),\end{aligned}$$

where the last term holds when n is even.

Consequently, if $f(x) = (1-\|x\|^2)\, g(x) \in \mathcal{U}(W_0;\mathbf{B}^d)$, then

$$\begin{aligned}\mathrm{proj}_n^{\nabla,0} f(x) &= (1-\|x\|^2)\, \mathrm{proj}_{n-2}^1 g(x) \\ &\quad + \frac{(n+d-2)\left(n+\frac{d-2}{2}\right)(n+d)}{8\,\sigma_{d-1}} \frac{H^1_{\frac{n}{2},n}}{b_1}\, \widehat{g}^{\,n,1}_{\frac{n}{2},1}\, Q^{n,0}_{\frac{n}{2},1}(x).\end{aligned}$$

Proof. From Proposition (2.3.1), we have $\widehat{f}^{\,n,0}_{0,\nu}(\nabla) = \langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}$. Then,

$$\sum_{\nu=1}^{a_n^d} \widehat{f}^{\,n,0}_{0,\nu}(\nabla)\, Y_\nu^n(x) = \sum_{\nu=1}^{a_n^d} \langle f, Y_\nu^n\rangle_{\mathbf{S}^{d-1}}\; Y_\nu^n(x) = \mathrm{proj}_{\mathcal{H}_n^d} f(x).$$

Again, from Proposition (2.3.1), we have that for $1 \leqslant j \leqslant \frac{n-1}{2}$,

$$\widehat{f}^{\,n,0}_{j,\nu}(\nabla) = -\frac{n+\frac{d-2}{2}}{j} \left\langle f, Y_\nu^{n-2j} \right\rangle_{\mathbf{S}^{d-1}}$$

$$+\frac{2\left(n+\frac{d-2}{2}\right)\left(n-j+\frac{d-2}{2}\right)}{j\,\sigma_{d-1}}\int_{\mathbf{B}^d} f(x)\,P^{n-2,1}_{j-1,\nu}(x)\,dx.$$

Therefore,

$$\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a^d_{n-2j}}\widehat{f}^{\,n,0}_{\,j,\nu}(\nabla)\,Q^{n,0}_{j,\nu}(x)$$

$$=-\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{j}P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\,\mathrm{proj}_{\mathcal{H}^d_{n-2j}}f(x)$$

$$+2\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{\nu=1}^{a^d_{n-2-2j}}\frac{n-j-1+\frac{d-2}{2}}{j+1}$$

$$\times\frac{1}{\sigma_{d-1}}\int_{\mathbf{B}^d} f(y)P^{n-2,1}_{j,\nu}(y)\,P^{n-2,1}_{j,\nu}(x)\,dy,$$

where we have made the change $j-1\to j$ in the last line. Using

$$H^1_{j,n-2}=\frac{(j+1)\,(d/2)\,(d/2+1)}{(n+\frac{d-2}{2})\,(n-1-j+\frac{d-2}{2})},$$

we obtain

$$\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{\nu=1}^{a^d_{n-2j}}\widehat{f}^{\,n,0}_{\,j,\nu}(\nabla)\,Q^{n,0}_{j,\nu}(x)$$

$$=-\left(n+\frac{d-2}{2}\right)(1-\|x\|^2)\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{j}P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\,\mathrm{proj}_{\mathcal{H}^d_{n-2j}}f(x)$$

$$+\frac{d\,(d/2+1)}{\sigma_{d-1}}(1-\|x\|^2)\int_{\mathbf{B}^d} f(y)\mathbb{P}^1_{n-2}(x,y)\,dy.$$

Note that

$$b_1=\frac{d\,(d/2+1)}{\sigma_{d-1}},$$

is the normalization constant for $W_1(x)$.

□

The study of the interaction between differentiation and the partial sums $S_n^{\nabla,0}$ depends on the following proposition. First, we recall integration by parts in higher dimensions. From the Divergence Theorem and the product rule, if u is a real valued function and $\mathbf{V}$ is a vector field, then

$$\int_{\mathbf{S}^{d-1}} u\,\mathbf{V}\cdot\xi\,d\sigma(\xi) = \int_{\mathbf{B}^d} \nabla\cdot(u\,\mathbf{V})\,dx = \int_{\mathbf{B}^d} u\,\nabla\cdot\mathbf{V}\,dx + \int_{\mathbf{B}^d} \nabla u\cdot\mathbf{V}\,dx.$$

Therefore, the integration by parts formula on the unit ball is

$$\int_{\mathbf{B}^d} u\,\nabla\cdot\mathbf{V}\,dx = \int_{\mathbf{S}^{d-1}} u\,\mathbf{V}\cdot\xi\,d\sigma(\xi) - \int_{\mathbf{B}^d} \nabla u\cdot\mathbf{V}\,dx.$$

Proposition 2.4.4. For $f\in\mathcal{U}(W_0;\mathbf{B}^d)$ and $m\geqslant 1$, we have

$$\partial_i\,\mathrm{proj}_m^{\nabla,0} f(x)\in\mathcal{V}_{m-1}^d(W_0),\quad 1\leqslant i\leqslant d,$$

and

$$D_{i,j}\,\mathrm{proj}_m^{\nabla,0} f(x)\in\mathcal{V}_m^d(\nabla,W_0),\quad 1\leqslant i<j\leqslant d.$$

Proof. By the definition of $\mathrm{proj}_m^{\nabla,0} f(x)$, it is sufficient to show that $\partial_i\,Q_{j,\nu}^{m,0}(x)\in\mathcal{V}_{m-1}^d(W_0)$ for $0\leqslant j\leqslant\frac{m}{2}$, $1\leqslant\nu\leqslant a_{m-2j}^d$, and $1\leqslant i\leqslant d$.

Fix $i\in\{1,2,\dots,d\}$. For $j=0$, we have

$$\partial_i Q_{0,\nu}^{m,0}(x) = \partial_i\,Y_\nu^m(x).$$

We compute

$$\left\langle\partial_i\,Y_\nu^m, P_{\ell,\eta}^{k,0}\right\rangle_0 = b_0\int_{\mathbf{B}^d}\partial_i\,Y_\nu^m(x)\,P_{\ell,\eta}^{k,0}(x)\,dx.$$

Applying the integration by parts formula to $P_{\ell,\eta}^{k,0}(x)$ and the vector field $Y_\nu^m(x)\,\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical vector in $\mathbb{R}^n$, we obtain

$$\begin{aligned}\int_{\mathbf{B}^d}\partial_i\,Y_\nu^m(x)\,P_{\ell,\eta}^{k,0}(x)\,dx =& P_\ell^{(0,k-2\ell+\frac{d-2}{2})}(1)\int_{\mathbf{S}^{d-1}}\xi_i\,Y_\eta^{k-2\ell}(\xi)\,Y_\nu^m(\xi)\,d\sigma(\xi)\\ &-\int_{\mathbf{B}^d}\partial_i P_{\ell,\eta}^{k,0}(x)\,Y_\nu^m(x)\,dx.\end{aligned}$$

The integral over $\mathbf{S}^{d-1}$ vanishes for $k\leqslant m-2$. Moreover, since $Y_\eta^m(x) = P_{0,\eta}^{m,0}(x)$, then the second integral on the right also vanishes for $k\leqslant m-2$. Therefore,

$$\left\langle\partial_i\,Y_\nu^m, P_{\ell,\eta}^{k,0}\right\rangle_0 = 0,\quad k\leqslant m-2.$$

Consequently, $\partial_i Q_{0,\nu}^{m,0}(x)\in\mathcal{V}_{m-1}^d(W_0)$.

For $1 \leqslant j \leqslant \frac{m-1}{2}$, we have

$$\partial_i Q_{j,\nu}^{m,0}(x) = \partial_i \,(1-\|x\|^2)\, P_{j-1,\nu}^{m-2,1}(x).$$

From the integration by parts formula and the fact that $1-\|x\|^2$ vanishes on the sphere, we get

$$\int_{\mathbf{B}^d} \partial_i \,(1-\|x\|^2)\, P_{j-1,\nu}^{m-2,1}(x)\, P_{\ell,\eta}^{k,0}(x)\, dx = -\int_{\mathbf{B}^d} \partial_i P_{\ell,\eta}^{k,0}(x)\, P_{j-1,\nu}^{m-2,1}(x)\, W_1(x)\, dx.$$

Hence,

$$\left\langle \partial_i \,(1-\|x\|^2)\, P_{j-1,\nu}^{m-2,1}(x), P_{\ell,\eta}^{k,0}(x) \right\rangle_0 = 0, \quad k \leqslant m-2,$$

and, thus, $\partial_i \, Q_{j,\nu}^{m,0}(x) \in \mathcal{V}_{m-1}^d(W_0)$.

Finally, from (2.5), we have

$$\begin{aligned}\int_{\mathbf{B}^d} \partial_i Q_{\frac{m}{2},1}^{m,0}(x)\, P_{\ell,\eta}^{k,0}(x)\, dx &= 4 \int_{\mathbf{B}^d} x_i \, P_{\frac{m}{2}-1}^{(0,\frac{d}{2})}(2\|x\|^2-1)\, P_{\ell,\eta}^{k,0}(x)\, dx \\ &= 4 \int_{\mathbf{B}^d} P_{\frac{m}{2}-1,i}^{m-1,0}(x)\, P_{\ell,\eta}^{k,0}(x)\, dx,\end{aligned}$$

where we have used the fact that $Y_i^1(x) = x_i$. Therefore,

$$\left\langle \partial_i Q_{\frac{m}{2},1}^{m,0}, P_{\ell,\eta}^{k,0} \right\rangle_0 = 0, \quad k \leqslant m-2.$$

Hence, we conclude that $\partial_i Q_{\frac{m}{2},1}^{m,0}(x) \in \mathcal{V}_{m-1}^d(W_0)$.

Now, $D_{i,j}$ maps $\mathcal{H}_n^d$ to itself, and

$$D_{i,j}\, Q_{\frac{m}{2},1}^{m,0}(x) = 0, \quad 1 \leqslant i < j \leqslant d,$$

since $Q_{\frac{m}{2},1}^{m,0}(x)$ is a radial function. This implies that $D_{i,j}\, \mathrm{proj}_m^{\nabla,0} f(x) \in \mathcal{V}_{m-1}^d(\nabla, W_0)$. □

We use the previous result to show that differentiation commutes with the partial Fourier sum $S_n^{\nabla,0}$.

Proposition 2.4.5. Let $\mu = 0$. Then,

$$\partial_i \, S_n^{\nabla,0} f \;=\; S_{n-1}^{0}(\partial_i f), \quad 1 \leqslant i \leqslant d,$$

and

$$D_{i,j}\, S_n^{\nabla,0} f \;=\; S_n^{\nabla,0}(D_{i,j}\, f), \quad 1 \leqslant i < j \leqslant d.$$

Proof. By its definition, $f - S_n^{\nabla,0} f = \sum_{m=n+1}^{+\infty} \mathrm{proj}_m^{\nabla,0} f$. From Proposition 2.4.4 we get that $\langle \partial_i (f - S_n^{\nabla,0} f), P\rangle_0 = 0$ for all $P \in \Pi_{n-1}^d$. Consequently, $S_{n-1}^0(\partial_i f - \partial_i S_n^{\nabla,0} f) = 0$. Since S_{n-1}^0 reproduces polynomials of degree at most $n-1$, then $S_{n-1}^0(\partial_i S_n^{\nabla,0} f) = \partial_i S_n^{\nabla,0} f$, which implies that

$$0 = S_{n-1}^0(\partial_i f - \partial_i S_n^{\nabla,0} f) = S_{n-1}^0(\partial_i f) - \partial_i S_n^{\nabla,0} f,$$

and the first communation relation is proved. The second relation can be established in a similar way. □

The relation in the proposition above passes down to the Fourier coefficients.

Proposition 2.4.6. Let $f \in \mathcal{U}(W_0; \mathbf{B}^d) \cap \mathcal{W}_2^2(W_1; \mathbf{B}^d)$. Then,

$$\begin{aligned}
\widehat{\Delta f}\,{}_{j,\nu}^{n-2,1} &= -4\,(j+1)\left(n - j - 1 + \frac{d-2}{2}\right) \widehat{f}\,{}_{j+1,\nu}^{n,0}(\nabla), \quad 0 \leqslant j \leqslant \frac{n-3}{2},\\
\widehat{\Delta f}\,{}_{\frac{n-2}{2},1}^{n-2,1} &= 4\,(n+d)\,(n+d-1)\,\widehat{f}\,{}_{\frac{n}{2},1}^{n,0}(\nabla),
\end{aligned}$$

where the last relation holds only when n is even. Moreover,

$$\begin{aligned}
\widehat{\Delta_0 f}\,{}_{j,\nu}^{n,0}(\nabla) &= -(n-2j)\,(n-2j+d-2)\,\widehat{f}\,{}_{j,\nu}^{n,0}(\nabla), \quad 0 \leqslant j \leqslant \frac{n-1}{2},\\
\widehat{\Delta_0 f}\,{}_{\frac{n}{2},1}^{n,0}(\nabla) &= 0.
\end{aligned}$$

Proof. From $\mathrm{proj}_n^{\nabla,0} f = S_n^{\nabla,0} f - S_{n-1}^{\nabla,0} f$, Proposition 2.4.5, and Theorem 1.6.1, we obtain $\Delta\,\mathrm{proj}_m^{\nabla,0} f = \mathrm{proj}_{m-2}^1(\Delta\, f)$.

On the other hand, we have

$$\begin{aligned}
\Delta\,\mathrm{proj}_n^{\nabla,0} &= \sum_{\nu=1}^{a_{n-2j}^d} \widehat{f}\,{}_{0,\nu}^{n,0}(\nabla)\,\Delta Y_\nu^n(x)\\
&\quad + \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\nu} \widehat{f}\,{}_{j,\nu}^{n,0}(\nabla)\,\Delta(1 - \|x\|^2)\,P_{j-1,\nu}^{n-2,1}(x)\\
&\quad + \frac{4}{n+d-2}\,\widehat{f}\,{}_{\frac{n}{2},1}^{n,0}(\nabla)\,\Delta\,P_{\frac{n}{2}}^{(-1,\frac{d-2}{2})}(2\|x\|^2 - 1).
\end{aligned}$$

Using $\Delta Y_\nu^n(x) = 0$, together with (2.9) and

$$\Delta(1-\|x\|^2)\, P_{j-1,\nu}^{n-2,1}(x) = -4\, j \left(n - j + \frac{d-2}{2}\right) P_{j-1,\nu}^{n-2,1}(x), \quad ([52])$$

we obtain

$$\begin{aligned}\Delta\, \mathrm{proj}_n^{\nabla,0} = &- 4 \sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} \sum_{\nu} (j+1) \left(n - j - 1 - \frac{d-2}{2}\right) \widehat{f}\,{}^{n,0}_{j+1,\nu}(\nabla)\, P_{j,\nu}^{n-2,1}(x)\\ &+ 4(n+d)\,(n+d-1)\, \widehat{f}\,{}^{n,0}_{\frac{n}{2},1}(\nabla)\, P_{\frac{n}{2}-1,1}^{n-2,1}(x).\end{aligned}$$

Hence, by $\Delta\, \mathrm{proj}_m^{\nabla,0} f = \mathrm{proj}_{m-2}^{1}(\Delta\, f)$, the first result follows.

Similarly, using $D_{i,j}\, S_n^{\nabla,0} f = S_n^{\nabla,0}(D_{i,j}\, f)$, we get $\Delta_0 \mathrm{proj}_n^{\nabla,0} f = \mathrm{proj}_n^{\nabla,0}(\Delta_0\, f)$. Then, using

$$\Delta_0\, Y(\xi) = -n\,(n+d-2)\, Y(\xi), \quad \forall Y \in \mathcal{H}_n^d, \quad \xi \in \mathbf{S}^{d-1}.$$

we get the second result. □

The main results of this section are the following theorems.

Theorem 2.4.7. Let $s \geqslant 1$ be an integer and $f \in \mathcal{U}(W_0; \mathbf{B}^d) \cap \mathcal{W}_2^{2s}(W_1, \mathbf{B}^d)$. Then, for $n \geqslant 2s+2$,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}} \Big[\mathcal{E}_{n-2s-2}(\Delta^s f)_{2s+1} + \mathcal{E}_n(\Delta_0^s f)_{\nabla,0}\Big],$$

and, consequently,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}} \Big[\|\Delta^s f\|_{2s+1} + \|\Delta_0^s f\|_{\nabla,0}\Big].$$

Proof. The Parseval identity reads,

$$\mathcal{E}_n(f)^2_{\nabla,0} = \|f - S_n^{\nabla,0} f\|^2_{\nabla,0} = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left|\widehat{f}\,{}^{m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m} = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where we split the sum as

$$\Sigma_1 = \sum_{m=n+1}^{\infty} \sum_{j=\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{\nu} \left|\widehat{f}\,{}^{m,0}_{j,\nu}(\nabla)\right|^2 \widetilde{H}^0_{j,m},$$

$$\Sigma_2 = \sum_{m=\lfloor \frac{n+2}{2} \rfloor}^{\infty} \left| \widehat{f}^{\,m,0}_{\frac{m}{2},1}(\nabla) \right|^2 \widetilde{H}^0_{\frac{m}{2},m},$$

$$\Sigma_3 = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{4} \rfloor - 1} \sum_{\nu} \left| \widehat{f}^{\,m,0}_{j,\nu}(\nabla) \right|^2 \widetilde{H}^0_{j,m}.$$

We estimate Σ_1 first. Using Proposition 2.4.6, we get

$$\left| \widehat{f}^{\,m,0}_{j,\nu}(\nabla) \right|^2 = \frac{1}{16\, j^2\, (m - j + \frac{d-2}{2})^2} \left| \widehat{\Delta f}^{\,m-2,1}_{j-1,\nu} \right|^2,$$

and iterating the first identity in Theorem 1.6.2, we obtain

$$\left| \widehat{f}^{\,m,0}_{j,\nu}(\nabla) \right|^2 = \frac{1}{16\, j^2\, (m - j + \frac{d-2}{2})^2} \prod_{i=1}^{s-1} \left(\kappa^{2i-1}_{m-j-1} \right)^{-2} \left| \widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{j-s-1,\nu} \right|^2.$$

For $\lfloor \frac{m}{4} \rfloor \leqslant j \leqslant \lfloor \frac{m}{2} \rfloor$, we have $j \sim m$, and, thus

$$\left| \widehat{f}^{\,m,0}_{j,\nu}(\nabla) \right|^2 \sim \frac{1}{m^{4s}} \left| \widehat{\Delta^s f}^{\,m-2s-2,2s+1}_{j-s-1,\nu} \right|^2.$$

Furthermore,

$$\frac{\widetilde{H}^0_{j,m}}{H^{2s+1}_{j-s-1,m-2s-2}} = \frac{\widetilde{H}^0_{j,m}}{H^0_{j,m}} \frac{H^0_{j,m}}{H^1_{j-1,m-2}} \frac{H^1_{j-1,m-2}}{H^{2s+1}_{j-s-1,m-2s-2}}.$$

From (1.11) and (2.7), we have

$$\frac{\widetilde{H}^0_{j,m}}{H^0_{j,m}} = \frac{4\,\lambda\,\sigma_{d-1}\,(m + \frac{d}{2})\,(m - j + \frac{d}{2})\,j^2}{d\,(m + \frac{d-2}{2})\,(m - j + \frac{d}{2})},$$

$$\frac{H^0_{j,m}}{H^1_{j-1,m-2}} = \frac{(m - 2 + \frac{d}{2})\,(m - j + \frac{d-2}{2})}{(\frac{d}{2} + 1)\,(m + \frac{d}{2})\,j},$$

$$\frac{H^1_{j-1,m-2}}{H^{2s+1}_{j-s-1,m-2s-2}} = \frac{(2)_{2s}\,(m - j - s + \frac{d-2}{2})_s\,(m - j + \frac{d+2}{2})_s}{(1 + \frac{d+2}{2})_{2s}\,(j - s)_s\,(j + 1)_s}.$$

It is easy to verify that when $j \sim m$,

$$\frac{\widetilde{H}^0_{j,m}}{H^{2s+1}_{j-s-1,m-2s-2}} \sim m^2.$$

Consequently, it follows that

$$\Sigma_1 \leqslant c \sum_{m=n+1}^{\infty} \sum_{j=\lfloor \frac{m}{4} \rfloor}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{\nu} m^{-4s+2} \left| \widehat{\Delta^s f}_{j-s-1,\nu}^{m-2s-2,2s+1} \right|^2 H_{j-s-1,m-2s-2}^{2s+1}.$$

Similarly, we obtain

$$\Sigma_2 \leqslant c \sum_{m=\lfloor \frac{n+2}{2} \rfloor}^{\infty} m^{-4s+2} \left| \widehat{\Delta^s f}_{\frac{m}{2}-s-1,1}^{m-2s-2,2s+1} \right|^2 H_{\frac{m}{2}-s-1,m-2s-2}^{2s+1}.$$

Next, we estimate Σ_3. Iterating the identities involving Δ_0 in Proposition 2.4.6, we obtain

$$\begin{aligned} \left| \widehat{f}_{j,\nu}^{m,0}(\nabla) \right|^2 &= \frac{1}{(m-2j)^{2s}\,(m-2j+d-2)^{2s}} \left| \widehat{\Delta_0^s f}_{j,\nu}^{m,0}(\nabla) \right|^2 \\ &\sim \frac{1}{m^{4s}} \left| \widehat{\Delta_0^s f}_{j,\nu}^{m,0}(\nabla) \right|^2, \end{aligned}$$

for $0 \leqslant j \leqslant \lfloor \frac{m}{4} \rfloor$. Consequently, it follows that

$$\begin{aligned} \Sigma_3 &\leqslant c \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{4} \rfloor - 1} \sum_{\nu} m^{-4s} \left| \widehat{\Delta_0^s f}_{j,\nu}^{m,0}(\nabla) \right|^2 \widetilde{H}_{j,m}^{0} \\ &\leqslant \frac{c}{n^{4s}}\, \mathcal{E}_n(\Delta_0^s f)_{\nabla,0}^2. \end{aligned}$$

Putting these estimates together completes the proof of the theorem.

□

2.4.1 Approximation behavior in terms of the fractional Laplace-Beltrami operator

In the proof of Theorem 3.3.5, we do not need to specify the basis of spherical harmonics in the definition of $Q_{j,\nu}^{n,0}$. It is far more complicated to give a bound for the error $\mathcal{E}_n(f)_{\nabla,0}$ in terms of derivatives of odd order involving Δ and Δ_0, for which we do need to specify the basis as in [38]. Thus, here we shall choose a more convenient distributional differential operator in order to avoid having to specify a basis.

Recall that the space $\mathcal{H}_n^d$ of spherical harmonics can be characterized as the eigenfunction space of the Laplace-Beltrami operator Δ_0 on $\mathbf{S}^{d-1}$:

$$\mathcal{H}_n^d = \left\{ f \in C^2(\mathbf{S}^{d-1}) : -\Delta_0 f = n(n+d-2) f \right\}.$$

Therefore, we can define the fractional powers of $-\Delta_0$.

Definition 2.4.8. *For $\alpha \in \mathbb{R}$, we define*

$$(-\Delta_0)^{\alpha/2} f = \sum_{n=0}^{\infty} (n(n+d-2))^{\alpha/2} \operatorname{proj}_{\mathcal{H}_n^d} f.$$

It is shown in [13] that

$$\left\| (-\Delta_0)^{1/2} f \right\|_{\mathbf{S}^{d-1}} = \|\nabla_0 f\|_{\mathbf{S}^{d-1}},$$

where $\|\cdot\|_{\mathbf{S}^{d-1}}$ is the norm induced by $\langle \cdot, \cdot \rangle_{\mathbf{S}^{d-1}}$ and ∇_0 denotes the tangential gradient defined as

$$\nabla_0 f = \nabla F|_{\mathbf{S}^{d-1}} \quad \text{with} \quad F(x) = f\left(\frac{x}{\|x\|}\right), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Theorem 2.4.9. Let $s \geqslant 1$ be an integer and $f \in \mathcal{U}(W_0; \mathbf{B}^d) \cap \mathcal{W}_2^{2s}(W_1; \mathbf{B}^d)$. Then, for $n \geqslant 2s+3$,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}} \left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-3}(\partial_i \Delta^s f)_{2s+2} + \mathcal{E}_n\left((-\Delta_0)^{1/2} \Delta_0^s f \right)_{\nabla,0} \right].$$

Consequently,

$$\mathcal{E}_n(f)_{\nabla,0} \leqslant \frac{c}{n^{2s-1}} \left[\sum_{i=1}^{d} \|\partial_i \Delta^s f\|_{2s+2} + \|(-\Delta_0)^{1/2} \Delta_0^s f\|_{\nabla,0} \right].$$

Proof. On one hand, from Lemma 2.3.6, we have

$$\mathcal{E}_{n-2s-2}(\Delta^s f)^2_{2s+1} \leqslant \frac{c}{n} \mathcal{E}_{n-2s-2}(\Delta^s f)^2_{\nabla,2s+2} \leqslant \frac{c}{n} \left[\sum_{i=1}^{d} \mathcal{E}_{n-2s-3}(\partial_i \Delta^s f)^2_{2s+2} \right].$$

On the other hand, we have

$$\mathcal{E}_n(\Delta_0^s f)^2_{\nabla,0} = \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left| \widehat{\Delta_0^s f}_{j,\nu}^{m,0}(\nabla) \right|^2 \widetilde{H}_{j,m}^0(\nabla)$$

$$\leqslant \sum_{m=n+1}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} (m-2j)\,(m-2j+d-2) \left| \widehat{\Delta_0^s f}_{j,\nu}^{m,0}(\nabla) \right|^2 \widetilde{H}_{j,m}^0(\nabla)$$
$$= \mathcal{E}_n \left((-\Delta_0)^{1/2} \Delta_0^s f \right)_{\nabla,0}^2 .$$

The result follows from combining the inequalities above and Theorem 3.3.5.
□

2.5 Numerical experiments

In this section we present numerical experiments to compare the approximation behavior of Fourier orthogonal expansions with respect to classical and Sobolev ball polynomials with $d = 2$ variables. To this end, we consider different functions defined on $\mathbf{B}^2$. For each function $f(x, y)$, we compute $S_n^\mu f$ and $S_n^{\nabla,0} f$ for different values of μ and n. The two approximations were compared by computing their respective root mean square error (RMSE) as follows. We generate a circular mesh consisting of 1441 points

$$\{(r_i \cos(\theta_j), r_i \sin(\theta_j)) : \; r_i = i/20, \; \theta_j = j\pi/36, \; 0 \leqslant i \leqslant 20, \; 0 \leqslant j \leqslant 71\}. \tag{2.10}$$

We set $z_{i,j} = f(r_i \cos(\theta_j), r_i \sin(\theta_j))$, and $\hat{z}_{i,j}$ equal to the value of the approximation (classical or Sobolev) at the same point, and compute the RMSE as:

$$\mathrm{RMSE}(S) = \left(\frac{(z_{0,0} - \hat{z}_{0,0})^2 + \sum_{i=1}^{20} \sum_{j=0}^{71} (z_{i,j} - \hat{z}_{i,j})^2}{1441} \right)^{1/2} . \tag{2.11}$$

where S denotes either $S_n^\mu f(x, y)$ or $S_n^{\nabla,0} f(x, y)$.

We consider three different continuous functions and provide figures showing their approximation overlapped with their graph. We also provide tables with the approximation error of S_n^μ and $S_n^{\nabla,0}$ for different values of μ and n. The figures and errors were obtained using Wolfram Mathematica®. To gain a more comprehensive perspective when comparing calculated errors, we have selected functions that are combinations of infinite class functions, with the exception of the last example. We point out that the approximation error in the Sobolev case seems to be smaller than the classical approximation error as the value of n gets large.

Example 1

First, we consider the function

$$f(x, y) = x \sin(5x - 6y) + y.$$

The graph of $f(x, y)$ is shown in Figure 2.1, and the approximations $S_{20}^{1} f(x, y)$ and $S_{20}^{\nabla,1} f(x, y)$ are shown in Figure 4.5. We list the RMSE of both approximations for different values of n and μ in Table 2.1. For $\mu > 0$, a seemingly faster rate of convergence can be observed for the Sobolev approximation. Nevertheless, for $\mu = 0$, the rate of convergence of the Sobolev expansion does not seem to be much faster than the classical one. This is consistent with the theoretical rates of convergence that appear in Section 3.3.

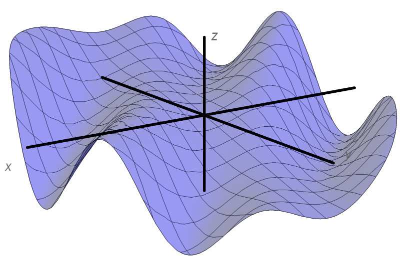

Figure 2.1: Graph of $f(x, y) = x \sin(5x - 6y) + y$.

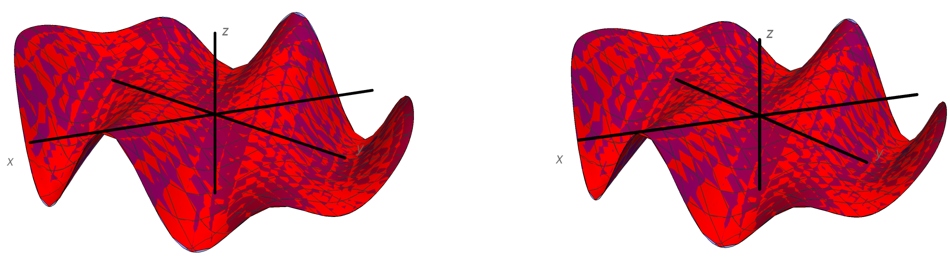

Figure 2.2: Approximations overlapped with the graph of $f(x, y)$. Left: $S_{20}^{1} f(x, y)$. Right: $S_{20}^{\nabla,1} f(x, y)$.

μ	n	RMSE($S_n^\mu f(x,y)$)	RMSE($S_n^{\nabla,\mu} f(x,y)$)
0	5	0.29919	0.29001
	10	0.01235	0.01704
	15	1.26607×10^{-4}	1.74117×10^{-4}
	20	8.9264×10^{-7}	5.35549×10^{-7}
1	5	0.30721	0.29003
	10	0.01913	0.01196
	15	2.65266×10^{-4}	1.23246×10^{-4}
	20	3.54923×10^{-6}	9.9071×10^{-7}
1.5	5	0.32634	0.29171
	10	0.02730	0.01365
	15	4.22629×10^{-4}	1.63851×10^{-4}
	20	9.80412×10^{-6}	4.10402×10^{-6}
2	5	0.35935	0.30226
	10	0.03816	0.01882
	15	6.42325×10^{-4}	2.62762×10^{-4}
	20	4.32089×10^{-6}	6.58669×10^{-6}
3.5	5	0.53568	0.41311
	10	0.08591	0.05138
	15	1.73557×10^{-3}	9.31567×10^{-4}
	20	2.1150×10^{-4}	1.10783×10^{-4}

Table 2.1: Approximation errors for $f(x,y)$.

Example 2

Now, we consider the continuous sinusoidal function

$$g(x,y) = \sin(10x + y).$$

Its graph is shown in Figure 2.3, and the approximations $S_{20}^\mu f(x,y)$ and $S_{20}^{\nabla,\mu} f(x,y)$ are shown in Figure 2.4. We note that the approximation error in the classical and Sobolev case seems to be larger at the maximum and minimum values of the function. Table 2.2 shows the errors corresponding to the approximations of $g(x,y)$. Again, the rates of convergence seem to corresponding to the theoretical rates.

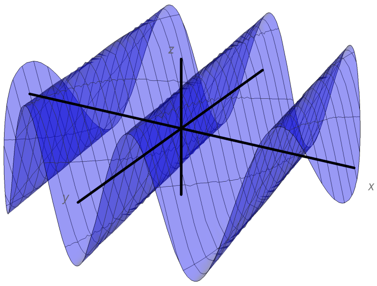

Figure 2.3: Graph of $g(x, y) = \sin(10x + y)$

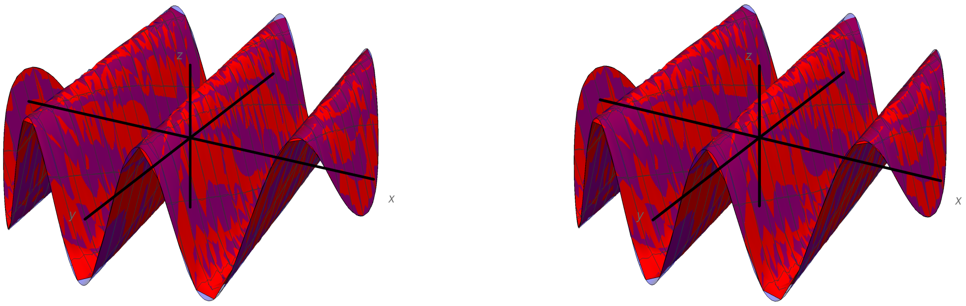

Figure 2.4: Approximations overlapped with the graph of $g(x, y)$. Left: $S^1_{20} g(x, y)$. Right: $S^{\nabla,1}_{20} g(x, y)$

Example 3

Here, we consider the continuous function

$$h(x, y) = e^{x^2 - y^2} - xy,$$

whose graph is shown in Figure 2.7. Both approximations are shown in Figure 4.9, and their respective RSME are listed in Table 2.4. Observe that, in this case, the RSME for both approximations are significantly smaller than in the previous examples. Note that the errors apparently do not change for a large n but this may be due to rounding errors.

μ	n	RMSE($S_n^\mu g(x,y)$)	RMSE($S_n^{\nabla,\mu} g(x,y)$)
0	5	0.69811	0.75427
	10	0.17290	0.20945
	15	7.21219×10^{-4}	8.09666×10^{-4}
	20	4.35377×10^{-6}	4.70028×10^{-6}
1	5	0.80258	0.69811
	10	0.32401	0.17290
	15	0.00210	0.00071
	20	1.59163×10^{-5}	4.35377×10^{-6}
1.5	5	1.01491	0.70814
	10	0.48232	0.21956
	15	0.00356	0.00117
	20	2.87009×10^{-5}	1.08525×10^{-5}
2	5	1.34532	0.80258
	10	0.69211	0.32401
	15	0.00568	0.00210
	20	5.24189×10^{-5}	1.59163×10^{-5}
3.5	5	2.89127	1.77811
	10	1.62760	0.95300
	15	0.01719	0.00856
	20	3.98356×10^{-4}	1.1693×10^{-4}

Table 2.2: Errors of the expansions of $g(x,y)$

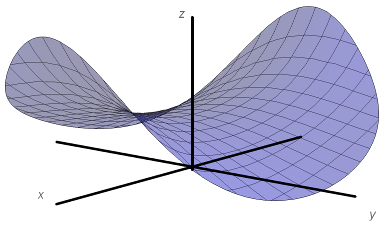

Figure 2.5: Graph of $h(x,y) = e^{x^2-y^2} - xy$

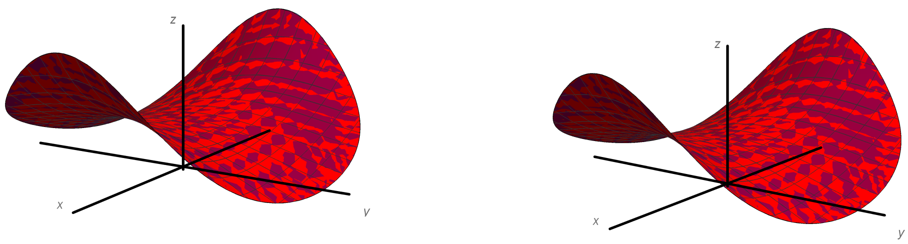

Figure 2.6: Approximations overlapped with the graph of $h(x,y)$. Left: $S_{20}^1 h(x,y)$. Right: $S_{20}^{\nabla,1} h(x,y)$

μ	n	RMSE($S_n^\mu h(x,y)$)	RMSE($S_n^{\nabla,\mu} h(x,y)$)
0	5	0.01063	0.01141
	10	8.52813×10^{-6}	1.80036×10^{-5}
	15	8.41258×10^{-7}	1.50115×10^{-5}
	20	9.02632×10^{-7}	1.5485×10^{-5}
1	5	0.01097	0.01061
	10	8.7175×10^{-6}	8.5254×10^{-6}
	15	1.80786×10^{-7}	6.95056×10^{-7}
	20	1.42097×10^{-6}	6.95056×10^{-7}
1.5	5	0.01182	0.01098
	10	8.98758×10^{-6}	1.57057×10^{-5}
	15	2.99213×10^{-6}	1.3121×10^{-5}
	20	1.29664×10^{-5}	1.75648×10^{-5}
2	5	0.01218	0.01126
	10	9.2924×10^{-6}	8.7124×10^{-6}
	15	1.74896×10^{-6}	1.57222×10^{-6}
	20	2.61056×10^{-6}	1.57222×10^{-6}
3.5	5	0.01342	0.01235
	10	1.12976×10^{-5}	9.79411×10^{-6}
	15	1.27364×10^{-5}	9.79411×10^{-6}
	20	1.9156×10^{-5}	2.72889×10^{-5}

Table 2.3: Errors of the expansions of $h(x,y)$

Example 4

Here, we consider the following univariate $\mathcal{C}^2$ spline defined on $[0,1]$ by:

$$q(t) = \begin{cases} -1.50391(t+0.8)^3 + 3.96995(t+0.8)^2 - 2.22067(t+0.8) + 0.35, & 0 \leqslant t \leqslant 0.8, \\ \\ 5.41466(t-0.8)^3 - 3.2488(t-0.8)^2 - 1.06683(t-0.8) + 0.8, & 0.8 \leqslant t \leqslant 1. \end{cases}$$

Then, we construct the radially symmetric function $h(x,y) = q(x^2+y^2)$ defined on $\mathbf{B}^2$ whose graph is shown in Figure 2.7. The classical and Sobolev approximations are shown in Figure 4.9, and their respective RSME are listed in Table 2.4. We remark that, contrary to the previous examples, $h(x,y)$ is not an analytic function. In spite of this, the approximation errors appear to behave similarly than in the previous examples.

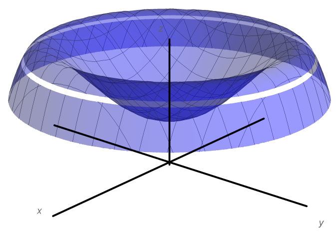

Figure 2.7: Graph of $h(x,y) = q(x^2 + y^2)$

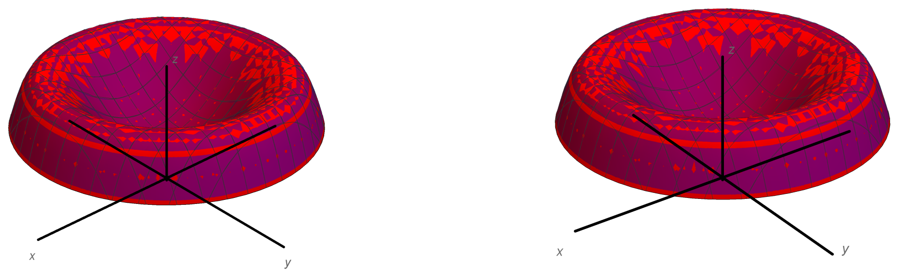

Figure 2.8: Approximations overlapped with the graph of $h(x,y)$. Left: $S_{20}^{1} h(x,y)$. Right: $S_{20}^{\nabla,1} h(x,y)$

μ	n	RMSE($S_n^{\mu} f(x,y)$)	RMSE($S_n^{\nabla,\mu} f(x,y)$)
0	5	0.032115	0.105139
	10	1.69904×10^{-3}	7.90909×10^{-3}
	15	2.81159×10^{-4}	8.211484×10^{-4}
	20	7.60244×10^{-5}	1.19557×10^{-4}
1	5	0.03692	0.06976
	7	9.17275×10^{-3}	7.2697×10^{-3}
	10	3.88087×10^{-3}	3.06606×10^{-3}
	15	2.59168×10^{-4}	4.45460×10^{-4}
	20	2.53382×10^{-4}	2.18760×10^{-4}
1.5	5	0.047003	0.060414
	10	5.42702×10^{-3}	2.71346×10^{-3}
	15	2.34768×10^{-4}	5.35595×10^{-4}
	20	6.85998×10^{-3}	1.81918×10^{-4}
2	5	0.059031	0.058599
	10	6.93792×10^{-3}	3.79857×10^{-3}
	15	9.35486×10^{-4}	4.16972×10^{-4}
	20	1.35988×10^{-3}	2.93696×10^{-4}
3.5	5	0.09692	0.08112
	10	0.010308	8.25693×10^{-3}
	15	4.58444×10^{-3}	1.9499×10^{-3}
	20	3.93952×10^{-3}	2.22019×10^{-3}

Table 2.4: Approximation errors for $h(x,y)$.

Chapter 3

Simultaneous approximation via Laplacians on the unit ball

We study the orthogonal structure on the unit ball $\mathbf{B}^d$ of $\mathbb{R}^d$ with respect to the Sobolev inner product

$$\langle f, g\rangle_\Delta = \lambda \int_{\mathbf{S}^{d-1}} f(\xi)\, g(\xi)\, d\sigma(\xi) + \int_{\mathbf{B}^d} \Delta[(1-\|x\|^2) f(x)]\, \Delta[(1-\|x\|^2) g(x)]\, dx,$$

where $\lambda > 0$, σ denotes the surface measure on the unit sphere $\mathbf{S}^{d-1}$, and Δ is the usual Laplacian operator. Our main contribution consists in the study of orthogonal polynomials associated with $\langle \cdot, \cdot \rangle_\Delta$, proving that they satisfy a fourth-order partial differential equation. We also study the approximation properties of the Fourier sums with respect to these orthogonal polynomials and, in particular, we estimate the error of simultaneous approximation of a function, its partial derivatives, and its Laplacian in the $L^2(\mathbf{B}^d)$ space. The results of this Chapter are contained in [32].

3.1 Introduction

In [2], Atkinson and Hansen studied the problem of finding the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ with zero boundary conditions on the unit ball $\mathbf{B}^d$ on $\mathbb{R}^d$, and asked the question of finding an explicit orthogonal basis for the Sobolev inner product

$$\langle f,g\rangle_\Delta = \frac{1}{\pi}\int_{\mathbf{B}^d} \Delta[(1-\|x\|^2)f(x)]\,\Delta[(1-\|x\|^2)g(x)]\,dx, \qquad (3.1)$$

where Δ denotes the usual Laplace operator.

Y. Xu answered that question in [51], where he constructed such basis in terms of spherical harmonics and classical Jacobi polynomials of varying parameter. In addition, he studied the orthogonal expansion of a function in that basis, and proved that it can be computed without using the derivatives of the function.

Our main objective is to study the influence of the additional term in the study of the basis and its impact into the Fourier coefficients and the errors for a given function.

The work is structured in the following way. Section 3.2 describes the first considered Sobolev inner product, deducing a Sobolev basis for that inner product, and proving that the polynomials satisfy a partial differential equation. Section 3.3 is devoted to analyze the Sobolev Fourier orthogonal expansions and approximation, giving explicit bounds for the errors.

3.2 Sobolev orthogonal polynomials

This section is devoted to the study of the orthogonal structure on the unit ball with respect to the *Sobolev inner product*

$$\begin{aligned}\langle f,g\rangle_\Delta = &\frac{\lambda}{\sigma_{d-1}}\int_{\mathbf{S}^{d-1}} f(\xi)\,g(\xi)\,d\sigma(\xi)\\ &+\frac{1}{8\,\sigma_{d-1}}\int_{\mathbf{B}^d} \Delta[(1-\|x\|^2)f(x)]\,\Delta[(1-\|x\|^2)g(x)]\,dx, \quad \lambda>0.\end{aligned} \qquad (3.2)$$

The normalization constants are chosen to simplify expressions in the sequel. Let us denote by $\mathcal{V}_n^d(\Delta)$ the space of Sobolev orthogonal polynomials of degree n with respect to (3.2). We point out that when $\lambda = 0$, we recover the inner product studied in [51] up to a normalization constant.

For our purpose, we need the following lemma.

Lemma 3.2.1. Let $\beta = n - 2j + \frac{d-2}{2}$ and $Y_\nu^{n-2j} \in \mathcal{H}_{n-2j}^d$. Then, for any polynoimal $q(s)$,

$$\Delta[(1-\|x\|^2)\,q(\|x\|^2)\,Y_\nu^{n-2j}(x)] = 4\,(\mathcal{J}_\beta q)(\|x\|^2)\,Y_\nu^{n-2j}(x), \qquad (3.3)$$

where

$$(\mathcal{J}_\beta q)(s) = s\,(1-s)\,q''(s) + (\beta+1-(\beta+3)\,s)q'(s) - (\beta+1)\,q(s).$$

Proof. Using spherical-polar coordinates, we can use (1.5) and (1.7) for the radial and spherical part of Δ, respectively. After a tedious calculation, we get that

$$\Delta\left[(1-\|x\|^2)\,q(\|x\|^2)Y_\nu^{n-2j}(x)\right] = \Delta\left[(1-r^2)\,q(r^2)\,r^{n-2j}\,Y_\nu^{n-2j}(\xi)\right]$$
$$= 4\left[r^2(1-r^2)q''(r^2) + (\beta+1-(\beta+3)\,r^2)q'(r^2) - (\beta+1)\,q(r^2)\right]Y_\nu^{n-2j}(x).$$

Setting $s \mapsto r^2$ gives the desired result.

□

We use the univariate Jacobi polynomials to construct the following multivariate polynomials defined on $\mathbf{B}^d$.

Definition 3.2.2. *For $n \geqslant 0$ and $0 \leqslant j \leqslant \frac{n}{2}$, let $\{Y_\nu^{n-2j}(x) : 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ denote an orthonormal basis of $\mathcal{H}_{n-2j}^d$. We define the polynomials*

$$Q_{0,\nu}^n(x) := Y_\nu^n(x),$$
$$Q_{j,\nu}^n(x) := (1-\|x\|^2)\,P_{j-1}^{(2,n-2j+\frac{d-2}{2})}(2\|x\|^2-1)\,Y_\nu^{n-2j}(x), \quad 1 \leqslant j \leqslant \frac{n}{2}.$$

It turns out that these polynomials are eigenfunctions of a fourth order linear partial differential operator.

Proposition 3.2.3. The polynomials in Definition 3.2.2 satisfy

$$\Delta\Big[(1-\|x\|^2)\,Q_{j,\nu}^n(x)\Big] \;=\; c_{j,n}\,P_{j,\nu}^{n,0}, \quad n \geqslant 0, \qquad (3.4)$$

with

$$c_{j,n} = \begin{cases} -4\left(n+\frac{d}{2}\right), & j = 0, \\ 4\,j\,(j+1), & 1 \leqslant j \leqslant \frac{n}{2}, \end{cases}$$

and

$$(1-\|x\|^2)\,\Delta\, P^{n,0}_{j,\nu}(x) \;=\; d_{j,n}\, Q^n_{j,\nu}(x), \quad n \geqslant 0, \tag{3.5}$$

with

$$d_{j,n} = \begin{cases} 0, & j=0,\\ 4\left(n-j+\frac{d}{2}\right)\left(n-j+\frac{d-2}{2}\right), & 1\leqslant j \leqslant \frac{n}{2}, \end{cases}$$

Proof. For $j=0$, by (3.3), we have

$$\Delta\Big[(1-\|x\|^2)\, Q^n_{0,\nu}(x)\Big] = \Delta\Big[(1-\|x\|^2)\, Y^n_\nu(x)\Big] = -4\left(n+\frac{d}{2}\right) Y^n_\nu(x).$$

Now, we deal with $1 \leqslant j \leqslant \frac{n}{2}$. The Jacobi polynomials satisfy the following property ([46, p. 71]):

$$(1-s)\, P^{(2,\beta)}_{j-1}(2s-1) = \frac{1}{2j+\beta+1}\left[(j+1)\, P^{(1,\beta)}_{j-1}(2s-1) - j\, P^{(1,\beta)}_{j}(2s-1)\right]. \tag{3.6}$$

Furthermore, the Jacobi polynomials $P^{(1,\beta)}_{j-1}(2s-1)$ satisfy the differential equation

$$s\,(1-s)\, y'' + (\beta+1-(\beta+3)\, s) y' = -(j-1)\,(j+\beta+1)\, y.$$

Using these two facts, we can easily deduce that

$$\begin{aligned}
&(2j+\beta+1)\,\mathcal{J}_\beta\Big[(1-s)\, P^{(2,\beta)}_{j-1}(2s-1)\Big]\\
&= (j+1)\,\mathcal{J}_\beta P^{(1,\beta)}_{j-1}(2s-1) - j\,\mathcal{J}_\beta P^{(1,\beta)}_{j}(2s-1)\\
&= (j+1)\Big[-(j-1)\,(j+\beta+1) - (\beta+1)\Big]\, P^{(1,\beta)}_{j-1}(2s-1)\\
&\quad - j\Big[-j(j+\beta+2) - (\beta+1)\Big] P^{(1,\beta)}_{j}(2s-1)\\
&= -j(j+1)\Big[(j+\beta)\, P^{(1,\beta)}_{j-1}(2s-1) - (j+\beta+1)\, P^{(1,\beta)}_{j}(2s-1)\Big].
\end{aligned}$$

We need yet another formula for Jacobi polynomials ([1, p. 782, (22.7.18)]):

$$(2j+\beta+1)\, P^{(0,\beta)}_{j}(2s-1) = (j+\beta+1)\, P^{(1,\beta)}_{j}(2s-1) - (j+\beta)\, P^{(1,\beta)}_{j-1}(2s-1),$$

which implies immediately that

$$\mathcal{J}_\beta\Big[(1-s)\, P^{(2,\beta)}_{j-1}(2s-1)\Big] = j\,(j+1)\, P^{(0,\beta)}_{j}(2s-1). \tag{3.7}$$

Then, by (3.3) and (3.7) with $\beta = n - 2j + \frac{d-2}{2}$,

$$\Delta\left[(1-\|x\|^2)\, Q^n_{j,\nu}(x)\right] = 4\, j\, (j+1)\, P^{n,0}_{j,\nu}(x).$$

This proves (3.4).

Using $\Delta\, Y^n_\nu(x) = 0$, we get

$$(1-\|x\|^2)\,\Delta\, P^{n,0}_{0,\nu} \;=\; 0 \;=\; d_{0,n}\, Q^n_{0,\nu}(x).$$

From (1.32) and Definition 3.2.2, we obtain

$$(1-\|x\|^2)\,\Delta\, P^{n,0}_{j,\nu} = d_{j,n}\,(1-\|x\|^2)\, P^{n-2,2}_{j-1,\nu}(x) = d_{j,n}\, Q^n_{j,\nu}(x),$$

proving (3.5). □

Combining (3.4) and (3.5), we get a partial differential equation for the Sobolev orthogonal polynomials.

Corollary 3.2.4. The polynomials in Definition 3.2.2 satisfy

$$(1-\|x\|^2)\,\Delta^2\left[(1-\|x\|^2)\, Q^n_{j,\nu}(x)\right] = \varpi_{n,j}\, Q^n_{j,\nu}(x), \tag{3.8}$$

where

$$\varpi_{n,j} = 16\, j\, (j+1)\left(n - j + \frac{d}{2}\right)\left(n - j + \frac{d-2}{2}\right).$$

The following relation follows readily from (1.10) and (3.6).

Proposition 3.2.5. The polynomials in Definition 3.2.2 satisfy

$$\begin{aligned} Q^n_{0,\nu}(x) &= P^{n,1}_{0,\nu}(x),\\ Q^n_{j,\nu}(x) &= \frac{1}{n+\frac{d}{2}}\left[(j+1)\, P^{n-2,1}_{j-1,\nu}(x) - j\, P^{n,1}_{j,\nu}(x)\right], \quad 1 \leqslant j \leqslant \frac{n}{2}. \end{aligned} \tag{3.9}$$

In the following proposition, we show that the polynomials in Definition 3.2.2 constitute a mutually orthogonal basis with respect to the inner product (3.2).

Proposition 3.2.6. For $n \geqslant 0$, $\{Q^n_{j,\nu} \;:\; 0 \leqslant j \leqslant \frac{n}{2},\; 1 \leqslant \nu \leqslant a^d_{n-2j}\}$ constitutes a mutually orthogonal basis of $\mathcal{V}^d_n(\Delta)$. Moreover,

$$\left\langle Q^n_{j,\nu}, Q^m_{k,\eta}\right\rangle_\Delta \;=\; \widetilde{H}^\Delta_{j,n}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta},$$

where,

$$\widetilde{H}_{j,n}^{\Delta} = \begin{cases} \lambda + n + \frac{d}{2}, & j = 0, \\ \dfrac{j^2\,(j+1)^2}{2^{n-2j+\frac{d}{2}}}\, h_j^{(0,n-2j+\frac{d-2}{2})}, & 1 \leqslant j \leqslant \frac{n}{2}. \end{cases} \tag{3.10}$$

Proof. If $j = k = 0$, then using the formula

$$\int_{\mathbf{B}^d} f(x)\,dx = \int_0^1 r^{d-1} \int_{\mathbf{S}^{d-1}} f(r\,\xi)\,d\sigma(\xi)\,dr,$$

and (3.3), we get

$$\langle Q_{0,\nu}^n, Q_{0,\eta}^m \rangle_\Delta = \delta_{n,m}\,\delta_{\nu,\eta}\left[\lambda + (\beta+1)^2 \int_0^1 s^\beta ds\right] = \delta_{n,m}\,\delta_{\nu,\eta}\left[\lambda + \beta + 1\right],$$

where $\beta = n - 2j + \frac{d-2}{2}$. If $j = 0$ and $k \geqslant 1$, then

$$\langle Q_{0,\nu}^n, Q_{k,\eta}^m \rangle_\Delta = -\,(\beta+1)\,k\,(k+1)\,\delta_{n,m-2k}\,\delta_{\nu,\eta} \int_0^1 P_k^{(0,\beta)}(2s-1) s^\beta ds = 0,$$

since the factor $(1 - \|x\|^2)$ vanishes on $\mathbf{S}^{d-1}$.

For $1 \leqslant j, k \leqslant \frac{n}{2}$, applying Green's identity

$$\int_{\mathbf{B}^d} (u\Delta v - v\Delta u)\,dx = \int_{\mathbf{S}^{d-1}} \left(\frac{\partial v}{\partial n} u - \frac{\partial u}{\partial n}\, v\right)\,d\sigma(\xi) \tag{3.11}$$

with $v(x) = (1 - \|x\|^2)\,Q_{j,\nu}^n(x)$ and $u = (1 - \|x\|^2)\,Q_{k,\eta}^m(x)$, we get

$$\begin{aligned} \langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta =& \frac{\lambda}{\sigma_{d-1}} \int_{\mathbf{S}^{d-1}} Q_{j,\nu}^n(\xi)\,Q_{k,\eta}^m(\xi)\,d\sigma(\xi) \\ &+ \frac{1}{8\,\sigma_{d-1}} \int_{\mathbf{B}^d} (1 - \|x\|^2)\,Q_{k,\eta}^m(x)\,\Delta^2\Big[(1 - \|x\|^2) Q_{j,\nu}^n(x)\Big]\,dx. \end{aligned}$$

Then, by (3.8) and Definition 3.2.2, we can write

$$\begin{aligned} \langle Q_{j,\nu}^n, Q_{k,\eta}^m \rangle_\Delta &= \frac{\varpi_{n,j}}{8\,\sigma_{d-1}} \int_{\mathbf{B}^d} P_{j-1,\nu}^{n-2,2}(x)\,P_{k-1,\eta}^{m-2,2}(x)\,W_2(x)\,dx \\ &= \frac{\varpi_{n,j}}{8\,\sigma_{d-1}\,b_2} H_{j-1,n-2}^2\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta}. \end{aligned}$$

Using (1.11), we get

$$\frac{H_{j-1,n-2}^2}{H_{j,n}^0} = \frac{1}{2} j\,(j+1) \frac{(\frac{d}{2}+1)\,(\frac{d}{2}+2)}{(n - j + \frac{d}{2})\,(n - j + \frac{d-2}{2})}.$$

Moreover, using (1.12) and the fact that $b_2 = (\frac{d}{2}+1)\,(\frac{d}{2}+2)\,b_0$, we obtain

$$\langle Q^n_{j,\nu}, Q^m_{k,\eta}\rangle_\Delta = \frac{j^2\,(j+1)^2}{2^{\beta+1}} h_j^{(0,\beta)}\,\delta_{n,m}\,\delta_{j,k}\,\delta_{\nu,\eta}.$$

□

Corollary 3.2.7. For $n \geqslant 2$,

$$\mathcal{V}^d_n(\Delta) = \mathcal{H}^d_n \oplus (1-\|x\|^2)\,\mathcal{V}^d_{n-2}(W_2).$$

3.3 Sobolev Fourier orthogonal expansions and approximation

Consider the Sobolev space

$$\mathtt{H}^s(\mathbf{B}^d) = \left\{ f \in C(\mathbf{B}^d);\, \partial^{\mathbf{m}} f \in L^2(\mathbf{B}^d),\, |\mathbf{m}| \leqslant s,\, \mathbf{m} \in \mathbb{N}^d_0 \right\},$$

where $L^2(\mathbf{B}^d) = L^2(W_0; \mathbf{B}^d)$.

For $f \in \mathtt{H}^2(\mathbf{B}^d)$, let us denote by $\widehat{f}^{\,n,\Delta}_{j,\nu}$ the Fourier coefficients with respect to the basis of $\mathcal{V}^d_n(\Delta)$ defined in (3.2.2), that is,

$$\widehat{f}^{\,n,\Delta}_{j,\nu} = \frac{1}{\widetilde{H}^\Delta_{j,n}} \left\langle f, Q^n_{j,\nu} \right\rangle_\Delta,$$

where $\widetilde{H}^\Delta_{j,n}$ is given in (3.10).

Let $\mathrm{proj}^\Delta_m : \mathtt{H}^2(\mathbf{B}^d) \to \mathcal{V}^d_m(\Delta)$ and $S^\Delta_n : \mathtt{H}^2(\mathbf{B}^d) \to \Pi^d_n$ denote the projection operator and partial sum operators

$$\mathrm{proj}^\Delta_m f(x) = \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a^d_{m-2j}} \widehat{f}^{\,m,\Delta}_{j,\nu}\, Q^m_{j,\nu}(x) \quad \text{and} \quad S^\Delta_n f(x) = \sum_{m=0}^{n} \mathrm{proj}^\Delta_m f(x).$$

We denote by $\|\cdot\|_\Delta$ the norm induced by the inner product (3.2), and by $\mathcal{E}_n(f)_\Delta$ the error of best approximation in $\mathtt{H}^2(\mathbf{B}^d)$ given by

$$\mathcal{E}_n(f)_\Delta = \|f - S^\Delta_n f\|_\Delta.$$

It turns out that the orthogonal expansion can be computed without involving the derivatives of f.

Proposition 3.3.1. For $j \geqslant 1$, let $\beta_j = n - 2j + \frac{d-2}{2}$. Then,

$$\widehat{f}_{j,\nu}^{n,\Delta} = \frac{2\,j\,(j+1)}{\sigma_{d-1}\widetilde{H}_{j,n}^{\Delta}}\Bigg[(\beta_j + j)\,(\beta_j + j + 1)\int_{\mathbf{B}^d} f(x)\,Q_{j,\nu}^n(x)\,dx - \frac{1}{2}\int_{\mathbf{S}^{d-1}} f(\xi)\,Y_\nu^{n-2j}(\xi)\,d\sigma(\xi)\Bigg],$$

furthermore, for $j = 0$,

$$\widehat{f}_{0,\nu}^{n,\Delta} = \frac{1}{\sigma_{d-1}}\int_{\mathbf{S}^{d-1}} f(\xi)\,Y_\nu^n(\xi)\,d\sigma(\xi).$$

Proof. Applying Green's identity (3.11) with $v(x) = (1 - \|x\|^2)\,f(x)$ and $u = \Delta\Big[(1 - \|x\|^2)\,Q_{j,\nu}^n(x)\Big] = 4\,j\,(j+1)\,P_{j,\nu}^{n,0}(x)$, $j \geqslant 1$, shows

$$\begin{aligned}\widehat{f}_{j,\nu}^{n,\Delta} &= \frac{1}{\sigma_{d-1}\widetilde{H}_{j,n}^{\Delta}}\Bigg[\lambda\int_{\mathbf{S}^{d-1}} f(\xi)\,Q_{j,\nu}^n(\xi)\,d\sigma(\xi) \\ &\qquad + \frac{1}{8}\int_{\mathbf{B}^d}\Delta\Big[(1 - \|x\|^2)\,f(x)\Big]\,\Delta\Big[(1 - \|x\|^2)\,Q_{j,\nu}^n(x)\Big]\,dx\Bigg] \\ &= \frac{1}{8\,\sigma_{d-1}\widetilde{H}_{j,n}^{\Delta}}\Bigg[\int_{\mathbf{B}^d}(1 - \|x\|^2)\,f(x)\,\Delta^2\Big[(1 - \|x\|^2)\,Q_{j,\nu}^n(x)\Big]\,dx \\ &\qquad - 8\,j\,(j+1)\int_{\mathbf{S}^{d-1}} f(\xi)\,Y_\nu^{n-2j}(\xi)\,d\sigma(\xi)\Bigg],\end{aligned}$$

where we have used (3.3) and $P_j^{(0,\beta)}(1) = 1$. The stated result for $j \geqslant 1$ follows from (3.8). The proof of $j = 0$ is similar but easier, in which we need to use $\Delta\Big[(1 - \|x\|^2)\,Y_\nu^n(x)\Big] = -4\,(n + \frac{d}{2})\,Y_\nu^n(x)$ and $\widetilde{H}_{0,n}^{\Delta} = \lambda + n + \frac{d}{2}$. □

We point out that the linear operator $\mathcal{D}$ defined by

$$\mathcal{D}[P(x)] = \Delta\Big[(1 - \|x\|^2)\,P(x)\Big], \quad P \in \Pi^d,$$

is a bijection on Π^d. Indeed, by Proposition 3.2.3, $\mathcal{D}[\,Q_{j,\nu}^n(x)] = c_{j,n}\,P_{j,\nu}^{n,0}$ with $c_{j,n} \neq 0$. Therefore, for each $n \geqslant 0$, $\mathcal{D}$ is a one-to-one correspondence between the classical basis for $\mathcal{V}_n^d(W_0)$ and the Sobolev basis for $\mathcal{V}_n^d(\Delta)$ defined in Definition 3.2.2.

The following results will be used to estimate the error of approximation of the Sobolev orthogonal expansion with respect to the basis in Definition 3.2.2.

Proposition 3.3.2. For $f \in \mathtt{H}^2(\mathbf{B}^d)$ and $m \geqslant 0$, we have

$$\mathcal{D}\, \mathrm{proj}_m^{\Delta} f(x) \in \mathcal{V}_m^d(W_0).$$

Furthermore, for $m \geqslant 0$, we have

$$\Delta_0\, \mathrm{proj}_m^{\Delta} f(x) \in \mathcal{V}_m^d(\Delta).$$

Proof. By the definition of $\mathrm{proj}_m^{\Delta} f(x)$ and (3.4), we have

$$\begin{aligned}\mathcal{D}\, \mathrm{proj}_m^{\Delta} f(x) &= \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a_{m-2j}^d} \widehat{f}_{j,\nu}^{\,m,\Delta}\, \Delta \left[(1-\|x\|^2)\, Q_{j,\nu}^m(x) \right] \\ &= \sum_{j=0}^{\frac{m}{2}} \sum_{\nu=1}^{a_{m-2j}^d} c_{n,j}\, \widehat{f}_{j,\nu}^{\,m,\Delta}\, P_{j,\nu}^{m,0}(x).\end{aligned}$$

Therefore, $\mathcal{D}\, \mathrm{proj}_m^{\Delta} f(x) \in \mathcal{V}_m^d(W_0)$.

The second part of the proposition is a direct consequence of identity (1.7). □

We use the previous result to show that $\mathcal{D}$ commutes with the partial Fourier sum S_n^{Δ}.

Proposition 3.3.3. For $f \in \mathtt{H}^2(\mathbf{B}^d)$,

$$\mathcal{D}\, S_n^{\Delta} f \;=\; S_n^0(\mathcal{D} f) \quad \text{and} \quad \Delta_0 S_n^{\Delta} f \;=\; S_n^{\Delta}(\Delta_0 f).$$

Proof. By its definition, $f - S_n^{\Delta} f = \sum_{m=n+1}^{+\infty} \mathrm{proj}_m^{\Delta} f$. From Proposition 3.3.2 we get that $\langle \mathcal{D}\,[f - S_n^{\Delta} f], P\rangle_0 = 0$ for all $P \in \Pi_n^d$. Consequently, $S_n^0(\mathcal{D} f - \mathcal{D}\, S_n^{\Delta} f) = 0$. Since S_n^0 reproduces polynomials of degree at most n, then $S_n^0(\mathcal{D}\, S_n^{\Delta} f) = \mathcal{D}\, S_n^{\Delta} f$, which implies that

$$0 = S_n^0(\mathcal{D} f - \mathcal{D}\, S_n^{\Delta} f) = S_n^0(\mathcal{D} f) - \mathcal{D}\, S_n^{\Delta} f,$$

and the communation relation is proved.

The second part can be established in a similar way taking into account that Δ_0 maps $\mathcal{H}_n^d$ to itself. □

The relation in the proposition above passes down to the Fourier coefficients.

Proposition 3.3.4. For $f \in \mathtt{H}^2(\mathbf{B}^d)$,

$$\widehat{\mathcal{D}f}\,^{n,0}_{0,\nu} = -4\left(n + \frac{d}{2}\right)\widehat{f}\,^{n,\Delta}_{0,\nu},$$

$$\widehat{\mathcal{D}f}\,^{n,0}_{j,\nu} = 4\,j\,(j+1)\,\widehat{f}\,^{n,\Delta}_{j,\nu}, \quad 1 \leqslant j \leqslant \frac{n}{2},$$

and

$$\widehat{\Delta_0 f}\,^{n,\Delta}_{j,\nu} = -(n-2j)\,(n-2j+d-2)\,\widehat{f}\,^{n,\Delta}_{j,\nu}, \quad 0 \leqslant j \leqslant \frac{n}{2}.$$

Proof. From $\operatorname{proj}_n^{\Delta} f = S_n^{\Delta} f - S_{n-1}^{\Delta} f$ and Proposition 3.3.3 we obtain $\mathcal{D}\operatorname{proj}_n^{\Delta} f = \operatorname{proj}_n^0(\mathcal{D}\,f)$. Then the first identity follows from (3.4). The second result follows directly from (1.7). □

Theorem 3.3.5. For $f \in \mathtt{H}^2(\mathbf{B}^d)$,

$$\mathcal{E}_n(f)_\Delta = c\,\mathcal{E}_n(\mathcal{D}f)_0, \quad n \geqslant 0.$$

Proof. The Parseval identity reads,

$$\mathcal{E}_n(f)^2_\Delta = \|f - S_n^\Delta f\|^2_\Delta = \sum_{m=n+1}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu}\left|\widehat{f}\,^{m,\Delta}_{j,\nu}\right|^2 \widetilde{H}^\Delta_{j,m} = \Sigma_1 + \Sigma_2,$$

where we split the sum as

$$\Sigma_1 = \sum_{m=n+1}^{\infty}\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor}\sum_{\nu}\left|\widehat{f}\,^{m,\Delta}_{j,\nu}\right|^2 \widetilde{H}^\Delta_{j,m},$$

$$\Sigma_2 = \sum_{m=n+1}^{\infty}\sum_{\nu}\left|\widehat{f}\,^{m,\Delta}_{0,\nu}\right|^2 \widetilde{H}^\Delta_{0,m}.$$

We estimate Σ_1 first. Using Proposition 3.3.4, we get

$$\left|\widehat{f}\,^{m,\Delta}_{j,\nu}\right|^2 = \frac{1}{16\,j^2\,(j+1)^2}\left|\widehat{\mathcal{D}\,f}\,^{m,0}_{j,\nu}\right|^2.$$

Furthermore,

$$\frac{\widetilde{H}^{\Delta}_{j,m}}{H^0_{j,m}} = \frac{\dfrac{j^2\,(j+1)^2}{2^{n-2j+\frac{d}{2}}}\,h_j^{(0,n-2j+\frac{d-2}{2})}}{\dfrac{b_0\,\sigma_{d-1}}{2^{n-2j+\frac{d}{2}+1}}\,h_j^{(0,n-2j+\frac{d-2}{2})}} = \frac{2\,j^2\,(j+1)^2}{b_0\,\sigma_{d-1}}.$$

Consequently, it follows that

$$\Sigma_1 = \frac{1}{8\,d} \sum_{m=n+1}^{\infty} \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \sum_{\nu} \left| \widehat{\mathcal{D} f}^{\,m,0}_{\,j,\nu} \right|^2 H^0_{j,m}.$$

Next, we estimate Σ_2. Using Proposition 3.3.4 again, we obtain

$$\left| \widehat{f}^{\,m,\Delta}_{\,0,\nu} \right|^2 = \frac{1}{16\,(m+\frac{d}{2})^2} \left| \widehat{\mathcal{D} f}^{\,m,0}_{\,0,\nu} \right|^2.$$

Moreover,

$$\frac{\widetilde{H}^{\Delta}_{0,m}}{H^0_{0,m}} = \frac{(m+\frac{d}{2})\,(m+\frac{d}{2}+\lambda)}{\frac{d}{2}}.$$

Consequently, it follows that

$$\Sigma_2 = c \sum_{m=n+1}^{\infty} \sum_{\nu} \left| \widehat{\mathcal{D} f}^{\,m,0}_{\,j,\nu} \right|^2 H^0_{j,m}.$$

Putting these estimates together completes the proof of the theorem. □

The main result of this section is stated in the following theorem.

Theorem 3.3.6. For $f \in \mathtt{H}^2(\mathbf{B}^d)$ and $n \geqslant 1$,

$$\left\| \mathcal{D} f - \mathcal{D}\, S^{\Delta}_n f \right\|_0 = \mathcal{E}_n(\mathcal{D} f)_0, \tag{3.12}$$

$$\left\| \partial_i \left[(1-\|x\|^2) \left(f - S^{\Delta}_n f \right) \right] \right\|_0 \leqslant \frac{c}{n}\, \mathcal{E}_n(\mathcal{D} f)_0, \quad 1 \leqslant i \leqslant d,$$

$$\left\| (1-\|x\|^2) \left(f - S^{\Delta}_n f \right) \right\|_0 \leqslant \frac{c}{n^2} \mathcal{E}_n(\mathcal{D} f)_0. \tag{3.13}$$

Proof. By Proposition 3.3.3,

$$\|\mathcal{D} f - \mathcal{D}\, S^{\Delta}_n f\|_0 = \|\mathcal{D} f - S^0_n(\mathcal{D} f)\|_0 = \mathcal{E}_n(\mathcal{D} f)_0,$$

which proves (3.12).

Now, we deal with (3.13). We use the well-known duality argument, the so called Aubin-Nietsche technique ([12]). We use the characterization

$$\left\|(1-\|x\|^2)\left(f-S_n^{\Delta}f\right)\right\|_0 = \sup_{\|g\|_0\neq 0} \frac{|\langle g,(1-\|x\|^2)\left(f-S_n^{\Delta}f\right)\rangle_0|}{\|g\|_0}. \tag{3.14}$$

We introduce the following auxiliary boundary value problem:

$$\begin{cases} \Delta^2\varphi_g = g, & \text{in } \mathbf{B}^d,\\ \Delta\varphi_g = 0, & \text{on } \mathbf{S}^{d-1},\\ \varphi_g = 0. & \text{on } \mathbf{S}^{d-1}. \end{cases} \tag{3.15}$$

Observe that, by Green's identity, for $h \in \mathtt{H}^2(\mathbf{B}^d)$, we have

$$\begin{aligned}\frac{1}{8\,\sigma_{d-1}}\int_{\mathbf{B}^d}\Delta\Big[(1-\|x\|^2)\,h\Big]\,\Delta\varphi_g\,dx &= \frac{1}{8\,\sigma_{d-1}}\int_{\mathbf{B}^d}(1-\|x\|^2)\,h\,\Delta^2\varphi_g\,dx\\ &= \frac{1}{8\,\sigma_{d-1}}\int_{\mathbf{B}^d}(1-\|x\|^2)\,h\,g(x)\,dx\\ &= \frac{1}{8\,d}\langle g,(1-\|x\|^2)\,h\,\rangle_0,\end{aligned} \tag{3.16}$$

where we have used $b_0 = d/\sigma_{d-1}$. Moreover, since $\varphi_g = 0$ on $\mathbf{S}^{d-1}$, there is a function $\widetilde{\varphi}_g$ such that $\varphi_g = (1-\|x\|^2)\,\widetilde{\varphi}_g$. If $g=0$, then $\|\widetilde{\varphi}_g\|_{\Delta}^2 = \langle\widetilde{\varphi}_g,\widetilde{\varphi}_g\rangle_{\Delta} = 0$, which implies that $\widetilde{\varphi}_g \equiv 0$. This shows that the homogeneous version of (3.15) has a unique solution $\varphi_g = 0$ and, by the linearity of the problem (3.15), we also have that the non-homogeneous problem with $g \in L^2(\mathbf{B}^d)$ has a unique solution.

Using (3.16), we get

$$\left\langle f-S_n^{\Delta}f,\widetilde{\varphi}_g\right\rangle_{\Delta} = \frac{1}{8\,d}\left\langle g,(1-\|x\|^2)\,(f-S_n^{\Delta}f)\,\right\rangle_0.$$

Since S_n^{Δ} reproduces polynomials of degree n, it follows that

$$\langle f-S_n^{\Delta}f,\,S_n^{\Delta}\widetilde{\varphi}_g\rangle_{\Delta} = 0.$$

Consequently,

$$|\langle g,(1-\|x\|^2)\left(f-S_n^{\Delta}f\right)\rangle_0| \leqslant \langle f-S_n^{\Delta}f,\,\widetilde{\varphi}_g-S_n^{\Delta}\widetilde{\varphi}_g\rangle_{\Delta} \leqslant \|f-S_n^{\Delta}f\|_{\Delta}\|\widetilde{\varphi}_g-S_n^{\Delta}\widetilde{\varphi}_g\|_{\Delta}.$$

Therefore, by (3.14), we have

$$\left\|(1-\|x\|^2)\left(f-S_n^{\Delta}f\right)\right\|_0 = \|f-S_n^{\Delta}f\|_{\Delta}\left(\sup_{\|g\|_0\neq 0}\frac{\|\widetilde{\varphi}_g - S_n^{\Delta}\widetilde{\varphi}_g\|_{\Delta}}{\|g\|_0}\right).$$

Moreover, by Theorem 3.3.5 and (1.26),

$$\begin{aligned}\|\widetilde{\varphi}_g - S_n^{\Delta}\widetilde{\varphi}_g\|_{\Delta} &= c\,\mathcal{E}_n(\mathcal{D}\widetilde{\varphi}_g)_0 = c\,\mathcal{E}_n(\Delta\,\varphi_g)_0\\ &\leqslant \frac{c}{n^2}\Big[\mathcal{E}_{n-2}(\Delta^2\,\varphi_g)_2 + \mathcal{E}_n(\Delta_0\,\Delta\,\varphi_g)_0\Big].\end{aligned} \tag{3.17}$$

Let us bound the term $\mathcal{E}_n(\Delta_0\Delta\,\varphi_g)_0$. Since $P^{n,0}_{0,\nu}$ and $P^{n,0}_{\frac{n}{2},1}$ are harmonic and radial functions, respectively, then $\Delta_0\,\Delta P^{n,0}_{0,\nu}=0$ and $\Delta_0\,\Delta P^{n,0}_{\frac{n}{2},1}=0$. This means that

$$\mathcal{E}_n(\Delta_0\Delta\,\varphi_g)_0^2 = \sum_{m=n+1}^{\infty}\sum_{j=1}^{\lfloor\frac{m-2}{2}\rfloor}\sum_{\nu=1}^{a^d_{m-2j}}\left|\widehat{\Delta_0\Delta\,\varphi_g}^{m,0}_{j,\nu}\right|^2 H^0_{j,m}.$$

We need the following identities (Theorem (1.6.2))

$$\widehat{\Delta_0\Delta\,\varphi_g}^{m,0}_{j,\nu} = \lambda_{m-2j}\,\widehat{\Delta\,\varphi_g}^{m,0}_{j,\nu} \quad\text{and}\quad \widehat{\Delta^2\,\varphi_g}^{m-2,2}_{j-1,\nu} = \kappa^0_{m-j}\,\widehat{\Delta\,\varphi_g}^{m,0}_{j,\nu},$$

for $0\leqslant j\leqslant \frac{m}{2}$ in the first identity and $0\leqslant j\leqslant\frac{m-2}{2}$ in the second identity, where λ_{m-2j} and κ^0_{m-j} are defined in 1.32. Using these identities we get

$$\mathcal{E}_n(\Delta_0\Delta\,\varphi_g)_0^2 = \sum_{m=n+1}^{\infty}\sum_{j=1}^{\lfloor\frac{m-2}{2}\rfloor}\sum_{\nu=1}^{a^d_{m-2j}}\frac{|\lambda_{m-2j}|^2}{|\kappa^0_{m-j}|^2}\left|\widehat{\Delta^2\,\varphi_g}^{m-2,2}_{j-1,\nu}\right|^2 H^0_{j,m}.$$

Moreover,

$$\frac{H^0_{j,m}}{H^2_{j-1,m-2}} = \frac{2\,(m-j+\frac{d-2}{2})\,(m-j+\frac{d+2}{2})}{(\frac{d+2}{2})\,j\,(j+1)}.$$

Therefore, for $1\leqslant j\leqslant\frac{m-2}{2}$,

$$\frac{|\lambda_{m-2j}|^2}{|\kappa^0_{m-j}|^2}\,\frac{H^0_{j,m}}{H^2_{j-1,m-2}} \leqslant c\,\frac{m+d}{m+\frac{d-2}{2}}.$$

Consequently,

$$\begin{aligned}\mathcal{E}_n(\Delta_0\Delta\,\varphi_g)_0^2 &\leqslant c\sum_{m=n+1}^{\infty}\frac{m+d}{m+\frac{d-2}{2}}\sum_{j=1}^{\lfloor\frac{m-2}{2}\rfloor}\sum_{\nu=1}^{a^d_{m-2j}}\left|\widehat{\Delta^2\varphi_g}^{m-2,2}_{j-1,\nu}\right|^2 H^2_{j-1,m-2}\\ &\leqslant c\,\mathcal{E}_{n-2}(\Delta^2\varphi_g)_2^2.\end{aligned} \tag{3.18}$$

Putting together (3.17) and (3.18), we get

$$\|\widetilde{\varphi}_g - S_n^{\Delta}\widetilde{\varphi}_g\|_{\Delta} \leqslant \frac{c}{n^2}\mathcal{E}_{n-2}(\Delta^2\varphi_g)_2 \leqslant \frac{c}{n^2}\|\Delta^2\varphi_g\|_2 \leqslant \frac{c}{n^2}\|\Delta^2\varphi_g\|_0 = \frac{c}{n^2}\|g\|_0,$$

where we have used (3.15) and the fact that $(1-\|x\|^2)^2 \leqslant 1$ on $\mathbf{B}^d$. This implies that

$$\sup_{\|g\|_0\neq 0} \frac{\|\widetilde{\varphi}_g - S_n^{\Delta}\widetilde{\varphi}_g\|_{\Delta}}{\|g\|_0} \leqslant \frac{c}{n^2},$$

and, therefore, by Theorem 3.3.5 again, we have

$$\left\|(1-\|x\|^2)\left(f - S_n^{\Delta} f\right)\right\|_0 \leqslant \frac{c}{n^2}\mathcal{E}_n(\mathcal{D}f)_0,$$

which proves (3.13).

The intermediate case follows from the multivariate Landau-Kolmogorov inequality ([6]): for $i = 1, 2, \ldots, d$,

$$\left\|\partial_i\left[(1-\|x\|^2)\left(f - S_n^{\Delta} f\right)\right]\right\|_0 \leqslant c\left\|(1-\|x\|^2)\left(f - S_n^{\Delta} f\right)\right\|_0^{1/2}\left\|\mathcal{D}\, f - \mathcal{D}\, S_n^{\Delta} f\right\|_0^{1/2}.$$

□

3.4 Numerical Experiments

In this section we present numerical experiments to compare the approximation behavior of Fourier orthogonal expansions with respect to classical and Sobolev ball polynomials with $d = 2$ variables. To this end, we consider different functions defined on $\mathbf{B}^2$. For each function $f(x, y)$, we compute $S_n^0 f$, $S_n^{\nabla,1} f$ (defined in Chapter 2) and $S_n^{\Delta} f$ for different values of n. The three approximations were compared by computing their respective root mean square error RMSE(S^*) defined in (2.11), where S^* denotes either $S_n^0 f(x,y)$, $S_n^{\nabla,1} f(x,y)$ or $S_n^{\Delta} f(x,y)$. We will use the circular mesh consisting of 1441 points defined in (2.10).

We consider two different continuous functions and provide figures showing their approximation overlapped with their graph. We also provide a table with the approximation error of S_n^0, $S_n^{\nabla,1}$ and S_n^{Δ} for different values of n.

The figures and errors were obtained using Wolfram Mathematica®. We have selected test functions from the virtual library of simulation experiments (see [45]). We would like to highlight that, as the value of n increases, the approximation error in the Sobolev cases appears to be smaller than the classical approximation error.

Example 1

First, we consider the continuous function called Franke's function ([45]):

$$f(x,y) = f_1((x+1)/2,(y+1)/2),$$

where

$$\begin{aligned} f_1(x,y) = &-0.2\exp\left(-(9x-4)^2-(9y-7)^2\right)\\ &+0.5\exp\left(-\frac{1}{4}(9x-7)^2-\frac{1}{4}(9y-3)^2\right)\\ &+0.75\exp\left(-\frac{1}{4}(9x-2)^2-\frac{1}{4}(9y-2)^2\right)\\ &+0.75\exp\left(-\frac{1}{49}(9x+1)^2-\frac{1}{10}(9x+1)\right). \end{aligned}$$

The graph of $f(x,y)$ and the approximation $S_{20}^{\Delta}f(x,y)$ are shown in Figure 3.1. We list the RMSE of the three approximations for different values of n in Table 3.1.

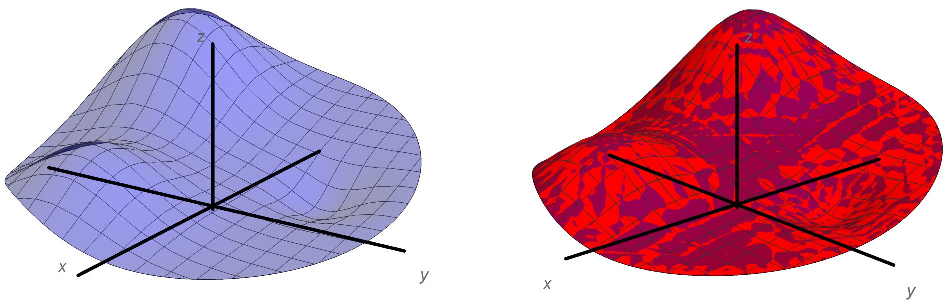

Figure 3.1: Left: $f(x,y)$. Right: Approximations $S_{30}^{\Delta}f(x,y)$ overlapped with the graph of $f(x,y)$.

n	RMSE($S_n^0 f(x,y)$)	RMSE($S_n^{\nabla,1} f(x,y)$)	RMSE($S_n^{\Delta} f(x,y)$)
5	0.033664	0.03698	0.03864
10	0.004819	0.005102	0.00474
15	0.000902	0.001062	0.000811
20	1.05053×10^{-4}	1.05892×10^{-4}	8.99766×10^{-5}
30	6.21477×10^{-6}	6.37865×10^{-6}	2.51665×10^{-5}

Table 3.1: Approximation errors for $f(x, y)$.

Example 2

Now, we consider the Gramacy & Lee (2008) function ([45]):

$$g(x, y) = 2x \exp\left(-4x^2 - 4y^2\right).$$

Figure 3.2 shows both the function and its approximation, and their respective RSME are listed in Table 3.2. Note that the errors apparently do not change for a large n but this may be due to rounding errors.

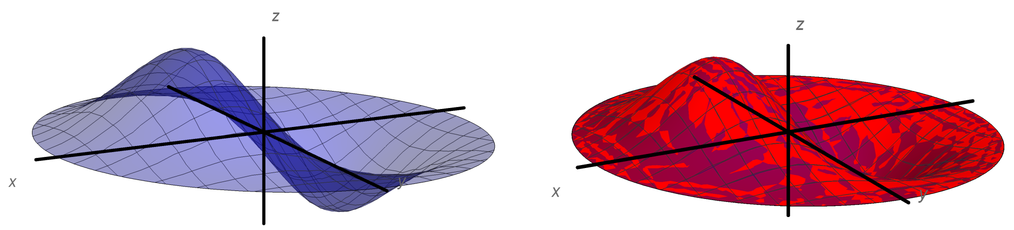

Figure 3.2: Left: $g(x, y)$. Right: Approximations $S_{30}^{\Delta} g(x, y)$ overlapped with the graph of $g(x, y)$.

n	RMSE($S_n^0 g(x,y)$)	RMSE($S_n^{\nabla,1} g(x,y)$)	RMSE($S_n^{\Delta} g(x,y)$)
5	0.025847	0.025847	0.028736
10	0.001384	0.001384	0.00139
15	4.40898×10^{-5}	4.40895×10^{-6}	3.95585×10^{-6}
20	5.31368×10^{-5}	5.03082×10^{-8}	4.24346×10^{-8}
30	1.82081×10^{-8}	5.03082×10^{-8}	7.29434×10^{-9}

Table 3.2: Approximation errors for $g(x, y)$.

Chapter 4

A class of Bernstein-type operators on the unit disk

We construct and study sequences of operators of Bernstein type acting on bivariate functions defined on the unit disk. To this end, we study Bernstein-type operators under a domain transformation, we analyze the bivariate Bernstein-Stancu operators, and we introduce Bernstein-type operators on disk quadrants by means of continuously differentiable transformations of the function. We state convergence results for continuous functions and we estimate the rate of convergence. Finally some interesting numerical examples are given, comparing approximations using the shifted Bernstein-Stancu and the Bernstein-type operator on disk quadrants. The results of this Chapter are presented in [40].

4.1 Introduction

In this Chapter, we are interested in finding an extension of the Bernstein operator to approximate functions defined on the unit disk. In this way, we will consider two kinds of modifications: by transformation of the argument of the function to be approximated, and by definition of an adequate basis of functions as (1.34). We present and study two Bernstein-type approximants, and we compare them by means of several examples.

The structure of this Chapter is as follows. In Section 4.2 we extend the Bernstein operator through a domain transformation. In Section 4.3 and Section 4.4, we define the *shifted n-th Bernstein-Stancu operator* and the *shifted n-th Bernstein-type operator*, and study their respective approximation properties. Section 4.5 is devoted to describing an extension of certain linear combinations of univariate Bernstein operators that give a better order of approximation. The last section is devoted to analyzing several examples, comparing the approximation results for both Bernstein-type operators on the disk, and the linear combinations introduced in Section 4.5.

4.2 Bernstein-type operators under a domain transformation

In the previous chapter we defined the Bernstein-Stancu operator $\mathscr{B}_n[f(x,y),\Omega]$ as in (1.38). Writing $\mathscr{B}_n[f(x,y),\Omega]$ explicitly, we have

$$\mathscr{B}_n[f(x,y),\Omega] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} F\left(\frac{k}{n},\frac{j}{n_k}\right) p_{n,k}(x)\, p_{n_k,j}\left(\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\right), (x,y)\in\Omega,$$

where $F(x,t) = f(x,(\phi_2(x)-\phi_1(x))\,t+\phi_1(x))$.

One way to extend the Bernstein operator on the unit square $\mathbf{Q}$ to another bounded domain $\Omega \subseteq \mathbb{R}^2$ is through an appropriate transformation or change of variables. In this section, we study several cases.

1. Let $\widehat{\mathbf{Q}} = [-1,1]\times[-1,1]$. The operator defined as

$$\widehat{\mathscr{B}}_n[f(x,y),\widehat{\mathbf{Q}}] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\frac{2k-n}{n},\frac{2j-n_k}{n_k}\right) p_{n,k}\left(\frac{x+1}{2}\right) p_{n_k,j}\left(\frac{y+1}{2}\right),$$

is a Bernstein operator on $\widehat{\mathbf{Q}}$. Indeed, for every function f defined on $\widehat{\mathbf{Q}}$, we define the function $F:\mathbf{Q}\to\mathbb{R}$ as

$$F(u,v) = f(2\,u-1, 2\,v-1), \quad (u,v)\in\mathbf{Q}.$$

Then, using the transformation $x = 2u - 1$ and $y = 2v - 1$ which maps $\mathbf{Q}$ into $\widehat{\mathbf{Q}}$, we get

$$\begin{aligned}\widehat{\mathscr{B}}_n[f(x,y),\widehat{\mathbf{Q}}] =& \sum_{k=0}^{n}\sum_{j=0}^{n_k} F\left(\frac{k}{n},\frac{j}{n_k}\right) p_{n,k}\,(u)\,p_{n_k,j}\,(v)\\ =& \mathscr{B}_n[F(u,v),\mathbf{Q}], \quad (x,y)\in\widehat{\mathbf{Q}}.\end{aligned}$$

2. An alternative way to obtain the Bernstein-Stancu operator on the simplex $\mathbf{T}^2$ is by considering the Duffy transformation

$$x = u,\;\; y = v(1-u), \quad (u,v)\in\mathbf{Q},$$

which maps $\mathbf{Q}$ into $\mathbf{T}^2$. Let f be a function defined on $\mathbf{T}^2$. We can define the function $F : \mathbf{Q} \to \mathbb{R}$ as

$$F(u,v) \;=\; f(u, v\,(1-u)), \quad (u,v)\in\mathbf{Q}.$$

Then, the operator

$$\widehat{\mathscr{B}}_n[f(x,y),\mathbf{T}^2] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\frac{k}{n},\frac{j}{n_k}\left(1-\frac{k}{n}\right)\right) p_{n,k}(x)\,p_{n_k,j}\left(\frac{y}{1-x}\right),$$

is a Bernstein-type operator on the simplex since, using the Duffy transformation, we get

$$\widehat{\mathscr{B}}_n[f(x,y),\mathbf{T}^2] = \mathscr{B}_n[F(u,v),\mathbf{Q}], \quad (x,y)\in\mathbf{T}^2.$$

Observe that $\widehat{\mathscr{B}}_n[f(x,y),\mathbf{T}^2]$ is not a polynomial unless $n - k - n_k \geqslant 0$. We recover the usual Bernstein-Stancu operator on the simplex by setting $n_k = n - k$.

3. Consider the unit ball in $\mathbb{R}^2$:

$$\mathbf{B}^2 \;=\; \{(x,y)\in\mathbb{R}^2 :\; x^2+y^2\leqslant 1\},$$

and the transformation $x = 2u-1,\; y = (2v-1)\sqrt{1-(2u-1)^2}$ which maps the square $\mathbf{Q}$ into $\mathbf{B}^2$. For every function f defined on $\mathbf{B}^2$, we can define the function $F : \mathbf{Q} \to \mathbb{R}^2$ as

$$F(u,v) \;=\; f\left(-1,(2v-1)\sqrt{1-(2u-1)^2}\right), \quad (u,v)\in\mathbf{Q}.$$

The operator

$$\widehat{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\frac{2k-n}{n}, \frac{2j-n_k}{n_k}\frac{2\sqrt{k\,(n-k)}}{n}\right) p_{n,k}\left(\frac{x+1}{2}\right) p_{n_k,j}\left(\frac{\frac{y}{\sqrt{1-x^2}}+1}{2}\right),$$

is a Bernstein operator on the unit ball since

$$\widehat{\mathscr{B}}_n[f(x,y),\mathbf{B}^2] \,=\, \mathscr{B}_n[F(u,v),\mathbf{Q}], \quad (x,y)\in\mathbf{B}^2.$$

Observe that, in this case,

$$p_{n,k}\left(\frac{x+1}{2}\right) p_{n_k,j}\left(\frac{\frac{y}{\sqrt{1-x^2}}+1}{2}\right)$$
$$= \binom{n}{k}\binom{n_k}{j}\frac{(1+x)^k(1-x)^{n-k}\left(\sqrt{1-x^2}+y\right)^j\left(\sqrt{1-x^2}-y\right)^{n_k-j}}{2^{n+n_k}(\sqrt{1-x^2})^{n_k}}.$$

In contrast with the previous two cases, there is no obvious choice of n_k such that $\widehat{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ is a polynomial. Nevertheless, notice that for $y=0$, we have

$$p_{n,k}\left(\frac{x+1}{2}\right) p_{n_k,j}\left(\frac{1}{2}\right) = \frac{1}{2^{n+n_k}}\binom{n}{k}\binom{n_k}{j}(1+x)^k(1-x)^{n-k},$$

and for $x=0$ we have

$$p_{n,k}\left(\frac{1}{2}\right) p_{n_k,j}\left(\frac{y+1}{2}\right) = \frac{1}{2^{n+n_k}}\binom{n}{k}\binom{n_k}{j}\,(1+y)^j\,(1-y)^{n_k-j}\,.$$

Therefore, $\widehat{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ is a polynomial on the x and y axes for any choice of n_k.

In Figure 4.1, the representation of the mesh in this case for $n=n_k=20$ is given.

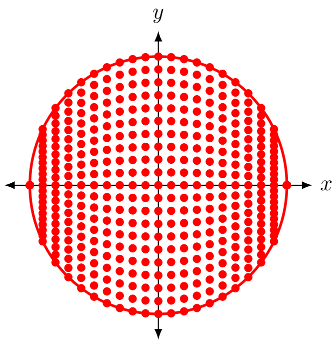

Figure 4.1: Mesh corresponding to case 3 for $n = n_k = 20$ for $0 \leqslant k \leqslant n$.

4. Let

$$B_1 = \{(x,y) \in \mathbf{B}^2 : x \geqslant 0, y \geqslant 0\}, \quad B_2 = \{(x,y) \in \mathbf{B}^2 : x \leqslant 0, y \geqslant 0\},$$
$$B_3 = \{(x,y) \in \mathbf{B}^2 : x \leqslant 0, y \leqslant 0\}, \quad B_4 = \{(x,y) \in \mathbf{B}^2 : x \geqslant 0, y \leqslant 0\},$$

denote the four quadrants of $\mathbf{B}^2$, and consider the transformation

$$u = x^2, \quad v = \frac{y^2}{1-x^2}, \quad (x,y) \in \mathbf{B}^2,$$

which maps each quadrant to $\mathbf{Q}$. The corresponding Bernstein operators on the quadrants are:

$$\widehat{\mathscr{B}}_n[f(x,y), B_1] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\sqrt{\frac{k}{n}}, \sqrt{\frac{j(n-k)}{n_k\, n}}\right) p_{n,k}(x^2)\, p_{n_k,j}\left(\frac{y^2}{1-x^2}\right),$$
$$\widehat{\mathscr{B}}_n[f(x,y), B_2] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(-\sqrt{\frac{k}{n}}, \sqrt{\frac{j(n-k)}{n_k\, n}}\right) p_{n,k}(x^2)\, p_{n_k,j}\left(\frac{y^2}{1-x^2}\right),$$
$$\widehat{\mathscr{B}}_n[f(x,y), B_3] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(-\sqrt{\frac{k}{n}}, -\sqrt{\frac{j(n-k)}{n_k\, n}}\right) p_{n,k}(x^2)\, p_{n_k,j}\left(\frac{y^2}{1-x^2}\right),$$
$$\widehat{\mathscr{B}}_n[f(x,y), B_4] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\sqrt{\frac{k}{n}}, -\sqrt{\frac{j(n-k)}{n_k\, n}}\right) p_{n,k}(x^2)\, p_{n_k,j}\left(\frac{y^2}{1-x^2}\right).$$

Indeed, for every function f defined on $\mathbf{B}^2$, we can define the functions on $\mathbf{Q}$:

$$F_1(u,v) = f\left(\sqrt{u}, \sqrt{v\,(1-u)}\right), \qquad F_2(u,v) = f\left(-\sqrt{u}, \sqrt{v\,(1-u)}\right),$$

$$F_3(u,v) = f\left(-\sqrt{u}, -\sqrt{v\,(1-u)}\right), \quad F_4(u,v) = f\left(\sqrt{u}, -\sqrt{v\,(1-u)}\right).$$

Then,

$$\widehat{\mathscr{B}}_n[f(x,y),B_1] = \mathscr{B}_n[F_1(u,v),\mathbf{Q}], \quad \widehat{\mathscr{B}}_n[f(x,y),B_2] = \mathscr{B}_n[F_2(u,v),\mathbf{Q}],$$
$$\widehat{\mathscr{B}}_n[f(x,y),B_3] = \mathscr{B}_n[F_3(u,v),\mathbf{Q}], \quad \widehat{\mathscr{B}}_n[f(x,y),B_4] = \mathscr{B}_n[F_4(u,v),\mathbf{Q}].$$

If we choose $n_k = n-k$, we have that $\widehat{\mathscr{B}}_n[f(x,y),B_i]$, $i=1,2,3,4$, are polynomials of degree $2n$ since

$$p_{n,k}(x^2)\,p_{n-k,j}\left(\frac{y^2}{1-x^2}\right) = \binom{n}{k\ \ j} x^{2k}y^{2j}(1-x^2-y^2)^{n-k-j}.$$

In this case, observe that for $k=0$, the mesh corresponding to B_1 and B_2, and similar to B_3 and B_4, coincide on the y axis (see Figure 4.2). Moreover, for $j=0$, the mesh corresponding to adjacent quadrants coincide on the x axis. Therefore, we can define a *piece-wise* Bernstein operator on $\mathbf{B}^2$ as follows:

$$\overline{\mathscr{B}}_n[f(x,y),\mathbf{B}^2] = \begin{cases} \widehat{\mathscr{B}}_n[f(x,y),B_1], & (x,y)\in B_1,\\ \widehat{\mathscr{B}}_n[f(x,y),B_2], & (x,y)\in B_2,\\ \widehat{\mathscr{B}}_n[f(x,y),B_3], & (x,y)\in B_3,\\ \widehat{\mathscr{B}}_n[f(x,y),B_4], & (x,y)\in B_4. \end{cases} \tag{4.1}$$

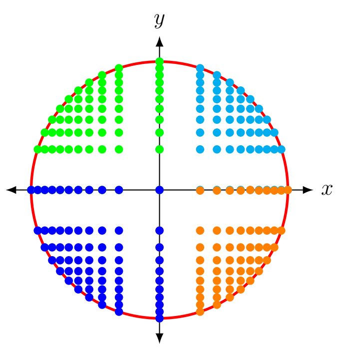

Figure 4.2: Circular mesh after applying the transformation $(u,v)\mapsto \left(\sqrt{u}, \sqrt{v\,(1-u)}\right)$ for $(u,v)\in\mathbf{Q}$ with $n=10$ and $n_k = n-k$, for $0\leqslant k\leqslant n$.

Proposition 4.2.1. For any function f on $\mathbf{B}^2$, $\overline{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ is a continuous function on $\mathbf{B}^2$.

Proof. Clearly, $\overline{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ is continuous on the interior of each quadrant. For $x=0$,

$$\begin{aligned}\widehat{\mathscr{B}}_n[f(x,y),B_1]\Big|_{x=0} &= \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(\sqrt{\frac{k}{n}},\sqrt{\frac{j(n-k)}{n_k\,n}}\right) p_{n,k}(0)\,p_{n_k,j}\left(y^2\right),\\ &= \sum_{j=0}^{n_0} f\left(0,\sqrt{\frac{j}{n_0}}\right) p_{n_0,j}\left(y^2\right)\\ &= \widehat{\mathscr{B}}_n[f(x,y),B_2]\Big|_{x=0},\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathscr{B}}_n[f(x,y),B_3]\Big|_{x=0} &= \sum_{k=0}^{n}\sum_{j=0}^{n_k} f\left(-\sqrt{\frac{k}{n}},-\sqrt{\frac{j(n-k)}{n_k\,n}}\right) p_{n,k}(0)\,p_{n_k,j}\left(y^2\right),\\ &= \sum_{j=0}^{n_0} f\left(0,-\sqrt{\frac{j}{n_0}}\right) p_{n_0,j}\left(y^2\right)\\ &= \widehat{\mathscr{B}}_n[f(x,y),B_4]\Big|_{x=0}.\end{aligned}$$

Similarly, for $y=0$

$$\widehat{\mathscr{B}}_n[f(x,y),B_1]\Big|_{y=0} = \widehat{\mathscr{B}}_n[f(x,y),B_4]\Big|_{y=0},$$

and

$$\widehat{\mathscr{B}}_n[f(x,y),B_2]\Big|_{y=0} = \widehat{\mathscr{B}}_n[f(x,y),B_3]\Big|_{y=0}.$$

Therefore, $\overline{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ is continuous on the x and y axes. □

4.3 Shifted Bernstein-Stancu operators

Motivated by the examples of Bernstein operators on different domains introduced in the previous section, we define the *shifted n-th Bernstein-Stancu operator* and study its approximation properties.

Let ϕ_1 and ϕ_2 be two continuous functions, and let $I=[a,b]$ be an interval such that $\phi_1<\phi_2$ on I. Let $\Omega\subset\mathbb{R}^2$ be the domain bounded by the curves $y=\phi_1(x)$, $y=\phi_2(x)$, and the straight lines $x=a$, $x=b$. Observe that for a fixed $x\in I$, the polynomials $\widetilde{p}_{n,k}(y;[\phi_1(x),\phi_2(x)])$, $n\geqslant 0$, $0\leqslant k\leqslant n$, constitute a univariate shifted Bernstein basis on the interval $[\phi_1(x),\phi_2(x)]$.

For every function $f(x,y)$ defined on Ω, define the function

$$\widetilde{F}(u,v;\Omega) \;=\; f\left((b-a)\,u+a,(\widetilde{\phi}_2(u)-\widetilde{\phi}_1(u))\,v+\widetilde{\phi}_1(u)\right), \tag{4.2}$$

where

$$\widetilde{\phi}_i(u)=\phi_i((b-a)\,u+a), \quad i=1,2,$$

$0\leqslant u\leqslant 1$, and $0\leqslant v\leqslant 1$.

The shifted n-th Bernstein-Stancu operator is defined as

$$\widetilde{\mathscr{B}}_n[f(x,y),\Omega]=\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k};\Omega\right)\,\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]), \quad (x,y)\in\Omega,$$

where $n_k=n-k$ or $n_k=k$ for all $0\leqslant k\leqslant n$. Written in terms of the univariate Bernstein basis, we get

$$\widetilde{\mathscr{B}}_n[f(x,y),\Omega]=\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k};\Omega\right)\,p_{n,k}\left(\frac{x-a}{b-a}\right)\,p_{n_k,j}\left(\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\right).$$

The following result plays an important role when studying the convergence of the shifted Bernstein-Stancu operator.

Lemma 4.3.1. Let ϕ_1 and ϕ_2 be two continuous functions, and let $I=[a,b]$ be an interval such that $\phi_1<\phi_2$ on I. Let $\Omega\subset\mathbb{R}^2$ be the domain bounded by the curves $y=\phi_1(x)$, $y=\phi_2(x)$, and the straight lines $x=a$, $x=b$. Then:

(i) $\widetilde{\mathscr{B}}_n[1,\Omega]=1$,

(ii) $\widetilde{\mathscr{B}}_n[x,\Omega]=x$,

(iii) $\widetilde{\mathscr{B}}_n[y,\Omega]\to y$ as $n\to+\infty$ uniformly on $[a,b]$,

(iv) $\widetilde{\mathscr{B}}_n[x^2,\Omega]=x^2+(x-a)\,(b-x)/n$,

(v) $\widetilde{\mathscr{B}}_n[y^2,\Omega]\to y^2$ as $n\to+\infty$ uniformly on $[a,b]$.

Proof. (i) Obviously $\widetilde{\mathscr{B}}_n[1,\Omega]=1$.

(ii) We compute

$$\widetilde{\mathscr{B}}_n[x,\Omega]=\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\left((b-a)\frac{k}{n}+a\right)$$

$$
\begin{aligned}
&=(b-a)\sum_{k=0}^{n}\widetilde{p}_{n,k}(x;I)\frac{k}{n}\left(\sum_{j=0}^{n_k}\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\right)+a\,\widetilde{\mathscr{B}}_n[1,\Omega]\\
&=(b-a)\sum_{k=0}^{n}\widetilde{p}_{n,k}(x;I)\frac{k}{n}+a\\
&=(b-a)\frac{x-a}{b-a}\sum_{k=0}^{n-1}\widetilde{p}_{n-1,k}(x;I)+a\\
&=x.
\end{aligned}
$$

(iii) Observe that

$$
\begin{aligned}
\sum_{j=0}^{n_k}\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\frac{j}{n_k}&=\sum_{j=1}^{n_k}\binom{n_k}{j}\frac{(y-\phi_1(x))^j\,(\phi_2(x)-y)^{n_k-j}}{(\phi_2(x)-\phi_1(x))^{n_k}}\,\frac{j}{n_k}\\
&=\sum_{j=0}^{n_k-1}\binom{n_k-1}{j}\frac{(y-\phi_1(x))^{j+1}\,(\phi_2(x)-y)^{n_k-1-j}}{(\phi_2(x)-\phi_1(x))^{n_k}}\\
&=\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}.
\end{aligned}
\tag{4.3}
$$

Therefore, applying the linearity, we get

$$
\begin{aligned}
\widetilde{\mathscr{B}}_n[y,\Omega]&=\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\left[\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}+\widetilde{\phi}_1\left(\frac{k}{n}\right)\right]\\
&=\left[\sum_{k=0}^{n}\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\widetilde{p}_{n,k}(x;I)\right]\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\\
&\quad+\sum_{k=0}^{n}\widetilde{\phi}_1\left(\frac{k}{n}\right)\widetilde{p}_{n,k}(x;I)\\
&=\widetilde{B}_n[\phi_2-\phi_1,I]\,\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}+\widetilde{B}_n[\phi_1,I],
\end{aligned}
$$

where $\widetilde{B}_n$ denotes the univariate shifted Bernstein operator acting on the variable x. Since $\widetilde{B}_n$ converges uniformly for a continuous function, we have

$$
\begin{aligned}
\lim_{n\to+\infty}\widetilde{\mathscr{B}}_n[y,\Omega]&=\lim_{n\to+\infty}\widetilde{B}_n[\phi_2-\phi_1,I]\,\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}+\lim_{n\to+\infty}\widetilde{B}_n[\phi_1,I]\\
&=[\phi_2(x)-\phi_1(x)]\,\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}+\phi_1(x)\\
&=y.
\end{aligned}
$$

(iv) We compute

$$
\begin{aligned}
\widetilde{\mathscr{B}}_n[x^2,\Omega] =& \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\, \left((b-a)\frac{k}{n}+a\right)^2\\
=&(b-a)^2\sum_{k=0}^{n}\widetilde{p}_{n,k}(x;I)\frac{k^2}{n^2} + 2\,a\,(b-a)\frac{x-a}{b-a}\sum_{k=0}^{n-1}\widetilde{p}_{n-1,k}(x;I)+a^2\\
=&(b-a)^2\left(\frac{n-1}{n}\left(\frac{x-a}{b-a}\right)^2\sum_{k=0}^{n-2}\widetilde{p}_{n-2,k}(x;I)+\frac{1}{n}\frac{x-a}{b-a}\sum_{k=0}^{n-1}\widetilde{p}_{n-1,k}(x;I)\right)\\
&+2\,a\,(x-a)+a^2\\
=&x^2+\frac{(x-a)(b-x)}{n}.
\end{aligned}
$$

(v) Finally, if $f(x,y)=y^2$ in (4.2), we get

$$
\begin{aligned}
\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k};\Omega\right) =& \left[\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}+\widetilde{\phi}_1\left(\frac{k}{n}\right)\right]^2\\
=&\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)^2\frac{j^2}{n_k^2}\\
&+2\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\widetilde{\phi}_1\left(\frac{k}{n}\right)\frac{j}{n_k}+\widetilde{\phi}_1\left(\frac{k}{n}\right)^2.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\widetilde{\mathscr{B}}_n[y^2,\Omega] =& \sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)^2\frac{j^2}{n_k^2}\\
&+2\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{p}_{n,k}(x;I)\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\widetilde{\phi}_1\left(\frac{k}{n}\right)\frac{j}{n_k}\\
&+\sum_{k=0}^{n}\sum_{j=0}^{n_k}\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\widetilde{\phi}_1\left(\frac{k}{n}\right)^2.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\sum_{j=0}^{n_k}\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\,\frac{j^2}{n_k^2} =& \frac{n_k-1}{n_k}\left(\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\right)^2\sum_{j=0}^{n_k-2}\widetilde{p}_{n_k-2,j}(y;[\phi_1,\phi_2])\\
&+\frac{1}{n_k}\frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\sum_{j=0}^{n_k-1}\widetilde{p}_{n_k-1,j}(y;[\phi_1,\phi_2])
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{n_k - 1}{n_k} \left(\frac{y - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \right)^2 + \frac{1}{n_k} \frac{y - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \\
&= \left(\frac{y - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \right)^2 + \frac{(y - \phi_1(x))\,(\phi_2(x) - y)}{n_k\,(\phi_2(x) - \phi_1(x))^2}.
\end{aligned}
$$

Together with (4.3), we get

$$
\begin{aligned}
\widetilde{\mathscr{B}}_n[y^2, \Omega] = &\left(\frac{y - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \right)^2 \widetilde{B}_n[\phi_2 - \phi_1, I]^2 \\
&+ \frac{(y - \phi_1(x))\,(\phi_2(x) - y)}{n\,(\phi_2(x) - \phi_1(x))^2} \sum_{k=0}^{n} \widetilde{p}_{n,k}(x; I) \left(\widetilde{\phi}_2\left(\frac{k}{n}\right) - \widetilde{\phi}_1\left(\frac{k}{n}\right) \right)^2 \frac{1}{n_k/n} \\
&+ 2 \left(\frac{y - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \right) \widetilde{B}_n[(\phi_2 - \phi_1)\,\phi_1, I] + \widetilde{B}_n[\phi_1^2, I].
\end{aligned}
$$

If $n_k = n - k$, then

$$
\sum_{k=0}^{n} \widetilde{p}_{n,k}(x; I) \left(\widetilde{\phi}_2\left(\frac{k}{n}\right) - \widetilde{\phi}_1\left(\frac{k}{n}\right) \right)^2 \frac{1}{n_k/n} \;=\; \widetilde{B}_n \left[\frac{(\phi_2(x) - \phi_1(x))^2}{1 - \frac{x-a}{(b-a)}}, I \right],
$$

and if $n_k = k$, then

$$
\sum_{k=0}^{n} \widetilde{p}_{n,k}(x; I) \left(\widetilde{\phi}_2\left(\frac{k}{n}\right) - \widetilde{\phi}_1\left(\frac{k}{n}\right) \right)^2 \frac{1}{n_k/n} \;=\; \widetilde{B}_n \left[\frac{(\phi_2(x) - \phi_1(x))^2}{\frac{x-a}{(b-a)}}, I \right].
$$

In either case, $\widetilde{\mathscr{B}}_n[y^2, \Omega] \to y^2$ as $n \to +\infty$. □

The convergence of the operator is clear from Lemma 4.3.1 and Volkov's theorem ([49]).

Now, we study the approximation properties of the shifted Bernstein-Stancu operators.

Definition 4.3.2 ([42])**.** *Let f be a function defined on Ω. The modulus of continuity of f is defined by*

$$
\omega(\delta_1, \delta_2) \;=\; \sup |f(x'', y'') - f(x', y')|,
$$

where $\delta_1, \delta_2 > 0$ are real numbers, whereas (x', y') and (x'', y'') are points of Ω such that $|x'' - x'| \leqslant \delta_1$ and $|y'' - y'| \leqslant \delta_2$.

Theorem 4.3.3. Let f be a continuous function on Ω. Then,

$$\lim_{n\to+\infty} \widetilde{\mathscr{B}}_n[f(x,y),\Omega] = f(x,y),$$

uniformly on Ω.

Proof. Let $\delta_1, \delta_2 > 0$ be real numbers.
Note that on Ω we have $\widetilde{\mathscr{B}}_n[1,\Omega] = 1$,

$$\widetilde{p}_{n,k}(x; I)\, \widetilde{p}_{n_k,j}(y; [\phi_1, \phi_2]) \geqslant 0, \quad 0 \leqslant k \leqslant n, \quad 0 \leqslant j \leqslant n_k,$$

and

$$|f(x'', y'') - f(x', y')| \leqslant \omega(|x'' - x'|, |y'' - y'|) \leqslant w(\delta_1, \delta_2).$$

Taking into account the inequality (see, for instance, [42, 44])

$$\omega(c_1\, \delta_1, c_2\, \delta_2) \leqslant (c_1 + c_2 + 1)\, \omega(\delta_1, \delta_2), \quad c_1, c_2 > 0,$$

we compute

$$\begin{aligned}
&\left| f(x,y) - \widetilde{F}\left(\frac{k}{n}, \frac{j}{n_k}; \Omega\right)\right| \\
&\leqslant \omega\left(\left|x - (b-a)\frac{k}{n} - a\right|, \left|y - \left(\widetilde{\phi}_2\left(\frac{k}{n}\right) - \widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k} - \widetilde{\phi}_1\left(\frac{k}{n}\right)\right|\right) \\
&\leqslant (\lambda_1 + \lambda_2 + 1)\, \omega(\delta_1, \delta_2),
\end{aligned}$$

where

$$\lambda_1 \equiv \lambda_1(x, n, k, \delta_1, a, b) = \frac{1}{\delta_1}\left|x - (b-a)\frac{k}{n} - a\right|,$$

and

$$\lambda_2 \equiv \lambda_2(x, n, k, n_k, \delta_2, \phi_1, \phi_2) = \frac{1}{\delta_2}\left|y - \left(\widetilde{\phi}_2\left(\frac{k}{n}\right) - \widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k} - \widetilde{\phi}_1\left(\frac{k}{n}\right)\right|.$$

Therefore,

$$\begin{aligned}
&|f(x,y) - \widetilde{\mathscr{B}}_n[f(x,y),\Omega]| \\
&\leqslant \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x; I)\, \widetilde{p}_{n_k,j}(y; [\phi_1, \phi_2]) \left| f(x,y) - \widetilde{F}\left(\frac{k}{n}, \frac{j}{n_k}; \Omega\right)\right| \\
&\leqslant \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x; I)\, \widetilde{p}_{n_k,j}(y; [\phi_1, \phi_2])(\lambda_1 + \lambda_2 + 1)\, \omega(\delta_1, \delta_2).
\end{aligned}$$

We will deal with each term in the last inequality separately.

Since $\mathscr{B}[1,\Omega]=1$, $0 \leqslant \widetilde{p}_{n,k}(x;I) \leqslant 1$, $0 \leqslant \widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \leqslant 1$, and $x \mapsto x^{1/2}$ is a concave function, by Jensen's inequality, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \left| x-(b-a)\frac{k}{n}-a \right| \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \left[\left(x-(b-a)\frac{k}{n}-a \right)^2\right]^{1/2} \\
&\leqslant \left[\sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \left(x-(b-a)\frac{k}{n}-a \right)^2\right]^{1/2}.
\end{aligned}$$

Using (i), (ii), and (iv) in Lemma 4.3.1, we get

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \left| x-(b-a)\frac{k}{n}-a \right| \\
&= \left[x^2\,\widetilde{\mathscr{B}}_n[1,\Omega] - 2\,x\,\widetilde{\mathscr{B}}_n[x,\Omega] + \widetilde{\mathscr{B}}_n[x^2,\Omega]\right]^{1/2} \to 0 \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

uniformly since $\widetilde{\mathscr{B}}_n[1,\Omega]=1$, $\widetilde{\mathscr{B}}_n[x,\Omega]=x$, and $\lim\limits_{n\to+\infty} \widetilde{\mathscr{B}}_n[x^2,\Omega]=x^2$.

Similarly, from Jensen's inequality, and using (i), (iii), and (vi) in Lemma 4.3.1, we get

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;\phi_1,\phi_2) \left| y-\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}-\widetilde{\phi}_1\left(\frac{k}{n}\right) \right| \\
&\leqslant \left[\sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]) \left(y-\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}-\widetilde{\phi}_1\left(\frac{k}{n}\right) \right)^2\right]^{1/2} \\
&= \left[y^2\,\widetilde{\mathscr{B}}_n[1,\Omega] - 2\,y\,\widetilde{\mathscr{B}}_n[y,\Omega] + \widetilde{\mathscr{B}}_n[y^2,\Omega]\right]^{1/2} \to 0 \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

uniformly since $\widetilde{\mathscr{B}}_n[1,\Omega]=1$, $\lim\limits_{n\to+\infty} \widetilde{\mathscr{B}}_n[y,\Omega]=y$, and $\lim\limits_{n\to+\infty} \widetilde{\mathscr{B}}_n[y^2,\Omega]=y^2$.

Finally, choosing $\delta_1=\delta_2=1/\sqrt{n}$, then $\omega(1/\sqrt{n},1/\sqrt{n}) \to 0$ as $n \to +\infty$, and, thus, $\widetilde{\mathscr{B}}_n[f(x,y),\Omega]$ converges uniformly to $f(x,y)$ on Ω. □

Recall that the univariate shifted Bernstein operator satisfy the following Voronowskaya type asymptotic formula: Let $f(x)$ be bounded on the interval

I, and let $x_0 \in I$ at which $f''(x_0)$ exists. Then,

$$\widetilde{B}_n[f(x), I]\Big|_{x=x_0} - f(x_0) = \mathcal{O}(n^{-1}). \tag{4.4}$$

Now, we give an analogous result for the Bernstein-Stancu operator.

Theorem 4.3.4. Let $f(x,y)$ be a bounded function on $\Omega = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b,\ \phi_1(x) \leqslant y \leqslant \phi_2(x)\}$, and let $(x_0, y_0) \in \Omega$ be a point at which $f(x,y)$ admits second order partial derivatives, and $\phi_i''(x_0)$, $i = 1, 2$, exist. Then,

$$\widetilde{\mathscr{B}}_n[f(x,y), \Omega]\Big|_{(x,y)=(x_0,y_0)} - f(x_0, y_0) = \mathcal{O}\left(\frac{1}{n}\right).$$

Proof. Let us write the Taylor expansion of $f(u,v)$ at the point (x_0, y_0):

$$\begin{aligned} f(u,v) =& f(x_0,y_0) + (u-x_0)\, f_x(x_0,y_0) + (v-y_0)\, f_y(x_0,y_0) \\ &+ \frac{(x-x_0)^2}{2} f_{xx}(x_0,y_0) + \frac{(u-x_0)(v-y_0)}{2}\left(f_{xy}(x_0,y_0) + f_{yx}(x_0,y_0)\right) \\ &+ \frac{(v-y_0)^2}{2}\, f_{yy}(x_0,y_0) + \|(u,v)-(x_0,y_0)\|^2\, h(u,v), \end{aligned}$$

where $h(u,v)$ is a bounded function such that $h(u,v) \to 0$ as $(u,v) \to (x_0,y_0)$. Applying $\widetilde{\mathscr{B}}_n$ to both sides, we get:

$$\begin{aligned} &\widetilde{\mathscr{B}}_n[f(u,v)] = f(x_0,y_0) + f_x(x_0,y_0)\, \widetilde{\mathscr{B}}_n[u-x_0] + \widetilde{\mathscr{B}}_n[v-y_0]\, f_y(x_0,y_0) \\ &+ \frac{1}{2} f_{xx}(x_0,y_0)\, \widetilde{\mathscr{B}}_n[(u-x_0)^2] + \frac{1}{2}\left(f_{xy}(x_0,y_0) + f_{yx}(x_0,y_0)\right) \widetilde{\mathscr{B}}_n[(u-x_0)(v-y_0)] \\ &+ \frac{1}{2}\, f_{yy}(x_0,y_0)\, \widetilde{\mathscr{B}}_n[(v-y_0)^2] + \widetilde{\mathscr{B}}_n\left[\|(u,v)-(x_0,y_0)\|^2\, h(u,v)\right], \end{aligned}$$

where we have omitted Ω for brevity. We deal with each term separately.

From Lemma 4.3.1 (ii), we get $\widetilde{\mathscr{B}}_n[u-x_0]\Big|_{u=x_0} = 0$. Next, from the proof of Lemma 4.3.1 (iii), we have

$$\widetilde{\mathscr{B}}_n[v-y_0] = \widetilde{B}_n[\phi_2(x) - \phi_1(x), I]\, \frac{y-\phi_1(x)}{\phi_2(x)-\phi_1(x)} + \widetilde{B}_n[\phi_1(x), I] - y_0.$$

But using (4.4), we get

$$\widetilde{\mathscr{B}}_n[v-y_0]\Big|_{(u,v)=(x_0,y_0)} = \left(\phi_2(x_0) - \phi_1(x_0) + \mathcal{O}\left(\frac{1}{n}\right)\right) \frac{y_0-\phi_1(x_0)}{\phi_2(x_0)-\phi_1(x_0)}$$

$$+\,\phi_1(x_0) + \mathcal{O}\left(\frac{1}{n}\right) - y_0 \;=\; \mathcal{O}\left(\frac{1}{n}\right).$$

Similarly,

$$\begin{aligned}
\widetilde{\mathscr{B}}_n[(u-x_0)^2]\Big|_{(u,v)=(x_0,y_0)} &= \mathcal{O}\left(\frac{1}{n}\right),\\
\widetilde{\mathscr{B}}_n[(u-x_0)(v-y_0)]\Big|_{(u,v)=(x_0,y_0)} &= \mathcal{O}\left(\frac{1}{n}\right),\\
\widetilde{\mathscr{B}}_n[(v-y_0)^2]\Big|_{(u,v)=(x_0,y_0)} &= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Now we deal with the last term

$$\begin{aligned}
&\widetilde{\mathscr{B}}_n\left[\|(u,v)-(x_0,y_0)\|^2\, h(u,v)\right]\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\,\widetilde{H}\left(\frac{k}{n},\frac{j}{n_k}\right)\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2]),
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\\
&= \left((b-a)\frac{k}{n}+a-x_0\right)^2 + \left(\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}+\widetilde{\phi}_1\left(\frac{k}{n}\right)-y_0\right)^2,
\end{aligned}$$

and

$$\widetilde{H}\left(\frac{k}{n},\frac{j}{n_k}\right) = h\left((b-a)\frac{k}{n}+a,\left(\widetilde{\phi}_2\left(\frac{k}{n}\right)-\widetilde{\phi}_1\left(\frac{k}{n}\right)\right)\frac{j}{n_k}+\widetilde{\phi}_1\left(\frac{k}{n}\right)\right).$$

Fix a real number $\varepsilon > 0$. Then there is a real number $\delta > 0$ such that if $\|(u,v)-(x_0,y_0)\| < \delta$, then $|h(u,v)| < \varepsilon$. Let S_δ be the set of k and j such that $\frac{1}{\delta^2}\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right) > 1$. Then,

$$\begin{aligned}
&\sum_{(k,j)\in S_\delta} \widetilde{p}_{n,k}(x_0;I)\,\widetilde{p}_{n_k,j}(y_0;[\phi_1,\phi_2])\\
&< \frac{1}{\delta^2}\sum_{(k,j)\in S_\delta} \widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\widetilde{p}_{n,k}(x_0;I)\,\widetilde{p}_{n_k,j}(y_0;[\phi_1,\phi_2])\\
&\leqslant \frac{1}{\delta^2}\left(\widetilde{\mathscr{B}}_n[(u-x_0)^2]+\widetilde{\mathscr{B}}_n[(v-y_0)^2]\right)\Big|_{(u,v)=(x_0,y_0)}\\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\sum_{(k,j)\notin S_\delta} \widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\left|\widetilde{H}\left(\frac{k}{n},\frac{j}{n_k}\right)\right|\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\\
&< \varepsilon \sum_{(k,j)\notin S_\delta} \widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\widetilde{p}_{n,k}(x_0;I)\,\widetilde{p}_{n_k,j}(y_0;[\phi_1,\phi_2]) \leqslant \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\widetilde{\mathscr{B}}_n\left[\|(u,v)-(x_0,y_0)\|^2\,h(u,v)\right]\right|\\
&\leqslant \sum_{(k,j)\in S_\delta}\left|\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\,\widetilde{H}\left(\frac{k}{n},\frac{j}{n_k}\right)\right|\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\\
&+\sum_{(k,j)\notin S_\delta}\widetilde{F}\left(\frac{k}{n},\frac{j}{n_k}\right)\left|\widetilde{H}\left(\frac{k}{n},\frac{j}{n_k}\right)\right|\widetilde{p}_{n,k}(x;I)\,\widetilde{p}_{n_k,j}(y;[\phi_1,\phi_2])\\
&\leqslant M\sum_{(k,j)\in S_\delta}\widetilde{p}_{n,k}(x_0;I)\,\widetilde{p}_{n_k,j}(y_0;[\phi_1,\phi_2]) + \mathcal{O}\left(\frac{1}{n}\right)\\
&\leqslant \mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}$$

where

$$M = \sup_{(u,v)\in\Omega}\left|\|(u,v)-(x_0,y_0)\|^2\,h(u,v)\right|.$$

Putting all the above together, we get

$$\left|\widetilde{\mathscr{B}}_n[f(x,y),\Omega]\Big|_{(x,y)=(x_0,y_0)} - f(x_0,y_0)\right| \leqslant \mathcal{O}\left(\frac{1}{n}\right),$$

and the result follows. □

4.4 Shifted Bernstein-type operators

We will use the following Bernstein-type operator studied in [10] and [21]:

$$C_n^\tau f = B_n\left(f\circ\tau^{-1}\right)\circ\tau,$$

where τ is any function continuously differentiable as many times as necessary, such that $\tau(0)=0$, $\tau(1)=1$, and $\tau'(x)>0$ for $x\in[0,1]$. Throughout this work, it will be sufficient for τ to be continously differentiable.

In [10], the following identities were given:

$$C_n^\tau 1 \;=\; 1, \quad C_n^\tau \tau \;=\; \tau, \quad C_n^\tau \tau^2 \;=\; \left(1 - \frac{1}{n}\right)\tau^2 + \frac{\tau}{n}.$$

We have the following result.

Proposition 4.4.1. Let f be a continuous function on $[0,1]$ and τ is any function that is continuously differentiable, such that $\tau(0) = 0$, $\tau(1) = 1$, and $\tau'(x) > 0$ for $x \in [0,1]$. Then,

$$\lim_{n\to\infty} C_n^\tau f(x) \;=\; f(x).$$

That is, $C_n^\tau f(x)$ converges uniformly to f on $[0,1]$.

Proof. Set $u = \tau(x)$. We compute

$$\begin{aligned} C_n^\tau f(x) &= \sum_{k=0}^{n} f\left(\tau^{-1}\left(\frac{k}{n}\right)\right) p_{n,k}(\tau(x)) \\ &= \sum_{k=0}^{n} f\left(\tau^{-1}\left(\frac{k}{n}\right)\right) p_{n,k}(u) \\ &= B_n f\left(\tau^{-1}(u)\right). \end{aligned}$$

Since $B_n f(\tau^{-1}(u)) \to f(\tau^{-1}(u)) = f(x)$ as $n \to +\infty$, the result follows from taking the limit on both sides of $C_n^\tau f(x) = B_n f(x)$. □

We also introduce the following shifted Bernstein-type operator

$$\widetilde{C}_n^\tau[f(x),[\alpha,\beta]] = \sum_{k=0}^{n} f \circ \tau^{-1}\left((\beta-\alpha)\frac{k}{n} + \alpha\right) \widetilde{p}_{n,k}(\tau(x);[\alpha,\beta]), \quad \alpha \leqslant x \leqslant \beta,$$

where $\tau(x)$ is any function that is continuously differentiable, such that $\tau(\alpha) = \alpha$, $\tau(\beta) = \beta$, and $\tau'(x) > 0$ for $x \in [\alpha,\beta]$.

Proposition 4.4.2. Let f be a continuous function on $[\alpha,\beta]$ and $\tau(x)$ is any function that is continuously differentiable, such that $\tau(\alpha) = \alpha$, $\tau(\beta) = \beta$, and $\tau'(x) > 0$ for $x \in [\alpha,\beta]$. Then,

$$\lim_{n\to\infty} \widetilde{C}_n^\tau[f(x),[\alpha,\beta]] \;=\; f(x).$$

Proof. Set $u = \tau(x)$. We compute

$$\begin{aligned}
\widetilde{C}_n^{\tau}[f(x),[\alpha,\beta]] &= \sum_{k=0}^{n} f\left(\tau^{-1}\left((\beta-\alpha)\frac{k}{n}+\alpha\right)\right) \widetilde{p}_{n,k}(\tau(x);[\alpha,\beta]) \\
&= \sum_{k=0}^{n} f\left(\tau^{-1}\left((\beta-\alpha)\frac{k}{n}+\alpha\right)\right) \widetilde{p}_{n,k}(u;[\alpha,\beta]) \\
&= \widetilde{B}_n[f\left(\tau^{-1}(u)\right),[\alpha,\beta]].
\end{aligned}$$

Since $\widetilde{B}_n[f(\tau^{-1}(u)),[\alpha,\beta]] \to f(\tau^{-1}(u)) = f(x)$ as $n \to +\infty$, the result follows from taking the limit on both sides of $\widetilde{C}_n^{\tau}[f(x),[\alpha,\beta]] = \widetilde{B}_n[f(x),[\alpha,\beta]]$. □

We define the *shifted* bivariate Bernstein-type operator. Let ϕ_1 and ϕ_2 be two continuous functions, and let $I = [a,b]$ be an interval such that $\phi_1 < \phi_2$ on I. Let $\Omega \subset \mathbb{R}^2$ be the domain bounded by the curves $y = \phi_1(x)$, $y = \phi_2(x)$, and the straight lines $x = a$, $x = b$. Let

$$T(x,y) \;=\; (\tau(x),\sigma_x(y)), \quad (x,y) \in \Omega,$$

where τ is any continuously differentiable function on I, such that $\tau(a) = a$, $\tau(b) = b$, and $\tau'(x) > 0$ for $x \in I$, and for each fixed $x \in I$, σ_x is any continuously differentiable function on $[\phi_1(x),\phi_2(x)]$, such that $\sigma_x(\phi_1(x)) = \phi_1(x)$, $\sigma_x(\phi_2(x)) = \phi_2(x)$, and $\sigma_x'(y) > 0$ for $y \in [\phi_1(x),\phi_2(x)]$.

For every function $f(x,y)$ defined on Ω, define the function

$$\widetilde{F}^T(u,v;\Omega) \;=\; f \circ T^{-1}\left((b-a)\,u + a, (\widetilde{\phi}_2(u) - \widetilde{\phi}_1(u))\,v + \widetilde{\phi}_1(u)\right),$$

for $0 \leqslant u \leqslant 1$, and $0 \leqslant v \leqslant 1$, where $\widetilde{\phi}_i$, $i = 1,2$, are defined in (4.2).

The shifted bivariate Bernstein-type operator is defined as

$$\mathscr{C}_n^T[f(x,y),\Omega] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{F}^T\left(\frac{k}{n},\frac{j}{n_k};\Omega\right) \widetilde{p}_{n,k}(\tau(x);I)\,\widetilde{p}_{n_k,j}(\sigma_x(y);[\phi_1,\phi_2]),$$

for $(x,y) \in \Omega$, where $n_k = n-k$ or $n_k = k$ for $0 \leqslant k \leqslant n$.

Written in terms of the univariate classical Bernstein basis, we get

$$\mathscr{C}_n^T[f(x,y),\Omega] = \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{F}^T\left(\frac{k}{n},\frac{j}{n_k};\Omega\right) p_{n,k}\left(\frac{\tau(x)-a}{b-a}\right) p_{n_k,j}\left(\frac{\sigma_x(y)-\phi_1(x)}{\phi_2(x)-\phi_1(x)}\right).$$

Proposition 4.4.3. For every function $f(x,y)$ defined on Ω,

$$\lim_{n\to+\infty} \widetilde{\mathscr{C}}_n^T[f(x,y),\Omega] \;=\; f(x,y).$$

Proof. Let $u=\tau(x)$ and, for each $x\in I$, $v=\sigma_x(y)$. Then,

$$\begin{aligned}\widetilde{\mathscr{C}}_n^T[f(x,y),\Omega] &= \sum_{k=0}^{n}\sum_{j=0}^{n_k} \widetilde{F}\left(\frac{k}{n},\frac{j}{n_k};a,b\right)\,\widetilde{p}_{n,k}(u;I)\,\widetilde{p}_{n_k,j}(v;[\phi_1,\phi_2])\\ &= \widetilde{\mathscr{B}}_n[(f\circ T)(u,v),\Omega].\end{aligned}$$

From Theorem 4.3.3, we have $\widetilde{\mathscr{B}}_n[(f\circ T)(u,v),\Omega]=\widetilde{\mathscr{B}}_n[f(x,y),\Omega]$ converges uniformly to $f(x,y)$. Hence, $\widetilde{\mathscr{C}}_n^T[f(x,y),\Omega]$ converges uniformly to $f(x,y)$. □

Now, we study shifted Bernstein-type operators defined on each quadrant of $\mathbf{B}^2$, denoted by B_i for $i=1,2,3,4$. We will choose T and n_k such that, for any function, the approximation given by Bernstein-type operators on each quadrant is a polynomial.

(i) For $x\in[0,1]$, let $\tau(x)=x^2$ and, for each fixed value of x, let $\sigma_x(y)=y^2/\sqrt{1-x^2}$. Let $n_k=n-k$, $\phi_1(x)=0$, and $\phi_2(x)=\sqrt{1-x^2}$. Then,

$$\widetilde{p}_{n,k}(x^2;[0,1])=\binom{n}{k}x^{2\,k}\,(1-x^2)^{n-k},$$

$$\widetilde{p}_{n-k,j}(\sigma_x(y);[\phi_1,\phi_2])=\frac{1}{(1-x^2)^{n-k}}\binom{n-k}{j}y^{2\,j}\,(1-x^2-y^2)^{n-k-j},$$

$$\widetilde{F}^T(u,v;B_1)\;=\;f\left(\sqrt{u},\,\sqrt{(1-u)\,v}\right),$$

where $B_1=\{(x,y)\in\mathbb{R}^2:\,x^2+y^2\leqslant 1,\,x,y\geqslant 0\}$. Then,

$$\widetilde{\mathscr{C}}_n^T[f(x,y),B_1]=\sum_{k=0}^{n}\sum_{j=0}^{n-k} f\left(\sqrt{\frac{k}{n}},\sqrt{\frac{j}{n}}\right)\binom{n}{k\;\;j}x^{2k}y^{2j}(1-x^2-y^2)^{n-k-j}.$$

(i) For $x\in[-1,0]$, let $\tau(x)=-x^2$ and, for each fixed value of x, let $\sigma_x(y)=y^2/\sqrt{1-x^2}$. Let $n_k=k$, $\phi_1(x)=0$, and $\phi_2(x)=\sqrt{1-x^2}$. Then,

$$\widetilde{p}_{n,k}(-x^2;[-1,0])=\binom{n}{k}(1-x^2)^k\,x^{2n-2k},$$

$$\widetilde{p}_{k,j}(\sigma_x(y);[\phi_1,\phi_2]) = \frac{1}{(1-x^2)^k}\binom{k}{j} y^{2j}\left(1-x^2-y^2\right)^{k-j},$$
$$\widetilde{F}^T(u,v;B_2) = f(-\sqrt{1-u},\ \sqrt{u\,v}),$$

where $B_2 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 1,\ x\leqslant 0,\ y\geqslant 0\}$. Then,

$$\mathscr{C}_n^T[f(x,y),B_2] = \sum_{k=0}^{n}\sum_{j=0}^{k} f\left(-\sqrt{1-\frac{k}{n}},\sqrt{\frac{j}{n}}\right)\binom{n}{k}\binom{k}{j} x^{2n-2k}y^{2j}(1-x^2-y^2)^{k-j}.$$

(iii) For $x\in[-1,0]$, let $\tau(x)=-x^2$ and, for each fixed value of x, let $\sigma_x(y) = -y^2/\sqrt{1-x^2}$. Let $n_k = k$, $\phi_1(x) = -\sqrt{1-x^2}$, and $\phi_2(x)=0$. Then,

$$\widetilde{p}_{n,k}(-x^2;[-1,0]) = \binom{n}{k}(1-x^2)^k\, x^{2n-2k},$$

$$\widetilde{p}_{k,j}(\sigma_x(y);[\phi_1,\phi_2]) = \frac{1}{(1-x^2)^k}\binom{k}{j}\left(1-x^2-y^2\right)^j y^{2k-2j},$$
$$\widetilde{F}^T(u,v;B_3) = f\left(-\sqrt{1-u},-\sqrt{u\,(1-v)}\right),$$

where $B_3 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 1,\ x,y\leqslant 0\}$. Then,

$$\mathscr{C}_n^T[f(x,y),B_3] = \sum_{k=0}^{n}\sum_{j=0}^{k} f\left(-\sqrt{1-\frac{k}{n}},-\sqrt{\frac{k-j}{n}}\right)\binom{n}{k}\binom{k}{j} x^{2n-2k}y^{2k-2j}(1-x^2-y^2)^j.$$

(iv) For $x\in[0,1]$, let $\tau(x)=x^2$ and, for each fixed value of x, let $\sigma_x(y) = -y^2/\sqrt{1-x^2}$. Let $n_k = n-k$, $\phi_1(x) = -\sqrt{1-x^2}$, and $\phi_2(x)=0$. Then,

$$\widetilde{p}_{n,k}(x^2;[0,1]) = \binom{n}{k} x^{2k}\,(1-x^2)^{n-k},$$

$$\widetilde{p}_{n-k,j}(\sigma_x(y);[\phi_1,\phi_2]) = \frac{1}{(1-x^2)^{n-k}}\binom{n-k}{j}(1-x^2-y^2)^j\, y^{2n-2k-2j},$$
$$\widetilde{F}^T(u,v;B_4) = f\left(\sqrt{u},\ -\sqrt{(1-u)\,(1-v)}\right),$$

where $B_4 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leqslant 1,\ x\geqslant 0,\ y\leqslant 0\}$. Then,

$$\mathscr{C}_n^T[f(x,y),B_4] = \sum_{k=0}^{n}\sum_{j=0}^{n-k} f\left(\sqrt{\frac{k}{n}},-\sqrt{1-\frac{k+j}{n}}\right)\binom{n}{k\ \ j} x^{2k}(1-x^2-y^2)^j\, y^{2n-2k-2j}.$$

Similar to (4.1), we can define a piece-wise Bernstein-type operator on $\mathbf{B}^2$ as follows:

$$\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2] = \begin{cases} \widetilde{\mathscr{C}}_n^T[f(x,y),B_1], & (x,y)\in B_1, \\ \widetilde{\mathscr{C}}_n^T[f(x,y),B_2], & (x,y)\in B_2, \\ \widetilde{\mathscr{C}}_n^T[f(x,y),B_3], & (x,y)\in B_3, \\ \widetilde{\mathscr{C}}_n^T[f(x,y),B_4], & (x,y)\in B_4. \end{cases} \tag{4.5}$$

The proof of the following proposition is similar to that of Proposition 4.2.1.

Proposition 4.4.4. For any function f on $\mathbf{B}^2$, $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$ is a continuous function on $\mathbf{B}^2$.

4.5 Better order of approximation

In [8], Butzer studied a certain linear combination of univariate Bernstein operators that, under certain conditions, give a better order of approximation compared with the classical operators. For a bounded function $f(x)$ defined on $[0,1]$, Butzer considers the linear combination of Bernstein polynomials

$$\mathfrak{L}_n^{[2k]}f(x) = \alpha_k\, B_{2^k n}f(x) + \alpha_{k-1}\, B_{2^{k-1}n}f(x) + \cdots + \alpha_0\, B_n f(x), \quad k \geqslant 0,$$

where the constants $\alpha_j = \alpha_j(k)$ satisfy

$$\alpha_k + \alpha_{k-1} + \cdots + \alpha_0 = 1.$$

The polynomials $\mathfrak{L}_n^{[2k]}f(x)$ satisfy the recurrence relation

$$\begin{aligned} &(2^k-1)\mathfrak{L}_n^{[2k]}f(x) = 2^k\mathfrak{L}_{2n}^{[2k-2]}f(x) - \mathfrak{L}_n^{[2k-2]}f(x), \\ &\mathfrak{L}_n^{[0]}f(x) = B_n f(x), \end{aligned} \tag{4.6}$$

and, if $f^{(2k)}$ exists at a point $x \in [0,1]$, then

$$\mathfrak{L}_n^{[2k-2]}f(x) - f(x) = \mathcal{O}\left(n^{-k}\right), \quad k \geqslant 0.$$

Using (4.6), we can obtain the following explicit expressions for the constants α_j's,

$$\alpha_j = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{2^j}{2^j - 2^i}, \quad 0 \leqslant j \leqslant k. \tag{4.7}$$

In [35], May considers a slightly more general operator $\mathcal{S}_n^{[k]}[f(x); d_0, d_1, \ldots, d_k] \equiv \mathcal{S}_n^{[k]}[f(x); d_j]$ given by

$$\begin{aligned} \mathcal{S}_n^{[k]}[f(x); d_j] &= \sum_{j=0}^{k} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}\ B_{d_j n} f(x), \\ \mathcal{S}_n^{[0]}[f(x); d_j] &= B_{d_0 n} f(x), \end{aligned} \tag{4.8}$$

where $d_0, d_1, \ldots, d_k$ are positive integers. Notice that if $d_j = 2^j$, then $\mathcal{S}_n^{[k]}[f(x); d_j] = \mathfrak{L}_n^{[2k]} f(x)$. Moreover, if $d_j = j+1$, then $\mathcal{S}_n^{[2^k-1]}[f(x); j+1]$ is a polynomial of degree $2^k n$. However, May proved that if $f^{(2^{k+1})}$ exists, then $\mathcal{S}_n^{[2^k-1]}[f(x), j+1]$ converges to $f(x)$ at a rate of n^{-2^k} and, hence, $\mathcal{S}_n^{[2^k-1]}[f(x), j+1]$ and $\mathfrak{L}_n^{[2k]} f(x)$ are polynomials of the same degree, but $\mathcal{S}_n^{[2^k-1]}[f(x), j+1]$ has a faster rate of convergence than $\mathfrak{L}_n^{[2k]} f(x)$. Observe that for $k = 1$, we have $\mathcal{S}_n^{[2^k-1]}[f(x); j+1] = \mathcal{S}_n^{[k]}[f(x); 2^j]$.

Motivated by the construction in (4.8), we define the following bivariate operators

$$\begin{aligned} \widetilde{\mathcal{S}}_n^{[k]}[f(x,y); d_j] &= \sum_{j=0}^{k} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}\ \widetilde{\mathscr{B}}_{d_j n}[f(x,y), \mathbf{B}^2], \\ \widetilde{\mathcal{S}}_n^{[0]}[f(x,y); d_j] &:= \widetilde{\mathscr{B}}_{d_0 n}[f(x,y), \mathbf{B}^2], \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \widetilde{R}_n^{[k]}[f(x,y), d_j] &:= \sum_{j=0}^{k} \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}\ \overline{\mathscr{C}}_{d_j n}[f(x,y), \mathbf{B}^2], \\ \widetilde{\mathcal{R}}_n^{[0]}[f(x,y); d_j] &:= \overline{\mathscr{C}}_{d_0 n}[f(x,y), \mathbf{B}^2]. \end{aligned} \tag{4.10}$$

Although we do not study the approximation behavior of these operators here, the numerical experiments in the following section suggest a better rate of convergence than $\widetilde{\mathscr{B}}_n$ and $\overline{\mathscr{C}}_n$.

4.6 Numerical experiments

In this section, we present numerical experiments where we compare the shifted Bernstein-Stancu operator $\widetilde{\mathscr{B}}_n$ on $\mathbf{B}^2$, and the shifted Bernstein-type operator $\overline{\mathscr{C}}_n$ in (4.5). To do this, we consider different functions defined on $\mathbf{B}^2$. For each function $f(x,y)$, we compute $\widetilde{\mathscr{B}}_n[f(x,y), \mathbf{B}^2]$ and

$\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$. We use a set of points randomly distributed on the unit disk (generated by *mesh* function in Mathematica®) to compare the function to its approximations. For $\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$, we use 630 points (x_i,y_i). We set $z_i = f(x_i,y_i)$, $1 \leqslant i \leqslant 630$, and $\hat{z}_i$ equal to the value of $\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ at the respective point (x_i,y_i), and compute the root mean square error (RMSE) as follows:

$$\mathrm{RMSE}(f,\widetilde{\mathscr{B}}_n) = \sqrt{\sum_{i=1}^{630} \frac{(z_i-\hat{z}_i)^2}{630}}.$$

Similarly, for $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$, we use randomly distributed 1082 points $(\bar{x}_j,\bar{y}_j)$. We set $w_j = f(\bar{x}_j,\bar{y}_j)$, $1 \leqslant j \leqslant 1082$, and $\bar{w}_j$ equal to the value of $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$ at the respective point $(\bar{x}_j,\bar{y}_j)$, and compute the RMSE as follows:

$$\mathrm{RMSE}(f,\overline{\mathscr{C}}_n) = \sqrt{\sum_{j=1}^{1082} \frac{(w_j-\bar{w}_j)^2}{1082}}.$$

In each case, we plot the RSME for increasing values of n using Mathematica®. For each operator, the set of points used to compute the RSME consists of a fixed number of points. On the other hand, the number of mesh points used to represent each operator depends on n. We represent $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$ on each quadrant using different colors as shown in Figure 4.3. We take $n = 100$, then the mesh required to obtain the operator for each quadrant consists 20200 points. For $\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$, we take $n = 200$. Then the mesh required to obtain the operator for all the unit disk consists of 40401 points.

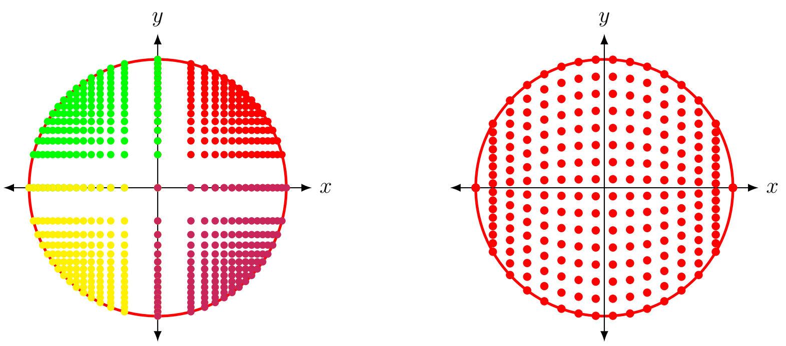

Figure 4.3: Left: Mesh for $\overline{\mathscr{C}}_n$ with $n = 15$. Color code for disk quadrants (B_1 red; B_2 green; B_3 yellow; B_4 purple). Right: Mesh for $\widetilde{\mathscr{B}}_n$ with $n = 15$.

We note that the operator $\overline{\mathscr{C}}_n$ requires two evaluations at the mesh points on the common boundaries of two adjacent quadrants. Therefore, the operator $\widetilde{\mathscr{B}}_n$ needs a smaller number of evaluations than the operator $\overline{\mathscr{C}}_n$ since, for a fixed n, $\widetilde{\mathscr{B}}_n$ and $\overline{\mathscr{C}}_n$ are composed of $(n+1)^2$ and $2\,(n+1)\,(n+2)$ evaluations, respectively. Additionally, we compute the RMSE for $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y),2^j]$ and $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^j]$ using the same set of randomly distributed points as before.

From definitions (4.9) and (4.10), we have

$$\begin{aligned}\widetilde{\mathcal{S}}_n^{[1]}[f(x,y);2^j] &= 2\,\widetilde{\mathscr{B}}_{2n}[f(x,y),\mathbf{B}^2]-\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2],\\ \widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^2] &= 2\,\overline{\mathscr{C}}_{2n}[f(x,y),\mathbf{B}^2]-\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2].\end{aligned}$$

For a fixed n, the mesh required to obtain $\widetilde{\mathscr{B}}_{2n}$ (respectively, $\overline{\mathscr{C}}_{2n}$) is a refinement of the mesh required for $\widetilde{\mathscr{B}}_n$ (respectively, $\overline{\mathscr{C}}_n$). Therefore, $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^j]$ and $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y),2^j]$ require $(2n+1)^2$ and $2\,(2n+1)\,(2n+2)$ evaluations, respectively.

Example 1

First, we consider the continuous function

$$f(x,y)=x\sin(5x-6y)+y,\quad (x,y)\in\mathbf{B}^2.$$

The graph of $f(x,y)$ is shown in Figure 4.4, and the approximations $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$ and $\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$ are shown in Figure 4.5. We list the RSME of both approximations for different values of n in Table 4.1 and plot them together in Figure 4.6, where the characteristic slow convergence inherited from the univariate Bernstein operators is observed. Moreover, the corresponding RSME are shown in Table 4.2 and Figure 4.7 for $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y),2^j]$ and $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^j]$, where a seemingly better approximation behavior can be observed.

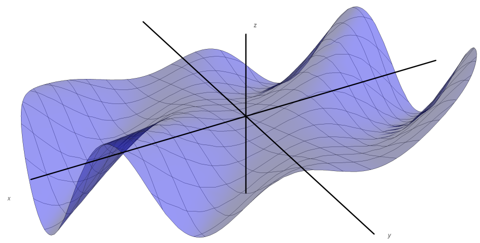

Figure 4.4: Graph of $f(x,y) = x\sin(5x-6y)+y$ on $\mathbf{B}^2$.

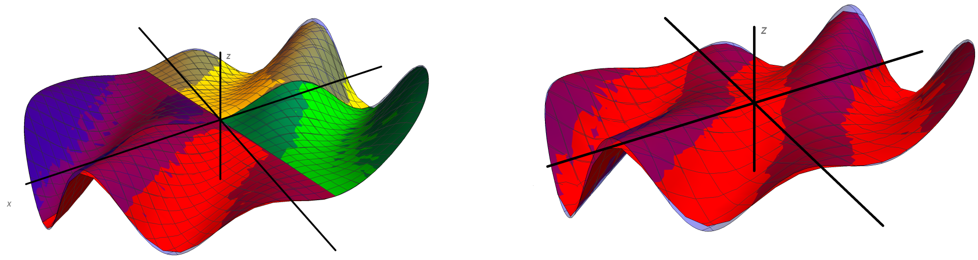

Figure 4.5: Approximations overlapped with the graph of $f(x,y)$. Left: $\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$. Right: $\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$.

n	$\overline{\mathscr{C}}_n[f(x,y),\mathbf{B}^2]$	$\widetilde{\mathscr{B}}_n[f(x,y),\mathbf{B}^2]$
10	0.191411	0.30623
20	0.117881	0.209091
30	0.0860663	0.16182
40	0.0682511	0.132416
50	0.0568288	0.112151
60	0.0488602	0.0972969
70	0.0429694	0.0859318
80	0.0384267	0.0769527

Table 4.1: RMSE for different values of n.

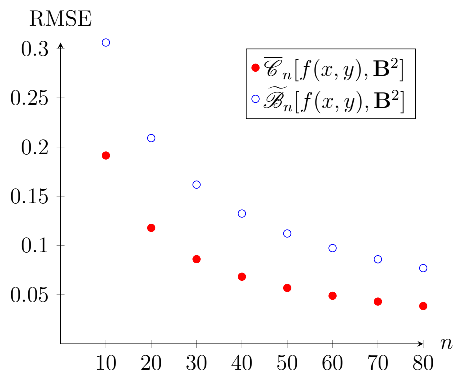

Figure 4.6: Plot of RMSE in Table 4.1.

n	$\widetilde{\mathcal{R}}_n^{[1]}[f(x,y);2^j]$	$\widetilde{\mathcal{S}}_n^{[1]}[f(x,y);2^j]$
10	0.0592126	0.14241
20	0.0306588	0.0666483
30	0.0217412	0.0389884
40	0.0171251	0.0258265
50	0.014188	0.0185806
60	0.0121117	0.0141754
70	0.0105426	0.0112949
80	0.00929964	0.009302

Table 4.2: RMSE for different values of n.

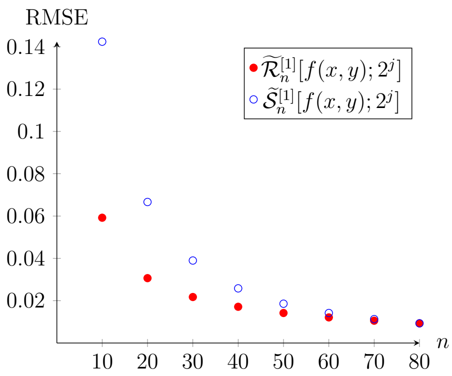

Figure 4.7: Plot of RMSE in Table 4.2.

Example 2

Now, we consider the continuous periodic function

$$g(x,y) = \sin(10x + y), \quad (x,y) \in \mathbf{B}^2.$$

Its graph is shown in Figure 4.8. It can be observed in Figure 4.9 that the approximation error for both operators is larger at the maximum and minimum values of the function. Table 4.3 and Figure 4.10 contain further evidence of this larger error. Moreover, in comparison with the previous example, it seems that the rate convergence of $\overline{\mathscr{C}}_n[g(x,y),\mathbf{B}^2]$ is significantly faster than the rate of convergence of $\widetilde{\mathscr{B}}_n[g(x,y),\mathbf{B}^2]$. Table 4.4 and Figure 4.11 show the errors corresponding to $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^j]$ and $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y),2^j]$. In comparison to $\widetilde{\mathscr{B}}_n$ and $\overline{\mathscr{C}}_n$, $\widetilde{\mathcal{R}}_n^{[1]}$ and $\widetilde{\mathcal{S}}_n^{[1]}$ appear to have a better approximation behavior.

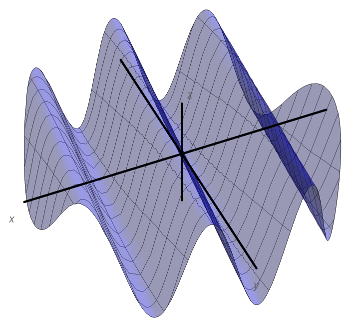

Figure 4.8: Graph of $g(x,y) = \sin(10x + y)$ on $\mathbf{B}^2$.

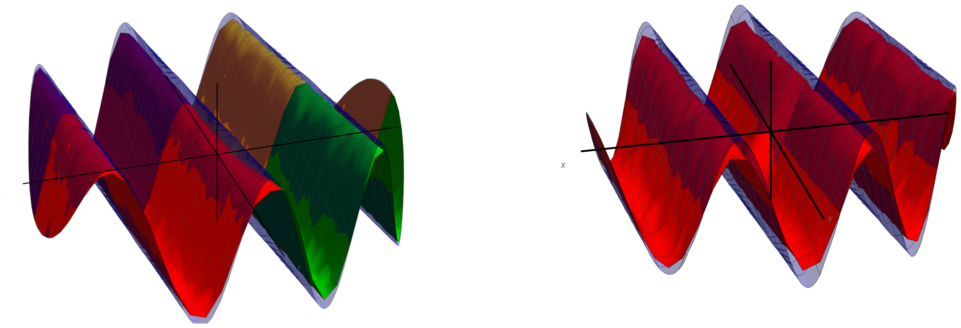

Figure 4.9: Approximations overlapped with the graph of $g(x,y)$. Left: $\overline{\mathscr{C}}_n[g(x,y),\mathbf{B}^2]$. Right: $\widetilde{\mathscr{B}}_n[g(x,y),\mathbf{B}^2]$.

n	$\overline{\mathscr{C}}_n[g(x,y),\mathbf{B}^2]$	$\widetilde{\mathscr{B}}_n[g(x,y),\mathbf{B}^2]$
10	0.535344	0.700146
20	0.366915	0.613427
30	0.278477	0.526227
40	0.225091	0.454904
50	0.189454	0.398559
60	0.163967	0.353775
70	0.144812	0.317628
80	0.129872	0.287968

Table 4.3: RMSE for different values of n.

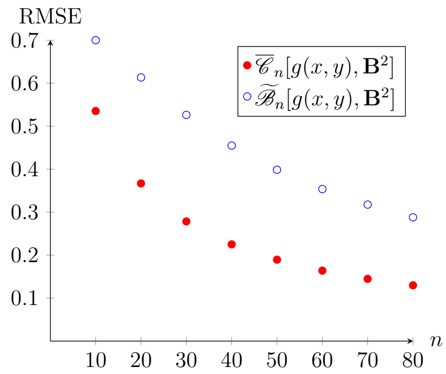

Figure 4.10: Plot of RMSE in Table 4.3.

n	$\widetilde{\mathcal{R}}_n^{[1]}[g(x,y);2^j]$	$\widetilde{\mathcal{S}}_n^{[1]}[g(x,y);2^j]$
10	0.231422	0.53511
20	0.116893	0.300211
30	0.0799078	0.18369
40	0.061706	0.122657
50	0.0506578	0.0873721
60	0.043083	0.0652876
70	0.0374756	0.0505948
80	0.0331074	0.0403428

Table 4.4: RMSE for different values of n.

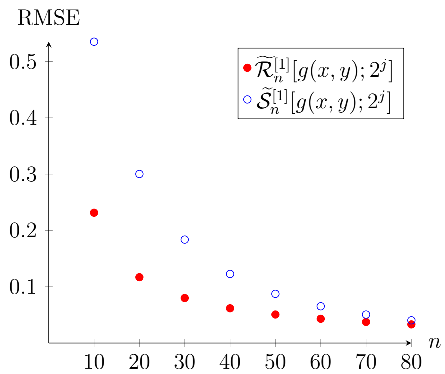

Figure 4.11: Plot of RMSE in Table 4.4.

Example 3

Here, we consider the continuous function

$$h(x, y) = e^{x^2 - y^2} - xy, \quad (x, y) \in \mathbf{B}^2,$$

(see Figure 4.12). Both approximations are shown in Figure 4.13, and their respective RSME are listed in Table 4.5 and plotted in Figure 4.14. Observe that, in this case, the RSME for both approximations are significantly smaller than in the previous examples. Moreover, based on Figure 4.14, it seems that for sufficiently large values of n, the rate of convergence of both approximations is considerably similar to each other. Table 4.6 and Figure 4.15 also show similar approximation behavior between $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y), 2^j]$ and $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y), 2^j]$.

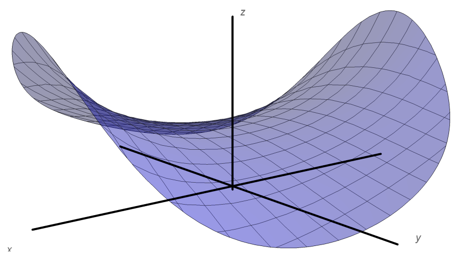

Figure 4.12: Graph of $h(x, y) = e^{x^2 - y^2} - xy$.

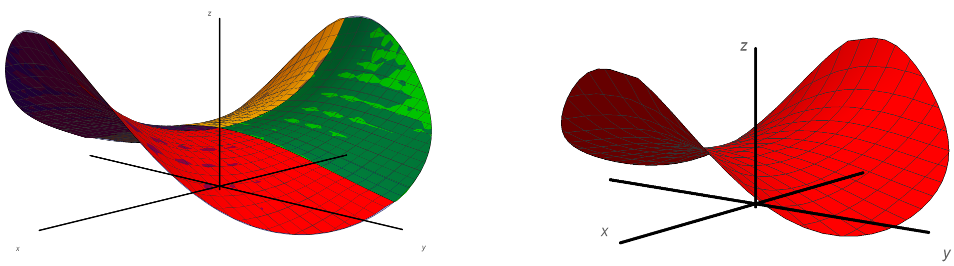

Figure 4.13: Approximations overlapped with the graph of $h(x, y)$. Left: $\overline{\mathscr{C}}_n[h(x, y), \mathbf{B}^2]$. Right: $\widetilde{\mathscr{B}}_n[h(x, y), \mathbf{B}^2]$.

n	$\overline{\mathscr{C}}_n[h(x,y),\mathbf{B}^2]$	$\widetilde{\mathscr{B}}_n[h(x,y),\mathbf{B}^2]$
10	0.0505862	0.140837
20	0.0293585	0.0685387
30	0.0213945	0.0455634
40	0.017105	0.0342737
50	0.0143844	0.0275514
60	0.0124871	0.0230843
70	0.0110789	0.0198962
80	0.00998647	0.0175041

Table 4.5: RMSE for different values of n.

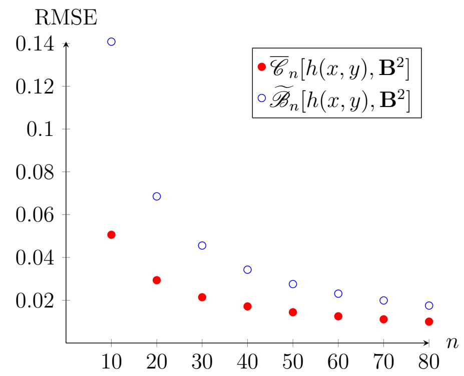

Figure 4.14: Plot of RSME in Table 4.5.

n	$\widetilde{\mathcal{R}}_n^{[1]}[h(x,y);2^j]$	$\widetilde{\mathcal{S}}_n^{[1]}[h(x,y);2^j]$
10	0.015677	0.0188267
20	0.00931191	0.0087729
30	0.0068375	0.00610222
40	0.00545586	0.00479186
50	0.00455264	0.00399096
60	0.00390752	0.00343958
70	0.00341937	0.0030301
80	0.00303472	0.00270966

Table 4.6: RMSE for different values of n.

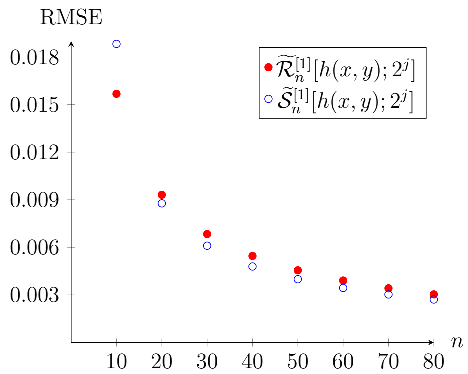

Figure 4.15: Plot of RMSE in Table 4.6.

Example 4

In this numerical example, we are interested in observing the behavior of Bernstein-type and Bernstein-Stancu operators at jump discontinuities.

Let us consider the following discontinuous function:

$$\eta(x,y)=\begin{cases}1, & \text{if } x^2+y^2<0.5,\\ 0, & \text{if } 0.5\leqslant x^2+y^2\leqslant 0.8,\\ 0.5, & \text{if } 0.8<x^2+y^2\leqslant 1.\end{cases}$$

The graph of $\eta(x,y)$ is shown in Figure 4.16 and the approximations are shown in Figure 4.13. It is interesting to observe the behavior of the approximations at the points of jump discontinuities and, thus, we have included Figure 4.18, where we show a cross sectional view of the approximations with increasing values of n. As in the univariate case, it seems that the Gibbs phenomenon does not occur. Finally, Table 4.7 and Figure 4.19 expose a significantly slow convergence rate for this discontinuous function in comparison with the previous continuous examples. As can be seen in Table 4.8 and Figure 4.20, it seems that a better approximation can be obtained with the operators $\widetilde{\mathcal{S}}_n^{[1]}[f(x,y),2^j]$ and $\widetilde{\mathcal{R}}_n^{[1]}[f(x,y),2^j]$.

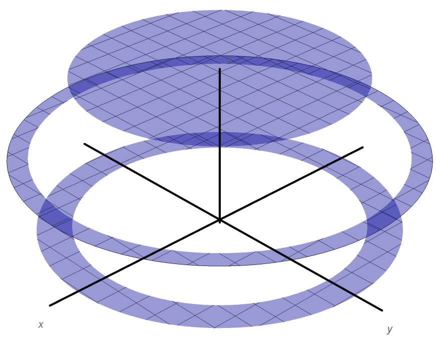

Figure 4.16: Graph of $\eta(x,y)$.

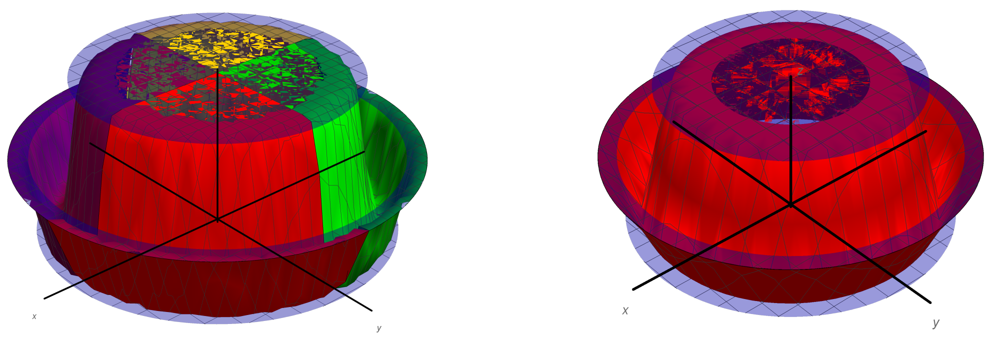

Figure 4.17: Approximations overlapped with the graph of $\eta(x,y)$. Left: $\overline{\mathscr{C}}_n[\eta(x,y),\mathbf{B}^2]$. Right: $\widetilde{\mathscr{B}}_n[\eta(x,y),\mathbf{B}^2]$.

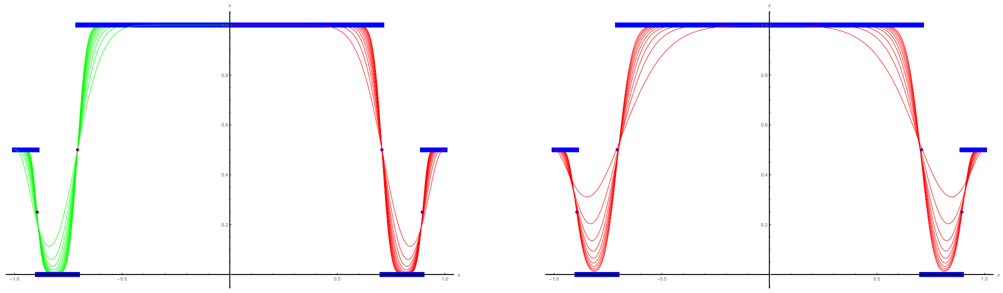

Figure 4.18: Cross sectional view of the approximations for increasing values of n. Left: $\overline{\mathscr{C}}_n[\eta(x,y),\mathbf{B}^2]$. Right: $\widetilde{\mathscr{B}}_n[\eta(x,y),\mathbf{B}^2]$.

n	$\overline{\mathscr{C}}_n[\eta(x,y),\mathbf{B}^2]$	$\widetilde{\mathscr{B}}_n[\eta(x,y),\mathbf{B}^2]$
10	0.216588	0.270366
20	0.175754	0.243468
30	0.156563	0.223305
40	0.144805	0.210916
50	0.136559	0.205949
60	0.130305	0.192988
70	0.125319	0.193887
80	0.121205	0.187675

Table 4.7: RMSE for different values of n.

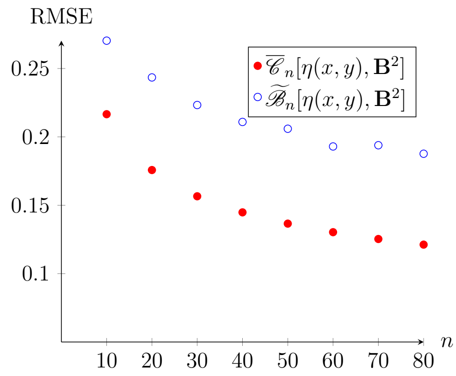

Figure 4.19: Plot of RSME in Table 4.7.

n	$\widetilde{\mathcal{R}}_n^{[1]}[\eta(x,y);2^j]$	$\widetilde{\mathcal{S}}_n^{[1]}[\eta(x,y);2^j]$
10	0.173582	0.203794
20	0.132669	0.149787
30	0.121201	0.140103
40	0.108429	0.122498
50	0.102915	0.120874
60	0.0959724	0.111816
70	0.0923721	0.111529
80	0.0878217	0.105521

Table 4.8: RMSE for different values of n.

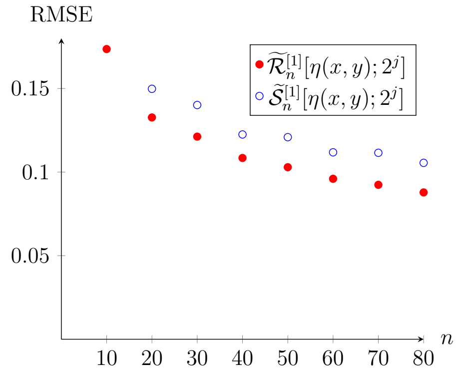

Figure 4.20: Plot of RMSE in Table 4.8.

Future work

Finally, we present several open problems that we intend to address in future research. We have listed these problems and provided a brief description of each.

P1 In Chapter 2, the inner product

$$\langle f,g\rangle_{\nabla,\mu}=f(0)g(0)+\lambda\int_{\mathbf{B}^d}\nabla f(x)\cdot\nabla g(x)\,(1-\|x\|^2)^\mu\,dx,$$

is explored, under the restrictions $\mu\geqslant 1$ and $\mu=0$. Several results are obtained, but a question that arises is whether similar results can be obtained in the case where $0<\mu<1$.

P2 In Chapter 2, is it possible to establish an estimate for $\|f-S_n^{\nabla,\mu}f\|_\mu$ or perhaps $\|f-S_n^{\nabla,\mu}f\|_{\mu+1}$?

P3 The following inner products may be considered.

$$\langle f,g\rangle_{\Delta,0}=f(0)g(0)+\int_{\mathbf{B}^d}\Delta[(1-\|x\|^2)f(x)]\,\Delta[(1-\|x\|^2)g(x)]\,dx,$$

and

$$\begin{aligned}\langle f,g\rangle_{\Delta,\mu}=&\lambda\int_{\mathbf{S}^{d-1}}f(\xi)\,g(\xi)\,d\sigma(\xi)\\&+\int_{\mathbf{B}^d}\Delta[(1-\|x\|^2)f(x)]\,\Delta[(1-\|x\|^2)g(x)](1-\|x\|^2)^\mu\,dx.\end{aligned}$$

The preservation of the form of the bases with respect to these inner products compared to the Sobolev bases being analyzed is a natural question. Additionally, exploring the possibility of obtaining similar error bounds as those previously studied is worthwhile.

P4 In Chapter 4, Bernstein operators that are introduced in (4.9) and (4.10) seem to provide a better order of approximation, but a formal proof of this claim is necessary.

P5 The Bernstein-Jacobi operators is defined as ([41]),

$$B_n^{(\alpha,\beta)} f(x) = \sum_{k=0}^{n} \frac{\langle f, p_{n,k} \rangle_{(\alpha,\beta)}}{\langle 1, p_{n,k} \rangle_{(\alpha,\beta)}} p_{n,k}(x).$$

Can this operator be extended to some inner product on the ball?

P6 Study the approximations through numerical experiments using data collected from optical measurement devices.

P7 The concept of a coherent pair was initially introduced by Iserles et al. ([23]), there are several generalizations of this concept, so it may be interesting to investigate whether coherent pairs exist with respect to the Sobolev inner products under study.

Appendix A

Mathematica® Codes

In this Section, we provide the Mathematica® codes used to generate the plots and calculate the errors presented earlier. These codes cover various aspects, such as Fourier series computation, polar mesh discretization, and numerical integration. The codes are presented in a clear and concise manner, with comments and explanations where necessary, to allow readers to easily understand and modify them for their own purposes. These codes have been made by using Mathematica® version 12.

A.1 Code used in Chapter 2

A.1.1 Code for $\mu > 0$

```
In[1]:= Clear[n]
In[2]:= m=5;
In[3]:= \[Mu]=1.5;
Function, g is the same as f used to evaluate the polar mesh.
In[4]:= f[x_,y_]:=x*Sin[5x-6y]+y
g[x_]:=f[x[[1]]*Cos[x[[2]]],x[[1]]*Sin[x[[2]]]]
Graph of the function over the unit disk.
In[6]:= PReal=Plot3D[f[x,y],{x,-1,1},{y,-1,1},AxesStyle->Thick,
    PlotRange-> All,RegionFunction->Function[{x,y,z},x^2+y^2<=
```

```
    1],Boxed->False,  AxesStyle->Thick,  AxesOrigin ->{0,0,0},
    AxesLabel->{x,y,z},PlotStyle->Directive[Opacity[0.4],Blue,
    Specularity[Black,30]],Ticks-> None]
Basis L2(W).
In[8]:= expn[j_,n_]:=If[n==2j,0,1];
In[9]:= P1[r_,\[Theta]_,n_,\[Mu]_,j_]:=Piecewise[{{r^(n-2j)*
    JacobiP[j,\[Mu],n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Cos[(n-2j)
    *\[Theta]],n!=2j},{JacobiP[j,\[Mu],0,2r^2-1],n==2j}}]
In[10]:= P2[r_,\[Theta]_,n_,\[Mu]_,j_]:=r^(n-2j)*JacobiP[j,\[Mu
    ],n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Sin[(n-2j)*\[Theta]]
Norm
In[11]:= h[j_,n_,\[Mu]_]:=Pochhammer[\[Mu]+1,j]*Pochhammer[1,n-j
    ](n-j+\[Mu]+1)/(j!*Pochhammer[\[Mu]+2,n-j]*(n+\[Mu]+1))
In[12]:= hs[j_,n_,\[Mu]_]:=(Pi/(\[Mu]))*(4j(n-j+\[Mu])+2(n-2j)
    *\[Mu])*h[j,n,\[Mu]-1]
Fourier coefficient L2(W)
The integral is numerically computed in polar coordinates.
In[13]:= S1[n_,j_,\[Mu]_]:=(1+\[Mu])/(Pi*h[j,n,\[Mu]])*Chop[
    NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*(1-r^2)^\[Mu
    ]*P1[r,\[Theta],n,\[Mu],j],{r,0,1},{\[Theta],0,2*Pi},Method->
    "LocalAdaptive",AccuracyGoal->8],10^-8];
The Fourier coefficients are stored in a matrix initially filled
     with zeros.
In[14]:= Coef1=Table[0,{j,0,m},{i,0,m/2}];
Calculation of the coefficients (first part)
In[15]:= For[i=0,i<=m,i=i+1,For[j=0,j<= i/2,j=j+1,Coef1[[i+1,j
    +1]]=S1[i,j,\[Mu]]]]//Timing
Out[15]= {5.9375,Null}
In[16]:= MatrixForm[Coef1]

In[17]:= S2[n_,j_,\[Mu]_]:=(1+\[Mu])/(Pi*h[j,n,\[Mu]])*Chop[
    NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*(1-r^2)^\[Mu
    ]*P2[r,\[Theta],n,\[Mu],j],{r,0,1},{\[Theta],0,2*Pi},Method->
    "LocalAdaptive",AccuracyGoal->8],10*^-8];
In[18]:= Coef2=Table[0,{j,0,m},{i,0,m/2}];
Calculation of the coefficients (second part)
In[19]:= For[i=0,i<=m,i=i+1,For[j=0,j<i/2,j=j+1,Coef2[[i+1,j
    +1]]=S2[i,j,\[Mu]]]]//Timing
Out[19]= {5.875,Null}
In[20]:= MatrixForm[Coef2]
```

```
Discretization of mesh to calculate errors.
In[21]:= xx={};
For[jj=0,jj<2*Pi,jj=jj+1.0*Pi/36,For[ii=0,ii<1,ii=ii+0.05;xx=
    Join[xx,{{ii,jj}}]]];
In[23]:= Length[xx];
In[24]:= zzReal=g/@xx;
Fourier series (double summation with coefficients multiplying
    the basis elements)
In[25]:= Sx[r_]:=Sum[Sum[Coef1[[n+1,j+1]]*P1[r[[1]],r[[2]],n,\[
    Mu],j],{j,0,n/2}],{n,0,m}]+Sum[Sum[Coef2[[n+1,j+1]]*P2[r
    [[1]],r[[2]],n,\[Mu],j],{j,0,n/2}],{n,0,m}]
Evaluation of the series on the polar mesh
In[26]:= aa=f[0,0]-Limit[Sx[{x,y}],{x->0,y-> 0}];
In[27]:= zz1=Sx/@xx;
In[28]:= bb1=zz1;
In[29]:= For[i=1,i<=Length[zz1],i++,bb1[[i]]=Join[{xx[[i,1]]*Cos
    [xx[[i,2]]],xx[[i,1]]*Sin[xx[[i,2]]]},{zz1[[i]]}]];
In[30]:= Imag1=ListPlot3D[bb1,InterpolationOrder->8,PlotRange->
    All,PlotStyle->Red,Boxed->False, AxesStyle->Thick,
    RegionFunction->Function[{x,y,z},x^2+y^2<= 1],AxesOrigin
    ->{0,0,0},Ticks-> None,AxesLabel->{x,y,z}];
Plotting of the function with the series
In[31]:= Show[Imag1,PReal]
Out[31]=
Sobolev Basis
In[32]:= Q1[r_,\[Theta]_,n_,\[Mu]_,j_]:=P1[r,\[Theta],n,\[Mu]-1,
    j]-P1[0,0,n,\[Mu]-1,j];
In[33]:= Q2[r_,\[Theta]_,n_,\[Mu]_,j_]:=P2[r,\[Theta],n,\[Mu]-1,
    j];
Fourier coefficient L2(\[Del],W)
In[34]:= S11[n_,j_,\[Mu]_]:=(\[Mu])/(Pi*h[j,n,\[Mu]-1])*Chop[
    NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*(1-r^2)^(\[Mu
    ]-1)*P1[r,\[Theta],n,\[Mu]-1,j],{r,0,1},{\[Theta],0,2*Pi},
    Method->"LocalAdaptive",AccuracyGoal->8],10*^-8];
In[35]:= Coef11=Table[0,{j,0,m},{i,0,m/2}];
In[36]:= For[i=1,i<=m,i=i+1,For[j=0,j<=i/2,j=j+1,Coef11[[i+1,j
    +1]]=S11[i,j,\[Mu]]]]//Timing
Out[36]= {7.67188,Null}
In[37]:= MatrixForm[Coef11]
```

```
In[38]:= S22[n_,j_,\[Mu]_]:=(\[Mu])/(Pi*h[j,n,\[Mu]-1])*Chop[
    NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*(1-r^2)^(\[Mu
    ]-1)*P2[r,\[Theta],n,\[Mu]-1,j],{r,0,1},{\[Theta],0,2*Pi},
    Method->"LocalAdaptive",AccuracyGoal->8],10*^-8];
In[39]:= Coef22=Table[0,{j,0,m},{i,0,m/2}];
In[40]:= For[i=1,i<=m, i=i+1,For[j=0,j<i/2,j=j+1,Coef22[[i+1,j
    +1]]=S22[i,j,\[Mu]]]]//Timing
Out[40]= {6.59375,Null}
In[41]:= MatrixForm[Coef22]

In[42]:= Sxx[r_]:=f[0,0]+Sum[Sum[Coef11[[n+1,j+1]]*Q1[r[[1]],r
    [[2]],n,\[Mu],j],{j,0,n/2}],{n,1,m}]+Sum[Sum[Coef22[[n+1,j
    +1]]*Q2[r[[1]],r[[2]],n,\[Mu],j],{j,0,n/2}],{n,1,m}]
In[43]:= zz2=Sxx/@xx;
In[44]:= bb2=zz2;
In[45]:= For[i=1,i<=Length[zz2],i++,bb2[[i]]=Join[{xx[[i,1]]*Cos
    [xx[[i,2]]],xx[[i,1]]*Sin[xx[[i,2]]]},{zz2[[i]]}]];
In[46]:= Imag2=ListPlot3D[bb2,InterpolationOrder->8,PlotRange->
    All,PlotStyle->Red,Boxed->False, AxesStyle->Thick,
    RegionFunction->Function[{x,y,z},x^2+y^2<= 1],AxesOrigin
    ->{0,0,0},Ticks-> None,AxesLabel->{x,y,z}];
In[47]:= Show[Imag2,PReal]
Out[47]=
Errors
In[48]:= aa=f[0,0]-Limit[Sxx[{x,y}],{x->0,y-> 0}];
In[49]:= aaa=f[0,0]-Limit[Sx[{x,y}],{x->0,y-> 0}];
In[50]:= Error1=(zz1-zzReal)^2;
In[51]:= Error2=(zz2-zzReal)^2;
In[52]:= Error1=Join[Error1,{aaa^2}];
In[53]:= Error2=Join[Error2,{aa^2}];
Comparison of errors
In[54]:= Sqrt[Total[(Error1)/(Length[Error1]) ]]
Out[54]= 0.292973
In[55]:= Sqrt[Total[(Error2)/(Length[Error2]) ]]
Out[55]= 0.291705
```

A.1.2 Code for $\mu = 0$

```
In[1]:= Clear[n]
In[2]:= m=5
Out[2]= 5
In[3]:= \[Lambda]=1;
Funcion, g es la misma f usada para evaluar el mallado en
    polares
In[4]:=
f[x_,y_]:=x*Sin[5x-6y]+y
g[x_]:=f[x[[1]]*Cos[x[[2]]],x[[1]]*Sin[x[[2]]]]
In[6]:=
Grafico de la funcion sobre el circulo unidad
In[7]:= PReal=Plot3D[f[x,y],{x,-1,1},{y,-1,1},AxesStyle->Thick,
    PlotRange-> All,RegionFunction->Function[{x,y,z},x^2+y^2<=
    1],Boxed->False, AxesStyle->Thick, AxesOrigin ->{0,0,0},
    AxesLabel->{x,y,z},PlotStyle->Directive[Opacity[0.4],Blue,
    Specularity[Black,30]],Ticks-> None]
Out[7]=
Exponente para generar el 2 y 1 que multiplica en el 2.3.6
In[8]:= expn[j_,n_]:=If[n==2j,0,1];
Base L2(W).
In[9]:= P1[r_,\[Theta]_,n_,\[Mu]_,j_]:=r^(n-2j)*JacobiP[j,\[Mu],
    n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Cos[(n-2j)*\[Theta]]
In[10]:= P11[r_,\[Theta]_,n_,j_]:=(1-r^2)*r^(n-2j)*JacobiP[j
    -1,1,n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Cos[(n-2j)*\[Theta]]
In[11]:= P2[r_,\[Theta]_,n_,\[Mu]_,j_]:=r^(n-2j)*JacobiP[j,\[Mu
    ],n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Sin[(n-2j)*\[Theta]]
In[12]:= P22[r_,\[Theta]_,n_,j_]:=(1-r^2)*r^(n-2j)*JacobiP[j
    -1,1,n-2j,2r^2-1]*(Sqrt[2^(expn[j,n])])*Sin[(n-2j)*\[Theta]]
In[13]:= P3[r_,n_]:=4/(n)*(JacobiP[n/2,-1,0,2r^2-1]-(-1)^(n/2)*
    Pochhammer[1,n/2]/(n/2)!)
Base esfericos armonicos
In[14]:= Y1[r_,\[Theta]_,n_,j_]:=r^(n-2j)*(Sqrt[2^(expn[j,n])])
    Cos[(n-2j)*\[Theta]]
In[15]:= Y2[r_,\[Theta]_,n_,j_]:=r^(n-2j)*(Sqrt[2^(expn[j,n])])*
    Sin[(n-2j)*\[Theta]]
In[16]:=
Coeficiente de Fourier L2(W,0)
La integral se realiza de forma numerica en coordenadas polares
```

```
In[17]:= C01[n_]:=1/(2Pi)*Chop[NIntegrate[f[Cos[\[Theta]],Sin[\[
    Theta]]]*Sqrt[2]*Cos[n*\[Theta]],{\[Theta],0,2Pi},Method->"
    LocalAdaptive",AccuracyGoal->8],10^-8];
In[18]:= C02[n_]:=1/(2Pi)*Chop[NIntegrate[f[Cos[\[Theta]],Sin[\[
    Theta]]]*Sqrt[2]*Sin[n*\[Theta]],{\[Theta],0,2Pi},Method->"
    LocalAdaptive",AccuracyGoal->8],10^-8];
In[19]:= C11[n_,j_]:=Chop[n/j*(2*(n-j)/(2Pi)*NIntegrate[f[r*Cos
    [\[Theta]],r*Sin[\[Theta]]]*r*P1[r,\[Theta],n-2,1,j-1],{r
    ,0,1},{\[Theta],0,2*Pi},Method->"LocalAdaptive",AccuracyGoal
    ->8]-1/(2Pi)*NIntegrate[f[Cos[\[Theta]],Sin[\[Theta]]]*Y1
    [1,\[Theta],n,j],{\[Theta],0,2Pi},Method->"LocalAdaptive",
    AccuracyGoal->8]),10^-8];

In[20]:= C12[n_,j_]:=Chop[n/j*(2*(n-j)/(2Pi)*NIntegrate[f[r*Cos
    [\[Theta]],r*Sin[\[Theta]]]*r*P2[r,\[Theta],n-2,1,j-1],{r
    ,0,1},{\[Theta],0,2*Pi},Method->"LocalAdaptive",AccuracyGoal
    ->8]-1/(2Pi)*NIntegrate[f[Cos[\[Theta]],Sin[\[Theta]]]*Y2
    [1,\[Theta],n,j],{\[Theta],0,2Pi},Method->"LocalAdaptive",
    AccuracyGoal->8]),10^-8];

In[24]:= C21[n_,j_]:=Chop[n/(4Pi)*(NIntegrate[f[Cos[\[Theta]],
    Sin[\[Theta]]],{\[Theta],0,2Pi},Method->"LocalAdaptive",
    AccuracyGoal->8]-n*NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta
    ]]]*r*P1[r,\[Theta],n-2,1,n/2-1],{r,0,1},{\[Theta],0,2*Pi},
    Method->"LocalAdaptive",AccuracyGoal->8]),10^-8];
In[25]:=
Los coeficientes se guardan en una matriz inicialmente de Ceros
In[26]:= Coef11=Table[0,{j,0,m},{i,0,m/2}];
In[27]:= Coef12=Table[0,{j,0,m},{i,0,m/2}];
In[28]:= Coef13={};
Elementos cuando j=0
In[29]:= For[i=1,i<=m, i=i+1,Coef11[[i+1,1]]=C01[i]]
In[30]:= For[i=1,i<=m, i=i+1,Coef12[[i+1,1]]=C02[i]]
In[31]:= MatrixForm[Coef11]
MatrixForm[Coef12]
Elementos 1<= j<= (n-1)/2
In[33]:= kk=0;
In[34]:= For[i=1,i<=m, i=i+1,For[j=1,j<=i/2,j=j+1,If[j==i/2,(
    Coef13=Join[Coef13,{C21[i,j]}];kk=kk+1),Coef11[[i+1,j+1]]=C11
    [i,j]]]];
```

```
In[35]:= MatrixForm[Coef11]

In[37]:= For[i=1,i<=m, i=i+1,For[j=1,j<i/2,j=j+1,Coef12[[i+1,j
    +1]]=C12[i,j]]];
In[38]:=
In[39]:= MatrixForm[Coef12]

Discretizacion para calcular los errores
In[40]:= xx={};
For[jj=0,jj<2*Pi,jj=jj+1.0*Pi/36,For[ii=0,ii<1,ii=ii+0.05;xx=
    Join[xx,{{ii,jj}}]]];
In[42]:= Length[xx]
Out[42]= 1440
In[43]:= zzReal=g/@xx;
Serie
In[44]:= Sxx[r_]:=f[0,0]+Sum[Coef11[[n+1,1]]*r[[1]]^n*Sqrt[2]*
    Cos[n*r[[2]]],{n,1,m}]+Sum[Coef12[[n+1,1]]*r[[1]]^n*Sqrt[2]*
    Sin[n*r[[2]]],{n,1,m}]+Sum[Sum[Coef11[[n+1,j+1]]*P11[r[[1]],r
    [[2]],n,j],{j,1,n/2}],{n,1,m}]+Sum[Sum[Coef12[[n+1,j+1]]*P22[
    r[[1]],r[[2]],n,j],{j,1,n/2}],{n,1,m}]+Sum[Coef13[[1,nn]]*P3[
    r[[1]],2*nn],{nn,1,kk}];
In[45]:= MatrixForm[Coef13]
Out[45]//MatrixForm= (0.156581
0.32778

)
In[46]:= zz2=Sxx/@xx;
In[47]:= bb2=zz2;
In[48]:= For[i=1,i<=Length[zz2],i++,bb2[[i]]=Join[{xx[[i,1]]*Cos
    [xx[[i,2]]],xx[[i,1]]*Sin[xx[[i,2]]]},{zz2[[i]]}]];
In[49]:= Imag2=ListPlot3D[bb2,InterpolationOrder->8,PlotRange->
    All,PlotStyle->Red,Boxed->False, AxesStyle->Thick,
    RegionFunction->Function[{x,y,z},x^2+y^2<= 1],AxesOrigin
    ->{0,0,0},Ticks-> None,AxesLabel->{x,y,z}];
In[50]:= Show[Imag2,PReal]
Out[50]=
In[51]:= Imag2
Out[51]=
In[52]:= aa=f[0,0]-Limit[Sxx[{x,y}],{x->0,y-> 0}];
In[53]:=
Errores
```

```
In[54]:= Error2=(zz2-zzReal)^2;
In[55]:= Error2=Join[Error2,{aa^2}];
Errores
In[56]:= Sqrt[Total[(Error2)/(Length[Error2])  ]]
Out[56]= 0.290001
```

A.2 Code used in Chapter 3

```
In[1421]:= Clear[n]
In[1422]:= m=5;
Funcion, g es la misma f usada para evaluar el mallado en
    polares
In[1423]:= f1[x_,y_]:=0.75*Exp[-1/4*(9x-2)^2-1/4*(9y-2)^2]+0.75*
    Exp[-1/49*(9x+1)^2-1/10*(9x+1)]+0.5*Exp[-1/4*(9x-7)^2-1/4*(9y
    -3)^2]-0.2*Exp[-(9x-4)^2-(9y-7)^2]
f2[x_,y_]:=x*Exp[-x^2-y^2]
f3[x_,y_]:=Cos[x+y]*Exp[x*y]
f[x_,y_]:=f3[(x+1)/2,(y+1)/2]
f[x_,y_]:=f3[(x+1)/2,(y+1)/2]
f[x_,y_]:=f1[(x+1)/2,(y+1)/2]
f[x_,y_]:=f2[2x,2y]
g[x_]:=f[x[[1]]*Cos[x[[2]]],x[[1]]*Sin[x[[2]]]]
In[1431]:=
Grafico de la funcion sobre el circulo unidad
In[1432]:= PReal=Plot3D[f[x,y],{x,-1,1},{y,-1,1},AxesStyle->
    Thick,PlotRange-> All,RegionFunction->Function[{x,y,z},x^2+y
    ^2<= 1],Boxed->False, AxesStyle->Thick, AxesOrigin ->{0,0,0},
    AxesLabel->{x,y,z},PlotStyle->Directive[Opacity[0.4],Blue,
    Specularity[Black,30]],Ticks-> None]

Out[1432]=
In[1433]:= expn[j_,n_]:=If[n==2j,0,1];
Norma
In[1434]:= hj[j_,a_,b_]:=2^(a+b+1)*Gamma[j+a+1]*Gamma[j+b+1]/((2
    j+a+b+1)*j!*Gamma[j+a+b+1])

In[1435]:= H[j_,n_]:=j^2*(j+1)^2*hj[j,0,n-2j]/(2^(n-2j+1))

Base Laplaciano
In[1436]:= Y1[r_,\[Theta]_,n_,j_]:=(Sqrt[2^(expn[j,n])])*r^(n-2j
    )*Cos[(n-2j)*\[Theta]]
In[1437]:= Y2[r_,\[Theta]_,n_,j_]:=(Sqrt[2^(expn[j,n])])*r^(n-2j
    )*Sin[(n-2j)*\[Theta]]
In[1438]:= Q1[r_,\[Theta]_,n_,j_]:=(1-r^2)*JacobiP[j-1,2,n-2j,2r
    ^2-1]*Y1[r,\[Theta],n,j]
In[1439]:= Q2[r_,\[Theta]_,n_,j_]:=(1-r^2)*JacobiP[j-1,2,n-2j,2r
```

```
^2-1]*Y2[r,\[Theta],n,j]
Coeficientes de Fourier : La integral se realiza de forma
    numerica en coordenadas polares
In[1440]:= C01[n_]:=Chop[1/(2Pi)*NIntegrate[f[Cos[\[Theta]],Sin
    [\[Theta]]]*Y1[1,\[Theta],n,0],{\[Theta],0,2Pi},Method->"
    LocalAdaptive",AccuracyGoal->8],10^-8];
In[1441]:= C02[n_]:=Chop[1/(2Pi)*NIntegrate[f[Cos[\[Theta]],Sin
    [\[Theta]]]*Y2[1,\[Theta],n,0],{\[Theta],0,2Pi},Method->"
    LocalAdaptive",AccuracyGoal->8],10^-8];
In[1442]:= C11[n_,j_]:=Chop[2j (j+1)/(2Pi*H[j,n])*(-1/2*
    NIntegrate[f[Cos[\[Theta]],Sin[\[Theta]]]*Y1[1,\[Theta],n,j
    ],{\[Theta],0,2Pi},Method->"LocalAdaptive",AccuracyGoal->8]+(
    n-j)*(n-j+1)*NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*
    Q1[r,\[Theta],n,j],{r,0,1},{\[Theta],0,2*Pi},Method->"
    LocalAdaptive",AccuracyGoal->8]),10^-8];
In[1443]:= C12[n_,j_]:=Chop[2j (j+1)/(2Pi*H[j,n])*(-1/2*
    NIntegrate[f[Cos[\[Theta]],Sin[\[Theta]]]*Y2[1,\[Theta],n,j
    ],{\[Theta],0,2Pi},Method->"LocalAdaptive",AccuracyGoal->8]+(
    n-j)*(n-j+1)*NIntegrate[f[r*Cos[\[Theta]],r*Sin[\[Theta]]]*r*
    Q2[r,\[Theta],n,j],{r,0,1},{\[Theta],0,2*Pi},Method->"
    LocalAdaptive",AccuracyGoal->8]),10^-8];
In[1444]:=
Los coeficientes se guardan en una matriz inicialmente de Ceros
In[1445]:= Coef11=Table[0,{j,0,m},{i,0,m/2}];
In[1446]:= Coef12=Table[0,{j,0,m},{i,0,m/2}];
Elementos cuando j=0
In[1447]:= For[i=0,i<=m, i=i+1,Coef11[[i+1,1]]=C01[i]]
In[1448]:= For[i=0,i<=m, i=i+1,Coef12[[i+1,1]]=C02[i]]
In[1449]:= MatrixForm[Coef11];
MatrixForm[Coef12];
Elementos 1<= j<= (n-1)/2
In[1451]:= For[i=0,i<=m, i=i+1,For[j=1,j<=i/2,j=j+1,Coef11[[i+1,j
    +1]]=C11[i,j]]];

In[1452]:= MatrixForm[Coef11];
In[1453]:= For[i=0,i<=m, i=i+1,For[j=1,j<i/2,j=j+1,Coef12[[i+1,j
    +1]]=C12[i,j]]];
In[1454]:= MatrixForm[Coef12];
Discretizacion para calcular los errores
In[1455]:= xx={};
For[jj=0,jj<2*Pi,jj=jj+1.0*Pi/36,For[ii=0,ii<1,ii=ii+0.05;xx=
```

```
    Join[xx,{{ii,jj}}]]];
In[1457]:=  Length[xx];
zzReal  = g  /@ xx;
Serie
In[1458]:=  Sxx[r_]:=Sum[Coef11[[n+1,1]]*Y1[r[[1]],r[[2]],n,0],{n
    ,0,m}]+Sum[Coef12[[n+1,1]]*Y2[r[[1]],r[[2]],n,0],{n,0,m}]+Sum
    [Sum[Coef11[[n+1,j+1]]*Q1[r[[1]],r[[2]],n,j],{j,1,n/2}],{n,0,
    m}]+Sum[Sum[Coef12[[n+1,j+1]]*Q2[r[[1]],r[[2]],n,j],{j,1,n
    /2}],{n,0,m}];
In[1459]:=  zz2=Sxx/@xx;
In[1460]:=  bb2=zz2;
In[1461]:=  For[i=1,i<=Length[zz2],i++,bb2[[i]]=Join[{xx[[i,1]]*
    Cos[xx[[i,2]]],xx[[i,1]]*Sin[xx[[i,2]]]},{zz2[[i]]}]];
In[1462]:=  Imag2=ListPlot3D[bb2,InterpolationOrder->8,PlotRange
    -> All,PlotStyle->Red,Boxed->False,  AxesStyle->Thick,
    RegionFunction->Function[{x,y,z},x^2+y^2<=  1],AxesOrigin
    ->{0,0,0},Ticks-> None,AxesLabel->{x,y,z}];
In[1463]:=  Show[Imag2,PReal]
Out[1463]=
In[1464]:=  Imag2
Out[1464]=
Errores
In[1465]:=  aa=f[0,0]-Limit[Sxx[{x,y}],{x->0,y-> 0}];
In[1466]:=  Error=(zz2-zzReal)^2;
In[1467]:=  Error=Join[Error,{aa^2}];
Errores
In[1468]:=  Sqrt[Total[(Error)/(Length[Error])  ]]
Out[1468]=  0.542849
```

A.3 Code used in Chapter 4

A.3.1 Code for errors of $\overline{\mathscr{C}}_n[f, \mathbf{B}^2]$ and $\widetilde{\mathscr{B}}_n[f, \mathbf{B}^2]$

```
In[1]:= m=30
Out[1]= 30
Definicion de las funciones
In[2]:= f[x_,y_]:=x*Sin[5x-6y]+y;
g[x_]:=x[[1]]*Sin[5*x[[1]]-6*x[[2]]]+x[[2]]

In[4]:= Binom1[n_,k_,j_]:=Binomial[n,k]*Binomial[n-k,j]*(f[Sqrt[
    k/n],Sqrt[j/n]]);
Binom2[n_,k_,j_]:=Binomial[n,k]*Binomial[n-k,j]*(f[-Sqrt[k/n],
    Sqrt[j/n]]);
Binom3[n_,k_,j_]:=Binomial[n,k]*Binomial[n-k,j]*(f[-Sqrt[k/n],-
    Sqrt[j/n]]);
Binom4[n_,k_,j_]:=Binomial[n,k]*Binomial[n-k,j]*(f[Sqrt[k/n],-
    Sqrt[j/n]]);
Binomdisc[n_,m_,k_,j_]:=2^(-m-n)*Binomial[n,k]*Binomial[m,j]*f[2
    k/n-1,(2 Sqrt[-((k (k-n))/1)] (2 j-m))/(m*n)];
Mallado
In[9]:= xx1=MeshCoordinates[DiscretizeRegion[Disk
    [],{{0,1},{0,1}}]];
xx2=MeshCoordinates[DiscretizeRegion[Disk[],{{-1,0},{0,1}}]];
xx3=MeshCoordinates[DiscretizeRegion[Disk[],{{-1,0},{-1,0}}]];
xx4=MeshCoordinates[DiscretizeRegion[Disk[],{{0,1},{-1,0}}]];
In[13]:= xx=Join[xx1,xx2,xx3,xx4];
In[14]:= xxx1={};
For[jj=Pi/36,jj<Pi,jj=jj+Pi/36,For[ii=0.1,ii<1,ii=ii+0.1;xxx1=
    Join[xxx1,{{ii*Cos[jj],ii*Sin[jj]}}]]];
xxx2={};
For[jj=Pi+Pi/36,jj<2*Pi,jj=jj+Pi/36,For[ii=0.1,ii<1,ii=ii+0.1;
    xxx2=Join[xxx2,{{ii*Cos[jj],ii*Sin[jj]}}]]];

In[18]:= xxx=Join[xxx1,xxx2];
In[19]:= zzReal1=g/@xx;
zzReal2=g/@xxx;
Operador on disk cuadrants
```

```
In[21]:= BST1[x_]:=Sum[Sum[Binom1[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)
    *(1-x[[1]]^2 - x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom1[n,k,0]x
    [[1]]^(2k)*(1-x[[1]]^2 - x[[2]]^2)^(n-k)+Binom1[n,k,n-k]*x
    [[1]]^(2k)*x[[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom1[n,0,j]*x
    [[2]]^(2j)*(1-x[[1]]^2 - x[[2]]^2)^(n-j),{j,1,n-1}]+Binom1[n
    ,0,0]*(1 - x[[1]]^2 - x[[2]]^2)^n+Binom1[n,0,n]*x[[2]]^(2n)+
    Binom1[n,n,0]x[[1]]^(2n);
BST2[x_]:=Sum[Sum[Binom2[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom2[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2 - x[[2]]^2)^(n-k)+Binom2[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom2[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-j),{j,1,n-1}]+Binom2[n,0,0]*(1 - x[[1]]^2 -
    x[[2]]^2)^n+Binom2[n,0,n]*x[[2]]^(2n)+Binom2[n,n,0]x[[1]]^(2n
    );
BST3[x_]:=Sum[Sum[Binom3[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom3[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2 - x[[2]]^2)^(n-k)+Binom3[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom3[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-j),{j,1,n-1}]+Binom3[n,0,0]*(1 - x[[1]]^2 -
    x[[2]]^2)^n+Binom3[n,0,n]*x[[2]]^(2n)+Binom3[n,n,0]x[[1]]^(2n
    );
BST4[x_]:=Sum[Sum[Binom4[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom4[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2 - x[[2]]^2)^(n-k)+Binom4[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom4[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2 - x[[2]]^2)^(n-j),{j,1,n-1}]+Binom4[n,0,0]*(1 - x[[1]]^2 -
    x[[2]]^2)^n+Binom4[n,0,n]*x[[2]]^(2n)+Binom4[n,n,0]x[[1]]^(2n
    );
In[25]:=
Operador on all disk
In[26]:= BSTD[x_]:=Sum[Sum[Binomdisc[n,n,k,j]*(1+x[[1]])^(k)*(1-
    x[[1]])^(n-k)*(1-x[[1]]^2)^(-(n)/2)*(Sqrt[1-x[[1]]^2]+x[[2]])
    ^(j)*(Sqrt[1-x[[1]]^2] - x[[2]])^(n-j),{j,0,n}],{k,0,n}];
In[27]:= error4={};
error1={};
tamano1={};
tamano4={};
Errores y Comparacion
In[31]:= For[n=10,n<=m,n=n+10,zz1=BST1/@xx1;zz2=BST2/@xx2;zz3=
    BST3/@xx3;zz4=BST4/@xx4;zz=Join[zz1,zz2,zz3,zz4];tamano1=Join
    [tamano1,List[n]];error4=Join[error4,List[Sqrt[Total[(zz-
```

```
    zzReal1)^2/Length[zz]  ]]]]]
In[32]:=  For[n=10,n<=m,n=n+10,zzz=BSTD/@xxx;tamano4=Join[tamano4
    ,List[n]];error1=Join[error1 ,List[Sqrt[Total[(zzz−zzReal2)^2/
    Length[zzz]  ]]]]]
ListPlot[{Transpose[{tamano4,error1}],Transpose[{tamano1,error4
    }]},AxesLabel−>{"n","RMSE"},PlotLegends−>Placed[LineLegend[{
    Row[{"Operator on all disk"}],Row[{"Operators on disk
    quadrants"}]},LegendFunction−>"Frame"],{0.7,0.85}],PlotStyle
    −>{PointSize[0.10],PointSize[0.10]},PlotMarkers−>{Graphics[{
    Blue,Circle[]},ImageSize−>8],Graphics[{Red,Disk[]},ImageSize
    −>8]}]
```

A.3.2 Code for errors of better approximations: $\widetilde{\mathcal{R}}_n^{[1]}[f;2^j]$ and $\widetilde{\mathcal{S}}_n^{[1]}[f;2^j]$

```
In[1]:=  m=10
Out[1]=  10
In[2]:=  f[x_,y_]:=Piecewise[{{−1,x<−0.5},{−0.5,−0.5<=x<0},{0,0<=
    x<0.5},{1,True}}]
g[x_]:=Piecewise[{{−1,x[[1]]<−0.5},{−0.5,−0.5<=x[[1]]<0},{0,0<=x
    [[1]]<0.5},{1,True}}]
In[4]:=  Binom1[n_,k_,j_]:=Binomial[n,k]∗Binomial[n−k,j]∗(f[Sqrt[
    k/n],Sqrt[j/n]]);
Binom2[n_,k_,j_]:=Binomial[n,k]∗Binomial[n−k,j]∗(f[−Sqrt[k/n],
    Sqrt[j/n]]);
Binom3[n_,k_,j_]:=Binomial[n,k]∗Binomial[n−k,j]∗(f[−Sqrt[k/n],−
    Sqrt[j/n]]);
Binom4[n_,k_,j_]:=Binomial[n,k]∗Binomial[n−k,j]∗(f[Sqrt[k/n],−
    Sqrt[j/n]]);
Binomdisc[n_,m_,k_,j_]:=2^(−m−n)∗Binomial[n,k]∗Binomial[m,j]∗f[2
    k/n−1,(2 Sqrt[−((k (k−n))/1)] (2 j−m))/(m∗n)];
In[10]:=  xx1=MeshCoordinates[DiscretizeRegion[Disk
    [],{{0,1},{0,1}}]];
xx2=MeshCoordinates[DiscretizeRegion[Disk[],{{−1,0},{0,1}}]];
xx3=MeshCoordinates[DiscretizeRegion[Disk[],{{−1,0},{−1,0}}]];
xx4=MeshCoordinates[DiscretizeRegion[Disk[],{{0,1},{−1,0}}]];
In[14]:=  xx=Join[xx1,xx2,xx3,xx4];
In[15]:=  xxx1={};
```

```
For[jj=Pi/36,jj<Pi,jj=jj+Pi/36,For[ii=0.1,ii<1,ii=ii+0.1;xxx1=
    Join[xxx1,{{ii*Cos[jj],ii*Sin[jj]}}]]];
xxx2={};
For[jj=Pi+Pi/36,jj<2*Pi,jj=jj+Pi/36,For[ii=0.1,ii<1,ii=ii+0.1;
    xxx2=Join[xxx2,{{ii*Cos[jj],ii*Sin[jj]}}]]];

In[19]:= xxx=Join[xxx1,xxx2];
In[20]:= zzReal1=g/@xx;
zzReal2=g/@xxx;
BST1[x_,n_]:=Sum[Sum[Binom1[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom1[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2-x[[2]]^2)^(n-k)+Binom1[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom1[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-j),{j,1,n-1}]+Binom1[n,0,0]*(1-x[[1]]^2-
    x[[2]]^2)^n+Binom1[n,0,n]*x[[2]]^(2n)+Binom1[n,n,0]x[[1]]^(2n
    );
BST2[x_,n_]:=Sum[Sum[Binom2[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom2[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2-x[[2]]^2)^(n-k)+Binom2[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom2[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-j),{j,1,n-1}]+Binom2[n,0,0]*(1-x[[1]]^2-
    x[[2]]^2)^n+Binom2[n,0,n]*x[[2]]^(2n)+Binom2[n,n,0]x[[1]]^(2n
    );
BST3[x_,n_]:=Sum[Sum[Binom3[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom3[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2-x[[2]]^2)^(n-k)+Binom3[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom3[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-j),{j,1,n-1}]+Binom3[n,0,0]*(1-x[[1]]^2-
    x[[2]]^2)^n+Binom3[n,0,n]*x[[2]]^(2n)+Binom3[n,n,0]x[[1]]^(2n
    );
BST4[x_,n_]:=Sum[Sum[Binom4[n,k,j]*x[[1]]^(2k)*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-k-j),{j,1,n-k-1}]+Binom4[n,k,0]x[[1]]^(2
    k)*(1-x[[1]]^2-x[[2]]^2)^(n-k)+Binom4[n,k,n-k]*x[[1]]^(2k)*x
    [[2]]^(2*(n-k)),{k,1,n-1}]+Sum[Binom4[n,0,j]*x[[2]]^(2j)*(1-x
    [[1]]^2-x[[2]]^2)^(n-j),{j,1,n-1}]+Binom4[n,0,0]*(1-x[[1]]^2-
    x[[2]]^2)^n+Binom4[n,0,n]*x[[2]]^(2n)+Binom4[n,n,0]x[[1]]^(2n
    );

Se define las mejores aproximaciones
In[27]:= BSbetter1[x_]:=2*BST1[x,2n]-BST1[x,n];
BSbetter2[x_]:=2*BST2[x,2n]-BST2[x,n];
```

```
BSbetter3[x_]:=2*BST3[x,2n]−BST3[x,n];
BSbetter4[x_]:=2*BST4[x,2n]−BST4[x,n];

In[31]:= BSTD[x_,n_]:=Sum[Sum[Binomdisc[n,n,k,j]*(1+x[[1]])^(k)
    *(1−x[[1]])^(n−k)*(1−x[[1]]^2)^(−(n)/2)*(Sqrt[1−x[[1]]^2]+x
    [[2]])^(j)*(Sqrt[1−x[[1]]^2] − x[[2]])^(n−j),{j,0,n}],{k,0,n}];
In[32]:= BSTDbetter[x_]:=2BSTD[x,2n]−BSTD[x,n];
In[33]:= error4={};
error1={};
tamano1={};
tamano4={};
In[37]:= For[n=10,n<=m,n=n+10,zz1=BSbetter1/@xx1;
zz2=BSbetter2/@xx2;
zz3=BSbetter3/@xx3;
zz4=BSbetter4/@xx4;zz=Join[zz1,zz2,zz3,zz4];tamano1=Join[tamano1
    ,List[n]];error4=Join[error4,List[Sqrt[Total[(zz−zzReal1)^2/
    Length[zz] ]]]]]
In[38]:= error4;
In[39]:= For[n=10,n<=m,n=n+10,zzz=BSTDbetter/@xxx;tamano4=Join[
    tamano4,List[n]];error1=Join[error1,List[Sqrt[Total[(zzz−
    zzReal2)^2/Length[zzz] ]]]]]
During evaluation of In[39]:= General::munfl: −3.68623*10^−61
    7.6274*10^−279 is too small to represent as a normalized
    machine number; precision may be lost.
During evaluation of In[39]:= General::munfl: −7.37246*10^−60
    1.22825*10^−264 is too small to represent as a normalized
    machine number; precision may be lost.
During evaluation of In[39]:= General::munfl: −7.00384*10^−59
    1.97788*10^−250 is too small to represent as a normalized
    machine number; precision may be lost.
During evaluation of In[39]:= General::stop: Further output of
    General::munfl will be suppressed during this calculation.
In[40]:= ListPlot[{Transpose[{tamano4,error1}],Transpose[{
    tamano1,error4}]},AxesLabel−>{"n","RMSE"},PlotLegends−>Placed
    [LineLegend[{Row[{"Operator on all disk"}],Row[{"Operators on
     disk quadrants"}]},LegendFunction−>"Frame"],{0.7,0.85}],
    PlotStyle−>{PointSize[0.10],PointSize[0.10]},PlotMarkers−>{
    Graphics[{Blue,Circle[]},ImageSize−>8],Graphics[{Red,Disk[]},
    ImageSize−>8]}]
```

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