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A Frobenius problem suggested by prime k-tuplets

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ABSTRACT

We study the Frobenius problem for certain k-tuplets, which include prime k-tuplets. Moreover, we analyze some properties of the numerical semigroups associated with these tuplets.

1. Introduction

A *prime triplet* is a sequence of three prime numbers (p_1, p_2, p_3) such that $p_1 < p_2 < p_3$ and $p_3 - p_1 = 6$. In particular, the sequences must be of the form $(p, p+2, p+6)$ or $(p, p+4, p+6)$. Since one of every three consecutive odd numbers is a multiple of three, and therefore not prime (except for three itself), we have that the prime triplets are the closest possible groupings of three prime numbers with the exceptions of $(2, 3, 5)$ and $(3, 5, 7)$.

Analogously, a *prime quadruplet* is a sequence (p_1, p_2, p_3, p_4), of four prime numbers, such that $p_1 < p_2 < p_3 < p_4$ and $p_4 - p_1 = 8$. In this case, the sequences must be of the form $(p, p+2, p+6, p+8)$ and are the closest possible groupings of four prime numbers with the exceptions of $(2, 3, 5, 7)$ and $(3, 5, 7, 11)$.

To formalize the concept of k-tuplet, let us follow [6]: if k is an integer greater than one, then $s(k)$ is the smallest number for which there exists a set of k ordered integers $B_k = \{b_1, \dots, b_k\}$ such that $b_1 = 0$, $b_k = s(k)$, and $\{b_1 \bmod q, \dots, b_k \bmod q\} \neq \{0, 1, \dots, q-1\}$ (that is, $\{b_1 \bmod q, \dots, b_k \bmod q\}$ is not a complete set of residues modulo q) for any prime number q. From here, a *prime k-tuplet* is a sequence of consecutive prime numbers, $P_k = (p_1, \dots, p_k)$, such that $p_i - p_1 = b_i$ for all $i \in \{1, \dots, k\}$ (observe that $p_k - p_1 = s(k)$). Thus, roughly speaking, a prime k-tuplet is a sequence of k consecutive prime numbers such that the difference between the first and the last (that is, $s(k)$) is as small as possible. Sometimes B_k is called an *admissible set* and P_k is named an *admissible constellation (of consecutive prime numbers)* (see [5]).

Let us observe that if we remove the condition on the function $s(k)$, we will have other admissible constellations. For example, $(p, p+4)$ (cousin primes), $(p, p+6)$ (sexy prime pairs), $(p, p+6, p+12)$ (sexy prime triplets), etcetera.

Let us also note that from the definition, it is easy to see that we can only obtain a finite number of sequences of k consecutive prime numbers $(p_1, \dots, p_k)$ such that $p_k - p_1 < s(k)$. In contrast, the generalized Hardy-Littlewood conjecture

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(see [8,9]) implies that, for every admissible set B_k, we can obtain infinitely many prime k-tuplets. In fact, all the current evidence confirm the conjecture for any admissible constellation (whether or not considering the condition on $s(k)$).

Let $(a_1, \ldots, a_e)$ be a sequence of positive integers such that $\gcd(a_1, \ldots, a_e) = 1$. Then a classical problem in additive number theory is the Frobenius problem: what is the greatest integer $\mathrm{F}(a_1, \ldots, a_e)$ that is not an element of the set $a_1\mathbb{N} + \ldots + a_e\mathbb{N}$? Although there exists a solution for this problem when $e = 2$ (see [18]), it is well-known that it is not possible to find a polynomial formula in order to compute $\mathrm{F}(a_1, \ldots, a_e)$ if $e \geq 3$ (see [4]). Therefore, many efforts have been devoted to obtaining partial results or developing algorithms to get the answer to this question (see [13]).

Among others, the main objective of this work is to find the solution to the Frobenius problem for (prime) triplets (Section 3) and (prime) quadruplets (Section 4). We also comment on some results for k-tuplets with $k \geq 5$ (Section 5). Moreover, from the computations, we conjecture that polynomials of degree two allow us to compute the Frobenius number of k-tuplets.

To achieve our purposes, we use the theory of numerical semigroups (closely related to the Frobenius problem) and, in particular, the Apéry set of a numerical semigroup (Section 2).

It is worth noting that the restriction on prime numbers is not essential to get the proposed results. In fact, in the statements on Apéry sets, Frobenius and pseudo-Frobenius numbers, and genus, we will consider k-tuples that only verify the condition of the admissible constellation; that is, if $\{p_1, p_2, \ldots, p_k\}$ is a k-tuplet, then $\{p_1 - p_1, p_2 - p_1, \ldots, p_k - p_1\}$ must be an admissible set.

2. Preliminaries (on numerical semigroups)

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N} = \{z \in \mathbb{Z} \mid z \geq 0\}$. A submonoid of $(\mathbb{N}, +)$ is a subset M of $\mathbb{N}$ that is closed under addition and contains the zero element. A *numerical semigroup* is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S = \{n \in \mathbb{N} \mid n \notin S\}$ is finite.

Let S be a numerical semigroup. Since $\mathbb{N} \setminus S$ is a finite set, we can define two invariants of S. Namely, the *Frobenius number of S* is the greatest integer that does not belong to S, denoted by $\mathrm{F}(S)$, and the *genus of S* is the cardinality of $\mathbb{N} \setminus S$, denoted by $\mathrm{g}(S)$. Let us note that, in number theory, it is common to use the term *Sylvester number* instead of the term genus (see [10]).

If X is a non-empty subset of $\mathbb{N}$, then we denote by $\langle X \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by X, that is,

$$\langle X \rangle = \left\{ \lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \setminus \{0\},\ x_1, \ldots, x_n \in X,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\}.$$

It is well-known (see Lemma 2.1 of [16]) that $\langle X \rangle$ is a numerical semigroup if and only if $\gcd(X) = 1$.

If S is a numerical semigroup and $S = \langle X \rangle$, then we say that X is a *system of generators of S*. Moreover, if $S \neq \langle Y \rangle$ for any subset $Y \subsetneq X$, then we say that X is a *minimal system of generators* of S. In Theorem 2.7 of [16], it is shown that each numerical semigroup admits a unique minimal system of generators and that such a system is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$, denoted by $\mathrm{e}(S)$, is the *embedding dimension* of S.

The (extended) Frobenius problem for a numerical semigroup S consists of finding formulas that allow us to compute $\mathrm{F}(S)$ and $\mathrm{g}(S)$ in terms of $\mathrm{msg}(S)$. As in the case of the Frobenius problem for sequences, such formulas are well-known for $\mathrm{e}(S) = 2$, but it is not possible to find polynomial formulas when $e(S) \geq 3$, except for particular families of numerical semigroups.

Now, let us define a useful tool to describe a numerical semigroup S. If $n \in S \setminus \{0\}$, then the *Apéry set of n in S* (named in honour of [1]) is $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$.

The following result is Lemma 2.4 of [16].

Proposition 2.1. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then the cardinality of* $\mathrm{Ap}(S, n)$ *is n. Moreover,*

$$\mathrm{Ap}(S, n) = \{w(0) = 0, w(1), \ldots, w(n-1)\},$$

where $w(i)$ is the least element of S congruent with i modulo n.

The knowledge of $\mathrm{Ap}(S, n)$ allows us to solve the problem of membership of an integer to the numerical semigroup S. In fact, if $x \in \mathbb{Z}$, then $x \in S$ if and only if $x \geq w(x \bmod n)$. Moreover, we have the following result from [3] (first formula) and [17] (second one).

Proposition 2.2. *Let S be a numerical semigroup and let $n \in S \setminus \{0\}$. Then*

1. $\mathrm{F}(S) = \max(\mathrm{Ap}(S, n)) - n$,
2. $\mathrm{g}(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$.

From this proposition, we have the solution to the Frobenius problem for S if we know an explicit description of $\mathrm{Ap}(S, n)$.

Remark 2.3. In [10,11], the author uses Apéry sets to compute the Frobenius number and the Sylvester number (that is, the genus) of numerical semigroups generated by arithmetic progressions and by arithmetic progressions with initial gaps. In particular, our corresponding results in Corollary 3.13 can be deduced from [11] (see Remarks 3.16 and 3.17).

Remark 2.4. We can consider different generalizations of the Frobenius number. For example, from [12] (and other references therein), it is possible to define the *p-Frobenius number* of a numerical semigroup $S = \langle a_1, a_2, \dots, a_e \rangle$ as the greatest integer that can be represented at most p ways by a linear combination with non-negative integer coefficients of $a_1, a_2, \dots, a_e$. Thus, the 0-Frobenius number is the classical Frobenius number. Since in [12] the author uses Apéry sets to obtain his results, a possible future work would be to compute the p-Frobenius number for k-tuplets.

Let S be a numerical semigroup. Following the notation introduced in [15], we say that an integer x is a *pseudo-Frobenius number of S* if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$. We denote by $\mathrm{PF}(S)$ the set of all the pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is a notable invariant of S (see [2]), the so-called *type* of S, denoted by $\mathrm{t}(S)$.

Let S be a numerical semigroup. We define over $\mathbb{Z}$ the following binary relation: $a \le_S b$ if $b - a \in S$. As stated in [16], it is clear that $\le_S$ is a non-strict partial order relation (that is, reflexive, transitive, and anti-symmetric).

The following result is Proposition 7 of [7] (see also Proposition 2.20 of [16]) and characterizes the pseudo-Frobenius numbers in terms of the maximal elements of $\mathrm{Ap}(S, n)$ with respect to the relation $\le_S$.

Proposition 2.5. *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\le_S}(\mathrm{Ap}(S, n))\}.$$

3. Triplets in the form of prime triplets

To have a well-posed Frobenius problem, we can consider six possibilities of triplets satisfying the condition of the admissible constellation (see [14]). However, since reasonings and calculations are similar, we will study in detail the two cases concerning prime triplets.

Let us recall that a prime triplet is of the form $(p, p+2, p+6)$ or of the form $(p, p+4, p+6)$. We improve this fact in the following proposition.

Proposition 3.1. *We have that:*

1. *If $(p, p+2, p+6)$ is a prime triplet, then $p = 6k+5$, with $k \in \mathbb{N}$.*
2. *If $(p, p+4, p+6)$ is a prime triplet, then $p = 6k+7$, with $k \in \mathbb{N}$.*

Proof. It is clear that if $p \in \mathbb{N} \setminus \{1\}$ is not divisible by two or three, then there exists $k \in \mathbb{N}$ such that $p = 6k+5$ or $p = 6k+7$.

1. If $p = 6k+7$, then $p+2 = 6k+9$ is multiple of three and can not be a prime number. Consequently, if $(p, p+2, p+6)$ is a prime triplet, then $p = 6k+5$.
2. The reasoning is similar to the above case. □

As a consequence of the previous proposition, a prime triplet is of one of the following two forms.

1. $(6k+5, 6k+7, 6k+11)$, with $k \in \mathbb{N}$.
2. $(6k+7, 6k+11, 6k+13)$, with $k \in \mathbb{N}$.

Since $\gcd(6k+5, 6k+7, 6k+11) = \gcd(6k+7, 6k+11, 6k+13) = 1$, we can define two families of numerical semigroups ($\mathcal{T}_1$ and $\mathcal{T}_2$).

- $S \in \mathcal{T}_1$ if $S = \langle 6k+5, 6k+7, 6k+11 \rangle$, with $k \in \mathbb{N}$.
- $S \in \mathcal{T}_2$ if $S = \langle 6k+7, 6k+11, 6k+13 \rangle$, with $k \in \mathbb{N}$.

It is easy to check that if $S \in \mathcal{T}_1 \cup \mathcal{T}_2$, then $\mathrm{e}(S) = 3$. Moreover, from [7], we deduce the following result.

Proposition 3.2. *Let S be a numerical semigroup such that $\mathrm{e}(S) = 3$. Then $\mathrm{t}(S) \in \{1, 2\}$. In addition, if $S = \langle n_1, n_2, n_3 \rangle$, where n_1, n_2, n_3 are pairwise relatively prime numbers, then $\mathrm{t}(S) = 2$.*

An immediate consequence of the above comments and results is the following proposition.

Proposition 3.3. *If $S \in \mathcal{T}_1 \cup \mathcal{T}_2$, then $\mathrm{e}(S) = 3$ and $\mathrm{t}(S) = 2$.*

3.1. First case (family $\mathcal{T}_1$)

In this subsection, we are interested in studying the numerical semigroups of the form $S = \langle 6k+5, 6k+7, 6k+11\rangle$, where $k \in \mathbb{N}$.

Straightforward computations lead to the following result.

Lemma 3.4. *If $k \in \mathbb{N}$, then we have the equalities*

1. $3(6k+7) = 2(6k+5) + 1(6k+11)$;
2. $(2k+2)(6k+11) = (2k+3)(6k+5) + 1(6k+7)$;
3. $2(6k+7) + (2k+1)(6k+11) = (2k+5)(6k+5)$.

Let us now see the key to this subsection.

Theorem 3.5. *If $k \in \mathbb{N}$ and $S = \langle 6k+5, 6k+7, 6k+11\rangle$, then*

$$\mathrm{Ap}(S, 6k+5) = \{a(6k+7) + b(6k+11) \mid (a,b) \in C\},$$

where $C = \big(\{0,1,2\} \times \{0,1,\dots,2k+1\}\big) \setminus \{(2, 2k+1)\}$.

Proof. From Lemma 3.4, we easily deduce that $\mathrm{Ap}(S, 6k+5)$ is a subset of $\{a(6k+7) + b(6k+11) \mid (a,b) \in C\}$. Now then, since the cardinality of C is less than or equal to $3(2k+2) - 1 = 6k+5$, and taking into consideration Proposition 2.1, we get that $\mathrm{Ap}(S, 6k+5) = \{a(6k+7) + b(6k+11) \mid (a,b) \in C\}$. □

Let us observe that we can rewrite several elements of $\mathrm{Ap}(S, 6k+5)$ as follows.

- $(6k+7) + b(6k+11) = (b+1)(6k+11) - 4$, for all $b \in \{0,1,\dots,2k+1\}$.
- $2(6k+7) + b(6k+11) = (b+2)(6k+11) - 8$, for all $b \in \{0,1,\dots,2k\}$.

Thus, we can easily describe the Apéry set by arranging its elements in increasing order.

Corollary 3.6. *If $k \in \mathbb{N}$ and $S = \langle 6k+5, 6k+7, 6k+11\rangle$, then*

$$\mathrm{Ap}(S, 6k+5) = \{0;\ (6k+11)-4, 6k+11;\ 2(6k+11)-8, 2(6k+11)-4, 2(6k+11);\ \dots;$$
$$(2k+1)(6k+11)-8, (2k+1)(6k+11)-4, (2k+1)(6k+11);\ (2k+2)(6k+11)-8, (2k+2)(6k+11)-4\}.$$

In this way, we can identify a pattern. In fact, in the previous corollary, we have used ";" to separate suitable groups of numbers.

As mentioned in Section 2, knowledge of the Apéry set allows us to obtain information about the numerical semigroup. Thus, in the current case, we have the following result. (Let us observe that, from Corollary 3.7, we recover Proposition 3.3.)

Corollary 3.7. *If $k \in \mathbb{N}$ and $S = \langle 6k+5, 6k+7, 6k+11\rangle$, then*

1. $\mathrm{PF}(S) = \{12k^2 + 28k + 9, 12k^2 + 28k + 13\}$;
2. $\mathrm{F}(S) = 12k^2 + 28k + 13$;
3. $\mathrm{g}(S) = 6k^2 + 16k + 8$.

Proof.
1. By Theorem 3.5, we have that $\mathrm{Maximals}_{\le_S}(\mathrm{Ap}(S, 6k+5)) = \{(6k+7) + (2k+1)(6k+11), 2(6k+7) + 2k(6k+11)\}$. Thereby, from Proposition 2.5, we can assert that $\mathrm{PF}(S) = \{(6k+7) + (2k+1)(6k+11) - (6k+5), 2(6k+7) + 2k(6k+11) - (6k+5)\} = \{12k^2 + 28k + 13, 12k^2 + 28k + 9\}$.
2. It is clear that $\mathrm{F}(S) = \max(\mathrm{PF}(S))$ and, therefore, $\mathrm{F}(S) = 12k^2 + 28k + 13$.
3. From Theorem 3.5, we have that

$$\mathrm{Ap}(S, 6k+5) = \{0, 6k+11, \dots, (1+2k)(6k+11), 6k+7, (6k+7)+(6k+11), \dots, (6k+7)+(1+2k)(6k+11),$$
$$2(6k+7), 2(6k+7) + (6k+11), \dots, 2(6k+7) + 2k(6k+11)\}.$$

Now, by Proposition 2.2, we get that

$$\mathrm{g}(S) = \frac{1}{6k+5}\Big(6k+11 + \cdots + (1+2k)(6k+11) + 6k+7 + (6k+7) + (6k+11) +$$

$$\cdots + (6k+7) + (1+2k)(6k+11) + 2(6k+7) + 2(6k+7) + (6k+11) +$$
$$\left.\cdots + 2(6k+7) + 2k(6k+11)\right) - \frac{(6k+5)-1}{2} = 6k^2 + 16k + 8. \quad \square$$

Remark 3.8. From Corollary 3.7, we deduce that if $p = 6k+5$ with $k \in \mathbb{N}$, then $\mathrm{F}(p, p+2, p+6) = \frac{p^2+4p-6}{3}$.

We finish with an illustrative example.

Example 3.9. If $k = 1$, then $S = \langle 11, 13, 17 \rangle$, $\mathrm{PF}(S) = \{49, 53\}$, $\mathrm{F}(S) = 53$, and $\mathrm{g}(S) = 30$ (by Corollary 3.7). Moreover, from Corollary 3.6, we know that $\mathrm{Ap}(S, 11) = \{0, 13, 17, 26, 30, 34, 43, 47, 51, 60, 64\}$.

3.2. Second case (family $\mathcal{T}_2$)

Now we study numerical semigroups of the form $S = \langle 6k+7, 6k+11, 6k+13 \rangle$, where $k \in \mathbb{N}$. The results are similar to those of Subsection 3.1. Therefore, we omit the proofs.

Lemma 3.10. *If $k \in \mathbb{N}$, then we have the equalities*

1. $3(6k+11) = 1(6k+7) + 2(6k+13)$;
2. $(2k+3)(6k+13) = (2k+4)(6k+7) + 1(6k+11)$;
3. $2(6k+11) + (2k+1)(6k+13) = (2k+5)(6k+7)$.

Theorem 3.11. *If $k \in \mathbb{N}$ and $S = \langle 6k+7, 6k+11, 6k+13 \rangle$, then*

$$\mathrm{Ap}(S, 6k+7) = \{a(6k+11) + b(6k+13) \mid (a,b) \in C\},$$

where $C = \big(\{0,1,2\} \times \{0,1,\ldots,2k+2\}\big) \setminus \{(2, 2k+1), (2, 2k+2)\}$.

Let us observe that

- $(6k+11) + b(6k+13) = (b+1)(6k+13) - 2$, for all $b \in \{0, 1, \ldots, 2k+2\}$;
- $2(6k+11) + b(6k+13) = (b+2)(6k+11) - 4$, for all $b \in \{0, 1, \ldots, 2k\}$.

Thus, we describe the Apéry set by arranging its elements in increasing order.

Corollary 3.12. *If $k \in \mathbb{N}$ and $S = \langle 6k+7, 6k+11, 6k+13 \rangle$, then*

$$\mathrm{Ap}(S, 6k+7) = \{0; (6k+13)-2, 6k+13; 2(6k+13)-4, 2(6k+13)-2, 2(6k+13); \ldots;$$
$$(2k+2)(6k+13)-4, (2k+2)(6k+13)-2, (2k+2)(6k+13); (2k+3)(6k+13)-2\}.$$

Corollary 3.13. *If $k \in \mathbb{N}$ and $S = \langle 6k+7, 6k+11, 6k+13 \rangle$, then*

1. $\mathrm{PF}(S) = \{12k^2 + 32k + 15, 12k^2 + 38k + 30\}$;
2. $\mathrm{F}(S) = 12k^2 + 38k + 30$;
3. $\mathrm{g}(S) = 6k^2 + 20k + 16$.

Remark 3.14. From Corollary 3.13, we have that if $p = 6k+7$ with $k \in \mathbb{N}$, then $\mathrm{F}(p, p+4, p+6) = \frac{p^2+5p+6}{3}$.

Let us see an example.

Example 3.15. If $k = 0$, then $S = \langle 7, 11, 13 \rangle$, $\mathrm{PF}(S) = \{15, 30\}$, $\mathrm{F}(S) = 30$, and $\mathrm{g}(S) = 16$ (by Corollary 3.13). Moreover, from Corollary 3.12, we get that $\mathrm{Ap}(S, 7) = \{0, 11, 13, 22, 24, 26, 37\}$.

Remark 3.16. The values of $\mathrm{F}(p, p+4, p+6)$ and $\mathrm{g}(p, p+4, p+6)$, for $p = 6k+7$ with $k \in \mathbb{N}$, can be obtained from Theorems 2 and 3 of [11], respectively. Indeed, it is enough to consider $a = p$, $K = 1$, $k = 3$, $d = 2$, and $r = 2$ in those theorems.

Remark 3.17. Let us recall that the *Sylvester sum* of a numerical semigroup $S = \langle a_1, a_2, \ldots, a_e\rangle$ is the value

$$s(S) = \sum_{x\in\mathbb{N}\setminus S} x.$$

In [11], the author computes the Sylvester sum of numerical semigroups generated by arithmetic progressions with initial gaps. To get the result, he again uses the Apéry set $\mathrm{Ap}(A, a_1) = \{w_0 = 0, w_1, \ldots, w_{a_1-1}\}$ by the formula

$$s(a_1, a_2, \ldots, a_e) = s(S) = \frac{1}{2a_1}\sum_{i=1}^{a_1-1} w_i^2 - \frac{1}{2}\sum_{i=1}^{a_1-1} w_i + \frac{a_1^2-1}{12}. \tag{1}$$

As a consequence, by Theorem 1 of [11], we have that if $p = 6k+7$ with $k \in \mathbb{N}$, then $s(p, p+4, p+6) = \frac{1}{108}(2p^4 + 24p^3 + 93p^2 + 202p + 435)$. Of course, we could directly apply (1) to the Apéry sets of the family $\mathcal{T}_2$ and obtain the same result. Even more, we could consider (1) for all the families we analyze in this work, but we prefer to leave it to the reader as a computation exercise.

4. Quadruplets in the form of prime quadruplets

As for triplets, there are several cases of quadruplets fulfilling the condition of the admissible constellation. Since the reasonings and computations are similar on all of them (see [14]), we will only study those directly associated with prime quadruplets, not forgetting that the results are, in fact, valid for any quadruplets satisfying the corresponding admissible constellation condition.

It is well-known that a prime quadruplet is of the form $(p, p+2, p+6, p+8)$. In the following result, we improve this expression.

Proposition 4.1. *We have that if $(p, p+2, p+6, p+8)$ is a prime quadruplet, then either $p = 4k+5$ or $p = 4k+7$ for some $k \in \mathbb{N}$.*

Proof. It is clear that if $p \in \mathbb{N}\setminus\{1\}$, then there exists $k \in \mathbb{N}$ such that $p = 4k+i$, with $i \in \{0, 1, 2, 3\}$. But, since $p, p+2, p+6, p+8$ are prime numbers, then $i \neq 0$ and $i \neq 2$. In addition, let us note that $(1, 3, 7, 9)$ and $(3, 5, 9, 11)$ are not prime quadruplets. □

In contrast to the case of triplets, a unique form defines the prime quadruplets. However, we need two different expressions to solve the Frobenius problem.

- $(4k+5, 4k+7, 4k+11, 4k+13)$, with $k \in \mathbb{N}$.
- $(4k+7, 4k+9, 4k+13, 4k+15)$, with $k \in \mathbb{N}$.

Thus, since $\gcd(4k+5, 4k+7) = \gcd(4k+7, 4k+9) = 1$ for all $k \in \mathbb{N}$, we have two families of numerical semigroups ($\mathcal{Q}_1$ and $\mathcal{Q}_2$).

- $S \in \mathcal{Q}_1$ if $S = \langle 4k+5, 4k+7, 4k+11, 4k+13\rangle$, with $k \in \mathbb{N}$.
- $S \in \mathcal{Q}_2$ if $S = \langle 4k+7, 4k+9, 4k+13, 4k+15\rangle$, with $k \in \mathbb{N}$.

Let us observe that $\langle 5, 7, 11, 13\rangle$ and $\langle 7, 9, 13, 15\rangle$ are numerical semigroups with embedding dimension equal to four. Moreover, since $4k+13 < 2(4k+5)$ and $4k+15 < 2(4k+7)$ for all $k \geq 1$, we can assert that $e(S) = 4$ for every $S \in \mathcal{Q}_1 \cup \mathcal{Q}_2$.

Remark 4.2. We have $S = \langle 1, 3, 7, 9\rangle = \langle 1\rangle$ and $S = \langle 3, 5, 9, 11\rangle = \langle 3, 5, 11\rangle$. Thus, $e(S) < 4$ in both cases. This is another reason for eliminating $(1, 3, 7, 9)$ and $(3, 5, 9, 11)$ as possibilities in Proposition 4.1.

Remark 4.3. It is possible to improve Proposition 4.1 a little more. Thus, if we assume that the elements on the quadruplet are not divisible by two or three, then we have that either $p = 12k+5$ or $p = 12k+11$ for some $k \in \mathbb{N}$. Even more, if we consider that the elements are not divisible by two, three or five, then all prime quadruplets (except $(5, 7, 11, 13)$) are of the form $p = 30k+11$ for some $k \in \mathbb{N}$. In any case, we obtain essentially the same conclusions for the families $\mathcal{Q}_1$ and $\mathcal{Q}_2$ associated with $p = 12k+5$ and $p = 12k+11$.

4.1. *First expression (family $\mathcal{Q}_1$)*

Let S be a numerical semigroup of the form $S = \langle 4k+5, 4k+7, 4k+11, 4k+13\rangle$, where $k \in \mathbb{N}$. Again we omit the proofs of the results because they are similar to those of Subsection 3.1.

Remark 4.4. The numerical semigroup $S = \langle 5, 7, 11, 13 \rangle$ behaves somewhat differently from the rest of the cases. In fact, $\mathrm{Ap}(S, 5) = \{0, 7, 11, 13, 14\}$ and

1. $\mathrm{PF}(S) = \{6, 8, 9\}$;
2. $\mathrm{F}(S) = 9$;
3. $\mathrm{g}(S) = 7$;
4. $\mathrm{t}(S) = 3$.

It is clear that except the value of $\mathrm{g}(S)$, Corollary 4.8 does not give these results.

Lemma 4.5. *If $k \in \mathbb{N} \setminus \{0\}$, then we have the equalities*

1. $3(4k + 7) = 2(4k + 5) + 1(4k + 11)$;
2. $3(4k + 11) = 1(4k + 7) + 2(4k + 13)$;
3. $(k + 2)(4k + 13) = (k + 3)(4k + 5) + 1(4k + 11)$;
4. $1(4k + 7) + 1(4k + 11) = 1(4k + 5) + 1(4k + 13)$;
5. $1(4k + 7) + (k + 1)(4k + 13) = (k + 4)(4k + 5)$;
6. $2(4k + 7) + 1(4k + 13) = 1(4k + 5) + 2(4k + 11)$;
7. $1(4k + 11) + (k + 1)(4k + 13) = (k + 2)(4k + 5) + 2(4k + 7)$;
8. $2(4k + 11) + k(4k + 13) = (k + 3)(4k + 5) + 1(4k + 7)$.

Theorem 4.6. *If $k \in \mathbb{N} \setminus \{0\}$ and $S = \langle 4k + 5, 4k + 7, 4k + 11, 4k + 13 \rangle$, then*

$$\mathrm{Ap}(S, 4k + 5) = \{a(4k + 7) + b(4k + 11) + c(4k + 13) \mid (a, b, c) \in C\},$$

where $C = C_a \cup C_b \cup C_c$, with $C_a = \big(\{1\} \times \{0\} \times \{0, 1, \ldots, k\}\big) \cup \{(2, 0, 0)\}$, $C_b = (\{0\} \times \{1, 2\} \times \{0, 1, \ldots, k\}) \setminus \{(0, 2, k)\}$, and $C_c = \{0\} \times \{0\} \times \{0, 1, \ldots, k + 1\}$.

Let us observe that

- $(4k + 7) + c(4k + 13) = (c + 1)(4k + 13) - 6$, for all $c \in \{0, 1, \ldots, k\}$;
- $(4k + 11) + c(4k + 13) = (c + 1)(4k + 13) - 2$, for all $c \in \{0, 1, \ldots, k\}$;
- $2(4k + 11) + c(4k + 13) = (c + 2)(4k + 15) - 4$, for all $c \in \{0, 1, \ldots, k - 1\}$.

Therefore, we can give the Apéry set by arranging its elements in increasing order.

Corollary 4.7. *If $k \in \mathbb{N} \setminus \{0\}$ and $S = \langle 4k + 5, 4k + 7, 4k + 11, 4k + 13 \rangle$, then*

$$\mathrm{Ap}(S, 4k + 5) = \{0; (4k + 13) - 6, (4k + 13) - 2, 4k + 13; 2(4k + 7);$$

$$2(4k + 13) - 6, 2(4k + 13) - 4, 2(4k + 13) - 2, 2(4k + 13); \ldots;$$

$$(k + 1)(4k + 13) - 6, (k + 1)(4k + 13) - 4, (k + 1)(4k + 13) - 2, (k + 1)(4k + 13)\}.$$

Corollary 4.8. *If $k \in \mathbb{N} \setminus \{0\}$ and $S = \langle 4k + 5, 4k + 7, 4k + 11, 4k + 13 \rangle$, then*

1. $\mathrm{PF}(S) = \{4k + 9, 4k^2 + 13k + 2, 4k^2 + 13k + 4, 4k^2 + 13k + 6, 4k^2 + 13k + 8\}$;
2. $\mathrm{F}(S) = 4k^2 + 13k + 8$;
3. $\mathrm{g}(S) = 2k^2 + 8k + 7$;
4. $\mathrm{t}(S) = 5$.

Remark 4.9. By Corollary 4.8, if $p = 4k + 5$ for some $k \in \mathbb{N} \setminus \{0\}$, then $\mathrm{F}(p, p + 2, p + 6, p + 8) = \frac{p^2+3p-8}{4}$.

We finish with an illustrative example.

Example 4.10. For $k = 24$ we have $S = \langle 101, 103, 107, 109 \rangle$ and, by Corollary 4.8, $\mathrm{PF}(S) = \{105, 2618, 2620, 2622, 2624\}$, $\mathrm{F}(S) = 2624$, and $\mathrm{g}(S) = 1351$. Moreover, by Corollary 4.7, $\mathrm{Ap}(S, 11) = \{0, 103, 107, 109, 206, 212, 214, 216, 218, \ldots, 2719, 2721, 2723, 2725\}$.

4.2. Second expression (family $\mathcal{Q}_2$)

Let S be a numerical semigroup of the form $S = \langle 4k+7, 4k+9, 4k+13, 4k+15\rangle$, where $k \in \mathbb{N}$.

Lemma 4.11. *If $k \in \mathbb{N}$, then we have the equalities*

1. $3(4k+9) = 2(4k+7) + 1(4k+13)$;
2. $3(4k+13) = 1(4k+9) + 2(4k+15)$;
3. $(k+2)(4k+15) = (k+3)(4k+7) + 1(4k+9)$;
4. $1(4k+9) + 1(4k+13) = 1(4k+7) + 1(4k+15)$;
5. $2(4k+9) + 1(4k+15) = 1(4k+7) + 2(40k+13)$;
6. $1(4k+13) + (k+1)(4k+15) = (k+4)(4k+7)$.

Theorem 4.12. *If $k \in \mathbb{N}$ and $S = \langle 4k+7, 4k+9, 4k+13, 4k+15\rangle$, then*

$$\mathrm{Ap}(S, 4k+7) = \{a(4k+9) + b(4k+13) + c(4k+15) \mid (a,b,c) \in C\},$$

where $C = C_a \cup C_b \cup C_c$, with $C_a = \big(\{1\} \times \{0\} \times \{0,1,\ldots,k+1\}\big) \cup \{(2,0,0)\}$, $C_b = \{0\} \times \{1,2\} \times \{0,1,\ldots,k\}$, and $C_c = \{0\} \times \{0\} \times \{0,1,\ldots,k+1\}$.

Let us observe that

- $(4k+9) + c(4k+15) = (c+1)(4k+15) - 6$, for all $c \in \{0,1,\ldots,k+1\}$;
- $(4k+13) + c(4k+15) = (c+1)(4k+15) - 2$, for all $c \in \{0,1,\ldots,k\}$;
- $2(4k+13) + c(4k+15) = (c+2)(4k+15) - 4$, for all $c \in \{0,1,\ldots,k\}$.

Consequently, we can arrange the elements of the Apéry set in increasing order.

Corollary 4.13. *If $k \in \mathbb{N}$ and $S = \langle 4k+7, 4k+9, 4k+13, 4k+15\rangle$, then*

$$\mathrm{Ap}(S, 4k+7) = \{0; (4k+15)-6, (4k+15)-2, 4k+15; 2(4k+9);$$
$$2(4k+15)-6, 2(4k+15)-4, 2(4k+15)-2, 2(4k+15); \ldots;$$
$$(k+1)(4k+15)-6, (k+1)(4k+15)-4, (k+1)(4k+15)-2, (k+1)(4k+15);$$
$$(k+2)(4k+15)-6, (k+2)(4k+15)-4\}.$$

Remark 4.14. If $S = \langle 7, 9, 13, 15\rangle$, then $\mathrm{Ap}(S, 7) = \{0, 9, 13, 15, 18, 24, 26\}$. Therefore, we can apply Corollary 4.13 if we only consider the five first values and the two last.

Corollary 4.15. *If $k \in \mathbb{N}$ and $S = \langle 4k+7, 4k+9, 4k+13, 4k+15\rangle$, then*

1. $\mathrm{PF}(S) = \{4k+11, 4k^2+19k+17, 4k^2+19k+19\}$;
2. $\mathrm{F}(S) = 4k^2+19k+19$;
3. $\mathrm{g}(S) = 2k^2+10k+12$;
4. $\mathrm{t}(S) = 3$.

Remark 4.16. By Corollary 4.15, if $p = 4k+7$ for some $k \in \mathbb{N}$, then $\mathrm{F}(p, p+2, p+6, p+8) = \frac{p^2+5p-8}{4}$.

We finish with an illustrative example.

Example 4.17. For $k = 1$ we have $S = \langle 11, 13, 17, 19\rangle$, $\mathrm{PF}(S) = \{15, 40, 42\}$, $\mathrm{F}(S) = 42$, and $\mathrm{g}(S) = 24$ (by Corollary 4.15). Moreover, from Corollary 4.13, we know that $\mathrm{Ap}(S, 11) = \{0, 13, 17, 19, 26, 32, 34, 36, 38, 51, 53\}$.

5. *k*-tuplets in the form of prime *k*-tuplets

From the contents of Sections 3 and 4, it looks like the problem becomes more and more longueur as soon as we consider larger and larger k-tuplets. In any case, it is not difficult to see what happens when $k \in \{5, 6, 7, 8\}$ and, in this way, to propose a conjecture (see [14]). Once again, there are several possibilities of k-tuplets satisfying the admissible constellation condition; moreover, for each k, the results are similar. Thus, we will only see those directly related to prime k-tuplets.

Quintuplets. There exist two families of quintuplets that include prime quintuplets. Namely, $(p, p+2, p+6, p+8, p+12)$ and $(p, p+4, p+6, p+10, p+12)$. Now, if $k \in \mathbb{N}$, these families are associated with the values $6k+5$ and $6k+7$, respectively. In addition, the Frobenius number is equal to

- $\frac{p^2+7p-12}{6}$ for $(p, p+2, p+6, p+8, p+12)$ with $p = 6k+5 \geq 11$,
- $\frac{p^2+11p+12}{6}$ for $(p, p+4, p+6, p+10, p+12)$ with $p = 6k+7 \geq 7$.

Furthermore, if we consider numerical semigroups generated by the above quintuplets, then the type is given by

- $\mathrm{t}(S) = 6$ for $S = \langle p, p+2, p+6, p+8, p+12 \rangle$ with $p = 6k+11 \geq 11$,
- $\mathrm{t}(S) = 4$ for $S = \langle p, p+4, p+6, p+10, p+12 \rangle$ with $p = 6k+7 \geq 13$.

Sextuplets. Although the expression $(p, p+4, p+6, p+10, p+12, p+16)$ is the unique possibility for the sextuplets, which include prime sextuplets, we have to consider four different families to study the Frobenius problem. Such families correspond to the values $p = 8k + r$ with $k \in \mathbb{N}$ and $r \in \{7, 9, 11, 13\}$. In this case, the Frobenius number is equal to

- $\frac{p^2+9p+16}{8}$ for $p = 8k+7$,
- $\frac{p^2+15p+16}{8}$ for $p = 8k+9$,
- $\frac{p^2+13p+16}{8}$ for $p = 8k+11$,
- $\frac{p^2+11p+16}{8}$ for $p = 8k+13$.

Moreover, for these families of numerical semigroups, the type is equal to 9 (of course, for $p = 8k+7$ when $k \geq 1$), 5, 5, and 7, respectively. It is interesting to observe that, although the value of the type is the same in the second and third cases, the structures of their sets of pseudo-Frobenius numbers are essentially different (see [14]).

Septuplets. For septuplets, which are associated with prime septuplets, we have two families: $(p, p+2, p+6, p+8, p+12, p+18, p+20)$ and $(p, p+2, p+8, p+12, p+14, p+18, p+20)$. Now, if $k \in \mathbb{N}$, then $p = 10k+11$ for the first case and $p = 10k+19$ for the second one. Thus, the pair (Frobenius number, type) is equal to $(\frac{p^2+9p-20}{10}, 13)$ (if $p \geq 31$) and $(\frac{p^2+11p-20}{10}, 11)$ (if $p \geq 19$), respectively ([14]).

Octuplets. Finally, there are three families of octuplets associated with prime octuplets: $(p, p+2, p+6, p+12, p+14, p+20, p+24, p+26)$, $(p, p+2, p+6, p+8, p+12, p+18, p+20, p+26)$, and $(p, p+6, p+8, p+14, p+18, p+20, p+24, p+26)$. As discussed in [14], we have to study 39 cases (13 for each family) to obtain formulas for the Frobenius number. We refer the reader interested in such results to [14].

A conjecture and future works. Based on all the above comments, we propose the following conjecture.

Conjecture 5.1. *Let us consider a k-tuplet satisfying the condition of the admissible constellation and suppose that p and $p+q$ are its first and last elements, respectively. Then we claim that:*

1. *The Frobenius number is given by a quadratic polynomial $a_2 p^2 + a_1 p + a_0$.*
2. *The leading coefficient is $a_2 = \frac{2}{q}$.*
3. *The constant term a_0 is an integer.*

Let us observe that we have got $a_0 = \pm 2$ when $1 \leq k \leq 7$. However, the values $a_0 = -4$, $a_0 = -6$, and $a_0 = 10$ appear in octuplets (see [14]). Thus, without going into further details, we have preferred to state that a_0 is an integer.

Remark 5.2. As we have commented several times above, not all the members of the k-tuples need to be prime numbers to get the given results here. Thus, new problems arise. For example, it would be interesting to know how the Frobenius numbers are given for other types of tuplets and whether there is any regularity among them. Among others, these will be the aims of future work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

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