PROBLEMAS SOBRE FLUJOS POR CURVATURA EXTRÍNSECA NO LINEALES

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Resumen y conclusiones

Un flujo geométrico consiste en encontrar soluciones de ecuaciones parabólicas en derivadas parciales que involucran cantidades geométricas de dos o más variedades dadas.
En el caso de los flujos geométricos extrínsecos, las soluciones corresponden a una familia de un parámetro de inmersiones cuyas deformaciones dependen de las curvaturas de la hipersuperficie en una variedad riemanniana objetivo.

El flujo geométrico extrínseco más estudiado en la literatura es el flujo por la curvatura media (FCM), ya que es el flujo gradiente del funcional de área.
En la Introducción de esta tesis presentamos una breve descripción del FCM actuando sobre hipersuperficies en $\mathbb{R}^{n+1}$. No obstante, queremos mencionar el trabajo pionero de Huisken en [Hui2] que demostró que cualquier hipersuperficie cerrada y convexa en $\mathbb{R}^{n+1}$, en el sentido de que las curvaturas principales son no negativas, se encoge hasta un punto en tiempo finito. Este fenómeno se suele denominar como desarrollo de una singularidad bajo el FCM, y es una tarea importante entender por qué aparecen, y cómo tratar con ellas luego de rescalar la hipersuperficie cerca de la singularidad.

Por otra parte, es natural preguntarse qué ocurre si consideramos otras funciones de curvatura en lugar de la curvatura media en el contexto de flujos geométricos extrínsecos. En este espíritu, los trabajos de B. Andrews y sus colaboradores en [And2], [And1], [AMZ] y [ALM] desarrollaron una potente teoría, similar a la dada para el FCM, para hipersuperficies estrictamente convexas en $\mathbb{R}^{n+1}$, donde la función de curvatura es convexa o cóncava. Por función de curvatura entendemos una función homogénea suave y simétrica cuyo dominio es un cono abierto de $\mathbb{R}^n$.

En esta tesis nos interesamos en soluciones eternas, soluciones que están definidas para todo tiempo, de flujos por curvatura extrínseca completamente no lineales tal que la evolución está dada por traslaciones en una dirección unitaria fija. Además, nos referimos a estas soluciones por solitones de traslación de la función de curvatura que estemos estudiando, y los abreviamos por trasladores.

Los solitones de traslación pueden verse como hipersuperficies en $\mathbb{R}^{n+1}$ que satisfacen una ecuación diferencial parcial de la forma $\gamma(\lambda) = \langle \nu, v \rangle$, donde $\gamma(\lambda)$ es la función de curvatura evaluada en las curvaturas príncipales de la hipersuperficie, ν es el vector normal unitario que apunta hacia afuera, y $v \in \mathbb{S}^n$ es la dirección de la traslación del flujo. Es importante destacar que esta ecuación es localmente uniforme elíptica cuando la curvaturas principales pertenece al cono $\Gamma := \left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \dfrac{\partial \gamma}{\partial \lambda_i} > 0 \right\}$. Esto nos permite estudiar los trasladores como en el contexto de la geometría diferencial clásica.

Por otro lado, cuando la función de curvatura es la curvatura media, los trasladores son un modelo para las singularidades de tipo II del FCM. Esto significa que el supremo de la norma de la segunda forma fundamental estalla en una tasa mayor que $O\left(\dfrac{1}{\sqrt{T-t}}\right)$, donde T es el tiempo máximo de existencia del flujo, y luego de escalar la hipersuperficie de manera adecuada, la evolución de la hipersuperficie converge a un H-traslador del espacio ambiente. Además, los trasladores del FCM son hipersuperficies mínimas en $(\mathbb{R}^{n+1}, e^{\langle x, v \rangle} dx^2)$. Este es un hecho notable para el estudio de estas soluciones ya que la teoría local de hipersuperficies mínimas se puede aplicar para construirlas y caracterizarlas.

Desgraciadamente, cuando la función de curvatura no es lineal, sólo se sabe que las singularidades de tipo II pueden modelarse mediante trasladores si además la función de curvatura es convexa como función definida en su dominio. Además, no tenemos esperanzas de que estas hipersuperficies sean mínimas en un espacio euclidiano conforme como en el caso de la curvatura media. Por ello, el estudio de los trasladores para funciones de curvatura no lineales es más complicado, y necesita que se desarrollen otro tipos de técnicas para el desarrollo de esta teoría.

Los resultados de esta tesis están en el espíritu de explotar el hecho de que la ecuación $\gamma(\lambda) = \langle \nu, v \rangle$ es localmente uniformemente elíptica cuando $\lambda \in \Gamma$. En particular, pudimos desarrollar propiedades geométricas para los trasladores contenidas en el capítulo 5.

Una de las propiedades geométricas que obtuvimos fue un principio de tangencia, y como corolarios, también obtuvimos un resultado de no existencia, y un teorema de unicidad cuando el solitón de traslación es un gráfico estrictamente convexo definido sobre una bola. Este último resultado se obtuvo mediante el método de los planos móviles de Alexandrov aplicado para esta ecuación.

Por otra parte, el resultado principal de esta tesis es una estimación de convexidad en el espíritu de [SS], donde los autores mostraron que un H-traslador 2-convexo con $H > 0$ en $\mathbb{R}^1$ con $n \geq 3$, es convexo. Hasta donde sabemos, sigue siendo un problema abierto si un γ-traslador que es un gráfico en $\mathbb{R}^{n+1}$ tal que $\lambda \in \Gamma$ y $\gamma : \Gamma \to \mathbb{R}$ es una función de curvatura cóncava es convexo o no. Afortunadamente, bajo las hipótesis del Teorema de estimación de convexidad 2.2.12, pudimos demostrar que estos solitones de traslación son asintóticamente convexos, en el sentido de que el mínimo de las curvaturas principales converge a cero para una secuencia de puntos en la hipersuperficie que tienden al infinito.

Además, para la familia de las funciones de curvatura $Q_k = \dfrac{S_{k+1}}{S_k}$, donde S_k denota el polinomio simétrico elemental de grado k en n-variables, mostramos en el Capítulo 3 estimaciones de gradiente y de segundo orden en el espíritu del trabajo de Ecker y Huisken en [EH1]. La principal contribución de este capítulo es un resultado de tipo Lioville Teorema 3.0.3 para Q_k-trasladores que son planos en el infinito.

Finalmente, también construimos trasladores rotacionalmente simétricos para la función de curvatura $\sqrt[n]{S_n}$ y Q_{n-1} en $\mathbb{R}^{n+1}$. Estas soluciones son de tipo "bowl" ya que son gráficos estrictamente convexas definidas en una bola o en todo $\mathbb{R}^n$.
Merece la pena mencionar un trabajo reciente de [Ren], en el que el autor construye soluciones de tipo "bowl" para una clase general de funciones de curvatura que son α-homogéneas con $\alpha \geq \frac{1}{2}$. Además, caracterizó cuándo la solución de tipo "bowl" estará definida en una bola o en todo el hiperplano en términos de la función de curvatura.

En conclusión, la teoría relacionada con los solitones de traslación para funciones de curvatura no lineales está todavía en sus primeras etapas. Queremos terminar este resumen con algunos problemas abiertos relacionados con esta teoría que causarán un gran impacto como en el caso de los solitones de traslación de la FCM:

1. ¿Existen ejemplos de solitones de traslación no convexos tales que $\lambda \in \Gamma$?
2. ¿Son las soluciones de tipo "bowl" los únicos solitones de traslación estrictamente convexos que son enteros o definidos en una bola?
3. ¿Son los solitones de traslación modelos de singularidades de tipo II de los flujos de curvatura cuando la función de curvatura es cóncava?

Abstract and conclusions

A geometric flow consists in finding solutions to parabolic partial differential equations that involve geometric quantities from given manifolds.
In the case of extrinsic geometric flows, solutions correspond to a one parameter family of immersions whose deformations depend on the curvatures of the hypersurface in a target riemannian manifold.

The most studied extrinsic geometric flow in the literature is the mean curvature flow (MCF), since it is the gradient flow of the area functional.
In the Introduction of this thesis we present a brief description of the MCF acting on hypersurface in $\mathbb{R}^{n+1}$. Nevertheless, we want to mention the pioneer work of Huisken in [Hui2] which showed that any closed and convex hypersurface in $\mathbb{R}^{n+1}$, in the sense of the principal curvatures are non-negative, will shrink to a point in finite time. This phenomena is usually called the development of a singularity under the MCF, and it is an important task to understand how the singularities appears and behave after rescalling the hypersurface.

On the other hand, it is natural to ask what happens if we considerate others curvature functions rather than the mean curvature in the extrinsic geometric flow setting. In this spirit, the works of Andrews and his collaborators in [And2], [And1], [AMZ] and [ALM] developed a powerful theory, similar to the one given for the MCF, for strictly convex hypersurface in $\mathbb{R}^{n+1}$ where the curvature function is convex or concave. By a curvature function we mean a smooth symmetric homogeneous function whose domain is an open cone of $\mathbb{R}^n$.

In this thesis we were concern in eternal solutions, solution that exist for all time, for fully non-linear curvature flows such that the evolution is given translations by a fixed unitary direction, and we refer to these solutions by translating solitons of the particular curvature function that we are studying.

Additionally, translating solitons can be seen as hypersurface in $\mathbb{R}^{n+1}$ which satisfy a partial differential equation of the form $\gamma(\lambda) = \langle \nu, v \rangle$, where $\gamma(\lambda)$ is the curvature function evalu-

ated in the princial curvatures, ν is te outward unit normal vector, and $v \in \mathbb{S}^n$ is the direction of the translation of the flow. It is important to remark that this equation is locally uniformly elliptic when the principal curvature belong to the cone $\Gamma := \left\{ \lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \ \dfrac{\partial \gamma}{\partial \lambda_i} > 0 \right\}$, this permit us to study translators as in the classical differential geometry context.

On the other hand, when the curvature function is the mean curvature of the hypersurface, translating solitons are model of type II singularities of the MCF. This means that the second fundamental of the evolution under the MCF blows up in a bigger rate than $O\left(\dfrac{1}{\sqrt{T-t}}\right)$, where T is the maximal time of existence of the flow.
Moreover, translating solitons of the MCF are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x,v \rangle} dx^2)$. This is a remarkable fact for the study of these solutions since the local theory of minimal hypersurfaces can be applied to construct and characterize them.

Unfortunately, when the curvature function is not linear, it is only known that type II singularities can be modeled by translating soliton if in addition the curvature function is convex. Furthermore, there is no hope for these hypersurface to be minimal in a conformal euclidean space as in the case of the mean curvature. That is why the study of translating soliton for non-linear curvature function is more difficult, and needs other type of techniques to be developed in the theory.

The results of this thesis are in the spirit of exploiting the fact equation $\gamma(\lambda) = \langle \nu, v \rangle$ is locally uniformly elliptic when $\lambda \in \Gamma$. In particular, we were able to developed geometric properties for translating solitons contained in Chapter 5.

One of the geometric properties that we obtained was a tangent principle, and as corollaries, we also obtained a non-existence result, and a uniqueness theorem when the translating soliton is a strictly convex graph defined on a ball. This last result was obtained by the method of moving planes of Alexandrov applied for this equation.

Moreover, the main result of this thesis is a convexity estimate in the spirits of [SS], where the authors showed that a 2-convex mean convex translating soliton of the MCF in $\mathbb{R}^{n+1}$ with

$n \geq 3$, is convex. To the best of our knowledge, it remains an open problem if a translating soliton which is a graph in $\mathbb{R}^{n+1}$ such that $\lambda \in \Gamma$ and $\gamma : \Gamma \to \mathbb{R}$ is a concave curvature function is convex or not. Fortunately, under the hypotheses of the convexity estimate Theorem 2.2.12, we were able to show that these translating soliton are asymptotically convex, in the sense that the minimum of the principal curvatures goes to zero for a sequence of points in the hypersurface that goes to infinity.

In addition, for the family of the curvature functions $Q_k = \dfrac{S_{k+1}}{S_k}$, where S_k denotes the elementary symmetric polynomial of degree k in n-variable, we showed in Chapter 3 gradient and second order estimates in the spirit of the work of Ecker and Huisken in [EH1]. The main contribution of this chapter is a Lioville's type result Theorem 3.0.3 for translating solitons of the Q_k curvature flow which are flat at infinity.

Finally, we also constructed rotationally symmetric translating solitons for the curvature function $\sqrt[n]{S_n}$ and Q_{n-1} in $\mathbb{R}^{n+1}$. These solution are of "bowl"-type since they are strictly convex graph defined in a ball or in the whole $\mathbb{R}^n$.
It is worth mentioning a recent paper [Ren], where the author construct "bowl"-type solution for a general class of curvature function which are α-homogeneous whit $\alpha \geq \frac{1}{2}$. Furthermore, he characterized when the "bowl"-type solution will be defined in a ball or in the whole hyperplane in terms of the curvature function.

In conclusion, the theory related to translation solitons for nonlinear curvature functions is still in its early stages. We want to finish this abstract with some open problems related to this theory which will cause a big impact as in the case of translating solitons of the MCF:

1. Are there example of non convex translating solitons such that $\lambda \in \Gamma$?

2. Are the "bowl"-type solutions the only strictly convex translating solitons which are entire or defined in a ball?

3. Are translating solitons models of type II singularities of curvature flows when the

curvature function is concave?

Contents

Chapter 1

Introducción

Un flujo geométrico es una ecuación diferencial parcial (EDP) dependiente del tiempo que involucra cantidades geométricas de una variedad dada. En las últimas décadas del siglo pasado, este tema cobró mucho interés en la comunidad matemática a partir de sus aplicaciones en Geometría Diferencial, Topología, Física y, en un contexto más aplicado, al Procesamiento de Imágenes.

Desde la perspectiva geométrica, se pueden distinguir dos tipos de flujos geométricos: intrínsecos y extrínsecos. Se entiende por un flujo geométrico intrínseco como una EDP que considera la evolución de la cantidades geométricas intrínsecas de una variedad riemanniana.

Un ejemplo importante de este tipo de flujos consiste en modificar una métrica riemanniana g_0 a otra con algún tensor de curvatura constante. Más concretamente, encontrar una familia de métricas de 1-parámetro $\{g(t)\}_{0\leq t<T}$ para una variedad M tal que

$$\begin{cases} \partial_t g = F(g) & , \ t \in (0,T) \\ g(0) = g_0. \end{cases},$$

donde F es un tensor que depende de la métrica $g(t)$ y sus derivadas. Por último, se estudia la métrica límite $g(T)$, donde $T \in (0, \infty]$ es el tiempo máximo de existencia del flujo, con la expectativa de que satisfaga la condición de curvatura constante.

En este contexto, el flujo intrínseco más notable y estudiado es $F(g) = -2\mathrm{Ric}(g)$, donde $\mathrm{Ric}(g)$ denota el tensor de curvatura de Ricci de $g(t)$, y se conoce como el flujo de Ricci.
El flujo de Ricci fue introducido por R. Hamilton en la década de 1980 como herramienta para demostrar la conjetura de geometrización de Thurston para 3-variedades cerradas. Esta conjetura establece que cualquier 3-variedad cerrada puede descomponerse canónicamente en piezas, donde cada pieza posee una de ocho tipos de estructuras geométricas diferentes. En 2003, gracias al trabajo de G. Perelman, se demostró finalmente esta conjetura y, como consecuencia, la conjetura de Poincaré, que era un problema abierto desde 1904. Remitimos a [CZ] para un excelente libro autocontenido sobre este tema.

Figure 1.1: Evolución de la superficie de una mancuerna bajo el flujo de Ricci. Imagen cortesía de Ignacio Mcmanus.

Por otro lado, un flujo geométrico extrínseco corresponde a una familia de inmersiones de un parámetro cuyas deformaciones dependen de las curvaturas extrínsecas de una variedad en una variedad riemanniana objetivo.
Más concretamente, en el caso de codimensión uno, si $F_0 : M^n \to N^{n+1}$ es una inmersión desde una variedad a una variedad riemanniana (N, g), uno quiere encontrar una familia de

1-parámetro de inmersiones $F : M \times (0,T) \to N$ tal que

$$\begin{cases} \partial_t F = -\gamma(\lambda)\nu, \ (x,t) \in M \times (0,T), \\ F(x,0) = F_0(x), \end{cases} \tag{1.0.1}$$

donde $\lambda = (\lambda_1, \ldots, \lambda_n)$ y ν son el vector de curvaturas principales y el vector normal unitario que apunta hacia afuera de $M_t = F(M,t)$ en N, respectivamente. En esta tesis nos referiremos a la ecuación 1.0.1 como un γ-flujo para abreviar.

El γ-flujo más estudiado en la literatura es el flujo de curvatura media, donde

$$\gamma = H = \lambda_1 + \ldots + \lambda_n.$$

Además, el H-flujo también aparece como el flujo de gradiente negativo del funcional de área de M en N, y en consecuencia, se puede utilizar para deformar hipersuperficies en hipersuperficies mínimas en diferentes espacios ambientes.

A continuación, queremos dar una breve descripción del análisis de las singularidades relacionadas con el H-flujo en $\mathbb{R}^{n+1}$ que están relacionadas con los resultados obtenidos en esta tesis.

En primer lugar, para contextualizar el análisis de las singularidades que aparecen bajo la evolución a través del H-flujo, queremos mencionar un resultado pionero de G. Huisken en [Hui2], que prueba que cualquier hipersuperficie cerrada y convexa[1] colapsará a un punto en tiempo finito bajo la evolucion del H-flujo.

Además, este fenómeno se caracterizó por la explosión de la cantidad

$$\max_{M_t} |A|,$$

[1]Las curvaturas principales de la hipersuperficie son no negativas.

donde $|A| = \sqrt{\lambda_1^2 + \ldots + \lambda_n^2}$, a medida que $t \to T$, donde T es el tiempo máximo de existencia del flujo. Para ejemplificar esto, la evolución de una esfera de radio R_0 bajo el H-flujo en $\mathbb{R}^{n+1}$ viene dada por

$$F(p,t) = R(t)p,$$

donde $R(t) = \sqrt{R_0^2 - 2nt}$ y $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. Observamos que el tiempo máximo de existencia viene dado por $T = \dfrac{R_0^2}{2n}$, y por tanto, se cumple que

$$|A|(t) = \frac{\sqrt{n}}{R(t)} = \frac{1}{\sqrt{2}\sqrt{T-t}}.$$

De hecho, las singularidades bajo el H-flujo en $\mathbb{R}^{n+1}$ se clasifican en dos tipos relacionados

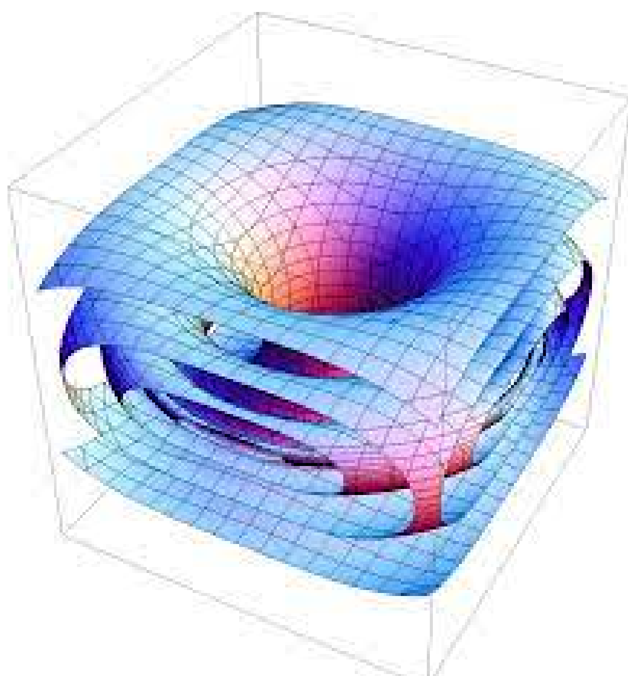


Figure 1.2: La evolución de una superficie bajo el flujo H en $\mathbb{R}^3$. Imagen por cortesía de Francisco Martín.

con la tasa de explosión de $\max\limits_{M_t} |A|$:

a) Tipo I: Existe una constante $C > 0$ tal que $\max_{M_t} |A| \leq \dfrac{C}{\sqrt{T-t}}$.

b) Tipo II: No es del tipo I, es decir: $\lim_{t \to T} \max_{M_t} |A| \sqrt{T-t} = \infty$.

La clasificación de estas singularidades es un problema abierto en toda su generalidad. Sin embargo, cuando los datos iniciales satisfacen $H > 0$, los autores en [HS1] demostraron que después de un reescalamiento parabólico adecuado, la singularidad puede ser modelada por una solución autosimilar[2] por el H-flujo. En efecto, estas soluciones son:

a) Auto-contractantes: Aparecen bajo un escalamiento parabólico alrededor de una singularidad de tipo I. Después de pasar al límite estas convergen a una solución antigua[3] autosimilar que evoluciona sólo por dilatación, es decir: $F(x,t) = \sqrt{-t}F_0(x)$ con $t \in (-\infty, 0]$.

   Además, cada corte de tiempo M_t es convexo y satisface la ecuación $H = -\langle p, \nu \rangle$ para todo $p \in M_t$.

b) Solitones de traslación: Aparecen bajo un escalamiento parabólico alrededor de una singularidad de tipo II. Después de pasar al límite estas convergen a una solución eterna[4] autosimilar que evoluciona por traslación en una dirección unitaria, es decir: $F(x,t) = F_0(x) + vt$ para $t \in \mathbb{R}$ y algún vector unitario $v \in Sp^n$.

   Además, cada corte de tiempo M_t puede escribirse como $\mathbb{R}^{n-k} \times S^k$ donde S^k es una hipersuperficie estrictamente convexa, y la ecuación $H = \langle \nu, v \rangle$ se cumple en cada M_t.

La herramienta principal utilizada en la demostración de esta clasificación fue la fórmula de monotonicidad de Huisken (véase [Hui1]), relacionada con la evolución de un funcional con

[2]Una solución al flujo H de la forma $F(x,t) = r(t)O(t)F_0(x) + V(t)$, donde $r(t) \geq 0$, $V(t) \in \mathbb{R}^{n+1}$ y $O(t) \in \mathcal{O}(n+1)$ es una matriz ortogonal. Visualmente esto significa que la hipersuperficie inicial F_0 evoluciona bajo una dilatación $r(t)$, una homotecia $O(t)$, y una traslación $V(t)$ en $\mathbb{R}^{n+1}$.

[3]Solución de la ecuación (2.0.1) que está definida para $(-\infty, a]$ para algún $a \in \mathbb{R}$.

[4]Solución de la ecuación (2.0.1) que está definida para todo $t \in (-\infty, \infty)$.

un núcleo de calor,

$$\frac{d}{dt}\int\limits_M \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}}dx = -\int\limits_M \frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}}\left|H+\frac{\langle x-p,\nu\rangle}{2|t_0-t|}\right|^2 dx \leq 0.$$

Por otro lado, queremos mencionar que existe un tipo adicional de solución autosimilar del H-flujo llamado auto-expansiva. En [EH1], los autores mostraron que la evolución de un gráfico entero Lipschitz existe para todo $t > 0$ y, después de una escalamiento adecuado, la evolución converge a una solución auto-expansiva como $t \to \infty$, es decir: el flujo límite converge a $F(x,t) = \sqrt{t}F_0(x)$ y cada trozo de tiempo M_t satisface $H = \langle p, \nu\rangle$. Para un estudio reciente de solcuiones auto-expansivas nos referimos a [Smo].

Finalmente, la clasificación de las soluciones autosimilares del H-flujo es una tarea importante para entender el comportamiento de las hipersuperficies bajo la evolución del H-flujo. Actualmente, este problema se ha completado para algunos casos particulares. Por ejemplo, en el caso de soluciones auto-contractantes convexas las posibles soluciones, módulo isometrías y difeomorfismos tangenciales, son esferas, cilindros o hipersuperficies de Abresch-Langer.

Por otra parte, recientemente se ha completado la clasificación de los solitones de traslación para grafos en $\mathbb{R}^3$, véase [HIMW2] para más detalles. Las posibles soluciones son el "bowl" solitón, que es un grafo entero y estrictamente convexo, los delta-wing, el griem-reaper y los griem-reapers inclinados, todas estas soluciones están definidas en bandas (véase la figura 1.3 a continuación).

Es importante mencionar que en la clasificación de estas soluciones autosimilares del H-flujo se utilizó la teoría de las superficies mínimas. Más concretamente, en [Ilm], T. Ilmanen demostró que estas soluciones también pueden verse como hipersuperficies mínimas en $(\mathbb{R}^{n+1}, g)$, donde g corresponde a $e^{-|p|^2}dx^2$ para los auto-contractantes, y $e^{-\langle p,v\rangle}dx^2$ para los solitones de traslación. También observamos que estas métricas no son completas, lo que significa que la teoría global de las superficies mínimas no se aplica a estas soluciones auto-similares, pero sí la teoría local.

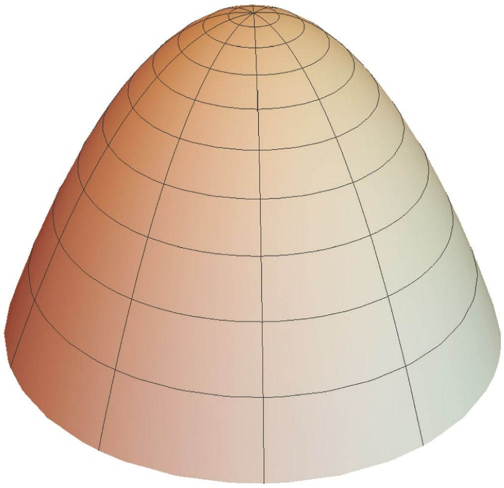
1.1 Objetivos

Los objetivos generales propuestos en esta tesis son los siguientes:

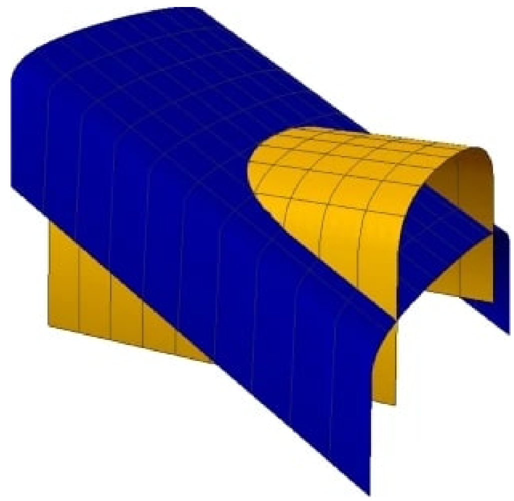
- Estudiar teoremas de clasificación para solitones por traslación de flujos por curvatura no lineales en espacios Euclideos.
- Estudiar comportamiento asintótico y principios de comparación en infinitos para dichos solitones.
- Caracterización de soluciones tipo “bowl” para flujos por curvatura no lineales.

Además, los objetivos específicos propuestos en esta tesis son los siguientes:

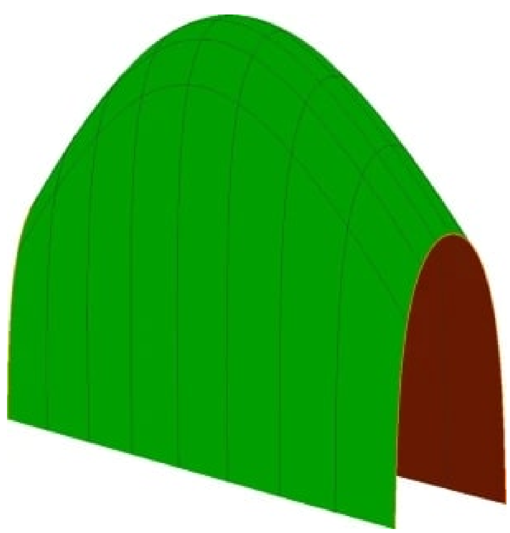
1. Encontrar estimaciones locales interiores hasta segundo orden para solitones por traslación del flujo Q_k. Ver siguiente sección para una definición de Q_k y S_k.
2. Construir ejemplos de solitones por traslación para los flujos dados por las funciones de curvatura $\sqrt[n]{S_n}$ y Q_{n-1} en espacion euclideanos.
3. Probar un principio de tangencia para solitones de traslación de γ-flujos en $\mathbb{R}^{n+1}$.
4. Probar que los únicos γ-trasladores que son grafos estrictamente convexos definidos en una bola en $\mathbb{R}^n$ son rotacionalmente simétricos asintóticos a cilindros, y por tanto únicos.
5. Encontrar y probar estimación de convexidad para γ-trasladores con γ es una función de curvatura concava, i.e: encontrar una cota inferior para el mínimo de las curvaturas principales en términos de la function velocidad γ y H.



(a) Bowl soliton en $\mathbb{R}^3$.



(b) Grim reaper y un grim reaper inclinado con ángulo $\theta = \dfrac{\pi}{6}$ in $\mathbb{R}^3$.



(c) Delta wing en $\mathbb{R}^3$.

Figure 1.3: Estas superficies están tomadas como soluciones de $H = \langle \nu, -e_3 \rangle$. Imágenes por cortesía de Francisco Martín.

1.2 Metodología y Resultados

Esta tesis trata de problemas relacionados con flujos geométricos extrínsecos (2.0.1) donde la función de curvatura $\gamma\Gamma \to \mathbb{R}$ es una función suave, simétrica y 1-homogénea definida en un cono abierto $\Gamma \subset \mathbb{R}^n$ que contiene al cono positivo $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0,\ i = 1, \ldots, n\}$.

Además, es importante mencionar que la metodología utilizada en esta tesis es la usual utilizada en investigación de matemáticas puras, o sea, revisión en la literatura relacionada con los objetivos propuesto, luego dar una formulación de los problemas observados, para finalmente dar un listado de teoremas y demostraciones que resuelven los problemas propuestos en la tesis.

Por otra parte, elegimos estudiar este tipo de funciones de curvatura porque son un análogo geométrico natural de la teoría de las EDP. De hecho, queremos remarcar la teoría desarrollada por Nirenberg, Caffarelli y Spruck en [CNS1] y [CNS2] para prescribir ecuaciones de curvatura completamente no lineales[5] en dominios acotados en $\mathbb{R}^n$ con condiciones de contorno de Dirichlet. Esta teoría sirvió de inspiración para el trabajo desarrollado en esta tesis.

Una clase importante de funciones de curvatura 1-homogéneas consideradas en esta tesis son de tipo cuociente hessiano[6], es decir

$$\sqrt[k]{S_k} \text{ and } Q_k = \frac{S_{k+1}}{S_k},\ \text{ o de manera general, } Q_{k,l} = \sqrt[k-l]{\frac{S_k}{S_l}}. \tag{1.2.1}$$

Recordemos que estas funciones se evalúan en las curvaturas principales de una hipersuper-

[5]Esto significa que en coordenadas $(x, u(x))$, la ecuación de curvatura prescrita es una EDP elíptica completamente no lineal

[6]La razón del nombre proviene de la teoría de prescripción de los polinomios simétricos elementales evaluados en los valores propios de $\mathrm{Hess}(u)$. También se conocen como de tipo Monge-Ampère ya que los polinomios simétricos elementales aparecen en expresiones de la forma $\det(\mathrm{Hess}(u) - \lambda \mathrm{Id})$.

ficie dada, y por definición, estos son los valores propios del operador de forma, es decir

$$\mathcal{W}_p : T_pM \to T_pM$$
$$\vec{v} \to -\nabla_{\vec{v}}\nu,$$

donde ∇ y ν son la conexión Levi-Civita de M y el vector normal unitario interior de M en $\mathbb{R}^{n+1}$. En coordenadas locales, el operador de forma puede escribirse por $h^i_j = g^{ik}h_{kj}$, donde g^{ij} denota los coeficientes de la inversa del tensor métrico g_{ij} de la hipersuperficie, y h_{ij} denota los coeficientes de la segunda forma fundamental de la hipersuperficie, respectivamente. Además, los coeficientes de la segunda forma fundamental corresponden a las componentes tangentes de la derivada covariante del campo normal unitario (módulo de orientación), es decir: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ donde $\{e_i\}$ son coordenadas locales del espacio tangente centradas en algún punto de la hipersuperficie.

Por otra parte, el símbolo

$$S_k(\lambda) = \sum_{1\leq i_1<\ldots<i_k\leq n} \lambda_{i_1}\ldots\lambda_{i_k} \tag{1.2.2}$$

denota el polinomio simétrico elemental de grado k en n-variables, y utilizamos la convención $S_0 = 1$ y $S_j = 0$ para $j > n$.
Nótese que la curvatura media $H = S_1(\lambda)$, la curvatura escalar $R_g = \dfrac{2}{n(n-1)}S_2(\lambda)$ y la curvatura guasiana $K = S_n(\lambda)$.

Por último, es importante destacar que estos flujos son parabólicos cuando las curvaturas principales de la evolución M_t pertenecen a los conos de Garlin

$$\Gamma_k := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : S_j(\lambda) > 0.j = 1, \ldots, k\}, \tag{1.2.3}$$

También mencionamos que para nosotros Γ_k siempre denotará la componente conexa que contiene el cono positivo Γ_+.

Ahora queremos discutir algunos trabajos y resultados anteriores relacionados con los flujos γ en la literatura.

En primer lugar, cuando los datos iniciales son compactos y estrictamente convexos, la evolución bajo un γ-flujo presenta un comportamiento similar al H-flujo. De hecho, en [And2] el autor demostró que bajo algunos supuestos naturales sobre γ (ver por ejemplo las Propiedades 1-7 en el capítulo 5) la evolución colapsará a un punto en tiempo finito, y después de un escalamiento parabólico adecuado, la solución converge a una esfera unitaria en $\mathbb{R}^{n+1}$.

Sin embargo, hasta el presente de esta tesis, el análisis de las singularidades relacionadas con la evolución bajo flujos γ no se entiende tan bien como en el caso del H-flujo. En efecto, uno de los principales problemas es la no linealidad relacionada con la geometría implicada. Por ejemplo, se dispone de la fórmula de monotonicidad de Huisken para describir las singularidades del H-flujo, mientras que no se conoce una fórmula similar para otras funciones de curvatura.

Alentadoramente, hay pocos resultados relacionados con el estudio de las singularidades bajo γ-flujo que describimos brevemente. En [Lyn3] el autor clasificó soluciones antiguas convexas uniformemente parabólicas a γ-flujo que presentan una singularidad de tipo I.
En concreto, estas soluciones son esferas o cilindros que se contraen. Un aspecto importante de su resultado es que incluye funciones de curvatura convexas y cóncavas que satisfacen propiedades de concavidad adicionales y admite un teorema de división.

Además, si la función de curvatura es convexa (y satisface las Propiedades (1)-(3) y 5 descritas en el capítulo 5), se conocen más resultados relacionados con las singularidades de Tipo II. En [ALM], los autores mostraron que las soluciones eternas que presentan una singularidad de Tipo II se descomponen como $M = \Sigma^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ y evolucionan bajo traslación por una dirección unitaria fija. Todavía es una problema abierto si este fenómeno se produce para las funciones cóncavas.

Por último, hay autores que definieron otro tipo de singularidad conocida por ser de Tipo 0, y está relacionada con la pérdida de convexidad de los datos iniciales bajo un flujo γ. Para describir este fenómeno mencionamos en primer lugar que si la hipersuperficie inicial satisface que las curvaturas principales pertenecen al cono de simetría

$$\left\{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \frac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0, \text{ for } i = 1, \ldots, n\right\}^7,$$

entonces bajo el γ-flujo esta propiedad también es satisfecha por la evolución M_t. De hecho, esto puede verse como una aplicación sencilla del principio de máximo fuerte aplicado a la ecuación

$$(\partial_t - \Delta_\gamma)\gamma = |A|^2_\gamma \gamma,$$

donde $\Delta_\gamma = \dfrac{\partial \gamma}{\partial h_{ij}} \nabla_i \nabla_j$ y $|A|^2_\gamma = \dfrac{\partial \gamma}{\partial h_{ij}} h_{ia} h_{aj}$. Nótese que para este argumento utilizamos que la matriz $\left(\dfrac{\partial \gamma}{\partial h_{ij}}\right)_{i,j}$ es una matriz localmente uniformemente definida positiva.

Sin embargo, hay ejemplos de funciones de curvatura que no preservan los conos positivos Γ_+ bajo la evolución del flujo.

En [AMZ], los autores construyeron una hipersuperficie convexa en $\mathbb{R}^{n+1}$ tal que la evolución bajo el flujo γ dado por

$$\gamma(\lambda) = \sum_{1 \leq i < j \leq n} \frac{\lambda_i \lambda_j}{\sqrt{\lambda_i^2 + \lambda_j^2}},$$

pierde convexidad para $t > 0$ pequeños. Nótese que $\gamma(\lambda)$ es positiva, estrictamente creciente

[7] Se trata de un cono abierto cuando γ es una función continua simétrica 1-homogénea. Además, si γ es cóncava, entonces este cono es también convexo. Esto se puede ver por el hecho de que bajo la hipótesis de 1-homogénea, ser cóncava es equivalente a ser subaditivo (es decir: $\gamma(a\lambda + b\mu) \geq a\gamma(\lambda) + b\gamma(\mu)$).

en cada variable, y cóncava en el cono $\{\lambda \in \mathbb{R}^n : \lambda_i + \lambda_j > 0\}$.

Además, los autores caracterizaron la pérdida de convexidad bajo la evolución de las funciones de curvatura cóncava si no se satisfacía la hipótesis de ser inversamente cóncavo en la restricción de la frontera de Γ_+. Esto significa que si $\gamma(1, ..., 1, 0) > 0$, entonces la función

$$\tilde{\gamma}(\lambda_1, \ldots, \lambda_{n-1}) = \gamma(\lambda_1^{-1}, \ldots, \lambda_{n-1}^{-1}, 0)^{-1}$$

es cóncava en el cono positivo $(n-1)$-dimensional $\lambda_+^{n-1} = \{\lambda \in \mathbb{R}^{n-1} : \lambda_i > 0\}$.

Por último, queremos mencionar que las singularidades de tipo 0 no se producen bajo la evolución del H-flujo. De hecho, en [HS2] los autores demostraron que todos los conos Γ_k se conservan bajo H-flujo, dando en particular que los datos iniciales no perderán su convexidad bajo el flujo. En consecuencia, el estudio de los γ-flujos para funciones de curvatura no lineales puede utilizarse para evolucionar hipersuperficies de forma que sólo algunos conos de convexidad se conserven bajo la evolución.

En esta tesis también nos interesamos en los solitones de traslación del γ-flujo en $\mathbb{R}^{n+1}$. Al igual que mencionamos para el H-flujo, se trata de soluciones autosimilares eternas del γ-flujo que evolucionan bajo traslaciones por el campo vectorial $V : \mathbb{R} \to \mathbb{S}^n$, es decir

$$F(x, t) = F_0(x) + V(t).$$

Además, podemos comprobar que $V(t) = \vec{v}t$ para algún vector constante $\vec{v} \in \mathbb{S}^n$. En efecto, observamos que

$$\partial_t V = \partial_t F(x, t) = -\gamma(\lambda)\nu.$$

Luego, como el operador de forma $\mathcal{W}_p$ es invariante bajo traslaciones, las curvaturas principales también son invariantes por traslaciones. En particular, $\partial_t\gamma(\lambda) = 0$, y esto implica que $\partial_t^2 V(t) = 0$, o, equivalentemente, $V(t) = \vec{v}t$ para algún vector constante $v \in \mathbb{S}^n$. Sin embargo, en esta tesis sólo consideraremos solitones de traslación del γ-flujo con vector de traslación $\vec{v} = e_{n+1}$ por simplicidad.

También observamos que los solitones de traslación del flujo γ, o γ-trasladores para abreviar, satisfacen en cada corte de tiempo M_t la ecuación

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle \,. \tag{1.2.4}$$

Esto significa que los γ-trasladores pueden ser estudiados tanto por la teoría de las ecuaciones parabólicas o con la teoría de las ecuaciones elípticas.

Los problemas que estudiamos en esta tesis están más relacionados con la perspectiva elíptica de los γ-trasladores. De hecho, los supuestos que asumiremos sobre la función de curvatura $\gamma : \Gamma \to \mathbb{R}$ (ver Sección 5.1) están en el espíritu de los problemas clásicos de geometría diferencial:

- γ se evalúa en el cono $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$.
- Para cada $i = 1, \dots, n$, $\dfrac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0$ tiene en Γ.

En coordenadas locales, esta propiedad da que la Ecuación (1.2.4) es localmente uniformemente elíptica, y además de que todo γ-traslador conexo y simplemente conexo es un grafo vertical sobre algún conjunto abierto conexo $\Omega \subset \mathbb{R}^n$.

Por otro lado, cuando consideramos funciones de curvatura que tienen una extensión continua en la frontera de $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$, como la familia $\sqrt[k]{S_k}$ soportada en Γ_k, tenemos como soluciones triviales a la ecuación (1.2.4) hiperplanos verticales y cilindros redondos (el vector normal unitario es ortogonal al hiperplano que contiene e_{n+1}). Nótese

que hay funciones de curvatura que no tienen esta propiedad, como Q_k soportada en Γ_k, dado que las soluciones triviales son cilindros redondos verticales.
Por último, queremos mencionar que la teoría de las EDP elípticas degeneradas es útil para el estudio de las funciones de curvatura que se anulan en la frontera de Γ. Remitimos al lector a ver [CNS1], [CNS2], [Ivo], [SUW] para ver buenos artículos introductorios sobre este tema.

Cabe mencionar que si consideramos el caso parabólico, existen estudios que permiten iniciar el flujo γ para funciones de curvatura γ que están soportadas en $\{\lambda \in \mathbb{R}^n : \gamma(\lambda) \geq 0\} \cap \overline{\Gamma}_+$. De hecho, esto lo hizo la autora en [Die] para el Q_k-flujo. Ella demostró que el flujo Q_k puede iniciarse para datos iniciales convexos cerrados tales que $Q_k \geq 0$ y las curvaturas principales pertenecen a Γ_{k-1}. Su principal contribución fue el uso de barreras cilíndricas que permiten iniciar el flujo H, dando lugar a que la evolución se vuelva estrictamente convexa para tiempos positivos, permitiendo iniciar el flujo Q_k. En particular, la evolución posee el mismo comportamiento que lo expuesto en [And1].

Sigue siendo un problema abierto si este procedimiento puede aplicarse para una función de curvatura diferente, o si puede extenderse a diferentes conos abiertos en lugar de Γ_{k-1}. Remitimos al lector a [ACGL] para un libro introductorio sobre los flujos γ en diferentes entornos.

1.2.1 Aproximación no variacional para Q_k-trasladores $\mathbb{R}^{n+1}$

En esta subsección vamos a demostrar que no existe una aproximación variacional para los Q_k-trasladores en $\mathbb{R}^{n+1}$. Recordemos que los Q_k-trasladores satisfacen

$$Q_k = \langle \nu, e_{n+1} \rangle \Leftrightarrow S_{k+1} - S_k \langle \nu, e_{n+1} \rangle = 0, \tag{1.2.5}$$

nótese que el si, y sólo si, ocurre cuando las curvaturas principales pertenecen a Γ_k.

Por tanto, para comprobar si podemos encontrar una funcional tal que la ecuación (1.2.5) aparezca como una ecuación de Euler-Lagrange, utilizaremos las fórmulas obtenidas en el capítulo 4 de [Ros].

En primer lugar, para una función suave dada $f : \mathbb{R}^{n+1} \to \mathbb{R}$, consideramos el funcional sobre variedades cerradas $M \subset \mathbb{R}^{n+1}$ dado por

$$\mathcal{F}(M) = \int_M f(S_1, \ldots, S_n, \langle X, e_{n+1} \rangle) dA,$$

donde S_i son los polinomios elementales simétricos evaluados en las curvaturas principales de $M \subset \mathbb{R}^{n+1}$ y X es el vector de posición en $\mathbb{R}^{n+1}$.

Entonces, eligiendo una variación normal $\varphi : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ con $\varphi_0 = \mathrm{Id}$ y $\left.\frac{d\varphi_t}{dt}\right|_{t=0} = g\nu$, donde $\varphi_t(x) = \varphi(x,t)$ y $g \in \mathcal{C}_c^\infty(M)$, podemos ver que

$$\left.\frac{d}{dt}\mathcal{F}(M_t)\right|_{t=0} = \int_M \left.\frac{d}{dt} f\right|_{t=0} dA + \left.\int_M f \frac{d}{dt} dA\right|_{t=0}, \tag{1.2.6}$$

donde $M_t = \varphi_t(M)$.

A continuación, calculamos cada término de la ecuación 1.2.6 por separado. Utilizaremos el teorema 4.1 de [Ros] para calcular el primer término de la ecuación anterior

$$\begin{aligned} \left.\frac{d}{dt} f\right|_{t=0} &= \sum_{i=1}^{n} D_i \left.\frac{d}{dt} S_i\right|_{t=0} + D_{n+1} f \left.\frac{d}{dt} \langle X, e_{n+1} \rangle\right|_{t=0} \\ &= \sum_{i=1}^{n} D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1) S_{i+1}) \right) + g D_{n+1} f \langle \nu, e_{n+1} \rangle, \end{aligned}$$

where $L_{i-1}(g) = \mathrm{div}_M (T_{i-1} \nabla g)$ and T_i is the ith-Newton transformation of the shape operator of M in $\mathbb{R}^{n+1}$ defined inductively by the relation

$$\begin{cases} T_i(\mathcal{W}_p) = S_i Id - \mathcal{W}_p T_{i-1}(\mathcal{W}_p), \\ T_0 = Id. \end{cases}$$

Por otro lado, es un resultado bien conocido que la fórmula de variación de la forma de volumen bajo variaciones normales viene dada por

$$\frac{d}{dt} dA \bigg|_{t=0} = -gH dA.$$

En consecuencia, obtenemos la siguiente fórmula para la primera variación de $\mathcal{F}$ dada por

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}(M_t)\bigg|_{t=0} \\ &= \int_M \sum_{i=1}^n D_i f \left(L_{i-1}(g) + g(S_1 S_i - (i+1)S_{i+1})\right) + gD_{n+1} f \left\langle \nu, e_{n+1} \right\rangle - gHf dA. \end{aligned} \tag{1.2.7}$$

Como hemos mencionado antes, los H-trasladores son puntos críticos del funcional

$$F_0(M) = \int_M e^{\langle X, e_{n+1}\rangle} dA.$$

Por lo tanto, tiene sentido considerar el siguiente funcional para Q_k-trasladores dada por

$$\mathcal{F}_k(M) = \int_M S_k e^{(k+1)\langle X, e_{n+1}\rangle} dA.$$

Entonces, como $f = x_k e^{(k+1)x_{n+1}}$ en $\mathcal{F}_k(M)$, la primera variación es fácil de calcular. En efecto, por la fórmula (1.2.7), resulta

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}_k(M_t)\Big|_{t=0} &(1.2.8)\\
&= \int_M -(k+1)g\left(S_{k+1} - S_k\langle \nu, e_{n+1}\rangle\right)e^{(k+1)\langle X, e_{n+1}\rangle} + L_{k-1}(g)e^{(k+1)\langle X, e_{n+1}\rangle}dA.
\end{aligned}$$

Remark 1.2.1. Obsérvese que el término $L_{k-1}(g)$ en (1.2.8) sólo desaparece cuando $k = 0$, ya que $T_{-1} = 0$ por definición, y esto concuerda con el hecho de que H-trasladores son hipersuperficies mínimas en $(\mathbb{R}^{n+1}, e^{\langle X, e_{n+1}\rangle}dx^2)$.

Además, para $k = 1$, $L_0(g) = \operatorname{div}_M(T_0\nabla g) = \Delta g$, donde Δ es el operador de Laplace-Beltrami de M. Por lo tanto, por el teorema de la divergencia, este término en la fórmula (1.2.8) se puede escribir por

$$\begin{aligned}
\int_M L_0(g)e^{2\langle X, e_{n+1}\rangle}dA &= \int_M g\Delta e^{2\langle X, e_{n+1}\rangle}dA\\
&= \int_M 2g\left(2|e_{n+1}^{\top}|^2 - H\langle \nu, e_{n+1}\rangle\right)e^{2\langle X, e_{n+1}\rangle}dA.
\end{aligned}$$

donde $e_{n+1}^{\top}$ es la proyección ortogonal sobre T_pM en $T_p\mathbb{R}^{n+1}$.

Por último, como podría sospechar el lector, la ecuación (1.2.5) no es la de Euler-Lagrange para $\mathcal{F}_k$ para $k \geq 1$, ya que en la ecuación de Euler-Lagrange aparecen más términos de curvatura. En consecuencia, no esperamos la existencia de ninguna funcional tal que un Q_k-translador aparezca como una hipersuperficie mínima en un espacio euclidiano ponderado.

1.2.2 Estimaciones para el operador Q_k

Desde la perspectiva de las EDP, las estimaciones de gradiente y de segundo orden son herramientas importantes para construir soluciones y analizar sus propiedades y comportamiento.

Como se mencionó anteriormente, en [EH1] y [EH2] los autores obtienen estimaciones locales de gradiente y curvatura para construir soluciones no compactas del H-flujo en $\mathbb{R}^{n+1}$. Estos resultados y técnicas motivaron el desarrollo de una teoría similar para ser aplicada a los flujos γ.

En este sentido, queremos mencionar el trabajo de [CD], donde los autores construyeron grafos convexos completos no compactos que evolucionan bajo el flujo Q_k en $\mathbb{R}^{n+1}$, véase la ecuación (1.2.1) para la definición. En su trabajo el uso de la convexidad fue un hecho importante.

En esta tesis nos ocupamos de relajar el supuesto de convexidad utilizado en [CD] para obtener estimaciones locales de gradiente y de segundo orden para el flujo Q_k, y también para los Q_k-trasladores, tales que las curvaturas principales pertenecen al cono Γ_{k+1}.

Más precisamente, si $F(x,t) = (x, u(x,t))$ where $u : \Omega \times [0,T) \to \mathbb{R}$ entonces la ecuación (2.0.1) con $\gamma = Q_k$, puede escribirse como

$$u_t = Q_k(Du, D^2u)\sqrt{1 + |Du|^2}, \tag{1.2.9}$$

y para Q_k-trasladores, if $F(x) = (x, u(x))$ con $u : \Omega \to \mathbb{R}$, la ecuación (2.2.1) puede escribirse como

$$Q_k(Du, D^2u) = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{1.2.10}$$

Por lo tanto, para obtener estimaciones de gradiente y de segundo orden sin la hipótesis de convexidad impuesta en los gráficos, tuvimos que entender mejor el término $Q_k(Du, D^2u)$[8] cuando las curvaturas principales de las gráficas pertenecen al cono Γ_{k+1}.

De hecho, gracias al trabajo realizado en [Hol] para el flujo $\sqrt[k]{S_k}$ cuando las curvaturas

[8]Observamos que estamos abusando con la notación al escribir $Q_k(Du, D^2u)$ en lugar de $Q_k(\lambda)$. La razón por la que elegimos escribir de esta manera es para enfatizar la dependencia en las variables del operador Q_k cuando se evalúa en las curvaturas principales de M_t o M.
Además, utilizamos una forma conveniente de trabajar con este término, véase la sección 3.2 en el capítulo 3 para más detalles

principales de M_t pertenecen al cono Γ_k, pudimos obtener la siguiente estimación para el flujo Q_k.

Theorem 1.2.1. *Sea $u : \Omega \to \mathbb{R}$ una solución de la ecuación* (2.2.6)*, donde $\Omega = B_r(0) \times (0,T]$ con $r > 0$. Supongamos que las curvaturas principales de la gráfica de u se encuentran en el cono Γ_{k+1} definido en* (2.1.3) *para cada t. Entonces, para $t > 0$, se cumple*

$$|Du(0,t)| \leq \exp\left(K + \frac{KMT}{rt}\frac{MrT}{t} + \frac{M^2}{t} + \frac{(T+1)r^2}{t} + \frac{TM^2}{tr^2} + \frac{TM}{tr}\right),$$

done $M = \sup\limits_{\Omega} u$ y $K = K(k,n)$.

Además, como hemos podido adaptar la técnica utilizada en el teorema anterior para los Q_k-trasladores, hemos obtenido la siguiente estimación del gradiente para esa ecuación.

Theorem 1.2.2. *Sea $r > 0$ y $u \in \mathcal{C}^3(B(0,r))$ una solución de* (2.2.7) *tal que las curvaturas principales de graph(u) está en Γ_{k+1}. Entonces, se cumple la siguiente estimación*

$$|Du(0)| \leq \exp\left(\frac{CM}{r} + \frac{CM^2}{r^2} + C\right),$$

donde $M = \sup\limits_{B(0,r)} u$ y $C = C(k,n)$.

Por otro lado, adaptando la demostración del teorema anterior, obtuvimos un teorema de tipo Liouviullie para Q_k-trasladores que dice lo siguiente.

Theorem 1.2.3. *No existen soluciones $u \in \mathcal{C}^3(\mathbb{R}^n)$ tal que*

1. *Las curvaturas principales de la gráfica de u están en Γ_{k+1}.*

2. *$u(x) = o(|x|)$ para $|x| \to \infty$.*

3. *La gráfica de u es Q_k-traslador en $\mathbb{R}^{n+1}$.*

Las estimaciones de segundo orden que obtenemos a continuación dependen de la suposición de que las curvaturas principales pertenecen al cono Γ_{k+2} con $k \geq 1$. La razón es que cualquier hipersuperficie con sus curvaturas principales situadas en Γ_{k+2} con $k \geq 1$, satisface $|A|^2 \leq H^2$ y $|A|_k^2 \leq CH^2$. Por lo tanto, dado que es más fácil estimar H que $|A|$ o $|A|_k^2$, aplicamos técnicas similares a las de [EH2] para obtener estimaciones de segundo orden para las soluciones de las ecuaciones (2.2.6) y (2.2.7), respectivamente.

Además, como en [EH2], suponemos que existe una función no negativa $h(x,t)$ tal que

$$|(\partial_t - \Box_k)h| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h,$$

donde $\Box_k f = \frac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$ y $|X|_k = \frac{\partial Q_k}{\partial h_{ij}} X_i X_j$.Véase la observación 3.3.2 de la sección 3.3 para un ejemplo de una función $h(x,t)$.

Theorem 1.2.4. *Sea $R > 0$ tal que $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ es un gráfico sobre una bola de radio R en el hiperplano ortogonal a $\vec{w} \in \mathbb{S}^n$ para todo $[0,T)$.*
Supongamos que M_t evoluciona bajo el Q_k-flujo y que las curvaturas principales pertenecen al cono Γ_{k+2} con $k \geq 1$. Entonces, para cualquier $t_0 \in (0,T)$ y $\theta \in [0,1)$, se cumple la siguiente estimación

$$\sup_{M_t} H^2 \leq \frac{c(k,n)}{(1-\theta)^2}\left(\frac{1}{t} + \frac{1}{R^2}\right) \sup_{[0,t_0]} \sup_{M_t} v^4,$$

donde $v = \langle \nu, \vec{w} \rangle^{-1}$.

Para los Q_k-trasladores, consideramos una función no negativa $h(x)$ con las propiedades anteriores pero cambiando la suposición $|(\partial_t - \Box_k)h| \leq C(k,n)$ por

$$|\Box_k h + \langle h, e_{n+1} \rangle| \leq C(k,n).$$

Theorem 1.2.5. *Sea $R > 0$ tal que $M = \{x \in M : h(x) \leq R^2\}$ es un gráfico sobre una bola de radio $R > 0$ en el hiperplano ortogonal a e_{n+1}.*
Supongamos que M es un Q_k-traslador tal que las curvatura principales pertenecen a la Γ_{k+2} con $k \geq 1$. Entonces, para cualquier $\theta \in [0, 1)$, se cumple la siguiente estimación

$$H^2 \leq \frac{c(k,n)}{(1-\theta)^2}\left(1 + \frac{1}{R^2}\right) \sup_{\{h \leq \theta R^2\}} v^4,$$

donde $v = \langle \nu, e_{n+1} \rangle^{-1}$.

1.2.3 γ-trasladores rotacionalmente simétricos en $\mathbb{R}^{n+1}$

En esta subsección queremos discutir sobre la construcción de γ-trasladores que son rotacionalmente simétricos en $\mathbb{R}^{n+1}$ para funciones de curvatura particulares.

En primer lugar, un γ-traslador gráfico rotacionalmente simétrico en $\mathbb{R}^{n+1}$ con $\gamma > 0$ corresponde al conjunto $\{(x, u(x)) \in \mathbb{R}^{n+1} : |x| < R\}$ donde $u(x) = \mathbf{u}(r)$, $\mathbf{u} : [0, R) \to \mathbb{R}$ y $r = |x|$.

Entonces, la ecuación (1.2.4) se convierte en una ecuación diferencial ordinaria (EDO) de la forma

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2) f\left(\frac{\dot{\mathbf{u}}}{r}\right),$$

donde la notación de puntos significa derivadas con respecto a r, y f es una función que aparece aplicando el teorema de la función implícita en las componentes conexas del conjunto de nivel $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$. Observamos que en este caso, las curvaturas principales son dos, y como $\gamma(\lambda)$ es creciente en cada variable, el teorema de la función implícita garantiza una expresión local de $f(x)$ en cada componente conexa de $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$.

En el caso de los H-trasladores, los autores en [AW] construyeron una solución estrictamente convexa definida para $R = \infty$ llamada el solitón “bowl” (ver Fig. 2.3a) que se comporta como un paraboloide en el infinito. Más precisamente, en [CSS], los autores mostraron que el solitón “bowl” en $\mathbb{R}^{n+1}$ se comporta en el infinito como

$$\frac{r^2}{n} - \ln(r) + O\left(\frac{1}{r}\right).$$

En esta tesis, construimos soluciones de tipo “bowl” para $\gamma = Q_{n-1}$ y $\gamma = \sqrt[n]{S_n}$ en $\mathbb{R}^{n+1}$. Estas soluciones son gráficas estrictamente convexas rotacionalmente simétricas sobre el hiperplano $\{x_1 = 0\}$, que se definen en bolas o en el hiperplano completo.

En particular, para $n = 2$, estas soluciones son explícitas y vienen dadas por

$$-\ln(1 - r^2) \text{ and } \int_0^r \sqrt{e^{s^2} - 1}ds$$

para Q_1 y $\sqrt{S_2}$, respectivamente. Observamos que el $\sqrt{S_2}$-traslador tipo “bowl” no es $\mathcal{C}^2$ en el origen, y el traductor Q_1 del tipo "bowl" es $\mathcal{C}^2$-asintótico[9] al cilindro $\mathbb{S}^1 \times \mathbb{R}$.

Theorem 1.2.6. *Para $n \geq 3$ existe un único traslador suave estrictamente convexo en $\mathbb{R}^{n+1}$ para las funciones de curvatura Q_{n-1} y $\sqrt[n]{S_n}$, tales trasladores son gráficos rotacionalmente simétricos. Además, se tiene*

1. *Para $n \geq 3$, el Q_{n-1}-traslador está definido en una bola de radio $Q_{n-1}(1, \ldots, 1) = \dfrac{1}{n}$ y es $\mathcal{C}^2$-asintótica al cilindro $\mathbb{S}^{n-1}\left(\frac{1}{n}\right) \times \mathbb{R}$.*

2. *Para $n \geq 3$, el $\sqrt[n]{S_n}$-traslador es entero.*

Además, también encontramos una familia de trasladores rotacionalmente simétricos $\sqrt{S_2}$ que son gráficos sobre el plano $\{x_1 = 0\}$ en $\mathbb{R}^3$ que satisfacen $H < 0$ y $K > 0$.

[9]Esto significa que las curvaturas principales $\lambda_1 \leq \ldots \leq \lambda_n$ satisfacen $\lambda_1 \to 0$ y $\lambda_i \to 1$ para $i \neq 1$ cuando $|p| \to \infty$.

Theorem 1.2.7. *Sea $a \in [0,1)$. Entonces, la familia de superficies*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R} \right\},$$

donde $z = \int_1^{r_a(z)} \sqrt{e^{s^2-a} - 1}ds$ con $r_a(0) = 1$, son $\sqrt{S_2}$-trasladores completos en $\mathbb{R}^3$ que satisfacen $H < 0$ y $K > 0$.

Más recientemente, en [Ren], el autor demostró la existencia de soluciones de tipo "bowl" para una clase general de funciones de curvatura simétrica α-homogéneas, sin ninguna suposición sobre convexidad o concavidad en la función de curvautra. Además caracterizó cuando la solución estará definida en una bola o será entera en términos de la función de curvatura, véase la introducción del capítulo 4 más adelante.

1.2.4 Propiedades geométricas de los γ-trasladores en $\mathbb{R}^{n+1}$

Otro aspecto importante que estudiamos en esta tesis son las propiedades geométricas de los γ-trasladores en $\mathbb{R}^{n+1}$. Entendemos por propiedades geométricas la aplicación de la teoría de las EDP aplicada en el contexto de la geometría diferencial: principios tangenciales, inexistencia, unicidad y estimaciones de convexidad. Uno de los ejemplos más importantes de estas aplicaciones son los principios del máximo aplicados a cantidades geométricas de una hipersuperficie.

Recordemos que los H-trasladores son hipersuperficies mínimas en $\left(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1}\rangle} dx^2\right)$. Por lo tanto, el principio tangencial, que permite comparar H-trasladores en función de cómo se tocan, es un resultado bien conocido de la teoría de superficies mínimas. Sin embargo, dado que los γ-trasladores no son hipersuperficies mínimas para $\gamma \neq H$, desarrollamos un principio de tangencia que dice lo siguiente.

Theorem 1.2.8. *Sean $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ dos γ-trasladores embebidos y conexos tales que*

1. $\gamma : \Gamma \to [0, \infty)$ *satisface las propiedades 1-4 y 7 dadas en la sección 5.1.*

2. Σ_1 *es estrictamente convexo.*

3. Σ_2 *es convexo.*

Entonces,

*a) (**Principio de tangencia interior**) Supongamos que existe un punto interior* $p \in \Sigma_1 \cap \Sigma_2$ *tal que los espacios tangentes coinciden en* p*. Si* Σ_1 *se encuentra en un lado de* Σ_2*, entonces ambas hipersuperficies coinciden.*

*b) (**Principio tangencial de frontera**) Supongamos que los bordes* $\partial\Sigma_i$ *se encuentran en el mismo hiperplano* Π*, y la intersección de* Σ_i *con* Π *es transversal. Si* Σ_1 *se encuentra en uno de los lados de* Σ_2 *y existen* $p \in \Sigma_1 \cap \Sigma_2$ *tales que los espacios tangentes coinciden, entonces ambas hipersuperficies coinciden.*

Además, gracias a los resultados expuestos en [Ren] (véanse los teoremas 1.3-1.4) también pudimos demostrar un resultado de no existencia para el γ-traductor tal que el γ-traductor de tipo "bowl" está definido en una bola redonda.

Corollary 1.2.9. *Supongamos que* $\gamma : \Gamma \to [0, \infty)$ *satisface las propiedades 1-4 y 7. Entonces, no hay* γ*-trasladores enteros convexos en* $\mathbb{R}^{n+1}$ *tales que el* γ*-traslador de tipo "bowl" esté definido en una bola.*

Este es un resultado curioso que podemos utilizar de la siguiente manera: si hay un grafo entero que evoluciona bajo un flujo γ tal que el γ- traslador de tipo "bowl" está definido en una bola, entonces este grafo entero no puede converger a un traslador cuando $t \to \infty$.

A continuación, aplicamos también el método de planos móviles de Alexandrov para obtener que el gráfico γ-traslador hereda las propiedades definidas en la frontera de su dominio.

Theorem 1.2.10. *Sea $\Sigma \subset \mathbb{R}^{n+1}$ un γ-traslador completo tal que*

1. *$\gamma : \Gamma \to [0, \infty)$ satisface las propiedades 1-4 y 7.*

2. *Σes un gráfico estrictamente convexo sobre una bola $B^n_r(0) \subset \mathbb{R}^n$.*

3. *Σ posee un único fin $\mathcal{C}^2$-asintótico al cilindro $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, es decir: las curvaturas principales de Σ satisfacen*

$$\begin{aligned} \min \{\lambda_i(p) : i = 1, \dots, n\} =& \lambda_1(p) \to 0, \\ \forall i \in \{2, \dots, n\}, \lambda_i(p) \to& \frac{1}{r}, \end{aligned}$$

cuando $|p| \to \infty$.

Entonces Σ es rotacionalmente simétrico con respecto al eje x_{n+1}.

Además, como corolario del teorema anterior y del resultado de unicidad expuesto en [Ren], obtenemos la siguiente propiedad de unicidad.

Corollary 1.2.11. *Sea Σ un γ-traslador como en el teorema anterior tal que el γ-traslador de tipo "bowl" está definido en una bola de radio $\gamma(1, \dots, 1)$. Entonces, Σ coincide con el γ-traslador de tipo "bowl".*

Finalmente, el resultado principal de esta tesis es una estimación de convexidad para los γ-trasladores en $\mathbb{R}^{n+1}$. Para dar un contexto de por qué este es el resultado principal de esta tesis, empezamos anunciando algunos hechos relacionados con los H-trasladores con $H > 0$. En primer lugar, en [SX], los autores demostraron que los H-trasladores con $H > 0$ en $\mathbb{R}^3$ son convexos (véase [HIMW3] para una demostración más breve). Este resultado tuvo una notable implicación en la clasificación de los H-trasladores gráficos en $\mathbb{R}^3$. Además, en [Has], el autor muestra que un H-traslador α-no colapsado y convexo debe ser el solitón "bowl". En resumen, puesto que los H-trasladores con $H > 0$ son convexos, se deduce que el único de ellos que es entero debe ser el solitón “bowl”. Remitimos al lector a [HIMW2] para la clasificación de los H-traductores que son grafos.

Por lo tanto, en el espíritu de mostrar que γ-traslador son convexos probamos la siguiente estimación de convexidad.

Theorem 1.2.12. *Sean $n \geq 3$, $\alpha, \delta > 0$ y $\Sigma \subset \mathbb{R}^{n+1}$ un γ-traslador inmerso completo orientable tal que*

a) $\gamma : \Gamma \to \mathbb{R}$ *satisface las propiedades 1-3 and 5-7.*

b) $\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : \alpha H \leq (\delta + 1)\gamma\}$ *está compactamente soportado en* $\Gamma \setminus Cyl_{n-1}$*, donde*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\}.$$

c) Existe una constante $\beta \in (0,1)$ *tal que* $\lambda_i + \lambda_j \geq \beta H$*, para todo* $1 \leq i < j \leq n$*.*

Entonces, $\lambda_1 \geq H - \alpha\gamma$ *en* Σ*, donde* $\lambda_1(p) = \min\{\lambda_i(p) : i = 1, \ldots, n\}$*.*

La demostración del Teorema 2.2.12 está inspirada en [SS], donde los autores muestran que un H-traslador con $H > 0$ y uniformemente 2-convexo es convexo. Observamos que nuestra demostración no funciona para los H-trasladores. La razón principal es que el cono $\Gamma_{\alpha,\delta}$ necesita estar soportado de forma compacta en $\Gamma_1 \setminus C_{n-1}$, pero para $\gamma = H$, $\Gamma_{\alpha,\delta} = \Gamma_1$. Por otro lado, la demostración dada en [SS] no puede adaptarse directamente a una función de curvatura cóncava γ, ya que los autores utilizan una aproximación cóncava para λ_1, y esta no se comporta de la misma manera para funciones de curvatura cóncavas.

Remark 1.2.2. Una estimación similar se demostró en [Lyn2] para una familia de funciones de curvatura de la forma

$$(1-c)H - c\left(\sum_{1=i_1<\ldots<i_k=n}\frac{1}{\lambda_{i_1}+\ldots+\lambda_{i_k}}\right)^{-1},\ c\in(0,1).$$

En contraste con nuestro resultado, su estimación es en el contexto parabólico compacto.

En consecuencia, como una aplicación de la estimación de convexidad, demostramos que bajo las mismas hipótesis del Teorema 1.2.12, los *gamma*-trasladores son asintóticamente convexo.

Corollary 1.2.13. *Sea $\Sigma \subset \mathbb{R}^{n+1}$ el γ-traslador donde $\gamma : \Gamma \to \mathbb{R}$ satisface la hipótesis del Teorema 1.2.12, entonces Σ es asintóticamente convexo.*

Chapter 2

Introduction

A geometric flow is a time dependent partial differential equation (PDE) which involves geometric quantities from a given manifold. In the last decades of the last century, this topic gained a lot of interest in the mathematical community from its applications in Differential Geometry, Topology, Physics, and, in a more applied context, Image Processing.

From the geometric perspective, one can distinguish two types of geometric flows: intrinsic and extrinsic. An intrinsic geometric flow is understood to be a PDE which considers the evolution of intrinsic geometric quantities from a given riemannian manifold.

An important example of these types of flows consists in modifying a riemannian metric g_0 into one with some constant curvature tensor. More precisely, find a 1-parameter family of metrics $\{g(t)\}_{0\leq t<T}$ for a manifold M such that

$$\begin{cases} \partial_t g = F(g) & , \; t \in (0,T) \\ g(0) = g_0. \end{cases},$$

where F is a tensor which depends on the metric $g(t)$ and its derivatives. Finally, one studies the limit metric $g(T)$, where $T \in (0,\infty]$ is the maximum existence time of the flow, with the expectation that it satisfies the certain curvature condition.

In this setting, the most remarkable and studied intrinsic flow is $F(g) = -2\mathrm{Ric}(g)$, where $\mathrm{Ric}(g)$ denote the Ricci curvature tensor of $g(t)$, and it is known as the Ricci Flow.

The Ricci Flow was introduced by R. Hamilton in the 1980's as a tool to prove the Thurston's geometrization conjecture for closed 3-manifolds. This problem states that any closed 3-manifold can be canonically decomposed into pieces, where each piece possesses one of eight different types of geometric structures. In 2003, thanks to the work of G. Perelman, this conjecture was finally proved and, as a consequence, the Poincaré conjecture, which was an open problem since 1904. We refer to [CZ] for an excellent self contained book about this topic.

Figure 2.1: Dumbbel's evolution under the Ricci flow. Image courtesy of Ignacio Mcmanus.

On the other hand, an extrinsic geometric flow corresponds to a one parameter family of immersions whose deformations depend on the extrinsic curvatures of a manifolds in a target riemannian manifold.

More precisely, in the codimension one case, if $F_0 : M^n \to N^{n+1}$ is an immersion from a manifold to a riemannian manifold (N, g), one wants to find a 1-parameter family of immersions $F : M \times (0,T) \to N$ such that

$$\begin{cases} \partial_t F = -\gamma(\lambda)\nu, \ (x,t) \in M \times (0,T), \\ F(x,0) = F_0(x), \end{cases} \tag{2.0.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and ν are the principal curvature vector and the outward unit normal vector of $M_t = F(M,t)$ in N, respectively. In this thesis we will refer to Equation 2.0.1 as a γ-flow for short.

The most studied γ-flow in the literature is the Mean Curvature Flow, where

$$\gamma = H = \lambda_1 + \ldots + \lambda_n.$$

The H-flow also appears as the negative gradient flow of the area functional of M in N, and consequently, it has been used to deform hypersurfaces into minimal hypersurfaces in different ambient spaces.

Next, we want to give a brief description of the analysis of singularities related to the H-flow in $\mathbb{R}^{n+1}$ that are related to the results obtained in this thesis.
Firstly, to contextualize the analysis of the singularities that appear under the evolution through the H-flow, we want to mention a pioneering result given by G. Huisken in [Hui2], which states that any closed and convex[1] hypersurface will collapse to a point in finite time. In addition, this phenomena was characterized by the blow up of the quantity

$$\max_{M_t} |A|,$$

where $|A| = \sqrt{\lambda_1^2 + \ldots + \lambda_n^2}$ as $t \to T$, where T is the maximal time of existence of the flow. To exemplify this, the evolution of a sphere of radius R_0 under the H-flow in $\mathbb{R}^{n+1}$ is given

[1]The principal curvatures of the hypersurface are non-negative.

by

$$F(p,t) = R(t)p,$$

where $R(t) = \sqrt{R_0^2 - 2nt}$ and $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. We note that the maximal time of existence is given by $T = \dfrac{R_0^2}{2n}$, and therefore, it holds that

$$|A|(t) = \frac{\sqrt{n}}{R(t)} = \frac{1}{\sqrt{2}\sqrt{T-t}}.$$

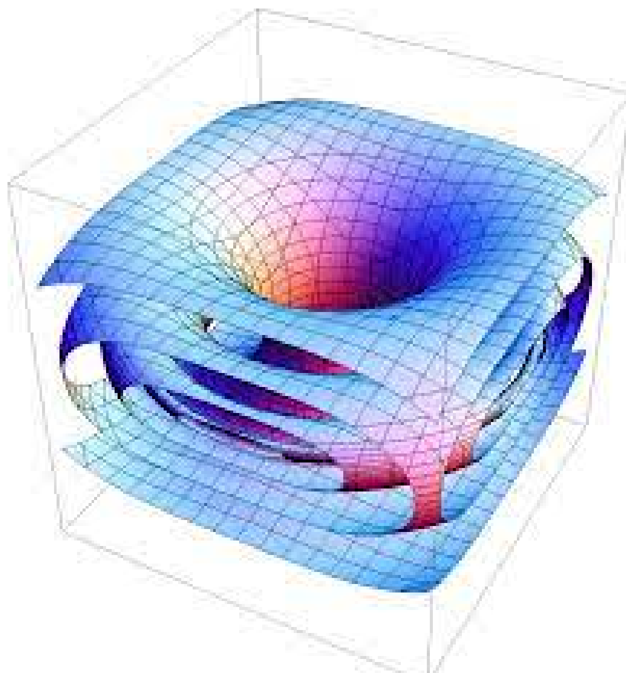


Figure 2.2: The evolution of a surface under the H-flow in $\mathbb{R}^3$. Image courtesy of Francisco Martín.

In fact, the singularities under the H-flow in $\mathbb{R}^{n+1}$ are classified into two types related to the blow up rate of $\max_{M_t} |A|$:

a) Type I: There exists a constant $C > 0$ such that $\max_{M_t} |A| \leq \dfrac{C}{\sqrt{T-t}}$.

b) Type II: It is not of type I, i.e: $\lim\limits_{t\to T}\max\limits_{M_t}|A|\sqrt{T-t}=\infty$.

The classification of these singularities is an open problem in all its generality. However, when the initial data is mean convex, i.e: $H>0$, the authors in [HS1] showed that after a suitable parabolic rescaling, the singularity can be modeled by a self-similar solution[2] of the H-flow. Indeed, these solutions are:

a) Self-shrinkers: They appear under a parabolic rescalling around a type I singularity. After passing to a limit they converge to an ancient[3] self-similar solution which evolves only by dilatation, ie: $F(x,t)=\sqrt{-t}F_0(x)$ with $t\in(-\infty,0]$.
In addition, each time slice is convex and satisfies the equation $H=-\langle p,\nu\rangle$ for $p\in M_t$.

b) Translating solitons: They appear under a parabolic rescalling around a type II singularity. After passing to a limit they converge to an eternal[4] self-similar solution which evolves by translation in an unitary direction, i.e: $F(x,t)=F_0(x)+vt$ for $t\in\mathbb{R}$ and some unit vector $v\in\mathbb{S}^n$.
In addition, each time slice can be written as $M_t=\mathbb{R}^{n-k}\times S^k$ where S^k is strictly convex hypersurface, and the equation $H=\langle\nu,v\rangle$ holds in M_t.

The main tool used in the proof of this classification was the Huisken's monoticity formula (see [Hui1]), related to the evolution of a heat kernel functional,

$$\frac{d}{dt}\int\limits_M\frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}}dx=-\int\limits_M\frac{e^{\frac{-|x-p_0|^2}{4|t_0-t|}}}{4\pi|t_0-t|^{\frac{n}{2}}}\left|H+\frac{\langle x-p,\nu\rangle}{2|t_0-t|}\right|^2dx\le 0.$$

On the other hand, we want to mention that there is an additional type of self-similar solution of the H-flow called self-expanders. In [EH1], the authors showed that the evolution

[2]A solution to the H-flow of the form $F(x,t)=r(t)O(t)F_0(x)+V(t)$, where $r(t)\ge 0$, $V(t)\in\mathbb{R}^{n+1}$ and $O(t)\in\mathcal{O}(n+1)$ is an orthogonal matrix. Visually this means that the initial hypersurface F_0 evolves under a dilatation $r(t)$, a homothety $O(t)$, and a translation $V(t)$ in $\mathbb{R}^{n+1}$.

[3]Solutions to Equation (2.0.1) which are defined for $(-\infty,a]$ for some $a\in\mathbb{R}$.

[4]Solutions to Equation (2.0.1) which are defined for $(-\infty,\infty)$.

of an entire Lipschitz graph exist for all $t > 0$ and, after a suitable rescalling, the evolution converges to a self-expanders as $t \to \infty$, ie: the limit flow converge to $F(x,t) = \sqrt{t}F_0(x)$ and each time slice satisfies $H = \langle p, \nu \rangle$. For a recent study of self-expanders we the refer to [Smo].

Finally, the classification of self-similar solutions of the H-flow is an important task to understand the behavior under the evolution of this flow. Actually, this problem has been completed for some particular cases.

In the case of convex self-shrinker the possible solutions, up to isometries and tangential diffeomorphisms, are spheres, cylinders or Abresch-Langer hypersurfaces.

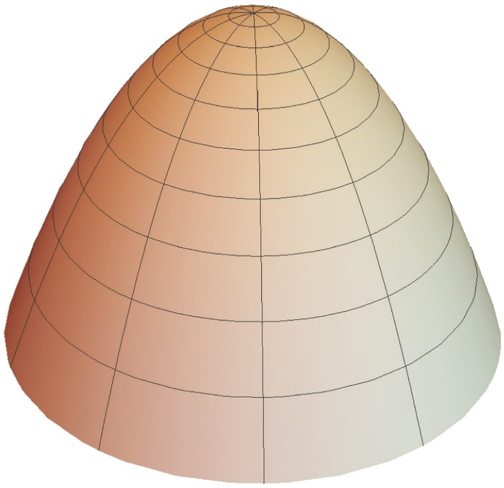
On the other hand, the classification of translating solitons have been recently completed for graphs in $\mathbb{R}^3$, see [HIMW2] for details. The possible solutions are the bowl soliton which is an entire and strictly convex graph, the delta wings, the grim-reaper and tilted griem-reapers, all these solutions are defined on strips (see Figure 2.3 below).

It is important to mention that in the classification of these self-similar solutions of the H-flow the theory of minimal surfaces was used.

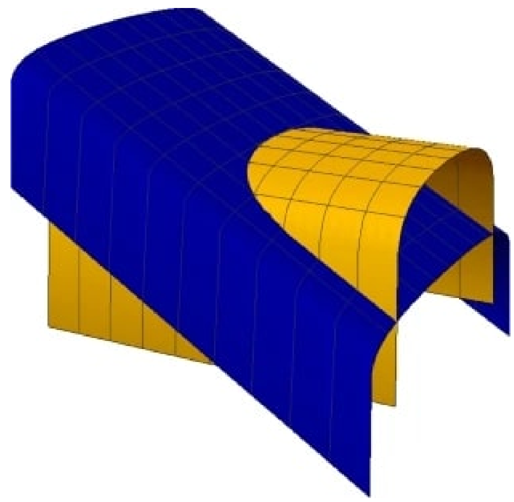
More precisely, in [Ilm], T. Ilmanen showed that these solutions can also be seen as minimal hypersurfaces in $(\mathbb{R}^{n+1}, g)$, where g correspond to $e^{-|p|^2}dx^2$ for self-shrinkers, and $e^{-\langle p,v \rangle}dx^2$ for translating solitons. We also note that these metrics are not complete, which means that the global theory of minimal surfaces does not apply for these self-similar solutions, but the local theory does.

2.1 Thesis Framework

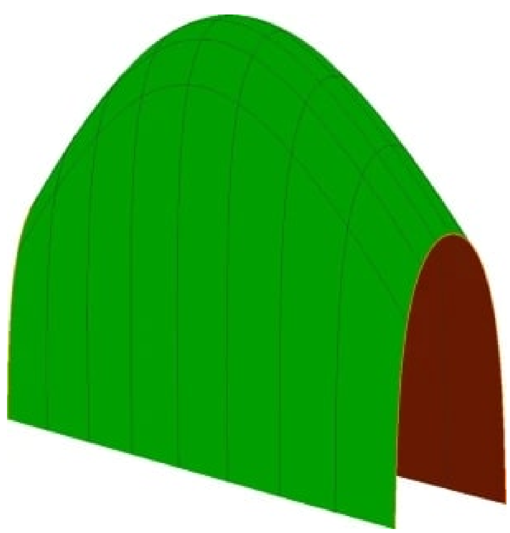
This thesis is about problems related to extrinsic geometric flow (2.0.1) for which the curvature function $\gamma : \Gamma \to \mathbb{R}$ is smooth, symmetric and 1-homogeneous function defined in an open cone $\Gamma \subset \mathbb{R}^n$ that contains the positive cone $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0,\ i = 1, \ldots, n\}$.



(a) Bowl soliton in $\mathbb{R}^3$.



(b) Grim reaper and the tilted grim reaper for $\theta = \dfrac{\pi}{6}$ in $\mathbb{R}^3$.



(c) Delta wing in $\mathbb{R}^3$.

Figure 2.3: These surfaces are taken as solutions to $H = \langle \nu, -e_3 \rangle$. Images courtesy of Francisco Martín.

We chose to study this type of curvature functions because they are a natural geometrical analogue from the theory of PDEs. In fact, we want to remark the theory developed by Nirenberg, Caffarelli and Spruck in [CNS1] and [CNS2] for prescribing fully nonlinear[5] curvature equations in bounded domains in $\mathbb{R}^n$ with Dirichlet boundary conditions. This theory served as inspiration for the work developed in this thesis.

Moreover, an important class of 1-homogeneous curvature function considered in this thesis are of quotient Hessian type[6], i.e:

$$\sqrt[k]{S_k} \text{ and } Q_k = \frac{S_{k+1}}{S_k}, \text{ or more generally, } Q_{k,l} = \sqrt[k-l]{\frac{S_k}{S_l}}. \tag{2.1.1}$$

Recall that these functions are evaluated in the principal curvatures of a given hypersurface, and by definition they are the eigenvalues of the shape operator, i.e:

$$\begin{aligned} \mathcal{W}_p :\ T_pM &\to T_pM \\ \vec{v} &\to -\nabla_{\vec{v}}\nu, \end{aligned}$$

where ∇ and ν are the Levi-Civita connection of M and the inward unit normal vector of M in $\mathbb{R}^{n+1}$. In local coordinates, the shape operator can be written by $h^i_j = g^{ik}h_{kj}$, where g^{ij} denotes the coefficients of the inverse of the metric tensor g_{ij} of the hypersurface, and h_{ij} denotes the coefficients of the second fundamental form of hypersurface, respectively.

In addition, the coefficients of the second fundamental form correspond to the tangent components of the covariant derivative of the unit normal map (modulo orientation), i.e.: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ where $\{e_i\}$ denotes a local coordinates of the tangent space centered at some point of the hypersurface.

[5]This means that in coordinates $(x, u(x))$, the prescribed curvature equation is a fully nonlinear elliptic PDE.

[6]The reason of the name comes from the theory of prescribing the elemental symmetric polynomials evaluated in the eigenvalues of $\text{Hess}(u)$. They are also known as of Monge–Ampère type since the elementary symmetric polynomials appear in expression of the form $\det(\text{Hess}(u) - \lambda \text{Id})$.

Furthermore, the symbol

$$S_k(\lambda) = \sum_{1 \le i_1 < \ldots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k} \tag{2.1.2}$$

denotes the elementary symmetric polynomial of grade k in n-variables, and we use the convention $S_0 = 1$ and $S_j = 0$ for $j > n$.

Note that the mean curvature $H = S_1(\lambda)$, the scalar curvature $R_g = \dfrac{2}{n(n-1)} S_2(\lambda)$ and the guassian curvature $K = S_n(\lambda)$.

Finally, is important to remark that these flows are parabolic when the principal curvatures of the evolution M_t belong to the Garlin cones

$$\Gamma_k := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : S_j(\lambda) > 0. j = 1, \ldots, k\}, \tag{2.1.3}$$

We also mention that for us Γ_k always will denote the connected component that contains the positive cone Γ_+.

Now we want to discuss some previous works and results related to γ-flows in the literature.

Firstly, when the initial data is compact and strictly convex, the evolution under a γ-flow presents a similar behavior to the H-flow. In fact, in [And2] the author showed that under some natural assumptions on γ (see for instance Properties 1-7 in Chapter 4) the evolution will collapse to a point in finite time, and after a suitable parabolic rescalling, the solution converge to a unit sphere in $\mathbb{R}^{n+1}$.

However, up to the present, the analysis of singularities related to the evolution under γ-flows is not as well understood as in the H-flow case. In fact, one of the main problems is the nonlinearity related to the geometry involved. For example, Huisken's monoticity formula is available to describe singularities of the H-flow, while a similar formula is not known for others curvature functions.

Encouragingly, there are few results related to the study of singularities under γ-flows that we briefly describe. In [Lyn3] the author classified uniformly parabolic convex ancient solutions to γ-flows which present a Type I singularity.
Namely, these solutions are shrinkers spheres or cylinders. One important aspect of his result is that includes convex and concave curvature functions which satisfy extra concavity properties and admits an splitting theorem.

Furthermore, if the curvature function is convex (and satisfies properties (1)-(3) and 5 describe in Chapter 5), more results are known related to Type II singularities. In [ALM], the authors showed that eternal solutions which present a Type II singularities split as $M = \Sigma^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and evolve under translation by a fixed unitary direction. It is still an open question if this phenomena occurs for concave functions.

Finally, some authors also defined another type of singularity which is known of Type 0, and it is related to the loss of convexity of the initial data under a γ-flow.
To describe this phenomena we firstly mention that if the initial hypersurface satisfies that the principal curvatures belong to the symmetric cone

$$\left\{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0, \frac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0, \text{ for } i = 1, \ldots, n\right\}^{7},$$

then under the γ-flow this property is also satisfied by the evolution M_t. In fact, this can be seen as an easy application of the strong maximum principle applied to the equation

$$(\partial_t - \Delta_\gamma)\gamma = |A|^2_\gamma \gamma,$$

[7]Note that this is an open cone when γ is a symmetric 1-homogeneous continuous function.
In addition, if γ is concave, then this cone is also convex. This can be seen by the fact under the 1-homogeneous hypothesis, concave is equivalent to being sub-additive (i.e: $\gamma(a\lambda + b\mu) \geq a\gamma(\lambda) + b\gamma(\mu)$).

where $\Delta_\gamma = \frac{\partial \gamma}{\partial h_{ij}} \nabla_i \nabla_j$ and $|A|^2_\gamma = \frac{\partial \gamma}{\partial h_{ij}} h_{ia} h_{aj}$. Note that for this argument we used that the matrix $\left(\frac{\partial \gamma}{\partial h_{ij}} \right)_{i,j}$ is a locally uniformly positive definite matrix.

Nevertheless, there are examples of curvature functions that do not preserve the positive cones Γ_+ under the evolution of the flow.

In [AMZ], the authors constructed a convex hypersurface in $\mathbb{R}^{n+1}$ such that the evolution under the γ-flow given by

$$\gamma(\lambda) = \sum_{1 \le i < j \le n} \frac{\lambda_i \lambda_j}{\sqrt{\lambda_i^2 + \lambda_j^2}},$$

loose it convexity for small $t > 0$. Note that $\gamma(\lambda)$ is positive, strictly increasing in each variable, and concave in the cone $\{\lambda \in \mathbb{R}^n : \lambda_i + \lambda_j > 0\}$.

Moreover, the authors characterized the loss of convexity under the evolution of concave curvature functions if the hypothesis of being inverse-concave at the restriction of the boundary of Γ_+ was not satisfied. This means that if $\gamma(1, ..., 1, 0) > 0$, then the function

$$\tilde{\gamma}(\lambda_1, \dots, \lambda_{n-1}) = \gamma(\lambda_1^{-1}, \dots, \lambda_{n-1}^{-1}, 0)^{-1}$$

is concave in the $(n-1)$-dimensional positive cone $\Gamma_+^{n-1} = \{\lambda \in \mathbb{R}^{n-1} : \lambda_i > 0\}$.

Finally, we want to mention that Type 0 singularities do not occurs under the evolution of the H-flow. In fact, in [HS2] the authors showed that all the Garlin cones Γ_k are preserved under H-flow, given in particular that the initial data will not lose its convexity under the flow. Consequently, the study of γ-flows for nonlinear curvature functions can be used to evolve hypersurfaces such that only some convexity cones are preserved under the evolution of the flow.

2.2 Curvature Problems and Results

Firstly, in this thesis we are also interested in translating solitons of the γ-flow in $\mathbb{R}^{n+1}$. As we mentioned for the H-flow, these are eternal self-similar solutions of (2.0.1) which evolve under translations by vector field $V : \mathbb{R} \to \mathbb{S}^n$, i.e:

$$F(x,t) = F_0(x) + V(t).$$

We can check that $V(t) = \vec{v}t$ for some constant vector $\vec{v} \in \mathbb{S}^n$.
Indeed, we note that

$$\partial_t V = \partial_t F(x,t) = -\gamma(\lambda)\nu.$$

Then, since the shape operator $\mathcal{W}_p$ is invariant under translation, the principal curvatures also are invariant. In particular $\partial_t \gamma(\lambda) = 0$, and this implies that $\partial_t^2 V(t) = 0$, or equivalently, $V(t) = \vec{v}t$ for some constant vector $v \in \mathbb{S}^n$. However, in this thesis we will only consider translating solitons of the γ-flow with translation vector $\vec{v} = e_{n+1}$ for simplicity.
Furthermore, we note that translating solitons of the γ-flow, or γ-translators for short, satisfy in each time slice M_t the equation

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle \,. \tag{2.2.1}$$

This means that γ-translators can be studied with the theory of parabolic equations, or with the theory of prescribing elliptic curvature equations.

The problems that we study in this thesis are more related with the elliptic setting of γ-translators. In fact, the assumptions rewarding the curvature function $\gamma : \Gamma \to \mathbb{R}$ (see Section 5.1) are in the spirit of the classical differential geometry problems:

- γ is evaluated in the cone $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$.
- For every $i = 1, \dots, n$, $\dfrac{\partial \gamma}{\partial \lambda_i}(\lambda) > 0$ holds in Γ.

In local coordinates, this property gives that Equation (2.2.1) is locally uniformly elliptic and every connected, and furthermore, each simple connected γ-translator is a vertical graph over some open connected set $\Omega \subset \mathbb{R}^n$.

On the other hand, when we consider curvature functions that have a continuous extension at the boundary of $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$, such as the family $\sqrt[k]{S_k}$ supported in Γ_k, we have as trivial solutions to Equation (2.2.1) vertical hyperplanes and round cylinders (the unit normal vector is orthogonal to the hyperplane which contains e_{n+1}). Note that there are curvature functions that do not have this property, such as Q_k supported in Γ_k, given that the trivial solution are vertical round cylinders.
Finally, we want to mention that the theory of degenerated elliptic PDE's is useful for the study of curvature functions that vanish in the boundary of Γ. We refer the reader to see [CNS1], [CNS2], [Ivo], [SUW] for nice introductory papers on this subject.

It is worth to mention that if we consider γ-translators in the parabolic setting, there are studies that one can initiate the γ-flow for curvature functions γ that are supported in $\{\lambda \in \mathbb{R}^n : \gamma(\lambda) \geq 0\} \cap \overline{\Gamma}_+$.
In fact, this was done by the author in [Die] for the Q_k-flow. She showed that the Q_k-flow can be initiated for closed convex initial data such that $Q_k \geq 0$ and the principal curvatures belong to Γ_{k-1}. Her main contribution was the use of cylindrical barriers that allow the H-flow to be initiated, giving that the evolution becomes strictly convex for positive times, allowing the Q_k-flow to be started. In particular, the evolution posses the same behavior as the exposed in [And1].

It remains an open problem if this procedure can be applied for different curvature function, or if it can be extended to different open cones rather than Γ_{k-1}. We refer the reader to [ACGL] for an introductory book about γ-flows in different settings.

2.2.1 Non-variational approach for Q_k-translators in $\mathbb{R}^{n+1}$

In this subsection we are going to show that there is no variational approach for Q_k-translators in $\mathbb{R}^{n+1}$. Recall Q_k-translators satisfy

$$Q_k = \langle \nu, e_{n+1} \rangle \Leftrightarrow S_{k+1} - S_k \langle \nu, e_{n+1} \rangle = 0, \tag{2.2.2}$$

note that the if, and only if, occurs when principal curvatures belong to Γ_k.

Therefore, to check if we can find a functional such that the equation (2.2.2) appears as an Euler-Lagrange equation, we will use the formulas obtained in Chapter 4 of [Ros].

Firstly, for a given smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we consider the functional over closed manifolds $M \subset \mathbb{R}^{n+1}$ given by

$$\mathcal{F}(M) = \int_M f(S_1, \dots, S_n, \langle X, e_{n+1} \rangle) dA,$$

where S_i are the symmetric elemental polynomials evaluated in the principal curvatures of $M \subset \mathbb{R}^{n+1}$ and X is the position vector in $\mathbb{R}^{n+1}$.

Then, by choosing a normal variation $\varphi : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ with $\varphi_0 = \mathrm{Id}$ and $\left.\frac{d\varphi_t}{dt}\right|_{t=0} = g\nu$, where $\varphi_t(x) = \varphi(x,t)$ and $g \in \mathcal{C}_c^{\infty}(M)$, we can see that

$$\left.\frac{d}{dt}\mathcal{F}(M_t)\right|_{t=0} = \int_M \left.\frac{d}{dt} f\right|_{t=0} dA + \int_M f \left.\frac{d}{dt} dA\right|_{t=0}, \tag{2.2.3}$$

where $M_t = \varphi_t(M)$.

Next, we calculate each term in Equation 2.2.3 separately.

We use Theorem 4.1 in [Ros] to calculate the first term in the above equation

$$\begin{aligned}\frac{d}{dt}f\bigg|_{t=0} &= \sum_{i=1}^{n} D_i \frac{d}{dt}S_i\bigg|_{t=0} + D_{n+1}f\ \frac{d}{dt}\left\langle X, e_{n+1}\right\rangle\bigg|_{t=0} \\ &= \sum_{i=1}^{n} D_i f\left(L_{i-1}(g) + g(S_1S_i - (i+1)S_{i+1})\right) + gD_{n+1}f\left\langle \nu, e_{n+1}\right\rangle,\end{aligned}$$

where $L_{i-1}(g) = \mathrm{div}_{_M}\left(T_{i-1}\nabla g\right)$ and T_i is the ith-Newton transformation of the shape operator of M in $\mathbb{R}^{n+1}$ defined inductively by the relation

$$\begin{cases} T_i(\mathcal{W}_p) = S_i Id - \mathcal{W}_p T_{i-1}(\mathcal{W}_p), \\ T_0 = Id. \end{cases}$$

On the other hand, it is a well know result that the variation formula of the volume form under normal variations is given by

$$\frac{d}{dt}dA\bigg|_{t=0} = -gHdA.$$

Consequently, we obtain the following formula for the first variation of $\mathcal{F}$ given by

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(M_t)\bigg|_{t=0} \qquad (2.2.4)\\ &= \int_M \sum_{i=1}^{n} D_i f\left(L_{i-1}(g) + g(S_1S_i - (i+1)S_{i+1})\right) + gD_{n+1}f\left\langle \nu, e_{n+1}\right\rangle - gHf dA.\end{aligned}$$

As we mentioned before, H-translators are critical points of the functional

$$F_0(M) = \int_M e^{\langle X, e_{n+1}\rangle} dA.$$

Therefore, it makes sense to consider the following function for Q_k-translator given by

$$\mathcal{F}_k(M) = \int_M S_k e^{(k+1)\langle X, e_{n+1}\rangle} dA.$$

Then, since $f = x_k e^{(k+1)x_{n+1}}$ in $\mathcal{F}_k(M)$, the first variation is easy to compute. Indeed, by Formula (2.2.4), it follows

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}_k(M_t)\Big|_{t=0} \\ &= \int_M -(k+1)g\left(S_{k+1} - S_k\langle \nu, e_{n+1}\rangle\right) e^{(k+1)\langle X, e_{n+1}\rangle} + L_{k-1}(g) e^{(k+1)\langle X, e_{n+1}\rangle} dA. \end{aligned} \tag{2.2.5}$$

Remark 2.2.1. Note that the term $L_{k-1}(g)$ in (2.2.5) only vanish when $k = 0$, since $T_{-1} = 0$ by definition, and this agree with the fact that H-translator are minimal hypersurface in $(\mathbb{R}^{n+1}, e^{\langle X, e_{n+1}\rangle} dx^2)$.

Moreover, for $k = 1$, $L_0(g) = \operatorname{div}_M(T_0 \nabla g) = \Delta g$, where Δ is the Laplace-Beltrami operator of M. Therefore, by the divergence theorem, this term in the formula (2.2.5) can be written by

$$\begin{aligned} \int_M L_0(g) e^{2\langle X, e_{n+1}\rangle} dA &= \int_M g \Delta e^{2\langle X, e_{n+1}\rangle} dA \\ &= \int_M 2g\left(2|e_{n+1}^\top|^2 - H\langle \nu, e_{n+1}\rangle\right) e^{2\langle X, e_{n+1}\rangle} dA. \end{aligned}$$

where $e_{n+1}^\top$ is the orthogonal projection onto T_pM in $T_p\mathbb{R}^{n+1}$.

Finally, as the reader could suspect, Equation (2.2.2) is not the Euler-Lagrange for $\mathcal{F}_k$ for $k \geq 1$, since more curvature terms will appears in the Euler-Lagrange equation. Consequently, we do not expect the existence of any functional such that a Q_k-translator appears as a minimal hypersurface in a weighted Euclidean space.

2.2.2 Gradient and second order estimates for Q_k-equations

From the PDE perspective, gradient and second-order estimates are important tools for constructing solutions, and analyzing their properties and behavior.

As mentioned above, in [EH1] and [EH2] the authors obtain local gradient and curvature estimates in order to construct noncompact solutions of the H-flow in $\mathbb{R}^{n+1}$. These results and techniques motivated the development of a similar theory to be applied for the γ-flows. In this sense, we want to mention the work of [CD], where the authors constructed complete non-compact convex graphs evolving under the Q_k-flow in $\mathbb{R}^{n+1}$, see Equation (2.1.1) for the definition. In their work the use of convexity was an important fact.

In this thesis we are concerned in relaxing the convexity assumption used in [CD] to obtain local gradient and second order estimates for the Q_k-flow, and also for Q_k-translators, such that the principal curvatures belong to the cone Γ_{k+1}.

More precisely, if $F(x,t) = (x, u(x,t))$ where $u : \Omega \times [0,T) \to \mathbb{R}$ then Equation (2.0.1) with $\gamma = Q_k$, can be written by

$$u_t = Q_k(Du, D^2u)\sqrt{1 + |Du|^2}, \tag{2.2.6}$$

and for Q_k-translators, if $F(x) = (x, u(x))$ with $u : \Omega \to \mathbb{R}$, Equation (2.2.1) can be written by

$$Q_k(Du, D^2u) = \frac{1}{\sqrt{1 + |Du|^2}}. \tag{2.2.7}$$

Therefore, to obtain gradient and second-order estimates without the hypothesis of convexity imposed in the graphs, we had to get a better understanding of the term $Q_k(Du, D^2u)$[8] when

[8] We note that we are abusing with the notation in writing $Q_k(Du, D^2u)$ instead of $Q_k(\lambda)$. The reason why we choose to wrote in this way is to emphasize the dependence in the variables of the operator Q_k when is evaluated at the principal curvatures of M_t or M.
In addition, we used a convenient way to work with this term, see Section 3.2 in Chapter 3 for more details.

the principal curvatures of the graphs belong to the cone Γ_{k+1}.

In fact, thank to the work realized in [Hol] for the $\sqrt[k]{S_k}$-flow when the principal curvatures of M_t belong to the cone Γ_k, we could obtain the following estimate for the Q_k-flow.

Theorem 2.2.1. *Let $u : \Omega \to \mathbb{R}$ be a solution to Equation (2.2.6), where $\Omega = B_r(0) \times (0, T]$ with $r > 0$. Assume that the principal curvatures of the graph of u lies in the cone Γ_{k+1} defined in (2.1.3) for each t. Then, for $t > 0$, it holds*

$$|Du(0,t)| \leq \exp\left(K + \frac{KMT}{rt}\frac{MrT}{t} + \frac{M^2}{t} + \frac{(T+1)r^2}{t} + \frac{TM^2}{tr^2} + \frac{TM}{tr}\right),$$

where $M = \sup\limits_{\Omega} u$ and $K = K(k,n)$.

Moreover, since we were able to adapt the technique used in the previous theorem for Q_k-translators, we obtained the following estimate of the gradient for that equation.

Theorem 2.2.2. *Let $r > 0$ and $u \in \mathcal{C}^3(B(0,r))$ be a solution (2.2.7) such that the principal curvatures of graph(u) lies in Γ_{k+1}. Then it holds*

$$|Du(0)| \leq \exp\left(\frac{CM}{r} + \frac{CM^2}{r^2} + C\right),$$

where $M = \sup\limits_{B(0,r)} u$ and $C = C(k,n)$.

On the other hand, by adapting the proof of the previous theorem, we obtained a Liouviullie type theorem for Q_k-translators which reads as follows.

Theorem 2.2.3. *There are no solutions $u \in \mathcal{C}^3(\mathbb{R}^n)$ such that*

1. *The principal curvatures of the graph of u lie in Γ_{k+1}.*

2. *$u(x) = o(|x|)$ as $|x| \to \infty$.*

3. The graph of u is Q_k-translator in $\mathbb{R}^{n+1}$.

The second order estimates that we obtained below depend on the assumption that the principal curvatures belong to the cone Γ_{k+2} for $k \geq 1$. The reason is that any hypersurface with their principal curvatures lying in Γ_{k+2} with $k \geq 1$, satisfies $|A|^2 \leq H^2$ and $|A|_k^2 \leq CH^2$. Therefore, since it is easier to estimate H than $|A|$ or $|A|_k^2$, we apply techniques similar to those in [EH2] to obtain second order estimate for solutions to equations (2.2.6) and (2.2.7), respectively.

In addition, as in [EH2], we assume that there is a non-negative function $h(x,t)$ such that

$$|(\partial_t - \Box_k)h| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h,$$

where $\Box_k f = \dfrac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$ and $|X|_k = \dfrac{\partial Q_k}{\partial h_{ij}} X_i X_j$. See Remark 3.3.2 in Section 3.3 for an example of a function $h(x,t)$.

Theorem 2.2.4. *Let $R > 0$ such that $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ is graph over a ball of radius R in the hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$ for all $[0,T)$.*
Assume that M_t evolves under the Q_k-flow and the principal curvatures belong to the cone Γ_{k+2} with $k \geq 1$. Then, for any $t_0 \in (0,T)$ and $\theta \in [0,1)$, the following estimate holds

$$\sup_{M_t} H^2 \leq \frac{c(k,n)}{(1-\theta)^2} \left(\frac{1}{t} + \frac{1}{R^2} \right) \sup_{[0,t_0]} \sup_{M_t} v^4,$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$.

For the Q_k-translator, we consider a non-negative function $h(x)$ with the above properties but changing the assumption $|(\partial_t - \Box_k)h| \leq C(k,n)$ to

$$|\Box_k h + \langle h, e_{n+1} \rangle| \leq C(k,n).$$

Theorem 2.2.5. *Let $R > 0$ such that $M = \{x \in M : h(x) \leq R^2\}$ is a graph over a ball of radius $R > 0$ in the hyperplane orthogonal to e_{n+1}.*
Assume that M is a Q_k-translator such that the principal curvatures belong to the Γ_{k+2} with $k \geq 1$. Then, for any $\theta \in [0, 1)$, the following estimate holds

$$H^2 \leq \frac{c(k,n)}{(1-\theta)^2}\left(1 + \frac{1}{R^2}\right) \sup_{\{h \leq \theta R^2\}} v^4,$$

where $v = \langle \nu, e_{n+1} \rangle^{-1}$.

2.2.3 Rotationally symmetric γ-ranslator in $\mathbb{R}^{n+1}$

In this subsection we want to discuss about the cunstruction of rotationally symmetric γ-translator in $\mathbb{R}^{n+1}$ for particular curvature functions.

Firstly, a graphical rotationally symmetric γ-translator in $\mathbb{R}^{n+1}$ with $\gamma > 0$ correspond to the set $\{(x, u(x)) \in \mathbb{R}^{n+1} : |x| < R\}$ where $u(x) = \mathbf{u}(r)$, $\mathbf{u} : [0, R) \to \mathbb{R}$ and $r = |x|$.

Then, Equation (2.2.1) turns into an ordinary differential equation (ODE) of the form

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2) f\left(\frac{\dot{\mathbf{u}}}{r}\right),$$

where the dot notation means derivatives with respect r and f is a function which appears by applying the implicit function theorem on the connected components of the level set $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$. We note that in this case, the principal curvatures are two, and since $\gamma(\lambda)$ is increasing in each variable, the implicit function theorem guarantees a local expression of $f(x)$ in each connected component of $\{\gamma(\lambda) = \langle \nu, e_{n+1} \rangle\}$.

In the case of the H-translators, the authos in [AW] constructed a strictly convex solution defined for $R = \infty$ called the "bowl" soliton (see Fig. 2.3a) which behaves like a paraboild at infinity. More precisely, in [CSS], the authors showed that the "bowl" soliton in $\mathbb{R}^{n+1}$

behaves at infinity as

$$\frac{r^2}{n} - \ln(r) + O\left(\frac{1}{r}\right).$$

In this thesis, we construct "bowl"-type solution for $\gamma = Q_{n-1}$ and $\gamma = \sqrt[n]{S_n}$ in $\mathbb{R}^{n+1}$. These solutions are strictly convex rotationally symmetric graph over the $\{x_{n+1} = 0\}$ hyperplane, which are defined in balls or the whole hyperplane.

Particularly, for $n = 2$, these solutions are explicit and they are given by

$$-\ln(1 - r^2) \text{ and } \int_0^r \sqrt{e^{s^2} - 1} ds$$

for Q_1 and $\sqrt{S_2}$, respectively. We note that the "bowl"-type $\sqrt{S_2}$-translator is not $\mathcal{C}^2$ at the origin, and the "bowl"-type Q_1-translator is $\mathcal{C}^2$-asymptotic[9] to the cylinder $\mathbb{S}^1 \times \mathbb{R}$.

Theorem 2.2.6. *For $n \geq 3$ there exists a unique smooth strictly convex translator in $\mathbb{R}^{n+1}$ for the curvature functions Q_{n-1} and $\sqrt[n]{S_n}$, such translators are rotationally symmetric graphs.*
In addition, it holds

1. *For $n \geq 3$, the Q_{n-1}-translator is defined in a ball of radio $Q_{n-1}(1, \ldots, 1) = \dfrac{1}{n}$ and is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}\left(\frac{1}{n}\right) \times \mathbb{R}$.*

2. *For $n \geq 3$, the $\sqrt[n]{S_n}$-translator is entire.*

Furthermore, we also found a family of rotationally symmetric $\sqrt{S_2}$-translator which are graph over the $\{x_1 = 0\}$ plane in $\mathbb{R}^3$ which satisfy $H < 0$ and $K > 0$.

Theorem 2.2.7. *Let $a \in [0, 1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{(r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R}\right\},$$

[9]This means that the principal curvatures $\lambda_1 \leq \ldots \leq \lambda_n$ satisfies $\lambda_1 \to 0$ and $\lambda_i \to 1$ for $i \neq 1$ as $|p| \to \infty$.

where $z = \int\limits_1^{r_a(z)} \sqrt{e^{s^2-a}-1}ds$ *with* $r_a(0) = 1$*, are complete* $\sqrt{S_2}$*-translators in* $\mathbb{R}^3$ *which satisfies* $H < 0$ *and* $K > 0$*.*

More recently, in [Ren], the author showed the existence of "bowl"-type solutions for a general class of α-homogeneous symmetric curvature functions, without any assumption about the convexity or concavity. In addition he characterized when the solution will be defined in a ball or entire in terms of the curvature function, see the introduction of Chapter 4 below.

2.2.4 Geometric Properties of γ-translators in $\mathbb{R}^{n+1}$

Another important aspect that we studied in this thesis are geometric properties of γ-translators in $\mathbb{R}^{n+1}$. Here we understand geometric properties as the application of PDE theory applied in the context of differential geometry: tangential principles, non-existence, uniqueness , and convexity estimates. One of the most important examples of such applications are maximum principles applied to geometric quantities of a hypersurface.

Recall that H-translators are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1}\rangle}dx^2)$. Therefore, the tangential principle, which allows comparing H-translators according to how they touch each other, is a well-known result from the minimal surface theory. Nevertheless, since γ-translators are not minimal hypersurfaces for $\gamma \neq H$, we develop a tangency principle which reads as follows.

Theorem 2.2.8. *Let* $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ *be two embedded connected* γ*-translators such that*

1. $\gamma : \Gamma \to [0, \infty)$ *satisfies properties 1-4 and 7 given in Section 5.1.*

2. Σ_1 *is strictly convex.*

3. Σ_2 *is convex.*

Then,

a) ***(Interior tangential principle)*** *Assume that there exists an interior point* $p \in \Sigma_1 \cap \Sigma_2$ *such that the tangent spaces coincide at* p*. If* Σ_1 *lies at one side of* Σ_2*, then both hypersurfaces coincide.*

b) ***(Boundary tangential principle)*** *Assume that the boundaries* $\partial\Sigma_i$ *lie in the same hyperplane* Π *and the intersection of* Σ_i *with* Π *is transversal. If* Σ_1 *lies at one side of* Σ_2 *and there exist* $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ *such that the tangent spaces coincide, then both hypersurfaces coincide.*

In addition, thanks to the results exposed in [Ren] (see Theorems 1.3-1.4) we also could show a non-existence result for γ-translator such that the "bowl"-type γ-translator is defined in a round ball.

Corollary 2.2.9. *Assume that* $\gamma : \Gamma \to [0, \infty)$ *satisfies Properties 1-4 and 7. Then, there are no convex entire* γ*-translators in* $\mathbb{R}^{n+1}$ *such that the "bowl"-type* γ*-translator is defined in a round ball.*

This is a curious result that we can use in the following way: if there is an entire graph evolving under a γ-flow such that the "bowl"-type γ-translator is defined in a ball, then this entire graph cannot converge to a translator as $t \to \infty$.

Next, we also apply the method of moving planes of Alexandrov to obtain that the graphical γ-translator inherits the properties defined in the boundary of its domain.

Theorem 2.2.10. *Let* $\Sigma \subset \mathbb{R}^{n+1}$ *be a complete* γ*-translator such that*

1. $\gamma : \Gamma \to [0, \infty)$ *satisfies properties 1-4 and 7.*

2. Σ *is strictly convex graph over a ball* $B_r^n(0) \subset \mathbb{R}^n$*.*

3. Σ posses a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, i.e: the principal curvatures of Σ satisfy

$$\min \{\lambda_i(p) : i = 1, \dots, n\} = \lambda_1(p) \to 0,$$
$$\forall i \in \{2, \dots, n\}, \lambda_i(p) \to \frac{1}{r},$$

as $|p| \to \infty$.

Then Σ is rotationally symmetric with respect the x_{n+1}-axis.

Moreover, as a corollary of the above theorem and the uniqueness result exposed in [Ren], we obtain the following uniqueness property.

Corollary 2.2.11. *Let Σ be a γ-translator as in the previous theorem such that the "bowl"-type γ-translator is defined in a ball of radius $\gamma(1, \dots, 1)$. Then, Σ coincide with the "bowl"-type γ-translator.*

Finally, the main result of this thesis is a convexity estimate for γ-translators in $\mathbb{R}^{n+1}$. To gives a context of why this is the main result of this thesis, we star by announcing some facts related to mean convex H-translators.
Firstly, in [SX], the authors shown that mean convex H-translators in $\mathbb{R}^3$ are convex (see [HIMW3] for a shorter proof). This result had a remarkable implication in the classification of graphical H-translators in $\mathbb{R}^3$. Moreover, in [Has], the author show that a α-noncollapse and convex H-translator must be the bowl soliton. In summary, since mean convex H-translators are convex, it follows that the only entire mean convex one must be the bowl soliton. We refer the reader to [HIMW2] for the classification of H-translators which are graphs.

Therefore, in the spirit of showing that γ-translator are convex we proved the following convexity estimate.

Theorem 2.2.12. *Let $n \geq 3$, $\alpha, \delta > 0$ and $\Sigma \subset \mathbb{R}^{n+1}$ be a complete, immersed, two-sided γ-translator such that*

a) $\gamma : \Gamma \to \mathbb{R}$ *satisfies Properties 1-3 and 5-7.*

b) $\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : \alpha H \leq (\delta+1)\gamma\}$ *which is compactly supported in* $\Gamma \setminus Cyl_{n-1}$*, where*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\}\,.$$

c) There exist a constant $\beta \in (0,1)$ *such that* $\lambda_i + \lambda_j \geq \beta H$*, for every* $1 \leq i < j \leq n$*.*

Then, $\lambda_1 \geq H - \alpha\gamma$ *in* Σ*, where* $\lambda_1(p) = \min\{\lambda_i(p) : i = 1, \ldots, n\}$*.*

The proof of Theorem 2.2.12 was inspired by [SS], where the authors show that a mean convex and uniform 2-convex translating soliton of the MCF is convex. We remark that our proof does not work for translators of the MCF. The main reason is that the cone $\Gamma_{\alpha,\delta}$ needs to be compactly supported in $\Gamma_1 \setminus C_{n-1}$, but for $\gamma = H$, $\Gamma_{\alpha,\delta} = \Gamma_1$.
On the other hand, the proof given in [SS] cannot be directly adapted to a concave speed function γ, since the authors use a concave approximation to λ_1 that cannot be used for concave curvature functions γ.

Remark 2.2.2. A similar estimate was proved in [Lyn2] for a family curvature functions of the form

$$(1-c)H - c\left(\sum_{1=i_1<\ldots<i_k=n} \frac{1}{\lambda_{i_1} + \ldots + \lambda_{i_k}}\right)^{-1},\ c \in (0,1).$$

In contrast with our result, his estimate is in the compact parabolic context.

In consequence, as an application of the convexity estimate, we were able to show that γ-translator are asymptotically convex.

Corollary 2.2.13. *Let* $\Sigma \subset \mathbb{R}^{n+1}$ *be* γ*-translator where* $\gamma : \Gamma \to \mathbb{R}$ *satisfies the hypothesis of Theorem 2.2.12, then* Σ *is asymptotically convex.*

Chapter 3

Estimates for Q_k-flow and Q_k-translators

In this chapter, we are interested in a particular class of degenerated geometric extrinsic evolution equations, where the speed of the flow is given by a 1-homogeneous function in the principal curvatures of the given initial hypersurface.

More precisely, given a manifold M^n and an immersion $F_0 : M^n \to \mathbb{R}^{n+1}$, we say that hypersurface $M_0 = F_0(M)$ evolves under the Q_k-flow if there exist a 1-parameter family of immersions $F : M \times [0,T) \to \mathbb{R}^{n+1}$ such that

$$\begin{cases} \left\langle \dfrac{\partial F}{\partial t}, -\nu \right\rangle = Q_k(\lambda), \text{ in } M \times (0,T), \\ F(\cdot, 0) = F_0(\cdot), \end{cases} \tag{3.0.1}$$

where ν is outward unit normal vector of $M_t = F(M,t)$ in $\mathbb{R}^{n+1}$,

$$Q_k = \frac{S_{k+1}}{S_k},$$

and $S_k(\lambda)$ denotes the elementary symmetric polynomial in n variables of degree k evaluated at the principal curvatures of $M_t = F(M,t)$, i.e:

$$S_{k+1}(\lambda) = \sum_{1 \le i_1 < i_2 < \ldots i_{k+1} \le n} \lambda_{i_1} \ldots \lambda_{i_{k+1}}. \tag{3.0.2}$$

Recall that the principal curvatures of a hypersurface M^n in $\mathbb{R}^{n+1}$ are the eigenvalues of the Weingarten map $h^i_j = g^{ik} h_{kj}$, where g^{ij} denotes the coefficients of the inverse of the metric tensor g_{ij}, and h_{ij} denotes the coefficients of the second fundamental form of M, respectively. The coefficients of the second fundamental form correspond to the tangent components of the covariant derivative of the unit normal map (modulo orientation), i.e.: $h_{ij} = \langle \nabla_i \nu, e_j \rangle$ where $\{e_i\}$ denotes local coordinates of T_pM.

In addition, by definition $S_0 = 1$ and $S_i = 0$ for $i > n$. We remark that $Q_0 = H$ is the scalar mean curvature, which has been widely studied in this century, see [CMP] for an introductory survey.

From the PDE perspective, Equation 3.0.1 is geometrically equivalent[1] to a locally uniformly parabolic equation when the principal curvature vector λ lie in the cone

$$\Gamma_{k+1} = \{\lambda \in \mathbb{R}^n : S_l(\lambda) > 0,\ l = 0, \ldots, k+1\}.$$

In particular, this fact motivates the study of the Q_k-flow in papers [And1], [Die] and [CD], which we describe in what follows. In [And1], the author shows classical results about existence, uniqueness and collapsing for closed strictly convex initial hypersurfaces. In [Die], the author uses a weaker condition on the convexity of the initial hypersurface to show similar results to [And1]. Namely, the author showed that one can start the Q_k-flow when $\lambda \in \Gamma_k \cap \{\lambda \ge 0\}$ instead of being strictly convex. This allows the function Q_k to vanish at

[1]This means that Equation (3.0.1) could have many different solutions by adding tangential components in the hypersurface parameterization, but from a geometric point of view these solutions are reparametrizations of each other.

some points of M_0. Finally, in [CD], the authors show an existence result of non-compact complete convex solutions in the spirit of [EH1] for the H-flow under the same assumptions of [Die].

In this chapter we focus on eternal solutions[2] of the Q_k-flow which evolve by translation on a fixed constant unitary direction, usually known as translating solitons, or Q_k-translators for short. These type of solutions are immersion of the form

$$F(x,t) = F_0(x) + tv,$$

where $v \in \mathbb{S}^n$ is a fixed direction. Since we are interested in the normal part, Q_k-translators can also be seen as hypersurfaces $M_0 \subset \mathbb{R}^{n+1}$ which satisfy the fully non-linear degenerated equation

$$Q_k(\lambda) = \langle \nu, v \rangle \,. \tag{3.0.3}$$

These type of solitons have been studied by many authors in the case $k = 0$, we recommend the reader to see the survey [HIMW3] as a guide in this area.

The interest on H-translators arises mainly due to two reasons: they appear as a model for type II singularities of the Mean Curvature Flow, and they are also minimal surfaces in weighted Euclidean spaces. See the Introduction 2 of this thesis for a brief review about this topic.

Finally, the work of this chapter is inspired by the seminal work of Ecker and Huisken [EH1] and [EH2], in obtaining gradient and second order local estimates for the study of the evolution of Lipschitz entire graphs under the H-flow. We obtain several results for the Q_k-flow and Q_k-translators in $\mathbb{R}^{n+1}$ that we summarize below.

Indeed, by adapting the gradient estimates given in [SUW] and [Hol] for the Q_k functions,

[2]Solutions which exist for all times.

we obtained local gradient estimates for these equations.

Theorem 3.0.1. *Let $r > 0$ and $u \in \mathcal{C}^{3,2}(B_r(0) \times (0,T])$ such that the graph of u evolves under the Q_k-flow and the principal curvatures of the graph of u lie in the cone Γ_{k+1} for each t. Then, for $t > 0$, it holds*

$$|Du(0,t)| \leq \exp\left(K + \frac{KMT}{rt} + \frac{M^2}{t} + \frac{(T+1)r^2}{t} + \frac{TM^2}{tr^2} + \frac{TM}{tr}\right), \tag{3.0.4}$$

where $M = \sup_{\Omega} u$ and $K = K(k,n)$.

Theorem 3.0.2. *Let $r > 0$ and $u \in \mathcal{C}^3(B(0,r))$ such that graph(u) is a Q_k-translator and the principal curvatures lie in the cone Γ_{k+1}. Then, it holds*

$$|Du(0)| \leq \exp\left(\frac{CM}{r} + \frac{CM^2}{r^2} + 1\right),$$

where $M = \sup_{B(0,r)} u$ and $C = C(k,n)$.

Next, by adapting the proof of the gradient estimate, we also obtained a Liouvillie's type result for Q_k-translators which reads as follow.

Theorem 3.0.3. *There are no functions $u \in \mathcal{C}^3(\mathbb{R}^n)$ such that*

1. *The principal curvatures of the graph of u lie in Γ_{k+1}.*
2. *$u(x) = o(|x|)$ as $|x| \to \infty$.*
3. *The graph of u is Q_k-translator in $\mathbb{R}^{n+1}$.*

Moreover, by assuming $k \geq 1$, any hypersurface M_0 whose principal curvatures lies in Γ_{k+2}, satisfies $|A|^2 \leq H^2$ and $|A|_k^2 \leq CH^2$ for some constant $C = C(k,n)$.
Therefore, by applying similar techniques as in [EH2], we obtain local second order estimates for solutions of Q_k-flows and Q_k-translators in $\mathbb{R}^{n+1}$, respectively.

In the statement of the following theorems we assume the existence of a non-negative function $h(x,t)$ such that

$$|(\partial_t - \Box_k)h| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h, \tag{3.0.5}$$

where $\Box_k f = \dfrac{\partial Q_k}{\partial h_{ij}} \nabla_i \nabla_j f$ and $|X|_k = \dfrac{\partial Q_k}{\partial h_{ij}} X_i X_j$. On the other hand, for Q_k-translator we will assume that there is a function $h(x)$ such that

$$|\Box_k h + \langle \nabla h, e_{n+1} \rangle| \leq C(k,n) \text{ and } |\nabla h|_k^2, |\nabla h|^2 \leq C(k,n)h. \tag{3.0.6}$$

In addition, as we will note later in this chapter, when the Q_k-flow is convex, the function

$$h(x,t) = |x|^2 - \langle x, \vec{w} \rangle^2 - \frac{2(n-k)}{k+1} t,$$

which measures the time dependent distance from the tangent space orthogonal to vector $\vec{w} \in \mathbb{S}^n$, satisfies the assumptions in (3.0.5). Meanwhile, for the Q_k-translators, we only work with the abstract function $h(x)$ since, as we will show later, $h(x) = |x|^2 - \langle x, e_{n+1} \rangle^2$ does not satisfies (3.0.6).

Theorem 3.0.4. *Let $R > 0$ and $w \in \mathbb{S}^n$ such that $M_t = \{x \in M_t : h(x,t) \leq R^2\}$ is graph over a ball of radius R in the hyperplane orthogonal to $\vec{w}$ for all $[0,T)$.*
Then, if the principal curvatures of M_t belong to the cone Γ_{k+2} with $k \geq 1$, for any $t_0 \in (0,T)$ and $\theta \in [0,1)$, the following estimate holds

$$\sup_{M_t} H^2 \leq \frac{c(k,n)}{(1-\theta)^2} \left(\frac{1}{t} + \frac{1}{R^2} \right) \sup_{[0,t_0]} \sup_{M_t} v^4,$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$.

Theorem 3.0.5. *Let $R > 0$ such that $\{x \in M : h(x) \leq R^2\}$ is a graph over a ball of radius*

R in the $\mathbb{R}^n$. Then, if the principal curvatures of M_t belong to the cone Γ_{k+2} with $k \geq 1$ for any $\theta \in [0,1)$, the following estimate holds

$$H^2 \leq \frac{c(k,n)}{(1-\theta)^2}\left(1+\frac{1}{R^2}\right)\sup_{\{h\leq \theta R^2\}} v^4,$$

where $v = \langle \nu, w\rangle^{-1}$.

All of these results are in the spirit of showing existence of entire graph solutions to the Q_k-flow and Q_k-translator in $\mathbb{R}^{n+1}$. Until now, the non-trivial examples for these geometric problems are of “bowl”-type, see [Ren] or Chapter 4 below for more details.
These solutions are called of “bowl type" because they resemble the “bowl" soliton of the H-flow. Indeed, they are complete strictly convex rotationally symmetric graph defined in $\mathbb{R}^n$ or in a ball of radio $Q_k(1,\ldots,1)$, and are asymptotic to paraboloids or round cylinders. Consequently, these estimates could be applied as in [CD] to find more examples of graphs solutions of the Q_k-flow.

On the other hand, these techniques can be applied to other type of 1-homogeneous curvature functions. More precisely, for any $\gamma : \Gamma \subset \mathbb{R}^n \to \mathbb{R}$ that satisfies

(a) $\Gamma \subset \mathbb{R}^n$ is a symmetric open convex cone and γ is symmetric[3], smooth and positive in Γ.

(b) γ is strictly increasing in each variable, i.e: $\dfrac{\partial \gamma}{\partial \lambda_i} > 0$ in Γ for every i.

(c) γ is 1-homogeneous, i.e: for every $c > 0$, $\gamma(c\lambda) = c\gamma(\lambda)$ in Γ.

The organization of this chapter goes as follows: In Section 3.1 we give some properties of the S_k and Q_k functions that we will use along the chapter. In Section 3.2 we develop the gradient estimates for graph solutions to the Q_k-flow and the Q_k-translator equation and the proof of Theorem (3.0.3). In Section 3.3 we prove second order estimates for both settings.

[3]Invariant under the n-permutation group in its variables.

3.1 Properties of Q_k functions

In this section we list some properties of S_k and Q_k functions in $\mathbb{R}^n$.

Definition 3.1.1. By setting $S_0 := 1$, $S_k := 0$ for $k > n$ and S_k as in formula (3.0.2) for $k = 1, \ldots, n$. We define the open convex cones

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \ S_i(\lambda) > 0, \ \text{for } i = 1, \ldots, k\}.$$

Example 1. The most common examples of Q_k functions corespond to

$$Q_0 = \lambda_1 + \ldots + \lambda_n \text{ and } Q_{n-1} = \left(\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n}\right)^{-1},$$

when we consider these expressions in (3.0.1) we have the mean and the inverse mean harmonic curvature flow, respectively.

Along this chapter we will denote by $S_{k,i}(\lambda)$ and $S_{k,i;j}(\lambda)$ the sum of all terms of $S_k(\lambda)$ which do not contain the factor λ_i and the factors λ_i and λ_j, respectively.

Lemma 3.1.1. *For any $k \in \{0, \ldots, n\}$, $i \in \{1, \ldots, n\}$ and $\lambda \in \mathbb{R}^n$ we have*

$$\frac{\partial S_{k+1}}{\partial \lambda_i} = S_{k,i}, \tag{3.1.1}$$

$$S_{k+1} = S_{k+1,i} + \lambda_i S_{k,i}, \tag{3.1.2}$$

$$\sum_{i=1}^{n} S_{k,i} = (n-k) S_k, \tag{3.1.3}$$

$$\sum_{i=1}^{n} \lambda_i S_{k,i} = (k+1) S_{k+1}, \tag{3.1.4}$$

$$\sum_{i=1}^{n} \lambda_i^2 S_{k,i} = S_1 S_{k+1} - (k+2) S_{k+2}, \tag{3.1.5}$$

$$\sum_{i=1}^{n} S_{k,i;j} = (n-k-1) S_{k,j}. \tag{3.1.6}$$

Proof. For a proof of (3.1.1)-(3.1.5) we refer to [HS1]. Note that Equation (3.1.6) follows by taking derivative with respect λ_j on (3.1.3). ☐

The next lemma contains similar formulas for Q_k rational functions.

Lemma 3.1.2. *For any $k \in \{0, \ldots, n\}$, $i \in \{1, \ldots, n\}$ and $\mu \in \mathbb{R}^n$ we have*

$$\sum_{i=1}^{n} \frac{\partial Q_k}{\partial \lambda_i} = (n-k) - (n-k+1)\frac{Q_k}{Q_{k-1}}, \tag{3.1.7}$$

$$\sum_{i=1}^{n} \lambda_i \frac{\partial Q_k}{\partial \lambda_i} = Q_k, \tag{3.1.8}$$

$$\sum_{i=1}^{n} \lambda_i^2 \frac{\partial Q_k}{\partial \lambda_i} = (k+1)Q_k^2 - (k+2)Q_{k+1}Q_k. \tag{3.1.9}$$

Proof. For Equation (3.1.7) we have,

$$\frac{\partial Q_k}{\partial \lambda_i} = \frac{1}{S_k^2}\left(S_k S_{k,i} - S_{k+1}S_{k-1,i}\right).$$

Then, by (3.1.3), it follows

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i} = \frac{1}{S_k^2}\big((n-k)S_k^2 - (n-k+1)S_{k+1}S_{k-1}\big).$$

For Equation (3.1.8), we use (3.1.4) to show that

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i}\lambda_i = \frac{1}{S_k^2}\left((k+1)S_{k+1}S_k - kS_{k+1}S_k\right).$$

Finally, for Equation (3.1.9) we use (3.1.5) to see that

$$\sum_i \frac{\partial Q_k}{\partial \lambda_i}\lambda_i^2 = \frac{1}{S_k^2}\big((k+1)S_{k+1}^2 - (k+2)S_{k+2}S_k\big).$$

☐

In addition, we have Newton's inequality for S_k polynomials.

Lemma 3.1.3. *For any $k \in \{1, \ldots, n-1\}$ and $\lambda \in \mathbb{R}^n$ we have*

$$(k+1)\,(n-(k-1))\,S_{k-1}S_{k+1} \leq k(n-k)S_k^2.$$

Equality holds if, and only if, all λ_i are equal.

Proof. For a proof we refer to [HLP]. □

In addition, by iterating Lemma 3.1.3, it can be shown a Newton Inequality for Q_k.

Corollary 3.1.4. *For any $l, k \in \{0, \ldots, n-1\}$ such that $l \leq k$ and $\lambda \in \Gamma_{k+1}$, we have*

$$Q_k \leq \frac{(l+1)(n-k)}{(k+1)(n-l)} Q_l. \tag{3.1.10}$$

Proof. First we note that if $\lambda \in \Gamma_{k+1}$, then it follows

$$Q_k \leq \frac{k(n-k)}{(k+1)(n-k+1)} Q_{k-1}.$$

Then, by iterating the above formula the result easily follows. □

3.2 Gradient Estimates and applications

In this section we derive local gradient estimates for solutions to equations (3.0.1) and (3.0.3). We first note that a smooth hypersurface can be locally written, as a graph, which after a rotation is $(x, u(x,t))$. In this setting, Equation (3.0.1) becomes

$$u_t = Q_k(Du, D^2u)\sqrt{1+|Du|^2}, \tag{3.2.1}$$

and Equation (3.0.3) becomes

$$Q_k(Du, D^2u) = \frac{1}{\sqrt{1+|Du|^2}}. \tag{3.2.2}$$

It is important to mention that we are abusing with the notation on the term $Q_k(Du, D^2u)$, since Q_k is a function in n-variables. In fact, we oftently will write

$$Q_k(\lambda) = Q_k(\lambda(g^{-1}II)) = Q_k(Du, D^2u)$$

when the hypersurface is given by the graph of a function. Note that $\lambda(g^{-1}II)$ denotes the principal curvatures (or eigenvalues) vector of the shape operator $g^{-1}II$, where g^{-1} is the inverse of the metric of the hypersurface, and II is the second fundamental form of the hypersurface in $\mathbb{R}^{n+1}$.

Moreover, we will use the Einstein summation convention of upper and lower index to write any tensor equation. In particular, in graphical local coordinates, the shape operator is written by $h^i_j = g^{ik}h_{kj}$, where

$$h_{ij} = \frac{u_{ij}}{w},\ g^{ij} = \delta_{ij} - \frac{u_i u_j}{w^2},$$

and $w = \sqrt{1+|Du|^2}$[4], are the coefficients of the second fundamental form and the inverse of the metric tensor $g_{ij} = \delta_{ij} + u_i u_j$, respectively. Here the subindex denote derivatives with respect to the corresponding variable.

However, the principal curvatures can also be obtained as the eigenvalues of the matrix

$$A_{ij} = w^{-1}\left(u_{ij} - \frac{u_i u_k u_{kj}}{w(1+w)} - \frac{u_j u_k u_{ki}}{w(1+w)} + \frac{u_i u_j u_k u_l u_{kl}}{w^2(1+w)^2}\right). \tag{3.2.3}$$

[4]Note that $w = v^{-1}$ in Theorem and 3.0.5.

In fact, $A_{ij} = \sqrt{g^{ik}} h_{kl} \sqrt{g^{lj}}$, where $\sqrt{g^{ij}}$ is the positive square root of g^{ij}. Then, since the determinant is a multiplicative function, it follows that

$$\det\left(h^i_j - \lambda Id\right) = 0 \Leftrightarrow \det\left(A_{ij} - \lambda Id\right) = 0,$$

which implies that the eigenvalues of h^i_j and A_{ij} coincides.

Finally, the reason to consider the matrix A_{ij} instead of h^i_j is only technical. The A_{ij} matrix is symmetric, and therefore, facilitates all calculations related to it. We refer the reader to Section 1 of [CNS1] for more details related to the matrix A_{ij}.
The method that we employ to obtain the estimate in Theorems 3.0.1 and 3.0.2 was first used in [CNS2] for deriving gradient estimates for Dirichlet Curvature PDEs. In [Hol], the author also used this method for deriving gradient estimates to the $\sqrt[k]{S_k}$-flow in $\mathbb{R}^{n+1}$.

Remark 3.2.1. The functions S_k can be evaluated in any symmetric matrix B by the relation

$$\det(tI + B) = t^n + S_1(B)t^{n-1} + \ldots + S_n(B).$$

Indeed, if we choose $\alpha \subset \{1, \ldots, n\}$ and denote $|\alpha|$ its cardinality. Then, by setting $B[\alpha]$ to be the principal submatrix of B[5] in rows and columns index by the elements in α, it follows that

$$S_k(B) = \sum_{\alpha \subset \{1,\ldots,n\}, |\alpha| = k} \det(B[\alpha]).$$

[5]The principal submatrix $B[\alpha]$ of a symmetric matrix B is given by the partition

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$$

where B_{11} is of size $|\alpha|$ is the principal submatrix and B_{22} is of size $n - |\alpha|$.

As an example to see this relation we choose

$$S_1(B) = \sum_{\alpha\subset\{1,\ldots,n\},|\alpha|=1} \det(B[\alpha]) = \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(B),$$

$$S_n(B) = \sum_{\alpha\subset\{1,\ldots,n\},|\alpha|=n} \det(B[\alpha]) = \det(B).$$

Recall that we are considering the symmetric elementary polynomial without being normalized. It is relevant to remark that the properties stated in Section 3.1 are valid for the eigenvalues of a given matrix (or equivalently, for diagonal matrices).

The following lemma was inspired by [Sae] and it is due to the shape of matrix $A := (A_{ij})$ after a change of coordinates for which $S_k(A)$ is easier to compute.

Lemma 3.2.1. *If the matrix A has the form,*

$$A_{11} = \frac{u_{11}}{w^3} < 0,\ A_{1j} = \frac{u_{1j}}{w^2} = A_{j1},\ A_{jj} = \frac{u_{jj}}{w} \text{ and } A_{ij} = A_{ji} = 0 \; (\text{for } 2 \le i \ne j \le n).$$

at some point (x_0, t_0) or x_0, where the principal value lies in Γ_{k+1}, the following equations hold:

(a) $\sum_{i=1}^{n} \frac{\partial Q_k}{\partial A_{ij}} A_{ij} = Q_k$, *this one hold for every symmetric matrix A_{ij}.*

(b) $S_l(\tilde{\lambda}) > 0,\ \frac{\partial S_l}{\partial A_{jj}}(\tilde{\lambda}) = S_{l-1}(\tilde{\lambda}|j) > 0$ *for $j = 2, \ldots n$ and $l = 1, \ldots, k+1$.*

(c) $S_l(\tilde{\lambda}) = S_l(\tilde{\lambda}|j) + A_{jj} S_{l-1}(\tilde{\lambda}|j)$.

(d) $S_l(\tilde{\lambda}) > S_l$.

(e) $S_l(A) = S_l(\tilde{\lambda}) + A_{11} S_{l-1}(\tilde{\lambda}) - \sum_{j>1} A_{1j}^2 S_{l-2}(\tilde{\lambda}|j)$.

(f) $S_l(A) = S_l(A|i) + A_{ii} S_{l-1}(A|i) - A_{1i}^2 S_{l-2}(A|1i)$.

(g) $\frac{\partial Q_k}{\partial A_{11}} \ge \frac{n}{(k+1)(n-k)},\ \frac{\partial Q_k}{\partial A_{ii}} > 0$ *for $i > 1$.*

(h) $\dfrac{\partial Q_k}{\partial A_{11}} \geq \dfrac{n}{(n-k)^2(k+1)} \sum_{i>1} \dfrac{\partial Q_k}{\partial A_{ii}}$.

Remark 3.2.2. Here we are using the following notation: $S_{l,ij} = \dfrac{\partial S_l}{\partial u_{ij}}$, $\tilde{\lambda} = \mathrm{diag}(A_{22}, \ldots, A_{nn}) = A|1$, $S_k(B|i)$ or $S_k(B|ij)$ means that the i-th row and i-th column resp. i,j-th row and i,j-th column are omitted from a matrix B.

Proof. We denote $Q_{k,ij} = \dfrac{\partial Q_k}{\partial u_{ij}}$. Then under this setting we have

$$Q_{k,ij} = \left(\frac{S_{k+1}}{S_k}\right)_{ij} = \frac{S_{k+1,ij}S_k - S_{k+1}S_{k,ij}}{S_k^2}.$$

Then, by combining the chain rule with the derivatives of the coefficients of A_{ij} given in (3.2.3), we have

$$\begin{aligned}
Q_{k,11} &= \frac{\partial Q_k}{\partial A_{11}} \frac{\partial A_{11}}{\partial u_{11}} = \frac{\partial Q_k}{\partial A_{11}} \frac{1}{w^3}, \\
Q_{k,1i} &= \frac{\partial Q_k}{\partial A_{1i}} \frac{\partial A_{1i}}{\partial u_{1i}} = \frac{\partial Q_k}{\partial A_{1i}} \frac{1}{w^2} = Q_{k,i1},\ i \neq 1 \\
Q_{k,ii} &= \frac{\partial Q_k}{\partial A_{ii}} \frac{\partial A_{ii}}{\partial u_{ii}} = \frac{\partial Q_k}{\partial A_{ii}} \frac{1}{w} \\
Q_{k,ij} &= \frac{\partial Q_k}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial u_{ij}} = \frac{\partial Q_k}{\partial A_{ij}} \frac{1}{w} = 0 = Q_{k,ji} \text{ (for } 2 \leq i \neq j \leq n).
\end{aligned}$$

Firstly, we note that Equation (a) follows by Equation (3.1.8) together with the formula $\sum_{i,j} \dfrac{\partial S_k}{\partial A_{ij}} A_{ij} = (k+1)S_{k+1}$.

Furthermore, since $A_{11} < 0$, it follows that $S_l(\tilde{\lambda}) > 0$ for $l = 1$. In particular, this fact implies that $\dfrac{\partial S_l}{\partial A_{jj}}(\tilde{\lambda}) = S_{l-1}(\tilde{\lambda}|j) > 0$.

Then, by iterating this process, we can get the same result for $j = 2, \ldots n$ and $l = 1, \ldots, k+1$. Consequently, by the shape of the matrix A_{ij} and the properties given in Lemma 3.1.1, Equations (c)-(f) hold.

Secondly, we analyze the diagonal terms $\frac{\partial Q_k}{\partial A_{ii}}$. We start with $i = 1$,

$$\frac{\partial Q_k}{\partial A_{11}} = \frac{1}{S_k^2}\left(\frac{\partial S_{k+1}}{\partial A_{11}} S_k - S_{k+1}\frac{\partial S_k}{\partial A_{11}}\right) = \frac{1}{S_k^2}\left(S_k(\tilde{\lambda})S_k - S_{k+1}S_{k-1}(\tilde{\lambda})\right).$$

Then, by combining the above equation with Equation (c), we obtain

$$\frac{\partial Q_k}{\partial A_{11}} = \frac{1}{S_k^2}\left(S_k(\tilde{\lambda})^2 - S_{k-1}(\tilde{\lambda})S_{k+1}(\tilde{\lambda}) + \sum_{j>1} A_{1j}^2 (S_{k-1}(\tilde{\lambda}|j)^2 - S_k(\tilde{\lambda}|j)S_{k-2}(\tilde{\lambda}|j))\right).$$

Note that by Lemma 3.1.3, the term in the sum is non-negative, and therefore we drop it from the inequality.

Consequently, by Equation (d), it follows that

$$\frac{\partial Q_k}{\partial A_{11}} \geq \frac{n}{(k+1)(n-k)}\left(\frac{S_k(\tilde{\lambda})}{S_k}\right)^2 \geq \frac{n}{(k+1)(n-k)}.$$

Next, we are going to show that $\frac{\partial Q_k}{\partial A_{ii}} > 0$ for $i > 1$. Indeed, by differentiating Equation (f) with respect A_{ii} with $i > 1$, we have

$$\begin{aligned}
\frac{\partial Q_k}{\partial A_{ii}} &= \frac{1}{S_k^2}\left(S_k \frac{\partial S_{k+1}}{\partial A_{ii}} - S_{k+1}\frac{\partial S_k}{\partial A_{ii}}\right) \\
&= \frac{1}{S_k^2}\left(S_k S_k(A|i) - S_{k+1}S_{k-1}(A|i)\right) \\
&= \frac{1}{S_k^2}\left(S_k(A|i)^2 - S_{k+1}(A|i)S_{k-1}(A|i)\right) + \frac{A_{1i}^2}{S_k^2}\left[S_{k-1}(A|i)S_{k-1}(A|1i) - S_k(A|i)S_{k-2}(A|1i)\right].
\end{aligned}$$

Now, we use Equation (e) in the second term from th above line to obtain

$$\begin{aligned}
\frac{\partial Q_k}{\partial A_{ii}} =& \frac{1}{S_k^2}\left(S_k(A|i)^2 - S_{k+1}(A|i)S_{k-1}(A|i)\right) + \frac{A_{1i}^2}{S_k^2}\left(S_{k-1}(\tilde{\lambda}|i)^2 - S_k(\tilde{\lambda}|i)S_{k-2}(\tilde{\lambda}|i)\right) \\
&+ \frac{A_{1i}^2}{S_k^2}\sum_{j>1, j\neq i} A_{1j}^2\left[S_{k-2}^2(\tilde{\lambda}|ij) - S_{k-3}(\tilde{\lambda}|ij)S_{k-1}(\tilde{\lambda}|ij)\right].
\end{aligned}$$

Finally, since Lemma 3.1.3 works for every symmetric matrix evaluated in the S_k polynomials, it follows that the three terms in the right hand side of the above line are positive, which means that $\frac{\partial Q_k}{\partial A_{ii}} > 0$.

Next, we study the sum the diagonal terms $\frac{\partial Q_k}{\partial A_{ii}}$.

Firstly, we write

$$\begin{aligned}\frac{\partial Q_k}{\partial A_{11}} =& \frac{S_k(\tilde{\lambda})}{S_k} + \frac{Q_k}{S_k} S_{k-1}(\tilde{\lambda}),\\ \frac{\partial Q_k}{\partial A_{ii}} =& \frac{S_k(A|i)}{S_k} + \frac{Q_k}{S_k} S_{k-1}(A|i)\\ =& \frac{1}{S_k}\left(S_k(\tilde{\lambda}|i) + A_{11}S_{k-1}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-2}(\tilde{\lambda}|ij)\right)\\ &+ \frac{Q_k}{S_k}\left(S_{k-1}(\tilde{\lambda}|i) + A_{11}S_{k-2}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-3}(\tilde{\lambda}|ij)\right).\end{aligned}$$

Then, it follows that

$$\begin{aligned}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}} =& \frac{1}{S_k}\left(S_k(\tilde{\lambda}) + \sum_{i>1}\left[S_k(\tilde{\lambda}|i) + A_{11}S_{k-1}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-2}(\tilde{\lambda}|ij)\right]\right) \quad (3.2.4)\\ &- \frac{Q_k}{S_k}\left(S_{k-1}(\tilde{\lambda}) + \sum_{i>1}\left[S_{k-1}(\tilde{\lambda|i}) + A_{11}S_{k-2}(\tilde{\lambda}|i) - \sum_{j>1, j\neq i} A_{1j}^2 S_{k-3}(\tilde{\lambda}|ij)\right]\right).\end{aligned}$$

Next, by applying Equation (3.1.3) on the terms $\sum_{i>1} S_l(\tilde{\lambda}|i)$ with $l = k-2, k-1$, we have

$$\sum_{i>1} S_l(\tilde{\lambda}|i) = (n-1-l)S_l(\tilde{\lambda}). \quad (3.2.5)$$

Moreover, applying Equation (3.1.6) on the terms $\sum_{i>1} S_l(\tilde{\lambda}|ij)$ with $l = k-3, k-2$, we have

$$\sum_{i>1} S_l(\tilde{\lambda}|ij) = (n-1-l-1)S_l(\tilde{\lambda}|j). \quad (3.2.6)$$

Consequently, by replacing Equations (3.2.5) and (3.2.6) in (3.2.4) we get

$$\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}} = \frac{(n-k)}{S_k}\left(S_k(\tilde{\lambda}) + A_{11}S_{k-1}(\tilde{\lambda}) - \sum_{j>1} A_{1j}^2 S_{k-2}(\tilde{\lambda}|j)\right)$$
$$- (n-k+1)\frac{Q_k}{S_k}\left(S_{k-1}(\tilde{\lambda}) + A_{11}S_{k-2}(\tilde{\lambda}) - \sum_{j>1} A_{1j}^2 S_{k-3}(\tilde{\lambda}|j)\right).$$

Finally, by applying Equation (e) over the terms in parenthesis of the above line we have

$$\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}} = (n-k) - (n-k+1)\frac{Q_k}{Q_{k-1}} \leq (n-k),$$

where in the last inequality we use that $\dfrac{Q_k}{Q_{k-1}} > 0$. Then, by combining this equation with Equation (g), Equation (h) holds. □

Now we prove the gradient estimate for the Q_k-flow.

Proof of Theorem 3.0.1. The proof is very similar to the given for Theorem 1.1 in [Hol]. For this reason we will use the same notation.

Moreover, by a translation on the x_{n+1}-axis, we may assume that $u > 0$. We also take the matrix A given in (3.2.3), and by a slight abuse of notation we set, $Q_k(A) = \dfrac{S_{k+1}(A)}{S_k(A)}$, where each function is evaluated in the eigenvalues of A.

Consequently, Equation (3.2.1) becomes

$$Q_k(A) = \frac{u_t}{w}, \text{ where } w = \sqrt{1+|Du|^2}. \tag{3.2.7}$$

Now we define the test function $G : \Omega \times \mathbb{S}^{n-1} \to \mathbb{R}$ given by

$$G(x,t,\xi) = t\rho(x)\varphi(u)\ln(u_\xi), \tag{3.2.8}$$

here ξ denote the direction derivative vector, $\rho(x) = 1 - \frac{|x|^2}{r^2}$ and $\varphi(u) = 1 + \frac{u}{M}$, where $M = \sup_\Omega u$. Furthermore, since ρ vanishes at $\partial B_r(0)$, we may assume that G attains its maximum at some point (x_0, t_0) with $t_0 > 0$ and $|x_0| < r$. Moreover, after a rotation, we may choose $\xi = \varepsilon_1$ where ε_i denotes the canonical euclidean base of $\mathbb{R}^{n+1}$.

Next, since $G(x, t, \xi)$ reach a maximum at (x_0, t_0) we have the following equations

$$0 = (\ln G)_i = \frac{\rho_i}{\rho} + \frac{\varphi'}{\varphi} u_i + \frac{u_{1i}}{u_1 \ln u_1}, \tag{3.2.9}$$

where the second derivatives satisfy

$$\begin{aligned}
0 &\geq (\ln G)_{ij} \\
&= \left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i \rho_j}{\rho^2}\right) + \left(\frac{\varphi''}{\varphi} - \left[\frac{\varphi'}{\varphi}\right]^2\right) u_i u_j + \frac{\varphi'}{\varphi} u_{ij} + \frac{u_{1ij}}{u_1 \ln u_1} - \left(1 + \frac{1}{\ln u_1}\right) \frac{u_{1i} u_{1j}}{u_1^2 \ln u_1} \\
&= \frac{\rho_{ij}}{\rho} + \frac{\varphi'}{\varphi} u_{ij} + \frac{\varphi'}{\varphi \rho}(\rho_i u_j + \rho_j u_i) + \frac{u_{ij1}}{u_1 \ln u_1} - \left(1 + \frac{2}{\ln u_1}\right) \frac{u_{1i} u_{1j}}{u_1^2 \ln u_1}.
\end{aligned} \tag{3.2.10}$$

In the third line we use Equation (3.2.9), $\varphi'' = 0$ and that u is of class $\mathcal{C}^3$. Furthermore, the time derivative satisfy

$$0 \leq (\ln G)_t = \frac{\varphi'}{\varphi} u_t + \frac{u_{1t}}{u_1 \ln u_1} + \frac{1}{t}. \tag{3.2.11}$$

Finally, we make another change of coordinates in Ω such that the following equations hold

$$\begin{aligned}
&u_i(x_0, t_0) = 0, \text{ for } i \neq 1; \; u_{ij}(x_0, t_0) = 0, \text{ for } i \neq j \text{ and } i, j \geq 2; \\
&u_{22}(x_0, t_0) \geq \ldots \geq u_{nn}(x_0, t_0).
\end{aligned} \tag{3.2.12}$$

Before using the equations in Lemma 3.2.1, we need to show $u_{11} < 0$ at (x_0, t_0). From now on all the quantities would be evaluated at (x_0, t_0).

Recall that $Du = (u_1, 0, \ldots, 0,)$, then

$$1 - \frac{u_1^2}{w(w+1)} = \frac{w^2 + w - u_1^2}{w(w+1)} = \frac{1}{w}.$$

In addition, we have that

$$A_{11} = \frac{u_{11}}{w^3},\ A_{1j} = \frac{u_{1j}}{w^2} = A_{j1},\ A_{jj} = \frac{u_{jj}}{w} \text{ and } A_{ij} = A_{ji} = 0 \text{ (for } 2 \leq i \neq j \leq n). \quad (3.2.13)$$

Furthermore, by Equation (3.2.9), it follows that

$$\frac{u_{11}}{u_1 \ln u_1} = -\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1, \quad (3.2.14)$$

$$\frac{u_{1i}}{u_1 \ln u_1} = -\frac{\rho_i}{\rho} \text{ (for } i \geq 2). \quad (3.2.15)$$

Next, we will assume that $G(x_0, t_0)$ is big enough such that $G = t\rho\varphi \ln u_1 \geq 16\frac{MT}{r}$. Consequently, we obtain

$$u_1 \geq \frac{8M}{r\rho} \text{ and } 16\frac{MT}{r} \geq \frac{8\varphi t}{\varphi' r}. \quad (3.2.16)$$

Then, for all j, we can estimate

$$\left| \frac{\rho_j}{\rho} \right| < \frac{1}{4} \frac{\varphi'}{\varphi} u_1, \quad (3.2.17)$$

Finally, by combining (3.2.14) and (3.2.17), we obtain

$$u_{11} = u_1 \ln u_1 \left(-\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1 \right) \leq -u_1^2 \frac{\varphi'}{2\varphi} \ln u_1 < 0. \quad (3.2.18)$$

Hence, $A_{11} < 0$ and we can use the equations from Lemma 3.2.1.

Then, by Equation (3.2.10), it follows

$$0 \geq Q_{k,ij}(\ln G)_{ij} = \underbrace{Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} + \frac{\varphi'}{\varphi}u_{ij} + \frac{\varphi'}{\varphi\rho}(\rho_i u_j + \rho_j u_i)\right)}_{B} + \underbrace{Q_{k,ij}\left(\frac{u_{ij1}}{u_1 \ln u_1} - \left(1 + \frac{2}{\ln u_1}\right)\frac{u_{1i}u_{1j}}{u_1^2 \ln u_1}\right)}_{C}. \tag{3.2.19}$$

We start with the term B in (3.2.19). Firstly, by Equation (3.2.12), we can split B in the following way

$$B = Q_{k,11}\left(\frac{\rho_{11}}{\rho} + \frac{\varphi'}{\varphi}u_{11} + \frac{2\varphi'}{\varphi\rho}\rho_1 u_1\right) + \sum_{i>1} Q_{k,ii}\left(\frac{\rho_{ii}}{\rho} + \frac{\varphi'}{\varphi}u_{ii}\right) + 2\sum_{j>1} Q_{k,1j}\left(\frac{\rho_{1j}}{\rho} + \frac{\varphi'}{\varphi}u_{1j} + \frac{\varphi'}{\varphi\rho}\rho_j u_1\right). \tag{3.2.20}$$

Secondly, by equations (a) and (3.2.7), the term $\sum_{i,j} Q_{k,ij}\frac{\varphi'}{\varphi}u_{ij}$ in (3.2.20) satisfies

$$\frac{\varphi'}{\varphi}\sum_{i,j} Q_{k,ij}u_{ij} = \frac{\varphi'}{\varphi}\left(Q_{k,11}u_{11} + 2\sum_{i>1} Q_{k,1i}u_{1i} + Q_{k,ii}u_{ii}\right) = \frac{\varphi'}{\varphi}\frac{u_t}{w}. \tag{3.2.21}$$

Moreover, by equation (3.2.15), the last term in (3.2.20) can be written by

$$\begin{aligned} 2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 &= -2\frac{u_1^2 \ln u_1}{\rho}\frac{\varphi'}{\varphi}\sum_{j>1}\frac{\partial Q_k}{\partial A_{1j}}A_{1j} \\ &= 2\frac{u_1^2 \ln u_1}{\rho}\frac{\varphi'}{\varphi}\sum_{j>1}\frac{A_{1j}^2}{S_k^2}\left(S_{k-1}(A|1j)S_k - S_{k+1}S_{k-2}(A|1j)\right). \end{aligned} \tag{3.2.22}$$

Then, by using (e), we can write the sum in the right hand side by the elementary symmetric polynomials evaluated in $\tilde{\lambda}$ as follows

$$\begin{aligned}2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 &= 2\frac{u_1^2 \ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_k(S_{k+1}(\tilde{\lambda}) + A_{11}S_k(\tilde{\lambda}) - S_{k+1}) - S_{k+1}(S_k(\tilde{\lambda}) + A_{11}S_{k-1}(\tilde{\lambda}) - S_k)}{S_k^2}\\ &= 2\frac{u_1^2 \ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_kS_{k+1}(\lambda) - S_{k+1}S_k(\lambda)}{S_k^2},\end{aligned}$$

and in the last line we used $S_l(\tilde{\lambda}) + A_{11}S_{l-1}(\tilde{\lambda}) = S_l(\lambda)$ with $l = k+1, k$. Recall that $S_l(\lambda)$ denote the l-elemental symmetric polynomial evaluated at the diagonal matrix $\lambda = (A_{ii})_{i\geq 1}$. Then, it follows that

$$2\sum_{j>1} Q_{k,1j}\frac{\varphi'\rho_j}{\varphi\rho}u_1 = 2\frac{u_1^2 \ln u_1}{\rho}\frac{\varphi'}{\varphi}\frac{S_k(\lambda)}{S_k}(Q_k(\lambda) - Q_k).$$

Following an idea of [Sae], we are going to use a first order Taylor expansion on Q_k, seeing as an $\mathbb{R}^{n^2}$ real function, around the "vector" $\lambda = (A_{ii})_{i\geq 1}$. Indeed, we have

$$Q_k = Q_k(\lambda) + \sum_{i,j\geq 1}\frac{\partial Q_k}{\partial A_{ij}}(\lambda)(A_{ij} - A_{ii}) + \frac{1}{2}\sum_{i,j,k,l\geq 1}\frac{\partial^2 Q_k}{\partial A_{ij}\partial A_{kl}}(\eta)(A_{ij} - A_{ii})(A_{kl} - A_{kk}),$$

where $\eta \in \mathbb{R}^{n^2}$ is close to λ. Then, by the shape of the matrix A_{ij} (3.2.13), we have

$$Q_k = Q_k(\lambda) + \sum_{j>1}\frac{\partial Q_k}{\partial A_{1j}}(\lambda)A_{1j} + \frac{1}{2}\sum_{i,j,k,l\geq 1}\frac{\partial^2 Q_k}{\partial A_{ij}\partial A_{kl}}(\eta)(A_{ij} - A_{ii})(A_{kl} - A_{kk}).$$

Finally, since Q_k is a concave function and for $j > 1$

$$\frac{\partial S_l}{\partial A_{1j}}(\lambda) = 0,$$

it follows that $\dfrac{\partial Q_k}{\partial A_{1j}}(\lambda) = 0$ for $j > 1$, and therefore $Q_k \leq Q_k(\lambda)$. Consequently, the whole term is non-negative and we may drop it from (3.2.20).

Thirdly, to estimate the term $2Q_{k,11}\dfrac{\varphi'\rho_1}{\varphi\rho}u_1$, we use Equations (g) and (3.2.14), to see that

$$\begin{aligned}2Q_{k,11}\frac{\varphi'\rho_1}{\varphi\rho}u_1 &= -2Q_{k,11}\frac{\varphi'}{\varphi}\left(\frac{u_{11}}{u_1\ln u_1}+\frac{\varphi'}{\varphi}u_1\right)u_1 \geq -\frac{\varphi'^2u_1^2}{2\varphi^2w^3}\frac{\partial Q_k}{\partial A_{11}}\\ &\geq -\frac{u_1^2}{2M^2w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.\end{aligned}$$

Finally, for the term $\sum_{i,j}Q_{k,ij}\dfrac{\rho_{ij}}{\rho}$ it follows that

$$\begin{aligned}\sum_{i,j}Q_{k,ij}\frac{\rho_{ij}}{\rho} &= -\frac{2}{r^2\rho}\sum_{i\geq 1}Q_{k,ii}\\ &= -\frac{2}{r^2\rho}\left(\frac{1}{w^3}\frac{\partial Q_k}{\partial A_{11}}+\frac{1}{w}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\right) \geq -\frac{2}{wr^2\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.\end{aligned} \tag{3.2.23}$$

Consequently we obtain,

$$B \geq \frac{\varphi'}{\varphi}\frac{u_t}{w} - \frac{2}{wr^2\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}} - \frac{u_1^2}{2M^2w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}. \tag{3.2.24}$$

Now we estimate the term C in Equation (3.2.19).

Firstly, by differentiating Equation (3.2.7) in the ε_1-direction, we obtain

$$\begin{aligned}\frac{u_{t1}}{w} - \frac{u_t}{w^3}u_1u_{11} &= \frac{\partial Q_k}{\partial A_{ij}}\frac{\partial A_{ij}}{\partial x_1}\\ &= \frac{\partial Q_k}{\partial A_{11}}\frac{\partial A_{11}}{\partial x_1} + 2\frac{\partial Q_k}{\partial A_{i1}}\frac{\partial A_{i1}}{\partial x_1} + \frac{\partial Q_k}{\partial A_{ii}}\frac{\partial A_{ii}}{\partial x_1}.\end{aligned} \tag{3.2.25}$$

Next, we estimate the right hand side of the above line explicitly. For this purpose, we may write

$$
\begin{aligned}
A_{11} =& \frac{1}{w}\left(u_{11} - \frac{2u_1u_ku_{1k}}{w(1+w)} + \frac{u_1^2u_ku_lu_{kl}}{w^2(1+w)^2}\right)\\
=& \frac{1}{w}\left(\frac{u_{11}}{w^2} - \frac{2u_1}{w(1+w)}\sum_{k>1}u_ku_{1k} + \frac{u_1^2}{w^2(1+w)^2}\sum_{k,l>1}u_ku_lu_{kl}\right),\\
A_{1i} =& \frac{1}{w}\left(u_{1i} - \frac{u_iu_ku_{1k}}{w(1+w)} - \frac{u_1u_ku_{ki}}{w(1+w)} + \frac{u_1u_iu_ku_lu_{kl}}{w^2(1+w)^2}\right)\\
=& \frac{1}{w}\left(\frac{u_{1i}}{w} - \frac{u_iu_1u_{11}}{w(1+w)} - \frac{u_iu_1u_{ii}}{w(1+w)} + \frac{u_iu_1^3u_{11}}{w^2(1+w)^2}\right.\\
&\left. - \frac{u_i}{w(1+w)}\sum_{k>1}u_ku_{1k} - \frac{u_1}{w(1+w)}\sum_{k>1,k\neq i}u_ku_{ki} + \sum_{k,l>1}\frac{u_1u_iu_ku_lu_{kl}}{w^2(1+w)^2}\right),\\
A_{ii} =& \frac{1}{w}\left(u_{ii} - \frac{2u_iu_ku_{1k}}{w(1+w)} + \frac{u_i^2u_ku_lu_{kl}}{w^2(1+w)^2}\right).
\end{aligned}
$$

Then, by denoting $A_{ij,1} = \dfrac{\partial A_{ij}}{\partial x_1}$, the following equations holds

$$
\begin{aligned}
A_{11,1} &= \frac{u_{111}}{w^3} - \frac{3u_1}{w^5}u_{11}^2 - \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2,\\
A_{1i,1} &= \frac{u_{1i1}}{w^2} - \frac{2u_1}{w^4}u_{11}u_{1i} - \frac{u_1}{w^2(1+w)}u_{1i}u_{ii} - \frac{u_1}{w^3(w+1)}u_{11}u_{1i},\\
A_{ii,1} &= \frac{u_{ii1}}{w} - \frac{u_1}{w^3}u_{11}u_{ii} - \frac{2u_1}{w^2(1+w)}u_{1i}^2.
\end{aligned}
$$

Here we used that $\dfrac{\partial w}{\partial x_1} = \dfrac{u_1u_{11}}{w}$. Recall that all these quantities are evaluated at (x_0, t_0) which Equation (3.2.13) holds.

Then, the following equation holds

$$
\begin{aligned}
&Q_{k,ij}u_{ij1}\\
&= \frac{\partial Q_k}{\partial A_{11}}\frac{u_{111}}{w^3} + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\frac{u_{1i1}}{w^2} + \sum_{i>1}\frac{\partial Q_k}{\partial A_{11}}\frac{u_{ii1}}{w}\\
&= \frac{\partial Q_k}{\partial A_{11}}\left(A_{11,1} + \frac{3u_1u_{11}^2}{w^5} + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right)\\
&\quad + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(A_{1i,1} + \frac{2u_1u_{11}u_{1i}}{w^4} + \frac{u_1u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1u_{11}u_{1i}}{w^3(1+w)}\right)\\
&\quad + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\left(A_{ii,1} + \frac{u_1u_{11}u_{ii}}{w^3} + \frac{2u_1u_{1i}^2}{w^2(1+w)}\right)\\
&= \frac{u_{t1}}{w} - \frac{u_t}{w^3}u_1u_{11} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{3u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right)\\
&\quad + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{2u_1u_{11}u_{1i}}{w^4} + \frac{u_1u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1u_{11}u_{1i}}{w^3(1+w)}\right)\\
&\quad + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{u_1u_{11}u_{ii}}{w^3} + \frac{2u_1u_{1i}^2}{w^2(1+w)}\right)\\
&= \frac{u_{t1}}{w} - \frac{u_1u_{11}}{w^2}Q_k + \left(\frac{\partial Q_k}{\partial A_{11}}A_{11} + 2\frac{\partial Q_k}{\partial A_{1i}}A_{1i} + \frac{\partial Q_k}{\partial A_{ii}A_{ii}}\right)\frac{u_1u_{11}}{w^2}\\
&\quad + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1u_{11}u_{1i}}{w^4} + \frac{u_1u_{1i}u_{ii}}{w^2(1+w)}\right.\\
&\quad \left.+\frac{u_1u_{11}u_{1i}}{w^3(w+1)}\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1u_{1i}^2}{w^2(1+w)}\\
&= \frac{u_{t1}}{w} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1u_{1i}^2}{w^2(1+w)}\\
&\quad + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1u_{11}u_{1i}}{w^4} + \frac{u_1u_{1i}u_{ii}}{w^2(1+w)} + \frac{u_1u_{11}u_{1i}}{w^3(w+1)}\right).
\end{aligned}
$$

In the third equality we used Equation (3.2.25), in the fourth equality we used Equation (3.2.7), and in the last equality we used Equation (a).

Now we replace the above equations in C to obtain

$$
\begin{aligned}
C =& Q_{k,ij}\left(\frac{u_{ij1}}{u_1\ln u_1}-\left(1+\frac{2}{\ln u_1}\right)\frac{u_{1i}u_{1j}}{u_1^2\ln u_1}\right)\\
=&\frac{u_{t1}}{wu_1\ln u_1}\\
&+\frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2u_{11}^2}{w^5}-\left(1+\frac{2}{\ln u_1}\right)\frac{u_{11}^2}{w^3}+\frac{2u_1^2}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right)\frac{1}{u_1^2\ln u_1}\\
&+2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\left[\frac{(2w+1)u_1^2}{w^2(1+w)}-\left(1+\frac{2}{\ln u_1}\right)\right]\frac{u_{11}u_{1i}}{w^2}+\frac{u_1^2u_{1i}u_{ii}}{w^2(1+w)}\right)\frac{1}{u_1^2\ln u_1}\\
&+\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{2u_1^2}{w^2(1+w)}-\left(1+\frac{2}{\ln u_1}\right)\frac{1}{w}\right)\frac{u_{1i}^2}{u_1^2\ln u_1}.
\end{aligned}
$$

Furthermore, we note that if u_1 is big enough, then the following inequalities holds

$$
\frac{(2w+1)u_1^2}{w^2(1+w)}=2-\frac{w+1}{w^2}\geq 1+\frac{2}{\ln u_1}=1+\frac{2}{\ln(\sqrt{w^2-1})}, \tag{3.2.26}
$$

$$
\frac{2u_1^2}{w^2(1+w)}=\frac{2}{w}-\frac{2}{w^2}\geq(1+\frac{2}{\ln u_1})\frac{1}{w}, \tag{3.2.27}
$$

Therefore, by removing the positives terms in C, we finally obtain

$$
C\geq\frac{u_{t1}}{wu_1\ln u_1}+\frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2}{w^2}-1-\frac{2}{\ln u_1}\right)\frac{u_{11}^2}{w^3u_1^2\ln u_1}+\frac{2}{w^2(1+w)\ln u_1}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii}. \tag{3.2.28}
$$

Note that the term $u_{11}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\frac{u_{1i}}{w^2}$ is non-negative.

Next, we analyze the third term in Equation (3.2.28).

We note that by differentiating Equation (f) we have

$$
\frac{\partial S_l}{\partial A_{1i}}=-2A_{1i}S_{l-2}(A|1i).
$$

Then, it follows that

$$\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} = -2 \frac{S_{k-1}(A|1i) S_k - S_{k+1} S_{k-2}(A|1i)}{S_k^2} A_{1i} u_{1i} u_{ii}.$$

Moreover, by using the shape of the coeficcient of the matrix A, Equation (3.2.13), we have

$$\begin{aligned}
\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &= 2 \frac{u_{1i}^2}{w S_k^2} \left(S_{k+1} A_{ii} S_{k-2}(A|1i) - S_k A_{ii} S_{k-1}(A|1i) \right) \\
&= 2 \frac{u_{1i}^2}{w S_k^2} \Big(S_{k+1}(S_{k-1}(\tilde{\lambda}) - S_{k-1}(\tilde{\lambda}|i)) - S_k(S_k(\tilde{\lambda}) - S_k(\tilde{\lambda}|i)) \Big) \\
&= 2 \frac{u_{1i}^2}{w S_k^2} \left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_k S_k(\tilde{\lambda}|i) - S_{k+1} S_{k-1}(\tilde{\lambda}|i) \right),
\end{aligned}$$

where in the second line we used Equation (c) with the term $A_{ii} S_l(A|1i) = A_{ii} S_l(\tilde{\lambda}|i)$ with $l = k-1, k-2$. In the last line we used that $\frac{\partial Q_k}{\partial A_{11}} = \frac{S_k(\tilde{\lambda}) S_k - S_{k+1} S_{k-1}(\tilde{\lambda})}{S_k^2}$ where the functions are evaluated in the matrix A_{ij}.

Now, we are going to show that the difference of the last two terms in the last line is non-negative. For this purpose we consider

$$\frac{\partial Q_k}{\partial A_{ii}}(\tilde{\lambda}) = \frac{S_k(\tilde{\lambda}|i) S_k(\tilde{\lambda}) - S_{k+1}(\tilde{\lambda}) S_{k-1}(\tilde{\lambda}|i)}{S_k^2(\tilde{\lambda})}.$$

Then, we note that we can write the term $S_k S_k(\tilde{\lambda}|i) - S_{k+1} S_{k-1}(\tilde{\lambda}|i)$ in the following manner

$$\begin{aligned}
S_k S_k(\tilde{\lambda}|i) - S_{k+1} S_{k-1}(\tilde{\lambda}|i) &= \frac{\partial Q_k}{\partial A_{ii}}(\tilde{\lambda}) S_k(\tilde{\lambda}) S_k + \frac{S_{k-1}(\tilde{\lambda}|i) S_{k+1}(\tilde{\lambda}) S_k}{S_k(\tilde{\lambda})} - S_{k+1} S_{k-1}(\tilde{\lambda}|i) \\
&\geq S_{k-1}(\tilde{\lambda}|j) S_k (Q_k(\tilde{\lambda}) - Q_k),
\end{aligned}$$

in the last line we dropped the first term because is non-negative.

Finally, putting all the above calculation together we obtain

$$
\begin{aligned}
\frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &\geq 2\frac{u_{1i}^2}{wS_k^2}\left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k (Q_k(\tilde{\lambda}) - Q_k)\right) \\
&= 2\frac{u_{1i}^2}{wS_k^2}\left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k \left[Q_k(\tilde{\lambda}) + \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11} - \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11} - Q_k\right]\right) \\
&= 2\frac{u_{1i}^2}{wS_k^2}\left(-S_k^2 \frac{\partial Q_k}{\partial A_{11}} + S_{k-1}(\tilde{\lambda}|i) S_k \left[Q_k(\lambda) - Q_k - \frac{\partial Q_k}{\partial A_{11}}(\tilde{\lambda}) A_{11}\right]\right) \\
&\geq -2\frac{\partial Q_k}{\partial A_{11}} \frac{u_{1i}^2}{w}, \qquad (3.2.29)
\end{aligned}
$$

where in the third line we used the 1-homogeneity of $Q_k(\lambda)$, and in the last line we used that $A_{11} < 0$ and $Q_k(\lambda) - Q_k \geq 0$.

Consequently, for the whole third term in (3.2.28) we have

$$
\begin{aligned}
\frac{2}{w^2(1+w)\ln u_1} \sum_{i>1} \frac{\partial Q_k}{\partial A_{1i}} u_{1i} u_{ii} &\geq -\frac{4}{w^3(1+w)} \frac{\partial Q_k}{\partial A_{11}} \sum_{i>1} \frac{u_{1i}^2}{\ln u_1} \\
&\geq -4\frac{u_1^2 \ln u_1}{w^3(1+w)} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}} \sum_{i>1} \frac{\rho_i^2}{\rho^2} \\
&\geq -16\frac{u_1^2 \ln u_1}{w^3(1+w)\rho^2} \sum_{j\geq 1} \frac{\partial Q_k}{\partial A_{jj}}, \qquad (3.2.30)
\end{aligned}
$$

where in the second line we used the estimate (3.2.29), in the third line we used Equation (3.2.15) and that $\sum_{i>1} \rho_i^2 \leq 4$.

Next, we study the second term in (3.2.28). We note that the term in parenthesis satisfies

$$
\frac{2u_1^2}{w^2} - \left(1 + \frac{2}{\ln u_1}\right) = 1 - \frac{2}{w^2} - 1 - \frac{2}{\ln u_1} > \frac{1}{2}, \qquad (3.2.31)
$$

if u_1 is big enough.

Therefore, by combining the last inequality with Equation (3.2.18), we have

$$\begin{aligned}
\frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2}{w^2}-\left(1+\frac{2}{\ln u_1}\right)\right)\frac{u_{11}^2}{w^3u_1^2\ln u_1} &\geq \frac{\ln u_1}{2w^3}\frac{\partial Q_k}{\partial A_{11}}\frac{u_{11}^2}{u_1^2\ln^2(u_1)}\\
&\geq \frac{u_1^2\ln u_1}{8w^3}\frac{\partial Q_k}{\partial A_{11}}\frac{\varphi'^2}{\varphi^2}\\
&\geq \frac{u_1^2\ln u_1}{32M^2w^3}\frac{\partial Q_k}{\partial A_{11}}\\
&\geq \frac{C(k,n)}{32M^2}\frac{u_1^2\ln u_1}{w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}, \qquad (3.2.32)
\end{aligned}$$

where $C(k,n)$ is the constant given in (h). Recall that we have chosen c_0 big enough such that $u_1 > c_0$ and the inequalities (3.2.16), (3.2.26), (3.2.27) and (3.2.31) hold.

Finally, the term C can be bounded from below in the following way

$$C \geq \sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{C(k,n)}{32M^2}\frac{u_1^2\ln u_1}{w^3} - C\frac{u_1^2\ln u_1}{w^3(w+1)}\frac{16}{\rho^2}\right) + \frac{u_{t1}}{wu_1\ln u_1}. \qquad (3.2.33)$$

Moreover, by combining the lower bounds from (3.2.24) and (3.2.33) for B and C defined in (3.2.19), we finally obtain

$$\begin{aligned}
0 \geq &\frac{u_{t1}}{wu_1\ln u_1} + u_t\frac{\varphi'}{w\varphi}\\
&+\sum_i \frac{\partial Q_k}{\partial A_{ii}}\left(-\frac{4r}{\rho w^3}\frac{u_1^2}{M} - \frac{2}{wr^2\rho} + \frac{C_2(k,n)}{64M^2}\frac{u_1^2\ln u_1}{w^3} - C\frac{u_1^2\ln u_1}{w^3(w+1)}\frac{16}{\rho^2}\right).
\end{aligned}$$

On the other hand, by Equation (3.2.11) we have

$$\frac{\varphi'}{\varphi}u_t + \frac{u_{t1}}{u_1\ln u_1} \geq -\frac{1}{t}. \qquad (3.2.34)$$

Then, after using Equation (3.2.34) and multiplying by $w^2 \left(\sum_{i\geq 1} \frac{\partial Q_k}{\partial A_{ii}} \right)^{-1}$, we obtain

$$M^2 \left(C \frac{\ln u_1}{(w+1)} \frac{16}{\rho} + \frac{k+1}{n-k} \frac{w^2 \rho}{u_1^2 t} + \frac{4r}{M} + \frac{2w^2}{u_1^2 r^2} \right) \geq \frac{C_2}{64} \rho \ln u_1.$$

Consequently, we obtain

$$M^2 \left(C \frac{16r}{M} + \left(1 + \frac{r^2}{M^2}\right) \frac{1}{c_3 t} + \frac{4r^2}{M^2} + \frac{1}{r^2}\left(1 + \frac{r}{M}\right) \right) \geq \frac{c_2}{64} \rho \ln u_1,$$

here we use the assumption (3.2.16). Note that this implies,

$$\rho \ln u_1 \leq K \left(Mr + \frac{1}{t}(M^2 + r^2) + r^2 + \frac{M^2}{r^2} + \frac{M}{r} \right),$$

for a universal constant $K = K(k, n)$. Then, we have

$$\begin{aligned} \ln |Du(0,t)| &\leq \frac{t_0}{t} \frac{\varphi(u(x_0,t_0))}{\varphi(u(0,t))} \frac{\rho(x_0) \ln u_1(x_0,t_0)}{\rho(0)} \\ &\leq K \left(\frac{MrT}{t} + \frac{M^2}{t} + \frac{r^2}{t} + \frac{Tr^2}{t} + \frac{TM^2}{tr^2} + \frac{TM}{tr} \right). \end{aligned}$$

Since we assumed that $u_1 \geq c_0$ and $G(x_0, t_0) \geq 16 \frac{MT}{r}$, we finally obtain

$$|Du(0,t)| \leq \exp\left(K + \frac{KMT}{rt} + \frac{M^2}{t} + \frac{(T+1)r^2}{t} + \frac{TM^2}{tr^2} + \frac{TM}{tr} \right).$$

□

Recall that a Q_k-translator is a surface that evolves by translations with unit speed, hence we may use the same method to obtain a local gradient estimate for graphical solutions to Equation (3.2.2).

Proof of Theorem 3.0.2. This proof is very similar to the one given in Theorem 3.0.1, for this reason we only point out the main differences from it.

First, we note that equation (3.2.2) can be written as

$$Q_k(A) = \frac{1}{w}, \tag{3.2.35}$$

where A is the matrix given in (3.2.3).

Secondly, we use the same test function $G(x, \xi)$ given in (3.2.8) without the time factor. We also change the cut off function ρ by

$$\rho(x) = r^2 - |x|^2.$$

As in the proof of Theorem 3.0.1 we may assume that the maximum of G is attached at some point $x_0 \in B_r(0)$. We also apply the same change of coordinates as we did before.

Now, if we want to use the equations from Lemma 3.2.1, we need to ensure that $u_{11} < 0$ at x_0. Recall, that the function $\varphi(u) = 1 + \frac{u}{M}$ where $M = \sup_{B(0,r)} u$.

Then, we will assume that

$$G = \rho\varphi \ln u_1 \geq 16rM. \tag{3.2.36}$$

Then, it follows that $u_1 \geq \frac{8rM}{\rho}$ and $\frac{\varphi'}{\varphi} \geq \frac{1}{2M}$, which also implies

$$\left|\frac{\rho_j}{\rho}\right| < \frac{2r}{\rho} \leq \frac{\varphi'}{2\varphi} u_1.$$

Finally we get,

$$u_{11} = u_1 \ln u_1 \left(-\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1\right) \leq -u_1^2 \frac{\varphi'}{2\varphi} \ln u_1 < 0.$$

Thirdly, by the same calculations as in the proof of Theorem 3.0.1, we will obtain the same B and C terms of Equation (3.2.19), which we will analyze in this setup.
We start with B and note that the only terms that change are equations (3.2.21) and (3.2.23). In this case, we have

$$\sum_{i,j\geq 1} Q_{k,ij}\frac{\varphi'}{\varphi}u_{ij} = \frac{\varphi'}{\varphi}Q_k = \frac{\varphi'}{w\varphi},$$

and

$$\sum_{i,j\geq 1} Q_{k,ij}\frac{\rho_{ij}}{\rho} = -\frac{2}{\rho}\sum_{i\geq 1} Q_{k,ii} = -\frac{2}{\rho}\left(\frac{1}{w^3}\frac{\partial Q_k}{\partial A_{11}} + \frac{1}{w}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\right) \geq -\frac{2}{w\rho}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.$$

Recall that the cut off function $\rho(x) = r^2 - |x|^2$ so that is why does not appear a r^{-1} factor in the last inequality.

Therefore,

$$B \geq \frac{\varphi'}{\varphi}\frac{1}{w} - \frac{2}{w\rho}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}} - \frac{u_1^2}{2M^2w^3}\sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}.$$

For the term C, we only need to estimate the therm $Q_{k,ij}u_{ij1}$.
We observe that

$$\begin{aligned} Q_{k,ij}u_{ij1} &= \frac{\partial Q_k}{\partial A_{11}}\frac{u_{111}}{w^3} + 2\sum_i \frac{\partial Q_k}{\partial A_{1i}}\frac{u_{1i1}}{w^2}\sum_i \frac{\partial Q_k}{\partial A_{11}}\frac{u_{ii1}}{w} \\ &= \underbrace{-\frac{u_1u_{11}}{w^3} + \frac{u_1u_{11}}{w^2}Q_k}_{=0} + \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1}{w^5}u_{11}^2 + \frac{2u_1}{w^3(1+w)}\sum_{k>1}u_{1k}^2\right) + \sum_{i>1}\frac{\partial Q_k}{\partial A_{ii}}\frac{2u_1}{w^2(1+w)}u_{1i}^2 \\ &\quad + 2\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}\left(\frac{u_1}{w^4}u_{11}u_{1i} + \frac{u_1}{w^2(1+w)}u_{1i}u_{ii} + \frac{u_1}{w^3(w+1)}u_{11}u_{i1}\right). \end{aligned}$$

Then, by using the same bounds given in Equation (3.2.28), it follows that

$$C \geq \frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2}{w^2}-\left(1+\frac{2}{\ln u_1}\right)\right)\frac{u_{11}^2}{w^3u_1^2\ln u_1}+\frac{2}{w^2(1+w)\ln u_1}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii}. \qquad (3.2.37)$$

Now, from Equation (3.2.30) we have

$$\frac{2}{w^2(1+w)\ln u_1}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii} \geq -\frac{4u_1^2\ln u_1}{w^3(1+w)}\sum_{i>1}\frac{\rho_i^2}{\rho^2}\sum_{j\geq 1}\frac{\partial Q_k}{\partial A_{jj}} \geq -\frac{16r^2u_1^2\ln u_1}{w^3(1+w)\rho^2}\sum_{j\geq 1}\frac{\partial Q_k}{\partial A_{jj}}.$$

In the las inequality we used that $\sum_{i>1}\frac{\rho_i^2}{\rho^2} \leq 4\frac{r^2}{\rho^2}$.
Furthermore, by Equation (3.2.32), it follows that

$$\frac{\partial Q_k}{\partial A_{11}}\left(\frac{2u_1^2}{w^2}-\left(1+\frac{2}{\ln u_1}\right)\right)\frac{u_{11}^2}{w^3u_1^2\ln u_1} \geq \frac{C(k,n)}{32M^2}\frac{u_1^2\ln u_1}{w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}},$$

where $C(k,n)$ is the constant in (h).

Hence, it follows that

$$C \geq \sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{C(k,n)}{32M^2}\frac{u_1^2\ln u_1}{w^3}-\frac{16u_1^2\ln u_1 r^2}{w^3(w+1)\rho^2}\right).$$

Finally, by adding the bounds from the estimates of B and C, we see that

$$\begin{aligned}0 &\geq B+C\\ &\geq \sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\left(\frac{C(k,n)}{32M^2}\frac{u_1^2\ln u_1}{w^3}-16\frac{u_1^2\ln u_1 r^2}{w^3(w+1)\rho^2}-\frac{2}{w\rho}-\frac{u_1^2}{2M^2w^3}\right),\end{aligned}$$

or equivalently,

$$\frac{C(k,n)}{32M^2}\ln u_1\rho \leq 16\frac{r^2\ln u_1}{\rho(w+1)}+\frac{2w^2}{u_1^2}+\frac{\rho}{2M^2}.$$

Then, by using Equation (3.2.36) we obtain

$$\rho \ln u_1 \le C\left(M^2 r^2 + r^2 + M^2\right).$$

Consequently, we have

$$\ln |Du(0)| \le \frac{\varphi(u(x_0))\rho(x_0)\ln u_1(x_0)}{\varphi(u(0))\rho(0)} \le C(k,n)\left(\frac{M^2}{r^2} + \frac{M}{r} + 1\right).$$

□

As a consequence of the proof given for Theorem 3.0.2 we also obtain a non-existence result for graphical Q_k-translator which behaves as $|x|$ at infinity.

Proof of Theorem 3.0.3. Let $u \in \mathcal{C}^3(\mathbb{R}^n)$ be a solution to Equation (3.2.2) such that satisfies Properties (1) and (2) in the statement of Theorem 3.0.3.

Then, by property 2, it follows that $\max_{B_r} |u| \le Cr$, for all $r > 1$. This fact together with Theorem 3.0.2 implies that $|Du(x)| \le C_1$ for all $x \in \mathbb{R}^n$.

Claim 3.2.1. *The function u is constant in $\mathbb{R}^n$.*

Proof. We argue this by contradiction, and without loss of generality we may assume that there is a $\delta > 0$ such that $|Du(0)| \ge \delta$.

In addition, we want to mention that the proof is very similar to the one given for Theorem (3.0.2). The reason of this is that Property 2 combined with Theorem 3.0.2 give us a control from below of the solution. In particular, by minor modifications in the test function, we were able to show a suitable estimate to conclude the proof of this theorem.

Let $r > 1$ and we consider the test function given by

$$G(x) = \rho(x)\varphi(u)|Du|,$$

where $\rho(x) = r^2 - |x|^2$, $\varphi(u) = \left(1 - \dfrac{u - \inf\limits_{B(0,r)} u}{M}\right)^{\beta}$, $M = 2(\sup\limits_{B(0,r)} u - \inf\limits_{B(0,r)} u)$ and the constant $\beta < 0$ is still to be fixed.

Moreover, we note that

$$M \leq 4\,||u||_{\infty}\,,\ \ \frac{1}{2} \leq 1 - \frac{u - \inf\limits_{B_r} u}{M} \leq 1,$$

and $G : \overline{B_r} \to \mathbb{R}$ attains its maximum at an interior point $x_0 \in B_r$.

In addition we will choose a coordinate system such that $u_{ij}(x_0)$ is a diagonal matrix for $2 \leq i, j \leq n$, $u_1(x_0) = |Du(x_0)|$ and $u_i(x_0) = 0$, $i \geq 2$.

Now let $\delta_1 \in [\delta, 1)$ such that $u_1(x_0) \geq \delta_1$, and since x_0 is an interior point of $B(0, r)$, we may shrink δ_1 to assume that

$$\rho(x_0) \geq \delta_1 r^2. \tag{3.2.38}$$

Note that Equation (3.2.38) is equivalent to write $(1 - \delta_1) \geq \dfrac{|x_0|^2}{r^2}$.

Next, we have the following equations at x_0.

First, for the gradient we have

$$0 = (\ln G)_i = \frac{\rho_i}{\rho} + \frac{\varphi'}{\varphi} u_i + \frac{u_{1i}}{u_1}. \tag{3.2.39}$$

Note that, by Equation (3.2.39), it follows that

$$\frac{u_{11}}{u_1} = -\frac{\rho_1}{\rho} - \frac{\varphi'}{\varphi} u_1 \text{ and } \frac{u_{1i}}{u_1} = -\frac{\rho_i}{\rho}. \tag{3.2.40}$$

Consequently, since $\beta < 0$ and $\dfrac{\varphi'}{\varphi} = -\dfrac{\beta}{M}\left(1 - \dfrac{u - \inf\limits_{B_r} u}{M}\right)^{-1} > \dfrac{|\beta|}{4\,||u||_{\infty}}$, we may enlarge r

such that $r > \dfrac{8\,||u||_\infty}{|\beta|\,\delta^2}$ and

$$\frac{u_{11}}{u_1} = \frac{2x_1}{\rho} - \frac{\varphi'}{\varphi}u_1 \leq \frac{2}{\delta_1 r} - \frac{|\beta|}{4\,||u||_\infty}\delta_1 < 0.$$

This fact allows us to use the equations from Lemma 3.2.1.

Moreover, the second derivatives satisfy

$$\begin{aligned} 0 &\geq Q_{k,ij}(\ln G)_{ij} \\ &= Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i\rho_j}{\rho^2} + \frac{\varphi'}{\varphi}u_{ij} + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right)u_i u_j + \frac{u_{1ij}}{u_1} - \frac{u_{1i}u_{1j}}{u_1^2}\right). \end{aligned}$$

We note that by Equation (3.2.40) the last term in the above equation satisfies

$$\begin{aligned} -\sum_{i,j} Q_{k,ij}\frac{u_{1i}u_{1j}}{u_1^2} &= -Q_{k,11}\frac{u_{11}^2}{u_1^2} - \sum_{i,j>1} Q_{k,ij}\frac{u_{1i}u_{1j}}{u_1^2} \\ &= -Q_{k,11}\left(\frac{\rho_1}{\rho} + \frac{\varphi'}{\varphi}u_1\right)^2 - \sum_{i,j>1} Q_{k,ij}\frac{\rho_i\rho_j}{\rho^2} \\ &= -2Q_{k,11}\frac{\rho_1}{\rho}\frac{\varphi'}{\varphi}u_1 - Q_{k,ij}\frac{\rho_i\rho_j}{\rho^2} - Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j \\ &\geq -Q_{k,11}\left(\frac{\rho_1^2}{\rho^2} + u_1^2\left(\frac{\varphi'}{\varphi}\right)^2\right) - Q_{k,ij}\frac{\rho_i\rho_j}{\rho^2} - Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j \\ &\geq -2Q_{k,ij}\frac{\rho_i\rho_j}{\rho^2} - 2Q_{k,ij}\left(\frac{\varphi'}{\varphi}\right)^2 u_i u_j. \end{aligned}$$

In the last line we use that $u_i(x_0) = 0$ for $i > 1$ and $Q_{k,ij}$ is a definite positive matrix.

Therefore, it follows that

$$0 \geq Q_{k,ij}\left(\frac{\rho_{ij}}{\rho} - 3\frac{\rho_i\rho_j}{\rho^2} + \frac{\varphi'}{\varphi}u_{ij} + \left(\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2\right)u_i u_j + \frac{u_{1ij}}{u_1}\right). \tag{3.2.41}$$

Now we are going to analyze the terms in (3.2.41).

Indeed, by Equation (3.2.13), the first and second term in (3.2.39) satisfy

$$\begin{aligned}\sum_{i,j\geq 1} Q_{k,ij}\left(\frac{\rho_{ij}}{\rho}-3\frac{\rho_i\rho_j}{\rho^2}\right) &= -2\sum_{i\geq 1}\frac{Q_{k,ii}}{\rho}-8\sum_{i,j\geq 1}Q_{k,ij}\frac{x_ix_j}{\rho^2}\\ &\geq -\frac{2}{\rho w}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}-12\sum_{i,j\geq 1}Q_{k,ii}\frac{|x_0|^2}{\rho^2}\\ &\geq \frac{-1}{wr^2}\left(\frac{2}{\delta_1}+\frac{12}{\delta_1^2}\right)\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.\end{aligned}$$

In the second line we used that $Q_{k,ij}$ is a definite positive matrix, which allows to estimate $\langle x_0, Q_{k,ij}x_0\rangle \leq \mathrm{Tr}(Q_{k,ij})|x_0|^2$. In the last line we used $|x|^2 \leq r^2$ and Equation (3.2.38).

Next, we estimate the third and fourth term from (3.2.41).

Indeed, if we choose $\beta = \dfrac{-1}{4}$, it follows that

$$\begin{aligned}Q_{k,ij}\left(u_{ij}\frac{\varphi'}{\varphi}+\left(\frac{\varphi''}{\varphi}-3\left(\frac{\varphi'}{\varphi}\right)^2\right)u_iu_j\right) &= \frac{\varphi'}{\varphi}\sum_{i,j\geq 1}\frac{\partial Q_k}{\partial A_{ij}}A_{ij}+Q_{k,11}\left(\frac{\varphi''}{\varphi}-3\left(\frac{\varphi'}{\varphi}\right)^2\right)u_1^2\\ &\geq \frac{1}{16M^2w^3}\frac{\partial Q_{11}}{\partial A_{11}}u_1^2\\ &\geq \frac{u_1^2}{16M^2w^3}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.\end{aligned}$$

In the second inequality we used (a) and

$$\frac{\varphi''}{\varphi}-3\left(\frac{\varphi'}{\varphi}\right)^2 = -\left(1-\frac{u-\inf_{B_r} u}{M}\right)^{-2}\frac{\beta(3\beta+1)}{M^2}\geq \frac{1}{16M^2}.$$

Finally, we estimate the fifth term from (3.2.41).

We note that Equation (3.2.37) applies in the following way

$$Q_{k,ij}\frac{u_{ij1}}{u_1}\geq \frac{2}{w^2(1+w)}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii},$$

where as in the proof of Theorem 3.0.2 we omitted all the non-negative terms.

In addition, Equation (3.2.30) can be applied in a similar manner

$$\begin{aligned}\frac{2}{w^2(1+w)}\sum_{i>1}\frac{\partial Q_k}{\partial A_{1i}}u_{1i}u_{ii} &\geq -\frac{4}{w^3(1+w)}\frac{\partial Q_k}{\partial A_{11}}\sum_{i>1}u_{1i}^2\\ &= -\frac{4u_1^2}{w^3(1+w)}\frac{\partial Q_k}{\partial A_{11}}\sum_{i>1}\frac{\rho_i^2}{\rho^2}\\ &\geq -\frac{16u_1^2r^2}{w^3(1+w)\rho^2}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}\\ &\geq -\frac{16u_1^2}{w^4r^2\delta_1^2}\sum_{i\geq 1}\frac{\partial Q_k}{\partial A_{ii}}.\end{aligned}$$

Consequently, by combining the above estimates, it holds

$$u_1^2\left(\frac{1}{16M^2}-\frac{16}{wr^2\delta_1^2}\right)\leq\frac{w^2}{r^2}\left(\frac{2}{\delta_1}+\frac{12}{\delta_1^2}\right).$$

Then, by enlarging r so that $r^2 \geq 16M^2\left(\frac{16}{\delta^2\sqrt{1+\delta^2}}+1\right)$, it follows that

$$\frac{1}{r^2}\leq\frac{1}{16M^2}-\frac{16}{\delta^2\sqrt{1+\delta^2}r^2}\leq\frac{1}{16M^2}-\frac{16}{wr^2\delta^2}\leq\left(\frac{1}{16M^2}-\frac{16}{wr^2\delta_1^2}\right).$$

Here we used that $\frac{1}{w}\leq\frac{1}{\sqrt{1+\delta^2}}$.

Therefore, this assumption implies that

$$u_1^2(x_0)\leq C\frac{M^2w^2}{r^2}\leq C'\frac{C_1^2\,||u||_\infty^2}{r^2},$$

where C and C' are constants that only depend on δ, and recall that C_1 is the constant such that $|Du|\leq C_1$.

Finally, since $G(x_0)$ is a maximum, it follows that

$$|Du(0)| \leq \frac{\rho(x_0)}{\rho(0)} \frac{\varphi(u(x_0))}{\varphi(u(0))} u_1(x_0) \leq C''(\delta) \frac{||u||_\infty}{r}$$

which implies that $|Du(0)| \to 0$ as $r \to \infty$, a contradiction with $|Du(0)| \geq \delta$. □

Therefore, u is constant but this fact contradicts property 1 from the statement of Theorem 3.0.3. □

3.3 Second Order Estimates

In this section we derive interior second order estimates for solutions to the Q_k-flow and Q_k-translator equation. For this purpose, we derive local uniform estimates for H^2 in both settings for cases $k \geq 1$.

The reason of this is that if M_0 satisfies $\lambda \in \Gamma_{k+2}$ for $k \geq 1$, then by Equations (3.1.9) and Corollary (3.1.4) it follows that

$$|A|_k^2 = (k+1)Q_k^2 - (k+2)Q_{k+1}Q_k \leq (k+1)Q_k^2 \leq \underbrace{\frac{(n-k)^2}{n^2(k+1)}}_{:=C(k,n)} H^2$$

and $|A|^2 \leq H^2$ holds on M_0. In addition, we will assume that M_0 is locally the graph of a function over a hyperplane orthogonal to the direction $\vec{w} \in \mathbb{S}^n$.

Lemma 3.3.1. *Let M_t be a solution to the Q_k-flow (3.0.1) and $\vec{w} \in \mathbb{S}^n$. Then, we have the* *following equations at $p \in M_t$:*

$$\left(\partial_t - \Box_k\right) u = 0, \text{ where } u = \langle p, \vec{w} \rangle, \tag{3.3.1}$$

$$\left(\partial_t - \Box_k\right) h_{ii} = \frac{\partial^2 Q_k}{\partial h_{cd} \partial h_{ab}} \nabla_i h_{cd} \nabla_i h_{ab} + |A|_k^2 h_{ij} - Q_k h_{il} h_j^l, \tag{3.3.2}$$

$$\left(\partial_t - \Box_k\right) \langle \nu, \vec{w} \rangle = |A|_k^2 \langle \nu, \vec{w} \rangle, \tag{3.3.3}$$

$$\left(\partial_t - \Box_k\right) v = -v|A|_k^2 - 2\,||\nabla v||_k^2\, v^{-1}, \text{ where } v = \langle \nu, \vec{w} \rangle^{-1}, \tag{3.3.4}$$

$$\left(\partial_t - \Box_k\right) H = \frac{\partial^2 Q_k}{\partial h_c^d \partial h_a^b} \nabla^i h_c^d \nabla_i h_a^b + |A|_k^2 H, \tag{3.3.5}$$

$$\left(\partial_t - \Box_k\right) Q_k = |A|_k^2 Q_k \tag{3.3.6}$$

$$\left(\partial_t - \Box_k\right) r^2 = 0, \text{ where } r^2 = |p|^2 + 2\frac{n-k}{k+1} t, \tag{3.3.7}$$

where $Q_{k,ij} = \dfrac{\partial Q_k}{\partial h_{ij}}$, $\Box_k f = \sum_{i,j} Q_{k,ij} \nabla_i \nabla_j f, ||X||_k^2 = \langle X, X \rangle_k$, $\langle X, Y \rangle_k = \sum_{i,j} Q_{k,ij} X^i Y^j$ *and* $|A|_k^2 = \sum_{i,j,l} Q_{k,ij} h_{il} h_{lj}$.

Proof. The proofs of Equations (3.3.1)-(3.3.4) can be found in [CD], and for Equations (3.3.5) and (3.3.6) the proofs can be found in [Die].

On the other hand, for Equation (3.3.7), we use a normal frame at $p \in M_t$ given by $e_i \in T_p M_t$. Then, it follows that

$$\partial_t r^2 = 2 \langle Q_k \nu, p \rangle + 2\frac{n-k}{k+1},$$

$$\Box_k r^2 = 2 Q_{k,ij} \delta_j^i + 2 \langle Q_k \nu, p \rangle.$$

Therefore, by Equation (3.1.7) together with Corollary 3.1.4, it follows that $(\partial_t - \Box_k) r = 0$ since $Q_{k,ij} \delta_j^i = \dfrac{n-k}{k+1}$. □

Remark 3.3.1. Note that when $\vec{w} = e_{n+1}$, then $v = \langle \nu, \vec{w} \rangle^{-1} = \sqrt{1 + |Du|^2} = w$ from the last section, and also we choose v rather than w since w is for an specific vector in $\mathbb{S}^n$.

In addition, Equation (3.3.4) implies that the assumption imposed on M_0, being locally a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^{n-1}$, is still valid under the Q_k-flow if the domain of v lies in the support of $R^2 - r^2$, where R depends on the domain of definition of u.

Remark 3.3.2. Let assume that M_t evolves by the Q_k-flow such that each M_t is convex and it can be written as a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$ for all $t \in [0, T)$.
Then, the function

$$h(x,t) = r^2 - u^2$$

satisfies $|(\partial_t - \Box_k)h| \leq C(k,n)$, $||\nabla h||_k^2 \leq Ch$ and $|\nabla h| \leq Ch$. To see this, we choose normal coordinate at $p \in M_t$, then by Equations (3.3.1) and (3.3.7), we have

$$(\partial_t - \Box_k)h = (\partial_t - \Box_k)r^2 - (\partial_t - \Box_k)u^2 = -2u(\partial_t - \Box_k)u + 2\,||\nabla u||_k^2 = 2\,||\nabla u||_k^2 .$$

In addition, since M_t is convex, it follows that $Q_{k,ij} = \dfrac{\partial Q_k}{\partial \lambda_i} \leq 1$. We refer to Proposition 2.2 in [CD] for a proof of this.
Furthermore, we have $\nabla u = \vec{w}^{\top}$, which implies that $||\nabla u||_k \leq 1$. In particular $|(\partial_t - \Box_k)h| \leq C$.
On the other hand,

$$|\nabla h|^2 = 4\left|(p - u\vec{w})^{\top}\right|^2 \leq 4\,|p - u\vec{w}|^2 = 4(|p|^2 - u^2) \leq 4h.$$

Finally, by the same argument, we also have $||\nabla h||_k^2 \leq Ch$.

In [CD] Theorem 2.4, the authors give a gradient estimate for the Q_k-flow of the form

$$\phi v \leq \sup_{M_0} \phi(p,0) v(p,0),$$

where $v = \langle \nu, \vec{w} \rangle^{-1}$ and $\phi = R^2 - r^2$, for a convex M_0 initial data such that $\lambda \in \Gamma_{k+1}$. Instead, we give a similar estimate without the convexity assumption.

Proposition 3.3.2. *Let M_t be a solution to* (3.0.1) *such that $\lambda \in \Gamma_{k+1}$. Then, for $R > 0$ and $x_0 \in \mathbb{R}^{n+1}$, the following estimate holds*

$$\phi_+(x,t)v(x,t) \leq \sup_{x \in M_0} \{\phi_+(x,0)v(x,0)\},$$

where $\phi_+ = \max\{\phi, 0\}$, $\phi(x,t) = R^2 - |x - x_0|^2 - 2\dfrac{n-k}{k+1}t$ and v is defined in the support of ϕ_+.

Proof. Firstly, the proof of this result is very similar to the one give in [EH2] for the mean curvature flow.

In addition, we note that without loosing generality its enough to consider $x_0 = 0$.

Let $\eta(r) = (R^2 - r^2)^2$, where $r^2 = |x|^2 + 2\dfrac{n-k}{k+1}t$ as in Equation (3.3.7). We note that $\dfrac{(\eta')^2}{\eta} = 4$ and $\eta'' = 2$, where $(\cdot)'$ denotes the derivative with respect r.

Next, we study the equation which satisfies ηv^2. In fact, we have

$$(\partial_t - \Box_k)(\eta v^2) = v^2 (\partial_t - \Box_k)\eta + 2v\eta (\partial_t - \Box_k) v - 2\eta \left|\left|\nabla v\right|\right|_k^2 - 4v \left\langle \nabla\eta, \nabla v \right\rangle_k. \qquad (3.3.8)$$

Note that by Equation (3.3.7) the term $(\partial_t - \Box_k)\eta$ satisfies

$$(\partial_t - \Box_k)\eta = -2(R^2 - r^2)(\partial_t - \Box_k) r^2 - 2\left|\left|\nabla r^2\right|\right|_k^2 \leq -2\left|\left|\nabla |x|^2\right|\right|_k^2.$$

Then, by applying this equation together with Equation (3.3.4) onto Equation (3.3.8), we obtain

$$(\partial_t - \Box_k)(\eta v^2) \leq -2v^2 \left|\left|\nabla |x|^2\right|\right|_k^2 - 6\eta \left|\left|\nabla v\right|\right|_k^2 - 2|A|_k^2 \eta v^2 - 4v \left\langle \nabla\eta, \nabla v \right\rangle_k. \qquad (3.3.9)$$

On the other hand, we have the term

$$-4v\left\langle\nabla\eta,\nabla v\right\rangle_k = -6v\left\langle\nabla\eta,\nabla v\right\rangle_k + \eta^{-1}\left\langle\nabla\eta,\nabla(\eta v^2)\right\rangle_k - \eta^{-1}v^2\left|\left|\nabla\eta\right|\right|_k^2.$$

Note that $\eta^{-1}v^2\left|\left|\nabla\eta\right|\right|_k^2 = 4v^2\left|\left|\nabla|x|^2\right|\right|_k^2$.

Then, by combining the above equation with Equation (3.3.9), it holds

$$(\partial_t - \Box_k)(\eta v^2) \leq -6v^2\left|\left|\nabla|x|^2\right|\right|_k^2 - 6\left\langle\nabla\eta,\nabla v\right\rangle_k - 6\eta\left|\left|\nabla v\right|\right|_k^2 + \eta^{-1}\left\langle\nabla\eta,\nabla(\eta v^2)\right\rangle_k, \tag{3.3.10}$$

note that we dropped the term $-2|A|_k^2\eta v^2$.

Now, let estimate the following term

$$\begin{aligned}
-6v\left\langle\nabla\eta,\nabla v\right\rangle_k &= 6Q_{k,ij}v\frac{|\eta'|}{\sqrt{\eta}}\nabla_i|x|^2\sqrt{\eta}\nabla_j v \\
&\leq 6Q_{k,ij}\left((\eta')^2\eta^{-1}\varepsilon\frac{\nabla_i|x|^2\nabla_j|x|^2}{2}v^2 + \eta\frac{\nabla_i v\nabla_j v}{2\varepsilon}\right) \\
&= 12\varepsilon v^2\left|\left|\nabla|x^2|\right|\right|_k + \frac{3}{\varepsilon}\eta\left|\left|\nabla v\right|\right|_k^2.
\end{aligned}$$

where in the second line we use Young's Inequality.

Then, by plugging this estimae onto (3.3.10) we get

$$(\partial_t - \Box_k)(\eta v^2) \leq (12\varepsilon - 6)v^2\left|\left|\nabla|x|^2\right|\right|_k^2 + \left(\frac{3}{\varepsilon} - 6\right)\eta\left|\left|\nabla v\right|\right|_k^2 + \eta^{-1}\left\langle\nabla\eta,\nabla(\eta v^2)\right\rangle_k.$$

Therefore, if we choose $\varepsilon = \frac{1}{2}$ it follows that

$$(\partial_t - \Box_k)(\eta v^2) \leq \eta^{-1}\left\langle\nabla\eta,\nabla(\eta v^2)\right\rangle_k.$$

Then, since $\lambda \in \Gamma_{k+1}$, the operator $\Box_k$ is locally uniformly elliptic. In particular, the weak parabolic maximum principle applied to ηv^2 yields the result. Moreover, since chaining η by ϕ_+ does not modify the estimate as long v is defined in the support of ϕ_+. $\square$

Remark 3.3.3. Note that this estimate as $R \to \infty$ gives a gradient global bound for entire graphs of the Q_k-flow with the principal curvatures belong at Γ_{k+1}.

Now we give the proof for the second order estimate for the Q_k-flow.

Proof of Theorem 3.0.4. The proof is very similar to the one given in [EH2] for the mean curvature flow. For completeness we give all the details here.

Let φ a real function to be chosen, and we consider

$$\begin{aligned}(\partial_t - \square_k)\left(H^2\varphi(v^2)\right) =& 2H\varphi\left(\partial_t - \square_k\right)H - 2\varphi\,||\nabla H||_k^2 + 2vH^2\varphi'\left(\partial_t - \square_k\right)v \\ &- 2H^2\varphi'\,||\nabla v||_k^2 - 2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k - 4H^2\varphi''v^2\,||\nabla v||_k^2 .\end{aligned}$$

Then, by replacing the above equation with Equations (3.3.4) and (3.3.5) together with the concavity of Q_k, it follows

$$\begin{aligned}(\partial_t - \square_k)\left(H^2\varphi\right) \leq& 2H^2\varphi|A|_k^2 - 2\varphi\,||\nabla H||_k^2 - 2v^2H^2\varphi'|A|_k^2 \\ &- 6H^2\varphi'\,||\nabla v||_k^2 - 2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k - 4H^2\varphi''v^2\,||\nabla v||_k^2 .\end{aligned} \tag{3.3.11}$$

Next, we estimate the term $-2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k$. Indeed, we have

$$\begin{aligned}-2\left\langle \nabla H^2, \nabla\varphi\right\rangle_k &= -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + H^2\varphi^{-1}\,||\nabla\varphi||_k^2 - 4H\varphi'v\left\langle \nabla H, \nabla v\right\rangle_k \\ &= -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + 4H^2\varphi^{-1}(\varphi')^2v^2\,||\nabla v||_k^2 - 4H\varphi'v\left\langle \nabla H, \nabla v\right\rangle_k \\ &\leq -\varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k + 6H^2\varphi^{-1}(\varphi')^2v^2\,||\nabla v||_k^2 + 2\varphi\,||\nabla H||_k^2 .\end{aligned}$$

In the last line we used $4H\varphi'v\left\langle \nabla H, \nabla v\right\rangle_k \leq 2\varphi\,||\nabla H||_k^2 + 2\varphi^{-1}(\varphi')^2v^2\,||\nabla v||_k$.

Then, by substituting the last equation onto (3.3.11), we obtain

$$\begin{aligned}(\partial_t - \Box_k)\, H^2\varphi(v^2) \leq & 2|A|_k^2 H^2(\varphi - v\varphi') - \varphi^{-1}\left\langle \nabla(H^2\varphi), \nabla\varphi\right\rangle_k \\ & - ||\nabla v||_k^2\, H^2\left(6\varphi'(1-\varphi'\varphi^{-1}v^2) + 4\varphi'' v^2\right).\end{aligned} \tag{3.3.12}$$

Now we let $\phi = H^2\varphi(v^2)$ and choose $\varphi(x) = \dfrac{x}{1-ax}$, where a is still to be chosen. In addition, since $\lambda \in \Gamma_{k+2}$, it follows that $|A|_k^2 \leq \tilde{C}(k,n)H^2$. Therefore, we have

$$\begin{aligned}(\partial_t - \Box_k)\, \phi \leq\ & 2\tilde{C}(k,n)\varphi^{-2}(\varphi - v^2\varphi')\phi^2 - \varphi^{-1}\left\langle \nabla\phi, \nabla\varphi\right\rangle_k \\ & - \phi\, ||\nabla v||_k^2 \left(6\varphi'(1-\varphi\varphi' v^2) + 4\varphi'' v^2\right),\end{aligned} \tag{3.3.13}$$

and

$$\begin{cases} \varphi - v^2\varphi' = -a\varphi^2, \\ \varphi^{-1}\left\langle \nabla\phi, \nabla\varphi\right\rangle_k = 2\varphi v^{-3}\nabla v, \\ (6\varphi'(1-\varphi'\varphi^{-1}v^2) + 4\varphi'' v^2) = \dfrac{2a}{(1-ax)^2}. \end{cases}$$

Consequently, by substituting these equations onto Equation (3.3.13), we obtain

$$(\partial_t - \Box_k)\, \phi \leq -2\tilde{C}(k,n)a\phi^2 - 2\varphi v^{-3}\left\langle \nabla\phi, \nabla\varphi\right\rangle_k - \frac{2a\phi}{(1-av^2)^2}\, ||\nabla v||_k^2. \tag{3.3.14}$$

On the other hand, we consider $\eta(x,t) = (R^2 - h(x,t))^2$ where $h(x,t)$ is the function given in hypothesis and satisfies (3.0.5).

Then, the evolution equation for η satisfies

$$(\partial_t - \Box_k)\, \eta \leq 2C(k,n)R^2 - 2\, ||\nabla h||_k^2. \tag{3.3.15}$$

Furthermore, the evolution equation of $\phi\eta$ is given by

$$(\partial_t - \Box_k)(\phi\eta) = \eta(\partial_t - \Box_k)\phi + \phi(\partial_t - \Box_k)\eta - 2\langle\nabla\eta, \nabla\phi\rangle_k. \tag{3.3.16}$$

Note that the term

$$-2\langle\nabla\eta, \nabla\phi\rangle_k = -2\eta^{-1}\langle\nabla\eta, \nabla\phi\eta\rangle_k + 8\phi\,||\nabla h||_k^2.$$

Then, by substituting Equations (3.3.14) and (3.3.15) with the this last equation onto Equation (3.3.16), we obtain

$$\begin{aligned}
&(\partial_t - \Box_k)(\phi\eta) \qquad (3.3.17)\\
&\leq -2a\tilde{C}(k,n)\phi^2\eta - \frac{2a\phi\eta}{(1-av^2)^2}\,||\nabla v||_k^2 + 2C(k,n)\phi R^2\\
&- 2\eta^{-1}\langle\nabla(\phi\eta), \nabla\eta\rangle + 6\phi\,||\nabla h||_k^2 - 2\varphi\eta v^{-3}\langle\nabla\phi, \nabla v\rangle_k.
\end{aligned}$$

Note that the term $-2\varphi\eta v^{-3}\langle\nabla\phi, \nabla v\rangle_k$ satisfies

$$\begin{aligned}
-2\varphi\eta v^{-3}\langle\nabla\phi, \nabla v\rangle_k &= -2v^{-3}\varphi\langle\nabla(\phi\eta), \nabla v\rangle_k + 4\phi\left\langle\eta^{1/2}\nabla v, v^{-3}\varphi\nabla h\right\rangle_k\\
&\leq -2v^{-3}\varphi\langle\nabla(\phi\eta), \nabla v\rangle_k + 2\phi\left(\eta\frac{||\nabla v||_k^2}{\varepsilon} + \varphi^2 v^{-6}\varepsilon\,||\nabla h||_k^2\right)\\
&= -2v^{-3}\varphi\langle\nabla v, \nabla(\phi\eta)\rangle_k + 2(\phi\eta)\frac{a}{(1-av^2)^2}\,||\nabla v||_k^2 + \frac{\phi}{v^2 a}\,||\nabla h||_k^2.
\end{aligned}$$

In second line we use Young's inequality, in third line we use $\varepsilon = \dfrac{(1-av^2)^2}{a}$ and $\varphi^2 v^{-6}\varepsilon = \dfrac{v^{-2}}{a}$.

Then, by applying the above equation onto Equation (3.3.17), it follows that

$$\begin{aligned}(\partial_t - \Box_k)(\phi\eta) \leq & -2a\tilde{C}(k,n)\eta\phi^2 + C(k,n)\phi\left(h\left(\frac{1}{v^2 a} + 1\right) + R^2\right) \\ & -2\left\langle \nabla(\phi\eta), v^{-3}\varphi\nabla v - 2\eta^{-1}\nabla\eta\right\rangle_k .\end{aligned} \tag{3.3.18}$$

Finally, we consider the test function $G = t\phi\eta$. Note that G reaches it maximum at $t_0 > 0$ and in a interior point of M_{t_0}. In particular, at this point we have that $\nabla G = t\nabla(\phi\eta) = 0$ and

$$0 \leq (\partial_t - \Box_k) G \leq -2a\tilde{C}(k,n)\phi^2\eta t_0 + C(k,n)\phi t_0\left(\frac{h}{v^2 a} + h + R^2\right) + \phi\eta. \tag{3.3.19}$$

Then, by multiplying $\dfrac{\eta t_0}{2a\tilde{C}(k,n)}$ to Equation (3.3.19), we see that

$$m(t_0)^2 \leq m(t_0)\frac{C(k,n)}{2a\tilde{C}(k,n)}\left(t_0\left(\frac{h}{v^2 a} + h + R^2\right) + \eta\right),$$

where $m(t_0) = \sup_{[0,t_0]}\sup_{M_t} G$.

Consequently, since $\eta \leq R^4$ and $M_t = \{h(x,t) \leq R^2\}$, it follows that

$$m(t_0) \leq c(k,n)\left(t_0 R^2 + R^4\right)\sup_{s\in[0,t_0]}\sup_{M_s} v^2,$$

provided that $a = \frac{1}{2}\inf_{s\in[0,t_0]}\inf_{M_s} v^{-2}$.

Finally, since $\varphi(v^2) \geq 1$ and $\eta > (1-\theta)^2 R^4$ in $\{x \in M_t : h(x,t) \leq \theta R^2\}$ for $\theta \in [0,1)$, the estimate from Theorem 3.0.4 follows since

$$H^2 \leq \frac{m(t_0)}{R^4 t(1-\theta)^2}, \text{ holds on } M_t \text{ with } t \leq t_0.$$

$\square$

Remark 3.3.4. By Remark 3.3.2 and Theorem 3.0.4, an analogous result to [EH1] and [EH2] can be proven for the Q_k-flow for convex initial data. But it is worth mentioning that these results were proven in [CD] using different techniques.

Now we develop a similar estimate for Equation (3.2.2). Recall that we are assuming that $M_0 = M$ is locally a graph over a hyperplane orthogonal to $\vec{w} \in \mathbb{S}^n$.

Lemma 3.3.3. *Let M be a Q_k-translator and $\vec{w} \in \mathbb{S}^n$. Then, we have the following equations at $p \in M$:*

$$\Box_k u = -Q_k \langle \nu, \vec{w} \rangle \ \text{ and } \nabla u = \vec{w}^\top := \langle e_i, \vec{w} \rangle e_i, \ \text{ where } u = \langle p, \vec{w} \rangle , \tag{3.3.20}$$

$$\Box_k r^2 = 2 \left(-Q_k \langle \nu, p \rangle + Q_{k,ij} \delta^i_j \right), \ \text{ where } r^2 = |p|^2, \tag{3.3.21}$$

$$\Box_k H + \sum_{i=1}^{n} Q_{k,ab;cd} \nabla^i h^b_a \nabla_i h^d_c + |A|^2_k H + \langle \nabla H, e_{n+1} \rangle = 0, \tag{3.3.22}$$

$$\Box_k Q_k + |A|^2_k Q_k + \langle \nabla Q_k, e_{n+1} \rangle = 0, \tag{3.3.23}$$

$$\Box_k v - v^2 \langle \nabla Q_k, \vec{w} \rangle - |A|^2_k v - 2v^{-1} ||\nabla v||^2_k = 0, \ \text{ where } v = \langle \nu, w \rangle^{-1} . \tag{3.3.24}$$

Proof. Firstly, we use normal coordinates centered at $p \in M$. This means that $\{e_i\}_{i=1}^n \subset T_p M$ is an orthogonal basis of eigenvalues of the second fundamental form (i.e: $h^j_i(p) = h_{ij}(p) = \lambda_i(p) \delta^i_j$ and λ_i is the principal curvature) at p.
We note that by the calculation given in the proof of Lemma 3.3.1 we only need to show Equations 3.3.20 (3.3.22) (3.3.23) (3.3.24).

Then, the height function $u = \langle p, \vec{w} \rangle$ satisfies

$$\nabla_i u = \langle e_i, \vec{w} \rangle ,$$
$$\nabla_j \nabla_i u = \langle \nabla_j e_i, \vec{w} \rangle = -h_{ij} \langle \nu, \vec{w} \rangle ,$$
$$\Box_k u = -Q_k \langle \nu, \vec{w} \rangle .$$

This proves Equation (3.3.20).

On the other hand, for the coefficient of the second fundamental form of M we have

$$\begin{aligned}\nabla_j\nabla_i Q_k =& \frac{\partial^2 Q_k}{\partial h_{cd}\partial h_{ab}}\nabla_j h_{cd}\nabla_i h_{ab} + \Box_k h_{ij} + |A|_k^2 h_{ij} \\ & - Q_k h_{il}h_{lj} + \frac{\partial Q_k}{\partial h_{ab}}\left(h_{ib}h_{am}h_{mj} - h_{im}h_{aj}h_{mb}\right).\end{aligned}$$

Here we used the Simons identities, we refer the reader to equations from Theorem 2.1 given in [HP].

Moreover, since $Q_k = \langle \nu, e_{n+1}\rangle$, it follows that

$$\nabla_j\nabla_i Q_k = -\langle \nabla h_{ij}, e_{n+1}\rangle - h_{il}h_{jl}Q_k,$$

where $e_{n+1} = (\underbrace{0,\ldots,0}_{n-\text{times}},1)$. Then, after combining both equations for $\nabla_j\nabla_i Q_k$, we obtain

$$\begin{aligned}\Box_k h_{ij} =& -\frac{\partial^2 Q_k}{\partial h_{cd}\partial h_{ab}}\nabla_j h_{cd}\nabla_i h_{ab} \\ & - |A|_k^2 h_{ij} - \langle \nabla h_{ij}, e_{n+1}\rangle - \frac{\partial Q_k}{\partial h_{ab}}\left(h_{ib}h_{am}h_{mj} - h_{im}h_{aj}h_{mb}\right).\end{aligned}$$

Consequently, Equation (3.3.22) follows by taking trace in the above equation.

For the function $\langle \nu, \vec{w}\rangle$, it follows that

$$\nabla_i \langle \nu, \vec{w}\rangle = -h_{il}\langle e_l, \vec{w}\rangle \text{ and } \nabla_j\nabla_i\langle \nu, \vec{w}\rangle = -\nabla_l h_{ij}\langle e_l, \vec{w}\rangle - h_{il}h_{lj}\langle \nu, \vec{w}\rangle,$$

note that we also use the Codazzi equations in the last line. Therefore, it holds

$$\Box_k\langle \nu, \vec{w}\rangle + |A|_k^2\langle \nu, \vec{w}\rangle + \langle \nabla Q_k, \vec{w}\rangle = 0. \tag{3.3.25}$$

Finally, for $v = \langle \nu, \vec{w} \rangle^{-1}$, we have

$$\nabla_i v = -v^2 \nabla_i \langle \nu, \vec{w} \rangle, \text{ and } \Box_k v = -v^2 \Box_k \langle \nu, \vec{w} \rangle + 2v^{-1} \|\nabla v\|_k^2 .$$

Then, by substituting the above equations with Equation (3.3.25), Equation (3.3.24) holds. Note that Equation (3.3.23) follows by taking $\vec{w} = e_{n+1}$. $\Box$

Now we derive an Ecker-Huisken type interior estimate for Q_k-translators with principal curvature vector $\lambda \in \Gamma_{k+2}$. As in the parabolic case, we assume that there is a positive function $h(x)$ which satisfies (3.0.6).

Proof of Theorem 3.0.5. The proof is very similar to the given in 3.0.4. Therefore, we only point put the main differences. We consider the test function

$$G(x) = H^2 \varphi(v^2) \eta(h),$$

where $\varphi(x) = \dfrac{x}{1 - ax}$, $a = \dfrac{1}{2} \inf v^{-2}$, $\eta = (R^2 - h)^2$ and h satisfies (3.0.6).

Let $\phi = H^2 \phi(v^2)$, then it follows that

$$\begin{aligned} -\Box_k \phi = & -2H\varphi \Box_k H - 2\varphi \|\nabla H\|_k^2 - 2vH^2 \varphi' \Box_k v \\ & - 2H^2 \varphi' \|\nabla v\|_k^2 - 2 \left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \|\nabla v\|_k^2 . \end{aligned}$$

Then, by replacing the above equation with Equations (3.3.22) and (3.3.24) together with the concavity of Q_k, it follows

$$\begin{aligned} -\Box_k \phi \leq & 2H^2 \varphi |A|_k^2 (\varphi - \varphi' v^2) - 2\varphi \|\nabla H\|_k^2 \\ & - 6H^2 \varphi' \|\nabla v\|_k^2 - 2 \left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \|\nabla v\|_k^2 \\ & - 2v^3 H^2 \varphi' \langle \nabla Q_k, e_{n+1} \rangle + 2H\varphi \langle \nabla H, e_{n+1} \rangle . \end{aligned}$$

We note that since $Q_k = \langle \nu, e_{n+1} \rangle$, it follows that $\nabla Q_k = -v^{-2}\nabla v$. Therefore, we have

$$\begin{aligned} -\Box_k \phi \leq & 2H^2 \varphi |A|_k^2 (\varphi - \varphi' v^2) - 2\varphi \left|\left|\nabla H\right|\right|_k^2 + \langle \nabla \phi, e_{n+1} \rangle \\ & - 6H^2 \varphi' \left|\left|\nabla v\right|\right|_k^2 - 2\left\langle \nabla H^2, \nabla \varphi \right\rangle_k - 4H^2 \varphi'' v^2 \left|\left|\nabla v\right|\right|_k^2 . \end{aligned}$$

In the first line we used that $\nabla \phi = 2H\varphi \nabla H + 2H^2 \varphi' v \nabla v$.

Therefore, by following the steps given in the proof of Theorem 3.0.4 it follows that

$$-\Box_k \phi \leq -2\tilde{C}(k,n) a \phi^2 - 2\varphi v^{-3} \langle \nabla \phi, \nabla \varphi \rangle_k - \frac{2a\phi}{(1-av^2)^2} \left|\left|\nabla v\right|\right|_k^2 + \langle \nabla \phi, e_{n+1} \rangle .$$

On the other hand, $\eta(x) = (R^2 - h(x))^2$ satisfies the following equation

$$-\Box_k \eta = 2\eta^{\frac{1}{2}} \Box_k h - 2\left|\left|\nabla h\right|\right|_k^2 \leq 2C(k,n) R^2 + \langle \nabla \eta, e_{n+1} \rangle - 2\left|\left|\nabla h\right|\right|_k^2 .$$

Therefore, for $G = \phi\eta$ holds the following estimate

$$\begin{aligned} -\Box_k G = & -\eta \Box_k \phi - \phi \Box_k \eta - 2\langle \nabla \eta, \nabla \phi \rangle_k . \\ \leq & -2\tilde{C}(k,n) a G \phi - 2\eta \varphi v^{-3} \langle \nabla \phi, \nabla \varphi \rangle_k - \frac{2aG}{(1-av^2)^2} \left|\left|\nabla v\right|\right|_k^2 + \langle \nabla \phi, e_{n+1} \rangle \\ & + 2C(k,n) R^2 \phi + \langle \nabla G, e_{n+1} \rangle - 2\phi \left|\left|\nabla h\right|\right|_k^2 - 2\langle \nabla \eta, \nabla \phi \rangle_k . \end{aligned}$$

Note that we can follows the same steps from the proof given in the parabolic case, this will yields to

$$\begin{aligned} -\Box_k G \leq & -2a\tilde{C}(k,n) G \phi + C(k,n) \phi \left(h \left(\frac{1}{v^2 a} + 1 \right) + R^2 \right) \\ & - 2\left\langle \nabla G, v^{-3} \varphi \nabla v - 2\eta^{-1} \nabla \eta \right\rangle_k + \langle \nabla G, e_{n+1} \rangle . \end{aligned}$$

Then, as in the parabolic case, G reaches it maximum at an interior point of M. In particular, at this point we have that $\nabla G = 0$ and

$$0 \leq -\Box_k G \leq -2a\tilde{C}(k,n)G\phi + C(k,n)\phi\left(h\left(\frac{1}{v^2 a}+1\right)+R^2\right)$$

Then, by multiplying $\dfrac{\eta}{2a(k+1)}$ in the last line, we see that

$$m^2 \leq m\frac{C(k,n)}{2a\tilde{C}(k,n)}\left(t_0\left(\frac{h}{v^2 a}+h+R^2\right)+\eta\right),$$

where $m = \sup_M G$.

Consequently, since $a = \inf_M v^{-2}$, $\eta \leq R^4$ and $M = \{h(x) \leq R^2\}$, it follows that

$$m \leq c(k,n)\left(R^2+R^4\right)\sup_M v^2.$$

Finally, since $\varphi(v^2) \geq 1$ and $\eta > (1-\theta)^2 R^4$ in $\{x \in M_t : h(x) \leq \theta R^2\}$ for $\theta \in [0,1)$, the estimate from Theorem 3.0.4 follows since

$$H^2 \leq \frac{m}{R^4(1-\theta)^2}, \text{ holds on } M.$$

□

Remark 3.3.5. We note that the function $h = r^2 - u^2$, where $r = |p|^2$ and $u = \langle p, e_{n+1}\rangle$, satisfies the following equations

$$\begin{aligned}
\nabla_i h &= \nabla_i r^2 - 2u\nabla_i u = 2\langle e_i, p - ue_{n+1}\rangle \Rightarrow \langle \nabla h, e_{n+1}\rangle = 2\langle e_i, p-ue_{n+1}\rangle \langle e_i, e_{n+1}\rangle,\\
\Box_k h &= 2(-Q_k\langle \nu, p\rangle + Q_{k,ij}\delta^i_j) + 2uQ_k\langle \nu, e_{n+1}\rangle - 2\,||\nabla u||_k^2\\
&= 2Q_k\langle \nu, p-ue_{n+1}\rangle + 2Q_{k,ij}\delta^i_j - 2\,||\nabla u||_k^2.
\end{aligned}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal base of T_pM of the second fundamental form of M and

$e_{n+1} = (0, \ldots, 0, 1)$. Then, since $Q_k = \langle \nu, e_{n+1} \rangle$, it follows that

$$
\begin{aligned}
|\Box_k h + \langle \nabla h, e_{n+1} \rangle| &\leq 2(n+1)|p - u e_{n+1}| + Q_{k,ij}\delta^i_j + 2\left|\left|\nabla u\right|\right|_k^2 \\
&= 2(n+1)h + 2\frac{n-k}{k+1} + \left|\left|e_{n+1}^\top\right|\right|_k^2 .
\end{aligned}
$$

Therefore, this function cannot be used in the statement of Theorem 3.0.5 without an estimate on h depending only on k and n.

Chapter 4

Rotationally Symmetric γ-Translators

From a historical point of view, rotationally symmetric hypersurfaces are often the first constructions made when one wants to prescribe a curvature function.
The most famous examples of these surfaces in $\mathbb{R}^3$ are the plane and the catenoid, which are the unique rotationally symmetric minimal surfaces, and spheres, which are the unique closed surface of constant mean curvature. Note that in these examples the mean curvature H is 0 in the former case and equal to a positive constant in the latter.

In relation with γ-translators, the "bowl" soliton is the unique H-translator which is a strictly convex entire graph in $\mathbb{R}^3$. That solution was found by Altschuler and Wu in [AW] in $\mathbb{R}^3$, and they noticed later that the same construction works in $\mathbb{R}^{n+1}$. In addition, this solution behaves at infinity like a paraboloid. More precisely, the authors in [CSS] showed that the "bowl" soliton in $\mathbb{R}^{n+1}$ has an asymptotic behavior described by

$$\frac{|x|^2}{2n} - \ln(|x|) + O\left(\frac{1}{|x|}\right), \text{ as } |x| \to \infty.$$

Regarding to the classification of H-translator in $\mathbb{R}^{n+1}$, the study of the "bowl" soliton led to important results in this area. For instance, in [Has], the author showed that a α-

noncollapse and convex H-translator must be the bowl soliton. Moreover, in [MSHS], the authors showed that a convex H-translator which is asymptotic to the "bowl" soliton, is in fact a “bowl” soliton.

Furthermore, the authors in [SX] showed that mean convex H-translators in $\mathbb{R}^3$ are in fact convex. Consequently, the "bowl" soliton is the unique mean convex entire H-translators in $\mathbb{R}^3$. We refer the reader to [HIMW2] and [HMW] for the classification of H-translators which are graphs and semigraphs, respectively.

Therefore, “bowl”-type solutions[1] can lead to important results in the classification of γ-translators in $\mathbb{R}^{n+1}$ as we will show in Chapter 5.

For this reason in this chapter we construct “bowl”-type γ-translators in $\mathbb{R}^{n+1}$ for the following families of curvature functions $\sqrt[n]{S_n}$ and Q_{n-1}, where $Q_k = \dfrac{S_{k+1}}{S_k}$ and $S_k(\lambda)$ denotes the elementary symmetric polynomial in n variables of degree k evaluated in the principal curvatures of the γ-translator, see Equation 2.1.2 in the Introduction for the definition of S_k.

Recall from the introduction that a γ-translator is a hypersurface $M \subset \mathbb{R}^{n+1}$ which satisfies the equation

$$\gamma(\lambda_1, \dots, \lambda_n) = \langle \nu, e_{n+1} \rangle \,,$$

where ν is the unit normal vector and λ_i is the principal curvature of M in $\mathbb{R}^{n+1}$, respectively. Note that these hypersurfaces evolve under translation in direction e_{n+1} when the γ-flow (2.0.1) is applied to them.

As we mentioned in the Introduction, the author in [Ren] constructed “bowl”-type γ-translators for a general class of curvature functions. Moreover, he showed they are asymptotic to paraboloids up to first order at infinity, and characterized when the graph will be entire or defined in round ball of radio $\gamma(1, \dots, 1)$ (see Remark 4.0.1 below) in terms of γ.

[1]Strictly convex rotationally symmetric graphs in $\mathbb{R}^{n+1}$ which are entire or defined in a round ball.

A summary of this dichotomy is characterized by the following hypotheses on γ (see Theorems 1.3-1.4 in [Ren]):

- If $\gamma(0, 1, \ldots, 1) > 0$, then the "bowl"-type solution is entire.
- If $\gamma(0, 1, \ldots, 1) = 0$, and, under the constraint $\gamma(x, y, \ldots, y) = 1$, $x \to L > 0$ as $y \to \infty$, then the "bowl"-type solution is defined in round ball.

  In addition, if $L = 0$, then we have the following extra conditions:

  - If $x = O(y^{-1})$, then the "bowl"-type solution is entire.
  - If there are positive constants C and $k \in (0, 1)$ such that $x \geq Cy^{-k}$ for big enough y, then the "bowl"-type solution is defined in a round ball.

On the other hand, we want to mention that in $\mathbb{R}^3$ the "bowl"-type solutions are explicit for the curvature functions Q_{n-1} and $\sqrt[n]{S_n}$ (see Fig. **??** below), and implicit for $n \geq 3$.

In addition, these curvature functions agree with the result exposed in [Ren] above: $\sqrt[n]{S_n}$ and Q_{n-1} both vanish at $(0, 1, \ldots, 1)$ and satisfy

$$\sqrt[n]{S_n}(x, y, \ldots, y) = 1 \Leftrightarrow x = \frac{1}{y^{n-1}} \to 0 \text{ as } y \to \infty,$$

$$Q_{n-1}(x, y, \ldots, y) = 1 \Leftrightarrow x = \frac{y}{y - \binom{n-1}{n-2}} \to 1 \text{ as } y \to \infty.$$

We note that this description for "bowl"-type $\sqrt[n]{S_n}$-translator in $\mathbb{R}^{n+1}$ only works for $n = 2$ since $x = O(y^{n-1})$.

The summary of the results of this chapter reads as follows.

Theorem 4.0.1. *For $n \geq 3$ there exists a unique smooth complete, strictly convex translator in $\mathbb{R}^{n+1}$ for the curvature functions Q_{n-1} and $\sqrt[n]{S_n}$, such translators are rotationally symmetric graphs.*
In addition, it holds

1. *For $n \geq 3$, the Q_{n-1}-translator is defined in a ball of radio $Q_{n-1}(1,\ldots,1)$ and is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(Q_{n-1}(1,\ldots,1)) \times \mathbb{R}$, i.e:*

$$\begin{aligned} \min\{\lambda_i(p) : i = 1,\ldots,n\} =& \lambda_n(p) \to 0, \\ \forall\, i \in \{1,\ldots,n-1\}\,,\, \lambda_i(p) \to& \frac{1}{Q_{n-1}(1,\ldots,1)}, \end{aligned}$$

as $|p| \to \infty$.

2. *For $n \geq 3$, the $\sqrt[n]{S_n}$-translator is entire.*

Now we briefly describe the strategy used in the proof of Theorem 4.0.1.
Firstly, we write Equation (4.0.2) in the form

$$\ddot{\mathbf{u}} = (1 + \dot{\mathbf{u}}^2) f\left(\frac{\dot{\mathbf{u}}}{r}\right), \tag{4.0.1}$$

where $f(x, y)$ is defined in an open cone in $\mathbb{R}^2$. We remark that the above equation can be obtained because the curvature functions are 1-homogeneous and strictly increasing in the positive cone $\{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$[2]. Secondly, to find the domain of $\mathbf{u}$, we find barriers solutions to a first order reduction of Equation (4.0.1) by setting $\mathbf{v} = \dot{\mathbf{u}}$ and $\mathbf{v}(0) = 0$. Finally, we use standard ODE theory (implicit function theorem, Arzela-Ascoli arguments, extensability

[2] An important fact is that we are not using the implicit function theorem for writing f. For general curvature functions, f can be written in terms of the connected components of the level set $\left\{\sqrt{1 + \dot{\mathbf{u}}^2}\gamma(\lambda) = 1\right\}$.

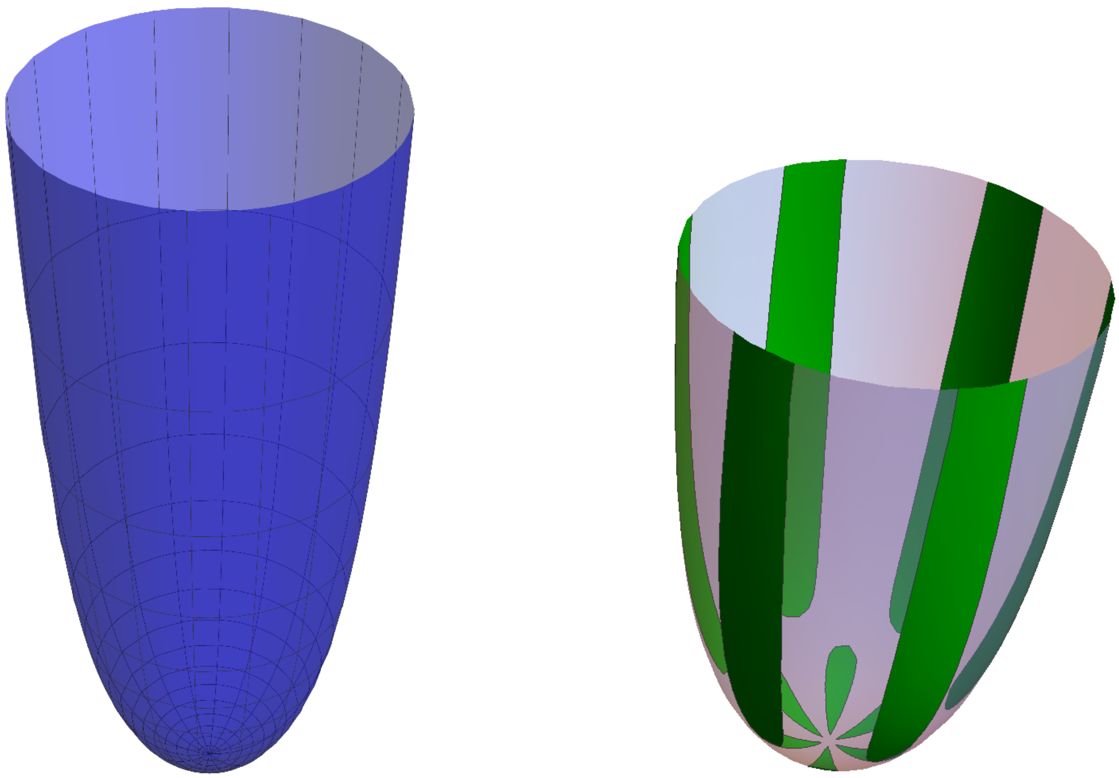


Figure 4.1: $\sqrt{S_2}$-translator given by $z = \int_0^{\sqrt{x^2+y^2}} \sqrt{e^{s^2}-1}ds$ (left) and Q_1-translator given by $z = -\ln(1-x^2-y^2)$ (right). Image courtesy of Francisco Martín.

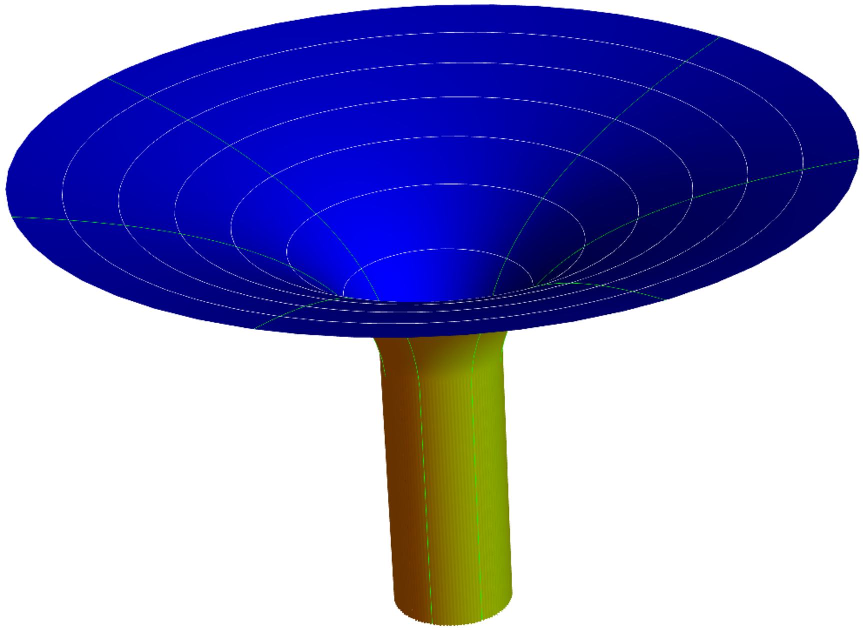


Figure 4.2: The $\sqrt{S_2}$-translator Σ_2 in $\mathbb{R}^3$. Image courtesy of Francisco Martín.

result, etc…) for first order ODEs to show that these unique solutions exist for $r \in [0, r_n)$ and they are smooth.

The last result of this chapter is an implicit "catenoid"-type solution constructed in the spirit of the "wing like" H-translators found in [CSS]. This solution is constructed as rotationally symmetric graph over the $\{y = 0\}$ plane in $\mathbb{R}^3$, see Fig. 4.2 above.

Theorem 4.0.2. *Let $a \in [0,1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R} \right\},$$

where $z = \int\limits_1^{r_a(z)} \sqrt{e^{s^2-a} - 1}\,ds$ with $r(0) = 1$, is a complete $\sqrt{S_2}$-translators in $\mathbb{R}^3$ which satisfies $H < 0$ and $K > 0$.

Remark 4.0.1. In Proposition 4.1.1 below, we show that the principal curvatures of a rotationally symmetric graph $M = \{(x, \mathbf{u}(|x|)), x \in [0, r_n)\} \subset \mathbb{R}^{n+1}$ are given by

$$\lambda_1 = \ldots = \lambda_{n-1} = \frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} \text{ and } \lambda_n = \frac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}},$$

where the dot derivative is taken with respect the variable $r = |x| = \sqrt{x_1^2 + \ldots + x_n^2}$. In particular, the γ-translator equation has the following form

$$\gamma(\lambda_1, \ldots, \lambda_n) = \langle \nu, e_{n+1} \rangle = \frac{1}{\sqrt{1+\dot{\mathbf{u}}^2}}. \tag{4.0.2}$$

Consequently, by assuming continuity in the second derivative of $\mathbf{u}$ together with the initial data $\dot{\mathbf{u}}(0) = 0$, we can see after taking limits when $r \to 0$ that

$$\gamma\left(\ddot{\mathbf{u}}(0), \ldots, \ddot{\mathbf{u}}(0)\right) = 1 \Leftrightarrow \ddot{\mathbf{u}}(0) = \frac{1}{\gamma(1, \ldots, 1)}.$$

In particular, with this information, we were able to find explicit barriers to describe our solutions.

The organization of this chapter goes as follows: In Section 4.1 we develop all the equations related to the geometry of rotationally symmetric γ-translator for $\gamma = Q_{n-1}$ and $\gamma = \sqrt[n]{S_n}$ in $\mathbb{R}^{n+1}$. In Section 4.2, we show the explicit γ-translators in $\mathbb{R}^3$. In section 4.3, we show the barriers and results related to the extension of the solution domain of these ODEs. In Section 4.4, we prove the existence part of Theorem 4.0.1 for the "bowl"-type $\sqrt[n]{S_n}$-translator. In Section 4.5 we prove the existence part of Theorem 4.0.1 for the "bowl"-type Q_{n-1}-translator. In Section 4.6, we show the uniqueness and regularity part of Theorem 4.0.1 for these curvature functions. In Section 4.7, we prove Theorem 4.0.2.

4.1 Definitions and Properties

A graphical rotationally symmetric hypersurface $M \subset \mathbb{R}^{n+1}$ can be written as

$$M = \left\{ (r\theta, \mathbf{u}(r)) : \theta \in \mathbb{S}^{n-1}, 0 \leq r < R \right\}, \tag{4.1.1}$$

where $\mathbf{u}(r) = u(|x|)$ is a smooth function $u : B_R(0) \to \mathbb{R}$ and $R \in [0, \infty]$. Note that we are using cylindrical coordinates $\theta = (\theta_1, \ldots, \theta_{n-1})$, $r = |x|$ and $s = x_{n+1}$, where θ_i are the canonical coordinates of $\mathbb{S}^{n-1}$ and $|\cdot|$ denotes the euclidean norm.

Remark 4.1.1. We will use the dot notation to denote radial derivatives $\dfrac{\partial}{\partial r}$ and bold letters for rotationally symmetric functions.

The following proposition contains geometric properties about these hypersurfaces in $\mathbb{R}^{n+1}$.

Proposition 4.1.1. *Let $M \in \mathbb{R}^{n+1}$ be a rotationally symmetric graph. Then, in the frame $\left\{\partial_{\theta_1}, \ldots, \partial_{\theta_{n-1}}, \partial_r, \partial_s\right\} \subset T_p\mathbb{R}^{n+1}$, we have the following quantities at $p \in M$:*

(a) Unit normal vector: $\nu = \dfrac{\partial_s - \dot{\mathbf{u}}\partial_r}{\sqrt{1+\dot{\mathbf{u}}^2}}$.

(b) Principal directions: ∂_{θ_i}, $\dot{\mathbf{u}}\partial_s + \partial_r$.

(c) Principal curvatures: $\lambda_i = \dfrac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}$, $\lambda_n = \dfrac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}$.

Proof. (a) Let $p \in M$, $p = (r\theta, \mathbf{u}(r))$. Then, a base for T_pM given by the cylindrical coordinates (θ, r, s) is $e_i = r\partial_{\theta_i}$ for $i = 1, \ldots, n-1$ and $e_n = \dot{\mathbf{u}}\partial_s + \partial_r$.

Indeed, it is not hard to check that the vector $\nu = \dfrac{\partial_s - \dot{\mathbf{u}}\partial_r}{\sqrt{1+\dot{\mathbf{u}}^2}}$ is unitary and $\nu \perp T_pM$.

Indeed, we have

$$\langle e_i, \nu \rangle = \frac{\langle r\partial_{\theta_i}, \partial_s - \dot{\mathbf{u}}\partial_r \rangle}{\sqrt{1+\dot{\mathbf{u}}^2}} = 0, \text{ for } i = 1, \ldots, n-1$$
$$\langle e_n, \nu \rangle = \frac{\langle \partial_r + \dot{\mathbf{u}}\partial_s, \partial_s - \dot{\mathbf{u}}\partial_r \rangle}{\sqrt{1+\dot{\mathbf{u}}^2}} = \frac{\dot{\mathbf{u}} - \dot{\mathbf{u}}}{\sqrt{1+\dot{\mathbf{u}}^2}} = 0.$$

(b,c) Firstly, we note that the Christoffel symbols at p satisfy

$$\Gamma^k_{ks} = \Gamma^s_{kk} = \Gamma^r_{rr} = \Gamma^r_{\theta_i r} = 0 \text{ and } \Gamma^r_{\theta_i\theta_j} = -\delta_{ij} r,$$

where δ_{ij} denotes the Kronecker delta function and the index k is respect $\varepsilon_k \in \left\{\partial_{\theta_1}, \ldots, \partial_{\theta_{n-1}}, \partial_r, \partial_s\right\}$. Therefore, the covariant derivatives in this coordinates are given by

$$\nabla_{e_i} e_j = \begin{cases} r^2\left(\Gamma^{\theta_k}_{\theta_i\theta_j}\partial_{\theta_k} - \delta_{ij} r\partial_r\right), \text{ for } i,j = 1\ldots,n-1, \\ r\left(\dot{\mathbf{u}}\left(\Gamma^{\theta_k}_{\theta_i s}\partial_{\theta_k} + \Gamma^s_{\theta_i s}\partial_s + \Gamma^r_{\theta_i s}\partial_r\right) + \left(\Gamma^{\theta_k}_{\theta_i r}\partial_{\theta_k} + \Gamma^s_{\theta_i r}\partial_s + \Gamma^r_{\theta_i r}\partial_r\right)\right), i \leq n-1 \text{ and } j = n, \\ \dot{\mathbf{u}}^2\Gamma^k_{ss}\varepsilon_k + \dot{\mathbf{u}}\Gamma^k_{sr}\varepsilon_k + \ddot{\mathbf{u}}\partial_s + \Gamma^k_{rr}\varepsilon_k, \text{ for } i = j = n. \end{cases}$$

In addition,, the metric in these coordinates is given by

$$g = \begin{pmatrix} r^2 g_{\mathbb{S}^{n-1}} & 0 \\ 0 & 1+\dot{\mathbf{u}}^2 \end{pmatrix},$$

where $g_{\mathbb{S}^{n-1}}$ denotes the round metric of $\mathbb{S}^{n-1}$.

Consequently, since the principal curvatures are the eigen values of the shape operator $A = g^{-1}II$, we calculate the terms of the second fundamental form of M

$$II_{ij} = \langle \nabla_{e_i} e_j, \nu\rangle = \begin{cases} \dfrac{\delta_{ij} r\dot{\mathbf{u}}}{\sqrt{1+\dot{\mathbf{u}}^2}} & , i,j = 1,\ldots,n-1, \\ 0 & , \ i \leq n-1, j = n, \\ \dfrac{\ddot{\mathbf{u}}}{\sqrt{1+\dot{\mathbf{u}}^2}} & , \ i = j = n. \end{cases}$$

Finally, the shape operator in this coordinates is given by

$$A(e_k, e_j) = g^{ik} II(e_k, e_j) = \begin{cases} 0 & , \text{ if } i \neq j, \\ \dfrac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} & , \text{ if } i = j,\ i = 1, \ldots, n-1, \\ \dfrac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}} & , \text{ if } i = j = n. \end{cases}.$$

We note that the shape operator A is diagonal, and the eigenvalues of this matrix correspond to the terms in the diagonal with eigenvectors given by $\{e_i\}_{i=1}^n$.

□

In the following proposition we calculate the curvature functions in terms of r, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$.

Proposition 4.1.2. *The functions S_k and Q_k evaluated in the principal curvatures of M are given by*

$$S_k = \frac{1}{k}\binom{n-1}{k-1}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^{k-1}\left[\frac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \frac{k\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right];$$
$$Q_k = \frac{(n-k)}{(k+1)}\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\frac{(n-k-1)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + (k+1)r\ddot{\mathbf{u}}}{(n-k)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + kr\ddot{\mathbf{u}}};$$

Proof. The proof is a direct computation from the last proposition. In fact, by Proposition 4.1.1, the principal curvatures of M are given by

$$\lambda_1 = \ldots = \lambda_{n-1} = \frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} \text{ and } \lambda_n = \frac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}.$$

Then, the function S_k evaluated at the principal curvatures is given by

$$
\begin{aligned}
S_k(\lambda) &= S_{k,n}(\lambda) + \lambda_n S_{k-1,n}(\lambda) \\
&= \binom{n-1}{k}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^k + \binom{n-1}{k-1}\frac{\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}}}\right)^{k-1} \\
&= \frac{1}{k}\binom{n-1}{k-1}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^{k-1}\left[\frac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \frac{k\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right].
\end{aligned}
$$

Finally, the function Q_k evaluated in the principal curvatures is given by

$$
\begin{aligned}
Q_k(\lambda) &= \frac{S_{k+1}(\lambda)}{S_k(\lambda)} \\
&= \frac{\frac{1}{k+1}\binom{n-1}{k}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^k\left[\frac{(n-k-1)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \frac{(k+1)\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right]}{\frac{1}{k}\binom{n-1}{k-1}\left(\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\right)^{k-1}\left[\frac{(n-k)\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}} + \frac{k\ddot{\mathbf{u}}}{(1+\dot{\mathbf{u}}^2)^{\frac{3}{2}}}\right]} \\
&= \frac{(n-k)}{(k+1)}\frac{\dot{\mathbf{u}}}{r\sqrt{1+\dot{\mathbf{u}}^2}}\frac{(n-k-1)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + (k+1)r\ddot{\mathbf{u}}}{(n-k)(1+\dot{\mathbf{u}}^2)\dot{\mathbf{u}} + kr\ddot{\mathbf{u}}}.
\end{aligned}
$$

□

As a corollary we obtain an ODE satisfied by these γ- translators.

Corollary 4.1.3. *1. Rotationally symmetric Q_k-translators satisfy the ODE*

$$
\begin{cases}
\ddot{\mathbf{u}} = F_{k,n}(r, \dot{\mathbf{u}}) \\
\mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0.
\end{cases}, \tag{4.1.2}
$$

where $F_{k,n}(x,y) = \dfrac{n-k}{k+1}(1+y^2)\dfrac{y}{x}\left(\dfrac{x(k+1)-(n-k-1)y}{(n-k)y-kx}\right).$

2. Rotationally symmetric $\sqrt[k]{S_k}$-translators satisfy the ODE

$$\begin{cases} \ddot{\mathbf{u}} = G_{k,n}(r, \dot{\mathbf{u}}) \\ \mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0. \end{cases}, \tag{4.1.3}$$

where $G_{k,n}(x,y) = \frac{y}{x}(1+y^2)\left(\frac{1}{\binom{n-1}{k-1}}\left(\frac{x}{y}\right)^k - \frac{(n-k)}{k}\right)$.

Proof. It is a straightforward calculation since the equations come from $\gamma(\lambda) = \frac{1}{\sqrt{1+\dot{\mathbf{u}}^2}}$. □

Remark 4.1.2. As we mentioned in the introduction of this chapter, all of the equations above can be seen as a first order ODE by considering $\mathbf{v} = \dot{\mathbf{u}}$. In particular, this fact simplifies finding barriers and the comparison methods that we show in Section 4.3.

4.2 Explicit Q_1 and $\sqrt{S_2}$ translators in $\mathbb{R}^3$

In this section we construct explicit examples of rotationally symmetric translators for the curvature function Q_1 and $\sqrt{S_2}$ in $\mathbb{R}^3$. For this purpose, we set $\mathbf{v} = \dot{\mathbf{u}}$, and therefore this γ-translators in $\mathbb{R}^{n+1}$ satisfies

$$\begin{cases} \dot{\mathbf{v}}(r) = F_{n-1,n}(r,\mathbf{v}) = \frac{\mathbf{v}(1+\mathbf{v}^2)}{\mathbf{v}-(n-1)r}, \\ \mathbf{v}(0) = 0, \end{cases} \tag{4.2.1}$$

for $\gamma = Q_{n-1}$, and for the $\gamma = \sqrt[n]{S_n}$-translators we have

$$\begin{cases} \dot{\mathbf{v}}(r) = G_{n,n}(r,\mathbf{v}) = (1+\mathbf{v}^2)\left(\frac{r}{\mathbf{v}}\right)^{n-1}, \\ \mathbf{v}(0) = 0, \end{cases} \tag{4.2.2}$$

In the following theorems we find explicit solutions for these ODEs.

Theorem 4.2.1. *The surface*

$$\left\{(r\theta, -\ln(1-r^2)) : \theta \in \mathbb{S}^1, r \in [0,1)\right\}$$

is a complete and strictly convex Q_1-translator $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$ in $\mathbb{R}^3$, i.e: $\lambda_2(p) \to 0$ and $\lambda_1(p) \to 1$ as $|p| \to \infty$.

Proof. The function $\mathbf{v}(r) = \dfrac{2r}{1-r^2}$ with $r \in (0,1)$ satisfies Equation (4.2.1). In fact, it is not hard to see that $\dot{\mathbf{v}} = \dfrac{2(1+r^2)}{(1-r^2)^2}$ and

$$F_{1,2}(r,\mathbf{v}) = \frac{2r}{1-r^2}\,\frac{\dfrac{(1+r^2)^2}{(1-r^2)^2}}{\dfrac{r(1+r^2)}{1-r^2}} = \frac{2(1+r^2)}{(1-r^2)^2} = \dot{\mathbf{v}}.$$

Then, by simple integration, the function $\mathbf{u} = \int_0^r \mathbf{v}(s)ds = -\ln(1-r^2)$ solves equation (4.1.2) for $k=1$ and $n=2$.

Moreover, this surface is complete since it is a graph, but non-entire since $\lim\limits_{r\to 1} \mathbf{u}(r) = \infty$, and it is strictly convex.

Finally, to see the asymptotic behavior we note that

$$\lambda_1 = \frac{\mathbf{v}}{r\sqrt{1+\mathbf{v}^2}} = \frac{2}{(1-r^2)\sqrt{1+\dfrac{4r^2}{(1-r^2)^2}}} = \frac{2}{\sqrt{(1-r^2)^2+4r^2}} \to 1,$$

$$\lambda_2 = \frac{\dot{\mathbf{v}}}{(1+\mathbf{v}^2)^{\frac{3}{2}}} = \frac{2(1+r^2)}{(1-r^2)^2}\,\frac{1}{\left(1+\dfrac{4r^2}{(1-r^2)^2}\right)^{\frac{3}{2}}} = \frac{2(1+r^2)(1-r^2)}{((1-r^2)^2+4r^2)^{\frac{3}{2}}} \to 0,$$

as $r \to 1$. □

Theorem 4.2.2. *The surface*

$$\left\{\left(r\theta, \int_0^r \sqrt{e^{s^2}-1}ds\right) : \theta \in \mathbb{S}^1, r \in [0,\infty)\right\}$$

is a complete strictly convex $\sqrt{S_2}$-translator in $\mathbb{R}^3$ which is smooth for $r \neq 0$ and 1-differentiable at $r = 0$.

Proof. First, we show that the function $\mathbf{v}(r) = \sqrt{e^{r^2}-1}$ satisfies Equation (4.2.2) n=2. In fact, we have that

$$\dot{\mathbf{v}} = \frac{re^{r^2}}{\sqrt{e^{r^2}-1}} = (1+\mathbf{v}^2)\frac{r}{\mathbf{v}} = G_{2,2}(r,\mathbf{v}),\ r \geq 0.$$

Finally, the result follows by taking $\mathbf{u} = \int_0^r \mathbf{v}(s)ds$. □

4.3 Barriers and Extension of solutions

In section we show explicit barriers solutions to Equations (4.2.1) and (4.2.2). Then, as an applications of ODE classic theory, we are able to extend the interval of existence of solutions to both equations in $\mathbb{R}^{n+1}$.

Definition 4.3.1. Firstly, we say that a function $\mathbf{w}(r)$ is a supersolution (subsolution respectively) to an initial value problem

$$\begin{cases} \dot{\mathbf{v}} = F(r,\mathbf{v}),\ \text{in } (r_0,R) \\ \mathbf{v}(r_0) = v_0, \end{cases} \tag{4.3.1}$$

if $\dot{\mathbf{w}} \geq F(r,\mathbf{w})$ with $\mathbf{w}(r_0) \geq v_0$ ($\dot{\mathbf{w}} \leq F(r,\mathbf{w})$ with $\mathbf{w}(r_0) \leq v_0$, respectively) at (r_0, R).

In addition, we say that the slope function F has a comparison principle if given a supersolution (subsolution respectively) $\mathbf{w}$ to the initial value problem (4.3.1) we have $\mathbf{v}(r) \leq \mathbf{w}(r)$ ($\mathbf{w}(r) \leq \mathbf{v}(r)$, respectively) at (r_0, R) for any solution $\mathbf{v}$ to Equation (4.3.1).

In the next proposition we show that Equations (4.2.1) and (4.2.2) posses comparison principles for any positive initial condition.

Proposition 4.3.1. *Let $r_0, v_0 > 0$, and $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function. Assume that $\mathbf{v}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ are a solution and a supersolution (resp. subsolution), respectively, to the initial value problem*

$$\begin{cases} \dot{\mathbf{v}} = (1+\mathbf{v}^2) f\left(\dfrac{\mathbf{v}}{r}\right) \\ \mathbf{v}(r_0) = v_0. \end{cases} \tag{4.3.2}$$

Then, $\mathbf{v}(r) \leq \mathbf{w}(r)$ (resp. $\mathbf{v}(r) \geq \mathbf{w}(r)$) holds in $[r_0, R)$.

Proof. We only show the case when $\mathbf{w}$ is a supersolution, since the subsolution case is analogous.

Let $\mathbf{v}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ be a solution and a supersolution to the initial value problem (4.3.2), respectively. Then, since $\mathbf{w}$ is a supersolution to Equation (4.3.2), we have $\mathbf{v}(r_0) = v_0 \leq w_0 = \mathbf{w}(r_0)$ and

$$\dot{\mathbf{w}} \geq (1+\mathbf{w}^2) f\left(\frac{\mathbf{w}}{r}\right) \Leftrightarrow \frac{d}{dr}\arctan(\mathbf{w}) \geq f\left(\frac{\mathbf{w}}{r}\right).$$

Now we assume the contrary of the result of this proposition. This means there exist $r_1 \in (r_0, R)$ such that $\mathbf{w}(r_1) < \mathbf{v}(r_1)$.

Next, we consider

$$d(r) = \arctan(\mathbf{w}) - \arctan(\mathbf{v}),$$

and we note that $d(r_0) \geq 0$, $d(r_1) < 0$. Then, by continuity, we have that $d(c) = 0$ where

$c = \inf \left\{ r \in (r_0, r_1) : \; d|_{[r, r_1]} < 0 \right\}$. Moreover, since $\tan(\cdot)$ is an increasing function, it holds that

$$0 \geq d(r) = \arctan(\mathbf{w}) - \arctan(\mathbf{v}) \Leftrightarrow \mathbf{v}(r) \geq \mathbf{w}(r) \text{ for } r \in [c, r_1].$$

In particular, it follows that

$$\dot{d} \geq f\left(\frac{\mathbf{w}}{r}\right) - f\left(\frac{\mathbf{v}}{r}\right) \geq 0,$$

since f is an decreasing function. Therefore, d is an increasing function at $[c, r_1]$. This means that $0 = d(r_2) \leq d(r_1) < 0$, a contradiction.

Consequently, $d \geq 0$ holds at $[r_0, R)$, and by composing with $\tan(\cdot)$, it holds that $\mathbf{v} \leq \mathbf{w}$ in $[r_0, R)$. □

Remark 4.3.1. We note that Proposition 4.3.1 applies for Equations (4.2.1) and (4.2.2) since the functions $f(x)$ are given by

$$\frac{x}{x - (n-1)} \text{ and } \frac{1}{x^{n-1}}$$

respectively.

Moreover, as we will see in the following sections, Proposition 4.3.1 can also be applied for initial condition $r_0 = v_0 = w_0 = 0$, since $f\left(\frac{\mathbf{v}}{r}\right)\Big|_{r=0} = f(\dot{\mathbf{v}}(0))$. In fact, the solutions to Equations (4.2.1) and (4.2.2) satisfy $\dot{\mathbf{v}}(0) = n$ and $\dot{\mathbf{v}}(0) = 1$, respectively, which means that the respective $f(x)$ of each ODE are well-defined continuous decreasing functions.

The following propositions provide explicit barrier solutions to Equations (4.2.1) and (4.2.2).

Proposition 4.3.2. *Let $\alpha \in (n-1, n]$, $\beta \geq n$ such that $\alpha r_0 \leq v_0 \leq \dfrac{\beta r_0}{\sqrt{1 - \beta^2 r_0^2}}$ for some*

$v_0 \in \mathbb{R}$. *Then, the functions*

$$\mathbf{w}_{0,\alpha}(r) = \alpha r \text{ and } \mathbf{w}_{1,\beta}(r) = \frac{\beta r}{\sqrt{1-\beta^2 r^2}}$$

defined for $r \in \mathbb{R}$ *and* $r \in \left[0, \frac{1}{\beta}\right)$, *respectively, are subsolution and supersolution, respectively, to the initial value problem*

$$\begin{cases} \dot{\mathbf{v}} = (1+\mathbf{v}^2)\dfrac{\frac{\mathbf{v}}{r}}{\frac{\mathbf{v}}{r}-(n-1)}, \\ \mathbf{v}(r_0) = v_0. \end{cases} \tag{4.3.3}$$

Proof. Firstly, we will show that $\mathbf{w}_{0,\alpha}(r)$ is a subsolution to Equation (4.3.3). We note $\mathbf{w}_{0,\alpha}(r_0) = r_0 \leq v_0 = \mathbf{v}(r_0)$ so the initial condition is satisfied.

Moreover, since $\alpha \in (n-1, n]$, it holds that

$$\begin{aligned} \frac{\dot{\mathbf{w}}_{0,\alpha}}{1+\mathbf{w}_{0,\alpha}^2} - \frac{\frac{\mathbf{w}_{0,\alpha}}{r}}{\frac{\mathbf{w}_{0,\alpha}}{r}-(n-1)} &= \frac{\alpha}{1+\alpha^2 r^2} - \frac{\alpha}{\alpha-(n-1)} \\ &= \frac{\alpha}{(\alpha-(n-1))(1+\alpha^2 r^2)}(\alpha - n - \alpha^2 r^2) \\ &\leq 0. \end{aligned}$$

This proves that $\mathbf{w}_{0,\alpha}$ with $\alpha \in (n-1, n]$ is a subsolution to Equation (4.3.3) for all $r \in \mathbb{R}$.
On the other hand, for $\mathbf{w}_{1,\beta}$ we note that $\mathbf{w}_{1,\beta}(r_0) = \dfrac{\beta r_0}{\sqrt{1-\beta^2 r_0^2}} \geq v_0 = \mathbf{v}(0)$.

In addition,

$$
\begin{aligned}
\dot{\mathbf{w}}_{1,\beta} &= \frac{\mathbf{w}_{1,\beta}}{r}(1+\mathbf{w}_{1,\beta}^2) \\
&= \frac{\frac{\mathbf{w}_{1,\beta}}{r}}{n-(n-1)}(1+\mathbf{w}_{1,\beta}^2) \\
&\geq \frac{\frac{\mathbf{w}_{1,\beta}}{r}}{\frac{\mathbf{w}_{1,\beta}}{r}-(n-1)}(1+\mathbf{w}_{1,\beta}^2),
\end{aligned}
$$

in the last inequality we use that $\frac{\mathbf{w}_{1,\beta}}{r} = \frac{\beta}{\sqrt{1-\beta^2 r^2}} \geq \beta \geq n$ at $\left[0, \frac{1}{\beta}\right]$. This shows that $\mathbf{w}_{1,\beta}$ with $\beta \geq n$ is a supersolution to Equation (4.3.3) at $\left[0, \frac{1}{\beta}\right]$. □

Proposition 4.3.3. *Let $n \geq 3$ and r_0, v_0 such that $r_0 \leq v_0 \leq e^{r_0} - 1$. Then, the functions*

$$\mathbf{w}_2(r) = r \text{ and } \mathbf{w}_3(r) = e^r - 1,$$

are subsolution and supersolution to the initial value problem

$$
\begin{cases}
\dot{\mathbf{v}} = (1+\mathbf{v}^2)\left(\frac{r}{\mathbf{v}}\right)^{n-1}, \\
\mathbf{v}(r_0) = v_0.
\end{cases}
\tag{4.3.4}
$$

for all $r \in \mathbb{R}$, respectively.

Proof. Firstly, we start showing that $\mathbf{w}_2$ is a subsolution to Equation (4.3.4) for all $r \in \mathbb{R}$. Then, by definition we have $\mathbf{w}_2(r_0) = r_0 \leq v_0 = \mathbf{v}(0)$, and

$$\frac{\dot{\mathbf{w}}_2}{1+\mathbf{w}_2^2} - \left(\frac{r}{\mathbf{w}_2}\right)^{n-1} = \frac{1}{1+r^2} - 1 \leq 0.$$

This shows that $\mathbf{w}_2$ is a subsolution to Equation (4.3.4).

On the other hand, for $\mathbf{w}_3$, we have by assumption that $\mathbf{w}_3(r_0) = e^{r_0} - 1 \geq v_0 = \mathbf{v}(0)$, and

$$\frac{\dot{\mathbf{w}}_3}{1+\mathbf{w}_3^2} = \frac{e^r}{2-2e^r+e^{2r}} \geq \frac{r^2}{(e^r-1)^2} \geq \left(\frac{r}{e^r-1}\right)^{n-1} = \left(\frac{r}{\mathbf{w}_3}\right)^{n-1},$$

which holds for all $r \geq 0$ and $n \geq 3$. To see that the first inequality holds, we note that

$$\frac{e^r}{2-2e^r+e^{2r}} \geq \frac{r^2}{(e^r-1)^2} \Leftrightarrow \frac{e^r}{r^2} \geq \frac{1}{(e^r-1)^2} + 1 \Leftrightarrow e^r - r^2 \geq \left(\frac{r}{e^r-1}\right)^2.$$

Then, $(e^r - r^2)|_{r=0} = 1 = \lim_{r\to 0} \left(\frac{r}{e^r-1}\right)^2$, but $e^r - r^2$ is increasing and $\left(\frac{r}{e^r-1}\right)^2$ is decreasing. This completes the proof that $\mathbf{w}_3$ is a super solution of the Equation (4.3.4). □

We now present a classical result from the theory of ODEs that allows us to extend solutions of initial value problems up to their maximal intervals of existence.

Lemma 4.3.4. *Let $\mathbf{v}(r)$ be a solution to a initial value problem* (4.3.1) *defined on the interval (r_-, r_+). Assume there is a compact set $[r_0, r_+] \times C \subset U \subset \mathbb{R}^2$ such that $\mathbf{v}(r_m) \in C$ for some sequence $r_m \in [r_0, r_+)$ converging to r_+ .*
Then, there exists $\varepsilon > 0$ and a function $\mathbf{v}_\varepsilon \in \mathcal{C}^1((r_-, r_+ + \varepsilon))$ such that $\mathbf{v}_\varepsilon|_{(r_-,r_+)} = \mathbf{v}$ and is a solution to the initial value problem (4.3.1) *with initial condition $\mathbf{v}_\varepsilon(r_+)$.*

Moreover, if there is such compact set C for every $r_+ > r_0$ (C might depend on r_+), then there exist a solution $\tilde{\mathbf{v}} \in \mathcal{C}^1((r_-, \infty))$ of the initial value problem (4.3.1) *with initial condition $\tilde{\mathbf{v}}(r_+)$ such that $\tilde{\mathbf{v}}|_{(r_-,r_+)} = \mathbf{v}$. The same result holds for r_-.*

Proof. We refer the reader to see the proof in [Tes] Corollary 2.15. □

Remark 4.3.2. The Lemma 4.3.4 is a powerful tool to find the maximal interval of existence of solution to an initial value problem of first order. Usually, the ingredients for using this result are:

- An ODE which has a comparison principle. This will ensure that the extension of the solution will be unique. We will see this fact in the section 4.6.

- Upper and lower barriers. This is also related to the previous point, but the important point is that the barriers will give an explicit construction of the compact set C to apply the Lemma 4.3.4.

In summary these assumptions are for producing unique solutions whose domain is the maximum existence interval associated with that of the initial value problem.

On the other hand, the negation of the Lemma statement 4.3.4 provides a method to find the maximum existence interval of the solution. In fact, if the supersolution and the subsolution tend to $\pm\infty$ at the extremes of the interval where they are defined, then the solution also will have this property. We emphasize that the existence interval of the solution must be contained in that of the barriers for this to take effect.

The following corollaries reveal the maximum existence interval for the solutions of our initial value problems.

Corollary 4.3.5. *Let $n \geq 3$, $0 < r_0 \leq v_0 \leq e^{r_0} - 1$ and $\mathbf{v} \in \mathcal{C}^1([r_0, R])$ be a solution to Equation* (4.3.4)*. Then, the solution $\mathbf{v}(r)$ exist for all $r \geq 0$.*

Proof. The proof consists in applying the Lemma 4.3.4 with the barriers found in Proposition 4.3.3. Firstly, we we note that the solution $\mathbf{v}$ verifies

$$\mathbf{w}_2(r_0) = r_0 \leq \mathbf{v}(r_0) \leq e^{r_0} - 1 = \mathbf{w}_3(r_0).$$

This implies that the function $\mathbf{w}_2$ and $\mathbf{w}_3$ are a subsolution and supersolution, respectively, to Equation (4.3.4). Then, by Proposition 4.3.1, $\mathbf{w}_2 \leq \mathbf{v} \leq \mathbf{w}_3$ holds in $[r_0, R]$.

On the other hand, we set $r_+ \geq R$ and $C = [v_0, \mathbf{w}_3(r_+)]$. Then, it follows that $\mathbf{v}(r_m) \in C$ for every sequence $r_m \to r_+$. In particular, by Lemma 4.3.4, the solution $\mathbf{v}(r)$ to Equation

(4.3.4) exists for all $r \geq r_0$. Finally, the same argument holds by replacing C with $[\mathbf{w}_2(r_-), v_0]$ where $0 \leq r_- \leq r_0$. Therefore, $\mathbf{v}(r)$ can be extended to $[0, \infty)$ as a solution to Equation (4.3.4). □

Theorem 4.3.6. *Let $n \geq 3$, $\varepsilon \in [0, 1)$, R, r_0 and v_0 such that*

$$R \in \left(0, \frac{1}{n}\right), \; r_0 > 0 \text{ and } (n - \varepsilon) r_0 \leq v_0 \leq \frac{(n + \varepsilon) r_0}{\sqrt{1 - (n + \varepsilon)^2 r_0^2}}.$$

Assume that $\mathbf{v} \in \mathcal{C}^1([r_0, R])$ is a solution to the initial value problem (4.3.3) with $v(r_0) = v_0$. Then, $\mathbf{v}(r)$ can be extended to $r \in \left[0, \frac{1}{n}\right)$ and $\lim_{r \to \frac{1}{n}} \mathbf{v}(r) = \infty$.

Proof. Firstly, we will show that any solution $\mathbf{v}(r)$ to Equation (4.3.3) can be extended to $\left[0, \frac{1}{n}\right)$. For this purpose we set $r_+ \in \left[R, \frac{1}{n}\right)$ and $C = [0, \mathbf{w}_{1,n+\varepsilon}(r_+)]$. Note that by hypothesis $\mathbf{w}_{0,n-\varepsilon}(r_0) \leq \mathbf{v}(r_0) \leq \mathbf{w}_{1,n+\varepsilon}(r_0)$.

Then, by Proposition 4.3.1 and Proposition 4.3.2, follows that $\mathbf{w}_{0,n-\varepsilon}$ and $\mathbf{w}_{1,n+\varepsilon}$ are a subsolution and a supersolutions, respectively, to Equation (4.3.3). This means that

$$(n - \varepsilon) r = \mathbf{w}_{0,n-\varepsilon}(r) \leq \mathbf{v} \leq \mathbf{w}_{n+\varepsilon}(r) = \frac{(n + \varepsilon) r}{\sqrt{1 - (n + \varepsilon)^2 r^2}}, \text{ for } r \in [r_0, R].$$

Therefore, for any sequence $r_m \to r_+$, we have $\mathbf{v}(r_m) \in [0, \mathbf{w}_{1,n+\varepsilon}(r_+)]$.

Next, by Lemma 4.3.4, any solution $\mathbf{v}(r)$ to Equation (4.3.3) exists up to $r < \frac{1}{n + \varepsilon}$. In particular, by taking $\varepsilon \to 0$, we can extended $\mathbf{v}(r)$ as a solution to Equation (4.3.3) up to $r < \frac{1}{n}$.

In addition, the same argument will hold by replacing C with $[\mathbf{w}_{0,n-\varepsilon}(r_-), R]$, where $0 \leq r_- \leq r_0$. This means that we can extend $\mathbf{v}$ to $r = 0$ as a solution to Equation (4.3.3).

On the other hand, since $v(r) \geq \mathbf{w}_{0,n} = nr$, the following inequalities holds

$$\begin{aligned}\dot{\mathbf{v}} &= (1+\mathbf{v}^2)\frac{\frac{\mathbf{v}}{r}}{\frac{\mathbf{v}}{r}-(n-1)} \\ &\geq (1+\mathbf{v}^2) \\ &\geq \mathbf{v}^2.\end{aligned}$$

In the second line we use that $\dfrac{x}{x-(n-1)} \geq 1$ for $x = \dfrac{\mathbf{v}}{r} \geq 0$.

Consequently, since $\mathbf{w}_4(r) = \dfrac{n}{1-nr}$ satisfies $\dot{\mathbf{w}}_4 = \mathbf{w}_4^2$, it follows that for r close to $\dfrac{1}{n}$, $\mathbf{v} \geq \dfrac{n}{1-nr}$. This gives that $\lim_{r\to\frac{1}{n}} \mathbf{v}(r) = \infty$ finalizing the proof. $\square$

4.4 $\sqrt[n]{S_n}$-translator in $\mathbb{R}^{n+1}$ for $n \geq 3$

In this section we construct a solutions to Equation (4.2.2) for $n \geq 3$.

The method that we employ is constructing a solution in a small interval containing $r = 0$. To accomplish this we use the implicit function theorem centered at a sequence of points $(a_m, b_m) \to (0,0)$ as $m \to \infty$ and an Arzela-Ascoli type argument for construting this solution. Finally, by the result of the previous section, we are able to extend this solution for $r \geq 0$.

Firstly, we note that Equation (4.2.2),

$$\dot{\mathbf{v}} = (1+\mathbf{v}^2)\left(\frac{r}{\mathbf{v}}\right)^{n-1},$$

can be solved implicitly. To see this we take $\mathbf{w} = \arctan(\mathbf{v})$, then $\mathbf{w}$ satisfies

$$\dot{\mathbf{w}}\tan^{n-1}(\mathbf{w}) = r^{n-1}.$$

Then, by integration we obtain

$$\int_0^r \tan^{n-1}(\mathbf{w})d\mathbf{w} = \frac{r^n}{n}.$$

We note that the right hand side of the above equation can be written recursively by

$$\begin{aligned}
I_{n-1}(\mathbf{w}) :&= \int_0^r \tan^{n-1}(\mathbf{w})d\mathbf{w} \\
&= \int_0^r \tan^{n-3}(\mathbf{w})(\sec^2(\mathbf{w}) - 1)d\mathbf{w} \\
&= \frac{\tan^{n-2}(\mathbf{w})}{n-2} - \int_0^r \tan^{n-3}(\mathbf{w})d\mathbf{w} \\
&= \frac{\tan^{n-2}(\mathbf{w})}{n-2} - I_{n-3}(\mathbf{w}).
\end{aligned}$$

Consequently, a solution to Equation (4.2.2) can be written implicitly by the relation

$$\mathcal{G}(r, \mathbf{v}) = \frac{\mathbf{v}^{n-2}}{n-2} - \int_0^{\mathbf{v}} \frac{t^{n-3}}{1+t^2}dt - \frac{r^n}{n} = 0. \tag{4.4.1}$$

Moreover, we note that the implicit curve (4.4.1) can be computed recursively by

$$I_0(\arctan(\mathbf{v})) = \arctan(\mathbf{v}) \text{ and } I_1(\arctan(\mathbf{v})) = \frac{1}{2}\ln(1+\mathbf{v}^2).$$

On the other hand, we already know that a solution $\mathbf{v}$ to Equation (4.2.2) can be written as an implicit curve $\mathcal{G}(r, \mathbf{v}) = 0$ such that $\mathbf{v}(0) = 0$. Unfortunately, for $n \geq 2$, we cannot apply directly the implicit function theorem to Equation (4.4.1) at $(0, 0)$ since

$$\partial_y \mathcal{G}(x, y)|_{(0,0)} = \left.\frac{y^{n-1}}{1+y^2}\right|_{(0,0)} = 0.$$

Nevertheless, we may apply an approximation procedure for constructing a solution to Equation (4.2.2).

Proposition 4.4.1. *For $n \geq 3$, let $b_m \leq \sqrt{\frac{n+2}{n}}$ be a sequence of positive numbers converging to 0 as $m \to \infty$. Then, by setting*

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt,$$

we have the following properties:

1. *The points (a_m, b_m) satisfy $\mathcal{G}(a_m, b_m) = 0$.*
2. *$0 < a_m \leq b_m$. Moreover, we have $\sqrt[n]{1 - \frac{n}{n+2} b_m^2} \leq \frac{a_m}{b_m} \leq 1$. In particular, it follows $\lim_{m\to\infty} \frac{b_m}{a_m} \to 1$ and $\lim_{m\to\infty} a_m = 0$.*

Proof. Firstly, we note that Property (1) follows by definition.

Secondly, since $b_m > 0$, we have

$$t^{n-3} - t^{n-1} \leq \frac{t^{n-3}}{1+t^2} \leq t^{n-3} - t^{n-1} + t^{n+1} \text{ for } t \in [0, b_m].$$

Then, it follows

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt \leq \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \left(t^{n-3} - t^{n-1}\right) dt = \frac{b_m^n}{n}.$$

For the other inequality, we have

$$\frac{a_m^n}{n} = \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} \frac{t^{n-3}}{1+t^2} dt \geq \frac{b_m^{n-2}}{n-2} - \int_0^{b_m} t^{n-3} - t^{n-1} + t^{n+1} dt = \frac{b_m^n}{n} - \frac{b_m^{n+2}}{n+2} \geq 0.$$

Finally, since the function $\frac{(\cdot)^n}{n}$ is 1-1 for positive numbers, it follows

$$b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2} \leq a_m \leq b_m.$$

$\square$

In the following theorem we construct a solution to Equation (4.2.2) for $n \geq 3$ by the implicit function theorem and a Arzela-Ascoli's argument.

Theorem 4.4.2. *For $n \geq 3$ there exist a non-constant solution $\mathbf{v} \in \mathcal{C}^1([0,\infty))$ to Equation (4.2.2).*

Proof. Firstly, we choose a sequence $b_m \leq \sqrt{\dfrac{n+2}{n}}$ of positive numbers converging to 0 as $m \to \infty$ such that $b_m \leq \exp\left(b_m \sqrt[n]{1 - \dfrac{n}{n+2} b_m^2}\right) - 1$ holds for all $m \gg 1$.
In fact, the function

$$T(x) = \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) - x - 1$$

satisfies $T(0) = 0$,

$$T'(0) = \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) \left[\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} - \frac{2x^2}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}}\right] - 1\Bigg|_{x=0} = 0,$$

$$\begin{aligned}
T''(0) = &\exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) \left[\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}} - \frac{2x^2}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}}\right]^2 \\
&+ \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) \left[\frac{4nx^3}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}-2} - \frac{6x}{n+2}\left(1 - \frac{n}{n+2}x^2\right)^{\frac{1}{n}-1}\right]\Bigg|_{x=0} \\
&= 1.
\end{aligned}$$

This means that $T'(x)$ is increasing for $x > 0$ close to 0, or $T'(x) > 0$ for $x > 0$. In particular, this means that $T(x)$ is increasing for $x > 0$ close to 0, and therefore $T(x) \geq 0$ for $x > 0$ close to 0.
Consequently, since $b_m \to 0$ as $m \to \infty$, there exist $M \gg 1$ such that

$$b_m \leq \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) - 1$$

for $m \geq M$.

Next, by setting a_m as in Proposition 4.4.1, we note that

$$\partial_y \mathcal{G}(x,y)\big|_{(a_m,b_m)} = \frac{b_m^{n-1}}{1+b_m^2} > 0.$$

Then, by the implicit function theorem there exist $\mathbf{v}_m(r) \in \mathcal{C}^\infty([a_m, R_m])$ such that $\mathcal{G}(r, \mathbf{v}_m(r)) = 0$ at $[a_m, R_m]$ with $\mathbf{v}_m(a_m) = b_m$ for some $R_m > 0$.

Moreover, since $\mathbf{v}_m$ is smooth, it also satisfies Equation (4.3.4) with initial data $\mathbf{v}_m(a_m) = b_m$. In addition, by Property 2 from Proposition 4.4.1, it follows that

$$\mathbf{w}_2(a_m) = a_m \leq b_m = \mathbf{v}_m(a_m) \leq \exp\left(b_m \sqrt[n]{1 - \frac{n}{n+2} b_m^2}\right) - 1 \leq e^{a_m} - 1 = \mathbf{w}_3(a_m)$$

Therefore, by Proposition 4.3.3, the functions $\mathbf{w}_2$ and $\mathbf{w}_3$ are a subsolution and supersolution, respectively, to the intial value problem (4.3.4) with initial data $\mathbf{v}_m(a_m) = b_m$.

Consequently, by Theorem 4.3.5, we can extend $\mathbf{v}_m$ up to $[0, R]$ for some $0 < R < \dfrac{\pi}{6}$ fixed.

Secondly, we are going to show that the sequence $\{\mathbf{v}_m\}_m$ is uniformly bounded in $\mathcal{C}^2([0, R])$. We start by showing the uniform bound in $\mathcal{C}([0, R])$. In fact, since

$$\dot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2)\left(\frac{r}{\mathbf{v}_m}\right)^{n-1} \geq (1 + \mathbf{v}_m^2)\left(\frac{r}{\mathbf{w}_{3_m}}\right)^{n-1} = (1 + \mathbf{v}_m^2)\left(\frac{r}{e^r - 1}\right)^{n-1} > 0,$$

it follows by the fundamental theorem of calculus that

$$\begin{aligned}
\arctan(\mathbf{v}_m(r)) &= \arctan(b_m) + \int_{a_m}^{r} \left(\frac{s}{\mathbf{v}_m(s)}\right)^{n-1} ds \\
&\leq \arctan(b_m) + \int_{a_m}^{r} \left(\frac{s}{\mathbf{w}_2(s)}\right)^{n-1} ds \\
&\leq \arctan(b_m) + r - a_m \\
&\leq \arctan(b_m) + R + a_m.
\end{aligned}$$

In the second line we use $\mathbf{w}_2 \leq \mathbf{v}_m$ for $r \geq 0$.

Moreover, since b_m and a_m tend to 0 as $m \to \infty$, it follows that

$$\mathbf{v}_m \leq \tan(3R). \tag{4.4.2}$$

for $m \gg M$. In particular, there exist C_1 such that $\sup_m ||\mathbf{v}_m||_\infty \leq C_1$.

Next, we are going to show a uniform bound for $\dot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by Equation (4.2.2), Property (2) and the uniform bound for $||\mathbf{v}_m||_\infty$, we have

$$\begin{aligned} 0 < \dot{\mathbf{v}}_m &= (1 + \mathbf{v}_m^2)\left(\frac{r}{\mathbf{v}_m}\right)^{n-1} \\ &\leq (1 + C_1^2)\left(\frac{r}{\mathbf{w}_2}\right)^{n-1} \\ &= (1 + C_1^2). \end{aligned}$$

Therefore, we can find a constant $C_2 > 0$ such that $\sup_m ||\dot{\mathbf{v}}_m||_\infty \leq C_2$.

Finally, we are going to show an uniform bound for $\ddot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by differentiating Equations (4.2.2), it follows

$$\ddot{\mathbf{v}}_m = (1 + \mathbf{v}_m^2)(n-1)\left(\frac{r}{\mathbf{v}_m}\right)^{n-2}\left(\frac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2}\right) + 2\mathbf{v}_m\dot{\mathbf{v}}_m\left(\frac{r}{\mathbf{v}_m}\right)^{n-1}.$$

First, we are going to estimate the term $\dfrac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2}$ in the above equation. In fact, we have

$$\begin{aligned}
\left|\frac{\mathbf{v}_m - r\dot{\mathbf{v}}_m}{\mathbf{v}_m^2}\right| &= \frac{\left|\mathbf{v}_m - r(1+\mathbf{v}_m^2)\left(\dfrac{r}{\mathbf{v}_m}\right)^{n-1}\right|}{\mathbf{v}_m^2} \\
&= \frac{1}{\mathbf{v}_m^2}\left|\frac{(\mathbf{v}_m - r)(\mathbf{v}_m^{n-1} + \ldots + r^{n-1})}{\mathbf{v}_m^{n-1}} - \frac{r^n}{\mathbf{v}_m^{n-3}}\right| \\
&\leq n\frac{\mathbf{v}_m - r}{\mathbf{v}_m^2} + \frac{r^n}{\mathbf{v}_m^{n-1}} \\
&\leq n\frac{\mathbf{w}_3 - r}{\mathbf{w}_2^2} + \frac{r^n}{\mathbf{w}_2^{n-1}} \\
&= n\frac{e^r - 1 - r}{r^2} + R \\
&\leq n\frac{e^R - 1 - R}{R^2} + R.
\end{aligned}$$

In the last line we use that the function $\dfrac{e^r - 1 - r}{r^2} = \sum_{k\geq 2}\dfrac{r^{k-2}}{k!}$ is increasing for all $r \geq 0$. Therefore, by the previous calculation, we obtain

$$||\ddot{\mathbf{v}}_m||_\infty \leq (1 + C_1^2)(n-1)\left(n\frac{e^R - 1 - R}{R^2} + R\right) + 2C_1C_2 = C_3.$$

Consequently, we have shown that $\sup_m ||\mathbf{v}||_{\mathcal{C}^2([a_m,\varepsilon'_m])} \leq C$ where $C = C_1 + C_2 + C_3$.

Finally, we have that $\{\mathbf{v}_m\}_m$ is uniformly bounded and equicontinuos in $\mathcal{C}^1([0,R])$ since $\mathbf{v}_m$ and $\dot{\mathbf{v}}_m$ are Lipschtiz functions with Lipschitz constant less that C. Therefore, by Arzela-Ascoli's theorem, there exist a subconvergent sequence in $\mathcal{C}^1$ topology of $\{\mathbf{v}_m\}$ to a function $\mathbf{v} \in \mathcal{C}^1([0,R])$. In addition, by continuity we have

$$\begin{aligned}
\mathbf{v}(0) &= \mathbf{v}\left(\lim_{m\to\infty} a_m\right) = \lim_{m\to\infty} b_m = 0, \\
\dot{\mathbf{v}}(0) &= \lim_{m\to\infty}(1 + b_m^2)\left(\frac{a_m}{b_m}\right)^{n-1} = 1.
\end{aligned}$$

Moreover, $\mathbf{v}$ satisfies $\mathbf{w}_2 \leq \mathbf{v} \leq \mathbf{w}_3$ at $[0, R]$. Then, by Theorem 4.3.5, $\mathbf{v}$ exist for all $r \geq 0$, finalizing the proof of this theorem. $\square$

4.5 Q_{n-1}-translator in $\mathbb{R}^{n+1}$ for $n \geq 3$

In this section we apply the same technique as the above section to construct a solutions to Equation (4.2.1) in small neighborhood containing $r = 0$. For this purpose we note that Equation (4.2.1),

$$\dot{\mathbf{v}} = (1+\mathbf{v}^2)\frac{\frac{\mathbf{v}}{r}}{\frac{\mathbf{v}}{r} - (n-1)},$$

can be solved implicitly. In fact, after some algebraic arrangements and multiplying by $\frac{\mathbf{v}^{n-2}}{(1+\mathbf{v}^2)^{\frac{n+1}{2}}}$, we see that Equation (4.2.1) is equivalent to

$$\frac{\mathbf{v}^{n-2}\dot{\mathbf{v}}}{(1+\mathbf{v}^2)^{\frac{n+1}{2}}}(\mathbf{v} - (n-1)r) - \left(\frac{\mathbf{v}}{\sqrt{1+\mathbf{v}^2}}\right)^{n-1} = 0.$$

Then, by the mothd of separation variables, we look for $\mathcal{F}(r, v)$ such that

$$\frac{\partial \mathcal{F}}{\partial r} = -\left(\frac{\mathbf{v}}{\sqrt{1+\mathbf{v}^2}}\right)^{n-1} \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = \frac{\mathbf{v}^{n-2}}{(1+\mathbf{v}^2)^{\frac{n+1}{2}}}(\mathbf{v} - (n-1)r).$$

Indeed, by an integration/differentiation procedure, we obtain an implicit curve given by

$$\mathcal{F}(r, \mathbf{v}) = -r\left(\frac{\mathbf{v}}{\sqrt{1+\mathbf{v}^2}}\right)^{n-1} + \int_0^{\mathbf{v}} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}}dt = 0. \tag{4.5.1}$$

Unfortunately, as in the previous section, we cannot apply directly the implicit function theorem to Equation (4.5.1) at $(0,0)$ since

$$\partial_y \mathcal{F}(x,y)|_{(0,0)} = \frac{y^{n-2}}{(1+y^2)^{\frac{n+1}{2}}}(y-(n-1)x)\Big|_{(0,0)} = 0.$$

Nevertheless, we may apply an approximation procedure for constructing a solution to Equation (4.2.1) which passes through $(0,0)$.

Proposition 4.5.1. *For $n \geq 3$, let $b_m \in (0,1)$ be a sequence of positive numbers converging to 0 as $m \to \infty$. Then, by setting*

$$a_m = \left(\frac{\sqrt{1+b_m^2}}{b_m}\right)^{n-1} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}} dt,$$

we have the following properties:

1. *The points (a_m, b_m) satisfy $\mathcal{F}(a_m, b_m) = 0$.*
2. *$a_m > 0$ and $\lim_{m\to\infty} a_m = 0$.*
3. *$\lim_{m\to\infty} \frac{a_m}{b_m} = \frac{1}{n}$.*

Proof. Firstly, we note that Property (1) and $a_m > 0$ follow by definition.
Moreover, the limit of $\frac{a_m}{b_m}$ can be calculated by L'Hôspital's rule. Indeed, we have

$$\begin{aligned}
\lim_{m\to\infty} \frac{a_m}{b_m} &= \lim_{m\to\infty} \frac{(1+b_m^2)^{\frac{n-1}{2}} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}} dt}{b_m^n} \\
&= \lim_{m\to\infty} \frac{b_m(n-1)(1+b_m^2)^{\frac{n-3}{2}} \int_0^{b_m} \frac{t^{n-1}}{(1+t^2)^{\frac{n+1}{2}}} dt + \frac{b_m^{n-1}}{(1+b_m^2)}}{nb_m^{n-1}} \\
&= \frac{1}{n}.
\end{aligned}$$

Finally, since $\lim_{m\to\infty} b_m = 0$ and $\lim_{m\to\infty} \frac{a_m}{b_m} = \frac{1}{n}$, it follows that $\lim_{m\to\infty} a_m = 0$. $\square$

In the following theorem we construct a solution to Equation (4.2.2) for $n \geq 3$ by the implicit function theorem and an Arzela-Ascoli's argument.

Theorem 4.5.2. *For $n \geq 3$ there exist a non-constant solution $\mathbf{v} \in \mathcal{C}^1([0, \frac{1}{n}))$ to Equation (4.2.1) such that $\lim_{r\to\frac{1}{n}} \mathbf{v}(r) = \infty$.*

Proof. Firstly, since $\frac{b_m}{a_m} \to n$ as $m \to \infty$, it holds

$$\partial_y \mathcal{F}(x,y)|_{(a_m,b_m)} = a_m \frac{b_m^{n-2}}{(1+b_m^2)^{\frac{n+1}{2}}} \left(\frac{b_m}{a_m} - (n-1) \right) > 0,$$

for $m \geq M$. Then, by the implicit function theorem there exists $\mathbf{v}_m(r) \in \mathcal{C}^\infty([a_m, R_m])$ for some $R_m > 0$ such that $\mathcal{F}(r, \mathbf{v}_m(r)) = 0$ at $[a_m, R_m]$ with $\mathbf{v}_m(a_m) = b_m$.

Moreover, since $\mathbf{v}_m$ is smooth, it also satisfies Equation (4.2.1) with initial data $\mathbf{v}_m(a_m) = b_m$. Therefore, since $\frac{b_m}{a_m} \to n$ as $m \to \infty$, it follows that

$$\mathbf{w}_{0,n-\varepsilon}(a_m) = (n-\varepsilon)a_m \leq b_m \leq a_m(n+\varepsilon) \leq \frac{(n+\varepsilon)a_m}{\sqrt{1-(n+\varepsilon)^2 a_m^2}} = \mathbf{w}_{1,n+\varepsilon}(a_m),$$

for some $\varepsilon \in (0,1)$ and $m \geq M$. In particular, by Theorem 4.3.6, we can extend $\mathbf{v}_m$ to $[0, R]$ for some $R < \frac{1}{n}$.

Secondly, we are going to show that the sequence $\{\mathbf{v}_m\}_m$ is uniformly bounded in $\mathcal{C}^2([0,R])$. For this purpose, we start by showing the uniform bound in $\mathcal{C}([0,R])$. Note that each $\mathbf{v}_m$ satisfies

$$\dot{\mathbf{v}}_m = (1+\mathbf{v}_m^2) \frac{\frac{\mathbf{v}_m}{r}}{\frac{\mathbf{v}_m}{r} - (n-1)} > 0.$$

Then, by the fundamental theorem of calculus together with that

$$\mathbf{w}_{0,n}(r) = nr \text{ and } \mathbf{w}_{1,n}(r) = \frac{nr}{\sqrt{1-n^2r^2}} \tag{4.5.2}$$

are subsolution and supersolution, respectively, to Equation (4.2.1), we have

$$\begin{aligned}
\arctan(\mathbf{v}_m) &= \arctan(b_m) + \int_{a_m}^{r} \frac{\frac{\mathbf{v}_m}{s}}{\frac{\mathbf{v}_m}{s} - (n-1)} ds \\
&\leq \arctan(b_m) + \int_{a_m}^{r} \frac{\frac{\mathbf{w}_{0,n}}{s}}{\frac{\mathbf{w}_{0,n}}{s} - (n-1)} ds \\
&= \arctan(b_m) + n(r - a_m) \\
&\leq \arctan(b_m) + n(R - a_m).
\end{aligned}$$

In the second line we use that the function $\frac{x}{x-(n-1)}$ is decreasing in $x = \frac{\mathbf{v}(r)}{r}$ for $r \geq a_m$ and $n = \frac{\mathbf{w}_{0,n}}{r} \leq \frac{\mathbf{v}_m}{r} \leq \frac{\mathbf{w}_{1,n}}{r} = \frac{n}{\sqrt{1-n^2r^2}}$.
Then, by taking limits on the right hand side of the last inequality we see that in converges to nR. Therefore, by composing with $\tan(\cdot)$, we can find a constant C_1 such that $\sup_m ||\mathbf{v}_m||_\infty \leq C_1$.

Next, we are going to show by the same methods a uniform bound for $\dot{\mathbf{v}}_m$ in $\mathcal{C}([0, R])$. In fact, by Equation (4.2.1), Property (3) in Proposition 4.5.1 and the estimate from above, it follows that

$$
\begin{aligned}
0 < \dot{\mathbf{v}}_m &= (1+\mathbf{v}_m^2)\frac{\dfrac{\mathbf{v}_m}{r}}{\dfrac{\mathbf{v}_m}{r}-(n-1)} \\
&\leq (1+C_1^2)\frac{\dfrac{\mathbf{w}_{0,n}}{r}}{\dfrac{\mathbf{w}_{0,n}}{r}-(n-1)} \\
&= n(1+C_1^2). \qquad (4.5.3)
\end{aligned}
$$

Recall that $\mathbf{w}_{0,n}$ and $\mathbf{w}_{1,n}$ are the function in (4.5.2).Therefore, we can find a constant $C_2 > 0$ such that $\sup_m ||\dot{\mathbf{v}}_m||_\infty \leq C_2$.

Finally, we are going to show an uniform bound for $\ddot{\mathbf{v}}_m$ in $\mathcal{C}(0,R])$. In fact, by differentiating Equations (4.2.2), and Property (2), it follows

$$
\ddot{\mathbf{v}}_m = -(1+\mathbf{v}_m^2)\frac{n-1}{\left(\dfrac{\mathbf{v}_m}{r}-(n-1)\right)^2}\left(\frac{\dot{\mathbf{v}}_m r-\mathbf{v}_m}{r^2}\right)+2\mathbf{v}_m\dot{\mathbf{v}}_m\frac{\dfrac{\mathbf{v}_m}{r}}{\dfrac{\mathbf{v}_m}{r}-(n-1)}
$$

From our previous bounds, we only need to estimate the term $\dfrac{|\dot{\mathbf{v}}_m r-\mathbf{v}_m|}{r^2}$ to control $|\ddot{\mathbf{v}}_m|$. We note that

$$
\begin{aligned}
\frac{|\dot{\mathbf{v}}_m r-\mathbf{v}_m|}{r^2} &= \frac{\left|(1+\mathbf{v}_m^2)\dfrac{\frac{\mathbf{v}_m}{r}}{\frac{\mathbf{v}_m}{r}-(n-1)}r-\mathbf{v}_m\right|}{r^2} \\
&= \frac{\mathbf{v}_m}{r^2\left(\dfrac{\mathbf{v}_m}{r}-(n-1)\right)}\left|n+\mathbf{v}_m^2-\frac{\mathbf{v}_m}{r}\right|.
\end{aligned}
$$

On the other hand, the term

$$
n+\mathbf{v}_m^2-\frac{\mathbf{v}_m}{r} \leq n+\mathbf{v}_m^2-\frac{\mathbf{w}_{0,n}}{r} = \mathbf{v}_m^2 \leq \mathbf{w}_{1,n}^2(r),
$$

and

$$\begin{aligned}\frac{\mathbf{v}_m}{r} - n - \mathbf{v}_m^2 &\leq \frac{\mathbf{w}_{1,n}}{r} - n - \mathbf{w}_{0,n}^2 \\ &= \frac{n}{\sqrt{1-n^2r^2}} - n - n^2r^2 \\ &= n^2r^2\left(\frac{n}{\sqrt{1-n^2r^2}(1+\sqrt{1-n^2r^2})} - 1\right) \\ &\leq n\mathbf{w}_{1,n}^2(r).\end{aligned}$$

Therefore,

$$\frac{|\dot{\mathbf{v}}_m r - \mathbf{v}_m|}{r^2} \leq n\frac{\mathbf{w}_{1,n}^3}{r^2} \leq \frac{n^3R}{\sqrt{1-n^2R^2}}.$$

Consequently,

$$|\ddot{\mathbf{v}}_m| \leq 2(n-1)(1+C_1^2)\frac{n^3R}{\sqrt{1-n^2R^2}} + 2nC_1C_2 = C_3.$$

This implies, $\sup_m ||\ddot{\mathbf{v}}_m|| \leq C_3$. In particular we have show that $\sup_m ||\mathbf{v}||_{\mathcal{C}^2([0,R])} \leq C$ where $C = C_1 + C_2 + C_3$.

Finally, since $\mathbf{v}_m$ and $\dot{\mathbf{v}}_m$ are Lipschtiz functions for every $m \geq M$, we obtain an uniformly bounded equicontinuous sequence in $\mathcal{C}^1([0,R])$. Therefore, by Arzela-Ascoli's theorem, there exists a subconvergent sequence in $\mathcal{C}^1$ topology of $\{\mathbf{v}_m\}$ to a function $\mathbf{v} \in \mathcal{C}^1([0,R])$. In addition, by continuity we have

$$\begin{aligned}\mathbf{v}(0) &= \mathbf{v}\left(\lim_{m\to\infty} a_m\right) = \lim_{m\to\infty} b_m = 0,\\ \dot{\mathbf{v}}(0) &= \lim_{m\to\infty}(1+b_m^2)\frac{\dfrac{b_m}{a_m}}{\dfrac{b_m}{a_m}-(n-1)} = n,\\ \mathbf{w}_{0,n}(r) &\leq \mathbf{v} \leq \mathbf{w}_{1,n}(r), \text{ for all } r \in [0,R],\end{aligned}$$

Therefore, by Theorem 4.3.6 the function $\mathbf{v}(r)$ can be extended to $r \in \left[0, \frac{1}{n}\right)$ as a solution of Equation (4.2.1) such that $\lim_{r\to\frac{1}{n}} \mathbf{v}(r) = \infty$. □

We finish this section by showing that the "bowl"-type Q_{n-1}-translator is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(Q_{n-1}(1,\dots,1)) \times \mathbb{R}$.

Theorem 4.5.3. *A "bowl"-type Q_{n-1}-translator $\Sigma \subset \mathbb{R}^{n+1}$ satisfies*

$$\begin{aligned}\min\{\lambda_i(p) : i = 1,\dots,n\} =& \lambda_n(p) \to 0,\\ \forall\, i \in \{1,\dots,n-1\}, \lambda_i(p) \to& \frac{1}{Q_{n-1}(1,\dots,1)},\end{aligned}$$

as $|p| \to \infty$ in Σ.

Proof. The proof is a consequence of the barriers of a "bowl"-type Q_{n-1}-translator. Indeed, we already know that $\lim_{r\to\frac{1}{n}} \mathbf{v}(p) = \infty$. Then, for the minimum of the principal curvatures of Σ we have

$$\lambda_n(p) = \frac{\dot{\mathbf{v}}}{(1+\mathbf{v}^2)^{\frac{3}{2}}} = \frac{f\left(\dfrac{\mathbf{v}}{r}\right)}{\sqrt{1+\mathbf{v}^2}},$$

where $f(x) = \dfrac{x}{x-(n-1)}$. Therefore, by the barrier $\mathbf{w}_{0,n}(r) \leq \mathbf{v}(r)$ it follows that $1 \leq f\left(\dfrac{\mathbf{v}}{r}\right) \leq n$, this means that $\lambda_n(p) \to 0$ as $r \to \frac{1}{n}$.

On the other hand, for the rest of the principal curvatures, we can use that $\mathbf{v}(r) \leq \mathbf{w}_4(r) = \dfrac{n}{1-nr}$ when r is close to $\frac{1}{n}$ to see that

$$\frac{n}{r\sqrt{(1-nr)^2+n^2}} = \frac{\mathbf{w}_4}{r\sqrt{1+\mathbf{w}_4^2}} \leq \lambda_i(p) = \frac{\mathbf{v}}{r\sqrt{1+\mathbf{v}^2}} \leq \frac{1}{r}.$$

Therefore, by taking limits in the last expresion as $r \to \frac{1}{n}$, we obtain $\lambda_i(p) \to n$. Recall that $\dfrac{1}{Q_{n-1}(1,\ldots,1)} = n$. □

4.6 Uniqueness and Regularity

In this section we show uniqueness and regularity results of solutions to Equations (4.2.1) and (4.2.2).

We start by showing Gronawall's inequalities for solutions to the initial value problems 4.3.4 and 4.3.3.

Lemma 4.6.1. *Let $n \geq 3$, $0 \leq r_0 < R \leq \infty$, $r_0 \leq w_0, u_0 \leq e^{r_0} - 1$ and $\mathbf{u}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ be solutions to Equation (4.2.2) with initial conditions $\mathbf{u}(r_0) = u_0$ and $\mathbf{w}(r_0) = w_0$, respectively. Then, the following inequality holds at (r_0, R)*

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(-\int_{r_0}^{r} \alpha(s)ds\right),$$

where $\alpha(r) = (n-3)\dfrac{r^{2n-5}}{(e^r-1)^{2n-6}}$.

Proof. Let $\mathbf{u}, \mathbf{w}$ be the functions from the statement of this lemma. Then, by setting $d =$

$\mathbf{u}-\mathbf{w}$, we see that

$$\begin{aligned}\dot{d} &= (1+\mathbf{u}^2)\left(\frac{r}{\mathbf{u}}\right)^{n-1} - (1+\mathbf{w}^2)\left(\frac{r}{\mathbf{w}}\right)^{n-1} \\ &= r^{n-1}\left(\frac{\mathbf{w}^{n-1}-\mathbf{u}^{n-1}}{(\mathbf{uw})^{n-1}} + \frac{\mathbf{w}^{n-3}-\mathbf{u}^{n-3}}{(\mathbf{uw})^{n-3}}\right) \\ &= -\left(\frac{r^{n-1}(\mathbf{w}^{n-2}+\ldots+\mathbf{u}^{n-2})}{(\mathbf{uw})^{n-1}} + \frac{r^{n-1}(\mathbf{w}^{n-4}+\ldots+\mathbf{u}^{n-4})}{(\mathbf{uw})^{n-3}}\right)d.\end{aligned}$$

Next, we want to estimate the term in parenthesis in the last line of the above equation. For this purpose we use the hypothesis $r_0 \leq w_0, u_0 \leq e^{r_0}-1$, then by Proposition (4.3.3), we have

$$\mathbf{w}_2(r) = r \leq \mathbf{w}(r), \mathbf{u}(r) \leq \mathbf{w}_3(r) = e^r - 1.$$

Then, we may estimate the term in parenthesis in the following way

$$\begin{aligned}&\frac{r^{n-1}(\mathbf{w}^{n-2}+\ldots+\mathbf{u}^{n-2})}{(\mathbf{uw})^{n-1}} + \frac{r^{n-1}(\mathbf{w}^{n-4}+\ldots+\mathbf{u}^{n-4})}{(\mathbf{uw})^{n-3}} \\ &\geq \frac{r^{n-1}(n-3)\mathbf{w}_2^{n-4}}{\mathbf{w}_3^{2(n-3)}} \\ &= (n-3)\frac{r^{2n-5}}{(e^r-1)^{2n-6}} = \alpha(r).\end{aligned}$$

In the second line we discard the first term and for the second term we applied the barriers $\mathbf{w}_2$ in the numerator and $\mathbf{w}_3$ in the denominator.

Therefore, we obtain that $\dot{d} \leq -\alpha(r)d$. This yields, by integration, that

$$|\mathbf{u}(r)-\mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(-\int_{r_0}^{r}\alpha(s)ds\right),$$

where $|\cdot|$ appears by interchanging $\mathbf{u}$ with $\mathbf{w}$ in the proof. □

Lemma 4.6.2. *Let* $0 \le r_0 < R \le \infty$, $nr_0 \le w_0, u_0 \le \dfrac{nr_0}{\sqrt{1-n^2r_0^2}}$ *and* $\mathbf{u}, \mathbf{w} \in \mathcal{C}^1([r_0, R))$ *be solutions to Equation* (4.2.1) *with initial conditions* $\mathbf{u}(r_0) = u_0$ *and* $\mathbf{w}(r_0) = w_0$*, respectively. Then, the following inequality holds at* (r_0, R)

$$|\mathbf{u}(r) - \mathbf{w}(r)| \le |u_0 - w_0| \exp\left(\int_{r_0}^{r} \alpha(s)ds\right),$$

where $\alpha(r) = \dfrac{2n^3r}{\sqrt{(1-n^2r^2)^3}}$.

Proof. Let $\mathbf{u}, \mathbf{w}$ be the functions from the statement of this lemma. Then, by setting $d = \mathbf{u} - \mathbf{w}$, we see that

$$\begin{aligned}
\dot{d} &= (1+\mathbf{u}^2)\frac{\frac{\mathbf{u}}{r}}{\frac{\mathbf{u}}{r}-(n-1)} - (1+\mathbf{w}^2)\frac{\frac{\mathbf{w}}{r}}{\frac{\mathbf{w}}{r}-(n-1)} \\
&= \frac{(n-1)\left(\frac{\mathbf{w}}{r}-\frac{\mathbf{u}}{r}\right)}{\left(\frac{\mathbf{u}}{r}-(n-1)\right)\left(\frac{\mathbf{w}}{r}-(n-1)\right)} + \frac{\left(\frac{\mathbf{u}^3\mathbf{w}}{r^2}-\frac{\mathbf{u}\mathbf{w}^3}{r^2}\right)+(n-1)\left(\frac{\mathbf{w}^3}{r}-\frac{\mathbf{u}^3}{r}\right)}{\left(\frac{\mathbf{u}}{r}-(n-1)\right)\left(\frac{\mathbf{w}}{r}-(n-1)\right)} \\
&= \frac{\left(\frac{\mathbf{u}\mathbf{w}}{r^2}(\mathbf{u}+\mathbf{w}) - \frac{(n-1)}{r}(1+\mathbf{w}^2+\mathbf{u}\mathbf{w}+\mathbf{u}^2)\right)}{\left(\frac{\mathbf{u}}{r}-(n-1)\right)\left(\frac{\mathbf{w}}{r}-(n-1)\right)} d
\end{aligned}$$

Next, we want to estimate the term that accompanies in the last line of the above equation. We already know by Proposition 4.3.1 that Equation (4.2.1) satisfies a comparison principle. Then, by Proposition 4.3.2, we may use the barriers $\mathbf{w}_{0,n} \le \mathbf{w}, \mathbf{u} \le \mathbf{w}_{1,n}$ to estimate the term in parenthesis in the following way

$$\frac{\left(\frac{\mathbf{u}\mathbf{w}}{r^2}(\mathbf{u}+\mathbf{w}) - \frac{(n-1)}{r}(1+\mathbf{w}^2+\mathbf{u}\mathbf{w}+\mathbf{u}^2)\right)}{\left(\frac{\mathbf{u}}{r}-(n-1)\right)\left(\frac{\mathbf{w}}{r}-(n-1)\right)} \le \frac{\frac{2\mathbf{w}_{1,n}^3}{r^2}}{\left(\frac{\mathbf{w}_{0,n}}{r}-(n-1)\right)^2} = \alpha(r).$$

Therefore, since $\dot{d} \leq \alpha(r)d$, it follows by integration that

$$|\mathbf{u}(r) - \mathbf{w}(r)| \leq |u_0 - w_0| \exp\left(\int_{r_0}^{r} \alpha(s)ds\right),$$

where $|\cdot|$ appears by interchanging $\mathbf{u}$ with $\mathbf{w}$ in the proof. □

Corollary 4.6.3. *For $n \geq 3$ the solutions to Equations (4.2.1) and (4.2.2) are unique in their respective domains. In addition, for $n = 2$, the solution $\mathbf{v} = \dfrac{2r}{1-r^2}$ to Equation (4.2.1) is unique.*

Proof. The proof follows by the estimates from Lemmas 4.6.1 and 4.6.2, since the boundary condition for two different solutions coincide. □

The last part of this section is devoted to the regularity of the solutions found in Theorems 4.4.2 and 4.5.2. We note that in the construction of these solutions, the regularity can be obtained by an inductive procedure on the estimates of the derivatives of the solutions $\mathbf{v}_m$, since each derivative can be obtained by a regular function of the previous derivatives of $\mathbf{v}_m$. On the other hand, we prefer to use a well-known regularity result related to the prescription of curvature functions. We refer the reader to [CNS2], [Tru], [SUW] and [AMZ] for a better knowledge on this theory.

First, we use a well know result about the regularity of solutions to an elliptic PDE.

Lemma 4.6.4. *Assume that $u \in \mathcal{C}^2(\Omega)$ satisfies*

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega,$$

where $F : \Gamma \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^{n\times n} \to \mathbb{R}$ is monotone increasing with respect to the symmetric matrix variable in $S^{n\times n}$. If $F \in \mathcal{C}^{k,\alpha}(\Gamma)$ for some $k \geq 1$ and $\alpha \in (0,1)$, then $u \in \mathcal{C}^{k+2,\alpha}(\Omega)$. In particular, if F is smooth, then so is u.

Proof. We refer the reader to [GT] Lemma 17.16. □

As an application of Lemma 4.6.4 we show that these γ-translators are smooth.

Corollary 4.6.5. *The γ-translators founded in Theorems 4.4.2 and 4.5.2 are smooth.*

Proof. First, we note that the solutions to Equation (4.1.2) with $k = n-1$ and Equation (4.1.3) with $k = n$, respectively, can be written $\mathbf{u}(r) = \int_0^r \mathbf{v}(s)ds$, where $\mathbf{v}$ is the unique solutions of class $\mathcal{C}^1$ to the Equations (4.2.1) and (4.2.2), respectively. In particular, $\mathbf{u}$ is of class $\mathcal{C}^2$.

Then, since the translators are graph of the form $M = \{(x, u(x)) : x \in B_R(0)\}$ where $R \in (0, \infty]$ and $u(x) = \mathbf{u}(|x|)$, it follows that the principal curvatures can be calculated as the eigenvalues of the symmetric matrix (3.2.3) used in Chapter 2. Recall that this matrix is defiened by

$$A = \sqrt{g^{-1}} II \sqrt{g^{-1}},$$

where $g = (\delta_{ij} + u_i u_j)dx^i dx^j$ is the metric of M, $g^{-1} = I - \dfrac{Du \otimes Du}{1 + |Du^2|}$ is the inverse of the metric g,

$$\sqrt{g^{-1}} = I - \frac{Du \otimes Du}{(1 + \sqrt{1 + |Du|^2})\sqrt{1 + |Du|^2}}$$

is the square root matrix of g^{-1}, and $II = \dfrac{D^2 u}{\sqrt{1 + |Du|^2}}$ is the second fundamental form of M. In addition, by the multiplicative properties of the determinant of a matrix, the eigenvalues of A coincides with the eigenvalues of $g^{-1}II$, which are the principal curvatures of M.

On the other hand, M is a γ-translator for $\gamma = S_{n-1}$ and Q_{n-1}, which means that

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle \text{ holds at } M, \tag{4.6.1}$$

where λ are the eigenvalues of A and $\nu = \dfrac{(1, -Du)}{1 + |Du|^2}$ is the inward unit normal. Now we are going to show that Equation (4.6.1) can be seen as an elliptic PDE of the form $F(Du, D^2u) = 0$ which satisfies the hypothesis of Lemma 4.6.4. In fact, we set

$$\tilde{\gamma}(Du, D^2u) - \frac{1}{\sqrt{1 + |Du|^2}} = 0,$$

where $\tilde{\gamma} : \mathbb{R}^n \times S_+^{n\times n} \to \mathbb{R}$ is defined by

$$\tilde{\gamma}(p, X) = \gamma\left(\left(I - \frac{p \otimes p}{\sqrt{1 + |p|^2}}\right) \frac{X}{\sqrt{1 + |p|^2}} \left(I - \frac{p \otimes p}{\sqrt{1 + |p|^2}}\right)\right),$$

and γ is treated as a function of a symmetric matrix Z by evaluating it on the eigenvalues $z_1, \dots, z_n$.

Moreover, $\tilde{\gamma}$ is increasing at X_{ij}. To see this we note

$$\frac{\partial \tilde{\gamma}}{\partial X_{ij}} \xi_i \xi_j = \frac{\partial \gamma}{\partial X_{pq}} \sqrt{g^{-1}}_{pi} \sqrt{g^{-1}}_{qj} \xi_i \xi_j > 0$$

holds for all $\xi \in \mathbb{R}^n \setminus \{0\}$, because $\sqrt{g^{-1}}$ is non-degenerate and the eigenvalues of the matrix $\dfrac{\partial \gamma}{\partial X_{pq}}$, which are equal to γ_{z_i} for $i = 1, \dots, n$, are positive by Lemmas 3.1.1 and 3.1.2 since M is strictly convex.

Finally, by Lemma 4.6.4, the solution u is smooth since S_n and Q_{n-1} are smooth in the positive cone Γ_n. □

4.7 Catenoid-like example

We end this chapter by giving an implicit formulation of a rotationally symmetric $\sqrt{S_2}$-translator over the plane $\{y = 0\}$ in $\mathbb{R}^3$.

Theorem 4.7.1. *Let $a \in [0,1)$. Then, the family of surfaces*

$$\Sigma_a = \left\{ (r_a(z)\cos(\theta), r_a(z)\sin(\theta), z) \in \mathbb{R}^3 : \theta \in [0, 2\pi),\ z \in \mathbb{R} \right\},$$

where $r_a(z)$ is giving implicitly by

$$z = \int_1^{r_a(z)} \sqrt{e^{\frac{s^2}{2} - a} - 1}\, ds \text{ with } r_a(0) = 1,$$

are $\sqrt{S_2}$-translators in $\mathbb{R}^3$. In addition, Σ_a are complete and satisfy $H < 0$ and $K > 0$.

Proof. The surface of the form

$$\Sigma = \{(r(z)\cos(\theta), r(z)\sin(\theta), z) : (\theta, z) \in [0, 2\pi] \times \mathbb{R}\},$$

where $r(z)$ is a function to be found, satisfies the following formulae

$$\lambda_1 = \frac{r''}{(1 + r'^2)^{\frac{3}{2}}},\ \lambda_2 = \frac{-1}{r\sqrt{1 + r'^2}} \text{ and } \nu = \frac{(-\cos(\theta), -\sin(\theta), r')}{\sqrt{1 + r'^2}}.$$

Then, equation $(S_2)^{\frac{1}{2}} = \langle \nu, e_3 \rangle$ reads as

$$\lambda_1 \lambda_2 = \langle \nu, e_3 \rangle^2 \Leftrightarrow r'' = -(1 + r'^2) r r'^2 \Leftrightarrow f f' = -(1 + f^2) r f^2,$$

where $r' = f(r)$ and $r'' = f'(r) r' = f f'$. Note that we are using r as an independent variable. Consequently, $f = 0$ is a trivial solution which gives that $r = C$ for some constant $C \in \mathbb{R}$. On the other hand, by assuming that $f \neq 0$ which obtain $f' = -(1 + f^2) r f$. It is easy to check that the solutions to this equation are $f = \dfrac{1}{\sqrt{e^{r^2 - a} - 1}}$ where $a \in \mathbb{R}$ is any constant.

Indeed, we have

$$-(1+f^2)rf = -\left(1+\frac{1}{e^{r^2-a}-1}\right)\frac{r}{\sqrt{e^{r^2-a}-1}} = -\frac{re^{r^2-a}}{(e^{r^2-a}-1)^{\frac{3}{2}}} = f'.$$

Finally, after integrating, we obtain the solutions $r(z) = C$ and

$$z = \int_1^{r_a(z)} \sqrt{e^{s^2-a}-1}\, ds \text{ with } r_a(0) = 1.$$

In particular, since we choose $r_a(0) = 1$, it follows that $a \in [0, 1)$ for r_a to be well defined as an increasing function.

Moreover, it is not hard to see that Σ_a satisfies $H < 0$ and $K > 0$. To see this we note that

$$r' = \frac{1}{\sqrt{e^{r_a^2-a}-1}} > 0 \text{ and } r'' = -\frac{e^{r_a^2-a}r_a' r_a}{(e^{r_a^2-a}-1)^{\frac{3}{2}}} = -\frac{e^{r_a^2-a}r_a}{\left(e^{r_a^2-a}-1\right)^2} < 0$$

Therefore, r is an increasing concave function. Then, by substituting in the principal curvatures, we see that λ_1 and λ_2 are negative, and therefore, $H < 0$ and $K > 0$. □

4.8 Appendix: "bowl"-type solution $\mathcal{C}^2$-asymptotics to cylinders

Proposition 4.8.1. *The "bowl"-type solution which is defined in the ball $B_{r_0}(0) \subset \mathbb{R}^n$ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$ where $r_0 = \gamma(1, \ldots, 1)$.*

Proof. Firstly, we note that the profile curve of the "bowl"-type γ-translator is given by

$$\left\{(r, \mathbf{u}(r)) \in \mathbb{R}^2 : r \in [0, r_0)\right\}.$$

Then, by setting $\mathbf{v} = \dot{\mathbf{u}}$, we have that

$$\dot{\mathbf{v}} = (1 + \mathbf{v}^2) f\left(\frac{\mathbf{v}}{r}\right), \tag{4.8.1}$$

where $f(x)$ appears in (4.8.1) by the implicit function theorem applied on the level set

$$\left\{(x, y) \in \mathbb{R}^2 : \gamma(x, y, \ldots, y) = 1\right\}.$$

Consequently, since the "bowl"-type solution is defined in a ball, it follows that $\lim_{x\to\infty} f(x) = L \geq 0$.

Next, the principal curvatures of the "bowl"-type solution satisfy

$$\begin{aligned} 0 < &\lambda_1 = \frac{\dot{\mathbf{v}}}{(1+\mathbf{v}^2)^{\frac{3}{2}}} = \frac{f\left(\frac{\mathbf{v}}{r}\right)}{\sqrt{1+\mathbf{v}^2}} \leq \frac{C}{\sqrt{1+\mathbf{v}^2}}, \\ &\lambda_i = \frac{\mathbf{v}}{r\sqrt{1+\mathbf{v}^2}}, \text{ for } i \in \{2, \ldots, n\}, \end{aligned}$$

In the first line we used that f is bounded as r approaches to r_0.

Therefore, since $\lim_{r\to r_0} \mathbf{v}(r) = \infty$, it follows that $\lambda_1 \to 0$ and $\lambda_i = \frac{1}{r_0}$ as $|p| \to \infty$. $\square$

Chapter 5

Geometric Properties of γ-Translators

Geometric properties of hypersurfaces in a $\mathbb{R}^{n+1}$ is one of the most important aspects for studying their classification. Roughly speaking, these properties appear when we apply PDE techniques over geometric quantities of a hypersurfaces.
The most remarkable PDE technique used in the literature is the to maximum principle. Usually, the maximum principle is applied to obtain inequalities related to geometric quantities of the hypersurface. More precisely, a maximum principle will state that if a geometric quantity attains its maximum or minimum in a interior point of a hypersurface and it satisfies an elliptic linear PDE, then the quantity will be constant.

In differential geometry, one of the famous application of maximum principles are the tangential or avoidance principle. This result states that if two connected hypersurface which satisfy a geometric equation, touch in a certain way, then both hypersurfaces must coincide. In particular, one of the oldest application of this type of result is that spheres are the only closed constant mean curvature surface in $\mathbb{R}^3$. That proof used the method of moving planes of Alexandrov, which implies that a closed constant mean curvature surface will be rotationally symmetric, by comparing it with its reflections over "moving" planes. Consequently, one can check that the only rotationally symmetric constant mean curvature surface is a sphere.

Finally, we also want to mention that maximum principle arguments have been generalized in many directions in recent years. We refer the reader to [AMR] for an excellent self contained book about this topic.

In this chapter we study related geometric properties of γ-translators in $\mathbb{R}^{n+1}$ for 1-homogeneous symmetric concave functions $\gamma : \Gamma \to [0, \infty)$, where $\Gamma \subset \mathbb{R}^n$ is open convex cone[1].

For this purpose we use maximum principles related to the translator equation

$$\gamma(\lambda) = \langle \nu, e_{n+1} \rangle ,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ denotes the principal curvature vector and ν is the outward unit normal of the hypersurface in $\mathbb{R}^{n+1}$, respectively.

Remark 5.0.1. The adjective "outward" on the unit normal vector ν only makes sense when the hypersurface is compact. As the reader can check in Proposition 5.0.1, there are no closed γ-translator in $\mathbb{R}^{n+1}$ whose principal curvatures belong to Γ. However, we refer to this adjective in the sense that the orientation imposed on ν agrees with the sign of being positive as in the closed case.

Moreover, a γ-translator $M_0 = F_0(M) \subset \mathbb{R}^{n+1}$ is also an eternal solution[2] to the γ-flow Equation (2.0.1) which evolves by translations in direction e_{n+1}. In fact, by setting

$$F(x, t) = F_0 + te_{n+1},$$

and taking time derivative and inner product with the unit normal vector ν of $M_t = F(M, t)$,

[1]See section 5.2 below for the precise hypothesis that we consider over $\gamma : \Gamma \to [0, \infty)$.

[2]A solution to Equation (2.0.1) which is defined for $t \in (-\infty, \infty)$.

we see that

$$\langle \partial_t F(x,t), \nu \rangle = \langle e_{n+1}, \nu \rangle = \gamma(\lambda).$$

Finally, by taking a normal reparametrization of $F(x,t)$, the γ-flow (2.0.1) holds for all $t \in \mathbb{R}$.

The first application of maximum principle that we give is about the nonexistence of closed γ-translators in $\mathbb{R}^{n+1}$ whose principal curvatures belong to Γ.

Proposition 5.0.1. *Let $\Gamma = \{\lambda \in \mathbb{R}^n : \gamma(\lambda) > 0\}$ and assume that $\gamma : \Gamma \to (0, \infty)$ satisfies properties 1-3. Then, there is no closed γ-translator in $\mathbb{R}^{n+1}$ such that its principal curvatures belong to the cone Γ.*

This result is a well know fact for H-translators, since they are minimal hypersurfaces in $(\mathbb{R}^{n+1}, e^{\langle x, e_{n+1}\rangle} dx^2)$. The main idea in our proof is the use of the maximum principle on the height function of the γ-translator.
Moreover, as an easy consequence of Proposition 5.0.1, we show that there is no totally umbilical strictly convex γ-translator in $\mathbb{R}^{n+1}$ without boundary.

Corollary 5.0.2. *Under the hypothesis of Theorem 5.0.1, there is no totally umbilical strictly convex γ-translator in $\mathbb{R}^{n+1}$ without boundary.*

Next, we consider the extra hypothesis on the curvature function γ to be also real analytic, locally uniformly elliptic in Γ_+. This will implies that in local coordinates of the form $(x, u(x))$ of a γ-translator, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the equation

$$F(x, Du, D^2u) = \gamma(\lambda) - \langle \nu, e_{n+1} \rangle = 0 \tag{5.0.1}$$

is uniformly elliptic and real analytic, in the sense that the function $F : \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is analytic, where $S(n)$ denotes the space of symmetric matrices of size n. We refer the reader to classic regularity results from PDE theory given in [Mor1] and [Mor2] (see [Bla] for a shorter proof) that a solution $u \in \mathcal{C}^\infty(\Omega)$ to Equation (5.0.1) is real analytic.

In particular, under this assumption, we have that a smooth γ-translator in $\mathbb{R}^{n+1}$ is real analytic.

Remark 5.0.2. We note that the part $\langle \nu, e_{n+1} \rangle = \dfrac{1}{\sqrt{1+|Du|^2}}$ in Equation (5.0.1) is always analytic in local coordinates.

Consequently, under the assumptions of γ being locally uniformly elliptic and real analytic in Γ_+ with $\gamma(0) = 0$, we develope an interior and a boundary tangential principle for γ-translators in $\mathbb{R}^{n+1}$ which reads as follows.

Theorem 5.0.3. *Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ be two embedded connected γ-translators such that*

1. *$\gamma : \Gamma \to [0, \infty)$ satisfies properties 1-4 and 7 given below.*
2. *Σ_1 is strictly convex.*
3. *Σ_2 is convex.*

Then,

a) ***(Interior tangential principle)*** *Assume that there exists an interior point $p \in \Sigma_1 \cap \Sigma_2$ such that the tangent spaces coincide at p. If Σ_1 lies at one side of Σ_2, then both hypersurfaces coincide.*

b) ***(Boundary tangential principle)*** *Assume that the boundaries $\partial\Sigma_i$ lie in the same hyperplane Π and the intersection of Σ_i with Π is transversal. If Σ_1 lies at one side of Σ_2 and there exist $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that the tangent spaces coincide, then both hypersurfaces coincide.*

In addition, as a corollary of the tangential principle, we show a non-existence results for entire γ-translators in $\mathbb{R}^{n+1}$ such that the "bowl"-type[3] γ-translators is defiened in a round

[3] A strictly convex rotationally symmetric γ-translators defined in round balls our in all $\mathbb{R}^n$

ball. We refer the reader to [Ren] Theorems 1.3-1.4 for the characterization of ‘bowl"-type γ-translators in $\mathbb{R}^{n+1}$.

Corollary 5.0.4. *Assume that $\gamma : \Gamma \to [0, \infty)$ satisfies Properties 1-4 and 7. Then, there are no convex entire γ-translators in $\mathbb{R}^{n+1}$ such that the "bowl"-type γ-translator is defined in a round ball.*

We also include in this chapter a uniqueness result for γ-translators in $\mathbb{R}^{n+1}$ which are strictly convex, defined over ball with a single end asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$.

Theorem 5.0.5. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a complete γ-translator such that*

1. *$\gamma : \Gamma \to [0, \infty)$ 1-4 and 7.*
2. *Σ is strictly convex graph over a ball $B^n_r(0) \subset \mathbb{R}^n$.*
3. *Σ posses a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, i.e: the principal curvatures of Σ satisfy*

$$\begin{aligned} \min \{\lambda_i(p) : i = 1, \ldots, n\} =& \lambda_1(p) \to 0, \\ \forall i \in \{2, \ldots, n\}, \lambda_i(p) \to& \frac{1}{r}, \end{aligned}$$

   as $|p| \to \infty$.

Then Σ is rotationally symmetric with respect the x_{n+1}-axis.

As a corollary, by the uniqueness result exposed in [Ren], we obtain the following.

Corollary 5.0.6. *Let Σ be a γ-translator as in Theorem 5.0.5 such that the "bowl"-type γ-translator is defined in a ball of radius $\gamma(1, \ldots, 1)$. Then, Σ coincide with the "bowl"-type γ-translator.*

Finally, the main result of this chapter is a convexity estimate for γ-translators in $\mathbb{R}^{n+1}$.

Theorem 5.0.7. *Let $n \geq 3$, $\alpha, \delta > 0$ and $\Sigma \subset \mathbb{R}^{n+1}$ be a complete, immersed, two-sided γ-translator such that*

a) $\gamma : \Gamma \to \mathbb{R}$ satisfies Properties 1-3 and 5-7.

b) $\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : (\delta+1)H \leq \alpha\gamma\}$ which is compactly supported in $\Gamma \setminus Cyl_{n-1}$, where

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\} .$$

c) There exist a constant $\beta \in (0,1)$ such that $\lambda_i + \lambda_j \geq \beta H$, for every $1 \leq i < j \leq n$.

Then, $\lambda_1 \geq H - \alpha\gamma$ in Σ, where $\lambda_1(p) = \min\{\lambda_i(p) : i = 1, \ldots, n\}$.

We use maximum principle arguments over the function $f = \dfrac{\gamma}{H - \lambda_1}$. In particular, we show under the assumptions of Theorem 5.0.7 that $f \geq \alpha^{-1}$ in Σ, which is equivalent to the convexity estimate. The main part of the proof uses the Omori-Yau maximum principle (see [AMR] Chapter 3), which gives suitable information to formulate a maximum principle at "infinity" of Σ.
The proof of this result was inspired by [SS], where the authors showed that a mean convex and uniform 2-convex H-translator is in fact convex.

Remark 5.0.3. In general Omori-Yau maximum principles require regularity on the solution to be applied. We note that this is possible since γ is concave. In fact, the Evans-Krylov theory for concave functions (see [Kry] or [GT] Theorem 17.14) can bee applied to give the suitable regularity.

Finally, as a consequence of Theorem 5.0.7, we were able to show that these γ-translators are asymptotically convex.

Corollary 5.0.8. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be γ-translator for which $\gamma : \Gamma \to \mathbb{R}$ satisfies the hypothesis of Theorem 5.0.7, then Σ is asymptotically convex.*

The organization of this chapter goes as follows: In Section 5.1, we give equations and estimates for γ-translators in $\mathbb{R}^{n+1}$ and the proofs of Proposition 5.0.1 and Corollary 5.0.2. In Section 5.2, we show the tangential principle, Theorem 5.0.3. In Section 5.2, we prove the non-existence theorems. In Section 5.4, we show the uniqueness results rewarding the "bowl"-type γ-translators. In Section 5.5, we give the proof of the convexity estimate Theorem 5.0.7.

5.1 Preliminaries

In this section we give some preliminaries with respect 1-homogeneous symmetric concave functions and the proofs of Proposition 5.0.1 and Corollary 5.0.2.
As we mentioned in the introduction of this chapter we require the following properties for $\gamma : \Gamma \to \mathbb{R}$.

1. $\Gamma \subset \mathbb{R}^n$ is a symmetric open convex cone which contains the positive cone $\Gamma_+ := \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$ and $\gamma : \Gamma \to \mathbb{R}$ is symmetric, smooth and positive.

2. γ is strictly increasing in each variable, i.e: $\frac{\partial \gamma}{\partial \lambda_i} > 0$ in Γ for every i.

3. γ is 1-homogeneous, i.e: for every $c > 0$, $\gamma(c\lambda) = c\gamma(\lambda)$ in Γ.

4. γ is real analytic, locally uniformly elliptic in Γ_+.

5. γ is strictly concave in off-radial direction, i.e: for every $\lambda \in \Gamma$ and $\xi \in \mathbb{R}^n$ it holds

$$\frac{\partial^2 \gamma}{\partial \lambda_i \partial \lambda_j}(\lambda)\xi_i\xi_j \leq 0,$$

   and equality yields if, and only if, ξ is a scalar multiple of λ.

6. Let $G : \{A \in \mathrm{Sym}(n) : \lambda(A) \in \Gamma\} \to \mathbb{R}$ be a smooth symmetric function such that $G(A) = \gamma(\lambda)$, when A is a diagonal matrix with entries $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then, there

exists a constant $C > 0$ such that

$$\frac{d}{ds}G(A + sB) \leq C\mathrm{Tr}(B),$$

whenever B is a symmetric 2-nonnegative matrix[4] with $\lambda(B) \in \Gamma$. Moreover, the inequality is strict unless $B = 0$.

7. γ vanishes at the boundary, i.e there exist a continuous extension of γ to $\overline{\Gamma}$ which vanishes identically at $\partial\Gamma$.

The reason why we impose these conditions for γ are the following:

- Property 1 is to simplify certain computations, since it is difficult to calculate derivatives on the eigenvalues rather than the entries of a matrix. Indeed, it is a well know result (see for instance [Gla], that there exists a smooth symmetric function

$$G : \{A \in \mathrm{Sym}(n) : \lambda(A) \in \Gamma\} \to \mathbb{R}$$

such that $G(A) = \gamma(\lambda)$ whence A is a diagonal matrix. See section 5.1 below for more details.

- Property 2 implies that γ-flow is weakly parabolic. For instance, if we write Equation (2.0.1) in local coordinates as a graph of a function under the exponential map defined in a ball in T_pM,

$$\partial_t u = \gamma(\nabla u, \nabla^2 u)\sqrt{1 + |\nabla u|^2},$$

Property 2 corresponds to a local ellipticity bound.

[4]The eigenvalues of B satisfy $\lambda_i + \lambda_j \geq 0$ for every $i \neq j$.

- Property 3 implies that the set of solutions of (2.0.1) is closed under parabolic scaling. This property is important for studying singularities that appear under these flows.

- Property 4 implies that in local coordinates of the form $(x, u(x))$ of a γ-translator, where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, the equation

$$F(x, Du, D^2u) = \gamma(\lambda) - \langle \nu, e_{n+1} \rangle = 0 \tag{5.1.1}$$

  is uniformly elliptic and real analytic, in the sense that the function $F : \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is analytic, where $S(n)$ denotes the space of symmetric matrices of size n. We refer the reader to classic regularity results from PDE theory given in [Mor1] and [Mor2] (see [Bla] for a shorter proof) that a solution $u \in \mathcal{C}^\infty(\Omega)$ to Equation (5.1.1) is real analytic.

  In particular, under this assumption, we have that a smooth γ-translator in $\mathbb{R}^{n+1}$ is real analytic.

  Remark 5.1.1. We note that the part $\langle \nu, e_{n+1} \rangle = \dfrac{1}{\sqrt{1 + |Du|^2}}$ in Equation (5.1.1) is always analytic in local coordinates.

- Properties 5 -7 are related to convexity estimates under the flow. Since the function γ is concave, a non-positive sign of the second order derivatives of γ (which appears frequently in the evolution equations under the γ-flow) would not give suitable information.

  Property 7 has been used in [Lyn2] and [LL] to preserve other convex cones.

  Furthermore, Property 6 ensures that 2-convexity[5] is preserved under the γ-flow.

  Finally, Property 7 is used to preserve the convexity of the initial hypersurface under the γ-flow.

[5] The principal curvatures of the hypersurface satisifies $\lambda_i + \lambda_j > 0$.

Regarding the notation that we use for the derivatives of symmetric functions in the principal curvatures of γ-translators along this chapter, we first mention that we abuse on the notation by setting $\varphi(A) = \varphi(\lambda(A))$ for a symmetric function φ, instead of using another function as in Remark 1. Here $A = (h_{ij})$ is a symmetric matrix such that $\lambda(A) \in \Gamma$.
Then, we write

$$\dot{\varphi}^{ab}(A) = \frac{\partial \varphi}{\partial h_{ab}}(A),\ \dot{\varphi}^{a}(\lambda) = \frac{\partial \varphi}{\partial \lambda_a}(\lambda), \ddot{\varphi}^{ab,cd}(A) = \frac{\partial^2 \varphi}{\partial h_{ab} \partial h_{cd}}(A),\ \ddot{\varphi}^{ab}(\lambda) = \frac{\partial^2 \varphi}{\partial \lambda_a \partial \lambda_b}(\lambda).$$

Furthermore, if φ is 1-homogeneous, then

$$\dot{\varphi}^{ab}(A) = \delta_{ab} \dot{\varphi}^{a}(\lambda),$$

for any diagonal matrix A.
Finally, if in addition the eigenvalues are simple, $\lambda_1 < \ldots < \lambda_n$, then we can write

$$\ddot{\varphi}^{ab,cd}(A) T_{ab} T_{cd} = \ddot{\varphi}^{ab}(\lambda) T_{aa} T_{bb} + 2 \sum_{a<b} \frac{\dot{\varphi}^{b}(\lambda) - \dot{\varphi}^{a}(\lambda)}{\lambda_b - \lambda_a} |T_{ab}|^2,$$

for every symmetric matrix T_{ab}.

Lemma 5.1.1. *Let Σ be a γ-translator. Then, we have the following equations at $p \in \Sigma$*

$$\Delta_\gamma h_{ij} + \ddot{\gamma}^{ab,cd} \nabla_i h_{ab} \nabla_j h_{cd} + |A|^2_\gamma h_{ij} + \langle \nabla h_{ij}, e_{n+1} \rangle = 0, \tag{5.1.2}$$

$$\Delta_\gamma \varphi + \left(\dot{\varphi}^{ij} \ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij} \ddot{\varphi}^{ab,cd} \right) \nabla_i h_{ab} \nabla_j h_{cd} + |A|^2_\gamma \varphi + \langle \nabla \varphi, e_{n+1} \rangle = 0, \tag{5.1.3}$$

$$\Delta_\gamma \gamma + \langle \nabla \gamma, e_{n+1} \rangle + |A|^2_\gamma \gamma = 0, \tag{5.1.4}$$

$$\Delta_\gamma u - |\nabla u|^2 + 1 = 0,\ u(p) = \langle p, e_{n+1} \rangle, \tag{5.1.5}$$

where $\Delta_\gamma = \dot{\gamma}^{ab} \nabla_a \nabla_b$, $\langle X, Y \rangle_\gamma = \dot{\gamma}^{ab} X_a Y_b$ *and* $|A|^2_\gamma = \dot{\gamma}^{ab} h_{ai} h_{ib}$.

Proof. The proof of these equations is very similar to that given in Lemma 3.3.3 for $\gamma = Q_k$. We only prove Equation (5.1.5) which has been written differently. In fact, substituting into Equation (3.3.20) with Q_k for γ and w for e_{n+1}, we have

$$\Delta_\gamma u = -\gamma \langle \nu, e_{n+1} \rangle = -\gamma^2,$$

here we are using a normal chart at $p \in \Sigma$ given by $\{e_1, \ldots, e_n\} \subset T_p\Sigma$ as an ortonormal base of eigenvalues of the second fundamnetal form of Σ at p.
On the other hand, we note that $\nabla_i u = \langle e_i, e_{n+1} \rangle$ which implies that $\nabla u = e_{n+1}^\top$, where $(\cdot)^\top$ denotes the tangential projection onto T_pM. Then, it follows that

$$\gamma^2 = \langle \nu, e_{n+1} \rangle^2 = 1 - \left| e_{n+1}^\top \right|^2 = 1 - |\nabla u|^2 .$$

Therefore, we finally obtain $\Delta_\gamma u - |\nabla u|^2 + 1 = 0$. □

The next lemma corresponds to a $D^2\gamma$ that we will use for the convex estimate.

Lemma 5.1.2. *Let Γ' be a symmetric closed cone compactly supported in $\Gamma \setminus Cyl_{n-1}$, where*

$$Cyl_j = \{\lambda(e_1 + \ldots + e_{n-j}) : \lambda > 0\} .$$

Then, there exist a constant $0 < C = C(n, \gamma, \Gamma')$ such that

$$C^{-1}\delta_{ab}\dot{\gamma}^a(\lambda) \leq \dot{\gamma}^{ab}(A) \leq C\delta_{ab}\dot{\gamma}^i(\lambda),$$
$$\sum_{i=1}^{n} \ddot{\gamma}^{ab,cd}(A)T_{iab}T_{icd} \leq -C\frac{|T|^2}{H},$$

where A is a diagonal matrix with eigenvalues $\lambda \in \Gamma'$ and T_{iab} is a totally symmetric tensor.

Proof. We refer the reader to Lemma 2.5 in [Lyn1]. □

Now we prove that there is no closed γ-translator in $\mathbb{R}^{n+1}$ such that its principal curvatures belong to Γ.

Proof of Proposition 5.0.1. Let us assume that Σ is a compact γ-translator without boundary in $\mathbb{R}^{n+1}$, and without loss of generality, we will also assume that $0 \in \Sigma$.
Then, by Equation (5.1.5), the height function $u(p) = \langle p, e_{n+1} \rangle$ satisfies

$$\Delta_\gamma u - |\nabla u|^2 + 1 = 0.$$

We note that property 2 implies that the operator Δ_γ is elliptic when the principal curvatures belong to Γ.
Consequently, by compactness and continuity, u must attains an interior minimum (recall that $\partial\Sigma = \emptyset$). However, this is impossible since $\Delta_\gamma u \geq 0$ and $\nabla u = 0$ at the point where the minimum of u is reached.
Therefore, there cannot exist any closed γ-translator such that its principal curvatures belong to Γ. □

To finalize this section, we use the above proposition to show that there is no totally umbilical γ-translator in $\mathbb{R}^{n+1}$ without boundary.

Proof of Corollary 5.0.2. Lets assume the opposite, which means there is a strictly convex totally umbilical γ-translator M without boundary in $\mathbb{R}^{n+1}$. It is a well know result (see for instance Ex. 8.6 in [dC]) that a totally umbilical hypersurface in $\mathbb{R}^{n+1}$ is piece of a n-hyperplane or a n-sphere. However, since the principal curvatures of M are positive, this implies that M is a piece of a n-sphere, in particular compact. This fact contradicts Proposition 5.0.1. □

5.2 Tangential Principle

The proof of Theorem 5.0.3 was inspired by the result in [Mø] for H-translators in $\mathbb{R}^{n+1}$.

Proof of Thoerem 5.0.3. Firstly, we mention that along this proof D_i, D_{ij} will denote the derivatives with respect to a local frame for each hypersurface.

Let $p \in \Sigma_1 \cap \Sigma_2$ be an interior point such that $T_p\Sigma_1 = T_p\Sigma_2$ and Σ_1 lies at one side of Σ_2. Then, after a rotation and a translation, there exists $r > 0$ such that each $M_i \cap B^{n+1}(p, r)$ is the graph of a smooth function $u_i : B^n_r(0) \subset \{x_{n+1} = 0\} \to \mathbb{R}$, where $p = (0, u_i(0))$. Therefore, since Σ_1 lies at one side of Σ_2, we may assume that $u_1 > u_2$ in $B^n_r(0) \setminus \{0\}$ and $u_1(0) = u_2(0)$.

Next, we consider a convex combination between u_1 and u_2 given by

$$u_s = (1 - s)u_1 + su_2, \text{ for } s \in [0, 1].$$

It is not hard to see that for each $s \in (0, 1)$, the graph of $u_s : B_r(0) \to \mathbb{R}$ is strictly convex. In fact, this holds since the convex combination of a positive definite matrix with a positive semi-definite matrix is positive definite. Here the involved matrices are the shape operators of the graphs of u_1 and u_2.

In particular, since for each $s \in (0, 1)$ the principal curvatures of the graph u_s lie in the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0\}$, it follows that u_s is an admissible family for the functional

$$E(s) = \gamma(h^i_j(s)) - \langle \nu(s), e_{n+1} \rangle .$$

Recall that the s dependence in $E(s)$ is related to the coefficients of the shape operator and the unit normal vector of u_s.

Furthermore, by the mean value theorem together with the fact that $E(1) = E(0) = 0$, it follows that

$$0 = E(1) - E(0) = \frac{\partial E}{\partial s}(s_0),$$

for some $s_0 \in (0,1)$.

On the other hand, we can explicitly calculate the term $\dfrac{\partial E}{\partial s}(s)$.
Indeed, by denoting $v = u_2 - u_1$ and $\dot{\gamma}^{\,ij}(s) = \dfrac{\partial \gamma}{\partial h^i_j}(h^i_j(s))$, we have

$$\begin{aligned}
\frac{\partial E}{\partial s}(s) =& \dot{\gamma}^{\,ij}(s)\frac{\partial h^i_j}{\partial s}(s) - \frac{\partial}{\partial s}\frac{1}{\sqrt{1+|Du_s|^2}} \\
=& \dot{\gamma}^{\,ij}(s)\frac{\partial}{\partial s}\left[\left(\frac{\delta_{ik}}{\sqrt{1+|Du_s|^2}} - \frac{D_i u_s D_k u_s}{(1+|Du_s|^2)^{3/2}}\right) D_{kj} u_s\right] + \frac{\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}} \\
=& \dot{\gamma}^{\,ij}(s)\left[\left(-\frac{\delta_{ik}\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}} - \frac{D_i v D_k u_s + D_k v D_i u_s}{(1+|Du_s|^2)^{\frac{3}{2}}} + 3\frac{D_i u_s D_k u_s \langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{5}{2}}}\right) D_{kj} u_s \right. \\
&\left. + \left(\frac{\delta_{ik}}{\sqrt{1+|Du_s|^2}} + \frac{D_i u_s D_k u_s}{(1+|Du_s|^2)^{\frac{3}{2}}}\right) D_{kj} v\right] + \frac{\langle Du_s, Dv\rangle}{(1+|Du_s|^2)^{\frac{3}{2}}}.
\end{aligned}$$

We claim that $v = 0$ in $B_r(0)$, and we will argue this by contradiction, i.e: we will assume that v is not a constant function such that $v(0) = 0$ and $v(x) < 0$ in $B^n_r(0) \setminus \{0\}$.

Firstly, we note that by continuity, v reaches a maximum in $\overline{B^n_r(0)}$.
Lets assume first that the maximum is reached at 0. Recall that v satisfies the linear elliptic PDE given by

$$\begin{aligned}
0 =& \dot{\gamma}^{\,ij}(s_0)\left(\frac{\delta_{ik}}{\sqrt{1+|Du_{s_0}|^2}} - \frac{D_i u_{s_0} D_k u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right) D_{kj} v \\
&+ \left\langle \left[\dot{\gamma}^{\,ij}(s_0)\left(-\frac{\delta_{ik} D_{kj} u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}} + 3\frac{D_i u_{s_0} D_k u_{s_0} D_{kj} u_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{5}{2}}}\right) + \frac{1}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right] Du_{s_0}, Dv \right\rangle \\
&- \dot{\gamma}^{\,ij}(s_0) D_{kj} u_{s_0} \frac{D_i u_{s_0} D_k v + D_k u_{s_0} D_i v}{(1+|Du_{s_0}|)^{\frac{3}{2}}}.
\end{aligned}$$

Then, since γ is a locally uniformly elliptic operator when the principal curvatures of the graph of u_{s_0} belong to Γ_+, it follows that the second order term in the above equation

$$\dot{\gamma}^{ij}(h^i_j(s_0))\left(\frac{\delta_{ik}}{\sqrt{1+|Du_{s_0}|^2}}-\frac{D_iu_{s_0}D_ku_{s_0}}{(1+|Du_{s_0}|^2)^{\frac{3}{2}}}\right)D_{kj}v$$

is uniformly elliptic in $B^n_r(0)$.

Consequently, the hypotheses of the strong maximum principle (Theorem 3.5 in [GT]) are satisfied, which means that v is constant in $B^n_r(0)$, giving a contradiction with our assumption.

On the other hand, let assume that the maximum of v is reached at some $x_0 \in \partial B^n_r(0)$. Then, since v cannot reach an interior maximum, it follows that $v(x_0) > 0$.

In particular, the hypotheses of Hopf's Lemma (Lemma 3.4 in [GT]) hold, which implies that $\frac{\partial v}{\partial N}(x_0) > 0$ where $N = \frac{x}{r}$ is the outward unit normal of $\partial B^n_r(0)$. This means that $\alpha(t) = v\left(\frac{x_0}{r}t\right)$ is an increasing function when t is close to r.

However, by continuity together with $v < 0$ in $B^n_r(0) \setminus \{0\}$, we can find a t_0 close to r such that $\alpha(t_0) = 0$. This contradicts the fact that v only vanishes at $x = 0$. This finishes the proof of the claim $v = 0$ at $B^n_r(0)$.

Therefore, we have that $\Sigma_1 \cap B^{n+1}(p,r) = \Sigma_2 \cap B^{n+1}(p,r)$. Finally, since both hypersurfaces are connected, we may apply the weak uniqueness continuation principle since each Σ_i is real analytic, to obtain that $\Sigma_1 = \Sigma_2$.

For the boundary tangency principle, we only need to change $B(p,r)$ with a hemisphere $B(p,r) \cap \{x_{n+1} \geq 0\}$, here we consider $\Pi = \{x_{n+1} = 0\}$.

Then, since the intersection $\Sigma_i \cap \Pi$ is transversal, the function $v = u_2 - u_1$ satisfies $\frac{\partial v}{\partial N}(p) = 0$ where $N = e_{n+1}$ is the normal unit vector of $\partial\Sigma_i$ at p.

On the other hand, the hypothesis of the Hopf's Lemma (Lemma 3.4 in [GT]) holds, $\frac{\partial v}{\partial N}(p) > 0$ which is a contradiction. Therefore, the same argument holds in that case to conclude the result. □

Remark 5.2.1. Note that without the hypothesis of γ being real analytic in Γ_+, we only would obtain that Σ_1 agrees with Σ_2 locally at p.

Remark 5.2.2. A possible generalization of Theorem 5.0.3 could be made by showing an analogous result related to the preservation of the local uniform ellipticity of γ when the principal curvatures of the graph of u_s belongs to other cones instead of Γ_+.

5.3 Non-existence result

In this section we prove a non-existence for γ-translators in $\mathbb{R}^{n+1}$ which are convex and entire graphs.

Proof of Corollary 5.0.4. We prove this by contradiction. Assume that there is an entire γ-translator Σ given by a function $u : \mathbb{R}^n \to \mathbb{R}$ such that its principal curvatures belong to the cone $\{\lambda \in \mathbb{R}^n : \lambda_i \geq 0\}$. This last property is equivalent to Σ being convex.

Let C_n be the "bowl"-type γ-translator in $\mathbb{R}^{n+1}$. Recall that C_n is a strictly convex rotationally symmetric graph defined in $B(0, r_0)$ which is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r_0) \times \mathbb{R}$ (see the Appendix 4.8 in Chapter 4 for a proof of this fact), where $r_0 = \gamma(1, \ldots, 1)$.

Then, translating suitably C_n over Σ, we can find a $t_0 > 0$ such that $C_n + te_{n+1}$ lies strictly above from Σ for $t \geq t_0$. Note that this can be done since C_n is not an entire graph. Now, we may translate $C_n + t\varepsilon_{n+1}$ downward until it touches Σ for the first time.
Finally, by the interior tangential principle Theorem 5.0.3, we obtain that $\Sigma = C_n$ which contradicts that Σ is entire. □

5.4 Uniqueness

In this section we proof the uniqueness result Theorem 5.0.5 and Corollary

Firstly, since the shape operator of a given hypersurface in $\mathbb{R}^{n+1}$ is invariant under isometries of $\mathbb{R}^{n+1}$, it follows that $\gamma(\tilde{\lambda}) = \gamma(\lambda)$ where $\tilde{\Sigma}$ is the image of any isometry applied to a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$.
In particular, if R is a rotational field which fixes the x_{n+1}-axis in $\mathbb{R}^{n+1}$, then

$$\langle \tilde{\nu}, e_{n+1} \rangle = \left\langle \nu, R^{-1} e_{n+1} \right\rangle = \langle \nu, e_{n+1} \rangle = \gamma(\lambda) = \gamma(\tilde{\lambda}).$$

This means that γ-translators remain as γ-translator after applying roational fields which fix the direction of translation under the γ-flow.

Consequently, under the hypothesis of Theorem 5.0.4, it is enough to show that Σ is symmetric along the plane $\{x_1 = 0\}$ to obtain that Σ is rotationally symmetric.
For this purpose we will adopt the following notations and definition used in [**?**] for the case when $\gamma = H$.
Let $A \subset \mathbb{R}^{n+1}$ and $t \in \mathbb{R}$, then we set:

- 1-parameter family of vertical hyperplanes $\Pi_t = \{x \in \mathbb{R}^{n+1} : \mathbf{p}(x) = t\}$, where

$$\mathbf{p}(x_1, \ldots, x_{n+1}) = x_1.$$

  In addition, we denote $\Pi = \Pi_0$.

- 1-parameter family of horizontal hyperhalfplanes $Z_t = \{x_{n+1} > t\}$.

- 1-parameter families of subsets of A given by

$$A_+(t) = \{x \in A : \mathbf{p}(x) \geq t\},$$
$$A_-(t) = \{x \in A : \mathbf{p}(x) \leq t\},$$
$$\delta_t(A) = A \cap \Pi_t.$$

  Note that $A_+(t)$ and $A_-(t)$ are the right hand side and the left hand side, respectively, along Π_t of A (see Definition 5.4.1 given below).

- 1-parameter families of right and left reflections of A, respectively, along the hyerplane Π_t given by

$$A^*_+(t) = \left\{(2t - x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \in A_+(t)\right\},$$
$$A^*_-(t) = \left\{(2t - x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_{n+1}) \in A_-(t)\right\},$$

  respectively.

- The orthogonal projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to the hyperplane Π given by

$$\pi(x_1, \ldots, x_{n+1}) = (0, x_2, \ldots, x_{n+1}).$$

Definition 5.4.1. Let A, B be two set of $\mathbb{R}^{n+1}$. We say that $B \leq A$ and it is read as "A is on the right side of B" if, and only if, for every $x \in \Pi$ such that

$$\pi^{-1}(\{x\}) \cap A \neq \emptyset \text{ and } \pi^{-1}(\{x\}) \cap B \neq \emptyset,$$

we have that

$$\sup\left\{\mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap B\right\} \leq \inf\left\{\mathbf{p}(p) : p \in \pi^{-1}(\{x\}) \cap A\right\}. \tag{5.4.1}$$

Note that for arbitrary sets, the relation $B \leq A$ is not a partial order, but for sets given by the graph of a function over the plane Π works as a partial order.

Remark 5.4.1. The method of moving planes requires specifically two things to be applied:

The first is that the hyperplanes (or any other hypersurface to take reflections along) to be solutions to Equation (2.2.1). To accomplish this, we add the hypothesis **??** on γ, which means that vertical hyperplanes (whose unit normal vector is different from $\pm e_{n+1}$) are γ-translators.

On the other hand, we need tangential interior and boundary principles, respectively, to decide what happens when the hypersurface intersects the reflection of the same hypersurface along a moving plane. In our case Theorem 5.0.3 gives us these results.
Recall that vertical hyperplanes are real analytic convex hypersurfaces, therefore the tangential principles will work this procedure.

Proof of Theorem 5.0.5. We consider the set

$$\mathcal{A} := \left\{t \in [0, t_0) : \Sigma_+(t) \text{ is a graph over } \Pi \text{ and } \Sigma_-(t) \leq \Sigma_+^*(t)\right\},$$

where $t_0 := \sup\{t > 0 : \Sigma \cap \Pi(t) \neq \emptyset\}$.

The set $\mathcal{A}$ will act as the index of the family of moving hyperplanes in the application of the moving plane method. For this reason, we want to show that $\mathcal{A}$ is the interval $[0, t_0)$. In fact, this will give us that $\Sigma_-(0) \leq \Sigma_+^*(0)$, and by analogous arguments we will obtain that $\Sigma_-^*(0) \leq \Sigma_+(0)$. We note that the combination of these two properties imply Σ is symmetric along the hyperplane Π.

Therefore, in the following claims we will show that $\mathcal{A} = [0, t_0)$ by arguing that $\mathcal{A}$ is not empty, open and closed set of $[0, t_0)$.

Claim 5.4.1. *The value t_0 lies in $(0, r]$.*

Proof. Let $p_0 = \Sigma \cap \Pi_{t_0}$ be a first order contact point[6] between Σ and a vertical hyperplane Π_t coming from $+\infty$. Recall that Σ is a complete graph defined over $B_r^n(0) \subset \mathbb{R}^n$ with a single end $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$.

Then, p_0 cannot be an interior point, since otherwise Σ would coincide with the plane Π_{t_0} by the interior tangential principle Theorem 5.0.3, which contradicts that Σ is a strictly convex graph.

Moreover, p_0 cannot be a boundary point, because the tangent boundary principle Theorem 5.0.3 would imply that Σ coincides with Π_{t_0}, obtaining the same contradiction as the previous case.

Therefore, p_0 must be a point at "infinity" i.e: $p_0 \in \Sigma \setminus B_R^{n+1}(0)$ for some $R >> 1$. Then, since Σ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, it follows that $t_0 \leq r$. □

Claim 5.4.2. *The set $\mathcal{A} \neq \emptyset$. In addition, for every $s \in \mathcal{A}$, $[s, t_0) \subset \mathcal{A}$.*

Proof. We will show in the following steps that there exist $\varepsilon \in (0, t_0)$, such that $(t_0 - \varepsilon, t_0) \subset \mathcal{A}$, giving that $\mathcal{A} \neq \emptyset$.

Step 5.4.1. *$\Sigma_+(t)$ is connected for every $t \in [0, t_0)$.*

Proof. Firstly, lets recall that Σ has a single end. This means that $\Sigma_+(t)$ has only one unbounded component for every $t \in [0, t_0)$.

Because otherwise, we can choose a compact Σ' component of $\Sigma_+(t)$. Then, we can translate a hyperplane Π_t until it touches Σ' at a first order contact point. Note that the tangential principles Theorem 5.0.3 imply that Σ' is flat, which contradicts that Σ is strictly convex

[6] This means that both hypersurfaces intersect at a point where the first derivatives coincide. In particular, the tangent planes coincides because the unit normal vectors of each hypersurface concide at this point.

graph.

Therefore, $\Sigma_+(t)$ is connected for every $t \in [0, t_0)$. $\square$

Step 5.4.2. *There exist $\varepsilon_0 \in (0, t_0)$ such that $\Sigma_+(t)$ is a graph over Π for $t \in (t_0 - \varepsilon_0, t_0)$.*

Proof. Firstly, we will show that there exist $\varepsilon_0 \in (0, t_0)$ such that

$$\langle \nu, e_1 \rangle = \frac{D_1 u}{\sqrt{1 + |Du|^2}} > 0$$

holds at $\Sigma_+(t)$ for all $t \in (t_0 - \varepsilon_0, t_0)$. Recall that Σ is a graph of the function $u : B_r^n(0) \subset \mathbb{R}^n \to \mathbb{R}$.

For this purpose let assume the contrary. This means that there exist an increasing sequence t_l converging to t_0 such that $D_1 u(x_1^l, \ldots, x_n^l) = 0$ for some $p_l = (x_1^l, \ldots, x_n^l, u(x_1^l, \ldots, x_n^l)) \in \Sigma_+(t_l)$.

Then, since $t_l \leq x_1^l < t_0$, it follows that $|p_l| \to \infty$. Recall that the end of Σ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^n(r) \times \mathbb{R}$, this means that $\nu(p_l) \to (1, 0 \ldots, 0)$ as $l \to \infty$. Note that this contradicts that $D_1 u(x_1^l, \ldots, x_n^l) = 0$ (see Fig 5.1).

Consequently, there exist $\varepsilon_0 \in (0, t_0)$ such that $\langle \nu, e_1 \rangle > 0$ holds at $\Sigma_+(t)$ for all $t \in (t_0 - \varepsilon, t_0]$. In particular, this fact together with Σ being a complete graph defined in $B_r^n(0)$[7] and $\Sigma_+(t)$ being connected, imply that $\Sigma_+(t)$ is a graph over Π for all $t \in (t_0 - \varepsilon_0, t_0]$. $\square$

Step 5.4.3. *Let $\varepsilon = \dfrac{\varepsilon_0}{2}$, then $\Sigma_-(t) \leq \Sigma_+^*(t)$ for every $t \in [t_0 - \varepsilon, t_0)$. Finalizing the first part of the proof of Claim5.4.2.*

Proof. Firstly, since $\Sigma_+(t)$ is a graph over Π for $t \in [t_0 - \varepsilon, t_0)$, the relation "being at the right side" (see Definition 5.4.1) is a partial order.

Consequently, it suffices to show that for $t_1 = t_0 - \dfrac{\varepsilon_0}{2}$, the condition $\Sigma_-(t_1) \leq \Sigma_+^*(t_1)$ (i.e:

[7] This means that the Gauss map $\nu : \Sigma \to \mathbb{S}^n$ is a proper local diffeomorphism which maps the boundary to the boundary of the image.

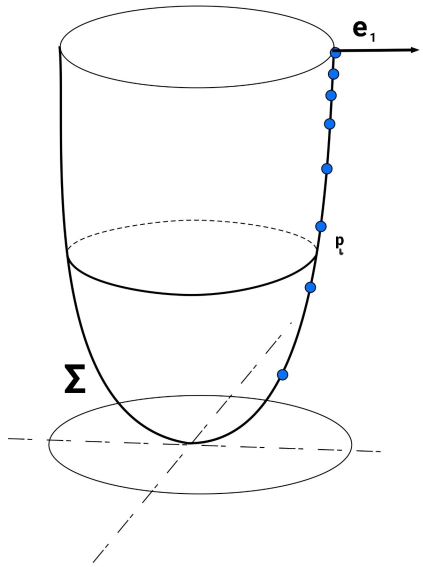


Figure 5.1: Sequence $\nu(p_l) \to e_1$ as $l \to \infty$ where $p_l \in \Sigma_+(t_l)$. Image courtesy of Francisco Martín.

the reflection of $\Sigma_+(t_1)$ along Π_{t_1} is at the right hand side of $\Sigma_-(t_1)$) is satisfied. Note that for this we are using that $\Sigma_-(t_1) \leq \Sigma_-(t)$ and $\Sigma_+^*(t) \leq \Sigma_+^*(t_1)$ for $t_1 < t$.

Next, if $\Sigma_-(t_1) \not\lesssim \Sigma_+^*(t_1)$, then we may find a first order contact point p_1 between $\Sigma_-(t_1)$ and $\Sigma_+^*(t_1)$. We note that $\mathbf{p}(p_1) \notin [t_0 - \varepsilon_0, t_1]$ since $\Sigma_+(t)$ is a graph over Π for $t \in [t_0 - \varepsilon, t_0)$, and this would imply that $\Sigma_+(\mathbf{p}(p_1))$ is not a graph over Π.

Therefore, $\mathbf{p}(p_1) < t_0 - \varepsilon_0$, and without loss of generality we may assume that $\mathbf{p}(p_1) \neq 0$, since otherwise Σ will be symmetric along Π finishing the proof of the Theorem 5.0.4. Then, the interior and boundary tangential principals Theorem 5.0.3 would give that $\Sigma_+^*(\mathbf{p}(p_1)) = \Sigma_-(\mathbf{p}(p_1))$, which means that Σ is symmetric along $\Pi_{\mathbf{p}(p_1)}$ with $\mathbf{p}(p_1) \neq 0$. This contradicts that the end of Σ is symmetric along the hyperplane Π. □

Step 5.4.4. *For every $s \in \mathcal{A}$, $[s, t_0) \subset \mathcal{A}$.*

Proof. The proof is straight forward from the definition of $\mathcal{A}$ together with the asymptotic behavior of the end of Σ. □

□

Claim 5.4.3. *The set $\mathcal{A}$ is closed in $[0, t_0)$.*

Proof. Let $s_k \in \mathcal{A}$ be a sequence of points such that $s_k \to s_0$. In the following two steps we are going to show by contradictions that $s_0 \in \mathcal{A}$.

Step 5.4.5. *Assume that $\Sigma_+(s_0)$ is not a graph over Π and $\Sigma_+^*(s_0) \leq \Sigma_-(s_0)$.*

Proof. The assumption of this step implies that we can find $p, q \in \Sigma_+(s_0)$ such that $p = (p_1, p_2, \ldots, p_{n+1})$ and $q = (q_1, p_2, \ldots, p_{n+1})$ with $p_1 < q_1$.

Then, by Claim 5.4.2, we have that $s > s_0$ for every $s \in \mathcal{A}$. In particular, it follows that follows that $p_1 = s_0$. Because otherwise $q \in \Sigma_+(p)$, and $\Sigma_+(p)$ would be a graph over Π which does not agree with our assumption.

Moreover, we note that

$$q_1 > s = \frac{q_1 + 3s_0}{4} > s_0 = p_1.$$

Then, since $q \in \Sigma_+(s)$, $p \in \Sigma_-(s)$ and

$$2s - q_1 = \frac{3s_0 - q_1}{2} = s_0 + \frac{s_0 - q_1}{2} < p_1,$$

it follows that $\Sigma_-(s) \not\leq \Sigma_+^*(s)$, which contradicts that $s \in \mathcal{A}$. □

Step 5.4.6. *Assume that $\Sigma_+(s_0)$ is a graph over Π and $\Sigma_-(s_0) \not\leq \Sigma_+^*(s_0)$.*

Proof. Firstly, since $\Sigma_-(s_0) \not\leq \Sigma_+^*(s_0)$, we may find a first order contact point in the intersection of these sets which is not in the hyperplane Π_{s_0}, i.e.:

$$p_0 \in (\Sigma_+^*(s_0) \cap \Sigma_-(s_0)) - \delta_{s_0}(\Sigma).$$

In particular, $\mathbf{p}(p_0) = 2s_0 - x_1$ for some $(x_1, \ldots, x_{n+1}) = q_0 \in \Sigma_+(s_0)$.

Moreover, since $\Sigma_+(s_0)$ is a graph over Π, we may write

$$x_1 = s_0 + f_{s_0}(0, x_2, \ldots, x_{n+1})$$

for some positive continuous function $f_{s_0} : \Pi \to \mathbb{R}$. The reason why f_{s_0} is positive is because $p_0 \notin \delta_{s_0}(\Sigma)$.

Next, we choose $x_1^k = s_k + f_{s_0}(0, x_2, \ldots, x_{n+1})$. Then, $q_k := (x_1^k, x_2, \ldots, x_{n+1}) \in \Sigma_+(s_k)$ and

$$2s_k - x_1^k = s_k - f_{s_0}(0, x_2, \ldots, x_{n+1}).$$

This means that $p_k := (2s_k - x_1^k, x_2, \ldots, x_{n+1}) \in \Sigma_+^*(s_k) \cap \Sigma_-(s_k)$.

Then, since $\Sigma_-(s_k) \leq \Sigma_+^*(s_k)$, it follows that $2s_k - x_1^k = s_k$ (see a proof in (5.4.4) below). In particular, $f_{s_0}(0, x_2, \ldots, x_{n+1}) = 0$, which means that $\mathbf{p}(p_0) = s_0 = x_1$, or equivalently, $p_0 \in \delta_{s_0}(\Sigma)$ given the desire contradiction. $\square$

Therefore, we have proven that $\Sigma_+(s_0)$ is a graph over Π and $\Sigma_+^*(s_0)$ is at the right hand side of $\Sigma_-(s_0)$, finalizing the proof of this claim. $\square$

Claim 5.4.4. *The set $\mathcal{A}$ is open in $[0, t_0)$.*

Proof. Let $t_1 \in \mathcal{A}$, we want to show that there exist $\varepsilon > 0$ such that $(t_1 - \varepsilon, t_1] \subset \mathcal{A}$. Then, by the Claim 5.4.2, the set $(t_1 - \varepsilon, t_1 + \varepsilon)$ will be an open neighborhood of t_1 contained in $\mathcal{A}$, or equivalently, $\mathcal{A}$ is an open subset of $[0, t_0)$.

We divide this proof in the following two steps.

Step 5.4.7. *There exist $\varepsilon_2 \in (0, t_1)$ such that $\Sigma_+(t_1 - \varepsilon_2)$ is a graph over Π.*

Proof. Since $t_1 \in \mathcal{A}$, it follows that $\Sigma_+(t_1)$ is a graph over Π. Let $h \gg 1$ such that $\Sigma_+(t_1) \cap Z_h$ is close to the cylinder $\mathbb{S}^n(r) \times \mathbb{R}$. Recall that $Z_h = \{x_{n+1} > h\}$.

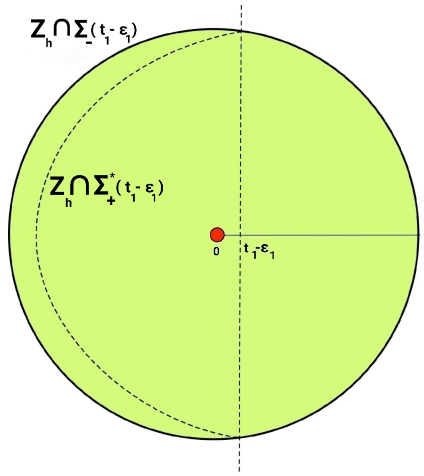


Figure 5.2: View from profile $x_{n+1} = h \gg 1$ with $\Sigma_+^*(t_1 - \varepsilon_1)$ is drawn in dashed lines. Image courtesy of Francisco Martín.

Then, if there is a point in $\Sigma_+(t_1) \cap Z_h$ such that the normal vector is included in Π, say it first coordinate is $\tilde{s} \in [t_1, t_0)$, we would have by the boundary tangential principle that $\Sigma_+^*(\tilde{s}) = \Sigma_-(\tilde{s})$ (see Fig 5.2.). This means that $\Pi_{\tilde{s}}$ is a hyperplane of symmetry of Σ, but this contradicts that the axis of symmetry of $\Sigma_+(s_1) \cap Z_h$ is the x_{n+1}-axis.

In particular, we have

$$\nu(\Sigma_+(t_1)) \cap \Pi = \emptyset.$$

Consequently, there exist an $\varepsilon_0 \in (0, t_1]$ such that

$$\nu(\Sigma_+(t)) \cap \Pi = \emptyset, \text{ holds for all } t \in (t_1 - \varepsilon_0, t_1).$$

Then, by combining this with the fact that Σ is embedded and $\Sigma_+(t)$ is connected for all $t \in [0, t_0)$, we obtain that $\Sigma_+(t) \cap Z_h$ can be written as a graph over Π for all $t \in (t_1 - \varepsilon_0, t_1)$. Furthermore, by taking $\varepsilon_1 = \frac{\varepsilon_0}{2}$ as we did in Claim 5.4.2, we will get that

$$\Sigma_-(t_1 - \varepsilon_1) \cap Z_h \leq \Sigma_+^*(t_1 - \varepsilon_1) \cap Z_h. \tag{5.4.2}$$

Next, we consider the compact set

$$K := \Sigma \cap \{x_{n+1} \leq h\}\,.$$

We note that

$$\langle \nu, e_1 \rangle = \frac{D_1 u}{\sqrt{1+|Du|^2}} > 0$$

holds in $K_+(t_1)$ since is a graph over Π.

Then, by compactness there exist a $\varepsilon_2 \in (0, \varepsilon_1)$ such that $D_1 u > 0$ holds in $K_+(t)$ for all $(t_1 - \varepsilon_2, t_1]$. Therefore, by combining this with the fact that Σ is embedded and $\Sigma_+(t)$ is connected, it follows that $K_+(t)$ can be written as a graph over Π for all $(t_1 - \varepsilon_2, t_1]$.

In summary, we have shown that there exist $\varepsilon_2 \in (0, t_1)$ such that $\Sigma_+(t)$ can be written as a graph over Π for all $t \in (t_1 - \varepsilon_2, t_1]$. □

Step 5.4.8. *There exist $\varepsilon_4 \in (0, \varepsilon_2)$, such that $\Sigma_-(t_1 - \varepsilon_4) \leq \Sigma_+^*(t_1 - \varepsilon_4)$.*

Proof. Firstly, since $\Sigma_+(t)$ is a graph over Π for all $t \in (t_1 - \varepsilon_2, t_1]$, we may find a $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\big(\Sigma_+^*(t) \cap \Sigma_-(t) \cap K\big) - \delta_t(\Sigma) \subset K_-(t - \varepsilon_3). \tag{5.4.3}$$

In fact, to see this we note that if

$$x = (x_1, \ldots, x_{n+1}) \in \Sigma_+^*(t) \cap \Sigma_-(t) \cap K \cap \delta_t(\Sigma)^c,$$

we have that $x_1 < t$ and $x_1 = 2t - y_1$ where $y_1 = f(0, x_2, \ldots, x_{n+1}) + t$ for some continuous function $f > 0$ defined over a compact set $\Omega_f \subset \Pi$.

Therefore, by compactness, we have that $y_1 - t \geq \varepsilon_3$ where

$$\varepsilon_3 = \min\left\{\varepsilon_2, \min_{\Omega_f} f\right\} > 0.$$

This means that for all $t \in (t_1 - \varepsilon_3, t_1]$ it follows that

$$t - x_1 = y_1 - t \geq \varepsilon_3 \Leftrightarrow x_1 \leq t - \varepsilon_3.$$

Moreover, since $\Sigma_-(t_1) \leq \Sigma_+^*(t_1)$, it follows that

$$\Sigma_+^*(t_1) \cap \Sigma_-(t_1) = \delta_{t_1}(\Sigma). \tag{5.4.4}$$

Because otherwise we could find a first order contact point

$$p_0 \in (\Sigma_+^*(t_1) \cap \Sigma_-(t_1)) - \delta_{t_1}(\Sigma).$$

Then, if p_0 is an interior point, the interior tangential principle Theorem 5.0.3 would implies that $\Pi_{\mathbf{p}(p_0)}$ is a hyperplane of symmetry of Σ, since $\Sigma_-(\mathbf{p}(p_0)) = \Sigma_+^*(\mathbf{p}(p_0))$.

Recall that the end of Σ is $\mathcal{C}^2$-asymptotic to the cylinder $\mathbb{S}^{n-1}(r) \times \mathbb{R}$, this means that $\mathbf{p}(p_0) = 0$ and we do not have anything to prove. On the other hand, if $\mathbf{p}(p_0) \neq 0$, then we would get a contradiction.

Moreover, if p_0 is boundary point the same conclusion will holds by using instead the boundary tangential principle Theorem 5.0.3. Consequently, or Σ is symmetric along the hyperplane Π or $\Sigma_-(t_1) \cap \Sigma_+^*(t_1) = \delta_{t_1}(\Sigma)$.

Next, we will show that there exist $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\Sigma_+^*(t) \cap \Sigma_-(t) \cap K = \delta_t(\Sigma) \cap K.$$

holds for all $t \in (t_1 - \varepsilon_4, t_1]$.

Indeed, if it not the case, we may find an increasing sequence t_l converging to t_1 such that

$$(\Sigma_+^*(t_l) \cap \Sigma_-(t_l) \cap K) - \delta_t(\Sigma) \neq \emptyset.$$

Let $(x_1^l, \ldots, , x_{n+1}^l) = p_l \in (\Sigma_+^*(t_l) \cap \Sigma_-(t_l) \cap K) - \delta_t(\Sigma)$. We note that by Equation (5.4.3) we have

$$x_1^l \leq t_l - \varepsilon_3. \tag{5.4.5}$$

Then, by compactness, we may assume, after taking a subsequence if it is necessary, that $p_l \to \tilde{p} = (\tilde{x}_1, \ldots, \tilde{x}_{n+1}) \in \Sigma_+^*(t_1) \cap \Sigma_-(t_1) \cap K$.
In particular, Equation(5.4.4) gives that $\tilde{x}_1 = t_1$. But, after taking limits on Equation (5.4.5), we see that $\tilde{x}_1 \leq t_1 - \varepsilon_3$, given a contradiction with $\tilde{x}_1 = t_1$.

Therefore, there exist $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\Sigma_+^*(t) \cap \Sigma_-(t) \cap K = \delta_t(\Sigma) \cap K$$

holds for all $t \in (t_1 - \varepsilon_4, t_1]$. This means that

$$\Sigma_-(t) \cap K \leq \Sigma_+^*(t) \cap K \text{ holds for all } t \in (t_1 - \varepsilon_3, t_1]$$

Thus, combining the above line with Equation (5.4.2), we finally obtain

$$\Sigma_-(t) \leq \Sigma_+^*(t) \text{ holds for all } t \in (t_1 - \varepsilon_4, t_1]$$

□

Therefore, by the Steps 5.4.7 and 5.4.8, it follows that $(t_1-\varepsilon_4, t_1] \subset \mathcal{A}$, and as we mention in the beginning of the proof of this claim, the set $\mathcal{A}$ is an open subset of $[0, t_0)$. $\square$

The proof finish by noting that $\mathcal{A}$ is a non empty open and closed subset of the connected set $[0, t_0)$. Therefore, $\mathcal{A} = [0, t_0)$. In particular, $\Sigma_-(0) \leq \Sigma_+^*(0)$.
An analogous argument will show that $\Sigma_-^*(0) \leq \Sigma_+(0)$. Therefore, Σ is symmetric with respect the hyperplane Π, finalizing the proof of Theorem 5.0.4. $\square$

Proof of Corllary 5.0.6 . Theorem 5.0.4 implies that Σ is a rotationally symmetric graph defined over the ball of radio r. Then, by the Theorems 1.3-1.4 in [Ren], it follows that $r = \sqrt[\alpha]{\gamma(1,\ldots,1)}$ and Σ is the "bowl"-type γ-translator in $\mathbb{R}^{n+1}$. $\square$

5.5 Convexity Estimate

In this section we prove the convexity estimate for γ-translators from Theorem 5.0.7 and, as a consequence, that these γ-translator are asymptotically convex.

Before giving the proof of Theorem 5.0.7, we describe the main tools used:

- Consider a good approximation for estimating $\lambda_1(p) = \{\lambda_i(p) : i = 1, \ldots, n\}$ using a maximum principle argument. For instance, in [SS] the authors used a δ-aproximation for the minimum between two elements of the form

$$f_\delta(x, y) = \frac{x+y}{2} - \sqrt{\frac{(x-y)^2}{4} + \delta xy},\ \delta \in \left(0, \frac{1}{2}\right).$$

  This δ-approximation is a symmetric 1-homogeneous concave function whenever $x+y > 0$, and satisfies nice properties when $\min(x, y) < 0$. In addition, the authors improved this δ-approximation to show that 2-convex and mean convex translators in $\mathbb{R}^{n+1}$ are convex. Nevertheless, the 2-convex hypothesis cannot be removed since it is necessary in the construction of the δ-approximation for the minimum in $n \geq 3$ variables.

- From the previous point, the approximating function will satisfy a maximum principle for which the minimum cannot be negative at an interior point of the γ-translator. A similar argument will be needed to study this behavior at infinity. For this purpose we need to consider an extra hypothesis to ensure a uniformly bounded second fundamental form of the γ-translator when we take limits of sequences of translations of the hypersurface.

  In particular, the 2-convexity for $n \geq 3$ gives a natural bound for mean convex H-translators. Indeed, we have that the only possible negative principal curvature is the minimum and

$$\begin{aligned} S_2(\lambda_1, \lambda_2, \lambda_3) &= \lambda_3\lambda_2 + \lambda_3(\lambda_1 + \lambda_2) > 0, \\ S_2(\lambda_1, \ldots, \lambda_n) &= S_{2;1} + \lambda_1 S_{1;1} \\ &= S_{2;1,2} + (\lambda_2 + \lambda_1) S_{1;1,2} + \lambda_1\lambda_2 > 0, \end{aligned}$$

  here we have arranged $\lambda_1 \leq \ldots \leq \lambda_n$. Note that for $n \geq 4$, it follows that $S_{2;1,2} \geq \lambda_1\lambda_2$. Consequently, for a H-translator follows

$$|A|^2 = H^2 - 2S_2 \leq H^2 = \langle \nu, e_{n+1} \rangle^2 \leq 1.$$

  On the other hand, when $\gamma \neq H$, other hypotheses are needed to ensure an uniform bound for $|A|$. In fact, if we restrict the domain of γ to the cone

$$\Gamma_{\alpha,\delta} = \left\{ \lambda \in \Gamma : H \leq \frac{\alpha}{\delta + 1}\gamma \right\},$$

  then a uniform bound follows by combining this with the 2-convexity of the γ-translator.

In fact, by these restrictions we obtain

$$|A|^2 \leq H^2 \leq \gamma^2 \left(\frac{\alpha}{\delta+1}\right)^2 \leq \left(\frac{\alpha}{\delta+1}\right)^2. \tag{5.5.1}$$

Moreover, the cone $\Gamma_{\alpha,\delta}$ is preserved under the γ-flow.
Indeed, the evolution equation of $\dfrac{H}{\gamma}$ is given by

$$(\partial_t - \Delta_\gamma)\frac{H}{\gamma} = \frac{g^{ij}}{\gamma}\frac{\partial^2\gamma}{\partial h_{ab} h_{cd}}\nabla_i h_{ab}\nabla_j h_{ab} + \frac{2}{\gamma}\left\langle \nabla\frac{H}{\gamma}, \nabla\gamma\right\rangle_\gamma, \tag{5.5.2}$$

where $\Delta_\gamma = \frac{\partial\gamma}{\partial h_{ij}}\nabla_i\nabla_j$ and $\langle X, Y\rangle = \frac{\partial\gamma}{\partial h_{ij}}X_i Y_j$. Then, an easy application of the maximum principle together with the fact that the second order term in (5.5.2) is non-positive, it follows that $\dfrac{H}{\gamma}$ remains bounded under the γ-flow.

- Thirdly, a uniform bound on the second fundamental form implies the existence of sub sequential limits of a sequence of hypersurfaces. Then, by considering a sequence of points such that $\inf \lambda_1$ is reached, we can translate the γ-translator with respect this sequence and obtain a sub-sequential limit which is a γ-translator where an Omori-Yau maximum principles applies at the limit hypersurface. Thus, the end of the argument is to assemble the internal maximum principle on the limiting hypersurface to ensure a contradiction and to obtain the estimate.

- Finally, if $\Gamma_{\alpha,\delta} \neq \emptyset$, then is a convex closed subset of Γ, which is compactly supported in Γ, i.e: the set

$$\overline{\Gamma}_{\alpha,\delta} \cap \partial B(0,1)$$

is compact in Γ. This fact implies an uniform estimate on the second order derivatives of γ, see Lemma 5.1.2.

Remark 5.5.1. It is still an open question which α and δ are optimal. For instance, one can restrict to

$$\frac{\alpha}{\delta+1} = \frac{H}{\gamma}\Big|_{\mathrm{Cyl}_j}$$

for some $\mathrm{Cyl}_j \subset \Gamma$.

On the other hand, if there is a Newton-Maclaurin Inequity (see Lemma 3.1.3) related to $\gamma : \Gamma \to (0,\infty)$, we could obtain that a closed γ-translator is totally umbilical a contradiction with Corollary 5.0.2.

Proof of Theorem 5.0.7. We consider on Σ the function

$$f(p) = \frac{\gamma(\lambda)}{S_{1,1}(\lambda)},$$

where $S_{1,1} = H - \lambda_1$ and $\lambda_1(p) = \min_{i=1,\dots,n}\{\lambda_i(p)\}$.

Therefore, the convexity estimate of Theorem 5.0.7 can be written by

$$f(p) \geq \frac{1}{\alpha}, \text{ for every } p \in \Sigma. \tag{5.5.3}$$

The proof will follows by contradiction, this means that we are going to assume

$$\inf_{\Sigma} f < \alpha^{-1}.$$

In particular, this assumption together with $\lambda \in \Gamma_{\alpha,\delta} = \{\lambda \in \Gamma : (\delta+1)H \leq \alpha\gamma\}$ gives an estimate of the form

$$\lambda_1 < H - \alpha\gamma \leq -\delta H, \tag{5.5.4}$$

for $p_N \in \Sigma$ such that $f(p_N) \to \inf_{\Sigma} f$. This is a key fact in our proof, since it permits to control some curvature terms in the equations.

Next, we calculate some equations related to f. For doing this, we use an orthonormal frame of principal directions $\{e_i\} \subset T_p\Sigma$ of Σ at p.

Then, by the equations in Lemma 5.1.1, it follows that

$$\nabla_a f = \frac{S_{1,1}\nabla_a \gamma - \gamma \nabla_a S_{1,1}}{(S_{1,1})^2}, \tag{5.5.5}$$

$$\Delta_\gamma f = \frac{f}{S_{1,1}} \left(\dot{S}_{1,1}^{ij} \ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij} \ddot{S}_{1,1}^{ab,cd} \right) \nabla_i h_{ab} \nabla_j h_{cd} - \nabla_{n+1} f - 2 \left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}} \right\rangle_\gamma . \tag{5.5.6}$$

We denote $\nabla_{n+1} f = \langle \nabla f, e_{n+1} \rangle$.

We divide the proof into several claims.

Claim 5.5.1. *Assume that there exist $p_0 \in \Sigma$, such that $f(p_0) = \inf_\Sigma f$. Then, $|A|_\gamma^2 \gamma = 0$ in Σ. In consequence, f cannot achieve a minimum in an interior point of Σ.*

Proof. Recall that by (5.5.4), $\lambda_1(p_0) < 0$.

Then, by rewriting Equation (5.5.6), we see that

$$\Delta_\gamma f + \nabla_{n+1} f + 2 \left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}} \right\rangle_\gamma = \frac{f}{S_{1,1}} \left(\dot{S}_{1,1}^{ij} \ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij} \ddot{S}_{1,1}^{ab,cd} \right) \nabla_i h_{ab} \nabla_j h_{cd} \leq 0. \tag{5.5.7}$$

In the right hand side of the previous equation we used the concavity of γ, the convexity of $S_{1,1}$, and the fact that $S_{1,1}$ and γ are increasing in each variable.

Furthermore, the left hand side in Equation (5.5.7) is locally uniformly elliptic as $\lambda \in \Gamma_{\alpha,\delta}$. Therefore, since $f(p_0)$ is a minimum, the strong maximum principle (see for instance Theorem 4.2 in [GP]) implies that the function $f(p)$ is constant in Σ.

Consequently, the right hand side of (5.5.7) satisfies

$$0=\left(\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd}-\dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\right)\nabla_i h_{ab}\nabla_j h_{cd}\Leftrightarrow \dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd}\nabla_i h_{ab}\nabla_j h_{cd}=\dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\nabla_i h_{ab}\nabla_j h_{cd}.$$

In particular, since $S_{1,1}$ is convex and γ is concave, it follows that the second derivatives of both terms must vanish at Σ. Furthermore, we note that

$$0=\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd}\nabla_i h_{ab}\nabla_j h_{cd}=(1-\delta_{1i})\ddot{\gamma}^{ab,cd}\nabla_i h_{ab}\nabla_i h_{cd}$$

combined with Lemma 5.1.2 implies that $\nabla_i h_{ab}=0$ holds on Σ for $i>1$ and all a,b.

Finally, the Codazzi Equations implies that $|\nabla S_{1,1}|=0$ holds on Σ. Then Equation (5.5.5) implies that $|\nabla\gamma|=0$ holds on Σ. Therefore, γ is constant in Σ.
Consequently, by Equation (5.1.4), it follows that $|A|_\gamma^2\gamma=0$ which cannot occur since at $\lambda(p_0)\in\Gamma_{\alpha,\delta}$ for which the functions γ and $|A|_\gamma^2$ are positive. $\square$

Now we focus on the case when $\inf\limits_{\Sigma} f(p)\in(0,\alpha^{-1})$ is attained at "infinity". We use the Omori-Yau maximum principle, which gives a sequence $p_N\in\Sigma$ such that $|p_N|\to\infty$, and the following equations holds

$$f(p_N)\to\inf_{\Sigma} f(p),\ |\nabla f(p_N)|<\frac{1}{N}\text{ and }\ \Delta_\gamma f(p_N)\geq-\frac{1}{N}. \tag{5.5.8}$$

We also consider the sequence of γ-translators given by

$$\Sigma_N=\Sigma-p_N.$$

Note that the second fundamental form of each Σ_N is uniformly bounded by the same constant since Σ_N is a translation of Σ by p_N. In addition, we note that $0\in\Sigma_N$ for each N. Therefore, by compactness (see [PR] for details), we can subtract a sub-sequence Σ_N' of Σ_N

and a γ-translator Σ'_∞ such that $\Sigma'_N \to \Sigma'_\infty$ uniformly smoothly on compacts subset of Σ'_∞. Finally, we denote by Σ_∞ the connected component of Σ'_∞ which contains the point 0.

Claim 5.5.2. *At $0 \in \Sigma'_\infty$, the function γ vanishes.*

Proof. If it is not the case, then $\gamma > 0$ at $0 \in \Sigma'_\infty$. But since $f(0) = \inf_\Sigma f$ we get a contradiction with Claim 5.5.1. □

Claim 5.5.3. *All the principal curvatures vanish at $0 \in \Sigma'_\infty$. Moreover, we have the following estimates at p_N*

$$f \geq \frac{1+\delta}{2\alpha},$$
$$\frac{\lambda_i}{S_{1,1}} \geq \frac{\beta}{2},$$
$$\frac{|\lambda_1|}{S_{1,1}} > \frac{\delta}{2}.$$

Proof. Firstly, recall that at p_n, $\lambda \in \Gamma_{\alpha,\delta}$. This means that $H(p_n) \to 0$ as $N \to \infty$ since $\gamma(p_N) \to 0$ as $N \to \infty$. In particular, by estimate (5.5.1), it follows that $|A|^2 \to 0$ giving that $\lambda_i(p_n) \to 0$ as $N \to \infty$.

Moreover, by (5.5.4) we have $\lambda_1(p_N) < 0$ holds for N big enough. Then, since $n \geq 3$, we have at p_N the following

$$H + \lambda_1 \geq (\lambda_3 + \lambda_1) + (\lambda_2 + \lambda_1),$$

which is positive by the 2-convexity. In particular, $H - \lambda_1 \leq 2H$ holds at p_N for N big enough.

Therefore, it follows that

$$f(p_N) = \frac{\gamma}{H - \lambda_1} \geq \frac{\gamma}{2H} \geq \frac{1+\delta}{2\alpha},$$

in the last inequality we use $\lambda \in \Gamma_{\alpha,\delta}$.

On the other hand, by the two uniform convexity, we obtain

$$\frac{\lambda_i}{H - \lambda_1} = \frac{\lambda_i + \lambda_1 - \lambda_1}{H - \lambda_1} \geq \frac{\beta H}{H - \lambda_1} \geq \frac{\beta}{2},$$

at p_N and $i > 1$. In the first inequality we use the 2-uniform convexity $\lambda_i + \lambda_1 \geq \beta H$ and the fact that $\lambda_1(p_n) < 0$.

Finally, for the last term, we have

$$\frac{|\lambda_1|}{H - \lambda_1} = \frac{-\lambda_1}{H - \lambda_1} > \frac{\alpha\gamma - H}{H - \lambda_1} \geq \frac{\delta H}{H - \lambda_1} \geq \frac{\delta}{2},$$

in the second inequality we use Equation (5.5.4). □

Next, we note that equation (5.5.5)

$$\begin{aligned} \nabla f &= \frac{\nabla \gamma}{S_{1,1}} - f \frac{\nabla S_{1,1}}{S_{1,1}} \\ &= \frac{\lambda_i \langle e_i, e_{n+1} \rangle}{S_{1,1}} e_i - f \frac{\nabla S_{1,1}}{S_{1,1}}, \end{aligned} \tag{5.5.9}$$

combined with Claim 5.5.3 gives that the term

$$\frac{\nabla_i \gamma}{S_{1,1}} = \frac{\lambda_i \langle e_i, e_{n+1} \rangle}{S_{1,1}} e_i$$

is uniformly bounded at p_N. In addition, since $\nabla f \to 0$ at $N \to \infty$, it follows that the term $\dfrac{\nabla S_{1,1}}{S_{1,1}}(p_N)$ is also bounded.

Claim 5.5.4. *The term*

$$\frac{\nabla S_{1,1}}{S_{1,1}}(p_N) \to 0, \ \ as\ N \to \infty.$$

Proof. We evaluate (5.5.6) at p_N. Then, by equation (5.5.8), we note

$$\frac{-C}{N} \leq \Delta_\gamma f + \nabla_{n+1} f + 2\left\langle \nabla f, \frac{\nabla S_{1,1}}{S_{1,1}} \right\rangle_\gamma = f \frac{\left(\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\right)}{S_{1,1}} \nabla_i h_{ab} \nabla_j h_{cd} \leq 0.$$

Then, since $\nabla f \to 0$, f and the term $\dfrac{\nabla S_{1,1}}{S_{1,1}}$ are bounded from below by a positive constant and from above as well, it follows that

$$\frac{1}{S_{1,1}}\left(\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\right) \nabla_i h_{ab} \nabla_j h_{cd} \to 0, \text{ as } N \to \infty. \tag{5.5.10}$$

On the other hand, we note that if $\lambda_a = \lambda_b$ at p_N for $a \neq b$, then $\nabla h_{ab}(p_N) = 0$. Indeed, to see this we choose normal coordinates centered at p_N. This means that we have an orthonormal base of principal directions $\{e_i\}_{i=1}^n$ of $T_{p_N}\Sigma$ such that $h_b^a(p_N) = h_{ab}(p_N) = \lambda_a(p_N)\delta_a^b = 0$ since $a \neq b$. Then, by taking derivatives, it follows that

$$0 = e_i(h_{ab}) = \nabla_i h_{ab} - A(\nabla_i e_a, e_b) - A(e_a, \nabla_i e_b) = h_{ab,i} + \Gamma_{ia}^b(\lambda_a - \lambda_b) = h_{ab,i}.$$

Therefore, we may assume that $\lambda_a \neq \lambda_b$ at p_N.

Moreover, for $i > 1$ and using $\lambda_1(p_N) < 0$ with Lemma 5.1.2, we see that

$$\left(\frac{\nabla_i h_{ab}}{S_{1,1}}\right)^2 \leq \frac{|\nabla_i h_{ab}|^2}{S_{1,1} H} \leq \frac{-1}{C S_{1,1}} \sum_{i>1} \ddot{\gamma}^{ab,cd} \nabla_i h_{ab} \nabla_i h_{cd}.$$

Then, by noting that $\dfrac{\partial S_{1,1}}{\partial \lambda_i} = (1 - \delta_{1i})$ and adding the term $\dfrac{\dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}}{S_{1,1}}$ in the last inequality, it yields

$$\frac{|\nabla_i h_{ab}|^2}{(S_{1,1})^2} \leq \frac{-1}{C S_{1,1}}\left(\dot{S}_{1,1}^{ij}\ddot{\gamma}^{ab,cd} - \dot{\gamma}^{ij}\ddot{S}_{1,1}^{ab,cd}\right) \nabla_i h_{ab} \nabla_j h_{cd}.$$

Therefore, by Equation (5.5.10), the term $\dfrac{\nabla_i h_{ab}}{S_{1,1}}$ goes to 0 as $N \to \infty$ for all a, b and $i > 1$. Then, the Codazzi Equations implies that $\dfrac{\nabla S_{1,1}}{S_{1,1}} \to 0$ as $N \to \infty$. □

We note that the last claim together with Equation (5.5.9) implies that

$$\left|\frac{\nabla_i \gamma}{S_{1,1}}\right| = \left|\frac{\lambda_i}{S_{1,1}}\right| |\langle e_i, e_{n+1}\rangle| \to 0 \text{ as } N \to \infty.$$

Finally, by Claim 5.5.3, which states a lower bound for each $\dfrac{\lambda_i}{S_{1,1}}(p_N)$, we have that $\langle e_i, e_{n+1}\rangle \to 0$, as well as $\langle \nu, e_{n+1}\rangle = \gamma \to 0$ when $N \to \infty$. In particular, we get that $e_{n+1} = \vec{0}$ at $0 \in \Sigma'_\infty$, which is a contradiction, finalizing the proof of Theorem 5.0.7. □

As an application of the convexity estimate we show the γ-translators are asymptotically convex.

Proof of Corollary 5.0.8. It is enough to prove that γ vanishes at infinity, since $0 \leq H \leq \frac{\alpha}{\delta+1}\gamma$. In fact, by the convexity estimate we have

$$\lambda_1 \geq H - \alpha\gamma.$$

Therefore, if γ vanish at infinity, it follows that $\lambda_1(p) \geq 0$ as $|p - p_0| \to \infty$ for a given point $p_0 \in \Sigma$,. □

Claim 5.5.5. *$\gamma(p) \to 0$ when $|p - p_0| \to \infty$.*

Proof. If it is not the case, there exists a sequence $p_i \in \Sigma$ such that $\liminf_{i\to\infty} \gamma(p_i) > 0$ and $|p_i - p_0| \to \infty$. In particular, we can extract a convergent subsequence of γ-translators $\Sigma_i = \Sigma - \{p_i\}$ which will converge to a γ-translator Σ_∞. Let $\Sigma'_\infty \subset \Sigma_\infty$ be the connected component containing p_0. In particular, γ will attains a positive minimum at $p_0 \in \Sigma'_\infty$.

Nevertheless, by Equation (5.1.4)

$$\Delta_\gamma \gamma + \langle \nabla \gamma, e_{n+1} \rangle + |A|_\gamma^2 \gamma = 0,$$

this minimum cannot be attained unless $|A|_\gamma^2 \gamma\big|_{p_0} = 0$ which implies that $\gamma(p_0) = 0$, given the desire contradiction. $\square$

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