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Gradual and Fuzzy Modules: Functor Categories †

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Abstract: The categorical treatment of fuzzy modules presents some problems, due to the well known fact that the category of fuzzy modules is not abelian, and even not normal. Our aim is to give a representation of the category of fuzzy modules inside a generalized category of modules, in fact, a functor category, $\mathbf{Mod}-\mathcal{P}$, which is a Grothendieck category. To do that, first we consider the preadditive category $\mathcal{P}$, defined by the interval $P = (0, 1]$, to build a torsionfree class $\mathcal{J}$ in $\mathbf{Mod}-\mathcal{P}$, and a hereditary torsion theory in $\mathbf{Mod}-\mathcal{P}$, to finally identify equivalence classes of fuzzy submodules of a module M with F-pair, which are pair (G, F), of decreasing gradual submodules of M, where G belongs to $\mathcal{J}$, satisfying $G = F^d$, and $\cup_\alpha F(\alpha)$ is a disjoint union of $F(1)$ and $F(\alpha)\setminus G(\alpha)$, where α is running in $(0, 1]$.

Keywords: fuzzy set; fuzzy module; gradual element; gradual module; gradual ring; functorial category

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1. Introduction

The behaviour of fuzzy ideals and modules is reflected in the category of fuzzy modules, but this category, as it was pointed out by Lopez–Permouth and Malik, in [1], has no relevant properties; for instance, it is not an abelian category. On the contrary, it seems more a kind of category of topological objects. Following the theory of gradual elements, introduced in [2], see also [3] and applied to subsets and subgroups in [4], our aim in this work is to expone a categorical framework in which to embed the theory of fuzzy modules. To do that, first we consider a preadditive category $\mathcal{A}$, and the additive functors from $\mathcal{A}$ to $\mathcal{Ab}$, the category of abelian groups. These functors are the objects of an abelian category, in fact a Grothendieck category, which we represent by $\mathcal{A}-\mathbf{Mod}$, each of these functors is called a left $\mathcal{A}$–module. The counterpart are the right $\mathcal{A}$–modules (the contravariant additive functors from $\mathcal{A}$ to $\mathcal{Ab}$). This framework can be extended to consider a commutative ring A and the category $A-\mathbf{Mod}$, of A–modules, instead of $\mathcal{Ab}$, or even more, to consider an arbitrary ring R; in this case we need to change the building method. In all these cases we have a Grothendieck category, but in the commutative case the Yoneda embedding provides some particular and interesting consequences, as the existence of enough projective and injective modules in $\mathcal{A}-\mathbf{Mod}$.

The category $\mathcal{A}-\mathbf{Mod}$ is well known, even if we do not impose any extra condition to $\mathcal{A}$, but such a general theory is not interesting for the applications we have in mint; for instance, the existence of simple modules or chain conditions properties are assured only in very restrictive cases. Otherwise, the category $\mathcal{A}-\mathbf{Mod}$ is a generalization of the category, $R-\mathbf{Mod}$, of left modules over a ring R; in fact $R-\mathbf{Mod}$ is an example of a functor category.

In the study of the category $R-\mathbf{Mod}$, left ideals are a fundamental tool; in the functor category $\mathcal{A}-\mathbf{Mod}$ left ideals are defined as submodules of the particular modules $\mathrm{Hom}_{\mathcal{A}}(X,-)$, for any object $X \in \mathrm{Obj}(\mathcal{A})$. Right ideals are defined, in the same way, using the contravariant functors $\mathrm{Hom}_{\mathcal{A}}(-,X)$. Our first goal in the study of general functor categories is to demonstrate the basic arithmetic properties of ideals and modules.

Once we have established the category $\mathcal{A}-\mathbf{Mod}$, of left $\mathcal{A}$–modules, as the framework in which develop the theory, we study some particular examples of preadditive categories $\mathcal{A}$; in particular, those defined by a poset P. Indeed, fixed a commutative ring A, we may associate to P several preadditive categories, one of them is denoted $\mathcal{P}$, which is defined by the hom–sets: for any $a, b \in P$, either $\mathrm{Hom}_{\mathcal{P}}(a,b) = Af_{a,b}$, (the free A–module on $\{f_{a,b}\}$), whenever $a \leq b$, or $\mathrm{Hom}_{\mathcal{P}}(a,b) = 0$, otherwise; and the composition in the obvious way. A left $\mathcal{P}$–module is an A–additive functor from $\mathcal{P}$ to $A-\mathbf{Mod}$. In addition, if P is a directed set, for any left $\mathcal{P}$–module F we can build a directed system of A–modules: $(\{F(a) \mid a \in P\}, \{F(f_{a,b}) \mid a,b \in P\})$, and its direct limit: $\varinjlim F$. In this way, every left $\mathcal{P}$–module has associated, uniquely, with an A–module. If P has a maximum, say 1, then $\varinjlim F = F(1)$, in other cases it is a module defined by the usual construction of the direct limit.

The particular case of $\mathrm{Hom}_{\mathcal{P}}(a,-)$ is of interest: since $f_{a,b}$ is both a monomorphism and an epimorphism, for every $a \leq b$, we have that $(f_{a,b})^* : \mathrm{Hom}_{\mathcal{P}}(b,x) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(a,x)$ and $(f_{a,b})_* : \mathrm{Hom}_{\mathcal{P}}(x,a) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(x,b)$ are always monomorphisms, and if we take direct limits, $\mathrm{Hom}_{\mathcal{P}}(x,a)$ is a submodule of $\varinjlim \mathrm{Hom}_{\mathcal{P}}(x,-)$. It happens that those $\mathcal{P}$–modules satisfying this properties have special properties.

The class $\mathcal{J}$ of all $\mathcal{P}$–modules F such that $F(a) \subseteq \varinjlim F$, i.e., $F(f_{a,b})$ is always a monomorphism is closed under: submodules, direct product, hence direct sums, and group–extensions.

This means that $\mathcal{J}$ is the torsionfree class for a torsion theory in $\mathcal{P}-\mathbf{Mod}$. On the other hand, this torsion theory defines a torsion class, i.e., a class of $\mathcal{P}$–modules closed under: quotients, direct sums, and group–extensions, and, in addition, it is closed under submodules.

Hence, $\mathcal{J}$ is the torsionfree class of a hereditary torsion theory; therefore, it is closed under essential extensions. Thus, $\mathcal{J}$ contains every module of the shape $\mathrm{Hom}_{\mathcal{P}}(x,-)$, all of them are projective, and for any module $F \in \mathcal{J}$, the injective hull $E(F)$ also belongs to $\mathcal{J}$. This means that $\mathcal{J}$ is an example of a class of modules that have been well studied. In particular, in this paper, we identify this hereditary torsion theory, and demonstrate that it is defined by the dense ideals.

In this case, to any $\mathcal{P}$–module F in $\mathcal{J}$ we define a new module F^d in such a way that the operator $F \mapsto F^d$ is an interior operator, and study the behaviour of this operator with respect to arithmetic properties of left ideals and modules; so later we verify that it defines a class of modules in $\mathcal{J}$ that allow us to define fuzzy submodules in a natural way.

To find, in this context, a representation of fuzzy modules, we consider F–pairs, i.e., pairs (G,F) of decreasing gradual submodules of a module M such that $G = F^d$ and M is the disjoint union of the family $\{F(\alpha) \setminus G(\alpha)\}$; see [4]. Our objective is to find a model of the fuzzy theory using a functor category, or equivalently gradual modules, and we can do that first considering algebraic operations on fuzzy submodules (the sum of two submodules, μ_1, μ_2, is the smallest submodule containing both submodules whenever $\mu_1(0) = \mu_2(0)$), hence we define an equivalence relation in the set of all fuzzy submodules of M, saying $\mu_1 \sim \mu_2$ whenever $\mu_1(x) = \mu_2(x)$ for any $0 \neq x \in M$; in this way every equivalence class contains a unique submodule μ^0 with $\mu^0(0) = 1$.

With this baggage we can establish a correspondence between F-pair on M and equivalence classes of fuzzy submodules of M. What is of interest in this correspondence is that we use the α–levels theory to associate a decreasing gradual submodule, $\sigma(\mu)$, the inverse is built using the property (F), or equivalently the F-pair theory. This correspondence is not a homomorphisms with respect to the sum of submodules; it has also a problem with respect to arbitrary unions. To solve this, we establish a new correspondence between fuzzy submodules and gradual submodules. Indeed, for any fuzzy submodule μ, we define a strictly decreasing gradual submodule; it is noting more that $\sigma(\mu)^d$, and demonstrates that the correspondence between equivalence classes of fuzzy submodules and strictly decreasing gradual submodules satisfying property (inf-F) is a bijection, and, in addition, it is a homomorphism with respect to sum, join and meet.

We organize this paper as follows. In Section 2, we introduce background notions on functor category defined by an preadditive category which includes Yoneda embedding, and demonstrate that it is a Grothendieck category, see [5,6]. We discuss the different rings we can use, starting from the ring $\mathbb{Z}$ of integer numbers, continuing with a commutative ring, and we collect the useful arithmetical notions on modules and ideals. In Section 3, we study the class of torsionfree modules when we particularize to the preadditive category defined by a directed poset; it is the torsionfree class of a hereditary torsion theory. In addition, we introduce an interior operator which will be of utility in studying fuzzy submodules. We introduce several elements associated to gradual submodules; in particular, decreasing gradual submodules, and related them with torsionfree modules, and strictly decreasing gradual submodules. To introduce them, we need to study an interior operator, and show that it defines a hereditary torsion class, which we relate with dense ideals. In Section 4, the relationship with the theory of fuzzy submodules is studied, where we establish a correspondence with strictly decreasing gradual submodules through a map which is a homomorphism with respect to the sum, product, union and intersection, and allows us to translate properties of fuzzy submodules and ideals to similar properties on gradual modules and submodules, and vice–versa.

2. Preadditive Categories

In this section, we introduce the basic notions of functor categories and modules over a preadditive category as background for studying gradual rings and modules.

2.1. Preadditive Categories

A category $\mathcal{A}$ is *preadditive* if it satisfies:

(i) $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is an abelian group for any $X,Y \in \mathrm{Obj}(\mathcal{A})$.
(ii) For any $X,Y,Z,T \in \mathrm{Obj}(\mathcal{A})$, and morphisms $f_1, f_2 \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{A}}(Z,X)$ and $h \in \mathrm{Hom}_{\mathcal{A}}(Y,T)$ we have:

$$(f_1+f_2)g = f_1g+f_2g \text{ and } h(f_1+f_2) = hf_1+hf_2. \tag{1}$$

In a category $\mathcal{C}$:

- an object T is *terminal* if for any object X there is only one morphism from X to T,
- an object I is *initial* if for any object X there is only one morphism from I to X,
- an object Z is *zero* if it is initial and terminal.

Lemma 1. *Let $\mathcal{A}$ be a preadditive category,*

(1) If $\mathrm{End}_{\mathcal{A}}(X)$ has only one element then X is a zero object.
(2) For any object $X \in \mathrm{Obj}(\mathcal{A})$ either X is the zero object or $\mathrm{Hom}_{\mathcal{A}}(X,X) = \mathrm{End}_{\mathcal{A}}(X)$ is a unitary ring.

Corollary 1. *If $\mathcal{A}$ is a preadditive category, for every nonzero objects $X,Y \in \mathrm{Obj}(\mathcal{A})$, we have that $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is*

(1) a right $\mathrm{End}_{\mathcal{A}}(X)$–module and
(2) a left $\mathrm{End}_{\mathcal{A}}(Y)$–module.

The following are examples of preadditive categories.

Example 1.

(1) The category $\mathcal{Ab}$ of all abelian groups and homomorphisms of abelian groups is a preadditive category, as is the category of left R–modules for any ring R.

(2) *If R is a ring and consider the category $\mathcal{R}$ with only one object, say $*$, whose endomorphisms are parameterized by R, then $\mathcal{R}$ is a preadditive category with $+$ the sum in R, and composition the product in R.*
(3) *If $\mathcal{A}$ is a preadditive category, the apposite category $\mathcal{A}^{op}$ is preadditive, being $\mathrm{Obj}(\mathcal{A}^{op}) = \mathrm{Obj}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(Y,X)$, for any $X, Y \in \mathrm{Obj}(\mathcal{A})$, and the composition $*^{op}$ defined $f *^{op} g = g \circ f$ for any f, g morphisms in $\mathcal{A}^{op}$ for which the composition is defined.*
(4) *For any partial ordered set P, and any ring R, we define a new category $\mathcal{P}$ whose objects are the elements of P, the homomorphism sets are*

$$\mathrm{Hom}_{\mathcal{P}}(a,b) = \begin{cases} \{f_{a,b}\}R \cong R, & \text{if } a \leq b, \text{ with neutral element written } 0_{a,b} \\ \{0_{a,b}\}, & \text{if } a \not\leq b. \end{cases} \tag{2}$$

and composition given by the following table, whenever $a \leq b \leq c$ (if $a \leq b \leq c$ is not satisfied the composition does not exist):

$0_{b,c} \circ 0_{a,b} = 0_{a,c}$,	$f_{b,c} \circ 0_{a,b} = 0_{a,c}$
$0_{b,c} \circ f_{a,b} = 0_{a,c}$,	$f_{b,c} \circ f_{a,b} = f_{a,c}$

(3)

For simplicity, we may write 0 by $0_{a,b}$. The sum in $\mathrm{Hom}_{\mathcal{P}}(a,b)$ is defined through the sum in R; if $b \in P$ is a bottom element, $b \in \mathrm{Obj}(\mathcal{P})$ is an initial object, and if $t \in P$ is a top element, $t \in \mathrm{Obj}(\mathcal{P})$ is a terminal object.

Given preadditive categories $\mathcal{A}$ and $\mathcal{B}$, a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *additive* whenever $F(f_1 + f_2) = F(f_1) + F(f_2)$ for any $f_1, f_2 \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$, and any pair $A_1, A_2 \in \mathrm{Obj}(\mathcal{A})$.

The following functors between preadditive categories will be additive unless the contrary is indicated.

2.2. Modules

Let $\mathcal{A}$ be a preadditive category. A *left $\mathcal{A}$–module* (or simply an *$\mathcal{A}$–module*) is an additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}b$, to the category of abelian groups.

If F is an $\mathcal{A}$–module, for any $X \in \mathrm{Obj}(\mathcal{A})$, any homomorphism f in the category $\mathcal{A}$, and any element $m \in F(X)$, we define the *dot–product*:

$$f \cdot m = \begin{cases} F(f)(m) \in F(Y), & \text{whenever } f \in \mathrm{Hom}_{\mathcal{A}}(X,Y), \\ 0 \in F(Y), & \text{otherwise, i.e., if } f \in \mathrm{Hom}_{\mathcal{A}}(Z,Y) \text{ and } Z \neq X. \end{cases} \tag{4}$$

The dot–product, for convenient f_i and m_j, satisfies the following properties:

(1) $f \cdot (m_1 + m_1) = f \cdot m_1 + f \cdot m_2$.
(2) $(f_1 + f_2) \cdot m = f_1 \cdot m + f_2 \cdot m$.
(3) $(f_1 f_2) \cdot m = f_1 \cdot (f_2 \cdot m)$.
(4) $\mathrm{id}_X \cdot m = m$.

Let G, F be $\mathcal{A}$–modules, a *homomorphism* from F to G is an abelian group natural transformation $\theta : F \longrightarrow G$.

Observe that an $\mathcal{A}$–module is a collection of abelian groups together with a family of homomorphisms satisfying the commutative properties induced by the commutative relations of $\mathcal{A}$.

In the following, $\mathcal{A}$ will be an *skeletally small preadditive category*; this means that the class of isomorphisms of $\mathcal{A}$ constitutes a set. We impose this condition to assume that in the category we shall construct the Hom's are sets.

In this case, the $\mathcal{A}$–modules and homomorphisms of $\mathcal{A}$–modules constitute a category, that we call $\mathcal{A}-\mathbf{Mod}$; indeed, it is an abelian category, as we are going to demonstrate later.

In the same way, we define *right $\mathcal{A}$–modules*, as contravariant additive functors from $\mathcal{A}$ to $\mathcal{A}b$; homomorphisms of right $\mathcal{A}$–modules, and the category $\mathbf{Mod}-\mathcal{A}$ of right $\mathcal{A}$–modules.

We can enrich the category $\mathcal{A}-\mathbf{Mod}$ whenever we consider a category $A-\mathbf{Mod}$ instead of $\mathcal{Ab}$, being A a commutative ring. In this case, we need the preadditive category $\mathcal{A}$ the Hom's sets to have an extra structure of A–module.

Let $\mathcal{A}$ be a preadditive category, and A be a commutative ring; we say $\mathcal{A}$ is a *preadditive A–category* (or simply an *A–category*) if

(1) any $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is an A–module, and
(2) for any $X, Y, Z \in \operatorname{Obj}(\mathcal{A})$ the map $\operatorname{Hom}_{\mathcal{A}}(Y, Z) \times \operatorname{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X, Z)$ is A–bilinear.

As a consequence, for any nonzero object $X \in \operatorname{Obj}(\mathcal{A})$ we have that $\operatorname{End}_{\mathcal{A}}(X)$ is an A–algebra.

2.3. Yoneda Embedding

Remember that if $\mathcal{A}$ is a preadditive A–category, and $X \in \operatorname{Obj}(\mathcal{A})$, then the induced functor $\operatorname{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \longrightarrow A - \mathbf{Mod}$ is an $\mathcal{A}$–module as it is A–additive. The image of any $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ is f_*, defined as:

$$f_*(g) = fg, \text{ for any } g \in \operatorname{Hom}_{\mathcal{A}}(A, X). \tag{5}$$

There is a map $\mathcal{Y} : \operatorname{Obj}(\mathcal{A}) \longrightarrow \operatorname{Obj}(\mathcal{A}-\mathbf{Mod})$, defined by $\mathcal{Y}(A) = \operatorname{Hom}_{\mathcal{A}}(A, -)$. For any homomorphism $g \in \operatorname{Hom}_{\mathcal{A}}(B, A)$, and any object $X \in \operatorname{Obj}(\mathcal{A})$, we have a homomorphism

$$g^* : \operatorname{Hom}_{\mathcal{A}}(A, X) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(B, X). \tag{6}$$

Therefore, we obtain a contravariant functor, $\mathcal{Y}$, from $\mathcal{A}$ to $\mathcal{A}-\mathbf{Mod}$; it is called the *Yoneda embedding*.

Lemma 2 (Yoneda lemma). *For every $X \in \operatorname{Obj}(\mathcal{A})$ there is an A–module isomorphism*

$$\omega : \operatorname{Hom}_{\mathcal{A}-\boldsymbol{Mod}}(\operatorname{Hom}_{\mathcal{A}}(X, -), F) \cong F(X). \tag{7}$$

Corollary 2. *The Yoneda embedding, $\mathcal{Y} : \mathcal{A} \longrightarrow \mathcal{A} - \boldsymbol{Mod}$, is a full and faithful contravariant functor.*

Ideals and Product of Ideals

Let $\mathcal{A}$ be a preadditive A–category. A *left ideal* of $\mathcal{A}$ is a submodule $\mathfrak{a}$ of an $\mathcal{A}$–module $\operatorname{Hom}_{\mathcal{A}}(X, -)$, let us call $i : \mathfrak{a} \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X, -)$ the natural transformation inclusion. In particular, for any homomorphism $f \in \operatorname{Hom}_{\mathcal{A}}(Y, Z)$, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{a}(Y) & \xrightarrow{\mathfrak{a}(f)} & \mathfrak{a}(Z) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_Z} \\ \operatorname{Hom}_{\mathcal{A}}(X, Y) & \xrightarrow[f_*]{} & \operatorname{Hom}_{\mathcal{A}}(X, Z). \end{array} \tag{8}$$

Hence for every $\alpha \in \mathfrak{a}(Y)$ and any $f \in \operatorname{Hom}_{\mathcal{A}}(Y, Z)$, we have $f_*(\alpha) = f\alpha \in \mathfrak{a}(Z)$.

In the same way, we may define a *right ideal* $\mathfrak{b}$ as a submodule of $\operatorname{Hom}_{\mathcal{A}}(-, X)$. In this case, for every $f \in \operatorname{Hom}_{\mathcal{A}}(Y, Z)$ and any $\beta \in \mathfrak{b}(Z)$ we have $f^*(\beta) = \beta f \in \mathfrak{b}(Y)$.

The *intersection* of a family of left ideals $\{\mathfrak{a}_i \mid i \in I\}$ is defined componentwise:

$$(\cap_i \mathfrak{a}_i)(Y) = \cap_i \mathfrak{a}_i(Y). \tag{9}$$

It is a left ideal.

The *sum* of a family of left ideals $\{\mathfrak{a}_i \mid i \in I\}$ is defined componentwise:

$$\left(\sum_i \mathfrak{a}_i\right)(Y) = \sum_i \mathfrak{a}_i(Y). \tag{10}$$

It is a left ideal.

The *product* of a left ideal $\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(X, -)$ and an $\mathcal{A}$–module F is defined as:

$$(\mathfrak{a}F)(Y) = \langle \{f \cdot m \mid f \in \mathfrak{a}(Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X, Y),\ m \in F(X)\}\rangle \tag{11}$$

It is a submodule of F.

3. Torsionfree Modules

In this section, we particularize to the case in which $\mathcal{A} = \mathcal{P}$, is the preadditive category defined by a poset P, which is, in addition, a directed set. We shall use the $\mathcal{P}$–module instead of left $\mathcal{P}$–module throughout this section if there is no risk of confusion.

3.1. Directed Posets

Let P be a poset, with minimum element 0; it is *directed* if for any $a, b \in P$ there exists $c \in P$ such that $a \leq c$ and $b \leq c$.

From the poset P, we build a category, $\mathcal{P}$, whose objects are the elements of P. For any $a, b \in P$, we define

$$\mathrm{Hom}_{\mathcal{P}}(a, b) = \begin{cases} \{0_{a,b}, f_{a,b}\}, & \text{if } a \leq b, \\ \{0_{a,b}\}, & \text{otherwise,} \end{cases} \tag{12}$$

with composition and addition given, for any $a, b, c \in P$, whenever $a \leq b \leq c$, by the rules:

$$\begin{array}{llll} 0_{b,c}0_{a,b} = 0_{a,c} & 0_{b,c}f_{a,b} = 0_{a,c}; & 0_{a,b} + 0_{a,b} = 0_{a,b} & 0_{a,b} + f_{a,b} = 0_{a,b}; \\ f_{b,c}0_{a,b} = 0_{a,c} & f_{b,c}f_{a,b} = f_{a,c}; & f_{a,b} + 0_{a,b} = 0_{a,b} & f_{a,b} + f_{a,b} = f_{a,b}. \end{array} \tag{13}$$

Let B be a ring. It is possible to modify the above category $\mathcal{P}$ to get a new preadditive B–category, also denoted by $\mathcal{P}$, in defining

$$\mathrm{Hom}_{\mathcal{P}}(a, b) = \begin{cases} \{f_{a,b}\}B =, & \text{if } a \leq b \\ \{0_{a,b}\}, & \text{otherwise,} \end{cases} \tag{14}$$

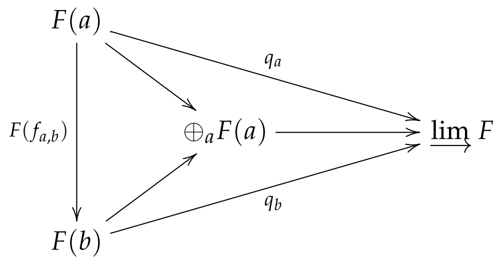
identifying $0_{a,b}$ with $f_{a,b}0$, and $0_{a,b}x$, for any $x \in B$, with addition defined following the addition in B, and composition using the former composition rules.

Lemma 3. *$\mathcal{P}$ is a preadditive B–category.*

Given a directed poset P, with minimum $0 \in P$, and a commutative ring A, consider the preadditive A–category $\mathcal{P}$, for any A–additive functor $F : \mathcal{P} \longrightarrow A - \mathbf{Mod}$, i.e., a left $\mathcal{P}$–module; we consider the family $\{F(a) \mid a \in P\}$, and, for any $a, b \in P$ the map $F(f_{a,b}) : F(a) \longrightarrow F(b)$, whenever it exists; this defines a directed system of A–modules:

$$(\{F(a) \mid a \in P\}, \{F(f_{a,b}) \mid a \leq b\}). \tag{15}$$

Since the existence of the direct limits in $A-\mathbf{Mod}$ is assured, we have an A–module: $\varinjlim F$, and homomorphisms, say $q_a : F(a) \longrightarrow \varinjlim F$, such that, for every pair $a \leq b$, the following diagram commutes.



(16)

Lemma 4. *Given $a, b \in P$ if $a \leq b$ then $f_{a,b}$ is an epimorphism and a monomorphism in $\mathcal{P}$.*

Proof. Indeed, if $f_{b,c}(n_1 f_{a,b}) = f_{b,c}(n_2 f_{a,b})$, then $n_1 f_{a,c} = n_2 f_{a,c}$, hence $n_1 = n_2$. In the same way, we prove $f_{b,c}$ is a monomorphism. □

Let $x \in P$, if we consider the $\mathcal{P}$–module $\mathrm{Hom}_{\mathcal{P}}(x, -)$, for any pair $a \leq b$ we have a module map $(f_{a,b})_* : \mathrm{Hom}_{\mathcal{P}}(x, a) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(x, b)$, which is a monomorphism. In general, $(f_{a,b})_*$ is not an epimorphism because if $a \leq b$ and $0_{x,b} \neq f \in \mathrm{Hom}_{\mathcal{P}}(x, b)$, then $x \leq b$, but it may be $x \not\leq a$, hence $\mathrm{Hom}_{\mathcal{P}}(x, a) = \{0_{x,a}\}$.

The same holds if we consider the right $\mathcal{P}$–module $\mathrm{Hom}_{\mathcal{P}}(-, x)$.

Proposition 1. *In the diagram (16), taking $F = \mathrm{Hom}_{\mathcal{P}}(x, -)$, every map $F(f_{a,b})$ is a monomorphism. Therefore, each map q_a is a monomorphism, i.e., each $\mathrm{Hom}_{\mathcal{P}}(x, a)$ is a submodule of $\varinjlim \mathrm{Hom}_{\mathcal{P}}(x, -)$.*

Proof. It is a consequence of being $f_{a,b}$ an epimorphism. Since, in the case $F(f_{a,b}) = (f_{a,b})_*$ is a monomorphism, the construction of the direct limit in the category of A–modules, as a quotient of a direct sum, implies that each q_a is a monomorphism. Indeed, if $q_a(x) = 0$, there exists $b \geq a$ such that $(f_{a,b})_*(x) = 0$, hence $x = 0$. □

The construction of $\mathrm{Hom}_{\mathcal{P}}(x, -)$ implies that we may identify $\mathrm{Hom}_{\mathcal{P}}(x, a)$ and $A f_{x,a}$ as A–modules, because both of them are isomorphic to A. Otherwise, if $f \in \mathrm{Hom}_{\mathcal{P}}(x, a)$, there exists $n \in A$ such that $f = n f_{x,a}$. Hence, if $x \leq a$, then $(f_{x,a})_* : \mathrm{Hom}_{\mathcal{P}}(x, x) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(x, a)$, and $f = n f_{x,a} = n f_{x,a} f_{x,x} = f f_{x,x} = f \cdot f_{x,x}$. Therefore, $f_{x,x}$ generates $\mathrm{Hom}_{\mathcal{P}}(x, -)$, i.e., $\langle f_{x,x} \rangle = \mathrm{Hom}_{\mathcal{P}}(x, -)$.

Proposition 2. *Each $\mathrm{Hom}_{\mathcal{P}}(x, -)$ is a cyclic $\mathcal{P}$–module with generator $f_{x,x}$.*

3.2. Torsionfree $\mathcal{P}$–Modules

In the category $\mathcal{P}-\mathbf{Mod}$, we shall collect in a class all $\mathcal{P}$–modules satisfying the property given in Proposition (1). Let F be a $\mathcal{P}$–module, we say F is *torsionfree*, if $F(f_{a,b})$ is a monomorphism for every $a \leq b$, and denote by $\mathcal{J}$ the class of all torsionfree $\mathcal{P}$–modules.

Proposition 3. *The class $\mathcal{J}$ satisfies the following properties:*

(1) It is closed under monomorphisms.
(2) It is closed under direct sums and direct products.
(3) It is closed under group–extension.

Proof. (1). Let F be a torsionfree $\mathcal{P}$–module, and $j : G \rightarrowtail F$ be a monomorphism, if $a \leq b$ we have a commutative diagram (sometimes, when working with commutative diagrams, we use the following notation: a monomorphism is represented by an arrow such as $\rightarrowtail$, and an epimorphism by $\twoheadrightarrow$).

$$\begin{array}{ccc} G(a) & \stackrel{j_a}{\rightarrowtail} & F(a) \\ {\scriptstyle G(f_{a,b})}\downarrow & & \downarrow{\scriptstyle F(f_{a,b})} \\ G(b) & \stackrel{j_b}{\rightarrowtail} & F(b) \end{array} \tag{17}$$

Hence $G(f_{a,b})$ is a monomorphism.

(2). Let $\{F_i \mid i \in I\}$ be a family of torsionfree $\mathcal{P}$–modules; for any index $i \in I$ we have a commutative diagram

$$\begin{array}{ccc} F_i(a) & \stackrel{(j_i)_a}{\rightarrowtail} & \oplus_i F_i(a) \\ {\scriptstyle F_i(f_{a,b})}\downarrow & & \downarrow{\scriptstyle \oplus_i F_i(f_{a,b})} \\ F_i(b) & \stackrel{(j_i)_b}{\rightarrowtail} & \oplus_i F_i(b) \end{array} \tag{18}$$

and the kernel of $\oplus_i F_i(f_{a,b})$ is zero. The direct product case is similar.

(3). Let F_1, F_2, F_3 be $\mathcal{P}$–modules; if $F_1, F_2 \in \mathcal{J}$ and $0 \to F_1 \to F_2 \to F_3 \to 0$ is a short exact sequence, we have a commutative diagram whenever $a \leq b$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1(a) & \rightarrowtail & F_2(a) & \twoheadrightarrow & F_3(a) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_1(f_{a,b})} & & \downarrow{\scriptstyle F_2(f_{a,b})} & & \downarrow{\scriptstyle F_3(f_{a,b})} & & \\ 0 & \longrightarrow & F_1(b) & \rightarrowtail & F_2(b) & \twoheadrightarrow & F_3(b) & \longrightarrow & 0 \end{array} \tag{19}$$

By the hypothesis $F_1(f_{a,b})$ and $F_3(f_{a,b})$ are monomorphisms, then $F_2(f_{a,b})$ is a monomorphism. □

In particular, the class $\mathcal{J}$ is the torsionfree class of a torsion theory in $\mathcal{P} - \mathbf{Mod}$. To find this torsion theory, for any $\mathcal{P}$–module, F, and any $a \in P$, we define

$$\eta(F)(a) = \{u \in F(a) \mid \text{ exists } b \in P, a \leq b, \text{ such that } F(f_{a,b})(u) = 0\}. \tag{20}$$

Lemma 5. *$\eta(F)$ is a submodule of F, and $F/\eta(F)$ is torsionfree.*

Proof. We have $\eta(F)(a)$, which is a submodule of $F(a)$; indeed, if $u_1, u_2 \in \eta(F)(a)$, there exist $b_1, b_2 \in P$, such that $F(f_{a,b_i})(u_i) = 0$, and there exists $c \in P$ such that $f_{a,c} = f_{b_i,c} f_{a,b_i}$, then $u_1 + u_2 \in \eta(F)(a)$. Otherwise, for any $u \in \eta(F)(a)$ and any $n \in A$, there exists $b \in P$, such that $F(f_{a,b})(nu) = n f_{a,b}(u) = 0$, then $nu \in \eta(F)(a)$.

We have $\eta(F)$ is a submodule of F; indeed, for any $f \in \mathrm{Hom}_{\mathcal{P}}(a,b)$, and any $u \in \eta(F)(a)$, there exists $n \in A$ such that $f = n f_{a,b}$, and $c \in P$ such that $F(f_{a,c})(u) = 0$. There exists $d \in P$ such that $f_{a,d} = f_{b,d} f_{a,b} = f_{c,d} f_{a,c}$, and $f \cdot u \in \eta(F)(b)$.

If $(F/\eta(F))(f_{a,b})(\overline{u}) = 0$, then $F(f_{a,b})(u) \in \eta(F)(b)$, and $u \in \eta(F)$.

$$\begin{array}{ccccc} \eta(F)(a) & \longrightarrow & F(a) & \longrightarrow & (F/\eta(F))(a) \\ {\scriptstyle \eta(F)(f_{a,b})}\downarrow & & \downarrow{\scriptstyle F(f_{a,b})} & & \downarrow{\scriptstyle (F/\eta(F))(f_{a,b})} \\ \eta(F)(b) & \longrightarrow & F(b) & \longrightarrow & (F/\eta(F))(b) \end{array} \tag{21}$$

□

A $\mathcal{P}$–module F such that $F = \eta(F)$ is called a *torsion* $\mathcal{P}$–module. We may characterize the $\mathcal{P}$–modules, which are torsion:

Lemma 6. *A $\mathcal{P}$–module F is torsion ($F = \eta(F)$) if, and only if, $\varinjlim F = 0$.*

Proof. Let F be a torsion $\mathcal{P}$–module, for any $x \in F(a)$ there exists $b \in P$ such that $F(f_{a,b})(x) = 0$. □

Presently, we can characterize the class $\mathcal{T}$ of all torsion $\mathcal{P}$–modules. Indeed, $\mathcal{T}$ is a hereditary torsion class,

Lemma 7.

(1) Let G be a submodule of $\mathcal{P}$–module F, then $\eta(G) = \eta(F) \cap G$.
(2) Let $f : F \longrightarrow G$ be a homomorphism, then $f(\eta(F)) \subseteq \eta(G)$.
(3) For any $\mathcal{P}$–module F we have $\eta(F/\eta(F)) = 0$.

*In particular, η is a radical torsion and defines a hereditary torsion theory in $\mathcal{P}-$**Mod**.*

Proof. (1). We have $(\eta(F) \cap G)(a) = \eta(F)(a) \cap G(a) = \eta(G)(a)$.

(2). Let $u \in \eta(F)(a)$, there exists $b \in P$ such that $F(f_{a,b})(u) = 0$, hence

$$G(f_{a,b})f(a)(u) = f(b)F(f_{a,b})(u) = 0. \tag{22}$$

(3). Let $\overline{u} \in \eta(F/\eta(F))(a)$, there exists $b \in P$ such that $(F/\eta(F))(f_{a,b})(\overline{u}) = 0$, hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta(F)(a) & \rightarrowtail & F(a) & \twoheadrightarrow & (F/\eta(F))(a) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta(F)(f_{a,b})} & & \downarrow{\scriptstyle F(f_{a,b})} & & \downarrow{\scriptstyle (F/\eta(F))(f_{a,b})} & & \\ 0 & \longrightarrow & \eta(F)(b) & \rightarrowtail & F(b) & \twoheadrightarrow & (F/\eta(F))(b) & \longrightarrow & 0 \end{array} \tag{23}$$

$F(f_{a,b})(u) \in \eta(F)(b)$, and there exists $c \in P$ such that $F(f_{b,c})F(f_{a,b})(u) = 0$; therefore $u \in \eta(F)(a)$, hence $\overline{u} = 0$. □

Since η is a hereditary torsion radical in the category $\mathcal{P}-$**Mod**, the torsionfree modules are those F such that $\eta(F) = 0$, i.e., they satisfy that the image of any $f_{a,b}$ is a monomorphism; hence, they are the torsionfree $\mathcal{P}$–modules, previously introduced.

We write the following result as a paraphasic of the basic property of torsionfree objects in a Grothendieck category.

Lemma 8. *Let F be a $\mathcal{P}$–module, for any $\mathcal{P}$–module G such that $G(f_{a,b})$ is a monomorphism whenever $a \leq b$, and any homomorphism $f : F \longrightarrow G$ there exists a unique homomorphism $f' : F/\eta(F) \longrightarrow G$ such that $f = pf'$.*

$$\begin{array}{ccc} F & \xrightarrow{\;p\;} & F/\eta(F) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \exists_1 f'} \\ & & G \end{array} \tag{24}$$

3.3. Dense Ideals

Let F be an $\mathcal{P}$–module, $a \in P$, and $x \in F(a)$. For any submodule $F' \subseteq F$, we define $(F' : x)$ as follows; for any $b \in P$ we put

$$(F' : x)(b) = \{f \in \mathrm{Hom}_{\mathcal{P}}(a, b) \mid F(f)(x) \in F'(b)\}. \tag{25}$$

Lemma 9. *With the above notation $(F' : x) \subseteq \mathrm{Hom}_{\mathcal{P}}(a, -)$ is an ideal.*

We call $(F' : x)$ the *residual ideal* of x with respect to F'. We define the *annihilator*, $\mathrm{Ann}(x)$, of $x \in F(a)$ as the residual ideal $\mathrm{Ann}(x) = (0 : x)$.

For any element $a \in P$, a family of left ideals $\mathcal{L}(a)$ of $\mathrm{Hom}_{\mathcal{P}}(a,-)$ is a *filter* if it satisfies:

(1) If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\mathfrak{a}_1 \in \mathcal{L}(a)$, then $\mathfrak{a}_2 \in \mathcal{L}_a$, for every left ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of $\mathrm{Hom}_{\mathcal{P}}(a,-)$.

(2) If $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{L}(a)$, then $\mathfrak{a}_1 \cap \mathfrak{a}_2 \in \mathcal{L}(a)$, for every left ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of $\mathrm{Hom}_{\mathcal{P}}(a,-)$.

Presently, we are interested in the relationship of $\mathcal{L}(a)$ and $\mathcal{L}(b)$ whenever there exists a map $g : a \to b$ in $\mathcal{P}$. Therefore, we have:

(3) If $g \in \mathrm{Hom}_{\mathcal{P}}(a,b)$ and $\mathfrak{a} \in \mathcal{L}(a)$, then $(\mathfrak{a} : g) \in \mathcal{L}(b)$.

Observe that in this case we have:

$$\begin{aligned} (\mathfrak{a} : g)(c) &= \{f \in \mathrm{Hom}_{\mathcal{P}}(b,c) \mid \mathrm{Hom}_{\mathcal{P}}(b,f)(g) \in \mathfrak{a}(c)\} \\ &= \{f \in \mathrm{Hom}_{\mathcal{P}}(b,c) \mid fg \in \mathfrak{a}(c)\} \subseteq \mathrm{Hom}_{\mathcal{P}}(b,c). \end{aligned} \tag{26}$$

A family of filters $\mathcal{L} = \{\mathcal{L}(a) \mid a \in P\}$ satisfying properties (1), (2) and (3) is called *linear filter* in $\mathcal{P} - \mathbf{Mod}$.

Our objective is to parameterize hereditary torsion theories in $\mathcal{P} - \mathbf{Mod}$ using families of filters $\mathcal{L} = \{\mathcal{L}(a) \mid a \in P\}$; to do that we need a fourth property. A linear filter $\mathcal{L} = \{\mathcal{L}(a) \mid a \in P\}$ is a *Gabriel filter* if it satisfies the property:

(4) Let $\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(a,-)$ be an ideal, and $\mathfrak{b} \in \mathcal{L}(a)$; if for every $b \in P$ and every $x \in \mathfrak{b}(b)$ we have $(\mathfrak{a} : x) \in \mathcal{L}(b)$, then $\mathfrak{a} \in \mathcal{L}(a)$.

In [7], a correspondence is established between Gabriel filters and hereditary torsion theories that associates to any Gabriel filter $\mathcal{L} = \{\mathcal{L}(a) \mid a \in P\}$; the torsion class $\mathcal{T}(\mathcal{L})$:

$$\mathcal{T}(\mathcal{L}) = \{F \mid \mathrm{Ann}(x) \in \mathcal{L}(a) \text{ for every } a \in P \text{ and every } x \in F(a)\}. \tag{27}$$

Otherwise, to any hereditary torsion class $\mathcal{T}$ associates, the Gabriel filter $\mathcal{L}(\mathcal{T})$, such that, for any $a \in P$:

$$\mathcal{L}(\mathcal{T})(a) = \left\{\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(a,-) \mid \frac{\mathrm{Hom}_{\mathcal{P}}(a,-)}{\mathfrak{a}} \in \mathcal{T}\right\}. \tag{28}$$

We have studied the hereditary torsion class of all torsion $\mathcal{P}$–module, hence we are interested in determining the associated Gabriel filter. Since $\mathcal{T} = \{F \mid \varinjlim F = 0\}$, then $\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{L}}(a,-)$ is in the Gabriel filter if, and only if, $\varinjlim \frac{\mathrm{Hom}_{\mathcal{P}}}{\mathfrak{a}} = 0$, or equivalently, if $\varinjlim \mathrm{Hom}_{\mathcal{P}}(a,-) = \varinjlim \mathfrak{a}$. In consequence, the Gabriel filter $\mathcal{L}$, associated to the torsion class of all torsion $\mathcal{P}$–modules, satisfies

$$\mathcal{L}(a) = \{\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(a,-) \mid \varinjlim \mathrm{Hom}(a,-) = \varinjlim \mathfrak{a}\}. \tag{29}$$

Observe that this Gabriel filter $\mathcal{L}$ can be also described as the filter of all ideals $\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(a,-)$ satisfying that for every $b \in P$, and any $g \in \mathrm{Hom}_{\mathcal{P}}(a,b)$, there exist $c \in P$, and $h \in \mathrm{Hom}_{\mathcal{P}}(b,c)$ such that $hg \in \mathfrak{a}(c)$. These are the *dense ideals* in [7].

3.4. An Interior Operator

Let $F \in \mathcal{J}$ be a torsionfree $\mathcal{P}$–module, for any $a \in P$ we define

$$\begin{aligned} &F^d(0) = F(0), \\ &F^d(a) = \textstyle\sum\{F(b) \mid b < a\}, \text{ if } a \neq 0, \end{aligned} \tag{30}$$

where this sum is in $\varinjlim F$.

Lemma 10. *Let F be a torsionfree $\mathcal{P}$–module, then F^d defines a functor from $\mathcal{P}$ to $A - \mathbf{Mod}$, hence a $\mathcal{P}$–module, and a submodule of F, which is also torsionfree.*

Proof. It is obvious that $F^d(a) \subseteq F(a)$ is a submodule. On the other hand, for any $f \in \mathrm{Hom}_{\mathcal{P}}(a,c)$ there exists $n \in A$ such that $f = nf_{a,c}$, and for any $x \in F^d(a)$ there existe $b \in P$ such that $x \in \mathrm{Im}(F(f_{b,a}))$, say $x = F(f_{b,a})(x)$. Then, we have:

$$f \cdot x = F(nf_{a,c})(x) = nF(f_{a,c})F(f_{b,a})(x) = nF(f_{b,c})(x) \in F^d(c). \tag{31}$$

□

This means that the operator $d : \mathcal{J} \longrightarrow \mathcal{J}$, defined by $d(F) = F^d$, is an *interior operator*. Indeed, it satisfies the statements in the following Lemma.

Lemma 11.

(1) $d(F) \subseteq F$ *for any* $F \in \mathcal{J}$.
(2) $d(F_1) \subseteq d(F_2)$ *whenever* $F_1 \subseteq F_2$, *for any* $F_1, F_2 \in \mathcal{J}$.
(3) $d(F) = dd(F)$ *for any* $F \in \mathcal{J}$.

A torsionfree $\mathcal{P}$–module is *d–open* if $d(F) = F$.

Let us show some arithmetical properties of this interior operator, with respect to submodules.

Proposition 4.

(1) *Let* $\{F_i \mid i \in I\}$ *be a family of torsionfree submodules of a* $\mathcal{P}$*–module* F, *then*

$$\left(\sum_i F_i\right)^d = \sum_i F_i^d. \tag{32}$$

As a submodule of F^d. *Thus, the class of d–open submodules is closed under sums.*

(2) *Let* $F_1, F_2 \subseteq F$ *be torsionfree submodules of a* $\mathcal{P}$*–module* F, *then*

$$(F_1 \cap F_2)^d = F_1^d \cap F_2^d. \tag{33}$$

Thus, the class of d–open submodules is closed under finite intersections.

(3) *Let* $\mathfrak{a}$ *be a torsionfree left ideal, and* $G \subseteq F$ *be a submodule of a torsionfree* $\mathcal{P}$*–module* F, *then*

$$(\mathfrak{a}G)^d = \mathfrak{a}^d G^d. \tag{34}$$

Thus, the class of d–open left ideals is closed under products.

Proof. (1). Let $a \in P$, then

$$\left(\sum_i F_i\right)^d (a) = \sum_{b<a} \left(\sum_i F_i\right)(b) = \sum_{b<a}\sum_i F_i(b) = \sum_i \sum_{b<a} F_i(b) = \sum_i F_i^d(a). \tag{35}$$

(2). Let $a \in P$, then

$$\begin{aligned}(F_1 \cap F_2)^d(a) = \sum_{b<a} (F_1 \cap F_2)(b) = \sum_{b<a} (F_1(b) \cap F_2(b)) \\ = \sum_{b<a} F_1(b) \cap \sum_{b<a} F_2(b) = F_1^d(a) \cap F_2^d(a).\end{aligned} \tag{36}$$

Due to the upper–continuous property of the lattice of A–submodules.

(3). Let $a \in P$, then

$$(\mathfrak{a}G)^d(a) = \sum_{b<a} (\mathfrak{a}G)(b) = \sum_{b<a} \langle \{ f \cdot m \mid f \in \mathfrak{a}(b) \subseteq \mathrm{Hom}_{\mathcal{P}}(c,b),\, m \in G(c)\} \rangle \tag{37}$$

$$(\mathfrak{a}^d(a)G^d)(a) = \langle g\cdot n \mid g\in \mathfrak{a}^d(a)\subseteq \mathrm{Hom}(c,a),\ n\in G^d(c)\rangle$$
$$= \langle g\cdot n \mid g\in \sum_{b<a}\mathfrak{a}(b),\ n\in \sum_{e<c}G(e)\rangle = \sum\langle g\cdot n \mid g\in \mathfrak{a}(b)\subseteq \mathrm{Hom}(c,b),\ n\in G(c)\rangle, \quad (38)$$

and both are equal. □

Since F is torsionfree, we have $\varinjlim F = \cup\{F(a) \mid \ a \in P\}$. For any $a \in P$, we have a short exact sequence $0 \to F^d(a) \to F(a) \to \frac{F(a)}{F^d(a)} \to 0$, and taking direct limits, we also have a short exact sequence

$$0 \longrightarrow \varinjlim F^d \longrightarrow \varinjlim F \longrightarrow 0 \longrightarrow 0 \quad (39)$$

Hence $\varinjlim F^d \cong \varinjlim F$.

If we consider the inclusion $F^d(a) \subseteq F(a)$, and the difference set $F(a)\setminus F^d(a)$, in general, the union of all these difference sets does not coincide with $\varinjlim F$, i.e., $\cup\{F(a)\setminus F^d(a) \mid \ 0 \neq a \in P\}\cup F(0) \subseteq \varinjlim F$. We say F satisfies *property (F)* whenever the equality holds; in this case we can associate to each element $x \in \varinjlim F$ either 0 or a unique $a \in P\setminus\{0\}$ such that $x \in F(a)\setminus F^d(a)$. Observe that this is a property of sets and not a property of modules.

4. Gradual and Fuzzy Modules

Let M be an A–module, a *fuzzy submodule* is a map $\mu\colon M \longrightarrow [0,1]$ satisfying some extra properties; we are interested in associating to a fuzzy submodule a filtration of submodules: the α–level filtration, and establish properties of μ via properties of the α–level filtration, in order to have a useful theory of fuzzy modules inside the framework of functorial categories. Our objective in this section is to demonstrate that another different filtration to the α–level filtration picks up more efficiently the properties of μ.

In this section, we work with $P = (0,1]$ and the preadditive A–category $\mathcal{P}$; therefore, with the category $\mathbf{Mod}-\mathcal{P}$ of right $\mathcal{P}$–modules.

4.1. Fuzzy Ideals

Let A be a commutative, a fuzzy subset μ is a *fuzzy ideal* if for any $x, y \in A$ we have:

(1) $\mu(x-y) \geq \min\{\mu(x),\mu(y)\}$,
(2) $\mu(xy) \geq \max\{\mu(x),\mu(y)\}$ and
(3) $\mu(0) \neq 0$, to avoid the trivial case.

Lemma 12 ([8])**.** *If μ is a fuzzy ideal, then $\mu(0) \geq \mu(x)$ for any $x \in A$.*

Proof. Take $y = 0$ in (2). □

Remember, for any $\alpha \in [0,1]$, the *α–level* of a fuzzy ideal μ is defined as:

$$\mu_\alpha = \{x \in A \mid \ \mu(x) \geq \alpha\}. \quad (40)$$

Observe that $\mu_0 = A$; for that reason we shall use α–levels with $\alpha \in (0,1]$.

Lemma 13 ([8])**.** *Let μ be a fuzzy subset of a ring A; the following statements hold:*

(1) If μ is a fuzzy ideal, μ_α is an ideal for every $0 \leq \alpha \leq \mu(0)$.
(2) If for any $\alpha \in \mathrm{Im}(\mu)$, we have μ_α is an ideal, then μ is a fuzzy ideal.

Proof. (1). Let $x, y \in \mu_\alpha$, then $\mu(x-y) \geq \min\{\mu(x),\mu(y)\} \geq \alpha$, hence $x-y \in \mu_\alpha$. Otherwise, if $x \in \mu_\alpha$, for any $y \in A$ we have $\mu(xy) \geq \mu(x) \geq \alpha$, then $xy \in \mu_\alpha$.

(2). Since μ_α is an ideal then $0 \in \mu_\alpha$; this means that $\mu(0) \geq \alpha$ for any $\alpha \in \mathrm{Im}(\mu)$, hence $\mu(0) = \max(\mathrm{Im}(\mu))$, and $\mu(0) \geq \mu(x)$ for any $x \in A$.

Let $x, y \in A$ and $\alpha = \min\{\mu(x), \mu(y)\}$, then $x, y \in \mu_\alpha$, and $x - y \in \mu_\alpha$, hence $\mu(x - y) \geq \alpha = \min\{\mu(x), \mu(y)\}$. Otherwise, if $\mu(x) = \alpha$, for any $y \in A$ we have $xy \in \mu_\alpha$, hence $\mu(xy) \geq \alpha = \mu(x)$. In consequence, μ is a fuzzy ideal of A. □

Let us call a *decreasing gradual right ideal* of A a family of ideals $\{\mathfrak{a}_\alpha \mid \alpha \in (0,1]\}$ such that if $\alpha \leq \beta$, then $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$, for any $\alpha, \beta \in (0,1]$. An example of a decreasing gradual right ideal is given by the set of α–level of a fuzzy ideal.

Lemma 14. *Let μ be a fuzzy ideal of a ring A; if $\mu(x) = \mu(y) = \mu(0)$, then $\mu(x - y) = \mu(0)$.*

Proof. It is a direct consequence of the above lemma as $\mu_{\mu(0)}$ is an ideal. □

The problem of working with algebraic operations of fuzzy ideals is hard; as it is pointed out in ([9], (p. 78)), if μ_1 and μ_2 are fuzzy ideals, then $\mu_1 + \mu_2$ non–necessarily coincides with the smallest fuzzy ideal containing μ_1 and μ_2; one condition in order to have this property is that $\mu_1(0) = \mu_2(0)$.

A similar problem arose when associating a right $\mathcal{P}$–module to a fuzzy ideal μ. The natural candidate is $\sigma(\mu)$, defined $\sigma(\mu)(\alpha) = \mu_\alpha = \{x \in A \mid \mu(x) \geq \alpha\}$, the *$\alpha$–level* of μ, which is empty if $\alpha > \mu(0)$.

This second problem can be easily solved if we put $\sigma(\mu)(\alpha) = \{0\}$ whenever $\alpha > \mu(0)$, and this means that a plethora of fuzzy ideals μ have associated the same decreasing gradual right ideal: exactly those which coincides in $A \setminus \{0\}$. To organize all fuzzy ideals, we may define an equivalence relation $\sim$ on fuzzy ideals by $\mu_1 \sim \mu_2$ if $\mu_1(x) = \mu_2(x)$ for any $0 \neq x \in A$. Observe that in the equivalence class $[\mu]$ of μ there exists exactly one element, that attending to μ is denoted by μ^0, such that $\mu^0(0) = 1$, i.e., $\mu^0(x) = \begin{cases} \mu(x), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

Lemma 15 ([4])**.** *Let μ be a fuzzy ideal of a ring A, then μ^0 is a fuzzy ideal.*

As a consequence we may define a new sum operation on fuzzy ideals using equivalence classes: $[\mu_1] + [\mu_2] = [\mu_1^0 + \mu_2^0]$. Be careful, as the map $(-)^0$ is not necessarily a homomorphism with respect to the sum of fuzzy ideals. If necessary, either we avoid the use of parenthesis, or we adorne the sum symbol, as $[+]$, to indicate we are working with equivalence classes. For the two fuzzy ideals, μ_1 and μ_2, we simply write

$$([\mu_1] + [\mu_2])(x) = (\mu_1[+]\mu_2)(x) = \mathrm{Sup}\{\mu_1^0(y) \wedge \mu_2^0(z) \mid y + z = x\}. \tag{41}$$

In this case, associated to every class $[\mu]$, there exists a right $\mathcal{P}$–module $\sigma(\mu)$, which is a submodule of A, the constant right $\mathcal{P}$–module equal to A, which is identify with the contravariant functor $\mathrm{Hom}_{\mathcal{P}}(-, 1)$.

Remark 1. *Unfortunately, the map $[\mu] \mapsto \sigma(\mu)$ is not a homomorphism with respect to the sum of submodules. Indeed, we have:*

$$\begin{aligned} \sigma(\mu_1[+]\mu_2)(x) &= \{x \in A \mid (\mu_1[+]\mu_2)(x) \geq \alpha\} \\ &= \{x \in A \mid \mathrm{Sup}\{\mu_1^0(y) \wedge \mu_2^0(z) \mid y + z = x\} \geq \alpha\} \\ &\supseteq \{y + z \mid \mu_1^0(y) \geq \alpha, \mu_2^0(z) \geq \alpha\} \\ &= \{y \mid \mu_1^0(y) \geq \alpha\} + \{z \mid \mu_2^0(z) \geq \alpha\} \\ &= \sigma(\mu_1)(\alpha) + \sigma(\mu_2)(\alpha). \end{aligned} \tag{42}$$

The following example shows that there are examples in which $\sigma(\mu_1) + \sigma(\mu_2) \subsetneq \sigma(\mu_1 + \mu_2)$.

Example 2. *As we know, for every fuzzy submodules μ_1, μ_2 we always have an inclusion $\sigma(\mu_1) + \sigma(\mu_2) \subseteq \sigma(\mu_1 + \mu_2)$; let us show that this inclusion could be proper. We define fuzzy submodules μ_1 and μ_2 as follows:*

$$\mu_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \setminus 2\mathbb{Z}, \\ 1 - \frac{2^t}{3^t}, & \text{if } x \in 2^t\mathbb{Z} \setminus 2^{t+1}\mathbb{Z}, \\ 1, & \text{if } x = 0. \end{cases} \qquad \mu_2(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z} \setminus 3\mathbb{Z}, \\ \frac{1}{2} - \frac{1}{3^t}, & \text{if } x \in 3^t\mathbb{Z} \setminus 3^{t+1}\mathbb{Z}, \\ 1, & \text{if } x = 0. \end{cases} \tag{43}$$

We claim $(\mu_1 + \mu_2)(2) = \mathrm{Sup}\{\mu_1(y) \wedge \mu_2(2-y) \mid y \in \mathbb{Z}\} \leq \frac{1}{2}$. *Indeed, we have two possibilities:*

(1) $\mu_1(y) > \frac{1}{2}$, *then* $y \in 4\mathbb{Z}$, *i.e., there exists* $k \in \mathbb{Z}$ *such that* $y = 4k$. *Hence,* $\mu_2(2-y) = \mu_2(2-4k) = \mu_2(2(1-2k)) < \frac{1}{2}$ *as* $2 - y \neq 0$.

(2) $\mu_1(y) < \frac{1}{2}$.

In both cases, we have $\mu_1(y) \wedge \mu_2(2-y) < \frac{1}{2}$. *In addition, we can choose y, such that* $\mu_1(y) \wedge \mu_2(2-y)$ *is as closed to* $\frac{1}{2}$ *as we desire. For any* $2 \leq t, s \in \mathbb{N}$ *there exist* $k, h \in \mathbb{Z}$ *such that* $2^{t-1}k - 3^s h = 1$, *hence* $2 - 2^t k = 2(1 - 2^{t-1}k) = 3^s h$; *now, if we take* $y = 2^t k$, *then* $\mu_1(y) \geq 1 - \frac{1}{3^t}$ *and* $\mu_2(2-y) \geq \frac{1}{2} - \frac{1}{3^t}$. *In consequence,* $\frac{1}{2} > \mu_1(y) \wedge \mu_2(2-y) \geq \frac{1}{2} - \frac{1}{3^t}$, *which implies that* $(\mu_1 + \mu_2)(2) = \frac{1}{2}$, *and* $2 \in (\mu_1 + \mu_2)_{\frac{1}{2}}$. *On the other hand, we have* $(\mu_1)_{\frac{1}{2}} + (\mu_2)_{\frac{1}{2}} = 4\mathbb{Z}$, *and* $2 \notin (\mu_1)_{\frac{1}{2}} + (\mu_2)_{\frac{1}{2}}$.

Remark 2. *The intersection of two fuzzy ideals* μ_1, μ_2 *satisfies* $\sigma(\mu_1 \cap \mu_2) = \sigma(\mu_1) \cap \sigma(\mu_2)$, *and the intersection of a family* $\{\mu_i \mid i \in I\}$ *of fuzzy ideals satisfies* $\sigma(\cap_i \mu_i) = \cap_i \sigma(\mu_i)$. *We point out that the map* $(-)^0$ *is a homomorphism with respect to the intersection.*

Remark 3. *Since, for any fuzzy ideal μ and elements* $x, y \in A$ *we have* $\mu(xy) \geq \mu(x) \vee \mu(y)$, *hence* $\mu_a \cdot \mu_b \subseteq \mu_{a \vee b}$, *and this coincides with the multiplication of right ideals of* $\mathcal{P}$: *if* $\alpha \leq \beta$ *then* $\sigma(\alpha) \cdot \sigma(\beta) \subseteq \sigma(\beta)$ *as* $\sigma(\beta)$ *is an ideal; hence it is with the multiplication of right ideals in the functor category.*

Remark 4. *The product of two fuzzy ideals* μ_1, μ_2 *is defined as:*

$$(\mu_1\mu_2)(x) = \mathrm{Sup}\left\{ \wedge_i(\mu_1(x_i) \wedge \mu_2(y_i)) \mid x = \sum_i x_i y_i \right\}. \tag{44}$$

Moreover, the map $(-)^0$ *is a homomorphism with respect to this product. For any fuzzy ideals,* μ_1, μ_2, *we have:*

$$\begin{aligned} (\mu_1^0\mu_2^0)(x) &= \mathrm{Sup}\{\wedge_i(\mu_1^0(x_i) \wedge \mu_2^0(y_i)) \mid x = \textstyle\sum_i x_i y_i\} \\ &= \mathrm{Sup}\{\wedge_i(\mu_1(x_i) \wedge \mu_2(y_i)) \mid x = \textstyle\sum_i x_i y_i\} = (\mu_1\mu_2)^0(x), \end{aligned} \tag{45}$$

if $x \neq 0$. *On the other hand,*

$$\begin{aligned} \sigma(\mu_1\mu_2)(\alpha) &= \{x \mid \mu_1\mu_2(x) \geq \alpha\} \\ &= \{x \mid \mathrm{Sup}\{\wedge_i(\mu_1(x_i) \wedge \mu_2(y_i)) \mid x = \textstyle\sum_i x_i y_i\} \geq \alpha\} \\ &\supseteq \{x \mid x = \textstyle\sum_i x_i y_i,\ x_i \in \sigma(\mu_1)(\alpha),\ y_i \in \sigma(\mu_2)(\alpha)\} \\ &= \sigma(\mu_1)(\alpha)\sigma(\mu_2)(\alpha) \\ &= (\sigma(\mu_1)\sigma(\mu_2))(\alpha). \end{aligned} \tag{46}$$

and the equality does not necessarily hold.

It is very easy to build examples in which we have proper inclusion $\sigma(\mu_1)\sigma(\mu_2) \subsetneqq \sigma(\mu_1\mu_2)$. In the following, we show one.

Example 3. *Let K be a field,* $\{X, Y\} \cup \{X_n \mid n \in \mathbb{N}\}$ *be a family of indeterminates over K, and* $\mathfrak{a}$ *the ideal of the polynomial ring* $K[X, Y, X_0, \ldots]$ *generated by the set* $\{X - Y^n X_n \mid n \in \mathbb{N}\}$.

We denote by A the quotient ring $\dfrac{K[X,Y,X_0,\dots]}{\mathfrak{a}} = K[x,y,x_0,\dots]$, *satisfying the relations* $x - y^n x_n = 0$, *for every* $n \in \mathbb{N}$.

In A, we have a strictly descending chain of ideals:

$$A \supsetneqq (y,x_0,x_1,\dots) \supsetneqq (y^2,x_0,x_1,\dots,) \supsetneqq \cdots \supsetneqq (y^n,x_0,x_1,\dots) \supsetneqq \cdots \supsetneqq (x_0,x_1,\dots). \quad (47)$$

Therefore, there is a fuzzy ideal μ, *defined by*

$$\begin{array}{l} \mu((y^n,x_0,x_1,\dots)\setminus(y^{n+1},x_0,x_1,\dots)) = \frac{1}{2} - \frac{1}{2^n}, \text{ for every } n \in \mathbb{N}, \text{ and} \\ \mu(x_0,x_1,\dots) = 1. \end{array} \quad (48)$$

Observe that $\mu(x_0) = \mu(x) = 1$, *but* $(\mu\mu)(x) = \mathrm{Sup}\{\mu(y) \wedge \mu(z) \mid yz = x\} \leq \frac{1}{2}$. *Therefore,* $x \in \sigma(\mu)(1)\sigma(\mu)(1) = \mu_1\mu_1 \subsetneqq (\mu\mu)_1 = \sigma(\mu\mu)(1)$.

Up to the present, we considered the decreasing gradual right ideal $\sigma(\mu)$, *defined by the* α*–levels:* $\sigma(\mu)(\alpha) = \mu(\alpha)$, *for every* $\alpha \in (0,1]$. *On the other hand, if we consider* $\widetilde{\sigma}(\mu) = \sigma(\mu)^d$, *we obtain a decreasing gradual right ideal that satisfies* $\widetilde{\sigma}(\mu\mu) = \widetilde{\sigma}(\mu)\widetilde{\sigma}(\mu)$, *as we demonstrate that* $\widetilde{\sigma}$ *preserves the product. The same holds when we consider the sum.*

To establish a homomorphism with respect to the sum and the product, first, let us collect in the following proposition the behaviour of $(-)^0$ with respect to the usual operations of fuzzy ideals.

Proposition 5. *Let* μ_1,μ_2 *be fuzzy ideals, and the following statements hold:*

(1) *In general,* $(\mu_1+\mu_2)^0 \neq \mu_1^0 + \mu_2^0$, *hence we define* $[\mu_1]+[\mu_2] = [\mu_1^0+\mu_2^0]$.
(2) $(\mu_1 \cap \mu_2)^0 = \mu_1^0 \cap \mu_2^0$, *hence we may define* $[\mu_1]\cap[\mu_2] = [\mu_1 \cap \mu_2]$.
(3) $(\mu_1\mu_2)^0 = \mu_1^0\mu_2^0$, *hence we may define* $[\mu_1]\cdot[\mu_2] = [\mu_1 \cdot \mu_2]$.

Second, in order to arrange the drawback shows in Remarks (1) and (4): σ is not a homomorphism with respect to the sum and product. We modify the notion of α–levels in considering strict α–levels. Let μ be a fuzzy ideal of a ring A, for any $\alpha \in [0,1]$, the *strong* α*–level* $\widetilde{\mu}_\alpha$ is defined as

$$\widetilde{\sigma}(\mu)(\alpha) = \widetilde{\mu}_\alpha = \begin{cases} \{x \in A \mid \mu^0(x) > \alpha\}, & \text{if } \alpha < 1, \\ \{x \in A \mid \mu^0(x) = 1\}, & \text{if } \alpha = 1. \end{cases} \quad (49)$$

This definition can be extended to any fuzzy subset μ such that $\mathrm{Im}(\mu)$ has a maximum element α_0. In this case, we shall define the *strong* α*–level* $\widetilde{\mu}_\alpha$ of μ as:

$$\widetilde{\sigma}(\mu)(\alpha) = \widetilde{\mu}_\alpha = \begin{cases} \{x \in A \mid \mu(x) > \alpha\} \cup \{0\}, & \text{if } \alpha \in \mathrm{Im}(\mu)\setminus\{\alpha_0\}, \\ \{x \in A \mid \mu(x) = \alpha_0\} \cup \{0\}, & \text{if } \alpha \geq \alpha_0. \end{cases} \quad (50)$$

Lemma 16. *Let* μ *be a fuzzy subset of a ring A such that* $\mathrm{Im}(\mu)$ *has a maximum, the following statements hold,*

(1) *If* μ *is a fuzzy ideal, for any* $\alpha \in [0,1]$ *we have that* $\widetilde{\mu}_\alpha$ *is a decreasing gradual right ideal.*
(2) *If for any* $\alpha \in [0,1]$, *we have that* $\widetilde{\mu}_\alpha$ *is an ideal, and then* μ *is a fuzzy ideal.*

Proof. (1). By Lemma (14) we have that $\widetilde{\mu}_1$ is an ideal. If $\alpha < \mu(0)$ and $x,y \in \widetilde{\mu}_\alpha$, then $\mu(x),\mu(y) > \alpha$, hence $\mathrm{Min}\{\mu(x),\mu(y)\} > \alpha$ and $\mu(x-y) \geq \mathrm{Min}\{\mu(x),\mu(y)\} > \alpha$, i.e., $x-y \in \widetilde{\mu}_\alpha$. If $x \in \widetilde{\mu}_\alpha$, for any $y \in A$ we have $\mu(xy) \geq \mu(x) > \alpha$, i.e., $xy \in \widetilde{\mu}_\alpha$.

(2). Let $x,y \in A$ and $\beta = \mu(x) \leq \mu(y)$. For any $\alpha < \beta$ we have $x,y \in \widetilde{\mu}_\alpha$, hence $x-y \in \widetilde{\mu}_\alpha$ and $\mu(x-y) > \alpha$, hence $\mu(x-y) \geq \beta$. Otherwise, if $x,y \in A$ and $\mu(x) = \beta$, for any $\alpha < \beta$ we have $x \in \widetilde{\mu}_\alpha$, then $xy \in \widetilde{\mu}_\alpha$ and $\mu(xy) > \alpha$, hence $\mu(xy) \geq \beta = \mu(x)$. □

In this way, we have that there exists a right $\mathcal{P}$–ideal $\widetilde{\sigma}(\mu)$ associated to the fuzzy ideal μ,

$$\widetilde{\sigma}(\mu)(\alpha) = \widetilde{\mu}_\alpha, \text{ for any } \alpha \in (0,1]. \tag{51}$$

Observe that $\widetilde{\sigma}(\mu) \subseteq \sigma(\mu)$.

The definition of $\widetilde{\sigma}$ can be extended to equivalence classes of fuzzy ideals in the obvious way. In this case, the map $[\mu] \mapsto \widetilde{\sigma}(\mu^0)$ is a homomorphism with respect to: the sum, the intersection and the product of classes of fuzzy ideals. For simplicity, we consider fuzzy ideals μ such that $\mu = \mu^0$, and if by Proposition 5 we avoid the use of brackets, then we have the following proposition:

Proposition 6. *Let μ_1, μ_2 be fuzzy ideals, then we have:*

(1) $\widetilde{\sigma}(\mu_1[+]\mu_2) = \widetilde{\sigma}(\mu_1) + \widetilde{\sigma}(\mu_2)$.
(2) $\widetilde{\sigma}(\mu_1\mu_2) = \widetilde{\sigma}(\mu_1)\widetilde{\sigma}(\mu_2)$.
(3) $\widetilde{\sigma}(\mu_1 \cap \mu_2) = \widetilde{\sigma}(\mu_1) \cap \widetilde{\sigma}(\mu_2)$.

Proof. (1). We have:

$$\begin{aligned} \widetilde{\sigma}(\mu_1[+]\mu_2)(x) &= \{x \in A \mid (\mu_1[+]\mu_2)(x) > \alpha\} \\ &= \{x \in A \mid \operatorname{Sup}\{\mu_1^0(y) \wedge \mu_2^0(z) \mid y+z=x\} > \alpha\} \\ &= \{y+z \mid \mu_1^0(y) > \alpha, \mu_2^0(z) > \alpha\} \\ &= \{y \mid \mu_1^0(y) > \alpha\} + \{z \mid \mu_2^0(z) > \alpha\} \\ &= \widetilde{\sigma}(\mu_1)(\alpha) + \widetilde{\sigma}(\mu_2)(\alpha). \end{aligned} \tag{52}$$

The same holds if we consider either the product or the intersection. □

The proof of the following proposition is straightforward.

Proposition 7. *For any family of fuzzy ideals $\{\mu_i \mid i \in I\}$ we have:*

(1) $\widetilde{\sigma}(\sum_i \mu_i) = \sum_i \widetilde{\sigma}(\mu_i)$.
(2) $\widetilde{\sigma}(\cap_i \mu_i) \subseteq \cap_i \widetilde{\sigma}(\mu_i)$.

In a similar way, we can develop this theory for fuzzy submodules of A–modules.

4.2. How to Associate Fuzzy Ideals to Gradual Right Ideals

We have studied how to associate a right $\mathcal{P}$–ideal to each fuzzy ideal in such a way that we have homomorphism with respect to the sum, intersection and product. In addition, this association preserves arbitrary sums. Presently, we deal with the reciprocal problem: associate a fuzzy ideal to a gradual right ideal.

Indeed, for any fuzzy ideal μ (satisfying $\mu = \mu^0$) the right $\mathcal{P}$–modules $\sigma(\mu)$ and $\widetilde{\sigma}(\mu)$ below are torsionfree, i.e., they belong to the class $\mathcal{J}$ and, by [4], they can be identified with *decreasing gradual right ideals* of A, i.e., maps σ from $(0,1]$ to the lattice of all ideals $\mathcal{L}(A)$ of A such that $\sigma(\beta) \subseteq \sigma(\alpha)$ whenever $\alpha \leq \beta$.

The problem is that not every decreasing gradual right ideal come from a fuzzy ideal; hence, first we need to know how to characterize those which are images of a fuzzy ideal.

Let μ be a fuzzy ideal; for any $x \in A$ we have that $\mu(x) = \operatorname{Max}\{\beta \mid x \in \sigma(\mu)(\beta)\}$. For any decreasing gradual right ideal σ, we say σ satisfies *property (max–F)* if for any $x \in A$ there exists $\operatorname{Max}\{\beta \mid x \in \sigma(\beta)\}$.

In the same way, if we consider $\widetilde{\sigma}(\mu)$, for any $x \in A$ we have $\mu(x) = \operatorname{Inf}\{\gamma \mid x \notin \widetilde{\mu}_\alpha\}$, and it is not the minimum. Thus, for any decreasing gradual right ideal σ, we say σ satisfies property *property (inf–F)* if for any $x \in A$ there exists $\operatorname{Inf}\{\gamma \mid x \notin \sigma(\gamma)\}$, and it is not the minimum. See Lemma (22) below.

Let σ be a decreasing gradual right ideal satisfying property (max–F), we define a map $\mu(\sigma) : A \longrightarrow [0,1]$ as follows:

$$\mu(\sigma)(x) = \operatorname{Max}\{\alpha \mid x \in \sigma(\alpha)\}. \tag{53}$$

First, we demonstrate that $\mu(\sigma)$ is a fuzzy ideal.

Lemma 17. *If σ is a decreasing gradual right ideal satisfying property (max–F), then $\mu(\sigma)$ is a fuzzy ideal and $\mu(\sigma) = \mu^0$.*

Proof. Let $x_1, x_2 \in A$; if $\mathrm{Max}\{\gamma \mid x \in \sigma(\gamma)\} = \mu(\sigma)(x_1) = \alpha_1 \leq \alpha_2 = \mu(\sigma)(x_2) = \mathrm{Max}\{\beta \mid x_2 \in \sigma(\beta)\}$, then $x_2 \in \mu(\sigma)_{\alpha_2} \subseteq \mu(\sigma)_{\alpha_1}$, hence $x_1, x_2 \in \mu(\sigma)_{\alpha_1}$, and $x_1 - x_2 \in \mu(\sigma)_{\alpha_1}$, i.e., $\mu(\sigma)(x_1 - x_2) \geq \alpha_1$.

Otherwise, for any $x, y \in A$, if $\mu(\sigma)(x) = \alpha$, then $xy \in \mu(\sigma)_\alpha$, hence $\mu(\sigma)(xy) \geq \alpha$. □

Lemma 18. *With the above notation, the maps $\mu \mapsto \sigma(\mu)$ and $\sigma \mapsto \mu(\sigma)$ establish a bijective correspondence between*

(i) Equivalence classes of fuzzy ideals and
(ii) Decreasing gradual right ideals satisfying the property (max–F).

Proof. Let μ be a fuzzy ideal, and consider $\mu(\sigma(\mu))$; for any $x \in A$, we have:

$$\mu(\sigma(\mu))(x) = \mathrm{Max}\{\gamma \mid x \in \sigma(\mu)(\gamma)\} = \mathrm{Max}\{\gamma \mid \mu(x) \geq \gamma\} = \mu(x). \tag{54}$$

Otherwise, let σ be a decreasing gradual right ideal and consider $\sigma(\mu(\sigma))$; for any $\alpha \in (0, 1]$, we have:

$$\begin{aligned}\sigma(\mu(\sigma))(\alpha) = \{x \mid \mu(\sigma)(x) \geq \alpha\} = \{x \mid \mathrm{Max}\{\gamma \mid x \in \sigma(\gamma)\} \geq \alpha\} \\ = \{x \mid x \in \sigma(\alpha)\} = \sigma(\alpha).\end{aligned} \tag{55}$$

□

Unfortunately, these maps are not homomorphisms when we consider the sum or product of ideals.

In the following, we shall consider the map $\mu \mapsto \widetilde{\sigma}(\mu)$, which will be a homomorphism with respect to sum, product and intersection of ideals. We are working for building an inverse to the map $\widetilde{\sigma}$. Since $\widetilde{\sigma}(\mu)$ satisfies the property (inf-F), we can define

$$\widetilde{\mu}(\widetilde{\sigma}(\mu))(x) = \mathrm{Inf}\{\gamma \mid x \notin \widetilde{\sigma}(\mu)(\gamma)\}, \tag{56}$$

being $\mathrm{Inf}\,\varnothing = 1$.

Lemma 19. *Let μ be a fuzzy ideal, then*

(1) $\widetilde{\mu}(\widetilde{\sigma}(\mu))$ is a fuzzy ideal.
(2) $\mu^0 = \widetilde{\mu}(\widetilde{\sigma}(\mu))$.

Proof. For any $0 \neq x \in A$ we have:

$$\widetilde{\mu}(\widetilde{\sigma}(\mu))(x) = \mathrm{Inf}\{\gamma \mid x \notin \widetilde{\sigma}(\mu)(\gamma)\} = \mathrm{Inf}\{\gamma \mid \mu(x) \leq \gamma\} = \mu(x). \tag{57}$$

□

In particular, we can see the class of fuzzy ideals inside of the class of decreasing gradual right ideals via the map $\mu \mapsto \widetilde{\sigma}(\mu)$.

This means that $\mu \mapsto \widetilde{\sigma}(\mu)$ is an injective map from the set of equivalence classes of all fuzzy ideals into the set of all decreasing gradual right ideals satisfying the property (inf–F). Otherwise, for any fuzzy ideal μ, we have $\widetilde{\sigma}(\mu) \subseteq \sigma(\mu)$, and the equality does not necessarily hold.

The problem is to determine those decreasing gradual right ideals σ, such that $\sigma = \widetilde{\sigma}(\mu)$ for some fuzzy ideal μ: we know that σ satisfies the following property:

(1) satisfies property (inf–F).

To characterize them, we define an operator, $\sigma \mapsto \sigma^d$, in the set of all gradual right ideals, i.e., maps σ from $(0,1]$ to the lattice of all ideals of A together with maps $\sigma(f_{\alpha,\beta} : \sigma(\beta) \longrightarrow \sigma(\alpha)$, whenever $\alpha \leq \beta$, satisfying $\sigma(f_{\beta,\gamma})\sigma(f_{\alpha,\beta}) = \sigma(f_{\alpha,\gamma})$, if $\alpha \leq \beta \leq \gamma$. For any gradual right ideal σ, we define a new gradual subset σ^d as follows:

$$\sigma^d(\alpha) = \begin{cases} \sigma(1), & \text{if } \alpha = 1, \\ \sum\{\sigma(\beta) \mid \beta > \alpha\}, & \text{for any } \alpha \in (0,1). \end{cases} \tag{58}$$

Lemma 20. *If σ is a gradual right ideal, then*

(1) *σ^d is a decreasing gradual right ideal.*
(2) *Not necessarily, we have $\sigma \subseteq \sigma^d$.*
(3) *$\sigma^{dd} = \sigma^d$.*

Proof. (1). For every $\alpha \in (0,1)$ we have $\sigma^d(\alpha) = \sum\{\sigma(\beta) \mid \beta > \alpha\}$ as a sum of submodules.

(2). It is easy.

(3). For every $\alpha \in (0,1)$ we have:

$$\sigma^{dd}(\alpha) = \sum\{\sigma^d(\beta) \mid \beta > \alpha\} = \sum\{\sum\{\sigma(\gamma) \mid \gamma > \beta\} \mid \beta > \alpha\} \\ = \sum\{\sigma(\gamma) \mid \gamma > \beta > \alpha\} = \sigma^d(\alpha). \tag{59}$$

□

We name *strictly decreasing gradual right ideal* any gradual right ideal σ such that $\sigma^d = \sigma$.

Since $\sigma^{dd} = \sigma^d$, it seems that we have a kind of *closure operator*; we only need to be situated in the suitable framework. In this case, it is the set of all decreasing gradual right ideals. Indeed, we have:

Lemma 21. *Let σ be a decreasing gradual right ideal, then we have:*

(1) *$\sigma^d \subseteq \sigma$.*
(2) *$\sigma^{dd} = \sigma^d$.*

If $\sigma \subseteq \tau$ are decreasing gradual right ideals, in addition then

(3) *$\sigma^d \subseteq \tau^d$.*
(4) *σ^d is the largest strictly decreasing gradual right ideal contained in σ.*

In consequence, we have an *interior operator*, $\sigma \mapsto \sigma^d$, on decreasing gradual right ideals, which characterizes strictly decreasing gradual right ideals.

We shall relate the properties (max–F) and (inf–F) in the sense that one of them is the proper one of the decreasing gradual right ideals and the other of strictly decreasing gradual right ideals.

Lemma 22. *Let σ be a decreasing right gradual right ideal, and the following statements are equivalent:*

(a) *σ satisfies property (max–F),*
(b) *σ^d satisfies property (inf–F).*

Proof. (a) $\Rightarrow$ (b). Let $x \in A$, and $\alpha = \text{Inf}\{\gamma \mid x \notin \sigma^d(x)\} = \text{Inf}\{\gamma \mid x \notin \sigma(\gamma)\}$. Let $\beta = \text{Max}\{\gamma \mid x \in \sigma(\gamma)\}$; hence $\beta \leq \alpha$, for any $\beta \leq \delta < \alpha$ we have $x \in \sigma(\delta)$ as $\delta < \alpha$, hence $\beta = \delta$; as a consequence $\beta = \alpha$.

(b) $\Rightarrow$ (a). Let $x \in A$, and $\alpha = \text{Sup}\{\gamma \mid x \in \sigma(\gamma)\}$. If $x \notin \sigma(\alpha)$, then $\beta = \text{Inf}\{\gamma \mid x \notin \sigma(\gamma)\} \leq \alpha$. For any $\beta < \delta < \alpha$, we have $x \in \sigma(\delta)$, because $\delta < \alpha$, and $x \notin \sigma(\delta)$ because $\beta < \delta$, which is a contradiction. Therefore, $x \in \sigma(\alpha)$. □

Presently, the following result is immediate.

Lemma 23. *For any fuzzy ideal μ, we have $\sigma(\mu)^d = \widetilde{\sigma}(\mu)$.*

Our aim is to establish a bijective correspondence between the sets of equivalence classes of fuzzy ideals and strictly decreasing gradual right ideals satisfying property (inf–F). With this in mind, for any gradual right ideal σ, we define a fuzzy subset $\widetilde{\mu}(\sigma)$ as follows:

$$\widetilde{\mu}(\sigma)(x) = \mathrm{Inf}\{\gamma \mid x \notin \sigma(\gamma)\}. \tag{60}$$

To establish a well founded correspondence between fuzzy ideals and a class of decreasing gradual right ideals, we need another definition. We continue with a useful Lemma.

Lemma 24. *Let σ be a decreasing gradual right ideal, if $\left(\sigma(\alpha) \setminus \sigma^d(\alpha)\right) \cap \left(\sigma(\beta) \setminus \sigma^d(\beta)\right) \neq \varnothing$, then $\alpha = \beta$, for any $\alpha, \beta \in P$.*

Proof. Let us assume $\alpha < \beta$, and let x in the intersection, then $x \in \sigma(\alpha)$ and $x \notin \sigma^d(\alpha)$, i.e., $x \notin \sigma(\beta)$ for any $\beta > \alpha$, hence $x \notin \sigma(\beta) \setminus \sigma^d(\beta)$. □

Let σ, τ be a decreasing gradual right ideal, and the pair (σ, τ) is an *E–pair* if the following statements hold.

(i) $\sigma \subseteq \tau$, $\sigma(1) = \tau(1)$, and σ is d–open.
(ii) $\{\tau(\alpha) \setminus \sigma(\alpha) \mid 1 \neq \alpha \in P\} \cup \{\tau(1)\}$ is a set of *mutually disjoint subsets*, i.e., any two non–empty sets have an empty intersection.

Lemma 25. *For any E–pair (σ, τ) we have $\sigma \subseteq \tau^d$.*

Proof. We have $\sigma = \sigma^d \subseteq \tau^d \subseteq \tau$, hence $\sigma \subseteq \tau^d$. □

Proposition 8. *If (σ, τ) is an E-pair, then $\sigma = \tau^d$.*

Proof. We always have $\tau(\alpha) \setminus \tau^d(\alpha) \subseteq \tau(\alpha) \setminus \sigma(\alpha)$. Otherwise, if $x \in \tau(\alpha) \setminus \sigma(\alpha)$, and $x \notin \tau(\alpha) \setminus \tau^d(\alpha)$, then $x \in \tau^d(\alpha)$, and there exists $\beta > \alpha$ such that $x \in \tau(\beta) \subseteq \tau(\alpha)$. Since $x \notin \sigma(\alpha)$, then $x \notin \sigma(\beta)$, and $x \in \tau(\beta) \setminus \sigma(\beta)$. We have $(\tau(\alpha) \setminus \sigma(\alpha)) \cap (\tau(\beta) \setminus \sigma(\beta)) \neq \varnothing$, which is a contradiction by Lemma (24). As a consequence, $\tau(\alpha) \setminus \sigma(\alpha) = \tau(\alpha) \setminus \tau^d(\alpha)$, i.e., $\sigma(\alpha) = \tau^d(\alpha)$. □

An E–pair (σ, τ) is an *F–pair* if, in addition, it satisfies:

(iii) $\cup\{\tau(\alpha) \setminus \sigma(\alpha) \mid 1 \neq \alpha \in P\} \cup F(1) = \cup\tau_\alpha(\alpha)$.

Remark 5. *Observe that if σ is a d–open gradual right ideal, then (σ^d, σ) is an E–pair but it is not an F–pair.*

Let (σ, τ) be an F–pair, then for any $x \in \cup_\alpha \sigma(\alpha)$, we define

$$\mu_{(\sigma,\tau)}(x) = \begin{cases} 1, & \text{if } x \in \sigma(1), \\ \alpha, & \text{if } x \in \tau(\alpha) \setminus \sigma(\alpha). \end{cases} \tag{61}$$

Theorem 1. *For any F–pair (σ, τ) we have that $\mu_{(\sigma,\tau)}$ is a fuzzy ideal satisfying $\mu_{(\sigma,\tau)}(0) = 1$.*

Proof. It follows from the following fact: with this definition, for any $\alpha \in (0,1]$, the α–level of $\mu_{(\sigma,\tau)}$ is just $\tau(\alpha)$. □

Now we can build the announced correspondence.

Proposition 9. *Let A be a ring, then:*

(1) *If σ is a strictly decreasing gradual right ideal satisfying property (inf-F), then $\widetilde{\mu}(\sigma)$ is a fuzzy ideal.*
(2) *The maps $\mu \mapsto \widetilde{\sigma}(\mu)$ and $\sigma \mapsto \widetilde{\mu}(\sigma)$ establish a bijective correspondence between*
   (a) *Equivalence classes of fuzzy ideals.*
   (b) *Strictly decreasing gradual right ideals satisfying the property (inf–F).*
   (c) *Gradual right ideals satisfy (max–F).*
   (d) *F–pairs in A.*

Proof. Let $x_1, x_2 \in A$, if $\operatorname{Inf}\{\gamma \mid\ x_1 \notin \sigma(\gamma)\} = \widetilde{\mu}(\sigma)(x_1) = \alpha_1 \leq \alpha_2 = \widetilde{\mu}(\sigma)(x_2) = \operatorname{Inf}\{\beta \mid x_2 \notin \sigma(\beta)\}$, then $x_1 - x_2 \in \sigma(\alpha_1)$, and

$$\widetilde{\mu}(\sigma)(x_1 - x_2) = \operatorname{Inf}\{\gamma \mid\ x_1 - x_2 \notin \sigma(\gamma)\} \geq \alpha_1 = \alpha_1 \wedge \alpha_2 = \widetilde{\mu}(\sigma)(x_1) \wedge \widetilde{\mu}(\sigma)(x_2). \quad (62)$$

Let $x_1, x_2 \in A$, if $\operatorname{Inf}\{\gamma \mid\ x_1 \notin \sigma(\gamma)\} = \widetilde{\mu}(\sigma)(x_1) = \alpha_1$, then $x_1x_2 \in \sigma(\alpha_1)$, and $\widetilde{\mu}(\sigma)(x_1x_2) \geq \widetilde{\mu}(\sigma)(x_1)$. □

4.3. Fuzzy and Gradual Submodules

We have studied fuzzy ideals and related them with gradual right ideals; in the same way, we can perform these notions for fuzzy modules and gradual right modules.

Let M be an A–module, a fuzzy subset μ of M is a *fuzzy submodule* if for any $x, y \in M$ we have:

(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(xr) \geq \mu(x)$ for any $r \in A$ and
(3) $\mu(0) \neq 0$, to avoid the trivial case.

In the same way, we define α–levels, strong α–levels, and gradual right submodules associated to a given fuzzy submodule. In this way, we obtain a similar result to the case of fuzzy and gradual right ideals.

Proposition 10. *Let A be a ring, and M be an A–module, then:*

(1) *If μ is a fuzzy submodule of M, then $\sigma(\mu)$ and $\widetilde{\sigma}(\mu)$ are decreasing gradual right submodules of M.*
(2) *If σ is a decreasing gradual right submodule satisfying property (F), then $\mu(\sigma)$ is a fuzzy submodule.*
(3) *If σ is a strictly decreasing gradual right ideal satisfying property (inf-F), then $\widetilde{\mu}(\sigma)$ is a fuzzy ideal.*
(4) *The maps $\mu \mapsto \widetilde{\sigma}(\mu)$ and $\sigma \mapsto \widetilde{\mu}(\sigma)$ establish a bijective correspondence between*
   (i) *Equivalence classes of fuzzy ideals.*
   (ii) *Strictly decreasing gradual right ideals satisfying the property (inf–F).*
   (iii) *Decreasing gradual right ideals satisfy (max–F).*
   (iv) *F–pair in M.*
(5) *In the case of strictly decreasing gradual right submodules, the bijection defined by $\mu \mapsto \widetilde{\sigma}(\mu)$ and $\sigma \mapsto \widetilde{\mu}(\sigma)$ preserves sums, intersections and products by ideals.*

5. Conclusions

In this paper we have built several object and categories with the aim of providing a categorical framework for fuzzy modules.

(1) Starting from a directed poset P, we have constructed a preadditive category $\mathcal{P}$ and the functor category $\mathcal{P}-\mathbf{Mod}$ as ambient to study fuzzy submodules.
(2) If P is $(0, 1]$, this module category has a special class of objects: $\mathcal{J}$ which is a torsionfree class in $\mathcal{P}-\mathbf{Mod}$.
(3) Using this class, we have built a pair of modules, or equivalently, of gradual submodules.
(4) To assure that the sum of fuzzy submodules is a good operation, we have introduced an equivalence relation and demonstrated that we can restrict ourselves to consider operations on equivalence classes, to finally establish a bijective correspondence between equivalence classes of fuzzy submodules and certain pairs: the F-pairs.

(5) With this correspondence, we have shown that it is a homomorphism for the usual operations of fuzzy ideals: the sum, the product, the union and the intersection, improving the results obtained when using the usual α–levels associated to a fuzzy submodule.

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