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Fuzzy cardinality based evaluation of quantified sentences

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Abstract

Quantified statements are used in the resolution of a great variety of problems. Several methods have been proposed to evaluate statements of types I and II. The objective of this paper is to study these methods, by comparing and generalizing them. In order to do so, we propose a set of properties that must be fulfilled by any method of evaluation of quantified statements, we discuss some existing methods from this point of view and we describe a general approach for the evaluation of quantified statements based on the fuzzy cardinality and fuzzy relative cardinality of fuzzy sets. In addition, we discuss some concrete methods derived from the mentioned approach. These new methods fulfill all the properties proposed and, in some cases, they provide an interpretation or generalization of existing methods.

1. Introduction

Quantified sentences are used in a large number of applications for representing assertions and/or restrictions about the number or percentage of objects that verify a certain property. These assertions and/or restrictions are one of the most used by humans in their reasoning processes. Because of this, some

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authors have tried to define a mathematical model for the representation of this knowledge in the field of AI by using the theory of fuzzy sets. The first approach was described in [28] by Zadeh. Since then, quantified sentences have been used in the resolution of several problems. One of the fields where quantified sentences have been more applied is that of flexible database querying. There is a large amount of literature on this topic, such as [3,7,15,17]. Quantified sentences have been applied in other fields such as pattern recognition, inductive learning, aggregation and decision making among others. Papers as [28–30,18,8–11,25] are some examples. Applications of quantified sentences in the field of expert systems are discussed in [13], where there is a section devoted to the applications of quantified sentences. In the field of data mining, quantified sentences have been used for example in [22]. We will use quantified sentences to develop data mining applications. In general, quantified sentences are a useful approximate reasoning tool for solving problems where linguistic quantifiers and natural languages are used in the representation of our knowledge.

Quantified sentences are usually classified into two classes, called type I sentences and type II sentences. A type I sentence is a sentence of the form:

$$Q \text{ of } X \text{ are } A,$$

where $X = \{x_1, \ldots, x_n\}$ is a finite set, Q a linguistic quantifier and A a fuzzy property defined over X.

A type II sentence can be described in general as:

$$Q \text{ of } D \text{ are } A,$$

where D is also a fuzzy property over X. Obviously, type I sentences are a special case of type II sentences where $D = X$. The following are examples of each type of sentences:

Type I : Most of the students are young.

Type II : Most of the efficient students are young.

In these examples, the set X is a finite set of students, the quantifier is "Most", the set A is the property "young" and the set D is the property "efficient".

Two kinds of linguistic quantifiers are taken into consideration in the evaluation of quantified sentences: these are called *absolute* and *relative* quantifiers. They are defined as possibility distributions, over the non-negative integers and the real interval $[0, 1]$, respectively. Absolute quantifiers represent fuzzy integer quantities or intervals. Examples of this type are "approximately 5" and "between 2 and 4". Relative quantifiers represent fuzzy proportions. Some examples are "Many", "Most" and "All". Although relative quantifiers are defined over the real interval $[0, 1]$ for simplicity, in fact only values of the rational interval $[0, 1]$ are used in the evaluation.

There are four possible combinations (type of sentence, type of quantifier). Each possible combination is evaluated in a slightly different way by any existing method of evaluation. In [12] sentences and quantifiers are related as follows: (type I, absolute); (type II, relative). In fact, one type II sentence with an absolute quantifier can be transformed into an equivalent type I sentence in the following way: "Q of D are A" with Q an absolute quantifier is equivalent to "Q of X are $A \cap D$". But in our opinion, it is important to add the pair (type I, relative) to the problem of the evaluation of sentences. Some of the better studied methods of evaluation focus on this case.

Evaluation of quantified sentences tries to obtain an accomplishment degree in the real interval $[0, 1]$ for the sentence. Different methods have been proposed to perform the evaluation of quantified sentences following this approach. Methods for the evaluation of type I sentences are described in [30,18,19,1,2,4]. Methods for type II sentences are described in [30,21,17,4,5]. We will talk briefly about these methods in this paper. Some other methods obtain a real interval or a fuzzy set as the accomplishment degree for the sentences, see [14] for example, and will not be mentioned in this paper. Our first objective in this work is to define what we consider some appropriate properties for any method of evaluation of quantified sentences that obtains a real value as the accomplishment degree, and to study and compare the existing methods from this point of view. The final objective is to define new methods to perform the evaluation according to the properties defined.

The contents of the paper are structured as follows. In Section 2, we give a set of properties to be fulfilled by any method of evaluation. In Section 3, we show the existing methods for the evaluation of type I and type II sentences. Section 4 is devoted to the description of the approach we use to define new methods, along with previous definitions of cardinalities of fuzzy sets and their properties. Section 5 shows new methods for the evaluation of type I sentences. In Section 6, we define new methods for the evaluation of type II sentences. Section 7 contains our conclusions and future work.

2. Appropriate properties for sentence evaluation methods

Every existing method of evaluation is defined according to a different approach or measure. The validity of any method comes from the semantic validity of the selected approach or measure when performing the evaluation. Despite this, any method is required to "work well" in the sense that the results obtained are somehow appropriate and coherent with what we expect. In this section, we propose what we consider some appropriate properties to be fulfilled by any method of evaluation. They are not intended to be a closed set of properties but a collection of known cases and intuitive constraints.

2.1. Properties for the evaluation of type I sentences (Q of X are A)

Property 2.1.1 (Crisp case). *If A is crisp, then the (known) result of the evaluation must be*

$$Q\left(\frac{|A|}{|X|}\right)$$

if Q is relative, and

$$Q(|A|)$$

if Q is an absolute quantifier.

Property 2.1.2. *Evaluation must be coherent with fuzzy logic in the case of quantifiers "exist" and "all". The sentence "Q of X are A" with* $Q = \exists$ *can be represented and evaluated using fuzzy logic as*

$$\bigvee_{x_i \in X} A(x_i)$$

with the fuzzy union performed by a t-conorm (usually the maximum), and in the case $Q = \forall$ *the evaluation must be*

$$\bigwedge_{x_i \in X} A(x_i)$$

with the fuzzy intersection performed by a t-norm.

Property 2.1.3. *Evaluation must be coherent with quantifiers inclusion. Given* $Q \subseteq Q'$ *(Q is more restrictive than Q'),* Eval ("*Q of X are A*") $\leqslant$ Eval ("*Q' of X are A*"), *where* Eval (c) *with c a quantified sentence is the result obtained from the evaluation of c. Intuitively, it is more difficult to fit Q than to fit Q' in the evaluation.*

Property 2.1.4. *Evaluation must be time-efficient (as much as possible). We consider time-efficient an efficiency between* $O(n)$ *and* $O(n \log n)$, $n = |X|$.

Property 2.1.5. *Evaluation must not be too "strict", i.e. given a quantifier defined over the set* $H = \{p/q | p \in \{0, \ldots, n\},\ q \in \{1, \ldots, n\}\}$, *with* $Q \neq \emptyset$ *and* $Q \neq H$, *we must be able to find a fuzzy set A so that the evaluation of the sentence is not in* $\{0, 1\}$. *The convenience of this property and the problems that can be derived if we do not require, it can be seen in Section* 3.1.

Property 2.1.6. *Evaluation must allow us to use any quantifier, i.e. any possibility distribution over the non-negative integers or over the real interval* $[0, 1]$. *There are many quantifiers with clear semantics that fall outside the group of "coherent quantifiers" used by many methods (see Section* 3.2). *We shall see an example in this work.*

2.2. Properties for the evaluation of type II sentences (Q of D are A)

Property 2.2.1 (Crisp case). *If A and D are crisp, then the (known) result of the evaluation must be*

$$Q\left(\frac{|A \cap D|}{|D|}\right).$$

We assume Q is relative.

Property 2.2.2. *In the case $D = X$ and for relative quantifiers, the resulting evaluation method is a valid method for the evaluation of type* I *sentences. Type* I *sentences are, in fact, a special case of type* II *sentences where $D = X$, so in this case the evaluation method of type* II *sentences must be a valid evaluation method of type* I *sentences.*

Property 2.2.3. *Evaluation must be time-efficient (as much as possible), i.e.* $\mathrm{O}(n)$ *or* $\mathrm{O}(n \log n)$.

Property 2.2.4. *If $D \subseteq A$ and D is a normal set then the evaluation method must return the value $Q(1)$. This is an intuitive property (the percentage of D that are A is* 100%).

Property 2.2.5. *If $D \cap A = \emptyset$ then the evaluation method must return the value $Q(0)$. This is also an intuitive property (the percentage of D that are A is* 0%).

Property 2.2.6. *Evaluation must be coherent with fuzzy logic in the case of the quantifier "exist" and "all", giving*

$$\bigvee_{x_i \in X} (A(x_i) \wedge D(x_i))$$

and

$$\bigwedge_{x_i \in X} (D(x_i) \to A(x_i))$$

using some t-conorm for the union and a t-norm for the intersection, and $\to$ being a fuzzy implication.

Property 2.2.7. *Evaluation must allow us to use any quantifier, i.e. any possibility distribution over* $[0, 1]$.

Property 2.2.8. *Evaluation must not be too "strict", i.e. given a quantifier* Q *in the rational interval* $[0,1]$ *with* $Q \neq \emptyset$ *and* $Q \neq H = \{p/q$ *with* $p \in \{0,\ldots,n\}$ *and* $q \in \{1,\ldots,n\}\}$ *we must be able to find fuzzy sets A and D so that the evaluation of the sentence is not in* $\{0,1\}$.

3. Some existing methods for the evaluation of quantified sentences

Given a quantified sentence of type I "Q of X are A", with $X = \{x_1,\ldots,x_n\}$ a finite crisp set and A a fuzzy set over X, the following are some existing methods to perform the evaluation of the sentence.

3.1. Type I sentences

3.1.1. Zadeh's method

Zadeh's method [30] is based on the use of the non-fuzzy cardinality Σ-count, also called power. In the case of relative quantifiers, the final evaluation is

$$Z_Q(A) = Q\left(\frac{P(A)}{|X|}\right),$$

where $P(A)$ is the power of A defined in Section 4.1.1. For absolute quantifiers we have

$$Z_Q(A) = Q(\|P(A)\|),$$

where $|P(A)|$ is the integer part of the real number $P(A)$.

This method fulfills all properties of Section 2.1 except Properties 2.1.5 and 2.1.2. As a counterexample for Property 2.1.5, in the case of universally quantified sentences, the evaluation is 1 if and only if $A = X$, and 0 in any other case. The case of the quantifier $\exists$ is similar, the evaluation being 0 if and only if $A = \emptyset$. As a counterexample for Property 2.1.2 we have the case $A = \{0/x_1, 0.5/x_2\}$ for the quantifier exists. The result obtained using Zadeh's method is 1, but there is no t-conorm such that $0 \vee 0.5 = 1$ (every t-conorm verifies that $0 \vee a = a$). For the quantifier all, one counterexample is $A = \{1/x1, 0.9/x2\}$. The result obtained by Zadeh's method is 0, but there is no t-norm such that $1 \wedge 0.9 = 0$ (every t-norm verifies $1 \wedge a = a$). The efficiency of the method is $O(n)$, n being equal to $|X|$.

3.1.2. Yager's method based on OWA operators

This method defined in [19] only considers the case of relative and non-decreasing quantifiers verifying $Q(0) = 0$, $Q(1) = 1$ (the so-called "coherent quantifiers"). A coherent family of quantifiers is a set of quantifiers $\{Q_1,\ldots,Q_n\}$ verifying $Q_1 = \forall$, $Q_n = \exists$ and $Q_i \subset Q_{i+1}$. By Property 2.1.2, sentences with quantifiers $Q_1 = \forall$ and $Q_n = \exists$ are evaluated by means of a t-norm and a t-conorm, usually min and max. The evaluation of sentences with other quantifiers of a coherent family can be performed by means of an OWA operator. Any OWA operator gives a result between min and max, and the coefficients of the operator are obtained from the quantifier and the value $n = |X|$ in the following way, that guarantees Property 2.1.3:

$$w_i = Q(i/n) - Q((i-1)/n), \quad i \in \{1, \ldots, n\} \text{ and } Q(0) = 0.$$

Finally, the evaluation of the sentence is

$$Y_Q(A) = \sum_{i=1}^{n} w_i b_i,$$

where b_i is the ith largest value of belongingness to the fuzzy set A. In the following, this will be the meaning of b_i.

This method fulfills every property of Section 2.1 except Property 2.1.6. Property 2.1.4 is powered if we consider that for every quantifier Q of a coherent family and every value n we calculate and save the values of the coefficients w_i. If the values of A are arranged in descending order, the efficiency is $O(n)$. If not, the best efficiency is $O(n \log n)$. Although Property 2.1.6 is not verified by this method by the requirement of the quantifier to be coherent, sentences with some kind (not all) of non-coherent quantifiers can be evaluated by means of semantic equivalences with sentences where the quantifier is the "antonym" of the original quantifier and the fuzzy set is the complement of A. This method is described in [23,13].

3.1.3. Yager's non-OWA family of methods

Yager [18] proposes to perform the evaluation of type I sentences in the following way:

$$Y'_Q(A) = \max_{C \subseteq X} \wedge_1 \left(Q\left(\frac{|C|}{n}\right), \underset{x_i \in C}{\wedge_2} A(x_i) \right).$$

The last expression is applied in the case of relative quantifiers, and

$$Y'_Q(A) = \max_{C \subseteq X} \wedge_1 \left(Q(|C|), \underset{x_i \in C}{\wedge_2} A(x_i) \right)$$

for absolute quantifiers. In both cases, $\wedge_1$ and $\wedge_2$ are two t-norms.

Some concrete members of this family of methods are studied by Yager [18]. Among them, we can remark the case where $\wedge_1 = \wedge_2 = \min$. In this case, the method obtained for relative quantifiers is

$$Ym'_Q(A) = \max_{C \subseteq X} \min \left(Q\left(\frac{|C|}{n}\right), \min_{x_i \in C} A(x_i) \right).$$

The expression for absolute quantifiers is similar. The properties of this method will be discussed in the next section.

3.1.4. Methods based on the Choquet and the Sugeno integrals

The use of the Choquet and the Sugeno integrals for the evaluation of quantified sentences is described among others in [1,2]. As in previous methods, the evaluation is restricted to the case of coherent quantifiers. In the case of relative quantifiers, the method based on Choquet's integral is defined by the expression

$$C_Q(A) = \sum_{i=1}^{n} b_i \times (Q(i/n) - Q((i-1)/n)$$

and the method based on Sugeno's integral is expressed as

$$S_Q(A) = \max_{1 \leqslant i \leqslant n} \min(Q(i/n), b_i),$$

where, as in previous cases, b_i is the ith largest value of A.

The following properties hold:

Property 3.1.4.1. *The method based on the Choquet integral is the OWA-based method of Yager. This is obvious and is shown in* [1,2].

Property 3.1.4.2. *The method based on the Sugeno integral is the method Ym'_Q of Section* 3.3, *as shown in* [1], *and hence is a member of the family of methods* Y'_Q.

Proof (*Relative quantifier, for absolute quantifier is similar*).

$$\begin{aligned} Ym'_Q(A) &= \max_{C \subseteq X} \min\left(Q\left(\frac{|C|}{n}\right), \min_{x_i \in C} A(x_i)\right) \\ &= \max_{1 \leqslant i \leqslant n} \left\{ \max_{|C|=i} \min\left(Q\left(\frac{|C|}{n}\right), \min_{x_i \in C} A(x_i)\right)\right\} \\ &= \max_{1 \leqslant i \leqslant n} \min\left(Q\left(\frac{i}{n}\right), \max_{|C|=i}\left\{\min_{x_i \in C} A(x_i)\right\}\right) \\ &= \max_{1 \leqslant i \leqslant n} \min\left(Q\left(\frac{i}{n}\right), L(A,i)\right) \\ &= \max_{1 \leqslant i \leqslant n} \min\left(Q\left(\frac{i}{n}\right), b_i\right) \\ &= S_Q(A), \end{aligned}$$

where $L(A,i)$ is defined in [6] as the possibility that "the cardinality of A is at least i". In the same paper, we show that $L(A,i) = b_i$. We give the definition of $L(A,i)$ in Section 4.1.3, Property 4.1.3.1.

This method fulfills Properties 2.1.2–2.1.5. The efficiency is $O(n \log n)$ if A is not arranged in decreasing order, and $O(n)$ otherwise. Properties 2.1.1 and 2.1.6 are in conflict, because Property 2.1.1 is fulfilled only if the quantifier is coherent. The following is a counterexample: let $X = \{x_1, x_2, x_3\}$ and $A = \{1/x_1, 1/x_2, 0/x_3\}$ and let $Q(0) = 0$, $Q(1/3) = 1$, $Q(2/3) = 0$, $Q(1) = 0$. Clearly, Q is not non-decrecient, and hence Q is not coherent. We then have $S_Q(A) = \max\{\min(1,1), \min(0,1), \min(0,0)\} = 1$, while $|A| = 2$ and then the expected result by Property 2.1.1 must have been $Q(2/3) = 0$, so Property 2.1.1 is not fulfilled when Q is not coherent.

3.2. Type II sentences

The evaluation of type II sentences is slightly more complex than the evaluation of type I ones. There are fewer methods for type II sentences than for type I sentences. Given a quantified sentence of type II Q of D are A, with D and A two fuzzy sets over X, X being a finite set, the following are some methods to perform the evaluation of the sentence.

3.2.1. Zadeh's method

Zadeh's method is described in [30], and can be seen as an application of the cardinality approach. The case $D = \emptyset$ is not evaluable. This method obtains the relative cardinality of A with respect to D as (see Section 4.2.1)

$$P(A/D) = \frac{P(A \cap D)}{P(D)},$$

where P is the power (Σ-count). The intersection is usually obtained via the minimum.

The evaluation of the sentence is, finally

$$Z_Q(A/D) = Q(P(A/D)) = Q\left(\frac{P(A \cap D)}{P(D)}\right).$$

It is easy to prove that Zadeh's method verifies Property 2.2.1. When $D = X$, we have $Z_Q(A/X) = Z_Q(A)$, so Property 2.2.2 is fulfilled. The efficiency of the method is $O(n)$, so Property 2.2.3 is also fulfilled. Properties 2.2.4, 2.2.5 and 2.2.7 are easy to prove. Property 2.2.6 is not fulfilled, because we have shown that in the case $D = X$, the remaining method for the evaluation of type I sentences is $Z_Q(A)$, and this method does not fulfill the coherency with logic. The same counterexamples used in Section 3.1.1 are valid here. For the same reason, the method does not fulfill Property 2.2.8. A counterexample similar to that of Zadeh's method for the evaluation of type I sentences can be shown using the quantifiers "exists" and "all" in the case $D = X$.

3.2.2. Yager's method based on OWA

Yager's proposal is described in [21] and is based on the OWA operator where the parameters w_i are calculated from Q and D. This method is defined for coherent and relative quantifiers. The parameters of the OWA operator are calculated as

$$w_i = Q(S_i) - Q(S_{i-1}) \quad i \in \{1, \ldots, n\},$$

where

$$S_i = \frac{1}{d}\sum_{j=1}^{i} e_i \quad \text{and} \quad d = \sum_{k=1}^{n} e_k$$

and e_k is the ith smallest value of belongingness to D and $S_0 = 0$.

The final evaluation of the sentence is

$$Y_Q(A/D) = \sum_{i=1}^{n} w_i c_i,$$

where c_i is the ith largest value of belongingness to the fuzzy set $\neg D \vee A$.

The method does not fulfill Property 2.2.5. As a counterexample, let $A = \{1/x_1, 0/x_2\}$ and $D = \{0/x_1, 0.7/x_2\}$. Let $Q = \exists$. The obtained result using Yager's method is 0.3, while the expected value was 0. Property 2.2.4 is not fulfilled. As a counterexample, let $A = \{1/x_1, 0.9/x_2\}$ and $D = \{1/x_1, 0.5/x_2\}$ and $Q(x) = x$. Clearly $D \subseteq A$ and D is normalized so the expected value is $Q(1) = 1$, but the result obtained by Y_Q is 0.93. Property 2.2.6 is not fulfilled by the method for the quantifier "exists", and the last counterexample is also a counterexample for this property (any t-norm t verifies $t(x, 0) = t(0, x) = 0$ and there is no t-conorm tc such that $\mathrm{tc}(0, 0) = 0.3$). Property 2.2.6 for the quantifier "all" is fulfilled by this method, using the minimum and the implication $\neg D \vee A$. Obviously, Property 2.2.7 is not fulfilled by this method. Yager's method fulfills the rest of properties of Section 2.2, although the efficiency must be improved by storing values of w_i for every tuple (Q, D, n).

3.2.3. Method of Vila, Cubero, Medina and Pons

The main advantage of this method, described in [17], is the efficiency $O(n)$, together with a non-strict evaluation. The method uses the degree of "orness" defined by Yager [19] for coherent quantifiers. This value is defined in the real interval $[0, 1]$ and provides the degree of neighborhood of one quantifier to the quantifier $\exists$. By definition, $\mathrm{orness}(\exists) = 1$ and $\mathrm{orness}(\forall) = 0$. Any coherent quantifier between $\exists$ and $\forall$ has associated a degree in $[0, 1]$. Using the orness and the logic evaluation of the sentences "$\forall$ of D are A" and "$\exists$ of D are A", the evaluation of "Q of D are A" is given by

$$V_Q(A/D) = o_Q \max_{x \in X}(D(x) \wedge A(x)) + (1 - o_Q)\min_{x \in X}(A(x) \vee (1 - D(x)),$$

where o_Q is the orness Q defined by

$$o_Q = \sum_{i=1}^{n} \left(\frac{n-i}{n-1} \right) \times (Q(i/n) - Q((i-1)/n)).$$

This method only fulfills Properties 2.2.3, 2.2.6 (easy to check), and 2.2.7.

4. The cardinality approach for the evaluation of sentences

Our approach for the evaluation of type I sentences is to obtain the accomplishment degree of a sentence by means of the degree of compatibility between the quantifier and the cardinality of the fuzzy set A. The mentioned approach for the evaluation of quantified sentences is used for the evaluation of type I sentences. Type II sentences can be evaluated by obtaining the compatibility between the "relative cardinality" of A with respect to D, and the relative quantifier Q. The crisp relative cardinality of A with respect to D is the percentage of elements of D that are elements of A. Some of the methods described in Section 3 can be interpreted in terms of this approach as we shall see in Section 4.4.

In this approach, one method of evaluation of quantified sentences is given by three elements: the schema of representation of the cardinality of a fuzzy set, the method of calculus of the cardinality, and the method for obtaining the compatibility between cardinality and quantifier. One usual way of representing the cardinality of a fuzzy set is by means of a scalar value, either integer or real. Another way of representation of the cardinality of a fuzzy set is the so-called "fuzzy cardinality". This consists of representing the cardinality as a fuzzy set over the non-negative integers. Several methods to calculate the cardinality using one or another of these schemas have been developed. In Section 4.1, we will look briefly at some of the most important existing methods related to sentence evaluation, and we also describe several new recently proposed methods. Methods for the representation and calculation of the relative cardinality of fuzzy sets are also described in Section 4.2. We will briefly talk about the calculus of the compatibility between cardinality and quantifier in Section 4.3.

4.1. Cardinality of a fuzzy set

The following are some measures of the cardinality of a fuzzy set.

4.1.1. Power (Σ-count)

This is an example of a scalar-valued measure of the cardinality of a fuzzy set. This measure was defined by De Luca and Termini. Given a fuzzy set A over a finite set $X = \{x_1, \dots, x_n\}$, the Power of A, $P(A)$, is defined as

$$P(A) = \sum_{x_i \in X} A(x_i).$$

4.1.2. Zadeh's first method

The fuzzy cardinality $Z(A)$ is defined as follows:

$$Z(A,k) = \begin{cases} 0 & \text{if does not exist } \alpha \mid |A_\alpha| = k, \\ \sup\{\alpha| \ |A_\alpha| = k\} & \text{otherwise.} \end{cases}$$

4.1.3. Method ED

This method is defined in [6] as a member of a more general family of cardinalities.

Definition 4.1.3.1. Let $X = \{x_1, \ldots, x_n\}$ and A a fuzzy set over X. First, we define the possibility that at least k elements of X belong to A, $L(A,k)$, as

$$L(A,k) = \begin{cases} 1, & k = 0 \\ 0, & k > n \\ \bigoplus_{(i_1,\ldots,i_k)\in I_k} (A(x_{i_1}) \otimes \cdots \otimes A(x_{i_k})), & 1 \leqslant k \leqslant n, \end{cases}$$

where I_k is the set of k-tuples of indexes defined by

$$I_k = \big\{(i_1,\ldots,i_k) \,|\, i_1 < i_2 < \cdots < i_k \text{ with } i_j \in \{1,\ldots,n\} \ \forall j \in \{1,\ldots,k\}\big\}$$

and $\oplus$ and $\otimes$ are a t-conorm and a t-norm, respectively.

Property 4.1.3.1. *Let $\oplus$ and $\otimes$ be the maximum and the minimum, respectively. Then*

$$L(A,k) = b_k \quad \forall k \in \{1,\ldots,n\},$$

where b_k is the kth largest value of belongingness of an element to the fuzzy set A.

Proof. Every t-norm is non-decreasing, so the largest value between the expressions $\otimes\lfloor A(x_{i_1}),\ldots,A(x_{i_k})\rfloor$ will be $\otimes\lfloor b_1,\ldots,b_k\rfloor = b_k$, because we are using the minimum as t-norm. We are using the maximum as t-conorm, so $L(A,k) = b_k$.

Definition 4.1.3.2. We define the possibility that exactly k elements of X belong to A, $E(A,k)$, as

$$E(A,k) = L(A,k) \otimes \overline{L(A,k+1)},$$

where $\otimes$ is any t-norm and the bar stands for a fuzzy complement.

The expression of E defines a family of fuzzy cardinalities. Some existing methods, such as the Dubois–Prade method, Zadeh's FECount and Ralescu's method, are proved to be members of the family E in [6]. The following is a new method of the family E defined in the same paper.

Definition 4.1.3.3. The fuzzy cardinality ED is a member of the family E that employs the maximum and minimum in the definition of L and Lukasiewicz's t-norm and the standard negation in the definition of E. The expression of the method is

$$\mathrm{ED}(A,k) = b_k - b_{k+1}$$

with $b_0 = 1$ and $b_{n+1} = 0$.

Proof. By Property 4.1.3.1 when using max–min with L, we have $L(A,k) = b_k$. Using in E Lukasiewicz's t-norm and the standard negation we have

$$\begin{aligned}\mathrm{ED}(A,k) &= \max\{b_k + (1 - b_{k+1}) - 1, 0\} \\ &= \max\{b_k - b_{k+1}, 0\} \\ &= b_k - b_{k+1}. \qquad \square\end{aligned}$$

As pointed out in [6], this method can be interpreted as a probabilistic measure of the cardinality of A, while other methods such as Zadeh's first method (see Section 4.1.2) are possibilistic measures.

Property 4.1.3.3. *The method ED verifies*

$$\sum_{i=0}^{n} \mathrm{ED}(A,i) = 1.$$

Proof.

$$\begin{aligned}\sum_{i=0}^{n} \mathrm{ED}(A,i) &= (b_0 - b_1) + (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_{n-2} - b_{n-1}) \\ &\quad + (b_{n-1} - b_n) + (b_n - b_{n+1}) \\ &= b_0 - b_{n+1} = 1. \qquad \square\end{aligned}$$

4.1.4. The Dubois–Prade method

Dubois and Prade define the set of crisp representatives of a fuzzy set as

$$\Re(A) = \{S \mid A_1 \subseteq S \subseteq \mathrm{Support}(A)\}.$$

This set is an alternative representation of a fuzzy set by a set of crisp sets different to that of the representation theorem based on α-cuts, but verifying the same properties. The degree of representativity of a given set S of $\Re(A)$ is

$$\overline{\pi_A}(S) = \begin{cases} \inf\{\mu_A(u) \mid u \in S\} & S \in \Re(A) \\ 0 & S \notin \Re(A). \end{cases}$$

Finally, the fuzzy cardinality of the fuzzy set A is given by

$$\mathrm{DP}(A,k) = \sup\{\overline{\pi_A}(S) \mid |S| = k\} \quad k \in \{1,\ldots,n\}.$$

In [6], we show that one alternative definition of DP is as follows:

$$\mathrm{DP}(A,k) = \begin{cases} 0, & k < |A_1|, \\ b_k, & k \geqslant |A_1|. \end{cases}$$

4.2. Relative cardinality of fuzzy sets

The relative cardinality of one set A with respect to a set D is a measure of the percentage of elements of D that are also elements of A. In general, it can be described as follows:

$$\text{Rel Card}\,(A/D) = \frac{\text{Card}\,(A \cap D)}{\text{Card}\,(D)}.$$

4.2.1. Zadeh's method

Zadeh defines the relative cardinality of a fuzzy set A with respect to a fuzzy set D as:

$$P(A/D) = \frac{P(A \cap D)}{P(D)},$$

where P is the Power defined in Section 4.1.1, and the intersection is performed by means of the minimum.

4.2.2. Method ES

This method is defined in [6]. Let A and D be two fuzzy sets over X, D being a normal fuzzy set. Let

$$M(A) = \{\alpha \in]0,1] \mid \exists x_i \in X \text{ such that } A(x_i) = \alpha\}$$

and let

$$M(A/D) = M(A \cap D) \cup M(D)$$

and let

$$\mathrm{CR}(A/D) = \left\{ \frac{|(A \cap D)_\alpha|}{|D_\alpha|} \text{ such that } \alpha \in M(A/D) \right\}.$$

Then, the relative cardinality of A with respect to D, $\mathrm{ES}(A/D)$, is defined as

$$\mathrm{ES}(A/D, c) = \max\left\{\alpha \in M(A/D) \mid c = \frac{|(A\cap D)_\alpha|}{|D_\alpha|}\right\} \quad \forall c \in \mathrm{CR}(A/D).$$

If D is not a normal fuzzy set, we first normalize D and scale the fuzzy set $A\cap D$ using the same factor used in the normalization of D, before we begin the process.

4.2.3. Method ER

This method is also defined in [6]. Let A and D be two fuzzy sets over X, D being a normal fuzzy set. Let $M(A/D) = \{\alpha_1, \ldots, \alpha_m\}$ be the set of representative α-cuts defined in the last section, with $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = 0$. Let

$$C(A/D, \alpha_i) = \frac{|(A\cap D)_{\alpha_i}|}{|D_{\alpha_i}|}.$$

Then, the relative cardinality of A with respect to D, $\mathrm{ER}(A/D)$, is defined as

$$\mathrm{ER}(A/D, c) = \sum_{c=C(A/D,\alpha_i)} (\alpha_i - \alpha_{i+1}) \quad \forall c \in \mathrm{CR}(A/D).$$

If D is not a normal fuzzy set, we first normalize D and scale the fuzzy set $A\cap D$ using the same factor used in the normalization of D, before we begin the process.

As pointed out in [6], this method can be interpreted as a probabilistic measure of the relative cardinality between A and D, while method ES is a possibilistic one.

4.3. Compatibility of fuzzy sets

The way we obtain the degree of compatibility between cardinality and quantifier depends on the schema we are using to represent the cardinality. When we use a scalar value, the compatibility is obtained by evaluating the quantifier at the point given by the cardinality. The way we perform this must take into account the type of the scalar value (integer or real) and the type of quantifier (absolute or relative). Absolute quantifiers can be evaluated for integer values, and relative quantifiers can be evaluated for real ones. However, we can obtain the compatibility between an integer cardinality c and a relative quantifier Q by evaluating $Q(c/|X|)$.

When we use a fuzzy set, the degree of compatibility between the cardinality and the quantifier is usually calculated as

$$\bigoplus_{i\in\{0,\ldots,n\}} (Q(i) \otimes C(A, i))$$

for type I sentences with absolute quantifiers, where $C(A,i)$ is the possibility that i is the cardinality of A, and $\otimes$ and $\oplus$ are a t-norm and a t-conorm, respectively. Similarly for type II sentences with relative quantifiers, we have

$$\bigoplus_{c=(p/q)\ p,q\in Z} (Q(c) \otimes C(A/D,c)),$$

where $C(A/D,c)$ is the possibility that c is the relative cardinality of A with respect to D, and

$$\bigoplus_{i\in\{0,\dots,n\}} (Q(i/n) \otimes C(A,i))$$

for type I sentences with relative quantifiers (we assume that $C(A/X,i/n) = C(A,i)$, with $n = |X|$).

4.4. Interpretation of some existing methods in terms of the cardinality approach

4.4.1. Zadeh's method for type I sentences

Zadeh's method represents the cardinality of A by means of a scalar value. The cardinality is calculated by means of the power of A (Section 4.1.1). Finally, the compatibility between cardinality and quantifier is obtained evaluating the quantifier at the point given by the cardinality.

4.4.2. Yager's method based on OWA for type I sentences

This method (and hence the method based on the Choquet fuzzy integral) can be interpreted in terms of the cardinality approach. The cardinality used is the method ED of Definition 4.1.3.3. We obtain the compatibility degree between the quantifier and the fuzzy cardinality ED by means of the Lukasiewicz t-conorm $l(x,y) = \min\{x+y,1\}$ and the product as the t-norm. The interpretation is shown in Section 5.2, Property 5.2.1 of this paper, and the obtained method is called GD. This method is equal to Yager's method only for coherent quantifiers, and is an extension that allows the use of any other quantifier.

4.4.3. Method based on the Sugeno fuzzy integral

This method can be interpreted as the compatibility degree between the fuzzy cardinality of Dubois–Prade (see Section 4.1.4) and the quantifier, by using max–min composition as follows

$$S_Q(A) = \max_{1\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,k)).$$

We assume, as does the method based on the Sugeno fuzzy integral, that the quantifier is coherent.

Proof. We shall discuss two cases:

(a) Let $|A_1| = 0$. Then, $\mathrm{DP}(A,k) = b_k$ for all $k \in \{1,\ldots,n\}$, so $\max_{1\leqslant i\leqslant n} \min(Q(i/n),\ \mathrm{DP}(A,i)) = \max_{1\leqslant i\leqslant n} \min(Q(i/n), b_i) = S_Q(A)$.

(b) Let $|A_1| = c > 0$. Then $\mathrm{DP}(A,c) = b_c = 1$ and $\mathrm{DP}(A,i) = 0$ for all $i < c$. Then

$$\begin{aligned}&\max_{1\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,i))\\ &\quad= \max\left\{\max_{1\leqslant i<c} \min(Q(i/n), \mathrm{DP}(A,i)), \max_{c\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,i))\right\}\\ &\quad= \max_{c\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,i)).\end{aligned}$$

On the other hand,

$$\begin{aligned}S_Q(A) &= \max_{1\leqslant i\leqslant n} \min(Q(i/n), b_i)\\ &= \max\left\{\max_{1\leqslant i<c} \min(Q(i/n), 1), \max_{c\leqslant i\leqslant n} \min(Q(i/n), b_i)\right\}.\end{aligned}$$

As Q is coherent, Q is non-decrecient and then

$$\max_{1\leqslant i<c} \min(Q(i/n), 1) < \min(Q(c/n), b_c) = \min(Q(c/n), 1),$$

so

$$\begin{aligned}S_Q(A) &= \max_{c\leqslant i\leqslant n} \min(Q(i/n), b_i)\\ &= \max_{c\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,i))\\ &= \max_{1\leqslant i\leqslant n} \min(Q(i/n), \mathrm{DP}(A,i)). \qquad \square\end{aligned}$$

4.4.4. Zadeh's method for type II sentences

Zadeh's method represents the relative cardinality of A with respect to D by means of a scalar value. The relative cardinality is calculated by means of the method described in Section 4.2.1. Finally, the compatibility between the relative cardinality and the quantifier is obtained by evaluating the quantifier at the point given by the cardinality, as was the case of Zadeh's method for type I sentences.

5. New methods for the evaluation of type I sentences

5.1. The family G of methods based on the cardinality approach

This family of methods is related to the family E of cardinalities described in Definition 4.1.3.2. The family G of methods of evaluation of type I sentences is given by

$$G_Q(A) = \bigoplus_{i\in\{0,\ldots,n\}} (E(A,i) \otimes Q(i))$$

for absolute quantifiers, and

$$G_Q(A) = \bigoplus_{i\in\{0,\ldots,n\}} (E(A,i) \otimes Q(i/n))$$

for relative quantifiers. This is a direct application of the cardinality approach.

Property 5.1.1. *The family G of methods verifies Property* 2.1.1 *of evaluation of type* I *sentences.*

Proof (*Absolute quantifiers, for relative quantifiers is similar*). If A is crisp, then by the property of any method of the family E (see [6])

$$E(A,i) = \begin{cases} 0, & i \neq |A|, \\ 1, & i = |A|, \end{cases}$$

and by the properties of every t-norm

$$E(A,i) \otimes Q(i) = \begin{cases} 0, & i \neq |A|, \\ Q(i), & i = |A|. \end{cases}$$

Finally, by the properties of every t-conorm

$$G_Q(A) = Q(|A|). \qquad \square$$

Property 5.1.2. *The family G of methods verifies Property* 2.1.6, *i.e. it does not require any property from the quantifier to be used in the evaluation.*

Property 5.1.3. *The family G of methods verifies Property* 2.1.3, *because any t-norm and any t-conorm are non-decreasing functions in their arguments.*

5.2. The method GD

This method belongs to the family G, and is based on the fuzzy cardinality method ED of the family E (see Definition 4.1.3.3). Using the product as the t-norm and the Lukasiewicz t-conorm in the expression of the compatibility G, the method GD is defined by

$$\mathrm{GD}_Q(A) = \sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i)$$

for absolute quantifiers, and

$$\mathrm{GD}_Q(A) = \sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i/n)$$

for relative quantifiers.

Proof. When using product and the Lukasiewicz t-conorm, we obtain the expression

$$\mathrm{GD}_Q(A) = \min\left\{\sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i),\ 1\right\}.$$

By Property 4.1.3.3 $\sum_{i=0}^{n} \mathrm{ED}(A,i) = 1$, and obviously $Q(i) \in [0,1]\ \forall i$ and $\mathrm{ED}(A,i) \in [0,1]\ \forall i$, so $\sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i) \leqslant 1$, and hence $\mathrm{GD}_Q(A) = \sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i)$. □

The proof is similar for relative quantifiers.

Property 5.2.1. *If Q is a relative and a coherent quantifier, then*

$$\mathrm{GD}_Q(A) = Y_Q(A) = C_Q(A),$$

i.e. the method GD is the method of Yager based on the OWA operator. As a consequence, we can see Yager's method based on OWA as belonging to the family G in the case of relative and coherent quantifiers. This allows us to give an interpretation of Yager's method in terms of the cardinality approach.

Proof.

$$\begin{aligned}
\mathrm{GD}_Q(A) &= \sum_{i=0}^{n} \mathrm{ED}(A,i) \times Q(i/n) \\
&= \sum_{i=0}^{n} (b_i - b_{i+1}) \times Q(i/n) \\
&= (b_0 - b_1) \times Q(0) + (b_1 - b_2) \times Q(1/n) + \cdots + (b_{n-1} - b_n) \\
&\quad \times Q((n-1)/n) + (b_n - b_{n+1}) \times Q(1) \\
&= \{\text{known } b_{n+1} = 0\} b_0 \times Q(0) + b_1 \times (Q(1/n) - Q(0)) + b_2 \\
&\quad \times (Q(2/n) - Q(1/n)) + \cdots + b_n \times (Q((n-1)/n) - Q(1)) \\
&= \{\text{known } Q \text{ is coherent and hence } Q(0) = 0\} \\
&= \sum_{i=1}^{n} b_i \times (Q(i/n) - Q((i-1)/n)) = Y_Q(A). \qquad \square
\end{aligned}$$

Hence the method GD can be seen as a generalization of Yager's method based on OWA that can be used with any quantifier (and hence fulfilling Property 2.1.6). This method also fulfills Properties 2.1.1, 2.1.3 and 2.1.6 because GD is a method of the family G, and it is easy to see that Property 2.1.5 is also fulfilled. The efficiency is $O(n \log n)$ if we do not have the fuzzy set A arranged in decreasing order, and $O(n)$ otherwise. Therefore, method GD fulfills all properties of Section 2.1.

5.3. The method ZS

This method is based on the cardinality approach. The fuzzy cardinality measure used is Zadeh's first method

$$Z(A,k) = \begin{cases} 0 & \text{if does not exist } \alpha \mid |A_\alpha| = k, \\ \sup\{\alpha \mid |A_\alpha| = k\} & \text{otherwise.} \end{cases}$$

The compatibility between cardinality and quantifier is calculated by means of the max–min composition using the cardinality approach in the following way:

$$\mathrm{ZS}_Q(A) = \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k), Q(k)\right)$$

for absolute quantifiers, and

$$\mathrm{ZS}_Q(A) = \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k), Q(k/n)\right)$$

for relative quantifiers.

It is easy to see that method ZS fulfills Properties 2.1.5 and 2.1.6. Method ZS fulfills every property of Section 2.1, as we show by the following properties.

Property 5.3.1. *The method ZS fulfills Property* 2.1.1.

Proof. Let A be a crisp set. Then

$$Z(A,k) = \begin{cases} 1, & k = |A|, \\ 0, & k \neq |A| \end{cases}$$

and then for absolute quantifiers we have

$$\mathrm{ZS}_Q(A) = Q(|A|)$$

and for relative quantifiers

$$\mathrm{ZS}_Q(A) = Q\left(\frac{|A|}{|X|}\right). \qquad \square$$

Property 5.3.2. *The method ZS fulfills Property* 2.1.2.

Proof. The quantifiers ∃ and ∀ are defined as follows:

$$\exists(x) = \begin{cases} 0, & x = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad \forall(x) = \begin{cases} 1, & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$
\begin{aligned}
(\forall)\ \mathrm{ZS}_{\forall}(A) &= \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k), \forall(k/n)\right)\\
&= Z(A,n)\\
&= \sup\{\alpha \mid |A_\alpha| = n\}\\
&= \min\{A(x_i) \mid x_i \in X\}.\\
(\exists)\ \mathrm{ZS}_{\exists}(A) &= \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k), \exists(k/n)\right)\\
&= \max_{k\in\{1,\dots,n\}} \min\left(Z(A,k), \exists(k/n)\right)\\
&= \max_{k\in\{1,\dots,n\}} Z(A,k)\\
&= \max_{k\in\{1,\dots,n\}} \sup\left(\alpha \mid |A_\alpha| = k\right)\\
&= \max_{k\in\{1,\dots,n\}} \max\left(A(x_i) \mid \left|A_{A(x_i)}\right| = k\right)\\
&= \max\{A(x_i) \mid x_i \in X\}. \qquad \square
\end{aligned}
$$

Property 5.3.3. *The method ZS fulfills Property* 2.1.3 *because max and min are non-decreasing functions in their arguments.*

Property 5.3.4. *Let*

$$M(A) = \{\alpha \in\,]0,1] \mid \exists x_i \in X \text{ such that } A(x_i) = \alpha\} \cup \{1\}.$$

Then the method ZS has the following alternative expression:

$$\mathrm{ZS}_Q(A) = \max_{\alpha\in M(A)} \min\left(\alpha\ ,\ Q(|A_\alpha|)\right)$$

for absolute quantifiers, *and*

$$\mathrm{ZS}_Q(A) = \max_{\alpha\in M(A)} \min\left(\alpha\ ,\ Q\left(\frac{|A_\alpha|}{|X|}\right)\right)$$

for relative quantifiers. This equivalence avoids obtaining $Z(A)$ explicitly.

Proof. Let $S(A)$ be the support of $Z(A)$, i.e.

$$S(A) = \{k \in \{0,\dots,n\} \mid \exists\alpha \in M(A) \text{ such that } |A_\alpha| = k\}.$$

Then

$$
\begin{aligned}
\mathrm{ZS}_Q(A) &= \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k), Q(k)\right)\\
&= \max_{k\in S(A)} \min\big(\max\{\alpha \mid |A_\alpha| = k\},
\end{aligned}
$$

$$Q(k)) = \max_{k \in S(A)} \left\{ \max_{|A_\alpha| = k} \min(\alpha,\ Q(k)) \right\}$$
$$= \max_{\alpha \in M(A)} \min(\alpha,\ Q(|A_\alpha|)). \qquad \square$$

The proof for relative quantifiers is similar. The efficiency of the calculation of the equivalent expression is $O(n \log n)$ if A is not arranged and $O(n)$ otherwise. Hence, method ZS fulfills Property 2.1.4.

Property 5.3.5. *If Q is a coherent quantifier, then the method ZS is the method S based on the Sugeno integral, i.e.*

$$\mathrm{ZS}_Q(A) = S_Q(A) = Ym'_Q(A).$$

Hence, the method ZS can be seen as a generalization of the method S that can be used with any quantifier (and hence fulfilling Property 2.1.6) *without any conflict with Property* 2.1.1.

Proof. We shall discuss two cases:

1. Let $b_i \neq b_j \quad \forall i, j \in \{1, \ldots, n\}$. Then it is easy to see that $Z(A, k) = b_k \forall k \in \{1, \ldots, n\}$. Under these conditions,

$$\mathrm{ZS}_Q(A) = \max_{k \in \{0, \ldots, n\}} \min(Z(A, k), Q(k/n))$$
$$= \max_{k \in \{1, \ldots, n\}} \min(Z(A, k), Q(k/n))$$
$$\{\text{because } Q \text{ is coherent and hence } Q(0) = 0\}$$
$$= \max_{k \in \{1, \ldots, n\}} \min(b_k, Q(k/n)) = S_Q(A). \qquad \square$$

2. Let h be the number of groups of repeated values of b_i. Let l_i be the length of the group i, $i \in \{1, \ldots, h\}$. Let c_i be the first index of the group i. Then the values of belongingness to A arranged in decreasing order are as follows:

$$\{b_1, \ldots, b_{c_1}, \ldots, b_{c_1 + l_1}, \ldots, b_{c_2}, \ldots, b_{c_2 + l_2}, \ldots, b_{c_h}, \ldots, b_{c_h + l_h}, \ldots, b_n\}$$

with $b_{c_i} = b_{c_i + 1} = \cdots = b_{c_i + l_i - 1} = b_{c_i + l_i} \quad \forall i \in \{1, \ldots, h\}$.

It is easy to see that

$$Z(A, k) = \begin{cases} b_k, & k \in \mathrm{BH} = \{1, \ldots, c_1 + l_1, c_2 + l_2, \ldots \\ & (\text{only } c_i + l_i) \ldots, c_h + l_h, \ldots, n\}, \\ 0 & \text{otherwise}, \end{cases}$$

because in fact, $Z(A, k)$ calculate the cardinality of every α-cut of A and assigns to that cardinality the possibility α, and all possible α-cuts of A are of the form $A_{b_j}$with $j \in \mathrm{BH}$.

Then, we have

$$\begin{aligned}
S_Q(A) &= \max_{k\in\{1,..,n\}} \min(Q(k/n), b_k) \\
&= \max_{k\in \mathrm{BH}} \min(Q(k/n), b_k) \\
&\qquad \{\text{because } Q \text{ is coherent and hence } Q \text{ is non-decrecient}\} \\
&= \max_{k\in \mathrm{BH}} \min(Q(k/n), Z(A,k)) \\
&= \max_{k\in\{1,\dots,n\}} \min(Q(k/n), Z(A,k)) \\
&= \{\text{because if } Z(A,k)=0, \min(Q(k/n), Z(A,k)) = 0 \\
&\quad \text{and the final maximum calculation does not change}\} \\
&= \max_{k\in\{0,\dots,n\}} \min(Q(k/n), Z(A,k)) \\
&= \{\text{because } Q \text{ is coherent and hence} Q(0) = 0, \\
&\quad \text{and therefore the maximum does not change}\} \\
&= \mathrm{ZS}_Q(A). \qquad \square
\end{aligned}$$

The following is a counterexample for the case where Q is not a coherent quantifier. Let $A = \{1/x_1, 1/x_2, 0.2/x_3\}$ and $Q(0) = 0, Q(1/3) = 0.8, Q(2/3) = 0.5, Q(1) = 0.2$. Clearly, Q is not coherent. Then $Z(A) = \{0/0, 0/1, 1/2, 0.2/3\}$. It is easy to check that $\mathrm{ZS}_Q(A) = 0.5$ while $S_Q(A) = 0.8$.

5.4. Some examples of application of the discussed methods

We shall use the following five quantifiers (Fig. 1), that can be defined as follows:

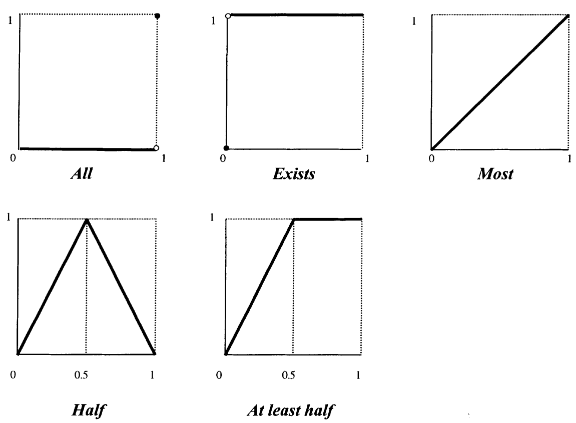

Fig. 1. Five relative quantifiers.

$$\mathrm{All}(x) = \begin{cases} 0, & x < 1, \\ 1, & x = 1, \end{cases} \quad \mathrm{Exists}(x) = \begin{cases} 0, & x = 0, \\ 1, & x > 0, \end{cases} \quad \mathrm{Most}(x) = x.$$

$$\mathrm{Half}(x) = \begin{cases} 2 \times x, & x < 0.5, \\ 2 \times (1 - x), & x \geqslant 0.5, \end{cases}$$

$$\text{At Least Half}(x) = \begin{cases} 2 \times x, & x < 0.5, \\ 1, & x \geqslant 0.5. \end{cases}$$

The quantifier Half must be interpreted as "approximately half", and is not coherent. The remaining four quantifiers are coherent. Because of this, methods that work with coherent quantifiers will not be used for the quantifier Half.

Example 5.4.1. Let A be the fuzzy set of Fig. 2. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 1.

$$A = \{0.8/x_1, 0.66/x_2, 0.58/x_3, 0.54/x_4, 0.43/x_5, 0.4/x_6\}.$$

We can see that Z_Q is a strict method for the quantifiers Exists and All. The value for S_Q is calculated for the quantifier Half although this method is not defined for working with non-coherent quantifiers. Method Y_Q cannot be evaluated for the quantifier Half. Moreover, we can see that for $Q = \mathrm{Most}$, $Z_Q(A) = Y_Q(A) = \mathrm{GD}_Q(A)$. We can show that this equality always holds.

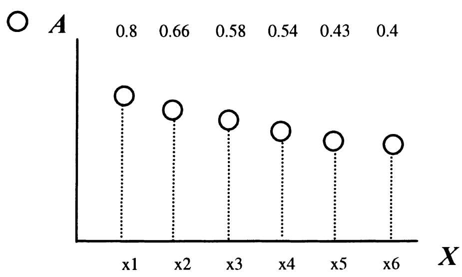

Fig. 2. Fuzzy set A for Example 5.4.1.

Table 1
Evaluation of Example 5.4.1

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A)$	0	1	0.5683	0.863	1
$Y_Q(A) = C_Q(A)$	0.4	0.8	0.5683	–	0.68
$\mathrm{GD}_Q(A)$	0.4	0.8	0.5683	0.223	0.68
$Ym'_Q(A) = S_Q(A)$	0.4	0.8	0.54	0.66	0.66
$\mathrm{ZS}_Q(A)$	0.4	0.8	0.54	0.66	0.66

Proof. First, we have shown that if Q is coherent, $Y_Q(A) = \mathrm{GD}_Q(A)$. As "Most" is coherent, this part is done. Secondly, let $Q = $ Most, i.e. $Q(x) = x$. Then we have $Z_Q(A) = (P(A)/n) = P(A)/n$ and

$$Y_Q(A) = \sum b_i w_i$$

and

$$w_i = Q(i/n) - Q((i-1)/n) = (i/n) - ((i-1)/n) = 1/n \forall i,$$

so

$$Y_Q(A) = (1/n) \sum b_i = P(A)/n = Z_Q(A). \qquad \square$$

Example 5.4.2. Let A be the fuzzy set defined in Fig. 3. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 2.

$$A = \{1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5, 0.9/x_6\}.$$

We have calculated again the value for $S_Q(A)$ with $Q = $ Half and in this example we can see that this method is not appropriate for evaluating non-coherent quantifiers like Half, because it is obvious that there are more than three elements (the half) in A, so the evaluation cannot be 1.

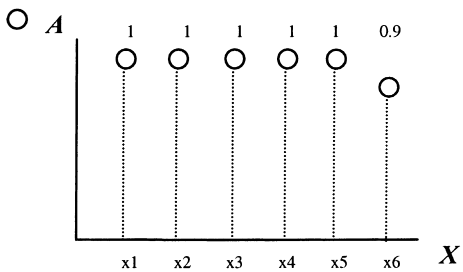

Fig. 3. Fuzzy set A for Example 5.4.2.

Table 2
Evaluation of Example 5.4.2

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A)$	0	1	0.983	0.033	1
$Y_Q(A) = C_Q(A)$	0.9	1	0.983	–	1
$\mathrm{GD}_Q(A)$	0.9	1	0.983	0.033	1
$Ym'_Q(A) = S_Q(A)$	0.9	1	0.9	1	1
$\mathrm{ZS}_Q(A)$	0.9	1	0.9	0.33	1

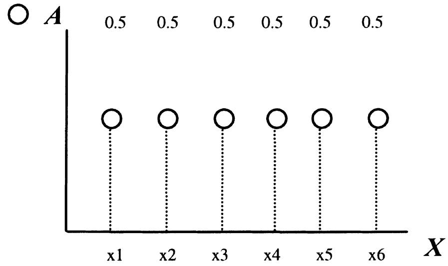

Fig. 4. Fuzzy set A for Example 5.4.3.

Table 3
Evaluation of Example 5.4.3

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A)$	0	1	0.5	1	1
$Y_Q(A) = C_Q(A)$	0.5	0.5	0.5	–	0.5
$\mathrm{GD}_Q(A)$	0.5	0.5	0.5	0	0.5
$Ym'_Q(A) = S_Q(A)$	0.5	0.5	0.5	0.5	0.5
$\mathrm{ZS}_Q(A)$	0.5	0.5	0.5	0	0.5

Example 5.4.3. Let A be the fuzzy set defined in Fig. 4. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 3.

$$A = \{0.5/x_1, 0.5/x_2, 0.5/x_3, 0.5/x_4, 0.5/x_5, 0.5/x_6\}.$$

This example shows that Z_Q can offer a strict evaluation even with quantifiers distinct of All and Exists. In this case, for the quantifier Half we have $P(A)/n = 3/6 = 0.5$ and $\mathrm{Half}(0.5) = 1$, but it seems to be clear that the fuzzy set A can only have 0 or 6 elements, because if we consider that for example $x_1 \in A$, then $x_2 \in A$ because $A(x_1) = A(x_2)$, $x_3 \in A$ because $A(x_1) = A(x_3)$, etc. and the same consideration holds if $x_1 \notin A$, so we cannot say that there are exactly 3 elements of A. One interpretation of this behaviour of Z_Q can be that the cardinality $P(A)$ has problems when adding several little values, because it obtains a value of cardinality that can be unrealistic (in this case, $P(A) = 3$). These problems can affect the behaviour of Z_Q. As the fuzzy cardinalities ED and ES used by GD_Q and ZS_Q only consider 0 and 6 as possible cardinalities of A and $Q(0) = Q(6) = 0$ when $Q = \mathrm{Half}$, then the evaluation obtained is 0, and that seems to be reasonable. For more discussion about the properties of the discussed cardinalities see [6].

We can also see that Z_Q is strict for the coherent quantifier "At least the half" in this example, due to the same reasons as in the case of "Half". We can see again that S_Q is not appropriate when using $Q =$ Half.

6. New methods for the evaluation of type II sentences

6.1. Generalization of the method GD to type II sentences

A first attempt to generalize the method GD to type II sentences was proposed in [4] and it was called *G*. The aim of this method is to provide a non-strict method (Property 2.2.8) that fulfills Properties 2.2.2 (when $D = X$, we have the method GD for type I sentences), 2.2.3 ($\mathrm{O}(n \log n)$), and 2.2.6 (coherent with logic). First, we define the fuzzy set $f(A/D, Q)$ as

$$\mu_{f(A/D,Q)}(x_i) = o_Q(D(x_i) \wedge A(x_i)) + (1 - o_Q)(D(x_i) \rightarrow A(x_i))$$

and the evaluation of the sentence Q of D are A is obtained from the evaluation of the sentence "Q of X are $f(A/D, Q)$" using the method GD of Section 4.2, i.e.

$$G_Q(A/D) = \mathrm{GD}_Q(f(A/D, Q)).$$

However, this method does not fulfill all the properties of Section 2.2. Moreover, this method was not based on the cardinality approach.

The generalized method GD is defined as the compatibility between the fuzzy relative cardinality ER and the quantifier Q by means of the product and the Lukasiewicz's t-conorm as follows:

$$\mathrm{GD}_Q(A/D) = \sum_{c \in \mathrm{CR}(A/D)} \mathrm{ER}(A/D, c) \times Q(c).$$

Property 6.1.1. *The method GD verifies Property* 2.2.1.

Proof. Let A and D be two crisp sets. Then, by definition $M(A/D) = \{1\}$ so $m = 1$ and $\alpha_1 = 1$ and $\alpha_2 = 0$ and then $\mathrm{CR}(A/D) = \{C(A/D, 1)\}$ i.e.

$$CR(A/D) = \left\{ \frac{|A \cap D|}{|D|} \right\}.$$

Therefore, $\mathrm{ER}(A/D, c) = 0 \ \forall c \neq C(A/D, 1)$ and

$$\mathrm{ER}(A/D) = \left\{ 1 \bigg/ \frac{|A \cap D|}{|D|} \right\}$$

and finally

$$\mathrm{GD}_Q(A/D) = Q\left(\frac{|A \cap D|}{|D|} \right). \qquad \square$$

Property 6.1.2. *The method GD for the evaluation of type* II *sentences in the case* $D = X$ *is the method GD for the evaluation of type* I *sentences described in Section* 5.2, *and hence fulfills Property* 2.2.2.

Proof. We have proved in [6] that if $D = X$ then

$$\mathrm{ER}\left(A/X, \frac{k}{n}\right) = \mathrm{ED}(A, k).$$

Then we have

$$\begin{aligned}
\mathrm{GD}_Q(A/X) &= \sum_{c \in \mathrm{CR}(A/D)} \mathrm{ER}(A/X, c) \times Q(c) \\
&= \sum_{k \in \{0,\dots,n\}} \mathrm{ER}(A/X, k/n) \times Q(k/n) \\
&\qquad \{\text{because } |X| = n \text{ and if } c = k/n \notin \mathrm{CR}(A/X) \\
&\qquad \text{then } \mathrm{ER}(A/X, c) = 0\} \\
&= \sum_{k \in \{0,\dots,n\}} \mathrm{ED}(A, k) \times Q(k/n) = \mathrm{GD}_Q(A). \qquad \square
\end{aligned}$$

Property 6.1.3. *The method GD has the equivalent expression*

$$\mathrm{GD}_Q(A/D) = \sum_{\alpha_i \in M(A/D)} (\alpha_i - \alpha_{i+1}) \times Q(C(A/D, \alpha_i)$$

so the efficiency of the method is O(n) *if A and D are arranged in decreasing order*, *and* O$(n \log n)$ *otherwise. This equivalent expression avoids calculating ER*(*A*/*D*) *explicitly*; *moreover*, *GD fulfills Property* 2.2.3.

Proof.

$$\begin{aligned}
\mathrm{GD}_Q(A/D) &= \sum_{c \in \mathrm{CR}(A/D)} \mathrm{ER}(A/D, c) \times Q(c) \\
&= \sum_{c \in \mathrm{CR}(A/D)} \left(\sum_{c = C(A/D, \alpha_i)} (\alpha_i - \alpha_{i+1}) \right) \times Q(c) \\
&= \sum_{c \in \mathrm{CR}(A/D)} \sum_{c = C(A/D, \alpha_i)} (\alpha_i - \alpha_{i+1}) \times Q(C(A/D, \alpha_i))
\end{aligned}$$

$$= \{\text{by definition of CR}\,(A/D),$$
$$\text{every } c \text{ has associated at least one } \alpha_i\}$$
$$= \sum_{\alpha_i \in M(A/D)} (\alpha_i - \alpha_{i+1}) \times Q(C(A/D, \alpha_i)). \qquad \square$$

Property 6.1.4. *The method GD fulfills Property* 2.2.4.

Proof. Let $D \subseteq A$. Then $A \cap D = D$ and hence $M(A/D) = M(D)$ and $1 \in M(D)$ (D is normalized). Moreover, $\mathrm{CR}(A/D) = \{1\}$ and hence $\mathrm{ER}(A/D) = \{1/1\}$, so finally $\mathrm{GD}_Q(A/D) = Q(1)$. $\square$

Property 6.1.5. *The method GD fulfills Property* 2.2.5.

Proof. Let $A \cap D = \emptyset$. Then, $M(A/D) = M(D)$ and $1 \in M(D)$ (D is normalized). Moreover, $\mathrm{CR}(A/D) = \{0\}$ and hence $\mathrm{ER}(A/D) = \{1/0\}$, so finally $\mathrm{GD}_Q(A/D) = Q(0)$. $\square$

Property 6.1.6. *The method GD fulfills Property* 2.2.6 *for the quantifier* $\exists$ *by means of the t-conorm* max *and the t-norm* min, *i.e.*

$$\mathrm{GD}_\exists(A/D) = \max_{x_i \in X} \min\,(A(x_i), D(x_i)).$$

Proof. We shall proceed in two steps:

($\exists$.1) Firstly, we will show that

$$\mathrm{ER}(A/D, 0) = 1 - \max_{x_i \in X} \min\,(A(x_i), D(x_i)).$$

By definition of $M(A/D)$, $\max_{x_i \in X} \min\,(A(x_i), D(x_i)) = \max(A \cap D) \in M(A/D)$. So, $\exists j \in \{1, \ldots, m\}$ such that $\max(A \cap D) = \alpha_j$.

Let $i < j$. Then, $(A \cap D) = \emptyset$, so $\left|(A \cap D)_{\alpha_j}\right| = 0$ and hence $C(A/D, \alpha_j) = 0$.

Let $i \geqslant j$. Then $(A \cap D) \neq \emptyset$ and hence $C(A/D, \alpha_j) > 0$. Then

$$\mathrm{ER}(A/D, 0) = \sum_{C(A/D, \alpha_i) = 0} (\alpha_i - \alpha_{i+1}) = \sum_{i<j} (\alpha_i - \alpha_{i+1})$$
$$= (\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) + \cdots + (\alpha_{j-1} - \alpha_j) = (\alpha_1 - \alpha_j)$$
$$= 1 - \alpha_j = 1 - \max(A \cap D)$$
$$= 1 - \max_{x_i \in X} \min\,(A(x_i), D(x_i)).$$

(∃.2) Secondly, we have

$$\begin{aligned}\mathrm{GD}_{\exists}(A/D) &= \sum_{c\in \mathrm{CR}(A/D)} \mathrm{ER}(A/D,c)\times \exists(c)\\ &= \sum_{c\in \mathrm{CR}(A/D),\ c>0} \mathrm{ER}(A/D,c)\\ &= \left(\sum_{c\in \mathrm{CR}(A/D)} \mathrm{ER}(A/D,c)\right) - \mathrm{ER}(A/D,0)\\ &= \{\mathrm{ER\ is\ a\ probabilistic\ measure}\}\\ &= 1-\left(1-\max_{x_i\in X}\min\,(A(x_i),D(x_i))\right)\\ &= \max_{x_i\in X}\min\,(A(x_i),D(x_i)). \qquad \square\end{aligned}$$

For the quantifier ∀, we have

$$\mathrm{GD}_{\forall}(A/D) = \sum_{C(A/D,\alpha_i)=1} (\alpha_i - \alpha_{i+1}),$$

i.e. the probability that the relative cardinality of A with respect to D is 1, and hence the probability that $D\subseteq A$. At this moment, we conjecture that this value can be expressed as Property 2.2.6 requires, we hope to offer a proof of this conjecture in a future paper.

Property 6.1.7. *The method GD fulfills Property* 2.2.7, *i.e. any quantifier can be used and there is no conflict with any other property.*

Property 6.1.8. *The method GD fulfills Property* 2.2.8.

Proof. We must find fuzzy sets A and D so that the evaluation of the sentence is not crisp.

1. If Q is not crisp, then two integers exist $p<q$ so that $0<Q(p/q)<1$. Let $c=p/q$. We define A and D as follows: $A=\{1/x_1,\ldots,\ 1/x_p, 0/x_{p+1},\ldots,0/x_n\}$ and $D=\{1/x_1,\ldots,1/x_P,1/x_{p+1},\ldots,1/x_q,\ 0/x_{q+1},\ldots,0/x_n\}$. Then $M(A/D)=\{1\}$, $\mathrm{CR}(A/D)=\{c\}$, $\mathrm{ER}(A/D)=\{1/c\}$ and $\mathrm{GD}_Q(A/D)=Q(c)$ with $0<Q(c)<1$.
2. If Q is crisp, then let $Q(0)=w\in\{0,1\}$. As $Q\neq\emptyset$ and $Q\neq[0,1]$, then two integers exist $p<q$ so that $Q(p/q)=1-w$ (i.e. if $Q(0)=0$ then $Q(p/q)=1$ and if $Q(0)=1$ then $Q(p/q)=0$). Let $0<a<1$ and let $c=p/q$. Then, let A and D be the following: $A=\{a/x_1,\ldots,a/x_p,\ 0/x_{p+1},\ldots,0/x_n\}$ and $D=\{1/x_1,\ldots,1/x_p,\ 1/x_{p+1},\ldots,1/x_q,\ 0/x_{q+1},\ldots,$

$0/x_n\}$. Then, $M(A/D) = \{1, a\}$, and $\mathrm{CR}(A/D) = \{0, p/q\}$, and finally $\mathrm{ER}(A/D) = \{(1-a)/0, a/c\}$, so finally $\mathrm{GD}_Q(A/D) = w \times (1-a) + (1-w) \times a$, i.e. if $w = 0$ then $\mathrm{GD}_Q(A/D) = a$, and if $w = 1$ then $\mathrm{GD}_Q(A/D) = 1 - a$, with $0 < a < 1$ and $0 < (1-a) < 1$. □

6.2. A possibilistic method for the evaluation of type II sentences

This method was described in [5]. It is based on the cardinality approach, and uses the relative cardinality ES described in Section 4.2.2. The resulting method, also called ZS, is defined as follows

$$\mathrm{ZS}_Q(A/D) = \max_{c \in \mathrm{CR}(A/D)} \min\left(\mathrm{ES}(A/D, c), Q(c)\right)$$

i.e. the cardinality approach using max–min composition to obtain the compatibility between the (fuzzy) relative cardinality and the quantifier.

Property 6.2.1. *The method ZS verifies Property* 2.2.1.

Proof. Let A and D be two crisp sets. Then, by definition $M(A/D) = \{1\}$ and then

$$\mathrm{CR}(A/D) = \left\{ \frac{|A \cap D|}{|D|} \right\}$$

therefore

$$\mathrm{ES}(A/D) = \left\{ 1 \Big/ \frac{|A \cap D|}{|D|} \right\}$$

and finally

$$\mathrm{ZS}_Q(A/D) = Q\left(\frac{|A \cap D|}{|D|} \right).$$

Property 6.2.2. *The method ZS for the evaluation of type* II *sentences in the case* $D = X$ *is the method ZS for the evaluation of type* I *sentences described in Section* 5.3, *and hence fulfills Property* 2.2.2.

Proof. We have proved in [6] that if $D = X$ then

$$\mathrm{ES}\left(A/X, \frac{k}{n} \right) = Z(A, k).$$

Let $S(A)$ be the support of A. Then

$$\begin{aligned}
\mathrm{ZS}_Q(A/X) &= \max_{c\in \mathrm{CR}(A/X)} \min\left(\mathrm{ES}(A/X,c),Q(c)\right)\\
&= \max_{k\in S(A)} \min\left(\mathrm{ES}\left(A/X,\frac{k}{n}\right),Q(k/n)\right)\\
&= \max_{k\in S(A)} \min\left(Z(A,k),Q(k/n)\right)\\
&= \max_{k\in\{0,\dots,n\}} \min\left(Z(A,k),Q(k/n)\right) = \mathrm{ZS}_Q(A). \qquad \square
\end{aligned}$$

Property 6.2.3. *The method ZS has the equivalent expression*

$$\mathrm{ZS}_Q(A/D) = \max_{\alpha\in M(A/D)} \min\left(\alpha, Q\left(\frac{\left|(A\cap D)_\alpha\right|}{|D_\alpha|}\right)\right)$$

so the efficiency of the method is $\mathrm{O}(n)$ *if A and D are arranged in decreasing order, and* $\mathrm{O}(n\log n)$ *otherwise. This equivalent expression avoids calculating ES(A/D) explicitly; moreover, ZS fulfills Property* 2.2.3.

Proof. Analogous to Property 5.3.4.

$$\begin{aligned}
\mathrm{ZS}_Q(A/D) &= \max_{c\in \mathrm{CR}(A/D)} \min\left(\mathrm{ES}(A/D,c),Q(c)\right)\\
&= \max_{c\in \mathrm{CR}(A/D)} \min\left(\max\left\{\alpha\in M(A/D)\,\middle|\, c=\frac{\left|(A\cap D)_\alpha\right|}{|D_\alpha|}\right\},\ Q(c)\right)\\
&= \max_{c\in CR(A/D)} \max_{c=\frac{|(A\cap D)_\alpha|}{|D_\alpha|}} \min\left(\alpha,Q(c)\right)\\
&= \max_{\alpha\in M(A/D)} \min\left(\alpha, Q\left(\frac{\left|(A\cap D)_\alpha\right|}{|D_\alpha|}\right)\right). \qquad \square
\end{aligned}$$

Property 6.2.4. *The method ZS fulfills Property* 2.2.4.

Proof. Let $D\subseteq A$. Then $A\cap D = D$ and hence $M(A/D) = M(D)$ and $1\in M(D)$ (D is normalized). Moreover, $\mathrm{CR}(A/D) = \{1\}$ and hence $\mathrm{ES}(A/D) = \{1/1\}$, so finally $\mathrm{ZS}_Q(A/D) = Q(1)$. $\square$

Property 6.2.5. *The method ZS fulfills Property* 2.2.5.

Proof. Let $A\cap D = \emptyset$. Then, $M(A/D) = M(D)$ and $1\in M(D)$ (D is normalized). Moreover, $\mathrm{CR}(A/D) = \{0\}$ and hence $\mathrm{ES}(A/D) = \{1/0\}$, so finally $\mathrm{ZS}_Q(A/D) = Q(0)$. $\square$

Property 6.2.6. *The method ZS fulfills Property* 2.2.6 *for the quantifiers* $\exists$ *and* $\forall$. *In the first case, by means of the t-conorm* max *and the t-norm* min, *i.e.*

$$\mathrm{ZS}_{\exists}(A/D) = \max_{x_i \in X} \min\,(A(x_i), D(x_i)).$$

In the second case, by means of the t-norm min *and a family of implications*

$$\mathrm{ZS}_{\forall}(A/D) = \alpha = \min_{x_i \in X} I_{\alpha}(D(x_i), A(x_i)),$$

where $I_\alpha(d,a)$ is defined by

$$I_{\alpha}(d,a) = \begin{cases} 1, & d \leqslant \max(a,\alpha),\ d < 1, \\ \alpha, & a \leqslant \alpha < d < 1, \\ a & \textit{otherwise}. \end{cases}$$

We will show that I_α is a fuzzy implication for all $\alpha \in [0,1]$ in Appendix A.

Proof.

$$\begin{aligned} (\exists)\ \mathrm{ZS}_{\exists}(A/D) &= \max_{\alpha \in M(A/D)} \min\left(\alpha\ ,\ \exists\left(\frac{\left|(A\cap D)_{\alpha}\right|}{|D_{\alpha}|}\right)\right) \\ &= \max\left\{\alpha \in M(A/D)\ \middle|\ (A\cap D)_{\alpha} \neq \emptyset\right\} \\ &= \max_{x_i \in X}\{(A\cap D)(x_i)\} \\ &= \max_{x_i \in X}\{\min\,(A(x_i), D(x_i))\}. \end{aligned}$$

($\forall$) Let $X1 = \{x \in X \mid A(x) < D(x)\}$ and let $X2 = X \setminus X1$. We will make the proof in four steps :

($\forall$.1) Firstly, we will show that $\exists x_0 \in X$ so that $\mathrm{ZS}_{\forall}(A/D) = A(x_0)$.

- Let $\mathrm{ZS}_{\forall}(A/D) = 1$. Then $A_1 = D_1 \neq \emptyset$ and $\exists x_0 \in X$ so that $A(x_0) = D(x_0) = 1 = \mathrm{ZS}_{\forall}(A/D)$.
- Let $\mathrm{ZS}_{\forall}(A/D) < 1$. Then $X1 \neq \emptyset$ (if $X1 = \emptyset$ then $D \subseteq A$ and by Property 6.2.4, $\mathrm{ZS}_{\forall}(A/D) = \forall(1) = 1$) and

$$\begin{aligned} \mathrm{ZS}_{\forall}(A/D) &= \max_{\alpha \in M(A/D)} \min\left(\alpha, \forall\left(\frac{\left|(A\cap D)_{\alpha}\right|}{|D_{\alpha}|}\right)\right) \\ &= \max\left\{\alpha \in M(A/D) \middle|\ (A\cap D)_{\alpha} = D_{\alpha}\right\} \\ &= \max_{x\in X}\left\{\alpha \in \{a(x), d(x)\} \text{ such that } (A\cap D)_{\alpha} = D_{\alpha}\right\} \\ &= \max_{x\in X}\left\{\alpha \in \{a(x), d(x)\} \text{ such that } (A|_{X1}\cap D|_{X1})_{\alpha} \cup (A|_{X2}\cap D|_{X2})_{\alpha}\right. \\ &\qquad \left. = (D|_{X1})_{\alpha} \cup (D|_{X2})_{\alpha}\right\}. \end{aligned}$$

By definition of $X2$, $A|_{X2} \cap D|_{X2} = D|_{X2}$ so we can say $\mathrm{ZS}_{\forall}(A/D) = \max_{x\in X1}\{\alpha \in \{a(x), d(x)\}$ such that $(A|_{X1} \cap D|_{X1})_{\alpha} = (D|_{X1})_{\alpha}\}$. We know $A(x) < D(x) \forall x \in X1$, so $\mathrm{ZS}_{\forall}(A/D) = \max_{x\in X1}\{\alpha = a(x)$ such that $(A|_{X1} \cap D|_{X1})_{\alpha} = (D|_{X1})_{\alpha}\}$ and $x_0 = \max_{x\in X1}\{a(x) | (A|_{X1} \cap D|_{X1})_{\alpha} = (D|_{X1})_{\alpha}\}$.

($\forall$.2) Secondly, we will show that if $\mathrm{ZS}_\forall(A/D)=\alpha$ then $\min_{x\in X}\{I_\alpha(D(x),A(x))\}\geqslant\alpha$.

Let us suppose $\min_{x\in X}\{I_\alpha(D(x),A(x))\}=A(x_1)<\alpha$. Then by definition of I_α, $D(x_1)=1$. But then $x_1\in D_1$ and hence $x_1\in D_\alpha$. By definition of α, $A_\alpha\cap D_\alpha=D_\alpha$, so $x_1\in A_\alpha$, and hence $A(x_1)\geqslant\alpha$ (contradiction), so we conclude $\min_{x\in X}\{I_\alpha(D(x),A(x))\}\geqslant\alpha$.

($\forall$.3) Thirdly, we will show that $\exists x_0\in X$ so that if $\mathrm{ZS}_\forall(A/D)=\alpha$ then $I_\alpha(D(x_0),A(x_0))=\alpha$.

- Let $\alpha=1$. Then by ($\forall$.1) $\exists x_0\in X$ so that $A(x_0)=D(x_0)=1$. Then, $I_\alpha(D(x_0),A(x_0))=I_1(1,1)=1=\alpha$.
- Let $\alpha<1$. Then by ($\forall$.1) $\exists x_0\in X1\subseteq X$ so that $A(x_0)=\alpha$. As $x_0\in X1$, $A(x_0)<D(x_0)$, so we have $A(x_0)=\alpha<D(x_0)$.
  1. Let $D(x_0)=1$. Then, $I_\alpha(D(x_0),A(x_0))=I_\alpha(1,\alpha)=\alpha$.
  2. Let $D(x_0)<1$. Then we have $A(x_0)=\alpha<D(x_0)<1$, so by definition $I_\alpha(D(x_0),A(x_0))=\alpha$.

($\forall$.4) Finally, we have shown in ($\forall$.2) that $\min_{x\in X}\{I_\alpha(D(x),A(x))\}\geqslant\alpha$ and in ($\forall$.3) that $\exists x_0\in X$ so that if $\mathrm{ZS}_\forall(A/D)=\alpha$ then $I_\alpha(D(x_0),A(x_0))=\alpha$, so we have $\mathrm{ZS}_\forall(A/D)=\alpha=\min_{x_i\in X}I_\alpha(D(x_i),A(x_i))$. □

Property 6.2.7. *The method ZS fulfills Property* 2.2.7, *i.e. any quantifier can be used and there is no conflict with any other property*, *except in the case of a crisp quantifier verifying* $Q(0)=Q(1)=1$, *as will be discussed in Property* 6.2.8.

Property 6.2.8. *The method ZS fulfills Property* 2.2.8 *except in the case that Q is a crisp quantifier and* $Q(0)=Q(1)=1$.

Proof. We must find fuzzy sets A and D so that the evaluation of the sentence is not crisp.

1. If Q is not crisp, then two integers exist $p<q$ so that $0<Q(p/q)<1$. Let $c=p/q$. We define A and D as follows: $A=\{1/x_1,\ldots,1/x_p,0/x_{p+1},\ldots,0/x_n\}$ and $D=\{1/x_1,\ldots,1/x_p,1/x_{p+1},\ldots,1/x_q,0/x_{q+1},\ldots,0/x_n\}$. Then $M(A/D)=\{1\}$, $\mathrm{CR}(A/D)=\{c\}$, $\mathrm{ES}(A/D)=\{1/c\}$ and $\mathrm{ZS}_Q(A/D)=Q(c)$.
2. If Q is crisp and $Q(0)<1$, then we consider two cases:
   (a) $Q(0)>0$. Then we define $A=\emptyset$ and $D=X=\{1/x_1,\ldots,1/x_n\}$. Hence $M(A/D)=\{1\}$, $\mathrm{CR}(A/D)=\{0\}$, $\mathrm{ES}(A/D)=\{1/0\}$ and $\mathrm{ZS}_Q(A/D)=Q(0)$.
   (b) $Q(0)=0$. By hypothesis of Property 2.2.8, $Q\neq\emptyset$ and hence two integers exist $p<q$ so that $Q(p/q)=1$ (Q is crisp). Let $c=p/q$ and let $0<a<1$. We define A and D as follows: $A=\{a/x_1,\ldots,a/x_p,0/x+p+1,\ldots,0/x_n\}$ and $D=\{1/x_1,a/x_2,\ldots,a/x_p,a/x_{p+1},\ldots,a/x_q,0/x_{q+1},\ldots,0/x_n\}$. Then $M(A/D)=\{1,a\}$, $\mathrm{CR}(A/D)=\{0,c\},\mathrm{ES}(A/D)=\{1/0,a/c\}$ and $\mathrm{ZS}_Q(A/D)=\max\{\min(1,Q(0)),\min(a,Q(c))\}=\min(a,Q(c))=a$.

3. If Q is crisp and $Q(1) < 1$ then we consider two cases:
   (a) $Q(1) > 0$. Then we define $A = D = X\{1/x1, \ldots, 1/xn\}$. Then $M(A/D) = \{1\}$, $\mathrm{CR}(A/D) = \{1\}$, $\mathrm{ES}(A/D) = \{1/1\}$ and $\mathrm{ZS}_Q(A/D) = Q(1)$.
   (b) $Q(1) = 0$. By hypothesis of Property 2.2.8, $Q \neq \emptyset$ and hence two integers exist $p < q$ so that $Q(p/q) = 1$ (Q is crisp). Let $c = p/q$ and let $0 < a < 1$. We define A and D as follows: $A = \{1/x_1, a/x_2 \ldots, a/x_p, 0/x_{p+1}, \ldots, 0/x_n\}$ and $D = \{1/x_1, a/x_2, \ldots, a/x_p, a/x_{p+1}, \ldots, a/x_q, 0/x_{q+1}, \ldots, 0/x_n\}$. Then $M(A/D) = \{1, a\}$, $\mathrm{CR}(A/D) = \{1, c\}$, $\mathrm{ES}(A/D) = \{1/1, a/c\}$ and $\mathrm{ZS}_Q(A/D) = \max\{\min(1, Q(1)), \min(a, Q(c))\} = \min(a, Q(c)) = a$. □

The question is, what could be the semantic of a crisp quantifier having $Q(0) = Q(1) = 1$? We think that this strange case will not be used in practice, so that this "exception" does not affect the fulfillment of Property 6.2.8 by the method ZS.

6.3. Some examples of application of the discussed methods

We shall use the quantifiers of Fig. 1, as in Section 5.4.

Example 6.3.1. Let A and D be the fuzzy sets defined in Fig. 5. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 4.

$$A = \{1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5, 0.9/x_6\}$$
$$D = \{0.3/x_1, 0.4/x_2, 0.8/x_3, 1/x_4, 0.1/x_5, 0.2/x_6\}.$$

In this example, $D \subseteq A$ and D is normalized, so for every quantifier Q the expected result is $Q(1)$, i.e. All(1) = 1, Exists(1) = 1, Most(1) = 1, Half(1) = 0 and At Least Half(1) = 1. We can see that only Z_Q, GD_Q and ZS_Q verify this property in general. The rest of the methods fail in this example for quantifiers All and Most.

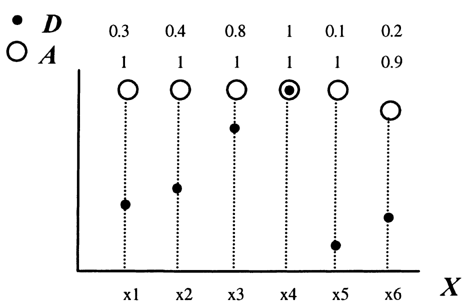

Fig. 5. Fuzzy sets A and D of Example 6.3.1.

Table 4
Evaluation of Example 6.3.1

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A/D)$	1	1	1	0	1
$Y_Q(A/D)$	0.9	1	0.964	–	1
$V_Q(A/D)$	0.9	1	0.95	–	0.98
$\mathrm{GD}_Q(A/D)$	1	1	1	0	1
$\mathrm{ZS}_Q(A/D)$	1	1	1	0	1

Example 6.3.2. Let A and D be the fuzzy sets defined in Fig. 6. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 5.

$$A = \{0/x_1, 0/x_2, 0/x_3, 1/x_4, 1/x_5, 1/x_6\},$$
$$D = \{1/x_1, 1/x_2, 0.1/x_3, 0/x_4, 0/x_5, 0/x_6\}.$$

In this example, we can see that method Y_Q does not fulfill Properties 2.2.6-$\exists$ and 2.2.5, because as $A \cap D = \emptyset$ then the evaluation must be $Q(0)$, and we have All (0) = Exists (0) = Most (0) = Half (0) = At Least Half (0) = 0.

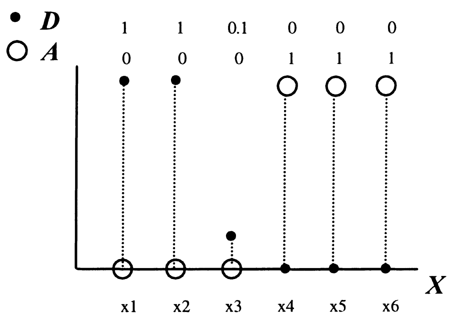

Fig. 6. Fuzzy sets A and D of Example 6.3.2.

Table 5
Evaluation of Example 6.3.2

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A/D)$	0	0	0	0	0
$Y_Q(A/D)$	0	0.9	0.042	–	0.085
$V_Q(A/D)$	0	0	0	–	0
$\mathrm{GD}_Q(A/D)$	0	0	0	0	0
$\mathrm{ZS}_Q(A/D)$	0	0	0	0	0

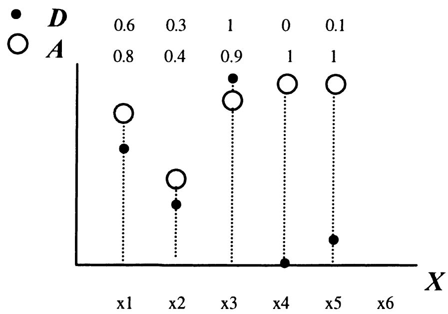

Fig. 7. Fuzzy sets A and D of Example 6.3.3.

Table 6
Evaluation of Example 6.3.3

Method	Quantifier				
	All	Exists	Most	Half	At Least Half
$Z_Q(A/D)$	0	1	0.95	0.1	1
$Y_Q(A/D)$	0.7	1	0.775	–	0.85
$V_Q(A/D)$	0.7	0.9	0.8	–	0.86
$\mathrm{GD}_Q(A/D)$	0.9	0.9	0.9	0	0.9
$\mathrm{ZS}_Q(A/D)$	0.9	0.9	0.9	0	0.9

Example 6.3.3. Let A and D be the fuzzy sets defined in Fig. 7. This example has been extracted from [21]. The results obtained using some of the methods and the five quantifiers defined before are shown in Table 6.

$$A = \{0.8/x_1, 0.4/x_2, 0.9/x_3, 1/x_4, 1/x_5\},$$
$$D = \{0.6/x_1, 0.3/x_2, 1/x_3, 0/x_4, 0.1/x_5\}.$$

In this example we can see that ZQ is too strict with the quantifier All.

7. Conclusions and future works

In this paper, we propose a (not closed) set of appropriate properties to be fulfilled by any method of evaluation of type I and type II sentences. We have discussed some existing methods from this point of view. For type I sentences, some of the existing methods are satisfactory with respect of most of the properties related to the evaluation, although they are restricted to relative and coherent quantifiers. For type II sentences, the existing methods are not satisfactory in the evaluation, and are also restricted to coherent quantifiers (as we discussed in Section 1, type II sentences are evaluated only for relative quantifiers). We have chosen the cardinality approach to obtain new methods that fulfill all the properties we have considered, and we have also interpreted these methods in terms of the cardinality approach.

We have shown that our method GD for type I sentences is a generalization of Yager's method based on OWA (that is, the method based on the Choquet fuzzy integral) that allows the use of any quantifier, wether coherent or not, and that can be used with absolute quantifiers. We have also shown that our method ZS for type I sentences is a generalization of the method based on the Sugeno fuzzy integral that allows using the same quantifiers as GD. Both methods GD and ZS fulfill Properties 2.1.1–2.1.6 for every absolute and relative quantifier. These methods use some new definitions of fuzzy cardinality defined in [6]. Another contribution has been an interpretation of the method based on the Sugeno integral by means of the cardinality approach using Dubois–Prade fuzzy cardinality and max–min composition.

Our contribution for the evaluation of type II sentences are the methods ZS and GD for type II sentences, which are efficient and non-strict methods of evaluation, fulfilling Properties 2.2.1–2.2.8 that we propose for every relative quantifier. They are also based on the cardinality approach, using new definitions of fuzzy relative cardinality proposed in [6]. Tables 7 and 8 show the methods discussed and proposed in this paper together with the properties fulfilled by each one. An "X" means that the method fulfills the property.

Type I sentences: As we discussed before, the efficiency of the methods GD and ZS can be improved to $O(n)$ if the fuzzy set A is arranged in non-increasing order.

Table 7
Properties of type I sentences evaluation methods

Method	2.1.1	2.1.2	2.1.3	2.1.4	2.1.5	2.1.6
Zadeh	X		X	$O(n)$		X
Yager-OWA	X	X	X	$O(n \log n)$	X	
GD	X	X	X	$O(n \log n)$	X	X
ZS	X	X	X	$O(n \log n)$	X	X

Table 8
Properties of type II sentences evaluation methods

Method	2.2.1	2.2.2	2.2.3	2.2.4	2.2.5	2.2.6.-$\exists$	2.2.6.-$\forall$	2.2.7	2.2.8
Zadeh	X	X	$O(n)$	X	X			X	
Yager	X	X	$O(n \log n)$				X		X
Vila			$O(n)$			X	X		X
GD	X	X	$O(n \log n)$	X	X	X	?	X	X
ZS	X	X	$O(n \log n)$	X	X	X	X	X	X

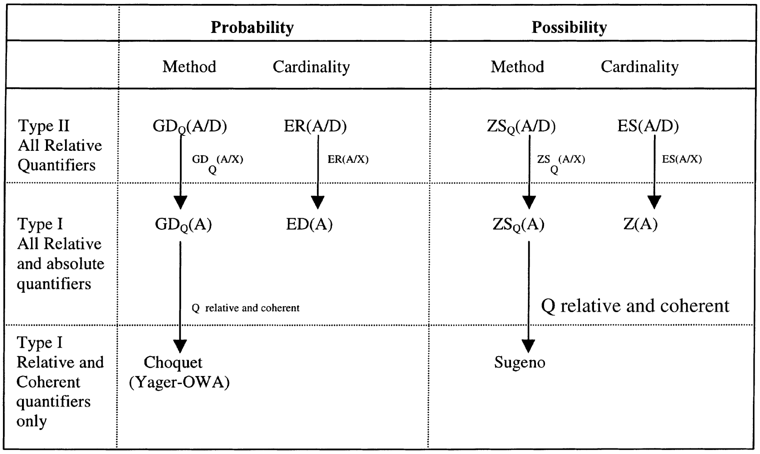

Fig. 8. Relations among methods of evaluation, with their corresponding cardinalities.

Type II sentences: The same consideration can be made with respect to the efficiency and the ordering of the fuzzy sets A and D. The efficiency of the methods GD and ZS is $O(n)$ if D and A are arranged in non-increasing order.

We have developed two methods for every type of quantified sentence, one probabilistic method (GD) and one possibilistic method (ZS) that generalize the existing probabilistic (Choquet) and possibilistic (Sugeno) approaches for the evaluation of type I sentences to type II sentences and any type of quantifier. These methods are based on new definitions of fuzzy cardinalities that are also related in terms of generalization. Schema (Fig. 8) shows the relation between the methods and between the cardinalities. The meaning of the arrows $X \to Y$ is "X generalizes Y".

Future works will focus on the efficient implementation of the new methods proposed and its use in database tasks and applications such as flexible query and data mining, where some new models which we are developing are based on the evaluation of quantified sentences. Another future work will be to find the relation between method GD and Property 2.2.6-$\forall$.

Appendix A

We will show that the family of functions I_α defined in Property 6.2.6 is a family of fuzzy implications. One fuzzy implication is a function with $[0,1] \times [0,1] \to [0,1]$ verifying the following five properties (see [16]):

1. Let $c \leqslant b$. Then, $I(c,x) \geqslant I(b,x)$;
2. Let $c \leqslant b$. Then, $I(x,c) \leqslant I(x,b)$;
3. $I(0,x) = 1$;

4. $I(1,x) = x$;
5. $I(b,I(c,x)) = I(c,I(b,x))$.

The family I_α is defined as follows:

$$I_\alpha(d,a) = \begin{cases} 1, & d \leqslant \max(a,\alpha),\ d < 1, \\ \alpha, & a \leqslant \alpha < d < 1, \\ a & \text{otherwise}, \end{cases}$$

where $\alpha \in [0,1]$.

Property A.1. *Let $c \leqslant b$. Then, we consider three cases*:

(a) *Let $I_\alpha(c,x) = 1$. Then it is obvious that $I(c,x) \geqslant I(b,x)$.*

(b) *Let $I_\alpha(c,x) = \alpha$. Then, $x \leqslant \alpha < c < 1$,* so $x \leqslant \alpha < b \leqslant 1$.
If $b = 1$ then $I_\alpha(b,x) = x \leqslant \alpha = I_\alpha(c,x)$.
If $b < 1$ then $x \leqslant \alpha < b < 1$ so $I_\alpha(b,x) = \alpha = I_\alpha(c,x)$.

(c) *Let $I_\alpha(c,x) = x$. There are only two possibilities*:
If $c = 1$ then $b = 1$ and then $I_\alpha(b,x) = x = I_\alpha(c,x)$.
If $c < 1$ and $c > \alpha$ and $x > \alpha$, then $b > \alpha$ and then $I_\alpha(b,x) = x = I_\alpha(c,x)$.

Property A.2. *Let $c \leqslant b$. Then, we consider three cases*:

(a) *Let $I_\alpha(x,b) = 1$. Then it is obvious that $I(x,b) \geqslant I(x,c)$.*

(b) *Let $I_\alpha(x,b) = \alpha$. Then $c \leqslant b \leqslant \alpha < x < 1$, so $I_\alpha(x,b) = \alpha = I_\alpha(x,c)$.*

(c) *Let $I_\alpha(x,b) = b$. Then there are three possibilities*:
If $x = 1$ then $I_\alpha(x,c) = c \leqslant b = I_\alpha(x,b)$.
If $x < 1$ and $x > \alpha$ and $b \geqslant c > \alpha$ then $I_\alpha(x,c) = c \leqslant b = I_\alpha(x,b)$.
If $x < 1$ and $x > \alpha$ and $b \geqslant \alpha \geqslant c$ then $I_\alpha(x,c) = \alpha \leqslant b = I_\alpha(x,b)$.

Property A.3. $I_\alpha(0,x) = 1$ *because* $0 \leqslant \max(x,\alpha)$ *and* $0 < 1$.

Property A.4. *$I_\alpha(1,x) = x$ because if $d = 1$ then $I_\alpha(d,a) = a$.*

Property A.5. *We must prove that $I_\alpha(b,I_\alpha(c,x)) = I_\alpha(c,I_\alpha(b,x))$. We shall consider six cases*:

1. *Let $I_\alpha(c,x) = I_\alpha(b,x) = 1$. Then, $I_\alpha(b,I_\alpha(c,x)) = I_\alpha(b,1) = 1$* and *$I_\alpha(c,I_\alpha(b,x)) = I_\alpha(c,1) = 1$.*

2. *Let $I_\alpha(c,x) = 1$ and let $I_\alpha(b,x) = x < 1$. Then, $I_\alpha(b,I_\alpha(c,x)) = I_\alpha(b,1) = 1$ and $I_\alpha(c,I_\alpha(b,x)) = I_\alpha(c,x) = 1$.*

3. *Let $I_\alpha(c,x) = I_\alpha(b,x) = x < 1$. Then $I_\alpha(b,I_\alpha(c,x)) = I_\alpha(b,x) = x$ and $I_\alpha(c,I_\alpha(b,x)) = I_\alpha(c,x) = x$.*

4. *Let $I_\alpha(c,x) = 1$ and let $I_\alpha(b,x) = \alpha < 1$. Then, $I_\alpha(b,I_\alpha(c,x)) = I_\alpha(b,1) = 1$. On the other hand, $I_\alpha(c,x) = 1$ so $c \leqslant \max(x,\alpha)$ and $c < 1$. Moreover,*

$I_\alpha(b,x) = \alpha$ so $x \leqslant \alpha < b < 1$ and hence $\max(x,\alpha) = \alpha < 1$, *so we have* $c \leqslant \max(\alpha,\alpha)$ *and $c < 1$ so $I_\alpha(c, I_\alpha(b,x)) = I_\alpha(c,\alpha) = 1$.*

5. *Let $I_\alpha(c,x) = I_\alpha(b,x) = \alpha < 1$. Then $x \leqslant \alpha < b < 1$ and $x \leqslant \alpha < c < 1$ so $\alpha = \alpha < b < 1$ and $\alpha = \alpha < c < 1$ and hence $I_\alpha(c, I_\alpha(b,x)) = I_\alpha(c,\alpha) = \alpha$ and $I_\alpha(b, I_\alpha(c,x)) = I_\alpha(b,\alpha) = \alpha$.*

6. *Let $I_\alpha(c,x) = x < 1$ and let $I_\alpha(b,x) = \alpha < 1$. Then $x \leqslant \alpha < b < 1$ and $I_\alpha(b, I_\alpha(c,x)) = I_\alpha(b,x) = \alpha$. On the other hand, as $I_\alpha(c,x) = x < 1$ then we have either $c >$* $\max(x,\alpha)$ *or $c = 1$.*

Let $c = 1$. Then $I_\alpha(c, I_\alpha(b,x)) = I_\alpha(c,\alpha) = I_\alpha(1,\alpha) = \alpha$.

Let $c >$ $\max(x,\alpha)$ *and $c < 1$. Then we have $x \leqslant \alpha < c < 1$ (we knew $x \leqslant \alpha < b < 1$) and hence $I_\alpha(c,x) = \alpha$, so $x = \alpha$ and hence $I_\alpha(c, I_\alpha(b,x)) = I_\alpha(c,\alpha) = I_\alpha(c,x) = x = \alpha$.*

Table 9 is a summary of the six cases discussed before.

The rest of the cases are symmetrical with respect to these six cases (interchanging b and c).

This family of implications also verifies another property:

Property A.6. $I_\alpha(x,x) = 1$ (*identity principle*). *We shall discuss two cases*:

Let $x < 1$. Then $x \leqslant$ $\max(x,\alpha)$ *and $x < 1$* so $I_\alpha(x,x) = 1$.

Let $x = 1$. Then by Property A.4, $I_\alpha(1,1) = 1$.

Special cases are as follows:

1. If $\alpha = 0$ then we obtain

$$I_0(d,a) = \begin{cases} 1, & d \leqslant a, \\ a, & d > a, \end{cases}$$

which is called a Gödel implication. This is an R-implication. An R-implication can be defined by means of the expression

Table 9
Possible cases for Property A.5

$I_\alpha(c,x)$	$I_\alpha(b,x)$	$I_\alpha(b, I_\alpha(c,x))$	$I_\alpha(c, I_\alpha(b,x))$
1	1	1	1
1	x	1	1
x	x	x	x
1	α	1	1
α	α	α	α
x	α	α	α

$$I(d,a) = \sup\{x | t(d,x) \leqslant a\},$$

t being a continuous t-norm. For this Gödel implication, t is the minimum. Since the greatest t-norm is the minimum, this implication is the greatest lower bound of R-implications.

2. If $\alpha = 1$, then we obtain the implication

$$I_1(d,a) = \begin{cases} 1, & d < 1, \\ a, & d = 1, \end{cases}$$

which is considered as the least upper bound of the class of R-implications (see [20]) although it cannot be defined using a continuous t-norm by means of the expression defined before. Nevertheless, a non-continuous t-norm exists so that I_1 can be defined by means of the expression for R-implications. This is the drastic intersection

$$t(x,y) = \begin{cases} x, & y = 1, \\ y, & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, it can be shown that any implication of the family I_α can be defined by means of the expression for R-implications using the following family of (non-continuous in general) t-norms:

$$t_\alpha(x,y) = \begin{cases} y, & x = 1, \\ x, & y = 1, \\ \min(x,y), & \alpha < x, y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

so that $I_\alpha(d,a) = \sup\{x | t_\alpha(d,x) \leqslant a\}$.

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