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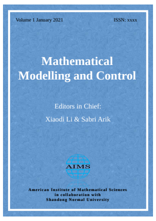

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Research article

Periodic solutions of the L_p-Minkowski problem with indefinite weight

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Abstract: We provide a new sufficient condition for the existence of a periodic solution of the singular differential equation

$$u'' + u = \frac{h(t)}{u^\rho},$$

which is associated with the planar L_p-Minkowski problem. A similar result is valid for the conformal version of the problem.

Keywords: L_p-Minkowski problem; singular differential equation; indefinite weight; periodic solution

1. Introduction

The L_p-Minkowski problem, first introduced by Lutwak in [1], asks about the existence of a closed convex hypersurface with support function u whose reciprocal Gauss curvature is $h(x)u^{p-1}$, where h is a prescribed function. In the planar case, this problem is equivalent to the existence of a periodic solution of the equation

$$u'' + u = \frac{h(t)}{u^\rho}, \tag{1.1}$$

where $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and $\rho := 1 - p$. By this reason, many researchers have considered this problem and there are a good number of papers providing sufficient conditions for the existence of a T-periodic solution of Eq (1.1), see for instance [2–9] and the references therein. However, in spite of the interest generated, the problem is far from being completely studied, specially in the case when h may change its sign. A related equation is the planar conformal curvature problem

$$-u'' + u = \frac{h(t)}{u^\rho}, \tag{1.2}$$

introduced by Loewner and Nirenberg [10]. The first result was given in [5], later improved in [11]. They stated that a valid sufficient condition for the existence of a T-periodic solution of eq. (1.2) is that $T \le 2\pi$, $h \in L^1(\mathbb{R}/T\mathbb{Z})$ is positive a.e. and $\rho > 1$ (see [11, Theorem 1.3]). Up to the date, it seems that there are no results for functions $h(t)$ of indefinite sign.

The objective of this paper is to consider Eqs (1.1) and (1.2) in a unified way and then to derive a new sufficient condition when $h(t)$ is of indefinite sign. It is worth to mention that related equations with a singular term appear in many applications (see [12] and the list of references) and consequently there are a variety of mathematical methods available, mainly of topological or variational nature. In our case, the main tool is a well-known fixed point theorem for Banach operators on conical shells due to Krasnoselskiĭ and the positivity of the associated Green function. This technique has been used before in related problems (see [13]), but in this case the consideration of a weight $h(t)$ of indefinite sign supposes the main novelty and a considerable technical difficulty as well.

2. Preliminaries and notations

Our proof is based on the following Krasnoselskiĭ-Guo fixed point theorem.

Lemma 2.1. *(Krasnoselskiĭ's-Guo fixed point theorem [14]) Let X be a Banach space and $\mathcal{K}$ a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let*

$$\Phi : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that one of the following conditions holds:
(i) $\|\Phi u\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi u\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$;
(ii) $\|\Phi u\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi u\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$.
Then Φ has a fixed point in the set $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To write the periodic problem as an equivalent fixed point problem, we use the notion of Green function. The following two lemmas are well known and can be found in many related papers (see for instance [15, Lemmas 2.1-2.5]. In this context, the general mechanism for the construction of Green functions in this context is described in [16].

Lemma 2.2. *If $M > 0$ is such that $M \neq \frac{2k\pi}{T}$ for any natural k, then for any $f \in L^1(\mathbb{R}/T\mathbb{Z})$ the equation*

$$u'' + M^2 u = f(t)$$

has a unique T-periodic solution given by

$$u(t) = \int_0^T G_1(t,s) f(s) ds,$$

where the Green's function $G_1(t,s)$ has the following form

$$G_1(t,s) = \begin{cases} \dfrac{\cos M(t-s-\frac{T}{2})}{2M \sin \frac{MT}{2}}, & 0 \leq s \leq t \leq T, \\ \dfrac{\cos M(t-s+\frac{T}{2})}{2M \sin \frac{MT}{2}}, & 0 \leq t < s \leq T. \end{cases}$$

Moreover, if $M < \frac{\pi}{T}$, then $G_1(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ and $\int_0^T G_1(t,s) M^2 ds \equiv 1$.

Lemma 2.3. *If $M > 0$, then for any $f \in L^1(\mathbb{R}/T\mathbb{Z})$ the equation*

$$-u'' + M^2 u = f(t)$$

has a unique T-periodic solution given by

$$u(t) = \int_0^T G_2(t,s) f(s) ds,$$

$$G_2(t,s) = \begin{cases} \dfrac{\exp(-M(s-t)) + \exp(M(s-t-T))}{2M(1-\exp(-MT))}, \\ \qquad 0 \leq s \leq t \leq T, \\ \dfrac{\exp(-M(s-t+T)) + \exp(M(s-t))}{2M(1-\exp(-MT))}, \\ \qquad 0 \leq t < s \leq T. \end{cases}$$

Moreover, $G_2(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ and $\int_0^T G_2(t,s) M^2 ds \equiv 1$.

We will use the notations

$$\begin{aligned} \mathcal{A}_1 &:= \min_{0 \leq s,t \leq T} G_1(t,s) = \frac{1}{2M} \cot \frac{MT}{2}, \\ \mathcal{B}_1 &:= \max_{0 \leq s,t \leq T} G_1(t,s) = \frac{1}{2M \sin \frac{MT}{2}}, \\ \mathcal{A}_2 &:= \min_{0 \leq s,t \leq T} G_2(t,s) = \frac{\exp(\frac{-MT}{2})}{M(1-\exp(-MT))}, \\ \mathcal{B}_2 &:= \max_{0 \leq s,t \leq T} G_2(t,s) = \frac{1+\exp(-MT)}{2M(1-\exp(-MT))}, \\ \sigma_1 &:= \frac{\mathcal{A}_1}{\mathcal{B}_1}, \quad \sigma_2 := \frac{\mathcal{A}_2}{\mathcal{B}_2}. \end{aligned} \tag{2.1}$$

Define

$$\mathcal{K}_i := \{u \in C_T : \min_{t\in\mathbb{R}} u(t) \geq \sigma_i \|u\|, \ \text{ for all } \ t \in \mathbb{R}\}, \ \ i = 1, 2,$$

where $C_T := \{u \in C(\mathbb{R}, \mathbb{R}), \ u(t + T) \equiv u(t), \ \text{for all } t \in \mathbb{R}\}$ with norm $\|u\| := \max_{t\in\mathbb{R}} |u(t)|$. It is easy to verify that $\mathcal{K}_1$ and $\mathcal{K}_2$ are cones in C_T.

Finally, for a given T-periodic function $h(t)$, we denote

$$\begin{aligned} &h^+(t) := \max\{h(t), 0\}, h^-(t) := -\min\{h(t), 0\},\\ &\overline{h} := \frac{1}{T}\int_0^T h(t)dt. \end{aligned}$$

3. Main results

The main result of the paper is the following one.

Theorem 3.1. *Fix $0 < T < \pi$ and $h \in L^1(\mathbb{R}/T\mathbb{Z})$. Assume that there exists $1 < M < \frac{\pi}{T}$ such that*

$$\frac{1}{\mathcal{A}_1 T \sigma_1^{1+\rho}}\left(\frac{\|h^-\|}{M^2 - 1}\right) < \overline{h^+}. \tag{3.1}$$

Then Eq (1.1) *admits at least one T-periodic solution.*

Proof. Writing eq. (1.1) as

$$u'' + M^2 u = \frac{h(t)}{u^\rho} + (M^2 - 1)u, \tag{3.2}$$

a T-periodic solution of eq. (3.2) is just a fixed point of the map Φ defined by

$$(\Phi u)(t) := \int_0^T G_1(t, s)\left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s)\right)ds,$$

and we know that $G_1(t, s) > 0$ for all $(t, s) \in [0, T] \times [0, T]$ by Lemma 2.2.

Let us note that since $M > 1$, we have the inequality

$$\mathcal{A}_1 T \sigma_1^{2+\rho} < \mathcal{A}_1 T \leq \int_0^T G_1(t, s)ds \equiv \frac{1}{M^2} < 1. \tag{3.3}$$

Now, we define two open sets

$$\Omega_1 := \{u \in C_T : \|u\| < r\} \quad \text{and} \quad \Omega_2 := \{u \in C_T : \|u\| < R\}.$$

Note that Φ is well-defined in the set $\mathcal{K}_1 \cap (\overline{\Omega}_2\backslash\Omega_1)$, and it is a completely continuous operator by a standard application of Ascoli-Arzela Theorem. Our intention is to apply Lemma 2.1.

By (3.1) and (3.3), the positive constants r and R can be fixed such that

$$R > \frac{1}{\sigma_1}\left(\frac{\overline{h^+}}{\sigma_1}\right)^{\frac{1}{1+\rho}} \geq r = (\mathcal{A}_1 T \overline{h^+})^{\frac{1}{1+\rho}} > \frac{1}{\sigma_1}\left(\frac{\|h^-\|}{M^2 - 1}\right)^{\frac{1}{1+\rho}}.$$

First, we claim that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2\backslash\Omega_1)) \subset \mathcal{K}_1$. In fact, for any $u \in \mathcal{K}_1 \cap (\overline{\Omega}_2\backslash\Omega_1)$,

$$\sigma_1 r < u(t) \leq R, \quad \text{for all } t \in \mathbb{R}.$$

From $M > 1$ and $r > \frac{1}{\sigma_1}\left(\frac{\|h^-\|}{M^2-1}\right)^{\frac{1}{1+\rho}}$, we see that

$$\begin{aligned} \frac{h(t)}{u^\rho(t)} + (M^2 - 1)u(t) =& \frac{h^+(t)}{u^\rho(t)} - \frac{h^-(t)}{u^\rho(t)} + (M^2 - 1)u(t)\\ >& -\frac{h^-(t)}{u^\rho(t)} + (M^2 - 1)u(t)\\ >& -\frac{\|h^-\|}{(\sigma_1 r)^\rho} + (M^2 - 1)\sigma_1 r\\ >&0, \end{aligned} \tag{3.4}$$

for all $t \in \mathbb{R}$. It follows from (3.4) that

$$\begin{aligned} \min_{t\in\mathbb{R}}(\Phi u)(t) =& \min_{t\in\mathbb{R}}\int_0^T G_1(t, s)\left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s)\right)ds\\ \geq& \mathcal{A}_1\int_0^T\left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s)\right)ds\\ =& \sigma_1\mathcal{B}_1\int_0^T\left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s)\right)ds\\ \geq& \sigma_1\max_{t\in\mathbb{R}}\int_0^T G_1(t, s)\left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s)\right)ds\\ =& \sigma_1\|\Phi u\|, \end{aligned}$$

which implies $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2\backslash\Omega_1)) \subset \mathcal{K}_1$.

Next, we prove that

$$\|\Phi u\| \leq \|u\|, \ \text{ for } \ u \in \mathcal{K}_1 \cap \partial\Omega_2. \tag{3.5}$$

In fact, for any $u \in \mathcal{K}_1 \cap \partial\Omega_2$, it is clear that $\|u\| = R$ and

$$\sigma_1 R \leq u(t) \leq R, \ \text{ for all } t \in \mathbb{R}.$$

Since $R > r > \frac{1}{\sigma_1}\left(\frac{\|h^-\|}{M^2-1}\right)^{\frac{1}{1+\rho}}$, from (3.2), we get

$$\begin{aligned}\frac{h(t)}{u^\rho(t)} + (M^2-1)u(t) >& -\frac{h^-(t)}{u^\rho(t)} + (M^2-1)u(t)\\ >& -\frac{\|h^-\|}{(\sigma_1 R)^\rho} + (M^2-1)(\sigma_1 R)\\ >&0,\end{aligned}$$

for all $t \in \mathbb{R}$. Then,

$$\begin{aligned}(\Phi u)(t) =& \int_0^T G_1(t,s)\left(\frac{h(s)}{u^\rho(s)} + (M^2-1)u(s)\right)ds\\ =& \int_0^T G_1(t,s)\left(\frac{h^+(s)}{u^\rho(s)} - \frac{h^-(s)}{u^\rho(s)} + (M^2-1)u(s)\right)ds\\ \leq& \frac{\mathcal{B}_1 T\overline{h^+}}{(\sigma_1 R)^\rho} + R\int_0^T G_1(t,s)M^2 ds - \mathcal{A}_1 T(\sigma_1 R)\\ =& \frac{\mathcal{B}_1 T\overline{h^+}}{(\sigma_1 R)^\rho} - \mathcal{A}_1 T(\sigma_1 R) + R,\end{aligned} \tag{3.6}$$

where we use the fact $\int_0^T G_1(t,s)M^2 ds \equiv 1$. From $R > \frac{1}{\sigma_1}\left(\frac{\overline{h^+}}{\sigma_1}\right)^{\frac{1}{1+\rho}}$ and $\sigma_1\mathcal{B}_1 = \mathcal{A}_1$, we get

$$\frac{\mathcal{B}_1 T\overline{h^+}}{(\sigma_1 R)^\rho} < \mathcal{A}_1 T(\sigma_1 R). \tag{3.7}$$

Applying (3.7) to (3.6),

$$(\Phi u)(t) \leq R, \quad \text{for all } t \in \mathbb{R},$$

and therefore (3.5) holds.

Finally, let us prove that

$$\|\Phi u\| \geq \|u\|, \quad \text{for } u \in \mathcal{K}_1 \cap \partial\Omega_1. \tag{3.8}$$

In fact, any $u \in \mathcal{K}_1 \cap \partial\Omega_1$ verifies $\|u\| = r$ and

$$\sigma_1 r \leq u(t) \leq r, \ \text{for all } t \in \mathbb{R}.$$

From $r > \frac{1}{\sigma_1}\left(\frac{\|h^-\|}{M^2-1}\right)^{\frac{1}{1+\rho}}$ and $M > 1$, we get

$$\begin{aligned}\frac{h(t)}{u^\rho(t)} + (M^2-1)u(t) >& -\frac{h^-(t)}{u^\rho(t)} + (M^2-1)u(t)\\ >& -\frac{\|h^-\|}{(\sigma_1 r)^\rho} + (M^2-1)(\sigma_1 r)\\ >&0,\end{aligned}$$

for all $t \in \mathbb{R}$. Then,

$$\begin{aligned}(\Phi u)(t) =& \int_0^T G_1(t,s)\left(\frac{h(s)}{u^\rho(s)} + (M^2-1)u(s)\right)ds\\ =& \int_0^T G_1(t,s)\left(\frac{h^+(s)}{u^\rho(s)} - \frac{h^-(s)}{u^\rho(s)} + (M^2-1)u(s)\right)ds\\ \geq& \int_0^T G_1(t,s)\frac{h^+(s)}{u^\rho(s)}ds\\ \geq& \frac{\mathcal{A}_1 T\overline{h^+}}{r^\rho} = r,\end{aligned}$$

since $r = (\mathcal{A}_1 T\overline{h^+})^{\frac{1}{1+\rho}}$ from definition of r. Hence, (3.8) holds. The proof is completed. □

Remark 3.2. *In view of* (2.1)*, it is important to note that* $\mathcal{A}_1$ *and* σ_1 *are functions of* M *and* T. *In fact, the sufficient condition* (3.1) *has an explicit expression as*

$$\frac{M}{M^2-1}\sin^{1+\rho}\left(\frac{MT}{2}\right)\|h^-\| < T\overline{h^+}\cos^{2+\rho}\left(\frac{MT}{2}\right). \tag{3.9}$$

From here it is easy to construct explicit examples. For instance, taking $T = \frac{\pi}{4}$, $M = 2$, $\rho = 1$ *and the function*

$$h(t) = \begin{cases} 5\pi\sin 8t, & t \in [0, \frac{\pi}{8}],\\ \sin 8t, & t \in [\frac{\pi}{8}, \frac{\pi}{4}],\end{cases}$$

some straightforward computations show that (3.9) *holds and then Eq* (1.1) *admits at least one* $\frac{\pi}{4}$*-periodic solution. Up to our knowledge, this example is not covered by any of the results available in the literature.*

Remark 3.3. *In the recent reference [17], it is proved that all the solutions of* (1.1) *are unbounded if* $\rho = 3$ *and* $h(t)$ *is a piece-wise constant* π*-periodic function. This suggests that the period value* $T = \pi$ *is critical, although in this example it is crucial the role of the cubic singularity, which is associated to the isochronicity of the center of the autonomous equation.*

Concerning the conformal Eq (1.2), we have a similar result to Theorem 3.1, in this case without any restriction on the period $T > 0$.

Theorem 3.4. *For a given* $h \in L^1(\mathbb{R}/T\mathbb{Z})$*, assume that there exists* $M > 1$ *such that*

$$\frac{1}{\mathcal{A}_2 T \sigma_2^{1+\rho}} \left(\frac{\|h^-\|}{M^2 - 1} \right) < \overline{h^+}. \tag{3.10}$$

Then Eq (1.2) *admits at least one* T*-periodic solution.*

Proof. If we write Eq (1.2) as

$$-u'' + M^2 u = \frac{h(t)}{u^\rho} + (M^2 - 1)u(t), \tag{3.11}$$

then a T-periodic solution of Eq (3.11) is equivalent to a fixed point of the map

$$(\Psi u)(t) := \int_0^T G_2(t,s) \left(\frac{h(s)}{u^\rho(s)} + (M^2 - 1)u(s) \right) ds.$$

From this point, the proof follows the same steps as Theorem 3.1. □

Remark 3.5. *As in Theorem 3.1, the sufficient condition* (3.10) *can be written explicitly as*

$$\frac{M(1 - \exp(-MT))(1 + \exp(-MT))^{2+\rho}}{2^{2+\rho} T \left(\exp\left(-\frac{MT}{2}\right) \right)^{3+\rho}} \left(\frac{\|h^-\|}{M^2 - 1} \right) < \overline{h^+}.$$

Again, we can use this condition to construct easily explicit examples.

Remark 3.6. *It is interesting to note that the condition* $\overline{h} > 0$ *is necessary for the existence of a* T*-periodic solution of Eq* (1.2)*. In fact, if* $v(t)$ *is a* T*-periodic solution, multiplying Eq* (1.2) *by* $v^\rho(t)$ *and integrating from 0 to* T*, we obtain*

$$\begin{aligned} &\int_0^T h(s)ds \\ &= -\int_0^T v''(s)v^\rho(s)ds + \int_0^T v^{1+\rho}(s)ds \\ &= \rho \int_0^T v^{\rho-1}(s)(v'(s))^2 ds + \int_0^T v^{1+\rho}(s)ds \\ &> 0. \end{aligned}$$

Of course, this argument is not valid for Eq (1.1). *Our conjecture is that* $\overline{h} > 0$ *is a necessary and sufficient condition, it remains as an open problem.*

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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