Letter

Positivity bounds in the standard model effective field theory beyond tree level

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Focusing on four-Higgs interactions, we analyze the robustness of tree-level-derived positivity bounds on standard model effective field theory (SMEFT) operators under quantum corrections. Among other results, we demonstrate that: (i) Even in the simplest extensions of the Standard Model, e.g., with one new scalar singlet or with a neutral triplet, some positivity bounds are strictly violated; (ii) the mixing of the dimension-eight operators under renormalization, which we compute here for the first time, can drive them out of their positivity region; (iii) the running of the dimension-eight interactions triggered by solely dimension-six terms respects the positivity bounds. Our results suggest, on one hand, that departures from positivity within the SMEFT, if ever found in the data, do not necessarily imply the breaking of unitarity or causality, nor the presence of new light degrees of freedom. On the other hand, they lead to strong constraints on the form of certain anomalous dimensions.

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I. INTRODUCTION

Effective field theories (EFT) are the right tool to describe particle physics in the presence of significant mass gaps. In particular, the Standard Model EFT (SMEFT), see [1] for a review, is a very promising candidate at energies 100 GeV $\lesssim E \lesssim$ TeV, given that no new resonances have been found in this regime. (The existence of weakly coupled light degrees of freedom, which would require extending the SMEFT, cannot be discarded, though.)

The parameters of the SMEFT Lagrangian have been subject of experimental scrutiny for many years. By now, many directions in the SMEFT, although certainly not all, have been severely constrained; see for example [2–6].

More recently, though, there has been a huge progress in narrowing the SMEFT landscape purely from theoretical arguments [7–19]. These rely on the basic principles of quantum mechanics and relativity, and in particular on the analyticity and unitarity of the S-matrix [20]. The corresponding bounds, typically derived from the EFT at tree level, appear in the form of constraints on the sign of certain combinations of Wilson coefficients, and they are commonly known as *positivity bounds*. Often, they are complementary to current experimental limits [11].

The possibility that running effects can modify the conclusions of positivity within the SMEFT has been only considered [21] in the presence of quantum gravity corrections (with the interesting conclusion that positivity constraints remain valid in this case); as it has been widely believed, since the seminal work of [20], that the bounds derived from the EFT at tree level should hold even in the presence of loops of massless scalars and vectors. (It has since been suggested that any eventual departure from positivity should be either ascribed to the failure of the EFT description or more ambitiously to the breakdown of quantum field theory; see for example [12] and references therein.)

Focusing on dimension-eight operators with four Higgs fields, in this paper we investigate under which conditions positivity bounds are violated by *running and threshold* effects of scalars and gauge bosons, even in the simplest extensions of the SM, and derive new constraints on the shape of certain anomalous dimensions.

II. POSITIVITY BOUNDS AT TREE LEVEL

To fix notation, let us first write the SM Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm SM} = &-\frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} - \frac{1}{4}W^I_{\mu\nu}W^{\mu\nu I} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\\ &+ (D_\mu H)^\dagger(D^\mu H) + \mu^2 H^\dagger H - \lambda(H^\dagger H)^2\\ &+ i(\bar{q}\not{D}q + \bar{u}\not{D}u + \bar{d}\not{D}d + \bar{l}\not{D}l + \bar{e}\not{D}e)\\ &- (\bar{q}Y_d H d + \bar{q}Y_u\tilde{H}u + \bar{l}Y_e H e + \text{H.c.}). \qquad (1)\end{aligned}$$

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We have introduced l and e for the left-handed (LH) and right-handed (RH) leptons, respectively; and q and u, d for the LH and RH quarks, respectively. G, W and B represent the gauge bosons of $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$, and $H = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2, \phi_3 + i\phi_4)^T$ stands for the Higgs doublet. We have also defined $\tilde{H} = \epsilon H$, with ϵ being the fully antisymmetric tensor.

Our convention for the covariant derivative is

$$D_\mu = \partial_\mu - ig_1 Y B_\mu - ig_2 \frac{\sigma^I}{2} W^I_\mu - ig_3 \frac{\lambda^A}{2} G^A_\mu. \qquad (2)$$

Y stands for the hypercharge, g_1, g_2, and g_3 represent the $U(1)_Y$, $SU(2)_L$, and $SU(3)_c$ gauge couplings; and σ^I and λ^A denote the Pauli and Gell-Mann matrices, respectively.

We disregard the Yukawa couplings throughout this paper, since they do not play any role in any of our discussions. Likewise, we work in the approximation $\mu^2 \to 0$. All our results are then valid up to μ^2/Λ^2 corrections, where Λ represents the SMEFT cutoff. Moreover, unless otherwise stated, we also assume $g_1, g_2, g_3 \to 0$, as our points are made clearer within this approximation.

The SMEFT extends the SM Lagrangian with operators of dimension higher than four, suppressed by increasing powers of the cutoff Λ. We neglect operators of dimension higher than eight as well as lepton- and baryon-number violating interactions. This leaves us with operators of dimension six and eight only, that we choose to describe using the Warsaw basis [22] and the basis of interactions reported in [23],[1] respectively.

Let us consider the process $\phi_i\phi_j \to \phi_i\phi_j$. At low energies, it can be described by four-Higgs interactions:

$$\mathcal{L} = \cdots - \lambda |H|^4 + \frac{c^{(i)}_{H^4D^2}}{\Lambda^2} \mathcal{O}^{(i)}_{H^4D^2} + \frac{c^{(j)}_{H^4D^4}}{\Lambda^4} \mathcal{O}^{(j)}_{H^4D^4}; \qquad (3)$$

see Table I for the definition of the operators. The most common nomenclature for the dimension-six operators is $\mathcal{O}_{\phi\Box}$ (for $i = 1$) and $\mathcal{O}_{\phi D}$ (for $i = 2$) [22]; we find however convenient to work with the different naming for clarity of the exposition.

The usual derivation of bounds on the dimension-eight coefficient works as follows. First, we *assume* that in the UV the forward scattering amplitude $\mathcal{A}(s) \equiv \mathcal{A}(s, t = 0)$ for the process of interest is analytic in the complex plane with, at most, branch cuts in the real s axis starting at $|s| > 0$; see Fig. 1. Note that this implicitly assumes that the running of the EFT Wilson coefficients, and therefore a branch cut all the way to the origin, is either absent or negligible. In this sense, we can talk about *the* EFT Wilson coefficients, without mention to any renormalization scale.

[1] See [24] for a different basis of dimension-eight interactions.

TABLE I. Independent four-Higgs operators at dimension six (top) and dimension eight (bottom).

$\mathcal{O}^{(1)}_{H^4D^2}$	$(H^\dagger H)\Box(H^\dagger H)$
$\mathcal{O}^{(2)}_{H^4D^2}$	$(H^\dagger D^\mu H)^*(H^\dagger D_\mu H)$
$\mathcal{O}^{(1)}_{H^4D^4}$	$(D_\mu H^\dagger D_\nu H)(D^\nu H^\dagger D^\mu H)$
$\mathcal{O}^{(2)}_{H^4D^4}$	$(D_\mu H^\dagger D_\nu H)(D^\mu H^\dagger D^\nu H)$
$\mathcal{O}^{(3)}_{H^4D^4}$	$(D_\mu H^\dagger D^\mu H)(D^\nu H^\dagger D_\nu H)$

We then consider the integral $\mathcal{I} = \oint \mathcal{A}(s)/s^3$ around a small circular path enclosing $s = 0$. By Cauchy's theorem, $\mathcal{I}$ is fixed by the residue of the integrand at the origin.

Now, the circular path can be deformed to an infinitely large contour as the one shown in the figure. The contribution from the circular sectors to $\mathcal{I}$ vanishes because the amplitude falls fast enough at infinity [25,26], while the contribution from the discontinuities can be related to the imaginary part of the forward amplitude [20], which by virtue of the optical theorem is positive. Altogether, we obtain that the residue of $\mathcal{A}(s)/s^3$ at the origin is positive, or in other words:

$$\left.\frac{d^2\mathcal{A}(s)}{ds^2}\right|_{s=0} > 0. \qquad (4)$$

This residue can be computed in the EFT. Using Eq. (3) for the process $\phi_1\phi_2 \to \phi_1\phi_2$ *at tree level*, we find that

$$\mathcal{A}(s) = -2\lambda + c^{(2)}_{H^4D^4}\frac{s^2}{\Lambda^4}, \qquad (5)$$

which gives

$$c^{(2)}_{H^4D^4} > 0. \qquad (6)$$

The processes $\phi_1\phi_3 \to \phi_1\phi_3$ and $\phi_1\phi_1 \to \phi_1\phi_1$ imply the constraints:

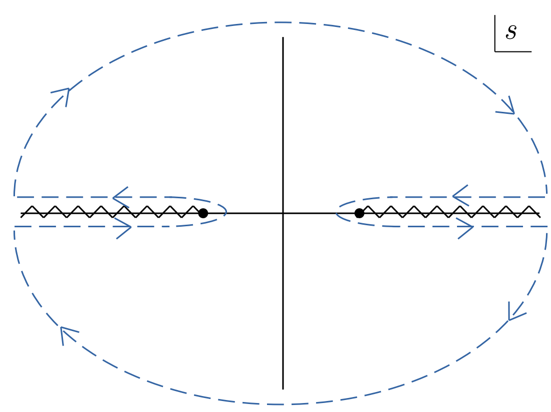

FIG. 1. Structure of singularities of the forward two-to-two amplitude $\mathcal{A}(s)$ in the complex plane of s. We also show the contour of integration used in the derivation of positivity bounds.

$$c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} > 0, \tag{7}$$

$$c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4} > 0, \tag{8}$$

respectively; see [11]. Compatible bounds were also obtained in [10] in the broken phase of the Higgs.

III. POSITIVITY BOUNDS AT ONE LOOP

In order to analyze the fate of the above results when moving to one loop, we must first notice that the Wilson coefficients of the operators, at the EFT cutoff scale Λ, admit themselves a perturbative expansion. To make it clear, we introduce a weak coupling g and write:

$$c^{(j)}_{H^4D^4} = g c^{(j)\,\mathrm{tree}}_{H^4D^4} + g^2 c^{(j)\,\mathrm{loop}}_{H^4D^4} + \cdots, \tag{9}$$

and similarly for $c^{(i)}_{H^4D^2}$ and λ. Thus, the forward amplitude for $\phi_1\phi_2 \to \phi_1\phi_2$ scattering to order $\mathcal{O}(g^2)$ in a neighborhood of $s = 0$ reads:

$$\mathcal{A}(s) \sim -2g\lambda^{\mathrm{tree}} + g^2\left[-2\lambda^{\mathrm{loop}} + \frac{3}{2\pi^2}(\lambda^{\mathrm{tree}})^2 \log\frac{\Lambda^2}{s}\right] + \left(g c^{(2)\,\mathrm{tree}}_{H^4D^4} + g^2 c^{(2)\,\mathrm{loop}}_{H^4D^4} - \frac{\beta^{(2)}_{H^4D^4}}{2}\log\frac{\Lambda^2}{s}\right)\frac{s^2}{\Lambda^4}, \tag{10}$$

up to finite terms proportional to the tree level Wilson coefficients. The $\beta^{(2)}_{H^4D^4}$ stands for the β function of $c^{(2)}_{H^4D^4}$, defined by $\mu dc^{(2)}_{H^4D^4}/d\mu = \beta^{(2)}_{H^4D^4}$. This function receives two contributions, corresponding to the renormalization triggered by pairs of dimension-six interactions and to that driven by dimension-eight operators via λ. Schematically:

$$\frac{1}{g^2}\beta^{(2)}_{H^4D^4} \sim \gamma'_{ij} c^{(i)\,\mathrm{tree}}_{H^4D^2} c^{(j)\,\mathrm{tree}}_{H^4D^2} + \gamma_i \lambda^{\mathrm{tree}} c^{(i)\,\mathrm{tree}}_{H^4D^4}, \tag{11}$$

where γ' and γ are anomalous dimensions. Several interesting conclusions can be derived from considering Eq. (10) in different limits.

To start with, let us assume that none of the effective interactions is generated at tree level, and $\lambda^{\mathrm{tree}} = 0$ as well. Then, ignoring s-independent terms, $\mathcal{A}(s)$ is simply:

$$\mathcal{A}(s) \sim c^{(2)\,\mathrm{loop}}_{H^4D^4}\frac{s^2}{\Lambda^4}. \tag{12}$$

Upon making the same reasoning leading to Eq. (4), we obtain $c^{(2),\mathrm{loop}}_{H^4D^4} > 0$.

Let us now turn our attention to the case in which all effective operators but $\mathcal{O}^{(2)}_{H^4D^4}$ can arise at tree level, and still $\lambda^{\mathrm{tree}} = 0$. The amplitude $\mathcal{A}(s)$ near $s = 0$ is

$$\mathcal{A}(s) \sim g^2\left(c^{(2)\,\mathrm{loop}}_{H^4D^4} - \frac{\gamma'_{ij}}{2} c^{(i)\,\mathrm{tree}}_{H^4D^2} c^{(j)\,\mathrm{tree}}_{H^4D^2}\log\frac{\Lambda^2}{s}\right)\frac{s^2}{\Lambda^4}. \tag{13}$$

In this case, the branch cut all the way to $s = 0$ originated by the logarithm prevents using the argument outlined in the previous section. To circumvent this obstacle, one can include a small mass m for the Higgs, thus generating an analytic region around $s = 0$. This amounts to deforming the logarithm $\log\Lambda^2/s \to \log[\Lambda^2/(s+m^2)]$. We can subsequently study the limit $m^2 \to 0$ upon expanding the logarithm in powers of s/m^2; see [20,27]. In such limit, the dominant contribution is

$$\mathcal{A}(s) \sim -g^2\frac{\gamma'_{ij}}{2}\log\frac{\Lambda^2}{m^2} c^{(i)\,\mathrm{tree}}_{H^4D^2} c^{(j)\,\mathrm{tree}}_{H^4D^2}\frac{s^2}{\Lambda^4} + \mathcal{O}(s^3). \tag{14}$$

A first implication of this result is that $c^{(2)\,\mathrm{loop}}_{H^4D^4}$ can have either sign without affecting the positivity of the forward amplitude. Therefore, the bound obtained in the previous section, $c^{(2)}_{H^4D^4} > 0$, does not necessarily hold in models in which this coefficient arises only in loops, provided that other operators are generated at tree level.

Further, requiring the second derivative of $\mathcal{A}(s)$ to be positive at the origin implies very severe constraints on the running of $c^{(2)}_{H^4D^4}$ triggered by pairs of dimension-six interactions, namely:

$$\gamma'_{ij} c^{(i)\,\mathrm{tree}}_{H^4D^2} c^{(j)\,\mathrm{tree}}_{H^4D^2} < 0. \tag{15}$$

Note that, because arbitrary values of the dimension-six Wilson coefficients are compatible with the assumption $c^{(2)\,\mathrm{tree}}_{H^4D^4} = 0$ [28], and given that λ can be always made zero by just tuning the renormalizable Lagrangian, the bound above is completely general. (Up to gauge corrections which, however, as we show below, are not present in this case.) Moreover, using the exact same reasoning one concludes that this inequality is valid even when including fermionic dimension-six operators, the relevant of which are $\mathcal{O}_{H\psi_R} = (H^\dagger i\overleftrightarrow{D}_\mu H)(\overline{\psi_R}\gamma^\mu\psi_R)$, $\mathcal{O}^{(1)}_{H\psi_L} = (H^\dagger i\overleftrightarrow{D}_\mu H)(\overline{\psi_L}\gamma^\mu\psi_L)$, $\mathcal{O}^{(3)}_{H\psi_L} = (H^\dagger i\overleftrightarrow{D}^I_\mu H)(\overline{\psi_L}\gamma^\mu\sigma_I\psi_L)$, as well as $\mathcal{O}_{Hud} = (\tilde{H} iD_\mu H)(u\gamma^\mu d) + \mathrm{H.c.}$, with $\psi_R = e, u, d$ and $\psi_L = l, q$.

Conversely, the renormalization of $c^{(2)}_{H^4D^4}$ driven by λ can take it out of its positivity region. To show why, let us now consider the limit of negligible dimension-six terms (this is only for simplicity of the exposition) and also $c^{(2)\,\mathrm{tree}}_{H^4D^4} = 0$. Upon deforming again the logarithm, we obtain:

$$\mathcal{A}(s) \sim \frac{g^2}{2}\left[\frac{3}{2\pi^2}(\lambda^{\mathrm{tree}})^2\frac{\Lambda^4}{m^4} - \gamma_i\lambda^{\mathrm{tree}} c^{(i)\,\mathrm{tree}}_{H^4D^4}\log\frac{\Lambda^2}{m^2}\right]\frac{s^2}{\Lambda^4}. \tag{16}$$

In the limit $m^2 \to 0$, the first term dominates and therefore $\gamma_i c^{(i)\,\mathrm{tree}}_{H^4D^4}$ is not necessarily negative for arbitrary values of the Wilson coefficients. This conclusion still holds if dimension-six operators are not ignored, precisely because they do not contribute to $\mathcal{A}(s)$ at tree level, and because they fulfill Eq. (15).

All these observations hold still in the presence of gauge couplings, although the proof is less straightforward. (For example, the massless gauge bosons induce poles at $s = 0$ even at tree level.) Likewise, analyses analogous to the one we just did for $\phi_1\phi_2 \to \phi_1\phi_2$ but applied to $\phi_1\phi_3 \to \phi_1\phi_3$ and $\phi_1\phi_1 \to \phi_1\phi_1$ reveal that the bounds in Eqs. (7) and (8) could be also violated at the loop level.

In summary, we can conclude that:

(i) If effective interactions can arise at tree level, but either $c^{(2)}_{H^4D^4}$ or the combination $c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4}$ or $c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4}$ vanishes accidentally at this order, then the constraints in Eqs. (6), (7), or (8) can be broken, respectively.

(ii) If no operator can be generated at tree level, then all bounds in Eqs. (6)–(8) are satisfied at one loop.

(iii) The renormalization of $c^{(j)}_{H^4D^4}$ by pairs of dimension-six operators maintain those Wilson coefficients within their (tree-level) positivity region.

(iv) The renormalization of $c^{(j)}_{H^4D^4}$ by relevant couplings (including mixing with other dimension-eight operators) can drive these Wilson coefficients out of their positivity region.

IV. ONE-LOOP MATCHING OF UV MODELS

In the reminder of this paper, we show that our previous arguments, albeit somewhat heuristic, are in fact realized in minimal extensions of the SM. Technical details on the following computations will be thoroughly explained elsewhere [29].

First, let us extend the SM with a heavy scalar neutral singlet $\mathcal{S}$ of mass $M = \Lambda$, with interaction Lagrangian:

$$\mathcal{L}_{\mathcal{S}} = \kappa_{\mathcal{S}} \mathcal{S} H^\dagger H. \tag{17}$$

This is obviously not the most generic Lagrangian, but it suffices to illustrate our point.

At tree level, we obtain: $c^{(1)\,\mathrm{tree}}_{H^4D^4} = c^{(2)\,\mathrm{tree}}_{H^4D^4} = 0$, $c^{(3)\,\mathrm{tree}}_{H^4D^4} = 2\frac{\kappa_{\mathcal{S}}^2}{M^2}$. At one loop and at the matching scale[2] $\mu = M$, we get instead: $c^{(1)\,\mathrm{loop}}_{H^4D^4} = -\frac{39}{144\pi^2}\frac{\kappa_{\mathcal{S}}^4}{M^4}$, $c^{(2)\,\mathrm{loop}}_{H^4D^4} = -\frac{39}{144\pi^2}\frac{\kappa_{\mathcal{S}}^4}{M^4}$, $c^{(3)\,\mathrm{loop}}_{H^4D^4} = -\frac{187}{720\pi^2}\frac{\kappa_{\mathcal{S}}^4}{M^4}$.

Therefore, $c^{(2)}_{H^4D^4} = -\frac{39}{144\pi^2}\frac{\kappa_{\mathcal{S}}^4}{M^4} < 0$, and $c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} = -\frac{39}{72\pi^2}\frac{\kappa_{\mathcal{S}}^4}{M^4} < 0$, and then both Eqs. (6) and (7) are violated within this model.

Let us now consider the SM extended with a scalar real triplet Ξ of mass M, too. The relevant Lagrangian is

$$\mathcal{L}_{\Xi} = \kappa_{\Xi} H^\dagger \Xi^I \sigma^I H. \tag{18}$$

When integrating Ξ out up to one loop, we obtain: $c^{(1)}_{H^4D^4} = 4\frac{\kappa_{\Xi}^2}{M^2} - \frac{107}{144\pi^2}\frac{\kappa_{\Xi}^4}{M^4}$, $c^{(2)}_{H^4D^4} = -\frac{61}{144\pi^2}\frac{\kappa_{\Xi}^4}{M^4}$, and $c^{(3)}_{H^4D^4} = -2\frac{\kappa_{\Xi}^2}{M^2} - \frac{271}{720\pi^2}\frac{\kappa_{\Xi}^4}{M^4}$. The tree and loop contributions are manifest. Once again, $c^{(2)}_{H^4D^2} < 0$.

In both cases, it can be checked by explicit computation that gauge corrections do not restore positivity provided $g \lesssim \kappa/M$.

Let us now turn our attention to three scalar extensions of the SM which *do not* generate any four-Higgs operators at tree level (including those of dimension six). These involve adding a heavy doublet with $Y = 1/2$ (φ) and adding heavy quadruplets with $Y = 1/2$ (Θ_1) and $Y = 3/2$ (Θ_3), respectively. We obtain: $c^{(1)}_{H^4D^4} = \frac{|\lambda_\varphi|^2}{24\pi^2}$, $c^{(2)}_{H^4D^4} = \frac{|\lambda_\varphi|^2}{24\pi^2}$, $c^{(3)}_{H^4D^4} = \frac{|\lambda_\varphi|^2}{6\pi^2}$; as well as $c^{(1)}_{H^4D^4} = \frac{|\lambda_{\Theta_1}|^2}{9\pi^2}$, $c^{(2)}_{H^4D^4} = \frac{|\lambda_{\Theta_1}|^2}{36\pi^2}$, $c^{(3)}_{H^4D^4} = -\frac{|\lambda_{\Theta_1}|^2}{18\pi^2}$; and $c^{(1)}_{H^4D^4} = 0$, $c^{(2)}_{H^4D^4} = \frac{|\lambda_{\Theta_3}|^2}{4\pi^2}$, $c^{(3)}_{H^4D^4} = 0$; where λ_φ, λ_{Θ_1} and λ_{Θ_3} are the unique linear couplings between one heavy field and three H bosons that can be written at the renormalizable level in each case; see [30].

In all these cases, as we already anticipated, the conditions in Eqs. (6)–(8) do hold.

V. RENORMALIZATION GROUP EVOLUTION

Let us now focus on the running of the Wilson coefficients $c^{(j)}_{H^4D^4}$. The contribution triggered by pairs of dimension-six operators, computed in [28], reads:

$$\begin{aligned} 16\pi^2 \beta^{(1)}_{H^4D^4} = \frac{8}{3}\Big[&-2(c^{(1)}_{H^4D^2})^2 - \frac{11}{8}(c^{(2)}_{H^4D^2})^2 + 4c^{(1)}_{H^4D^2}c^{(2)}_{H^4D^2} \\ &\underline{+3c^2_{Hd}}\, \underline{\underline{+c^2_{He}}}\, \underline{\underline{\underline{+2(c^{(1)}_{Hl})^2}}} - 2(c^{(3)}_{Hl})^2 \underline{\underline{\underline{\underline{+6(c^{(1)}_{Hq})^2}}}} - 6(c^{(3)}_{Hq})^2 + \underline{\underline{\underline{\underline{3c^2_{Hu}}}}} - 3c^2_{Hud}\Big], \end{aligned} \tag{19}$$

[2]At lower scales, the matching corrections involve logarithmic terms $\sim \log M/\mu$. These can be reproduced from the running triggered by dimension-six and dimension-eight terms (these latter ones proportional to the quartic coupling generated at tree level) that we work out in the subsequent section. The same applies to the triplet model discussed next.

$$
\begin{aligned}
16\pi^2\beta^{(2)}_{H^4D^4} = \frac{8}{3}\Big[&-2(c^{(1)}_{H^4D^2})^2 - \frac{5}{8}(c^{(2)}_{H^4D^2})^2 - 2c^{(1)}_{H^4D^2}c^{(2)}_{H^4D^2} \\
&\underline{-3c^2_{Hd}}\ \underline{\underline{-c^2_{He}}}\ \underline{\underline{\underline{-2(c^{(1)}_{Hl})^2}}} - 2(c^{(3)}_{Hl})^2 \underline{\underline{\underline{\underline{\underline{-6(c^{(1)}_{Hq})^2}}}}} - 6(c^{(3)}_{Hq})^2 \underline{\underline{\underline{\underline{\underline{\underline{-3c^2_{Hu}}}}}}}\Big],
\end{aligned} \tag{20}
$$

$$
16\pi^2\beta^{(3)}_{H^4D^4} = \frac{8}{3}\left[-5(c^{(1)}_{H^4D^2})^2 + \frac{7}{8}(c^{(2)}_{H^4D^2})^2 - 2c^{(1)}_{H^4D^2}c^{(2)}_{H^4D^2} + 4(c^{(3)}_{Hl})^2 + 12(c^{(3)}_{Hq})^2 + 3c^2_{Hud}\right]. \tag{21}
$$

(The Wilson coefficients of the fermionic operators are matrices in flavor space, so c^2 must be interpreted as the trace $\mathrm{Tr}[c^\dagger c]$.)

It can be trivially seen, for example, that $\beta^{(2)}_{H^4D^4} < 0$, implying

$$
c^{(2)}_{H^4D^4}(\mu) \sim \beta^{(2)}_{H^4D^4} \log\frac{\mu}{\Lambda} > 0, \tag{22}
$$

given that $\mu/\Lambda < 1$ within the region of validity of the EFT.

Likewise, we have that $\beta^{(1)}_{H^4D^4} + \beta^{(2)}_{H^4D^4} < 0$ as well as $\beta^{(1)}_{H^4D^4} + \beta^{(2)}_{H^4D^4} + \beta^{(3)}_{H^4D^4} < 0$, and therefore all bounds in Eqs. (6)–(8) are respected by dimension-six quantum corrections at all scales.

For illustration, in the equations above we have marked which positive (fermionic) coefficients in $\beta^{(1)}_{H^4D^4}$ are canceled by $\beta^{(2)}_{H^4D^4}$, forcing the second inequality even though it could well be that $\beta^{(1)}_{H^4D^4} > 0$. Note also that $\beta^{(3)}_{H^4D^4}$ can be positive in a plethora of cases, for example simply if $c^{(3)}_{Hl}$ is the only nonvanishing Wilson coefficient. The same holds for $\beta^{(1)}_{H^4D^4} + \beta^{(3)}_{H^4D^4}$ as well as for $\beta^{(2)}_{H^4D^4} + \beta^{(3)}_{H^4D^4}$. That is, the cancellations are in place precisely in those combinations given by the positivity bounds.

However, the mixing of the three $c^{(j)}_{H^4D^4}$ due to renormalizable terms drive the former away from their positivity region. Indeed, upon a thorough computation including the (in this case dominant) contribution from gauge couplings, we obtain that:

$$
\begin{aligned}
16\pi^2\beta^{(1)}_{H^4D^4} = \frac{1}{6}[&(30c^{(1)}_{H^4D^4} + 41c^{(2)}_{H^4D^4} + 15c^{(3)}_{H^4D^4})g_2^2 \\
&- (16c^{(1)}_{H^4D^4} + 7c^{(2)}_{H^4D^4} + 15c^{(3)}_{H^4D^4})g_1^2 \\
&+ 16(3c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4})\lambda],
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
16\pi^2\beta^{(2)}_{H^4D^4} = \frac{1}{6}[&(28c^{(1)}_{H^4D^4} + 43c^{(2)}_{H^4D^4} + 15c^{(3)}_{H^4D^4})g_2^2 \\
&+ (14c^{(1)}_{H^4D^4} + 33c^{(2)}_{H^4D^4} + 15c^{(3)}_{H^4D^4})g_1^2 \\
&+ 16(c^{(1)}_{H^4D^4} + 3c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4})\lambda],
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
16\pi^2\beta^{(3)}_{H^4D^4} = -\frac{1}{3}[&(36c^{(1)}_{H^4D^4} + 29c^{(2)}_{H^4D^4} + 42c^{(3)}_{H^4D^4})g_2^2 \\
&+ (8c^{(1)}_{H^4D^4} + 2c^{(2)}_{H^4D^4} + 9c^{(3)}_{H^4D^4})g_1^2 \\
&- 16(3c^{(1)}_{H^4D^4} + 2c^{(2)}_{H^4D^4} + 5c^{(3)}_{H^4D^4})\lambda].
\end{aligned} \tag{25}
$$

(We do not include fermionic dimension-eight operators, because they do not arise in the models in which we later use these expressions.) It is clear that $\beta^{(2)}_{H^4D^4}$ is not necessarily negative; likewise for the other positivity relations.

As a matter of example, let us assume that $c^{(2)}_{H^4D^4}(\mu = M) = 0$. Then, we have:

$$
\begin{aligned}
c^{(2)}_{H^4D^4}(\mu) \sim -\frac{1}{96\pi^2}[&(28g_2^2 + 14g_1^2 + 16\lambda)c^{(1)}_{H^4D^4}(M) \\
&+ (15g_2^2 + 15g_1^2 + 16\lambda)c^{(3)}_{H^4D^4}(M)]\log\frac{M}{\mu}.
\end{aligned} \tag{26}
$$

If $c^{(1)}_{H^4D^4}(\mu = M) \geq 0$ and $c^{(3)}_{H^4D^4}(\mu = M) \geq 0$, as predicted for example in the neutral singlet scalar extension of the SM, then $c^{(2)}_{H^4D^4}(\mu)$ is strictly negative, in conflict with Eq. (6). As we discussed above, this does not contradict the positivity of the forward scattering amplitude *around* $s = 0$, as this is dominated by the running of the relevant couplings. Note, though, that these latter SM contributions are of completely different relevance at the much larger s tested in typical experiments (where incidentally they are absorbed in the background), and in particular they do not necessarily restore any positivity in that case.

Similar violations of the positivity bounds occur in many other models for which some of the combinations of Wilson coefficients entering the inequalities in Eqs. (6)–(8) vanish at tree level. To mention a few of the simplest ones:

$$
\mathcal{S} \sim (1,1)_0 \mapsto c^{(1,2,3)}_{H^4D^4} \sim (0,0,1), \tag{27}
$$

$$
\Xi \sim (1,3)_0 \mapsto c^{(1,2,3)}_{H^4D^4} \sim (2,0,-1), \tag{28}
$$

$$
\mathcal{B} \sim (1,1)_0 \mapsto c^{(1,2,3)}_{H^4D^4} \sim (-1,1,0), \tag{29}
$$

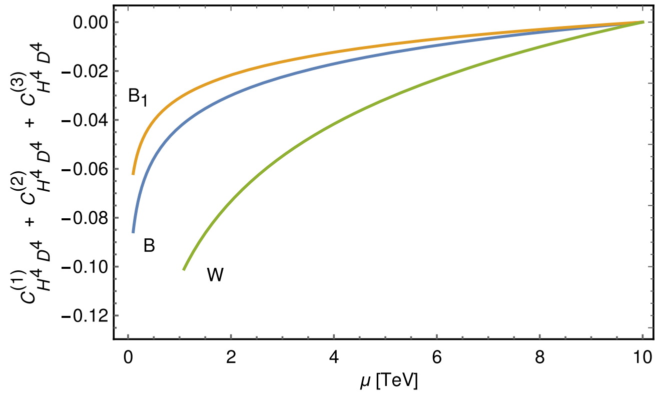

FIG. 2. Evolution of $c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4}$ in different models; see the text for details.

$$\mathcal{B}_1 \sim (1,1)_1 \mapsto c^{(1,2,3)}_{H^4D^4} \sim (1,0,-1), \tag{30}$$

$$\mathcal{W} \sim (1,3)_0 \mapsto c^{(1,2,3)}_{H^4D^4} \sim (1,1,-2). \tag{31}$$

The first two fields are scalars, while the last three are vectors. The first numbers in parentheses and the subscript represent the $SU(3)_c \times SU(2)_L$ quantum numbers and the hypercharge, respectively. The last numbers in parentheses contain the ratios of the $c^{(1,2,3)}_{H^4D^4}$ Wilson coefficients at tree level.

For the sake of example, we plot the evolution of $c^{(1)}_{H^4D^4} + c^{(2)}_{H^4D^4} + c^{(3)}_{H^4D^4}$ in Fig. 2. For each model, we assume that the Wilson coefficients are fixed to the values given in Eqs. (27)–(31) at the matching scale $M = 10$ TeV. All curves include also the (subleading) contribution of dimension-six terms to the running.

VI. CONCLUSIONS

We have argued that tree-level-derived positivity bounds on the Wilson coefficients $c^{(j)}_{H^4D^4}$ of four-Higgs dimension-eight operators within the SMEFT do not necessarily hold at one loop.

First, they can be violated at the matching scale. We have underpinned this statement with explicit calculations. In particular, we have computed the one-loop matching of the singlet and triplet scalar extensions of the SM (which are not only among the simplest ones but they are also of great phenomenological interest [31–37]) onto the SMEFT to dimension eight, demonstrating the violation of two of the three positivity bounds. And second, all positivity bounds can be broken by their running triggered by renormalizable interactions such as the Higgs quartic or the gauge couplings.

In this respect, it would be interesting to study modified scale-dependent constraints, relying on s-dependent integration contours such as the *arcs* discussed in [38]; see also Sec. 11 of [27].

Conversely, we have shown that the renormalization of $c^{(j)}_{H^4D^4}$ driven by dimension-six terms does not break positivity. This implies strong constraints on the form of the corresponding anomalous dimensions. In turn, this observation provides new nonrenormalization results, which we are currently exploring.

Finally, let us note that the experimental program for measuring (or bounding) the Wilson coefficients $c^{(j)}_{H^4D^4}$ in multi-boson final states (they modify the quartic gauge couplings) is already ongoing [39]. The feasibility for disentangling the three different couplings relies on the fact that they enter through different combinations in different channels; for example in ZZ, WW and WZ [10].

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