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# Spectral analysis of multifractional LRD functional time series

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**Abstract**

Long Range Dependence (LRD) in functional sequences is characterized in the spectral domain under suitable conditions. Particularly, multifractionally integrated functional autoregressive moving averages processes can be introduced in this framework. The convergence to zero in the Hilbert-Schmidt operator norm of the integrated bias of the periodogram operator is proved. Under a Gaussian scenario, a weak-consistent parametric estimator of the long-memory operator is then obtained by minimizing, in the norm of bounded linear operators, a divergence information functional loss. The results derived allow, in particular, to develop inference from the discrete sampling of the Gaussian solution to fractional and multifractional pseudodifferential models introduced in Anh et al. (Fract Calc Appl Anal 19(5):1161-1199, 2016; 19(6):1434–1459, 2016) and Kelbert (Adv Appl Probab 37(1):1–25, 2005).

**Keywords** Minimum contrast parameter estimation · Multifractional functional ARIMA models · Multifractional in time evolution equations · Spatial-varying long-range dependence range

**Mathematics Subject Classification** 60G10 · 60G12 · 60G18 · 60G20 · 60G22 (primary) · 60G60

## 1 Introduction

One can find evidence of Long Range Dependence (LRD) in time series data arising in several areas like agriculture, environment, economics, finance, geophysics, just to mention a few. Indeed, a huge amount of literature on this topic has been developed over the last few decades (cf., [6, 8, 18, 30, 35]). This framework allows the description of

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processes with long persistence in time. In the stationary case, LRD is characterized by a slow decay of the covariance function, and an unbounded spectral density, typically at zero frequency. In the real-valued process framework, we refer to the reader to the papers [1, 4, 16, 17, 19, 21, 25, 36], among others.

Special attention has been paid to the self-similar asymptotic behavior of the second-order moments of the Gaussian solution to fractional and multifractional linear pseudodifferential equations (see, e.g., [2, 3, 22]). Particularly, the analysis of LRD phenomena in an infinite-dimensional framework is a challenging topic where several problems remain open. Only a few contributions can be found on this topic in functional processes. That is the case of time-varying isotropic vector random fields on the sphere introduced in [28], that were also analyzed by [27] in the framework of compact two-points homogeneous spaces.

On the other hand, LRD in functional sequences is characterized by the non-summability in time of the nuclear norms of the associated family of covariance operators. In the linear case, a variable-order fractional power law usually characterizes the asymptotic behavior in time of the norms of the functional parameters, given by bounded linear operators. That is the case of the approaches in the current literature based on operator-valued processes. A fractional Brownian motion with values in a Hilbert space, involving an operator-valued Hurst coefficient, is considered in [33] (see also [32] on the functional analytical tools applied). In [14], a central and functional central limit theorems are obtained under non-summability of the operator norm sequence. The limit process in this functional central limit result is a self-similar process, characterized by an operator defining the self-similarity exponent. Note that the LRD models introduced in these papers in the linear setting are characterized and analyzed in the time domain. Recently, in [15], for LRD linear processes in a separable Hilbert space, a stochastic-integral based approach is adopted to representing the limiting process of the sample autocovariance operator in the space of Hilbert–Schmidt operators.

A semiparametric linear framework has been adopted to analyze LRD in functional sequences in [26]. The functional dependence structure is specified via the projections of the curve process onto different sub-spaces, spanned by the eigenvectors of the long-run covariance function. A Central Limit Theorem is derived under suitable regularity conditions. Functional Principal Component Analysis is applied in the consistent estimation of the orthonormal functions spanning the dominant subspace, where the projected curve process displays the largest dependence range. The memory parameter and the dimension of the dominant subspace are estimated as well. The conditions assumed are satisfied, in particular, by a functional version of fractionally integrated autoregressive moving averages processes. Some interesting applications to US stock prices and age specific fertility rates are also provided.

As follows from the above cited references, the spectral domain has not been exploited yet in the formulation and estimation of LRD in stationary functional time series. Furthermore, LRD functional time series models have mainly been introduced in the linear setting. Our paper attempts to cover these gaps. To this aim, the spectral representation of a self-adjoint operator on a separable Hilbert space, in terms of a spectral family of projection operators, is considered. Suitable conditions are then assumed on the symbol defining such a representation, for the spectral density operator family at a neighborhood of zero frequency. Specifically, the behavior of the spectral density operator at zero frequency is characterized by a bounded symmetric positive operator family, whose operator norm slowly varies at zero frequency, composed with an unbounded operator at zero frequency involving the long-memory operator. The corresponding covariance operator family displays a heavy tail behavior in time as proved in Proposition 1. As an interesting special case, we refer to a family of fractionally integrated functional autoregressive moving averages processes of variable order (see also Remark 9 in [26]). Several additional examples can be found by tapering, in the frequency domain, the symbols of the spectral density operator family, associated with infinite-dimensional stationary LRD processes in continuous time. Particularly, we consider the case of fractional integration of variable order of functional processes with rational spectral density operator (see, e.g., [2, 3, 22]). The convergence to zero, in the Hilbert–Schmidt operator norm, of the integrated periodogram bias operator is derived, under the square integrability in the frequency domain of the Hilbert–Schmidt operator norm of the spectral density operator family. This condition holds under mild conditions, in our case, under the second-order property of the functional process, assuming the integrability in the frequency domain of the operator norm of the spectral density operator family. The weak consistency of the proposed parametric estimator of the long-memory operator then follows in the Gaussian case, extending Theorem 3 in [4].

Note that the parametric estimation approach in the spectral domain has not been exploited yet in the functional time series context. Under short-range dependence (SRD), [31] adopts a nonparametric framework. Specifically, a weighted average of the functional values of the periodogram operator is considered as an estimator of the spectral density operator. This methodology is not applicable when one wants to approximate the behavior of the spectral density operator at zero frequency in the presence of LRD. In this paper, we consider a parametric estimator of the long-memory operator, computed by minimizing the operator norm of a weighted Kullback–Leibler divergence operator. This operator compares the behavior at a neighborhood of zero frequency of the true spectral density operator, underlying to the curve data, with the possible semiparametric candidates. On the other hand, this functional is linear with respect to the periodogram operator. This is an important advantage of the proposed estimation methodology in relation to nonparametric kernel estimation.

The outline of the paper is the following. Preliminary definitions, results and first conditions are established in Sect. 2. The main assumptions are formulated in Sect. 3. Under this setting of conditions, LRD is characterized in the functional spectral domain. The heavy tail behavior in time of the associated covariance operator family is obtained in Proposition 1. Some examples are provided as well. In Sect. 4, the convergence to zero of the Hilbert-Schmidt operator norm of the integrated bias of the periodogram operator is proved in Theorem 1. Under a Gaussian scenario, Theorem 2 in Sect. 5 derives the consistent parametric estimation of the long-memory operator in the functional spectral domain. Some final comments are given in Sect. 6.

## 2 Preliminaries

In what follows, $(\Omega, \mathcal{A}, P)$ denotes the basic probability space. Let $H$ be a real separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ , and $\widetilde{H} = H + iH$, its complex version, whose elements are functions of the form

$$\psi = \varphi_1 + i\varphi_2, \quad \varphi_i \in H, \quad i = 1, 2.$$

Its inner product is given by

$$\begin{aligned} &\langle \varphi_1 + i\varphi_2, \phi_1 + i\phi_2 \rangle_{\widetilde{H}} \\ &\quad = \langle \varphi_1, \phi_1 \rangle_H - \langle \varphi_2, \phi_2 \rangle_H + i \left( \langle \varphi_2, \phi_1 \rangle_H \right. \\ &\qquad \left. + \langle \varphi_1, \phi_2 \rangle_H \right). \end{aligned} \tag{2.1}$$

Recall that $\mathcal{L}^2_{\widetilde{H}}(\Omega, \mathcal{A}, P)$ denotes the space of second-order zero-mean $\widetilde{H}$-valued random variables on $(\Omega, \mathcal{A}, P)$, with the norm $\|X\|^2_{\mathcal{L}^2_{\widetilde{H}}(\Omega, \mathcal{A}, P)} = E[\|X\|^2_{\widetilde{H}}]$, for every $X \in \mathcal{L}^2_{\widetilde{H}}(\Omega, \mathcal{A}, P)$.

In the following, fix an orthonormal basis $\{\varphi_k, \ k \geq 1\}$ of $H$, and consider

$$\{\psi_k = (1/2)\left[\varphi_k + i\varphi_k\right], \ k \geq 1\}, \tag{2.2}$$

as an orthonormal basis of $\widetilde{H}$. All the subsequent identities involving operator norms can be expressed in terms of such an orthonormal basis, allowing the interpretation of $H$ as a closed subspace of $\widetilde{H}$. Particularly, the nuclear $\|\cdot\|_{L^1(\widetilde{H})}$ , and the Hilbert–Schmidt $\|\cdot\|_{\mathcal{S}(\widetilde{H})}$ operator norms on $\widetilde{H}$ are defined as follows:

$$\begin{aligned} \|\mathcal{A}\|_{L^1(\widetilde{H})} &= \sum_{k \geq 1} \left\langle \left[\mathcal{A}^\star \mathcal{A}\right]^{1/2} (\psi_k), \psi_k \right\rangle_{\widetilde{H}}, \\ \|\mathcal{A}\|_{\mathcal{S}(\widetilde{H})} &= \left[ \sum_{k \geq 1} \left\langle \mathcal{A}^\star \mathcal{A}(\psi_k), \psi_k \right\rangle_{\widetilde{H}} \right]^{1/2} \\ &= \sqrt{\|\mathcal{A}^\star \mathcal{A}\|_{L^1(\widetilde{H})}}, \end{aligned} \tag{2.3}$$

with $\{\psi_k, \ k \geq 1\}$ being an orthonormal basis of $\widetilde{H}$ as given in (2.2).

We denote by $\| \cdot \|_{\mathcal{L}(\widetilde{H})}$ the norm in the space of bounded linear operators on $\widetilde{H}$, i.e., $\|\mathcal{A}\|_{\mathcal{L}(\widetilde{H})} = \sup_{\psi \in \widetilde{H}; \ \|\psi\|=1} \|\mathcal{A}(\psi)\|_{\widetilde{H}}$. This norm is also usually referred as the operator norm (or uniform operator norm). Through the paper we consider the equality between operators on $\widetilde{H}$ (respectively, on $H$) in the norm of the space $\mathcal{L}(\widetilde{H})$ (respectively, of the space $\mathcal{L}(H)$) implying the pointwise identity of such operators over the functions on $\widetilde{H}$ (respectively on $H$). Otherwise, the norm with respect to which the identity considered holds is established.

For simplicity of notation, in the subsequent development, the letter $\mathcal{K}$ will refer to a positive constant whose specific value may vary from one to another inequality or identity.

Let $\{X_t,\ t \in \mathbb{Z}\}$ be a strictly stationary functional time series with zero mean $E[X_t] = 0$, and functional variance $\sigma_X^2 = E[\|X_t\|_H^2] = E[\|X_0\|_H^2] = \|R_0\|_{L^1(H)}$, for every $t \in \mathbb{Z}$. Also,

$$\mathcal{R}_t = E[X_{s+t} \otimes X_s] = E[X_t \otimes X_0], \quad \forall t, s \in \mathbb{Z},$$
$$\mathcal{R}_t(g)(h) = E[X_{s+t}(h)X_s(g)] = E\left[\langle X_{s+t}, h\rangle_H \langle X_s, g\rangle_H\right], \quad \forall h, g \in H. \tag{2.4}$$

Note that, $E[\|X_0\|_H^2] < \infty$ implies $P[X_t \in H] = 1$, for all $t \in \mathbb{Z}$.

Let $\mathcal{F}_\omega$ be the spectral density operator on $\widetilde{H}$, defined by the following identity in the $\mathcal{L}(\widetilde{H})$ norm, for $\omega \in [-\pi, \pi]\backslash\{0\}$ :

$$\mathcal{F}_\omega \underset{\mathcal{L}(\widetilde{H})}{=} \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \exp(-i\omega t)\, \mathcal{R}_t. \tag{2.5}$$

***Remark 1*** In [31], convergence of series (2.5) holds in the nuclear norm for SRD functional sequences. Here, a weaker convergence is assumed. Indeed, identity (2.5) could hold for $\omega \in [-\pi, \pi]\backslash\Lambda_0$, where $\int_{\Lambda_0} d\omega = 0$. In our case, $\Lambda_0 = \{0\}$ for the characterization of LRD in **Assumption II** below.

For simplicity, in the following, we will omit the reference to the set $[-\pi, \pi]\backslash\Lambda_0$, when the identities hold almost surely in the frequency domain. That is the case of the identities for a spectral density operator family involving an unbounded spectral density operator at zero-frequency (see Eq. (3.1) below).

The functional Discrete Fourier Transform (fDFT) $\widetilde{X}^{(T)}$ of the functional data $\{X_t,\ t = 1, \ldots, T\}$ is defined as

$$\widetilde{X}_\omega^{(T)}(\cdot) \underset{\widetilde{H}}{=} \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} X_t(\cdot) \exp(-i\omega t), \quad \omega \in [-\pi, \pi], \tag{2.6}$$

where $\underset{\widetilde{H}}{=}$ denotes the equality in $\widetilde{H}$ norm. Hence, $\widetilde{X}_\omega^{(T)}$ is $2\pi$-periodic and Hermitian with respect to $\omega \in [-\pi, \pi]$.

***Remark 2*** Under the condition $E[\|X_0\|_{\widetilde{H}}^2] < \infty$, applying triangle inequality,

$$E\left[\|\widetilde{X}_\omega^{(T)}\|_{\widetilde{H}}\right] \leq \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} E\|X_t(\cdot)\|_{\widetilde{H}} < \infty,$$

for every $\omega \in [-\pi, \pi]$. The fDFT $\widetilde{X}_\omega^{(T)}$ defines a random element in $\widetilde{H}$, and $P\left[\widetilde{X}_\omega^{(T)}(\cdot) \in \widetilde{H}\right] = 1.$ Hence, $\mathcal{F}_\omega^{(T)} = E\left[\widetilde{X}_\omega^{(T)} \otimes \overline{\widetilde{X}_\omega^{(T)}}\right] \in L^1(\widetilde{H})$, for $\omega \in [-\pi, \pi]$.

The periodogram operator $p_{\omega}^{(T)} = \widetilde{X}_{\omega}^{(T)} \otimes \overline{\widetilde{X}_{\omega}^{(T)}}$ is an empirical operator, with mean $E[p_{\omega}^{(T)}] = E[\widetilde{X}_{\omega}^{(T)} \otimes \widetilde{X}_{-\omega}^{(T)}] = \mathcal{F}_{\omega}^{(T)}$. Particularly, under (2.5), for any $T \geq 2$, the following identity holds in $\mathcal{L}(\widetilde{H})$ :

$$\mathcal{F}_{\omega}^{(T)} = E\left[p_{\omega}^{(T)}\right] = \frac{1}{2\pi T}\left[\sum_{t=1}^{T}\sum_{s=1}^{T}\exp\left(-i\omega(t-s)\right)E[X_t \otimes X_s]\right]$$
$$= \frac{1}{2\pi}\sum_{u=-(T-1)}^{T-1}\exp\left(-i\omega u\right)\frac{(T-|u|)}{T}\mathcal{R}_u. \tag{2.7}$$

Let $F_T$ be the Féjer kernel, given by

$$F_T(\omega) = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\exp\left(-i(t-s)\omega\right), \quad \omega \in [-\pi, \pi], \quad T \geq 2. \tag{2.8}$$

Applying the Fourier Transform Inversion Formula in the space $\mathcal{L}(\widetilde{H})$, from Eqs. (2.7) and (2.8), for each $\omega \in [-\pi, \pi]$,

$$\mathcal{F}_{\omega}^{(T)} = [F_T * \mathcal{F}_{\bullet}](\omega)$$
$$= \int_{-\pi}^{\pi} F_T(\omega - \xi)\mathcal{F}_{\xi} d\xi, \quad T \geq 2. \tag{2.9}$$

### 2.1 Preliminaries on spectral analysis of self-adjoint operators

This section presents some preliminary elements on spectral theory of self-adjoint operators on a separable Hilbert space (see, e.g., [12], pp. 112–140).

It is well-known that, for a self-adjoint operator $\mathcal{D}$ on a separable Hilbert space $\widetilde{H}$, there exists a family of projection operators $\{E_{\lambda},\ \lambda \in \Lambda \subseteq \mathbb{R}\}$, also called the spectral family of $\mathcal{D}$, such that the following identity holds:

$$\mathcal{D} = \int_{\Lambda} \lambda dE_{\lambda}. \tag{2.10}$$

This family of projection operators satisfies the following properties:

(i) $E_{\lambda}E_{\mu} = E_{\inf\{\lambda,\mu\}}$;
(ii) $\lim_{\widetilde{\lambda}\to\lambda;\ \widetilde{\lambda}\geq\lambda} E_{\widetilde{\lambda}} = E_{\lambda}$;
(iii) $\lim_{\lambda\to-\infty} E_{\lambda} = 0$; $\lim_{\lambda\to\infty} E_{\lambda} = I_{\widetilde{H}}$, where $I_{\widetilde{H}}$ denotes the identity operator on $\widetilde{H}$.
(iv) The domain of $\mathcal{D}$ is defined as

$$\mathrm{Dom}(\mathcal{D}) = \left\{h \in \widetilde{H} : \int_{\Lambda} |\lambda|^2 d\left\langle E_{\lambda}(h), h\right\rangle_{\widetilde{H}} < \infty\right\}. \tag{2.11}$$

(v) A continuous function $G(\mathcal{D})$ admits the representation

$$G(\mathcal{D}) = \int_{\Lambda} G(\lambda) dE_{\lambda}, \tag{2.12}$$

and

$$\mathrm{Dom}(G(\mathcal{D})) = \left\{ h \in \widetilde{H} : \int_{\Lambda} |G(\lambda)|^2 \, d\left\langle E_{\lambda}(h), h \right\rangle_{\widetilde{H}} < \infty \right\}.$$

The operator integrals (2.10) and (2.12) are understood as improper operator Stieltjes integrals which converge strongly (see, e.g., Section 8.2.1 in [34]). Let $\Delta = (a, b]$, $-\infty < a < b < \infty$, $E_{\Delta} := E_b - E_a$.
The family $E_{\Delta}$ of self-adjoint bounded non-negative operators from the Borel sets $\Delta \subseteq \mathbb{R}$ into the space $\mathcal{L}(\widetilde{H})$ of bounded linear operators on a Hilbert space $\widetilde{H}$ is called an operator measure if

$$E_{\left[\cup_{j=1}^{\infty} \Delta_j\right]} = \sum_{j=1}^{\infty} E_{\Delta_j},$$

where the limit at the right-hand side is understood in the sense of weak-convergence of operators, with $\Delta_i \cap \Delta_j = \varnothing$, $i \neq j$, $E_{\varnothing} = 0$.

From (iii), for every $g, h \in \widetilde{H}$,

$$\begin{aligned} \int_{\Lambda} d\left\langle E_{\lambda}(g), h \right\rangle_{\widetilde{H}} &= \langle g, h \rangle_{\widetilde{H}} \\ \int_{\Lambda} d\left\langle E_{\lambda}(h), h \right\rangle_{\widetilde{H}} &= \|h\|_{\widetilde{H}}^2 . \end{aligned} \tag{2.13}$$

Thus, $\{E_{\lambda}, \ \lambda \in \Lambda \subseteq \mathbb{R}\}$ provides a resolution of the identity.

Note that, from (2.10) (see (i)–(v)), for every $\psi \in \mathrm{Dom}(\mathcal{D}) \subseteq \widetilde{H}$,

$$\begin{aligned} \|\mathcal{D}(\psi)\|_{\widetilde{H}}^2 &= \langle \mathcal{D}(\psi), \mathcal{D}(\psi) \rangle_{\widetilde{H}} \\ &= \langle \mathcal{D}\mathcal{D}(\psi), \psi \rangle_{\widetilde{H}} \\ &= \int_{\Lambda} |\lambda|^2 d\left\langle E_{\lambda}(\psi), \psi \right\rangle_{\widetilde{H}} \\ &= \left\langle \left[ \mathcal{D}\left( \int_{\Lambda} \lambda dE_{\lambda} \right) \right](\psi), \psi \right\rangle_{\widetilde{H}} = \left\langle \left[ \left( \int_{\Lambda} \lambda dE_{\lambda} \right) \mathcal{D} \right](\psi), \psi \right\rangle_{\widetilde{H}} . \end{aligned} \tag{2.14}$$

If $\mathcal{D} \in \mathcal{L}(\widetilde{H})$, hence, Eq. (2.11) holds for every $\psi \in \widetilde{H}$, and from (2.14),

$$\begin{aligned}
&\left\| \left[ \mathcal{D} - \int_{\Lambda} \lambda dE_{\lambda} \right] (\psi) \right\|_{\widetilde{H}}^{2} \\
&= \left\langle \mathcal{D}(\psi) - \int_{\Lambda} \lambda dE_{\lambda}(\psi), \mathcal{D}(\psi) - \int_{\Lambda} \lambda dE_{\lambda}(\psi) \right\rangle_{\widetilde{H}} \\
&= \int_{\Lambda} |\lambda|^{2} d\left\langle E_{\lambda}(\psi), \psi \right\rangle_{\widetilde{H}} + \int_{\Lambda} |\lambda|^{2} d\left\langle E_{\lambda}(\psi), \psi \right\rangle_{\widetilde{H}} - 2\int_{\Lambda} |\lambda|^{2} d\left\langle E_{\lambda}(\psi), \psi \right\rangle_{\widetilde{H}} = 0.
\end{aligned} \tag{2.15}$$

Thus, $\left\| \mathcal{D} - \int_{\Lambda} \lambda dE_{\lambda} \right\|_{\mathcal{L}(\widetilde{H})} = 0$, and the weak-sense representation (2.10) also holds in $\mathcal{L}(\widetilde{H})$-norm. In particular, for $\mathcal{D} \in L_0(\widetilde{H})$, with $L_0(\widetilde{H})$ denoting the class of compact operators on $\widetilde{H}$, the mapping $\lambda \longrightarrow E_{\lambda}$ has discontinuities at the points given by the eigenvalues $\{\lambda_k(\mathcal{D}),\ k \geq 1\}$, with

$$E_{\lambda_k} - \lim_{\widetilde{\lambda} \to \lambda_k(\mathcal{D});\ \widetilde{\lambda} < \lambda_k(\mathcal{D})} E_{\widetilde{\lambda}} = P_k,$$

where $P_k$ is the projection operator onto the eigenspace generated by the eigenvectors associated with the eigenvalue $\lambda_k(\mathcal{D})$, for every $k \geq 1$.

Let us now consider the following assumption:
**Assumption I**. Assume that

$$\int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{L}(\widetilde{H})} d\omega < \infty. \tag{2.16}$$

***Remark 3*** **Assumption I** holds, for instance, when the family $\{\mathcal{F}_{\omega},\ \omega \in [-\pi, \pi]\}$ is a.s. continuous in $\omega \in [-\pi, \pi]$, with respect to $\mathcal{L}(\widetilde{H})$-norm, since applying reverse triangle inequality, $\|\mathcal{F}_{\omega}\|_{\mathcal{L}(\widetilde{H})}$ is a.s. continuous in $\omega \in [-\pi, \pi]$.

***Remark 4*** Note that, under **Assumption I**, for every $t \in \mathbb{Z}$,

$$\|\mathcal{R}_t\|_{\mathcal{L}(\widetilde{H})} = \left\| \int_{-\pi}^{\pi} \exp(i\omega t)\, \mathcal{F}_{\omega} d\omega \right\|_{\mathcal{L}(\widetilde{H})} \leq \int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{L}(\widetilde{H})} d\omega < \infty. \tag{2.17}$$

The next preliminary result will be applied in the subsequent development.

**Lemma 1** *Under **Assumption I**,*

$$\sum_{t \in \mathbb{Z}} \|\mathcal{R}_t\|_{\mathcal{S}(\widetilde{H})}^{2} = \int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{S}(\widetilde{H})}^{2} d\omega < \infty. \tag{2.18}$$

***Proof*** Given an orthonormal basis $\{\psi_k,\ k \geq 1\}$ of $\widetilde{H}$, under **Assumption I**, $\mathcal{F}_{\omega}$ is a.s. bounded in $\omega \in [-\pi, \pi]$. In particular,

$$\int_{-\pi}^{\pi}\langle\mathcal{F}_{\omega}(\psi_k),\psi_l\rangle_{\widetilde{H}}\,d\omega \leq \|\psi_l\|_{\widetilde{H}}\int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}(\psi_k)\|_{\widetilde{H}}d\omega \leq \int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\|_{\mathcal{L}(\widetilde{H})}\,d\omega<\infty, \tag{2.19}$$

for every $k, l \geq 1$. Hence, from (2.5) and (2.19), for every $t \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi}\exp(it\omega)\,\langle\mathcal{F}_{\omega}(\psi_k),\psi_l\rangle_{\widetilde{H}}\,d\omega = \langle\mathcal{R}_t(\psi_k),\psi_l\rangle_{\widetilde{H}}\,, \quad k,l\geq 1. \tag{2.20}$$

From (2.20), applying Fourier transform inversion formula,

$$\begin{aligned}
\sum_{t\in\mathbb{Z}}\|\mathcal{R}_t\|^2_{\mathcal{S}(H)} &= \sum_{t\in\mathbb{Z}}\sum_{k,l\geq 1}|\mathcal{R}_t(\psi_k)(\psi_l)|^2\\
&= \sum_{t\in\mathbb{Z}}\sum_{k,l\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp(it(\omega-\xi))\,\mathcal{F}_{\omega}(\psi_k)(\psi_l)\overline{\mathcal{F}_{\xi}(\psi_k)(\psi_l)}d\xi d\omega\\
&= \sum_{k,l\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\sum_{t\in\mathbb{Z}}\exp(it(\omega-\xi))\right]\mathcal{F}_{\omega}(\psi_k)(\psi_l)\overline{\mathcal{F}_{\xi}(\psi_k)(\psi_l)}d\xi d\omega\\
&= \sum_{k,l\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\delta(\omega-\xi)\mathcal{F}_{\omega}(\psi_k)(\psi_l)\overline{\mathcal{F}_{\xi}(\psi_k)(\psi_l)}d\xi d\omega\\
&= \int_{-\pi}^{\pi}\sum_{k,l\geq 1}|\mathcal{F}_{\omega}(\psi_k)(\psi_l)|^2\,d\omega\\
&= \int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\|^2_{\mathcal{S}(\widetilde{H})}\,d\omega.
\end{aligned} \tag{2.21}$$

From Eqs. (2.20) and (2.21), keeping in mind that $\mathcal{F}_{\omega}$ is nonnegative symmetric operator,

$$\begin{aligned}
\sum_{t\in\mathbb{Z}}\|\mathcal{R}_t\|^2_{\mathcal{S}(H)} &= \int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\|^2_{\mathcal{S}(\widetilde{H})}\,d\omega = \int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\mathcal{F}_{\omega}\|_{L^1(\widetilde{H})}\,d\omega\\
&\leq \int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\|_{L^1(\widetilde{H})}\,d\omega = \int_{-\pi}^{\pi}\sum_{k\geq 1}\left\langle\left[\mathcal{F}^{\star}_{\omega}\mathcal{F}_{\omega}\right]^{1/2}(\psi_k),\psi_k\right\rangle_{\widetilde{H}}d\omega\\
&= \sum_{k\geq 1}\int_{-\pi}^{\pi}\langle\mathcal{F}_{\omega}(\psi_k),\psi_k\rangle_{\widetilde{H}}\,d\omega\\
&= \sum_{k\geq 1}\langle\mathcal{R}_0(\psi_k),\psi_k\rangle_{\widetilde{H}} = \|\mathcal{R}_0\|_{L^1(\widetilde{H})} = \sigma_X^2<\infty.
\end{aligned} \tag{2.22}$$

□

***Remark 5*** Under **Assumption I**, from Lemma 1, $\mathcal{F}_{\omega}\in\mathcal{S}(\widetilde{H})$, for $\omega\in[-\pi,\pi]\backslash\Lambda_0$, with, as before, $\int_{\Lambda_0}d\omega = 0$. Also, $\|\mathcal{F}_{\omega}\|_{\mathcal{S}(\widetilde{H})}\in L^2([-\pi,\pi],\mathbb{C})$.

## 3 Spectral analysis of LRD functional time series

As commented in Introduction, the literature on LRD modeling in functional sequences has been mainly developed in the time domain, under the context of linear processes in Hilbert spaces (see, e.g., [14, 26, 32, 33]), paying special attention to the theory of operator self-similar processes (see [10, 23, 24, 29], among others).

The next condition characterizes the unbounded behavior at zero frequency of the spectral density operator family.

**Assumption II**. Let $\{\mathcal{A}_\theta,\ \theta\in\Theta\}$ be a parametric family of positive bounded self-adjoint long-memory operators, with $\Theta$ denoting the parameter space. For each $\theta\in\Theta$, assume that as $\omega\to 0$:

$$\left\|\mathcal{F}_{\omega,\theta}|\omega|^{\mathcal{A}_\theta}\mathcal{M}_{\omega,\mathcal{F}}^{-1}-I_{\widetilde{H}}\right\|_{\mathcal{L}(\widetilde{H})}\to 0, \tag{3.1}$$

where $I_{\widetilde{H}}$ denotes the identity operator on $\widetilde{H}$, and $\{\mathcal{M}_{\omega,\mathcal{F}},\ \omega\in[-\pi,\pi]\}$ is a family of bounded positive self-adjoint operators.

For $\omega\in[-\pi,\pi]$ and $\theta\in\Theta$, the spectral representation of $\mathcal{M}_{\omega,\mathcal{F}}$ and $\mathcal{A}_\theta$ in terms of a common spectral family $\{E_\lambda,\ \lambda\in\Lambda\}$ of projection operators (see Sect. 2.1) is considered in the next assumption.

**Assumption III**. Assume that $\mathcal{A}_\theta$ and $\mathcal{M}_{\omega,\mathcal{F}}$ admit the following spectral representations:

$$\begin{aligned}\mathcal{A}_\theta &\underset{\mathcal{L}(\widetilde{H})}{=}\int_\Lambda \alpha(\lambda,\theta)dE_\lambda,\quad \theta\in\Theta,\\ \mathcal{M}_{\omega,\mathcal{F}} &\underset{\mathcal{L}(\widetilde{H})}{=}\int_\Lambda M_{\omega,\mathcal{F}}(\lambda)dE_\lambda,\quad \omega\in[-\pi,\pi].\end{aligned} \tag{3.2}$$

We refer to $\{\alpha(\lambda,\theta),\ \lambda\in\Lambda\}$ and $\{M_{\omega,\mathcal{F}}(\lambda),\ \lambda\in\Lambda\}$ as the respective symbols of the self-adjoint operators $\mathcal{A}_\theta$ and $\mathcal{M}_{\omega,\mathcal{F}}$.

***Remark 6*** From (3.2), operators $|\omega|^{-\mathcal{A}_\theta}$ and $\mathcal{M}_{\omega,\mathcal{F}}$ commute, for any $\theta\in\Theta$, and $\omega\in[-\pi,\pi]\backslash\Lambda_0$ (see, e.g., [12]).

Under **Assumptions II–III**, as $\omega\to 0$,

$$\left\|\mathcal{F}_{\omega,\theta}\int_\Lambda\frac{|\omega|^{\alpha(\lambda,\theta)}}{M_{\omega,\mathcal{F}}(\lambda)}dE_\lambda-I_{\widetilde{H}}\right\|_{\mathcal{L}(\widetilde{H})}\to 0,\quad \forall\theta\in\Theta. \tag{3.3}$$

**Assumption IV**. $\{\alpha(\lambda,\theta),\ \lambda\in\Lambda\}$ and $\{M_{\omega,\mathcal{F}}(\lambda),\ \lambda\in\Lambda\}$ satisfy:

(i) For $(\lambda,\theta)\in\Lambda\times\Theta$, there exist $l_\alpha(\theta)$ and $L_\alpha(\theta)$ such that

$$\begin{aligned}&0<l_\alpha(\theta)\leq\alpha(\lambda,\theta)\leq L_\alpha(\theta)<1\\ &l_\alpha(\theta)=\inf_{\psi\in\widetilde{H};\ \|\psi\|_{\widetilde{H}}=1}\langle\mathcal{A}_\theta(\psi),\psi\rangle_{\widetilde{H}},\ L_\alpha(\theta)\\ &\quad=\sup_{\psi\in\widetilde{H};\ \|\psi\|_{\widetilde{H}}=1}\langle\mathcal{A}_\theta(\psi),\psi\rangle_{\widetilde{H}}.\end{aligned} \tag{3.4}$$

(ii) For each $\lambda_0 \in \Lambda$, $M_{\omega,\mathcal{F}}(\lambda_0)$ is slowly varying function at $\omega = 0$ in Zygmund's sense (see, e.g., Definition 6.6 in [7]). Furthermore, we also assume that there exist positive constants $m$ and $M$ such that, for every $\omega \in [-\pi, \pi]$,

$$\begin{aligned} & m \leq M_{\omega,\mathcal{F}}(\lambda) \leq M, \quad \forall \lambda \in \Lambda \\ & m < \inf_{\psi \in \widetilde{H};\ \|\psi\|_{\widetilde{H}}=1} \left\langle \mathcal{M}_{\omega,\mathcal{F}}(\psi), \psi \right\rangle_{\widetilde{H}} < \sup_{\psi \in \widetilde{H};\|\psi\|_{\widetilde{H}}=1} \\ & \quad \left\langle \mathcal{M}_{\omega,\mathcal{F}}(\psi), \psi \right\rangle_{\widetilde{H}} < M. \end{aligned} \tag{3.5}$$

### 3.1 LRD characterization in the time domain

The next proposition shows the heavy-tail behavior in time of the inverse functional Fourier transform of the spectral density operator family satisfying the above formulated conditions.

**Proposition 1** *Let* $\left\{\mathcal{F}_{\omega,\theta},\ (\omega,\theta) \in [-\pi,\pi] \times \Theta\right\}$ *be the semiparametric family of spectral density operators satisfying* ***Assumptions I–IV***. *Consider*

$$\mathcal{R}_{t,\theta} \underset{\mathcal{L}(\widetilde{H})}{=} \int_{-\pi}^{\pi} \exp(i\omega t)\, \mathcal{F}_{\omega,\theta} d\omega, \quad t \in \mathbb{Z},\ \theta \in \Theta. \tag{3.6}$$

*Then,*

$$\left\| \mathcal{R}_{t,\theta} \left[ \widetilde{\mathcal{M}}_{t,\mathcal{F},\mathcal{A}_\theta} t^{\mathcal{A}_\theta - I_{\widetilde{H}}} \right]^{-1} - I_{\widetilde{H}} \right\|_{\mathcal{L}(\widetilde{H})} \to 0, \quad t \to \infty, \tag{3.7}$$

*with, as before,* $I_{\widetilde{H}}$ *denoting the identity operator on* $\widetilde{H}$. *Here, for each* $\theta \in \Theta$, $\widetilde{\mathcal{M}}_{t,\mathcal{F},\mathcal{A}_\theta}$ *admits the representation*

$$\begin{aligned} \widetilde{\mathcal{M}}_{t,\mathcal{F},\mathcal{A}_\theta} & \underset{\mathcal{L}(\widetilde{H})}{=} \int_{\Lambda} 2\Gamma(1-\alpha(\lambda,\theta)) \sin\left((\pi/2)\alpha(\lambda,\theta)\right) M_{1/t,\mathcal{F}}(\lambda) dE_\lambda \\ & = \int_{\Lambda} \widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda) dE_\lambda, \end{aligned} \tag{3.8}$$

*where symbols* $\alpha(\lambda,\theta)$ *and* $M_{\omega,\mathcal{F}}(\lambda)$ *satisfy* ***Assumptions III-IV***. *Thus,* $\{X_t,\ t \in \mathbb{Z}\}$ *displays LRD.*

***Proof*** For each $\lambda \in \Lambda$, under **Assumption IV**, from Theorem 6.5 in [7], as $t \to \infty$,

$$\left| \left[ \int_{-\pi}^{\pi} \exp(it\omega) \frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)}} d\omega \right] \left[ \widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda) t^{\alpha(\lambda,\theta)-1} \right]^{-1} - 1 \right| \to 0. \tag{3.9}$$

Under **Assumption IV**, for each $\theta \in \Theta$, from Eq. (3.8), the sequence

$$\left\{ \left| \left[ \int_{-\pi}^{\pi} \exp(in\omega) \frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)}} d\omega \right] \left[ \widetilde{M}_{n,\mathcal{F},\mathcal{A}_\theta}(\lambda) t^{\alpha(\lambda,\theta)-1} \right]^{-1} - 1 \right|,\ n \in \mathbb{N} \right\}$$

is uniformly bounded in $\lambda \in \Lambda$. Thus, we can apply Bounded Convergence Theorem to obtain, from the pointwise convergence (3.9),

$$\lim_{t\to\infty}\int_{\Lambda}\left|\left[\int_{-\pi}^{\pi}\exp(it\omega)\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)}}d\omega\right]\left[\widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda)t^{\alpha(\lambda,\theta)-1}\right]^{-1}-1\right|dE_\lambda$$
$$\underset{\mathcal{L}(\widetilde{H})}{=}\int_{\Lambda}\lim_{t\to\infty}\left|\left[\int_{-\pi}^{\pi}\exp(it\omega)\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)}}d\omega\right]\right.$$
$$\left.\left[\widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda)t^{\alpha(\lambda,\theta)-1}\right]^{-1}-1\right|dE_\lambda=0. \tag{3.10}$$

From Remark 4, under **Assumption I**, $\mathcal{R}_t$ is bounded, for every $t \in \mathbb{Z}$. From (3.10), under **Assumptions II-III**, keeping in mind (2.5), we obtain

$$\begin{aligned}\lim_{t\to\infty}\int_{\Lambda}&\widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda)t^{\alpha(\lambda,\theta)-1}dE_\lambda\\ &=\lim_{t\to\infty}\int_{\Lambda}\left[\int_{-\pi}^{\pi}\exp(it\omega)\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)}}d\omega\right]dE_\lambda\\ &=\lim_{t\to\infty}\int_{-\pi}^{\pi}\exp(it\omega)\mathcal{F}_\omega d\omega\\ &=\lim_{t\to\infty}\mathcal{R}_{t,\theta},\end{aligned} \tag{3.11}$$

in the bounded operator norm.

Thus, Eq. (3.7) holds. Therefore, for $M > 0$, sufficiently large,

$$\begin{aligned}\sum_{t\in\mathbb{Z}}\left\|\mathcal{R}_{t,\theta}\right\|_{L^1(H)}&\geq\sum_{|t|>M}\left\|\mathcal{R}_{t,\theta}\right\|_{\mathcal{L}(H)}\\ &\geq\sum_{|t|>M}\left\|\int_{\Lambda}\widetilde{M}_{t,\mathcal{F},\mathcal{A}_\theta}(\lambda)dE_\lambda\right\|_{\mathcal{L}(H)}|t|^{l_\alpha(\theta)-1}\not<\infty,\end{aligned} \tag{3.12}$$

as we wanted to prove.

## 3.2 Examples

Some special cases of the LRD family of functional sequences introduced in the spectral domain under **Assumptions I–IV** are now analyzed.

## 3.3 Example 1. Multifractionally integrated functional autoregressive moving averages processes

We consider here, in the stationary case, an extended family (see Remark 9 in [26]) of multifractionally integrated functional autoregressive moving averages models.

Let $B$ be a difference operator such that

$$E\|B^j X_t - X_{t-j}\|_H^2 = 0, \quad \forall t, j \in \mathbb{Z}. \tag{3.13}$$

Consider the state equation

$$(1-B)^{\mathcal{A}_\theta/2}\Phi_p(B)X_t \underset{\mathcal{L}^2_H(\Omega,\mathcal{A},P)}{=} \Psi_q(B)\eta_t, \quad \forall t \in \mathbb{Z}, \tag{3.14}$$

where equality holds in the norm of the space $\mathcal{L}^2_H(\Omega, \mathcal{A}, P)$. Here, $\{\eta_t,\ t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed random curves such that $E[\eta_t] = 0$, and $E[\eta_t \otimes \eta_s] = \delta_{t,s}\mathcal{R}_0^\eta$, with $\mathcal{R}_0^\eta \in L^1(H)$, and $\delta_{t,s} = 0$, for $t \neq s$, and $\delta_{t,s} = 1$, for $t = s$. In particular,

$$\left\| \mathcal{R}_0^\eta(h) - \sum_{l\geq 1}^{\infty} \lambda_l(\mathcal{R}_0^\eta)\,\langle \phi_l, h\rangle_H\, \phi_l \right\|_H = 0, \quad \forall h \in H, \tag{3.15}$$

where $\{\phi_n,\ n \geq 1\}$ is an orthonormal basis of eigenvectors in $H$, associated with the eigenvalues $\{\lambda_n(\mathcal{R}_0^\eta),\ n \geq 1\}$. Here,

$$\Phi_p(B) = 1 - \sum_{j=1}^{p} \varphi_j B^j, \; \Psi_q(B) = \sum_{j=1}^{q} \psi_j B^j,$$

where operators $\varphi_j,\ j = 1, \ldots, p$, and $\psi_j,\ j = 1, \ldots, q$, are assumed to be positive self-adjoint bounded operators on $H$, admitting the following diagonal spectral decompositions:

$$\varphi_j = \sum_{l\geq 1} \lambda_l(\varphi_j)\phi_l \otimes \phi_l, = 1, \ldots, p$$
$$\psi_j = \sum_{l\geq 1} \lambda_l(\psi_j)\phi_l \otimes \phi_l,\ j = 1, \ldots, q. \tag{3.16}$$

Also, for each $l \geq 1$, $\Phi_{p,l}(z) = 1 - \sum_{j=1}^{p} \lambda_l(\varphi_j) z^j$ and $\Psi_{q,l} = \sum_{j=1}^{q} \lambda_l(\psi_j) z^j$ have not common roots, and their roots are outside of the unit circle (see also Corollary 6.17 in [7]).

We also assume that, for each $\theta \in \Theta$, operator $\mathcal{A}_\theta$ admits the diagonal spectral representation:

$$\mathcal{A}_\theta = \sum_{l\geq 1} \alpha(l,\theta)\phi_l \otimes \phi_l, \tag{3.17}$$

and

$$\int_{-\pi}^{\pi} \sup_{l\geq 1} \frac{\lambda_l(\mathcal{R}_0^\eta)}{2\pi} \left| \frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))} \right|^2 |1 - \exp(-i\omega)|^{-\alpha(l,\theta)}\, d\omega < \infty. \tag{3.18}$$

Thus, **Assumption I** holds. Note that, for each $l \geq 1$ and $\theta \in \Theta$,

$$(1 - \exp(-i\omega))^{-\alpha(l,\theta)/2} = \sum_{j=0}^{\infty} a_j(l) \exp(-ij\omega)$$
$$a_j(l) = \frac{\Gamma(j + \alpha(l,\theta)/2)}{\Gamma(j+1)\Gamma(\alpha(l,\theta)/2)}, \quad j \geq 0.$$

Assume that, for each $l \geq 1$ and $\theta \in \Theta$,

$$\sum_{j=0}^{\infty} b_{j,\theta}(l) z^j = (1 - \exp(-iz))^{-\alpha(l,\theta)/2} \frac{\Psi_{q,l}(z)}{\Phi_{p,l}(z)}, \quad z \in \mathbb{C}. \tag{3.19}$$

From Eqs. (3.14)–(3.19), applying Corollary 6.17 in [7],

$$X_t(\phi_l) \underset{\mathcal{L}^2(\Omega,\mathcal{A},P)}{=} \left( \sum_{j=0}^{\infty} b_j(l) B^j \right) \eta_t(\phi_l), \quad l \geq 1, \tag{3.20}$$

and from Corollary 6.18 in [7], for each $l \geq 1$ and $\theta \in \Theta$, there exists $\widehat{f}(\omega, l, \theta)$ such that

$$\begin{aligned} \widehat{f}(\omega, l, \theta) &= \frac{\lambda_l(\mathcal{R}_0^{\eta})}{2\pi} \left| \sum_{j=0}^{\infty} b_j(l) \exp(-ij\omega) \right|^2 \\ &= \frac{\lambda_l(\mathcal{R}_0^{\eta})}{2\pi} \left| \frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))} \right|^2 |1 - \exp(-i\omega)|^{-\alpha(l,\theta)}. \end{aligned} \tag{3.21}$$

Thus, for each $l \geq 1$ and $\theta \in \Theta$,

$$\left\langle \mathcal{R}_{t,\theta}(\phi_l), \phi_l \right\rangle_{\widetilde{H}} = \int_{-\pi}^{\pi} \exp(i\omega t)\, \widehat{f}(\omega, l, \theta) d\omega, \tag{3.22}$$

and under **Assumption I** (see Eq. (3.18)), we obtain, for each $\theta \in \Theta$,

$$\begin{aligned} \mathcal{R}_{t,\theta} \underset{\mathcal{L}(H)}{=} \int_{-\pi}^{\pi} \exp(i\omega t) &\left[ \sum_{l \geq 1} \frac{\lambda_l(\mathcal{R}_0^{\eta})}{2\pi} \left| \frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))} \right|^2 \right. \\ &\times \left. |1 - \exp(-i\omega)|^{-\alpha(l,\theta)} \phi_l \otimes \phi_l \right] d\omega, \end{aligned}$$

under the condition (see Eq. (2.22) in Lemma 1),

$$\sigma_{X,\theta}^2 = \sum_{l \geq 1} \left| \int_{-\pi}^{\pi} \frac{\lambda_l(\mathcal{R}_0^{\eta})}{2\pi} \left| \frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))} \right|^2 |1 - \exp(-i\omega)|^{-\alpha(l,\theta)} d\omega \right| < \infty.$$

Note that, in this example, our operator measure satisfies

$$dE(l)(\psi)(\varphi) = d\langle E_{\lambda}(\psi), \varphi\rangle = \phi_l(\psi)\phi_l(\varphi), \quad \forall \psi, \varphi,\ l \geq 1.$$

Thus, we are considering a discrete (or point) operator measure, defined from the common system of eigenvectors $\{\phi_l,\ l \geq 1\}$. Equivalently, the spectral family $\{E_{\lambda_l},\ l \geq 1\}$ admits a representation in terms of a spectral kernel $\widetilde{\Phi}$, defined from the eigenvectors $\{\phi_l,\ l \geq 1\}$, and a point spectral measure (see, e.g., Section 8.2.1 in [34]):

$$E_{\lambda_l} = \sum_{k=1}^{l} \phi_k \otimes \phi_k = \sum_{k=1}^{l} \widetilde{\Phi}_k, \quad l \geq 1. \tag{3.23}$$

Note that, since $\sin(\omega) \sim \omega$, $\omega \to 0$,

$$\begin{aligned} &|1 - \exp(-i\omega)|^{-\mathcal{A}_\theta} \\ &\quad = [4\sin^2(\omega/2)]^{-\mathcal{A}_\theta/2} \sim |\omega|^{-\mathcal{A}_\theta}, \quad \omega \to 0, \end{aligned} \tag{3.24}$$

where the frequency varying operator $|1 - \exp(-i\omega)|^{-\mathcal{A}_\theta/2}$ is interpreted as in [10, 14, 32, 33].

Keeping in mind (3.24), the following identifications are obtained in relation to (3.1)–(3.3),

$$\begin{aligned} M_{\omega,\mathcal{F}}(\lambda) = M_{\omega,\mathcal{F}}(l) = \frac{\lambda_l(\mathcal{R}_0^\eta)}{2\pi}\left|\frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))}\right|^2, \\ \omega \in [-\pi, \pi],\ |1 - \exp(-i\omega)|^{-\alpha(l,\theta)} \sim |\omega|^{-\alpha(l,\theta)}, \omega \to 0, \quad l \geq 1. \end{aligned} \tag{3.25}$$

Hence, as $\omega \to 0$, for each $l \geq 1$,

$$\begin{aligned} &\widehat{f}(\omega, l, \theta) \sim \mathcal{K}_l|\omega|^{-\alpha(l,\theta)}, \mathcal{K}_l = \frac{\lambda_l(\mathcal{R}_0^\eta)}{2\pi}\left|\frac{\Psi_{q,l}(1)}{\Phi_{p,l}(1)}\right|^2, \\ &\mathcal{K} = \sup_{l \geq 1} \mathcal{K}_l < \infty. \end{aligned} \tag{3.26}$$

Assume that $\Psi_{q,l}$ and $\Phi_{p,l}$, $l \geq 1$, are such that

$$M_{\omega,\mathcal{F}}(l) = \frac{\lambda_l(\mathcal{R}_0^\eta)}{2\pi}\left|\frac{\Psi_{q,l}(\exp(-i\omega))}{\Phi_{p,l}(\exp(-i\omega))}\right|^2$$

satisfies the conditions given in **Assumption IV(ii)**. Hence, from Proposition 1, the extended class of fractionally integrated functional autoregressive moving averages models analyzed here displays LRD (see also Remark 9 in [26]). Indeed, $\mathcal{A}_\theta/2$ defines the multifractional order of integration.

### 3.4 Example 2. Discrete sampling of multifractional *H*-valued processes in continuous time

Let $H = L^2(\mathbb{R}, \mathbb{R})$, and $\widetilde{H} = L^2(\mathbb{R}, \mathbb{C})$. Consider

$$d\langle E_\lambda(\varphi), \psi\rangle_{\widetilde{H}} = \int_{\mathbb{R}} \widehat{\varphi}(\lambda)\widehat{\psi}(\lambda)d\lambda, \tag{3.27}$$

$$\widehat{\psi}(\lambda) = \int_{\mathbb{R}} \exp\left(-i\langle\lambda, z\rangle\right)\psi(z)dz, \quad \psi \in L^1(\mathbb{R}),$$

$$\widehat{\varphi}(\lambda) = \int_{\mathbb{R}} \exp\left(-i\langle\lambda, z\rangle\right)\varphi(z)dz, \quad \varphi \in L^1(\mathbb{R}). \tag{3.28}$$

With this particular choice, for $(\lambda, \omega) \in \mathbb{R}^2$, assume that the symbol $f(\omega, \lambda, \theta)$ of the spectral density operator $\mathcal{F}_\omega$, with respect to the spectral family $\{E_\lambda,\ \lambda \in \mathbb{R}\}$ introduced in (3.27)–(3.28) is defined as follows:

$$f(\omega, \lambda, \theta) = |\omega|^{-\alpha(\lambda,\theta)} N_\omega(\lambda)h(\omega), \tag{3.29}$$

where $\alpha(\lambda, \theta)$ satisfies **Assumption IV(i)**, and $h$ is a positive even taper function of bounded variation, with bounded support is the interval $[-\pi, \pi]$, with $h(-\pi) = h(\pi) = 0$ (see, e.g., [20]). We also assume that $h$ is Lipschitz-continuous function, and $N_\omega$ is such that $M_{\omega,\mathcal{F}}(\lambda) = N_\omega(\lambda)h(\omega)$ satisfies **Assumption IV(ii)**. Furthermore, for $\omega \in [-\pi, \pi]\backslash\{0\}$,

$$\sup_{\lambda\in\mathbb{R}} |f(\omega, \lambda, \theta)| < \infty. \tag{3.30}$$

As special case of (3.29), we can consider the tapered continuous version of Example 1 in Section 3.3

$$f(\omega, \lambda, \theta) = |\omega|^{-\alpha(\lambda,\theta)} \frac{P(\lambda, \omega)}{Q(\lambda, \omega)} h(\omega)1_{[-\pi,\pi]}(\omega),\ (\lambda, \omega) \in \mathbb{R}^2,$$

where the taper function satisfies the above required conditions, and $P$ and $Q$ are positive polynomials such that **Assumption IV**(ii) holds. Particularly, when discrete sampling of the solution to fractional and multifractional pseudodifferential evolution equations with Gaussian functional innovations is considered, one can implement inference tools from this framework (see, e.g., [2, 3, 22]).

## 4 The convergence to zero in $\mathcal{S}(\widetilde{H})$ norm of the bias of the integrated periodogram operator

Theorem 1 provides the convergence to zero, in the Hilbert–Schmidt operator norm, of the integrated bias of the periodogram operator. Note that, in [9], weak-convergence of the covariance operator of the fDFT to the spectral density operator, and the convergence of their respective traces is proved. The next result provides convergence in $\mathcal{S}(\widetilde{H})$ norm of the integrated covariance operator of the fDFT to the integrated spectral density operator, in the frequency domain, beyond the SRD condition assumed in [9].

**Theorem 1** *Under **Assumption I**, the following limit holds:*

$$\left\| \int_{-\pi}^{\pi} \left[ \mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] d\omega \right\|_{\mathcal{S}(\widetilde{H})} \to 0, \quad T \to \infty.$$

***Proof*** Let $\{\psi_k,\ k \geq 1\}$ be an orthonormal basis of $\widetilde{H}$. Under **Assumption I**,

$$\begin{aligned} &\left\| \int_{-\pi}^{\pi} \left[ \mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] d\omega \right\|_{\mathcal{S}(\widetilde{H})}^{2} \\ &= \sum_{k \geq 1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[ \mathcal{F}_{\xi} \mathcal{F}_{\omega} - \mathcal{F}_{\xi} \mathcal{F}_{\omega}^{(T)} - \mathcal{F}_{\xi}^{(T)} \mathcal{F}_{\omega} \right. \\ &\quad \left. + \mathcal{F}_{\xi}^{(T)} \mathcal{F}_{\omega}^{(T)} \right] (\psi_k)(\psi_k) d\omega d\xi. \end{aligned} \tag{4.1}$$

From Lemma 1 (see Eq. (2.22)), for every $k \geq 1$, $\mathcal{F}_{\omega}(\psi_k)(\psi_k) \in L^1([-\pi, \pi])$. Hence, for each $k \geq 1$,

$$\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_k) \to \mathcal{F}_{\omega}(\psi_k)(\psi_k), \quad T \to \infty, \ \omega \in [-\pi, \pi] \backslash \Lambda_0. \tag{4.2}$$

Applying triangle inequality, for every $T \geq 2$,

$$\begin{aligned} &\left| \left[ \mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] (\psi_k)(\psi_k) \right| \leq 2 \left\| \mathcal{F}_{\omega} \right\|_{L^1(\widetilde{H})} < \infty, \\ &\omega \in [-\pi, \pi] \backslash \Lambda_0, \end{aligned} \tag{4.3}$$

since from (2.22),

$$\int_{-\pi}^{\pi} \left\| \mathcal{F}_{\omega} \right\|_{L^1(\widetilde{H})} d\omega < \infty.$$

Hence, from Eqs. (4.2)–(4.3), keeping in mind (2.22), Dominated Convergence Theorem leads to

$$\begin{aligned} &\lim_{T \to \infty} \int_{-\pi}^{\pi} \left| \left[ \mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] (\psi_k)(\psi_k) \right| d\omega \\ &= \int_{-\pi}^{\pi} \lim_{T \to \infty} \left| \left[ \mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] (\psi_k)(\psi_k) \right| d\omega = 0, k \geq 1. \end{aligned} \tag{4.4}$$

Note also that, from Lemma 1, $\left\| \mathcal{F}_{\omega} \right\|_{\mathcal{S}(\widetilde{H})} \in L^2\left([-\pi, \pi], \mathbb{C}\right)$. Hence, for every $k, l \geq 1$, $\mathcal{F}_{\omega}(\psi_k)(\psi_l) \in L^2([-\pi, \pi], \mathbb{C})$. Particularly, from Young's convolution inequality with $p = 2$, for each $k, l \geq 1$,

$$\int_{-\pi}^{\pi} |\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_l)|^2 d\omega \leq \int_{-\pi}^{\pi} |\mathcal{F}_{\omega}(\psi_k)(\psi_l)|^2 d\omega. \tag{4.5}$$

Hence, under **Assumption I**, applying the Cauchy–Schwarz and Jensen's inequalities, and (4.5), from Lemma 1 (see Eq. (2.22)), we obtain

$$\begin{aligned}
&\sum_{k\geq 1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\xi}^{(T)}\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_k) d\omega d\xi \\
&= \sum_{k\geq 1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left\langle \mathcal{F}_{\omega}^{(T)}(\psi_k), \mathcal{F}_{\xi}^{(T)}(\psi_k)\right\rangle_{\widetilde{H}} d\omega d\xi \\
&\leq \sum_{k\geq 1} \left[\int_{-\pi}^{\pi} \left\|\mathcal{F}_{\omega}^{(T)}(\psi_k)\right\|_{\widetilde{H}} d\omega\right]\left[\int_{-\pi}^{\pi} \left\|\mathcal{F}_{\xi}^{(T)}(\psi_k)\right\|_{\widetilde{H}} d\xi\right] \\
&\leq 4\pi^2 \sum_{k\geq 1} \sqrt{\left[\int_{-\pi}^{\pi} \mathcal{F}_{\omega}^{(T)}\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_k) d\omega\right]}\sqrt{\left[\int_{-\pi}^{\pi} \mathcal{F}_{\xi}^{(T)}\mathcal{F}_{\xi}^{(T)}(\psi_k)(\psi_k) d\xi\right]} \\
&\leq 4\pi^2 \sqrt{\sum_{k\geq 1} \int_{-\pi}^{\pi} \left\|\mathcal{F}_{\omega}^{(T)}(\psi_k)\right\|_{\widetilde{H}}^2 d\omega}\sqrt{\sum_{k\geq 1} \int_{-\pi}^{\pi} \left\|\mathcal{F}_{\xi}^{(T)}(\psi_k)\right\|_{\widetilde{H}}^2 d\xi} \\
&= 4\pi^2 \sqrt{\sum_{k,l\geq 1} \int_{-\pi}^{\pi} |\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_l)|^2 d\omega}\sqrt{\sum_{k,l\geq 1} \int_{-\pi}^{\pi} |\mathcal{F}_{\xi}^{(T)}(\psi_k)(\psi_l)|^2 d\xi} \\
&\leq 4\pi^2 \sqrt{\sum_{k,l\geq 1} \int_{-\pi}^{\pi} |\mathcal{F}_{\omega}(\psi_k)(\psi_l)|^2 d\omega}\sqrt{\sum_{k,l\geq 1} \int_{-\pi}^{\pi} |\mathcal{F}_{\xi}(\psi_k)(\psi_l)|^2 d\xi} \\
&= 4\pi^2 \int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{S}(\widetilde{H})}^2 d\omega < \infty.
\end{aligned} \tag{4.6}$$

Following similar steps to (4.6),

$$\begin{aligned}
&\sum_{k\geq 1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\xi}^{(T)}\mathcal{F}_{\omega}(\psi_k)(\psi_k) d\omega d\xi \\
&\quad \leq 4\pi^2 \int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{S}(\widetilde{H})}^2 d\omega < \infty,
\end{aligned} \tag{4.7}$$

as well as

$$\begin{aligned}
&\sum_{k\geq 1} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\xi}\mathcal{F}_{\omega}^{(T)}(\psi_k)(\psi_k) d\omega d\xi \\
&\quad \leq 4\pi^2 \int_{-\pi}^{\pi} \|\mathcal{F}_{\omega}\|_{\mathcal{S}(\widetilde{H})}^2 d\omega < \infty.
\end{aligned} \tag{4.8}$$

Furthermore, from Lemma 1 (see Eq. (2.22)), $\mathcal{F}_{\omega}(\psi_k)(\psi_k) \in L^1([-\pi, \pi])$, for every $k \geq 1$. Thus, we can consider Young's convolution inequality with $p = 1$ leading to

$$\int_{-\pi}^{\pi} \mathcal{F}_{\xi}^{(T)}(\psi_k)(\psi_k)d\xi \leq \int_{-\pi}^{\pi} \mathcal{F}_{\xi}(\psi_k)(\psi_k)d\xi, \quad k \geq 1.$$

Therefore,

$$\int_{-\pi}^{\pi} \|\mathcal{F}_{\xi}^{(T)}\|_{\mathcal{L}(\widetilde{H})}d\xi \leq \int_{-\pi}^{\pi} \|\mathcal{F}_{\xi}\|_{\mathcal{L}(\widetilde{H})}d\xi. \tag{4.9}$$

From Eqs. (4.6)–(4.8), we can apply Dominated Convergence Theorem, and keeping in mind Eqs. (4.4) and (4.9), we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \left\| \int_{-\pi}^{\pi} \left[\mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)}\right] d\omega \right\|^2_{\mathcal{S}(\widetilde{H})} \\
&= \sum_{k\geq 1} \lim_{T\to\infty} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left[\mathcal{F}_{\xi}\mathcal{F}_{\omega} - \mathcal{F}_{\xi}\mathcal{F}_{\omega}^{(T)} - \mathcal{F}_{\xi}^{(T)}\mathcal{F}_{\omega} + \mathcal{F}_{\xi}^{(T)}\mathcal{F}_{\omega}^{(T)}\right](\psi_k)(\psi_k)d\omega d\xi. \\
&\leq \sum_{k\geq 1} 2\left[\int_{-\pi}^{\pi} \|\mathcal{F}_{\xi}\|_{\mathcal{L}(\widetilde{H})}d\xi\right] \lim_{T\to\infty}\int_{-\pi}^{\pi} \left|\left[\mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)}\right](\psi_k)(\psi_k)\right| d\omega = 0.
\end{aligned} \tag{4.10}$$

□

## 5 Semiparametric estimation in the spectral domain

This section introduces the estimation methodology adopted in the functional spectral domain. Theorem 2 derives the weak consistency of the formulated parametric estimator of the long-memory operator.

Under **Assumptions I-IV**, let $\Theta \subset \mathbb{R}^p$, $p \geq 1$, be a compact subset of $\mathbb{R}^p$. Assume that the true parameter value $\theta_0$ lies in the interior of $\Theta$, denoted as int $\Theta$. The symbol $\alpha : \mathbb{R}\times\Theta \longrightarrow (0, 1)$ is such that $\alpha(\cdot, \theta_1) \neq \alpha(\cdot, \theta_2)$, for $\theta_1 \neq \theta_2$, for every $\theta_1, \theta_2 \in \Theta$. Thus, under (3.1), we get indentifiability in the semiparametric model. Denote by $\widehat{\theta}_T$ the estimator of the true parameter value $\theta_0$, based on a functional sample of size $T$. Hence, $\widehat{\alpha}_T(\lambda, \theta) = \alpha(\lambda, \widehat{\theta}_T)$ provides the parametric estimator of the symbol $\alpha(\lambda, \theta)$ of $\mathcal{A}_{\theta}$.

Let now introduce the elements involved in the definition of our operator loss function, to compute the minimum contrast estimator $\widehat{\theta}_T$, under suitable conditions. Specifically, for each $\omega \in [-\pi, \pi]$, the weighting operator $\mathcal{W}_{\omega}$ is introduced as a bounded positive self-adjoint operator admitting the following spectral representation:

$$\mathcal{W}_{\omega}=\int_{\Lambda}W(\omega,\lambda,\beta)dE_{\lambda},=\int_{\Lambda}\widetilde{W}(\lambda)|\omega|^{\beta}dE_{\lambda},\quad \beta>0. \tag{5.1}$$

In particular, the symbol $W(\omega,\lambda,\beta)$ of operator $\mathcal{W}_{\omega}$ factorizes, in terms of $\widetilde{W}(\lambda)$ and $|\omega|^{\beta}$, with $\widetilde{W}$ defining the symbol of a positive self-adjoint operator $\widetilde{\mathcal{W}}\in\mathcal{L}(\widetilde{H})$ such that

$$\begin{aligned} m_{\widetilde{\mathcal{W}}} &= \inf_{\psi\in\widetilde{H};\ \|\psi\|_{\widetilde{H}}=1}\left\langle\widetilde{\mathcal{W}}(\psi),\psi\right\rangle_{\widetilde{H}}, \\ M_{\widetilde{\mathcal{W}}} &= \sup_{\psi\in\widetilde{H};\ \|\psi\|_{\widetilde{H}}=1}\left\langle\widetilde{\mathcal{W}}(\psi),\psi\right\rangle_{\widetilde{H}}. \end{aligned} \tag{5.2}$$

For each $\theta\in\Theta$, the normalizing operator $\sigma_{\theta}^{2}$ is computed as follows:

$$\sigma_{\theta}^{2}=\int_{-\pi}^{\pi}\mathcal{F}_{\omega,\theta}\mathcal{W}_{\omega}d\omega=\int_{-\pi}^{\pi}\int_{\Lambda}\frac{M_{\omega,\mathcal{F}}(\lambda)\widetilde{W}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)-\beta}}dE_{\lambda}d\omega. \tag{5.3}$$

Thus, the symbol $\Sigma_{\theta}^{2}$ of $\sigma_{\theta}^{2}$ is given by

$$\Sigma_{\theta}^{2}(\lambda)=\int_{-\pi}^{\pi}\frac{M_{\omega,\mathcal{F}}(\lambda)\widetilde{W}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)-\beta}}d\omega,\ \forall\lambda\in\Lambda. \tag{5.4}$$

Under **Assumption IV(ii)** (see Eq. (3.5)), and (5.2), for every $\lambda\in\Lambda$,

$$\begin{aligned} & mm_{\widetilde{\mathcal{W}}}\left(\left[\int_{-\pi}^{-1}+\int_{1}^{\pi}\right]|\omega|^{-L(\theta)+\beta}d\omega+\int_{-1}^{1}|\omega|^{-l(\theta)+\beta}d\omega\right) \\ &= mm_{\widetilde{\mathcal{W}}}\left[\frac{(-\pi)^{1+\beta-L(\theta)}-(-1)^{1+\beta-L(\theta)}}{1+\beta-L(\theta)}+\frac{(\pi)^{1+\beta-L(\theta)}-1}{1+\beta-L(\theta)}+\frac{(-1)^{1-l(\theta)+\beta}}{1-l(\theta)+\beta}\right. \\ &\left.+\frac{1}{1-l(\theta)+\beta}\right] \\ &\leq \Sigma_{\theta}^{2}(\lambda)\leq MM_{\widetilde{\mathcal{W}}}\left(\left[\int_{-\pi}^{-1}+\int_{1}^{\pi}\right]|\omega|^{-l(\theta)+\beta}d\omega+\int_{-1}^{1}|\omega|^{-L(\theta)+\beta}d\omega\right) \\ &= MM_{\widetilde{\mathcal{W}}}\left[\frac{(-\pi)^{1+\beta-l(\theta)}-(-1)^{1+\beta-l(\theta)}}{1+\beta-l(\theta)}+\frac{(\pi)^{1+\beta-l(\theta)}-1}{1+\beta-l(\theta)}\right. \\ &\quad\left.+\frac{(-1)^{1-L(\theta)+\beta}}{1-L(\theta)+\beta}+\frac{1}{1-L(\theta)+\beta}\right]. \end{aligned} \tag{5.5}$$

Thus, $\sigma_{\theta}^{2}$ is a bounded operator. The symbol of $[\sigma_{\theta}^{2}]^{-1}$ is given by

$$\left[\Sigma_{\theta}^{2}(\lambda)\right]^{-1}=\left[\int_{-\pi}^{\pi}\frac{M_{\omega,\mathcal{F}}(\lambda)\widetilde{W}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)-\beta}}d\omega\right]^{-1},\lambda\in\Lambda. \tag{5.6}$$

From (5.5), for every $\lambda \in \Lambda$,

$$\left\{ mm_{\widetilde{\mathcal{W}}} \left[ \frac{(-\pi)^{1+\beta-L(\theta)} - (-1)^{1+\beta-L(\theta)}}{1+\beta-L(\theta)} \right.\right.$$
$$\left.\left. + \frac{(\pi)^{1+\beta-L(\theta)} - 1}{1+\beta-L(\theta)} + \frac{(-1)^{1-l(\theta)+\beta}}{1-l(\theta)+\beta} + \frac{1}{1-l(\theta)+\beta} \right] \right\}^{-1}$$
$$\geq \left[ \Sigma_{\theta}^{2}(\lambda) \right]^{-1} \geq \left\{ MM_{\widetilde{\mathcal{W}}} \left[ \frac{(-\pi)^{1+\beta-l(\theta)} - (-1)^{1+\beta-l(\theta)}}{1+\beta-l(\theta)} \right.\right.$$
$$\left.\left. + \frac{(\pi)^{1+\beta-l(\theta)} - 1}{1+\beta-l(\theta)} + \frac{(-1)^{1-L(\theta)+\beta}}{1-L(\theta)+\beta} + \frac{1}{1-L(\theta)+\beta} \right] \right\}^{-1}. \tag{5.7}$$

Hence, $[\sigma_{\theta}^{2}]^{-1}$ is strictly positive and bounded.

From (5.3), we can consider the following factorization of the spectral density operator, for $(\omega, \theta) \in [-\pi, \pi]\backslash\{0\} \times \Theta$,

$$\mathcal{F}_{\omega,\theta} = \sigma_{\theta}^{2} \Upsilon_{\omega,\theta} = \Upsilon_{\omega,\theta} \sigma_{\theta}^{2}, \tag{5.8}$$

where, for each $\theta \in \Theta$, and $\omega \in [-\pi, \pi]$, $\omega \neq 0$,

$$\Upsilon_{\omega,\theta} = \int_{\Lambda} \Upsilon(\omega, \lambda, \theta) dE_{\lambda} = \int_{\Lambda} \frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta)} \Sigma_{\theta}^{2}(\lambda)} dE_{\lambda}. \tag{5.9}$$

From Eqs. (5.1)–(5.9), for each $\theta \in \Theta$, and any $\varrho, \psi \in \widetilde{H}$,

$$\int_{-\pi}^{\pi} \Upsilon_{\omega,\theta} \mathcal{W}_{\omega}(\varrho)(\psi) d\omega = \int_{\Lambda} d \left\langle E_{\lambda}(\varrho), \psi \right\rangle_{\widetilde{H}} = \langle \varrho, \psi \rangle_{\widetilde{H}}. \tag{5.10}$$

Equivalently, $\int_{-\pi}^{\pi} \Upsilon_{\omega,\theta} \mathcal{W}_{\omega} d\omega$ coincides with the identity operator $I_{\widetilde{H}}$ on $\widetilde{H}$, for each $\theta \in \Theta$.

Let us now consider the empirical operator $\mathbf{U}_{T,\theta}$ given by, for each $\theta \in \Theta$,

$$[\mathbf{U}_{T,\theta}] = - \int_{-\pi}^{\pi} p_{\omega}^{(T)} \ln\left(\Upsilon_{\omega,\theta}\right) \mathcal{W}_{\omega} d\omega, \tag{5.11}$$

where $T$ denotes as before the sample size. Its theoretical counterpart $U_{\theta}$ is defined, for each $\theta \in \Theta$, as

$$\begin{aligned} U_{\theta} &= - \int_{-\pi}^{\pi} \mathcal{F}_{\omega,\theta_0} \ln\left(\Upsilon_{\omega,\theta}\right) \mathcal{W}_{\omega} d\omega \\ &= - \int_{-\pi}^{\pi} \int_{\Lambda} \frac{M_{\omega,\mathcal{F}}(\lambda) \widetilde{W}(\lambda)}{|\omega|^{\alpha(\lambda,\theta_0)-\beta}} \ln\left(\Upsilon(\omega, \lambda, \theta)\right) dE_{\lambda} d\omega. \end{aligned} \tag{5.12}$$

***Remark 7*** Note that, under **Assumption IV(i)**, for each $\theta \in \Theta$, $U_\theta \in \mathcal{L}(\widetilde{H})$ for any $\beta > 0$. Specifically, for every $\lambda \in \Lambda$, $\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta_0)}} \in L^1([-\pi,\pi])$, and $\ln(\Upsilon(\omega,\lambda,\theta))\, W(\omega,\lambda,\beta) \in L^1([-\pi,\pi])$, with

$$\sup_{\lambda\in\Lambda}\left|-\int_{-\pi}^{\pi}\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta_0)}}\ln(\Upsilon(\omega,\lambda,\theta))\,W(\omega,\lambda,\beta)d\omega\right| < \infty. \tag{5.13}$$

In addition, for $T$ large, $\mathbf{U}_{T,\theta} \in \mathcal{L}(\widetilde{H})$ a.s. (see Theorem 2 below).

We now consider the loss operator $\mathcal{K}(\theta_0,\theta)$ to be minimized, with respect to $\theta$, in the operator norm. Specifically, consider, for each $\theta \in \Theta$,

$$[\mathcal{K}(\theta_0,\theta)] = \int_{-\pi}^{\pi}\mathcal{F}_{\omega,\theta_0}\ln\left(\Upsilon_{\omega,\theta_0}\Upsilon_{\omega,\theta}^{-1}\right)\mathcal{W}_\omega d\omega = [U_\theta - U_{\theta_0}]. \tag{5.14}$$

From Remark 7, for each $\theta \in \Theta$, $\mathcal{K}(\theta_0,\theta) \in \mathcal{L}(\widetilde{H})$. Furthermore, the symbol of $\mathcal{K}(\theta_0,\theta)$ is given by

$$\begin{aligned}&\int_{-\pi}^{\pi}\frac{M_{\omega,\mathcal{F}}(\lambda)}{|\omega|^{\alpha(\lambda,\theta_0)}}\ln\left(\frac{\Upsilon(\omega,\lambda,\theta_0)}{\Upsilon(\omega,\lambda,\theta)}\right)W(\omega,\lambda,\beta)d\omega,\\ &\lambda\in\Lambda, \quad \theta\in\Theta.\end{aligned} \tag{5.15}$$

The operator $[\sigma^2_{\theta_0}]^{-1}\mathcal{K}(\theta_0,\theta)$ could be interpreted as a weighted Kullback–Leibler divergence operator, measuring the discrepancy between the two semiparametric functional spectral models $\Upsilon_{\omega,\theta_0}$ and $\Upsilon_{\omega,\theta}$, for each $\theta \in \Theta$ (see, e.g., [11]). Note that, from Eqs. (5.3)–(5.14), applying Jensen's inequality, for every $k \geq 1$, and $\theta \in \Theta$,

$$\begin{aligned}-[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k) &\leq \left\|\sigma^2_\theta\right\|_{\mathcal{L}(\widetilde{H})}\\ &\ln\left(\int_{-\pi}^{\pi}\int_\Lambda \Upsilon(\omega,\lambda,\theta)W(\omega,\lambda,\beta)d\left\langle E_\lambda(\psi_k),\psi_k\right\rangle_{\widetilde{H}}d\omega\right)\\ &= \left\|\sigma^2_\theta\right\|_{\mathcal{L}(\widetilde{H})}\ln\left(\int_{-\pi}^{\pi}\Upsilon_{\omega,\theta}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega\right)\\ &= \left\|\sigma^2_\theta\right\|_{\mathcal{L}(\widetilde{H})}\ln\left(\|\psi_k\|^2_{\widetilde{H}}\right) = 0.\end{aligned} \tag{5.16}$$

for any orthonormal basis $\{\psi_k,\ k \geq 1\}$ of $\widetilde{H}$. From Eq. (5.16), for every $k \geq 1$, and $\theta \in \Theta$, $[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k) \geq 0$. Thus, $\{\mathcal{K}(\theta_0,\theta),\ \theta \in \Theta\}$ is a parametric family of non-negative self-adjoint bounded operators such that

$$\begin{aligned}\|\mathcal{K}(\theta_0,\theta)\|_{\mathcal{L}(\widetilde{H})} &= \sup_{k\geq 1}[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k) > 0, \quad \theta \neq \theta_0\\ \|\mathcal{K}(\theta_0,\theta)\|_{\mathcal{L}(\widetilde{H})} &= \sup_{k\geq 1}[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k) = 0 \Leftrightarrow \theta = \theta_0.\end{aligned} \tag{5.17}$$

Hence, from Eqs. (5.17),

$$\begin{aligned}\theta_0 &= \arg \min_{\theta\in\Theta} \|[\mathcal{K}(\theta_0,\theta)]\|_{\mathcal{L}(\widetilde{H})} \\ &= \arg \min_{\theta\in\Theta} \sup_{k\geq 1} \mathcal{K}(\theta_0,\theta)(\psi_k)(\psi_k) \\ &= \arg \min_{\theta\in\Theta} \sup_{k\geq 1} U_\theta(\psi_k)(\psi_k). \end{aligned} \tag{5.18}$$

We then consider the following estimator $\widehat{\theta}_T$ computed from the empirical contrast operator $U_{T,\theta}$ in (5.11), and a given orthonormal basis $\{\psi_k,\ k\geq 1\}$ of $\widetilde{H}$ :

$$\widehat{\theta}_T = \arg \min_{\theta\in\Theta} \sup_{k\geq 1} U_{T,\theta}(\psi_k)(\psi_k). \tag{5.19}$$

**Theorem 2** *Let $\{X_t,\ t\in\mathbb{Z}\}$ be a stationary zero-mean Gaussian functional sequence satisfying **Assumptions I–IV**. Consider in **Assumption IV(ii)** the particular case where $\mathcal{M}_{\omega,\mathcal{F}}$ satisfies, for any $\xi>0$,*

$$\lim_{\omega\to 0}\left\|\mathcal{M}_{\omega/\xi,\mathcal{F}}\mathcal{M}^{-1}_{\omega,\mathcal{F}} - I_{\widetilde{H}}\right\|_{\mathcal{L}(\widetilde{H})} = 0. \tag{5.20}$$

*Under the conditions reflected in Eqs. (5.1)–(5.16), for $\beta>1$, we then have*

$$E\left\|\int_{-\pi}^{\pi}\left[p_\omega^{(T)} - \mathcal{F}_{\omega,\theta_0}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})} \to 0, \quad T\to\infty, \tag{5.21}$$

*where, for $(\omega,\theta)\in[-\pi,\pi]\times\Theta$,*

$$\mathcal{W}_{\omega,\theta} = \ln\left(\Upsilon_{\omega,\theta}\right)\mathcal{W}_\omega. \tag{5.22}$$

*Furthermore, the estimator $\widehat{\theta}_T$ in (5.19) satisfies*

$$\widehat{\theta}_T \to_P \theta_0, \quad T\to\infty,$$

*where $\to_P$ denotes convergence in probability.*

***Proof*** The operator $\mathcal{W}_{\omega,\theta}$ introduced in (5.22) admits the spectral representation

$$\begin{aligned}\mathcal{W}_{\omega,\theta} = \int_\Lambda &\Big[\ln\left(M_\omega(\lambda)\right) - \ln\left(\Sigma^2_\theta(\lambda)\right) \\ &-\alpha(\lambda,\theta)\ln\left(|\omega|\right)\Big]\widetilde{W}(\lambda)|\omega|^\beta dE_\lambda,\end{aligned} \tag{5.23}$$

for $\omega \in [-\pi, \pi]$, and $\theta \in \Theta$. From (5.23),

$$
\begin{aligned}
\|\mathcal{W}_{\omega,\theta}\|_{\mathcal{L}(\widetilde{H})} &\leq \left\|\ln\left(\mathcal{M}_{\omega,\mathcal{F}}\right)|\omega|^{\beta}\widetilde{\mathcal{W}}_{\omega}\right\|_{\mathcal{L}(\widetilde{H})} \\
&\quad + \left\|\mathcal{A}_{\theta}\ln\left(|\omega|\right)|\omega|^{\beta}\widetilde{\mathcal{W}}_{\omega}\right\|_{\mathcal{L}(\widetilde{H})} + \left\|\ln\left(\sigma_{\theta}^{2}\right)|\omega|^{\beta}\widetilde{\mathcal{W}}_{\omega}\right\|_{\mathcal{L}(\widetilde{H})} \\
&\leq \ln(M)\pi^{\beta}M_{\widetilde{\mathcal{W}}} + L(\theta)\ln(\pi)\pi^{\beta}M_{\widetilde{\mathcal{W}}} + \left\|\ln\left(\sigma_{\theta}^{2}\right)\right\|_{\mathcal{L}(\widetilde{H})}\pi^{\beta}M_{\widetilde{\mathcal{W}}}, \quad (5.24)
\end{aligned}
$$

for every $\theta \in \Theta$, $\omega \in [-\pi, \pi]$, and $\beta > 0$.

From (5.24),

$$
\begin{aligned}
\sup_{\omega\in[-\pi,\pi]}\|\mathcal{W}_{\omega,\theta}\|_{\mathcal{L}(\widetilde{H})} &\leq \ln(M)\pi^{\beta}M_{\widetilde{\mathcal{W}}} + L(\theta)\ln(\pi)\pi^{\beta}M_{\widetilde{\mathcal{W}}} \\
&\quad + \left\|\ln\left(\sigma_{\theta}^{2}\right)\right\|_{\mathcal{L}(\widetilde{H})}\pi^{\beta}M_{\widetilde{\mathcal{W}}} = \mathcal{H}(\theta). \quad (5.25)
\end{aligned}
$$

Thus, the family $\left\{\mathcal{W}_{\omega,\theta},\ \omega \in [-\pi, \pi]\right\}$ is equicontinuous, for any $\theta \in \Theta$. We first prove that the following limits hold, for each $\theta \in \Theta$,

$$
\left\|\int_{-\pi}^{\pi}\left[\mathcal{F}_{\omega,\theta_0}^{(T)} - \mathcal{F}_{\omega,\theta_0}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})} \to 0, \quad T \to \infty \quad (5.26)
$$

$$
E\left\|\int_{-\pi}^{\pi}\left[p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})}^{2} \to 0, \quad T \to \infty, \quad (5.27)
$$

where $E\left(p_{\omega}^{(T)}\right) = \mathcal{F}_{\omega,\theta_0}^{(T)}$.

From Theorem 1, and Eq. (5.26),

$$
\begin{aligned}
&\left\|\int_{-\pi}^{\pi}\left[E\left(p_{\omega}^{(T)}\right) - \mathcal{F}_{\omega,\theta_0}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})} \\
&\quad \leq \mathcal{H}(\theta)\left\|\int_{-\pi}^{\pi}\left[\mathcal{F}_{\omega,\theta_0}^{(T)} - \mathcal{F}_{\omega,\theta_0}\right]d\omega\right\|_{\mathcal{S}(\widetilde{H})} \to 0, \quad T \to \infty. \quad (5.28)
\end{aligned}
$$

Under the Gaussian distribution of $\{X_t,\ t \in \mathbb{Z}\}$, applying Fourier Transform Inversion Formula, we obtain

$$
\begin{aligned}
&E\left\|\int_{-\pi}^{\pi}\left[p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})}^{2} \\
&= \sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[E\left[p_{\xi}^{(T)}p_{\omega}^{(T)}\right] + \mathcal{F}_{\xi,\theta_0}^{(T)}\mathcal{F}_{\omega,\theta_0}^{(T)} - \mathcal{F}_{\xi,\theta_0}^{(T)}E\left[p_{\omega}^{(T)}\right]\right. \\
&\quad \left. - E\left[p_{\xi}^{(T)}\right]\mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\xi,\theta}^{\star}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega d\xi \\
&= \sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[E\left[p_{\xi}^{(T)}p_{\omega}^{(T)}\right] - \mathcal{F}_{\xi,\theta_0}^{(T)}\mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\xi,\theta}^{\star}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi T)^2}\sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\sum_{t_1,s_1,t_2,s_2=1}^{T}\exp\left(-i\omega(t_1-s_1)-i\xi(t_2-s_2)\right)\right.\\
&\quad\times\left[E\left[X_{t_1}\otimes X_{s_1}\otimes X_{t_2}\otimes X_{s_2}\right]-E\left[X_{t_1}\otimes X_{s_1}\right]E\left[X_{t_2}\otimes X_{s_2}\right]\right]\Big]\\
&\quad\times\mathcal{W}^{\star}_{\xi,\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega d\xi\\
&= \frac{1}{(2\pi T)^2}\sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\sum_{t_1,s_1,t_2,s_2=1}^{T}\exp\left(-i\omega(t_1-s_1)-i\xi(t_2-s_2)\right)\right.\\
&\quad\times\left[E\left[X_{t_1}\otimes X_{t_2}\right]E\left[X_{s_1}\otimes X_{s_2}\right]+E\left[X_{t_1}\otimes X_{s_2}\right]E\left[X_{t_2}\otimes X_{s_1}\right]\right]\Big]\\
&\times\mathcal{W}^{\star}_{\xi,\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega d\xi\\
&= \frac{2\pi}{T}\sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathcal{F}_{\widetilde{\omega},\theta_0}\mathcal{F}_{\widetilde{\xi},\theta_0}\left[\frac{1}{[2\pi]^3T}\sum_{t_1,s_1,t_2,s_2=1}^{T}\exp\left(it_1(\widetilde{\omega}-\omega)\right)\right.\\
&\quad\times\exp\left(is_1(\omega+\widetilde{\xi})+it_2(-\xi-\widetilde{\omega})+is_2(\xi-\widetilde{\xi})\right)\\
&\quad\left.+\exp\left(it_1(\widetilde{\omega}-\omega)+is_1(\omega+\widetilde{\xi})+it_2(-\xi-\widetilde{\xi})+is_2(\xi-\widetilde{\omega})\right)\right]\\
&\times\mathcal{W}^{\star}_{\xi,\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega d\xi d\widetilde{\omega}d\widetilde{\xi}\\
&= \frac{2\pi}{T}\sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{f_1(\omega)}^{f_2(\omega)}\int_{g_1(\omega)}^{g_2(\omega)}\int_{h_1(\omega,u_1)}^{h_2(\omega,u_1)}\Phi_T^4(u_1,u_2,u_3)\mathcal{F}_{u_1+\omega,\theta_0}\mathcal{F}_{u_2-\omega,\theta_0}\\
&\quad\times\mathcal{W}^{\star}_{-(u_1+u_3+\omega),\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)du_3du_2du_1d\omega\\
&\qquad+\frac{2\pi}{T}\sum_{k\geq 1}\int_{-\pi}^{\pi}\int_{f_1(\omega)}^{f_2(\omega)}\int_{g_1(\omega)}^{g_2(\omega)}\int_{\widetilde{h}_1(\omega,\widetilde{u}_1)}^{\widetilde{h}_2(\omega,\widetilde{u}_1)}\Phi_T^4(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)\mathcal{F}_{\widetilde{u}_1+\omega,\theta_0}\mathcal{F}_{\widetilde{u}_2-\omega,\theta_0}\\
&\quad\times\mathcal{W}^{\star}_{\widetilde{u}_3-\widetilde{u}_1-\omega,\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\widetilde{u}_3d\widetilde{u}_2d\widetilde{u}_1d\omega\\
&\quad= \frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{f_1(\omega)}^{f_2(\omega)}\int_{g_1(\omega)}^{g_2(\omega)}\int_{h_1(\omega,u_1)}^{h_2(\omega,u_1)}\Phi_T^4(u_1,u_2,u_3)\\
&\qquad\times\left\langle\mathcal{F}_{u_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{u_2-\omega,\theta_0}\mathcal{W}_{-(u_1+u_3+\omega),\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}du_3du_2du_1d\omega\\
&\qquad+\frac{2\pi}{T}\int_{-\pi}^{\pi}\int_{f_1(\omega)}^{f_2(\omega)}\int_{g_1(\omega)}^{g_2(\omega)}\int_{\widetilde{h}_1(\omega,\widetilde{u}_1)}^{\widetilde{h}_2(\omega,\widetilde{u}_1)}\Phi_T^4(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)\\
&\qquad\times\left\langle\mathcal{F}_{\widetilde{u}_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{\widetilde{u}_2-\omega,\theta_0}\mathcal{W}_{\widetilde{u}_3-\widetilde{u}_1-\omega,\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\widetilde{u}_3d\widetilde{u}_2d\widetilde{u}_1d\omega\\
&\qquad\leq\frac{\mathcal{K}\pi}{T}\int_{[-\pi,\pi]^4}\Phi_{4T}^4(u_1,u_2,u_3)\\
&\qquad\quad\times\left\langle\mathcal{F}_{2u_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2u_2-\omega,\theta_0}\mathcal{W}_{-(2u_1+4u_3+\omega),\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega du_1du_2du_3\\
&\qquad\quad+\frac{\mathcal{K}\pi}{T}\int_{[-\pi,\pi]^4}\Phi_{4T}^4(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)\\
&\qquad\quad\times\left\langle\mathcal{F}_{2\widetilde{u}_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2\widetilde{u}_2-\omega,\theta_0}\mathcal{W}_{4\widetilde{u}_3-2\widetilde{u}_1-\omega,\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega d\widetilde{u}_1d\widetilde{u}_2d\widetilde{u}_3, \qquad (5.29)
\end{aligned}
$$

where, for $\omega\in[-\pi,\pi]$, $f_1(\omega)=-\pi-\omega$, $f_2(\omega)=\pi-\omega$, $g_1(\omega)=-\pi+\omega$, $g_2(\omega)=\pi+\omega$, $h_1(\omega,u_1)=-\pi-u_1-\omega$, $h_2(\omega,u_1)=\pi-u_1-\omega$, $\widetilde{h}_1(\omega,\widetilde{u}_1)=$

$-\pi+\widetilde{u}_1+\omega$, $\widetilde{h}_2(\omega,\widetilde{u}_1)=\pi+\widetilde{u}_1+\omega$. For $v_4=-(v_1+v_2+v_3)$, $v_j\in[-\pi,\pi]$, $j=1,2,3,4$, in (5.29), the multidimensional kernel $\Phi_T^4$ of Féjer type is defined as follows:'

$$\Phi_T^4(v_1,v_2,v_3,v_4)=\Phi_T^4(v_1,v_2,v_3)$$
$$=\frac{1}{(2\pi)^3T}\sum_{t_1,s_1,t_2,s_2=1}^{T}\exp\left(i(t_1v_1+s_1v_2+t_2v_3+s_2v_4)\right)$$
$$=\frac{1}{(2\pi)^3T}\prod_{j=1}^{4}\frac{\sin(Tv_j/2)}{\sin(v_j/2)} \tag{5.30}$$

(see, e.g., Eq. (6.6) in [5]).

Denote in Eq. (5.29), for each $k\geq 1$, and $u_i\in[-\pi,\pi]$, $i=1,2,3$, $\theta\in\Theta$,

$$G_{k1,\theta}(u_1,u_2,u_3)=\int_{-\pi}^{\pi}\left\langle\mathcal{F}_{2u_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta}(\psi_k),\mathcal{F}_{2u_2-\omega,\theta_0}\mathcal{W}_{-(2u_1+4u_3+\omega),\theta}(\psi_k)\right\rangle_{\widetilde{H}}d\omega$$
$$G_{k2,\theta}(u_1,u_2,u_3)=\int_{-\pi}^{\pi}\left\langle\mathcal{F}_{2\widetilde{u}_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta}(\psi_k),\mathcal{F}_{2\widetilde{u}_2-\omega,\theta_0}\mathcal{W}_{4\widetilde{u}_3-2\widetilde{u}_1-\omega,\theta}(\psi_k)\right\rangle_{\widetilde{H}}d\omega$$
$$\sum_{k\geq 1}G_{k1,\theta}(u_1,u_2,u_3)=\int_{-\pi}^{\pi}\left\langle\mathcal{F}_{2u_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2u_2-\omega,\theta_0}\mathcal{W}_{-(2u_1+4u_3+\omega),\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega$$
$$\sum_{k\geq 1}G_{k2,\theta}(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)=\int_{-\pi}^{\pi}\left\langle\mathcal{F}_{2\widetilde{u}_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2\widetilde{u}_2-\omega,\theta_0}\mathcal{W}_{4\widetilde{u}_3-2\widetilde{u}_1-\omega,\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega. \tag{5.31}$$

From Eqs. (5.24) and (5.25), for each $\theta\in\Theta$, considering $\gamma=\beta-1>0$,

$$\left\|\mathcal{F}_{\xi}\mathcal{F}_{\omega}\mathcal{W}_{\widetilde{\xi},\theta}\mathcal{W}_{\widetilde{\omega},\theta}\right\|_{\mathcal{L}(\widetilde{H})}$$
$$\leq M^2\left[\pi^{2(1-l(\theta))}\right]\left[[\ln(M)]^2\pi^{2\gamma}M_{\widetilde{W}}^2+\left[L(\theta)\ln(\pi)\pi^{\gamma}M_{\widetilde{W}}\right]^2\right.$$
$$\left.+\left\|\ln\left(\sigma_\theta^2\right)\right\|_{\mathcal{L}(\widetilde{H})}^2\left(\pi^{\gamma}M_{\widetilde{W}}\right)^2\right],\quad\forall\xi,\omega,\widetilde{\omega},\widetilde{\xi}\in[-\pi,\pi]. \tag{5.32}$$

Thus, we can apply Bounded Convergence Theorem to obtain, for each $k\geq 1$,

$$\lim_{u_i\to 0,\ i=1,2,3}G_{k1,\theta}(u_1,u_2,u_3)=\int_{-\pi}^{\pi}\lim_{u_i\to 0,\ i=1,2,3}\mathcal{F}_{2u_1+\omega,\theta_0}\mathcal{F}_{2u_2-\omega,\theta_0}$$
$$\times\mathcal{W}^{\star}_{-(2u_1+4u_3+\omega),\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega$$
$$=G_{k1,\theta}(0,0,0)$$
$$=\lim_{\widetilde{u}_i\to 0,\ i=1,2,3}G_{k2,\theta}(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)=\int_{-\pi}^{\pi}\lim_{\widetilde{u}_i\to 0,\ i=1,2,3}\mathcal{F}_{2\widetilde{u}_1+\omega,\theta_0}\mathcal{F}_{2\widetilde{u}_2-\omega,\theta_0}$$
$$\times\mathcal{W}^{\star}_{4\widetilde{u}_3-2\widetilde{u}_1-\omega,\theta}\mathcal{W}_{\omega,\theta}(\psi_k)(\psi_k)d\omega$$
$$=G_{k2,\theta}(0,0,0), \tag{5.33}$$

which means that $G_{ki}$, $i = 1, 2$, are continuous at zero, and uniform convergence holds in the limits of their convolutions with multidimensional Féjer kernel. Particularly,

$$\lim_{T\to\infty}\int_{[-\pi,\pi]^3}\Phi_{4T}^4(v_1, v_2, v_3)G_{ki,\theta}(v_1, v_2, v_3)dv_1dv_2dv_3 = G_{ki,\theta}(0, 0, 0), \quad (5.34)$$

for each $k \geq 1$, $i = 1, 2$, and $\theta \in \Theta$.

Furthermore, the absolute integrability of the functions

$$\mathcal{G}_1(u_1, u_2, u_3) = \sum_{k\geq 1} G_{k1,\theta}(u_1, u_2, u_3),$$
$$\mathcal{G}_2(u_1, u_2, u_3) = \sum_{k\geq 1} G_{k2,\theta}(u_1, u_2, u_3)$$

over $[-\pi, \pi]^3$ holds. Specifically, under **Assumptions I–IV**, and Eq. (5.20), keeping in mind Eqs. (5.24) and (5.25), we obtain

$$\begin{aligned}
&\int_{[-\pi,\pi]^3}|\mathcal{G}_1(u_1, u_2, u_3)|\prod_{i=1}^{3}du_i\\
&\leq \int_{[-\pi,\pi]^3}\sum_{k\geq 1}\left|G_{k1,\theta}(u_1, u_2, u_3)\right|\prod_{i=1}^{3}du_i\\
&= \int_{[-\pi,\pi]^3}\sum_{k\geq 1}\int_{\Lambda}\int_{-\pi}^{\pi}\frac{M_{2u_1+\omega,\mathcal{F}}(\lambda)}{|2u_1+\omega|^{\alpha(\lambda,\theta_0)}}\frac{M_{2u_2-\omega,\mathcal{F}}(\lambda)}{|2u_2-\omega|^{\alpha(\lambda,\theta_0)}}\\
&\quad\times\Big|\ln\left(M_{-(2u_1+4u_3+\omega)}(\lambda)\right)-\ln\left(\Sigma_{\theta}^2(\lambda)\right)\\
&\quad-\alpha(\lambda,\theta)\ln\left(|-(2u_1+4u_3+\omega)|\right)\Big|\,\widetilde{W}(\lambda)|-(2u_1+4u_3+\omega)|^{\beta}\\
&\quad\times\Big|\ln\left(M_{\omega}(\lambda)\right)-\ln\left(\Sigma_{\theta}^2(\lambda)\right)\\
&\quad-\alpha(\lambda,\theta)\ln\left(|\omega|\right)\Big|\,\widetilde{W}(\lambda)|\omega|^{\beta}d\omega d\left\langle E_{\lambda}(\psi_k),\psi_k\right\rangle_{\widetilde{H}}\prod_{i=1}^{3}du_i\\
&\leq [\mathcal{H}(\theta)]^2\pi^3\int_{-\pi}^{\pi}\|\mathcal{F}_{\omega}\|^2_{\mathcal{S}(\widetilde{H})}d\omega < \infty. \quad (5.35)
\end{aligned}$$

Similarly, we can prove that $\mathcal{G}_2(u_1, u_2, u_3) \in L^1([-\pi, \pi]^3)$. Thus, the following limits are obtained from the convolution of functions $\mathcal{G}_1(u_1, u_2, u_3)$, and $\mathcal{G}_2(u_1, u_2, u_3)$ with Féjer kernel in (5.29):

$$\begin{aligned}
&\lim_{T\to\infty}\frac{\mathcal{K}\pi}{T}\int_{[-\pi,\pi]^4}\Phi_{4T}^4(u_1, u_2, u_3)\\
&\quad\times\left\langle\mathcal{F}_{2u_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2u_2-\omega,\theta_0}\mathcal{W}_{-(2u_1+4u_3+\omega),\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega du_1du_2du_3
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{T\to\infty}\frac{\mathcal{K}\pi}{T}\int_{[-\pi,\pi]^4}\Phi_{4T}^4(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3)\\
&\times\left\langle \mathcal{F}_{2\widetilde{u}_1+\omega,\theta_0}\mathcal{W}_{\omega,\theta},\mathcal{F}_{2\widetilde{u}_2-\omega,\theta_0}\mathcal{W}_{4\widetilde{u}_3-2\widetilde{u}_1-\omega,\theta}\right\rangle_{\mathcal{S}(\widetilde{H})}d\omega d\widetilde{u}_1 d\widetilde{u}_2 d\widetilde{u}_3\\
=&\lim_{T\to\infty}\frac{\mathcal{K}\pi}{T}\left[\mathcal{G}_1(0,0,0)+\mathcal{G}_2(0,0,0)\right]. \qquad (5.36)
\end{aligned}
$$

From Eqs. (5.29)–(5.36), as $T\to\infty$,

$$
E\left\|\int_{-\pi}^{\pi}\left[p_{\omega}^{(T)}-\mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})}^2=\mathcal{O}\left(\frac{1}{T}\right).
$$

Applying Jensen's inequality,

$$
\begin{aligned}
&E\left\|\int_{-\pi}^{\pi}\left[p_{\omega}^{(T)}-\mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})}\\
&\leq\sqrt{E\left\|\int_{-\pi}^{\pi}\left[p_{\omega}^{(T)}-\mathcal{F}_{\omega,\theta_0}^{(T)}\right]\mathcal{W}_{\omega,\theta}d\omega\right\|_{\mathcal{S}(\widetilde{H})}^2}\to 0,\quad T\to\infty. \qquad (5.37)
\end{aligned}
$$

From (5.28) and (5.37), applying triangle inequality, Eq. (5.21) holds. In particular,

$$
\left\|U_{T,\theta}-U_{\theta}\right\|_{\mathcal{S}(\widetilde{H})}\to_P 0,\quad T\to\infty,\quad \forall\theta\in\Theta. \qquad (5.38)
$$

Therefore, for each $\theta\in\Theta$, as $T\to\infty$,

$$
\begin{aligned}
&\left\|U_{T,\theta}-U_{T,\theta_0}-\mathcal{K}(\theta_0,\theta)\right\|_{\mathcal{S}(\widetilde{H})}\\
&=\left[\sum_{k,l\geq 1}\left|[U_{T,\theta}-U_{T,\theta_0}](\psi_k)(\psi_l)-[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_l)\right|^2\right]^{1/2}\to_P 0,
\end{aligned}
\qquad (5.39)
$$

implying that, as $T\to\infty$,

$$
\sup_{k\geq 1}\left|[U_{T,\theta}-U_{T,\theta_0}](\psi_k)(\psi_k)-[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k)\right|\to_P 0. \qquad (5.40)
$$

From the reverse triangle inequality, denoting

$$
\begin{aligned}
L_T(\theta)&=\sup_{k\geq 1}\left|[U_{T,\theta}-U_{T,\theta_0}](\psi_k)(\psi_k)\right|\\
\text{and}\ \ \mathcal{L}(\theta)&=\sup_{k\geq 1}\left|[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k)\right|,
\end{aligned}
$$

we have

$$
L_T(\theta)\to_P\mathcal{L}(\theta),\quad T\to\infty,\quad\forall\theta\in\Theta. \qquad (5.41)
$$

From Eqs. (5.16)–(5.18),

$$\mathcal{L}(\theta) = \sup_{k\geq 1}[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k) > 0, \quad \theta \neq \theta_0,$$
$$\theta_0 = \arg\min_{\theta\in\Theta}\mathcal{L}(\theta) = \arg\min_{\theta\in\Theta}\sup_{k\geq 1}[\mathcal{K}(\theta_0,\theta)](\psi_k)(\psi_k), \tag{5.42}$$

for any orthonormal basis $\{\psi_k,\ k\geq 1\}$ of $\widetilde{H}$.

To prove the consistency of the estimator $\widehat{\theta}_T$ in (5.19), we first show that the convergence (5.41) holds uniformly in $\theta\in\Theta$. Specifically, for any $\theta_1,\theta_2\in\Theta$, from Eq. (5.9), considering the triangle inequality, and the fact that $p_\omega^{(T)}$ and $\mathcal{W}_\omega$ are non-negative operators for every $\omega\in[-\pi,\pi]$, we obtain, for each $k\geq 1$,

$$\begin{aligned}
&\left|U_{T,\theta_1} - U_{T,\theta_2}(\psi_k)(\psi_k)\right| \\
&\leq \int_{-\pi}^{\pi}\left|p_\omega^{(T)}\ln\left(\Upsilon_{\omega,\theta_2}\Upsilon_{\omega,\theta_1}^{-1}\right)\mathcal{W}_\omega(\psi_k)(\psi_k)\right|d\omega \\
&= \int_{-\pi}^{\pi}\left|\ln\left(\sigma_{\theta_1}^2[\sigma_{\theta_2}^2]^{-1}\right) + \left(\mathcal{A}_{\theta_1}-\mathcal{A}_{\theta_2}\right)\ln(|\omega|)\right| \\
&\quad\times\left|p_\omega^{(T)}\mathcal{W}_\omega(\psi_k)(\psi_k)\right|d\omega \\
&\leq \left\|\ln\left(\sigma_{\theta_1}^2[\sigma_{\theta_2}^2]^{-1}\right)\right\|_{\mathcal{L}(\widetilde{H})}\int_{-\pi}^{\pi}p_\omega^{(T)}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega \\
&\quad+\left\|\mathcal{A}_{\theta_1}-\mathcal{A}_{\theta_2}\right\|_{\mathcal{L}(\widetilde{H})}\int_{-\pi}^{\pi}|\ln(|\omega|)|\,p_\omega^{(T)}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega.
\end{aligned}\tag{5.43}$$

From (5.43), to prove the convergence (5.41) holds uniformly in $\theta\in\Theta$, we only need to show that, for any $k\geq 1$,

$$\int_{-\pi}^{\pi}p_\omega^{(T)}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega = \mathcal{O}_P(1), \quad T\to\infty \tag{5.44}$$
$$\int_{-\pi}^{\pi}|\ln(|\omega|)|\,p_\omega^{(T)}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega = \mathcal{O}_P(1), \quad T\to\infty \tag{5.45}$$

(see Theorems 21.9 and 21.10 in [13]). Note that, for $k\geq 1$,

$$\sigma_{\theta_0}^2(\psi_k)(\psi_k) = \int_{-\pi}^{\pi}\mathcal{F}_{\omega,\theta_0}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega \leq \|\sigma_{\theta_0}^2\|_{\mathcal{L}(\widetilde{H})} < \infty \tag{5.46}$$

$$\begin{aligned}
&\int_{-\pi}^{\pi}|\ln(|\omega|)|\,\mathcal{F}_{\omega,\theta_0}\mathcal{W}_\omega(\psi_k)(\psi_k)d\omega \leq 2\pi\sup_{(\omega,\lambda)\in[-\pi,\pi]\times\Lambda}|\ln(|\omega|)|/|\omega|^{\alpha(\lambda,\theta_0)-\beta} \\
&\quad\times\sup_{\omega\in[-\pi,\pi]}\|\widetilde{W}\mathcal{M}_{\omega,\mathcal{F}}\|_{\mathcal{L}(\widetilde{H})} < \infty,
\end{aligned}\tag{5.47}$$

where, for $\beta > 1$,

$$\mathcal{C} = 2\pi\sup_{(\omega,\lambda)\in[-\pi,\pi]\times\Lambda}|\ln(|\omega|)|/|\omega|^{\alpha(\lambda,\theta_0)-\beta} < \infty.$$

From Theorem 1, as $T \to \infty$,

$$\left\| \int_{-\pi}^{\pi} \left[ E\left[ p_{\omega}^{(T)} \right] - \mathcal{F}_{\omega,\theta_0} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})} \to 0 \tag{5.48}$$

$$\left\| \int_{-\pi}^{\pi} |\ln(|\omega|)| \left[ E\left[ p_{\omega}^{(T)} \right] - \mathcal{F}_{\omega,\theta_0} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})} \to 0. \tag{5.49}$$

In a similar way to Eqs. (5.29)–(5.38), it can also be proved that

$$E \left\| \int_{-\pi}^{\pi} \left[ p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0}^{(T)} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})}^{2} \to 0, \ T \to \infty \tag{5.50}$$

$$E \left\| \int_{-\pi}^{\pi} |\ln(|\omega|)| \left[ p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0}^{(T)} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})}^{2} \to 0, \ T \to \infty. \tag{5.51}$$

From Eqs. (5.46)–(5.51), as $T \to \infty$,

$$\left\| \int_{-\pi}^{\pi} \left[ p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})} \to_P 0 \tag{5.52}$$

$$\left\| \int_{-\pi}^{\pi} |\ln(|\omega|)| \left[ p_{\omega}^{(T)} - \mathcal{F}_{\omega,\theta_0} \right] \mathcal{W}_{\omega} d\omega \right\|_{\mathcal{S}(\widetilde{H})} \to_P 0. \tag{5.53}$$

From (5.52) and (5.53), Eqs. (5.44) and (5.45) are satisfied uniformly in $k \geq 1$. Thus, (5.41) holds uniformly in $\theta \in \Theta$.

To prove $\widehat{\theta}_T$ is weakly consistent, consider that $\widehat{\theta}_T$ does not converge in probability to $\theta_0$. Hence, there exists a subsequence $\{\widehat{\theta}_{T_m}, \ m \in \mathbb{N}\}$ such that $\widehat{\theta}_{T_m} \to_P \theta' \neq \theta_0$, as $T_m \to \infty$, when $m \to \infty$. From (5.42), for $\tau > 0$ satisfying $0 < \nu < \mathcal{L}(\theta') - \tau$, for certain $\nu > 0$, applying uniform convergence in $\theta \in \Theta$, in Eq. (5.41), there exists $m_0$ such that for $m \geq m_0$,

$$P\left[ \inf_{l \geq m} L_{T_l}(\widehat{\theta}_{T_l}) \geq \mathcal{L}(\theta') - \tau > \nu > 0 \right] \geq p_0 > 1/2. \tag{5.54}$$

From Eqs. (5.39), (5.40) and (5.42), for $T$ sufficiently large,

$$U_{T,\theta} - U_{T,\theta_0}(\psi_k)(\psi_k) \geq 0, \quad \forall k \geq 1.$$

Then, from definition of the estimator $\widehat{\theta}_T$ in (5.19), and uniform convergence in probability in (5.41), that also holds in the $\mathcal{S}(\widetilde{H})$ norm (see Eqs. (5.52)–(5.53)), there exists $m_0^{\star}$ such that for $m \geq m_0^{\star}$,

$$P\left[ \sup_{l \geq m} L_{T_l}(\widehat{\theta}_{T_l}) \leq \inf_{\theta \in \Theta} \mathcal{L}(\theta) = \mathcal{L}(\theta_0) = 0 \right] \geq p_0 > 1/2, \tag{5.55}$$

which, in particular, implies

$$P\left[\inf_{l\geq m} L_{T_l}(\widehat{\theta}_{T_l}) \leq \inf_{\theta\in\Theta} \mathcal{L}(\theta) = \mathcal{L}(\theta_0) = 0\right] \geq p_0 > 1/2. \tag{5.56}$$

For $m \geq \max\{m_0, m_0^{\star}\}$, Eqs. (5.54)–(5.56) lead to a contradiction. Thus, $\widehat{\theta}_T \to_P \theta_0$, as $T \to \infty$. □

***Remark 8*** The multifractionally integrated functional autoregressive moving averages process family introduced in Section 3.3 satisfies the conditions assumed in Theorem 2, for a suitable choice of the polynomial sequence $\left\{\Phi_{p,l},\ \Psi_{q,l},\ l \geq 1\right\}$.

## 6 Final comments

The spectral analysis of SRD functional time series has been currently achieved in several papers. Particularly, in Introduction, we have referred to the pioneer contribution in [31]. This paper constitutes a first attempt in the spectral analysis of stationary functional time series beyond the SRD condition. Specifically, this paper applies spectral theory of self-adjoint operators on a separable Hilbert space to characterize LRD in functional time series in the spectral domain, under **Assumptions I–IV** (see Proposition 1). As special cases, multifractionally integrated functional ARMA processes are considered (see Sect. 3.3). Their tapered continuous version in the spectral domain is also analyzed in Sect. 3.4. This second example allows the implementation of parametric estimation techniques in the functional spectral domain, from the discrete sampling in time of the solution to fractional and multifractional pseudodifferential models introduced in [2, 3, 22]. Our main results, Theorems 1 and 2, respectively provide the convergence to zero in $\mathcal{S}(\widetilde{H})$ norm of the bias of the integrated periodogram operator, and the weak consistent estimation of the LRD operator, in a parametric framework in the spectral domain. Note that Theorem 1 holds beyond the linear and Gaussian case, under our LRD setting, while Theorem 2 is proved under a LRD Gaussian scenario.

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## Declaration

**Conflict of interest** The author declares to have no conflict of interest.

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