EXPLORING NEW SOLUTIONS TO TINGLEY'S PROBLEM FOR FUNCTION ALGEBRAS

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ABSTRACT. In this note we present two new positive answers to Tingley's problem in certain subspaces of function algebras. In the first result we prove that every surjective isometry between the unit spheres, S(A) and S(B), of two uniformly closed function algebras A and B on locally compact Hausdorff spaces can be extended to a surjective real linear isometry from A onto B. In a second goal we study surjective isometries between the unit spheres of two abelian JB*-triples represented as spaces of continuous functions of the form

 $C_0^{\mathbb{T}}(X) := \{ a \in C_0(X) : a(\lambda t) = \lambda a(t) \text{ for every } (\lambda, t) \in \mathbb{T} \times X \},\$

where X is a (locally compact Hausdorff) principal \mathbb{T} -bundle. We establish that every surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ admits an extension to a surjective real linear isometry between these two abelian JB^{*}-triples.

1. Preliminaries

The problem of extending a surjective isometry between the unit spheres of two Banach spaces –named *Tingley's problem* after the contribution of D. Tingley in [45]– is nowadays a trending topic in functional analysis (see a representative sample in the references [6, 14, 15, 19, 20, 21, 22, 23, 34, 35, 38, 9] and the surveys [47, 37]). This isometric extension problem remains open for Banach spaces of dimension bigger than or equal to 3 though. In fact, it has not been until recently that a complete positive solution for 2-dimensional Banach spaces was obtained by T. Banakh in [2], a result culminating a tour-de-force by several researchers (cf. [1, 3, 6]).

In the last years, a growing interest on Tingley's problem for surjective isometries between the unit spheres of certain function algebras has attracted different specialists to approach this problem. The pioneering paper by R. Wang [46] inspired many subsequent results. O. Hatori, S. Oi and R.S. Togashi proved that each surjective isometry between the unit spheres of two uniform algebras can be always extended to a surjective real linear isometry between the uniform algebras (cf. [29]). We recall

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that a uniform algebra is a closed subalgebra of C(K) which contains constants and separates the points of K, where the latter is a compact Hausdorff space and C(K)denotes the Banach algebra of all complex-valued continuous functions on K. This conclusion was improved by O. Hatori by showing that each uniform algebra \mathcal{A} satisfies the complex Mazur-Ulam property, that is, every surjective isometry from its unit sphere onto the unit sphere of another Banach space E extends to a surjective real linear isometry from \mathcal{A} onto E (see [27, Theorem 4.5]).

This note is aimed to present our recent advances on Tingley's problem for some Banach spaces which are representable as certain function spaces. More concretely, in section 2 we study Tingley's problem in the case of a surjective isometry between the unit spheres of two uniformly closed function algebras. Note that uniformly closed function algebras constitute a strictly wider class than that given by uniform algebras. Indeed, we begin with a locally compact Hausdorff space X. A uniformly closed function algebra A on X is a uniformly closed and strongly separating (i.e. for each $x \in X$ and $y, z \in X$ with $y \neq z$, there exist $f, g \in A$ such that $f(x) \neq 0$ and $g(y) \neq g(z)$) subalgebra of the algebra, $C_0(X)$, of all continuous complex-valued functions on X vanishing at infinity. We can obviously regard A as a subalgebra of $C_0(X \cup \{\infty\})$, where $X \cup \{\infty\}$ denotes the one-point compactification of X. However, it is worth observing that, under such an identification, A never contains the constant functions, so, it is not a uniform algebra.

The main conclusion of section 2 proves that surjective isometry $\Delta: S(A) \to S(B)$ between the unit spheres of two uniformly closed function algebras A and B, extends to a surjective real linear isometry $T: A \to B$ (see Theorem 2.1). Our arguments are based on an appropriate use of the Choquet boundary of each uniformly closed function algebra, the existence of Urysohn's lemma type properties for this Choquet boundary (as in [39, 40, 12, 33]) and a good description of the elements in the image of Δ at points in the Choquet boundary. The proof of the already mentioned result by Hatori, Oi and Togashi in [29] is inspired by some of Wang's original tools in [46]. In this manuscript we apply similar techniques, however, the arguments here provide a different point of view, and are not mere extensions to the case of uniformly closed function algebras as non-unital versions of uniform algebras.

The third section of this note is focused on the study of Tingley's problem for a surjective isometry between the unit spheres of two abelian JB*-triples. As it is well-known, and explained in section 3, JB*-triples are precisely those complex Banach spaces whose open unit ball is a bounded symmetric domain (cf. [31]). A JBW*-triple is a JB*-triple which is also a dual Banach space. It has been recently shown (cf. [4, 30]) that every surjective isometry from the unit sphere of a JBW*triple onto the unit sphere of another Banach space extends to a surjective real linear isometry between the spaces. Few or nothing is known for general JB*-triples. The elements in the subclass of abelian JB^{*}-triples can be identified, thanks to a Gelfand representation theory, with subspaces of continuous functions. Indeed, let X be a principal T-bundle (i.e. a subset of a Hausdorff locally convex complex space such that $0 \notin X, X \cup \{0\}$ is compact, and $\mathbb{T}X \subseteq X$, where $\mathbb{T} = S(\mathbb{C})$). When X is regarded as a locally compact Hausdorff space, the closed subspace of $C_0(X)$ defined by

$$C_0^{\mathbb{T}}(X) := \{ a \in C_0(X) : a(\lambda t) = \lambda a(t) \text{ for every } (\lambda, t) \in \mathbb{T} \times X \},\$$

is not, in general, a subalgebra of $C_0(X)$ but it is closed for the triple product $\{a, b, c\} = a\overline{b}c$ $(a, b, c \in C_0^{\mathbb{T}}(X))$. The Gelfand representation theory affirms that each abelian JB*-triple is isometrically isomorphic to some $C_0^{\mathbb{T}}(X)$ for a suitable principal \mathbb{T} -bundle X (see [31, Corollary 1.11]). These spaces are also related to Lindenstrauss spaces (cf. [36, Theorem 12]).

The main conclusion in section 3 establishes that each surjective isometry Δ : $S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$, with X and Y being two principal T-bundles, admits an extension to a surjective real linear isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ (see Theorem 3.6). This statement is complemented with Lemma 3.2 where it is shown that for each surjective real linear isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ there exist a T-invariant clopen subset $D \subseteq X$ and a T-equivariant homeomorphism $\phi : Y \to X$ satisfying

$$T(a)(s) = a(\phi(s))$$
, for all $a \in C_0^{\mathbb{T}}(X)$ and for all $s \in \phi^{-1}(D)$,

and

 $T(a)(s) = \overline{a(\phi(s))}$, for all $a \in C_0^{\mathbb{T}}(X)$ and for all $s \in \phi^{-1}(X \setminus D)$.

Tingley's problem for surjective isometries between the unit spheres of function spaces deserves its own attention, and a self-contained treatment. In a forthcoming paper we shall explore the Mazur–Ulam property for the function spaces studied in this note.

2. TINGLEY'S PROBLEM FOR UNIFORMLY CLOSED FUNCTION ALGEBRAS

Let X be a locally compact Hausdorff space. Along this note we denote by $C_0(X)$ the set of all continuous complex-valued functions f on X, which vanish at infinity in the usual sense: for each $\varepsilon > 0$ the set $\{x \in X : |f(x)| \ge \varepsilon\}$ is a compact subset of X. Then $C_0(X)$ is a commutative Banach algebra under pointwise operations and the supremum norm $||f|| = \sup_{x \in X} |f(x)|$ $(f \in C_0(X))$. A subset B of $C_0(X)$ is said to be strongly separating, if for each $x \in X$ and $y, z \in X$ with $y \neq z$, there exist $f, g \in B$ such that $f(x) \neq 0$ and $g(y) \neq g(z)$. A uniformly closed function algebra A on X is a uniformly closed and strongly separating subalgebra of $C_0(X)$.

For each function $f \in A$ the symbol $\operatorname{Ran}(f)$ will stand for the range of f. We set $\operatorname{Ran}_{\pi}(f) = \{z \in \operatorname{Ran}(f) : |z| = ||f||\}$ $(f \in A)$. A peaking function g for A is a function of A with $\operatorname{Ran}_{\pi}(g) = \{1\}$; that is, if $g \in A$ satisfies ||g|| = 1 and |g(x)| = 1

for $x \in X$, then g(x) = 1. A compact subset $\mathcal{P} \subset X$ is called a *peak set* of A if there exists a peaking function $f \in A$ for which $\mathcal{P} = \{x \in X : f(x) = 1\}$. A subset which coincides with an intersection of a family of peak sets of A is called a *weak peak set* of A. A *peak point* (respectively, a *weak peak point*) of A is an element $x \in X$ satisfying that $\{x\}$ is a peak set (respectively, a *weak peak set*) of A. The *Choquet boundary* or the *strong boundary* for A, denoted by Ch(A), is the set of all weak peak points of A. It is shown in [40, Theorem 2.1] (see also [39]) that Ch(A) is precisely the set of all $x \in X$ such that the evaluation functional at the point x, δ_x , is an extreme point of the unit ball of the dual space of A (cf. [12, Definition 2.3.7]). It is well known that Ch(A) is indeed a boundary (norming set) for A; furthermore, Ch(A) satisfies the following properties (see, for example, [33, Propositions 2.2 and 2.3]):

- (1) For each $f \in A$ there exists $x \in Ch(A)$ such that |f(x)| = ||f||;
- (2) For each $\varepsilon > 0$, $x \in Ch(A)$ and each open subset O in X with $x \in O$ there exists a peaking function $u \in A$ such that u(x) = 1 and $|u| < \varepsilon$ on $X \setminus O$.

The following is the main result of this section.

Theorem 2.1. Let S(A) and S(B) be unit spheres of two uniformly closed function algebras A and B, respectively. If $\Delta: S(A) \to S(B)$ is a surjective isometry, then there exists a surjective, real linear isometry $T: A \to B$ such that $T = \Delta$ on S(A).

Remark 1. Let $T: A \to B$ be a surjective real linear isometry. In [33, Theorem 1.1], such an isometry T was characterized as a weighted composition operator, more concretely, there exist a continuous function $\kappa : \operatorname{Ch}(B) \to \mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, a (possibly empty) clopen subset K of $\operatorname{Ch}(B)$, and a homeomorphism $\varphi : \operatorname{Ch}(B) \to \operatorname{Ch}(A)$ such that

$$T(f)(y) = \begin{cases} \kappa(y)f(\varphi(y)), & \text{for } y \in K, \\ \kappa(y)\overline{f(\varphi(y))}, & \text{for } y \in \operatorname{Ch}(B) \backslash K, \end{cases}$$

for all $f \in A$.

Notation. Under the previous assumptions, for each $f \in S(A)$, we write $|f|^{-1}(1)$ for the set $\{x \in X : |f(x)| = 1\}$, and we set

$$M_f = |f|^{-1}(1) \cap \operatorname{Ch}(A).$$

For $x \in Ch(A)$, we denote by P_x the set of all peaking functions f for A with f(x) = 1. Define $\lambda P_x = \{\lambda f : f \in P_x\}$ for each $\lambda \in \mathbb{T}$. In the same way, we define Q_y the set of all peaking functions u for B with u(y) = 1. Here we note that

$$M_f = \{ z \in Ch(A) : |f(z)| = 1 \} = \{ z \in Ch(A) : f(z) = \lambda \}$$

for all $f \in \lambda P_x$ and $\lambda \in \mathbb{T}$.

For each $\lambda \in \mathbb{T}$ and $x \in Ch(A)$, we define

$$\lambda V_x = \{ f \in S(A) : f(x) = \lambda \}.$$

We see that $\lambda P_x \subset \lambda V_x$. In the same way, we define $\mu W_y = \{u \in S(B) : u(y) = \mu\}$ for each $\mu \in \mathbb{T}$ and $y \in Ch(B)$.

Lemma 2.2. Let $f, g \in S(A)$ and $x_0 \in M_f$. If $f(x_0) \neq g(x_0)$, then there exists $h \in S(A)$ such that ||f - h|| = 2 > ||g - h||.

Proof. Note first that $|f(x_0)| = 1$, since $x_0 \in M_f$. Set $2\delta = |f(x_0) - g(x_0)|$, and then $\delta > 0$. Define the open neighborhood O of x_0 by $O = \{x \in X : |g(x) - g(x_0)| < \delta\}$. Since $x_0 \in M_f \subset Ch(A)$, there exists $u \in P_{x_0}$ such that $|u| < 2^{-1}$ on $X \setminus O$. We set $h = -f(x_0)u \in S(A)$. We have

$$2 = |2f(x_0)| = |f(x_0) - h(x_0)| \le ||f - h|| \le 2,$$

and thus ||f - h|| = 2.

Take an arbitrary $x \in X$. We shall prove that |g(x) - h(x)| < 2. If $x \in O$, then we observe that |g(x) - h(x)| < 2. Indeed, if |g(x) - h(x)| = 2, then g(x) = -h(x)and |h(x)| = 1, since $g, h \in S(A)$. This implies that |u(x)| = |h(x)| = 1. Since u is a peaking function for A, we obtain u(x) = 1, and hence $g(x) = -h(x) = f(x_0)$. Since $x \in O$, we get $2\delta = |f(x_0) - g(x_0)| = |g(x) - g(x_0)| < \delta$, a contradiction. We have proved that |g(x) - h(x)| < 2 for all $x \in O$. Suppose now that $x \in X \setminus O$. Then $|u(x)| < 2^{-1}$. It follows that

$$|g(x) - h(x)| \le |g(x)| + |f(x_0)u(x)| \le 1 + \frac{1}{2} < 2.$$

Hence, |g(x) - h(x)| < 2, and consequently, ||g - h|| < 2.

In the rest of this section, we assume that A and B are uniformly closed function algebras on locally compact Hausdorff spaces X and Y, respectively, and that $\Delta: S(A) \to S(B)$ is a surjective isometry with respect to the supremum norms.

Lemma 2.3. Let $f, g \in S(A)$. If f = g on M_f , then $\Delta(f) = \Delta(g)$ on $M_{\Delta(f)}$.

Proof. Arguing by contradiction we suppose the existence of $y_0 \in M_{\Delta(f)}$ such that $\Delta(f)(y_0) \neq \Delta(g)(y_0)$. Applying Lemma 2.2 to $\Delta(f), \Delta(g) \in S(B)$ and $y_0 \in M_{\Delta(f)}$, we can choose $h \in S(A)$ so that $\|\Delta(f) - \Delta(h)\| = 2 > \|\Delta(g) - \Delta(h)\|$, here we have used that Δ is surjective. Since Δ is an isometry, we have $\|f - h\| = 2 > \|g - h\|$. Recall that Ch(A) is a boundary for A, and thus there exists $x_0 \in Ch(A)$ with $|f(x_0) - h(x_0)| = 2$, and by the other condition $|g(x_0) - h(x_0)| < 2$. Since $f, h \in S(A)$, we get $|f(x_0)| = 1$, which implies $x_0 \in M_f$. Consequently, $f(x_0) \neq g(x_0)$ for $x_0 \in M_f$, which is impossible.

Lemma 2.4. Let $x \in Ch(A)$, $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$. If $f_j \in \lambda P_x$ for each $j \in \mathbb{N}$ with $1 \leq j \leq n$, then $g = n^{-1} \sum_{j=1}^n f_j \in A$ satisfies $g \in \lambda P_x$ with $M_g \subset \bigcap_{j=1}^n M_{f_j}$.

Proof. Since $f_j \in \lambda P_x$ for j = 1, 2, ..., n, then $f_j(x) = \lambda$ and $||f_j|| = 1$ for all $j \in \{1, 2, ..., n\}$. We have

$$n = |n\lambda| = \left|\sum_{j=1}^{n} f_j(x)\right| \le \sum_{j=1}^{n} |f_j(x)| \le \sum_{j=1}^{n} ||f_j|| = n.$$

Hence, $g = n^{-1} \sum_{j=1}^{n} f_j \in A$ satisfies $\overline{\lambda}g(x) = 1 = ||g||$.

We shall prove that $g \in \lambda P_x$. Suppose that $|\overline{\lambda}g(x')| = 1$ for $x' \in X$, and then $|\sum_{j=1}^n f_j(x')| = n$. Since $|f_j(x')| \leq 1$, it follows that $|\overline{\lambda}f_j(x')| = |f_j(x')| = 1$ for all $j \in \{1, 2, ..., n\}$, which implies that $\overline{\lambda}f_j(x') = 1$ because $\overline{\lambda}f_j \in P_x$, and thus $\overline{\lambda}g(x') = 1$. This shows that $\overline{\lambda}g \in P_x$ (consequently, $g \in \lambda P_x$) and $M_g \subset \bigcap_{j=1}^n M_{f_j}$.

The characterization of compactness in terms of the finite intersection property is employed in our next result.

Lemma 2.5. The intersection $\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)$ is non-empty for all λ in \mathbb{T} and x in Ch(Λ)

 $\operatorname{Ch}(A).$

Proof. First, we note that $|u|^{-1}(1)$ is a compact subset of Y for each $u \in \Delta(\lambda P_x)$. So, by the characterization of compactness in terms of the finite intersection property, it is enough to show that $\bigcap_{j=1}^{n} |u_j|^{-1}(1) \neq \emptyset$ for each $n \in \mathbb{N}$ and $u_j \in \Delta(\lambda P_x)$ with $j = 1, 2, \ldots, n$.

Let $n \in \mathbb{N}$ and $u_j \in \Delta(\lambda P_x)$ for j = 1, 2, ..., n. Choose $f_j \in \lambda P_x$ so that $u_j = \Delta(f_j)$, and set $g = n^{-1} \sum_{j=1}^n f_j \in A$. We see that $g \in \lambda P_x$ with $M_g \subset \bigcap_{j=1}^n M_{f_j}$ by Lemma 2.4. We shall prove that $M_g = \bigcap_{j=1}^n M_{f_j}$. Here, we recall that

(2.1)
$$M_f = \{z \in Ch(A) : |f(z)| = 1\} = \{z \in Ch(A) : f(z) = \lambda\}$$

for all $f \in \lambda P_x$. Let $x_0 \in \bigcap_{j=1}^n M_{f_j}$. Since $f_j \in \lambda P_x$, we have $f_j(x_0) = \lambda$ for all $j \in \{1, 2, \ldots, n\}$ by (2.1). It follows that $g(x_0) = n^{-1} \sum_{j=1}^n f_j(x_0) = \lambda$. Since $g \in \lambda P_x$, equality (2.1) shows that $x_0 \in M_g$, and consequently, $\bigcap_{j=1}^n M_{f_j} \subset M_g$. Therefore, we conclude that $M_g = \bigcap_{j=1}^n M_{f_j}$, as claimed.

For each $z \in M_g = \bigcap_{j=1}^n M_{f_j}$, we have $g(z) = \lambda = f_j(z)$, that is, $g = f_j$ on M_g for each j = 1, 2, ..., n. If we apply Lemma 2.3, we deduce that $\Delta(g) = \Delta(f_j) = u_j$ on $M_{\Delta(g)}$. Then $|u_j(\zeta)| = |\Delta(g)(\zeta)| = 1$ for each $\zeta \in M_{\Delta(g)}$, and consequently $\bigcap_{i=1}^n |u_j|^{-1}(1) \neq \emptyset$, as claimed. \Box We explore next the intersection of the non-empty set in the previous lemma with the Choquet boundary of B.

Lemma 2.6. The intersection $\operatorname{Ch}(B) \cap \left(\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)\right)$ is non-empty for each $\lambda \in \mathbb{T}$ and each $x \in \operatorname{Ch}(A)$.

Proof. Let $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. There exists $y_0 \in \bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)$ by Lemma 2.5. Take an arbitrary $u \in \Delta(\lambda P_x)$ (in particular, $|u(y_0)| = 1 = ||u||$). Define the function $\tilde{u} \in B$ by

$$\tilde{u}(y) = \left(\overline{u(y_0)}^2 u^2(y) + \overline{u(y_0)}u(y)\right)/2, \ (y \in Y).$$

We observe that $\tilde{u} \in Q_{y_0}$. Namely, $1 = \tilde{u}(y_0) \leq ||\tilde{u}|| \leq 1$, and thus $\tilde{u} \in W_{y_0}$. Suppose now that $|\tilde{u}(y)| = 1$ for some $y \in Y$, and then $|\overline{u(y_0)}u^2(y) + u(y)| = 2$. It follows that

$$2 \le |\overline{u(y_0)}u^2(y)| + |u(y)| \le 2$$

which shows that |u(y)| = 1. Hence $|\overline{u(y_0)}u(y)+1| = 2$, and consequently $\overline{u(y_0)}u(y) = 1$. This implies that $\tilde{u}(y) = 1$, and we have therefore proven that $\tilde{u} \in Q_{y_0}$. We see that $(\tilde{u})^{-1}(1)$ is a peak set for B with

$$(\tilde{u})^{-1}(1) = u^{-1}(u(y_0)) \subset |u|^{-1}(1).$$

By the arbitrariness of $u \in \Delta(\lambda P_x)$, we get $y_0 \in \bigcap_{u \in \Delta(\lambda P_x)}(\tilde{u})^{-1}(1)$. It is known that every non-empty weak peak set for B contains a weak peak point, that is, $\operatorname{Ch}(B) \cap \left(\bigcap_{u \in \Delta(\lambda P_x)}(\tilde{u})^{-1}(1)\right) \neq \emptyset$ (see, for example, [33, Proposition 2.1]). This shows that

$$\operatorname{Ch}(B) \cap \left(\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1) \right) \neq \emptyset.$$

In the next result we replace λP_x with λV_x .

Lemma 2.7. The intersection $\operatorname{Ch}(B) \cap \left(\bigcap_{v \in \Delta(\lambda V_x)} |v|^{-1}(1)\right)$ is non-empty for each $\lambda \in \mathbb{T}$ and each $x \in \operatorname{Ch}(A)$.

Proof. By Lemma 2.6, there exists $y \in Ch(B) \cap \left(\bigcap_{u \in \Delta(\lambda P_x)} |u|^{-1}(1)\right)$. Take an arbitrary $v \in \Delta(\lambda V_x)$. We shall prove that |v(y)| = 1. Let $f \in \lambda V_x$ be such that $\Delta(f) = v$, and then $f(x) = \lambda$ and ||f|| = 1. Define the function $\tilde{f} \in S(A)$ by

$$\tilde{f}(z) = (\overline{\lambda}^2 f^2(z) + \overline{\lambda} f(z))/2, \ (z \in X).$$

We see that $\tilde{f} \in P_x$ with

$$M_{\tilde{f}} = \{ z \in Ch(A) : |\tilde{f}(z)| = 1 \} = \{ z \in Ch(A) : f(z) = \lambda \}.$$

Recall that $M_{\tilde{f}} = \{z \in Ch(A) : \tilde{f}(z) = 1\}$, since $\tilde{f} \in P_x$. For each $z \in M_{\tilde{f}}$, we have $\lambda \tilde{f}(z) = \lambda = f(z)$, and thus $\lambda \tilde{f} = f$ on $M_{\tilde{f}} = M_{\lambda \tilde{f}}$. Lemma 2.3 shows that

$$\Delta(\lambda f) = \Delta(f) \quad \text{on } M_{\Delta(\lambda \tilde{f})}.$$

Since $\lambda \tilde{f} \in \lambda P_x$, we obtain $|\Delta(\lambda \tilde{f})(y)| = 1$, that is, $y \in M_{\Delta(\lambda \tilde{f})}$. It follows that $v(y) = \Delta(f)(y) = \Delta(\lambda \tilde{f})(y)$, and consequently, $|v(y)| = |\Delta(\lambda \tilde{f})(y)| = 1$. Hence $y \in |v|^{-1}(1)$. We conclude from the arbitrariness of $v \in \Delta(\lambda V_x)$ that $y \in Ch(B) \cap (\bigcap_{v \in \Delta(\lambda V_x)} |v|^{-1}(1))$.

We determine next the behaviour of Δ on sets of the form λP_x .

Lemma 2.8. For each $(\lambda, x) \in \mathbb{T} \times Ch(A)$, there exists a couple (μ, y) in $\mathbb{T} \times Ch(B)$ such that $\Delta(\lambda P_x) \subset \mu W_y$.

Proof. Let us fix $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. By Lemma 2.7, there exists $y \in Ch(B) \cap (\bigcap_{v \in \Delta(\lambda V_x)} |v|^{-1}(1))$. For each $f \in \lambda P_x$, we have $|\Delta(f)(y)| = 1$ by the choice of y. Since $f \in \lambda P_x \subset S(A)$, we obtain $||\Delta(f)|| = 1$. Hence, $\Delta(f) \in \mu W_y$ with $\mu = \Delta(f)(y) \in \mathbb{T}$.

Now, we prove that $\Delta(f)(y) = \Delta(g)(y)$ for all $f, g \in \lambda P_x$. Set $h = (f+g)/2 \in A$, and then $h \in \lambda P_x$ by Lemma 2.4. We observe that

$$M_h = h^{-1}(\lambda) \cap \operatorname{Ch}(A) = f^{-1}(\lambda) \cap g^{-1}(\lambda) \cap \operatorname{Ch}(A),$$

since $f, g, h \in \lambda P_x$, where $k^{-1}(\lambda) = \{z \in X : k(z) = \lambda\}$ for $k \in \lambda P_x$. Therefore, we have f = h = g on M_h . We derive from Lemma 2.3 that $\Delta(f) = \Delta(h) = \Delta(g)$ on $M_{\Delta(h)}$. Since $\Delta(h) \in \Delta(\lambda V_x)$, we get $|\Delta(h)(y)| = 1$ by the choice of y. Thus, $y \in M_{\Delta(h)}$, and consequently $\Delta(f)(y) = \Delta(g)(y)$.

The above arguments show that $\Delta(f) \in \mu W_y$ for all $f \in \lambda P_x$, where $\mu = \Delta(f)(y)$ is independent of the choice of $f \in \lambda P_x$. This shows that $\Delta(\lambda P_x) \subset \mu W_y$ for some $\mu \in \mathbb{T}$ and $y \in Ch(B)$.

Lemma 2.9. For each $(\lambda, x) \in \mathbb{T} \times Ch(A)$, there exists a couple (μ, y) in $\mathbb{T} \times Ch(B)$ such that $\Delta(\lambda V_x) \subset \mu W_y$.

Proof. Fix λ, x as in the statement. By Lemma 2.8, there exist $\mu \in \mathbb{T}$ and $y \in Ch(B)$ such that $\Delta(\lambda P_x) \subset \mu W_y$. Let $v \in \Delta(\lambda V_x)$. We shall prove that $v \in \mu W_y$. Let $f \in \lambda V_x$ be such that $\Delta(f) = v$. Define the function $\tilde{f} \in A$ by

$$\tilde{f}(z) = (\overline{\lambda}^2 f^2(z) + \overline{\lambda} f(z))/2, \quad (z \in X).$$

We see that $\tilde{f} \in P_x$ with

$$M_{\tilde{f}} = \{ z \in \operatorname{Ch}(A) : \tilde{f}(z) = 1 \} = f^{-1}(\lambda) \cap \operatorname{Ch}(A).$$

For each $z \in M_{\tilde{f}}$, we have $\lambda \tilde{f}(z) = \lambda = f(z)$, and hence $\lambda \tilde{f} = f$ on $M_{\tilde{f}} = M_{\lambda \tilde{f}}$. Lemma 2.3 shows that $\Delta(\lambda \tilde{f}) = \Delta(f)$ on $M_{\Delta(\lambda \tilde{f})}$. Since $\tilde{f} \in P_x$, we have $\Delta(\lambda \tilde{f}) \in \Delta(\lambda P_x) \subset \mu W_y$. Thus $\Delta(\lambda \tilde{f}) \in \mu W_y$, that is, $\Delta(\lambda \tilde{f})(y) = \mu$. This implies that $|\Delta(\lambda \tilde{f})(y)| = 1$, which yields $y \in M_{\Delta(\lambda \tilde{f})}$. Therefore, $v(y) = \Delta(f)(y) = \Delta(\lambda \tilde{f})(y) = \mu$, and consequently $v \in \mu W_y$. This shows that $\Delta(\lambda V_x) \subset \mu W_y$.

We shall discuss next the uniqueness of the couple (y, μ) in previous lemmata.

Lemma 2.10. If $\lambda V_x \subset \lambda' V_{x'}$ holds for some $\lambda, \lambda' \in \mathbb{T}$ and $x, x' \in Ch(A)$, then $\lambda = \lambda'$ and x = x'.

Proof. Suppose, on the contrary, that $x \neq x'$. There exists $f \in P_x \subset V_x$ such that |f(x')| < 1 (cf. the properties in page 4). Then $\lambda f \in \lambda V_x \setminus (\lambda' V_{x'})$, since $|\lambda f(x')| < 1$. This contradicts $\lambda V_x \subset \lambda' V_{x'}$. Hence, we obtain x = x', and thus $\lambda V_x \subset \lambda' V_x$ by the hypothesis. For each $g \in V_x$, we have $\lambda g \in \lambda' V_x$, which shows that $\lambda = \lambda g(x) = \lambda'$. We thus conclude that $\lambda = \lambda'$.

Lemma 2.11. For each $(\lambda, x) \in \mathbb{T} \times Ch(A)$, there exists a unique couple (μ, y) in $\mathbb{T} \times Ch(B)$ such that $\Delta(\lambda V_x) = \mu W_y$.

Proof. Let us fix $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. By Lemma 2.9 there exist $\mu \in \mathbb{T}$ and $y \in Ch(B)$ such that $\Delta(\lambda V_x) \subset \mu W_y$. Another application of Lemma 2.9, with $\mu \in \mathbb{T}$, $y \in Ch(B)$ and Δ^{-1} , shows the existence of $\lambda' \in \mathbb{T}$ and $x' \in Ch(A)$ such that $\Delta^{-1}(\mu W_y) \subset \lambda' V_{x'}$. Thus, we have $\Delta(\lambda V_x) \subset \mu W_y \subset \Delta(\lambda' V_{x'})$, and hence $\lambda V_x \subset \lambda' V_{x'}$. Therefore, we obtain $\lambda = \lambda'$ and x = x' by Lemma 2.10, which shows that $\Delta(\lambda V_x) = \mu W_y$.

Suppose that $\Delta(\lambda V_x) = \mu' W_{y'}$ for some $\mu' \in \mathbb{T}$ and $y' \in Ch(B)$. Then $\mu W_y = \Delta(\lambda V_x) = \mu' W_{y'}$, and hence $\mu W_y = \mu' W_{y'}$. Lemma 2.10 shows that $\mu = \mu'$ and y = y', which proves the uniqueness of $\mu \in \mathbb{T}$ and $y \in Ch(B)$.

We are now in a position to define the key functions describing the behaviour of Δ on sets of the form λV_x .

Definition 2.1. By Lemma 2.11, there exist well-defined maps $\alpha \colon \mathbb{T} \times Ch(A) \to \mathbb{T}$ and $\phi \colon \mathbb{T} \times Ch(A) \to Ch(B)$ with the following property:

$$\Delta(\lambda V_x) = \alpha(\lambda, x) W_{\phi(\lambda, x)} \qquad (\lambda \in \mathbb{T}, x \in Ch(A)).$$

Our next goal will consist in isolating the key properties of the just defined maps.

Lemma 2.12. For each $\mu, \mu' \in \mathbb{T}$ and $y, y' \in Ch(B)$ with $y \neq y'$, there exist $u \in \mu Q_y$ and $v \in \mu' Q_{y'}$ such that $||u - v|| < \sqrt{2}$. *Proof.* Choose disjoint open sets $O, O' \subset Y$ so that $y \in O$ and $y' \in O'$. There exist $u \in \mu Q_y$ and $v \in \mu' Q_{y'}$ such that |u| < 1/3 on $Y \setminus O$ and |v| < 1/3 on $Y \setminus O'$. For $z \in O$, we have $|u(z) - v(z)| \le 1 + 1/3 < \sqrt{2}$, since $O \cap O' = \emptyset$. For $z \in Y \setminus O$, we obtain $|u(z) - v(z)| \le 1/3 + 1 < \sqrt{2}$ by the choice of u. We thus conclude $||u - v|| < \sqrt{2}$, as is claimed.

Lemma 2.13. If $\lambda \in \mathbb{T}$ and $x \in Ch(A)$, then $\phi(\lambda, x) = \phi(-\lambda, x)$.

Proof. Let $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. We set $\mu = \alpha(\lambda, x)$, $\mu' = \alpha(-\lambda, x)$, $y = \phi(\lambda, x)$ and $y' = \phi(-\lambda, x)$. Then $\Delta(\lambda V_x) = \mu W_y$ and $\Delta((-\lambda)V_x) = \mu' W_{y'}$. Suppose, on the contrary, that $y \neq y'$. Lemma 2.12 assures the existence of $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$. By the choice of \tilde{u} and \tilde{v} , we see that $\Delta^{-1}(\tilde{u}) \in \Delta^{-1}(\mu Q_y) \subset$ $\Delta^{-1}(\mu W_y) = \lambda V_x$ and $\Delta^{-1}(\tilde{v}) \in \Delta^{-1}(\mu' W_{y'}) \subset (-\lambda) V_x$. Then $\Delta^{-1}(\tilde{u})(x) = \lambda$ and $\Delta^{-1}(\tilde{v})(x) = -\lambda$, and therefore

$$2 = |2\lambda| = |\Delta^{-1}(\tilde{u})(x) - \Delta^{-1}(\tilde{v})(x)| \le ||\Delta^{-1}(\tilde{u}) - \Delta^{-1}(\tilde{v})||$$

= $||\tilde{u} - \tilde{v}|| < \sqrt{2},$

which is a contradiction. Consequently, we have y = y', and hence $\phi(\lambda, x) = \phi(-\lambda, x)$.

Lemma 2.14. If $\lambda \in \mathbb{T}$ and $x \in Ch(A)$, then $\phi(\lambda, x) = \phi(1, x)$; hence, the point $\phi(\lambda, x)$ is independent of the choice of $\lambda \in \mathbb{T}$.

Proof. Let $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. Set $\mu = \alpha(\lambda, x)$, $\mu' = \alpha(1, x)$, $y = \phi(\lambda, x)$ and $y' = \phi(1, x)$. Then $\Delta(\lambda V_x) = \mu W_y$ and $\Delta(V_x) = \mu' W_{y'}$. We shall prove that y = y'. Suppose that $y \neq y'$. Under this assumption, there exist $\tilde{u} \in \mu Q_y$ and $\tilde{v} \in \mu' Q_{y'}$ such that $\|\tilde{u} - \tilde{v}\| < \sqrt{2}$ (cf. Lemma 2.12). By the choice of \tilde{u} and \tilde{v} , we obtain $\Delta^{-1}(\tilde{u}) \in \lambda V_x$ and $\Delta^{-1}(\tilde{v}) \in V_x$. Thus $\Delta^{-1}(\tilde{u})(x) = \lambda$ and $\Delta^{-1}(\tilde{v})(x) = 1$. If $\operatorname{Re} \lambda \leq 0$, then $|\lambda - 1| \geq \sqrt{2}$, which shows that

$$\begin{split} \sqrt{2} &\leq |\lambda - 1| = |\Delta^{-1}(\tilde{u})(x) - \Delta^{-1}(\tilde{v})(x)| \leq ||\Delta^{-1}(\tilde{u}) - \Delta^{-1}(\tilde{v})| \\ &= ||\tilde{u} - \tilde{v}|| < \sqrt{2}. \end{split}$$

We arrive at a contradiction, which yields y = y' if $\operatorname{Re} \lambda \leq 0$. Now we consider the case when $\operatorname{Re} \lambda > 0$. Note that $\phi(-\lambda, x) = \phi(\lambda, x) = y$ by Lemma 2.13. Hence, $\Delta((-\lambda)V_x) = \nu W_y$ for some $\nu \in \mathbb{T}$. Since $\operatorname{Re}(-\lambda) < 0$, the above arguments can be applied to $\Delta((-\lambda)V_x) = \nu W_y$ and $\Delta(V_x) = \mu' W_{y'}$ to deduce that y = y'. Then we get y = y' even if $\operatorname{Re} \lambda > 0$.

Definition 2.2. By Lemma 2.14, we may and do write $\phi(\lambda, x) = \phi(x)$. We will also write $\alpha(\lambda, x) = \alpha_x(\lambda)$ for each $\lambda \in \mathbb{T}$ and $x \in Ch(A)$. Then we obtain

(2.2)
$$\Delta(\lambda V_x) = \alpha_x(\lambda) W_{\phi(x)} \qquad (\lambda \in \mathbb{T}, \ x \in Ch(A)).$$

The arguments above can be applied to the surjective isometry Δ^{-1} from S(B) onto S(A). Then there exist two maps $\beta \colon \mathbb{T} \times Ch(B) \to \mathbb{T}$ and $\psi \colon Ch(B) \to Ch(A)$ such that

(2.3)
$$\Delta^{-1}(\mu W_y) = \beta_y(\mu) V_{\psi(y)} \qquad (\mu \in \mathbb{T}, \ y \in \operatorname{Ch}(B)),$$

where $\beta_y(\mu) = \beta(\mu, y)$ for each $\mu \in \mathbb{T}$ and $y \in Ch(B)$. We may regard α_x and β_y as maps from \mathbb{T} into itself for each $x \in Ch(A)$ and $y \in Ch(B)$.

Lemma 2.15. The maps $\alpha_x \colon \mathbb{T} \to \mathbb{T}$, for each $x \in Ch(A)$, and $\phi \colon Ch(A) \to Ch(B)$ are both bijective with $\alpha_x^{-1} = \beta_{\phi(x)}$ and $\phi^{-1} = \psi$.

Proof. Let $x \in Ch(A)$. We will prove that α_x and ϕ are injective. Take $\lambda \in \mathbb{T}$ arbitrarily. If we apply (2.3) with $\mu = \alpha_x(\lambda)$ and $y = \phi(x)$, then we get

$$\Delta^{-1}(\alpha_x(\lambda)W_{\phi(x)}) = \beta_{\phi(x)}(\alpha_x(\lambda))V_{\psi(\phi(x))}.$$

Combining the equality above with (2.2), we obtain

$$\lambda V_x = \Delta^{-1}(\alpha_x(\lambda)W_{\phi(x)}) = \beta_{\phi(x)}(\alpha_x(\lambda))V_{\psi(\phi(x))}$$

Lemma 2.10 shows that $\lambda = \beta_{\phi(x)}(\alpha_x(\lambda))$ and $x = \psi(\phi(x))$; since $\lambda \in \mathbb{T}$ is arbitrary, the first equality shows that α_x is injective. The second one shows that ϕ is injective, since $x \in Ch(A)$ is arbitrary.

Now we prove that α_x and ϕ are both surjective. Let $\mu \in \mathbb{T}$ and $y \in Ch(B)$. Applying (2.2) with $\lambda = \beta_y(\mu)$ and $x = \psi(y)$, we get $\Delta(\beta_y(\mu)V_{\psi(y)}) = \alpha_{\psi(y)}(\beta_y(\mu))W_{\phi(\psi(y))}$. The last equality, together with (2.3), shows that

$$\mu W_y = \alpha_{\psi(y)}(\beta_y(\mu))W_{\phi(\psi(y))}.$$

According to Lemma 2.10, we have

(2.4)
$$\mu = \alpha_{\psi(y)}(\beta_y(\mu))$$

and $y = \phi(\psi(y))$; since $y \in Ch(B)$ is arbitrary, the second equality shows that ϕ is surjective. Then there exists ϕ^{-1} : $Ch(B) \to Ch(A)$. We obtain $\phi(\phi^{-1}(y)) =$ $y = \phi(\psi(y))$, which yields $\phi^{-1}(y) = \psi(y)$. We conclude, from the arbitrariness of $y \in Ch(B)$, that $\phi^{-1} = \psi$. Since ψ is bijective with $\psi^{-1} = \phi$, for each $x \in Ch(A)$ there exists $y \in Ch(B)$ such that $x = \psi(y)$. By (2.4), $\mu = \alpha_{\psi(y)}(\beta_y(\mu)) = \alpha_x(\beta_{\phi(x)}(\mu))$ holds for all $\mu \in \mathbb{T}$. This implies that α_x is surjective for each $x \in Ch(B)$. There exists α_x^{-1} , and then $\alpha_x(\alpha_x^{-1}(\mu)) = \mu = \alpha_x(\beta_{\phi(x)}(\mu))$ for all $\mu \in \mathbb{T}$. This shows that $\alpha_x^{-1} = \beta_{\phi(x)}$ for each $x \in Ch(A)$.

Lemma 2.16. For each $x \in Ch(A)$, the map $\alpha_x \colon \mathbb{T} \to \mathbb{T}$ is a surjective isometry.

Proof. Let $x \in Ch(A)$ and $\lambda_1, \lambda_2 \in \mathbb{T}$. Note that $\Delta(\lambda f)(\phi(x)) = \alpha_x(\lambda)$ for all $\lambda \in \mathbb{T}$ and $f \in V_x$ by (2.2). For each $f \in V_x$, we have

$$\begin{aligned} |\alpha_x(\lambda_1) - \alpha_x(\lambda_2)| &= |\Delta(\lambda_1 f)(\phi(x)) - \Delta(\lambda_2 f)(\phi(x))| \\ &\leq ||\Delta(\lambda_1 f) - \Delta(\lambda_2 f)|| = ||\lambda_1 f - \lambda_2 f|| \\ &= |\lambda_1 - \lambda_2|. \end{aligned}$$

Hence, $|\alpha_x(\lambda_1) - \alpha_x(\lambda_2)| \leq |\lambda_1 - \lambda_2|$. By applying the same argument to Δ^{-1} we deduce that β_y also is a contractive mapping. Having in mind that $\alpha_x^{-1} = \beta_{\phi(x)}$ (cf. Lemma 2.15), we obtain that α_x and $\beta_{\phi(x)}$ are surjective isometries on \mathbb{T} .

Fix $x \in Ch(A)$. Since $\alpha_x : \mathbb{T} \to \mathbb{T}$ is a surjective isometry on the unit sphere of the complex plane, and Tingley's problem admits a positive solution in this case, α_x admits an extension to a surjective real linear isometry on \mathbb{C} , therefore one of the following statements hold:

(2.5)
$$\alpha_x(\lambda) = \alpha_x(1)\lambda \text{ for all } \lambda \in \mathbb{T}, \text{ or } \alpha_x(\lambda) = \alpha_x(1)\overline{\lambda} \text{ for all } \lambda \in \mathbb{T}.$$

One final technical result separates us from the main goal of this section.

Lemma 2.17. Let $f \in S(A)$ and $x_0 \in Ch(A)$ be such that $|f(x_0)| < 1$. We set $\lambda = f(x_0)/|f(x_0)|$ if $f(x_0) \neq 0$, and $\lambda = 1$ if $f(x_0) = 0$. For each r with 0 < r < 1, there exists $g_r \in V_{x_0}$ such that

$$rf + (1 - r|f(x_0)|)\lambda g_r \in \lambda V_{x_0}$$

Proof. Note first that $1 - |f(x_0)| > 0$. We set

$$F_0 = \left\{ x \in X : |f(x) - f(x_0)| \ge \frac{1 - |f(x_0)|}{2} \right\}, \text{ and}$$
$$F_m = \left\{ x \in X : \frac{1 - |f(x_0)|}{2^{m+1}} \le |f(x) - f(x_0)| \le \frac{1 - |f(x_0)|}{2^m} \right\}$$

for each $m \in \mathbb{N}$. We see that F_n is a closed subset of X with $x_0 \notin F_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $x_0 \in Ch(A)$, for each $n \in \mathbb{N} \cup \{0\}$ there exists $f_n \in P_{x_0}$ such that

(2.6)
$$|f_n| < \frac{1-r}{1-r|f(x_0)|}$$
 on F_n .

We set $g_r = f_0 \sum_{n=1}^{\infty} f_n/2^n$ (we note that the series converges in A). We observe that

$$1 = g_r(x_0) \le ||g_r|| \le ||f_0|| \sum_{n=1}^{\infty} \frac{||f_n||}{2^n} = 1,$$

and hence $g_r \in V_{x_0}$. Set $h_r = rf + (1 - r|f(x_0)|)\lambda g_r \in A$. We shall prove that $h_r \in \lambda V_{x_0}$. Since $g_r(x_0) = 1$ and $f(x_0) = |f(x_0)|\lambda$, we have $h_r(x_0) = \lambda$. Take $x \in X$

arbitrarily. To prove that $|h_r(x)| \leq 1$, we will consider three cases. If $x \in F_0$, then (2.6) shows that

$$|g_r(x)| \le |f_0(x)| \sum_{n=1}^{\infty} \frac{|f_n(x)|}{2^n} < \frac{1-r}{1-r|f(x_0)|}$$

We obtain

$$|h_r(x)| \le r|f(x)| + (1 - r|f(x_0)|)|\lambda g_r(x)| < r + (1 - r) = 1$$

Hence, $|h_r(x)| < 1$ if $x \in F_0$.

Suppose that $x \in F_m$ for some $m \in \mathbb{N}$. Then $|f(x) - f(x_0)| \le (1 - |f(x_0)|)/2^m$ by the choice of F_m . We get

$$|f(x)| \le |f(x_0)| + \frac{1 - |f(x_0)|}{2^m} = \left(1 - \frac{1}{2^m}\right)|f(x_0)| + \frac{1}{2^m}.$$

We derive from (2.6) that

$$|g_r(x)| \le |f_0(x)| \left(\frac{|f_m(x)|}{2^m} + \sum_{n \ne m} \frac{|f_n(x)|}{2^n} \right)$$

$$< \frac{1}{2^m} \frac{1-r}{1-r|f(x_0)|} + 1 - \frac{1}{2^m}.$$

It follows that

$$|(1-r|f(x_0)|)\lambda g_r(x)| < \frac{1-r}{2^m} + \left(1-\frac{1}{2^m}\right)(1-r|f(x_0)|).$$

We infer from these inequalities that

$$\begin{aligned} |h_r(x)| &\leq r|f(x)| + |(1 - r|f(x_0)|)\lambda g_r(x)| \\ &< r\left(1 - \frac{1}{2^m}\right)|f(x_0)| + \frac{r}{2^m} + \frac{1 - r}{2^m} + \left(1 - \frac{1}{2^m}\right)(1 - r|f(x_0)|) \\ &= 1, \end{aligned}$$

and hence, $|h_r(x)| < 1$ for $x \in \bigcup_{n=1}^{\infty} F_n$.

Now we consider the case in which $x \notin \bigcup_{n=0}^{\infty} F_n$. Then $x \in \bigcap_{n=0}^{\infty} (X \setminus F_n)$, which implies that $f(x) = f(x_0)$. We have

$$|h_r(x)| \le r|f(x_0)| + 1 - r|f(x_0)| = 1,$$

and we thus conclude that $|h(x)| \leq 1$ for all $x \in X$, and consequently $h_r \in \lambda V_{x_0}$. \Box

We have alrady gathered the tools to prove Theorem 2.1.

Proof of Theorem 2.1. Let $f \in S(A)$ and $y \in Ch(B)$. To simplify the notation we set $x = \psi(y)$ and $\lambda = f(x)/|f(x)| \in \mathbb{T}$ if $f(x) \neq 0$, and $\lambda = 1$ if f(x) = 0, where $\psi = \phi^{-1}$ as in Lemma 2.15. We first prove that $|\Delta(f)(y)| = |f(x)|$. If |f(x)| = 1, then $f \in \lambda V_x$ and thus

$$|\Delta(f)(y)| = |\Delta(f)(\phi(x))| = |\alpha_x(\lambda)| = 1 = |f(x)|$$

by (2.2). We need to consider the case when |f(x)| < 1. By Lemma 2.17, for each r with 0 < r < 1 there exists $g_r \in V_x$ such that $h_r = rf + (1 - r|f(x)|)\lambda g_r \in \lambda V_x$. We obtain

$$||h_r - f|| = ||(r - 1)f + (1 - r|f(x)|)\lambda g_r||$$

$$\leq (1 - r) + 1 - r|f(x)| = 2 - r - r|f(x)|.$$

Since $h_r \in \lambda V_x$, we have $\Delta(h_r)(y) = \Delta(h_r)(\phi(x)) = \alpha_x(\lambda)$ by (2.2). Therefore, we get

$$1 - |\Delta(f)(y)| = |\alpha_x(\lambda)| - |\Delta(f)(y)| \le |\alpha_x(\lambda) - \Delta(f)(y)|$$

= $|\Delta(h_r)(y) - \Delta(f)(y)|$
 $\le ||\Delta(h_r) - \Delta(f)|| = ||h_r - f||$
 $\le 2 - r - r|f(x)|.$

Since r with 0 < r < 1 is arbitrary, we get

(2.7)
$$1 - |\Delta(f)(y)| \le |\alpha_x(\lambda) - \Delta(f)(y)| \le 1 - |f(x)|,$$

which shows that $|f(x)| \leq |\Delta(f)(y)| = |\Delta(f)(\phi(x))|$. By similar arguments, applied to Δ^{-1} instead of Δ , we have $|u(y)| \leq |\Delta^{-1}(u)(\psi(y))|$ for all $u \in S(B)$. In particular, $|\Delta(f)(y)| \leq |\Delta^{-1}(\Delta(f))(\psi(y))| = |f(x)|$, and consequently

$$|\Delta(f)(y)| = |f(\psi(y))|$$
, for all $y \in Ch(B)$, $f \in A$.

We shall prove that

(2.8)
$$\Delta(f)(y) = \alpha_x(\lambda)|f(x)|.$$

Since $|\Delta(f)(y)| = |f(x)|$, we need to consider the case when $\Delta(f)(y) \neq 0$. It follows from (2.7) that

$$1 = |\alpha_x(\lambda)| \le |\alpha_x(\lambda) - \Delta(f)(y)| + |\Delta(f)(y)|$$

$$\le (1 - |f(x)|) + |f(x)| = 1,$$

which shows that

$$|\alpha_x(\lambda)| = |\alpha_x(\lambda) - \Delta(f)(y)| + |\Delta(f)(y)|.$$

By the equality condition for the triangle inequality, there exists $t \ge 0$ such that $\alpha_x(\lambda) - \Delta(f)(y) = t\Delta(f)(y)$. Hence, we have $\Delta(f)(y) = \alpha_x(\lambda)/(1+t)$. On the other hand,

$$|f(x)| = |\Delta(f)(y)| = \left|\frac{\alpha_x(\lambda)}{1+t}\right| = \frac{1}{1+t},$$

which yields $\Delta(f)(y) = \alpha_x(\lambda)|f(x)|$.

Now, having in mind (2.5), we define two subsets K_+ and K_- of Ch(A) by

$$K_{+} = \{ x \in Ch(A) : \alpha_{x}(\lambda) = \alpha_{x}(1)\lambda, \text{ for all } \lambda \}, \text{ and}$$
$$K_{-} = \{ x \in Ch(A) : \alpha_{x}(\lambda) = \alpha_{x}(1)\overline{\lambda} \text{ for all } \lambda \}.$$

We see that Ch(A) is the disjoint union of K_+ and K_- (cf. (2.5)). Recall that $\lambda = f(x)/|f(x)|$ if $f(x) \neq 0$, and $\lambda = 1$ if f(x) = 0. We derive from (2.8) that

(2.9)
$$\Delta(f)(y) = \alpha_x(\lambda)|f(x)| = \begin{cases} \alpha_x(1)f(x), & \text{if } x \in K_+ \\ \alpha_x(1)\overline{f(x)}, & \text{if } x \in K_- \end{cases}$$
$$= \begin{cases} \alpha_{\psi(y)}(1)f(\psi(y)), & \text{if } y \in \psi^{-1}(K_+) \\ \alpha_{\psi(y)}(1)\overline{f(\psi(y))}, & \text{if } y \in \psi^{-1}(K_-) \end{cases}$$

Set $L_+ = \psi^{-1}(K_+)$ and $L_- = \psi^{-1}(K_-)$. We deduce from the bijectivity of ψ that $\operatorname{Ch}(B)$ is the disjoint union of L_+ and L_- . Consider finally, the positive homogenous extension $T: A \to B$ defined by

$$T(g) = \begin{cases} \|g\| \Delta\left(\frac{g}{\|g\|}\right), & \text{if } g \in A \setminus \{0\}\\ 0, & \text{if } g = 0 \end{cases}$$

Clearly, T is a surjective mapping. The identity in (2.9) shows that

$$T(g)(y) = \begin{cases} \alpha_{\psi(y)}(1)g(\psi(y)), & \text{if } y \in L_+ \\ \alpha_{\psi(y)}(1)\overline{g(\psi(y))}, & \text{if } y \in L_- \end{cases} \quad (g \in A).$$

Since $\psi \colon \operatorname{Ch}(B) \to \operatorname{Ch}(A)$ is bijective, the previous identity shows that T is a surjective isometry. Namely, pick g_1, g_2 in A. It follows from the previous identity and the surjectivity of ψ that

$$||T(g_1) - T(g_2)|| = \sup_{y \in Ch(B)} |(T(g_1) - T(g_2))(y)| = \sup_{y \in Ch(B)} |(g_1 - g_2)\psi(y)|$$

=
$$\sup_{x \in Ch(A)} |(g_1 - g_2)(x)| = ||g_1 - g_2||$$

where in the first and fourth equalities we applied that Ch(B) and Ch(A) are norming sets for B and A, respectively (see page 4). Therefore T is a real linear isometry by the Mazur–Ulam theorem.

The final argument in the proof of Theorem 2.1 can be also deduced from [35, Lemma 6] or [18, Lemma 2.1], the identity in (2.9) and the fact that Choquet boundaries are boundaries, and thus norming sets.

Although we do not make any use of the maximal convex subsets of the unit sphere of a uniformly closed function algebra, nor of the deep result asserting that a surjective isometry between the unit spheres of two Banach spaces maps maximal convex subsets to maximal convex subsets (see [10, Lemma 5.1] and [41, Lemma 3.5]), the conclusion in [42, Lemma 3.3] (see also [29, Lemma 3.1]) can be applied to deduce that every maximal convex subset C of the unit sphere of uniformly closed function algebra A on a locally compact Hausdorff space X is of the form

$$\mathcal{C} = \lambda V_x = \{ f \in S(A) : f(x) = \lambda \},\$$

for some $\lambda \in \mathbb{T}$ and $x \in Ch(A)$ (this can be compared with [29, Lemma 3.2]).

3. TINGLEY'S PROBLEM FOR COMMUTATIVE JB*-TRIPLES

Despite of having their own right to be studied as main protagonists, there exist certain function spaces which are also employed in other branches. An example appears in the Gelfand representation for commutative JB*-triples. As a brief introduction we shall mention that these complex spaces arose in holomorphic theory, in the study and classification of bounded symmetric domains in arbitrary complex Banach spaces. These domains are the appropriate substitutes of simply connected domains to extend the Riemann mapping theorem to dimension greater than or equal to 2 (cf. [31] or the detailed presentation in [8, §5.6]).

For the sake of brevity, we shall omit a detailed presentation of the theory for general JB*-triples. However, for the purpose of this note, it is worth recalling that by the Gelfand theory of JB*-triples, the elements in the subclass of commutative JB*-triples can be represented as spaces of continuous functions (cf. [31], [48], [5, §3], [7, §4.2.1]). Indeed, let X be a subset of a Hausdorff locally convex complex space such that $0 \notin X, X \cup \{0\}$ is compact, and $\mathbb{T}X \subseteq X$, where $\mathbb{T} := \{\lambda \in \mathbb{T} : |\lambda| = 1\}$. Let us observe that under these hypotheses, $\lambda x = \mu x$ for $x \in X, \lambda, \mu \in \mathbb{T}$ implies $\lambda = \mu$. The space X is called a *principal* \mathbb{T} -bundle in [31].

A locally compact \mathbb{T}_{σ} -space is a locally compact Hausdorff space X together with a continuous mapping $\mathbb{T} \times X \to X$, $(\lambda, t) \mapsto \lambda t$, satisfying $\lambda(\mu t) = (\lambda \mu)t$ and 1t = t, for all $\lambda, \mu \in \mathbb{T}, t \in X$. Each principal \mathbb{T} -bundle X is a locally compact \mathbb{T}_{σ} -space. We can extend the product by elements in \mathbb{T} to the one-point compactification $X \cup \{\omega\}$ of X by setting $\lambda \omega = \omega$ ($\lambda \in \mathbb{T}$). We now consider the following subspace of continuous functions on a locally compact \mathbb{T}_{σ} -space X

$$C_0^{\mathbb{T}}(X) := \{ a \in C_0(X) : a(\lambda t) = \lambda a(t) \text{ for every } (\lambda, t) \in \mathbb{T} \times X \}.$$

We shall regard $C_0^{\mathbb{T}}(X)$ as a norm closed subspace of $C_0(X)$ with the supremum norm. We observe that every $C_0(L)$ space is a $C_0^{\mathbb{T}}(X)$ space (cf. [36, Proposition 10]). However, there exist examples of principal \mathbb{T} -bundles X for which the space $C_0^{\mathbb{T}}(X)$ is not isometrically isomorphic to a $C_0(L)$ space (cf. [31, Corollary 1.13 and subsequent comments]). $C_0^{\mathbb{T}}(X)$ spaces, with X a locally compact \mathbb{T}_{σ} -space, are directly related to Lindenstrauss spaces (see [36, Theorem 12]).

Let us now fix a locally compact \mathbb{T}_{σ} -space X. Although $C_0^{\mathbb{T}}(X)$ need not be a subalgebra of $C_0(X)$, it is closed for the (pointwise) triple product defined by $\{a, b, c\} := ab^*c = a\overline{b}c \ (a, b, c \in C_0^{\mathbb{T}}(X))$. We shall write $a^{[1]} = a, a^{[3]} = \{a, a, a\}$ and $a^{[2n+1]} = \{a, a, a^{[2n-1]}\}$ for all natural n. For each $t_0 \in X$, the mapping $\delta_{t_0} :$ $C_0^{\mathbb{T}}(X) \to \mathbb{C}, \ \delta_{t_0}(a) = a(t_0)$ is a functional in the closed unit ball of the dual of $C_0^{\mathbb{T}}(X)$. Motivated by the classical results for $C_0(L)$ spaces the reader might think that each δ_{t_0} is an extreme point of the closed unit ball of $C_0^{\mathbb{T}}(X)^*$, however this is not always the case, some elements $t_0 \in X$ produce zero functionals. For example if $t_0 \in X$ satisfies that $t_0 \in (\mathbb{T} \setminus \{1\})t_0$ (i.e. t_0 is in the \mathbb{T} -orbit of itself by an element which is not 1), it is easy to check that $\delta_{t_0} = 0$ as a functional in $C_0^{\mathbb{T}}(X)^*$. By [36, Lemma 11] the extreme points of the closed unit ball of $C_0^{\mathbb{T}}(X)^*$ are precisely those δ_{t_0} which are non-zero, that is,

(3.1)
$$\partial_e \left(\mathcal{B}_{C_0^{\mathbb{T}}(X)^*} \right) = \{ \delta_{t_0} : t_0 \notin (\mathbb{T} \setminus \{1\}) t_0 \}.$$

Henceforth the extreme points of the closed unit ball, \mathcal{B}_E , of a Banach space E will be denoted by $\partial_e(\mathcal{B}_E)$. Clearly, the set $\partial_e\left(\mathcal{B}_{C_0^{\mathbb{T}}(X)^*}\right)$ is norming and a kind of Choquet boundary for $C_0^{\mathbb{T}}(X)$.

Those complex Banach spaces called JB*-triples are precisely the complex Banach spaces whose unit ball is a bounded symmetric domain, and were introduced by W. Kaup in [31] to classify these domains, and to establish a generalization of Riemann mapping theorem in dimension ≥ 2 . A JB*-triple is a complex Banach space E admitting a continuous triple product $\{\cdot, \cdot, \cdot\}: E \times E \times E \to E$, which is symmetric and linear in the outer variables, conjugate-linear in the middle one, and satisfies the following axioms:

- (a) L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) L(x,L(b,a)y), for all a, b, x, yin E, where L(a,b) is the operator on E given by $L(a,b)x = \{a, b, x\}$;
- (b) For all $a \in E$, L(a, a) is a hermitian operator with non-negative spectrum;
- (c) $||\{a, a, a\}|| = ||a||^3$, for all $a \in E$.

The class of JB*-triples includes all C*-algebras and all JB*-algebras (cf. [31, pages 522, 523 and 525]).

A JB*-triple E is called *abelian* or *commutative* if the set $\{L(a, b) : a, b \in E\}$ is a commutative subset of the space $\mathcal{B}(E)$ of all bounded linear operators on E (cf. [31, §1], [7, §4.1.47] or [25, §4] where commutative JB*-triples are called "associative"). Despite of the technical definition, every commutative JB*-triple can be isometrically represented, via a triple isomorphism (that is, a linear bijection preserving the triple product), as a space of the form $C_0^{\mathbb{T}}(X)$ for a suitable principal \mathbb{T} -bundle X (see [31, Corollary 1.1], [7, Theorem 4.2.7], see also the interesting representation theorems in [24, §3] and [25, §4]).

Let X and Y be two (locally compact and Hausdorff) principal T-bundles. Each surjective linear isometry T from $C_0^{\mathbb{T}}(X)$ onto $C_0^{\mathbb{T}}(Y)$ is a triple isomorphism (i.e., it preserves the triple product seen above). Furthermore, that is the case, if and only if, there exists a T-equivariant homeomorphism $\phi : Y \to X$ (i.e., $\phi(\lambda s) =$ $\lambda\phi(s)$, for all $(\lambda, s) \in \mathbb{T} \times Y$) such that $T(a)(s) = a(\phi(s))$, for all $s \in Y$ and $a \in C_0^{\mathbb{T}}(X)$ (see [31, Proposition 1.12]). That is, surjective linear isometries and triple isomorphisms coincide, and they are precisely the composition operators with a T-equivariant homeomorphism between the principal T-bundles.

In some of the result of this section, we can apply tools and techniques in the theory of general JB^{*}-triples. However, since the commutative objects of this category admit a concrete representation as function spaces, we strive for presenting basic arguments which do not require any knowledge on the general theory.

Our next goal will consist in determining the explicit form of all real linear isometries between $C_0^{\mathbb{T}}(X)$ spaces for principal \mathbb{T} -bundles (i.e. commutative JB*-triples), a description which materializes and concretizes the theoretical conclusions for real linear surjective isometries on C*-algebras and JB*-triples in [13, 16].

Let us begin by determining the triple ideals of an abelian JB*-triple $C_0^{\mathbb{T}}(X)$ which are *M*-summands. This basic theory is probably known by experts but we have been unable to find a explicit reference. So, we strove for presenting a complete argument with auxiliary references. The reference book on *M*-summands and *M*-ideals is [26]. A linear projection *P* on a real or complex Banach space *E* is called an *M*-projection if

$$||x|| = \max\{||P(x)||, ||x - P(x)||\}, \text{ for all } x \in E.$$

A closed subspace $F \subseteq E$ is called an *M*-summand if it is the range of an *M*-projection. For a locally compact Hausdorff space *L*, the *M*-summands of $C_0(L)$ are described in [26, Example I.1.4(*a*) and Lemma I.1.5], and they correspond to subspaces of the form

 $I = \{ a \in C_0(L) : a(t) = 0 \text{ for all } t \in D \},\$

where D is a clopen subset of L. Similar arguments to those used in the quoted reference can be applied to deduce our next lemma.

Lemma 3.1. Let X be a principal \mathbb{T} -bundle. Then the M-summands of $C_0^{\mathbb{T}}(X)$ are precisely the subspaces of the form

$$I = \{ a \in C_0^{\mathbb{T}}(X) : a(t) = 0 \text{ for all } t \in D \},\$$

where D is a \mathbb{T} -invariant (i.e. $\mathbb{T}D = D$) clopen subset of X.

Proof. Since X is a principal \mathbb{T} -bundle, the extreme points of the closed unit ball of $C_0^{\mathbb{T}}(X)^*$ are those in the set $\{\delta_{t_0} : t_0 \in X\}$ (cf. (3.1)). Suppose I is a closed subspace of $C_0^{\mathbb{T}}(X)$ which is an M-summand for an M-projection P. Let $J = (Id-P)(C_0^{\mathbb{T}}(X))$. Clearly $C_0^{\mathbb{T}}(X) = I \oplus^{\ell_{\infty}} J$, and $C_0^{\mathbb{T}}(X)^* = J^{\circ} \oplus^{\ell_1} I^{\circ}$, where I° stands for the polar of I in $C_0^{\mathbb{T}}(X)^*$, and similarly for J. Let us define $D := \{t \in X : \delta_t \in I^{\circ}\}$. Clearly, D is a \mathbb{T} -invariant closed subset of X and $I \subseteq I_D := \{a \in C_0^{\mathbb{T}}(X) : a(t) = 0 \text{ for all } t \in D\}$. The equality $I = I_D$ will follow from the Hahn-Banach and Krein-Milman theorems as soon as we prove that $\partial_e(\mathcal{B}_{I^{\circ}}) \subseteq I_D^{\circ}$, but this is clear from the well known fact that

$$X \equiv \partial_e \left(\mathcal{B}_{C_0^{\mathbb{T}}(X)^*} \right) = \partial_e \left(\mathcal{B}_{I^\circ} \right) \cup \partial_e \left(\mathcal{B}_{J^\circ} \right).$$

Similarly, $J = \{a \in C_0^{\mathbb{T}}(X) : a(t) = 0 \text{ for all } t \in \widehat{D}\}$ for a T-invariant closed subset \widehat{D} of X. Since D and \widehat{D} are disjoint with $D \cup \widehat{D} = X$, we deduce that D is clopen. \Box

It should be noticed that the set D in the previous lemma might be compact.

Let us now determine the surjective real linear isometries between abelian JB*triples.

Lemma 3.2. Let X and Y be two principal \mathbb{T} -bundles. Then for each surjective real linear isometry $T: C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ there exist a \mathbb{T} -invariant clopen subset $D \subseteq X$ and a \mathbb{T} -equivariant homeomorphism $\phi: Y \to X$ satisfying

$$T(a)(s) = a(\phi(s)), \text{ for all } a \in C_0^{\mathbb{T}}(X) \text{ and for all } s \in \phi^{-1}(D),$$

and

$$T(a)(s) = \overline{a(\phi(s))}, \text{ for all } a \in C_0^{\mathbb{T}}(X) \text{ and for all } s \in \phi^{-1}(X \setminus D).$$

Moreover, the set $\tilde{D} := \phi^{-1}(D)$ is \mathbb{T} -invariant and clopen in Y, $T_1 := T|_{C_0^{\mathbb{T}}(D)} : C_0^{\mathbb{T}}(D) \to C_0^{\mathbb{T}}(\tilde{D})$ is a complex linear surjective isometry given by the composition operator of the mapping $\phi|_{\tilde{D}}$, $T_2 := T|_{C_0^{\mathbb{T}}(X \setminus D)} : C_0^{\mathbb{T}}(X \setminus D) \to C_0^{\mathbb{T}}(Y \setminus \tilde{D})$ is a conjugate-linear surjective isometry given by the pointwise conjugation and the composition operator of the mapping $\phi|_{Y \setminus \tilde{D}}$, $C_0^{\mathbb{T}}(X) = C_0^{\mathbb{T}}(D) \oplus^{\ell_{\infty}} C_0^{\mathbb{T}}(X \setminus D)$, and $T = T_1 \oplus T_2$.

Proof. Since the unique Cartan factor of rank 1 which can be an M-summand in the second dual of an abelian JB*-triple is the trivial one (\mathbb{C}), it follows from [16, Theorem 3.1] that there exist two closed subtriples I and J of $C_0^{\mathbb{T}}(X)$ such that $C_0^{\mathbb{T}}(X) = I \oplus^{\ell_{\infty}} J, T_1 := T|_I : I \to T(I)$ is a surjective complex linear isometry and $T_2 := T|_J : J \to T(J)$ is a surjective conjugate-linear isometry.

By Lemma 3.1 there exist \mathbb{T} -invariant clopen subsets $D \subseteq X$ and $\tilde{D} \subseteq Y$ such that $I = \{a \in C_0^{\mathbb{T}}(X) : a(t) = 0 \text{ for all } t \in X \setminus D\} \cong C_0^{\mathbb{T}}(D)$ and $T(I) = \{a \in C_0^{\mathbb{T}}(Y) : a(t) = 0 \text{ for all } t \in Y \setminus \tilde{D}\} \cong C_0^{\mathbb{T}}(\tilde{D})$. Since $T|_I : I \to T(I)$ is a surjective complex linear isometry, we derive from Kaup's theorem (cf. [31, Proposition 1.12]) that there exists a \mathbb{T} -equivariant homeomorphism $\phi_1 : \tilde{D} \to D$ satisfying $T(a)(s) = a(\phi_1(s))$ for all $a \in I, s \in \tilde{D}$.

Since, clearly, $J = \{a \in C_0^{\mathbb{T}}(X) : a(t) = 0 \text{ for all } t \in D\} \cong C_0^{\mathbb{T}}(X \setminus D)$ and $T(J) = \{a \in C_0^{\mathbb{T}}(Y) : a(t) = 0 \text{ for all } t \in \tilde{D}\} \cong C_0^{\mathbb{T}}(Y \setminus \tilde{D}), \text{ and the mapping}$ $\overline{\cdot} \circ T_2 = T|_J : J \to T(J), a \mapsto \overline{T(a)} \ (a \in J) \text{ is a surjective complex linear isometry,}$ Kaup's theorem gives a \mathbb{T} -equivariant homeomorphism $\phi_2 : Y \setminus \tilde{D} \to X \setminus D$ satisfying $T(a)(s) = \overline{a(\phi_2(s))}$ for all $a \in J, s \in Y \setminus \tilde{D}$.

Finally taking $\phi: Y \to X$, $\phi(s) = \phi_1(s)$ for $s \in \tilde{D}$ and $\phi(s) = \phi_2(s)$ for $s \in Y \setminus \tilde{D}$ we get the desired T-equivariant homeomorphism from Y onto X.

Let X be a locally compact \mathbb{T}_{σ} -space. Let us consider the commutative C*-algebra

$$C_0^{hom}(X) = \{ b \in C_0(X) : b(\lambda x) = b(x) \ \forall x \in X, \lambda \in \mathbb{T} \} \equiv C_0(X/\mathbb{T}),$$

equipped with the pointwise product and involution and the supremum norm. The commutative JB*-triple $C_0^{\mathbb{T}}(X)$ is a Banach $C_0^{hom}(X)$ -bimodule under the pointwise product.

Let us continue with a rudimentary continuous triple functional calculus in our setting. Let $\mathcal{B}_{\mathbb{C}}$ denote the closed unit ball of \mathbb{C} , regarded as principal \mathbb{T} -bundle. For each $a \in C_0^{\mathbb{T}}(X)$ with $||a|| \leq 1$ and each f in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}}) = \{f \in C(\mathcal{B}_{\mathbb{C}}) : f(0) = 0, f(\lambda\zeta) = \lambda f(\zeta), \lambda \in \mathbb{T}, \zeta \in \mathcal{B}_{\mathbb{C}}\}$, the composition $f \circ a$ lies in $C_0^{\mathbb{T}}(X)$, and it will be denoted by $f_t(a) = f \circ a$. This coincides with the so-called continuous triple functional calculus in the wider setting of JB*-triples. Let us observe that for $f_n(\zeta) = |\zeta|^{2n}\zeta$ with $n \in \mathbb{N} \cup \{0\}$ ($\zeta \in \mathcal{B}_{\mathbb{C}}, f_n \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$), and each a in the closed unit ball of $C_0^{\mathbb{T}}(X)$, we have $(f_n)_t(a) = a^{[2n+1]}$.

The next result is a type of concretized version of [17, Lemma 3.3].

Lemma 3.3. Let F be a norm closed face of the closed unit ball of $C_0^{\mathbb{T}}(X)$, where X is a principal \mathbb{T} -bundle, and let f be a function in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ such that f is the identity on \mathbb{T} . Then for each a in F, the element $f_t(a)$ lies in F.

Proof. Since each $f \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ satisfies $f(\lambda\zeta) = \lambda f(\zeta)$ ($\lambda \in \mathbb{T}, \zeta \in \mathcal{B}_{\mathbb{C}}$), the values of f on the interval [0, 1] determine the whole function f. We can now repeat, almost literally the argument in [17, Lemma 3.3]. Fix $a \in F$ and a positive $\varepsilon < 1/2$, let f_{ε} and g_{ε} denote the functions in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose restrictions to [0, 1] are given by

$$f_{\varepsilon}(t) = \begin{cases} 0, & 0 \le t \le \varepsilon/2, \\ \text{affine,} & \varepsilon/2 \le t \le \varepsilon, \\ f(t), & \varepsilon \le t < 1 - \varepsilon, \\ \text{affine,} & 1 - \varepsilon \le t \le 1 - \varepsilon/2, \\ 1, & 1 - \varepsilon/2 \le t \le 1 \end{cases}$$

and

$$g_{\varepsilon}(t) = (1 - \varepsilon/2)^{-1} (t - \varepsilon/2f_{\varepsilon}(t)), \ t \in [0, 1],$$

respectively. According to this definition, $||f - f_{\varepsilon}||_{\infty}$ tends to zero when ε tends to zero, and $(1 - \varepsilon/2)g_{\varepsilon}(t) + (\varepsilon/2)f_{\varepsilon}(t) = t$, for all $t \in [0, 1]$, witnessing that $a = (1 - \varepsilon/2)(g_{\varepsilon})_t(a) + (\varepsilon/2)(f_{\varepsilon})_t(a)$. Applying now that F is a face, we conclude that $(f_{\varepsilon})_t(a) \in F$ for all $0 < \varepsilon < 1/2$. Letting ε tend to zero we get $f_t(a) \in F$. \Box

We fix now an arbitrary locally compact \mathbb{T}_{σ} -space X. Our space $C_0^{\mathbb{T}}(X)$ lacks of peaking functions, since for each $a \in S(C_0^{\mathbb{T}}(X))$, we have $\mathbb{T} \subseteq a(X)$. We can combine the description of the set of extreme points of the closed unit ball of $C_0^{\mathbb{T}}(X)^*$ given in (3.1) with the facial theory of JB*-triples in [17] to determine the maximal proper faces of the closed unit ball of $C_0^{\mathbb{T}}(X)$, however we prioritize a self-contained argument for function spaces more accessible for a wider audience.

We recall a fundamental property of $C_0^{\mathbb{T}}(X)$. Let $\boldsymbol{\mu}$ denote the unit Haar measure on \mathbb{T} . For each $a \in C_0(X)$ we consider a function $\pi_{\mathbb{T}}(a) : X \to \mathbb{C}$ defined by

$$\pi_{\mathbb{T}}(a)(t) = \int \lambda^{-1} a(\lambda t) d\boldsymbol{\mu}, \quad (t \in X).$$

It is known that $\pi_{\mathbb{T}}$ is a contractive projection of $C_0(X)$ onto $C_0^{\mathbb{T}}(X)$ (cf. [36]).

Pick $t_0 \in X$ with $t_0 \notin (\mathbb{T} \setminus \{1\}) t_0$ and $\mu \in \mathbb{T}$. We shall define the set

$$F_{t_0,\mu} = F_{t_0,\mu}^X := \{ a \in \mathcal{B}_{C_0^{\mathbb{T}}(X)} : a(t_0) = \mu \}.$$

At this stage it is not at all clear that $F_{t_0,\mu}$ is non-empty, an statement which is straightforward in the case of $C_0(L)$ spaces thanks to Urysohn's lemma.

Remark 3.4. Suppose X is a locally compact \mathbb{T}_{σ} -space. Let W be a \mathbb{T} -invariant open neighbourhood of t_0 in X which is contained in a compact \mathbb{T} -invariant subset. We consider the following continuous function

$$\mathbb{T}t_0 \cup (X \setminus W) \to \mathbb{C},$$
$$\lambda t_0 \mapsto \lambda, \text{ and } t \mapsto 0 \text{ for all } t \in X \setminus W.$$

Find, via Tiezte's theorem, a continuous function $\tilde{h} \in C_0(X)$ extending the previous mapping. Let $h := \pi_{\mathbb{T}}(\tilde{h}) \in C_0^{\mathbb{T}}(X)$. It is easy to check that $h(t_0) = 1$ and h(t) = 0 for all $t \in X \setminus W$. Clearly, $\mu h \in F_{t_0,\mu}$. This construction, which was already considered in [36, Proof of Lemma 11], is a kind of Urysohn's lemma for $C_0^{\mathbb{T}}(X)$ spaces, and will be employed along this section.

Let E be a Banach space. As observed by R. Tanaka in [42, Lemma 3.3] and [43, Lemma 3.2], Eidelheit's separation theorem or the geometric Hahn-Banach theorem can be employed to deduce that a convex subset $C \subseteq S(E)$ is a maximal convex subset if and only if it is a maximal norm closed proper face of the closed unit ball, \mathcal{B}_E , of E.

We give next a concrete description of the norm closed faces of $\mathcal{B}_{C_0^{\mathbb{T}}(X)}$. The conclusion can be also derived from the study of norm closed faces of the closed unit ball of a general JB*-triple [17] and a good knowledge on the minimal tripotents in the second dual and its relation with the extreme point of the closed unit ball of the first dual. For the sake of simplicity we include here an alternative argument with techniques of function algebras.

Lemma 3.5. Let X be a principal \mathbb{T} -bundle. Then every maximal convex subset of $S(C_0^{\mathbb{T}}(X))$ (equivalently, each maximal proper norm closed face of the closed unit ball of $C_0^{\mathbb{T}}(X)$) is of the form

$$F_{t_0,\mu} = F_{t_0,\mu}^X := \{ a \in \mathcal{B}_{C_0^{\mathbb{T}}(X)} : a(t_0) = \mu \}$$

for some $t_0 \in X$ and $\mu \in \mathbb{T}$.

Proof. Under these hypotheses $F_{t_0,\mu}$ is a non-empty proper face of the closed unit ball of $C_0^{\mathbb{T}}(X)$. Let F be a maximal convex (proper) subset of $S(C_0^{\mathbb{T}}(X))$ containing $F_{t_0,\mu}$. Fix $b \in F_{t_0,\mu}$. If there exists $a \in F \setminus F_{t_0,\mu}$ the function $c = \frac{a+b}{2} \in F$ satisfies $|c(t_0)| < \delta < 1$ for an appropriate δ . The set $U := \{t \in X : |c(t)| < \delta\}$ contains t_0 , is open and \mathbb{T} -invariant because $c \in C_0^{\mathbb{T}}(X)$. As we commented above, we can find $h \in C_0^{\mathbb{T}}(X)$ with ||h|| = 1, $h(t_0) = \mu$ and $h|_{X \setminus U} \equiv 0$.

Let k be the function in $C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose restriction to [0, 1] vanishes on $[0, \delta]$, takes value 1 at 1 and is affine on the rest. Lemma 3.3 assures that $k_t(c) = k \circ c \in F$. Since $k_t(c)|_U \equiv 0$ it can be easily seen that $k_t(c)h = 0$. Therefore $\left\|\frac{h+k_t(c)}{2}\right\| = \max\left\{\left\|\frac{h}{2}\right\|, \left\|\frac{k_t(c)}{2}\right\|\right\} = \frac{1}{2}$. Since, clearly, $\frac{h+k_t(c)}{2} \in F$, the intersection of F with the open unit ball of $C_0^{\mathbb{T}}(X)$ is non-empty, which contradicts that F is proper.

Labelling the proper maximal norm closed faces, $F_{t_0,\mu}$, of the closed unit ball of $C_0^{\mathbb{T}}(X)$ in terms of pairs (t_0,μ) with $t_0 \in X$ and $\mu \in \mathbb{T}$ does not produce an unambiguous association because $F_{t_0,\mu} = F_{\gamma t_0,\gamma\mu}$ for all $\gamma \in \mathbb{T}$. To avoid repetitions let us consider the following property: a non-empty subset S of a principal \mathbb{T} -bundle X satisfies the *non-overlapping property* if for each $t \in S$ we have $S \cap \mathbb{T}t = \{t\}$. Thanks to Zorn's lemma we can always find a maximal non-overlapping subset X_0 of X. Let us observe that in this case, $\mathbb{T}X_0 = X$, actually, for each $t \in X$ there exist unique $t_0 \in X_0$ and $\mu \in \mathbb{T}$ such that $t = \mu t_0$. Consequently, the set $\{\delta_{t_0} : t_0 \in X_0\}$ is norming. Furthermore, the set

$$\{F_{t_0,\mu}: \mu \in \mathbb{T}, t_0 \in X_0\}$$

covers all possible proper maximal norm closed faces of $\mathcal{B}_{C_0^{\mathbb{T}}(X)}$. Namely, given a maximal proper face of the form F_{s_0,μ_0} there exist unique $t_0 \in X_0$ and $\nu \in \mathbb{T}$ such that $s_0 = \nu t_0$, and thus

$$F_{s_0,\mu_0} = F_{\nu t_0,\mu_0} = F_{t_0,\overline{\nu}\mu_0}$$

Actually, for each proper maximal face F of $\mathcal{B}_{C_0^{\mathbb{T}}(X)}$

(3.2) there exist unique
$$t_0 \in X_0$$
 and $\mu \in \mathbb{T}$ such that $F = F_{t_0,\mu}^X$.

An alternative proof for Lemma 3.5 can be deduced from [42, Lemma 3.3] (see also [29, Lemma 3.1].

The main result of this section is a solution to Tingley's problem in the case of commutative JB*-triples.

Theorem 3.6. Let X and Y be two principal \mathbb{T} -bundles. Then each surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ admits an extension to a surjective real linear isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$.

Furthermore, there exist a \mathbb{T} -invariant clopen subset $D \subseteq X$ and a \mathbb{T} -equivariant homeomorphism $\phi: Y \to X$ satisfying

$$\Delta(a)(s) = a(\phi(s)), \text{ for all } a \in S(C_0^{\mathbb{T}}(X)) \text{ and for all } s \in \phi^{-1}(D),$$

and

$$\Delta(a)(s) = \overline{a(\phi(s))}, \text{ for all } a \in S(C_0^{\mathbb{T}}(X)) \text{ and for all } s \in \phi^{-1}(X \setminus D).$$

Consequently, there exists a surjective isometry $T : C_0^{\mathbb{T}}(X) \to C_0^{\mathbb{T}}(Y)$ such that $T|_{C_0^{\mathbb{T}}(D)}$ is complex linear, $T|_{C_0^{\mathbb{T}}(X\setminus D)}$ is conjugate-linear and $T(a) = \Delta(a)$ for all $a \in S(C_0^{\mathbb{T}}(X))$.

The proof will be given after a series of technical lemmata. Let us begin by recalling a key result in the techniques developed to study the problem of extension of isometries which is essentially due to L. Cheng and Y. Dong [10, Lemma 5.1] and R. Tanaka [42] (see also [41, Lemma 3.5], [44, Lemmas 2.1 and 2.2]).

Proposition 3.7. ([10, Lemma 5.1], [42, Lemma 3.3], [41, Lemma 3.5]) Let Δ : $S(E) \rightarrow S(F)$ be a surjective isometry between the unit spheres of two Banach spaces, and let \mathcal{M} be a convex subset of S(E). Then \mathcal{M} is a maximal proper face of \mathcal{B}_E (equivalently, a maximal convex subset of S(E)) if and only if $\Delta(\mathcal{M})$ is a maximal proper (closed) face of \mathcal{B}_F (equivalently, a maximal convex subset of S(F)).

The next corollary is a consequence of Proposition 3.7 and Lemma 3.5.

Corollary 3.8. Let X and Y be two principal \mathbb{T} -bundles, and let $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ be a surjective isometry. Then for each $t_0 \in X$ and each $\mu \in \mathbb{T}$ there exist elements $s_0 \in Y$ and $\nu \in \mathbb{T}$ satisfying

$$\Delta(F_{t_0,\mu}^X) = F_{s_0,\nu}^Y.$$

We have already given some arguments showing that the elements s_0 and ν in the conclusion of the previous corollary need not be unique. To avoid the problem we consider the next lemma.

Lemma 3.9. Let X and Y be two principal \mathbb{T} -bundles, and let $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$ be a surjective isometry. Let X_0 be a maximal non-overlapping subset of X. Then for each t_0 in X_0 there exists a unique $s_0 = \phi(t_0) \in Y$ satisfying

$$\Delta(F_{t_0,1}^X) = F_{\phi(t_0),1}^Y$$

The mapping $\phi = \phi_{X_0} : X_0 \to Y$ is well-defined and injective.

Proof. By Corollary 3.8 there exist $s \in Y$, $\mu \in \mathbb{T}$ such that

(3.3)
$$\Delta(F_{t_0,1}^X) = F_{s,\mu}^Y = F_{\overline{\mu}s,1}^Y.$$

Observe that the element $\overline{\mu}s$ satisfying the identity in (3.3) is unique. Indeed, if $F_{s_1,1}^Y = F_{s_2,1}^Y$ for some s_1, s_2 in Y with $s_2 \notin \mathbb{T}s_1$ we can find a function $a \in S(C_0^{\mathbb{T}}(Y))$ with $a(s_1) = 1$ and $a(s_2) = 0$ (cf. Remark 3.4), which is impossible. If $s_2 = \nu s_1$ for some $\nu \in \mathbb{T}$, for each $a \in F_{s_1,1}^Y = F_{s_2,1}^Y$, we have $1 = a(s_1) = a(s_2) = \nu a(s_1) = \nu$, witnessing that $s_1 = s_2$. The first conclusion follows by setting $\phi(t_0) = \overline{\mu}s$ for the element $\overline{\mu}s$ given by (3.3).

The rest is clear from the previous arguments.

Henceforth we fix a surjective isometry $\Delta : S(C_0^{\mathbb{T}}(X)) \to S(C_0^{\mathbb{T}}(Y))$, where X and Y are two principal T-bundles, $X_0 \subseteq X$ a maximal non-overlapping subset and $\phi = \phi_{X_0} : X_0 \to Y$ the injective mapping given by Lemma 3.9.

The next step in our strategy isolates a crucial property of ϕ .

Lemma 3.10. Let t_0 be an element in X_0 , and let a be an element in $S(C_0^{\mathbb{T}}(X))$ satisfying $a(t_0) = 0$. Then $\Delta(a)(\phi(t_0)) = 0$. *Proof.* Since a continuous function in $S(C_0^{\mathbb{T}}(X))$ vanishing at t_0 can be approximated in norm by functions in $S(C_0^{\mathbb{T}}(X))$ vanishing on a neighbourhood of t_0 , we can assume that *a* vanishes on a \mathbb{T} -invariant open neighbourhood *U* of t_0 .

Let us consider a function $b \in S(C_0^{\mathbb{T}}(X))$ satisfying $b(t_0) = 1$ and $b|_{X \setminus U} \equiv 0$. Clearly, $b \in F_{t_0,1}$, and by orthogonality of a and b we have $||a \pm b|| = 1$.

By [34, Proposition 2.3(a)] we have $\Delta(-F_{t_0,1}) = -\Delta(F_{t_0,1})$. Therefore, there exists $c \in F_{t_0,1}$ such that $-\Delta(c) = \Delta(-b)$, and hence $\Delta(-b)(\phi(t_0)) = -\Delta(c)(\phi(t_0)) = -1$. It follows from the assumptions that

$$1 = ||a + b|| = ||\Delta(a) - \Delta(-b)|| \ge |\Delta(a)(\phi(t_0)) - \Delta(-b)(\phi(t_0))|$$

= $|\Delta(a)(\phi(t_0)) + 1|,$
$$1 = ||a - b|| = ||\Delta(a) - \Delta(b)|| \ge |\Delta(a)(\phi(t_0)) - \Delta(b)(\phi(t_0))|$$

= $|\Delta(a)(\phi(t_0)) - 1|$

witnessing that $\Delta(a)(\phi(t_0)) = 0$.

Lemma 3.11. The set $Y_0 = \{\phi(t_0) : t_0 \in X_0\} = \phi(X_0) \subseteq Y$ is a maximal nonoverlapping subset of Y, and hence the set $\{\delta_s : s \in Y_0\}$ is norming in $C_0^{\mathbb{T}}(Y)$. Furthermore, the mapping $\phi : X_0 \to Y_0$ is a bijection satisfying

$$\Delta(F_{t_0,1}^X) = F_{\phi(t_0),1}^Y, \text{ for all } t_0 \in X_0,$$

and ϕ^{-1} is precisely the mapping given by Lemma 3.9 for Δ^{-1} and Y_0 .

Proof. We shall first show that Y_0 is non-overlapping. Since ϕ is injective we can suppose that we have $\phi(t_1) \neq \phi(t_2)$ in Y_0 with $t_1 \neq t_2$ in X_0 . Since X_0 is nonoverlapping we can find $a \in F_{t_1,1}^X$ with $a(t_2) = 0$ (cf. Remark 3.4). Lemma 3.10 implies that $\Delta(a)(\phi(t_2)) = 0$, and consequently $\phi(t_2) \notin \mathbb{T}\phi(t_1)$ because $\Delta(a) \in \Delta(F_{\phi(t_1),1}^Y)$. This shows that Y_0 is non-overlapping.

Let us find, via Zorn's lemma, a maximal non-overlapping subset of $Y, Y_0 \supset Y_0$. By applying Lemma 3.9 to Δ^{-1} and \tilde{Y}_0 we deduce the existence of an injective mapping $\psi: \tilde{Y}_0 \to X$ satisfying

$$\Delta^{-1}(F_{s_0,1}^Y) = F_{\psi(s_0),1}^X$$
, for all $s_0 \in \tilde{Y}_0$

In particular we have

$$\Delta(F_{\psi(\phi(t_0)),1}^X) = F_{\phi(t_0),1}^Y = \Delta(F_{t_0,1}^X), \text{ for all } t_0 \in X_0,$$

which implies that $\psi(\phi(t_0)) = t_0$ for all $t_0 \in X_0$.

By the first part of our argument, applied to Δ^{-1} and \tilde{Y}_0 , we know that $\psi(\tilde{Y}_0)$ must be a non-overlapping subset of X containing X_0 . The maximality of X_0 implies that $\psi(\tilde{Y}_0) = X_0$.

If there exists $s_3 \in \tilde{Y}_0 \setminus Y_0$, by the maximality of X_0 , and the definition of ψ , there exist $t_0 \in X_0$ and $\nu \in \mathbb{T}$ such that

$$\Delta^{-1}(F_{s_3,1}^Y) = F_{\psi(s_3),1}^X = F_{\nu t_0,1}^X.$$

Having in mind that \tilde{Y}_0 is non-overlapping, $s_3 \in \tilde{Y}_0 \setminus Y_0$ and $\phi(t_0) \in Y_0$, we can find $\tilde{a} \in F_{s_3,1}^Y$ vanishing at $\phi(t_0)$ (cf. Remark 3.4). Lemma 3.10, applied to Δ^{-1} , \tilde{a} and $\phi(t_0)$, implies that

$$0 = \Delta^{-1}(\tilde{a})(\psi\phi(t_0)) = \Delta^{-1}(\tilde{a})(t_0)$$

However $\Delta^{-1}(\tilde{a}) \in \Delta^{-1}(F_{s_{3,1}}^Y) = F_{\nu t_{0,1}}^X$, and hence $1 = \Delta^{-1}(\tilde{a})(\nu t_0) = \nu \Delta^{-1}(\tilde{a})(t_0) = 0$, leading to a contradiction.

Having in mind that X_0 and Y_0 are maximal non-overlapping subsets of X and Y, respectively, and considering the property we commented in (3.2), we deduce the next fact:

(3.4) for each
$$t_0 \in X_0$$
, and $\mu \in \mathbb{T}$ there exist unique $\phi(t_0, \mu) \in Y_0$
and $\sigma(t_0, \mu) \in \mathbb{T}$ satisfying $\Delta(F_{t_0, \mu}^X) = F_{\tilde{\phi}(t_0, \mu), \sigma(t_0, \mu)}^Y$.

The mappings $\tilde{\phi} : X_0 \times \mathbb{T} \to Y_0$ and $\sigma : X_0 \times \mathbb{T} \to \mathbb{T}$ are well-defined. We shall make use of these mappings in the subsequent results.

Lemma 3.12. The mappings σ and ϕ satisfy

$$\sigma(t_0, -\mu) = -\sigma(t_0, \mu), \text{ and } \tilde{\phi}(t_0, -\mu) = \tilde{\phi}(t_0, \mu),$$

for all $t_0 \in X_0$ and $\mu \in \mathbb{T}$.

Proof. A new application of [34, Proposition 2.3(a)] gives

$$F_{\tilde{\phi}(t_0,-\mu),\sigma(t_0,-\mu)}^Y = \Delta(F_{t_0,-\mu}^X) = \Delta(-F_{t_0,\mu}^X) = -\Delta(F_{t_0,\mu}^X)$$
$$= -F_{\tilde{\phi}(t_0,\mu),\sigma(t_0,\mu)}^Y = F_{\tilde{\phi}(t_0,\mu),-\sigma(t_0,\mu)}^Y.$$

If $\tilde{\phi}(t_0, -\mu) \neq \tilde{\phi}(t_0, \mu)$ in Y_0 , there exists a function \tilde{a} in $F^Y_{\tilde{\phi}(t_0, -\mu), \sigma(t_0, -\mu)}$ vanishing at $\tilde{\phi}(t_0, \mu)$ (cf. Remark 3.4), contradicting the previous identity. Therefore $\tilde{\phi}(t_0, -\mu) = \tilde{\phi}(t_0, \mu)$ and $-\sigma(t_0, \mu) = \sigma(t_0, -\mu)$.

Proposition 3.13. The identity

$$\tilde{\phi}(t_0,\mu) = \tilde{\phi}(t_0,1) = \phi(t_0)$$

holds for all $t_0 \in X_0$ and $\mu \in \mathbb{T}$.

Proof. The last equality follows from (3.4) and Lemma 3.9.

Let us observe that $\tilde{\phi}(t_0, \mu), \phi(t_0) \in Y_0$, and the latter is a maximal non-overlapping set of Y. Thus, if $\tilde{\phi}(t_0, \mu) \neq \phi(t_0)$, we can find two open disjoint neighbourhoods of these two points, and hence by Remark 3.4 there exist two orthogonal or disjoint functions $\tilde{a} \in F_{\tilde{\phi}(t_0,\mu),\sigma(t_0,\mu)}^Y$ and $\tilde{b} \in F_{\phi(t_0),1}^Y$, in particular $\|\tilde{a} \pm \tilde{b}\| = 1$. It follows from the defining properties of $\tilde{\phi}$ and ϕ that $a = \Delta^{-1}(\tilde{a}) \in F_{t_0,\mu}^X$ and $b = \Delta^{-1}(\tilde{b}) \in F_{t_0,1}^X$.

If $\operatorname{Re}(\mu) \leq 0$ we have

$$|1 - \mu| = |b(t_0) - a(t_0)| \le ||a - b|| = ||\Delta(a) - \Delta(b)|| = ||\tilde{a} - \tilde{b}|| = 1,$$

which is impossible.

If $\operatorname{Re}(\mu) > 0$, having in mind that, by Lemma 3.12, $\tilde{\phi}(t_0, \mu) = \tilde{\phi}(t_0, -\mu)$ with $\operatorname{Re}(-\mu) < 0$ and $\tilde{\phi}(t_0, \mu) = \tilde{\phi}(t_0, -\mu) \neq \phi(t_0)$, which is impossible by the conclusion of the previous case.

From now on we shall only work with the bijection $\phi: X_0 \to Y_0$ and its inverse (associated to Δ^{-1} , see Lemma 3.11). For each $t_0 \in X_0$, the mapping $\sigma(t_0, \cdot) = \sigma^{\Delta}(t_0, \cdot) : \mathbb{T} \to \mathbb{T}$ is well-defined. Furthermore, by applying the same arguments to Δ^{-1} , we prove the existence of a new mapping $\sigma^{\Delta^{-1}}(\cdot, \cdot) : Y_0 \times \mathbb{T} \to \mathbb{T}$ satisfying the appropriate properties described in the comments after Lemma 3.11.

Lemma 3.14. For each $t_0 \in X_0$ the mappings $\sigma^{\Delta}(t_0, \cdot), \sigma^{\Delta^{-1}}(\phi(t_0), \cdot) : \mathbb{T} \to \mathbb{T}$ are bijective and $\sigma^{\Delta^{-1}}(\phi(t_0), \cdot)$ is the inverse of $\sigma^{\Delta}(t_0, \cdot)$.

Proof. Fix an arbitrary $t_0 \in X_0$ and $\mu \in \mathbb{T}$. The conclusion follows almost straightforwardly from the identities

$$F_{t_0,\mu}^X = \Delta^{-1} \left(F_{\phi(t_0),\sigma^{\Delta}(t_0,\mu)}^Y \right) = F_{\phi^{-1}\phi(t_0),\sigma^{\Delta^{-1}}(\phi(t_0),\sigma^{\Delta}(t_0,\mu))}^X$$
$$= F_{t_0,\sigma^{\Delta^{-1}}(\phi(t_0),\sigma^{\Delta}(t_0,\mu))}^X$$

We can argue as in Lemma 2.16 to deduce that $\sigma^{\Delta}(t_0, \cdot)$ is an isometric mapping for each $t_0 \in X_0$.

Lemma 3.15. For each $t_0 \in X_0$, the mappings $\sigma^{\Delta}(t_0, \cdot), \sigma^{\Delta^{-1}}(\phi(t_0), \cdot) : \mathbb{T} \to \mathbb{T}$ are surjective isometries.

Proof. Fix μ_1, μ_2 in \mathbb{T} . Let us fix an element $a \in F_{t_0,1}$. Since, by definition, $\Delta(F_{t_0,\mu_j}^X) = F_{\phi(t_0),\sigma(t_0,\mu_j)}^Y$ it can be easily seen that $|\sigma(t_0,\mu_1) - \sigma(t_0,\mu_2)| = |\Delta(\mu_1 a)(\phi(t_0)) - \Delta(\mu_2 a)(\phi(t_0))|$

$$\begin{aligned} |\sigma(t_0,\mu_1) - \sigma(t_0,\mu_2)| &= |\Delta(\mu_1 a)(\phi(t_0)) - \Delta(\mu_2 a)(\phi(t_0))| \\ &\leq ||\Delta(\mu_1 a) - \Delta(\mu_2 a)|| = ||(\mu_1 - \mu_2)a|| = |\mu_1 - \mu_2|. \end{aligned}$$

This proves that $\sigma(t_0, \cdot) = \sigma^{\Delta}(t_0, \cdot)$ is contractive. Replacing Δ with Δ^{-1} , we deduce that $\sigma^{\Delta^{-1}}(\phi(t_0), \cdot)$ is contractive too. Since $\sigma^{\Delta^{-1}}(\phi(t_0), \cdot)$ is the inverse of $\sigma^{\Delta}(t_0, \cdot)$ (cf. Lemma 3.14), we can conclude that $\sigma^{\Delta}(t_0, \cdot)$ and $\sigma^{\Delta^{-1}}(\phi(t_0), \cdot)$ are isometries. \Box

It follows from the previous lemma that for each $t_0 \in X_0$, $\sigma^{\Delta}(t_0, \cdot) : \mathbb{T} \to \mathbb{T}$ is a surjective isometry. By some of the results commented in the introduction, for example, by the solution to Tingley's problem for $\mathbb{T} = S(\mathbb{C})$ [11], we deduce that

(3.5)
$$\sigma^{\Delta}(t_0,\mu) = \sigma^{\Delta}(t_0,1) \ \mu, \text{ or } \sigma^{\Delta}(t_0,\mu) = \sigma^{\Delta}(t_0,1) \ \overline{\mu} \ (\forall \mu \in \mathbb{T}).$$

The just stated property determines a partition $X_0 = X_0^+ \cup X_0^-$ with respect to the subsets

$$X_0^+ = \{ t_0 \in X_0 : \sigma^{\Delta}(t_0, \mu) = \sigma^{\Delta}(t_0, 1) \ \mu, \forall \mu \in \mathbb{T} \},\$$

and

$$X_0^- = \{ t_0 \in X_0 : \sigma^{\Delta}(t_0, \mu) = \sigma^{\Delta}(t_0, 1) \ \overline{\mu}, \forall \mu \in \mathbb{T} \}.$$

The continuous triple functional calculus explained before Lemma 3.3 is now applied in our next technical result.

Lemma 3.16. Let us fix $t_0 \in X_0$ and $a \in S(C_0^{\mathbb{T}}(X))$ with $|a(t_0)| < 1$. Set $\lambda = \frac{a(t_0)}{|a(t_0)|}$ if $a(t_0) \neq 0$ and $\lambda = 1$ otherwise. Then for each $\varepsilon > 0$ there exist $b_{\varepsilon} \in F_{t_0,1}^X$ and $a_{\varepsilon} \in S(C_0^{\mathbb{T}}(X))$ satisfying

$$ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon} \in \lambda F_{t_0,1}^X = F_{t_0,\lambda}^X, \text{ for all } 0 < r < 1,$$

 $a_{\varepsilon}(t_0) = a(t_0) \text{ and } ||a - a_{\varepsilon}|| < \varepsilon.$

Proof. The case for $a(t_0) = 0$ is easier. In such a case, by Remark 3.4 and a standard argument, for each ε there exists $a_{\varepsilon} \in S(C_0^{\mathbb{T}}(X))$ and $b_{\varepsilon} \in F_{t_0,1}^X$ which are orthogonal or disjoint and $||a - a_{\varepsilon}|| < \varepsilon$. Then clearly, $||ra_{\varepsilon} + b_{\varepsilon}|| = \max\{r, ||b_{\varepsilon}||\} = 1 = (ra_{\varepsilon} + b_{\varepsilon})(t_0)$.

Suppose next that $0 < |a(t_0)| < 1$, and fix a positive ε such that $|a(t_0)| + \varepsilon < 1$. Let us find a T-invariant open neighbourhood W_{ε} of t_0 contained in a T-invariant compact subset such that $|a(s)| < |a(t_0)| + \varepsilon/2$, for all $s \in W_{\varepsilon}$. Let us find, via Remark 3.4 a function $b_{\varepsilon} \in F_{t_0,1}^X$ such that $b_{\varepsilon}|_{X \setminus W_{\varepsilon}} \equiv 0$.

Let us consider the following $h_{\varepsilon} \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$ whose values on [0, 1] are the following:

$$h_{\varepsilon}(s) = \begin{cases} s, & 0 \le s \le |a(t_0)|, \\ \text{affine}, & |a(t_0)| \le s \le |a(t_0)| + \varepsilon/2, \\ |a(t_0)| - \varepsilon/2, & s = |a(t_0)| + \varepsilon/2, \\ \text{affine}, & |a(t_0)| + \varepsilon/2 \le s \le |a(t_0)| + \varepsilon, \\ s, & |a(t_0)| + \varepsilon \le s \le 1. \end{cases}$$

Let $\iota : \mathcal{B}_{\mathbb{C}} \hookrightarrow \mathbb{C}$ denote the inclusion mapping –note that $\iota \in C_0^{\mathbb{T}}(\mathcal{B}_{\mathbb{C}})$. Set $a_{\varepsilon} := (h_{\varepsilon})_t(a)$. Since, $||h_{\varepsilon} - \iota|| = \varepsilon$, it follows that

$$||a_{\varepsilon} - a|| = ||(h_{\varepsilon})_t(a) - \iota(a)|| \le \varepsilon.$$

Clearly $a_{\varepsilon}(t_0) = a(t_0)$. For $s \in X \setminus W_{\varepsilon}$ we have

$$|(ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon})(s)| = |ra_{\varepsilon}(s)| \le r \le 1.$$

For $s \in W_{\varepsilon}$, we have $|a(s)| < |a(t_0)| + \varepsilon/2$, and hence

$$a_{\varepsilon}(s) = h_{\varepsilon}(a(s)) = h_{\varepsilon}(e^{i\alpha_s}|a(s)|) = e^{i\alpha_s}h_{\varepsilon}(|a(s)|) \in e^{i\alpha_s} \ [0, |a(t_0)|].$$

Therefore,

$$|(ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon})(s)| \le r|a(t_0)| + 1 - r|a(t_0)| = 1.$$

Finally the identity

$$(ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon})(t_0) = ra_{\varepsilon}(t_0) + (1 - r|a(t_0)|)\frac{a_{\varepsilon}(t_0)}{|a_{\varepsilon}(t_0)|}$$
$$= \frac{a_{\varepsilon}(t_0)}{|a_{\varepsilon}(t_0)|} = \frac{a(t_0)}{|a(t_0)|} = \lambda,$$

proves that $ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon} \in F^X_{t_0,\lambda}$, as desired.

In the next proposition we shall determine the point-evaluations of elements in the image of Δ at points of the form $\phi(t_0)$.

Proposition 3.17. For each $t_0 \in X_0$ and each $a \in S(C_0^{\mathbb{T}}(X))$ we have

$$\Delta(a)(\phi(t_0)) = \sigma^{\Delta}\left(t_0, \frac{a(t_0)}{|a(t_0)|}\right)|a(t_0)| = \begin{cases} \sigma^{\Delta}(t_0, 1) a(t_0), & \text{if } t_0 \in X_0^+, \\ \sigma^{\Delta}(t_0, 1) \overline{a(t_0)}, & \text{if } t_0 \in X_0^-, \end{cases}$$

where X_0^+ and X_0^- are the subsets of X_0 introduced just before Lemma 3.16.

Proof. Let us fix $t_0 \in X_0$ and $a \in S(C_0^{\mathbb{T}}(X))$. The case $a(t_0) = 0$ follows from Lemma 3.10. If $|a(t_0)| = 1$ we have $a \in F_{t_0,a(t_0)}^X$ and thus

$$\Delta(a)(\phi(t_0)) = \sigma^{\Delta}(t_0, a(t_0)) = \begin{cases} \sigma^{\Delta}(t_0, 1) a(t_0), & \text{if } t_0 \in X_0^+, \\ \sigma^{\Delta}(t_0, 1) \overline{a(t_0)}, & \text{if } t_0 \in X_0^- \end{cases}$$

(cf. (3.4), Proposition 3.13 and (3.5)). We can therefore assume that $1 > |a(t_0)| > 0$. Set $\lambda = \frac{a(t_0)}{|a(t_0)|}$.

We shall first show that

(3.6)
$$|\Delta(a)(\phi(t_0))| = |a(t_0)|.$$

By Lemma 3.16 for each $\varepsilon > 0$ there exist $b_{\varepsilon} \in F_{t_0,1}^X$ and $a_{\varepsilon} \in S(C_0^{\mathbb{T}}(X))$ satisfying

$$c_{r,\varepsilon} = ra_{\varepsilon} + (1 - r|a(t_0)|)\lambda b_{\varepsilon} \in \lambda F_{t_0,1}^X = F_{t_0,\lambda}^X, \text{ for all } 0 < r < 1,$$

 $a_{\varepsilon}(t_0) = a(t_0)$ and $||a - a_{\varepsilon}|| < \varepsilon$. In particular,

$$\Delta(c_{r,\varepsilon})(\phi(t_0)) = \sigma^{\Delta}(t_0,\lambda),$$

and by definition,

$$\|c_{r,\varepsilon} - a_{\varepsilon}\| \le (1-r) + 1 - r|a(t_0)| = 2 - r - r|a(t_0)|.$$

On the other hand,

$$(3.7) 1 - |\Delta(a_{\varepsilon})(\phi(t_0))| = |\sigma^{\Delta}(t_0, \lambda)| - |\Delta(a_{\varepsilon})(\phi(t_0))| \\ \leq |\sigma^{\Delta}(t_0, \lambda) - \Delta(a_{\varepsilon})(\phi(t_0))| \\ = |\Delta(c_{r,\varepsilon})(\phi(t_0)) - \Delta(a_{\varepsilon})(\phi(t_0))| \\ \leq ||\Delta(c_{r,\varepsilon}) - \Delta(a_{\varepsilon})|| = ||c_{r,\varepsilon} - a_{\varepsilon}|| \\ \leq 2 - r - r|a(t_0)|,$$

witnessing that $r + r|a(t_0)| - 1 \leq |\Delta(a_{\varepsilon})(\phi(t_0))|$ (0 < r < 1). Letting $r \to 1$ we get $|a(t_0)| \leq |\Delta(a_{\varepsilon})(\phi(t_0))|$. Now, letting ε tend to zero, it follows from the continuity of Δ that

$$|a(t_0)| \le |\Delta(a)(\phi(t_0))|.$$

Applying the same argument to Δ^{-1} , ϕ^{-1} , $\Delta(a)$ and $\phi(t_0)$ in the roles of Δ , ϕ , a and t_0 we get

$$|\Delta(a)(\phi(t_0))| \le |\Delta^{-1}\Delta(a)(\phi^{-1}\phi(t_0))| = |a(t_0)|,$$

which concludes the proof of (3.6).

If we take limits $r \to 1$ and $\varepsilon \to 0$ in the inequalities given by the second and last lines of (3.7) we arrive to

(3.8)
$$\left| \sigma^{\Delta} \left(t_0, \frac{a(t_0)}{|a(t_0)|} \right) - \Delta(a)(\phi(t_0)) \right| \le 1 - |a(t_0)|.$$

Consequently, by (3.6), we get

$$1 = \left| \sigma^{\Delta} \left(t_0, \frac{a(t_0)}{|a(t_0)|} \right) \right| \le \left| \sigma^{\Delta} \left(t_0, \frac{a(t_0)}{|a(t_0)|} \right) - \Delta(a)(\phi(t_0)) \right| + |\Delta(a)(\phi(t_0))| \\ \le 1 - |a(t_0)| + |a(t_0)| = 1.$$

It then follows that the equality holds in a triangular inequality, so there exists a positive $\delta > 0$ such that $\delta \sigma^{\Delta} \left(t_0, \frac{a(t_0)}{|a(t_0)|} \right) = \Delta(a)(\phi(t_0))$, and in particular $\delta = 0$

 $|\Delta(a)(\phi(t_0))| = |a(t_0)|$. We have therefore proved that

$$\Delta(a)(\phi(t_0)) = \sigma^{\Delta}\left(t_0, \frac{a(t_0)}{|a(t_0)|}\right) |a(t_0)|.$$

The rest is clear from (3.5) and the subsequent comments.

For each t_0 in a principal \mathbb{T} -bundle X, we shall write $\overline{\delta_{t_0}}$ for the conjugate linear functional on $C_0^{\mathbb{T}}(X)$ defined by $\overline{\delta_{t_0}}(a) = \overline{a(t_0)}$ $(a \in C_0^{\mathbb{T}}(X))$.

Proof of Theorem 3.6. Let X_0 and Y_0 denote the maximal non-overlapping subsets employed in the previous arguments. Let $\phi : X_0 \to Y_0$ be the bijection presented in Lemmata 3.9 and 3.11. As we have already commented, the sets $\{\delta_{t_0} : t_0 \in X_0\}$ and $\{\delta_{\phi(t_0)} : t_0 \in X_0\}$ are norming in $C_0^{\mathbb{T}}(X)^*$ and $C_0^{\mathbb{T}}(Y)^*$, respectively. The same property holds for the set

$$\{\sigma^{\Delta}(t_0, 1) \,\delta_{\phi(t_0)} : t_0 \in X_0^+\} \cup \{\sigma^{\Delta}(t_0, 1) \,\overline{\delta_{\phi(t_0)}} : t_0 \in X_0^-\} \\ = \{\delta_{\sigma^{\Delta}(t_0, 1)\phi(t_0)} : t_0 \in X_0^+\} \cup \{\overline{\delta_{\sigma^{\Delta}(t_0, 1)\phi(t_0)}} : t_0 \in X_0^-\}.$$

By Proposition 3.17 for each $a \in S(C_0^{\mathbb{T}}(X))$ we have

(3.9)
$$\delta_{\phi(t_0)}(\Delta(a)) = \begin{cases} \delta_{\sigma^{\Delta}(t_0,1)\phi(t_0)}(a), & \text{if } t_0 \in X_0^+, \\ \\ \overline{\delta_{\sigma^{\Delta}(t_0,1)\phi(t_0)}}(a), & \text{if } t_0 \in X_0^-. \end{cases}$$

Lemma 6 in [35] (see also [18, Lemma 2.1]) implies that Δ admits an extension to a surjective real linear isometry. We can alternatively follow a similar argument to that in the proof of Theorem 2.1, and employ the identity in (3.9) to prove that the positive homogeneous extension of Δ is additive, and hence a real linear extension of Δ . The final conclusions are straightforward consequences of Lemma 3.2.

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