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Assessments in public procurement procedures☆

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ABSTRACT

In this paper we study how to assess the performance of a group of individuals according to their achievements in several attributes or categories by means of a scoring system. Such an assessment is the composition of two steps. First, each individual obtains a partial score in each category (that may potentially depend on her opponents' performance). And second, those partial scores are combined into a global assessment. The partial score in each attribute is upper bounded by an exogenous threshold or cap. Each problem is determined by four elements: a set of agents (or tenders), a set of attributes to be evaluated, a matrix of achievements that specified the score each agent has obtained in each attribute, and a vector of caps. By means of the axiomatic methodology, we identify the families of assessment functions that satisfy some natural requirements (*anonymity, continuity, monotonicity, null contribution, additivity*, and *separability*). Our findings state that these families are weighted averages of the attribute assessments. Finally, as an illustration, we analyze a public tender whose purpose was to carry out an accounts auditing of a public company. As a practical implication of our theoretical results, we show that truncation presents significant advantages with respect to other methods. Particularly, it avoids the exclusion paradox.

1. Introduction

In 2019 over 2000 billion euros of EU citizens' money was spent on public procurement procedures to provide goods and services to cities, regions, nations, and EU institutions. The destinations of most of the budget were medical equipment, pharmaceuticals and personal care products (329.3 million euros), IT services (375.4 million euros), and business services (297.3 million euros). The adjudication of contracts is made through a tendering process by which potential contestants submit their offers [20]. A committee assesses the offers and chooses the winner. Therefore, it is crucial to have good and appropriate scoring rules to assess bids, even more in those cases that may have particular relevance because of the budget, the economic consequences, or the political implications [14]. The purpose of this work is precisely that, we provide a mechanism to assess offers in a public procurement process. The method we propose is well founded from a social choice perspective, since we mathematically prove that it is the only one that satisfies a collection of suitable properties. In addition, it also avoids some of the malfunctioning of other methods. The comparisons are done by means of a detailed case study.

In public tenders, bidders submit their offers in detailed memos that explain several aspects of their proposals. Then, a committee of experts assesses those offers according to several attributes (price, technical quality, experience, proposed improvements,...) and assigns a score to each bidder in each attribute. Usually, each attribute has a cap, that is, a maximum threshold of the points that can be obtained in that attribute (40% for price, 30% for technical quality, 10% for experience, 10% for proposed improvements,...). The overall assessment of the bidder is the lump sum of points across attributes. The use of caps in the assessment of tenders is well settled by the EU regulations (Directives of the of the European Parliament and of the Council 2014/23/EU and 2014/24/EU [10], for example) but there exists a vivid controversy on how to apply those caps. Several resolutions of judicial courts have justified different methods such as *truncation, proportionality*, or *linearity*. As we will show in the case study we analyze in this work, the first method presents significant advantages. In addition, *truncation* is very well grounded from the perspective of the decision theory and social choice.

Tendering is just a particular case of a more general class of problems, where the performance of a group of agents must be

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assessed by means of a scoring process taking into consideration:[1] (i) their own performance in several attributes, (ii) the performance of their peers, and (iii) the existence of a cap that bounds the score of each attribute. In our setting, a *problem* has four elements: the set of agents, the set of attributes, the matrix of achievements that indicates the performance of each agent at each attribute, and the vector of caps that stipulates the maximum score any agent may obtain at each attribute. An *assessment function* is a mechanism to assess the agents considering all the elements of the problem. We consider assessment functions that are the composition of two other processes: a *partial assessment function* and an *aggregation function*. As in reality, a partial assessment functions assigns to each agent (each bidder, for example) a score that does not exceed the cap in each attribute (60 for price and 40 for technical quality, for instance). This score depends on the own performance and the other agents' achievements. An aggregation function summarizes, for each agent, the scores across attributes.

Since, in reality, the achievements are usually above the cap, a first adjustment needs to be made. The partial assessment function does this work. For each attribute, it assigns to each agent a score that fits within the cap. It depends potentially on her performance, the performance of her peers at that attribute, and the cap. The aggregation function aggregates the scores across attributes to determine the final assessment of the agents.

We analyze the problem from the perspective of the axiomatic methodology, by which the assessment functions are justified in terms of the axioms or properties they satisfy. In general, suitable combinations of properties are imposed as the desirable or minimal requirements that the assessment functions must satisfy. The goal is then to identify the solutions or unique solution that satisfy these axioms. For this purpose, we propose a collection of properties that are suitable for this problem.

The first group of properties we consider reflects basic principles of fairness. In particular, *monotonicity* states that the assessment of an individual cannot decrease if her performance increases. *Null contribution* requires that, if the achievement of an agent at one attribute is null, then zero points are scored. *Null agent* imposes that, if an agent is scored zero at all attributes, then her overall evaluation must also be zero. *Anonymity* says that the identity of the individuals cannot affect the assessment. The second group of axioms relates to the stability of the assessment with respect to changes in the achievements. Thus, *continuity* implies that small changes in the performance of individuals do not drastically alter the assessment. *Additivity* states that the assessment must be additive. This requirement has been widely applied for similar problems. However, we show that, in our framework, the null assessment function is the only solution that satisfies additivity.[2] This is due to the existence of caps. As an alternative, we propose *restricted additivity*, which follows the lines of additivity but it is not so demanding. In essence, restricted additivity requires additivity when there is no conflict with the caps. It is the furthest point to which we can extend this notion in our model. As an alternative to restricted additivity, we also analyze the separability principle, which has a long tradition in the decision-making literature [8,32]. We propose two properties: *attribute separability* and *agent separability*. The former states that difference in the evaluations due to a change in just one attribute is independent of the achievements in the rest of attributes. In the same line, *agent separability* requires that the difference in the evaluations due to a change in just one achievement of an agent is independent of the rest of achievements.

We show that the unique assessment functions that satisfy monotonicity, null contribution, continuity, and restricted additivity is the weighted sum of truncated partial assessment functions. These truncated partial assessment functions work as follows: for a given attribute, the score an agent obtains is the maximum between a linear transformation of her achievement and the cap. The aggregation function is simply the weighted sum of all those scores. The unique assessment method that is compatible with the previous properties is the composition of these two functions. The linear transformations and the weights may be different for different agents. If, in addition to the previous axioms, we also require anonymity, then the linear transformation and the weights must be the same for all individuals.

We also identify the unique assessment functions that satisfy null agent, monotonicity, null contribution, continuity, attribute separability, and agent separability. As in the family of the previous paragraph, the aggregation functions is a weighted sum of scores. However, for this combination of properties the partial assessment functions are continuous cumulative distribution functions, adjusted by the caps. The truncated partial assessment functions are just particular cases of the latter family.

1.1. Empirical illustration

To complete our analysis we propose a case study. In particular, we study the contract AD-13-009 of the Sociedad Urbanística Municipal de Vitoria - Gasteizco Udal Hirigintza Elkartea, whose task was to carry out the reports of accounts auditing and financial control actions of this public company. Six tenders submitted their offers, which were assessed according to six attributes, including price, experience, partner, etc. We show that, out of the three considered methods, truncation emerges as the most convenient. Its assessments are more stable (the relative position of two companies in the ranking is not affected by a third contestant) and it is the only one that avoids the exclusion paradox. Clauses to exclude abnormal or disproportionate bids are very usual in public tendering [11]. This prevents the adjudication to a tender whose bid is so low and unrealistic that the completion of the project is not guaranteed. When this happens, the tender is excluded from the process. What is the impact of the exclusion on the rest of the contestants? We say that a method suffers the exclusion paradox if the exclusion of a non-winning company changes the winner of the contract. If the method has this malfunction, the process would be easily manipulable; some tenders may have the incentive to invite third companies to participate with disproportionately low offers in order to alter the contest.

1.2. Related literature

One may interpret the problem we study as a situation in which several judges (the attributes) have opinions (agents' achievements) on some individuals (agents). The solution would be a social ordering or assessment of the individuals as a function of the judges' opinions. Several authors have analyzed the problem of information aggregation from different approaches. Arrow [2] demonstrated that when individuals' preferences are ordinal, there is no aggregation procedure that holds for a minimum set of reasonable conditions. Many other papers in the literature, following [21,22], or [23], among others, have focused on cardinal preferences. In our context, ordinality would allow for positive monotonic transformation of the achievements, that would not alter the social assessment. Cardinality, however, restricts the possibilities to only linear transformation. In our model, the existence of cap

[1] Even though public procurement is the main application of our mathematical model, other situations can also be embedded. Such is the case of the assessment of candidates for university positions, the performance of regions for the distribution of a federal budget, etc. With this in mind, we develop a theoretical framework general enough to encompass all these situations.

[2] The null assessment function is identically equal to zero, regardless the performance of the individuals.

precludes any possible transformation of the original achievements, since it would alter both the absolute and relative comparisons between the caps and the agents' goals.

The problem in hand can be analyzed from the perspective of multiple criteria decision analysis (MCDA), since we have several alternatives to evaluate according to various criteria. In fact, it is not difficult to classify this problem as a MCDA problem according to the the MCDA taxonomy in [7]. There are many MCDA method but not all are suitable for all decision problems. Watróbski et al. [44] study this problem of how to select a MCDA method for a particular decision situation and propose a methodological and practical framework for that. In this paper, we analyze the MCDA problem of public tendering and focus on the assessment functions most commonly used for this problem. In the MCDA literature we can find numerous papers on public tendering. A survey on this topic is [12]. Some recent references on public tendering or public procurement are the following. Mattar et al. [29] propose a method of exclusion and pairwise comparison as an alternative to the least-cost sufficient performance and the weighted utility optimization in procurement process in order to obtain a more robust procedure. Hatush and Skitmore [19] propose multi-criteria additive utility functions as method in contractor selection. Maybe one of the major problems of this approach is how to determine the utility functions associated with each criterion. Bana e Costa et al. [6] propose a methodology based on two phases: (1) determination, structuring and levelage of the criteria, (2) weighting of the criteria based on the MACBETH approach (see [4,5]). This methodology ends in an additive value model to assess the different proposals in a public call for tenders.

The most usual mistakes in practical public procurement are analyzed in [33]. Then they propose the use of a card system to determine the weights of the different criteria and the evaluation of the alternatives as a possible solution to avoid those mistakes. Mohemad et al. [31] propose an ontological extraction framework for the development of decision support systems to enhance the tender assessment process, in particular in construction tendering processes. Lorentziadis [26] studies the problem of the determination of the weights associated with the criteria after the opening of the sealed bids, instead of fixing the weights before. He analytically develops a variety of post-objective methods of weight determination. He proposes that, in the tendering process, the selected post-objective method for weight determination would be publicly announced with the call, so that the evaluation process would remain fair and objective. He justifies that as the exact weights are not known prior to the opening of the bids, corruption, specially in the preparation of the tender terms, might be significantly contained. Falagario et al. [16] propose the use of the cross efficiency evaluation based on the Data Envelopment Analysis (DEA), for evaluating different offers in a public tender awarded through the *Most Economically Advantageous Tender* (MEAT) criterion. To do that, some criteria are considered as inputs and the rest of criteria as outputs. As in [26], in this DEA-based approach there are also no weightings of the criteria before the sealed bids are opened. Vahdani et al. [42] and Diabagate et al. [9] use fuzzy logic for analyzing public procurement procedures. Vahdani et al. [42] propose the use of a compromise solution between the positive ideal solution and the negative ideal solution, in such a way that the chosen alternative to be as close as possible the best and as far away from the worst as possible, while [9] propose the use of the rule of proportion.

The problem of how to tackle with abnormally low bids in public tendering is studied in [3] and [18]. In particular, [3] analyze scoring formulas and abnormally low bids criteria in several countries, while [18] study how to assess abnormally low bids by means of a statistical approach. Mielcarz et al. [30] present a procedure of tender bids evaluation base on value management. This procedure consists of two steps. In the first step, the qualitative criteria are evaluated and the decision is simply whether bids are admisible or not. In the second step, the quantitative criteria are evaluated taking into account their impacts in the Net Present Value (NPV) of the project and the weights of the different criterio are obtained by means of an ordinary least square method. The bid with highest valuation in the quantitative criteria is the winner. Ek et al. [15] study the suitability of two MCDA methods to evaluate alternatives in the procurement phase of public works based on the MEAT criterion but incorporating sustainability criteria. Simoens and Cheung [41] review the literature on the inclusion of value-added services (VAS) in tendering for biosimilars. Lorentziadis [27] studies the behavior of bidders in MEAT public procurement when the bidders are asymmetric, and shows that equilibrium prices depend critically on the comparison of the cost difference with respect to the increment of the score attribute.

On the other hand, [17] analyze the impact of the number of offers submitted to public procurement tenders in the healthcare sector by means of generalized linear models and quantile regression. Vieira et al. [43] introduce the Collaborative Value Modeling framework to build evaluation models. This method combines web-Delphi and multi-criteria decision conferencing and allows to consider situations with many evaluation criteria and many stakeholders. This approach has also been proposed, for example, for the selection of contractors in the public sector. Marovic et al. [28] propose a methodology for the optimal constructor selection by means of combining the analytic hierarchy process (AHP) together with PROMETHEE. On the other hand, [24] propose a combination of Exploratory Factor Analysis, MACBETH and SMART for contractor selection in public sector construction.

Differently from most papers in the public procurement literature, we analyze the problem from the perspective of the axiomatic methodology.

The rest of the paper is structured as follows: In Section 2 we set the model. In Section 3 we present the properties we analyze. Section 4 is devoted to the main results and characterizations. In Section 5 we discuss the case study. Finally, Section 6 concludes. All proofs are included in an Appendix.

2. Model and assessment functions

Let us assume that we have a society consisting of n **agents**, $N = \{1, 2, \ldots, n\}$. We want to assess their individual performances as a function of their achievements with respect to a set of p **attributes**, $P = \{1, 2, \ldots, p\}$. A **matrix of achievements** is a matrix A with n rows (one for each agent) and p columns (one for each attribute). The element a_i^t of matrix A describes the achievement of agent i at attribute t:

$$A = \begin{pmatrix} a_1^1 & \dots & a_1^p \\ \vdots & \ddots & \vdots \\ a_n^1 & \dots & a_n^p \end{pmatrix} \in \mathbb{R}_+^n \times \mathbb{R}_+^p$$

We assume that each a_i^t is non-negative, but we do not impose any upper bound to its value. That is, a_i^t represents the original achievement, without any kind of truncation or normalization. We denote by a_i and a^t the ith row and tth column of the matrix A, respectively. We also denote by $\mathbb{A}$ the set of all possible matrices like A.

As explained in the Introduction, in practice, many performance assessments are based on a scoring system that includes an upper bound for the achievements in each attribute. We denote by $c = (c^1, \ldots, c^p) \in \mathbb{R}_{++}^p$ the vector of **caps**, where c^t is the maximum score that can be assigned to an agent in attribute t.

For the remainder of the paper we use the following notation. The null vector of size n is 0_n, while the null matrix of size $n \times p$ is $0_{n\times p}$. We use a_{-i} to denote the matrix A where we have removed the row corresponding to agent i. We use a^{-t} to denote the matrix A where we have removed the column corresponding to attribute t. Similarly, a_{-i}^{-t} results from A by removing the ith row and tth column. Analogous notation is used for the vector of caps.

An assessment problem with caps, or simply a **problem**, is a 4-tuple (N, P, A, c) consisting in a set of agents N, a set of attributes P, a matrix of achievements A and a vector of caps c. For our analysis we do not require changes in N, P, or c. Thus, for the sake of simplicity, we assume that those elements are fixed and describe a problem simply by A.

For each attribute $t \in P$ we define a **partial assessment function**, relative to this attribute, as a mapping $f^t : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$ that scores the performance of the agents according to their outcomes in attribute t. That is, $f(a^t) = \left(f_i^t(a^t)\right)_{i \in N}$ and each $f_i^t(a^t)$ is the partial score obtained by agent i at attribute t. Since c^t is the maximum partial score an agent can achieve, it must hold that $f_i^t(a^t) \leq c^t$ for all $i \in N$ and all $t \in P$. Notice that the partial assessment functions may differ both across agents and across attributes.

For each agent $i \in N$, an **aggregation function** is a mapping $F_i : [0, c^1] \times \ldots \times [0, c^p] \longrightarrow \mathbb{R}$ that aggregates the scores across attributes. An **assessment function** is a mapping $S : \mathbb{A} \longrightarrow \mathbb{R}^n$ that, for each problem $A \in \mathbb{A}$, assesses the performance of agents in N as the composition of the partial assessment functions and the aggregation functions, i.e., $S(A) = (S_i(A))_{i \in N}$, where

$$S_i(A) = F_i\left[f_i^1(a^1), f_i^2(a^2), \ldots, f_i^p(a^p)\right]$$

Next we present several assessment functions that can be applied to our model.

The first is straightforward, it states that the assessment of any agent is zero, regardless of the matrix of achievements.

Null assessment function. For each $A \in \mathbb{A}$ and each $i \in N$,

$$f_i^t(a^t) = 0 \, \forall t \in P, \text{ and } F_i(x^1, \ldots, x^p) = 0.$$

This is,

$$S_i(A) = 0.$$

The *truncation assessment function* works as follows. For each attribute, if the achievement of an agent is below the cap then the score equals the achievement, but if it is above the cap then it is truncated and the score is equal to the cap. The final assessment is simply the lump sum of scores.

Truncation assessment function. For each $A \in \mathbb{A}$ and each $i \in N$,

$$f_i^t(a^t) = \min\{c^t, a_i^t\} \, \forall t \in P \text{ and } F_i(x^1, \ldots, x^p) = \sum_{t=1}^{t} x^t.$$

This is,

$$S_i(A) = \sum_{t=1}^{p} \min\{c^t, a_i^t\}$$

The previous assessment function can be generalized easily by introducing weights into the aggregation function. In principle, we may weight attributes differently for different agents.

Weighted truncation assessment functions. Given the weights $(\beta_1^1, \ldots, \beta_n^p)$. For each $A \in \mathbb{A}$ and each $i \in N$,

$$f_i^t(a^t) = \min\{c^t, a_i^t\} \, \forall t \in P, \text{ and } F_i(x^1, \ldots, x^p) = \sum_{t=1}^{p} \beta_i^t x^t.$$

This is,

$$S_i(A) = \sum_{t=1}^{p} \beta_i^t \min\{c^t, a_i^t\}.$$

In the next method, the partial assessment function assigns the cap to the agent with the highest achievement in the attribute, and then rescales the rest of the agents proportionally. The aggregation function is the lump sum of scores.

Proportional assessment function. For each $A \in \mathbb{A}$ and each $i \in N$,

$$f_i^t(a^t) = \begin{cases} \frac{a_i^t}{\max_{j \in N} a_j^t} c^t & \text{if } \max_{j \in N} a_j^t > 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall t \in P$$

and $F_i(x^1, \ldots, x^p) = \sum_{t=1}^{p} x^t$. This is,

$$S_i(A) = \sum_{t=1}^{p} \frac{a_i^t}{\max_{j \in N} a_j^t} c^t$$

In the last case, the partial assessment function scores with the cap to the agent with the highest achievement and 0 to the agent with the lowest one. Then, the rest of agents are rescaled accordingly. The aggregation functions adds the scores.

Linear assessment function. For each $A \in \mathbb{A}$ and each $i \in N$,

$$f_i^t(a^t) = \begin{cases} \frac{a_i^t - \min_{j \in N} a_j^t}{\max_{j \in N} a_j^t - \min_{j \in N} a_j^t} c^t & \text{if } \max_{j \in N} a_j^t - \min_{j \in N} a_j^t \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall t \in P$$

and $F_i(x^1, \ldots, x^p) = \sum_{t=1}^{p} x^t$. This is,

$$S_i(A) = \sum_{t=1}^{p} \frac{a_i^t - \min_{j \in N} a_j^t}{\max_{j \in N} a_j^t - \min_{j \in N} a_j^t} c^t$$

3. Properties for the assessment functions

We now enumerate the axioms for assessment functions we consider reasonably relevant for public procurement procedures.

Monotonicity says that, if an agent increases her performance then the assessment (via the partial assessment and aggregation functions) should not decrease.

Monotonicity. For each $i \in N$, and each $A, \overline{A} \in \mathbb{A}$ such that $\overline{A} = (\overline{a}_i, a_{-i})$, if $a_i \geq \overline{a}_i$, then[3]

$$f_i^t(a_i^t, a_{-i}^t) \geq f_i^t(\overline{a}_i^t, a_{-i}^t) \quad \forall t \in P$$

and

$$F_i\left[f_i^1(a_i^1, a_{-i}^1), \ldots f_i^p(a_i^p, a_{-i}^p)\right] \geq F_i\left[f_i^1(\overline{a}_i^1, a_{-i}^1), \ldots f_i^p(\overline{a}_i^p, a_{-i}^p)\right].$$

The next requirement is sometimes needed for technical reasons. However, it makes considerable intuitive sense. *Continuity* states that small changes in the achievements of agents should not cause large changes in their assessment.

Continuity. For each sequence $\{A^\nu\}$ of problems in $\mathbb{A}$ and each $A \in \mathbb{A}$, if $\{A^\nu\}$ converges to A then $\{S(A^\nu)\}$ converges to $S(A)$.

The following property requires that, when an agent has no achievement at all in an attribute, the partial assessment (with respect to that attribute) must be zero.

Null contribution. For each $i \in N$ and each $t \in P$, if $a_i^t = 0$ then S must be such that $f_i^t(a^t) = 0$.

Notice that this property does not exclude the possibility of assigning null assessment to other achievements a_i^t different from zero. This makes sense in situations where very low performances should not even be taken into consideration.

The *null agent* principle is quite straightforward, it simply says that, if the achievements of an agent are all null, then her assessment must be zero.

Null agent. For each $i \in N$, $a_i = 0_P$ then $S_i(A) = 0$.

The next condition is a minimal and natural requirement of impartiality. It simply says that the identity of the agents should not play any role in the assessment of the performance. More precisely,

[3] We use the notation $a_i \geq \overline{a}_i$ when $a_i^t \geq \overline{a}_i^t$ for all $t \in P$.

if we permute the label of the agents, their assessments permute accordingly.

Anonymity. If π is a permutation of the set of agents, $S_i(A) = S_{\pi(i)}(\pi(A))$.

In many situations where performances must be assessed, *additivity* emerges as a desirable property. It states that the assessment must be additive with respect to A. In this case, if new achievements are obtained by the agents, we just need to add the new score to the existing one, without having to re-assess the whole situation from the beginning.

Additivity. For each $A, \overline{A} \in \mathbb{A}$ and each $i \in N$,

(i) If $f_i^t(a^t + \overline{a}^t) = f_i^t(a^t) + f_i^t(\overline{a}^t)$.

(ii) If $F_i\left[f_i^1(a^1) + f_i^1(\overline{a}^1), \ldots, f_i^p(a^p) + f_i^p(\overline{a}^p)\right] = F_i\left[f_i^1(a^1), \ldots, f_i^p(a^p)\right] + F_i\left[f_i^1(\overline{a}^1), \ldots, f_i^p(\overline{a}^p)\right]$.

As the following proposition illustrates, when caps exist for the partial assessment functions, additivity is extremely demanding. If it is required, the assessment function must be identically equal to zero.

Proposition 1. *If S satisfies additivity then, for each $i \in N$, $S_i(A) = 0$ for all $A \in \mathbb{A}$.*

We substitute additivity with a milder version (called *restricted additivity*) which essentially says that the assessment must be additive as long as the model (this is, the caps) allows this. This alternative version avoids the drawbacks presented in the proof of Proposition 1. The new condition is in the line of the spirit of additivity and, simultaneously, can be applied to our setting.

Restricted additivity. For each $A, \overline{A} \in \mathbb{A}$ and each $i \in N$

(i) If $f_i^t(a^t) + f_i^t(\overline{a}^t) \le c^t$, then

$$f_i^t(a^t + \overline{a}^t) = f_i^t(a^t) + f_i^t(\overline{a}^t).$$

(ii) If $f_i^t(a^t) + f_i^t(\overline{a}^t) \le c^t$ for all $t \in P$, then

$$F_i\left[f_i^1(a^1) + f_i^1(\overline{a}^1), \ldots, f_i^p(a^p) + f_i^p(\overline{a}^p)\right] = F_i\left[f_i^1(a^1), \ldots, f_i^p(a^p)\right] + F_i\left[f_i^1(\overline{a}^1), \ldots, f_i^p(\overline{a}^p)\right].$$

Next, we present an alternative to additivity. Imagine that, for a particular attribute $t \in P$, the achievements change from a^t to $\overline{a}^t$, while all the other attributes remain constant. *Attribute separability* requires that any eventual variation in the assessments only depends on the values of a^t and $\overline{a}^t$.[4] Formally,

Attribute separability. For each $i \in N$ and each $t \in P$, $F_i(x^{-t}, x^t) - F_i(x^{-t}, \bar{x}^t) = \varphi(x^t, \bar{x}^t)$, for all $x^{-t} \in [0, c^1] \times \ldots [0, c^{t-1}] \times [0, c^{t+1}] \ldots \times [0, c^p]$ and for all $x^t, \bar{x}^t \in [0, c^t]$.

Using a similar approach, we can also consider *agent separability*, which states, if the achievement changes from a_i^t to $\overline{a}_i^t$ (keeping all the others constant), any eventual variation in the assessments only depends on the values of a_i^t and $\overline{a}_i^t$.

Agent separability. For each $i \in N$ and each $t \in P$, $f_i^t(a_{-i}^t, a_i^t) - f_i^t(a_{-i}^t, \overline{a}_i^t) = \psi(a_i^t, \bar{a}_i^t)$, for all $a_{-i}^t \in \mathbb{R}_+^{n-1}$ and for all $a_i^t, \bar{a}_i^t \in \mathbb{R}_+$.

4. Main results: Characterizations of assessment functions

We now provide our main characterizations, which identify the set of assessment functions that uniquely satisfy the properties discussed in the previous section. These results are preceded by two technical lemmas.

[4] The notion of separability has a long tradition of use in decision making (e.g. [8] or [32]).

Lemma 1. *If an assessment function satisfies restricted additivity and null contribution then $F_i[0, \ldots, 0] = 0$ for any $i \in N$.*

The next lemma is a kind of counterpart of the Cauchy's functional equation when additivity is replaced by restricted additivity. In functional analysis it can be shown that a function $\phi : \mathbb{R}_+ \longrightarrow [0, c]$ satisfies additivity ($\phi(x + y) = \phi(x) + \phi(y)$) if and only if ϕ is linear. If we weaken the condition to restricted additivity the result does not then hold. In fact, as the following lemma illustrates, more requirements on the function ϕ are needed in order to recover the linearity, or truncate linearity, to be more precise.

Lemma 2. *A continuous function $\phi : \mathbb{R}_+ \longrightarrow [0, c]$ satisfies that*

(i) *If $x > y$ then $\phi(x) \ge \phi(y)$ for all $x, y \in \mathbb{R}_+$.*

(ii) *If $\phi(x) + \phi(y) \le c$ then $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}_+$.*

(iii) $\phi(0) = 0$.

if and only if

$$\phi(x) = 0 \quad \text{or} \quad \phi(x) = \min\left\{c, \frac{c}{z}x\right\},$$

where $z = \min\{x \in \mathbb{R}_+ | \phi(x) = c\}$.

Our main result states that, if continuity, restricted additivity, monotonicity, and null contribution are required, then the assessment function must work as follows. For each attribute, the partial assessment function is the minimum between the cap and a linear transformation of the achievement of the agent. The aggregation function is the weighted sum of the partial assessment functions. In principle, both the linear transformation and the weights may differ across attributes and across agents.

Theorem 1. *An assessment function satisfies continuity, restricted additivity, monotonicity, and null contribution if and only if, for each $i \in N$, there exist $(\beta_i^1, \ldots, \beta_i^p) \in \mathbb{R}_+^p$ and $(\lambda_i^1, \ldots, \lambda_i^p) \in \mathbb{R}_+^p$ such that:*

$$S_i(A) = F_i\left(f_i^1(a^1), f_i^2(a^2), \ldots, f_i^p(a^p)\right) = \sum_{t=1}^{p} \beta_i^t \min\left\{\lambda_i^t a_i^t, c^t\right\} \quad (1)$$

If, in addition to continuity, restricted additivity, monotonicity, and null contribution we also impose the assessment function to be anonymous, then the weights and the linear transformations of the previous theorem must be the same for all agents, although they may depend on the attribute.

Theorem 2. *A non-degenerated assessment function satisfies continuity, restricted additivity, monotonicity, null contribution, and anonymity if and only if there exist $(\beta^1, \ldots, \beta^p) \in \mathbb{R}_{++}^p$ and $(\lambda^1, \ldots, \lambda^p) \in \mathbb{R}_+^p$ such that, for each $i \in N$,*

$$S_i(A) = F_i\left(f_i^1(a^1), f_i^2(a^2), \ldots, f_i^p(a^p)\right) = \sum_{t=1}^{p} \beta^t \min\left\{\lambda^t a_i^t, c^t\right\}.$$

Next, we explore more general results. We find out that, if we replace restricted additivity by attribute separability and null agent, then the assessment function must be a weighted sum of the partial assessment functions. However, in contrast with the previous characterizations, these partial assessment function may be non-linear. Before presenting our characterizations, some technical lemmas are necessary.

Lemma 3. *If an assessment function satisfies monotonicity, null contribution, and null agent then $F_i(x^1, \ldots, x^p) \ge 0$, for each $x^t \in [0, c^t]$, $t \in P$, for all $i \in N$.*

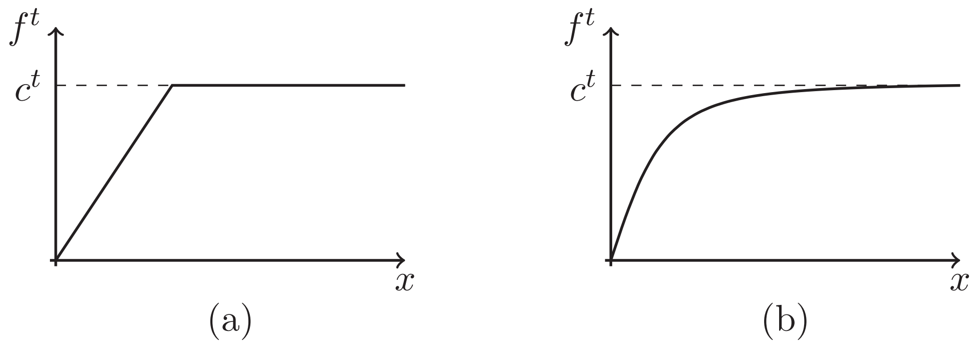

Fig. 1. Examples of partial assessment functions in Theorems 2 and 6.

Lemma 4. *If an assessment function S satisfies null contribution, null agent, and attribute separability then there exist functions* $\{H_i^t\}_{i\in N}^{t\in P}$ *defined on* $\mathbb{R}_+^n$*, such that, for each* $i \in N$*,*

$$S_i(A) = \sum_{t\in P} H_i^t(a^t), \text{ for all } A \in \mathbb{A},$$

where $H_i^t(a_{-i}^t, 0) = 0$.

The next result states that, if an assessment function satisfies monotonicity, continuity, null contribution, null agent, and attribute separability, then it must work as follows. For each attribute, the partial assessment function is the product of the cap and a continuous and non-decreasing function that varies between 0 and 1. The aggregation function is the weighted sum of those partial assessment functions. Formally,

Theorem 3. *A non-degenerated assessment function S satisfies monotonicity, continuity, null contribution, null agent, and attribute separability if and only if, for each* $i \in N$*, there exist* $(\beta_i^1, \dots, \beta_i^p) \in \mathbb{R}_+^p$ *and continuous and non-decreasing functions* $\{h_i^t\}^{t\in P}$ *defined on* $\mathbb{R}_+^n$ *with image on the interval [0,1] such that*

$$S_i(A) = \sum_{t\in P} \beta_i^t c^t h_i^t(a^t),$$

where $h_i^t(a_{-i}^t, 0) = 0$.

If, in addition to the previous properties, we also require anonymity, then we obtain the following characterization.

Theorem 4. *A non-degenerated assessment function S satisfies monotonicity, continuity, null contribution, null agent, attribute separability, and anonymity if and only if there exist* $(\beta^1, \dots, \beta^p) \in \mathbb{R}_+^p$ *and continuous and non-decreasing functions* $\{h^t\}^{t\in P}$ *defined on* $\mathbb{R}_+^n$ *with image on the interval [0,1] such that, for each* $i \in N$*,*

$$S_i(A) = \sum_{t\in P} \beta^t c^t h^t(a^t)$$

where $h^t(a_{-i}^t, 0) = 0$.

Now, we explore the implications of including agent separability as an additional requirement in Theorems 3 and 4.

Lemma 5. *If an assessment function S satisfies null contribution and agent separability then there exist functions* $\{g_i^t\}_{i\in N}^{t\in P}$*, such that, for each* $i \in N$*,*

$$f_i^t(a^t) = g_i^t(a_i^t).$$

The following theorem states that, if an assessment function satisfies monotonicity, continuity, null contribution, null agent, attribute separability, and agent separability then the aggregation function is the weighted sum of those partial assessment functions. And, for each attribute, the partial assessment function is the product of the cap and a continuous cumulative distribution function defined on $\mathbb{R}_+$. Formally,

Theorem 5. *A non-degenerated assessment function S satisfies monotonicity, continuity, null contribution, null agent, attribute separability, and agent separability if and only if, for each* $i \in$*, there exist* $(\beta_i^1, \dots, \beta_i^p) \in \mathbb{R}_+^p$ *and cumulative distribution functions* $\{h_i^t\}^{t\in P}$ *of absolutely continuous random variables with images on* $\mathbb{R}_+$ *such that*

$$S_i(A) = \sum_{t\in P} \beta_i^t c^t h_i^t(a_i^t),$$

If, in addition to the previous properties, we also require anonymity, then we obtain the following characterization.

Theorem 6. *A non-degenerated assessment function S satisfies monotonicity, continuity, null contribution, null agent, attribute separability, agent separability, and anonymity if and only if there exist* $(\beta^1, \dots, \beta^p) \in \mathbb{R}_+^p$ *and cumulative distribution functions* $\{h^t\}^{t\in P}$ *of absolutely continuous random variables with images on* $\mathbb{R}_+$ *such that*

$$S_i(A) = \sum_{t\in P} \beta^t c^t h^t(a_i^t).$$

It is worth noting that Theorem 6 (Theorem 5) is more general than Theorem 2 (Theorem 1). In both cases the aggregation function is a weighted average of the attributes. Nevertheless, the requirement of restricted additivity in Theorem 2 restricts the partial assessment functions to be essentially linear. If we replace restricted additivity by attribute separability and null agent, Theorem 6 shows that many more alternatives emerge. Fig. 1 illustrates the difference. While both Cases (a) and (b) (linear and non-linear) are admissible partial assessment functions in Theorem 6, only Case (a) (linear) is compatible with the properties in Theorem 2.

It is obvious that the properties in Theorem 6 expands the alternatives of the central authority to choose among, but it also makes more difficult to select just one of them.

5. Case study

The provision of many public services (including contracting works, goods, services and personnel) are made by means of a competition in public procurement, in which tenders submit their offers to be assessed according to several criteria. In order to determine the best option bids are assessed and scored in each criterion/attribute, and the overall assessment of the bidder is the sum of points across attributes. This process entails several difficulties, some of which have ended up in judicial courts. In order to reflect the relative relevance of a criterion, maximum values for the score are usually established. The caps in our theoretical model represent these maximum values. What to do when a bid exceeds the cap has been the main source of controversy. The three most applied alternatives are truncation, proportionality, and linearity, which have been formally introduced Section 2.

Notice that these three procedures correspond to three of the assessment functions defined in Section 2. In public tendering, the most applied methods are truncation and normalization, and there exists a significant controversy between the supporters of each them. To better understand this discussion, we highlight below the legal and judicial principles that are used to resolve for or against truncation.

Law 9/2017 [25], of November 8, on Public Sector Contracts, by which the Directives of the European Parliament and of the Council 2014/23/EU and 2014/24/EU, of February 26th, are transposed into the Spanish legal system, establishes in its Art [11]. 131 that "*The award will be made, ordinarily using a plurality of award criteria based on the principle of best quality/price ratio,...* ". These criteria can be objective, i.e., quantifiable or value judgment, that is, subjective assessment (Art. 145). Law 9/2017 [25] does establish in Art.146.3 the use of thresholds to indicate indirectly the relevance of each criterion in the adjudication of the contract. However, nothing is specified on how to operate with those thresholds/caps. Therefore, these rules and regulations (both at national and European level) leave the choice between truncation or normalization to the decision maker's interpretation.

Which are the arguments for and against truncation? In order to disentangle this issue, we use as a guide the seminar by Doménech Pascual [13]. The arguments against truncation are:

- It prevents more advantageous or better offers from obtaining higher scores (R/45/2016, R/40/2018) [34,35]. The only justification is to assume ex-ante that the offer is unreliable as abnormal or disproportionate.[5] But, for those cases, there already exists a procedure set forth in the Law (R/45/2016, R/40/2018, R/75/2020) [34–36].
- It breaks proportionality in score (R/40/2018, R/143/2019, R/75/2020) [34,36,37].
- It leads to the paradox that abnormal or disproportionate offers are awarded maximum score (R/40/2018) [34].
- It eliminates the real weight of the criteria because many ties may occur (R/143/2019, R/75/2020) [36,37].
- It implies that scores can be known ex ante (R/40/2018, R/75/2020) [34,36].
- It discourages competitiveness because the effort required to submit better offers does not result in extra score (R/45/2016, R/40/2018, R/75/2020) [34–36].

In summary, all previous resolutions consider that truncation breaks proportionality in the assessment and violates the principles of equal treatment and efficiency.

However, supported by the Directive 2014/24/UE which has been transposed into the Law 9/2017, recent resolutions R/976/2018 [40], R/484/2019 [38], R/853/2019 [39] establish that the criterion "price" in no way is always and in any case equivalent to "lower price". The criteria are related and linked to each other and are defined by the contracting authority, which can determine how they operate and are applied, and if the cost factor can take the form of a fixed price not subject to improvement due to a reduction according to the Directive 2014/24/UE, for a greater reason a non-fixed price must be admitted, but limited by a satiety threshold, which could be reduced beyond that limit, but without being favored by an increase of points in its valuation. Therefore, satiety thresholds are admissible when more than one criterion is used. Likewise, better offers in a criterion do not obtain fewer points than others. Furthermore, satiety thresholds can be considered as a complementary measure that discourage abnormal or disproportionate offers which can be excluded from the competition, because if you can do the most, you can do the least, which is, compared to the exclusion of the abnormal offer, the non-allocation of more points to the offers of price below the established threshold. Thus, if the contracting authority can reduce the weight of the price criterion with respect to the other objective and subjective criteria, with greater reason it can increase the weighting of that and set a maximum limit of the price reduction that the bidders can bid but from which they do not obtain additional points. Of course, when there is only one criterion then a satiety threshold makes no sense, since many ties may occur, leaving the decision to the subjective discretion of the decision maker. Even with all these arguments, R/976/2018 considers that the use of satiety thresholds is not the best practice [40], because it is preferable to let the prices offered by the different bidders be those that they freely decide, based on their forecast costs and expectations of profit.

Table 1
Attributes and caps.

Attribute	Description	Cap
Price (b)	Cost of the audit report. The reference bid price was 60,000 euros	51
Description (d)	Methodology, timeline, and planning	1
Partners (p)	Working hours by partners of the auditing firm	11.4
Others (o)	Working hours by other members of the team	7.6
Experience (e)	Experience in similar contracts	20
Improvements (i)	Proposals to enhance the contract	9

Table 2
Initial offers of the companies in the public tender AD-13-009. The bids come in euros, while the rest of attributes are points obtained according to the committee's assessment, which may or may not be above the cap.

	Attributes / Criteria					
Company	*b*	*d*	*p*	*o*	*e*	*i*
A	14,452	1	9.16	4.24	16	3
B	30,000	1	12.00	2.40	20	7
C	39,000	1	0.00	0.00	16	1
D	40,935	1	8.21	4.44	0	1
E	51,000	1	7.89	8.00	14	3
F	60,000	1	0.00	5.48	16	1

Now, we present the case study to make a comparative analysis of the three discussed methods (truncation, normalization, and range normalization). To this end we consider the public tender AD-13-009 of the Sociedad Urbanística Municipal de Vitoria - Gasteizco Udal Hirigintza Elkartea, Ensanche 21 Zabalgunea, S.A. The purpose of this contract was to carry out the reports of accounts auditing and financial control actions of the public company. The criteria used in the awarding and their corresponding caps are in Table 1. Six tenders submitted their offers. The awarding committee assigned scores to each of the applicants, resulting in the matrix of achievements represented in Table 2. The method applied to this particular public procurement process was proportionality.

Table 3 shows the results of the application of the truncation, proportional, and linear assessment functions for the public tender we study.[6] The first column is the name of the tender, Columns 2 to 7 are the result of applying the assessment functions

[5] See [18].

[6] The formal expressions of these functions are in Section 2. Since lower prices are preferred to higher prices, we have considered the inverse of the bid, so that, the best offer (the lowest price) gets the highest score in this attribute.

Table 3
Ranking comparisons of the initial offers depending on the methodology.

Company	Attributes / Criteria *b*	*d*	*p*	*o*	*e*	*i*	Total	Ranking
Truncation								
A	38.72	1.00	9.16	4.24	16.00	3.00	72.12	1
B	25.50	1.00	11.40	2.40	20.00	7.00	67.30	2
C	17.85	1.00	0.00	0.00	16.00	1.00	35.85	4
D	16.21	1.00	8.21	4.44	0.00	1.00	30.86	5
E	7.65	1.00	7.89	7.60	14.00	3.00	41.14	3
F	0.00	1.00	0.00	5.48	16.00	1.00	23.48	6
Proportionality								
A	51.00	1.00	8.70	4.03	16.00	3.86	84.59	1
B	24.57	1.00	11.40	2.28	20.00	9.00	68.25	2
C	18.90	1.00	0.00	0.00	16.00	1.29	37.18	4
D	18.01	1.00	7.80	4.22	0.00	1.29	32.31	6
E	14.45	1.00	7.50	7.60	14.00	3.86	48.41	3
F	12.28	1.00	0.00	5.21	16.00	1.29	35.78	5
Linearity								
A	51.00	1.00	8.70	4.03	16.00	3.00	83.73	1
B	33.59	1.00	11.40	2.28	20.00	9.00	77.27	2
C	23.51	1.00	0.00	0.00	16.00	0.00	40.51	4
D	21.35	1.00	7.80	4.22	0.00	0.00	34.37	5
E	10.08	1.00	7.50	7.60	14.00	3.00	43.18	3
F	0.00	1.00	0.00	5.21	16.00	0.00	22.21	6

Table 4
Ranking comparisons after Company A's update.

Company	Attributes / Criteria *b*	*p*	*t*	*t*	*e*	*i*	Total	Ranking
Truncation								
A	14.15	1.00	9.16	4.24	16.00	3.00	47.55	2
B	25.50	1.00	11.40	2.40	20.00	7.00	67.30	1
C	17.85	1.00	0.00	0.00	16.00	1.00	35.85	4
D	16.21	1.00	8.21	4.44	0.00	1.00	30.86	5
E	7.65	1.00	7.89	7.60	14.00	3.00	41.14	3
F	0.00	1.00	0.00	5.48	16.00	1.00	23.48	6
Proportionality								
A	35.29	1.00	8.70	4.03	16.00	3.86	68.88	2
B	51.00	1.00	11.40	2.28	20.00	9.00	94.68	1
C	39.23	1.00	0.00	0.00	16.00	1.29	57.52	4
D	37.38	1.00	7.80	4.22	0.00	1.29	51.68	5
E	30.00	1.00	7.50	7.60	14.00	3.86	63.96	3
F	25.50	1.00	0.00	5.21	16.00	1.29	49.00	6
Linearity								
A	28.31	1.00	8.70	4.03	16.00	3.00	61.04	2
B	51.00	1.00	11.40	2.28	20.00	9.00	94.68	1
C	35.70	1.00	0.00	0.00	16.00	0.00	52.70	3
D	32.41	1.00	7.80	4.22	0.00	0.00	45.43	5
E	15.30	1.00	7.50	7.60	14.00	3.00	48.40	4
F	0.00	1.00	0.00	5.21	16.00	0.00	22.21	6

(truncation, proportionality, or linearity) to each attribute. Column 8 is the lump sum of points, and Column 9 is the position in the ranking. As can be observed, these three methods lead to different orderings on the overall assessment of the tenders. Company A would win the contract with any of the methods employed. This is mostly due to the aggressive price offered by A, less than a half of the price proposed by the tender with the next lowest bid (Company B). Even though truncation and proportionality provides the same winner, the overall assessments of A and B are much closer to the truncation assessment function than to the proportional assessment function. By definition, both proportionality and linearity always assign the cap to the firm with the best offer, artificially overweighting the impact of, in this case, the price. Truncation, on the contrary, behaves differently, the cap may or may not be achieved (see Column *b* in Table 3) and thus the overweighting is less problematic.[7]

The EU regulations, for example, foresee the inclusion of clauses of abnormality or disproportionality so that an excessively low bid can be considered as reckless and excluded from the process [3]. In the case we study there was no such clause. However, the companies had the possibility to submit an updated offer within an established period. Company A did this, increasing the price from the original 14,452 euros to 43,350 euros (almost triple). No other modification was made, and the rest of the tenders kept their initial offers. Table 4 contains the results for the new situation. In this case truncation and proportionality provide the same ordering but linearity differs. In all three cases Company B is the new winner. After A's modification we arrive at two conclusions. One, under truncation the assessments of all companies except A remain invariant. And two, under proportionality and linearity, a change in A's offer alters the relative ordering of third tenders. Under proportionality, before A's modification, F's offer is better than D's offer (35.78 vs. 32.31), but after A's modification, D's offer is better than F's offer (51.68 vs. 49.00). An analogous argument applies to linearity, also affecting Companies D and F.

In general, it can be proved that truncation does not alter the assessments of tenders other than the one affected by the change

Table 5
Modified offers of the companies in the public tender AD-13-009. Changes with respect to the initial data are underlined.

Company	Attributes / Criteria *b*	*d*	*p*	*o*	*e*	*i*
A	14,452	1	4.10	3.03	4	0
B	30,000	1	12.00	2.40	16	3
C	39,000	1	0.00	0.00	16	1
D	40,935	1	10.00	6.20	20	8
E	51,000	1	7.89	8.00	10.00	3
F	60,000	1	0.00	5.48	16	1

in one of the attributes. As Table 4 shows, both proportionality and linearity do not. In the literature on social choice it is very usual to impose methods which satisfy *independence of third alternatives*. This property states that the relative position of two firms (that have not modified their offers) is not affected by a change in a third. Truncation does fulfill this requirement, but proportionality and linearity do not. As we will show below, the violation of this property may be specially harmful in public procurement processes.

Consider, just for illustration, the same public tender but with some, but realistic, changes in the offers of companies A, B and D. New data are in Table 5.

As mentioned above, clauses to exclude abnormal or disproportionate bids are very usual in public tender. One of its goals is to avoid the adjudication to a tender whose bid is so low and unrealistic that the completion of the project is not guaranteed. A common clause of abnormality or disproportionality would be the following: *An offer is considered abnormal if it is either 25% lower than the tender base price or 10% lower than the average of the bids. In this case, the bidding companies have to justify that they can carry out the contract with the bid submitted*. When this happens, tenders may end up being expelled from the process if she can not justify in a satisfactory manner the low level of price. What is the impact of the exclusion of a company on the rest of the contestants? We say that a method satisfies the *exclusion* property if the exclusion of a tender does not alter the ordering of the other tenders. In particular, if a non-winning company is excluded then the winner of the contract should not change. Otherwise, the process would be easily manipulable, some tenders may have the incentive to invite

[7] Notice that in Table 3 we apply the assessment functions as they are defined in Section 2. For sake of exposition we assume that all attributes are equally relevant. The same reasoning can be done with alternative sets of weights.

Table 6
Ranking comparisons for the new situation.

Company	Attributes / Criteria *b*	*p*	*t*	*t*	*e*	*i*	Total	Ranking
Truncation								
A	38.72	1.00	4.10	3.03	4.00	0.00	50.85	3
B	25.50	1.00	11.40	2.40	16.00	3.00	59.30	2
C	17.85	1.00	0.00	0.00	16.00	1.00	35.85	5
D	16.21	1.00	10.00	6.20	20.00	8.00	61.41	1
E	7.65	1.00	7.89	7.60	10.00	3.00	37.14	4
F	0.00	1.00	0.00	5.48	16.00	1.00	23.48	6
Proportionality								
A	51.00	1.00	3.90	2.88	4.00	0.00	62.77	2
B	24.57	1.00	11.40	2.28	16.00	3.38	58.62	3
C	18.90	1.00	0.00	0.00	16.00	1.13	37.02	5
D	18.01	1.00	9.50	5.89	20.00	9.00	63.40	1
E	14.45	1.00	7.50	7.60	10.00	3.38	43.93	4
F	12.28	1.00	0.00	5.21	16.00	1.13	35.62	6
Linearity								
A	51.00	1.00	3.90	2.88	0.00	0.00	58.77	3
B	33.59	1.00	11.40	2.28	15.00	3.38	66.65	2
C	23.51	1.00	0.00	0.00	15.00	1.13	40.64	4
D	21.35	1.00	9.50	5.89	20.00	9.00	66.74	1
E	10.08	1.00	7.50	7.60	7.50	3.38	37.05	5
F	0.00	1.00	0.00	5.21	15.00	1.13	22.34	6

Table 7
Ranking comparisons for the new situation after the exclusion of Company A.

Company	Attributes / Criteria *b*	*p*	*t*	*t*	*e*	*i*	Total	Ranking
Truncation								
A	-	-	-	-	-	-	-	-
B	25.50	1.00	11.40	2.40	16.00	3.00	59.30	2
C	17.85	1.00	0.00	0.00	16.00	1.00	35.85	4
D	16.21	1.00	10.00	6.20	20.00	8.00	61.41	1
E	7.65	1.00	7.89	7.60	10.00	3.00	37.14	3
F	0.00	1.00	0.00	5.48	16.00	1.00	23.48	5
Proportionality								
A	-	-	-	-	-	-	-	-
B	51.00	1.00	11.40	2.28	16.00	3.38	85.06	1
C	39.23	1.00	0.00	0.00	16.00	1.13	57.36	4
D	37.38	1.00	9.50	5.89	20.00	9.00	82.77	2
E	30.00	1.00	7.50	7.60	10.00	3.38	59.48	3
F	25.50	1.00	0.00	5.21	16.00	1.13	48.84	5
Linearity								
A	-	-	-	-	-	-	-	-
B	51.00	1.00	11.40	2.28	12.00	2.57	80.25	1
C	35.70	1.00	0.00	0.00	12.00	0.00	48.70	3
D	32.41	1.00	9.50	5.89	20.00	9.00	77.80	2
E	15.30	1.00	7.50	7.60	0.00	2.57	33.97	4
F	0.00	1.00	0.00	5.21	12.00	0.00	18.21	5

third companies to participate with disproportionately low offers in order to alter the contest.

Table 6 shows the application of truncation, proportionality, and linearity to data in Table 5. As we can observe, in all three cases Company D receives the adjudication of the contract, and, depending on the methods, the second tender in the ranking is either A or B. Now, notice that the price offered by A is unrealistically low, below the 25% of the tender reference price -60,000 euros- and less than a half of the next lowest price. According to a clause of abnormality or disproportionality, Company A would be excluded.[8] If that happens, all the scores must be recalculated (Table 7). Under proportionality, the winner of the contract has changed after the exclusion of A, Company D originally received the adjudication and now it goes to Company B. The same issue applies to linearity. Thus, the disqualification of a tender has modified the bidding of the contract. Therefore, both proportionality and linearity violate the exclusion property. Truncation, on the contrary, satisfies this requirement, always. In fact, among all the assessment functions presented in Section 2, truncation is the only one that fulfills exclusion and is immune to manipulations.

6. Final remarks

In this work, we have considered the problem of providing a cardinal assessment of the performance of a group of agents across several issues. As done in practice in many situations, we have focused on scoring methods that are the composition of two steps. First, each agent is scored in each attribute, with the limitation that the score cannot exceed an upper bound of points exogenously set (the cap). This score may potentially depend on the own performance and other individuals' achievements. In the second step, the scores are aggregated in order to obtain the agents' overall assessment. The first step is formalized by a *partial assessment function*, and the second by an *aggregation function*. An *assessment function* is simply the composition of a partial assessment and an aggregation function.

There are many possible partial assessment and aggregation functions, and therefore many more assessment functions. For instance, the partial assessment may simply truncate the achievement when it exceeds the cap, scoring all agents equally when they perform above the cap. As an alternative, we may also rescale the scores, in order to keep, up to certain level, the disparities in the achievements. As for the aggregation function, we may consider the arithmetic or geometric mean (weighted or unweighted), the maximum, etc. It is obvious that, depending on the choices we make, the assessment function will have more or less appealing properties.

We have analyzed this problem from an axiomatic perspective. In order to do that, we have presented several axioms that are suitable for this framework. Some of them relates to principles of fairness, while other applies notions of stability. In the first group we have *anonymity, monotonicity, null contribution*, and *null agent. Continuity, restricted additivity, attribute separability*, and *agent separability* are in the second group. We have two main characterizations. First, we find that, if we impose monotonicity, null contribution, continuity, and restricted additivity, then we must use a particular class of assessment functions. These assessment functions are very simple. For each agent, do the weighted sum of a linear transformation of her achievements, truncated if they exceed the caps. In this family of assessment functions, both the weights and the linear transformations are degrees of freedom, since they may vary across individuals and attributes. If, in addition to the previous axioms, we also require anonymity, we obtain that the weights and the linear transformation must be the same for all agents, although they may differ for attributes. Secondly, we show that the combination of monotonicity, null contribution, null agent, continuity, attribute separability, and agent separability also leads to assessment functions whose aggregate functions are weighted sums. However, the partial assessment functions are more general than in the previous case. If we also imposed anonymity, then the partial assessment functions are probability distribution functions adjusted by the caps. It still remains as an open question how to determine the weights for the aggregation function. However, Theorems 1 to 6 are general enough to leave the choice to the discretion of the central planner, who may accommodate, for each procurement process, the specificities and characteristics of considered attributes.

One may argue that, in practice, restricted additivity is less appealing than additivity. Proposition 1 states that if a procurement process imposes maximum thresholds (caps) in the evaluation of one or several attributes, then any assessment function (other than the null rule) will violate additivity. That is, the mere presence of

8 This would be an extreme but illustrative situation hat will only happen if Company A is not able to provide a satisfactory justification for the low price.

caps entails a reformulation of this property. In this line, restricted additivity reconciles the existence of caps with the principle underlying additivity. Restricted additivity states that the assessment function must be additive anywhere but in those situations where the caps apply.

In the case study we have analyzed a public procurement to make an audit on the financial situation of a public company. We have compared the three most applied assessment functions (truncation, proportionality, and linearity), and we conclude that truncation is the best option of all those three methods. It provides assessments that do not artificially outweigh some attributes, and it is immune to manipulations because it satisfies the exclusion property (the exclusion of a bidder with an abnormal offer does not change the winner of the contract).

In our characterizations the weights of the formula are not specified, leaving to the central authority the choice on which attributes should be more relevant in the scoring. One may wonder about the possibility of endogenizing those weights. Even though we did not carry out an exhaustive analysis, we have explored several methods, and we have obtained that none those are compatible with the properties in Theorems 1 and 5. Our findings suggest that none of the assessment functions in these results would admit endogenous weights. For sake of illustration, let us follow [26] to analyze the application of the *average least favorable* and *average most favorable* methods. Let us consider the following matrix of achievements, with three agents and three attributes:

$$A = \begin{pmatrix} 1 & 6 & 5 \\ 3 & 5 & 3 \\ 2 & 2 & 4 \end{pmatrix}$$

Let us suppose that $[W^1, W'^1] = [0.5, 1]$, $[W^2, W'^2] = [0, 2]$, and $[W^3, W'^3] = [0.2, 1]$ are the intervals for Attributes 1, 2, and 3, respectively. Then, the vectors of least favorable weights are $\boldsymbol{v_1} = (0, 0.2, 0.8)$, $\boldsymbol{v_2} = (0.5, 0, 0.5)$, and $\boldsymbol{v_1} = (0.5, 0.3, 0.2)$ for Agents 1, 2, and 3, respectively. In any average least favorable method $Q_1 = Q_2 = Q_3 = \frac{1}{3}$. Then, $v^1 = \frac{1}{3}$, $v^2 = \frac{0.5}{3}$, and $v^3 = 0.5$. Therefore, $h_1 = \frac{4.5}{3}$, $h_2 = \frac{11.5}{3}$, and $h_3 = \frac{12.5}{3}$. That is, the project is assigned to Agent 3. Now, imagine that the achievement of Agent 3 at Attribute 1 increases from 2 to 10. The new matrix of achievements is

$$A' = \begin{pmatrix} 1 & 6 & 5 \\ 3 & 5 & 3 \\ 10 & 2 & 4 \end{pmatrix}$$

The new vectors of least favorable weights are $\boldsymbol{v_1} = (0.8, 0, 0.2)$, $\boldsymbol{v_2} = (0.5, 0, 0.5)$, and $\boldsymbol{v_1} = (0.5, 0.3, 0.2)$. Then, $v^1 = 0.6$, $v^2 = 0.1$, and $v^3 = 0.3$. Therefore, $h_1 = 3.9$, $h_2 = 4.7$, and $h_3 = 4.5$. The project is now assigned to Agent 2. To summarize, Agent 1 has modified her achievements and, as a result, the winner has changed from Agent 2 to Agent 3, even though none of them have altered their scores. If, instead, we apply the average most favorable methods, we get similar behaviors.

On one hand, the previous endogenizing methods lead to the violation of the principle of *independence of third alternatives* (which may be a problem by itself). On the other hand, any assessment function in Theorems 1 and 5 satisfies this property. Therefore, the weights in the family of assessment functions characterized in Theorems 1 and 5 could not be endogenized. In other words, there is a trade-off between the properties required in Theorem 1 and 5 and the endogenous determination of the weights. We must choose either one of the other.

We acknowledge that there are still some open issues we do not address in this paper and deserve a deeper analysis in future works. We suggest two potential extensions to be explored. One, the properties on additivity, or separability, may not be adequate for several situations. Even though separability provides many more alternatives that additivity, in both cases the aggregation function is an additive mean (excluding the geometric mean, for example). Further research would address the question of identifying the family of assessment functions that satisfy monotonicity, null contribution, continuity, and some other appealing axioms, but without imposing restricted additivity or separability. And two, even though the proportional and linear assessment functions are not the focal mechanisms in our study, they are quite used in practice. Therefore, it is natural to wonder which are the properties that characterize these two assessment functions.

CRediT authorship contribution statement

Ricardo Martínez: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Funding acquisition. **Joaquín Sánchez-Soriano:** Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Funding acquisition. **Natividad Llorca:** Conceptualization, Methodology, Formal analysis, Investigation.

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Appendix A. Appendix

A1. Proof of Proposition 1

Proof. Suppose that there exist $A \in \mathbb{A}$, $i \in N$, and $t \in P$ such that $f_i^t(a^t) = x > 0$. On the one hand, by *additivity*, we know that $f_i^t(kA) = kx$ for any $k \in \mathbb{Z}_+$. If k is large enough the value of kx exceeds the cap c^t, which contradicts the definition of partial assessment function. Therefore, $f_i^t(a^t) = 0$ for any $a^t \in \mathbb{R}_+^n$ and any $i \in N$. On the other hand, it must happen that $F_i[0, \dots, 0] = 0$ for any $i \in N$. Indeed, let $x = F[0, \dots, 0]$. By *additivity*, $2x = F[2 \cdot 0, \dots, 2 \cdot 0] = F[0, \dots, 0] = x$. And thus, $x = 0$. □

A2. Proof of Lemma 1

Proof. Let us consider the null matrix $0_{n\times p} \in \mathbb{A}$. Let $i \in N$. Because of *null contribution* $f_i^t(0_n) = 0$ for all $t \in P$. Hence, in application of *restricted additivity*, $2F_i[0, \dots, 0] = F_i[0, \dots, 0]$. Therefore, $F_i[0, \dots, 0] = 0$. □

A3. Proof of Lemma 2

Proof. It is straightforward to check that the functions in the statement are continuous and satisfy Conditions (i) to (iii). We prove the converse. Let ϕ be a continuous function that fulfills the three conditions of the statement.

The function ϕ must satisfy that, either it is identically equal to zero, or there exists $\hat{x} \in \mathbb{R}_+$ such that $\phi(\hat{x}) > 0$. If the former happens, we have already concluded the proof and ϕ is of one of the types of the statement. If the latter happens, let us define the values z and w as

$$z = \min\{x \in \mathbb{R}_+ | \phi(x) = c\} \quad \text{and} \quad w = \min\left\{x \in \mathbb{R}_+ \middle| \phi(x) = \frac{c}{2}\right\}$$

Those values w and z exist. Indeed, we distinguish two cases:

- If $\hat{x}$ is such that $\phi(\hat{x}) \geq \frac{c}{2}$. Since ϕ is continuous and $\phi(0) = 0$ by Condition (iii), in application of the intermediate value theorem, there exists $w \in [0, \hat{x}]$ such that $\phi(w) = \frac{c}{2}$. As an immediate implication of Condition (ii), the value z must also exist.
- If $\hat{x}$ is such that $\phi(\hat{x}) < \frac{c}{2}$. By Condition (ii) we know that $\phi(2\hat{x}) = 2\phi(\hat{x})$. And now we repeat the argument, if $\phi(2\hat{x})$ is above $\frac{c}{2}$ the intermediate value theorem ensures the existence of w, but if $\phi(2\hat{x})$ is below we can again apply Condition (ii) to obtain that $\phi(4\hat{x}) = 4\phi(\hat{x})$. Applying this argument iteratively, there must be an iteration where we can use the previous case to conclude that w exists. Otherwise, we would obtain that $2^m\phi(\hat{x}) = \phi(2^m\hat{x}) < \frac{c}{2}$ for any $m \in \mathbb{Z}_{++}$. Or, equivalently, $\phi(\hat{x}) < \frac{c}{2^{m+a}}$ for any $m \in \mathbb{Z}_{++}$. But this is impossible for a positive integer m arbitrarily large (unless $\phi(\hat{x}) = 0$).

By Condition (ii) we know that $\phi(2w) = 2\phi(w) = c = \phi(z)$. Now, we show that it must happen that $z = 2w$. Supposing that it is not the case, there are only two possibilities

- If $z > 2w$. This contradicts the definition of z because $\phi(2w) = c$.
- If $z < 2w$. Notice that $z - w < w$, and hence $\phi(z - w) \leq \frac{c}{2}$. Thus, $\phi(w) + \phi(z - w) \leq c$. Condition (ii) implies that

$$\phi(z) = \phi(w) + \phi(z - w) \;\equiv\; c = \frac{c}{2} + \phi(z - w) \;\equiv\; \phi(z - w) = \frac{c}{2}$$

By definition of w, $z - w \geq w$, which contradicts the assumption that $z < 2w$.

Therefore, $z = 2w$. We distinguish now several cases:

- Let $x, y \in [0, w]$. Because of monotonicity, $\phi(x), \phi(y) \leq \phi(w) = \frac{c}{2}$. In application of Condition (ii) (since $\phi(x) + \phi(y) \leq \frac{c}{2} + \frac{c}{2} = c$) we obtain that, within this interval, $\phi(x + y) = \phi(x) + \phi(y)$. This is the Cauchy's equation of a continuous and non-decreasing function, and thus $\phi(x) = \lambda x$ for some $\lambda \in \mathbb{R}_+$.[9] Since $\phi(w) = \frac{c}{2}$ and $z > w$, we conclude that

$$\phi(x) = \frac{c}{z} \cdot x$$

- Let $x \in [w, 2w]$. Condition (ii) implies that $\phi(x) = \phi(w) + \phi(x - w)$ (because $x - w \in [0, w]$ and $\phi(w) + \phi(x - w) \leq c$). Hence,

$$\phi(x) = \phi(w) + \phi(x - w) = \frac{c}{2} + \frac{c}{z}(x - w) = \frac{c}{2} - \frac{c}{z}\left(x - \frac{z}{2}\right) = \frac{c}{z} \cdot x$$

- Let $x \in [z, +\infty[$. Since ϕ is non-decreasing and upper bounded by the value c we have that $\phi(x) = c$.

Considering all cases together, we conclude that

$$\phi(x) = \min\left\{c, \frac{c}{z}x\right\}.$$

To conclude the argument we need to show that the value z exists, which amounts to saying that w exists (since $z = 2w$). □

[9] See [1] for several results on Cauchy's equation.

A4. Proof of Theorem 1

Proof. It is clear from Eq. (1) that

$$F_i\Big(f_i^1(a^1), f_i^2(a^2), \ldots, f_i^p(a^p)\Big) = \sum_{t=1}^{p} \beta_i^t f_i^t(a^t)$$

and

$$f_i^t(a^t, c^t) = \min\left\{\lambda_i^t a_i^t, c^t\right\}$$

We first check that any assessment function in the statement of the problem satisfies the four properties.

- Continuity. It is obvious because both F and f_i^t are continuous.
- Monotonicity. It is obvious that functions f_i^t are monotonic with respect to a_i^t, and that the function F_i is also monotonic.
- Null contribution. If $a_i^t = 0$ then, by definition, $f_i^t((a_i^t, a_{-i}^t)) = 0$.
- Restricted additivity. Let us suppose that $f_i^t(a^t) + f_i^t(\overline{a}^t) \leq c^t$ for some $a^t, \overline{a}^t \in \mathbb{R}^n$. In such a case $f_i^t(a^t) = \lambda_i^t a_i^t$, $f_i^t(\overline{a}^t) = \lambda_i^t \overline{a}_i^t$, and $\lambda_i^t a_i^t + \lambda_i^t \overline{a}_i^t \leq c^t$. Thus, we have that

$$\begin{aligned} f(a^t + \overline{a}^t) &= \min\{\lambda_i^t(a_i^t + \overline{a}_i^t), c^t\} \\ &= \lambda_i^t(a_i^t + \overline{a}_i^t) = \min\{\lambda_i^t a_i^t, c^t\} + \min\{\lambda_i^t \overline{a}_i^t, c^t\} \\ &= f_i^t(a^t) + f_i^t(\overline{a}^t) \end{aligned}$$

Now, if $f_i^t(a^t) + f_i^t(\overline{a}^t) \leq c^t$ for all $t \in P$, then

$$\begin{aligned} &F_i\Big[f_i^1(a^1) + f_i^1(\overline{a}^1), \ldots, f_i^p(a^p) + f_i^p(\overline{a}^p)\Big] = \\ &\quad \sum_{t=1}^{p} \beta_i^t(f_i^t(a^t) + f_i^t(\overline{a}^t)) = \\ &\quad F_i\Big[f_i^1(a^1), \ldots, f_i^p(a^p)\Big] + F_i\Big[f_i^1(\overline{a}^1), \ldots, f_i^p(\overline{a}^p)\Big] \end{aligned}$$

Let us see the converse. We distinguish three possible cases depending on the structure of the matrix A.

Case (1) . Let $A_k^r \in \mathbb{A}$ be a matrix all whose entries but one (a_k^r) are null:

$$A_k^r = \begin{pmatrix} a_k^r; & 0; & 0; & 0_n; \end{pmatrix}$$

By definition,

$$\begin{aligned} S_i(A_k^r) &= F_i\Big[f_i^1\big(a^1\big), \ldots, f_i^r(a^r), \ldots, f_i^p(a^p)\Big] \\ &= F_i\Big[f_i^1(0_n), \ldots, f_i^r\big(0, \ldots, a_k^r, \ldots, 0\big), \ldots, f_i^p(0_n)\Big] \end{aligned}$$

In application of *null contribution*, we know that $f_i^t(0_n) = 0$ for all $t \in P$ and $f_i^r\big(0, \ldots, a_k^r, \ldots, 0\big) = 0$ when $i \neq k$. Therefore,

$$S_i(A_k^r) = \begin{cases} F_i\Big[0, \ldots, f_i^r\big(0, \ldots, a_k^r, \ldots, 0\big), \ldots, 0\Big] & \text{if } i = k \\ F_i[0, \ldots, 0] & \text{if } i \neq k \end{cases}$$

By applying Lemma 1, we obtain that

$$S_i(A_k^r) = \begin{cases} F_i\Big[0, \ldots, f_i^r\big(0, \ldots, a_k^r, \ldots, 0\big), \ldots, 0\Big] & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Now, let us define the functions ϕ_k^r and Φ_k^r as follows

$$\phi_k^r(a_k^r) = f_k^r\big(0, \ldots, a_k^r, \ldots, 0)\big)$$

and

$$\Phi_k^r(a_k^r) = F_k\Big[0, \ldots, \phi_k^r(a_k^r), \ldots, 0\Big]$$

Then,

$$S_i(A_k^r) = \begin{cases} \Phi_k^r(a_k^r) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Monotonicity implies that $0 \leq \Phi_k^r(a_k^r) \leq \gamma_k^r$, where $\gamma_k^r = F_k[0, \ldots, c^r, \ldots, 0]$. Let $z_k^r = \min\left\{x \in \mathbb{R}_+ | \phi_k^r(x) = c^r\right\}$ and $w_k^r = \min\left\{x \in \mathbb{R}_+ | \phi_k^r(x) = \frac{c^r}{2}\right\}$. Notice that $z_k^r = 2w_k^r$.[10] Since ϕ_k^r satisfies the conditions of Lemma 2, we know that either ϕ_k^r is identically equal to zero or[11]

$$\phi_k^r(a_k^r) = \min\left\{c^r, \frac{c^r}{z_k^r}a_k^r\right\} = \begin{cases} \frac{c^r}{z_k^r}a_k^r & \text{if } a_k^r \leq z_k^r \\ c^r & \text{if } a_k^r \geq z_k^r \end{cases}$$

- If $x, y \in [0, w_k^r]$. Notice that $\phi_k^r(x) + \phi_k^r(y) = \frac{c^r}{z_k^r}x + \frac{c^r}{z_k^r}y = \frac{x+y}{2w_k^r}c^r \leq c^r$. In application of *restricted additivity* we have that $\Phi_k^r(x+y) = \Phi_k^r(x) + \Phi_k^r(y)$. This is the Cauchy's functional equation, whose solution in this context is:

$$\Phi_k^r(x) = \frac{F_k\left[0, \ldots, \frac{c^r}{2}, \ldots, 0\right]}{w_k^r}x = \frac{S_k(\hat{A}_k^r, c)}{w_k^r}x,$$

where $\hat{A}_k^r$ is such that $\hat{a}_k^r = \frac{c^r}{2}$.
- If $x \in [w_k^r, 2w_k^r] = [w_k^r, z_k^r]$. Since $\phi_k^r(w_k^r) + \phi_k^r(x - w_k^r) \leq c^r$ (because of the definition of w_k^r and the fact that $x - w_k^r \in [0, w_k^r]$), *restricted additivity* implies that

$$\Phi_k^r(x) = \Phi_k^r(w_k^r) + \Phi_k^r(x - w_k^r) = \frac{F_k\left[0, \ldots, \frac{c^r}{2}, \ldots, 0\right]}{w_k^r}x = \frac{S_k(\hat{A}_k^r, c)}{w_k^r}x,$$

where $\hat{A}_k^r$ is such that $\hat{a}_k^r = \frac{c^r}{2}$.
- If $x \in [z_k^r, +\infty[$ then, by *monotonicity*, $\Phi_k^r(x) \geq \Phi_k^r(z_k^r) = \gamma_k^r$. Since Φ_k^r is upper bounded by γ_k^r, we conclude that in this case

$$\Phi_k^r(x) = \gamma_k^r,$$

being $\gamma_k^r = S_k(\overline{A}_k^r)$, where $\overline{A}_k^r$ is such that $\overline{a}_k^r = c^r$

Therefore,

$$S_i(A_k^r) = \begin{cases} \Phi_k^r(a_k^r) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases},$$

where

$$\Phi_k^r(a_k^r) = \begin{cases} \alpha_k^r a_k^r & \text{if } a_k^r \in [0, z_k^r] \\ \gamma_k^r & \text{if } a_k^r \in [z_k^r, +\infty[\end{cases}, \tag{A.1}$$

$\alpha_k^r = \frac{F_k\left[0, \ldots, \frac{c^r}{2}, \ldots, 0\right]}{w_k^r} = \frac{F_k\left[0, \ldots, \phi_k^r(w_k^r), \ldots, 0\right]}{w_k^r}$, and $\gamma_k^r = F_k[0, \ldots, c^r, \ldots, 0]$.

Case (2) . Let $A^r \in \mathbb{A}$ be a matrix all whose columns but one (a^r) are null:

$$A^r = \left(0_n; \quad a^r; \quad 0_n; \quad \right)$$

Using an argument similar to the previous case, we have that

$$S_i(A^r) = F_i\left[0, \ldots, f_i^r\left(a_i^r, \ldots, a_n^r\right)\right), \ldots, 0\right]$$

We can easily express A^r as a sum of matrices like those in Case (1): $A^r = \sum_{k=1}^n A_k^r$, where each A_k^r has all the entries equal to zero except, eventually, a_k^r. Notice that, because of *null contribution*, $\sum_{k=1}^n f_i^r\left(0, \ldots, a_k^r, \ldots, 0\right) = f_i^r\left(0, \ldots, a_i^r, \ldots, 0\right) \leq c^r$. Then, in application of *restricted additivity*, $S_i(A^r) = \sum_{k=1}^n S_i(A_k^r)$. Since, by Case (1), we already know the expression of each $S_i(A_k^r)$, we can write:

$$S_i(A^r) = \sum_{k=1}^n S_i(A_k^r) = S_i(A_i^r) = \Phi_i^r(a_i^r),$$

where Φ_i^r is given by Eq. (A.1).

Case (3) . Let $A \in \mathbb{A}$ be a general matrix without any restriction on its entries. Notice that

$$A = \sum_{r=1}^p A^r,$$

where each A^r is a matrix all whose columns, except a^r, are null. Notice that, because of *null contribution* $\sum_{r=1}^p f_i^t\left((A^r)^t\right) = f_i^t\left((A^t)^t\right) \leq c^t$ for all $t \in P$. Because of *restricted additivity*, $S_i(A) = \sum_{r=1}^p S_i(A^r)$. Since, by Case (2), we already know the expression of each $S_i(A^r)$, we can write:

$$S_i(A) = \sum_{r=1}^p S_i(A^r) = \sum_{r=1}^p S_i(A_i^r) = \sum_{r=1}^p \Phi_i^r(a_i^r),$$

where Φ_i^r is given by Eq. (A.1).

It remains to see what happens when ϕ_k^r is null in Case (1). If $\phi_k^r(a_k^r) = 0$ then $S_i(A_k^r) = \Phi_k^r(a_k^r) = 0$. Therefore, using the argument of Case (2), we have that for attribute r it holds that $S_i(A^r) = \Phi_i^r(a_i^r) = 0$.

Finally, let us define $\beta_i^r \in \mathbb{R}$ and $\lambda_i^r \in \mathbb{R}$ as follows

$$\beta_i^r \in \left\{0, \frac{\gamma_i^r}{c^r}\right\} \quad \text{and} \quad \lambda_i^r = \alpha_i^t \frac{c^r}{\gamma_i^r}$$

Then $\Phi_i^r(a_i^r) = \beta_i^r \cdot f_i^r(a^r, c^r)$ where

$$f_i^r(a^r, c^r) = \min\{\lambda_i^r a_i^r, c^r\} = \begin{cases} \lambda_i^r a_i^r & \text{if } a_i^r \leq z_i^r \\ c^r & \text{if } a_i^r \geq z_i^r \end{cases}$$

□

A5. Proof of Lemma 3

Proof. Le $i \in N$. By *null contribution* and *null agent*, $F_i(0, \ldots, 0) = 0$. Now, by *monotonicity* $F_i(x^1, \ldots, x^p) \geq 0$ for each $x^t \in [0, c^t], t \in P$. □

A6. Proof of Lemma 4

Proof. Since S satisfies *attribute separability*, we know that there exist functions G_i^t such that

$$F_i(x^1, \ldots, x^p) = \sum_{t \in P} G_i^t(x^t).$$

Therefore, we have that

$$S_i(A) = \sum_{t \in P} G_i^t(f_i^t(a^t)) = \sum_{t \in P} g_i^t(a^t),$$

where $g_i^t = G_i^t \circ f_i^t$.

Since S satisfies *null agent* and *null contribution*,

$$S_i((a_{-i}^t, 0)_{t \in P}) = \sum_{t \in P} g_i^t(a_{-i}^t, 0) = \sum_{t \in P} g_i^t(0^t) = 0.$$

We now define the following functions

$H_i^t(a^t) = g_i^t(a^t) - g_i^t(a_{-i}^t, 0).$

On one hand, it is obvious that $H_i^t(a_{-i}^t, 0) = 0$. On the other hand, we have that

$$S_i(A) = \sum_{t \in P} g_i^t(a^t) = \sum_{t \in P} g_i^t(a^t) - \sum_{t \in P} g_i^t(a_{-i}^t, 0) = \sum_{t \in P} H_i^t(a^t).$$

□

[10] The argument has already been showed in the proof of Lemma 2.

[11] We now focus on the case when ϕ_k^r is not null and will discuss the other possibility further down the proof.

A7. Proof of Theorem 3

Proof. We consider the following assessment function S:

$$S_i(A) = F_i\left(f_i^1(a^1), f_i^2(a^2), \dots, f_i^p(a^p)\right) = \sum_{t=1}^{p} \beta_i^t f_i^t(a^t),$$

and

$$f_i^t(a^t) = c^t h_i^t(a^t).$$

First, we check that any assessment function defined as in the statement satisfies the five properties.

- Monotonicity. It is obvious that functions f_i^t are monotonic with respect to a_i^t, and that the function F_i is also monotonic.
- Continuity. It is obvious because both F_i and f_i^t are continuous.
- Null contribution. If $a_i^t = 0$ then, by definition, $f_i^t(a_{-i}^t, 0) = c^t h_i^t(a_{-i}^t, 0) = 0$.
- Null agent. It is obvious because $h_i^t(a_{-i}^t, 0) = 0$, for all $i \in N$ and $t \in P$.
- Attribute separability. This is obvious because of the definition of F_i.

Let us see the converse. Let S be an assessment function satisfying the five properties with partial assessment functions f_i^t and aggregations functions F_i. Let $i \in N$. On the one hand, in application of *attribute separability*, there exist functions $\{G_i^t\}^{t\in P}$ such that

$$F_i(x^1, \dots, x^p) = \sum_{t\in P} G_i^t(x^t),$$

where each G_i^t is non-decreasing because of *monotonicity*. On the other hand, in application of Lemma 4 there exist functions $\{H_i^t\}^{t\in P}$ such that

$$S_i(A) = F_i(f_i^1(a^1), \dots, f_i^p(a^p)) = \sum_{t\in P} H_i^t(a^t),$$

where $H_i^t = G_i^t \circ f_i^t$ and $H_i^t(a_{-i}^t, 0) = 0$. Again, by *monotonicity*, we know that each H_i^t is non-decreasing. Since S satisfies *continuity*, these functions must be also continuous.

Now, let us define h_i^t as follows:

$$h_i^t(a^t) = \frac{H_i^t(a^t)}{G_i^t(c^t)}.$$

Notice that, *monotonicity* and the definition of partial assessment function imply that $H_i^t(a^t) \le G_i^t(c^t)$ for all $a^t \in \mathbb{R}_+^p$. That is, $h_i^t(a^t) \in [0,1]$ for all $a^t \in \mathbb{R}_+^p$. Finally, let us define $\beta_i^t = \frac{G_i^t(c^t)}{c^t}$. Then,

$$S_i(A) = \sum_{t\in P} H_i^t(a^t) = \sum_{t\in P} \beta_i^t c^t h_i^t(a^t)$$

□

A8. Proof of Lemma 5

Proof. On the one hand, since S satisfies *agent separability*, $f_i^t(a^t) = \sum_{j\in N} g_{ij}^t(a_i^t)$. On the other hand, since S satisfies *null contribution*, for all $a_{-i}^t \in \mathbb{R}_+^{n-1}$, we have that

$$f_i^t(a_{-i}^t, 0) = 0 = g_{ii}^t(0) + \sum_{j\in N\setminus i} g_{ij}^t(a_j^t).$$

This implies that $\sum_{j\in N\setminus i} g_{ij}^t(a_j^t) = g_{ii}^t(0)$, for all $a_{-i}^t \in \mathbb{R}_+^{n-1}$, therefore, we can rewrite f_i^t as follows

$$f_i^t(a^t) = g_{ii}^t(a_i^t) - g_{ii}^t(0).$$

Now, we define $g_i^t(a_i^t) = g_{ii}^t(a_i^t) - g_{ii}^t(0)$, for all $a_i^t \in \mathbb{R}_+$. □

A9. Proof of Theorem 5

Proof. We consider the following assessment function S:

$$S_i(A) = F_i\left(f_i^1(a^1), f_i^2(a^2), \dots, f_i^p(a^p)\right) = \sum_{t=1}^{p} \beta_i^t f_i^t(a^t),$$

and

$$f_i^t(a^t) = c^t h_i^t(a_i^t).$$

First, we check that any assessment function defined as in the statement satisfies the six properties.

- Monotonicity. It is obvious that functions f_i^t are monotonic with respect to a_i^t, and that the function F_i is also monotonic.
- Continuity. It is obvious because both F_i and f_i^t are continuous.
- Null contribution. If $a_i^t = 0$ then, by definition, $f_i^t(a_{-i}^t, 0) = c^t h_i^t(0) = 0$.
- Null agent. It is obvious because $h_i^t(0) = 0$, for all $i \in N$ and $t \in P$.
- Attribute separability. This is obvious because of the definition of F_i.
- Agent separability. It is obvious by definition of h_i^t, for all $i \in N$ and $t \in P$.

Let us see the converse. Let S be an assessment function satisfying the six properties with partial assessment functions f_i^t and aggregations functions F_i. Let $i \in N$. On the one hand, in application of *attribute separability*, we have that there exist functions $G_i^t, t \in P$ such that

$$F_i(x^1, \dots, x^p) = \sum_{t\in P} G_i^t(x^t),$$

where each G_i^t is non-decreasing because of *monotonicity*. On the other hand, in application of Lemma 5 there exist functions $\{g_i^t\}^{t\in P}$ such that

$$f_i^t(a^t) = g_i^t(a_i^t).$$

Again, by *monotonicity*, we know that each g_i^t is non-decreasing. Therefore, we have that

$$S_i(A) = \sum_{t\in P} G_i^t(g_i^t(a_i^t)).$$

Let us define $H_i^t = G_i^t \circ g_i^t$. Since S satisfies *continuity*, these functions must be also continuous. Furthermore, since S satisfies *null agent* and *monotonicity*, $H_i^t(0) = 0$ for any $t \in P$.

Now, let us define h_i^t as follows:

$$h_i^t(a_i^t) = \frac{H_i^t(a_i^t)}{F_i(0, \dots, 0, c^t, 0, \dots, 0)}.$$

Notice that, *monotonicity* and the definition of partial assessment function imply that $H_i^t(a_i^t) \le F_i(0, \dots, 0, c^t, 0, \dots, 0)$ for all $a_i^t \in \mathbb{R}_+$. That is, $h_i^t(a^t) \in [0,1]$ for all $a_i^t \in \mathbb{R}_+$. Finally, let us define $\beta_i^t = \frac{F_i(0,\dots,0,c^t,0,\dots,0)}{c^t}$. Then,

$$S_i(A) = \sum_{t\in P} H_i^t(a_i^t) = \sum_{t\in P} \beta_i^t c^t h_i^t(a_i^t),$$

where the β_i^t's and the functions h_i^t's are in the conditions of the statement of the theorem. □

References

[1] Aczel J. A short course on functional equations. D Reidel Publishing Company; 1987.

[2] Arrow K. A difficulty in the concept of social welfare. J Polit Economy 1950;58:328–46.

[3] Ballesteros-Pérez P, Skitmore M, Pellicer E, González-Cruz MC. Scoring rules and abnormally low bids criteria in construction tenders: a taxonomic review. Constr Manage Econ 2015;33:259–78.

[4] Bana e Costa CA, Vansnick JC. MACBETH - an interactive path towards the construction of cardinal value functions. Int Trans Oper Res 1994;1:489–500.
[5] Bana e Costa CA, Vansnick JC. Applications of the MACBETH approach in the framework of an additive aggregation model. J Multi-Criteria Decis Anal 1997;6:107–14.
[6] Bana e Costa CA, Corrêa EC, De Corte JM, Vansnick JC. Facilitating bid evaluation in public call for tenders: a socio-technical approach. Omega 2002;30:227–42.
[7] Cinelli M, Kadzinski M, Gonzalez M, Slowinski R. How to support the application of multiple criteria decision analysis? Let us start with a comprehensive taxonomy. Omega 2020;96:102261.
[8] Debreu G. Topological methods in cardinal utility theory. Mathematical methods in the social sciences.. Arrow K, Karlin S, Suppes P, editors. Stanford University Press; 1960.
[9] Diabagate A, Azmani A, El Harzli M. Tendering process: improvement of analysis and evaluation of tenders based on the use of fuzzy logic and rule of proportion. Int J Comput Appl 2014;101:43–50.
[10] Directive2014/23/EU of the European Parliament and of the Council of 26 February 2014 on the award of concession contracts Text with EEA relevance.
[11] Directive 2014/24/UE of the European Parliament and of the Council of 26 February 2014 on public procurement and repealing directive 2004/18/EC Text with EEA relevance.
[12] Domberger S, Rimmer S. Competitive tendering and contracting in the public sector: a survey. Int. J. Econ. Bus. 1994;1:439–53.
[13] Doménech Pascual G. Umbrales de saciedad. “Prohibidos o permitidos” Part of the course: Contratación y Compliance: Hacia una Contratación de Calidad, Innovadora e Integra. Fundación Universidad-Empresa de Valencia ADEIT, Valencia (Spain), March 9–10, 2020; 2020. (In Spanish)
[14] Dykstra M, van der Windt N. Beauty contest design. In: M004 report. SEOR-Erasmus competition & regulation institute (Janssen, coord.); 2002. p. 1–18.
[15] Ek KM, Rempling A, Rosén L, Claeson-Jonsson C, Brinkhoff P, Norin M. Multi--criteria decision analysis methods to support sustainable infrastructure construction 2019.
[16] Falagario M, Sciancalepore F, Costantino N, Pietroforte R. Using a DEA-cross efficiency approach in public procurement tenders. Eur J Oper Res 2012;218:523–9.
[17] Gavurova B, Mikeska M, Huculova E. Evaluation of selected determinants of public procurement in the health sector. Adm Manage Public 2020;34:45–63.
[18] Gunduz M, Karacan HV. Assessment of abnormally low tenders: a multinomial logistic regression approach. Technol Econ Dev Economy 2017;23:848–59.
[19] Hatush Z, Skitmore M. Contractor selection using multicriteria utility theory: an additive model. Build Environ 1998;33:105–15.
[20] Janssen MCW. Auctions and beauty contests. A policy perspective. In: M004 Report. SEOR-Erasmus competition & regulation institute; 2002. p. 258.
[21] Kalai E. Proportional solutions to bargaining situations: interpersonal utility comparisons. Econometrica 1977;45:1623–30.
[22] Keeney RL. A group preference axiomatization with cardinal utility. Manage Sci 1976;23:140–5.
[23] Keeney RL, Kirkwood CW. Group decision making using cardinal social welfare functions. Manage Sci 1975;22:430–7.
[24] Khoso AR, Yusof AM, Khahro SH, Abidin NIAB, Memon NA. Automated two-stage continuous decision support model using exploratory factor analysis-MACBETH-SMART: an application of contractor selection in public sector construction; 2021. Forthcoming
[25] Law 9/2017, of 8 November 2017, on public sector contracts, by which the directives of the European Parliament and of the Council 2014/23/EU and 2014/24/EU, of 26 February 2014, are transposed into the Spanish legal system. (In Spanish).
[26] Lorentziadis PL. Post-objective determination of weights of the evaluation factors in public procurement tenders. Eur J Oper Res 2010;200:261–7.
[27] Lorentziadis PL. Competitive bidding in asymmetric multidimensional public procurement. Eur J Oper Res 2020;282:211–20.
[28] Marovic I, Peric M, Hanak T. A multi-criteria decision support concept for selecting the optimal contractor. Appl Sci 2021;11:1660.
[29] Mattar SG, Macdonald RJ, Choo EU. Procurement process: decision by exclusion and pairwise comparisons. Omega 1992;20:705–12.
[30] Mielcarz P, Osiichuk D, Owczarkowski R. Tender bids evaluation in the context of value-based management. In: Dudycz T, Osbert-Pociecha G, Brycz B, editors. Efficiency in business and economics. Springer proceedings in business and economics. Springer; 2018.
[31] Mohemad R, Hamdan A, Othman Z, Maizura N. Decision support systems (DSS) in construction tendering processes. Int J Comput Sci Issues 2010;7:35–45.
[32] Moulin H. Axioms of cooperative decision making. Cambridge University Press; 1988.
[33] Pictet J, Bollinger D. Extended use of the cards procedure as a simple elicitation technique for MAVT. Eur J Oper Res 2008;185:1300–7.
[34] R/40/2018. Resolución 040/2018, de 22 de marzo de 2018, de la Titular del Órgano Administrativo de Recursos contractuales De la Comunidad Autónoma de Euskadi/Euskal Autonomia Erkidegoko Kontratuen Inguruko Errekurtsoen Administrazio Organoaren titularra. (In Spanish).
[35] R/45/2016. Resolución 045/2016, de 11 de abril de 2016, del titular del Órgano Administrativo de Recursos Contractuales de la Comunidad Autónoma de Euskadi/Euskal Autonomia Erkidegoko Kontratuen inguruko Errekurtsoen Administrazio Organoaren titularra. (In Spanish).
[36] R/75/2020. Resolució núm.: 75/2020, de 19 de febrer de 2020, del Tribunal Català de Contractes del Sector Públic. (In Catalan).
[37] R/143/2019. Acuerdo 143/2019, de 27 de diciembre de 2019, del Tribunal Administrativo de Contratos Públicos de Aragón (In Spanish).
[38] R/484/2019. Resolución no. 484/2019, de 9 de mayo de 2019, del Tribunal Administrativo Central de Recursos Contractuales. (In Spanish).
[39] R/853/2019. Resolución no. 853/2019, de 18 de julio de 2019, del Tribunal Administrativo Central de Recursos Contractuales. (In Spanish).
[40] R/976/2018. Resolución no. 976/2018, de 26 de octubre de 2018, del Tribunal Administrativo Central de Recursos Contractuales. (In Spanish).
[41] Simoens S, Cheung R. Tendering and biosimilars: what role for value-added services? J. Mark. Access Health Policy 2020;8:1705120.
[42] Vahdani B, Mousavi SM, Hashemi H, Mousakhani M, Tavakkoli-Moghaddam R. A new compromise solution method for fuzzy group decision-making problems with an application to the contractor selection. Eng Appl Artif Intell 2013;26:779–88.
[43] Vieira ACL, Oliveira MD, Bana e Costa CA. Enhancing knowledge construction processes within multicriteria decision analysis: the collaborative value modelling framework. Omega 2020;94:102047.
[44] Watróbski J, Jankowski J, Ziemba P, Karczmarczyk A, Ziolo M. Generalised framework for multi-criteria method selection. Omega 2019;86:107–24.