RENORMALIZATION GROUP AND SCATTERING-EQUIVALENT HAMILTONIANS ON A COARSE MOMENTUM GRID* **

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We consider the $\pi\pi$-scattering problem in the context of the Kadyshevsky equation. In this scheme, we introduce a momentum grid and provide an isospectral definition of the phase shift based on the spectral shift of a Chebyshev angle. We address the problem of the unnatural high momentum tails present in the fitted interactions which reaches energies far beyond the maximal center-of-mass energy of $\sqrt{s} = 1.4$ GeV. It turns out that these tails can be integrated out by using a block-diagonal generator of the SRG.

1. Introduction

Scattering phase shifts is one of the most important sources of information on the interaction between particles. In general, their theoretical calculation amounts to solving integral equations such as the Lippmann–Schwinger (LS) equation [1] in the non-relativistic case or the Bethe–Salpeter equation (BSE) [2] in the relativistic one. In any case, a potential describing the interaction is required and, in general, the integrals involved must be calculated numerically. The quality of the approximation implicit in the numerical method employed is of crucial relevance for ensuring the desired predictive power of the approach.

Hamiltonian methods are very useful when treating hadronic reactions as a few-body problem by means of the corresponding potentials. They have the advantage that the observables do not depend on the basis used as the spectrum is invariant under unitary transformations. Separable potentials provide numerous advantages when solving the corresponding equations. It often occurs, however, that separable potential that fits the interaction displays an annoying long momentum tail, not natural to the physical problem.

In this work, we consider the $\pi\pi$-scattering problem and handle the long-momentum tail issue by means of the similarity renormalization group (SRG) for Hamiltonians [3, 4]. We present, furthermore, a new alternative method for the calculation of phase shifts in a Hamiltonian framework [5].

2. Phase shifts and the spectral-shift method

Among the 3d-reductions of the BSE, the Kadyshevsky equation [6] is of particular interest in the context of $\pi\pi$ interactions, as it can be easily extended to the three-body problem. The reaction matrix in the Kadyshevsky scheme is given by

$$R_l\left(p',p,\sqrt{s}\,\right) = V_l\left(p',p\right) + \int\limits_0^\infty \mathrm{d}q\, \frac{q^2}{4E_q^2}\frac{V_l\left(p',q\right)R_l(q,p,\sqrt{s}\,)}{\sqrt{s}-2E_q}\,, \qquad (1)$$

where l is the orbital angular momentum, s is the Mandelstam variable, and the subscript q (or, analogously, p) in E_q indicates that $E_q = \sqrt{\vec{q}^2 + m^2}$. Scattering phase shifts $\delta_l(p)$ can be calculated from

$$-\tan\delta_l(p) = \frac{\pi}{8}\frac{p}{E_p}R_l\left(p,p,\sqrt{s}\,\right)\,. \qquad (2)$$

The Hamiltonian version of Eq. (1) is the eigenvalue equation

$$H\Psi_l(p) \equiv 2E_p\Psi_l(p) + \int\limits_0^\infty \mathrm{d}q\frac{q^2}{4E_q^2}V_l(p,q)\Psi_l(q) = E\,\Psi_l(p)\,. \qquad (3)$$

In general, it needs to be solved numerically. We choose the Gauss–Chebyshev quadrature

$$p_n = \frac{\Lambda_{\mathrm{num}}}{2}\left[1-\cos\phi_n\right]\,, \qquad w_n = \frac{\Lambda_{\mathrm{num}}}{2}\frac{\pi}{N}\phi_n\,, \qquad \phi_n = \frac{\pi}{N}\left(n-\frac{1}{2}\right)\,, \qquad (4)$$

where $n = 1,\dots,N$. The variable ϕ_n is the *Chebyshev angle*, and $\Lambda_{\mathrm{num}} = p_{\mathrm{max}}$ is a high-momentum UV cutoff.

We consider the separable model potential given in Ref. [7], which fits experimental data in the low-energy regime. One notices that such potentials present long-momentum tails that reach energies beyond the experimental region. This issue will be addressed in the next section.

On the momentum grid, Eq. (3) becomes

$$2E_n\Psi_n + \sum_k w_k \frac{p_k^2}{4E_k^2} V_{n,k}\Psi_k = \sqrt{s}\,\Psi_n\,. \tag{5}$$

It is already known that phase shifts can be determined from the spectrum of the Hamiltonian, as it has been explained by DeWitt [8], and Fukuda and Newton [9]. They related the shift produced in the spectrum after introducing the interaction to the scattering phase shifts appearing in the S-matrix. Such a relation is based on the use of an equidistant energy and momentum grids, respectively. Since our Gauss–Chebyshev grid is equidistant in the Chebyshev angle, the corresponding relation is [5, 10]

$$\delta_n = -\pi \frac{\Phi_n - \phi_n}{\Delta\phi_n} \equiv -\pi \frac{\Delta\Phi_n}{\Delta\phi_n}\,, \tag{6}$$
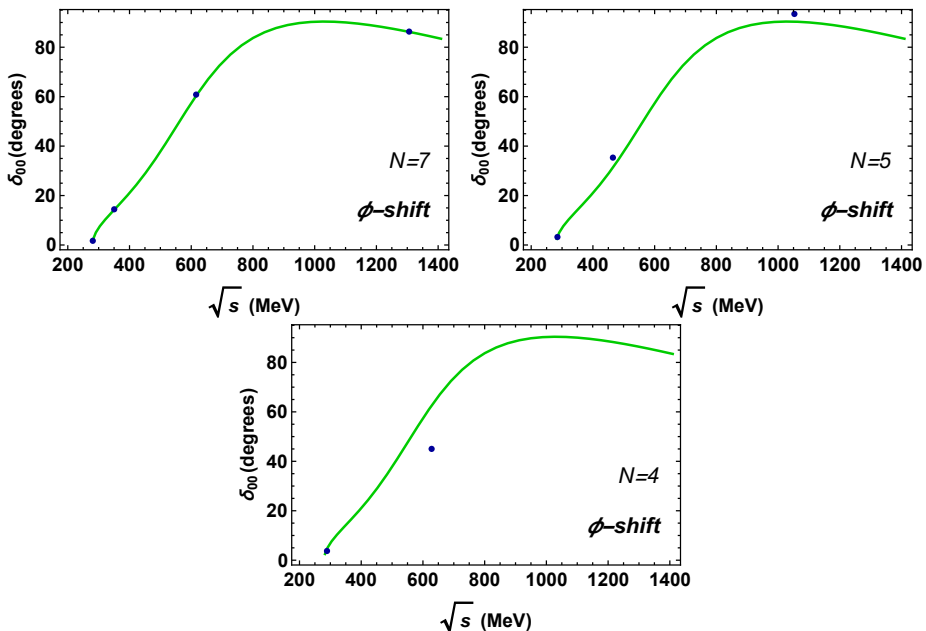

Fig. 1. (Colour on-line) $\pi\pi$-scattering phase shifts in the S_0 channel calculated with the ϕ-shift method using a grid of $N = 7$, 5, and 4 points (black/blue dots) compared with the model fit (grey/green line).

where $\Delta\phi_n = \frac{\pi}{N}$, and the "interacting" angles Φ_n are calculated form the Hamiltonian eigenvalue $\sqrt{s} = 2E_n = \sqrt{m^2 + P_n^2}$, inverting Eqs. (4) after replacing p_n by P_n.

Figure 1 shows scattering phase shifts in the $LI = 00$ channel calculated using Eq. (6) for different number of grid points, compared with the exact solution that fits the data in the inelastic range[1]. The calculation starts slightly differing form the exact solution for a grid of 5 points. This is an extremely small number as it can be noticed by comparing with results obtained using other standard methods, such as *e.g.* the Lippmann–Schwinger equation [10].

3. Similarity renormalization group and scattering

In order to calculate the phase shifts in the experimental range shown in figure 1, we used the separable model potentials given in Ref. [7], which fit the experimental data in the range of interest. The annoying fact is that such potentials present very long tails (even up to 30 GeV), which involve energies far beyond the physical problem (only ~ 1.5 GeV). In fact, in order to describe scattering phase shifts in the range given in the left panel of figure 2, one needs to calculate them in the entire range reached by the potential, as it is illustrated in the right panel of the same figure.
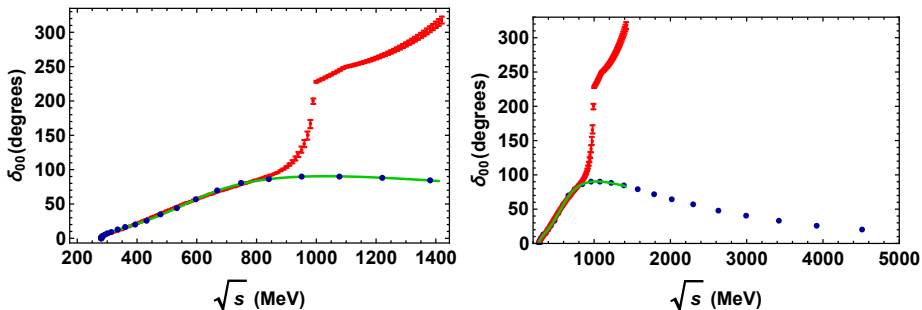

Fig. 2. (Colour on-line) Phase shifts calculated using the ϕ-shift method (black/blue dots) compared with the experimental data [11] (grey/red dots with error bars) and with the model fit (solid, light grey/green line). The left panel shows the experimental region. The right panel shows the energy region covered by the Hamiltonian.

This issue is not only unnatural from the physical point of view, it implies also an undesired computational load which does not provide any additional information about the interaction.

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[1] *Cf.* also the case with a larger number of points in the left panel in figure 2.

We address this issue by means of the SRG [3, 4, 12]. This procedure is based on the idea that it is possible to apply a (scale dependent) unitary transformation on a given initial Hamiltonian in such a way that after a continuous evolution along the running of the scale parameter, the Hamiltonian is transformed into a more convenient basis. Similarity transformations preserve the eigenvalues and, as a consequence, the scattering phase shifts obtained from the spectrum remain invariant. The SRG evolution of the Hamiltonian obeys

$$\frac{\mathrm{d}H_t}{\mathrm{d}t} = [[G_t, H_t], H_t],\qquad(7)$$

where t is the renormalization-group parameter and G_t is the generator which determines the basis into which the *running* Hamiltonian, H_t, is transformed[2]. Different choices of G_t involve different bases for the effective Hamiltonian. In this work, we consider a bloc-diagonal generator, with the structure $G = PHP + QHQ$, where P and Q are the orthogonal projectors,
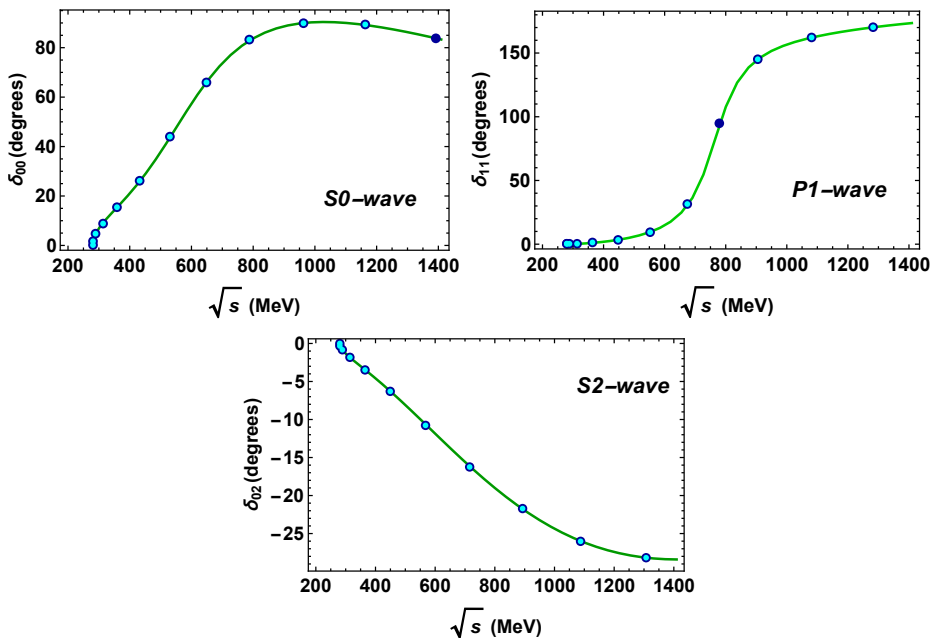

Fig. 3. (Colour on-line) Phase shifts in different δ_{lI} channels calculated with the ϕ-shift method using the original Hamiltonian (black/dark blue dots) and with the block below the cutoff (light-grey/light-blue dots), compared with the model fit (solid, green line). The calculation has been made with an initial grid of 50 points.

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[2] Although the SRG has been mostly used in nuclear physics, recent progress has been made in the last few years concerning its application to QCD [13–15].

$P = \theta(\Lambda - p)$ and $Q = \theta(p - \Lambda)$. This transformation defines two subspaces separated by a cutoff Λ, and converts the initial Hamiltonian into a block-diagonal matrix. The cutoff ($\Lambda \sim 1400$ GeV) is chosen in such a way that the energy range beyond the experimental region decouples form the other one. The matrix elements relevant to the physical problem remain now in the small block and the scattering phase shifts can be calculated from the eigenvalues of this reduced matrix.

Figure 3 shows the phase shifts calculated in channels $lI = 00$, 11, and 02, using the ϕ-shift method with the initial 50×50 matrix (black/dark blue dots) and with the reduced space (11×11 matrix) of the evolved Hamiltonian (light-grey/light-blue dots). While the results are the same, the number of computational operations needed to obtain them is much smaller in the latter case.

4. Conclusions

In this work, we have presented two different methods that we have combined in a convenient way in order to calculate scattering phase shifts in a remarkable simple fashion and with a minimal computational cost.

On the one hand, we have presented the newest version of the so-called spectral-shift methods for the calculation of scattering phase shifts. Alike the energy-shift [8] and the momentum-shift method [9], it is based on the displacement of the grid associated with the integration variable that occurs after the interaction, in an analogous way to what happens in the energy levels appearing in the scattering process in a box.

The ϕ-shift method [5], suitable to the Chebyshev momentum grid employed here, yields results with exceptional accuracy, even when one considers a grid with a small number o points. The method appears as a useful tool that is computationally cheaper than other conventional methods.

On the other hand, we have used the SRG method to transform the original Hamiltonian matrix into a similar one (with the same eigenvalues) with block-diagonal structure. This effective Hamiltonian distinguishes two energy regions separated by a cutoff. This allows us to select the experimental region and proceed with the calculation of scattering phase shifts using only the matrix elements of the block below the cutoff. This procedure simplifies even more the calculation.

To construct the Hamiltonian employed in all these calculations, we have chosen the Kadyshevsky scheme, in which the three-body problem is easy to implement. Once the $\pi\pi$ interaction is understood in this framework, one can start considering the study of $\pi\pi\pi$ interaction appearing, *i.e.* in the ω or the A_1 decays, among other problems.

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