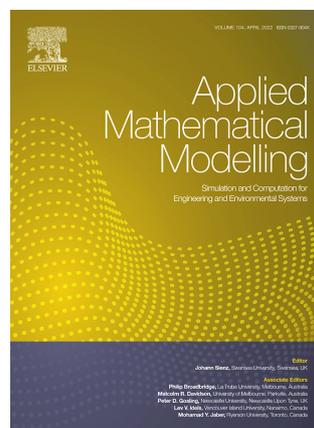


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On the characteristic functional of a doubly stochastic Poisson process: Application to a narrow-band process

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Abstract

The characteristic functional (c.fl.) of a doubly stochastic Poisson process (DSPP) is studied and it provides us the finite dimensional distributions of the process and so its moments. It is also studied the case of a DSPP which intensity is a narrow-band process. The Karhunen–Loève expansion of its intensity is used to obtain the probability distribution function and a decomposition of this Poisson process. The covariance derived from the general c.fl. is applied in this particular DSPP.

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Keywords: Doubly stochastic Poisson process; Characteristic functional; Narrow band process; Karhunen–Loève expansion

1. Introduction

A random point process is a mathematical model for numerous phenomena characterized by localized events distributed randomly in a continuous space. In practice, it can be of interest to count the number of points in subsets of the space given arise to the counting processes. The

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Poisson processes are suitable models for many real counting phenomena but they are insufficient in some cases because of the deterministic character of its intensity function. The doubly stochastic Poisson process (DSPP) is a generalization of the Poisson process when the intensity of the occurrence of the points is influenced by an external process called information process such that the intensity becomes a random process. This process was introduced by Cox [1].

DSPP has been deeply studied. Pointing out some examples, it has been studied from the probabilistic point of view by Grandell [2], Snyder and Miller [3], Valderrama et al. [17] etc. or from the martingale point of view by Brémaud [4], Last and Brandt [5] and Daley and Vere-Jones [6] etc. All of them have derived important tools to deal with the DSPP even the practical calculus for most particular examples are generally difficult if not impossible to evaluate. Even though, we can find that some of the mentioned authors and others, like Yue [7] or Bouzas et al. [8] have also treated with more general processes as compound DSPP or filtered compound DSPP.

On the other hand, it is not so usual to give explicit expressions for the moments of many example of DSPP's because of their difficult evaluations. Also, the characteristic functional is defined in very general stochastic processes [6] or for Poisson processes [3] but not for DSPP's.

We have found many examples in optical communications systems where the signal carrier arriving at a photodetector is Gaussian (i.e. $\cos[\omega t + \theta(t)]$) and so, the number of photoelectrons generated during $[t_0, t]$ is a Poisson process with a certain intensity (i.e. following the same example, $A\{1 + m \cos[\omega t + \theta(t)]\} + \lambda_0$, with $A, m =$ constants depending on the photodetector surface, Planck's constant, etc. and $\lambda_0 =$ constant that models extraneous counts due to background radiation and dark current). These counting processes are used further to estimate the electricity intensity modelling it as a filtered Poisson process. See Pratt [9] or Snyder and Miller [3] for more examples and deeper study of them. This paper tries to give a contribution on the study of such classes of counting processes so further applications of them can be better developed, studying a more general Poisson process but with a similar structure to those. This will be the DSPP with intensity of the form

$$\lambda(t, x(t)) = a[1 + mA \cos(2\pi\omega_0 t + \theta)] + \lambda_0, \quad t \in [0, T], \quad (1)$$

where A and θ are independent random variables such that A follows a Rayleigh distribution with parameter β and θ follows an Uniform distribution (i.e. $A \rightsquigarrow \mathcal{R}(\beta)$ and $\theta \rightsquigarrow U[0, 2\pi]$) so $A \cos[2\pi\omega_0 t + \theta]$ is the harmonic oscillator, and a, m and λ_0 are suitable constants. The intensity described above is obviously a stochastic process so, the point process could be a DSPP with such intensity process. It is known that the harmonic oscillator which appears within the intensity process is a narrow band and Gaussian process (see for example Ochi [10]). The fact of being a Gaussian distribution could be seen as an objection to the model proposed because an intensity cannot take negative values. For this reason the DSPP with this intensity does not exist but it can be seen as an acceptable approximation of a DSPP because the probability of taking negative values is negligible. We will discuss it later but as an introduction, we will remember that it is common to find cases in which this assumption is naturally made in time or space.

The exposed ideas will be presented along the paper on the following way. In Section 2, we will explicitly define the characteristic functional of a DSPP and derive from it the distribution of finite dimensional random vectors and its moments.

In Section 3, we study the particular case of DSPP which intensity is given by $a[1 + mA \cos(2\pi\omega_0 t + \theta)] + \lambda_0$, $t \in [0, T]$. Within Section 3, in Section 3.1 we derive the Karhunen–

Loève expansion of the intensity process and derived from it, the explicit expression of the parametric function as well as its distribution. We will also derive its probability density function which can be calculated by means of a recursive formula for the generating function of the parametric function. Section 3.2 gives the characteristic function of this DSPP. It supplies us a decomposition of the process in three other easier Poisson processes so, it can be seen as a Compound DSPP.

2. Characteristic functional of a DSPP and consequences

First of all, let us give a brief theoretical review about DSPP. A DSPP $\{N(t): t \geq t_0\}$ with intensity stochastic process $\{\lambda(t, x(t)): t \geq t_0\}$ is defined as a conditioned Poisson process which intensity is the process $\{\lambda(t, x(t)): t \geq t_0\}$ given the information process $\{x(t): t \geq t_0\}$.

Therefore, the probability that the number of points occurring in $[t_0, t]$ is n , using the conditioning method [3], is given by

$$P[N(t) = n] = E\{P[N(t) = n/x(\sigma) : t_0 \leq \sigma < t]\} \\ = E\left\{\frac{1}{n!} \left(\int_{t_0}^t \lambda(\sigma, x(\sigma)) d\sigma\right)^n \exp\left(-\int_{t_0}^t \lambda(\sigma, x(\sigma)) d\sigma\right)\right\}$$

for $n = 0, 1, 2, \dots$

Let $A(t, x(t)) \equiv A(t)$ and $\lambda(t, x(t)) \equiv \lambda(t)$ in order not to complicate the notation. $A(t)$ is the parametric function of the DSPP and $A(t) = \int_{t_0}^t \lambda(\sigma) d\sigma$. Then, it is clearly also influenced by the information process, so it is a process itself. Therefore, we can write

$$P[N(t) = n] = E\left\{\frac{1}{n!} A(t)^n e^{-A(t)}\right\} = \frac{1}{n!} G_{A(t)}^n(-1), \tag{2}$$

where $G_{A(t)}(s) \equiv G_A(s)$ is the moment generating function (g.f.) of $A(t)$.

The method of conditioning also allows us to have an expression of the characteristic function (c.f.) of the process

$$M_{N(t)}(iu) = E\left\{\exp\left[(e^{iu} - 1) \int_{t_0}^t \lambda(\sigma) d\sigma\right]\right\} = E\{\exp[(e^{iu} - 1)A(t)]\} = M_A(e^{iu} - 1), \tag{3}$$

where $M_A(s)$ is the c.f. of $A(t)$.

Until now, we have just studied characteristics of unidimensional distributions. We will now approach the study of finite-dimensional random vectors.

The characteristic functional (c.fl.) was first introduced by Kolmogorov [11]. It is a generalization of the characteristic function. Given a random variable X in a linear space L , the c.fl. of X is defined by $\Phi_X(iv) = E[e^{i\langle X, v \rangle}]$. We give an expression for the c.fl. of a DSPP based on the definition for a random measure η found in Daley and Vere-Jones [6], where $\Phi_\eta(iv) = E\{\exp[i\int v(\sigma) d(\eta)]\}$.

As the statistics of a DSPP are many times intractable, it is interesting to find any way of calculating them. We show that the c.fl. allows us to determine the joint c.f. of a finite set of variables

of the DSPP (i.e. $\{N(t_1), N(t_2), \dots, N(t_m)\}$). In particular, having the c.f. of $\{N(t_1), N(t_2)\}$ we can find the covariance of a DSPP. Finally, we will apply this result to the DSPP with intensity (1).

Definition 1. The characteristic functional of a DSPP is defined by

$$\Phi_N(iv) \equiv \exp \left[i \int_0^T v(\sigma) N(d\sigma) \right], \quad (4)$$

where v is a real-valued function and the integral is a counting integral with evaluation

$$\int_0^T v(\sigma) N(d\sigma) = \sum_{i=1}^{N(T)} v(\omega_i)$$

being ω_i the occurrence times of the DSPP.

See Itô stochastic calculus for a deeper study of this kind of integrals, for example in Snyder and Miller [3] or Kloeden and Platen [12].

Using the method of conditioning, the c.fl. of a DSPP becomes

$$\begin{aligned} \Phi_N(iv) &= E \left\{ \exp \left[i \int_0^T v(\sigma) N(d\sigma) / x(\sigma) : t_0 \leq \sigma < t \right] \right\} \\ &= E \left\{ \exp \left[\int_0^T \lambda(\sigma, x(\sigma)) (e^{iv(\sigma)} - 1) d\sigma \right] \right\}. \end{aligned} \quad (5)$$

Taking different functions $v(\sigma)$ we can get c.f.'s of variables related with the DSPP, as we can see in the following cases:

Case 1 (C.f. of the increments). Taking into account the following function:

$$v(\sigma) = \begin{cases} 0; & 0 \leq \sigma < t_1 \\ \alpha; & t_1 \leq \sigma < t_2, \quad 0 < t_1 < t_2 < T \\ 0; & t_2 \leq \sigma < T \end{cases}$$

we have that (4) becomes

$$\Phi_N(iv) = E \{ \exp [i\alpha N(t_1, t_2)] \} \equiv M_{N(t_1, t_2)}(i\alpha).$$

It is the c.f. of the increment of the DSPP between t_1 and t_2 .

We also have from (5) that

$$\Phi_N(iv) = E \left\{ \exp \left[(e^{i\alpha} - 1) \int_{t_1}^{t_2} \lambda(\sigma) d\sigma \right] \right\}$$

or in terms of the parametric function,

$$\Phi_N(iv) = E \{ \exp [(e^{i\alpha} - 1) (\Lambda(t_2) - \Lambda(t_1))] \} = M_{\Lambda(t_2) - \Lambda(t_1)}(e^{i\alpha} - 1),$$

being $M_{\Lambda(t_2) - \Lambda(t_1)}(is)$ the c.f. of $\Lambda(t_2) - \Lambda(t_1)$.

Case 2 (*C.f. of finite-dimensional random vectors*). Let us choose now the following $v(\sigma)$ function:

$$v(\sigma) = \begin{cases} \alpha_1 + \alpha_2 + \dots + \alpha_m; & 0 \leq \sigma < t_1 \\ \alpha_2 + \dots + \alpha_m; & t_1 \leq \sigma < t_2 \\ \vdots & \vdots \\ \alpha_m; & t_{m-1} \leq \sigma < t_m \\ 0; & t_m \leq \sigma < T, \end{cases}$$

where $0 < t_1 < t_2 < \dots < t_m < T$.

From the definition of the c.fl. (see (4)), it results that

$$\begin{aligned} \Phi_N(iv) &= E\{\exp [i(\alpha_1 + \dots + \alpha_m)N(t_1) + i(\alpha_2 + \dots + \alpha_m)(N(t_2) - N(t_1)) \\ &\quad + \dots + i\alpha_m(N(t_m) - N(t_{m-1}))]\} \\ &= E\{\exp [i\alpha_1 N(t_1) + \dots + i\alpha_m N(t_m)]\} \equiv M_{N(t_1), \dots, N(t_m)}(i\alpha_1, \dots, i\alpha_m), \end{aligned}$$

where $M_{N(t_1), \dots, N(t_m)}(i\alpha_1, \dots, i\alpha_m)$ is the joint c.f. of $(N(t_1), \dots, N(t_m))$.

From Eq. (5) we can also have

$$\begin{aligned} \Phi_N(iv) &= E\left\{ \exp \left[(e^{i(\alpha_1 + \dots + \alpha_m)} - 1) \int_0^{t_1} \lambda(\sigma) d\sigma + \dots + (e^{i\alpha_m} - 1) \int_{t_{m-1}}^{t_m} \lambda(\sigma) d\sigma \right] \right\} \\ &= E\{\exp[(e^{i(\alpha_1 + \dots + \alpha_m)} - e^{i(\alpha_2 + \dots + \alpha_m)})A(t_1) + \dots + (e^{i\alpha_m} - 1)A(t_m)]\} \\ &= M_{A(t_1), \dots, A(t_m)}(e^{i(\alpha_1 + \dots + \alpha_m)} - e^{i(\alpha_2 + \dots + \alpha_m)}, \dots, e^{i\alpha_m} - 1). \end{aligned}$$

Taking $m = 2$, the c.f. of the bidimensional distributions is obtained. Having the joint c.f. of $N(t_1)$ and $N(t_2)$, $t_1 < t_2$, it is possible to calculate the moments of $(N(t_1), N(t_2))$ as the covariance function. We know that

$$\begin{aligned} R_N(t_1, t_2) &= E[N(t_1) \cdot N(t_2)] - E[N(t_1)]E[N(t_2)] \\ &= \frac{\partial^2 M_{N(t_1), N(t_2)}(i\alpha_1, i\alpha_2)}{\partial i\alpha_1 \partial i\alpha_2} \Bigg|_{\alpha_1 = \alpha_2 = 0} - E[A(t_1)] \cdot E[A(t_2)]. \end{aligned} \tag{6}$$

Differentiating $\Phi_N(iv)$ ($m = 2$) and after some manipulations, the expression (6) becomes

$$R_N(t_1, t_2) = E[A(t_1) \cdot A(t_2)] + E[A(t_1)] - E[A(t_1)] \cdot E[A(t_2)]$$

so, we can conclude that the covariance function of a DSPP is

$$R_N(t_1, t_2) = R_A(t_1, t_2) + E[A(t_1)].$$

In terms of the intensity process, the covariance function can be written in the following way:

$$R_N(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} R_\lambda(u, v) dv du + \int_0^{t_1} E[\lambda(u)] du. \tag{7}$$

3. DSPP in which intensity is a narrow-band process

This section studies the particular DSPP which intensity is

$$\lambda(t) = a[1 + mA \cos(2\pi\omega_0 t + \theta)] + \lambda_0, \quad t \in [0, T], \tag{8}$$

where A and θ are independent random variables such that A follows a Rayleigh distribution with parameter β and θ follows an Uniform distribution (i.e. $A \rightsquigarrow \mathcal{R}(\beta)$ and $\theta \rightsquigarrow U[0, 2\pi]$). As mentioned in Section 1, this intensity is a more general process than the usual $A\{1 + m\cos[\omega t + \theta(t)]\} + \lambda_0$, with A , m and λ_0 constants. Otherwise, intensity given in Eq. (8) has its advantages and inconveniences as we will see and discuss in Section 3.1.

3.1. Probability distribution function

We will first derive the Karhunen–Loève expansion of the intensity stochastic process defined in (8), use it to find an explicit expression of the parametric function of the DSPP and find the expression of the probability distribution function (pdf) for the DSPP with the intensity proposed.

The mean of the intensity process is $\{\lambda(t, x(t)), t \in [0, T]\}$.

$$E[\lambda(t)] = a + amE\{A \cos[2\pi\omega_0 t + \theta]\} + \lambda_0 = a + \lambda_0.$$

On the other hand, defining the new process

$$\lambda^*(t) = \lambda(t) - a - \lambda_0, \tag{9}$$

it is clearly centered and also Gaussian and its covariance function must be the same as the one of $\lambda(t)$. The expression of its covariance function is

$$R_\lambda(t, s) = R_{\lambda^*}(t, s) = E[\lambda^*(t)\lambda^*(s)] = \frac{a^2 m^2 E[A^2]}{2} \cos[2\pi\omega_0(t - s)]. \tag{10}$$

As $\{\lambda^*(t), t \in [0, T]\}$ is a centered process and has a known covariance function, we can derive its Karhunen–Loève expansion (see for example Todorovic [13], or Wong [14] for theoretical details).

Considering $T = \frac{k}{\omega_0}$, $k \in \mathbb{Z}$, the Karhunen–Loève expansion of $\{\lambda^*(t), t \in [0, T]\}$ is

$$\lambda^*(t) = \sqrt{\frac{2}{T}} \sin(2\pi\omega_0 t) \xi_1 + \sqrt{\frac{2}{T}} \cos(2\pi\omega_0 t) \xi_2,$$

where ξ_1 and ξ_2 are independent identically centered Gaussian random variables with variance equal to α , $\alpha = \frac{a^2 m^2 E[A^2]}{4} T$. In practical applications, the Karhunen–Loève expansion must be truncated in order to obtain approximation of the stochastic process with a finite number of parameters. In our case, the intensity (9) has an exact expansion, because its covariance has an unique eigenvalue with multiplicity two so that the space of eigenfunctions is bidimensional. For this reason, the expansion is a linear combination of just the two elements of an orthonormal basis of the space (see Appendix A for further details).

As $\lambda^*(t) = \lambda(t) - a - \lambda_0$, we have that the Karhunen–Loève expansion of $\{\lambda(t), t \in [0, T]\}$, considering $T = \frac{k}{\omega_0}$, $k \in \mathbb{Z}$ is

$$\lambda(t) = \sqrt{\frac{2}{T}} \sin(2\pi\omega_0 t) \xi_1 + \sqrt{\frac{2}{T}} \cos(2\pi\omega_0 t) \xi_2 + a + \lambda_0,$$

where ξ_1 and ξ_2 are i.i.d. random variables with the Gaussian distribution $N(0, \sqrt{\alpha})$; $\alpha = \frac{a^2 m^2 E[A^2]}{4} T$.

Having this expansion of $\{\lambda(t), t \in [0, T]\}$, it is possible and simple to find the explicit expression of the parametric function of the DSPP, $A(t) = \int_{t_0}^t \lambda(\sigma) d\sigma$

$$A(t) = t(a + \lambda_0) + \sqrt{\frac{2}{T}} \left[\frac{1 - \cos(2\pi\omega_0 t)}{2\pi\omega_0} \xi_1 + \frac{\sin(2\pi\omega_0 t)}{2\pi\omega_0} \xi_2 \right]. \tag{11}$$

From Eq. (11), we observe that $A(t)$ is a linear combination of Gaussian random variables so, it is also Gaussian with

$$E[A(t)] = t(a + \lambda_0)$$

and using the independence between ξ_1 and ξ_2 , after few calculations we obtain

$$\text{Var}[A(t)] = \frac{a^2 m^2 E[A^2] [1 - \cos(2\pi\omega_0 t)]}{4\pi^2 \omega_0^2}.$$

As it is known, the parametric function must be positive, but in this case it could become negative with positive probability. Note that large values of t and an adequate Rayleigh distribution with no large $E[A^2]$ can make the probability of becoming negative to be negligible. The same problem is approached in a similar way by Boel and Beneš [15] when they study the intensity of the number of messages in a communication network. They suggest to multiply the intensity by an indicator function that is equal to zero for negative values, so that they do not take into account negative values. This consideration is made after the calculations because if the indicator function were introduced before it would have changed the stochastic process and its pdf to unknown ones and therefore no calculation could have been made.

Using the intensity of Eq. (8), we will calculate the probability that the intensity is negative. Let us observe that $\lambda(t) < 0$ if and only if

$$\cos(2\pi\omega_0 t + \theta) < 0 \quad \text{and} \quad A > \frac{-(\lambda_0 + a)}{am \cos(2\pi\omega_0 t + \theta)}$$

so using the probability density function of a Rayleigh of parameter β , we have

$$P[\lambda(t) < 0] = \frac{1}{2} \exp \left[\frac{-\left(\frac{\lambda_0 + a}{am |\cos(2\pi\omega_0 t + \theta)|} \right)^2}{2\beta^2} \right].$$

This probability can be made as small as we want. Choosing an integer k and after some calculations, we have got

$$P[\lambda(t) < 0] \leq 10^{-k} \iff \frac{\lambda_0 + a}{2am\beta} \geq \sqrt{k \ln 10 - \ln 2}, \quad \text{for all } t, \theta.$$

Therefore, choosing the parameter β or the constant a , we can make the probability that the intensity is negative, the error of using it, to be negligible.

It is usual to find in literature approximations or models that fit phenomena “well enough” or “reasonably”. In fact, the Gaussian distribution is an usual theoretical assumption or approximation even though many real random variables cannot take negative values.

Bouzas et al. [16] have already solved this problem of negativity with positive probability using truncated Gaussian distributions but the mathematical expressions are much more complicated. Also a process with truncated Gaussian distributions as marginal distributions has unknown finite dimensional distributions neither the covariance, so it is not possible to give explicit conclusions as we do in the case considered in this paper.

Using the distribution of $\Lambda(t)$ we have just calculated and the expression of the pdf of Eq. (2), we can now give the following proposition:

Proposition 2. *Let the process $\{N(t): t \geq t_0\}$ be a DSPP with the intensity process $\{a[1 + mA \cos(2\pi\omega_0 t + \theta)] + \lambda_0, t \in [0, T]\}$, where A and θ independent r.v.’s such that $A \rightsquigarrow \mathcal{R}(\beta)$ and $\theta \rightsquigarrow U[0, 2\pi]$, a, m and λ_0 constants and $T = \frac{k}{\omega_0}, k \in \mathbb{Z}$. Its pdf has the expression*

$$P[N(t) = n] = \frac{1}{n!} G_{\Lambda(t)}^{(n)}(-1), \quad n = 0, 1, 2, \dots, \tag{12}$$

where $G_{\Lambda(t,x(t))}^{(n)}(s)$ is the n th derivative of the generating function of the parametric function $\Lambda(t)$ distributed as a $N(t(a + \lambda_0), \frac{am}{2\pi\omega_0} \sqrt{E[A^2][1 - \cos(2\pi\omega_0 t)]})$.

It is not possible to find an explicit expression for $G_{\Lambda}^{(n)}(-1)$ but by means of a recursive formula (see Appendix B), it can be shown that

$$G_{\Lambda}^{(n)}(-1) = G_{\Lambda}^{(n-1)}(-1) \times \left[t(a + \lambda_0) - \frac{a^2 m^2 E[A^2][1 - \cos(2\pi\omega_0 t)]}{4\pi^2 \omega_0^2} \right] + (n - 1) G_{\Lambda}^{(n-2)}(-1) \\ \times \frac{a^2 m^2 E[A^2][1 - \cos(2\pi\omega_0 t)]}{4\pi^2 \omega_0^2}, \tag{13}$$

where we have that the function $G_{\Lambda}(-1) = \exp[\frac{a^2 m^2 E[A^2][1 - \cos(2\pi\omega_0 t)]}{8\pi^2 \omega_0^2} - t(a + \lambda_0)]$ and $G_{\Lambda}^{(n)}(-1) = 0, n < 0$.

3.2. Characteristic function

We will now calculate the c.f. of the DSPP with intensity process defined in (8) by two different ways, using the distribution of the parametric function $\Lambda(t)$ and using the form of $\Lambda(t)$ as a linear combination of random variables (see Eq. (11)).

Taking into account Eq. (3) and the expression of the c.f. of a Gaussian random variable, straightforward we deduce

$$M_{N(t)}(iu) = \exp \left\{ (e^{iu} - 1)t(a + \lambda_0) + (e^{iu} - 1) \frac{a^2 m^2 E[A^2][1 - \cos(2\pi\omega_0 t)]}{4\pi^2 \omega_0^2} \right\}$$

remembering the consideration made in Section 3.1 that $T = \frac{k}{\omega_0}, k \in \mathbb{Z}$.

The second way of calculating this c.f. leads us to the following proposition:

Proposition 3. Let the process $\{N(t): t \geq t_0\}$ be a DSPP with the intensity process $\{a[1 + mA \cos(2\pi\omega_0 t + \theta)] + \lambda_0, t \in [0, T]\}$, where A and θ independent r.v.'s such that $A \rightsquigarrow \mathcal{R}(\sigma)$ and $\theta \rightsquigarrow U[0, 2\pi]$, a , m and λ_0 constants and $T = \frac{k}{\omega_0}, k \in \mathbb{Z}$, then

$$N(t) = N_1(t) + N_2(t) + N_3(t),$$

where $N_1(t)$ is an inhomogeneous PP with parametric function $\Lambda_1(t) = t(a + \lambda_0)$, $N_2(t)$ is a DSPP with $\Lambda_2(t) = \sqrt{\frac{2}{T} \frac{1 - \cos(2\pi\omega_0 t)}{2\pi\omega_0}} \xi_1$ and $N_3(t)$ another DSPP with $\Lambda_3(t) = \sqrt{\frac{2}{T} \frac{\sin(2\pi\omega_0 t)}{2\pi\omega_0}} \xi_2$, ξ_1 and ξ_2 i.i.d. random variables with distribution $N(0, \sqrt{\alpha})$; $\alpha = \frac{a^2 m^2 E[A^2]}{4} T$.

Proof. As mentioned in Eq. (3) and introducing the form of $\Lambda(t)$ seen in Eq. (11) where ξ_1 and ξ_2 are independent, we can deduce

$$\begin{aligned} M_{N(t)}(iu) &= E\{\exp[(e^{iu} - 1)\Lambda(t)]\} \\ &= E\{\exp[(e^{iu} - 1)t(a + \lambda_0)]\} E\left\{\exp\left[(e^{iu} - 1)\sqrt{\frac{2}{T} \frac{1 - \cos(2\pi\omega_0 t)}{2\pi\omega_0}} \xi_1\right]\right\} \\ &\quad \times E\left\{\exp\left[(e^{iu} - 1)\sqrt{\frac{2}{T} \frac{\sin(2\pi\omega_0 t)}{2\pi\omega_0}} \xi_2\right]\right\}. \end{aligned}$$

This last expression can be rewritten as follows:

$$M_{N(t)}(iu) = M_{N_1(t)}(iu) \cdot M_{N_2(t)}(iu) \cdot M_{N_3(t)}(iu),$$

where $N_1(t)$, $N_2(t)$ and $N_3(t)$ are the stochastic processes mentioned in the proposition. $N_1(t)$ and $N_2(t)$ are the type of DSPP called as randomly scaled Poisson process.

A similar proposition could have been enunciated using the form of the intensity process. \square

3.3. Covariance

The covariance function of the DSPP studied in this paper which intensity process is described in (8), has been calculated from Eq. (10). First, knowing the intensity function of the process, Eq. (10) and with $t_1 < t_2$, it is obtained that

$$\begin{aligned} E\left[\int_0^{t_1} \int_0^{t_2} \lambda(u)\lambda(v) \, dv \, du\right] &= \int_0^{t_1} \int_0^{t_2} R_\lambda(u, v) \, dv \, du \\ &= \frac{a^2 m^2 E[A^2]}{2\pi^2 \omega_0^2} \cos[\pi\omega_0(t_1 - t_2)] \sin(\pi\omega_0 t_1) \sin(\pi\omega_0 t_2) \end{aligned}$$

so then, we finally have from Eq. (7) that

$$R_N(t_1, t_2) = t_1(a + \lambda_0) + \frac{a^2 m^2 E[A^2]}{2\pi^2 \omega_0^2} \cos[\pi\omega_0(t_1 - t_2)] \sin(\pi\omega_0 t_1) \sin(\pi\omega_0 t_2).$$

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Appendix A. Derivation of the Karhunen–Loève expansion for the intensity

As $\{\lambda^*(t), t \in [0, T]\}$ is a centered process and has a known covariance function, we can derive its Karhunen–Loève expansion as follows.

Let us solve the following integral equation:

$$\alpha\varphi(t) = \int_0^T R_{\lambda^*}(t, s)\varphi(s) ds \quad (\text{A.1})$$

with $R_{\lambda^*}(t, s)$ of Eq. (10) to find the eigenvalues and eigenfunctions of the covariance function.

Differentiating (A.1) twice with respect to t , we obtain

$$\alpha\varphi''(t) = -(2\pi\omega_0)^2\varphi(t).$$

As it is well known, the general solution of this second-order linear differential equation is

$$\varphi(t) = b_1 \sin(2\pi\omega_0 t) + b_2 \cos(2\pi\omega_0 t). \quad (\text{A.2})$$

Using this general solution (A.2) and introducing it in Eq. (A.1), after some calculations and considering $T = \frac{k}{\omega_0}, k \in \mathbb{Z}$, we have

$$\begin{aligned} \alpha\varphi(t) &= \alpha[b_1 \sin(2\pi\omega_0 t) + b_2 \cos(2\pi\omega_0 t)] = \frac{\alpha^2 m^2 E[A^2]}{4} T [b_1 \sin(2\pi\omega_0 t) + b_2 \cos(2\pi\omega_0 t)] \\ &= \frac{\alpha^2 m^2 E[A^2]}{4} T \varphi(t), \end{aligned}$$

so it resulted that the integral equation has a unique and double eigenvalue and it is $\alpha = \frac{\alpha^2 m^2 E[A^2]}{4} T$.

An orthonormal basis of the bidimensional solutions space is

$$\left\{ \sqrt{\frac{2}{T}} \sin(2\pi\omega_0 t), \sqrt{\frac{2}{T}} \cos(2\pi\omega_0 t) \right\} \quad (\text{A.3})$$

and both functions verify the integral equation so, they are eigenfunctions of the integral equation.

Having derived the eigenvalues and eigenfunctions of the integral equation, the Karhunen–Loève expansion of $\{\lambda^*(t, x(t)), t \in [0, T]\}$, considering $T = \frac{k}{\omega_0}, k \in \mathbb{Z}$ is

$$\lambda^*(t) = \sqrt{\frac{2}{T}} \sin(2\pi\omega_0 t) \xi_1 + \sqrt{\frac{2}{T}} \cos(2\pi\omega_0 t) \xi_2,$$

where, due that $\lambda^*(t)$ is Gaussian, ξ_1 and ξ_2 are independent Gaussian random variables and have identically centered Gaussian distribution and variance equal to α , it is $\xi_1, \xi_2 \rightsquigarrow N(0, \sqrt{\alpha})$; $\alpha = \frac{a^2 m^2 E[A^2]}{4} T$.

Appendix B. Derivation of recursive formula for the probability distribution function

In this appendix we will prove that the n th derivative of the g.f. of the parametric function evaluated in -1 , it is $G_A^n(-1)$, which appears in the expression of the pdf in Eq. (12) can be calculated by the recursive formula (13).

Remembering the distribution of $A(t)$ (see Section 2), its g.f. is the following

$$G_A(s) = \exp \left[\frac{1}{2} \sigma^2 s^2 + s\mu \right],$$

where $\sigma^2 = \text{Var}[A] = \frac{a^2 m^2 E[A^2][1 - \cos(2\pi\omega_0 t)]}{4\pi^2 \omega_0^2}$ and $\mu = E[A] = t(a + \lambda_0)$.

Let us calculate the consecutive derivatives of the g.f.:

$n = 1,$

$$G'_A(s) = \exp \left[\frac{1}{2} \sigma^2 s^2 + s\mu \right] [\sigma^2 s + \mu] = G_A(s) [\sigma^2 s + \mu]$$

$n = 2,$

$$G''_A(s) = G'_A(s) [\sigma^2 s + \mu] + G_A(s) \sigma^2$$

\vdots

$n-1$, let us suppose

$$G_A^{(n-1)}(s) = G_A^{(n-2)}(s) [\sigma^2 s + \mu] + (n-2)G_A^{(n-3)}(s) \sigma^2$$

n , and differentiating again, it results

$$G_A^{(n)}(s) = G_A^{(n-1)}(s) [\sigma^2 s + \mu] + (n-1)G_A^{(n-2)}(s) \sigma^2.$$

so, we have proved by inducting a recursive formula for $G_A^{(n)}(s)$.

Evaluating the function $G_A^{(n)}(s)$ in -1 and substituting μ and σ^2 by its values, it is proved that $G_A^{(n)}(-1)$ can be calculated with the recursive formula mentioned in order to calculate the pdf.

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