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Two-dimensional incompressible micropolar fluid models with singular initial data

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ABSTRACT

This paper deals with the interaction between microstructures and the appearance or persistence of singular configurations in the Cauchy problem for the two-dimensional model of incompressible micropolar fluids. We analyze the case of null angular viscosity and singular initial data, including the possibility of vortex sheets or measures as initial data in Morrey spaces. Through integral techniques we establish the existence of weak solutions local or global in time. In addition, the uniqueness and stability of these solutions is analyzed.

1. Introduction

The Navier–Stokes equations form the fundamental mathematical model for describing the motion of Newtonian fluids. Since a wide variety of important fluids are non-Newtonian, new theories and mathematical models taking into account the complexity of these fluids are necessary. According to the theory of Eringen for micropolar fluids [1], the interaction between the fluid motion and the rotational motion of micro-particles plays a key role. These fluids constitute a subclass of *simple microfluids* [2], which exhibit microrotational effects and microrotational inertia, and form part of the class of fluids that present asymmetric stress tensors. From a physical point of view [3], micropolar fluids are those consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, whose deformation is ignored. Since deformation of particles is not taken into account, the model is also suitable for turbulent motions where the motion of eddies is the dominant factor [4]. Moreover, the micropolar fluid models can be considered as an essential generalization of the Navier–Stokes equations in the sense that it allows considering some physical phenomena that cannot be treated by the classical Navier–Stokes approach, given that the particles of the fluid are subject to both translational and rotational motion.

The micropolar models describe accurately the behavior of fluids with microstructures such as polymeric suspensions, liquid crystals or biological fluids like blood [3,5,6]. The presence of singularities and deformations is common in the media which these fluids go through, where a buildup of white blood cells, cholesterol, or other particles can promote turbulent effects. Therefore, turbulences emerge and vorticity could be also generated in a very small region around these microstructures causing the appearance of vortex filaments or vortex sheets. This kind of phenomena make it necessary to consider micropolar fluid models with singular initial conditions. In this sense, the study of these fluids and their singularities or concentrations of vorticity is interesting since their evolution or generation is globally affected by their microstructures. This not only happens with certain biological fluids, as said before, but there are numerous industrial applications such as polymeric suspensions in aerosols (namely, medicines or the dissolution of fuel droplets in combustion engines) or liquid crystals that contain additives, among many others, see [3,5,6]. This functional framework escapes the context currently dealt with in the literature. Therefore, it becomes a relevant matter to incorporate to the model the aforementioned singularities, characterized as measures on curves in the case of vortex sheets. This type of vorticities is associated with velocities that are generalized double layer potentials on those curves defining the vortex sheet. Depending on the regularity of these

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curves, the functional space in which the velocities are initially defined has less regularity than that in which the problem of micropolar fluids has been studied to date. The aim of this paper is to shed some light on the possible appearance or persistence of singular solutions related to the dynamics of the microstructure, such as the configuration of vortex sheets, by analyzing the competition among diffusion, microstructure and convection in the case of singular configurations in micropolar fluids. The employed methodology is general enough to be applied to other problems such as aggregation or swarming. In addition the method is constructive, and can be a source of inspiration for the further development of an efficient numerical method. There are other approaches suited for describing the interaction between microstructures and fluids, which take into consideration new effects such as the aggregation or fragmentation of particles. These models give rise to strongly non-linear coupled systems consisting of the Navier–Stokes equation for the fluid and the Vlasov–Boltzmann equation for the particles, see for instance [7] and the references therein. Unlike them, the rotational motion of the microparticles is inherent to the problem of micropolar fluids and responds to strongly nonlinear phenomena. A problem of interest in this direction is how to combine the various phenomena: rotation, aggregation, and fragmentation in the micro and macro scales.

The micropolar fluid motion is described by the following nonlinear coupled system, where we use boldface letters to denote vector fields in $\mathbb{R}^n$, $n = 2, 3$,

$$\begin{aligned} \partial_t \boldsymbol{u} - (\nu+\kappa)\Delta\boldsymbol{u} - 2\kappa \operatorname{curl} \boldsymbol{b} + \nabla p + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \boldsymbol{f}, \\ \partial_t \boldsymbol{b} - \gamma\Delta\boldsymbol{b} - (\alpha+\beta)\nabla \operatorname{div}\boldsymbol{b} + 4\kappa\boldsymbol{b} - 2\kappa\operatorname{curl}\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} &= \boldsymbol{g}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{1.1}$$

together with initial conditions

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \qquad \boldsymbol{b}(\cdot,0) = \boldsymbol{b}_0, \tag{1.2}$$

and boundary conditions. In the whole space, which is the object of this work, we replace the boundary condition by the following condition of decay at infinity

$$|\boldsymbol{u}(\boldsymbol{x},t)| \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to +\infty, \tag{1.3}$$

where $|\cdot|$ denotes the norms of $\mathbb{R}^n$. In system (1.1), the unknowns $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), \cdots, u_n(\boldsymbol{x},t))$, $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x},t) = (b_1(\boldsymbol{x},t), \cdots, b_n(\boldsymbol{x},t))$ and $p = p(\boldsymbol{x},t)$ represent the linear velocity field, the microrotational field (interpreted as the angular velocity field of rotation of particles) and the pressure, respectively. The functions $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t) = (f_1(\boldsymbol{x},t), \cdots, f_n(\boldsymbol{x},t))$ and $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x},t) = (g_1(\boldsymbol{x},t), \cdots, g_n(\boldsymbol{x},t))$ are given and denote the density of external body forces per unit mass and a body source of moments, respectively. The constants $\nu, \kappa, \gamma, \alpha, \beta$ are viscosity coefficients, where ν is the usual Newtonian viscosity, κ is called the microrotational viscosity and γ, α, β are the angular viscosities. If the microrotation of the particles is neglected ($\kappa = 0$ and $\boldsymbol{b} = \boldsymbol{g} = 0$), we obtain the classical Navier–Stokes system.

The two-dimensional case can be understood as a special case of the three-dimensional model [8]. To this end, one considers that the flow itself and the external fields do not depend on the x_3-coordinate. Moreover, assume that the velocity component u_3 in the x_3-direction is zero and the axes of rotation of the particles are parallel to the x_3-axis, that is, $\boldsymbol{u} = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), 0)$, $\boldsymbol{b} = (0, 0, b_3(\boldsymbol{x},t))$, $\boldsymbol{f} = (f_1(\boldsymbol{x},t), f_2(\boldsymbol{x},t), 0)$ and $\boldsymbol{g} = (0, 0, g_3(\boldsymbol{x},t))$, with $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Then, using the notation $\boldsymbol{u} = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t))$, $b = b_3(\boldsymbol{x},t)$, $p = p(\boldsymbol{x},t)$, $\boldsymbol{f} = (f_1(\boldsymbol{x},t), f_2(\boldsymbol{x},t))$, $g = g_3(\boldsymbol{x},t)$ and

$$\operatorname{curl}\boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad \operatorname{div}\boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2},$$

$$\operatorname{curl} b = \left(\frac{\partial b}{\partial x_2}, -\frac{\partial b}{\partial x_1}\right),$$

we can replace $\boldsymbol{u}, b, \boldsymbol{f}$ and g as above into system (1.1) to obtain the system

$$\begin{aligned} \partial_t \boldsymbol{u} - (\nu+\kappa)\Delta\boldsymbol{u} - 2\kappa\operatorname{curl} b + \nabla p + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \boldsymbol{f}, \\ \partial_t b - \gamma\Delta b + 4\kappa b - 2\kappa\operatorname{curl}\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla) b &= g, \\ \operatorname{div}\boldsymbol{u} &= 0. \end{aligned} \tag{1.4}$$

Though it is a relatively recent field of research, there are several references in literature dealing with the mathematical analysis of micropolar fluids. In bounded regular domains we highlight the works [9,10], in which the existence of weak solutions in two and three dimensions was proved. In its turn, the existence and uniqueness of a local and a global strong solution was shown in [11,12], respectively. The existence of global strong solutions for small data of the problem (1.1) in bounded domains is proved in [13] by using a semigroup approach in L^p, $1 < p < \infty$, see also [14]. The case of unbounded domains is less studied. For the case of exterior domains we highlight [15], where the authors proved the existence and uniqueness of a strong solution. For two-dimensional unbounded domains we refer the work [16] (see also [17]). In the general case of a connected open set of $\mathbb{R}^n$, existence and uniqueness of weak solutions for the incompressible micropolar fluid equations are proved in [18].

The works mentioned above have in common the velocity–pressure formulation approach and the solution in the L^2 sense. However, in the context of fluid dynamics it is interesting to describe the dynamics in terms of the evolution of the vorticity field. The vorticity of the vector field $\boldsymbol{v}$ in $\mathbb{R}^n$ is defined by $\operatorname{curl}\boldsymbol{v}$ and represents the tendency of fluid to rotate around $\boldsymbol{x} \in \mathbb{R}^n$. For $n = 2$ the vorticity is a scalar real-valued function given by $\omega(\boldsymbol{x},t) = \operatorname{curl}\boldsymbol{u}(\boldsymbol{x},t)$, with $\boldsymbol{x} \in \mathbb{R}^2$ and $t \geq 0$. Under the perspective of computational methods, this approach gains greater importance, since the vorticity contains all the necessary information for the reconstruction of the velocity field from the Biot–Savart law:

$$\boldsymbol{u}(\boldsymbol{x},t) = (\boldsymbol{K} * \omega)(\boldsymbol{x},t) = \int_{\mathbb{R}^2} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y})\omega(\boldsymbol{y},t)\,d^2\boldsymbol{y}, \tag{1.5}$$

where $\boldsymbol{K}$ is the Biot–Savart kernel, that is,

$$\boldsymbol{K}(\boldsymbol{x}) = \frac{1}{2\pi}|\boldsymbol{x}|^{-2}(-x_2, x_1), \;\; \boldsymbol{x} \in \mathbb{R}^2. \tag{1.6}$$

The velocity fields given by (1.5) may include, in particular, the case of vortex sheets and point sources of vorticity.

Since the generation of vorticity usually occurs in very small regions, it is natural to consider the initial vorticity concentrated in a set of zero Lebesgue measure. As a matter of fact, it will be taken in L^1 or, more generally, assumed to be a finite measure. We refer to the initial configuration of the vorticity as singular initial data. Our main purpose is to understand how the vorticity evolves over time. In the Navier–Stokes case, there are several works dealing with singular initial data, see for instance [19–23] and the references therein. In [20] the authors constructed a smooth global solution for the Navier–Stokes model in $\mathbb{R}^2$ by considering the initial vorticity as a linear combination of Dirac masses whose total variation is small when compared to viscosity. This result was generalized for the case of a finite measure in [21,23]. On the other hand, in [19] a global result of well-posedness in $L^1(\mathbb{R}^2)$ was obtained. In [22], the uniqueness problem in 2D was analyzed for the case in which the initial vorticity is a finite measure. Here, the authors were also able to remove the hypothesis about its size. In 3D we refer to [24–26], where similar results to the 2D case were obtained under the hypothesis of small initial vorticity. In addition, an L^∞ estimate and decay in time of the velocity field were obtained, and the physically relevant case of vortex sheets was also discussed. We recall that the initial data has a vortex sheet structure if the vorticity can be written as $\omega_0 = \alpha\delta_S$, where α is the strength and δ_S is the Dirac measure located on the curve S. In all these works the parabolic character of the vorticity equation was of great importance so that the singular data are compatible with the considered model.

Turning our attention to the micropolar case, the parabolic character mentioned above was also a key argument in [27], in the particular case of null angular viscosity ($\gamma = 0$), to prove the global existence and uniqueness of smooth solutions to (1.4) with initial data in $H^s(\mathbb{R}^2)$, $s > 2$. The authors used an interesting new quantity, namely $Z = \omega - \dfrac{2\kappa}{\nu+\kappa} b$, with the goal of circumventing the recursive relationship that exists between the angular velocity b and the vorticity ω, and then obtained estimates in L^∞ for both.

The novelty of this paper lies in incorporating to this study the physically relevant case of vortex sheets by analyzing the Cauchy problem associated with the two dimensional micropolar models with zero partial viscosity ($\gamma = 0$). More precisely, we study the existence and uniqueness of mild solutions (see the definition of mild formulation in (2.10)–(2.11)) to the micropolar fluid models when the initial condition is chosen to be a bounded function in L^1 or even a measure supported on a curve (vortex sheet). Furthermore, results about the stability of the problem through quantitative estimates are included. We are also interested in the asymptotic behavior of the micropolar fluid motion with respect to time. In this sense, a time-decaying bound of order $t^{-1/2}$ is given in L^∞ for the velocity field. This type of study opens up new possibilities in the ambit of micropolar fluids with different interaction kernels.

Let us consider the following system in $\mathbb{R}^2$:

$$\begin{aligned} \partial_t\omega - (\nu+\kappa)\Delta\omega + (\boldsymbol{u}\cdot\nabla)\omega &= -2\kappa\Delta b + \operatorname{curl}\boldsymbol{f}, \\ \partial_t b + 4\kappa b + (\boldsymbol{u}\cdot\nabla) b &= 2\kappa\omega + g, \\ \boldsymbol{u} &= \boldsymbol{K} * \omega, \\ \omega(\cdot,0) &= \omega_0, \\ b(\cdot,0) &= b_0, \end{aligned} \tag{1.7}$$

where we have applied the curl operator to the first equation in (1.4), and taken into account that $\operatorname{curl}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = (\boldsymbol{u}\cdot\nabla)\omega$, $\operatorname{curl}(\Delta\boldsymbol{u}) = \Delta\omega$, $\operatorname{curl}(\nabla p) = 0$ along with the equivalence between (1.5) and the system

$$\begin{aligned} \operatorname{curl}\boldsymbol{u} &= \omega, \\ \operatorname{div}\boldsymbol{u} &= 0, \\ |\boldsymbol{u}(\boldsymbol{x},t)| \to 0 \quad &\text{as} \quad |\boldsymbol{x}| \to +\infty. \end{aligned}$$

We assume, without loss of generality, $\boldsymbol{f} = 0$ and $g = 0$. We also use the standard notation for the Lebesgue and Sobolev spaces. The usual norm in the space $L^p(\mathbb{R}^2)$ is denoted by $\|\cdot\|_p$. If X is a Banach space, T a positive real number or $T = +\infty$ and $1 \le p \le \infty$, we denote $L^p(0,T;X)$ the Banach space of all measurable functions $v : (0,T) \longrightarrow X$ such that $t \mapsto \|v(t)\|_X$ is in $L^p(0,T)$ with norm

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|_X^p dt\right)^{1/p}, \qquad \text{if} \quad 1 \le p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_X,$$

if $p = \infty$. The space $C([0,T],X)$ is understood in a similar manner. On the other hand, it is known that Morrey-type spaces of measures are suitable for the mathematical formulation of phenomena that involve, for example, mass concentration as the initial vorticity profile [28]. Let us recall the definition of these spaces. Let $B(\boldsymbol{x},R)$ be the euclidean ball of center $\boldsymbol{x}$ and radius $R > 0$. We define the Morrey space $\mathcal{M}^p(\mathbb{R}^n) = \mathcal{M}^p$ as the set of all measures μ satisfying

$$TV_{B(\boldsymbol{x},R)}(\mu) \le CR^{n/p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

independently of $\boldsymbol{x} \in \mathbb{R}^n$ and $R > 0$, where $TV_{B(\boldsymbol{x},R)}(\mu)$ is the total variation of μ in the $B(\boldsymbol{x},R)$. The Morrey space (of functions) $M^p(\mathbb{R}^n) = M^p$ is defined as the space of locally integrable functions f such that

$$\int_{B(\boldsymbol{x},R)} |f(\boldsymbol{y})|\, d^n\boldsymbol{y} < CR^{n/p'}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^n,\ R > 0,$$

where C is independent of $\boldsymbol{x}$ and R. We highlight that $\mathcal{M}^p$ is a Banach space when it is endowed with the norm

$$\|\mu\|_{\mathcal{M}^p} = \sup_{\boldsymbol{x}\in\mathbb{R}^n, R>0} R^{-n/p'} TV_{B(\boldsymbol{x},R)}(\mu).$$

In the same way, M^p is a Banach space under the norm

$$\|f\|_{M^p} = \sup_{\boldsymbol{x}\in\mathbb{R}^n, R>0} R^{-n/p'} \int_{B(\boldsymbol{x},R)} |f(\boldsymbol{y})|\, d^n\boldsymbol{y}.$$

Note that $\mathcal{M}^1$ coincides with the space of finite variation measures (here denoted $\mathcal{M}$), and $M^1 = L^1$. Moreover, $M^\infty = L^\infty$ with equivalent norms, and $L^p \subset M^p$, for $1 < p < \infty$, with continuous injection.

In what follows we use C to denote a generic constant independent of x and t, but still depending on the general data of the problem. When necessary we shall explicitly indicate certain dependencies such as $C = C(Z, R, \ldots)$.

We now list the main results of this paper:

Theorem 1.1. *Let $\omega_0, b_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then, the system (1.7) has a unique global mild solution and the inequality*

$$\|\boldsymbol{u}(\cdot,t)\|_\infty \le C\{(\nu+\kappa)t\}^{-1/2} \tag{1.8}$$

holds, with $t \in (0,T]$, for all $T > 0$. Moreover, assume that $(\omega, \boldsymbol{u}, b)$ and $(\hat{\omega}, \hat{\boldsymbol{u}}, \hat{b})$ are mild solutions of system (1.7) with initial data (ω_0, b_0) and $(\hat{\omega}_0, \hat{b}_0)$, respectively. Then, the following inequalities are verified

$$\|(\boldsymbol{u}-\hat{\boldsymbol{u}})(\cdot,t)\|_\infty \le Ct^{-1/2}, \tag{1.9}$$

$$\begin{aligned} \|(\omega-\hat{\omega})(\cdot,t)\|_1 + \|(\omega-\hat{\omega})(\cdot,t)\|_\infty + \|(b-\hat{b})(\cdot,t)\|_1 \\ + \|(b-\hat{b})(\cdot,t)\|_\infty \le C\hat{\Pi}, \end{aligned} \tag{1.10}$$

where $\hat{\Pi} = \hat{\Pi}(\omega_0, b_0, \hat{\omega}_0, \hat{b}_0) = \max\{\|\omega_0-\hat{\omega}_0\|_1, \|\omega_0-\hat{\omega}_0\|_\infty, \|b_0-\hat{b}_0\|_1, \|b_0-\hat{b}_0\|_\infty\}$ and $C > 0$ is a constant independent of $\hat{\Pi}$.

Theorem 1.2. *Let $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ and $b_0 \in \mathcal{M}(\mathbb{R}^2) \cap \mathcal{M}^p(\mathbb{R}^2)$, with $p > 2$. Then, the system (1.7) has a unique global weak solution such that*

$$\|\boldsymbol{u}(\cdot,t)\|_\infty \le C\{(\nu+\kappa)t\}^{-1/2}, \tag{1.11}$$

with $t \in (0,T]$, for all $T > 0$. Moreover, the solutions are stable in an analogous sense to (1.9)–(1.10), where now $\hat{\Pi}$ depends on the norms $\mathcal{M}(\mathbb{R}^2)$ and $\mathcal{M}^p(\mathbb{R}^2)$ of the initial data.

Below we summarize the general ideas used to demonstrate these results. First, we introduce a new quantity that relates the vorticity and the angular velocity, defined by

$$W = (\nu+\kappa)\omega - 2\kappa b. \tag{1.12}$$

Then, (1.7) is formally equivalent to the following system, which will be our main object of study in what follows,

$$\partial_t W - (\nu+\kappa)\Delta W + \frac{4\kappa^2}{\nu+\kappa} W = -(\boldsymbol{u}\cdot\nabla) W + \frac{8\kappa^2\nu}{\nu+\kappa} b,$$

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{K} * \left(\frac{1}{\nu+\kappa} W + \frac{2\kappa}{\nu+\kappa} b\right), \\ \partial_t b + \frac{4\kappa\nu}{\nu+\kappa} b + (\boldsymbol{u}\cdot\nabla) b &= \frac{2\kappa}{\nu+\kappa} W \\ W(\cdot,0) &= W_0 = (\nu+\kappa)\omega_0 - 2\kappa b_0, \\ b(\cdot,0) &= b_0. \end{aligned} \tag{1.13}$$

The next step consists in considering a regularized problem for (1.13), obtained by regularizing the initial data and introducing a time delay in the nonlinear terms. We then deal with mild solutions. More precisely, we prove uniform estimates for short times by integral equation techniques. In particular, we prove L^∞ estimates for the velocity field, which allow us to obtain the existence of solution by compactness arguments. Using the same ideas, uniqueness and stability are obtained.

Remark 1.1. The pressure field can be recovered by taking into consideration the incompressibility of the fluid and the velocity fields $(\boldsymbol{u}, b)$:

$$\Delta p = \operatorname{div}(\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\nu+\kappa)\Delta\boldsymbol{u} + 2\kappa \operatorname{curl} b),$$

when the derivatives make sense.

This work is organized as follows: In Section 2 we introduce some notations and basic results that allow to introduce the definition of weak solution and construct the regularized problem, as well as to obtain the existence of a regularized solution. In Sections 3 and 4 we prove our main results.

2. The regularized problem

The aim of this section is to introduce the notations and summarize the basic results that are necessary to follow the arguments. The following results will be used extensively in this work to obtain the *a priori* estimates, see [29,30].

Lemma 2.1. *If $\delta, \mu, \tau > 0$ and $t > 0$, then*

$$t^{1-\mu}\int_0^t (t-s)^{\mu-1} s^{\delta-1} \mathrm{e}^{-\tau s} ds \le C, \tag{2.1}$$

where $C = C(\mu,\delta,\tau) = \max\{1, 2^{1-\mu}\} G(\delta)(1+\delta/\mu)\tau^{-\delta}$ and G denotes the Gamma function.

Lemma 2.2. *Given $f \in W^{m,p}(\mathbb{R}^n)$ and $\theta : \mathbb{R}^n \to \mathbb{R}$ satisfying:*

1. $\int_{\mathbb{R}^n} \theta(\boldsymbol{x})\, d^n\boldsymbol{x} = 1,$
2. $\int_{\mathbb{R}^n} \boldsymbol{x}^\alpha \theta(\boldsymbol{x})\, d^n\boldsymbol{x} = 0, \quad \forall\ |\alpha| \le n-1,$
3. $\int_{\mathbb{R}^n} |\boldsymbol{x}|^n |\theta(\boldsymbol{x})|\, d^n\boldsymbol{x} < \infty.$

Set $\theta_\delta(\boldsymbol{x}) = \frac{1}{\delta^n}\theta\left(\frac{\boldsymbol{x}}{\delta}\right)$, $\delta > 0$. Then, $\|f - f*\theta_\delta\|_p \le C\delta^m \|f\|_{W^{m,p}}$.

Lemma 2.3. *Let $I : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map and $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ a bijective map such that $I - \phi \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} (\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{K}(\boldsymbol{x}-\phi(\boldsymbol{y})))\omega(\boldsymbol{y})\, d^2\boldsymbol{y}\right| \\ &\le C\|I-\phi\|_{L^\infty(\mathbb{R}^2)}(1 + |\log\ \|I-\phi\|_{L^\infty(\mathbb{R}^2)}|) \end{aligned} \tag{2.2}$$

holds for every $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, where $C = C(\|\omega\|_{L^1(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)})$.

We highlight more properties of the Biot–Savart kernel that will be useful later on. See [31, Chapter 6], [25, p. 588] and [32, p. 119] for different proofs.

Lemma 2.4.

1. *[Hardy–Littlewood–Sobolev inequality] Let $1 < q < p < \infty$ with $\frac{1}{p} + \frac{1}{n} = \frac{1}{q} + 1$ and $\mu \in \mathcal{M}^p(\mathbb{R}^n)$. Then, $\boldsymbol{K} * \mu \in M^q(\mathbb{R}^n)$ and*

   $$\|\boldsymbol{K} * \mu\|_{M^q} \le C(p)\|\mu\|_{\mathcal{M}^p}.$$

2. *Consider*

   $$\frac{1}{p} + \frac{1}{n} < 1 < \frac{1}{q} + \frac{1}{n},$$

   and $\mu \in \mathcal{M}^p(\mathbb{R}^n) \cap \mathcal{M}^q(\mathbb{R}^n)$. Then, there exists a constant $C > 0$ so that

   $$\|\boldsymbol{K} * \mu\|_\infty \le C\|\mu\|_{\mathcal{M}^p}^{\left(\frac{1}{n}-\frac{1}{q'}\right)/\left(\frac{1}{q}-\frac{1}{p}\right)} \|\mu\|_{\mathcal{M}^q}^{\left(\frac{1}{p'}-\frac{1}{n}\right)/\left(\frac{1}{q}-\frac{1}{p}\right)}.$$

System (1.7) couples the heat equation and the non-linear transport equation. Regarding the heat equation, the Duhamel Principle will be widely used here in order to define a mild solution to

$$\begin{aligned} \partial_t \boldsymbol{v} - (\nu+\kappa)\Delta\boldsymbol{v} &= \boldsymbol{h}, \quad \boldsymbol{x} \in \mathbb{R}^2,\ t > 0, \\ \boldsymbol{v}(\cdot,0) &= \boldsymbol{v}_0, \end{aligned}$$

which is explicitly given by

$$\boldsymbol{v}(\cdot,t) = \Gamma(\cdot,t) * \boldsymbol{v}_0 + \int_0^t \Gamma(\cdot,t-s) * \boldsymbol{h}(\cdot,s) ds,$$

where

$$\Gamma(\boldsymbol{x},t) = \frac{1}{4\pi(\nu+\kappa)t}\exp\left(-\frac{|\boldsymbol{x}|^2}{4(\nu+\kappa)t}\right)$$

is the Heat kernel.

The following result summarize some properties of the heat kernel Γ in Morrey spaces. See [25, p. 586–590] for the details.

Lemma 2.5. *Let $1 \le p \le q \le \infty$ and $\mu \in \mathcal{M}^p(\mathbb{R}^2)$. Then, there exists a constant $C > 0$ for which the inequality*

$$\|\partial_x^\beta \Gamma(\cdot,t) * \mu\|_{M^q} \le C\{(\nu+\kappa)t\}^{-|\beta|/2-(1/p-1/q)}\|\mu\|_{\mathcal{M}^p}, \tag{2.3}$$

holds, for all $t > 0$, where β is a multi-index. Moreover, $\Gamma(\cdot,t) * \mu \to \mu$, as $t \to 0$, weakly as measures on each fixed open ball and

$$\lim_{t\to 0}\|\Gamma(\cdot,t) * \mu\|_{M^p} = \|\mu\|_{\mathcal{M}^p}. \tag{2.4}$$

If $\mu \in \mathcal{M}^p \cap \mathcal{M}^q$, $1 \le p \le q \le \infty$, then we also have the following result of interpolation in Morrey spaces

$$\|\mu\|_{\mathcal{M}^r} \le \|\mu\|_{\mathcal{M}^p}^{1-\theta}\|\mu\|_{\mathcal{M}^q}^{\theta},$$

with $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$ and $0 \le \theta \le 1$. In particular, all the previous estimates also hold in L^p spaces.

Now, we recall the classical representation for solutions of the transport equation.

Lemma 2.6. *If $\boldsymbol{v} \in L^\infty(0,T; \boldsymbol{W}^{1,\infty}(\mathbb{R}^2))$ and $h \in L^1(0,T; L^p(\mathbb{R}^2))$, the following Cauchy problem for $\theta = \theta(\boldsymbol{x},t)$ with initial data $\theta_0 \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, has a unique solution*

$$\begin{aligned} \partial_t\theta + \operatorname{div}(\boldsymbol{v}\theta) &= h, \ \ \boldsymbol{x} \in \mathbb{R}^2,\ \ t \in [0,T), \\ \operatorname{div}\boldsymbol{v} &= 0, \\ \theta(\cdot,0) &= \theta_0, . \end{aligned}$$

This solution is defined by

$$\theta(\boldsymbol{x},t) = \theta_0(\boldsymbol{X}(0;\boldsymbol{x},t)) + \int_0^t h(\boldsymbol{X}(s;\boldsymbol{x},t),s) ds,$$

where the function $\boldsymbol{X}(t;\boldsymbol{x},t_0)$, $0 \le t_0 < T$, *denotes the characteristics related to* $\boldsymbol{v}$, *that is, the unique solution of the system*

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X}(t;\boldsymbol{x},t_0) = \boldsymbol{v}(\boldsymbol{X}(t;\boldsymbol{x},t_0),t), \; t \in [0,T), \\ \boldsymbol{X}(t_0;\boldsymbol{x},t_0) = \boldsymbol{x}. \end{cases}$$

In particular,

$$\|\theta(\cdot,t)\|_p \le \|\theta_0\|_p + \int_0^t \|h(\cdot,s)\|_p ds, \; t \in [0,T). \tag{2.5}$$

We omit the proof and refer to [33, Chapter 1] and [34, p. 232]. This result can be easily extended to Log-Lipschitz velocity fields

$$|\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{y},t) - \boldsymbol{v}(\boldsymbol{x},t)| \le C|\boldsymbol{y}|(1+|\ln|\boldsymbol{y}||), \tag{2.6}$$

for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$ and $t > 0$, which will be our object of study in this paper.

In order to construct a regularized system to (1.13), we consider two C^∞ positive functions $\rho(t)$ and $\psi(\boldsymbol{x})$ such that

(i) $\psi(|\boldsymbol{x}|) = \psi(\boldsymbol{x})$ and it is a nonincreasing function of $|\boldsymbol{x}|$,
(ii) $\int \rho dt = 1$ and $\int \psi \, d^2\boldsymbol{x} = 1$,
(iii) $\mathrm{supp}(\rho) \subset [1,2]$.

Given $\varepsilon > 0$, we set $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$, $\psi_\varepsilon(\boldsymbol{x}) = \varepsilon^{-2}\psi(\boldsymbol{x}/\varepsilon)$, $W_0^\varepsilon = W_0 * \psi_\varepsilon$ and $b_0^\varepsilon = b_0 * \psi_\varepsilon$, where $*$ denotes the convolution in $\mathbb{R}^2$. Note that $W_0^\varepsilon, \; b_0^\varepsilon \in W^{m,1}(\mathbb{R}^2) \cap W^{m,\infty}(\mathbb{R}^2)$, for all $m \in \mathbb{N}$.

We denote by $\Pi(W_0,b_0)$ a constant depending on the norms of the initial data, either in terms of the L^1 and L^∞ norms, as in Section 3, or in terms of the norms in Morrey spaces, as in Section 4.

Given a function f defined on $\mathbb{R}^2 \times \mathbb{R}$, we set a time-delay mollification of f, denoted by $M^\varepsilon(f)$, defined by

$$M^\varepsilon(f)(\boldsymbol{x},t) = \int_{2\varepsilon}^{+\infty} \rho_\varepsilon(t-s) f(\boldsymbol{x},s) ds. \tag{2.7}$$

By definition of $M^\varepsilon(f)$ we obtain

$$\begin{aligned} M^\varepsilon(f)(\boldsymbol{x},t) &= 0, \quad \text{if} \quad t \le 2\varepsilon, \\ \|M^\varepsilon(f)(\cdot,t)\|_X &\le \max_{t-2\varepsilon<s<t-\varepsilon} \|f(\cdot,s)\|_X \\ &\le \max_{t/2<s<t} \|f(\cdot,s)\|_X, \end{aligned} \tag{2.8}$$

for any normed space X.

We introduce the following regularized and linearized problem

$$\begin{aligned} \partial_t W^\varepsilon - (\nu+\kappa)\Delta W^\varepsilon + \frac{4\kappa^2}{\nu+\kappa} W^\varepsilon &= -M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon) + \frac{8\kappa^2\nu}{\nu+\kappa} M^\varepsilon(b^\varepsilon), \\ \boldsymbol{u}^\varepsilon &= \boldsymbol{K} * \left(\frac{1}{\nu+\kappa} W^\varepsilon + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)\right), \\ \partial_t b^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa} b^\varepsilon + (\boldsymbol{u}^\varepsilon\cdot\nabla)b^\varepsilon &= \frac{2\kappa}{\nu+\kappa} W^\varepsilon, \\ W^\varepsilon(\cdot,0) &= W_0^\varepsilon, \\ b^\varepsilon(\cdot,0) &= b_0^\varepsilon. \end{aligned} \tag{2.9}$$

The main result of this section is

Proposition 2.1. *The problem* (2.9) *admits a unique classical solution* $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)$. *This solution is* C^∞ *and* $\mathrm{div}\, \boldsymbol{u}^\varepsilon = 0$.

Proof. To prove that the above system is uniquely solvable in $\mathbb{R}^2 \times [0,+\infty)$ we proceed by induction. We consider the time interval $J_m = [m\varepsilon, (m+1)\varepsilon]$, $m \in \mathbb{N}$. By (2.7) the problem (2.9) is reduced to a linear problem on each time interval J_m. Then, through the semigroup theory, the smoothing effect of the heat equation and classic theory of transport equation, we obtain smooth solutions W^ε, $\boldsymbol{u}^\varepsilon$, b^ε in J_m. Since the nonlinear couplings associated to the system for $t \in J_{m+1}$ are delayed on their values in $t \in J_m$, the problem is now linear and we can conclude that there exists a unique smooth solution for the above regularized problem. Furthermore, $\boldsymbol{u}^\varepsilon$ is divergence-free as consequence of $\mathrm{div}\, \boldsymbol{K} = 0$.

Indeed, using (2.7), W^ε satisfies

$$\begin{aligned} \partial_t W^\varepsilon - (\nu+\kappa)\Delta W^\varepsilon + \frac{4\kappa^2}{\nu+\kappa} W^\varepsilon &= 0, \\ W^\varepsilon(\cdot,0) &= W_0^\varepsilon, \end{aligned}$$

for $t \le 2\varepsilon$. Then, since $W_0^\varepsilon \in C^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, we have by the classic theory for the heat equation that there exists a unique smooth solution $W^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R}^2)$ of the above system. Therefore, $\boldsymbol{u}^\varepsilon(\cdot,t) = \dfrac{1}{\nu+\kappa}\boldsymbol{K} * W^\varepsilon$ is well defined and is $C^\infty(\mathbb{R}^2)$ when $t \le 2\varepsilon$. Furthermore, the hypothesis $b_0^\varepsilon \in C^\infty(\mathbb{R}^2)$ implies that there exists a unique $b^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R}^2)$ satisfying the system

$$\begin{aligned} \partial_t b^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa} b^\varepsilon + (\boldsymbol{u}^\varepsilon\cdot\nabla)b^\varepsilon &= \frac{2\kappa}{\nu+\kappa} W^\varepsilon, \qquad t \le 2\varepsilon \\ b^\varepsilon(\cdot,0) &= b_0^\varepsilon. \end{aligned}$$

Assume that (2.9) has a unique smooth solution in $\mathbb{R}^2 \times J_i$, $i = 0,1,\ldots,m$. The continuity of this solution allows us to set the initial data in $t = (m+1)\varepsilon$ as the corresponding solution of the equivalent system at the upper end of the interval J_m. Moreover, the system for $t \in J_{m+1}$ is now linear,

$$\begin{aligned} \partial_t W^\varepsilon - (\nu+\kappa)\Delta W^\varepsilon + \frac{4\kappa^2}{\nu+\kappa} W^\varepsilon &= G^\varepsilon, \\ \boldsymbol{u}^\varepsilon &= \boldsymbol{K} * \left(\frac{1}{\nu+\kappa} W^\varepsilon + \frac{2\nu}{\nu+\kappa} H^\varepsilon\right), \\ \partial_t b^\varepsilon + \frac{4\kappa\nu}{\nu+\kappa} b^\varepsilon + (\boldsymbol{u}^\varepsilon\cdot\nabla)b^\varepsilon &= \frac{2\kappa}{\nu+\kappa} W^\varepsilon, \end{aligned}$$

where $G^\varepsilon(\cdot,t)$ and $H^\varepsilon(\cdot,t)$ are defined in a natural way through the solutions of the previous step. Then, by an induction process, we have that $G^\varepsilon, H^\varepsilon \in C^\infty$ that implies, via the Duhamel principle, the existence and uniqueness for $W^\varepsilon(\cdot,t)$, $\boldsymbol{u}^\varepsilon(\cdot,t)$ and $b^\varepsilon(\cdot,t)$ in J_{m+1}. Thus, the proposition is proved. ■

In order to define the mild formulation of (2.9), note that using $\mathrm{div}\, \boldsymbol{u}^\varepsilon = 0$, we have $M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon) = M^\varepsilon(\mathrm{div}\, \boldsymbol{u}^\varepsilon W^\varepsilon)$, and, since M^ε commute with the convolution and the differentiation with respect to x, we obtain

$$\begin{aligned} &-\Gamma(\cdot,t-s) * M^\varepsilon(\mathrm{div}\, \boldsymbol{u}^\varepsilon W^\varepsilon)(\cdot,s) \\ &= -\int_{\mathbb{R}^2} \Gamma(\cdot-\boldsymbol{y},t-s) \int_{2\varepsilon}^{+\infty} \rho_\varepsilon(s-\tau) \sum_j \frac{\partial(u_j^\varepsilon W^\varepsilon)}{\partial x_j}(\boldsymbol{y},\tau) d\tau \, d^2\boldsymbol{y} \\ &= \int_{2\varepsilon}^{+\infty} \rho_\varepsilon(s-\tau) \int_{\mathbb{R}^2} \sum_j \frac{\partial \Gamma}{\partial x_j}(\cdot-\boldsymbol{y},t-s)(u_j^\varepsilon W^\varepsilon)(\boldsymbol{y},\tau) \, d^2\boldsymbol{y} d\tau \\ &= \int_{\mathbb{R}^2} \sum_j \frac{\partial \Gamma}{\partial x_j}(\cdot-\boldsymbol{y},t-s) \int_{2\varepsilon}^{+\infty} \rho_\varepsilon(s-\tau)(u_j^\varepsilon W^\varepsilon)(\boldsymbol{y},\tau) d\tau \, d^2\boldsymbol{y} \\ &= \sum_j \frac{\partial \Gamma}{\partial x_j}(\cdot,t-s) * M^\varepsilon(u_j^\varepsilon W^\varepsilon)(\cdot,s). \end{aligned}$$

Therefore, we can write

$$-\Gamma(\cdot,t-s) * M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) := \nabla\Gamma(\cdot,t-s) * M^\varepsilon(\boldsymbol{u}^\varepsilon W^\varepsilon)(\cdot,s).$$

Then, the **mild formulation** of the problem (2.9) can be defined by using the Duhamel Principle and Lemma 2.6 as follows

$$W^\varepsilon(\cdot,t) = \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}t}\Gamma(\cdot,t) * W_0^\varepsilon + \int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * M^\varepsilon(\boldsymbol{u}^\varepsilon W^\varepsilon)(\cdot,s)ds + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\Gamma(\cdot,t-s) * M^\varepsilon(b^\varepsilon)(\cdot,s)ds, \tag{2.10}$$

$$\boldsymbol{u}^\varepsilon(\cdot,t) = \left(\boldsymbol{K} * \left\{\frac{1}{\nu+\kappa}W^\varepsilon + \frac{2\kappa}{\nu+\kappa}M^\varepsilon(b^\varepsilon)\right\}\right)(\cdot,t), \tag{2.11}$$

$$b^\varepsilon(\cdot,t) = \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{X}^\varepsilon(0;\cdot,t)) + \frac{2\kappa}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}W^\varepsilon(\boldsymbol{X}^\varepsilon(s;\cdot,t),s)ds, \tag{2.12}$$

where

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{X}^\varepsilon(t;\boldsymbol{x},t_0) = \boldsymbol{u}^\varepsilon(\boldsymbol{X}^\varepsilon(t;\boldsymbol{x},t_0),t), \ \ t\in[0,T), \\ \boldsymbol{X}^\varepsilon(t_0;\boldsymbol{x},t_0) = \boldsymbol{x}. \end{cases}$$

3. Case of initial data in $L^1 \cap L^\infty$

In this section, we will present the proof of Theorem 1.1. The proof is splitted into several parts dealing with existence, uniqueness and stability of solutions. Firstly, we establish the *a priori* estimates for the sequence $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)$ that was introduced in the previous section and analyze the convergence of this sequence.

3.1. A priori estimates

Let us see in the following proposition some properties of the sequence $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)$ for initial data $W_0, b_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$.

Proposition 3.1. *There exist $T^* = T^*(\nu,\kappa,\Pi(W_0,b_0)) > 0$ and positive constants C and $C_{\nu,\kappa}$, such that the following estimates*

$$\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le C\{(\nu+\kappa)t\}^{-1/2}, \tag{3.1}$$

$$\|W^\varepsilon(\cdot,t)\|_1 + \|W^\varepsilon(\cdot,t)\|_\infty + \|b^\varepsilon(\cdot,t)\|_1 + \|b^\varepsilon(\cdot,t)\|_\infty \le C_{\nu,\kappa}\Pi(W_0,b_0), \tag{3.2}$$

hold for any time $t\in(0,T^*]$. In addition, $\boldsymbol{u}^\varepsilon$ is uniformly bounded in $L^\infty(0,T^*;L^\infty(\mathbb{R}^2))$.

Remark 3.1. The dependence on ν and κ of the constant $C_{\nu,\kappa}$ is deduced throughout the proof of the proposition and can be expressed as

$$C_{\nu,\kappa} = \frac{1}{1-\tilde{C}_{\nu,\kappa}[2T^*+\tilde{C}]},$$

where $\tilde{C}_{\nu,\kappa} = \max\{\frac{C}{\nu+\kappa}, \frac{8\kappa^2\nu}{\nu+\kappa}, \frac{2\kappa}{\nu+\kappa}\}$, C is a positive constant and $\tilde{C}$ is given by (2.1).

Proof. In view of Lemma 2.4 and (2.11), we have

$$\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le \frac{C}{\nu+\kappa}\left\{\|W^\varepsilon(\cdot,t)\|_1 + \|W^\varepsilon(\cdot,t)\|_\infty + 2\kappa\left(\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1 + \max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_\infty\right)\right\}, \tag{3.3}$$

for all $t>0$.

Let $q = 1$ or ∞. Using (2.12) we have

$$\|b^\varepsilon(\cdot,t)\|_q \le \Pi(W_0,b_0) + \frac{2\kappa}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\|W^\varepsilon(\cdot,s)\|_q ds, \tag{3.4}$$

for all $t>0$.

Now, combining (2.10), Lemma 2.5 and Eq. (2.8) we have

$$\begin{aligned}\|W^\varepsilon(\cdot,t)\|_q &\le \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}t}\|\Gamma(\cdot,t)*W_0^\varepsilon\|_q \\ &+ \int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\nabla\Gamma(\cdot,t-s)*M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_q ds \\ &+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \parallel \Gamma(\cdot,t-s)*M^\varepsilon \\ &\times (b^\varepsilon)(\cdot,s)\parallel_q ds \\ &\le \|W_0^\varepsilon\|_q + C(\nu+\kappa)^{-1/2}\int_0^t (t-s)^{-1/2} \\ &\times \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_q ds \\ &+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|M^\varepsilon(b^\varepsilon)(\cdot,s)\|_q ds.\end{aligned}$$

Then, we find

$$\begin{aligned}\|W^\varepsilon(\cdot,t)\|_q &\le \Pi(W_0,b_0) + C(\nu+\kappa)^{-1/2}\int_0^t (t-s)^{-1/2}\mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \\ &\times \max_{s/2<\tau<s}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_q ds \\ &+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_q ds,\end{aligned} \tag{3.5}$$

for all $t>0$.

Combining (3.3)–(3.5), and setting

$$\lambda(t) = \sup_{s\le t}\left\{\|W^\varepsilon(\cdot,s)\|_1, \|W^\varepsilon(\cdot,s)\|_\infty, \|b^\varepsilon(\cdot,s)\|_1, \|b^\varepsilon(\cdot,s)\|_\infty\right\},$$

we find

$$\lambda(t) \le C_0\Pi(W_0,b_0) + C_1(\nu,\kappa)t\lambda(t) + C_2(\nu,\kappa)\sqrt{t}\lambda(t)^2,$$

or equivalently

$$0 \le C_0\Pi(W_0,b_0) + \big(C_1 t - 1\big)\lambda(t) + C_2\sqrt{t}\lambda(t)^2,$$

where $C_0 > 1$, $C_1(\nu,\kappa) = \dfrac{8\kappa^2\nu+2\kappa}{\nu+\kappa}$ and $C_2(\nu,\kappa) = C(\nu+\kappa)^{-3/2}(2\kappa+1)$. Then, choosing $T^*>0$ small enough such that

$$\begin{cases} T^* < \dfrac{1}{C_1}, \\ \sigma = (C_1T^*-1)^2 - 4C_0\Pi(W_0,b_0)C_2\sqrt{T^*} > 0, \end{cases} \tag{3.6}$$

and by continuity of λ, we obtain

$$0 \le \lambda(t) \le \frac{1-C_1t-\sqrt{\sigma}}{2C_2\sqrt{t}} \le \frac{1}{2C_2\sqrt{t}},$$

for $t\in(0,T^*]$. From this we deduce $\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le \dfrac{C(2\kappa+1)}{\nu+\kappa}\lambda(t)$, $t>0$, and the estimate (3.1) holds. Now, replacing (3.1) in (3.5) and adding (3.4) we obtain

$$\|W^\varepsilon(\cdot,t)\|_q + \|b^\varepsilon(\cdot,t)\|_q \le C_0\Pi(W_0,b_0) + C(\nu,\kappa)\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\} \times \left\{\max_{s/2<\tau<s}\|W^\varepsilon(\cdot,\tau)\|_q + \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_q\right\}ds.$$

If $\lambda_1(t) = \max_{s\le t}\{\|W^\varepsilon(\cdot,s)\|_q, \|b^\varepsilon(\cdot,s)\|_q\}$, we have

$$\begin{aligned}\lambda_1(t) &\le C_0\Pi(W_0,b_0) + C(\nu,\kappa)\lambda_1(t)\\ &\quad\times \int_0^t \left((t-s)^{-1/2}s^{-1/2} + \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right) ds\\ &\le C_0\Pi(W_0,b_0) + C_{\nu,\kappa}M(t)\lambda_1(t),\end{aligned}$$

with $M(t) = 2t + C$. Note that $C_{\nu,\kappa}$ fits the expression given in Remark 3.1. Then, if $C_{\nu,\kappa}M(T^*) < 1$, we have proved (3.2). Since we know that $\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_{L^\infty} \le \frac{C(2\kappa+1)}{\nu+\kappa}\lambda(t)$ and we have just proved that $\lambda(t)$ is uniformly bounded, we deduce that $\boldsymbol{u}^\varepsilon(x,t)$ is uniformly bounded in L^∞. ■

3.2. Passage to the limit and existence of weak solution

The aim of this section is to prove that the *a priori* estimates obtained in the preceding section allow to obtain weak solutions as limit of the solutions of regularized problem (2.9), as $\varepsilon \to 0$. The existence of the limit is based on the Ascoli–Arzelà Theorem (we refer to [31, Chapter 5]) and the weak-compactness in L^p standards.

Proposition 3.2. *The sequence of solutions $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$ to the system (2.9) admit a subsequence, still denoted by $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$, such that*

$$W^\varepsilon \rightharpoonup W \text{ weak-}* \text{ in } L^\infty(0,T^*;L^\infty(\mathbb{R}^2)), \tag{3.7}$$

$$b^\varepsilon \rightharpoonup b \text{ weak-}* \text{ in } L^\infty(0,T^*;L^\infty(\mathbb{R}^2)), \tag{3.8}$$

$$\boldsymbol{u}^\varepsilon \longrightarrow \boldsymbol{u} \text{ uniformly in compact set of } \mathbb{R}^2\times[0,T^*]. \tag{3.9}$$

Proof. Through the estimate (3.2), one gets $\{W^\varepsilon\}_{\varepsilon>0}$ and $\{b^\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(0,T^*;L^\infty(\mathbb{R}^2))$. Therefore, there exist a subsequence and W, b such that (3.7) and (3.8) hold.

In order to obtain (3.9), we use (3.1) and (3.2) to find that $\{\boldsymbol{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(0,T^*;L^\infty(\mathbb{R}^2))$. Now, we need to ensure that the family $\{\boldsymbol{u}^\varepsilon\}_{\varepsilon>0}$ is equicontinuous in time and space. By (2.6) we obtain the spatial equicontinuity for fixed time, that is, for all $\beta > 0$, there exist $\delta > 0$ such that if $|(\boldsymbol{x},t)-(\boldsymbol{y},t)| < \delta$, then $|\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)-\boldsymbol{u}^\varepsilon(\boldsymbol{y},t)| < \beta$, for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^2$ and $t\in[0,T^*]$. In order to deal with the equicontinuity in time, let us consider $t, t+h \in [0,T^*]$ with $h>0$. We can write

$$\begin{aligned}&\boldsymbol{u}^\varepsilon(\boldsymbol{x},t+h) - \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\\ &= \frac{1}{\nu+\kappa}\boldsymbol{K}*\left\{W^\varepsilon(\boldsymbol{x},t+h) - W^\varepsilon(\boldsymbol{x},t)\right\}\\ &\quad+ \frac{2\kappa}{\nu+\kappa}\boldsymbol{K}*\left\{M^\varepsilon(b^\varepsilon)(\boldsymbol{x},t+h) - M^\varepsilon(b^\varepsilon)(\boldsymbol{x},t)\right\}.\end{aligned} \tag{3.10}$$

By definition of M^ε and combining (2.10) and (2.12), we have

$$\begin{aligned}&\boldsymbol{K}*\left\{W^\varepsilon(\boldsymbol{x},t+h) - W^\varepsilon(\boldsymbol{x},t)\right\}\\ &= \boldsymbol{K}*\left\{\Gamma(\boldsymbol{x},h)*W^\varepsilon(\boldsymbol{x},t) - W^\varepsilon(\boldsymbol{x},t)\right\}\\ &+ \int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(\boldsymbol{x},t+h-s)*\boldsymbol{K}*M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\boldsymbol{x},s)ds\\ &+ \frac{8\kappa^2\nu}{\nu+\kappa}\int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(\boldsymbol{x},t+h-s)*\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\boldsymbol{x},s)ds\end{aligned}$$

and

$$\begin{aligned}&\boldsymbol{K}*\left\{M^\varepsilon(b^\varepsilon)(\boldsymbol{x},t+h) - M^\varepsilon(b^\varepsilon)(\boldsymbol{x},t)\right\}\\ &= \int_{2\varepsilon}^{+\infty} \rho_\varepsilon(t-s)\boldsymbol{K}*\left\{b^\varepsilon(\boldsymbol{x},s+h) - b^\varepsilon(\boldsymbol{x},s)\right\}ds,\end{aligned}$$

with

$$\begin{aligned}&\boldsymbol{K}*\left\{b^\varepsilon(\boldsymbol{x},s+h) - b^\varepsilon(\boldsymbol{x},s)\right\}\\ &= \boldsymbol{K}*\left\{\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon(X^\varepsilon(s,\boldsymbol{x},s+h),s) - b^\varepsilon(\boldsymbol{x},s)\right\}\\ &+ \frac{2\kappa}{\nu+\kappa}\int_s^{s+h} \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau-s)}\boldsymbol{K}*W^\varepsilon(X^\varepsilon(\tau,\boldsymbol{x},\tau+h),\tau)d\tau.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&|\boldsymbol{K}*\{W^\varepsilon(\boldsymbol{x},t+h) - W^\varepsilon(\boldsymbol{x},t)\}|\\ &\le |\boldsymbol{K}*\{\Gamma(\boldsymbol{x},h)*W^\varepsilon(\boldsymbol{x},t) - W^\varepsilon(\boldsymbol{x},t)\}|\\ &+ \left|\boldsymbol{K}*\left\{\int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(\boldsymbol{x},t+h-s)*M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\boldsymbol{x},s)ds\right\}\right|\\ &+ \left|\boldsymbol{K}*\left\{\frac{8\kappa^2\nu}{\nu+\kappa}\int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(\boldsymbol{x},t+h-s)*M^\varepsilon(b^\varepsilon)(\boldsymbol{x},s)ds\right\}\right|\\ &:= W_1 + W_2 + \frac{8\kappa^2\nu}{\nu+\kappa}W_3.\end{aligned}$$

In order to estimate W_1, applying Lemma 2.2 with $\theta = \Gamma$ we derive,

$$W_1 = |\boldsymbol{K}*\{\Gamma(\boldsymbol{x},h)*W^\varepsilon(\boldsymbol{x},t) - W^\varepsilon(\boldsymbol{x},t)\}| \le C\sqrt{h}, \tag{3.11}$$

where, for small ε, we have chosen the ε^{-1} associated with the $W^{1,p}$ norm of $W^\varepsilon(\cdot,t)$ on the order of $\varepsilon = \sqrt{h}$, while for ε large it is enough to apply the mean value theorem. Thanks to Proposition 3.1, W_2 and W_3 can be estimated as follows

$$\begin{aligned}W_2 &= \left|\boldsymbol{K}*\left\{\int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\nabla\Gamma(\boldsymbol{x},t+h-s)*M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\boldsymbol{x},s)ds\right\}\right|\\ &\le \int_t^{t+h} \|\nabla\Gamma(\cdot,t+h-s)\|_1\|M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_\infty^{1/2}\\ &\quad\times \|M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_1^{1/2}ds\\ &\le C\int_t^{t+h}(t+h-s)^{-1/2}ds \le Ch^{1/2},\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}W_3 &= \left|\boldsymbol{K}*\left\{\int_t^{t+h} \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t+h-s)}\Gamma(\boldsymbol{x},t+h-s)*M^\varepsilon(b^\varepsilon)(\boldsymbol{x},s)ds\right\}\right|\\ &\le \int_t^{t+h} \|\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\cdot,s)\|_\infty ds\\ &\le \int_t^{t+h} \max_{s/2\le\tau\le s}\|b^\varepsilon(\cdot,\tau)\|_1^{1/2}\|b^\varepsilon(\cdot,\tau)\|_\infty^{1/2}ds \le Ch.\end{aligned} \tag{3.13}$$

On the other hand, in order to estimate the second term of (3.10) note that

$$\begin{aligned}&|\boldsymbol{K}*\{b^\varepsilon(\boldsymbol{x},s+h) - b^\varepsilon(\boldsymbol{x},s)\}|\\ &\le \left|\boldsymbol{K}*\left\{\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)}b^\varepsilon\,(X^\varepsilon(s,\boldsymbol{x},s+h),s) - b^\varepsilon(\boldsymbol{x},s)\right\}\right|\\ &+ \left|\frac{2\kappa}{\nu+\kappa}\int_s^{s+h} \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau-s)}\boldsymbol{K}*W^\varepsilon(X^\varepsilon(\tau,\boldsymbol{x},\tau+h),\tau)d\tau\right|\\ &:= B_1 + \frac{2\kappa}{\nu+\kappa}B_2.\end{aligned}$$

Now, in order to bound the first term we find

$$\begin{aligned}B_1 &= \Big|\boldsymbol{K} * \Big\{ e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}\,(X^{\varepsilon}(s,\boldsymbol{x},s+h),s) - b^{\varepsilon}(\boldsymbol{x},s) \\ &\quad + e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}(\boldsymbol{x},s) - e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}(\boldsymbol{x},s)\Big\}\Big| \\ &\le \Big|\boldsymbol{K} * \Big\{ e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}\,(X^{\varepsilon}(s,\boldsymbol{x},s+h),s) - e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}(\boldsymbol{x},s)\Big\}\Big| \\ &\quad + \Big|\boldsymbol{K} * \Big\{ e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}(\boldsymbol{x},s) - b^{\varepsilon}(\boldsymbol{x},s)\Big\}\Big| \\ &:= B_{11} + B_{12}.\end{aligned}$$

Then, we have by Lemma 2.3

$$\begin{aligned}B_{11} &= \Big| \int_{\mathbb{R}^2} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y}) e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}\,(X^{\varepsilon}(s,\boldsymbol{y},s+h),s)\; d^2\boldsymbol{y} \\ &\quad - \int_{\mathbb{R}^2} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{z}) e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} b^{\varepsilon}(\boldsymbol{z},s)\, d^2\boldsymbol{z}\Big| \\ &= \Big| \int_{\mathbb{R}^2} [\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{K}(\boldsymbol{x} - X^{\varepsilon}(s,\boldsymbol{y},s+h))] e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} \\ &\quad \times b^{\varepsilon}\,(X^{\varepsilon}(s,\boldsymbol{y},s+h),s)\; d^2\boldsymbol{y}\Big| \\ &\le C\|I(\cdot) - X^{\varepsilon}(s,\cdot,s+h)\|_{\infty}\,(1 + |\log\|I(\cdot) - X^{\varepsilon}(s,\cdot,s+h)\|_{\infty}|)\,,\end{aligned}$$

where $C = C(\|b^{\varepsilon}(\cdot,s)\|_1 + \|b^{\varepsilon}(\cdot,s)\|_{\infty}) > 0$. Since

$$\begin{aligned}\|I(\cdot) - X^{\varepsilon}(s,\cdot,s+h)\|_{\infty} &= \sup_{\boldsymbol{y}} |X^{\varepsilon}(s,\boldsymbol{y},s+h) - I(\boldsymbol{y})| \\ &= \sup_{\boldsymbol{y}} |X^{\varepsilon}(s,\boldsymbol{y},s+h) - X^{\varepsilon}(s+h,\boldsymbol{y},s+h)| \\ &= \sup_{\boldsymbol{y}} \Big| \int_s^{s+h} \boldsymbol{u}^{\varepsilon}(X^{\varepsilon}(\xi,\boldsymbol{y},s+h),\xi) d\xi \Big| \le Ch,\end{aligned}$$

we obtain

$$B_{11} \le Ch\,[1 + \log\,(Ch)]\,, \tag{3.14}$$

and

$$\begin{aligned}B_{12} &= \Big|\boldsymbol{K} * \Big\{\Big(e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} - 1\Big)\, b^{\varepsilon}(\boldsymbol{y},s)\Big\}\Big| \\ &\le C\,\Big| e^{-\frac{4\kappa\nu}{\nu+\kappa}(s+h)} - 1\Big|\; \|b^{\epsilon}(\cdot,s)\|_1^{1/2} \|b^{\epsilon}(\cdot,s)\|_{\infty}^{1/2} \\ &\le Ch.\end{aligned} \tag{3.15}$$

Switching integrals in time with the convolution, taking into account the bound estimates of W^{ε} and the area preservation properties of the mapping flow X^{ε}, we also deduce

$$B_2 = \Big|\boldsymbol{K} * \Big\{ \int_s^{s+h} e^{-\frac{4\kappa\nu}{\nu+\kappa}(\tau - s)} W^{\varepsilon}(X^{\varepsilon}(\tau,\boldsymbol{x},\tau+h),\tau) d\tau \Big\}\Big| \le Ch. \tag{3.16}$$

Hence, combining (3.11)–(3.16), the sequence $t \mapsto \boldsymbol{u}^{\varepsilon}(\boldsymbol{x},t)$ is equicontinuous in time allowing to apply Ascoli–Arzelá theorem to deduce the existence of a subsequence of $\{\boldsymbol{u}^{\varepsilon}\}_{\varepsilon>0}$ such that (3.9) holds. ■

Let us see now that $(W, \boldsymbol{u}, b)$ is a mild solution of system (1.13):

Definition 3.1. $(W, \boldsymbol{u}, b)$ is a mild solution to (1.13) if it satisfies the initial data weak-$\star$ as $t \to 0$ and

1.

$$\begin{aligned}W(\boldsymbol{x},t) &= e^{-\frac{4\kappa^2}{\nu+\kappa}t}(\Gamma(\cdot,t) * W_0)(\boldsymbol{x}) \\ &\quad - \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot,t-s) * (\boldsymbol{u}W)(\boldsymbol{x},s)ds \\ &\quad + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \Gamma(\cdot,t-s) * b(\boldsymbol{x},s)ds,\end{aligned}$$

2.

$$\begin{aligned}b(\boldsymbol{x},t) &= e^{-\frac{4\kappa\nu}{\nu+\kappa}t} b_0(\boldsymbol{X}(0;\boldsymbol{x},t)) \\ &\quad + \frac{2\kappa}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} W(\boldsymbol{X}(s;\boldsymbol{x},t),s)ds,\end{aligned}$$

3.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\nu+\kappa}\boldsymbol{K} * W(\boldsymbol{x},t) + \frac{2\kappa}{\nu+\kappa}\boldsymbol{K} * b(\boldsymbol{x},t),$$

hold, where the equality is understood in the sense of $L^{\infty}(0,T^*; L^p(\mathbb{R}^2))$ functions, for some appropriate p.

Proposition 3.3. *The triple $(W, \boldsymbol{u}, b)$ given by Proposition 3.2 is a mild solution of system (1.13).*

Proof. Let $\varphi \in C_0^{\infty}(\mathbb{R}^2 \times [0,T^*))$ and $\Omega = \mathrm{supp}(\varphi)$. (2.10) leads to

$$\begin{aligned}&\int_0^{T^*}\int_{\mathbb{R}^2} W^{\varepsilon}(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt \\ &\quad = \int_0^{T^*}\int_{\mathbb{R}^2} e^{-\frac{4\kappa^2}{\nu+\kappa}t}\Gamma(\boldsymbol{x},t) * W_0^{\varepsilon}(\boldsymbol{x})\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt \\ &\quad - \int_0^{T^*}\int_{\mathbb{R}^2}\Big[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \\ &\qquad \times\ \nabla\Gamma(\cdot,t-s) * M^{\varepsilon}(\boldsymbol{u}^{\varepsilon}W^{\varepsilon})(\boldsymbol{x},s)ds\Big]\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt \\ &\quad + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}\Big[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \\ &\qquad \times \Gamma(\cdot,t-s) * M^{\varepsilon}(b^{\varepsilon})(\boldsymbol{x},s)ds\Big]\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.\end{aligned}$$

For the convergence of the l.h.s it is sufficient to consider the weak-$*$ convergence of W^{ε} and for the first term of r.h.s it is possible to take the limit by definition of mollifier sequence.

Now, we prove the convergence for the second term of the r.h.s., and the same idea can be used to prove the convergence for the last term. Thus, we consider

$$\begin{aligned}&\int_0^{T^*}\int_{\mathbb{R}^2}\Big[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * M^{\varepsilon}(\boldsymbol{u}^{\varepsilon}W^{\varepsilon})(\boldsymbol{x},s)ds\Big] \\ &\quad \times\ \varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt := I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_0^{T^*}\int_{\mathbb{R}^2}\Big[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (M^{\varepsilon}(\boldsymbol{u}^{\varepsilon}W^{\varepsilon}) \\ &\quad -\ \boldsymbol{u}^{\varepsilon}W^{\varepsilon})(\boldsymbol{x},s)ds\Big]\,\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt\end{aligned}$$

and

$$\begin{aligned}I_2 &= \int_0^{T^*}\int_{\mathbb{R}^2}\Big[\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (\boldsymbol{u}^{\varepsilon}W^{\varepsilon})(\boldsymbol{x},s)ds\Big] \\ &\quad \times\ \varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.\end{aligned}$$

We start proving the convergence to 0 of I_1.

$$\begin{aligned}|I_1| &= \Big|\int_0^{T^*}\int_0^t\int_{\mathbb{R}^2} e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) * (M^{\varepsilon}(\boldsymbol{u}^{\varepsilon}W^{\varepsilon}) \\ &\quad -\ \boldsymbol{u}^{\varepsilon}W^{\varepsilon})(\boldsymbol{x},s)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,ds\,dt\Big| \\ &\le C(\nu,\kappa,T^*)\|\varphi\|_{L^{\infty}(0,T^*;L^{\infty}(\Omega))} \\ &\quad \times\ \|M^{\varepsilon}(\boldsymbol{u}^{\varepsilon}W^{\varepsilon}) - \boldsymbol{u}^{\varepsilon}W^{\varepsilon}\|_{L^{\infty}(0,T^*;L^1(\Omega))} \to 0,\end{aligned}$$

where we have used that $\|M^\varepsilon(\boldsymbol{u}^\varepsilon W^\varepsilon)-\boldsymbol{u}^\varepsilon W^\varepsilon\|_{L^\infty(0,T^*;L^1(\Omega))}\le C\varepsilon$. Now, we focus on I_2. Let us write $I_2=I_{21}+I_{22}$, where

$$I_{21}=\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s)*(\boldsymbol{u}^\varepsilon W^\varepsilon - \boldsymbol{u}W^\varepsilon)(\boldsymbol{x},s)ds\right]\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.$$

and

$$I_{22}=\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s) *\boldsymbol{u}W^\varepsilon(\boldsymbol{x},s)ds\right]\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.$$

We have that

$$\begin{aligned}|I_{21}|&=\left|\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s)*(\boldsymbol{u}^\varepsilon W^\varepsilon -\boldsymbol{u}W^\varepsilon)(\boldsymbol{x},s)ds\right]\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt\right|\\
&\le \|\varphi\|_{L^\infty(0,T^*;L^\infty(\mathbb{R}^2))}\int_0^{T^*}\int_0^t \times\ \|\nabla\Gamma(\cdot,t-s)*(\boldsymbol{u}^\varepsilon-\boldsymbol{u})W^\varepsilon(\cdot,s)\|_{L^1(\Omega)}\,ds\,dt\\
&\le C(\nu,\kappa,T^*)\|\varphi\|_{L^\infty(0,T^*;L^\infty(\Omega))}\|\boldsymbol{u}^\varepsilon-\boldsymbol{u}\|_{L^\infty(0,T^*;L^\infty(\Omega))} \times\ \|W^\varepsilon\|_{L^\infty(0,T^*;L^1(\Omega))}\to 0.\end{aligned}$$

On the other hand, by using Fubini theorem and the fact that $W^\varepsilon \rightharpoonup W$ weak-$*$ in $L^\infty(0,T^*;L^\infty(\mathbb{R}^2))$, we find

$$I_{22}\to\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s)*(\boldsymbol{u}W)(\boldsymbol{x},s)ds\right] \times\ \varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.$$

These arguments along with the Fundamental Lemma of Calculus of Variations lead to

$$\begin{aligned}W(\boldsymbol{x},t)&=\mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}t}\Gamma(\cdot,t)*W_0(\boldsymbol{x})\\
&-\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\nabla\Gamma(\cdot,t-s)*(\boldsymbol{u}W)(\boldsymbol{x},s)\,ds\\
&+\frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\Gamma(\cdot,t-s)*b(\boldsymbol{x},s)\,ds.\end{aligned}\tag{3.17}$$

The same strategy applied to (2.12) yields

$$\begin{aligned}\int_0^{T^*}\int_{\mathbb{R}^2} b^\varepsilon(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt&=\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon \times\ (\boldsymbol{X}^\varepsilon(0;\boldsymbol{x},t))\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt\\
&+\frac{2\kappa}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2}\left[\int_0^t \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}W^\varepsilon(\boldsymbol{X}^\varepsilon(s;\boldsymbol{x},t),s)ds\right] \times\ \varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.\end{aligned}$$

We focus the analysis of the convergence on the term

$$J=\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{X}^\varepsilon(0;\boldsymbol{x},t))\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt,$$

since for the rest of the terms it is analogous. Using the change of variables $\boldsymbol{X}^\varepsilon(0;\boldsymbol{x},t)=\boldsymbol{y}$, we can write it as follows

$$\begin{aligned}&\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{X}^\varepsilon(0;\boldsymbol{x},t))\varphi(\boldsymbol{x},t)d^2\boldsymbol{x}\,dt\\
&=\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{y})\varphi(\boldsymbol{X}^\varepsilon(0;\boldsymbol{y},t),t)\,d^2\boldsymbol{y}\,dt\\
&:=J_1+J_2,\end{aligned}$$

where

$$J_1=\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{y})(\varphi(\boldsymbol{X}^\varepsilon(0;\boldsymbol{y},t),t)-\varphi(\boldsymbol{X}(0;\boldsymbol{y},t),t))\,d^2\boldsymbol{y}\,dt,$$

and

$$J_2=\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{y})\varphi(\boldsymbol{X}(0;\boldsymbol{y},t),t)\,d^2\boldsymbol{y}\,dt.$$

The term J_1 converges to 0 due to the continuity of φ (uniform on compact sets):

$$\begin{aligned}|J_1|&=\left|\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0^\varepsilon(\boldsymbol{y})(\varphi(\boldsymbol{X}^\varepsilon(0;\boldsymbol{y},t),t)-\varphi(\boldsymbol{X}(0;\boldsymbol{y},t),t))\,d^2\boldsymbol{y}\,dt\right|\\
&\le\|b_0^\varepsilon\|_{L^\infty}\int_0^{T^*}\int_{\Omega}|\varphi(\boldsymbol{X}^\varepsilon(0;\boldsymbol{y},t),t)-\varphi(\boldsymbol{X}(0;\boldsymbol{y},t),t)|\ d^2\boldsymbol{y}\,dt\to 0.\end{aligned}$$

Since $\|b_0^\varepsilon-b_0\|_1\to 0$, we find

$$J_2\to\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0(\boldsymbol{y})\varphi(\boldsymbol{X}(0;\boldsymbol{y},t),t)\,d^2\boldsymbol{y}\,dt.$$

By reverting the change of variable, we obtain

$$J\to\int_0^{T^*}\int_{\mathbb{R}^2}\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0(\boldsymbol{X}(0;\boldsymbol{x},t))\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt.$$

Then, we conclude

$$\begin{aligned}b(\boldsymbol{x},t)&=\mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}t}b_0(\boldsymbol{X}(0;\boldsymbol{x},t))\\
&+\frac{2\kappa}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}W(\boldsymbol{X}(s;\boldsymbol{x},t),s)\,ds.\end{aligned}\tag{3.18}$$

We finish the proof dealing with (2.11)

$$\begin{aligned}&\int_0^{T^*}\int_{\mathbb{R}^2}\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt\\
&=\int_0^{T^*}\int_{\mathbb{R}^2}\frac{1}{\nu+\kappa}\boldsymbol{K}*W^\varepsilon(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt\\
&+\int_0^{T^*}\int_{\mathbb{R}^2}\frac{2\kappa}{\nu+\kappa}\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,d^2\boldsymbol{x}\,dt:=N_1+N_2.\end{aligned}$$

We prove the convergence of N_1 using [35, Proposition 4.16], while for N_2 we proceed in the same way after adding and subtracting b^ε in the second member of the convolution. Then, the proof is completed by combining the above inequalities. ■

In order to obtain global existence of mild solution, one can then proceed by steps of size T^* to obtain the above estimates for all $t>0$. To do this, we shift the initial time to $t=\delta<T^*$, and take $T^*>0$ as in Proposition 3.1. Then, we have that (3.1) and (3.2) hold in $[\delta,T^*+\delta]$ and so on. Then, we have the following result

Corollary 3.1. *If $W_0,b_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, then the system (1.13) has a global mild solution, such that for any $T>0$*

$$\{(\nu+\kappa)t\}^{1/2}\boldsymbol{u}\in L^\infty(0,T;L^\infty(\mathbb{R}^2)).$$

Remark 3.2. Note that in each step described above, the initial data is more regular and thus, we have a gain in regularity for the solution obtained (see [36]).

3.3. Uniqueness and stability of solution

Let us take advantage of the results of the previous section to provide the uniqueness and asymptotic stability results for the solution of the problem (1.13) with respect to disturbances of the initial data. Since for both results the techniques are similar to those used previously, we focus on the uniqueness result and just state the stability one.

Proposition 3.4. *Let $W_0,b_0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then, there exist a unique global solution of (1.13) satisfying (1.8).*

Proof. We assume that for $i = 1, 2$, $(W^i, \boldsymbol{u}^i, b^i)$, are solutions of (1.13). We will show that $W^1 = W^2$ and $b^1 = b^2$ on $\mathbb{R}^2\times(0, +\infty)$. We set

$$Z(\cdot, t) = (W^1 - W^2)(\cdot, t),$$
$$E(\cdot, t) = (b^1 - b^2)(\cdot, t).$$

Note that (Z, E) satisfy the system

$$\begin{aligned}
\partial_t Z - (\nu+\kappa)\Delta Z + \frac{4\kappa^2}{\nu+\kappa} Z &= -\left(\left(\boldsymbol{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E\right)\right)\cdot\nabla\right) \\
&\quad\times W^1 - (u^2\cdot\nabla)Z + \frac{8\kappa^2\nu}{\nu+\kappa}E \\
\partial_t E + \frac{4\kappa\nu}{\nu+\kappa}E + (\boldsymbol{u}^2\cdot\nabla)E &= -\left(\left(\boldsymbol{K} * \left(\frac{1}{\nu+\kappa} Z + \frac{2\kappa}{\nu+\kappa} E\right)\right)\cdot\nabla\right) b^1 \\
&\quad + \frac{2\kappa}{\nu+\kappa} Z \\
Z(\cdot, 0) &= 0, \\
E(\cdot, 0) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
Z(\cdot, t) &= \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) \\
&\quad * \left\{W^1\left(\boldsymbol{K} * \left(\frac{1}{\nu+\kappa}Z + \frac{2\kappa}{\nu+\kappa}E\right)\right)\right\}(\cdot, s)ds \\
&\quad + \int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) * (Z\boldsymbol{u}^2)(\cdot, s)ds \\
&\quad + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \Gamma(\cdot, t-s) * E(\cdot, s)ds, \\
E(\cdot, t) &= \int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} \nabla\Gamma(\cdot, t-s) \\
&\quad * \left\{b^1\left(\boldsymbol{K} * \left(\frac{1}{\nu+\kappa}Z + \frac{2\kappa}{\nu+\kappa}E\right)\right)\right\}(\cdot, s)ds \\
&\quad + \frac{2\kappa}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)} Z(\cdot, s)ds.
\end{aligned}$$

Then, we find

$$\begin{aligned}
\|Z(\cdot, t)\|_1 &\le \int_0^t \|\, \nabla\Gamma(\cdot, t-s) \\
&\quad * \left\{W^1\left(\boldsymbol{K} * \left(\frac{1}{\nu+\kappa}Z + \frac{2\kappa}{\nu+\kappa}E\right)\right)\right\}(\cdot, s)\,\|_1\, ds \\
&\quad + \int_0^t \|\nabla\Gamma(\cdot, t-s) * (Z\boldsymbol{u}^2)(\cdot, s)\|_1 ds \\
&\quad + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \|\Gamma(\cdot, t-s) * E(\cdot, s)\|_1 ds \\
&\le C_0(\nu+\kappa)^{-3/2}\int_0^t (t-s)^{-1/2}s^{-1/2} \\
&\quad \times \left\{\|Z(\cdot, s)\|_1 + 2\kappa\|E(\cdot, s)\|_1\right\}ds \\
&\quad + C_0\int_0^t (t-s)^{-1/2}s^{-1/2}\|Z(\cdot, s)\|_1 ds \\
&\quad + \frac{4\kappa\nu}{\nu+\kappa}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} 2\kappa\|E(\cdot, s)\|_1\, ds,
\end{aligned}$$

that is,

$$\begin{aligned}
\|Z(\cdot, t)\|_1 &\le C_1\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\right\} \\
&\quad \times \left\{\|Z(\cdot, s)\|_1 + 2\kappa\|E(\cdot, s)\|_1\right\}ds,
\end{aligned}$$

with $C_1 = \max\left\{C_0(\nu+\kappa)^{-3/2} + C_0,\ \frac{4\kappa\nu}{\nu+\kappa}\right\}$. Similarly, we have

$$\begin{aligned}
2\kappa\|E(\cdot, t)\|_1 &\le C_2\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\} \\
&\quad \times \left\{\|Z(\cdot, s)\|_1 + 2\kappa\|E(\cdot, s)\|_1\right\}ds,
\end{aligned}$$

with $C_2 = \max\left\{2\kappa C_0(\nu+\kappa)^{-3/2},\ \frac{4\kappa^2}{\nu+\kappa}\right\}$. Therefore, we obtain

$$\begin{aligned}
&\|Z(\cdot, t)\|_1 + 2\kappa\|E(\cdot, t)\|_1 \\
&\quad \le C_3\int_0^t \left\{(t-s)^{-1/2}s^{-1/2} + e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} + e^{-\frac{4\kappa\nu}{\nu+\kappa}(t-s)}\right\} \\
&\quad \times \left\{\|Z(\cdot, s)\|_1 + 2\kappa\|E(\cdot, s)\|_1\right\}ds,
\end{aligned}$$

where $C_3 = \max\{C_1,\ C_2\}$. Setting $\Upsilon(t) = \sup_{s\le t}\left\{\|Z(\cdot, s)\|_1 + 2\kappa\|E(\cdot, s)\|_1\right\}$, $t > 0$, and following as in Proposition 3.1, we obtain $\Upsilon(t) = 0$ and the solution of problem (1.13) is unique. ■

Finally, let us summarize the stability result for solutions in the following proposition.

Proposition 3.5. *Let $W_0, b_0, \hat{W}_0, \hat{b}_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Assume that the $(W, \boldsymbol{u}, b)$ and $(\hat{W}, \hat{\boldsymbol{u}}, \hat{b})$ are solution of problem (1.13) with initial data (W_0, b_0) and $(\hat{W}_0, \hat{b}_0)$, respectively. Then, the inequalities below are satisfied for $t > 0$*

$$\|(\boldsymbol{u} - \hat{\boldsymbol{u}})(\cdot, t)\|_\infty \le Ct^{-1/2}, \tag{3.19}$$

$$\begin{aligned}
\|(W-\hat{W})(\cdot, t)\|_1 + \|(W-\hat{W})(\cdot, t)\|_\infty + \|(b-\hat{b})(\cdot, t)\|_1 \\
+ \|(b-\hat{b})(\cdot, t)\|_\infty \le C\hat{\Pi},
\end{aligned} \tag{3.20}$$

where $\hat{\Pi} = \hat{\Pi}(W_0, b_0, \hat{W}_0, \hat{b}_0) = \max\{\|W_0 - \hat{W}_0\|_1, \|W_0 - \hat{W}_0\|_\infty, \|b_0 - \hat{b}_0\|_1, \|b_0 - \hat{b}_0\|_\infty\}$ and $C > 0$ is a constant independent of $\hat{\Pi}$ (see Remark 3.1).

Proof. We consider $Z = W - \hat{W}$, $E = b - \hat{b}$, $\boldsymbol{U} = \boldsymbol{u} - \hat{\boldsymbol{u}}$ and the system below

$$\begin{aligned}
\partial_t Z - (\nu+\kappa)\Delta Z + \frac{4\kappa^2}{\nu+\kappa}Z &= -(\boldsymbol{U}\cdot\nabla)W - (\hat{\boldsymbol{u}}\cdot\nabla)Z + \frac{8\kappa^2\nu}{\nu+\kappa}E, \\
\boldsymbol{U} &= \boldsymbol{K} * \left(\frac{1}{\nu+\kappa}Z + \frac{2\kappa}{\nu+\kappa}E\right), \\
\partial_t E + \frac{4\kappa\nu}{\nu+\kappa}E + (\hat{\boldsymbol{u}}\cdot\nabla)E &= -(\boldsymbol{U}\cdot\nabla)b + \frac{2\kappa}{\nu+\kappa}Z, \\
Z(\cdot, 0) &= Z_0 = W_0 - \hat{W}_0, \\
E(\cdot, 0) &= E_0 = b_0 - \hat{b}_0.
\end{aligned}$$

Then, we can proceed as previously. ■

4. Case of measures as initial data

This section deals with *a priori* estimates for the case $W_0 \in \mathcal{M}(\mathbb{R}^2)$ and $b_0 \in \mathcal{M}(\mathbb{R}^2)\cap\mathcal{M}^p(\mathbb{R}^2)$, with $p > 2$ fixed, which will lead us to a theory of existence of weak solutions for the system (1.13). The main difference with respect to the case of initial data in $L^1\cap L^\infty$ is how one derives the appropriate *a priori* estimates.

First, observe that by $W^\varepsilon(\cdot,t), b^\varepsilon(\cdot,t) \in C^\infty(\mathbb{R})^2 \cap L^1(\mathbb{R}^2)$ and $\boldsymbol{u}^\varepsilon = \boldsymbol{K} * \left(\frac{1}{\nu+\kappa} W^\varepsilon + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)\right)$. We have

$$\operatorname{curl} \boldsymbol{u}^\varepsilon(\cdot,t) = \frac{1}{\nu+\kappa} W^\varepsilon(\cdot,t) + \frac{2\kappa}{\nu+\kappa} M^\varepsilon(b^\varepsilon)(\cdot,t),$$

that is,

$$W^\varepsilon(\cdot,t) = (\nu+\kappa)\operatorname{curl} \boldsymbol{u}^\varepsilon(\cdot,t) - 2\kappa M^\varepsilon(b^\varepsilon)(\cdot,t).$$

Therefore, for $0<s<t$, we can write

$$-\Gamma(\cdot,t-s) * M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) = \nabla\Gamma(\cdot,t-s) * M^\varepsilon(W^\varepsilon \boldsymbol{u}^\varepsilon)(\cdot,s) \tag{4.1}$$

and

$$\begin{aligned} &-\Gamma(\cdot,t-s) * M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s) \\ &= -(\nu+\kappa)\operatorname{curl}\nabla\Gamma(\cdot,t-s) * M^\varepsilon(\boldsymbol{u}^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s) \\ &- 2\kappa\nabla\Gamma(\cdot,t-s) * M^\varepsilon(M^\varepsilon(b^\varepsilon)\boldsymbol{u}^\varepsilon)(\cdot,s), \end{aligned} \tag{4.2}$$

where $\boldsymbol{u}^\varepsilon\boldsymbol{u}^\varepsilon = \sum_{i,j}^{2} u_i^\varepsilon u_j^\varepsilon$. We have the following result:

Proposition 4.1. *There exists $T^* = T^*(\nu,\kappa,\Pi(W_0,b_0)) > 0$ and positive constants C and $C_{\nu,\kappa}$ such that*

$$\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le C\{(\nu+\kappa)t\}^{-1/2},$$
$$\|W^\varepsilon(\cdot,t)\|_1 + \|b^\varepsilon(\cdot,t)\|_1 \le C_{\nu,\kappa}\Pi(W_0,b_0),$$
$$\{(\nu+\kappa)t\}^{1-\frac{1}{p}}\|W^\varepsilon(\cdot,t)\|_{M^p} + \|b^\varepsilon(\cdot,t)\|_{M^p} \le C_{\nu,\kappa}\Pi(W_0,b_0),$$

hold for all $t\in(0,T^*]$.

Remark 4.1. The dependence on ν and κ of the constant $C_{\nu,\kappa}$ is deduced explicitly throughout the proof, and is similar to that derived in Remark 3.1.

Proof. Note that

$$\begin{aligned} \|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le{}& \frac{\mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}t}}{\nu+\kappa}\|(\boldsymbol{K}*\Gamma(\cdot,t))*W_0^\varepsilon\|_\infty \\ &+ \frac{1}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\boldsymbol{K}*(\Gamma(\cdot,t-s)*M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon))(\cdot,s)\|_\infty ds \\ &+ \frac{8\kappa^2\nu}{(\nu+\kappa)^2}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\Gamma(\cdot,t-s)*(\boldsymbol{K}*M^\varepsilon(b^\varepsilon))(\cdot,s)\|_\infty ds \\ &+ \frac{2\kappa}{\nu+\kappa}\|\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty \\ \le{}& \frac{C}{\nu+\kappa}\{(\nu+\kappa)t\}^{-1/2}\|W_0\|_{\mathcal{M}} \\ &+ \frac{C}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\Gamma(\cdot,t-s)*M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_1^{1/2} \\ &\quad\times\|\Gamma(\cdot,t-s)*M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_\infty^{1/2} ds \\ &+ \frac{8\kappa^2\nu}{(\nu+\kappa)^2}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\|\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\cdot,s)\|_\infty ds \\ &+ \frac{2\kappa}{\nu+\kappa}\|\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty. \end{aligned}$$

Applying Lemma 2.5, we obtain the estimates

$$\begin{aligned} &\|\Gamma(\cdot,t-s)*M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_1 \\ &= \|\nabla\Gamma(\cdot,t-s)*M^\varepsilon(W^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_1 \\ &\le C\{(\nu+\kappa)(t-s)\}^{-1/2}\max_{s/2<\tau<s}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1, \end{aligned}$$

and

$$\begin{aligned} &\|\Gamma(\cdot,t-s)*M^\varepsilon((\boldsymbol{u}^\varepsilon\cdot\nabla)W^\varepsilon)(\cdot,s)\|_\infty \\ &\le (\nu+\kappa)\|\operatorname{curl}\nabla\Gamma(\cdot,t-s)*M^\varepsilon(\boldsymbol{u}^\varepsilon\boldsymbol{u}^\varepsilon)(\cdot,s)\|_\infty \\ &+ 2\kappa\|\nabla\Gamma(\cdot,t-s)*M^\varepsilon(M^\varepsilon(b^\varepsilon)\boldsymbol{u}^\varepsilon)(\cdot,s)\|_\infty \\ &\le C\{(\nu+\kappa)(t-s)\}^{-1}\max_{s/2<\tau<s}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty^2 \\ &+ 2\kappa C\{(\nu+\kappa)(t-s)\}^{-1/2-1/p} \\ &\qquad\times\max_{s/2<\tau<s}\left\{\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\left(\max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}\right)\right\}. \end{aligned}$$

On the other hand, by Lemma 2.4 we have

$$\begin{aligned} \|\boldsymbol{K}*M^\varepsilon(b^\varepsilon)(\cdot,t)\|_\infty &\le C\|M^\varepsilon(b^\varepsilon)(\cdot,s)\|_1^\alpha\|M^\varepsilon(b^\varepsilon)(\cdot,s)\|_{M^p}^{1-\alpha} \\ &\le C\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha}, \end{aligned}$$

where $\alpha = \frac{p-2}{2(p-1)} \in (0,1)$.

Therefore, combining the above estimates we obtain the following estimate for $\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty$,

$$\begin{aligned} \|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le{}& \frac{C}{\nu+\kappa}\{(\nu+\kappa)t\}^{-1/2}\|W_0\|_{\mathcal{M}} \\ &+ \frac{C}{(\nu+\kappa)^{1/2}}\int_0^t\{(\nu+\kappa)(t-s)\}^{-3/4} \\ &\times\max_{s/2<\tau<s}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty^{3/2}\|W^\varepsilon(\cdot,\tau)\|_1^{1/2}ds \\ &+ \frac{(2\kappa)^{1/2}C}{\nu+\kappa}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\{(\nu+\kappa)(t-s)\}^{-1/2-1/2p} \\ &\qquad\times\max_{s/2<\tau<s}\left\{\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1^{1/2}\right. \\ &\times\left.\left(\max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}^{1/2}\right)\right\}ds \\ &+ \frac{8\kappa^2\nu C}{(\nu+\kappa)^2}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \\ &\times\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1^\alpha\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p}^{1-\alpha}ds \\ &+ \frac{2\kappa C}{\nu+\kappa}\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha}. \end{aligned}$$

Thus,

$$\begin{aligned} &\{(\nu+\kappa)t\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \\ &\le C(\nu+\kappa)^{-1}\|W_0\|_{\mathcal{M}} \\ &+ C(\nu+\kappa)^{-3/2}t^{1/2}\int_0^t(t-s)^{-3/4}s^{-3/4} \\ &\quad\times\max_{s/2<\tau<s}(\{(\nu+\kappa)\tau\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty)^{3/2}\|W^\varepsilon(\cdot,\tau)\|_1^{1/2}ds \\ &+ (2\kappa)^{1/2}C(\nu+\kappa)^{-3/2-1/2p}t^{1/2}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}(t-s)^{-1/2-1/2p}s^{-1/2} \\ &\quad\times\max_{s/2<\tau<s}\left\{\{(\nu+\kappa)\tau\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1^{1/2}\right. \\ &\quad\times\left.\left(\max_{\tau/2<\eta<\tau}\|b^\varepsilon(\cdot,\eta)\|_{M^p}^{1/2}\right)\right\}ds \\ &+ 8\kappa^2\nu C(\nu+\kappa)^{-3/2}t^{1/2}\int_0^t \mathrm{e}^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)} \\ &\quad\times\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1^\alpha\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p}^{1-\alpha}ds \\ &+ 2\kappa C(\nu+\kappa)^{-1/2}t^{1/2}\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_1^\alpha\max_{t/2<s<t}\|b^\varepsilon(\cdot,s)\|_{M^p}^{1-\alpha}. \end{aligned} \tag{4.3}$$

Analogously, we obtain the following estimates for W^ε and b^ε, where we assume $1 < 2p(p+2)^{-1} < r < 2$ and $0\le\beta\le 1$ such that $\frac{1}{r} = \beta + \frac{1-\beta}{p}$:

$$\|b^\varepsilon(\cdot,t)\|_1 \le \|b_0\|_{\mathcal{M}} + 2\kappa(\nu+\kappa)^{-1}\int_0^t \|W^\varepsilon(\cdot,s)\|_1 ds, \tag{4.4}$$

$$\|b^\varepsilon(\cdot,t)\|_{M^p} \le \|b_0\|_{\mathcal{M}^p} + 2\kappa(\nu+\kappa)^{-2+1/p} \times \int_0^t s^{-1+1/p}\{(\nu+\kappa)s\}^{1-1/p}\|W^\varepsilon(\cdot,s)\|_{M^p} ds, \tag{4.5}$$

$$\begin{aligned}\|W^\varepsilon(\cdot,t)\|_1 \le{}& \|W_0\|_{\mathcal{M}} + C(\nu+\kappa)^{-1} \\ &\times \int_0^t (t-s)^{-1/2}s^{-1/2}\max_{s/2<\tau<s}\{(\nu+\kappa)\tau\}^{1/2} \\ &\times \|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty \|W^\varepsilon(\cdot,\tau)\|_1 ds \\ &+ 8\kappa^2\nu(\nu+\kappa)^{-1}\int_0^t \max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_1 ds,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\{(\nu+\kappa)t\}^{1-1/p}\|W^\varepsilon(\cdot,t)\|_{M^p} \\ &\le C\|W_0\|_{\mathcal{M}} \\ &+ C(\nu+\kappa)^{-1}t^{1-1/p}\int_0^t (t-s)^{-1/2-1/r+1/p}s^{-3/2+1/r} \\ &\times \max_{s/2<\tau<s}\Big\{\{(\nu+\kappa)\tau\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,\tau)\|_\infty\|W^\varepsilon(\cdot,\tau)\|_1^\beta \\ &\qquad \times(\{(\nu+\kappa)\tau\}^{1-1/p}\|W^\varepsilon(\cdot,\tau)\|_{M^p})^{1-\beta}\Big\}ds \\ &+ 8\kappa^2\nu(\nu+\kappa)^{-1/p}t^{1-1/p}\int_0^t e^{-\frac{4\kappa^2}{\nu+\kappa}(t-s)}\max_{s/2<\tau<s}\|b^\varepsilon(\cdot,\tau)\|_{M^p} ds.\end{aligned} \tag{4.7}$$

Setting $\Pi(W_0,b_0) = \max\{\|W_0\|_{\mathcal{M}}, \|b_0\|_{\mathcal{M}}, \|b_0\|_{\mathcal{M}^p}\}$,

$$\begin{aligned}\lambda(t) = \sup_{s\le t}\Big\{&\{(\nu+\kappa)s\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,s)\|_\infty, \|W^\varepsilon(\cdot,s)\|_1, \\ &\{(\nu+\kappa)s\}^{1-1/p}\|W^\varepsilon(\cdot,s)\|_{M^p}, \|b^\varepsilon(\cdot,s)\|_1, \|b^\varepsilon(\cdot,s)\|_{M^p}\Big\},\end{aligned}$$

and taking into account Lemma 2.1, we can rewrite the estimates (4.3)–(4.7) in the following way

$$\begin{aligned}\{(\nu+\kappa)t\}^{1/2}\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_\infty \le{}& C(\nu+\kappa)^{-1}\Pi(W_0,b_0) \\ &+ C\Big((\nu+\kappa)^{-3/2} \\ &\quad + (2\kappa)^{1/2}(\nu+\kappa)^{-3/2-1/2p}\Big)\lambda(t)^2 \\ &+ C(\nu+\kappa)^{1/2}t^{1/2}\lambda(t),\end{aligned}$$

$$\|b^\varepsilon(\cdot,t)\|_1 \le \Pi(W_0,b_0) + 2\kappa(\nu+\kappa)^{-1}t\lambda(t),$$

$$\|b^\varepsilon(\cdot,t)\|_{M^p} \le \Pi(W_0,b_0) + 2\kappa(\nu+\kappa)^{-2+1/p}t^{1/p}\lambda(t),$$

$$\|W^\varepsilon(\cdot,t)\|_1 \le \Pi(W_0,b_0) + C(\nu+\kappa)^{-1}\lambda(t)^2 + 8\kappa^2\nu(\nu+\kappa)^{-1}t\lambda(t),$$

and

$$\{(\nu+\kappa)t\}^{1-1/p}\|W^\varepsilon(\cdot,t)\|_{M^p} \le C\Pi(W_0,b_0) + C(\nu+\kappa)^{-1}\lambda(t)^2 + 2\nu(\nu+\kappa)^{1-1/p}t^{1-1/p}\lambda(t).$$

Therefore,

$$\begin{aligned}0 \le \lambda(t) \le{}& C_0(\nu,\kappa)\Pi(W_0,b_0) + C_1(\nu,\kappa,p) \\ &\times\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big]\lambda(t) + C_2(\nu,\kappa,p)\lambda(t)^2,\end{aligned}$$

or equivalently

$$0 \le C_0\Pi(W_0,b_0) + \Big\{C_1\Big[t+t^{1/2}+t^{1/p}+t^{(p-1)/p}\Big]-1\Big\}\lambda(t) + C_2\lambda(t)^2,$$

where C_0, C_1 and C_2 are given by

$$C_0 = C(\nu+\kappa)^{-1} + C + 3,$$

$$\begin{aligned}C_1 = \max\Big\{&C(\nu+\kappa)^{1/2}, 2\kappa(\nu+\kappa)^{-1} + 8\kappa^2\nu(\nu+\kappa)^{-1}, \\ &2\kappa(\nu+\kappa)^{-2+1/p}, 2\nu(\nu+\kappa)^{1-1/p}\Big\}\end{aligned}$$

and

$$C_2 = C((\nu+\kappa)^{-3/2} + (2\kappa)^{1/2}(\nu+\kappa)^{-3/2-1/2p}) + 2C(\nu+\kappa)^{-1}.$$

Thus, for initial data such that $4C_0C_2\Pi(W_0,b_0) < 1$, there exists $T^* > 0$ for which

$$\begin{cases} C_1\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big] - 1 < 0, \\ \sigma = \Big\{C_1\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big] - 1\Big\}^2 \\ -4C_0C_2\Pi(W_0,b_0) > 0, \end{cases} \tag{4.8}$$

hold for all $t \in [0,T^*]$. Indeed, since $\lim_{t\to 0^+} C_1\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big] = 0$, there exists $\delta > 0$ such that $t \in (0,\delta)$ implies $C_1\Big[t + t^{1/2} + t^{1/p} + t^{(p-1)/p}\Big] < 1 - 2\sqrt{C_0C_2\Pi(W_0,b_0)}$, proving inequality (4.8) for $0 < T^* \le \delta$. Therefore, since λ is continuous, we conclude that $0 \le \lambda \le \frac{1}{2C_2}$. This completes the proof. ■

With these estimates, all the results from the previous section can be extended to the case under discussion considering $t > 0$. However, this solution must be understood in a weak sense, although a bootstrap a posteriori argument provides that the solutions will be regular, see [36].

Definition 4.1. Given $W_0, b_0 \in \mathcal{M}(\mathbb{R}^2)$ and $T > 0$, we say that $(W, \boldsymbol{u}, b)$, satisfying the initial conditions, is a weak solution of the system (1.13) if

(i) $\boldsymbol{u} = \boldsymbol{K} * \Big(\frac{1}{\nu+\kappa}W + \frac{2\kappa}{\nu+\kappa}b\Big)$,

(ii) The following equalities

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^2}\Big(\partial_t\varphi - (\nu+\kappa)\Delta\varphi + \frac{4\kappa^2}{\nu+\kappa}\varphi + (\boldsymbol{u}\cdot\nabla)\varphi\Big)\,dW \\ &\qquad = -\int_{\mathbb{R}^2}\varphi(\cdot,0)\,dW_0 + \frac{8\kappa^2\nu}{\nu+\kappa}\int_0^T\int_{\mathbb{R}^2}\varphi\,db, \\ &\int_0^T\int_{\mathbb{R}^2}\Big(\partial_t\phi + \frac{4\kappa\nu}{\nu+\kappa}\phi + (\boldsymbol{u}\cdot\nabla)\phi\Big)\,db \\ &\qquad = -\int_{\mathbb{R}^2}\phi(\cdot,0)\,db_0 + \frac{2\kappa}{\nu+\kappa}\int_0^T\int_{\mathbb{R}^2}\phi\,dW.\end{aligned} \tag{4.9}$$

hold for every $\varphi \in C_0^2(\mathbb{R}^2\times[0,T))$ and $\phi \in C_0^1(\mathbb{R}^2\times[0,T))$.

Indeed, taking into account that every uniformly bounded sequence in L^1 admits a subsequence which converges weak-$*$ as measures in $L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2))$ together with Proposition 4.1, then an identical argument to that of Proposition 3.2 allows to conclude the following result.

Proposition 4.2. *The sequence $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$ of solutions to system (2.9) admit a subsequence, still denoted by $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$,* *such that*

$$W^\varepsilon \rightharpoonup W \text{ weak-}* \text{ as measures in } L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2)), \tag{4.10}$$

$$b^\varepsilon \rightharpoonup b \text{ weak-}* \text{ as measures in } L^\infty(0,T^*;\mathcal{M}(\mathbb{R}^2)), \tag{4.11}$$

$$\boldsymbol{u}^\varepsilon \longrightarrow \boldsymbol{u} \text{ uniformly in compact set of } \mathbb{R}^2 \times (0,T^*). \tag{4.12}$$

Proposition 4.3. *The triple $(W, \boldsymbol{u}, b)$ given by Proposition 4.2 is a weak solution of system (1.13).*

Proof. The regularized solutions $(W^\varepsilon, \boldsymbol{u}^\varepsilon, b^\varepsilon)_{\varepsilon>0}$ satisfy the system in the weak form

$$\int_0^{T^*}\int_{\mathbb{R}^2}\left(\partial_t\varphi - (\nu+\kappa)\Delta\varphi + \frac{4\kappa^2}{\nu+\kappa}\varphi\right)W^\varepsilon d^2\boldsymbol{x}\,dt$$
$$+\int_0^{T^*}\int_{\mathbb{R}^2} M^\varepsilon\left(\left((\boldsymbol{u}^\varepsilon\cdot\nabla)\varphi(\boldsymbol{x},t)\right)W^\varepsilon\right)\,d^2\boldsymbol{x}\,dt$$
$$= -\int_{\mathbb{R}^2}\varphi(\cdot,0)W_0^\varepsilon\,d^2\boldsymbol{x}$$
$$+\frac{8\kappa^2\nu}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2} M^\varepsilon(b^\varepsilon)\varphi\,d^2\boldsymbol{x}\,dt,$$
$$\int_0^{T^*}\int_{\mathbb{R}^2}\left(\partial_t\phi + \frac{4\kappa\nu}{\nu+\kappa}\phi + (\boldsymbol{u}^\varepsilon\cdot\nabla)\phi\right)b^\varepsilon\,d^2\boldsymbol{x}\,dt$$
$$= -\int_{\mathbb{R}^2}\phi(\cdot,0)b_0^\varepsilon\,d\boldsymbol{x} + \frac{2\kappa}{\nu+\kappa}\int_0^{T^*}\int_{\mathbb{R}^2} W^\varepsilon\phi\,d^2\boldsymbol{x}\,dt.$$

Note that the explicit dependence of the variables $(\boldsymbol{x}, t)$ in the first equation for φ indicates that the function is not affected by the time-delay of M^ε. From the estimates for W^ε and b^ε in Morrey spaces (Proposition 4.1), we deduce that the velocity is equicontinuous in space and time for $t > 0$. Furthermore, the estimates in Morrey spaces of functions imply that there are no concentrations in the diagonal ($\boldsymbol{x} = \boldsymbol{y}$) towards Dirac masses at the limit of $M^\varepsilon((\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y})W^\varepsilon(\boldsymbol{y},t))W^\varepsilon)(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{x},t)$, in compact sets of $\mathbb{R}^2\times\mathbb{R}^2\times(0,T^*)$, as $\varepsilon \to 0$. Then, the following convergence property for the non-linear terms

$$\int_0^{T^*}\int_{\mathbb{R}^2} M^\varepsilon\left(\left((\boldsymbol{u}^\varepsilon\cdot\nabla)\varphi(\boldsymbol{x},t)\right)W^\varepsilon\right)\,d^2\boldsymbol{x}\,dt$$
$$\to \int_0^{T^*}\int_{\mathbb{R}^2}(\boldsymbol{u}\cdot\nabla)\varphi\,dW$$

hold true, see [37,38] for similar arguments. Passing to the limit in the linear terms does not present additional difficulty. Thus, we have proved the proposition. ■

The nonconcentration argument from the preceding proof and the convolution properties in Morrey spaces with singular kernels allow us to deduce the following result.

Corollary 4.1. *If $W_0 \in \mathcal{M}(\mathbb{R}^2)$ and $b_0 \in \mathcal{M}(\mathbb{R}^2)\cap\mathcal{M}^p(\mathbb{R}^2)$, with $p > 2$, then the system (1.13) has a global weak solution such that for any $T > 0$,*

$$\{(\nu+\kappa)t\}^{1/2}\boldsymbol{u} \in L^\infty(0,T;L^\infty(\mathbb{R}^2)).$$

Since the uniqueness and stability of solution are entirely analogous to the previous case, we have proved Theorem 1.2.

In the introduction to this paper we have mentioned the case of vortex sheets as a motivation to consider initial data in Morrey spaces. Classically this configuration is associated with vorticity. Indeed, as mentioned before, let δ_S be the Dirac measure located on the curve S in $\mathbb{R}^2$, with no end points parametrized by a piecewise C^1 function $\zeta : I \to \mathbb{R}^2$ being I an open real interval, and α a function on the curve S which represents its density or strength. In the case of vortex sheet structure of vorticity, the initial data is a measure of the type $\alpha\delta_S$. These measures belong to $\mathcal{M}^s(\mathbb{R}^2)$, where s depends on the regularity of α and on the regularity of the tangent τ to S according to the following result.

Lemma 4.1. *If the initial vorticity ω_0 is given by $\alpha\delta_S$ and $\tau\alpha \in L^p(S)^2$, with $p \geq 1$, then $\omega_0 \in \mathcal{M}^s(\mathbb{R}^2)$, with $s = \dfrac{2p'}{2p'-1}$ and p' is defined as $\frac{1}{p'}+\frac{1}{p} = 1$.*

Proof. By definition of Dirac delta on a curve, we have

$$\langle\alpha\delta_S, \boldsymbol{\varphi}\rangle = \int_I \boldsymbol{\varphi}(\zeta(\xi))\cdot\tau(\zeta(\xi))\alpha(\zeta(\xi))\,d\xi,\ \ \forall\ \boldsymbol{\varphi}\in C_0(\mathbb{R}^2)^2,$$

where $\cdot$ is the inner product. Therefore

$$TV_{B(\boldsymbol{x},R)}(\alpha\delta_S) \leq \int_{\zeta(I)\cap B(\boldsymbol{x},R)} |\boldsymbol{\varphi}(\zeta(\xi))|\,|\tau(\zeta(\xi))\alpha(\zeta(\xi))|\,d\xi$$
$$\leq \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p}\,|\zeta(I)\cap B(\boldsymbol{x},R)|^{1/p'} \leq \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p}R^{1/p'}.$$

Then, for $s' = 2p'$ we have $\dfrac{1}{s}+\dfrac{1}{s'} = 1$ and we conclude that $TV_{B(\boldsymbol{x},R)}(\alpha\delta_S) \leq \|\boldsymbol{\varphi}\|_{L^\infty}\|\tau\alpha\|_{L^p}R^{2/s'}$ and $\alpha\delta_S \in \mathcal{M}^s(\mathbb{R}^2)$. ■

5. Conclusions

We have studied the well-posedness and asymptotic behavior of a two-dimensional incompressible micropolar fluid model with null angular viscosity in terms of the evolution of the singular initial vorticity. Using a new quantity relating the vorticity and the angular velocity as well as integral techniques, we establish the existence of weak solutions local or global in time. Similar arguments lead to prove uniqueness and stability of solutions. The physically relevant case of vortex sheets and, more generally, the case of measures as initial data in Morrey spaces are included in this analysis. The combination of diffusion and non-linear phenomena such as transport and reaction at macro scales, as well as the influence of the microstructure, is interesting when studying the propagation or persistence of singularities. In this case, viscosity helps to regularize macroscopic singular structures such as vortex sheets, although the microstructure, governed by a transport equation, is not regularized.

Note that in the particular case of Lipschitz sheets, $p = \infty$ in Lemma 4.1, the initial vorticity belongs to $\mathcal{M}^2(\mathbb{R}^2)$. Also, we can consider extending the structure of vortex sheet to the initial configuration of W, which is included in the functional framework studied in this paper. This possibility involves not only the vorticity, but also affects the microstructure represented by b. The question of verifying whether conditions for persistence in time might exist for this structure, or in general analyzing its dynamics on the micro-scale associated with b is of great interest, but requires some different analysis.

Finally, we mention that this work opens various possibilities for future research when the interaction kernel is replaced by others that include different effects. In particular, it might be interesting to study those models that take into account rotational or potential kernels instead of the Biot–Savart one, as done in [39,40] for the Euler equations. This analysis, together with appropriate numerical simulations, allows us to include some phenomena like aggregation or dispersion which are very common in nature, for instance, in swarming or soft microstructure non-linear phenomena. In the context of swarming, the evolution of the population density is usually modeled by a convection–diffusion equation, while for dynamic models we find a coupled equation of the same type for the velocity field. Thus, we are faced with systems that allow a similar treatment to that of the micropolar fluid models studied here. In particular, after considering the estimates obtained above and some of the ideas presented in [39,40], we can study two-dimensional swarming problems by extending the arguments developed in this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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