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ORIGINAL PAPER

On the pillars of Functional Analysis

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Abstract
Many authors consider that the main pillars of Functional Analysis are the Hahn–Banach Theorem, the Uniform Boundedness Principle and the Open Mapping Principle. The first one is derived from Zorn's Lemma, while the latter two usually are obtained from Baire's Category Theorem. In this paper we show that these three pillars should be either just two or at least eight, since the Uniform Boundedness Principle, the Open Mapping Principle and another five theorems are equivalent, as we show in a very elemental way. Since one can give an almost trivial proof of the Uniform Boundedness Principle that does not require the Baire's theorem, we conclude that this is also the case for the other equivalent theorems that, in this way, are simultaneously proved in a simple, brief and concise way that sheds light on their nature.

Keywords Uniform boundedness · Open mapping · Closed graph · Banach isomorphism · Norms theorem · Sum theorem · Closed range

Mathematics Subject Classification Primary 46-01 · 46-02

1 Introduction

Many authors point out that the core of the Functional Analysis lies in three pillars, also known the "big three" (see for instance [4], [7]), namely the Uniform Boundedness Principle, the Open Mapping Theorem, and the Hahn–Banach Theorem ((HBT) for short). As there are so many references corroborating it, we only give a few sample items: [1, Chapter 2], [2], [5, Chapter 4], [10, p. 97], [12]. In some texts the Closed Graph Theorem is added to that list (see [3, p. 215] or [8], for instance). In most books on Functional Analysis, the first two theorems are proved (independently) from Baire's Category Theorem while the (HBT) is derived from Zorn's Lemma.

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The aim of this paper is to prove the equivalence between the seven theorems that make up the Theorem 1.1 below. Thus, the mentioned three pillars of the Functional Analysis should be only two or at least eight, depending on whether we consider the (HBT) and the Theorem 1.1 or, alternatively, the (HBT) and the seven theorems involved in the statement of Theorem 1.1. The proof of Theorem 1.1 is done in a very concise and simple way. Since a direct and elementary proof of the Uniform Boundedness Principle that does not require Baire's Theorem can be given, we conclude that this is also the case for all the results involved in Theorem 1.1. Thus, beyond providing a short and simultaneous proof of all of them, we also show that they have the same relevance because they are logically equivalent as Theorem 1.1 establishes. In fact, the proof of this result reveals how close the seven theorems that it involves are to each other.

As usual, if X and Y are normed spaces over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) then, $L(X, Y)$ will denote the normed space of all bounded linear operators from X into Y.

Theorem 1.1 *The following statements are equivalent:*

(i) *Uniform Boundedness Principle (UBP): Let $\{T_i\}_{i\in I}$ be a family of bounded linear maps from a Banach space X into a normed space Y_i. If $\{T_i\}_{i\in I}$ is pointwise bounded then* $\sup_{i\in I} \|T_i\| < \infty$.

(ii) *Open Mapping Theorem (OMT): Let X and Y be Banach spaces. If $T \in L(X, Y)$ is surjective then T is open.*

(iiii) *Open Mapping Theorem (bis) (OMTbis): Let X be a Banach space, Y a normed space, and $T \in L(X, Y)$ a surjective map. Then, T is open if and only if Y is complete.*

(iv) *Banach Isomorphism Theorem (BIT): Let X and Y be Banach spaces. If $T \in L(X, Y)$ is bijective, then T^{-1} is continuous.*

(v) *Norms Theorem (NT): Let $\|\cdot\|$ and $|\cdot|$ be complete norms on a linear space X such that they are comparable. Then $\|\cdot\|$ and $|\cdot|$ are equivalent.*

(vi) *Closed Graph Theorem (CGT): If X and Y are Banach spaces then, a linear operator $T : X \to Y$ is continuous if and only if its graph is closed (i. e. the separating subspace of T is zero).*

(vii) *Sum Theorem (ST): Let M and N be closed subspaces of a Banach space X. Then, $M + N$ is closed if, and only, if the map $(m, n) \to m + n$ from $M \times N$ into $M + N$ is open.*

(viii) *Closed Range Theorem (CRT): Let X and Y be Banach spaces and $T : \mathcal{D}(T) \subseteq X \to Y$ a closed linear operator whose domain $\mathcal{D}(T)$ is dense in X. Let $\mathcal{R}(T)$ be the range of T and $T^* : \mathcal{D}(T^*) \subseteq Y^* \to \mathcal{R}(T^*) \subseteq X^*$ the transpose of T. Then, the following assertions are equivalent:*

(a) *$\mathcal{R}(T)$ is closed in Y,*
(b) *$\mathcal{R}(T^*)$ is closed in X^*,*
(c) $\mathcal{R}(T^*) = (\ker T)^{\perp}$,
(d) $\mathcal{R}(T) = (\ker T^*)^{\top}$,
(e) *$T : \mathcal{D}(T) \to \mathcal{R}(T)$ is open,*
(f) *$T^* : \mathcal{D}(T^*) \to \mathcal{R}(T^*)$ is open.*

We will prove this theorem in Sect. 3.

2 The background

The aim of this section is to establish Proposition 2.1 (whose proof is straightforward) and Theorem 2.5 (which is a direct consequence of the Bipolar Theorem and, therefore, of the (HBT)).

Let X and Y be normed spaces. A linear map $T : X \to Y$ is *open* if T, maps open sets of X into open sets of Y. Note that such a T must be surjective. Indeed if, for a normed space Z, we denote by $B_Z(z, r)$ the open ball centered in $z \in Z$ with radius $r > 0$ then, obviously, T is open if and only if there exists $r > 0$ such that

$$B_Y(0, r) \subseteq T(B_X(0, 1)), \tag{2.1}$$

and the surjectivity of T follows. Moreover, T is open if and only if there exists $K > 0$ satisfying that, for every $y \in Y$, there exists $x \in X$ such that $T(x) = y$ with $\|x\| \leq K \|y\|$. (See for instance [10, p. 99]).

Throughout this paper, if M and N are subspaces of X then the space $M \times N$ will be considered provided with the norm

$$\|(M, N)\| := \max\{\|m\|, \|n\|\}.$$

Therefore, the map $M \times N \to M + N$ given by $(m, n) \to m + n$ is open if, and only if, there exists $K > 0$ satisfying that, for every $x \in M + N$, there exists $(m, n) \in M \times N$ with $x = m + n$ such that

$$\|(m, n)\| = \max\{\|m\|, \|n\|\} \leq K \|x\|. \tag{2.2}$$

From now on, the graph of a linear map T will be denoted by $G(T)$.

Proposition 2.1 *Let X and Y be normed spaces. For every densely defined linear operator $T : D(T) \subseteq X \to R(T) \subseteq Y$, the following assertions are equivalent:*

(i) *$T : D(T) \to R(T)$ is open.*
(ii) *For $M = G(T)$ and $N = X \times \{0\}$, the map $M \times N \to M+N$ given by $(m, n) \to m+n$ is open.*

Proof If $M = G(T)$ and $N = X \times \{0\}$, then $M + N = X \times R(T)$ and, by (2.2), the map $(m, n) \to m + n$ is open if, and only if, there exists $K > 0$ such that for every $(x, y) \in X \times R(T)$ there exists $w \in X$ satisfying that $Tw = y$ and

$$\max\{\|(w, Tw)\|, \|(x - w, 0)\|\} \leq K \|(x, y)\|. \tag{2.3}$$

To show (ii) $\Rightarrow$ (i), let $y \in R(T)$. Since $(0, y) \in X \times R(T)$, by (2.3), there exists $(w, T(w)) \in G(T)$ (which means that $w \in D(T)$ and $Tw = y$) such that

$$\|w\| \leq \max\{\|(w, Tw)\|, \|(-w, 0)\|\} \leq K \|(0, y)\| = K \|y\|.$$

Therefore, T is open.

To prove (i) $\Rightarrow$ (ii) suppose that T is open and let $\widetilde{K} > 0$ be such that, for every $y \in R(T)$, there exists $w \in D(T)$ with $Tw = y$ satisfying $\|w\| \leq \widetilde{K} \|y\|$. Assume that $\widetilde{K} > 1$ (replace $\widetilde{K}$ with $\widetilde{K} + 1$ if necessary). If $(x, y) \in X \times R(T)$ then, there exists $w \in D(T)$ such that $Tw = y$ and $\|w\| \leq \widetilde{K} \|y\|$. Consequently,

$$\|(x - w, 0)\| \leq \|(x, 0)\| + \|(w, 0)\| = \|x\| + \|w\| \leq \widetilde{K} \|x\| + \widetilde{K} \|y\| \leq 2\widetilde{K} \|(x, y)\|.$$

Moreover, $\|(w, Tw)\| = \|(w, y)\| = \max\{\|w\|, \|y\|\} \leq \widetilde{K} \|y\| \leq 2\widetilde{K} \|(x, y)\|$. Thus, for $K = 2\widetilde{K}$ we have $\max\{\|(w, Tw)\|, \|(x - w, 0)\|\} \leq K \|(x, y)\|$ with $Tw = y$, which proves (2.3) and hence (ii) □

From now on, if X is a normed space and if M is a subspace of X then, we denote by B_M the open unit ball of M and by $\overline{B}_M$ the closed one. Similarly, $B_M(0,r) := \{x \in M : \|x\| < r\}$ (for $r > 0$), while $\overline{B}_M(0,r)$ means the corresponding closed ball.

Note that from (2.1) the map $M \times N \to M + N$ given by $(m,n) \to m + n$ is open if there exists $r > 0$ such that

$$B_{M+N}(0,r) \subseteq B_M + B_N$$

(as $T(B_{M\times N}) = B_M + B_N$, where $T(m,n) = m + n$). The next result characterizes this property.

Lemma 2.2 *Let M and N be subspaces of a normed space X, and $r > 0$.*

(i) *If $B_{M+N}(0,r) \subseteq B_M + B_N$ then $B_{\overline{M+N}}(0,r) \subseteq \overline{B_M + B_N}$.*
(ii) *If X is complete, if M and N are closed, and if $B_{\overline{M+N}}(0,r) \subseteq \overline{B_M + B_N}$, then $B_{M+N}(0,\widetilde{r}) \subseteq B_M + B_N$, for every $\widetilde{r} < r$.*

Proof Assertion (i) is trivial. To prove (ii) suppose that $B_{\overline{M+N}}(0,r) \subseteq \overline{B_M + B_N}$, and note that if $\delta > 0$ then,

$$B_{\overline{M+N}}(0,r) \subseteq B_M + B_N + B_{\overline{M+N}}(0,\delta).$$

Let $y \in B_{M+N}(0,r)$. Put $\delta = r\varepsilon$ with $0 < \varepsilon < 1$ to obtain $m_1 \in B_M$, $n_1 \in B_N$, and $y_1 \in B_{\overline{M+N}}(0,r\varepsilon)$ such that $y = m_1 + n_1 + y_1$. Since $\frac{y_1}{\varepsilon} \in B_{\overline{M+N}}(0,r)$, we have that $y = \sum_{k=1}^{k=2} \varepsilon^{k-1} m_k + \sum_{k=1}^{k=2} \varepsilon^{k-1} n_k + y_2$ with $m_2 \in B_M$, $n_2 \in B_N$, and $y_2 \in B_{\overline{M+N}}(0,r\varepsilon^2)$. Iterating, we obtain $y = \sum_{k=1}^{k=n} \varepsilon^{k-1} m_k + \sum_{k=1}^{k=n} \varepsilon^{k-1} n_k + y_n$, with $m_k \in B_M$, $n_k \in B_N$, $y_k \in B_{\overline{M+N}}(0,r\varepsilon^k)$, and $k = 1,\cdots,n$. Since X is complete, $m := \sum_{k=1}^{\infty} \varepsilon^{k-1} m_k$ and $n := \sum_{k=1}^{\infty} \varepsilon^{k-1} n_k$ belong to X. In fact, $m \in B_M(0,\frac{1}{1-\varepsilon})$, $n \in B_N(0,\frac{1}{1-\varepsilon})$, and $y = m + n$. Therefore $B_{M+N}(0,r) \subseteq B_M(0,\frac{1}{1-\varepsilon}) + B_N(0,\frac{1}{1-\varepsilon})$, and the rest is clear. □

Next, we characterize again the openness of the map $M \times N \to M + N$ given by $(m,n) \to m + n$.

Lemma 2.3 *Let M and N be subspaces of a normed space X. Then the following assertions are equivalent:*

(i) *There exists $r > 0$ such that $B_{M+N}(0,r) \subseteq B_M + B_N$.*
(ii) *There exists $r > 0$ such that $(B_X(0,r) + N) \cap M \subseteq B_X + (M \cap N)$.*
(iii) *There exists $r > 0$ such that $(\overline{B}_X(0,r) + N) \cap M \subseteq \overline{B}_X + (M \cap N)$.*
(iv) *There exists $r > 0$ such that $B_{M+N}(0,r) \subseteq B_M + N$.*

Proof If (i) holds for $0 < r < 1$, then (ii) holds for this r as well. Indeed, if $m := x + n \in (B_X(0,r) + N) \cap M$ then $x = m - n \in B_{M+N}(0,r)$, so that there exists $\widetilde{m} \in B_M$ and $\widetilde{n} \in B_N$ such that $x = \widetilde{m} + \widetilde{n}$. Therefore, $m = x + n = \widetilde{m} + \widetilde{n} + n$ with $x \in B_X(0,1)$ and $\widetilde{n} + n = m - \widetilde{m} \in M \cap N$. this proves (i) $\Rightarrow$ (ii).

The assertion (ii) $\Rightarrow$ (iii) is trivial. In fact, if $r > 0$ satisfies (ii) then every $\widetilde{r} < r$ satisfies (iii). To prove (iii) $\Rightarrow$ (iv), consider $r > 0$ satisfying (iii) and let us show that $B_{M+N}(0,\widetilde{r}) \subseteq B_M + N$, for every $\widetilde{r} < r$. If $x := m + n \in M + N$ is such that $\|x\| < \widetilde{r}$ then, for $\varepsilon > 0$ with $\|(1+\varepsilon)x\| < r$, we have

$$(1+\varepsilon)m = (1+\varepsilon)x - (1+\varepsilon)n \in (\overline{B}_X(0,r) + N) \cap M,$$

and by (iii) there exists $y \in \overline{B}_X$ and $w \in (M \cap N)$ such that $(1+\varepsilon)m = y + w$. Thus $x = m + n = \frac{y}{1+\varepsilon} + (\frac{w}{1+\varepsilon} + n) \in B_M + N$ which proves (iii) $\Rightarrow$ (iv).

Finally (iv) $\Rightarrow$ (i) is obvious because if $r > 0$ is such that $B_{M+N}(0,r) \subseteq B_M + N$ then, $B_{M+N}(0,r) \subseteq B_M + B_N(0,r+1)$, and hence (i) is fulfilled for $\frac{r}{1+r}$. □

Let X^* be the topological dual of X. For $A \subseteq X$ and $F \subseteq X^*$ let A^0 and ${}^0F$ denote the *polar* of A and the *prepolar* of F, respectively. Thus

$$A^0 := \{x^* \in X^* : \sup_{a\in A} \left|x^*(a)\right| \leq 1\},$$
$${}^0F := \{x \in X : |f(x)| \leq 1, \forall f \in F\}.$$

For $A, B \subseteq X$ and $F, G \subseteq X^*$ basic properties are the following:

(i) If $A \subseteq B$ then $B^0 \subseteq A^0$; and if $F \subseteq G$ then ${}^0G \subseteq {}^0F$,
(ii) $({}^0(A^0))^0 = A^0$ and, consequently, $A^0 \subseteq B^0$ if and only if ${}^0(B^0) \subseteq {}^0(A^0)$.

The well-known Bipolar Theorem states that *if A is a subset of a normed space X then,* ${}^0(A^0)$ *is the closure of the absolute convex hull of A*. This is an immediate (and trivial) consequence of the geometrical version of the Hahn–Banach Theorem (See [12, Theorem 15.5] for details).

On the other hand the *orthogonal* of $A \subseteq X$ and the *preorthogonal* of $F \subseteq X^*$ are the sets

$$A^\perp := \{f \in X^* : f(x) = 0, \forall x \in A\}.$$
$$F^\top := \{x \in X : f(x) = 0, \forall f \in F\}.$$

It is obvious that if $0 \in A \cap B$ then $(A + B)^\perp = A^\perp \cap B^\perp$.

Lemma 2.4 *For closed subspaces M and N of a Banach space X, and $r > 0$,*

$$\overline{B}_{X^*}(0, r) + N^\perp = B_N(0, \tfrac{1}{r})^0$$
$$(B_N(0, r) + M)^0 = B_N(0, r)^0 \cap M^\perp.$$

Proof If $f \in B_N(0, \frac{1}{r})^0$ then $\left\| f_{/N} \right\| \leq r$ and, if $g \in X^*$ is an extension of $f_{/N}$ with $\|g\| = \left\| f_{/N} \right\|$, then $f = g + (f - g) \in \overline{B}_{X^*}(0, r) + N^\perp$ which shows that $B_N(0, \frac{1}{r})^0 \subseteq \overline{B}_{X^*}(0, r) + N^\perp$. The inclusion $\overline{B}_{X^*}(0, r) + N^\perp \subseteq B_N(0, \frac{1}{r})^0$ is trivial, as well as $B_N(0, r)^0 \cap M^\perp = (B_N(0, r) + M)^0$. In fact, if $0 \in A \subseteq X$ then $(A + M)^0 = A^0 \cap M^\perp$. □

Theorem 2.5 *Let M and N, be closed subspaces of a Banach space X. The following assertions are equivalent:*

(i) *The map $(m^*, n^*) \to m^* + n^*$ from $M^\perp \times N^\perp$ into $M^\perp + N^\perp$ is open.*
(ii) *The map $(m, n) \to m + n$, from $M \times N$ into $M + N$ is open.*

Proof The map $(m^*, n^*) \to m^* + n^*$ from $M^\perp \times N^\perp$ into $M^\perp + N^\perp$ is open if, and only if, there exists $r > 0$ such that $B_{M^\perp + N^\perp}(0, r) \subseteq B_{M^\perp} + B_{N^\perp}$. By Lemma 2.3, this is equivalent to the existence of $r > 0$ such that,

$$(\overline{B}_{X^*}(0, r) + N^\perp) \cap M^\perp \subseteq \overline{B}_{X^*} + (M^\perp \cap N^\perp). \tag{2.4}$$

Since $\overline{B}_{X^*}(0, r) = B_X(0, \frac{1}{r})^0$, by Lemma 2.4 we have

$$(\overline{B}_{X^*}(0, r) + N^\perp) \cap M^\perp = (B_N(0, \tfrac{1}{r}))^0 \cap M^\perp = \left(B_N(0, \tfrac{1}{r}) + M\right)^0 \text{ and}$$
$$\overline{B}_{X^*} + (M^\perp \cap N^\perp) = \overline{B}_{X^*} + (M + N)^\perp = (B_{M+N})^0.$$

Therefore, (2.4), and hence (i), means that $\left(B_N(0, \frac{1}{r}) + M\right)^0 \subseteq (B_{M+N})^0$ which is equivalent to

$${}^0\left((B_{M+N})^0\right) \subseteq {}^0\left((B_N(0, \tfrac{1}{r}) + M)\right)^0. \tag{2.5}$$

Since $\overline{B_{M+N}} = {}^0((B_{M+N}))^0$ and ${}^0\left((B_N(0,\frac{1}{r}) + M)\right)^0 = \overline{B_N(0,\frac{1}{r}) + M}$, by the Bipolar Theorem (as these sets are absolutely convex), we obtain that (2.5) is equivalent to the fact that $\overline{B_{M+N}(0,r)} \subseteq \overline{B_N + B_M}$ for some $r > 0$. By Lemma 2.2 this means that $B_{M+N}(0,r) \subseteq B_N + B_M$ for some $r > 0$, which is (ii). □

We conclude this section with some remarks on the dual of a densely defined operator.

Let X and Y be Banach spaces and $T : \mathcal{D}(T) \subseteq X \to Y$ be a densely defined linear operator. We define

$$\mathcal{D}(T^*) := \{g \in Y^* : gT : \mathcal{D}(T) \to \mathbb{K} \text{ is continuous}\}.$$

Let $\widehat{gT} \in X^*$ be the unique continuous extension to X that the operator gT has. We define the adjoint (or dual) operator of T as the operator $T^* : \mathcal{D}(T^*) \to X^*$ given by $T^*g = \widehat{gT}$, for every $g \in \mathcal{D}(T^*)$. Note that $G(T^*)$ is a subspace of $Y^* \times X^*$. Moreover the spaces $Y^* \times X^*$ and $(X \times Y)^*$ are canonically identified by defining $(g, f)(x, y) := f(x) + g(y)$, for every $g \in Y^*$, $f \in X^*$ and $(x, y) \in X \times Y$. Therefore, since $(g, f)(x, -Tx) = f(x) - g(Tx)$, we have

$$\begin{aligned} G(-T)^{\perp} \; : &= \{(g, f) \in Y^* \times X^* : f = gT \text{ in } \mathcal{D}(T)\} = \\ &\{(g, f) \in Y^* \times X^* : f = \widehat{gT} \text{ in } Y^*\} = \{(g, T^*g) : g \in Y^*\} = G(T^*). \end{aligned} \tag{2.6}$$

For the sake of completeness (as we will not need it) note that if, additionally, T is continuous then $\mathcal{D}(T^*) = Y^*$, trivially, and T^*, is continuous with $\|T^*\| \leq \|T\|$. Moreover, for every $x \in \mathcal{D}(T)$, there exists $g \in Y^*$ with $\|g\| = 1$ such that $|g(Tx)| = \|Tx\|$, so that $\|Tx\| = \|T^*g(x)\| \leq \|T^*g\| \, \|x\| \leq \|T^*\| \, \|x\|$, and hence $\|T\| = \|T^*\|$. In fact, we have that T is continuous if and only $\mathcal{D}(T^*) = Y^*$ and $T^* : Y^* \to X^*$ is continuous, in which case $\|T\| = \|T^*\|$.

3 The proof of the main theorem

Next we prove Theorem 1.1. Also we will provide a proof of the Uniform Boundedness Principle that does not need to use Baire's Theorem. For the sake of completeness we include the proofs of all the assertions involved being aware of that some of them are well known.

Proof of Theorem 1.1 (i) $\Rightarrow$ (ii). **[(UBP) $\Rightarrow$ (OMT)]**. Let X and Y be Banach spaces, and $T \in L(X,Y)$ a surjective operator. For $n \geq 1$, define

$$\|y\|_n := \inf\{\|u\| + n\,\|v\| : u \in X, v \in Y, \text{ and } Tu + v = y\}.$$

Then $\|\cdot\|_n$ is a norm on Y and $\|y\|_n \leq n\,\|y\|$, for every $y \in Y$ (take $u = 0$ and $v = y$ as $T(0) + y = y$). Let Z be the space of all the sequences in Y having a finite number of non-zero entries, provided with the norm

$$\|\{z_n\}\| := \max_{n \in \mathbb{N}} \|z_n\|_n \,, \text{ for } \{z_n\} \in Z.$$

Let $T_n : Y \to Z$ be the map $T_n(y) = \{z_k\}_{k \in \mathbb{N}}$ where $z_k = 0$ if $k \neq n$ and $z_n = y$. Obviously $T_n \in L(Y,Z)$, (as $\|T_n(y)\| = \|y\|_n \leq n\,\|y\|$). In fact, $\|T_n\| \leq n$. On the other hand, $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded because if $y \in Y$, then there exists $x \in X$ such that $Tx = y$, so that $\|T_n(y)\| = \|y\|_n \leq \|x\|$, for every $n \geq 1$. By the (UBP) there exists $M > 0$ such that $\|T_n\| \leq M$, for every $n \in \mathbb{N}$. Thus,

$$\|y\|_n = \|T_n(y)\| \leq M\,\|y\|\,,$$

for every $y \in Y$. Consequently, if $\|y\| < \frac{1}{M}$ then $\|y\|_n < 1$, and there exists $u_n \in X$, and $v_n \in Y$, such that $Tu_n + v_n = y$ and

$$\|u_n\| + n\|v_n\| \to \|y\|_n < 1.$$

Therefore, $\|u_n\| < 1$ and $n\|v_n\| < 1$. Since $\|v_n\| < \frac{1}{n}$ we have $Tu_n \to y$ so that $y \in \overline{T(B_X)}$. This shows that, for $r = \frac{1}{M}$,

$$B_Y(0, r) \subseteq \overline{T(B_X)}. \tag{3.1}$$

We claim now that $B_Y(0, \frac{r}{2}) \subseteq T(B_X)$. From (3.1), if we rescale then,

$$B_Y(0, \tfrac{r}{2^n}) \subseteq \overline{T(B_X(0, \tfrac{1}{2^n}))}.$$

Let $y \in B_Y(0, \frac{r}{2})$. Since $y \in \overline{T(B_X(0, \frac{1}{2}))}$, there exists $x_1 \in B_X(0, \frac{1}{2})$ satisfying that $\|y - Tx_1\| < \frac{r}{2^2}$, so that $y - Tx_1 \in B_Y(0, \frac{r}{2^2}) \subseteq \overline{T(B_X(0, \frac{1}{2^2}))}$. Therefore there exists $x_2 \in (B_X(0, \frac{1}{2^2})$ such that $\|y - Tx_1 - Tx_2\| < \frac{r}{2^3}$. Iterating, we obtain $x_n \in B_X(0, \frac{1}{2^n})$ satisfying $\left\|y - \sum_{k=1}^{n} Tx_k\right\| < \frac{r}{2^{n+1}}$, for every $n \in \mathbb{N}$. Since X is complete and $\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^n} = 1$, we have that $x = \sum_{k=1}^{\infty} x_k \in X$ and $\|x\| < 1$. Moreover, $Tx = T(\sum_{k=1}^{\infty} x_k) = \sum_{k=1}^{\infty} Tx_k = y$, so that $B_Y(0, \frac{r}{2}) \subseteq T(B_X)$ and hence T is open.

(ii) $\Rightarrow$ (iii). [**(OMT)** $\Rightarrow$ **(OMTbis)**]. Let X be a Banach space, Y a normed space and $T \in L(X,Y)$ a surjective map. If Y is complete then, T is open by the (OMT). Conversely, that T is open means that $\widehat{T}^{-1}$ is continuous, where $\widehat{T} : X/\ker T \to Y$ is the canonical factorization of T. Hence $\widehat{T}$ is bicontinuous and the complete norm of $X/\ker T$ induces a norm on Y, namely $|||y||| = \left\|\widehat{T}^{-1}(y)\right\|$, which is equivalent to the original one, and consequently Y is complete.

(iii) $\Rightarrow$ (iv). [**(OMTbis)** $\Rightarrow$ **(BIT)**]. If X and Y are Banach spaces and if $T \in L(X,Y)$ is bijective, then T^{-1} is continuous as, by the (OMTbis), T is open and this nothing but the continuity of T^{-1}.

(iv) $\Rightarrow$ (v). [**(BIT)** $\Rightarrow$ **(NT)**]. If $\|\cdot\|$ and $|||\cdot|||$ are complete norms on a linear space X and if $K > 0$ satisfies $\|\cdot\| \leq K\,|||\cdot|||$ then, by the (BIT), the identity map $i : (X, |||\cdot|||) \to (X, \|\cdot\|)$ is bicontinous, and hence $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

(v) $\Rightarrow$ (vi). [(**NT**) $\Rightarrow$ **(CGT)**]. Let X and Y be Banach spaces and $T : X \to Y$ be a linear map. Then, $G(T)$ is closed in $X \times Y$ if and only if the norm

$$|||x||| = \max\{\|x\|, \|Tx\|\}, \quad (x \in X)$$

is complete. Since $|||\cdot||| \leq \|\cdot\|$, by the (NT) we obtain that $G(T)$ is closed if and only if these norms are equivalent which is nothing but the continuity of T.

(vi) $\Rightarrow$ (vii). [(**CGT**) $\Rightarrow$ **(ST)**]. Let X be a Banach space and M and N be closed subspaces of X. Let $S : M \times N \to M + N$ be the mapping given by $S(m, n) = m + n$. Since S is continuous, $\ker S := S^{-1}(0)$ is a closed subspace of $M \times N$ and the factorization $\widehat{S} : (M \times N)/\ker S \to M + N$ is a linear, bijective, and continuous map from the Banach space $(M \times N)/\ker S$ into the normed space $M + N$. Since the graph of $\widehat{S}^{-1}$ is closed if and only if $M + N$ is closed, by the (CGT) we conclude that S is open (i.e. $\widehat{S}^{-1}$ is continuous) if and only if $M + N$ is closed (i.e. $\widehat{S}^{-1}$ has closed graph).

(vii) $\Rightarrow$ (viii). [**(ST)** $\Rightarrow$ **(CRT)**]. Let $T : \mathcal{D}(T) \subseteq X \to Y$ be a closed linear operator whose domain $\mathcal{D}(T)$ is dense in the Banach space X. Let $M =: G(T)$ and $N := X \times \{0\}$. Then $M^{\perp} = G(T)^{\perp} \equiv G(-T^*)$ by (2.6), and $N^{\perp} = Y^* \times \{0\}$ trivially. Moreover $M + N = X \times \mathcal{R}(T)$ and $M^{\perp} + N^{\perp} = Y^* \times \mathcal{R}(T^*)$.

By Proposition 2.1, the map $T : \mathcal{D}(T) \to \mathcal{R}(T)$ is open if, and only if, the sum map $(m, n) \to m + n$ from $M \times N$ into $M + N$ is open and, by the (ST), this is satisfied if, and only if, $M + N$, or equivalently $\mathcal{R}(T)$, is closed.

Similarly, by Proposition 2.1 we obtain that $T^* : \mathcal{D}(T^*) \to \mathcal{R}(T^*)$ is open if and only if $(m^*, n^*) \to m^* + n^*$, from $\widetilde{M} \times \widetilde{N}$ into $\widetilde{M} + \widetilde{N}$, is open where $\widetilde{M} := G(T^*)$ and $\widetilde{N} := \overline{\mathcal{D}(T^*)} \times \{0\}$. By the (ST), this is satisfied if, and only if, $\widetilde{M} + \widetilde{N} = \overline{\mathcal{D}(T^*)} \times \mathcal{R}(T^*)$ is closed which means that $\mathcal{R}(T^*)$ is closed. But this is equivalent to the fact that $M^\perp + N^\perp = Y^* \times \mathcal{R}(T^*)$ is closed which means, again by the (ST), that the map $M^\perp \times N^\perp \to M^\perp + N^\perp$, given by $(m^*, n^*) \to m^* + n^*$, is open. Finally note that (by Theorem 2.5) this last map is open if and only if the sum map $(m, n) \to m + n$ from $M \times N$ into $M + N$ is open, which concludes the proof of the (CRT) taking into account the trivial equalities $\overline{\mathcal{R}(T^*)} = (\ker T)^\perp$ and $\overline{\mathcal{R}(T)} = (\ker T^*)^\top$.

(viii) $\Rightarrow$ (i). [**(CRT) $\Rightarrow$ (UBP)].** Let X and $\{Y_i\}$ be a Banach spaces (complete Y_i if necessary) and let $\{T_i : X \to Y_i\}_{i \in I}$ be a pointwise bounded family of bounded linear operators. Let $\bigoplus_\infty Y_i$ be the Banach space of all the families $\{y_i\}_{i \in I}$ with $\sup_{i \in I}\{\|y_i\|\} < \infty$, provided with the norm $\|\cdot\|_\infty$. Let $T : X \to \bigoplus_\infty Y_i$ be the linear operator given by $Tx = \{T_i x\}_{i \in I}$ (note that T is well defined as $\{T_i\}_{i \in I}$ is pointwise bounded). Moreover, $G(T)$ is closed as we can check directly. Thus, the projection $P_X : G(T) \to X$, is a bijective bounded linear operator (between Banach spaces) whose range is closed. Therefore P_X is open, by the (CRT), which means that P_X^{-1} is continuous. Since the projection $P_Y : G(T) \to Y$ is obviously continuous we conclude that $T := P_Y P_X^{-1}$ is continuous which is nothing but $\sup_{i \in I} \|T_i\| < \infty$. □

The equivalence (i) $\Leftrightarrow$ (v) in the statement of the (CRT) asserts that

$$\begin{aligned} &\text{A densely defined linear operator } T : D(T) \subseteq X \to R(T) \subseteq Y, \\ &\text{with closed graph, is open if and only if } R(T) \text{ is closed.} \end{aligned} \tag{3.2}$$

This is a generalization of the (OMTbis), to the wider class of densely defined operators with closed graph, that can be deduced from the (ST) joint with Proposition 2.1. Moreover, (3.2) is enough to obtain the (UBP) as showed in the assertion (CRT) $\Rightarrow$ (UBP). Therefore, the role of the Bipolar Theorem is just to show that T is open if and only if T^* is open. Consequently, if in Theorem 1.1 we replace the (CRT) statement with (3.2) then, Section 2 can be reduced to Proposition 2.1.

The proof of (UBP) $\Rightarrow$ (OMT) follows [6], and some ideas from [9] have been useful for our approach in the proof of (ST) $\Rightarrow$ (CRT).

For the sake of completeness we include a proof of the (UBP) that does not require Baire's Theorem, from [11]. Others proofs of the (UBP) that do not Baire's Theorem are known; for instance the one given by Hahn applying the gliding hump argument [8, Exercise 1.76].

Proof of the Uniform boundedness principle.

First of all, we establish the following trivial result.

Lemma 3.1 *Let X and Y be normed spaces and let $T \in L(X,Y)$. Then, for every $x_0 \in X$ and every $r > 0$,*

$$\sup_{x \in B(x_0, r)} \|T(x)\| \geq r \|T\|.$$

Proof For every $x \in X$ we have

$$\|Tx\| \leq \frac{1}{2}(\|T(x - x_0)\| + \|T(x + x_0)\|) \leq \max\{\|T(x - x_0)\|, \|T(x + x_0)\|\},$$

and the result follows. □

In the proof of (UBP) ⇒ (OMT) we have applied the (UBP) to a particular family $\{T_i\}_{i\in I}$ where all the Y_i where equal. Anyway, embedding Y_i in $\bigoplus_\infty Y_i$ if necessary, it is not restrictive to assume in the (UBP) that $Y = Y_i$ for every $i \in I$.

Theorem 3.2 (Uniform Boundedness Principle). *Let X be a Banach space, Y a normed space and $\{T_i\}_{i\in I}$ a family of operators in $L(X,Y)$. If $\sup_{i\in I} \|T_i(x)\| < \infty$, for every $x \in X$, then* $\sup_{i\in I} \|T_i\| < \infty$.

Proof Suppose that $\sup_{i\in I} \|T_i\| = \infty$ and choose a sequence T_n in $\{T_i\}_{i\in I}$ such that $\|T_n\| \geq 4^n$. Put $x_0 = 0$ and apply inductively the Lemma 3.1 to get a sequence x_n such that $\|x_n - x_{n-1}\| \leq \frac{1}{3^n}$ and $\|T_n x_n\| \geq \frac{2}{3}\frac{1}{3^n}\|T_n\|$. Since x_n is a Cauchy sequence there exists $x \in X$ such that $x_n \to x$, and if $m > n$, then

$$\|x_n - x_m\| \leq \sum_{k=n+1}^{m} \|x_k - x_{k-1}\| \leq \sum_{k=n+1}^{m} \frac{1}{3^k} = \frac{3}{2}\left(\frac{1}{3^{n+1}} - \frac{1}{3^{m+1}}\right).$$

Letting $m \to \infty$ we obtain that $\|x - x_n\| \leq \frac{1}{2}\frac{1}{3^n}$. Consequently,

$$\|T_n(x)\| \geq \|T_n(x_n)\| - \|T_n(x - x_n)\| \geq \frac{1}{6}\frac{1}{3^n}\|T_n\| \geq \frac{1}{6}\frac{4^n}{3^n} \to \infty,$$

which shows that $\{T_i\}_{i\in I}$ is not pointwise bounded, as desired. □

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