The embedded Calabi-Yau conjecture for finite genus

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Abstract

Suppose M is a complete, embedded minimal surface in $\mathbb{R}^3$ with an infinite number of ends, finite genus and compact boundary. We prove that the simple limit ends of M have properly embedded representatives with compact boundary, genus zero and with constrained geometry. We use this result to show that if M has at least two simple limit ends, then M has exactly two simple limit ends. Furthermore, we demonstrate that M is properly embedded in $\mathbb{R}^3$ if and only if M has at most two limit ends if and only if M has a countable number of limit ends.

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1 Introduction.

The Calabi-Yau conjectures refer to a series of conjectures concerning the nonexistence of a complete, minimally immersed surface $f\colon M \to \mathbb{R}^3$ whose image $f(M)$ is constrained to be contained in a particular region of $\mathbb{R}^3$ (see Calabi [2], page 212 in Chern [3], problem 91 in Yau [45] and page 360 in Yau [46]). Calabi's original conjectures [2] state that a complete, nonflat minimal surface cannot be contained in the unit ball $\mathbb{B}(1) = \{x \in \mathbb{R}^3 \mid |x| < 1\}$ or even in a halfspace of $\mathbb{R}^3$. Among the positive results on the Calabi-Yau conjectures, we mention that the Strong Halfspace Theorem [17] implies the validity of the conjectures for properly immersed minimal surfaces in a closed halfspace. A spectacular positive result by Colding and Minicozzi [9] is that any complete, embedded minimal surface M in $\mathbb{R}^3$ with finite topology is proper, and so the Halfspace Theorem (Hoffman and Meeks [17]) implies that M cannot be contained in a halfspace unless it is a finite number of parallel planes. In contrast to Colding and Minicozzi's properness result for the finite topology embedded Calabi-Yau problem, Ferrer, Martín, Meeks and Nadirashvili have conjectured that there is a particular bounded domain Ω in $\mathbb{R}^3$ (see [14] for a description of Ω), which is smooth except at one

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point and satisfies the following property: *Every open surface with compact (possibly empty) boundary whose ends have infinite genus admits a complete, proper minimal* embedding *into* Ω. We refer the reader to Section 2 for a brief elementary topological discussion of the notions of end, the genus of an end, limit end, simple limit end and end representative for any noncompact surface, terms we will use freely in this manuscript.

The theory developed in this paper represents the first step in resolving the following fundamental conjecture, which gives a strong converse to the just mentioned existence conjecture of Ferrer, Martín, Meeks and Nadirashvili for open surfaces with compact boundary.

Conjecture 1.1 (Embedded Calabi-Yau Conjecture for Finite Genus) *Every connected, complete embedded minimal surface $M \subset \mathbb{R}^3$ of finite genus and compact (possibly empty) boundary is properly embedded in $\mathbb{R}^3$.*

Corollary 1 in [30] implies Conjecture 1.1 under the additional hypothesis that M is a leaf of a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$, or equivalently, when M has locally bounded Gaussian curvature in $\mathbb{R}^3$. As mentioned above, Colding and Minicozzi [9] have proved Conjecture 1.1 under the additional assumption that M has finite topology. In [36], Meeks and Rosenberg proved that connected, complete embedded minimal surfaces in $\mathbb{R}^3$ with positive injectivity radius are proper; their theorem is a generalization of the properness result of Colding and Minicozzi since complete, embedded finite topology minimal surfaces in $\mathbb{R}^3$ have positive injectivity radius.

These results, together with others by Bernstein and Breiner [1], Collin [11], Meeks and Pérez [23] and Meeks and Rosenberg [35], imply that a complete, nonflat embedded minimal surface $M \subset \mathbb{R}^3$ with finite topology has annular ends which are asymptotic to ends of planes, catenoids or M has just one end which is asymptotic to the end of a helicoid. In all of these cases, M is proven to be conformally a compact Riemann surface $\overline{M}$ punctured in a finite number of points (in particular, M is recurrent for Brownian motion), and the embedding of M into $\mathbb{R}^3$ can be expressed analytically in terms of meromorphic data defined on $\overline{M}$. In the case that a complete embedded minimal surface M of finite topology in $\mathbb{R}^3$ has nonempty compact boundary, a similar description of its conformal structure (∂M has full harmonic measure) and of its asymptotic behavior (a few more asymptotic types arise than in the case without boundary) hold, see [23] for details. Concerning conformal questions, one consequence of the results in this paper is Corollary 1.8, which states that if a properly embedded minimal surface in $\mathbb{R}^3$ has a limit end of genus zero, then it is recurrent; this can be viewed as a generalization of our previous result [31] that any properly embedded minimal surface of finite genus in $\mathbb{R}^3$ is recurrent.

Using the techniques developed by Colding and Minicozzi [5, 6, 7, 8, 9, 10], Meeks and Rosenberg [36] and those in our papers [26, 28, 30, 31, 34], we shall prove here that if a complete, embedded minimal surface of finite genus in $\mathbb{R}^3$ has a countable number of limit ends, then it is properly embedded in $\mathbb{R}^3$ (see Theorem 1.3 below). By the main result of Collin, Kusner, Meeks and Rosenberg in [12], any properly embedded minimal surface in $\mathbb{R}^3$ must have a countable number of ends, even if it does not have finite genus. More generally, the results in [12] imply that a properly embedded minimal surface with compact boundary in $\mathbb{R}^3$ can have at most two limit ends, and that if it has empty boundary and two limit ends, then it is recurrent.

Our first key partial result on Conjecture 1.1 is the following theorem, which is the main result in Section 3 (see Remark 3.4).

Theorem 1.2 *Let $M \subset \mathbb{R}^3$ be a complete embedded minimal surface of finite genus with compact boundary and exactly one limit end. Then M is properly embedded in $\mathbb{R}^3$.*

More generally, we have the following extension of the above result, which is proved in Section 5.

Theorem 1.3 *Suppose $M \subset \mathbb{R}^3$ is a complete, connected, embedded minimal surface of finite genus, an infinite number of ends and compact boundary (possibly empty). Then:*

1. *M has at most two simple limit ends.*
2. *M has exactly one or two limit ends if and only if M is proper in $\mathbb{R}^3$.*
3. *Suppose M has a countable number of limit ends. Then:*
   - *3-A. M has one or two limit ends.*
   - *3-B. M is proper in $\mathbb{R}^3$.*
   - *3-C. If M has two limit ends, then its annular ends are planar.*
   - *3-D. If $\partial M = \emptyset$, then M has exactly two limit ends and M is recurrent for Brownian motion.*
   - *3-E. If $\partial M \neq \emptyset$, then ∂M has full harmonic measure.*

Remark 1.4 In contrast to item 3-D of Theorem 1.3, we note that Traizet [43] has constructed a complete embedded minimal surface in $\mathbb{R}^3$ of infinite genus, with empty boundary, one limit end and infinitely many catenoidal type ends.

The proof of Theorem 1.3 depends on Theorem 1.6 below, which describes the geometry, topology and conformal structure of certain representatives of a simple limit end of genus zero for a complete embedded minimal surface in $\mathbb{R}^3$; see Figure 1 for a suggestive picture describing the key geometric features of such a representative.

Before stating Theorem 1.6, we will need the following definition.

Definition 1.5 Let E be a complete embedded minimal surface in $\mathbb{R}^3$ with nonempty compact boundary ∂E. We define the *flux vector of E* as

$$F_E = \int_{\partial E} \eta \in \mathbb{R}^3, \qquad (1)$$

where η denotes the inward pointing unit conormal vector to E along ∂E.

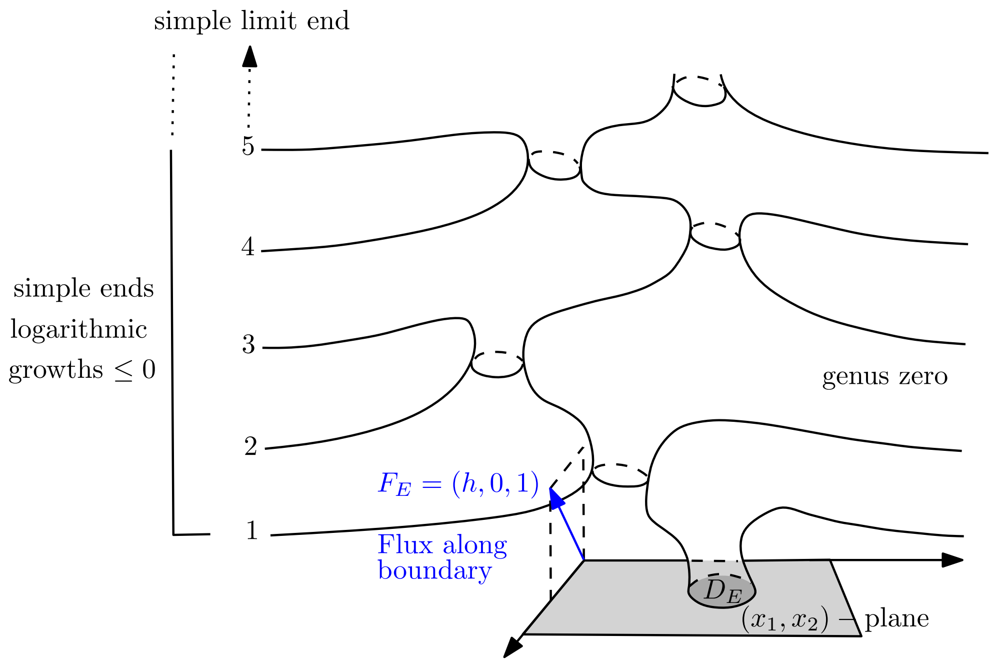

Figure 1: A graphical representation of the end representative E in Theorem 1.6.

Theorem 1.6 *Suppose* $\mathbf{e}$ *is a simple limit end of genus zero of a complete, connected, embedded minimal surface* $M \subset \mathbb{R}^3$ *with compact (possibly empty) boundary. Then* $\mathbf{e}$ *can be represented by a subdomain* $E \subset \text{Int}(M)$ *that is properly embedded in* $\mathbb{R}^3$ *and, after a translation, rotation and homothety of* M*,* E *satisfies the following statements:*

1. *The annular ends of* E *have nonpositive logarithmic growths.*

2. E *has genus zero and one limit end, which, in the natural ordering of the ends of* E *given by the Ordering Theorem*[1] *in [16], is the top end of* E*.*

3. *The boundary* ∂E *is a simple closed curve in the* (x_1, x_2)*-plane, and the flux vector* F_E *of* E *defined as in (1) is* $(h, 0, 1)$ *for some* $h > 0$*. Furthermore,* ∂E *bounds a convex disk* $D_E \subset \{x_3 = 0\}$ *whose interior is disjoint from* E*, see Figure 1.*

4. *There exists an orientation preserving diffeomorphism* $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $f(\mathcal{R}_+) = E$*, where* $\mathcal{R}_+$ *is the top half of a Riemann minimal example*[2] *with boundary circle in the* (x_1, x_2)*-plane.*

5. E *has bounded Gaussian curvature.*

6. E *is conformally diffeomorphic to the closed punctured disk* $\{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$ *minus a sequence of points converging to* 0*. In particular,* ∂E *has full harmonic measure.*

[1] Observe that the Ordering Theorem stated in [16] also holds for properly embedded minimal surfaces in $\mathbb{R}^3$ with compact boundary.

[2] See [29, 33] for a discussion of the singly-periodic, genus-zero, Riemann minimal examples.

Remark 1.7 If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface of finite genus and infinite topology, then M has exactly two limit ends $e_{-\infty}, e_\infty$ which are simple limit ends of genus zero, and which admit representatives $E_{-\infty}, E_\infty$ that satisfy the conclusions of Theorem 1.6, see [30, 31]. In this case where M has no boundary, then Theorem 8.1 in [33] implies that the asymptotic behavior of each of its two limit ends can be described by the geometry of the ends of a Riemann minimal example.

Crucial ingredients in the proof of Theorem 1.6 are the Limit Lamination Closure Theorem (Theorem 1 in [36]), the Local Picture Theorem on the Scale of Topology (Theorem 1.1 in [27]) and Theorem 2.2 in [26] on the structure of certain possibly singular lamination limits of certain sequences of minimal surfaces in $\mathbb{R}^3$. These ingredients, as well as many arguments in this paper, rely heavily on Colding-Minicozzi theory.

Theorems 1.3 and 1.6 not only play an important theoretical role in our strategy to prove Conjecture 1.1, but they also have important consequences for properly embedded minimal surfaces, such as the one given in the next corollary; this corollary follows from the more general result Corollary 6.1.

Corollary 1.8 *If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with a limit end of genus zero, then M is recurrent for Brownian motion.*

Some of the results in this paper were announced by the authors at a conference in Paris in 2004. Our proofs use results of Colding-Minicozzi theory that led us to develop a detailed study of minimal laminations with singularities and subsequent applications. The present paper can be considered as a culmination of a long term project in the understanding of complete embedded minimal surfaces of finite genus in $\mathbb{R}^3$.

2 Preliminaries on the ends of a noncompact surface.

Given $p \in \mathbb{R}^3$ and $R > 0$, we denote by $\mathbb{B}(p, R)$ the open ball centered at p of radius R. When $p = \vec{0}$, we let $\mathbb{B}(R) = \mathbb{B}(\vec{0}, R)$. If $\Sigma \subset \mathbb{R}^3$ is a surface and $p \in \Sigma$, then $K_\Sigma, d_\Sigma, I_\Sigma, B_\Sigma(p, R)$ and $T_p\Sigma$ respectively stand for the Gaussian curvature function of Σ, its intrinsic distance function, its injectivity radius function, the intrinsic ball centered at p of radius $R > 0$ and the tangent plane to Σ at p. Also, $\overline{\mathbb{D}} = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$ stands for the closed unit disk.

Throughout the paper, $M \subset \mathbb{R}^3$ will denote a connected, complete embedded minimal surface with compact boundary (possibly empty).

We next recall the notion of end of M. Consider the set

$$\mathcal{A} = \{\alpha \colon [0, \infty) \to M \ \mid \ \alpha \text{ is a proper arc}\}.$$

In $\mathcal{A}$, we define the equivalence relation $\alpha_1 \sim \alpha_2$ if for every compact set $C \subset M$, α_1, α_2 lie eventually (outside a compact subset of the parameter domain $[0, \infty)$) in the same component of $M - C$.

Definition 2.1 Each equivalence class in $\mathcal{E}(M) = \mathcal{A}/_\sim$ is called an *end* of M. If $\mathbf{e} \in \mathcal{E}(M)$, $\alpha \in \mathbf{e}$ is a proper arc and $E \subset M$ is a proper connected subdomain with compact boundary such that $\alpha([t_0, \infty)) \subset E$ for some $t_0 \geq 0$, then we say that E *represents* the end $\mathbf{e}$.

The space $\mathcal{E}(M)$ has the following natural Hausdorff topology. For each proper subdomain $E \subset M$ with compact boundary, we define the basis open set $B(E) \subset \mathcal{E}(M)$ to be those equivalence classes in $\mathcal{E}(M)$ which have representative proper arcs contained in E. With this topology, $\mathcal{E}(M)$ is a totally disconnected compact space which embeds topologically as a subspace of $[0,1] \subset \mathbb{R}$ (see pages 288-289 of [24] for a proof of this property). In the sequel, we will view $\mathcal{E}(M)$ as a subset of $[0,1]$ endowed with the induced metric topology.

Note that every *simple end* $\mathbf{x}$ of M (i.e., $\mathbf{x}$ is an isolated point of $\mathcal{E}(M)$) with genus zero can be represented by a proper annulus $E_{\mathbf{x}} \subset M$ which is homeomorphic to $\mathbb{S}^1 \times [0, \infty)$.

Next consider a *simple limit end* $\mathbf{e}$ of M, i.e., there exists a neighborhood $O(\mathbf{e}) \subset \mathcal{E}(M)$ such that $O(\mathbf{e}) - \{\mathbf{e}\}$ consists of simple ends and $\mathbf{e}$ is a limit point of a sequence of simple ends $\{\mathbf{x}_n\}_n \subset O(\mathbf{e}) - \{\mathbf{e}\}$. Suppose that the simple limit end $\mathbf{e}$ has genus zero, i.e., $\mathbf{e}$ admits a representative of genus zero. By the classification of genus-zero surfaces and after a possible replacement by a smaller neighborhood $O(\mathbf{e})$ of $\mathbf{e}$ in $\mathcal{E}(M)$, there exists a proper subdomain E of M satisfying:

(A1) E is diffeomorphic to $\overline{\mathbb{D}}(*) = \overline{\mathbb{D}} - [\{0\} \cup \{\frac{1}{2n}\}_{n \in \mathbb{N}}]$. Furthermore, $\partial E \cap \partial M = \varnothing$.

(A2) E represents all the ends in $O(\mathbf{e})$, and the equivalence class under $\sim$ of every proper arc in E represents a unique end in $O(\mathbf{e})$.

3 Simple limit ends of genus zero can be represented by properly embedded surfaces.

We begin this section with several key notions that are closely tied to obtaining properness results for minimal surfaces, including Theorem 1.2 which will be proved here.

Definition 3.1 1. An embedded surface with boundary (possibly empty) $\Sigma \subset \mathbb{R}^3$ is said to be *locally simply connected in* $\mathbb{R}^3$ if for every $p \in \mathbb{R}^3$, there exists $r = r(p) > 0$ such that the closure of each component of $\Sigma \cap \mathbb{B}(p,r)$ that is disjoint from $\partial\Sigma$, is a compact disk with boundary in $\partial\mathbb{B}(p,r)$. Σ has *locally positive injectivity radius away from* $\partial\Sigma$ if for every $p \in \mathbb{R}^3$, there exists $r = r(p) > 0$ such that the injectivity radius function I_Σ of Σ is bounded away from zero on the union of the components of $\Sigma \cap \overline{\mathbb{B}}(p,r)$ that are disjoint from $\partial\Sigma$.

2. Let $A \subset \mathbb{R}^3$ be an open set and $\{\Sigma_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ be a sequence of embedded surfaces (possibly with boundary). The sequence $\{\Sigma_n\}_n$ is called *locally simply connected* in A if for every $p \in A$, there exists $r = r(p) > 0$ such that $\mathbb{B}(p,r) \subset A$ and for n sufficiently large, $\mathbb{B}(p,r)$ intersects Σ_n in components that are disks with boundaries in $\partial\mathbb{B}(p,r)$. $\{\Sigma_n\}_n$ is said to have *locally positive injectivity radius in A,* if for every $p \in A$, there exists $\varepsilon_p > 0$

and $n_p \in \mathbb{N}$ such that for $n > n_p$, the restricted functions $(I_{\Sigma_n})|_{\Sigma_n \cap \mathbb{B}(p,\varepsilon_p)}$ are uniformly bounded away from zero.

Remark 3.2 With the notation of item 2 of Definition 3.1, if the surfaces Σ_n have nonempty boundaries and $\{\Sigma_n\}_n$ has locally positive injectivity radius in A, then for any $p \in A$ there exists $\varepsilon_p > 0$ and $n_p \in \mathbb{N}$ such that $\partial \Sigma_n \cap \mathbb{B}(p,\varepsilon_p) = \emptyset$ for $n > n_p$, i.e., points in the boundary of Σ_n must eventually diverge in space or converge to a subset of $\mathbb{R}^3 - A$.

By Proposition 1.1 in [9], if $M \subset \mathbb{R}^3$ is an embedded minimal surface, then the property that M is locally simply connected in $\mathbb{R}^3$ is equivalent to the property that M has locally positive injectivity radius away from ∂M. The same proposition gives that a sequence of embedded minimal surfaces $\{M_n\}_n$ has locally positive injectivity radius in an open set $A \subset \mathbb{R}^3$ if and only if $\{M_n\}_n$ is locally simply connected in A.

Theorem 2 in [36] implies that if an embedded, complete, nonflat minimal surface in $\mathbb{R}^3$ (with empty boundary) has positive injectivity radius, then it is proper. Although not stated explicitly in [36], the following result is an immediate consequence of the proof of Theorem 2 in [36] and other arguments therein.

Theorem 3.3 *Let $M \subset \mathbb{R}^3$ be a complete, connected, embedded minimal surface with compact boundary. If the injectivity radius function I_M of M is bounded away from zero outside of some intrinsic ε-neighborhood of ∂M, then M is proper in $\mathbb{R}^3$. Furthermore, if M has finite topology, then I_M is bounded away from zero outside of some intrinsic ε-neighborhood of ∂M, and so, M is proper in $\mathbb{R}^3$.*

Remark 3.4 Theorem 3.5 below implies the main properness statement in Theorem 1.6. It also implies Theorem 1.2 by the following reasoning. Suppose that $M \subset \mathbb{R}^3$ is a complete embedded minimal surface of finite genus with compact boundary and exactly one limit end $\mathbf{e}$, which must therefore be a simple limit end. If E is the proper representative of $\mathbf{e}$ given in the next theorem, then the surface $M - \text{Int}(E)$ has finite topology and must therefore be proper by Theorem 3.3; hence, $M = E \cup (M - \text{Int}(E))$ is also proper in $\mathbb{R}^3$.

Theorem 3.5 *Suppose that $M \subset \mathbb{R}^3$ is a complete embedded minimal surface with possibly empty compact boundary. Every simple limit end $\mathbf{e} \in \mathcal{E}(M)$ of genus zero can be represented by a subdomain $E \subset M$ with compact boundary whose injectivity radius is bounded away from zero outside each compact neighborhood of its boundary. In particular, E is proper in $\mathbb{R}^3$.*

Proof. Let $\mathbf{e} \in \mathcal{E}(M)$ be a simple limit end of genus zero. Consider a proper subdomain $E \subset M$ satisfying properties (A1) and (A2) stated in the preliminaries section. With a slight abuse of notation, we identify E with the parameter domain $\overline{\mathbb{D}}(*) = \overline{\mathbb{D}} - [\{0\} \cup \{\frac{1}{2n}\}_{n \in \mathbb{N}}]$, see property (A1). The proof of Theorem 3.5 will be divided into several statements; more precisely, Lemmas 3.7 and 3.11 and Propositions 3.8 and 3.12. As the proof develops, we will replace E by similar proper subdomains of E and $O(\mathbf{e})$ by the open subset of ends of the replaced E, but will continue to label these objects by the same letters.

We first deal with the (simple) annular ends in E. For $n \in \mathbb{N}$, let $\mathbb{S}^1_n$ denote the circle of center $0 \in \mathbb{C}$ and radius $\frac{1}{2n+1}$. Let E_n be the proper subdomain of E bounded by $\partial E \cup \mathbb{S}^1_n$. Since E_n has finite topology and compact boundary, then Theorem 3.3 applied to E_n insures that E_n is proper in $\mathbb{R}^3$. As each of the (finitely many) ends of E_n is an annular end, then Collin's theorem [11] implies that each end of E_n has finite total curvature and is asymptotic to an end of a plane or catenoid. After a rigid motion in $\mathbb{R}^3$, we may assume that:

(B1) The annular ends of E are represented by graphs over their projections to $\{x_3 = 0\}$ with logarithmic growth (which is zero when the end is asymptotic to the end of a plane).

The proof of Theorem 3.5 is by contradiction. Hence **assume there is no end representative E of e which is a proper surface**. By Theorem 3.3, the injectivity radius function I_E of every such a representative has the property that I_E fails to be bounded away from zero outside some small ε-neighborhood of ∂E. Therefore, there exists a sequence of points $q_n \in E$ such that $d_E(q_n, \partial E)$ is bounded away from zero and $I_E(q_n) \to 0$ as $n \to \infty$. Clearly, the q_n diverge in E. As I_E becomes unbounded when approaching each of the simple ends of E, we deduce that the q_n converge to the origin when viewed inside $\overline{\mathbb{D}}(*)$.

Since E has genus zero, then the Local Picture Theorem on the Scale of Topology (see Theorem 1.1, Proposition 4.20 and Remark 4.32 in [27]) implies that we can find a divergent sequence of points $p_n \in E$ (called *points of almost minimal injectivity radius for* E) and positive numbers $\varepsilon_n \to 0$, such that $d_E(p_n, q_n) \to 0$ as $n \to \infty$ and:

(C1) The closure M_n of the component of $\mathbb{B}(p_n, \varepsilon_n) \cap E$ that contains p_n is compact with boundary $\partial M_n \subset \partial \mathbb{B}(p_n, \varepsilon_n)$. Furthermore, M_n is disjoint from ∂E for n large enough (this follows from the fact that p_n is divergent in E).

(C2) Let $\lambda_n = 1/I_{M_n}(p_n)$, where I_{M_n} denotes the injectivity radius function of E restricted to M_n. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n}$ in M_n and $\lambda_n \varepsilon_n \to \infty$.

Furthermore, exactly one of the following two cases occurs after extracting a subsequence.

(C3) The surfaces $\lambda_n(M_n - p_n)$ have uniformly bounded Gaussian curvature on compact subsets of $\mathbb{R}^3$. In this case, there exists a connected, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$, such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n(M_n - p_n)$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.

(C4) After possibly a rotation in $\mathbb{R}^3$, the surfaces $\lambda_n(M_n - p_n)$ converge to a minimal parking garage structure[3] of $\mathbb{R}^3$ consisting of a foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes, with two columns l_1, l_2 such that the associated highly sheeted, double multivalued graphs forming in $\lambda_n(M_n - p_n)$ around l_1, l_2 for n sufficiently large, are oppositely handed. Furthermore, after relabeling, l_1 intersects $\overline{\mathbb{B}}(1)$ and l_2 is at distance 1 from l_1.

[3] We refer the reader to Section 3 in [27] for the definition of parking garage structure of $\mathbb{R}^3$.

In order to finish the proof of Theorem 3.5, we must find a contradiction in each of the Cases (C3), (C4) above.

Suppose first that Case (C3) holds. As the $\lambda_n(M_n - p_n)$ all have genus zero, then M_∞ has genus zero as well. By classification results for properly embedded minimal surfaces of genus zero (Collin [11], López-Ros [18], Meeks-Pérez-Ros [33]), M_∞ is a catenoid or a Riemann minimal example. Let $\gamma \subset M_\infty$ be the waist circle if M_∞ is a catenoid, and in the case M_∞ is a Riemann minimal example, then let γ be a simple closed planar curve (actually a circle) which separates the two limit ends of M_∞.

Remark 3.6 In the sequel, we will need the notion of *flux vector* of a minimal surface along a closed curve Γ once we have chosen a unit conormal vector η along Γ; this flux is the vector in $\mathbb{R}^3$ given by the integral of η along Γ, which clearly is defined up to a sign. This ambiguity still lets us make sense of when this flux is nonzero, or when it is vertical.

Since γ has nonzero flux, then for n large, γ is approximated by the image by the composition of a translation by vector $-p_n$ with a homothety by λ_n of a simple closed planar curve $\gamma_n \subset M_n$ also with nonzero flux.

Lemma 3.7 *$\gamma \subset M_\infty$ has vertical flux. Furthermore, after choosing a subsequence, each curve γ_n also has vertical flux.*

Proof. It suffices to prove that for n large, $\gamma_n \subset E$ has vertical flux. If the subdisk in $\overline{\mathbb{D}}$ bounded by γ_n does not contain $0 \in \overline{\mathbb{D}}$, then γ_n is homologous to a finite number of loops around the annular ends of E, and so, γ_n has vertical flux by property (B1). Otherwise, after replacing by a subsequence, we may assume that γ_n is topologically parallel to γ_{n+k} and to ∂E in $\overline{\mathbb{D}} - \{0\}$ for $n, k \in \mathbb{N}$, n large. Hence for any $k \in \mathbb{N}$, γ_n is homologous in E to the union of γ_{n+k} with a finite number of loops around annular ends of E, and so, the flux along γ_n is equal to a vertical vector minus the flux along γ_{n+k}. Since the flux along γ_{n+k} goes to zero as $k \to \infty$ (because length$(\gamma_{n+k}) \to 0$), then the flux of E along γ_n is vertical. □

Proposition 3.8 *M_∞ is not a Riemann minimal example.*

Proof. Arguing by contradiction, assume M_∞ is a Riemann minimal example. Since $\gamma \subset M_\infty$ has nonzero vertical flux by Lemma 3.7, then Theorem 6 in [30] implies that the planar ends of M_∞ are not horizontal.

Let Q be the plane passing through the origin in $\mathbb{R}^3$ that is parallel to the planar ends of M_∞ (equivalently, Q is the limit tangent plane at infinity of M_∞). Observe that planes parallel to Q at heights (with respect to Q) different from the heights corresponding to the planar ends of M_∞, intersect M_∞ transversely in simple closed curves (actually in circles). Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be four such circles on M_∞, chosen so that the cycles $\Gamma_1 \cup \Gamma_2$, $\Gamma_2 \cup \Gamma_3$ and $\Gamma_3 \cup \Gamma_4$ each bound a noncompact subdomain $\Omega_{1,2}, \Omega_{2,3}, \Omega_{3,4}$ respectively of M_∞, each containing exactly two planar ends and such that $\Omega_{1,2} \cap \Omega_{2,3} = \Gamma_2$ and $\Omega_{2,3} \cap \Omega_{3,4} = \Gamma_3$, see Figure 2 top. Observe

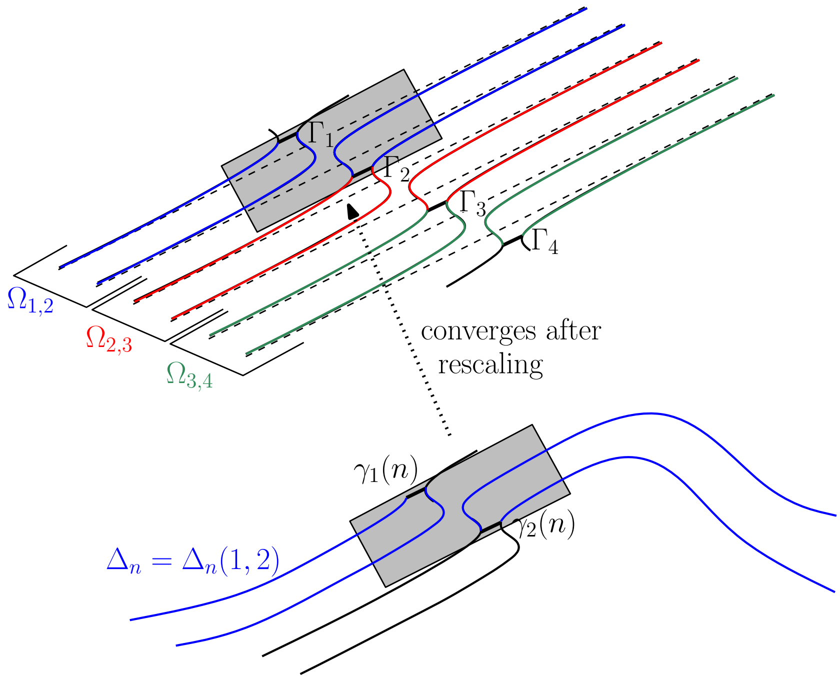

Figure 2: Top: The (tilted) limit minimal example M_∞. Below: compact portions Δ_n of E inside the shaded box which converges after expanding to a related compact portion of M_∞.

that there exists a compact arc $c\colon [1,4] \to M_\infty$ such that $c(i) \in \Gamma_i$, $i = 1,2,3,4$, and c intersects transversely exactly once each of the curves Γ_i.

For n large, let $\gamma_1(n), \gamma_2(n), \gamma_3(n), \gamma_4(n)$ be related simple closed planar curves in M_n, in the sense that $\lambda_n(\gamma_i(n) - p_n)$ converges as $n \to \infty$ to Γ_i, $i = 1,2,3,4$. We can also assume that the $\gamma_i(n)$ are contained in planes parallel to Q. Similarly, c is the limit of related compact arcs $\lambda_n(c_n - p_n)$, where $c_n\colon [1,4] \to E$ satisfies $c_n(i) \in \gamma_i(n)$, $i = 1,2,3,4$, and c_n intersects exactly once each of the curves $\gamma_i(n)$.

To proceed with the proof of Proposition 3.8, we will need two assertions.

Assertion 3.9 *After possibly reindexing, there is a domain $\Delta_n \subset E$ of finite topology such that $\partial\Delta_n = \gamma_1(n) \cup \gamma_2(n)$.*

Proof. When considered to be curves in $\overline{\mathbb{D}}$, the closed curves $\gamma_i(n)$ all separate $\overline{\mathbb{D}}$. Therefore, $\overline{\mathbb{D}} - \cup_{i=1}^4 \gamma_i(n)$ consists of five components. As the compact arc $c_n([1,4])$ intersects exactly three of these five components in open intervals of the form $c_n((j,j+1))$ ($j = 1,2,3$), then at least two of these components are annuli disjoint from $\partial\overline{\mathbb{D}}$; of these two annuli, at least one, called A, is disjoint from the limit end 0 of E. Hence, if we remove from A the annular ends of E, then we obtain a planar domain with finite topology, which we take as Δ_n. Now the assertion follows. □

For $i = 1,2$, let $\gamma_i'(n) \subset E - \Delta_n$ be planar curves (contained in planes parallel to Q) close to and topologically parallel to $\gamma_i(n)$, let $A_i(n) \subset E - \Delta_n$ be the open annulus with compact closure bounded by $\gamma_i(n) \cup \gamma_i'(n)$, and let $D_i(n), D_i'(n) \subset \mathbb{R}^3$ be the corresponding compact planar disks bounded by $\gamma_i(n), \gamma_i'(n)$ respectively. Finally, define $B_i(n)$ to be the compact domain in $\mathbb{R}^3$ with boundary $A_i(n) \cup D_i(n) \cup D_i'(n)$, for $i = 1, 2$, see Figure 3.

Since Δ_n has finite topology and compact boundary, then it is properly embedded in $\mathbb{R}^3$ (note that Δ_n is not compact by the convex hull property). Without loss of generality, we may assume that for $i = 1, 2$, each of the interiors of $D_i(n), D_i'(n)$ intersects Δ_n transversely in a finite (possibly empty) collection of simple closed curves (recall that M_∞ has been obtained as a limit after an intrinsic blow-up procedure, rather than an extrinsic one). Let $\Delta_1(n)$ denote the closure of the component of $\Delta_n \cap \left[\mathbb{R}^3 - (B_1(n) \cup B_2(n))\right]$ that contains $\partial\Delta_n$. Since $X_n = \mathbb{R}^3 - (B_1(n) \cup B_2(n))$ is simply connected and $\Delta_1(n) - \partial\Delta_1(n)$ is properly embedded in X_n, then $\Delta_1(n) - \partial\Delta_1(n)$ separates X_n into two subdomains. Let $F_i(n)$ denote the planar domain in $D_i(n)$ with boundary $\Delta_1(n) \cap D_i(n)$ and let $F_i'(n)$ denote the planar domain in $D_i'(n)$ with boundary $\Delta_1(n) \cap D_i'(n)$, for $i = 1, 2$. Hence,

$$\Delta_2(n) = \Delta_1(n) \cup F_1(n) \cup F_1'(n) \cup F_2(n) \cup F_2'(n)$$

is a properly embedded, piecewise smooth surface that bounds an open region R_n of $\mathbb{R}^3$ such that the boundary of R_n is a good barrier for solving least-area problems in it (the smooth part of ∂R_n is minimal and the interior angles are convex); see Figure 3.

Now choose a simple closed curve $\alpha \subset \Omega_{1,2} \subset M_\infty$ which bounds one of the annular ends of $\Omega_{1,2}$. For n sufficiently large, let α_n denote a related simple closed curve on Δ_n such that the $\lambda_n(\alpha_n - p_n)$ converge smoothly to α as $n \to \infty$ and

$$\alpha_n \subset \text{Int}[\Delta_1(n)] \subset \Delta_2(n) = \partial R_n.$$

We can also assume that the curves α and α_n are chosen so that $\alpha_n \cup D_1'(n) \cup D_2'(n)$ lies on the boundary of its convex hull.

Assertion 3.10 *Consider a (possibly empty) collection $\mathcal{T}$ of closed curves in*

$$\cup_{i=1}^2 (D_i(n) \cup D_i'(n)) \cap \Delta_1(n).$$

Then, for n sufficiently large, $\alpha_n \cup \mathcal{T}$ does not bound a compact minimal surface in the closure $\overline{R_n}$ of R_n.

Proof. Assume by contradiction that such a surface S exists. Since S is compact and minimal, the convex hull property implies that S is contained in the convex hull of its boundary $\partial S = \alpha_n \cup \mathcal{T}$; thus S lies in the closed slab containing the disks $D_1'(n) \cup D_2'(n)$. Note that for n large, there exists a path $\beta_n \subset \Delta_n - \alpha_n$ joining $\gamma_1(n)$ to $\gamma_2(n)$. After adding two arcs $c_{n,1}, c_{n,2}$ to β_n such that $c_{n,i} \subset A_i(n)$, we obtain an embedded arc $\widehat{\beta}_n$ that joins $\gamma_1'(n)$ to $\gamma_2'(n)$, see Figure 3 top. Let $\widehat{\beta}_n'$ be a path parallel and close to $\widehat{\beta}_n$, lying outside $\overline{R_n}$ in $\mathbb{R}^3$, and with the same end points as $\widehat{\beta}_n$. Observe that $\widehat{\beta}_n'$ lies the closed slab containing $D_1'(n) \cup D_2'(n)$. Since $\widehat{\beta}'{}_n$ does not intersect $\overline{R_n}$, then $\widehat{\beta}_n'$ has zero intersection number with S. On the other hand, this

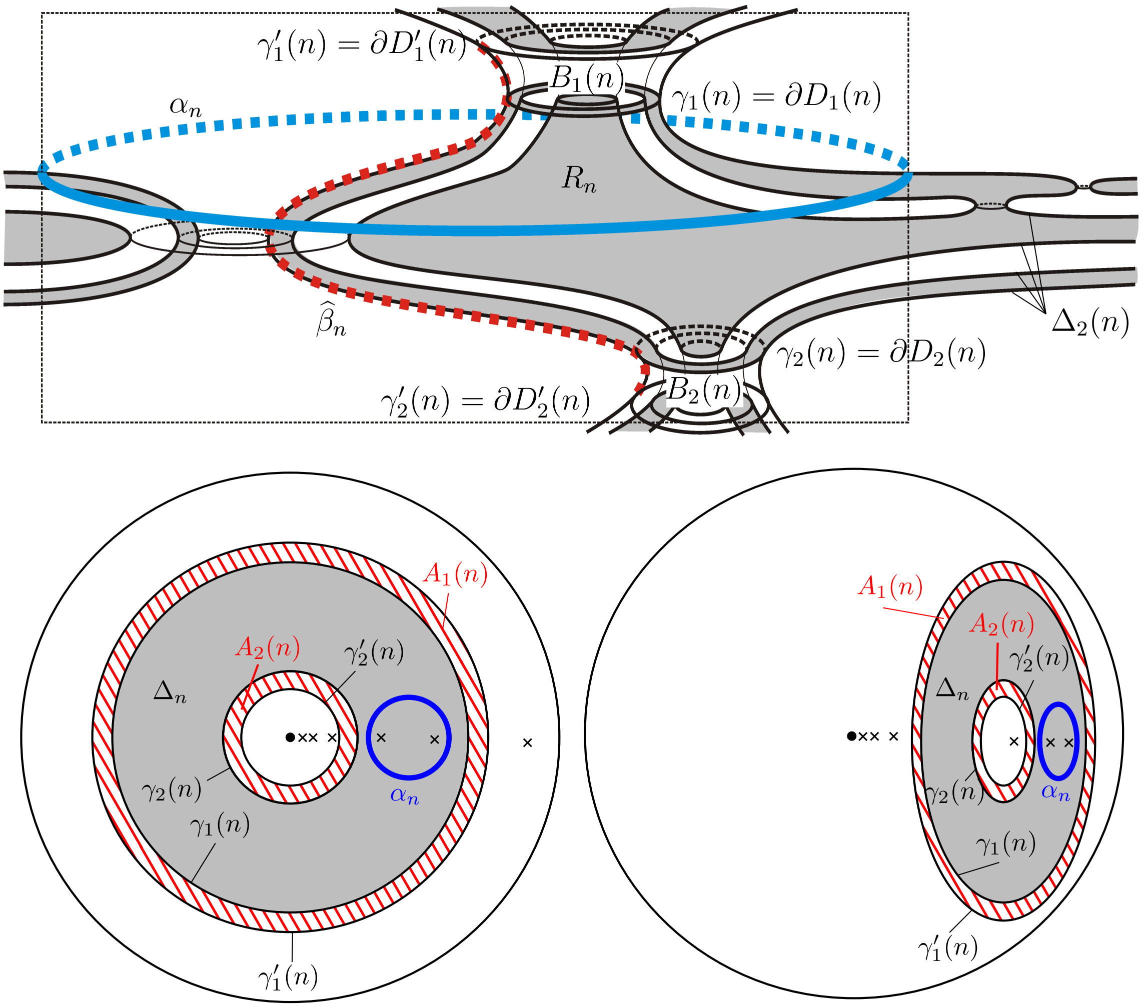

Figure 3: Top: The portion of E inside the dotted box converges as $n \to \infty$ to a compact portion of a Riemann minimal example M_∞; note that the limit tangent plane of M_∞ is represented as horizontal in the figure. Bottom: Two topological configurations for the subdomain with finite topology $\Delta_n \subset E$, depending on whether or not the curves $\gamma_i(n)$ wind around the limit end of E, $i = 1, 2$.

intersection number can be computed (mod 2) as the sum of the linking numbers (mod 2) of $\widehat{\beta}'_n$ with the boundary components of S. Since ∂S can be assumed to lie on the boundary of a convex body disjoint from the end points of $\widehat{\beta}'_n$, S is contained in this convex body, and $\widehat{\beta}'_n$ does not link any of the curves in $\mathcal{T}$ but it has linking number one with α_n, then $\widehat{\beta}'_n$ must have odd intersection number with S, which is a contradiction. This proves Assertion 3.10. $\Box$

Note that α_n separates the planar domain Δ_n into two closed components, where one of these components S'_n is a planar domain with $\partial S'_n = \alpha_n$, see Figure 3. Also α_n separates the planar domain $\Delta_1(n)$ into two closed components, where one of these components S_n satisfies $S_n \subset S'_n$. The boundary of S_n consists of α_n together with a collection $\mathcal{T}_n$ of closed planar curves in $\cup_{i=1}^2 (D_i(n) \cup D'_i(n)) \cap \Delta_1(n)$. Let $S_n(1) \subset S_n(2) \subset \ldots$ be a compact exhaustion of S_n by smooth connected subdomains with $\partial S_n \subset \partial S_n(1)$ and let $\widehat{S_n}(k)$ be an area-minimizing compact surface in $\overline{R_n}$ with $\partial \widehat{S_n}(k) = \partial S_n(k)$ in the relative $\mathbb{Z}_2$-homology class of $S_n(k)$, for all $k \in \mathbb{N}$; $\widehat{S_n}(k)$ is orientable since either $S_n(k) \cup \widehat{S_n}(k)$ is the piecewise-smooth boundary of a connected compact region of $\mathbb{R}^3$, or else $S_n(k) \cup \widehat{S_n}(k)$ is the union of some components of $S_n(k)$ and a piecewise-smooth compact region of $\mathbb{R}^3$. A limit of some subsequence of $\{\widehat{S_n}(k)\}_k$ produces a properly embedded, oriented stable minimal surface $\widehat{S_n}(\infty) \subset \overline{R_n}$ with boundary $\alpha_n \cup \mathcal{T}_n$, see [37] for these standard arguments. By Assertion 3.10, $\alpha_n \cup \mathcal{T}_n$ does not bound a compact minimal surface in $\overline{R_n}$. Therefore, the component $S_n(\infty)$ of $\widehat{S_n}(\infty)$ which contains α_n is noncompact. Let $G_n \colon S_n(\infty) \to \mathbb{S}^2(1)$ be the Gauss map of $S_n(\infty)$.

Given $n \in \mathbb{N}$, consider the dilation (i.e., the composition of a translation and a homothety) $f_n(x) = \lambda_n(x - p_n)$, $x \in \mathbb{R}^3$. Let $R_0 > 0$, $n_0 \in \mathbb{N}$ be sufficiently large so that the following properties hold for all $n \geq n_0$:

1. $f_n(\partial\Delta_1(n)) \subset \overline{\mathbb{B}}(R_0)$ and, without loss of generality, we may assume that the closed curves $f_n(\alpha_n) \subset f_n(\Delta_1(n)) \cap \partial\mathbb{B}(R_0)$ converge to $\alpha = M_\infty \cap \partial\mathbb{B}(R_0)$ as $n \to \infty$.

2. There exists an increasing sequence of numbers $R_n > R_0$ that diverge to infinity and such that for every $n \in \mathbb{N}$, the component Σ_n of $f_n(\Delta_1(n)) \cap [\mathbb{B}(R_n) - \mathbb{B}(R_0)]$ that contains $f_n(\alpha_n)$ is a graph over its projection to the plane Q, and the Σ_n converge smoothly on compact sets of $\mathbb{R}^3$ as $n \to \infty$ to the annular end of M_∞ bounded by α.

Note that $f_n(S_n(\infty)) \cap [\mathbb{B}(R_n) - \overline{\mathbb{B}}(R_0)]$ is either contained in Σ_n, or it is disjoint from Σ_n. By curvature estimates for stable minimal surfaces and after choosing a subsequence, the surfaces

$$\frac{1}{\sqrt{R_n}} \left([\Sigma_n \cup (f_n(S_n(\infty))] \cap \left[\mathbb{B}(R_n) - \overline{\mathbb{B}}(R_0)\right] \right)$$

converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$ with quadratic decay of curvature, which contains the leaf $Q - \{\vec{0}\}$. By the Local Removable Singularity Theorem (Theorem 1.1 in [34]), $\mathcal{L}$ extends to a minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$ with quadratic decay of curvature. As $\overline{\mathcal{L}}$ contains Q, Corollary 6.3 in [34] implies that all leaves of $\overline{\mathcal{L}}$ are flat, and hence, they are planes parallel to Q.

Back to the scale of E, consider the compact subdomain $S'_n(\infty) = S_n(\infty) \cap f_n^{-1}(\overline{\mathbb{B}}(\sqrt{R_n}))$ of $S_n(\infty)$. Then, the normal lines to the boundary of $S_n(\infty) - S'_n(\infty)$ make arbitrarily small

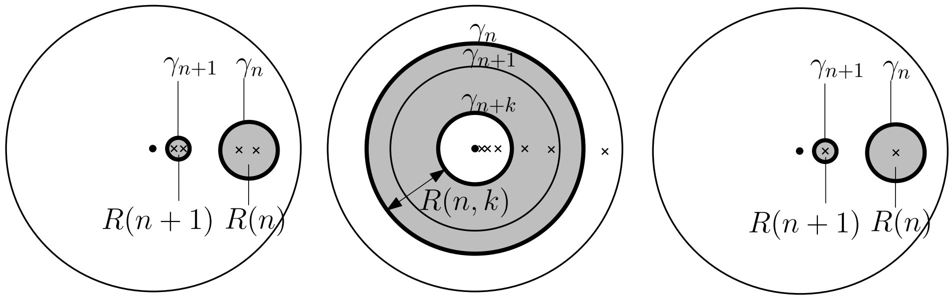

Figure 4: Cases (D1) (left), (D2) (center) and (D3) (right) for Proposition 3.12.

angles with the normal line to the plane Q for n sufficiently large. Pick a component K_n of $S_n(\infty)-S'_n(\infty)$ that intersects the boundary of $S_n(\infty)-S'_n(\infty)$. Since K_n is stable with finite total curvature [15], then, for n sufficiently large, the Gaussian image of K_n must be arbitrarily close to one of the two unit normal vectors $\pm V_Q$ to Q, considered to be points of $\mathbb{S}^2(1)$. As K_n lies in the closure of a complement of E, then property (B1) implies that the planar and catenoid-type ends of K_n have limiting Gaussian images contained in the set $\{(0,0,\pm 1)\} \subset \mathbb{S}^2(1)$. But this last set is a positive distance from $\{\pm V_Q\}$, which is a contradiction. This contradiction completes the proof of Proposition 3.8. □

Lemma 3.11 *Case (C4) cannot occur.*

Proof. Since Case (C4) forces the surfaces $\lambda_n(M_n - p_n)$ to have the appearance for n large of a properly embedded, minimal planar domain with two limit ends, large curvature and fixed size "horizontal" flux[4] (see Traizet and Weber [44], or [25, 27]), the proof of this lemma follows from a straightforward adaptation of the proof of Proposition 3.8. □

Proposition 3.12 *M_∞ is not a catenoid.*

Proof. Reasoning by contradiction, assume M_∞ is a catenoid. By Lemma 3.7, M_∞ has a vertical axis and the simple closed curves $\gamma_n \subset E$ defined just before Lemma 3.7 can be chosen to be horizontal convex curves with vertical flux. For n large, we can choose a compact unstable annulus $C_n \subset E$ with $\gamma_n \subset \text{Int}(C_n)$ so that C_n is arbitrarily close to a rescaling of a fixed, large, compact unstable piece C of a vertical catenoid. We may also assume that ∂C_n consists of two convex curves in horizontal planes. Let $D_n \subset \mathbb{R}^3$ denote the open convex horizontal disk with $\partial D_n = \gamma_n$.

There are three different possible topological configurations for γ_n in E, after choosing a subsequence (see Figure 4).

[4]By "horizontal flux" we mean the nonzero component of the flux vector of $\lambda_n(M_n - p_n)$ that is parallel to the planes of the limit parking garage structure associated to Case (C4).

(D1) Each γ_n is the boundary of a proper subdomain $R(n) \subset E$ with a finite number of annular ends greater than 1.

(D2) When considered to lie in $\overline{\mathbb{D}} - \{0\}$, each γ_n is homologous to ∂E. Hence, for k large, the annular domain $R(n,k) \subset E$ bounded by $\gamma_n \cup \gamma_{n+k}$ has a finite positive number of annular ends.

(D3) Each γ_n bounds a proper annulus $R(n) \subset E$ with $\partial R(n) = \gamma_n$.

The proof of Proposition 3.12 will be a case-by-case elimination of each of these three possibilities (for n sufficiently large).

We first check that Case (D1) does not occur. In this case, γ_n bounds a proper, finite topology domain $R(n) \subset E$ with more than one end and vertical flux.

Assertion 3.13 *The open planar disks $D_1(n), D_2(n) \subset \mathbb{R}^3$ bounded by the curves in ∂C_n, are disjoint from $R(n)$.*

Proof. If not, the proper surface $R(n)$ intersects the compact region $W_n \subset \mathbb{R}^3$ bounded by $C_n \cup D_1(n) \cup D_2(n)$ in a compact component $\Omega(n)$ with boundary in $D_1(n) \cup D_2(n)$. Observe that $\Omega(n)$ intersects both $D_1(n)$ and $D_2(n)$ by the maximum principle for minimal surfaces. Let $\widehat{W}_n$ be the closure of the component of $W_n - \Omega(n)$ that contains C_n in its boundary. Since $\partial D_1(n) \cup \partial D_2(n)$ bounds the annulus C_n in $\widehat{W}_n$ and $\partial D_1(n)$ is homotopically nontrivial in $\widehat{W}_n$, then the Geometric Dehn Lemma for Planar Domains in Theorem 5 in [39] (as adapted in the more general boundary setting of [40]) implies that $\partial D_1(n) \cup \partial D_2(n)$ is the boundary of an embedded, least-area minimal annulus in $\widehat{W}_n$. But $\partial D_1(n) \cup \partial D_2(n)$ also bounds a stable minimal annulus in the outer side of C_n, since C_n is a good barrier that is an unstable minimal annulus. This contradicts Theorem 1.1 in [38] which states that a pair of convex curves in parallel planes can bound at most one compact stable minimal annulus. This contradiction proves Assertion 3.13. □

Once we know that $D_i(n) \cap R(n) = \text{Ø}$ for $i = 1, 2$, then $\gamma_n = R(n) \cap \overline{D_n}$, which implies that $R(n) \cup D_n$ is a properly embedded surface in $\mathbb{R}^3$. Hence, $R(n) \cup D_n$ separates $\mathbb{R}^3$ into two components. In this situation, for n large the standard López-Ros argument can be applied to $R(n)$ (since it is a complete embedded minimal surface with finite total curvature, vertical flux and convex planar boundary which is the boundary of an open convex planar disk disjoint from the surface, see Theorem 2 in [41] for a similar argument), to conclude that $R(n)$ is an annulus. **Thus, Case (D1) does not occur.**

We will use the following property when ruling out Cases (D2) and (D3).

Assertion 3.14 *Suppose after choosing a subsequence, that $\{p_n\}_n$ converges to some point $p_\infty \in \mathbb{R}^3$ and Case (D3) holds for γ_n for all $n \in \mathbb{N}$. Then, the horizontal plane $L(p_\infty) \subset \mathbb{R}^3$ passing through p_∞ satisfies that $E \cap L(p_\infty) = \text{Ø}$, after removing any small compact neighborhood of ∂E.*

Proof. Since we are in Case (D3), then γ_n bounds a proper annulus $R(n) \subset E$. After replacing γ_n by one of the boundary curves of the almost perfectly formed catenoid C_n, we have that the new annulus $R(n) \subset E$ with $\partial R(n)$ the replaced boundary curve, is disjoint from Int(C_n), and thus, we can assume that the total absolute curvature of $R(n)$ is arbitrarily small for n sufficiently large. Since the Gauss map of $R(n)$ is open, almost vertical along $\partial R(n)$ (by Lemma 3.7), the image of this Gauss map has a limiting value $(0, 0, \pm 1)$ at the end of $R(n)$, and the spherical image of the Gauss map of $R(n)$ is arbitrarily small, then we deduce that $R(n)$ is the graph of a function defined on the projection of $R(n)$ to the (x_1, x_2)-plane, and this graph has arbitrarily small gradient.

As we can assume that $\gamma_n \to p_\infty$ as $n \to \infty$, it follows that the graphical annuli $R(n)$ converge smoothly away from p_∞ to the horizontal plane $L(p_\infty)$ passing through p_∞. To finish the proof of the assertion, it only remains to show that Int$(E) \cap L(p_\infty) = \text{Ø}$. Arguing by contradiction, suppose that $L(p_\infty)$ intersects E at an interior point. Since $L(p_\infty)$ is not contained in E, then $L(p_\infty)$ intersects E transversely at some interior point of E. This implies that for n sufficiently large, $R(n)$ intersects $E - R(n)$, which is impossible since E is embedded. Now the assertion is proved. □

We next check that Case (D2) does not occur for n large. Arguing by contradiction, assume that n is large and (D2) holds. Notice that for n fixed and for $k \geq 1$, the proper subdomains $R(n, k)$ bounded by $\gamma_n \cup \gamma_{n+k}$ give rise to an proper exhaustion of the representative of the limit end of E whose boundary is γ_n. Rather than choosing γ_n near the waist circle of the forming unstable compact catenoid piece C_n, we choose γ_n to be a curve contained in a horizontal plane at a height so that for each k, the (noncompact) subdomain $R(n, k) \subset E$ contains two unstable, pairwise disjoint, compact almost-catenoidal pieces, also denoted by C_n, C_{n+k}, near γ_n and γ_{n+k} respectively, so that C_n is an annular neighborhood of γ_n (resp. C_{n+k} is a neighborhood of γ_{n+k}) in the new proper domain $R(n, k)$. We may assume that both boundary curves of C_n and of C_{n+k} are convex horizontal curves for all k. Also, n can be chosen so that for all k sufficiently large, the almost-catenoid C_{n+k} is much smaller than the scale of the almost-catenoid C_n, see Figure 5.

Given $n \in \mathbb{N}$, let $D'_n \subset \mathbb{R}^3$ be the horizontal open disk bounded by $\partial C_n - \gamma_n$ (recall that D_n is the horizontal open disk bounded by γ_n). We next analyze the intersection of $R(n, k)$ with $D_n, D'_n, D_{n+k}, D'_{n+k}$.

(D2-a) We may assume that D_{n+k}, D'_{n+k} are disjoint from C_n (because the scale of C_{n+k} is much smaller than the scale of C_n, and both C_n, C_{n+k} are inside E which is an embedded surface).

(D2-b) An analogous reasoning as in the proof of Assertion 3.13 shows that both D_{n+k}, D'_{n+k} are disjoint from $R(n, k)$. Observe that the boundary curve γ_{n+k} must be contained in the compact region $W_n \subset \mathbb{R}^3$ bounded by $C_n \cup D_n \cup D'_n$ as in Figure 5 (otherwise the arguments in the proof of Assertion 3.13 lead to a contradiction).

The maximum principle and the fact that the scale of C_{n+k} is much smaller than the scale of

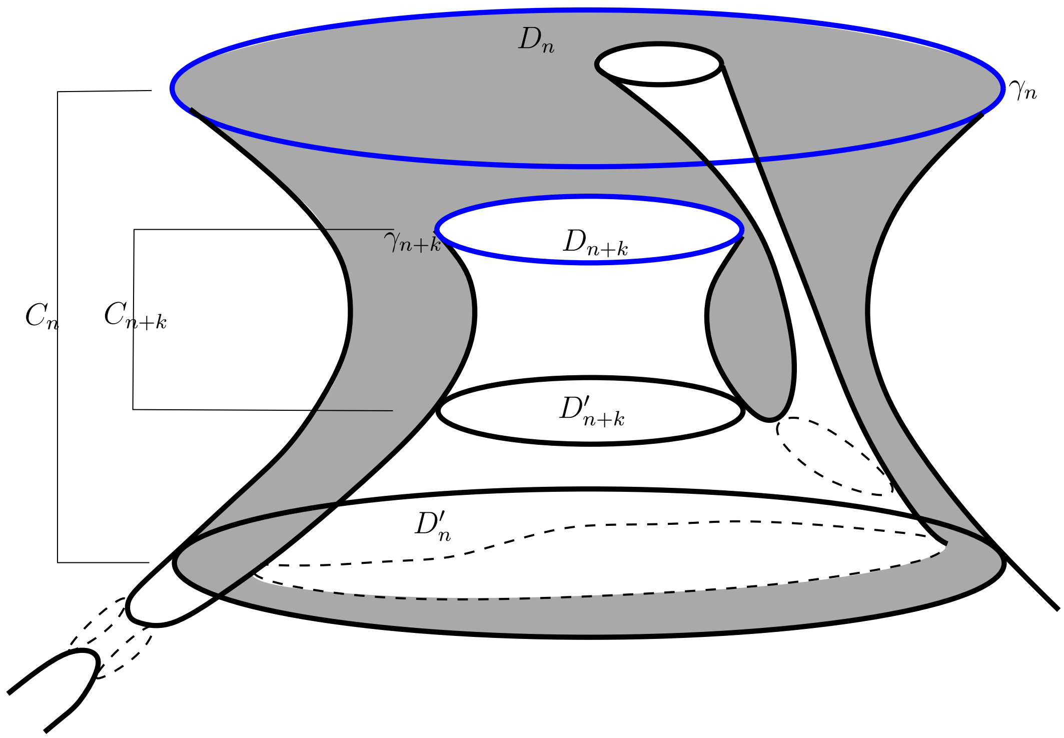

Figure 5: The boundary curve γ_{n+k} of $R(n,k)$ must be contained in the compact region $W_n \subset \mathbb{R}^3$ bounded by $C_n \cup D_n \cup D'_n$.

C_n imply that C_{n+k} is contained in the interior of W_n. Therefore, the topological balls W_n can be assumed to be concentric, in the following sense:

($\star$) After replacing by a subsequence and re-indexing, $W_{n+1} \subset \text{Int}(W_n)$.

Since the scales of the catenoids C_n are converging to zero as $n \to \infty$, Property ($\star$) implies that the W_n converge to a point $c_\infty \in \mathbb{R}^3$, which satisfies $\{c_\infty\} = \bigcap_{n\in\mathbb{N}} W_n \subset \text{Int}(W_1)$. Without loss of generality, we may assume that $\partial E \cap W_1 = \emptyset$.

We next prove that the surface $E(W_1) := E \cap [\text{Int}(W_1) - \{c_\infty\}]$ has locally positive injectivity radius in $\text{Int}(W_1) - \{c_\infty\}$. Otherwise, there is a point $q \in \text{Int}(W_1) - \{c_\infty\}$ and a sequence of points $q_j \in E(W_1)$, $j \in \mathbb{N}$, of almost minimal injectivity radius for $E(W_1)$ in the sense of the Local Picture Theorem on the Scale of Topology, that diverge in $E(W_1)$ but converge to q as $j \to \infty$. After blowing up $E(W_1)$ around the points q_j on the scale of the injectivity radius, we find a limit which is a catenoid (i.e., the other possibilities given by the Local Picture Theorem on the Scale of Topology are not possible by the arguments in Proposition 3.8 and Lemma 3.11). In particular, the catenoid which is forming nearby q_j inside $E(W_1)$ for j large, is of one of the types (D1), (D2) or (D3); in this case we will simply say that Case (D1), (D2) or (D3) holds for q_j. Case (D1) for q_j is not possible by our previous arguments based on Assertion 3.13 and the López-Ros deformation. Also observe that Case (D2) cannot occur at q_j for j large, because the q_j are converging to $q \neq c_\infty$, which implies that q_j does not lie in W_n for n large but fixed, in contradiction with Property ($\star$). This implies that for j large, Case (D3) holds for q_j. Since the q_j converge to q and Case (D3) holds for q_j for every j, then Assertion 3.14 insures that the horizontal plane $L(q)$ passing through q in disjoint from E after removing any small compact neighborhood of ∂E. This is impossible, since $L(q)$ intersects C_1. This contradiction proves that $E(W_1)$ has locally positive injectivity radius in $\text{Int}(W_1) - \{c_\infty\}$.

Since $E(W_1)$ has locally positive injectivity radius in $\text{Int}(W_1) - \{c_\infty\}$, Remark 2 in [36] ensures that the closure of $E(W_1)$ in $\text{Int}(W_1) - \{c_\infty\}$ is a minimal lamination $\mathcal{L}$ of $\text{Int}(W_1) - \{c_\infty\}$ that contains $E(W_1)$ as a subcollection of leaves.

We next prove that $\mathcal{L}$ has no limit leaves in some neighborhood of c_∞. Otherwise, the sublamination $\mathcal{L}'$ of limit leaves of $\mathcal{L}$ is not empty, and $\mathcal{L}'$ consists of stable leaves by Theorem 1 in [32]. By Corollary 7.1 in [34], $\mathcal{L}'$ extends across c_∞ to a lamination of $\text{Int}(W_1)$. Thus, there exists a stable minimal surface $L_1 \subset \text{Int}(W_1)$ passing through c_∞ such that $L_1 - \{c_\infty\}$ is a leaf of $\mathcal{L}'$. Since L_1 is stable and C_n is unstable, then L_1 is disjoint from C_n for all $n \geq 2$. Therefore, for $\varepsilon > 0$ small enough, the ball $\mathbb{B}(c_\infty, \varepsilon)$ of center c_∞ and radius ε intersects L_1 in a component Ω_1 which is a disk that separates $\mathbb{B}(c_\infty, \varepsilon)$. Take $n \in \mathbb{N}$ large enough so that $W_n \subset \mathbb{B}(c_\infty, \varepsilon)$, which exists since $\{c_\infty\} = \bigcap_{n\in\mathbb{N}} W_n$. As Ω_1 contains $c_\infty \in \text{Int}(W_n)$ but $\Omega_1 \cap C_n = \emptyset$ and $W_n \cap \partial\Omega_1 = \emptyset$, then $\Omega_1 \cap (D_n \cup D_n')$ is nonempty. Without loss of generality, we may assume that Ω_1 intersects $D_n \cup D_n'$ transversely and so, there exists a simple closed curve β in $\Omega_1 \cap (D_n \cup D_n')$. This contradicts the maximum principle applied to the subdisk of Ω_1 bounded by β. This contradiction proves that $\mathcal{L}$ has no limit leaves in some neighborhood of c_∞.

Since $\mathcal{L}$ has no limit leaves in some neighborhood of c_∞, we may assume that in some small compact neighborhood N of c_∞ in $\mathbb{R}^3$, $\mathcal{L} \cap N = [E - \{c_\infty\}] \cap N$ and $[E - \{c_\infty\}] \cap N$ is a properly embedded minimal surface in $N - \{c_\infty\}$ of genus zero. But properly embedded minimal surfaces of finite genus in a punctured Riemannian ball extend smoothly across the puncture (see for example, Corollary 2.7 in [28] for this minimal lamination extension result). This is clearly not possible because the Gaussian curvature of E is not bounded in any neighborhood of c_∞. **This contradiction proves that Case (D2) does not occur for n large.**

Finally we check that Case (D3) does not occur, which will finish the proof of Proposition 3.12. By Lemmas 3.7, 3.11 and Proposition 3.8 and from the previously considered cases, we may assume that all local pictures M_n of E on the scale of topology (defined by properties (C1)-...-(C4)) produce, after blowing-up, limiting catenoids with vertical axes, and the horizontal almost waist circles $\gamma_n \subset E$ are in Case (D3) for all $n \in \mathbb{N}$ (after passing to a subsequence). Consider the related sequences $\{M_n\}_n$, $\{\gamma_n\}_n$. We can assume that for all n, M_n contains a compact piece of an almost perfectly formed unstable catenoid C_n containing γ_n, where C_n is a shrunken image of a large compact portion of an almost-catenoid whose boundary consists of simple closed convex horizontal planar curves. Since we are in Case (D3), then γ_n bounds a proper annulus $R(n) \subset E$. After replacing γ_n by one of the boundary curves of the almost perfectly formed catenoid C_n, we have that the new annulus $R(n) \subset E$ bounded by γ_n satisfies the following properties (see the proof of Assertion 3.14):

(E1) $R(n)$ is the graph of a function defined on the projection of $R(n)$ to the (x_1, x_2)-plane, and this graph has arbitrarily small gradient.

(E2) Length$(\gamma_n) \to 0$ as $n \to \infty$.

We will next show that Assertion 3.13 holds in this new setting.

Assertion 3.15 *After extracting a subsequence and possibly replacing E by another end representative, for every $n \in \mathbb{N}$, the open planar disks $D_1(n), D_2(n) \subset \mathbb{R}^3$ bounded by the curves in ∂C_n, are disjoint from E.*

Proof. Let $W_n \subset \mathbb{R}^3$ be the compact region bounded by $C_n \cup D_1(n) \cup D_2(n)$. After choosing a subsequence and removing a small neighborhood of ∂E from E, we may assume that $W_n \cap \partial E = \emptyset$. Observe that $E \cap \text{Int}(W_n)$ is locally simply connected: otherwise, there exists some point $p_\infty \in E \cap \text{Int}(W_n)$ where Case (D3) holds for γ_m for all $m \in \mathbb{N}$ sufficiently large (m larger than n); in this case, Assertion 3.14 ensures that the horizontal plane $L(p_\infty)$ passing through p_∞ is disjoint from E after removing any compact neighborhood of ∂E, which is impossible since $L(p_\infty) \cap C_n \neq \emptyset$. Thus, $E \cap \text{Int}(W_n)$ is locally simply connected.

The arguments in the previous paragraph and Assertion 3.14 ensure that there exists an open set $U \subset \mathbb{R}^3$ such that $C_n \subset U$ and the restriction of the injectivity radius function of E to $E \cap U$ is bounded away from zero. Therefore, the closure of $E \cap U$ relative to the open set U is a minimal lamination of U. As $E \cap \text{Int}(W_n)$ is locally simply connected, the closure of $E \cap \text{Int}(W_n)$ relative to $\text{Int}(W_n)$ is a minimal lamination of $\text{Int}(W_n)$. Consequently, the closure of $E \cap [U \cup \text{Int}(W_n)]$ is a minimal lamination of $U \cup \text{Int}(W_n)$. Since C_n is unstable, then C_n

is not contained in a limit leaf of this lamination, which implies that the distance from C_n to the closure $\overline{E \cap \text{Int}(W_n)}$ of $E \cap \text{Int}(W_n)$ is positive.

As C_n is unstable, we can find a compact unstable subannulus $C'_n \subset \text{Int}(C_n)$ such that $\partial C'_n$ consists of two convex horizontal curves that bound open planar disks $D'_1(n), D'_2(n) \subset \mathbb{R}^3$. Let $W'_n \subset W_n$ be the compact region bounded by $C'_n \cup D'_1(n) \cup D'_2(n)$. It follows from the previous paragraph that the closure of $E \cap \text{Int}(W_n)$ relative to $\text{Int}(W_n)$ is a minimal lamination of $\text{Int}(W_n)$, that is at a positive distance from C_n. In particular, the closure of $E \cap \text{Int}(W_n)$ relative to $\text{Int}(W_n)$ intersected with W'_n is a compact, possibly empty, set X in W'_n.

Suppose the assertion fails for some n, that is, E intersects $D_1(n) \cup D_2(n)$. Then, $E \cap \text{Int}(W_n) \neq \emptyset$ and thus, we can assume $E \cap \text{Int}(W'_n) \neq \emptyset$ by choosing C'_n sufficiently close to C_n. In particular, $X \neq \emptyset$. As X is a compact union of minimal surfaces in W'_n, then the maximum principle applied to x_3 gives that each component of X intersects both disks $D'_1(n), D'_2(n)$. Since X is a good barrier for solving Plateau type problems in W'_n, and $\partial C'_n$ does not bound minimal disks in $W'_n - X$, then there exists a least area annulus $A' \subset W'_n$ with boundary $\partial A = \partial C'_n$. This is impossible, by the same reasoning as in the proof of Assertion 3.13. This completes the proof of Assertion 3.15. □

Arguing by contradiction, assume that Case (D3) occurs for all n. By our earlier considerations, there would exist an infinite collection of pairwise-disjoint almost-catenoids C_n forming on E of the type described in Case (D3) and that satisfy the conclusions of Assertion 3.15. Also, we can assume that the logarithmic growths of the associated graphs $R(n)$ all have the same sign, say negative.

Consider the piecewise smooth graphical planes $P_n = D_2(n) \cup R(n)$, where $D_2(n)$ is the lower open disk given in Assertion 3.15. Note that as $D_2(n) \cap E = \emptyset$, then $E - R(n)$ is contained in the component of $\mathbb{R}^3 - P_n$ above P_n. It follows that the connected surface $E - \cup_n R(n)$ must lie above each of the P_n. By elementary separation properties, this situation is not possible as it would imply that P_1 lies above P_2 and P_2 lies above P_1. **This contradiction completes the proof that Case (D3) does not occur.** So, Proposition 3.12 is proved. □

By Lemma 3.11, Propositions 3.8, 3.12 and the paragraph before Remark 3.6, we conclude that the injectivity radius function I_E is bounded away from zero outside of some (and thus, every) intrinsic ε-neighborhood of ∂E. Therefore, Theorem 3.3 insures that E is properly embedded in $\mathbb{R}^3$, which completes the proof of Theorem 3.5. □

4 The proof of Theorem 1.6.

Let $\mathbf{e}$ be a simple limit end of genus zero of a complete, embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary (possibly $\partial M = \emptyset$). By Theorem 3.5, we can choose a representative E of $\mathbf{e}$ such that E is properly embedded in $\mathbb{R}^3$. The arguments at the end of Section 2 show that after relabeling, properties (A1), (A2) hold for E. As explained in the second paragraph of the proof of Theorem 3.5, each simple end of E has and an annular end representative with finite total curvature and is asymptotic to an end of a plane or catenoid, which after a fixed rotation of M in $\mathbb{R}^3$, is a graph over its projection to the (x_1, x_2)-plane. Since E is properly embedded in $\mathbb{R}^3$, it follows from the Ordering Theorem [16] and Theorem 1.1 in [12] that the limit end of E, after a possible rotation by π around the x_1-axis, is the top end of E.

Lemma 3.6 in [12] implies that a limit end of a properly embedded minimal surface with compact boundary in $\mathbb{R}^3$ cannot have a representative that lies above the end of a catenoid with positive logarithmic growth. Therefore, since the limit end of E is its top end and the middle ends of E are asymptotic to planes and catenoidal ends, none of the catenoidal ends in E have positive logarithmic growth. This proves items 1 and 2 of Theorem 1.6.

Lemma 4.1 *There exists a divergent sequence of points $q_n \in E$ such that $\frac{I_E(q_n)}{|q_n|} \to 0$ as $n \to \infty$, where I_E is the injectivity radius function of E.*

Proof. Otherwise, there exists $c > 0$ such that $I_E(\cdot) \geq c|\cdot|$ in E, away from a compact neighborhood of ∂E. Since ∂E is compact, E is properly embedded and E does not have finite total curvature, then Theorem 1.2 in [34] implies that there exists a divergent sequence of points $y_n \in E$ such that $K_E(y_n)|y_n|^2 \to -\infty$ as $n \to \infty$. Consider the sequence of positive numbers $\sigma_n = \frac{1}{|y_n|} \to 0$. Since

$$\frac{I_{\sigma_n E}(\sigma_n x)}{|\sigma_n x|} = \frac{I_E(x)}{|x|},$$

we conclude that the sequence of surfaces $\{\sigma_n E\}_n$ has locally positive injectivity radius in the open set $\mathbb{R}^3 - \{\vec{0}\}$ in the sense of Definition 3.1, or equivalently, the sequence of compact genus-zero minimal surfaces $\{(\sigma_n E) \cap \overline{\mathbb{B}}(n)\}_n$ is locally simply connected in $\mathbb{R}^3 - \{0\}$, see the first paragraph after Remark 3.2. Since the surfaces $(\sigma_n E) \cap \overline{\mathbb{B}}(n)$ have genus zero with compact boundary and the Gaussian curvature of $(\sigma_n E) \cap \overline{\mathbb{B}}(n)$ at the point $\sigma_n y_n \in \partial \mathbb{B}(1)$ diverges as $n \to \infty$, then item 2 of Theorem 2.2 in [26] implies that after passing to a subsequence, $\{(\sigma_n E) \cap \overline{\mathbb{B}}(n)\}_n$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{\vec{0}\}$, outside of a nonempty singular set of convergence $S(\mathcal{L}) \subset \mathcal{L}$ (this is the closed subset of points $x \in \mathcal{L}$ such that the supremum of the absolute Gaussian curvature of $(\sigma_n E) \cap \mathbb{B}(x, \varepsilon)$ is not bounded in n, for any $\varepsilon > 0$), and the following property holds:

(F) The closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ in $\mathbb{R}^3$ is a foliation of $\mathbb{R}^3$ by planes, and the closure $\overline{S(\mathcal{L})}$ of $S(\mathcal{L})$ consists of one or two complete lines orthogonal to the planes in $\overline{\mathcal{L}}$.

Since the limit end of E is its top end and its annular ends are catenoidal with nonpositive logarithmic growth, it follows that $\mathcal{L}$ is contained in the closed upper halfspace $\{x_3 \geq 0\}$ minus the origin. This contradicts property (F) above, which completes the proof of Lemma 4.1. $\square$

Consider the divergent sequence $\{q_n\}_n \subset E$ given by Lemma 4.1. We next apply a similar rescale-by-topology argument as as we did in the proof of Theorem 3.5 just after property (B1), but instead of using the Local Picture Theorem on the Scale of Topology as we did there, we will use the following extrinsic argument. Given $n \in \mathbb{N}$ large so that the boundary of E lies in

$\mathbb{B}(|q_n|/2)$, consider the continuous, nonnegative function $h_n : \overline{\mathbb{B}}(q_n, |q_n|/2) \cap E \to \mathbb{R}$ given by

$$h_n(x) = \frac{\mathrm{dist}_{\mathbb{R}^3}(x, \partial\mathbb{B}(q_n, |q_n|/2))}{I_E(x)}.$$

h_n vanishes at $\partial\overline{\mathbb{B}}(q_n, |q_n|/2)$. Let p_n be a maximum of h_n. Observe that

$$h_n(p_n) \geq h_n(q_n) = \frac{|q_n|}{2I_E(q_n)} \to \infty,$$

and define

$$r_n = \frac{1}{2}\mathrm{dist}_{\mathbb{R}^3}\left(p_n, \partial\mathbb{B}(q_n, |q_n|/2)\right) = \frac{1}{2}h_n(p_n)I_E(p_n).$$

Then, the sequence of embedded minimal surfaces of genus zero and compact boundary

$$\widetilde{E}_n = \lambda_n\left[E \cap \overline{\mathbb{B}}(p_n, r_n) - p_n\right] \tag{2}$$

is uniformly locally simply connected in $\mathbb{R}^3$, where $\lambda_n = 1/I_E(p_n)$ (in fact, $\widetilde{E}_n$ has boundary in the sphere centered at the origin with radius $\frac{1}{2}h_n(p_n) \to \infty$ and the injectivity radius function of $\widetilde{E}_n$ is at least $1/2$ at points at least at distance $1/2$ from its boundary). By Theorem 2.2 in [26] applied to this sequence of surfaces, we deduce that there exists a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ and a closed subset $S(\mathcal{L}) \subset \mathcal{L}$ such that $\{\widetilde{E}_n\}_n$ converges C^β, for all $\beta \in (0,1)$, on compact subsets of $\mathbb{R}^3 - S(\mathcal{L})$ to $\mathcal{L}$; here $S(\mathcal{L})$ is the singular set of convergence of the $\widetilde{E}_n$ to $\mathcal{L}$. Furthermore, exactly one of the two following cases holds:

(G1) The surfaces $\widetilde{E}_n$ have uniformly bounded Gaussian curvature on compact subsets of $\mathbb{R}^3$. In this case, $S(\mathcal{L}) = \text{Ø}$ and either $\mathcal{L}$ is a collection of planes (this case cannot occur since the injectivity radius function of $\widetilde{E}_n$ at the origin is 1 for each $n \in \mathbb{N}$), or $\mathcal{L}$ consists of a single leaf M_∞, which is properly embedded in $\mathbb{R}^3$ with genus zero. Furthermore, in this last case $\widetilde{E}_n$ converges smoothly on compact sets in $\mathbb{R}^3$ to M_∞ with multiplicity one and exactly one of the following three cases holds for M_∞:

- (a) M_∞ has one end and it is asymptotic to a helicoid (in this case, Theorem 0.1 in [35] insures that M_∞ is a helicoid). Again, this case cannot occur as the injectivity radius function of $\widetilde{E}_n$ at the origin is 1 for each $n \in \mathbb{N}$.
- (b) M_∞ has nonzero finite total curvature. In this case, M_∞ is a catenoid by the main result in [18].
- (c) M_∞ has two limit ends. In this case, M_∞ is a Riemann minimal example by [33].

(G2) $\mathcal{L}$ has the structure of a limiting parking garage in the following sense: $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ by parallel planes and $S(\mathcal{L})$ consists of one or two lines orthogonal to the planes in $\mathcal{L}$ (called columns of the limiting parking garage structure), and as $n \to \infty$, a pair of highly sheeted multivalued graphs forms inside $\widetilde{E}_n$ around each of the lines in $S(\mathcal{L})$. Furthermore, if $S(\mathcal{L})$ consists of two lines l, l', then l intersects $\overline{\mathbb{B}}(1)$, l' is at distance 1 from l and the pairs of multivalued graphs inside the $\widetilde{E}_n$ around different lines are oppositely handed. In fact, $S(\mathcal{L})$ cannot consist of a single line; a proof of this property can be found by a direct adaptation of the second paragraph of the proof of Lemma 3.4 in [26].

4.1 Finding horizontal planes P_n and "concentric" curves $\widehat{\Gamma}(n) \subset E \cap P_n$.

Lemma 4.2 *After possibly replacing E by another end representative, there exists a sequence $\{P_n\}_{n\in\mathbb{N}\cup\{0\}}$ of horizontal planes with $x_3(P_n) < x_3(P_{n+1})$ and $x_3(P_n) \to \infty$, such that each P_n intersects E transversely and $P_n \cap E$ contains a simple closed curve $\widehat{\Gamma}(n)$ with the following properties:*

1. *$\partial E = \widehat{\Gamma}(0) \subset P_0$.*
2. *When viewed in $\overline{\mathbb{D}} - \{0\}$, each $\widehat{\Gamma}(n)$ with $n \in \mathbb{N}$ is topologically parallel to ∂E.*
3. *Given $n \in \mathbb{N}$, let $\Omega_n \subset \overline{\mathbb{D}}(*)$ be the finite topology subdomain whose boundary is $\widehat{\Gamma}(n) \cup \partial E$. Then, $\Omega_n \subset \Omega_{n+1}$ for all n.*
4. *When viewed in $\mathbb{R}^3$, Ω_n lies below the plane P_n.*
5. *E lies locally above P_0 along ∂E.*
6. *If Case (G1) occurs then:*
   - *(a) For each $n \in \mathbb{N} \cup \{0\}$, $\widehat{\Gamma}(n)$ bounds a compact convex disk $D_n \subset P_n$ whose interior is disjoint from E. Furthermore, the D_n all lie in the same side of E.*
   - *(b) The limit tangent plane at infinity of M_∞ is horizontal.*
7. *If Case (G2) occurs, then the planes in the limit parking garage structure are horizontal.*

Proof. We first claim that if P is a horizontal plane such that $\partial E \subset \{x_3 < x_3(P)\}$, then $P \cap E$ contains exactly one compact component that is nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$ ($P \cap E$ might contain infinitely many compact components that bound disks in $\overline{\mathbb{D}} - \{0\}$, each one containing finitely many annular ends of E). To see this, note that $P \cap E$ contains at least one compact component that is nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$ since ∂E lies below P, the limit end of E is its top end and E is connected. If $P \cap E$ contains two compact components both nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$, then we can choose two of such components Γ, Γ' satisfying that $\Gamma \cup \Gamma'$ is the boundary of a compact annulus $A(\Gamma, \Gamma') \subset \overline{\mathbb{D}} - \{0\}$ such that $\text{Int}(A(\Gamma, \Gamma')) \cap x_3^{-1}(x_3(P))$ does not contain components which are nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$ and when viewed in $\mathbb{R}^3$, $A(\Gamma, \Gamma') \cap E$ locally lies above P along $\Gamma \cup \Gamma'$. Observe that $A(\Gamma, \Gamma')$ contains finitely many (annular) ends of E, each of which has nonpositive logarithmic growth. Therefore, $A(\Gamma, \Gamma') - x_3^{-1}(-\infty, x_3(P))$ is a parabolic surface with boundary, and $x_3|_{A(\Gamma,\Gamma') - x_3^{-1}(-\infty, x_3(P))}$ is a bounded nonconstant harmonic function with constant boundary values, which is impossible. This proves our claim.

Assume that Case (G2) occurs for the limit of the $\widetilde{E}_n$. Recall that a limiting parking garage structure in $\mathbb{R}^3$ with two oppositely handed vertical columns closely resembles geometrically and topologically a Riemann minimal example with almost horizontal flux vector and finite positive injectivity radius; we refer the reader to the paper [27] for further explanations.

Let l, l' be the straight lines which are the columns of the limiting parking garage structure, and let $\widetilde{c}_n = \lambda_n(c_n - p_n) \subset \widetilde{E}_n$ be a connection loop for the forming parking garage structure;

this means that $\widetilde{c}_n$ is a closed curve, which approximates arbitrarily well (for n large enough) a path that starts at a point in the first column, travels on one level of the limiting parking garage to the second column, goes "up" one level (remember that we do not know that the columns l, l' are vertical) and then travels back again on this level "over" the previous arc until arriving at the first forming column, and then goes "down" one level until it closes up.

We claim that when viewed in $\overline{\mathbb{D}} - \{0\}$, c_n cannot bound a disk; to see this, note that if c_n bounds a disk in $\overline{\mathbb{D}} - \{0\}$, then c_n bounds a finite topology domain Δ_n in E with vertical flux. Since for n large the flux of $\widetilde{E}_n$ along $\widetilde{c}_n$ is arbitrarily close to a nonzero vector orthogonal to l, we conclude that l, l' are horizontal. This implies that there are points in the interior of Δ_n whose heights are strictly greater than the maximum height of c_n. Since the ends of Δ_n are graphical with nonpositive logarithmic growth, we find a contradiction with the maximum principle for $x_3|_{\Delta_n}$. Therefore, our claim holds.

We next prove that l, l' are vertical lines. Pick a plane $\widetilde{P}$ in the limiting parking garage structure, orthogonal to l, l' and for n large, let P_n be a plane such that $\lambda_n(P_n - p_n)$ converges to $\widetilde{P}$ as $n \to \infty$, such that the height of P_n does not coincide with the height of any planar end of E. Choose two connection loops $c_n, c'_n \subset E$ lying at different sides of P_n. Since both c_n, c'_n are homologically nontrivial in $\overline{\mathbb{D}} - \{0\}$ by the last paragraph, then c_n, c'_n are topologically parallel in $\overline{\mathbb{D}} - \{0\}$ and thus, there exists an annular domain $A(c_n, c'_n) \subset \overline{\mathbb{D}} - \{0\}$ bounded by $c_n \cup c'_n$. Observe that we can choose c_n, c'_n so that $A(c_n, c'_n)$ contains annular ends of E (by the convex hull property). If l, l' were not vertical, then for n large $A(c_n, c'_n) \cap E$ would contain interior points whose heights are strictly greater than the maximum height of $c_n \cup c'_n$, which is a contradiction as in the previous paragraph. Therefore, l, l' are vertical lines, which proves item 7 of the lemma.

We continue assuming that Case (G2) occurs. By Sard's theorem, we can assume that P_n intersects transversely E. Identifying $A(c_n, c'_n) \cap E$ with its image minimal surface in $\mathbb{R}^3$, we deduce that the intersection set $A(c_n, c'_n) \cap x_3^{-1}(x_3(P_n))$ consists of a nonzero finite number of Jordan curves contained in the interior of $A(c_n, c'_n)$. By elementary separation properties, there exists at least one component $\widehat{\Gamma}(n)$ of $A(c_n, c'_n) \cap x_3^{-1}(x_3(P_n))$ which is topologically parallel to c_n in $A(c_n, c'_n)$; in fact, $\widehat{\Gamma}(n)$ is unique by the arguments in the first paragraph of this proof. Thus, $\widehat{\Gamma}(n) \subset E$ satisfies item 2 of the lemma.

Note that the curves $\widehat{\Gamma}(n)$ can be chosen (after passing to a subsequence) so that the finite topology domains $\Omega_n \subset \overline{\mathbb{D}}(*)$ bounded by $\widehat{\Gamma}(n) \cup \partial E$ satisfy $\Omega_n \subset \Omega_{n+1}$ for all n, so item 3 of the lemma holds by construction. Without loss of generality, we may assume that $c_n \subset \Omega_n$. Given $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, the annulus $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+k)) \subset \overline{\mathbb{D}} - \{0\}$ bounded by $\widehat{\Gamma}(n) \cup \widehat{\Gamma}(n+k)$ satisfies that $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+k)) \cap E$ is a finitely punctured annulus and $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+k)) \cap E$ lies below the horizontal plane at height $\max\{x_3(\widehat{\Gamma}(n)), x_3(\widehat{\Gamma}(n+k))\}$ (by the maximum principle applied to $x_3|_{A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+k)) \cap E}$, since the annular ends of E have nonpositive logarithmic growth). As E contains points of arbitrarily large heights because the limit end of E is its top end, we conclude that the heights of the planes P_n are not bounded from above. After passing to a subsequence, we can assume that $x_3(P_n) < x_3(P_{n+1})$ and $x_3(P_n) \to \infty$ as $n \to \infty$. This implies that after replacing E by a representative of the same limit end bounded by the curve $\widehat{\Gamma}(0)$, we can assume that item 1 of the lemma holds provided that Case (G2) occurs.

Observe that the finite topology domain Ω_n equals $A(\widehat{\Gamma}(0), \widehat{\Gamma}(n))$, hence item 4 holds by the last paragraph. By transversality, this implies that $E - \Omega_n$ lies locally above P_n along $\widehat{\Gamma}(n)$. In particular, E lies locally above $P_0 = \{x_3 = x_3(\partial E)\}$ along ∂E and item 5 of the lemma holds provided that Case (G2) occurs. Thus, the proof of Lemma 4.2 is finished if Case (G2) holds.

Next assume that Case (G1) occurs for the limit of the $\widetilde{E}_n$ with M_∞ being a Riemann minimal example. The previous arguments can be adapted to prove that:

- If $\widetilde{c}_n = \lambda_n(c_n - p_n) \subset \widetilde{E}_n$ converges to a circle C in the Riemann minimal example M_∞, then c_n winds once around 0 in $\overline{\mathbb{D}} - \{0\}$ (adapt the arguments in the fourth paragraph of the present proof and use that if the flux of a Riemann minimal example is vertical, then its planar ends are not horizontal).

- The limit tangent plane at infinity for M_∞ is vertical (adapt the arguments in the fifth paragraph of the present proof).

- There exists a sequence of horizontal planes P_n such that $\{\lambda_n(P_n - p_n\}\}_n$ converges to $\{x_3 = x_3(C)\}$, and compact components $\widehat{\Gamma}(n)$ of $E \cap P_n$ that are Jordan curves which, when viewed in $\overline{\mathbb{D}} - \{0\}$, wind once around 0 (adapt the arguments in the sixth paragraph above).

- The finite topology domain $\Omega_n \subset \overline{\mathbb{D}}(*)$ bounded by $\widehat{\Gamma}(n) \cup \partial E$ can be chosen so that $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$, and all of the remaining properties of Lemma 4.2 hold (follow verbatim the arguments in the seventh paragraph of this proof).

Finally suppose that Case (G1) occurs for the limit of the $\widetilde{E}_n$ with M_∞ being a catenoid. Let $\widetilde{P}, P_n \subset \mathbb{R}^3$ be parallel planes so that $\widetilde{P}$ intersects M_∞ in its waist circle $\widetilde{\gamma}$, and for each n $P_n \cap E$ contains a convex Jordan curve γ_n such that $\{\lambda_n(\gamma_n - p_n)\}_n$ converges to $\widetilde{\gamma}$ as $n \to \infty$.

Claim 4.3 *For n sufficiently large, γ_n is nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$.*

Proof. Assume that γ_n bounds a disk Δ in $\overline{\mathbb{D}} - \{0\}$. By the convex hull property, Δ contains a finite positive number of ends of E, all of which are annular with finite total curvature and vertical (possibly zero) flux. As γ_n is convex, a standard application of the López-Ros deformation argument shows that Δ contains exactly one end of E. This annular end of E has negative logarithmic growth for n sufficiently large, as the flux of M_∞ along $\widetilde{\gamma}$ is nonzero. The same reason gives that M_∞ is a vertical catenoid, and thus, $\widetilde{P}, P_n$ are horizontal planes. For n sufficiently large, consider a compact annular neighborhood $A(\gamma_n)$ of γ_n in E with the following properties:

(H1) $A(\gamma_n)$ is bounded by two compact, convex curves in horizontal planes and the lower boundary curve of $A(\gamma_n)$ bounds an annular end $R(n)$ of E of catenoidal type (with negative logarithmic growth).

(H2) $A(\gamma_n)$ is unstable and the sequence $\lambda_n(A(\gamma_n)-p_n)$ converges smoothly with multiplicity one to a large compact piece of M_∞ containing $\widetilde{\gamma}$.

Let D_n (resp. D'_n) be the compact horizontal disk in $\mathbb{R}^3$ whose boundary is the lower (resp. upper) boundary component of $A(\gamma_n)$. Thus, $\partial D_n = \partial R(n)$. By the same arguments as in the proof of Assertion 3.13, the compact region $W_n \subset \mathbb{R}^3$ bounded by $A(\gamma_n) \cup D_n \cup D'_n$, satisfies that $W_n \cap E = A(\gamma_n)$ (note that we can assume that n is sufficiently large so that ∂E does not intersect W_n). As E is connected and proper, we deduce that $E - R(n)$ is disjoint from the piecewise smooth, properly embedded topological plane $R(n) \cup D_n$. As the limit end of E is its top end, we deduce that $E - R(n)$ lies entirely above $R(n) \cup D_n$. In particular, $R(n)$ is the lowest end of E. As this can only happen once for the γ_n, this proves Claim 4.3. □

We continue assuming that Case (G1) occurs with M_∞ being a catenoid. By Claim 4.3, we can assume that γ_n is nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$ for each $n \in \mathbb{N}$. Let $\Omega_n \subset \overline{\mathbb{D}}(*)$ be the subdomain with finite topology and $\partial\Omega_n = \partial E \cap \gamma_n$. Adapting the arguments in the fifth paragraph of this proof (with Ω_n instead of $A(c_n, c'_n)$) we conclude that the catenoid M_∞ is vertical, and thus, $\widetilde{P}, P_n$ are horizontal planes. As for n large we can assume that ∂E lies below P_n, the claim in the first paragraph of the proof of Lemma 4.2 shows that γ_n is the unique compact component of $P_n \cap E$ that is nonzero in $H_1(\overline{\mathbb{D}} - \{0\})$. We now define $\widehat{\Gamma}(n) := \gamma_n$. Once here, items 1-6 in Lemma 4.2 are easy to prove by direct adaptation of the arguments in paragraphs six and seven above. We leave the details to the reader. □

For the remainder of this section, we will assume that E satisfies the properties stated in Lemma 4.2.

Definition 4.4 Since E is proper, Theorem 3.1 in [12] implies that $(x_3|_E)^{-1}([t,\infty))$ is a parabolic manifold with boundary, i.e., it has full harmonic measure on its boundary. In this situation, the Algebraic Flux Lemma for parabolic manifolds (Meeks [21]) ensures that if we define

$$V_E := \int_{\{x_3=t\}} \frac{\partial x_3}{\partial \eta} \in [0,\infty], \tag{3}$$

where η is the inward pointing conormal to $(x_3|_E)^{-1}([t,\infty))$, then V_E is independent of $t \geq \max(x_3|_{\partial E}) = x_3(P_0)$, where P_0 is the horizontal plane defined in Lemma 4.2. We call V_E the *vertical flux component* of E.

In what follows, we will use the notation

$$T = T_H + T_V \tag{4}$$

for the decomposition of a vector $T \in \mathbb{R}^3$ in its horizontal and vertical components.

Corollary 4.5 (Flux Estimates) *Let Ω_n be the subdomains of E defined in Lemma 4.2, and let $\beta_n \in (-\infty, 0]$ be the sum of the (nonpositive) logarithmic growths of the simple ends of Ω_n. Let η denote the outward pointing conormal vector to Ω_n along $\widehat{\Gamma}(n)$ and define the associated flux vector*

$$F(\widehat{\Gamma}(n)) := \int_{\widehat{\Gamma}(n)} \eta.$$

Then, for each $n \in \mathbb{N}$:

1. *$F(\widehat{\Gamma}(n)) = F_E - 2\pi\beta_n e_3$, where F_E is the flux of E given in (1) and $e_3 = (0,0,1)$.*

2. *$F(\widehat{\Gamma}(n))_H = (F_E)_H$. Furthermore, after a normalization of E by replacing it by its image under a rotation around the x_3-axis, $F_E = (h,0,\tau)$ for some $h,\tau \in (0,\infty)$, where $h = |(F_E)_H|$ and $\tau = |(F_E)_V|$.*

3. *Case (G2) does not occur.*

4. *Let $\beta_\infty = \lim_{n\to\infty} \beta_n \in [-\infty, 0]$. If β_∞ is finite, then $V_E\, e_3 = (F_E)_V - 2\pi\beta_\infty e_3$, where V_E is defined in (3).*

5. *Case (G1-c) (i.e., M_∞ is a Riemann minimal example) occurs if and only if β_∞ is finite. In this case, $\lambda_\infty = \lim_{n\to\infty} \lambda_n$ exists and is a positive number, and M_∞ is the scaled Riemann minimal example with horizontal limit tangent plane at infinity that has injectivity radius 1 and flux vector $\lambda_\infty(h, 0, \tau - 2\pi\beta_\infty)$.*

6. *Case (G1-b) (i.e., M_∞ is a catenoid) occurs if and only if $\beta_\infty = -\infty$ (equivalently, $\lim_{n\to\infty} \lambda_n = 0$).*

Proof. Item 1 follows from the divergence theorem applied to the harmonic coordinate functions of E, using the fact that the flux contributions for catenoidal ends of Ω_n are all vertical with negative logarithmic growth.

The first statement in item 2 follows from taking horizontal components in item 1; we next prove the second statement in item 2. First suppose that Case (G1) holds. By item (6-a) of Lemma 4.2, the boundary curves of Ω_n are convex planar curves that bound horizontal disks D_n whose interiors are disjoint from E, and the D_n all lie on the same side of E. Since for n large Ω_n is not an annulus, then if $(F_E)_H = 0$, then the López-Ros deformation argument applied to Ω_n would lead to a contradiction. The fact that $(F_E)_V \neq 0$ follows directly from the maximum principle for x_3, since E is not contained in a horizontal plane. This proves the second statement in item 2 when Case (G1) holds. Thus item 2 will hold once we prove item 3.

In the case that (G2) holds, the connection loop $c_n \subset \Omega_n$ (defined in the proof of Lemma 4.2) is homologous in $\overline{\mathbb{D}} - \{0\}$ to ∂E. Since the planes in the limiting parking garage are horizontal by item 7 of Lemma 4.2, then the ratio $\frac{|F(c_n)_V|}{|F(c_n)_H|}$ of the length of the vertical component $F(c_n)_V$ over the length of the horizontal component $F(c_n)_H$ of the flux vector $F(c_n)$ converges to zero as $n \to \infty$. As $F(c_n)_H = (F_E)_H$ by the divergence theorem, then $|F(c_n)_V|$ tends to 0 as $n \to \infty$. This is impossible, since the arguments in obtaining item 1 show that $|F(c_n)_V| \geq |(F_E)_V| > 0$. This contradiction gives that items 2 and 3 hold.

We next prove item 4. Taking vertical components in the equality of item 1 and using that the limit β_∞ of the β_n is assumed to be finite, we have that $\lim_n F(\widehat{\Gamma}(n))_V$ exists and equals $(F_E)_V - 2\pi\beta_\infty e_3$. Hence it remains to show that

$$V_E\, e_3 = \lim_n F(\widehat{\Gamma}(n))_V. \tag{5}$$

To see this, we will describe $E \cap P_n$ for $n \in \mathbb{N}$ given. Observe that if C is a noncompact component of $E \cap P_n$, then C is a noncompact embedded arc and each of the two ends of C diverges to the same annular end of E, which is therefore a planar end asymptotic to P_n. Hence, after moving slightly the height of P_n, we can assume that every component of $E \cap P_n$ is compact. Next consider a (compact) component C of $E \cap P_n$. By item 4 of Lemma 4.2, $C \subset E - \text{Int}(\Omega_n)$. By the claim in the first paragraph of the proof of Lemma 4.2, either $C = \widehat{\Gamma}(n)$ or C bounds a disk in $\overline{\mathbb{D}} - \{0\}$.

Assume that E contains a planar annular end. By embeddedness of E, all annular ends above E (with the ordering given by the Ordering Theorem) must be also planar. After replacing E by another end representative of its limit end, we can assume that all the ends of E are planar. In this case, $E \cap P_n = \widehat{\Gamma}(n)$ (otherwise, there exists a component C of $E \cap P_n$ such that C bounds a disk Δ_C in $\overline{\mathbb{D}} - \{0\}$ by the last paragraph, and we contradict the maximum principle applied to $x_3|_{\Delta_C}$ as all the (finitely many) ends of E in Δ_C are planar). Since $E \cap P_n = \widehat{\Gamma}(n)$ for each $n \in \mathbb{N}$, then (3) computed for $t = x_3(P_n)$ gives that $V_E\, e_3 = F(\widehat{\Gamma}(n))_V$ for each $n \in \mathbb{N}$, from where (5) follows directly.

By the arguments in the last paragraph, we can assume that all the annular ends of E have negative logarithmic growth. Fix $n \in \mathbb{N}$. As $E - \Omega_n$ lies locally above P_n along $\widehat{\Gamma}(n)$ (by item 4 of Lemma 4.2) and every annular end of E in $E - \Omega_n$ is represented by a punctured disk that lies entirely below P_n, then we conclude that $E \cap P_n$ consists of $\widehat{\Gamma}(n)$ together with infinitely many compact components $C_i(n)$, $i \in \mathbb{N}$, each of which bounds a disk $\Delta_{C_i(n)}$ in $\overline{\mathbb{D}} - \{0\}$ that contains a finite positive number of catenoidal type ends of E. Therefore, (3) computed for $t = x_3(P_n)$ gives that

$$V_E\, e_3 = F(\widehat{\Gamma}(n))_V + \sum_{i \in \mathbb{N}} F(C_i(n)), \tag{6}$$

where $F(C_i(n))$ is the (vertical) flux vector of E along $\partial \Delta_{C_i(n)}$ computed with the unit conormal vector that points outwards from $\Delta_{C_i(n)}$ along its boundary. Observe that given $n, i \in \mathbb{N}$, the divergence theorem gives that $F(C_i(n))$ equals e_3 times a finite sum of logarithmic growths of annular ends of E. As the sequence of domains $\{\Omega_n\}_n$ forms an increasing exhaustion of E, then given $n, i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ sufficiently large so that all annular ends in $\Delta_{C_i(n)}$ lie in the closure of Ω_{n+k} in $\overline{\mathbb{D}} - \{0\}$. This observation and (6) imply that (5) holds, and the proof of item 4 is complete.

We next show items 5 and 6. Using item 2 we have

$$\lambda_n F(\widehat{\Gamma}(n))_H = \lambda_n (F_E)_H = \lambda_n (h, 0, 0)$$

for each $n \in \mathbb{N}$. If Case (G1-c) occurs, then the left-hand-side of the last equation tends to the nonzero horizontal component of the flux $F(M_\infty)$ of M_∞, which implies that the λ_n converge to a finite positive number λ_∞. Taking vertical components in item 1 we have

$$\lambda_n F(\widehat{\Gamma}(n))_V = \lambda_n [(F_E)_V - 2\pi \beta_n e_3]. \tag{7}$$

Taking $n \to \infty$ in (7), we obtain $\langle F(M_\infty), e_3 \rangle = \lambda_\infty (\tau - 2\pi \beta_\infty)$, hence β_∞ is finite (and negative, as β_n is nonpositive for every n) and $F(M_\infty) = \lambda_\infty (h, 0, \tau - 2\pi \beta_\infty)$.

If Case (G1-b) happens, then the horizontal component of the flux of M_∞ is zero and thus, a similar reasoning shows that $\frac{|F(\widehat{\Gamma}(n))_H|}{|F(\widehat{\Gamma}(n))_V|} \to 0$, hence the β_n diverge to $-\infty$ and the λ_n converge to zero. This finishes the proof of the corollary. $\square$

In the remainder of this section, we will assume that E satisfies the normalization stated in Corollary 4.5, and we will also use the notation in that corollary.

Lemma 4.6 *Suppose $\{p'_n\}_n \subset E$ is a divergent sequence such that $\{I_E(p'_n)\}_n$ is bounded. Then, β_∞ is finite and a subsequence of the surfaces $E - p'_n$ converges smoothly on compact sets of $\mathbb{R}^3$ with multiplicity one to the Riemann minimal example with horizontal ends and flux vector $(h, 0, \tau - 2\pi\beta_\infty)$.*

Proof. First assume that the Gaussian curvature of the sequence $\{E - p'_n\}_n$ is locally bounded in $\mathbb{R}^3$. Then, a subsequence of $\{E - p'_n\}_n$ converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with a nonsimply connected leaf L passing through the origin and genus zero. By Theorem 7 in [30], L is proper. By the Halfspace Theorem, L is the unique leaf of $\mathcal{L}$. Since L has genus zero, then L is either a catenoid or a Riemann minimal example; in particular, the convergence of $E - p'_n$ to L is of multiplicity one. Similar arguments as those in the proof of Lemma 4.2 imply that:

(I1) If L is a catenoid (resp. a Riemann minimal example), then the waist curve of L (resp. each circle contained in L) is the limit as $n \to \infty$ of closed curves $\alpha_n \subset \overline{\mathbb{D}}(*)$ that wind once around the limit end $\vec{0}$ of E in the parameter domain $\overline{\mathbb{D}}(*)$ of $E - p'_n$.

(I2) The annular ends of L are horizontal.

Note that the horizontal component $F(\alpha_n)_H$ of the flux of $E - p'_n$ along α_n is independent of n and nonzero (by item 2 of Corollary 4.5), which is clearly impossible if L is a vertical catenoid. This proves that L is a Riemann minimal example. The fact that the flux of L is $(h, 0, \tau - 2\pi\beta_\infty)$ comes from taking limits in the fluxes of the curves α_n and using the arguments in the proof of item 5 of Corollary 4.5. This completes the proof of the lemma provided that the Gaussian curvature of $\{E - p'_n\}_n$ is locally bounded in $\mathbb{R}^3$.

Now assume that the Gaussian curvature of $\{E - p'_n\}_n$ fails to be locally bounded in $\mathbb{R}^3$. As I_E is bounded away from zero outside every ε-neighborhood of ∂E by Theorem 3.5, then Theorem 2.2 in [26] ensures that after choosing a subsequence, $\{E - p'_n\}_n$ converges to a minimal parking garage structure with two columns (the one-column case of a limiting parking garage structure is ruled out because $I_E(p'_n)$ is bounded from above by assumption). Similar arguments as in the proof of item 3 of Corollary 4.5 lead to a contradiction, which completes the proof of the lemma. $\square$

4.2 Analysis of the Case (G1) when M_∞ is a Riemann minimal example.

In this section, we will prove that Theorem 1.6 holds provided that Case (G1) occurs and that the limit surface M_∞ of the surfaces $\widetilde{E}_n$ given by (2) is a Riemann minimal example.

Let $\mathcal{R}$ be the Riemann example with horizontal ends and flux vector $(h, 0, \tau - 2\pi\beta_\infty)$, which is just a fixed rescaling of M_∞ by item 5 of Corollary 4.5. Recall that $\mathcal{R}$ is invariant under the π-rotation about infinitely many horizontal straight lines L_k, $k \in \mathbb{Z}$, that intersect the surface orthogonally (the lines L_k are parallel to the lines in which $\mathcal{R}$ intersects horizontal planes at the heights of its planar ends, and the heights of L_k are ordered by $k \in \mathbb{Z}$). Given $k \in \mathbb{Z}$, let $A_1(k), A_2(k) \in \mathcal{R}$ the two points in which L_k intersects $\mathcal{R}$. For $i = 1, 2$, let $J_i^{\mathcal{R}} \subset \mathcal{R}$ be the integral curve of the gradient of the third coordinate function x_3 of $\mathcal{R}$, passing through the points $A_i(k)$ for all $k \in \mathbb{N}$. $J_2^{\mathcal{R}}$ is the reflected image of $J_1^{\mathcal{R}}$ with respect to the vertical plane of symmetry of $\mathcal{R}$, and both $J_1^{\mathcal{R}}, J_2^{\mathcal{R}}$ are properly embedded, periodic Jordan arcs, see Figure 6 for a picture in a fundamental region of $\mathcal{R}$. If we parameterize $\mathcal{R}$ conformally by a cylinder $\mathbb{S}^1 \times \mathbb{R}$ so that x_3 corresponds to the projection over the second factor, then $J_i^{\mathcal{R}}$ corresponds to $\{\theta_0\} \times \mathbb{R}$ for certain $\theta_0 \in \mathbb{S}^1$. Observe that the image of $J_1^{\mathcal{R}}$ through the Gauss map $N_{\mathcal{R}}$ of $\mathcal{R}$ is a simple closed curve $C \subset \mathbb{S}^2$, and if we parameterize $J_1^{\mathcal{R}}$ by x_3, then the derivative of the argument of $g_{\mathcal{R}}(J_1^{\mathcal{R}}(x_3))$ is a positive (or negative) periodic function, where $g_{\mathcal{R}}$ denotes the stereographic projection of $N_{\mathcal{R}}$ from the north pole of $\mathbb{S}^2$; this last property follows from the well-known fact that the Gauss map of a minimal surface and its conjugate minimal surface are the same, and the conjugate surface of a Riemann example is another Riemann example, where the integral curves of the gradient of x_3 correspond to circles in the conjugate surface.

Proposition 4.7 *Let $C \subset \mathbb{S}^2$ be closed curve defined in the last paragraph. Then, after replacing E by another end representative of the limit end, the inverse image of C through the Gauss map of E consists of two disjoint, proper Jordan arcs J_1, J_2 satisfying the following properties:*

1. *I_E restricted to $J_1 \cup J_2$ is bounded from above, and*

$$\limsup_{x \in J_1 \cup J_2} I_E(x) = \limsup_{x \in J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}} I_{\mathcal{R}}(x) < \infty.$$

2. *When viewed in $\mathbb{R}^3$, the unit tangent vector along $J_1 \cup J_2$ makes an angle with the horizontal planes which is bounded away from zero.*

Proof. After a small perturbation of C (by the Sard-Smale theorem), we can assume that the Gauss map N of E is transverse to C. In particular, $N^{-1}(C)$ consists of a proper (possibly disconnected) 1-dimensional submanifold of E; after replacing E by a subend, we may assume that the geometry of E near ∂E is close to the one of $\mathcal{R}$ and thus, $N^{-1}(C)$ intersects ∂E transversely at two points. Observe that the tangent plane to E along $N^{-1}(C)$ is bounded away from the horizontal. The proposition will be a consequence of three assertions.

Assertion 4.8 *If $I_E(p'_n) \to \infty$ for a sequence of points $p'_n \in N^{-1}(C)$, then for n large $N^{-1}(C)$ makes an angle with the horizontal at p'_n which is bounded away from zero.*

Proof. Let $J(n)$ denote the component of $N^{-1}(C)$ that contains p'_n. Arguing by contradiction, we may assume that the tangent line to $J(n)$ at p'_n makes an angle less than $\frac{1}{n}$ with the horizontal and $I_E(p'_n)$ is much greater than n. Since $I_E(p'_n) \to \infty$, Proposition 1.1 in [9] ensures that after replacing by a subsequence, we may assume that p'_n lies in a compact minimal disk $D_n \subset \overline{\mathbb{B}}(p'_n, n) \cap E$ with $\partial D_n \subset \partial\overline{\mathbb{B}}(p'_n, n)$. Since the vertical component of the flux $V_E = \tau - 2\pi\beta_\infty$ is finite and the tangent plane to E along $N^{-1}(C)$ is bounded away from the horizontal, then for n large, there exist constants an $C, R_0 \in (0, n/3)$ depending on V_E and there is a point $q'_n \in \mathbb{B}(p'_n, R_0) \cap E$ where the absolute Gaussian curvature of E is at least C. It then follows from Theorem 0.1 in [8] that a subsequence of the disks

$$\Sigma_n = \frac{1}{\sqrt{n}}(D_n - p'_n) \subset \overline{\mathbb{B}}(\sqrt{n})$$

converges on compact subsets of $\mathbb{R}^3$ to a minimal parking garage structure $\mathcal{F}$ with a single column being a straight line L passing through the origin and orthogonal to the planes in $\mathcal{F}$, see also Meeks [20, 22]. Since V_E is finite, L is the x_3-axis.

It follows from [19] that for n large, $\Sigma_n(1) = \Sigma_n \cap \overline{\mathbb{B}}(\vec{0}, 1)$ contains a compact arc α_n along which $\Sigma_n(1)$ has vertical tangent spaces, and α_n is converging C^1 to the line segment $\{(0,0,t) \mid t \in [-1,1]\}$; this last result can be found in Meeks [19]. Some further refinements by Meeks, Pérez and Ros (Corollary 4.27 in [27]) give that at every point $x \in \alpha_n$ and for n large enough, $\Sigma_n(1)$ nearby x can be closely approximated in the C^2-norm by compact domains of a shrunk vertical helicoid H_x whose axis contains x, and in the complement of a small tube T_n around α_n containing the forming vertical (scaled) helicoids, the remaining surface $\Sigma_n(1) - T_n$ consists of almost horizontal multigraphs with an arbitrarily large number of sheets for n large. In particular, the corresponding curve $\frac{1}{\sqrt{n}}(J(n) - p'_n)$ lies in T_n. Since the inverse image J_H of $C \subset \mathbb{S}^2$ by the Gauss map on a vertical helicoid makes an angle with the horizontal that is bounded away from zero (because if we use conformal coordinates $\rho e^{i\theta}$ defined on $\mathbb{C} - \{0\}$ for the helicoid so that the polar angle θ corresponds to height in $\mathbb{R}^3$, then J_H can be parameterized by θ, and the angle of $J_H(\theta)$ with the horizontal has constant positive derivative with respect to θ). Thus, for n sufficiently large the same property holds for $J(n)$ near p'_n. This contradiction completes the proof of Assertion 4.8. □

Let $c_0, c_1 \subset \mathcal{R}$ be the horizontal circles passing through the points $A_1(0), A_1(1)$ defined just before the statement of Proposition 4.7. Let *Cyl* be a solid, compact vertical cylinder whose axis passes through the branch point of $N_{\mathcal{R}}$ at height $\frac{1}{2}(x_3(c_0) + x_3(c_1))$, of radius $r > 0$ large enough so that $[J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}] \cap x_3^{-1}((x_3(c_0), x_3(c_1)))$ is contained in the interior of *Cyl*, and such that the top and bottom disks in ∂*Cyl* contain the circles c_1, c_0, respectively, see Figure 6. By construction, the side of *Cyl* intersects $\mathcal{R}$ in an almost horizontal closed curve γ (for r large enough) that winds once around the axis of *Cyl*, and $N_{\mathcal{R}}(\gamma)$ is a closed spherical curve arbitrarily close to $(0,0,1) \in \mathbb{S}^2$ that winds twice around $(0,0,1)$ (we can assume that $(0,0,1)$ is the extended value of $N_{\mathcal{R}}$ at the planar end of $\mathcal{R}$ between the heights of c_0 and c_1). Let $\mathcal{R}_0$ be the compact subdomain $\mathcal{R} \cap$ *Cyl* of $\mathcal{R}$.

Given $\varepsilon > 0$ small, let $\mathcal{R}_0(\varepsilon) \subset E$ be a compact subdomain which is ε-close to $\mathcal{R}_0$ in the Hausdorff distance in $\mathbb{R}^3$, which exists since the smooth limit of translations of E is $\mathcal{R}$. We may assume that $\partial\mathcal{R}_0(\varepsilon)$ consists of three components $c_0(\varepsilon), c_1(\varepsilon), \gamma(\varepsilon)$, so that $c_0(\varepsilon), c_1(\varepsilon)$

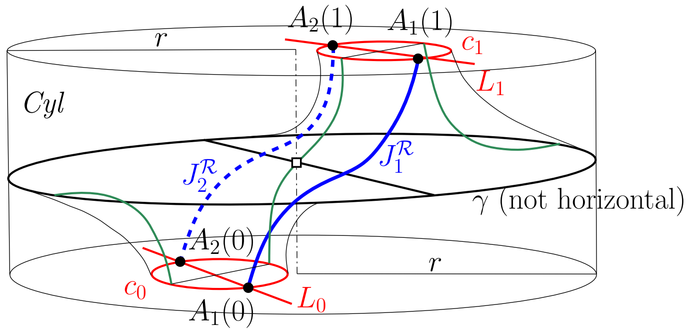

Figure 6: The small square in the center of the figure represents the branch point of the Gauss map $N_{\mathcal{R}}$ of $\mathcal{R}$ whose height is the average of the heights of the circles c_0, c_1. The green curves represent the intersection of $\mathcal{R} \cap$ *Cyl* with the symmetry plane of $\mathcal{R}$.

are horizontal convex curves and $\gamma(\varepsilon)$ is a Jordan curve whose image by N is at positive spherical distance from C. We may take ε sufficiently small so that $N^{-1}(C) \cap \mathcal{R}_0(\varepsilon)$ consists of two disjoint arcs, each one joining $c_0(\varepsilon)$ to $c_1(\varepsilon)$. The complement of $\mathcal{R}_0(\varepsilon)$ in $\overline{\mathbb{D}} - \{0\}$ consists of three components, namely two annular components of which one contains ∂E and another one contains 0, and a disk component Δ. By the convex hull property, Δ contains a finite positive number of annular ends of E. Observe that we can choose an infinite sequence of pairwise disjoint domains of the type $\mathcal{R}_0(\varepsilon)$ in E, so that the sequence collapses to the origin when viewed in $\overline{\mathbb{D}} - \{0\}$.

Assertion 4.9 *After replacing E by a limit subend, every such a domain Δ contains exactly one annular end of E.*

Proof. Arguing by contradiction, assume that we have a sequence Δ_n of such domains so that Δ_n contains at least two annular ends of E. As the limiting normal vector of E at its annular ends is vertical and $N^{-1}(C) \cap \partial\Delta_n = \emptyset$, then a simple continuity argument gives that $N^{-1}(C) \cap \Delta_n$ contains a finite positive number of components, each of which is a Jordan curve. Choose one of these Jordan curves $\beta_n \subset N^{-1}(C) \cap \Delta_n$. If for each $n \in \mathbb{N}$ there exists a point $p'_n \in \beta_n$ so that the sequence $\{I_E(p'_n)\}_n$ is bounded, then Lemma 4.6 implies that after extracting a subsequence, the $E - p'_n$ converge smoothly with multiplicity one to $\mathcal{R}$. Note that the sequence $\{(\Delta_n \cap E) - p'_n\}_n$ also converges to $\mathcal{R}$ (because the intrinsic distance from $\gamma_n(\varepsilon)$ to β_n goes to infinity as $n \to \infty$). This is impossible, as every closed curve in $\Delta_n \cap E$ has vertical flux but $\mathcal{R}$ does not have this property. Therefore, the sequence of numbers $\{\min I_E(x) \mid x \in \beta_n\}_n$ goes to ∞. In this situation, Assertion 4.8 gives a contradiction by taking for each $n \in \mathbb{N}$ a point $x_n \in \beta_n$ of maximum height in $\mathbb{R}^3$ (which exists since β_n is a Jordan curve). This finishes the proof of Assertion 4.9. □

By Assertion 4.9, after replacing E by a limit subend, we assume that every Δ-domain as defined in the description just before the statement of Assertion 4.9, contains one end of E. By the last sentence before Assertion 4.9, these Δ-domains occur in a sequence collapsing to the limit end of E. By the Gauss-Bonnet formula, the total Gaussian curvature of the annulus $\Delta \cap E$ is arbitrarily small by choosing $\mathcal{R}_0(\varepsilon)$ appropriately. Therefore, $\Delta \cap E$ is a graph over its projection into the (x_1, x_2)-plane, of a function with small length of its gradient (the maximum of the length of the gradient of the graphing function occurs at $\partial\Delta$). By gluing $(\Delta \cap E) \cup \mathcal{R}_0(\varepsilon)$ with the two planar disks bounded by $c_0(\varepsilon) \cup c_1(\varepsilon)$, we obtain a piecewise smooth topological plane Π, which is properly embedded in $\mathbb{R}^3$. Take a maximal collection $\{\Pi_n\}_n$ of such topological planes, so that $\Pi_n \cap \Pi_m = \emptyset$ if $n \neq m$. $\mathbb{R}^3 - \bigcup_{n\in\mathbb{N}} \Pi_n$ consists of a countable union of open components, each of which is a topological slab S_n.

Given such a slab S_n, observe that the closure of $N^{-1}(C) \cap S_n$ is a compact 1-manifold with four boundary points. Therefore, $N^{-1}(C) \cap S_n$ consists of a finite number of Jordan curves plus two arcs. We first check that for n sufficiently large, $N^{-1}(C) \cap S_n$ does not contain Jordan curve components. Otherwise, there exists a sequence of points $p'_n \in N^{-1}(C) \cap S_n$ where the tangent line to $N^{-1}(C) \cap S_n$ is horizontal. By Assertion 4.8, $I_E(p'_n)$ must be bounded. Thus, Lemma 4.6 gives that after extracting a subsequence, the $E - p'_n$ converge smoothly to $\mathcal{R}$, which is impossible since on $\mathcal{R}$ the corresponding set $N_{\mathcal{R}}^{-1}(C) = J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}$ satisfies that the angle with the horizontal planes is bounded away from zero. Therefore, $N^{-1}(C) \cap S_n$ does not contain Jordan curve components that for n sufficiently large, and the same argument proves that the two compact arcs in the closure of $N^{-1}(C) \cap S_n$ make an angle with the horizontal planes which is bounded away from zero; in particular, each of these arcs joins two boundary components of S_n. After replacing E by a subend, we can assume that $N^{-1}(C)$ consists of two proper arcs J_1, J_2 satisfying item 2 of the proposition. In particular, each J_i can be parameterized by the x_3-coordinate, $i = 1, 2$.

Assertion 4.10 *Given $\delta > 0$, there exists $x_{3,0} = x_{3,0}(\delta) \in \mathbb{R}$ such that if $x_3 \geq x_{3,0}$, then for $i = 1, 2$ it holds*

$$I_E(J_i(x_3)) \leq \delta + \limsup_{x \in J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}} I_{\mathcal{R}}(x).$$

Proof. Arguing by contradiction, suppose that the assertion fails. Then, there exists $\delta > 0$ and a sequence of heights $t_n \to \infty$ so that $I_E(J_i(t_n)) > \delta + c$, where $c = \limsup_{x \in J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}} I_{\mathcal{R}}(x)$. After passing to a subsequence, we can assume that given $n \in \mathbb{N}$, there exists $t'_n \in (t_n, t_{n+1})$ so that the related point $J_i(t'_n)$ lies in a region of the form $\mathcal{R}_0(\varepsilon)$ where E is ε-close to a compact portion of $\mathcal{R}$. Then, after taking ε much smaller than δ, we can assume that $I_E(J_i(t'_n)) \leq \frac{\delta}{2} + c$. By continuity of $I_E \circ J_i$, there exists $t''_n \in (t_n, t'_n]$ such that $I_E(J_i(t''_n)) = \frac{\delta}{2} + c$ for each $n \in \mathbb{N}$. Applying Lemma 4.6 to $p_n := J_i(t''_n)$ we deduce that the $E - J_i(t''_n)$ converge (after extracting a subsequence) smoothly to $\mathcal{R}$, which is impossible since the value of the injectivity radius of $E - J_i(t''_n)$ at the origin is $\frac{\delta}{2} + c$ and the injectivity radius is a continuous function with respect to smooth limits (see e.g. Ehrlich [13] and Sakai [42]). Now the assertion is proved. □

Finally, item 1 of Proposition 4.7 follows directly from Assertion 4.10. Item 2 of Proposition 4.7 follows from Assertion 4.10, Lemma 4.6 and the fact that the unit tangent vector along the curves $J_1^{\mathcal{R}} \cup J_2^{\mathcal{R}}$ makes an angle with the horizontal planes which is bounded away from zero. This completes the proof of the proposition. □

A direct consequence of Lemma 4.6 and Assertion 4.10 is that for every divergent sequence of points $p'_n \in J_1 \cup J_2$, the surfaces $E - p'_n$ converge smoothly to $\mathcal{R}$ after passing to a subsequence. This property together with Assertion 4.9 imply that after replacing E by a subend, E consists of an infinite number of noncompact pieces M_n, each of which is has the topology of a pair of paints with a point removed (this puncture is one annular end of E), and the two compact boundary components of $c_{0,n}, c_{1,n}$ of M_n can be taken arbitrarily close to translated copies of the horizontal circles $c_0, c_1 \subset \mathcal{R}$ defined in the paragraph just after the proof of Assertion 4.8. Furthermore, $c_{1,n} = c_{0,n+1}$ and $M_n \cap M_{n+1} = c_{1,n}$ for all $n \in \mathbb{N}$.

We next explain why Theorem 1.6 holds in the Case (G1) when M_∞ is a Riemann minimal example. The main properness statement of Theorem 1.6 was proven in Section 3. Items 1, 2 of Theorem 1.6 were proven in the second paragraph of this section 4. Item 3 of Theorem 1.6 follows from Lemma 4.2 and Corollary 4.5. In particular, items 1, 2, 3 of Theorem 1.6 also hold in the Case (G1) when M_∞ is a vertical catenoid. Assume from now on that Case (G1) occurs and M_∞ is a Riemann minimal example. Item 4 of Theorem 1.6 is a consequence of the last paragraph. The same description of E as a union of domains M_n implies that the Gaussian curvature of E is bounded, which is item 5 of Theorem 1.6. The next proposition completes the proof of Theorem 1.6 in the Case (G1) when M_∞ is a Riemann minimal example.

Proposition 4.11 *If Case (G1) occurs and M_∞ is a Riemann minimal example, then item 6 of Theorem 1.6 holds.*

Proof. Suppose that the proposition fails. As each of the annular ends of E has finite total curvature, E is conformally diffeomorphic to $\widehat{D} = \overline{\mathbb{D}} - \{x \in \mathbb{D} \mid |x| \leq a\}$ for some $a \in (0,1)$, with a countable discrete set of points $\{e_n\}_{n\in\mathbb{N}}$ removed and where $|e_n| \searrow a$ as $n \to \infty$.

By the above decomposition of E as a countable union of regions M_n, there exists $\delta > 0$ and a sequence $f_n \colon \mathbb{S}^1 \times [0,\delta] \to E$ (here $\mathbb{S}^1$ is the unit circle) of conformal embeddings with $f_n(\mathbb{S}^1 \times [0,\delta])$ being arbitrarily close to a region $\mathcal{R}_0(\varepsilon)$ of 'Riemann type' to which one attaches an annular end of E (that might have negative logarithmic growth, arbitrarily close to zero). Observe that the f_n have pairwise disjoint images in $\mathbb{R}^3$ for different values of n.

Consider on $\widehat{D}$ the usual flat metric g_0. Next we will show that the g_0-area of $f_n(\mathbb{S}^1 \times [0,\delta])$ is at least $2\pi a^2 \delta$, which gives the desired contradiction since the g_0-area of $\widehat{D}$ is finite and we have an infinite number of such pairwise disjoint embeddings f_n in $\widehat{D}$.

To compute the g_0-area of $f_n(\mathbb{S}^1 \times [0,\delta])$, we will apply the coarea formula to the smooth function $h_n \colon f_n(\mathbb{S}^1 \times [0,\delta]) \to \mathbb{R}$ that satisfies

$$(h_n \circ f_n)(\theta, t) = t, \quad \text{for all } (\theta, t) \in \mathbb{S}^1 \times [0,\delta]. \tag{8}$$

Thus,

$$\text{Area}(f_n(\mathbb{S}^1 \times [0,\delta]), g_0) = \int_0^\delta \left(\int_{h_n^{-1}(t)} \frac{ds_t}{|\nabla_0 h_n|} \right) dt, \tag{9}$$

where ds_t, $|\nabla_0 h_n|$ denote respectively the length element of the simple closed curve $h_n^{-1}(t) = f_n(\mathbb{S}^1 \times \{t\})$ and the gradient of h_n, both computed with respect to g_0. Since f_n is a conformal diffeomorphism onto its image endowed with g_0, we deduce that

$$v := \frac{1}{\left|\frac{\partial f_n}{\partial t}(\theta,t)\right|}\frac{\partial f_n}{\partial t}(\theta,t) \tag{10}$$

is a unit normal vector to the curve $f_n(\mathbb{S}^1\times\{t\})$ at the point $f_n(\theta,t)$. Hence, (8) and (10) give

$$|\nabla_0 h_n|(f_n(\theta,t)) = (dh_n)_{f_n(\theta,t)}(v) = \frac{1}{\left|\frac{\partial f_n}{\partial t}(\theta,t)\right|},$$

which implies that the right-hand-side of (9) equals

$$\int_0^\delta\left(\int_{f_n(\mathbb{S}^1\times\{t\})}\left|\frac{\partial f_n}{\partial t}\right| ds_t\right)dt = \int_0^\delta\left(\int_{\mathbb{S}^1\times\{t\}}\left|\frac{\partial f_n}{\partial t}\right|\left|\frac{\partial f_n}{\partial \theta}\right| d\theta\right)dt. \tag{11}$$

Using again the conformality of f_n in the right-hand-side of (11) and the Cauchy-Schwarz inequality, we obtain

$$\text{Area}(f_n(\mathbb{S}^1\times[0,\delta]),g_0) = \int_0^\delta\left(\int_{\mathbb{S}^1\times\{t\}}\left|\frac{\partial f_n}{\partial \theta}\right|^2 d\theta\right)dt \geq \frac{1}{2\pi}\int_0^\delta\left(\int_{\mathbb{S}^1\times\{t\}}\left|\frac{\partial f_n}{\partial \theta}\right| d\theta\right)^2 dt$$

$$= \frac{1}{2\pi}\int_0^\delta[\text{length}(f_n(\mathbb{S}^1\times\{t\}))]^2 dt \overset{(\star)}{\geq} \frac{1}{2\pi}\int_0^\delta(2\pi a)^2 dt = 2\pi a^2\delta,$$

where in $(\star)$ we have used that $f_n(\mathbb{S}^1\times\{0\})$ is a loop in $\widehat{D}$ that is parallel to $\partial\widehat{D}$. This completes the proof of the proposition. □

4.3 Analysis of the Case (G1) when M_∞ is a catenoid.

We will devote this section to prove Theorem 1.6 provided that Case (G1) occurs and that the limit surface M_∞ of the surfaces $\widetilde{E}_n$ given by (2) is a vertical catenoid.

Without loss of generality, we will assume that the waist circle of M_∞ is the unit circle in the (x_1,x_2)-plane. Recall the following properties demonstrated above for each $n\in\mathbb{N}$:

(J1) There exists a horizontal plane P_n so that $P_n\cap E$ contains a convex Jordan curve $\widehat{\Gamma}(n)$ and the $\lambda_n(\widehat{\Gamma}(n)-p_n)$ converge as $n\to\infty$ to the waist circle $\widetilde{\gamma}$ of M_∞. Moreover, the heights of P_n diverge increasingly to ∞.

(J2) $\widehat{\Gamma}(0)=\partial E$, $\widehat{\Gamma}(n)$ is topologically parallel to ∂E in $\overline{\mathbb{D}}-\{0\}$ (items 1 and 2 of Lemma 4.2) and $\widehat{\Gamma}(n)$ is the unique compact component of $P_n\cap E$ that is topologically parallel to ∂E in $\overline{\mathbb{D}}-\{0\}$ (claim in the first paragraph of the proof of Lemma 4.2).

(J3) When viewed in $\overline{\mathbb{D}}(*)$, $\widehat{\Gamma}(n)$ bounds a noncompact domain $E(\widehat{\Gamma}(n))$ which is an end representative of the limit end of E; we take $E(\widehat{\Gamma}(n))$ as the closure of the component of $E-\widehat{\Gamma}(n)$ such that $E(\widehat{\Gamma}(n))\cap\partial E=\emptyset$.

(J4) When viewed in $\mathbb{R}^3$, $\widehat{\Gamma}(n)$ bounds a compact convex disk $D_n \subset P_n$ whose interior is disjoint from E, and the D_n all lie in the same side of E (item (6a) of Lemma 4.2). We will denote by W the closure of the component of $\mathbb{R}^3 - (E \cup D_0)$ that contains D_n for $n \geq 1$.

(J5) $F(\widehat{\Gamma}(n)) = F_E - 2\pi\beta_n e_3$ and $F(\widehat{\Gamma}(n))_H = (F_E)_H$ is not zero (items 1 and 2 of Corollary 4.5).

(J6) $V_E = \infty$, $\beta_\infty = \sum_n \beta_n = -\infty$ and $\lambda_n \to 0$ as $n \to \infty$ (items 4 and 6 of Corollary 4.5). In particular, the annular ends of E all have strictly negative logarithmic growths (because if one of these ends were asymptotic to a plane, then all annular ends of E above this last one would be planar as well, and $F(\widehat{\Gamma}(n))_V$ would then be independent of n, which contradicts that $\beta_\infty = -\infty$ after taking vertical components in the first formula of (J5)).

(J7) $\{I_E(p'_n)\}_n$ is unbounded for every divergent sequence $\{p'_n\}_n \subset E$ (Lemma 4.6).

Proposition 4.12 *Given $\varepsilon > 0$ small, there exist compact annular subdomains $\Delta_n = \Delta_n(\varepsilon) \subset E$ bounded by horizontal convex curves, such that for n sufficiently large:*

1. *There exist numbers $\lambda'_n > 0$ converging to zero and points $p'_n \in \mathbb{R}^3$ such that the Hausdorff distance between $\lambda'_n(\Delta_n - p'_n)$ and $M_\infty(\varepsilon) = \{x \in M_\infty : |x_3| \leq 1/\varepsilon\}$ is less than ε, and $\lambda'_n(\Delta_n - p'_n)$ can be written as a normal graph over its projection to M_∞ with C^2-norm less than ε and the boundary curves of $\lambda'_n(\Delta_n - p'_n)$ are contained in the planes $\{x_3 = \pm 1/\varepsilon\}$.*

2. *For all $n \in \mathbb{N}$, the boundary curves of Δ_n are topologically parallel to ∂E in $\overline{\mathbb{D}} - \{0\}$. Therefore, we may assume that the Δ_n are ordered so that for each n, Δ_n is contained in the component of $\overline{\mathbb{D}}(*) - \Delta_{n+1}$ that contains ∂E.*

3. *The closed horizontal slab in $\mathbb{R}^3$ that contains Δ_n is strictly below the one that contains Δ_{n+k} for all $n, k \in \mathbb{N}$, $k \neq 0$.*

4. *Except for a finite number of components, each component Ω of $E - \bigcup_{n\in\mathbb{N}} \Delta_n$ is topologically a plane with two disks removed, and Ω is the graph of a function u defined over the projection of Ω to the (x_1, x_2)-plane, with $|\nabla u| < 1$.*

5. *The Gaussian curvature K_E of E is asymptotically zero.*

Proof. Items 1 and 2 follow from the facts that the sequence $\{\widetilde{E}_n\}_n$ defined by (2) converges smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity 1 to the vertical catenoid M_∞, that $\lambda_n \to 0$, and that the convex horizontal curves $\widehat{\Gamma}(n)$ defined in (J1) are topologically parallel to ∂E in $\overline{\mathbb{D}} - \{0\}$.

We next prove item 3. Let $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1))$ be the subdomain of E bounded by $\widehat{\Gamma}(n) \cup \widehat{\Gamma}(n+1)$. Since the ends of $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1))$ have negative logarithmic growths and $\{x_3(P_n)\}_n$ is increasing, the maximum principle applied to the function $x_3|_{A(\widehat{\Gamma}(n),\widehat{\Gamma}(n+1))}$ implies that

$A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1))$ is contained in the halfspace $\{x_3 \leq x_3(\widehat{\Gamma}(n+1))\}$. As $A(\widehat{\Gamma}(n-1), \widehat{\Gamma}(n))$ is contained in the halfspace $\{x_3 \leq x_3(\widehat{\Gamma}(n))\}$ and $A(\widehat{\Gamma}(n-1), \widehat{\Gamma}(n)) \cap A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1)) = \widehat{\Gamma}(n)$, then we conclude that $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1))$ locally lies above P_n along $\widehat{\Gamma}(n)$. Observe that $A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1))$ contains the portion Δ_n^+ of Δ_n that lies above P_n, and it also contains the portion Δ_{n+1}^- of Δ_{n+1} that lies below P_{n+1}. By the maximum principle applied to $x_3|_{A(\widehat{\Gamma}(n), \widehat{\Gamma}(n+1)) - [\Delta_n^+ \cup \Delta_{n+1}^-]}$, we deduce that the lower boundary curve of Δ_{n+1} lies strictly above the upper boundary curve of Δ_n, which implies that item 3 holds.

Next we prove item 4 of the proposition. For $\varepsilon > 0$ small and fixed, choose a maximal collection of pairwise disjoint domains $\{\Delta_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ which satisfy items 1, 2 and 3. After replacing E by a subend representative of the limit end, we may assume that ∂E is thre bottom boundary component of Δ_0.

We will first show that if a component Ω of $E - \bigcup_{n\in\mathbb{N}\cup\{0\}} \Delta_n$ is topologically a plane with two disks removed, then Ω is the graph of a function u defined over the projection of Ω to the (x_1, x_2)-plane, with $|\nabla u| < 1$. To see this, note that if $\varepsilon > 0$ is sufficiently small, the total geodesic curvature of E along each of the two components of $\partial\Omega$ is arbitrarily close to -2π. As we are assuming that Ω has exactly one end and this end has finite total curvature, then the Gauss-Bonnet formula gives that Ω has arbitrarily small total Gaussian curvature by taking ε sufficiently small. Therefore, the Gaussian image of Ω lies in a small neighborhood of one of the poles, say the north pole, of the unit sphere. Hence, the projection of Ω to the (x_1, x_2)-plane is a proper submersion which is injective on each of the two boundary components of Ω. In this setting, a straightforward covering space type argument implies that Ω is a graph of a smooth function u defined over the projection of Ω to the (x_1, x_2)-plane. The fact that $|\nabla u| < 1$ follows from the fact that the Gaussian image of Ω lies in a small neighborhood of the north pole. **Therefore, to prove item 4 we must show that except for a finite number of components of $E - \bigcup_{n\in\mathbb{N}\cup\{0\}} \Delta_n$, all these components have the topology of a plane minus two disks.** Observe that item 2 of this proposition implies that every component of $E - \bigcup_{n\in\mathbb{N}\cup\{0\}} \Delta_n$ is a planar domain with two boundary components and a finite number of annular ends with negative logarithmic growth.

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be the collection of components of $E - \bigcup_{n\in\mathbb{N}\cup\{0\}} \Delta_n$. Enumerate these components so that Ω_n is the component of $E - \bigcup_{n\in\mathbb{N}\cup\{0\}} \Delta_n$ with boundary components

$$\alpha_n = \Omega_n \cap \Delta_{n-1}, \quad \beta_n = \Omega_n \cap \Delta_n. \tag{12}$$

Fix for each $n \in \mathbb{N}$ a dilation $f_n \colon \mathbb{R}^3 \to \mathbb{R}^3$ so that the Hausdorff distance between $f_n(\Delta_n)$ and $M_\infty(\varepsilon)$ is minimized, and so, by the definition of Δ_n, this Hausdorff distance is less than ε.

Lemma 4.13 *After passing to a subsequence, the surfaces $f_n(\Omega_n)$ converge with multiplicity one to the representative $M_\infty \cap \{x_3 \leq -1/\varepsilon\}$ of the bottom end of M_∞. In fact, the surfaces $f_n(E)$ converge smoothly on compact subsets of $\mathbb{R}^3$ to M_∞.*

Proof. We first show that if the sequence of curves $\{f_n(\alpha_n)\}_n$ diverges to infinity in $\mathbb{R}^3$, then the lemma holds. Following the notation in property (J3) above, we denote by $E(\alpha_n)$ the closure of the component of $E-\alpha_n$ such that $E(\alpha_n)\cap\partial E=\emptyset$. By construction, after passing to a subsequence, we may assume that the curves $f_n(\beta_n)$ converge in the C^2-norm to a convex horizontal curve $\widehat{\beta}$. Take a divergent sequence $\{R_n\}_n$ of positive numbers so that for each n, the boundary $f_n(\alpha_n)$ of $f_n(E(\alpha_n))$ lies outside of the closed ball $\overline{\mathbb{B}}(R_n)$ centered at the origin. Then, item 3 of Theorem 2.2 in [26] applied to the sequence of compact minimal surfaces $\{f_n(E(\alpha_n))\cap\overline{\mathbb{B}}(R_n)\}_n$ implies that after extracting a subsequence, the $f_n(E(\alpha_n))\cap\overline{\mathbb{B}}(R_n)$ converge smoothly on compact subsets of $\mathbb{R}^3$ with multiplicity one to a connected, properly embedded, nonflat minimal surface $\widehat{M_\infty}$ of genus zero, that is either a catenoid, a helicoid or a Riemann minimal example. Clearly, $\widehat{M_\infty}$ contains the curve $\widehat{\beta}$. $\widehat{M_\infty}$ cannot be a helicoid since $\widehat{M_\infty}$ has nonzero vertical flux along $\widehat{\beta}$; the same argument shows that either $\widehat{M_\infty}$ is a vertical catenoid or a Riemann minimal example with vertical flux. Our earlier arguments imply that $\widehat{M_\infty}$ must be a vertical catenoid. Since the limit set of the sequence $\{f_n(E(\alpha_n))\cap\overline{\mathbb{B}}(R_n)\}_n$ equals the limit set of $\{f_n(E)\}_i$ (this follows from the properties that $f_n(E-E(\alpha_n))$ lies in the halfspace $\{x_3\leq x_3(f_n(\alpha_n))\}_n$ and $x_3(f_n(\alpha_n))\to-\infty$ as $n\to\infty$), then we conclude the Lemma in the special case that the $f_n(\alpha_n)$ diverge to infinity in $\mathbb{R}^3$.

We next divide the proof into two parts: in the first one, we will prove the lemma assuming that, after choosing a subsequence, Ω_n contains just one annular end for every n. In the second part we will suppose that, after choosing a subsequence, Ω_n contains more than one annular end for every n.

Assume that Ω_n contains just one annular end for every n. We will demonstrate that the curves $f_n(\alpha_n)$ diverge to infinity in $\mathbb{R}^3$. Arguing by contradiction, assume that after choosing a subsequence, $f_n(\alpha_n)$ lies in a compact set of $\mathbb{R}^3$ independently of $n\in\mathbb{N}$. Recall that the logarithmic growths of the annular ends of E are bounded (between the negative logarithmic growth of the lowest end of E and zero), and that the dilation f_n has homothetic factor going to zero as $n\to\infty$. Therefore, the logarithmic growths of the unique annular end of $f_n(\Omega_n)$ is arbitrarily small in absolute value for n sufficiently large. Since the flux of $f_n(\Omega_n)$ along $f_n(\beta_n)$ is converging to the nonzero flux of M_∞ along $\widehat{\beta}$ and the flux of $f_n(\Omega_n)$ along its annular end is arbitrarily small, then the divergence theorem implies that the flux of $f_n(\Omega_n)$ along $f_n(\alpha_n)$ converges to the negative of the flux of M_∞ along $\widehat{\beta}$, that is nonzero. This property and the fact that the convex planar curve $f_n(\alpha_n)$ lies in a compact set independent of n, imply that the $f_n(\alpha_n)$ converge (after passing to a subsequence) to a convex, horizontal planar curve $\widehat{\alpha}$ as $n\to\infty$. Also recall that $f_n(\Omega_n)$ is a graph over its projection to the (x_1,x_2)-plane, by the fourth paragraph in the proof of Proposition 4.12. Therefore, curvature estimates for stable minimal surfaces imply that the $f_n(\Omega_n)$ converge to a minimal graph over the complement in the plane $\{z=0\}$ of the two disks bounded by $\Pi(\widehat{\alpha}),\Pi(\widehat{\beta})$, where $\Pi(x,y,z)=(x,y,0)$. As this minimal graph has vertical flux and two convex boundary curves, a standard application of the López-Ros deformation argument leads to contradiction. This contradiction shows that the $f_n(\alpha_n)$ diverge to infinity in $\mathbb{R}^3$. By the discussion in the first paragraph of this proof, we now conclude that Lemma 4.13 holds if Ω_n has one end for every n (after choosing a subsequence).

Next assume that Ω_n has always at least two ends. Again by the discussion in the first paragraph of the proof of Lemma 4.13, it remains to show that the curves $f_n(\alpha_n)$ diverge to infinity in $\mathbb{R}^3$. Assume this last property fails to hold. Since the diameter of the sets $f_n(\alpha_n)$ are uniformly bounded (because the diameter of $f_n(\Delta_{n-1})$ is bounded as the homothetic factor of the dilation f_n is going to zero and the diameter of $f_{n-1}(\Delta_{n-1})$ is comparable to the one of $M_\infty(\varepsilon)$), we may assume from this point on that the curves $f_n(\alpha_n)$ all lie in a fixed bounded subset of $\mathbb{R}^3$. This bounded set must lie below the plane $\{x_3 = -1\}$ if ε is chosen sufficient small (by the already proven item 3 of Proposition 4.12). We will find the desired contradiction by analyzing each of the following two mutually exclusive situations (after passing to a subsequence):

(K1) The diameters of the curves $f_n(\alpha_n)$ are not bounded away from zero.

(K2) The diameters of the curves $f_n(\alpha_n)$ are bounded away from zero.

Suppose that Case (K1) holds. Then, after choosing a subsequence, we may assume that the curves $f_n(\alpha_n)$ converge to a point $p \in \mathbb{R}^3$ that lies below the plane $\{x_3 = -1\}$. Consider the sequence of compact, embedded, minimal planar domains $\{f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)\}_n$. We claim that the $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ form a locally simply connected sequence of minimal planar domains in $\mathbb{R}^3 - \{p\}$. Otherwise, our previous arguments show that we can produce, after blowing-up by topology, a new limit of dilations of the $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ which is a vertical catenoid. This means that after extracting a subsequence and for n sufficiently large, $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ contains a compact subdomain C_n which is arbitrarily close to a homothetically shrunk copy of a large compact region of a vertical catenoid, where the homothetic factor can be taken arbitrarily small. Note that C_n cannot lie in $f_n(\Omega_n)$ because this contradicts the maximality of the family $\{\Delta_m\}_m$. Since $f_n(\Delta_n)$ is ε-close to $M_\infty(\varepsilon)$, we deduce that C_n must lie in $f_n(E(\alpha_{n+1}))$. To see that this is impossible, first observe that for n sufficiently large, the generator of the homology group $H_1(C_n)$ of C_n is topologically parallel to $f_n(\beta_n)$ modulo annular ends of $f_n(E)$ (adapt the arguments as in the proof of Claim 4.3). As the vertical component of the flux vector of $f_n(\Delta_n)$ along $f_n(\beta_n)$ is larger than some positive number in absolute value (namely, one half of the vertical flux of M_∞) and the annular ends of $f_n(E)$ all have negative logarithmic growths, we deduce from the divergence theorem that the vertical component of the flux vector of $f_n(C_n)$ is positive and bounded away from zero (see the last paragraph of the proof of Corollary 4.5), which contradicts that the length of a generator of $H_1(C_n)$ tends to zero as $n \to \infty$. Therefore, the sequence $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ is locally simply connected sequence in $\mathbb{R}^3 - \{p\}$. In fact, this argument shows that for all $n \in \mathbb{N}$ and given a regular neighborhood $U_n(\delta)$ of the boundary of $f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)$ in $f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)$ with radius $\delta > 0$, the restriction of the injectivity radius function of $f_n(E)$ to $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - U_n(\delta)$ is uniformly bounded away from zero (independently of n).

In this setting, item 3 of Theorem 2.2 in [26] ensures that after passing to a subsequence, the surfaces $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - \{p\}$ whose closure $\overline{\mathcal{L}}$ in $\mathbb{R}^3$ consists of a single leaf which is a properly embedded minimal surface L_1 of genus zero that is either a helicoid, a catenoid or a Riemann minimal example. Furthermore, the convergence of the $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ to L_1 is smooth on compact sets of $\mathbb{R}^3 - \{p\}$. Our previous arguments show that L_1 is the vertical catenoid M_∞. As p is a point in L_1, then M_∞ must pass through p. We next analyze the intersection of $f_n(E(\alpha_n))$ with a ball $\mathbb{B}(p, \delta)$

of small radius $\delta > 0$ so that $\mathbb{B}(p,\delta) \cap M_\infty$ is a graphical disk with boundary Γ_∞. For n large, we can assume that $f_n(\alpha_n) \subset \mathbb{B}(p,\delta)$. Since $f_n(E(\alpha_n))$ is a properly embedded surface of genus zero and $f_n(E(\alpha_n)) \cap \partial\mathbb{B}(p,\delta)$ consists of a single curve Γ_n such that $\{\Gamma_n\}_n \to \Gamma_\infty$, then we conclude that $f_n(\alpha_n) \cup \Gamma_n$ bounds a compact annulus in $f_n(E(\alpha_n))$; in fact, $f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(p,\delta)$ is this annulus.

We now arrive at the desired contradiction as follows. Consider a horizontal plane Π strictly below the height of p. Since M_∞ is the smooth limit of the $[f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)] - \{p\}$ away from p, we conclude that the horizontal circle $M_\infty \cap \Pi$ is arbitrarily close to a simple closed, planar convex curve $c_n \subset f_n(E(\alpha_n))$. Observe that c_n can be joined to both $f_n(\alpha_n)$ and $f_n(\beta_n)$ by arcs that do not intersect $f_n(\alpha_n) \cup f_n(\beta_n) \cup c_n$ except at their extrema. Therefore, $f_n(\alpha_n) \cup f_n(\beta_n) \cup c_n$ is the boundary of a compact planar domain in $f_n(E(\alpha_n))$. Since α_n and β_n are both nontrivial in $\overline{\mathbb{D}} - \{0\}$, then c_n must bound a disk in $\overline{\mathbb{D}} - \{0\}$ and therefore, c_n bounds in E a punctured disk T_n with the number of punctures being positive and finite (depending on n) by the convex hull property. Recall that the logarithmic growths of the annular ends of $f_n(E_n)$ are arbitrarily small in absolute value for n sufficiently large. As the flux of $f_n(E(\alpha_n))$ along c_n is bounded away from zero, we conclude that the number of punctures in T_n is unbounded as $n \to \infty$. Since $f_n(T_n)$ lies below the plane Π, $c_n = \partial[f_n(T_n)]$ is a convex horizontal curve inside Π and the flux of any closed curve in $f_n(T_n)$ is vertical, then the López-Ros deformation argument implies that T_n contains just one puncture, which is a contradiction for n large. This contradiction proves that Case (K1) does not occur.

Finally suppose that Case (K2) occurs. This assumption implies that after passing to a subsequence, the following properties hold.

(L1) The surfaces $f_n(\Delta_{n-1})$ converge to a compact minimal annulus Δ_∞ bounded by two horizontal simple closed convex curves, and Δ_∞ is close in the Hausdorff distance to a compact piece of a vertical catenoid. Similarly, $f_n(\Delta_n)$ converge to a compact minimal annulus Δ^∞ bounded by two horizontal simple closed convex curves, and Δ^∞ is ε-close in the Hausdorff distance to $M_\infty(\varepsilon)$.

(L2) The curves $f_n(\alpha_n)$ converge to the top boundary component $\widehat{\alpha}$ of Δ_∞.

(L3) For all $n \in \mathbb{N}$, the restriction of the injectivity radius function of $f_n(E)$ to $f_n(E(\alpha_n)) \cap \overline{\mathbb{B}}(n)$ is uniformly bounded away from zero (independently of n; this is a consequence of the arguments in the first paragraph of the proof of Case (K1)).

We now divide the argument of why Case (K2) leads to contradiction into two subcases, depending on whether or not the sequence $\{f_n(\Omega_n)\}_n$ has locally bounded second fundamental form in $\mathbb{R}^3$.

First suppose that $\{f_n(\Omega_n)\}_n$ has locally bounded second fundamental form in $\mathbb{R}^3$. By the arguments in the proof of Lemma 1.1 in [35], after passing to a subsequence, the surfaces $f_n(\Omega_n)$ converge to a minimal lamination $\mathcal{L}_1$ of $\mathbb{R}^3 - (\widehat{\alpha} \cup \widehat{\beta})$. In fact, property (L1) above together with the definition of f_n imply that $\{f_n(\text{Int}(\Omega_n \cup \Delta_{n-1} \cup \Delta_n))\}_n$ converges to a minimal lamination $\mathcal{L}_2$ of $\mathbb{R}^3 - (\widehat{\alpha}_1 \cup \widehat{\beta}_1)$, where

$$\widehat{\alpha}_1 = \partial\Delta_\infty - \widehat{\alpha}, \quad \widehat{\beta}_1 = \partial\Delta^\infty - \widehat{\beta},$$

and $\mathcal{L}_2$ contains the interior of both $\Delta_\infty, \Delta^\infty$ as portions of its leaves. We will call $L(\Delta_\infty)$ (resp. $L(\Delta^\infty)$) the leaf of $\mathcal{L}_2$ that contains the interior of Δ_∞ (resp. of Δ^∞). Note that $L(\Delta_\infty)$ might coincide with $L(\Delta^\infty)$. Also observe that neither $L(\Delta_\infty)$ nor $L(\Delta^\infty)$ are stable, as both Δ_∞, Δ^∞ can be assumed to be unstable by choosing ε in the statement of Proposition 4.12 sufficiently small. As $L(\Delta_\infty), L(\Delta^\infty)$ are not stable, Theorem 1 in [32] implies that $L(\Delta_\infty), L(\Delta^\infty)$ are not limit leaves of $\mathcal{L}_2$. Also note that every limit leaf of $\mathcal{L}_1$ is contained in a limit leaf of $\mathcal{L}_2$, and so, limit leaves of $\mathcal{L}_1$ are complete stable minimal surfaces, which are planes. In particular, limit leaves of $\mathcal{L}_1$ and of $\mathcal{L}_2$ are the same. This implies that the closure in $\mathbb{R}^3$ of each nonflat leaf of $\mathcal{L}_2$ is proper in $\mathbb{R}^3$, in a halfspace or in a slab with boundary being limit leaves of $\mathcal{L}_1$. In particular, the following surfaces with compact boundary are proper in $\mathbb{R}^3$, proper in an open halfspace or proper in an open slab:

$$\begin{cases} L(\Delta_\infty) \cup \widehat{\alpha}_1, \quad L(\Delta^\infty) \cup \widehat{\beta}_1 & \text{if } L(\Delta_\infty) \neq L(\Delta^\infty), \\ L(\Delta_\infty) \cup \widehat{\alpha}_1 \cup \widehat{\beta}_1 & \text{if } L(\Delta_\infty) = L(\Delta^\infty). \end{cases}$$

Suppose that $L(\Delta_\infty) \neq L(\Delta^\infty)$. In this setting, property (L3) above and the intrinsic version of the one-sided curvature estimates by Colding and Minicozzi (Corollary 0.8 in [9]) imply that $L(\Delta_\infty) \cup \widehat{\alpha}_1$ has bounded Gaussian curvature in any small regular neighborhood of the limit set of $L(\Delta_\infty) \cup \widehat{\alpha}_1$ (see Lemma 1.2 in [35] for a similar argument using the extrinsic version of the one-sided curvature estimates by Colding and Minicozzi). Therefore, $L(\Delta_\infty) \cup \widehat{\alpha}_1$ is proper in $\mathbb{R}^3$, and the same holds for $L(\Delta^\infty) \cup \widehat{\beta}_1$ by similar arguments. In the case that $L(\Delta_\infty) = L(\Delta^\infty)$, the same reasoning gives that $L(\Delta_\infty) \cup \widehat{\alpha}_1 \cup \widehat{\beta}_1$ is proper in $\mathbb{R}^3$.

Observe that $L(\Delta_\infty)$ has genus zero and one or two boundary curves, each of which bounds an open convex horizontal disk disjoint from $L(\Delta_\infty)$ (this follows from the arguments in the proof of Assertion 3.13). If $L(\Delta_\infty)$ has one boundary curve, then we contradict the Halfspace Theorem, as $L(\Delta_\infty)$ lies in the halfspace $\{x_3 \leq 0\}$ (because $f_n(\Omega_n)$ has the same property for all n) and $L(\Delta_\infty)$ contains interior points with heights strictly greater than its boundary curve. Therefore, $L(\Delta_\infty)$ has two boundary curves (equivalently, $L(\Delta_\infty) = L(\Delta^\infty)$). By the convex hull property, $L(\Delta_\infty)$ is noncompact. As $L(\Delta_\infty) - \Delta^\infty$ is contained in $\{x_3 \leq 0\}$, then $L(\Delta_\infty)$ has horizontal limit tangent plane at infinity. Since both α_n, β_n have the same horizontal component of their fluxes, and the homothetic factors λ_n in (2) tend to zero, then we deduce that the fluxes of $L(\Delta_\infty)$ along its boundary curves are vertical. This implies that $L(\Delta_\infty)$ has vertical flux, since it has genus zero. Therefore, $L(\Delta_\infty)$ cannot have a finite positive number of ends by the López-Ros deformation argument.

The previous paragraph implies that $L(\Delta_\infty)$ has infinitely many ends. Let D be a positive number such that the boundary of $L(\Delta_\infty)$ is contained in the ball $\mathbb{B}(D)$ of radius D centered at the origin. We claim that for every sequence $\{\mu_n\}_n$ of positive numbers going to zero, the restriction to $\mu_n[L(\Delta_\infty) - \mathbb{B}(2D)]$ of the injectivity radius function of $\mu_n L(\Delta_\infty)$ is greater that some positive constant (independent of n) times the distance to the origin. Otherwise, there exists a sequence of points $x_n \in \mu_n[L(\Delta_\infty) - \mathbb{B}(2D)]$ such that

$$\frac{I_{\mu_n L(\Delta_\infty)}(x_n)}{|x_n|} \to 0 \quad \text{as } n \to \infty,$$

where $I_{\mu_n L(\Delta_\infty)}$ stands for the injectivity radius function of $\mu_n L(\Delta_\infty)$. Since the last quotient is invariant under rescaling, the sequence $x_n/|x_n|$ lies in the unit sphere and the boundary of $\mu_n L(\Delta_\infty)$ shrinks to the origin as $n \to \infty$, we produce a sequence of blow-up points on the scale of topology on Ω_n, which is impossible by previous arguments (maximality of $\{\Delta_m\}_m$). This proves our claim.

Since $L(\Delta_\infty)$ has infinitely many ends, then $L(\Delta_\infty)$ has infinite total curvature. As $L(\Delta_\infty)$ has compact boundary, then Theorem 1.2 in [34] ensures that $L(\Delta_\infty)$ does not have quadratic decay of curvature, i.e., there exists a divergent sequence $y_n \in L(\Delta_\infty)$ such that

$$|K_{L(\Delta_\infty)}|(y_n) \cdot |y_n|^2 \to \infty \quad \text{as } n \to \infty,$$

where $K_{L(\Delta_\infty)}$ denotes the Gaussian curvature of $L(\Delta_\infty)$. Taking $\mu_n = 1/|y_n|$ and using the claim in the last paragraph, we conclude by Theorem 2.2 in [26] that after passing to a subsequence, the $\mu_n L(\Delta_\infty)$ converge to a minimal parking garage structure of $\mathbb{R}^3$. This is impossible, since the limit set of the $\mu_n L(\Delta_\infty)$ lies in $\{x_3 \leq 0\}$. This contradiction implies that $L(\Delta_\infty)$ cannot have infinitely many ends, and thus, case (K2) does not occur in the special case that $\{f_n(\Omega_n)\}_n$ has locally bounded second fundamental form in $\mathbb{R}^3$.

By the last sentence, it remains to prove that the sequence $\{f_n(\Omega_n)\}_n$ has locally bounded second fundamental form in $\mathbb{R}^3$ provided that case (K2) happens. By property (L3) above and the 1-sided curvature estimates by Colding-Minicozzi, we conclude that the norms of the second fundamental forms of the $f_n(\Omega_n \cup \Delta_{n-1} \cup \Delta_n)$ are bounded on some small fixed compact regular neighborhood W of $\Delta_\infty \cup \Delta^\infty$. Arguing by contradiction, suppose, after extracting a subsequence, that there is a sequence of points $q_n \in f_n(\Omega_n) - W$ that converges to a point $q \in \mathbb{R}^3 - W$ where the norms of the second fundamental forms of the $f_n(\Omega_n)$ are greater than n. By property (L3), Colding-Minicozzi theory in [7] (see also Figure 2 in [28]) implies that after extracting a subsequence, the following properties hold:

(M1) There is a positive number δ less than one half of the distance in $\mathbb{R}^3$ from q to W, a relatively closed subset $\mathcal{S}_{q,\delta}$ of $\mathbb{B}(q,\delta)$ and a minimal lamination $\mathcal{L}_{q,\delta}$ of $\mathbb{B}(q,\delta) - \mathcal{S}_{q,\delta}$ such that $\mathbb{B}(q,\delta) \cap f_n(\Omega_n)$ consists of disks with their boundary curves in $\partial\mathbb{B}(q,\delta)$ and a subsequence of these disks converges C^α, $\alpha \in (0,1)$, to $\mathcal{L}_{q,\delta}$ in $\mathbb{B}(q,\delta) - \mathcal{S}_{q,\delta}$.

(M2) For each point $s \in \mathcal{S}_{q,\delta}$, there is a limit leaf of $\mathcal{L}_{q,\delta}$ that is a minimal disk punctured at s, and the closure in $\mathbb{B}(q,\delta)$ of the collection of all the limit leaves of $\mathcal{L}_{q,\delta}$ forms a minimal lamination $\mathcal{F}_{q,\delta}$ of $\mathbb{B}(q,\delta)$.

We refer the reader to description (D) in Section 3 of [28] for details on observations (M1), (M2). Furthermore, straightforward diagonal arguments using this just described local structure of the limit set of the $f_n(\Omega_n)$ near points in $\mathbb{R}^3$ where the norms of their second fundamental forms are becoming unbounded, demonstrate that there exists a possibly singular minimal lamination $\mathcal{L}'$ of $\mathbb{R}^3 - W$ and a relatively closed set $\mathcal{S} \subset \mathcal{L}'$ in $\mathbb{R}^3 - W$ (the set of points where $\mathcal{L}'$ fails to admit a local lamination structure) such that after extracting a subsequence, the $f_n(\Omega_n)$ converge C^α $(0 < \alpha < 1)$ on compact subsets of $\mathbb{R}^3 - [W \cup \mathcal{S} \cup S(\mathcal{L}')]$ to $\mathcal{L}'$, where $S(\mathcal{L}') \subset \mathcal{L}' - \mathcal{S}$ is the set of points where $\mathcal{L}'$ admits a local lamination structure but the second fundamental forms of the surfaces $f_n(\Omega_n)$ blow up as $n \to \infty$ ($S(\mathcal{L}')$ is called the singular set of convergence of the sequence). In fact, since the second fundamental forms of the $f_n(\Omega_n \cup \Delta_n \cup \Delta_{n-1})$ are uniformly bounded in W, we conclude that $\mathcal{L}'$ can be extended to a possibly singular minimal lamination $\mathcal{L}_1'$ of $\mathbb{R}^3-(\widehat{\alpha}_1\cup\widehat{\beta}_1)$ and the surfaces $f_n(\Omega_n\cup\Delta_n\cup\Delta_{n-1})$ converge C^α to $\mathcal{L}_1'$ in $\mathbb{R}^3 - [\widehat{\alpha}_1 \cup \widehat{\beta}_1 \cup \mathcal{S} \cup S(\mathcal{L})]$. Furthermore, the singular set (resp. the singular set of convergence) of $\mathcal{L}_1'$ equals the singular set $\mathcal{S}$ (resp. the singular set of convergence $S(\mathcal{L}')$) of $\mathcal{L}'$. Additionally, the closure in $\mathbb{R}^3 - [\widehat{\alpha}_1 \cup \widehat{\beta}_1]$ of the sublamination of limit leaves of $\mathcal{L}_1'$ is a (regular) minimal lamination $\mathcal{F}$ of $\mathbb{R}^3 - [\widehat{\alpha}_1 \cup \widehat{\beta}_1]$ with $\mathcal{S} \cup S(\mathcal{L}') \subset \mathcal{F}$. Observe that the leaves $L(\Delta_\infty), L(\Delta^\infty)$ of $\mathcal{L}_1'$ that contain respectively Int(Δ_∞), Int(Δ^∞), are unstable and thus, they are not leaves of $\mathcal{F}$. In fact, $\Delta_\infty \cup \Delta^\infty$ can be assumed to lie at a positive distance from $\mathcal{F}$ after slightly changing the compact domains $\Delta_\infty, \Delta^\infty$. This implies that the leaves of $\mathcal{F}$ are complete in $\mathbb{R}^3$, and since they are stable, then these leaves are planes.

Since $\mathcal{F}$ contains $\mathcal{S}\cup S(\mathcal{L}')$, then the norms of the second fundamental forms of the surfaces $f_n(\Omega_n)$ are locally bounded in the open set $\mathbb{R}^3 - \mathcal{F}$, which is a countable union of open slabs and open halfspaces. As the top boundary component of $L(\Delta^\infty)$ is $\widehat{\beta}_1$, it follows that $L(\Delta^\infty)$ is contained in the halfspace $\{x_3 \leq x_3(\widehat{\beta}_1)\}$ and $L(\Delta^\infty)$ is proper in the open slab A of $\mathbb{R}^3$ with boundary $\{x_3 = x_3(\widehat{\beta}_1)\} \cup P$, where P is the plane in $\mathcal{F}$ with largest x_3-coordinate (which exists since $\mathcal{S} \cup S(\mathcal{L}') \neq \emptyset$). Observe that $L(\Delta^\infty)$ is proper in A (otherwise there exists a plane in $\mathcal{F} \cap A$).

Next we will show that $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ is incomplete. Arguing by contradiction, suppose that $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ is complete. As the injectivity radius function of $f_n(\Omega_n) \cup \Delta_n \cup \Delta_{n-1}$ restricted to $f_n(\Omega_n)$ is uniformly bounded away from zero (otherwise we could find a sequence of blow-up points in Ω_n, which is impossible) and $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ is assumed to be complete, then the injectivity radius function of $L(\Delta^\infty)$ is bounded away from zero outside any neighborhood of its boundary. In this setting, Theorem 1.2 implies that $L(\Delta^\infty)$ is proper in $\mathbb{R}^3$. To find the desired contradiction, we distinguish two cases; first suppose that $\partial L(\Delta^\infty) = \widehat{\beta}_1$. In this case, $L(\Delta^\infty) \cup \widehat{\beta}_1$ has full harmonic measure by Lemma 2.2 in [12]. But the third coordinate function of $L(\Delta^\infty)\cup\widehat{\beta}_1$ is a bounded harmonic function with constant boundary values $x_3(\widehat{\beta}_1)$ and values at interior points strictly below $x_3(\widehat{\beta}_1)$, which is a contradiction. Second, suppose that $\partial L(\Delta^\infty) = \widehat{\alpha}_1 \cup \widehat{\beta}_1$; in this case, $L(\Delta^\infty)$ has finitely many ends by the same Lemma 2.2 in [12], and thus, these ends are asymptotic to horizontal planes. Now the López-Ros deformation argument applied to $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ leads to contradiction as this noncompact embedded minimal surface has vertical flux and two convex horizontal boundary components. Therefore, $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ must be incomplete.

Since $L(\Delta^\infty) \cup \partial L(\Delta^\infty)$ is incomplete and $L(\Delta^\infty)$ is proper in A, then $L(\Delta^\infty)$ contains a proper arc $\tau\colon [0,1) \to L(\Delta^\infty)$ of finite length with its limiting endpoint $q \in \mathcal{S}\cap P$; previous arguments also imply that $L(\Delta^\infty)$ has vertical flux. If there exists another point $q' \in (\mathcal{S} \cap P) - \{q\}$ where $L(\Delta^\infty)$ fails to be complete, one can construct a sequence of connection loops $\sigma_k \subset L(\Delta^\infty)$, $k \in \mathbb{N}$, that converge as $k \to \infty$ with multiplicity 2 away from $\{q, q'\}$ to an compact embedded arc σ in $P-(\mathcal{S}-\{q,q'\})$ that joins q to q', and such that the fluxes of these connection loops on $L(\Delta^\infty)$ converge to a nonzero horizontal vector, which contradicts that $L(\Delta^\infty)$ has vertical flux. Therefore, $L(\Delta^\infty)$ only has $q \in \mathcal{S}\cap P$ as a point of incompleteness.

By the extrinsic 1-sided curvature estimates of Colding-Minicozzi, there is an $\varepsilon' > 0$ small such that the intersection of $L(\Delta^\infty)$ with the ε'-neighborhood $P(\varepsilon')$ of P is a disk that contains a pair of disjoint ∞-valued graphs Σ_1, Σ_2 (with respect to polar coordinates in P centered at q, each Σ_i is an ∞-valued graph over an annulus in P centered at q with inner radius 1 and arbitrarily large radius, $i = 1, 2$) and both Σ_1, Σ_2 spiral together into P by above. Furthermore, Σ_1 and Σ_2 can be joined by curves in $L(\Delta^\infty) \cap P(\varepsilon')$ with uniformly bounded length. In this setting, Corollary 1.2 in [4] (see especially the paragraph just after this corollary) leads to a contradiction. This contradiction completes the proof that Case (K2) cannot occur.

Since we have discarded Cases (K1) and (K2) above, then the curves $f_n(\alpha_n)$ diverge to infinity in $\mathbb{R}^3$. Thus, the first paragraph in the proof of Lemma 4.13 ensures that the conclusions of Lemma 4.13 hold. □

Recall that we had called Ω_n, $n \in \mathbb{N}$, to the components of $E - \bigcup_{m\in\mathbb{N}\cup\{0\}} \Delta_m$, where the index n is chosen so that (12) holds, and that in order to prove item 4 of Proposition 4.12, it suffices to find a contradiction with the following assumption:

(♣) The number of components Ω_n with has at least two annular ends is infinite.

Suppose that (♣) holds, and let $\Omega_{n(i)}$, $i \in \mathbb{N}$, denote the subsequence of the Ω_n with at least two annular ends each. Since any path in $\Omega_{n(i)}$ joining two consecutive annular ends intersects the inverse image by the Gauss map of E of the horizontal equator in the sphere, we conclude that there exists some point $x_i \in \Omega_{n(i)}$ where the tangent plane to $\Omega_{n(i)}$ is vertical. We can assume that x_i is chosen so that it is an extrinsically closest such point to the upper boundary component $\beta_{n(i)}$ of $\Omega_{n(i)}$. Let $d_{\overline{\partial}}(i) > 0$ be the extrinsic distance from $f_{n(i)}(x_i)$ to $f_{n(i)}(\beta_{n(i)})$, where f_n is the dilation defined just before Lemma 4.13. Since $f_{n(i)}(\Omega_{n(i)})$ converges as $i \to \infty$ to $M_\infty \cap \{x_3 \leq \frac{-1}{\varepsilon}\}$ by Lemma 4.13 (recall that M_∞ is the vertical catenoid whose waist circle is the unit circle in the (x_1, x_2)-plane) and the tangent plane to $f_{n(i)}(\Omega_{n(i)})$ at $f_{n(i)}(x_i)$ is vertical, it follows that $d_{\overline{\partial}}(i) \to \infty$ as $i \to \infty$.

Now we apply a homothety to obtain the surface

$$\overline{\Theta}_{n(i)} = \frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(\Omega_{n(i)}) - f_{n(i)}(b_i)], \tag{13}$$

where b_i is a closest point to x_i in $\beta_{n(i)}$. The surface $\overline{\Theta}_{n(i)}$ is an embedded, minimal planar domain passing through the origin, with two horizontal, almost circular boundary components and a positive number of ends (at least two), all with negative logarithmic growth; after extracting a subsequence, let $\mathbf{x} \in \mathbb{R}^3$ be the limit of the points $\frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(x_i) - f_{n(i)}(b_i)]$ and note that $\mathbf{x}$ is at a distance 1 from the origin. Let

$$\overline{\alpha}_{n(i)} = \frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(\alpha_{n(i)}) - f_{n(i)}(b_i)], \quad \overline{\beta}_{n(i)} = \frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(\beta_{n(i)}) - f_{n(i)}(b_i)]$$

be the respective lower and upper boundary components of $\overline{\Theta}_{n(i)}$.

Since the lengths of $\overline{\alpha}_{n(i)}, \overline{\beta}_{n(i)}$, are shrinking to zero, then after extracting a subsequence, the $\overline{\beta}_{n(i)}$ converge to $\vec{0}$ and the $\overline{\alpha}_{n(i)}$ either converge to a point $q(\alpha) \in \mathbb{R}^3$ or they diverge in $\mathbb{R}^3$.

It follows that $\{\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)\}_i$ is a sequence of compact genus zero minimal surfaces which is locally simply connected in $\mathbb{R}^3 - W$, where $W = \{\vec{0}, q(\alpha)\}$ in the case that $q(\alpha)$ exists and $W = \{\vec{0}\}$ otherwise. By Theorem 2.2 in [26], after choosing a subsequence, the surfaces $\overline{\Theta}_n \cap \overline{\mathbb{B}}(n)$ converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - W$ and $\mathcal{L}$ extends to a minimal lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$. Notice that $\overline{\mathcal{L}}$ contains a complete leaf $L_{\mathbf{x}}$ passing through $\mathbf{x}$. Since $\overline{\Theta}_{n(i)}$ is contained in the halfspace $\{x_3 \leq 1\}$ for i large (since $\frac{1}{d_{\overline{\partial}}(i)} f_{n(i)}(\Delta_{n(i)})$ shrinks to $\vec{0}$), then Theorem 2.2 in [26] implies that all of the leaves in $\overline{\mathcal{L}}$ are horizontal planes and that the sequence of norms of the second fundamental forms of the surfaces $\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)$ is locally bounded in $\mathbb{R}^3 - W$. In particular, $L_{\mathbf{x}}$ is a horizontal plane. Since the tangent plane of $\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)$ is vertical at x_i for each i, the sequence $\{\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)\}_i$ cannot have uniformly bounded curvature in any fixed sized neighborhood of $\mathbf{x}$, which implies that $\mathbf{x} = q(\alpha)$ (in particular, $q(\alpha)$ exists).

We next explain how to refine the arguments in the last paragraph to conclude the following property.

Claim 4.14 *Once we restrict to the subsequence $\{\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)\}_i$ that limits to $\mathcal{L}$, for any sequence of points $y_i \in \Omega_{n(i)}$ with vertical tangent plane (not necessarily the closest such points in $\Omega_{n(i)}$ to the upper boundary component $\beta_{n(i)}$), the points $\frac{1}{d_{\overline{\partial}}(i)} f_{n(i)}(y_i)$ converge to* $\mathbf{x}$.

Proof. Let $d_{\overline{\partial}}(y_i, i)$ be the extrinsic distance from $f_{n(i)}(y_i)$ to $f_{n(i)}(\beta_{n(i)})$, which is attained at some point $f_{n(i)}(b_i')$ with $b_i' \in \beta_{n(i)}$. Observe that $d_{\overline{\partial}}(y_i, i) \geq d_{\overline{\partial}}(i)$ and the arguments before Claim 4.14 prove that after choosing a subsequence, the curves

$$\frac{1}{d_{\overline{\partial}}(y_i, i)}[f_{n(i)}(\alpha_{n(i)}) - f_{n(i)}(b_i')]$$

converge to the same (subsequential) limit $\mathbf{y}$ of the points

$$\frac{1}{d_{\overline{\partial}}(y_i, i)}[f_{n(i)}(y_i) - f_{n(i)}(b_i')],$$

which in turn is a point in the unit sphere. Since the curves $\frac{1}{d_{\overline{\partial}}(y_i,i)}[f_{n(i)}(\alpha_{n(i)}) - f_{n(i)}(b_i')]$ converge to the same limit point as the $\overline{\alpha}_{n(i)}$ (this last limit was called $q(\alpha)$ in the preceeding paragraph), we conclude that $\mathbf{y} = q(\alpha) = \mathbf{x}$ and that $\frac{d_{\overline{\partial}}(y_i,i)}{d_{\overline{\partial}}(i)}$ tends to 1 as $n \to \infty$, from where we obtain that the $\frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(y_i) - f_{n(i)}(b_i')]$ converge to $\mathbf{x}$. The fact that the whole original sequence $\{\frac{1}{d_{\overline{\partial}}(i)}[f_{n(i)}(y_i) - f_{n(i)}(b_i')]\}_i$ converges to $\mathbf{x}$ (i.e., we do not need to pass to a subsequence of the y_i once we restrict to the subsequence that produces the convergent sequence $\{\overline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)\}_i$ to $\mathcal{L}$) follows from arguing by contradiction and passing to further subsequences. □

Now return to the definition of the point $x_i \in \Omega_{n(i)}$. Define the related point z_i as an extrinsically farthest point in $\Omega_{n(i)}$ to its lower boundary component $\alpha_{n(i)}$ where the tangent plane to $\Omega_{n(i)}$ is vertical; since the set of all such points z is compact in $\Omega_{n(i)}$ and a positive distance from $\alpha_{n(i)}$, the point z_i exists. We now apply our previous arguments with z_i in place of x_i: consider for each $i \in \mathbb{N}$ the related surface

$$\underline{\Theta}_{n(i)} = \frac{1}{d_{\underline{\partial}}(z_i, i)}[f_{n(i)}(\Omega_{n(i)}) - f_{n(i)}(a_i)],$$

where $d_{\underline{\partial}}(z_i, i) > 0$ is the extrinsic distance from $f_{n(i)}(z_i)$ to $f_{n(i)}(\alpha_{n(i)})$ and a_i is a point in $\alpha_{n(i)}$ closest to z_i. Observe that we do not know if $d_{\underline{\partial}}(z_i, i) \to \infty$ as $i \to \infty$ but we do know (from the previous paragraph) that the distances in $\mathbb{R}^3$ from the origin to the top boundary component $\frac{1}{d_{\underline{\partial}}(z_i,i)} \left(f_{n(i)}(\beta_{n(i)}) - f_{n(i)}(a_i)\right)$ of $\underline{\Theta}_{n(i)}$ diverges to infinity as $i \to \infty$.

Claim 4.15 *The sequence $d_{\underline{\partial}}(z_i, i)$ is bounded independently of i.*

Proof. Arguing by contradiction, suppose after choosing a subsequence, $d_{\underline{\partial}}(z_i, i) \geq 2i$. By Claim 4.14, the extrinsic distance from z_i to $\beta_{n(i)}$ is much larger than the extrinsic distance from z_i to a_i. Therefore, we can assume that the curve

$$\frac{1}{d_{\underline{\partial}}(z_i, i)}[f_{n(i)}(\beta_{n(i)}) - f_{n(i)}(a_i)]$$

lies outside the ball $\mathbb{B}(i^2)$. Consider the compact minimal surfaces $\underline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)$, which form a locally simply connected sequence of surfaces in $\mathbb{R}^3 - \{\vec{0}\}$ by our previous arguments. After extracting a subsequence, let $\mathbf{z} \in \mathbb{R}^3$ be the limit of the points

$$\frac{1}{d_{\underline{\partial}}(z_i, i)}[f_{n(i)}(z_i) - f_{n(i)}(a_i)],$$

which is a point in the unit sphere. As before, Theorem 2.2 in [26] implies that after passing to a subsequence, the $\underline{\Theta}_{n(i)} \cap \overline{\mathbb{B}}(i)$ converge to a minimal lamination $\mathcal{L}_+$ of $\mathbb{R}^3 - \{\vec{0}\}$, which extends to a lamination $\overline{\mathcal{L}_+}$ of $\mathbb{R}^3$ with a leaf $L_{\mathbf{z}}$ passing through $\mathbf{z}$.

Now consider the surfaces

$$\Sigma_i = \frac{1}{d_{\underline{\partial}}(z_i, i)} \left[f_{n(i)}(\Omega_{n(i)} \cup \Delta_{n(i)-1} \cup \Omega_{n(i)-1}) - f_{n(i)}(a_i)\right], \quad i \in \mathbb{N}.$$

For each i, Σ_i is a noncompact planar domain bounded by two convex horizontal curves, that we call

$$\partial\Sigma_i^+ = \frac{1}{d_{\underline{\partial}}(z_i, i)}[f_{n(i)}(\beta_{n(i)}) - f_{n(i)}(a_i)], \quad \partial\Sigma_i^- = \frac{1}{d_{\underline{\partial}}(z_i, i)}[f_{n(i)}(\alpha_{n(i)-1}) - f_{n(i)}(a_i)],$$

and $x_3(\partial\Sigma_i^-) < x_3(\partial\Sigma_i^+)$. Previous arguments show that the sequence of curves $\{\partial\Sigma_i^-\}_i$ either converges to some point $q_- \in \mathbb{R}^3$ (that would then lie in $\{x_3 \leq 0\}$, possibly being $\vec{0}$), or else $\{\partial\Sigma_i^-\}_i$ diverges in $\mathbb{R}^3$. Maximality of the family $\{\Delta_m\}_m$ implies as above that the sequence of surfaces $\{\Sigma_i \cap \overline{\mathbb{B}}(i)\}_n$ is locally simply connected in $\mathbb{R}^3 - W$ where $W = \{\vec{0}, q_-\}$ if q_- exists, and $W = \{\vec{0}\}$ otherwise. Therefore, Theorem 2.2 in [26] implies that after passing to a subsequence, the Σ_i converge to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3 - W$, which extends to a lamination $\overline{\mathcal{L}}$ of $\mathbb{R}^3$. Note that $\overline{\mathcal{L}}$ contains $\overline{\mathcal{L}}_+$ as a sublamination. Also observe that the same arguments applied to the sequence of surfaces

$$\frac{1}{d_{\underline{\partial}}(z_i, i)} \left[f_{n(i)}(\Omega_{n(i)-1}) - f_{n(i)}(a_i)\right].$$

give that the surfaces $\frac{1}{d_{\underline{\partial}}(z_i,i)}\left[f_{n(i)}(\Omega_{n(i)-1})-f_{n(i)}(a_i)\right]\cap\overline{\mathbb{B}}(i)$ converge to a minimal lamination $\mathcal{L}_-$ of $\mathbb{R}^3-W$, which extends to a lamination $\overline{\mathcal{L}_-}$ of $\mathbb{R}^3$. Moreover, $\overline{\mathcal{L}_-}$ contains a leaf that passes through the origin, and $\overline{\mathcal{L}}_-$ is a sublamination of $\overline{\mathcal{L}}$. By construction, the $\frac{1}{d_{\underline{\partial}}(z_i,i)}\left[f_{n(i)}(\Omega_{n(i)-1})-f_{n(i)}(a_i)\right]$ lie in $\{x_3\leq 0\}$ for each $i\in\mathbb{N}$, and thus, $\overline{\mathcal{L}_-}$ is also contained in $\{x_3\leq 0\}$. In particular, $\{x_3=0\}$ is a leaf of $\overline{\mathcal{L}}_-$. In this setting, Theorem 2.2 in [26] implies that all leaves of $\mathcal{L}_-$ are horizontal planes, and thus, the same theorem gives that all leaves of $\overline{\mathcal{L}}$ are horizontal planes. In particular, $L_{\mathbf{z}}$ is a horizontal plane.

Since the tangent plane to $\Omega_{n(i)}$ at z_i is vertical, then the convergence of the $\underline{\Theta}_{n(i)}\cap\overline{\mathbb{B}}(i)$ to $L_{\mathbf{z}}$ cannot be smooth around $\mathbf{z}$. This property and Theorem 2.2 in [26] imply that $\overline{\mathcal{L}}_+$ is a foliation of $\mathbb{R}^3$ by horizontal planes and the $\underline{\Theta}_{n(i)}\cap\overline{\mathbb{B}}(i)$ converge to $\overline{\mathcal{L}}$ outside the origin and one or two vertical lines (this is the singular set of convergence), one of which passes through $\mathbf{z}$. This is impossible, since the compact surfaces $\underline{\Theta}_{n(i)}\cap[\overline{\mathbb{B}}(i^2/2)-\mathbb{B}(2)]$ have uniformly bounded Gaussian curvature (this follows since the last surfaces do not have vertical tangent planes, and so they are locally graphical hence stable, and by curvature estimates for stable minimal surfaces). Now Claim 4.15 is proved. □

As a consequence of Lemma 4.13, the diameter of the compact surface

$$\frac{1}{d_{\underline{\partial}}(z_i,i)}\left[f_{n(i)}(\Delta_{n(i)-1})-f_{n(i)}(a_i)\right]$$

tends to zero as $i\to\infty$. In particular, the diameter of the top boundary curve of the last surface tends to zero, which implies that

$$\frac{1}{d_{\underline{\partial}}(z_i,i)}\left[f_{n(i)}(\alpha_{n(i)})-f_{n(i)}(a_i)\right]\to\vec{0}\quad\text{as } i\to\infty.$$

On the other hand, Claim 4.14 implies that

$$\frac{1}{d_{\underline{\partial}}(z_i,i)}\left[f_{n(i)}(\beta_{n(i)})-f_{n(i)}(a_i)\right]\quad\text{diverges in } \mathbb{R}^3 \text{ as } i\to\infty.$$

Therefore, Theorem 2.2 in [26] implies that after passing to a subsequence, the $\underline{\Theta}_{n(i)}\cap\overline{\mathbb{B}}(i)$ converge to a minimal lamination $\mathcal{L}_+$ of $\mathbb{R}^3-\{\vec{0}\}$. From this point, we can repeat verbatim the arguments in the proof of Claim 4.15 to obtain a contradiction. This contradiction shows that property (♣) cannot hold, and so, item 4 of Proposition 4.12 is proven.

Finally, item 5 of Proposition 4.12 follows from the fact that the Gaussian curvature functions of the domains Ω_n and Δ_n become uniformly small as $n\to\infty$. Now the proof of Proposition 4.12 is complete. □

Proposition 4.16 *Items 4, 5 and 6 of Theorem 1.6 hold in the Case (G1) when M_∞ is a catenoid. In particular, Theorem 1.6 holds in this case.*

Proof. Recall that in the paragraph just before Proposition 4.11, we explained that items 1, 2, 3 of Theorem 1.6 hold in the Case (G1) when M_∞ is a catenoid. The description of E as a union of domains Δ_n and Ω_n as given in Proposition 4.12 implies that item 4 of Theorem 1.6 holds. Item 5 of the same theorem follows from item 5 of Proposition 4.12. Finally, the arguments in the proof of Proposition 4.11 can be easily adapted to the current situation, with the only change of the annular regions of "Riemann type" by similar annular regions of "catenoid type", namely regions of the type of the compact annuli $\Delta_n = \Delta_n(\varepsilon)$ that appear in Proposition 4.12, each of which contains the image of a conformal embedding $f_n(\mathbb{S}^1 \times [0, \delta])$ for some $\delta > 0$ independent of n (here we are using the notation in the proof of Proposition 4.11). This finishes the proof of Proposition 4.16. $\square$

5 The proof of Theorem 1.3.

Suppose $M \subset \mathbb{R}^3$ is a complete, embedded minimal surface with finite genus, an infinite number of ends and compact boundary.

We first check that M has at most two simple limit ends. Arguing by contradiction, suppose M has at least three simple limit ends, say $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$. By Theorem 1.6, we can choose three pairwise disjoint, properly embedded representatives $E_1, E_2, E_3 \subset M$, representing $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ respectively, such that each one satisfies, after a possible rotation, the conclusions of Theorem 1.6. Embeddedness of M implies that after a rotation, the annular ends of E_1, E_2, E_3 may be assumed to be asymptotic to ends of horizontal planes and catenoids with vertical axes. Furthermore, after a possible reindexing, we may assume that the ends E_1, E_2 are simple top limit ends, that ∂E_1 is a simple closed curve in the (x_1, x_2)-plane and that ∂E_2 has constant positive x_3-coordinate.

Let $D_{E_1} \subset \{x_3 = 0\}$ be the planar disk with $\partial D_{E_1} = \partial E_1$ and let X_1 be the closure of the component of $\mathbb{R}^3 - (E_1 \cup D_{E_1})$ that lies above D_{E_1} locally near D_{E_1}. Similarly, we can define a horizontal disk D_{E_2} with $\partial D_{E_2} = \partial E_2$ and the related closed component X_2 of $\mathbb{R}^3 - (E_2 \cup D_{E_2})$ above D_{E_2}.

An elementary topological analysis applied to the topological picture of E_1 and E_2 given in item 4 of Theorem 1.6 shows, after possibly reindexing E_1 and E_2 and replacing E_1 and E_2 by representing subends, that $D_{E_2} \cap E_1 \neq \emptyset$ and X_2 contains a representative $E_1' \subset E_1$ of the limit end of E_1 with $\partial E_1' \subset D_{E_2} \subset \partial X_2$. Let X_3 be the closure of the component of $X_2 - E_1'$ which has ∂E_2 in its boundary. The piecewise smooth surface ∂X_3 is a good barrier for solving least-area problems in X_3 (Meeks and Yau [40]), see Figure 7.

Let $\widetilde{E}_2$ be a noncompact, properly embedded surface of least-area in X_3 with $\partial \widetilde{E}_2 = \partial E_2 \subset \partial X_3$. By a result of Fischer-Colbrie [15], the orientable, stable minimal surface $\widetilde{E}_2$ has finite total curvature. Since $\widetilde{E}_2$ is contained in X_2, it must have a finite number of ends, all of which are annuli and which are parallel to the planar and catenoidal ends of E_2. Since points of $\widetilde{E}_2$ near D_{E_2} have x_3-coordinates which are larger than the constant value $x_3(D_{E_2})$, $\widetilde{E}_2$ must have a highest end which has positive logarithmic growth by the maximum principle applied to the harmonic function $x_3|_{\widetilde{E}_2}$. Hence, $\widetilde{E}_2$ has a catenoid-type end representative F of positive logarithmic growth. Since the annular ends of E_1 have nonpositive logarithmic growth, then

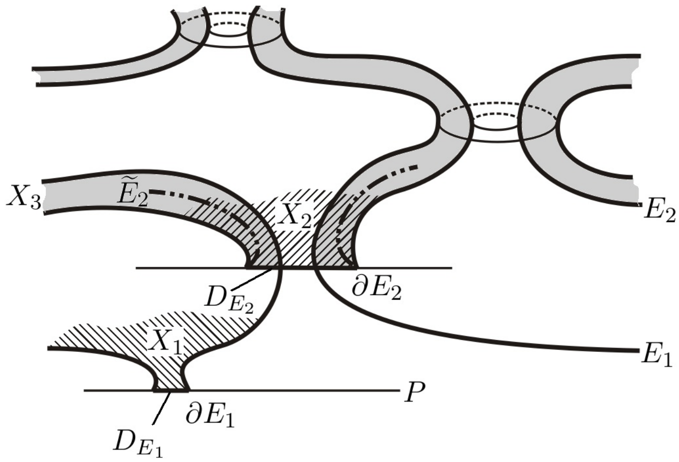

Figure 7: The area-minimizing surface $\widetilde{E}_2$ is trapped between two simple top limit ends E_1, E_2.

none of the annular ends of E_1 lie above F. This implies that E_1 lies below the region of $\mathbb{R}^3$ bounded by the union of F and a horizontal disk with boundary in F. Since E_1 also lies above some catenoid end of negative logarithmic growth, the results in [12] imply that E_1 has quadratic area growth. By the monotonicity formula, each annular end in E_1 contributes with at least $\frac{\pi}{2}R^2$ to the area growth of E_1 in each ball $\mathbb{B}(R)$ for $R > 0$ large. Hence, E_1 has a finite number of ends. This contradiction shows that M cannot have more than two simple limit ends, which is item 1 of Theorem 1.3.

Next we prove item 2 of Theorem 1.3. If M is properly embedded in $\mathbb{R}^3$, then the results in [12] imply M has one or two limit ends, which are the top and/or bottom ends in the ordering of the ends of M. On the other hand, if M has one or two limit ends, then these limit ends are simple limit ends, and so, these limit ends have proper representatives by Theorem 1.6. The remaining finite number of ends of M are then annuli, each of which is proper (see Theorem 3.3 in Section 4). Hence, M is proper, which proves item 2 in Theorem 1.3.

Concerning item 3, suppose now that M has a countable number of limit ends. A result proven in pages 288, 289 of [24] states that the space of ends of M embeds topologically as a totally disconnected, closed subset A of the closed unit interval $I = [0, 1]$. Since the set of limit points L_A of A is a closed countable subspace of the metric space I (and hence L_A is complete), Baire's theorem implies that L_A contains a countable dense set of isolated points (see Lemma 5.1 below). In particular, if L_A has at least three points, then M has at least three simple limit ends. Since M cannot have more than two simple limit ends by item 1 of Theorem 1.3, then L_A consists of one or two points, and so M has one or two limit ends, each of which is a simple limit end. Hence, part 3-A of Theorem 1.3 holds. As M has at most two limit ends, then item 2 of Theorem 1.3 implies that M is properly embedded in $\mathbb{R}^3$, which is part 3-B.

If M has exactly two limit ends, then these limit ends are simple. By the proof of item 1 of Theorem 1.3, we deduce that after a rotation of M, these simple limit ends have representatives

E_1, E_2, where E_1 is a top limit end of M and E_2 is a bottom limit end of M. By item 1 of Theorem 1.6, the annular ends of E_1 have nonpositive logarithmic growths and the annular ends of E_2 have nonnegative logarithmic growths. Thus, the embeddedness of M implies that all the annular ends of M must have zero logarithmic growth, which means that they are planar, and item 3-C is proved.

Now assume $\partial M = \emptyset$. Since M has finite genus, then the main result in [31] insures that M has two limit ends and is recurrent for Brownian motion, which is part 3-D.

We finally prove item 3-E of Theorem 1.3. Assume $\partial M \neq \emptyset$. If the annular ends of M are horizontal and planar, there exists a horizontal plane P that intersects M transversely in a finite number of simple closed curves, and P can be chosen to lie above ∂M. Hence, the closure Σ of each component of $M - P$ is a properly embedded minimal surface with compact boundary and Σ is contained in a closed halfspace of $\mathbb{R}^3$. Theorem 3.1 in [12] implies that such a Σ is a parabolic surface with boundary. Since there are a finite number of such closed components Σ, and the union of these components along related compact boundary components is M, we conclude that ∂M has full harmonic measure, and so item 3-E holds provided that all of the annular ends of M are horizontal and planar.

If there exists an annular end with nonzero (say negative) logarithmic growth, then this end is asymptotic to the end of a negative half catenoid, and so, there exists a horizontal plane P whose intersection with this catenoidal end is an almost circle, and the end has a representative E with $\partial E \subset P$ such that E is graphical over the outside of the open planar disk $D \subset P$ whose boundary is ∂E. We may assume that P is at height zero. The complement of the topological plane $E \cup D$ in M consists of several components, each one with nonempty boundary contained in $M \cap D$. Since M is proper, then $M \cap D$ is compact. In particular, $M - (E \cup D)$ has a finite number of components. Let $\Sigma_1, \ldots, \Sigma_k$ be the components of $M - (E \cup D)$ which lie below $E \cup D$. For each $i = 1, \ldots, k$, the surface with boundary Σ_i is parabolic, since its third coordinate function is a proper negative harmonic function. By items 3-A and 3-C, the surface M has exactly one limit end. Since a limit end of a properly embedded minimal surface in $\mathbb{R}^3$ cannot lie below a catenoidal end of negative logarithmic growth (see Lemma 3.6 in [12]), then the limit end of M has a representative of genus zero E_T which lies above $E \cup D$. In particular, the limit end of M is its top end. By item 6 of Theorem 1.6, the representative E_T is parabolic. Let Ω be the closure of one of the (finitely many) components of $M - (E_T \cup \Sigma_1 \cup \ldots \cup \Sigma_k)$. Since Ω has a finite number of ends, each of which is asymptotic to an end of a plane or half catenoid, then Ω has quadratic area growth. Therefore, Ω is also a parabolic surface with boundary. As M is a finite union of parabolic surfaces with boundary along their common compact boundaries, we deduce that M has full harmonic measure on its boundary. This finishes the proof of Theorem 1.3. $\square$

For the sake of completeness, we prove the following elementary fact which was needed in the above proof.

Lemma 5.1 *Suppose X is a complete countable metric space, $L(X) \subset X$ is the set of limit points of X and $S(X) = X - L(X)$ is the open set of nonlimit points of X. Then:*

1. *$S(X)$ is dense in X.*

2. *$L(X)$ is a complete countable metric space, and so, its set $S(L(X))$ of isolated points is dense in $L(X)$.*

Proof. Let $L(X) = \{p_1, ..., p_n, ...\}$ be a listing, possibly finite or empty, of the set of limit points of X. If $L(X) = \emptyset$, then $\overline{S(X)} = S(X) = X$, and so, item 1 holds. Otherwise, consider the subsets $X_n = X - \{p_1, ..., p_n\}$ and note that each X_n is an open dense subset of X. The intersection of this countable collection of sets is equal to $S(X)$ and must be dense in X by Baire's theorem. Hence, $S(X)$ is dense in X, which proves item 1 in the lemma.

Since $S(X)$ is an open set and X is a complete countable metric space, then $L(X) = X - S(X)$ is a closed countable set which is complete in the induced metric. Hence, by item 1, $S(L(X))$ is dense in $L(X)$. $\square$

6 The proof of Corollary 1.8.

This last section is devoted to the following result, which has Corollary 1.8 stated in the Introduction as a special case.

Corollary 6.1 *Suppose that $M \subset \mathbb{R}^3$ is a connected properly embedded minimal surface with compact boundary and a limit end of genus zero. Then M is recurrent for Brownian motion if its boundary is empty, and otherwise its boundary has full harmonic measure.*

Proof. Suppose for the moment that the corollary holds when the surface M has nonempty boundary. Then, in the special case that the boundary of M is empty, consider a compact disk $D \subset M$ and note that $M - \text{Int}(D)$ has full harmonic measure by assumption, which implies that M is recurrent for Brownian motion. Thus, it suffices to prove the corollary in the special case that M has nonempty boundary.

Assume now that $\partial M \neq \emptyset$. Let $\widehat{E} \subset M$ be an end representative of a limit end of M of genus zero. Since $\widehat{E}$ is proper in $\mathbb{R}^3$ with compact boundary, then item 2 of Theorem 1.3 implies that $\widehat{E}$ has one or two simple limit ends. Let $E \subset \widehat{E}$ be an end representative of a simple limit end of M of genus zero. After a fixed rotation of M and a replacement of E by a subend representative of its limit end, we may assume that E satisfies the conclusions of Theorem 1.6, and $\partial M \subset \{x_3 < 0\}$.

We claim that there exist a pair of horizontal open disks $D_1, D_2 \subset \mathbb{R}^3 - E$ with the following properties.

(N1) $\partial D_i \subset E$, $i = 1, 2$, and $0 \leq x_3(D_1) < x_3(D_2)$.

(N2) $D_1 \cap E = \emptyset$ and if we denote by X_1 the closure of the mean convex region of $\mathbb{R}^3 - (E \cup D_1)$, then $D_2 \subset \mathbb{R}^3 - X_1$. In particular, $D_2 \cap E = \emptyset$.

(N3) Define X_2 as the closure of the mean convex region of $\mathbb{R}^3 - (E \cup D_2)$. Then, $M - E$ is disjoint from $X_1 \cup X_2$. In particular, $M - E$ is contained in the halfspace $\{x_3 \leq x_3(D_2)\}$.

To prove the claim and following the discussion in Sections 4.2 and 4.3, we will explain how to construct the disks D_1, D_2 in each of the cases given by (G1) with M_∞ being a Riemann minimal example with horizontal limit tangent plane at infinity, or M_∞ being a vertical catenoid. In the first case, we simply take D_1, D_2 as the horizontal disks bounded by almost-circles $c_0(\varepsilon), c_1(\varepsilon)$ contained in the boundary of a piece $\mathcal{R}_0(\varepsilon) \subset E$ as defined in the paragraph just before Assertion 4.9. In the case (G1) with M_∞ being a vertical catenoid, we take D_1, D_2 as the convex horizontal disks bounded by α_n and β_n, respectively (here we are using the notation in (12)). Properties (N1), (N2) hold from item 4 of Theorem 1.6. Concerning item (N3), if $M - E$ intersects X_1 then one can find a contradiction by adapting the arguments in paragraphs four and five of the proof of Theorem 1.3. Hence $M - E$ is disjoint from X_1 and similarly, $M - E$ is disjoint from X_2.

As $M - \text{Int}(E)$ is contained in a closed halfspace by item (N3) and $M - \text{Int}(E)$ is proper, then $M - \text{Int}(E)$ is a parabolic surface with compact boundary by Theorem 3.1 in [12]. By item 6 in Theorem 1.6, E is also a parabolic surface with compact boundary. Therefore, $M = (M - \text{Int}(E)) \cup E$ is a parabolic surface with compact boundary, i.e., ∂M has full harmonic measure. □

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