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Lie Derivatives of the Shape Operator of a Real Hypersurface in a Complex Projective Space

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Abstract. We consider real hypersurfaces M in complex projective space equipped with both the Levi-Civita and generalized Tanaka–Webster connections. Associated with the generalized Tanaka–Webster connection we can define a differential operator of first order. For any nonnull real number k and any symmetric tensor field of type (1,1) B on M, we can define a tensor field of type (1,2) on M, $B_T^{(k)}$, related to Lie derivative and such a differential operator. We study symmetry and skew symmetry of the tensor $A_T^{(k)}$ associated with the shape operator A of M.

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Keywords. kth g-Tanaka–Webster connection, complex projective space, real hypersurface, shape operator, Lie derivatives.

1. Introduction

We will denote by $\mathbb{C}P^m$, $m \geq 2$, the complex projective space equipped with the Kählerian structure (J, g), J being the complex structure and g the Fubini-Study metric with constant holomorphic sectional curvature 4. Take a connected real hypersurface without boundary M in $\mathbb{C}P^m$ whose local normal unit vector field is N. Take $\xi = -JN$. Then ξ is a tangent vector field to M that we call the Reeb vector field (or the structure vector field) on M. For any tangent vector field X on M, we write $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX and $\eta(X) = g(X, \xi)$. Then (ϕ, ξ, η, g) defines on M an almost contact metric structure [1], where g is the induced metric on M.

Takagi, see Refs. [5,8–10], classified homogeneous real hypersurfaces of $\mathbb{C}P^m$ into six types. All of them are Hopf, that is, their structure vector fields are principal ($A\xi = \alpha\xi$, for a function α on M). Denote by $\mathbb{D}$ the maximal holomorphic distribution on M: at any point $p \in M$, $\mathbb{D}_p = \{X \in T_pM | g(X, \xi_p) = 0\}$. Kimura [5] proved that any Hopf real hypersurface M in $\mathbb{C}P^m$ whose principal curvatures are constant belongs to Takagi's list.

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The unique real hypersurfaces in $\mathbb{C}P^m$ with two distinct principal curvatures are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, see Ref. [2]. Their principal curvatures are $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $cot(r)$ with eigenspace $\mathbb{D}$.

The canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold was defined, independently, by Tanaka [11], and Webster [13], and it is known as the Tanaka–Webster connection. For contact metric manifolds, Tanno [12] introduced a generalized Tanaka–Webster connection.

For a real hypersurface M of $\mathbb{C}P^m$ and any nonnull real number k, Cho, see [3,4], generalized Tanno's definition to the concept of kth generalized Tanaka–Webster connection by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any X, Y tangent to M. Then the four elements of the almost contact metric structure on M are parallel for this connection and if the shape operator of the real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the real hypersurface is contact and the kth generalized Tanaka–Webster connection coincides with the Tanaka–Webster connection.

We define the kth Cho operator on M associated with the tangent vector field X by $F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any Y tangent to M. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M. We also define the kth torsion operator associated with the tangent vector field X by $T^{(k)}_X Y = T^{(k)}(X, Y)$ for any Y tangent to M.

Let $\mathcal{L}$ denote the Lie derivative on M. Then $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M. On M we can also define a differential operator of first order associated with the kth generalized Tanaka–Webster connection by $\mathcal{L}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X = \mathcal{L}_X Y + T^{(k)}_X Y$, for any X, Y tangent to M.

Let now B be a symmetric tensor of type (1,1) defined on M. We can associate with B a tensor field of type (1,2) $B^{(k)}_T$ by $B^{(k)}_T(X, Y) = [T^{(k)}_X, B]Y = T^{(k)}_X BY - BT^{(k)}_X Y$, for any X, Y tangent to M.

Consider the condition $\mathcal{L}^{(k)} B = \mathcal{L}B$ for some nonnull real number k. This means that for any X, Y tangent to M $(\mathcal{L}^{(k)}_X B)Y = (\mathcal{L}_X B)Y$. This is equivalent to having $B^{(k)}_T = 0$.

Generalizing this we can consider that the tensor $B^{(k)}_T$ is symmetric, that is, $B^{(k)}_T(X, Y) = B^{(k)}_T(Y, X)$ for any X, Y tangent to M. This is equivalent to have the following Codazzi-type condition

$$\left(\left(\mathcal{L}^{(k)}_X - \mathcal{L}_X\right) B\right) Y = \left(\left(\mathcal{L}^{(k)}_Y - \mathcal{L}_Y\right) B\right) X \tag{1.2}$$

for any X, Y tangent to M.

On the other hand, we can suppose that $B^{(k)}_T$ is skew symmetric, that is, $B^{(k)}_T(X, Y) = -B^{(k)}_T(Y, X)$, for any X, Y tangent to M. This is equivalent to the following Killing-type condition:

$$\left(\left(\mathcal{L}_X^{(k)} - \mathcal{L}_X\right) B\right) Y + \left(\left(\mathcal{L}_Y^{(k)} - \mathcal{L}_Y\right) B\right) X = 0 \tag{1.3}$$

for any X, Y tangent to M.

In the particular case of $B = A$, the shape operator of M, in Ref. [7] the first author proved non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, satisfying $\mathcal{L}^{(k)}A = \mathcal{L}A$, that is, $A_T^{(k)} = 0$, for any nonnull real number k.

The purpose of the present paper is to study real hypersurfaces M in $\mathbb{C}P^m$ such that the shape operator satisfies either (1.2) or (1.3). In fact, we will obtain the following.

Theorem 1. *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that, for some nonnull real number k, $A_T^{(k)}$ is symmetric.*

In the case of $A_T^{(k)}$ being skew symmetric, we have a very different situation given by the

Theorem 2. *Let M be a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then the tensor field $A_T^{(k)}$ is skew symmetric if and only if M is locally congruent to a geodesic hypersphere of radius r, $0 < r < \frac{\pi}{2}$, such that $cot(r) = k$.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kählerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M, see Ref. [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any vectors X, Y tangent to M. From (2.1), we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J, we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equation of Codazzi is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.5}$$

for any tangent vector fields X, Y to M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}_p = \{X \in$

$T_pM|g(X,\xi_p)=0\}$. We will say that M is Hopf if ξ is principal, that is, $A\xi=\alpha\xi$ for a certain function α on M.

In the sequel, we need the following result, which consists of a combination of the Lemmas 2.1, 2.2 and 2.4 in Ref. [6].

Theorem 2.1. *If ξ is a principal curvature vector with corresponding principal curvature α, this is locally constant and if $X\in\mathbb{D}$ is principal with principal curvature λ, then $2\lambda-\alpha\neq 0$ and ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3. Proof of Theorem 1

If M satisfies (1.2) for $B=A$, we get $\mathcal{L}_X^{(k)}AY-\mathcal{L}_XAY-A\mathcal{L}_X^{(k)}Y+A\mathcal{L}_XY=\mathcal{L}_Y^{(k)}AX-\mathcal{L}_YAX-A\mathcal{L}_Y^{(k)}X+A\mathcal{L}_YX$ for any X,Y tangent to M. Therefore, we have $F_X^{(k)}AY-F_{AY}^{(k)}X-2AF_X^{(k)}Y+2AF_Y^{(k)}X=F_Y^{(k)}AX-F_{AX}^{(k)}Y$, for any X,Y tangent to M. This yields

$$\begin{aligned}
&2g(\phi AX,AY)\xi-\eta(AY)\phi AX-k\eta(X)\phi AY-g(\phi A^2Y,X)\xi\\
&\quad+\eta(X)\phi A^2Y+k\eta(AY)\phi X\\
&\quad-2g(\phi AX,Y)A\xi+2\eta(Y)A\phi AX+2k\eta(X)A\phi Y+2g(\phi AY,X)A\xi\\
&\quad-2\eta(X)A\phi AY-2k\eta(Y)A\phi X\\
&=-\eta(AX)\phi AY-k\eta(Y)\phi AX-g(\phi A^2X,Y)\xi\\
&\quad+\eta(Y)\phi A^2X+k\eta(AX)\phi Y
\end{aligned}\tag{3.1}$$

for any X,Y tangent to M. If we suppose that $X,Y\in\mathbb{D}$, (3.1) becomes

$$\begin{aligned}
&2g(\phi AX,AY)\xi-\eta(AY)\phi AX-g(\phi A^2Y,X)\xi+k\eta(AY)\phi X\\
&\quad-2g(\phi AX,Y)A\xi+2g(\phi AY,X)A\xi\\
&=-\eta(AX)\phi AY-g(\phi A^2X,Y)\xi+k\eta(AX)\phi Y
\end{aligned}\tag{3.2}$$

for any $X,Y\in\mathbb{D}$. If M is Hopf, that is $A\xi=\alpha\xi$, then (3.2) gives $2g(\phi AX,AY)\xi-g(\phi A^2Y,X)\xi-2\alpha g(\phi AX,Y)\xi+2\alpha g(\phi AY,X)\xi=-g(\phi A^2X,Y)\xi$ for any $X,Y\in\mathbb{D}$. This yields $2A\phi AX+A^2\phi X-2\alpha\phi AX-2\alpha A\phi X=-\phi A^2X$ for any $X\in\mathbb{D}$. Let us suppose that $X\in\mathbb{D}$ satisfies $AX=\lambda X$. From Theorem 2.1, we have $A\phi X=\mu\phi X$ with $\mu=\frac{\alpha\lambda+2}{2\lambda-\alpha}$. Therefore, we obtain $2\lambda\mu+\mu^2-2\alpha(\lambda+\mu)+\lambda^2=0$. That is, $(\lambda+\mu)(\lambda+\mu-2\alpha)=0$.

If $\lambda+\mu=0$, as $\mu=\frac{\alpha\lambda+2}{2\lambda-\alpha}$, we obtain $\frac{2\lambda^2+2}{2\lambda-\alpha}=0$. This yields $\lambda^2+1=0$, which is impossible.

If $\lambda+\mu=2\alpha$, we should have $\lambda^2-2\alpha\lambda+\alpha^2+1=0$. This gives $\lambda=\alpha\pm\sqrt{-1}$, which is also impossible. Thus, our real hypersurface must be non-Hopf. This means that ξ is not principal. Therefore, we can write $A\xi=\alpha\xi+\beta U$ at least on a neighborhood of a point of M, where U is a unit vector field in $\mathbb{D}$ and β a nonvanishing function on such a neighborhood. From now on, we will denote $\mathbb{D}_U=\{X\in\mathbb{D}|g(X,U)=g(X,\phi U)=0\}$ and make the calculations on that neighborhood. Then (3.2) becomes

$$2g(\phi AX,AY)\xi-\beta g(Y,U)\phi AX-g(\phi A^2Y,X)\xi+k\beta g(Y,U)\phi X$$

$$-2g(\phi AX,Y)A\xi + 2g(\phi AY,X)A\xi$$
$$= -\beta g(X,U)\phi AY - g(\phi A^2X,Y)\xi + k\beta g(X,U)\phi Y \tag{3.3}$$

for any $X, Y \in \mathbb{D}$. The scalar product of (3.3) and ϕU yields $-\beta g(Y,U)g(AX, U) + k\beta g(Y,U)g(X,U) = -\beta g(X,U)g(AY,U) + k\beta g(X,U)g(Y,U)$, for any $X, Y \in \mathbb{D}$. As $\beta \neq 0$, we obtain

$$g(Y,U)g(AX,U) = g(X,U)g(AY,U) \tag{3.4}$$

for any $X, Y \in \mathbb{D}$. If in (3.4) we take $X = U$, $Y \in \mathbb{D}_U$ we obtain $g(AU,Y) = 0$ for any $Y \in \mathbb{D}_U$, and if we take $X = U$, $Y = \phi U$ we get $g(AU,\phi U) = 0$. Therefore, we have

$$AU = \beta\xi + \gamma U \tag{3.5}$$

for a certain function γ.

The scalar product of (3.3) and U yields $-\beta g(Y,U)g(\phi AX,U) + k\beta g (Y,U)g(\phi X,U) - 2\beta g(\phi AX,Y) + 2\beta g(\phi AY,X) = -\beta g(X,U)g(\phi AY,U) + k\beta g(X,U)g(\phi Y,U)$, for any $X, Y \in \mathbb{D}$. If $Y = U$ it follows $-\beta g(\phi AX,U) + k\beta g(\phi X,U) - 2\beta g(\phi AX,U) + 2\beta g(\phi AU,X) = 0$ for any $X \in \mathbb{D}$. That is, $3g(A\phi U,X) - kg(\phi U,X) + 2\gamma g(\phi U,X) = 0$. Therefore,

$$A\phi U = \frac{k-2\gamma}{3}\phi U. \tag{3.6}$$

Take now $X = \xi$, $Y \in \mathbb{D}$ in (3.1). We obtain

$$2\beta g(A\phi U,Y)\xi - \beta\eta(AY)\phi U - k\phi AY + \phi A^2Y - 2\beta g(\phi U,Y)A\xi + 2kA\phi Y$$
$$-2A\phi AY = -\alpha\phi AY + \alpha\beta g(U,\phi Y)\xi + \beta g(AU,\phi Y)\xi + k\alpha\phi Y \tag{3.7}$$

for any $Y \in \mathbb{D}$. Its scalar product with ξ gives, being $\beta \neq 0$, $2g(A\phi U,Y) - 2\alpha g(\phi U,Y) + 2kg(\phi Y,U) - 2g(\phi AY,U) = \alpha g(U,\phi Y) + \gamma g(U,\phi Y)$, for any $Y \in \mathbb{D}$. Therefore, we have $4A\phi U = (\alpha + 2k - \gamma)\phi U$, which is equivalent to

$$A\phi U = \frac{\alpha+2k-\gamma}{4}\phi U. \tag{3.8}$$

From (3.6) and (3.8), it follows

$$3\alpha + 2k + 5\gamma = 0. \tag{3.9}$$

If we take $Y = U$ in (3.7), it follows $-\beta^2\phi U - k\phi AU + \phi A^2U + 2kA\phi U - 2A\phi AU = -\alpha\phi AU + k\alpha\phi U$. That is, $-\beta^2 - k\gamma + \beta^2 + \gamma^2 + 2k\gamma' - 2\gamma\gamma' = -\alpha\gamma + k\alpha$, where $\gamma' = \frac{\alpha+2k-\gamma}{4}$. This yields $\gamma^2 - (2\gamma' - \alpha + k)\gamma + k(2\gamma' - \alpha) = 0$. Therefore, $\gamma = \frac{2\gamma'-\alpha+k\pm\sqrt{(2\gamma'-\alpha-k)^2}}{2}$. From this, either $\gamma = k$ or $\gamma = 2\gamma' - \alpha$.

Suppose now $X, Y \in \mathbb{D}_U$. Then (3.2) yields $2g(\phi AX, AY)\xi - g(\phi A^2Y,X) \xi - 2g(\phi AX,Y)A\xi + 2g(\phi AY,X)A\xi = -g(\phi A^2Y,X)\xi$. Its scalar product with U gives $-2\beta g(\phi AX,Y) + 2\beta g(\phi AY,X) = 0$. Then $g((\phi A + A\phi)X,Y) = 0$ for any $X, Y \in \mathbb{D}_U$. This implies that $(\phi A + A\phi)X = 0$ for any $X \in \mathbb{D}_U$. If we suppose that $AX = \lambda X$ we obtain that $A\phi X = -\lambda\phi X$. If we take such an X in (3.7) we get $-k\phi AX + \phi A^2X + 2kA\phi X - 2A\phi AX = -\alpha\phi AX + k\alpha\phi X$. Therefore, $-k\lambda + \lambda^2 - 2k\lambda + 2\lambda^2 = -\alpha\lambda + k\alpha$. This yields $3\lambda^2 - (3k - \alpha)\lambda - k\alpha = (\lambda - k)(\lambda + \frac{\alpha}{3}) = 0$. Thus, either $\lambda = k$ or $\lambda = -\frac{\alpha}{3}$.

From (3.6) and (3.9) if $\gamma = k$, a constant, $\gamma' = -\frac{k}{3}$ is constant and $\alpha = -\frac{7}{3}$ is also constant. Furthermore, all principal curvatures on $\mathbb{D}_U$ are also constant.

If $\gamma = 2\gamma' - \alpha = \frac{2k-4\gamma}{3} - \alpha$, we obtain $3\gamma = 2k - 4\gamma - 3\alpha$. Then $2k - 7\gamma - 3\alpha = 0$ and from (3.9) $4k - 2\gamma = 0$. This yields $\gamma = 2k$ is constant, $\alpha = -4k$ and $\gamma' = -k$ are also constant. As above, all principal curvatures in $\mathbb{D}_U$ are constant.

Take a unit $X \in \mathbb{D}_U$ such that $AX = \lambda X$. The Codazzi equation gives $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. That is, $\nabla_X(\alpha\xi + \beta U) - A\phi AX - \nabla_\xi(\lambda X) + A\nabla_\xi X = -\phi X$. Therefore, $\alpha\lambda\phi X + X(\beta)U + \beta\nabla_X U + \lambda^2\phi X - \lambda\nabla_\xi X + A\nabla_\xi X = -\phi X$. If we take ϕX instead of X, we have similarly $\alpha\lambda X + (\phi X)(\beta)U + \beta\nabla_{\phi X}U - \lambda^2 X + \lambda\nabla_\xi\phi X + A\nabla_\xi\phi X = X$. In both cases, taking the scalar product with ξ, we have

$$g(\nabla_\xi X, U) = g(\nabla_\xi \phi X, U) = 0. \tag{3.10}$$

The scalar product with U of the expression for X yields $X(\beta) - \lambda g(\nabla_\xi X, U) + g(\nabla_\xi X, \beta\xi + \gamma U) = 0$ In the case of ϕX we obtain $(\phi X)(\beta) + \lambda g(\nabla_\xi \phi X, U) + g(\nabla_\xi \phi X, \beta\xi + \gamma U) = 0$. Bearing in mind (3.10), we conclude that

$$Z(\beta) = 0 \tag{3.11}$$

for any $Z \in \mathbb{D}$.

On the other hand, we have $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$. That is, $\nabla_{\phi U}(\alpha\xi + \beta U) - A\phi A\phi U - \nabla_\xi(\gamma'\phi U) + A\nabla_\xi\phi U = U$. This implies

$$\alpha\phi A\phi U + \beta\nabla_{\phi U}U + (\phi U)(\beta)U + \gamma' AU - \gamma'\nabla_\xi\phi U + A\nabla_\xi\phi U = U. \tag{3.12}$$

Its scalar product with ξ gives $\beta g(A\phi U, \phi U) + \beta\gamma' + \gamma' g(A\xi, U) + g(\nabla_\xi\phi U, \alpha\xi + \beta U) = 0$. That is, $3\beta\gamma' - \alpha\beta + \beta g(\nabla_\xi\phi U, U) = 0$. Therefore,

$$g(\nabla_\xi\phi U, U) = -3\gamma' + \alpha. \tag{3.13}$$

The scalar product of (3.12) with U gives $-\alpha g(A\phi U, \phi U) + (\phi U)(\beta) + \gamma\gamma' - \gamma' g(\nabla_\xi\phi U, U) + g(\nabla_\xi\phi U, \beta\xi + \gamma U) = 1$. From (3.13) this yields $-\alpha\gamma' + (\phi U)(\beta) + \gamma\gamma' - \gamma'(-3\gamma' + \alpha) - \beta^2 - 3\gamma\gamma' + \alpha\gamma = 1$. Thus, we obtain

$$(\phi U)(\beta) = 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 + \beta^2 - \alpha\gamma. \tag{3.14}$$

Now $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$ implies

$$\alpha\phi AU + U(\beta)U + \beta\nabla_U U - \gamma\gamma'\phi U - \xi(\beta)\xi - \beta\phi A\xi - \gamma\nabla_\xi U + A\nabla_\xi U = -\phi U. \tag{3.15}$$

Its scalar product with ξ gives $-\beta g(U, \phi AU) - \xi(\beta) + \gamma g(U, \phi A\xi) + g(\nabla_\xi U, \alpha\xi + \beta U) = 0$. From this we have

$$\xi(\beta) = 0. \tag{3.16}$$

The scalar product of (3.15) with U yields $U(\beta) + g(\nabla_\xi U, \beta\xi + \gamma U) = 0$. This gives

$$U(\beta) = 0. \tag{3.17}$$

From (3.11), (3.14), (3.16) and (3.17) we obtain

$$grad(\beta) = (\beta^2 + 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma)\phi U. \tag{3.18}$$

We will call $\omega = \beta^2 + 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma$. We know that $g(\nabla_X grad(\beta), Y) = g(\nabla_Y grad(\beta), X)$ for any X, Y tangent to M. In our case we have $X(\omega)g(\phi U, Y) + \omega g(\nabla_X \phi U, Y) = Y(\omega)g(\phi U, X) + \omega g(\nabla_Y \phi U, X)$. If we take $X = \xi$, from (3.16) and the fact that the all the elements different from β appearing in ω are constant, we have $\xi(\omega) = 0$. Thus, we get $-\omega g(U, AY) = \omega g(\nabla_\xi \phi U, Y)$ for any Y tangent to M. Taking now $Y = U$, bearing in mind (3.13) we arrive to $-\omega\gamma = \omega(-3\gamma' + \alpha)$. If we suppose $\omega \neq 0$ it follows $-\gamma = -3\gamma' + \alpha$. If $\gamma = k$, $\gamma' = -\frac{k}{3}$ and $\alpha = -\frac{7k}{3}$. Therefore, $-k = k - \frac{7k}{3}$ implies $k = 0$, which is impossible. In the other possible case $\gamma' = -k$, $\gamma = 2k$ and $\alpha = -4k$. Then $-2k = 3k - 4k$ gives also a contradiction.

Thus, we have proved that $\omega = 0$. Then $1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma + \beta^2 = 0$. If $\gamma = k$, $\gamma' = -\frac{k}{3}$ and $\alpha = -\frac{7k}{3}$. This yields $\beta^2 + 1 + \frac{4}{3}k^2 = 0$, which is impossible. Then $\gamma = 2k$, $\alpha = -4k$ and $\gamma' = -k$. Thus $\beta^2 + 9k^2 + 1 = 0$, also impossible, and we have finished the proof. □

4. Proof of Theorem 2

If M satisfies (1.3) for $B = A$ and any X, Y tangent to M we obtain

$$\begin{aligned} &-\eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2Y + k\eta(AY)\phi X \\ &-\eta(AX)\phi AY - k\eta(Y)\phi AX - g(\phi A^2X, Y)\xi + \eta(Y)\phi A^2X + k\eta(AX)\phi Y = 0. \end{aligned} \tag{4.1}$$

for any X, Y tangent to M. If $X, Y \in \mathbb{D}$ (4.1) becomes

$$\begin{aligned} -\eta(AY)\phi AX - g(\phi A^2Y, X)\xi + k\eta(AY)\phi X - \eta(AX)\phi AY \\ -g(\phi A^2X, Y)\xi + k\eta(AX)\phi Y = 0 \end{aligned} \tag{4.2}$$

for any $X, Y \in \mathbb{D}$. If in (4.1) we take $X = \xi$, $Y \in \mathbb{D}$, we obtain

$$-\eta(AY)\phi A\xi - k\phi AY + \phi A^2Y - \eta(A\xi)\phi AY - g(\phi A^2\xi, Y)\xi + k\eta(A\xi)\phi Y = 0 \tag{4.3}$$

for any $Y \in \mathbb{D}$.

Let us suppose that M is Hopf and $A\xi = \alpha\xi$. From (4.2) we obtain $-g(\phi A^2Y, X)\xi - g(\phi A^2X, Y)\xi = 0$ for any $X, Y \in \mathbb{D}$. Therefore, $A^2\phi X = \phi A^2X$ for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, we know that $A\phi X = \mu\phi X$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Thus, we have $\lambda^2 = \mu^2$ and either $\lambda = \mu$ or $\mu = -\lambda$.

If $\frac{\alpha\lambda+2}{2\lambda-\alpha} = -\lambda$ we obtain $\alpha\lambda + 2 = -2\lambda^2 + \lambda\alpha$. This implies $\lambda^2 + 1 = 0$, which is impossible. Therefore, $\lambda = \mu$. Taking such a Y in (4.3), we have $-k\phi AY + \phi A^2Y - \alpha\phi AY + k\alpha\phi Y = 0$. That is, $-k\lambda + \lambda^2 - \alpha\lambda + k\alpha = 0$. This gives $\lambda^2 - (\alpha + k)\lambda + k\alpha = (\lambda - k)(\lambda - \alpha) = 0$, and the possible solutions are either $\lambda = k$ or $\lambda = \alpha$. Then M has two distinct constant principal curvatures and from Ref. [2] M must be locally congruent to a geodesic hypersphere whose principal curvature on $\mathbb{D}$ is $cot(r) = k$.

If M is non-Hopf, as in the previous section, we write $A\xi = \alpha\xi + \beta U$, with the same conditions. From (4.2), it follows

$$\begin{aligned} -\beta g(U, Y)\phi AX - g(\phi A^2Y, X)\xi + k\beta g(U, Y)\phi X \\ -\beta g(U, X)\phi AY - g(\phi A^2X, Y)\xi + k\beta g(U, X)\phi Y = 0 \end{aligned} \tag{4.4}$$

for any $X, Y \in \mathbb{D}$. Its scalar product with U yields

$$\begin{aligned}\beta g(U,Y)g(A\phi U,X)-k\beta g(U,Y)g(\phi U,X)+\beta g(U,X)g(A\phi U,Y)\\ -k\beta g(U,X)g(\phi U,Y)=0\end{aligned}\tag{4.5}$$

for any $X,Y\in\mathbb{D}$. If in (4.5) we take $X\in\mathbb{D}_U$, $\beta g(U,Y)g(A\phi U,X)=0$. As $\beta\neq 0$, if $Y=U$ we obtain $g(A\phi U,X)=0$ for any $X\in\mathbb{D}_U$. If in (4.5) we take $X=Y=U$ we have $2\beta g(A\phi U,U)=0$. This implies $g(A\phi U,U)=0$. Finally, taking $Y=\phi U$ in (4.5) we get $\beta g(U,X)g(A\phi U,\phi U)-k\beta g(U,X)=0$ for any $X\in\mathbb{D}$. For $X=U$ we obtain $g(A\phi U,\phi U)=k$. Therefore, we have seen that

$$A\phi U=k\phi U.\tag{4.6}$$

The scalar product of (4.4) and ϕU gives $-\beta g(Y,U)g(AU,X)+k\beta g(U,Y)g(U,X)-\beta g(U,X)g(AU,Y)+k\beta g(U,X)g(U,Y)=0$ for any $X,Y\in\mathbb{D}$. Taking $X=Y=U$ we have $-2\beta g(AU,U)+2k\beta=0$ and then $g(AU,U)=k$. On the other hand, (4.3) yields $-\beta^2 g(Y,U)\phi U-k\phi AY+\phi A^2Y-\alpha\phi AY+\alpha g(A\xi,\phi Y)\xi+\beta g(AU,\phi Y)\xi+k\alpha\phi Y=0$ for any $Y\in\mathbb{D}$. Its scalar product with ξ implies

$$\alpha g(A\xi,\phi Y)+\beta g(AU,\phi Y)=0\tag{4.7}$$

for any $Y\in\mathbb{D}$. If $Y=\phi X$, $X\in\mathbb{D}_U$, we obtain $\beta g(AU,X)=0$ for any $X\in\mathbb{D}_U$ and if $Y=\phi U$ in (4.7) it follows $-\alpha\beta-\beta g(AU,U)=0$. Therefore, $g(AU,U)=-\alpha$ and we get

$$\alpha=-k\tag{4.8}$$

and

$$AU=\beta\xi+kU.\tag{4.9}$$

Let $X,Y\in\mathbb{D}_U$. From (4.4) we have $-g(\phi A^2Y,X)-g(\phi A^2X,Y)=0$. From (4.6) and (4.9) $\mathbb{D}_U$ is A-invariant and we obtain $\phi A^2X=A^2\phi X$ for any $X\in\mathbb{D}_U$. Let us suppose that $Y\in\mathbb{D}_U$ satisfies $AY=\lambda Y$. From (4.3) we get $-k\lambda\phi Y+\lambda^2\phi Y-\alpha\lambda\phi Y+k\alpha\phi Y=0$. From (4.8) it follows $\lambda^2\phi Y-k^2\phi Y=0$. Thus, $\lambda^2=k^2$ and λ is constant.

For such a $Y\in\mathbb{D}_U$ the Codazzi equation gives $(\nabla_YA)\xi-(\nabla_\xi A)Y=-\phi Y$. Therefore, $\nabla_Y(-k\xi+\beta U)-A\phi AY-\nabla_\xi(\lambda Y)+A\nabla_\xi Y=-\phi Y$. Then $-k\phi AY+Y(\beta)U+\beta\nabla_YU-A\phi AY-\lambda\nabla_\xi Y+A\nabla_\xi Y=-\phi Y$. Its scalar product with U yields $Y(\beta)-\lambda g(\nabla_\xi Y,U)+g(\nabla_\xi Y,\beta\xi+kU)=0$ and

$$Y(\beta)=(\lambda-k)g(\nabla_\xi Y,U).\tag{4.10}$$

On the other hand, $(\nabla_YA)U-(\nabla_UA)Y=0$. From this we obtain $\nabla_Y(\beta\xi+kU)-A\nabla_YU-\nabla_U(\lambda Y)+A\nabla_UY=0$. That is, $Y(\beta)\xi+\beta\phi AY+k\nabla_YU-A\nabla_YU-\lambda\nabla_UY+A\nabla_UY=0$. Its scalar product with ξ gives $Y(\beta)-kg(U,\phi AY)-g(\nabla_YU,\alpha\xi)+\lambda g(Y,\phi AU)+g(\nabla_UY,\alpha\xi+\beta U)=0$. Then

$$Y(\beta)=-\beta g(\nabla_UY,U).\tag{4.11}$$

Its scalar product with U implies $-g(\nabla_YU,\beta\xi)-\lambda g(\nabla_UY,U)+g(\nabla_UY,\beta\xi+kU)=0$. This yields $(\lambda-k)g(\nabla_YU,Y)=0$. If $g(\nabla_UY,U)=0$ from (4.11) we get $Y(\beta)=0$. If $g(\nabla_UY,U)\neq 0$, $\lambda=k$ and from (4.10) again

$$Y(\beta)=0\tag{4.12}$$

for any $Y\in\mathbb{D}_U$.

Moreover $(\nabla_UA)\xi-(\nabla_\xi A)U=-\phi U$ implies $\nabla_U(-k\xi+\beta U)-A\phi AU-\nabla_\xi(\beta\xi+kU)+A\nabla_\xi U=-\phi U$. Then $-k\phi AU+U(\beta)U+\beta\nabla_UU-A\phi AU-$

$\xi(\beta)\xi - \beta\phi A\xi - k\nabla_\xi U + A\nabla_\xi U = -\phi U$ and its scalar product with ξ gives $-\beta g(U, \phi AU) - \xi(\beta) + kg(U, \phi A\xi) + g(\nabla_\xi U, \alpha\xi + \beta U) = 0$. Therefore,

$$\xi(\beta) = 0 \tag{4.13}$$

and its scalar product with U yields $U(\beta) + g(\nabla_\xi U, \beta\xi) = 0$. That is

$$U(\beta) = 0. \tag{4.14}$$

Now we develop $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$. Then $\nabla_{\phi U}(-k\xi + \beta U) - A\phi A\phi U - \nabla_\xi(k\phi U) + A\nabla_\xi \phi U = U$ that implies $-k\phi A\phi U + (\phi U)(\beta)U + \beta\nabla_{\phi U}U - A\phi A\phi U - k\nabla_\xi\phi U + A\nabla_\xi\phi U = U$. Its scalar product with U yields

$$(\phi U)(\beta) = 1 + \beta^2 - 2k^2. \tag{4.15}$$

Its scalar product with ξ gives $\beta g(A\phi U, \phi U) + \beta g(A\phi U, \phi U) + kg(\phi U, \phi A\xi) + g(\nabla_\xi\phi U, -k\xi + \beta U) = 0$. Therefore,

$$g(\nabla_\xi\phi U, U) = -4k \tag{4.16}$$

and

$$grad(\beta) = \omega\phi U \tag{4.17}$$

where $\omega = 1 + \beta^2 - 2k^2$. Now, as in previous section, $g(\nabla_X(\omega\phi U), Y) = g(\nabla_Y(\omega\phi U), X)$ for any X, Y tangent to M. This yields $X(\omega)g(\phi U, Y) + \omega g(\nabla_X\phi U, Y) = Y(\omega)g(\phi U, X) + \omega g(\nabla_Y\phi U, X)$. If $X = \xi$ we get $\omega g(\nabla_\xi\phi U, Y) = \omega g(\nabla_Y\phi U, \xi) = -\omega g(\phi U, \phi AY) = -\omega g(U, AY)$. Take $Y = U$. Then $\omega g(\nabla_\xi\phi U, U) - k\omega$. This and (4.16) give $\omega = 0$ and, therefore, β is constant and equals $2k^2 - 1$.

Now $(\nabla_U A)\phi U - (\nabla_{\phi U}A)U = -2\xi$. Then $\nabla_U(k\phi U) - A\nabla_U\phi U - \nabla_{\phi U}(\beta\xi + kU) + A\nabla_{\phi U}U = -2\xi$, that is, $k\nabla_U\phi U - A\nabla_U\phi U - \beta\phi A\phi U - k\nabla_{\phi U}U + A\nabla_{\phi U}U = -2\xi$. If we take its scalar product with U we obtain $3k\beta = 0$, which is impossible and finishes the proof. □

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