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The impact of words in mathematics education – Case of symbols

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ABSTRACT

The development of standardized forms of symbolic notation is a long and complicated process and the selection of symbolic notation is frequently based on factors other than merit. As multiple representations are common in math education, we examined a different representation of the symbol in mathematics education. In a teaching experiment, we investigated whether participants benefit from words instead of standard symbols and whether the different representation affects performance. For the teaching experiment, students were divided into two groups: the "word" group, and the "sign" group. We utilized words instead of the formal symbol in the "word" group and examined student's understanding and mathematics reasoning. Results indicate that the multiple symbolic representations (word instead of sign) are helpful. The study also indicates that some math symbols may mislead students. According to sociocultural perspectives and linguistic approaches, our study illustrates that using words instead of formal symbols improves student's understanding and reasoning and indicates that the use of words can block common errors made by students. This research emphasizes the importance of various forms of multiple representations in mathematics learning.

Keywords: mathematics education; words; Multiple Representation; Sociocultural Perspective.

INTRODUCTION

As Presmeg mentioned (Presmeg et al, 2019), the study of signs includes a long and rich history. Many researchers who are interested in understanding the processes of learning and teaching mathematics (e.g., Anderson) have concentrated on semiotics. In Vygotsky's tradition, semiotics has been employed as a powerful research lens (e.g. Mariotti and Bartolini Bussi, 1998). According to semiotics, if the person chooses Peirce's theory, learning mathematics mostly begins with the interpretation of the use of visible signs. The researchers have frequently investigated the students' interpretation of signs. Paul, for example, argued that students often interpret the equals sign as an operator symbol, and this is particularly true regarding those who have difficulties dealing with mathematics (Powell, 2014). Priss, by presenting some examples, stated that "mathematical language can be imperfect, multipurpose, full of synonyms, and devoid of signs" (Priss, 2018). Sociocultural perspectives in mathematics education emphasize the social and cultural dimensions of symbols (Presmeg et al, 2018).Lerrman pointed out that:

"*Culture, language, and meaning precede us. We are born into a world already formed discursively. The reality or otherwise of the world or the certainty of our knowledge of it are not the issue: the issue is that we receive all knowledge of the world through language and other forms of communication. What things signify is learned by us as we grow into our cultures, the plurality arising from the multiple situations that constitute us: gender; class; ethnicity; color; religion, and so on*"(Lerman ,2001).

Undoubtedly, language is one of the important sociocultural features and the bridge to reality. Although many of the assumptions about language as a psychological marker are shared, the methods of studying language and word use have often been a battleground. Most narrative researchers assume that language is, by definition, contextual. Consequently, words, phrases, sentences, or entire texts must be considered within the context of the goals of the speaker and the relationship between the speaker and the audience(Pennebaker et al., 2003).In most cases, when two people interact they use words. If this means of interaction is distorted (for whatever reason), then the interaction is also distorted. Cajori pointed out that

"The development of standardized forms of symbolic notation was a long and complicated process, however, with the vast majority of symbols invented by mathematicians being discarded. Leibniz and Euler, for example,

are the only two mathematicians to invent more than two symbols which have been universally adopted in modern mathematics “(Cajori, 1993).

The selection of symbolic notation is frequently based on factors other than merit (OHalloran 2014). In this study, “left-hand limit”, “right-hand limit”, the symbols used for those concepts, and the students' perception of the symbol was investigated. We did not address the theoretical foundations of this subject; rather we examined an example according to the semiotic perspective, which, as well as affirming Priss's statements, suggests a proposal based on the socio-cultural standpoint and the linguistic approach. Although the structural dimension of semiotic systems is chiefly abstract and theoretical, the semiotic perspective of mathematical activity provides a method to conceptualize the teaching and learning of mathematics, which is mainly concentrated on signs and the use of them (Ernest, 2006). The example presented in this study demonstrated that the standard symbols used in textbooks might even mislead students in some cases. Considering that semiotics has been relevant to language for a long time (e.g., de Saussure, 1959), a symbol based on a linguistic approach was presented in this study. The mentioned suggestion could be an example that includes semiotic mediation (Vygotsky 1978), social semiotics, different representation theories (Goldin & Janvier, 1998), relationships between sign systems, and visualization theories including body movements as a means of signification (Radford, 2014).

THEORETICAL BACKGROUND

Mathematics language includes three semiotic components (linguistic, symbolic and visual resources) in the construction of mathematical meanings (OHalloran 2014) and concepts can be provided to students by different representational forms such as pictures or text. Studies on Multiple External Representations (MERs) (Mayer, 2005) and its impact on student’s learning have been considered in many researches related to mathematics education. Current theories of multimedia learning such as the Cognitive Theory of Multimedia Learning (CTML) (Mayer, 2010)and the Integrated Model of Text and Picture Comprehension (ITPC)(Schnotz and Bannert, 2003) provide substantiated explanations for the bene ts of certain types of MERs. The basic definition of MERs also encompasses combinations of representations. (Ott et al, 2018).

Theories of multimedia learning are usually explained based on assumptions regarding information processing and the construction of mental models. The CTML is based on the multi-store memory model and assumes that verbal (words) and non-verbal information (pictures) are processed in two different cognitive subsystems, resulting in two specific mental representations(Ott et al., 2018)&(Atkinson and Shiffrin, 1971) . These separate mental representations are integrated into a coherent mental model when appropriate prior knowledge is retrieved from long-term memory. The comprehension of the theory on registers of representation requires consideration of three key characteristics (Pino-Fan et al., 2015):

▪There is as many different semiotic representations of the same mathematical object, as semiotic registers utilized in mathematics.

▪Each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented; what is being explicitly stated is the content of the representation.

▪The content of semiotic representations must never be confused with the mathematical objects that these represent.

Characteristics of the theory on registers of representation can be manifested in the multimodal mathematics register. In this study we embrace OHallorans definition of multimodal (OHalloran,2014):

"The multimodal (or multisemiotic) makeup of mathematics means that three different meaning potentials are accessed to construct mathematical reality: namely, linguistic, symbolic and visual forms of representation, each of which have developed specific grammatical features to fulfill the functions they are required to serve."

Mathematics is a multimodal semiotic enterprise and semiotic representations depend on the organized system of signs such as language, numerical writing, symbolic writing, and Cartesian diagrams. The multimodal approach (linguistic, symbolic, and visual resources) has useful implications for student learning. Logical reasoning and the use of symbolic notation in geometry, arithmetic, algebra, and analysis evolve the mathematics. Mathematical symbolic notation developed as a central resource for the scientific method, expanding the experiential and logical realm beyond what was possible with natural language (OHalloran, 2014) but as Cajori pointed out, the choice of symbols was not always based on their efficiency. Other conditions were also involved in the selection of symbols.

THE RESEARCH QUESTION

The current research examines the relationship between the understanding of the students and the symbols. Specifically, we raised the question that does use words and phrases instead of formal math symbols improve students' understanding? We examined an example that is very challenging for high school students. We used the $x \to a^{\text{left}}$ and $x \to a^{\text{right}}$ symbols instead of $x \to a^{-}$ and $x \to a+$ symbols respectively. Then we addressed the following question: What is the effect of using the new symbol, which is based on the linguistic approach in the socio-cultural perspective, on the understanding of students regarding the limit concept?

METHOD

We used design-based research (Bakker & Eerde, 2015) to develop and carry out a classroom teaching experiment. The experiment involved three weeks of classroom instruction organized into an instructional sequence. The tasks designed for the teaching experiment were based on field-tested tasks developed in previous research by the project researchers. The same lessons were taught in all classrooms across grades K–11. In this teaching experiment, students were asked to answer three questions. We asked initially students to answer the questions in writing (in both group). Answers were coded and according to that, two students were selected for the interview.

Participants

Four classes participated in this study. Two classes were in Zanjan city (the capital of Zanjan province in Iran) and the two others were in a region of Zanjan province. The students of the latter classes were from neighboring rural regions. These four classes were divided into two groups:
"Sign" group: A class in Zanjan city and a class from the rural region (22 + 20)
"Word" group: A class in Zanjan city and a class from the rural region (25 + 24)
All four classes were 11^{th}-grade (17 years old) in the field of experimental sciences and the average score of students in 10^{th}-grade mathematics was 11.7 (out of 20).

Professional development of teacher

Two one-hour sessions were held with mathematics teachers during which we told them that our purpose is to investigate the effect of symbols, the "left-hand" and "right-hand" limits in particular, on the understanding of the students. We explained the design-based research method (Baker, 2015) according to the appendix and asked them to share the common "limit" mistakes of students with us. Teachers believed that since the answer to some of the limit questions can be determined using the plug and chug technique($\lim_{x \to a} f(x) = f(a)$), students generalize this techniques for most "limit" questions. Furthermore, teachers considered the lack of proper understanding of the left-hand and right-hand limits as one of the reasons regarding the mentioned issue. We decided to employ the "left" and "right" words in "Word" group instead of the standard symbols (- and +) considering "left-hand" and "right-hand" limits to examine whether the sign affects the understanding of the students. We utilize $x \to a^{left}$ and $x \to a^{right}$ symbols instead of $x \to a^{-}$ and $x \to a^{+}$.

Pre-interview and pre-test

Before starting the teaching experiment, some questions were provided for students in all classes to make sure they had the necessary knowledge to learn the concept of limit and cotangent. Asking questions was executed through a short interview. The questions were mainly about functions and their various types, especially piecewise functions such as absolute value function and the floor function. The questions also included the properties of the function, particularly increasing and decreasing function, properties of right triangle and the concept of infinity. The answers of the students indicated that they had great difficulty dealing with the piecewise functions, especially in terms of plotting such functions. The difficulties of the students regarding trigonometric functions were also highly considerable.

DATA COLLECTION AND ANALYSIS

The gathered data on which the results of this study are based include students' work, field notes, and voice records of class activities executed by teachers and researchers. An important part of the data was a set of short interviews with students. The approximate time of each interview was ten minutes, though it took more time in some cases it. Besides increasing our knowledge about the viewpoint of the students, such short interviews impacted the students' learning because they often corrected their viewpoints during the conversation. Nevertheless, it did not compromise the validity of the research (Brizuela et al, 2015), since the aim of the study was to realize how students discuss a symbol using a linguistic approach and a standard symbol. Once the data were selected, we began a line-by-line review of the transcripts (with periodic review of the voice records associated with the transcripts) and tagged any moments in the transcripts when references were made to conceptions. We tagged these moments using the constant comparative method (Glaser & Strauss, 1967) to identify key features in how students discussed and used conceptions. We used this approach to describe the range of understandings displayed by these students. Once these understandings had been listed, coded, and identified in the transcripts, we began to create a detailed narrative account that could help explain the contexts in which these understandings were manifested.

RESULTS

The results of the study are presented using tables and figures as in th following:

Table 1: student's response for question1

Response	"WORD" group $\lim_{x\to 3^{left}} \frac{1}{x-3} = ?$	"SIGN" group $\lim_{x\to 3^-} \frac{1}{x-3} = ?$
Correct response	55%	37%
Substituting x with "3" (incorrect response)	45%	46%
Substituting x with "-3" (incorrect response)	0%	17%

Table 2: student's response for question2

Response	"WORD" group $\lim_{x\to 3^{right}} \frac{1}{x-3} = ?$	"SIGN" group $\lim_{x\to 3^+} \frac{1}{x-3} = ?$
Correct response	55%	37%
Substituting x with "3" (incorrect response)	45%	63%

Table 3: student's response for question3

Response	"WORD" group $\lim_{x\to -1^{left}} \frac{[x]}{x} = ?$	"SIGN" group $\lim_{x\to -1^-} \frac{[x]}{x} = ?$
Correct response	33%	17%
Substituting x with "-1" (incorrect response)	67%	76%
Substituting x with "+1" (incorrect response)	0%	7%

Table 4: student's response for question4

Response	"WORD" group $\lim_{x\to -1^{right}} \frac{[x]}{x} = ?$	"SIGN" group $\lim_{x\to -1^+} \frac{[x]}{x} = ?$
Correct response	33%	17%
Inserting the -1 value instead of x	67%	83%

We mentioned those exercises again during the interviews and asked the students to explain their answers.

The interview with Reza

Interviewer: $\lim_{x\to 5^-} \frac{1}{x-5} = ?$

Reza: $\lim_{x\to 5^-} \frac{1}{x-5} = \frac{1}{-5-5} = \frac{1}{-10}$

Interviewer: How do you read the $x \to 5^-$ symbol?

Reza: X tends to the "negative five" (the number 5 which is negative)

Interviewer: Do you think that the "negative five" is the same as "-5"?

Reza: Yes.

Interviewer: 5^- have nothing to do with -5. $x \to 5^-$ means that x tends 5 from the left.

Reza: Ok.

Interviewer: Let me use another symbol for the next question. $\lim_{x\to 3^{left}} \frac{1}{x-3} = ?$

Reza: "When x tends to 3, x-3 tends to zero, and the fraction with the small denominator is the large fraction." (Reza made no reference to -3)

Interviewer: How large?

Reza: Apparently there should be no limitation.

Interviewer: Can x itself be 3?

Reza: Yes

Interviewer: What does the "left" word above the number 3 mean?

Reza: "It means that x is tending to 3 from the left."

Interviewer: Does tending different with equal?

Reza: "Oh, x isn't 3."

Interviewer: Does tending from the right or left make any difference?

Reza: Probably there should be a difference. (He tried to draw a thing like the table and interviewer helped Reza to construct a table and asked him complete the table 1).

Table 5:

x	2.9	2.99	2.999	….	$x \to 3^{\text{left}}$
$f(x) = \frac{x}{x-3}$	-29	-299	-2999	….	$f(x) \to -\infty$

The interview with Mohsen

Interviewer: $\lim_{x\to -1^-} \frac{[x]}{x} = ?$

Mohsen: $\lim_{x \to -1^-} \frac{[1]}{1} = \frac{1}{1} = 1$
Interviewer: why do you Substitute x with 1.
Mohsen: “Because of - ×- = +”
Interviewer: Do you mean -1×-1=1?
Mohsen: Yes.
Interviewer: $x \to -1^-$ means that x tends -1 from the left-hand.
Mohsen: “So why do we show it with a “- “sign?”
Interviewer: This is a contract. If you want I will show it as $x \to -1^{left}$.
Mohsen: “You mean, x is on the left -1”
Interviewer: Yes, and it tends to -1.
Mohsen: “So what number should I put instead of -1?”
Interviewer: x is variable and not fixed. x tends to -1.
Mohsen: “from left?”
Interviewer: Yes.
Mohsen: “-1.3?”
It seemed that using the “left” instead of “- “corrected Mohsen's idea (-×- = +).
Marina’s interview
Interviewer: $\lim_{x \to 5^{right}} \frac{1}{x-5} =?$
Marina: x ⟶ 5 ^{right} ⇒ x-5⟶ 0 ⇒ $\frac{1}{x-5}$ ⟶∞.
Interviewer: Does the word "right" refer to a concept?
Marina: “Yes, This indicates that x is approaching 5 from the right.”
Interviewer: Do you think it is different from approaching from the left?
Marina drew an axis of numbers and marked the number 5 on it and showed x on the right and one x on the left. She first substituted x with “4.5” then she substituted x with “5.5”.
Marina: “Yes, $\frac{1}{x-5}$ ⟶+∞”.

DISCUSSION AND CONCLUSION

The written answers of students and the answers they gave to the questions during the interviews indicated that the use of a symbol that conveys the concept or a part of it can positively affect the reasoning process of students. Some students attempted to answer the questions through substitution or by a computational operation (like simplifying algebraic expressions). They commonly did not mention the limit’s concept. However, the results illustrated that although some students could not properly calculate the value of the left-hand and right-hand limits, those who used the “left” and “right” symbols could employ more conceptual methods and understand the questions asked by interviewers. The students were able to mentally begin the visualization process by hearing the “left” or “right” terms (Reza's case, in which he mentioned that x was tending to 3). This study illustrated that when we use the left and right terms instead of the standard symbol, they create a sense perception, and we know now that sensory perception is one of the fundamental elements of cognition. As concepts are the main units for modeling thinking and language processing, the current research confirmed that language can also contribute to comprehend mathematical concepts. The students who used “left” and “right” symbols employed more correct arguments to answer left-hand and right-hand limit questions. When they used the “left” and “right” symbols, they could precisely visualize moving from left and moving from right, and their discussion was concentrated on the concept of tending to point (limit concept). The students totally understood the “left” symbol and were looking for something on the left side of the number 3 in the number axis and were able to have a better perception of the “limit” concept. They properly understood the meaning of tendency; however, it was difficult for them to relate that concept to a mathematical calculation. For answering the questions, they frequently drew “number axis”. Therefore, the interviewers could help them to discuss more accurately. Visualization of movement from the left and movement from the right led to the emergence of a perceptual evolution (Marina case). In contrast, when students used the”+” symbol, they confused it with “+3”, which caused them not to understand the concept of approaching from the right and approaching from the left. The same mistake appeared in the case of the “-“symbol with more clarity. Some students confused it with -3 and this never seen in word group (table1 and table3). Consequently, the concept of approaching from the left was ignored (Mohsen case).Mathematical diagrams, such as geometric figures, mathematical formulas, equations, and graphs, as epistemological materials help not only in the formulation and validation of conjectures but also in the creation of mathematical ideas. This study illustrated that words play a similar role (Ludlow, 2018). Perception without conception is simply blind, and conception without perception is simply absurd. A compelling finding from our study is illustrated that the word and vocabularies caused more discussion between students and teachers, and they could more easily express their ideas. Though their ideas could be wrong, and more discussion can be a worthwhile method to teach mathematics (Novaes, 2018). In many cases, the development of some symbols, their popularity day, and their eventual collapse creates an interesting story (Cajori, 2011), and it seems that continuing a part of this story is currently the responsibility of researchers in the mathematics teaching field. Our finding illustrate that using the "left" and "right" symbols instead of "-" and "+" blocks students from entering a detour. Also this result approved the linguistic approach in socio-cultural standpoints regarding semiotics (Ludlow, 2015). This study attempted to recall a problem in math classrooms and bring up the question that why is insisted on using a symbol in math textbook while students have a problem with it.

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APPENDIX

	General points	Examples
Introduction:	topic	The effect of words in mathematics education
	common problem	Students' problems in understanding limits.
	knowledge gap	How to use words to help students understand limits concepts
	mathematical learning goal	Understanding of limits
Literature review forms the basis for formulating the research aim (the research has to be anchored and relevant)		
Research aim:	Is aim descriptive explanatory, evaluative advisory etc.,	To examine the efficiency of words in teaching limits the (evaluative aim)
Research aim has to be narrowed down to a research question and possibly sub questions with the help)		
Literature review (theoretical background):	Orienting frameworks	Semiotics
	Frameworks for action	Cultural and social perspectives
	Domain-specific learning theories	Realistic Mathematics Education
With the help of theoretical constructs the research question(s) can be formulated (the formulation has to be precise)		
Research question:	What knowledge is required to achieve the research aim	Can the use of words be effective in understanding students?
It should be underpinned why this research question requires DBR (the method should be functional)		
Research approach:	The lack of the type of learning (it can be studied)	Formal symbols are unfamiliar to students in different areas and from different cultures, so it interferes with their learning.
Using a research method involves several research instruments and techniques		
Research instruments and techniques	Research instrument that connects different theories and concrete experiences in the form of testable hypotheses.	Series of hypothetical learning trajectories (HLTs)
	1. Identify students' prior knowledge	1. Prior interviews and pretest
	2. Professional development of teacher	2. Preparatory meetings with teacher
	3. Interview schemes and planning	3. Mini-interviews, observation scheme
	4. Intermediate feedback and reflection with teacher	4. Debrief sessions with teacher
	5. Determine learning yield	5. Posttest
Design	Design guidelines	Guided reinvention; Historical and didactical phenomenology
Data analysis	Hypotheses have to be tested by comparison of hypothetical and observed learning. Additional analyses may be necessary	
Results	Insights into patterns in learning and means of supporting such learning	Series of HLTs as progressive diagrammatic reasoning about growing samples
Discussion	Theoretical and practical yield	Concrete example of an historical and didactical phenomenology in mathematics education

		Application of semiotics in an educational domain
		Insights into computer use in the mathematics classroom
		Series of learning activities
		Improved computer tools
The aim, theory, question, method and results should be aligned (the research has to be **consistent**)		