Notes on Translating Solitons in Semi-Riemannian Manifolds

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Abstract

This is a summary of some works related to Translating Solitons in Semi-Riemannian Manifolds. We review known facts in Euclidean and Lorentz Spaces, and then introduce them in Semi-Riemannian Products. When they are graphs, we obtain the necessary PDE. In addition, if we consider the action of a Lie group by isometries, we reduce the PDE to an ODE. We focus on studying this ODE in an abstract way, from existence to extensions of solutions. Finally, we give some examples.

1 Introduction

A smooth submanifold either in the Euclidean Space $\mathbb{R}^n$ or in the Lorentzian Space $\mathbb{L}^n$ is called *translating soliton* when its mean curvature $\vec{H}$ satisfies the following equation:

$$\vec{H} = v^{\perp}, \tag{1}$$

for some constant unit vector $v \in \mathbb{R}^n$ or $v \in \mathbb{L}^n$. In the case of the Lorentz Space, it is assumed that v is not lightlike. In such case, if a submanifold $F : M \to \mathbb{R}^{n+1}$ satisfies this condition, then it is possible to define the forever flow $\Gamma : M \times [0,+\infty) \to \mathbb{R}^{n+1}$, $\Gamma(p,t) = \Gamma_t(p) = F(p) + tv$. Clearly,

$$\left(\frac{\partial}{\partial t}\Gamma_t\right)^{\perp} = v^{\perp} = \vec{H}.$$

This shows that, up to tangential diffeomorphisms, the submanifold is a *solution to the mean curvature flow.* This justifies our definition. In fact, it is a Type-II singularity, since it does not essentially change with time. Until now such solutions have been almost exclusively studied in the case where the ambient space is the Euclidean (or the Minkowski) space. For a good list of known examples, see [7]. Probably, the most famous examples are the Grim Reaper curve in $\mathbb{R}^2$ and the translating paraboloid and translating catenoid, [2]. Also, in [6], the authors studied those translating solitons in Minkowski 3-space with rotational symmetry.

If one wants to generalize (1), the simplest way is to choose a parallel vector field. Since manifolds admitting such a vector field are locally a product $M \times \mathbb{R}$, in [5], the authors introduce the notion of (graphical) translating solitons on a semi-Riemannian product $M \times \mathbb{R}$.

In this note, we recall some results from two papers, namely [4] and [5]. Firstly, in the Preliminaries Section we recall some basic facts that we will use later, where we find the first steps of translating solitons in product spaces $M \times \mathbb{R}$ with a product metric $g_M + \varepsilon dt^2$, with $\varepsilon = \pm 1$ and g_M the metric on M. Since the manifold M might not be complete, we can almost say that we are dealing with a *semi-Riemannian cohomogeneity of degree one* Σ*-manifold*, since there is a Lie group acting by isometries and the quotient is a 1-dimensional manifold (see [1].) In this setting, we can construct our translating solitons from the solutions to an ODE.

Section 3 is devoted to studing the already mentioned ODE, from two points of view. One of them is solving a boundary problem. Indeed, given $h \in C^1(a,b)$ such that $\lim_{s\to a} h(s) = +\infty$, and $\varepsilon, \tilde{\varepsilon} \in \{1,-1\}$, consider

$$w'(s) = (\tilde{\varepsilon} + \varepsilon w^2(s))(1 - w(s)h(s)), \quad w(a) = 0.$$

We show in Theorem 1 that there exists a solution under a not very restrictive condition on function h.

The solutions in Theorem 1 are just local, i. e., they are defined in a small interval $[a, a+\delta)$. Thus, the second point of view consist of the extension of our solutions. In this way, in Propositions 1 and 3 of Section 3, we show that for $\varepsilon\tilde{\varepsilon} = 1$ and $h > 0$ or $h < 0$, it is possible to extend the solution to the interval $[s_0, b)$, where $s_0 \in (a,b)$ is the chosen initial point. In Proposition

2, we show some reasonable conditions under which, the solutions defined on $[s_0, b)$ admit $\lim_{s \to b} w(s) \in \mathbb{R}$.

Section 4 is devoted to examples. On one hand, we exhibit translating solitons in $\mathbb{H}^n \times \mathbb{R}$ foliated by horospheres. Also, we make a study on the round sphere $\mathbb{S}^n \times \mathbb{R}$, where we obtain translating solitons defined on the whole sphere $\mathbb{S}^n$ but removing two points.

2 First Steps

The following results can be found in [5]. Assume that (M, g) is a connected semi-Riemannian manifold of dimension $n \geq 2$ and index $0 \leq \alpha \leq n-1$. Given $\varepsilon = \pm 1$, we construct the semi-Riemannian product $\widetilde{M} = M \times \mathbb{R}$ with metric $\langle , \rangle = g + \varepsilon dt^2$. The vector field $\partial_t \in \chi(\widetilde{M})$ is obviously Killing and unit, spacelike when $\varepsilon = +1$ and timelike when $\varepsilon = -1$. Now let $F : \Gamma \to \widetilde{M}$ be a submanifold with mean curvature vector $\vec{H}$. Denote by $\partial_t^{\perp}$ the normal component of ∂_t along F.

Definition A *With the previous notation, we will call F a (vertical) translating soliton of mean curvature flow, or simply, a translating soliton, if* $\vec{H} = \partial_t^{\perp}$.

In this paper, we will focus on graphical translating solitons. Namely, given $u \in C^2(M)$, we construct its graph map $F : M \to M \times \mathbb{R} =: \widetilde{M}$, $F(x) = (x, u(x))$. Let ν be the upward normal vector along F with $\varepsilon' = \mathrm{sign}(\langle \nu, \nu \rangle) = \pm 1$. Let us call ∇u the gradient of u.

Proposition 1 *In the above conditions, the corresponding partial differential equation that u must satisfy is*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{\varepsilon'(1 + \varepsilon |\nabla u|^2)}} \right) = \frac{1}{\sqrt{\varepsilon'(1 + \varepsilon |\nabla u|^2)}}.$$

Let Σ be a Lie group acting by isometries on M and $\pi : M \to I$ be a submersion, I and open interval, such that the fibers of π are orbits of the action. In addition, assume that π is a semi-Riemannian submersion with constant mean curvature fibers. For each $s \in I$, $\pi^{-1}\{s\} \cong \Sigma$ is a hypersurface with constant mean curvature. The value of the mean curvature of $\pi^{-1}\{s\}$ is denoted by $h(s)$. Then, we have a function $h : I \to \mathbb{R}$.

Theorem 1 *Let (M, g) be a connected semi-Riemannian manifold. Let Σ be a Lie group acting by isometries on M and $\pi : (M, g_M) \to (I, \widetilde{\varepsilon} ds^2)$*

be a semi-Riemannian submersion, I an open interval, such that the fibers of π are orbits of the action, and satisfying that for each $s \in I$, the fiber $\pi^{-1}\{s\} \subset M$ has constant mean curvature $h(s)$. Take $u \in C^2(M, \mathbb{R})$ and consider its graph map

$$F : M \to M \times \mathbb{R}, F(x) = (x, u(x))$$

for any $x \in M$. Then, F is a Σ-invariant translating soliton if, and only if, there exists a solution $f \in C^2(I, \mathbb{R})$ to

$$f^{''}(s) = (\widetilde{\varepsilon} + \varepsilon(f')^2(s))(1 - h(s)f'(s)) \tag{2}$$

such that $u = f \circ \pi$.

For the sake of clarity, we now introduce a method to construct a translating soliton in a manifold foliated by the orbits of the action of a Lie group acting by isometries.

Algorithm 1 Let (M, g) be semi-Riemannian manifold, Σ a Lie subgroup of $\mathrm{Iso}(M, g)$, I open interval, and $\varepsilon \in \{\pm 1\}$. The metric in $M \times \mathbb{R}$ is $\langle , \rangle = g + dt^2$.

1. Assume $\phi : M \to \Sigma \times I$ is a diffeomorphism such that its restriction $\pi : M \to I$ satisfies $|\nabla \pi|^2 = \tilde{\varepsilon} = \pm 1$.
2. For each $s \in I$, compute the mean curvature $h(s)$ of the fiber $\pi^{-1}\{s\} \subset M$.
3. Solve the following equation for some initial values in an interval $J \subset I$,
$$f^{''}(s) = (\widetilde{\varepsilon} + \varepsilon(f'(s))^2)(1 - f'(s)h(s)).$$
4. The translating soliton can be constructed by one of the following equivalent ways:
$$F : \Sigma \times J \to M \times \mathbb{R},\ F(\sigma, s) = (\phi^{-1}(\sigma, s), f(s)).$$
$$\overline{F} : \phi^{-1}(\Sigma \times J) \to M \times \mathbb{R},\ \overline{F}(x) = (x, f(\pi(x))).$$

3 Solution and Extension to a Boundary Problem With Singularity

Inspired by previous sections, we consider and solve the following boundary problem.

Theorem 1 *Given $a \in \mathbb{R}$, $b \leq +\infty$, $\varepsilon, \tilde{\varepsilon} \in \{1, -1\}$, choose $q \in C^1[a,b)$ such that $q(a) = 0$, $q(s) \neq 0$ for any $s > a$, $\tilde{\varepsilon} q'(a) \geq 0$, and define $h : (a,b) \to \mathbb{R}$ given by $h = 1/q$. Then, the boundary problem*

$$w'(s) = (\tilde{\varepsilon} + \varepsilon w^2(s))(1 - w(s)h(s)), \quad w(a) = 0 \tag{3}$$

has a solution $w : [a, a+\delta) \to \mathbb{R}$ for a suitable small $\delta > 0$.

Along this section, we will always assume the following:

(H) *Given $a < b \leq +\infty$, take $s_0 \in (a,b)$. Consider $h \in C^1(a,b)$ such that $h > 0$.*

Next result shows that this Condition (H) provides the hypothesis to extend the local solutions *until the end of time*. In other words, the local solution provided by the Lindelöf-Pickard Theorem can be extended up to b (the supremum of the interval.)

Proposition 1 *Assume (H).*

1. *For each $w_0 \in \mathbb{R}$, the initial value problem*

$$w'(s) = (1 + w^2(s))(1 - h(s)w(s)), \quad w(s_0) = w_0, \tag{4}$$

has a unique C^2-solution w on $(s_0 - \rho, b)$, for some $\rho > 0$.

2. *If $b = +\infty$, then $\lim\limits_{s \to b} h(s)w(s) = 1$.*

By Theorem 1 and Proposition 1, we obtain the following result.

Corollary 1 *Assume (H), and in addition $\lim\limits_{s \to a} h(s) = +\infty$ and $\lim\limits_{s \to a} \frac{h'(s)}{h^2(s)} = h_1 > 0$. Then, the boundary problem (3) has a unique globally defined solution $w \in C^1[a,b)$.*

Proposition 2 *Assuming (H), suppose in addition $b < +\infty$ and there exist $\lim\limits_{s \to b} h(s) = +\infty$.*

1. *If there exists $\lim\limits_{s \to b} \frac{h'(s)}{h^2(s)} = h_1 \in [0, +\infty)$, there is a solution to (4) for certain w_0 such that $\lim\limits_{s \to b} w(s) = 0$ and $\lim\limits_{s \to b} w'(s) = \frac{1}{1+h_1}$.*

2. *If for some $M > 0$ and $s_1 \in [s_0, b)$, it holds $w(s) \geq M$ for every $s \geq s_1$, then there exist $\lim\limits_{s \to b} w(s) = w_1 \geq M$ and $\lim\limits_{s \to b} w'(s) = -\infty$.*

3. *If there exist* $M < 0$, $s_1 \in [s_0, b)$ *such that for every* $s \geq s_1$, $w(s) \leq M$, *then there exist* $\lim_{s\to b} w(s) = w_1 \leq M$ *and* $\lim_{s\to b} w'(s) = +\infty$.

This proposition can be easilly proved.

The case $\varepsilon = \tilde{\varepsilon} = -1$ can be studied in a similar way. All ideas are already explained, so its proof is left to the reader.

Proposition 3 *Given* $h : [s_0, b) \to \mathbb{R}$, $h \in C^1[s_0, b)$ *and* $b \leq +\infty$ *such that* $h(s) < 0$.

1. *For each* $w_0 \in \mathbb{R}$, *the boundary value problem*

$$w'(s) = -(1 + w^2(s))(1 - h(s)w(s)), \quad w(s_0) = w_0, \tag{5}$$

   has a unique C^2*-solution* w *on* $[s_0, b)$.

2. *If* $b = +\infty$, *then* $\lim_{s\to+\infty} h(s)w(s) = 1$.

3. *Assume* $b < +\infty$ *and there exist the limits* $\lim_{s\to b} h(s) = -\infty$ *and* $\lim_{s\to b} \frac{h'(s)}{h^2(s)} = h_1 \in (-\infty, 0]$. *Then, for certain* $w_0 \in \mathbb{R}$, *there exist the limits* $\lim_{s\to b} w(s) = 0$ *and* $\lim_{s\to b} w'(s) = \frac{1}{1+h_1}$.

4. *Assume* $b < +\infty$ *and there exists* $\lim_{s\to b} h(s) = -\infty$. *If for some* $M > 0$ *and* $s_1 \in [s_0, b)$, *it holds* $w(s) \geq M$ *for every* $s \geq s_1$, *then there exist* $\lim_{s\to b} w(s) = w_1 \geq M$ *and* $\lim_{s\to b} w'(s) = -\infty$.

5. *Assume* $b < +\infty$ *and there exist* $M < 0$, $s_1 \in [s_0, b)$ *such that for every* $s \geq s_1$, $w(s) \leq M$, *then there exist* $\lim_{s\to b} w(s) = w_1 \leq M$ *and* $\lim_{s\to b} w'(s) = +\infty$.

4 Examples

Example 1 In $\mathbb{R}^n$, with the standard flat metric g_0, we consider the Poincare's Half hyperplane model of $\mathbb{H}^n$, namely

$$\mathbb{H}^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}, \quad g = \frac{1}{x_n^2} g_0.$$

Let Σ be the Lie group $\Sigma = (\mathbb{R}^{n-1}, +)$ acting by isometries on $\mathbb{H}^n$ as usual, namely

$$\Sigma \times \mathbb{H}^n \to \mathbb{H}^n, \quad (w, p) \to (p_1 + w_1, ..., p_{n-1} + w_{n-1}, p_n)$$

where $w = (w_1, w_2, ..., w_{n-1})$ and $p = (p_1, p_2, ..., p_n)$, respectively. Note that the orbits are the well-known *horospheres.*

We define the projection map, with its usual properties:

$$\bar{\tau} : \mathbb{H}^n \to \mathbb{R}, \tau(x_1, x_2, ..., x_n) = \ln(x_n).$$

Consider two local frames $(\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n})$ and $(E_i = x_n \partial_{x_i} : i = 1, \dots, n)$ of $T\mathbb{H}^n$. A straightforward computation shows

$$\nabla \bar{\tau} = E_n, \quad \mathrm{div}(\nabla \bar{\tau}) = -n + 1. \tag{6}$$

We arrive to the following initial value problem,

$$f^{''}(s) = (1 + (f'(s))^2)(1 + (n-1)f'(s)), \quad f'(s_0) = f_0, \quad f(s_0) = f_1, \tag{7}$$

where $s_0, f_0, f_1 \in \mathbb{R}$. By the easy change $f'(s) = w(s)$, we transform this problem in

$$w'(s) = (1 + w^2(s))(1 + (n-1)w(s)), \quad w(s_0) = f_0. \tag{8}$$

To solve this equation, we consider the map

$$F : \mathbb{R} \backslash \{-1/(n-1)\} \to \mathbb{R},$$
$$F(t) = \frac{1}{1 + (n-1)^2} \left[(n-1) \ln \left(\frac{|1 + (n-1)t|}{\sqrt{1 + t^2}} \right) + \arctan t \right] + C_0,$$

for some integration constant $C_0 \in \mathbb{R}$. From here we obtain 3 cases.
Case 1: $f_0 = -1/(n-1)$. Then, $f(s) = f_1 - s/(n-1)$ is a solution to (7).

Case 2: $f_0 > -1/(n-1)$. We restrict F, namely $F_1 : \left(\frac{-1}{n-1}, +\infty\right) \to \mathbb{R}$. In this case, $F' > 0$, so that F is injective. To compute its image, we see

$$\begin{aligned} \lim_{t \to +\infty} F_1(t) &= \left(\frac{1}{1 + (n-1)^2} \right) \left((n-1)\ln(n-1) + \frac{\pi}{2} \right) + C_0 =: K_0, \\ \lim_{t \to \frac{-1}{n-1}^+} F_1(t) &= = -\infty. \end{aligned}$$

We obtain that $F_1 : \left(\frac{-1}{n-1}, +\infty\right) \to (-\infty, K_0)$ is bijective, and there exists its inverse function

$$F_1^{-1} : (-\infty, K_0) \to \left(\frac{-1}{n-1}, +\infty \right).$$

Now, we recover $w(s) = F_1^{-1}(s)$, $w(s_0) = F_1^{-1}(s_0) = w_0 > \frac{-1}{n-1}$, and $\lim_{s\to-\infty} w(s) = -1/(n-1)$, $\lim_{s\to K_0} w(s) = +\infty$. Finally,

$$f : (-\infty, K_0) \to \mathbb{R}, \quad f(s) = f_1 + \int_{s_0}^{s} w(u)du.$$

Then, we obtain lim $\lim_{s\to-\infty} f(s) = -\infty$ and $\lim_{s\to K_0} f(s) = +\infty$. Thus, function f has a finite time blow up.

Case 3: $f_0 < -1/(n-1)$. Similarly to case 2 we can easilly obtain f function has a finite time blow up.

Next, for each case, we resort to Algorithm 1 to obtain our translating solitons.

Example 2 In $\mathbb{R}^{n+1}$, $n \geq 1$, with its standard flat metric g, consider a round $n-$sphere of radius 1 centered at 0, namely $\mathbb{S}^n$. Now, the Lie group $O(n-1)$ acts by isometries on $\mathbb{S}^n$ as usual:

$$O(n-1) \times \mathbb{S}^n \to \mathbb{S}^n, \;\; (A, x) \to A.x = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} A(x_1, ..., x_n)^t \\ x_{n+1} \end{pmatrix}$$

We restrict our study to $M = \mathbb{S}^n \backslash \{N, S\}$, i. e., we remove the North and South Poles. In this way, the space of orbits can be identified by the following projection map

$$\tau : M \to (-\pi/2, \pi/2), \quad \tau(x) = -\arcsin(x_{n+1}).$$

We obtain that the mean curvature of the orbits is given by $h(s) = (1-n)\tan(s)$. Thus, we consider the following differential equation:

$$f''(s) = \big(1 + f'(s)^2\big)\big(1 - (n-1)\tan(s) f'(s)\big), \tag{9}$$

which we reduce in a first step to ($w = f'$),

$$w'(s) = \big(1 + w^2(s)\big)\big(1 - (n-1)\tan(s) w(s)\big). \tag{10}$$

Assume a solution $w : (-\delta, \delta) \to \mathbb{R}$ such that $w(0) = w_o$. Since $h > 0$ on $(0, \pi/2)$, by Proposition 1, we can extend to $w : (-\delta, \pi/2) \to \mathbb{R}$. Now, by taking $z : (-\pi/2, \delta) \to \mathbb{R}$, $z(u) = -w(-u)$, it is clear that z is another solution to (10). By Proposition 1, we can extend $z : (-\pi/2, \pi/2) \to \mathbb{R}$. This means that each solution to (10) can be globally defined $w : (-\pi/2, \pi/2) \to \mathbb{R}$. Clearly, for each $f_0 \in \mathbb{R}$, we construct a solution $f(s) = \int w(x)dx + f_0$, $f : (-\pi/2, \pi/2) \to \mathbb{R}$. Now, by using Algorithm 1, given a solution f, we obtain a translating soliton defined on the sphere except two points, namely $\mathbb{S}^n \backslash \{N, S\}$.

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