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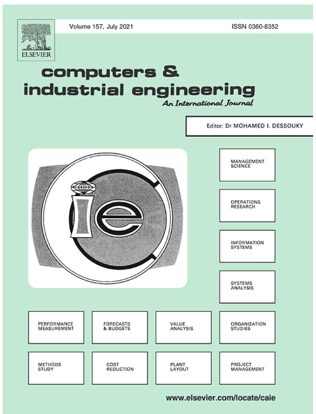

A multi-state warm standby system with preventive maintenance, loss of units and an indeterminate multiple number of repairpersons

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Abstract

A Markovian Arrival Process with Marked arrivals is used to model a discrete-time complex warm standby multi-state system in a well-structured way. The online unit is subject to internal failure, repairable or non-repairable, and/or external shocks. These shocks can produce total failure, modification of the internal performance or cumulative damage. To avoid serious damage and considerable financial loss, random inspection is performed. Internal degradation and cumulative external damage are partitioned into minor and major states. Both are inspected and if a major state is observed the unit is sent to the repair facility for preventive maintenance. Each warm standby unit may undergo a repairable failure at any time. Thus, three different time distributions are applicable to the repairpersons: corrective repair for the online unit and the warm standby units, and preventive maintenance. The repair facility is composed of multiple and variable repairpersons. When a non-repairable failure occurs, the system continues working with one less unit and the number of repairpersons may be modified. The system continues working with fewer units as long as this is possible. Measures of interest in the reliability field are calculated. Costs and rewards are included in the model. A numerical example shows how the optimum system may be achieved, according to the number of repairpersons and the preventive maintenance performed. The model is built in an algorithmic form, which facilitates its computational implementation.

Keywords. Warm standby system, MAP, preventive maintenance, loss of units, multiple repairpersons

1. Introduction

Redundant systems and preventive maintenance, to improve overall reliability, prevent system failures and reduce costs, are of considerable research interest. To avoid serious damage and major financial losses, various reliability-enhancing methods can be applied, such as redundancy and periodic maintenance. In this respect, cold, hot and warm redundant standby and k-out-of-n systems were proposed. In a cold standby system, the redundant units are not subject to failure before being put into full operation, whereas in hot standby all units can fail at any time, with an identical pattern of behaviour. In warm standby the units can fail at any time but at different rates. In the first case, the redundant units are not exposed to wear, whereas in the second and third the failure rate of redundant units is greater than zero. In a warm standby system, the failure rate of a unit in online mode is usually higher than that of one in warm standby mode.

Warm standby systems are commonly examined in reliability literature. Thus, Singh (1989) used differential equations to model a system with multiple operating units, in conjunction with warm standby units. Levitin et al. (2013) considered an optimal standby element sequencing problem (SESP) for 1-out-of-N: G heterogeneous warm-standby systems, seeking to determine the initiation sequence of the system elements that would minimise the mission cost of the system. Li et al. (2016) studied a warm standby repairable system in which the concept of vacation time in the repair facility was included. A Markovian system with a Laplace transform was considered by Sadeghi and Roghanian (2017) in the analysis of a warm standby repairable system that incorporated an imperfect switching mechanism. Zhai et al. (2015) built a multi-valued decision diagram to analyse a demand-based warm standby system, in which each component had a nominal capacity and if the total capacity of the working components did not correspond to the system, it would fail. Recently, Goel and Kumar (2018) analysed a two-unit cold standby system, considering general distributions for the embedded random variables. In another paper, the probability of mission success for an arbitrary redundancy level was evaluated by Levitin et al. (2017). Other approaches have been adopted by Wang et al. (2018) who studied a redundancy allocation problem for cold-standby systems with degrading components, and by Jia et al. (2017), who considered a general demand-based warm standby system subject to a component degradation process.

Preventive maintenance is intended to improve system reliability and thus increase the benefits produced. This question was discussed by Osaki and Asakura (1970), regarding the behaviour of a two-unit standby redundant system. Maintenance policies for reliability systems have also been addressed by Nakagawa (2005). Zhong and Jin (2014) analyzed, using semi-Markovian processes, a cold standby two-component system with preventive maintenance. Redundant systems with preventive maintenance were also described by Ruiz-Castro (2013, 2014). Finally, Qiu et al. (2018) developed reliability and maintenance models for a single-unit system subject to hard failures within an environment of random external shocks.

1.1. Multi-state systems (MSS)

A system that has a finite number of performance levels and various failure modes with different effects on the entire system performance is called a multi-state system (MSS). This concept was first discussed by Murchland (1975), and the question of MSS reliability has since been developed extensively. Important early achievements (up to the mid-1980s) have been reviewed by Natvig (1985) and by El-Neweihi and Proschan (1984), and Lisnianski et al. (2010) performed a comprehensive analysis of MSS. These systems can also be studied using Markov and semi-Markov models (Ruiz-Castro and Dawabsha (2018), Li et al. (2017), Peng et al. (2017)). In this field, Ruiz-Castro (2016b) analysed a complex multi-state system, based on Markov counting and reward processes. Lisnianski and Frenkel (2012), too, have used Markov processes to analyse MSS, reporting favourably on this approach. Finally in this regard, Yu et al. (2018) presented a continuous-time Markov process-based model for evaluating time-dependent reliability indices of multi-state degraded systems.

1.2. Phase-type distributions (PH) and Markovian Arrival Processes (MAP)

For reliability, several distributions are frequently used, including the exponential, Erlang and Weibull distributions. However, the latter involves calculations that may become unmanageable, due to the analytic expression required. Phase-type distributions (PH) play an important role in this respect. This class of distribution was introduced by Neuts (1975), and subsequently described in greater detail by the same author, Neuts (1981); it has been applied in fields such as reliability and queuing theory. Phase-type distributions have been modelled using multiply-redundant MSS (Ruiz-Castro and Li (2011)).

A Markovian Arrival Process (MAP) is a counting process where PH distributions play an important role. Neuts (1979) introduced them and Artalejo et al. (2010) provided a comprehensive review on MAPs. MAPs are used to model MSS in a well-structured form and count events associated with the system both algorithmically and computationally. A special case of this process is that of the Marked MAP, which enables several types of arrivals to be counted. The arrival rates (or probabilities for the discrete case) of events can be customised for different situations. Important results associated with MAPs were presented recently by He (2014) and Alfa (2016). These studies develop MAP theory in an intuitive way, observing that MAPs with marked arrivals, or MMAPs, are an extension of MAPs when marked arrivals occur. This approach is of interest in fields such as telecommunications. Many reliability systems have inputs to the system over time, such as a repairable failure, a non-repairable failure, inspections or an external shock. In this respect, Ruiz-Castro (2016a, 2016b) used a MMAP to analyse redundant standby systems.

1.3. Discrete time systems

Most reliability models discussed in the literature have been considered in continuous time. However, discrete time should also be taken into account due to the way in which certain systems perform or because some events, such as periodic inspections, only occur in discrete time. The behaviour of devices in fields such as civil and aeronautical engineering was analysed through reliability systems that evolve in discrete time. Thus, Shatnawi (2016) used them in software reliability engineering. Discrete time systems have been modelled by using semi-Markov processes (Barbu and Limnios (2008); Georgiadis and Limnios (2014)), and redundant Markovian MSS (Li et al. (2017); Ruiz-Castro and Li (2011)).

1.4. Multiple and variable repairpersons

In reliability literature, the repair facility is usually composed of a single repairperson, when systems are subject to repairable failures and/or preventive maintenance. However, this assumption is not always realistic and it would also be useful to model systems with multiple and variable repairpersons over time. Recently, Ruiz-Castro et al. (2018) modelled multi-state complex cold standby systems subject to multiple events with loss of units with a variable number of repairpersons. This study showed that the optimum system obtained depended on the number of repairpersons available and on the preventive maintenance performed.

It is normally assumed that when a system unit undergoes a non-repairable failure it is replaced by a new one within a negligible time. This assumption, however, is not always realistic. Another setting considered is that of redundant systems, in which a unit that undergoes a non-repairable failure is not replaced as long as the system remains operational. This situation has been analysed for different redundant multi-state systems (Ruiz-Castro (2018), Ruiz-Castro and Fernández-Villodre (2012), Ruiz-Castro et al. (2018)).

1.5. Main differences between cold and warm standby system: main contributions

In the present paper, we develop a complex redundant MSS that evolves in discrete time, in a well-structured way, using MAPs with Marked arrivals (MMAPs), thus extending the system given by Ruiz-Castro et al. (2018) to the warm standby case. Here, the online unit is subject to similar events but in addition failures may occur in the standby units. This new system is not an immediate consequence of the cold standby system (Ruiz-Castro et al. (2018)). It has the following main features and contributions: a) a lifetime distribution is introduced for warm standby units. In consequence, the complex behaviour of warm standby units is modelled. The lifetime of each warm standby unit follows a geometric distribution; b) new measures associated with warm standby units, such as the expected duration of working in warm standby and the expected number of warm standby failures, are obtained; c) a different corrective repair time distribution for the warm standby units is considered, such that three different types of tasks may be required of the repairpersons. Therefore, the modelling of the repair facility has been extended, as has the state-space; d) rewards and costs are introduced in this new model, according to the operational phases, number of repairable failures from warm standby and from the online unit, and the different types of repair performed. The expected cost from corrective repair of a unit that failed from warm standby is taken into account; e) when warm standby units are considered, multiple units may fail simultaneously. The system is modelled assuming that both the online unit and one or more of the warm standby units can fail simultaneously; f) a new MMAP is constructed and measures of warm standby failures are recorded. The structure of this new MMAP is totally different from the cold case; with new transitions between macro-states given that multiple failures can occur simultaneously and a new type of repair for units that failed from warm standby units. The new transition probabilities are built from the MMAP according to the new state space; g) a numerical example extends the cold standby case to that of warm standby and a new optimum system is obtained, depending on the number of repairpersons available and the preventive maintenance performed.

As in the cold standby case, a variable number of repairpersons is considered. The MMAP built for the warm standby system is completely different from that used in the cold case. The state-space is different and so, therefore, are the transition probabilities. The lifetime of each warm standby unit is included in the model and consequently new events, such as the failure of a warm standby unit, are considered. Multiple interdependent time distributions are embedded in the system (covering events such as internal failure, external shock, inspection time, corrective repair time and preventive maintenance) and multiple events can occur simultaneously. This fact complicates the modelling and makes the algorithm much more necessary. MMAPs enable us to model complex systems in a matrix-algorithmic way. All of these distributions are embedded in the proposed MMAP, which contains the interdependence of these distributions.

1.6. Some application examples

In the present paper, we present and discuss a real life system. In computer engineering, the backup server is periodically inspected by an installed monitoring program that analyses logic and physics parameters to detect possible errors arising from internal and/or external events. Data are regularly mirrored and replicated to all warm servers using disk-based replication or shared disks. In industrial engineering, any facility that requires a reliable electrical supply must have available generating sets capable of generating electricity in case of need. For instance, the power station of a ship consists of three generator sets. During cruising and berthing, the power station normally keeps one generator set for power supply, and another two in standby state. In civil engineering, in a fluid transfer system, if a valve fails to operate, the pump can be disabled and the system shut down. Warm redundancy systems, thus, are designed to prevent absolute, fatal failure.

The paper has the following organization. Section 2 is focused in the system, it is described and assumptions are given. In Section 3 the state-space is built, the modelling of the online and warm standby units and the repair facility is developed and The MMAP is constructed. Section 4 is focused on building interesting measures in transient and stationary regime. The mean number of events is then calculated. Rewards are introduced in Section 5. A numerical example illustrates the versatility of the model, and the optimum system is obtained in Section 6. Finally, the main conclusions drawn from this study are presented in Section 7.

2. Describing the system: Assumptions

In this paper, we consider a complex warm standby system composed of K units, one of which is online, while the remainder are in warm standby. Warm standby units are expected to operate in warm standby mode for a period of time until failure, until the on line unit enters a failure state or until a major damage stage is inspected. At this point, a warm standby unit enters online mode, replacing the faulty unit. The online unit has features similar to those of the online unit described in Ruiz-Castro et al. (2018). Thus, it is a multi-state unit partitioned between major and minor damage, which is subject to internal failure, repairable or not, and/or external shocks, and is inspected at random intervals. The internal operational time is PH distributed with representation $\left(\boldsymbol{\alpha},\mathbf{T}\right)$ and two types of internal failure may occur, repairable or non-repairable. Whenever a repairable failure occurs, the unit is sent to the repair facility. The internal repair time distribution is PH with representation $\left(\boldsymbol{\beta}^1,\mathbf{S}_1\right)$.

When a non-repairable failure occurs, the online unit is removed and the number of repairpersons may be modified. In this system, therefore, units may be lost and the number of repairpersons is variable.

The time distribution between two consecutive external shocks is PH distributed with representation $\left(\boldsymbol{\gamma},\mathbf{L}\right)$. External shocks may produce multiple consequences, including total failure, altered internal performance or cumulative external damage. The cumulative external damage is also differentiated into minor and major states. The transition probability matrix for the transitions between external damage states is denoted by $\mathbf{D}$ (initially the state is one, no damage). When the cumulative external damage reaches a given threshold, a non-repairable failure occurs. The cumulative external damage may increase each time that an external shock occurs and total failure occurs after an external shock with a probability equal to ω^0. After an external shock, the internal behaviour of the online unit may also be altered. This probability is governed by the matrix **W**.

Random inspections are conducted, during which the internal behaviour and the cumulative external damage of the online unit are observed. The time between two consecutive inspections is PH distributed with representation $(\boldsymbol{\eta}, \mathbf{M})$. Whenever major damage (internal and/or external) is observed, preventive maintenance is carried out. Preventive maintenance time is PH distributed with representation $(\boldsymbol{\beta}^2, \mathbf{S}_2)$. At any time, one or more of the warm standby units may undergo a repairable failure with a probability equal to *p* and multiple events can occur simultaneously.

The system verifies the assumptions (1-12) given for the online unit of the cold standby system described in Ruiz-Castro et al. (2018) and commented above. In this work the system incorporates the following three assumptions associated to the warm standby units.

***Assumption* (a).** Each warm standby unit may undergo a repairable failure at any time with a probability equal to *p*.

***Assumption* (b).** The distribution of the corrective repair time for a unit that failed from warm standby is phase-type with representation $(\boldsymbol{\beta}^0, \mathbf{S}_0)$. The number of corrective repair phases in this case is equal to z_0.

***Assumption* (c).** Multiple failures may occur simultaneously. The priority in the repair facility is the following: corrective repair for the online unit, preventive maintenance and corrective repair for warm standby units.

Figure 1 shows the general behaviour of the system.

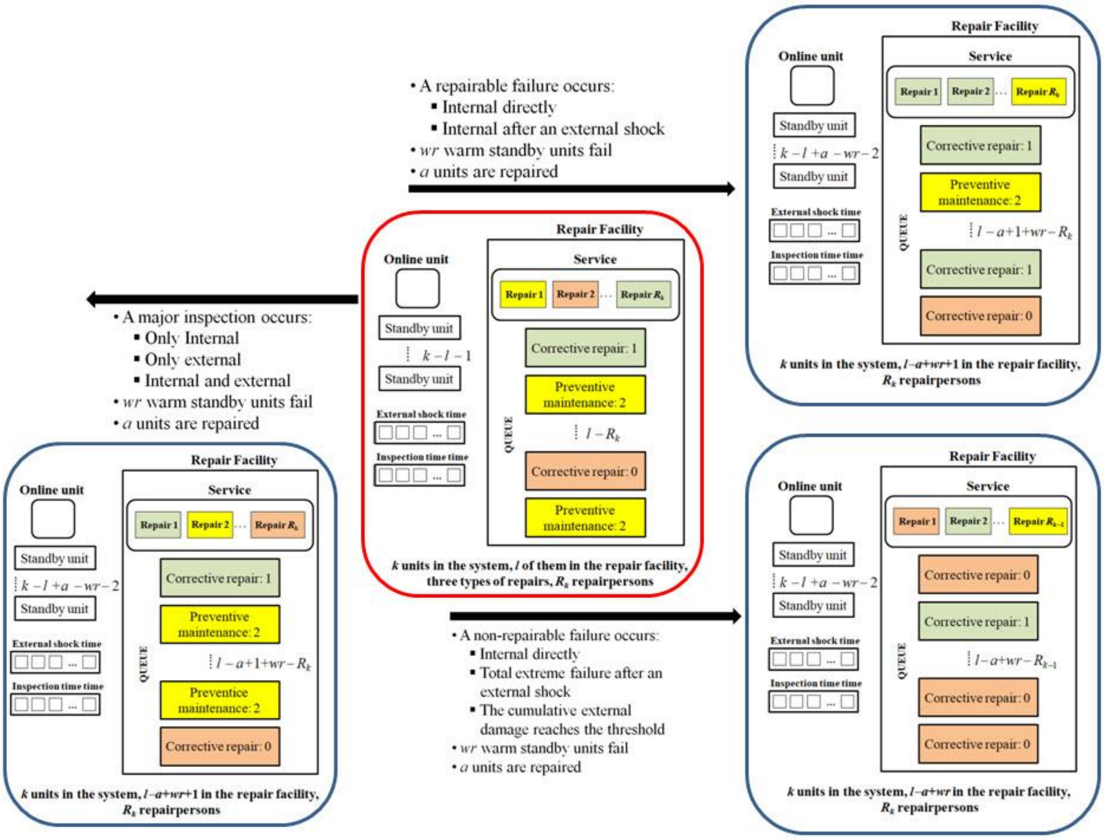

Figure 1. General behaviour of the system

3 The model

To model this complex system several steps are given in this section. Firstly, the online unit, the warm standby units and the repair facility are modelled, algorithmically and computationally. Then, it will be used to model the system through a Marked Markovian Arrival Process (MMAP).

3.1. The state-space

The system is governed by a discrete Markov process vector with the following state-space. The state-space of the system has three levels of nested macro-states and is given by $S=\left\{\mathbf{U}^K,\mathbf{U}^{K-1},\ldots,\mathbf{U}^1\right\}$, where $\mathbf{U}^k$ is the third level that contains the phases when there are k units in the system, for $k=1,\ldots,K$. The second level of macro-states is given by $\mathbf{U}^k=\left\{\mathbf{E}_0^k,\mathbf{E}_1^k,\ldots,\mathbf{E}_k^k\right\}$. The macro-state $\mathbf{E}_s^k$ is composed of the phases when the number of the units in the system is k and s of them are non-operational (in the repair facility). The Figure 2 shows the transition diagram for this macro-state. There are three different types of actions by repairpersons, corrective for online and warm standby units and preventive maintenance. All of them must be saved in memory in order to be addressed. Thus, the macro-state $\mathbf{E}_s^k$ is composed of the macro-states $\mathbf{E}_{i_1,\ldots,i_s}^k$ (first level), k units in the system of which s are in the repair facility. The ordered sequence $i_{1,\ \ldots\ ,}\ i_s$ indicate the types of repair where i_l is equal to 0 (the unit comes from warm standby), 1 (the unit comes from online place) or 2 (preventive maintenance). The description of the macro-states can be seen in Ruiz-Castro et al. (2018). The phase $\left(k,s;i,j,u,m,r_1,\ldots,r_{\min\{s,R_k\}}\right)$ indicates:

- k: units in the system
- R_k: repairpersons when there are k units in the system
- s: units in the repair facility
- i: internal performance phase
- j: external shock phase
- u: cumulative damage phase
- m: phase of the inspection phase
- r_h: corrective repair/preventive maintenance phase for the h-th unit being repaired

Although the state-spaces of cold and warm standby systems are similar, the possible transitions are different because of, in the first case, at most only one failure can occur at a certain time. If the cold standby system is considered, the only transitions from macro-state $\mathbf{E}_s^k$ (k units in the system, s of them in the repair facility) is to $\mathbf{E}_{s-\min\{s,R_k\}}^k$, $\mathbf{E}_{s-\min\{s,R_k\}+1}^k$,..., $\mathbf{E}_{s+1}^k$ and to $\mathbf{E}_{s-\min\{s,R_k\}}^{k-1}$, $\mathbf{E}_{s-\min\{s,R_k\}+1}^{k-1}$,..., $\mathbf{E}_s^{k-1}$. Given that multiple failures can occur for the warm standby case, the possible transitions between macro-states are more complex. . The Figure 2 shows the transition diagram from macro-state $\mathbf{E}_s^k$.

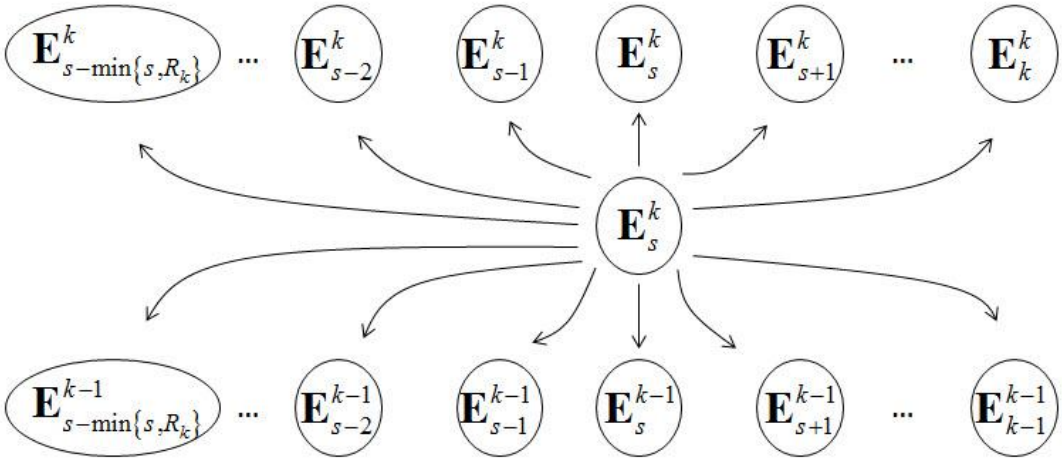

Figure 2. Transition diagram from the macro-state $\mathbf{E}_s^k$

3.2. The online unit

The online unit may undergo the following types of event; repairable failure, non-repairable failure or major inspection, whether or not an external shock occurs. We denote them as,

A_1: Internal repairable failure because of wearing out (*rep*, 1)

A_2: Internal repairable failure as a consequence of an external shock (*rep*, 2)

B_1: Major revision 1 (only major internal damage observed) (*mr*, 1)

B_2: Major revision 2 (only major external damage observed) (*mr*, 2)

B_3: Major revision 3 (internal and external damage observed) (*mr*, 3)

C_1: Non-repairable failure because of wearing out (*nrep*, 1)

C_2: Non-repairable failure as a consequence of an external shock (*nrep*, 2)

O: No events (0)

These events are interdependent and the dependency relationship is embedded in the model. Ruiz-Castro and Dawabsha (2019) describe the dependence relationship between the events in a discrete general MMAP.

The case B_1 is developed in detail in this subsection. The rest is given in Appendix A.

B_1: *Major revision in response to major internal damage*

[1] A major revision is always performed after an inspection $\left(\mathbf{M}^0\eta\right)$, whether or not there has been an external shock. In this case, the event is only counted when major internal damage is observed.

Auxiliary matrices **V** and **U** are constructed to identify cumulative external damage and internal degradation, respectively. Subscript 1 or 2 indicates the type of damage observed, minor or major, respectively. These matrices are

$$U_1(s,t)\begin{cases}1 & ; \quad 1\le s=t\le n_1\\ 0 & ; \quad \text{otherwise}\end{cases},\quad U_2(s,t)\begin{cases}1 & ; \quad s=t>n_1\\ 0 & ; \quad \text{otherwise}\end{cases}$$

$$V_1(s,t)\begin{cases}1 & ; \quad 1\le s=t\le d_1\\ 0 & ; \quad \text{otherwise}\end{cases},\quad V_2(s,t)\begin{cases}1 & ; \quad s=t>d_1\\ 0 & ; \quad \text{otherwise.}\end{cases}$$

Thus, two possibilities may arise

- Inspection reveals major internal damage but no internal failure $\left(\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\right)$; there is no external shock ($\mathbf{L}$) and inspection reveals minor cumulative external damage $\left(\mathbf{V}_1\mathbf{e}\boldsymbol{\omega}\right)$. Then,

$$\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{V}_1\mathbf{e}\boldsymbol{\omega}.$$

- An external shock occurs $\left(\mathbf{L}^0\boldsymbol{\gamma}\right)$, which may worsen the internal behaviour of the online unit but without provoking failure $\left(\mathbf{TWe}\boldsymbol{\alpha}\right)$. The external shock may provoke cumulative external damage without failure $\left(\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right)$. Inspection reveals only major internal damage. Then, for this case,

$$\mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right).$$

Therefore, for this case, the transition for the online unit is given by

$$\mathbf{H}_{\mathrm{mr},1}=\left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{V}_1\mathbf{e}\boldsymbol{\omega}+\mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta.$$

3.3. Warm standby system

In this section we now introduce warm standby units into the model. In a cold standby system only one unit can fail at a certain time, however in this new system, multiple warm standby units may fail simultaneously at a certain time. To model the failures of these warm standby units with that of the online unit a new matrix function is built.

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[1] Throughout the paper, given a matrix **A** the column vector $\mathbf{A}^0$ is equal to **e**–**Ae** where **e** is a column vector of 1's with appropriate order. If **e** has the subscript *r*, the order of this vector is *r*

If wr indicates the number of standby units which are in a fail state at a certain time, k is the number of units in the systems (l of them in the repair facility), then the number of possible combinations for this number of failures is given by $\binom{k-l-1}{wr}$.

The probability of having r failed standby units is $\binom{k-l-1}{wr} p^{wr}(1-p)^{k-l-1-wr}$.

Therefore, if the online unit is introduced, the following transition matrix for the online and warm standby units is defined by considering the different types of events

$$\mathbf{H}_{k,c,l,wr} = \binom{k-l-1}{wr} p^{wr}(1-p)^{k-l-1-wr}\,\mathbf{H}_c,$$

$l = 0,\dots, k-1$; $r \le k-l-1$, where $c = \{0, (rep,1), (rep,2), (mr,1), (mr,2), (mr,3), (nrep,1), (nrep,2)\}$.

The matrix $\mathbf{H}_{k,c,l,wr}$ contains the transition probabilities when:

- the system has k units
- l units are non-operational
- wr warm standby units break down at the next time point
- the online unit goes to situation c.

This matrix for the case, the online unit is broken and $wr = k-l-1$ is

$$\mathbf{H}'_{k,c,l,k-l-1} = p^{k-l-1}\mathbf{H}'_c, \quad \text{for } c = \{(rep,1), (rep,2), mr, (nrep,1), (nrep,2)\}.$$

3.4. Repair facility

Our aim in this section is to model the repair facility with multiple repairpersons, using various matrix auxiliary functions for this purpose. New functions and transitions for the repair facility are developed by considering that a new type of corrective repair is introduced for the failures from warm standby units. The MMAP that governs the system is built. The matrix blocks of this counting Markov process contain new transition probability matrices given that new transitions between macro-states can occur in the warm standby system.

3.4.1. Auxiliary functions

An interesting aspect of this situation is that if the number of repairpersons is fewer than units remaining to be repaired after a transition (it is possible because of the number of repairpersons can be modified after a non-repairable failure). If this fact occurs, the number of units that will be returned to the queue is

$$^{2}b = \max\left\{\min\left\{l, R_k\right\} - a - I_{\{nr=1\}} R_{k-1} - I_{\{nr=0\}} R_k, 0\right\},$$

where *nr* is the indicatory value which is equal to 1 if a non-repairable failure occurs and equal to 0 otherwise.

The following auxiliary matrix functions govern the behaviour of the units under repair,

- $C\left(k, l, a, b; k_1, \ldots, k_a; i_1, \ldots, i_{l-a+mr+wr}; j_1, \ldots, j_l\right)$. The output is the transition probability matrix associated with the units that are being repaired (the order of the repaired units is specified).
- $B\left(k, l, a, b; i_1, \ldots, i_{l-a+mr+wr}; j_1, \ldots, j_l\right)$. The response is the transition probability matrix of the units that are being repaired (the order of repaired units is not specified).

In these functions:

- a is the number of units for which the repair process has concluded.
- k_h denotes the ordinal of the h-th repairpersons who concluded the repair(s)
- i_h and j_h are the types of repair for the broken h-th unit, after and before the transition (corrective warm standby, 0, corrective online unit, 1, preventive maintenance, 2).
- *mr* is equal to 1 if the online unit goes to the repair facility at this transition and 0 otherwise.

These matrices are developed in Appendix B.

3.4.2. Transition for the repair facility

After a transition, the amount of new units (whether in the queue or not) that may enter for repair depending on the number of repairpersons available before and after the transition time (if one non-repairable failure has occurred) is

[2] The function $I_{\{\}}$ is the indicatory function where the output is 1 when the condition is true and zero otherwise

$$\varepsilon = \min\left\{\max\left\{l - R_k, 0\right\} + mr + wr, \max\left\{I_{\{nr=1\}} R_{k-1} + I_{\{nr=0\}} R_k - \min\left\{R_k, l\right\} + a, 0\right\}\right\}.$$

If there are l units in repair facility and a of them are repaired for $l>0$ and $a \neq l$, then the transition probability matrix is

$$E\left(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l;mr,wr,nr\right)=$$
$$\begin{cases} B\left(k,l,a,0;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right)\otimes \boldsymbol{\beta}^{i_{\min(l,R_k)-a+1}} \otimes \ldots \otimes \boldsymbol{\beta}^{i_{\min(l,R_k)-a+\varepsilon}} & ; \quad \varepsilon > 0 \\ B\left(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right) & ; \quad \varepsilon = 0 \\ \mathbf{0} & ; \quad \text{otherwise.} \end{cases}$$

If $l = 0$ or $a = l \neq 0$ with $l \leq R_k$ then

- For $l = a = 0$, $mr = 0$ and $nr = 0, 1$

$$E\left(k,l=0,a=0,b=0;0,\overset{s}{\ldots},0;mr=0,wr=s,nr=0,1\right)=I_{\{s>0\}}\left(\boldsymbol{\beta}^0 \otimes \overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots} \otimes \boldsymbol{\beta}^0\right)+I_{\{s=0\}},$$

- For $l = a = 0$, $mr = 1$ and $nr = 0$

$$E\left(k,l=0,a=0,b=0;i_{mr},0,\overset{s}{\ldots},0;mr=1,wr=s,nr=0\right)$$
$$=\boldsymbol{\beta}^{i_{mr}} \otimes \left(I_{\{s>0\}}\left(\boldsymbol{\beta}^0 \otimes \overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots} \otimes \boldsymbol{\beta}^0\right)+I_{\{s=0\}}\right),$$

- For $l = a \neq 0$, $mr = 0$ and $nr = 0, 1$

$$E\left(k,l,a=l,b=0;0,\overset{s}{\ldots},0;j_1,\ldots,j_l;mr=0,wr=s,nr=0,1\right)$$
$$=B\left(k,l,l,0;j_1,\ldots,j_l\right)\otimes \boldsymbol{\beta}^0 \otimes \overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots} \otimes \boldsymbol{\beta}^0,$$

- For $l = a \neq 0$, $mr = 1$ and $nr = 0$

$$E\left(k,l,a=l,b=0;i_{mr},0,\overset{s}{\ldots},0;j_1,\ldots,j_l;mr=1,wr=s,nr=0\right)$$
$$=B\left(k,l,l,0;j_1,\ldots,j_l\right)\otimes \boldsymbol{\beta}^{i_{mr}} \otimes \boldsymbol{\beta}^0 \otimes \overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots} \otimes \boldsymbol{\beta}^0.$$

3.5. The MMAP

The system is modelled by a MMAP, according to the events given in Section 3.2. The representation of this MMAP is ,

$$\left(\mathbf{D}^{O}, \mathbf{D}^{A_1}, \mathbf{D}^{A_2}, \mathbf{D}^{B_1}, \mathbf{D}^{B_2}, \mathbf{D}^{B_3}, \mathbf{D}^{C_1}, \mathbf{D}^{C_2}, \mathbf{D}^{FC_1}, \mathbf{D}^{FC_2}\right) .$$

The events FC_1 and FC_2 are the events C_1 and C_2, respectively, for the case only one unit in the system.

The superscript in $\mathbf{D}^Y$ indicates the type of event that has occurred. This matrix is composed of three matrix blocks levels. The last one corresponds to the transitions $\mathbf{U}^k \to \mathbf{U}^k$ or $\mathbf{U}^k \to \mathbf{U}^{k-1}$ and it is noted as $\mathbf{D}^{Y,k}$. In turn, the matrix $\mathbf{D}^{Y,k}$ is composed of the matrix blocks $\mathbf{D}_{lh}^{Y,k}$ which correspond to the transitions $\mathbf{E}_l^k \to \mathbf{E}_h^k$ or $\mathbf{E}_l^k \to \mathbf{E}_h^{k-1}$ (level 2). Finally, several types of repair may take place. The matrices $\mathbf{D}_{lh}^{Y,k}$ are composed of matrix blocks $\mathbf{D}_{lh}^{Y,k}\left(i_1,\ldots,i_h; j_1,\ldots,j_l\right)$ for the transitions $\mathbf{E}_{i_1,\ldots,i_l}^k \to$ $\mathbf{E}_{i_1,\ldots,i_h}^k$ or $\mathbf{E}_{i_1,\ldots,i_l}^k \to$ $\mathbf{E}_{i_1,\ldots,i_h}^{k-1}$ (level 1). This matrix block $\mathbf{D}_{lh}^{Y,k}\left(i_1,\ldots,i_h; j_1,\ldots,j_l\right)$ contains these transition probabilities, where the type of failure in the repair facility is sequenced for the case before and after transition. Case A_1, in which a repairable internal failure occurs without any previous external shock, is described in the following section. Appendix C contains the rest of the cases.

Matrix $\mathbf{D}^{A_1,k}$

We assume that the system has k units and one repairable failure occurs without a previous external shock. The elements of the matrix $\mathbf{D}^{A_1,k}$ for k = 1,..., K are given by

$$\mathbf{D}^{A_1,k} = \left(\mathbf{D}_{lh}^{A_1,k}\right)_{l,h=0,\ldots,k}$$

where $\mathbf{D}_{lh}^{A_1,k} = \mathbf{0}$ if $h < l+1-\min\{l, R_k\}$ or $l = k$.

- If the system is composed of k units, all of them operational, then the transition to h units in the repair facility, for h = 1, …, k, occurs because both the online and h–1 warm standby units undergo a repairable failure. Then,

$$\mathbf{D}_{0h}^{A_1,k}\left(1,0,\overset{h-1}{\ldots},0\right) = \left(\mathbf{H}_{k,(rep,1),0,h-1} I_{\{h<k\}} + \mathbf{H}'_{k,(rep,1),0,h-1} I_{\{h=k\}}\right)$$
$$\otimes E\left(k,0,0,0;1,0,\overset{h-1}{\ldots},0;1,h-1,0\right).$$

- If the system is composed of k units, l of them in the repair facility, wr warm standby units fail, a units are repaired and the online unit is broken type repairable, for $l=1,\dots,k-1$; $a=0,\dots,\min\{l, R_k\}$; $wr = 0,\dots, k-l-1$ with $k > 1$, then the transition probability matrix is

$$\mathbf{D}_{l,l+wr+1-a}^{A_1,k,wr}\left(i_1,\dots,i_{l-a},1,0,\overset{wr}{\dots},0;\, j_1,\dots,j_l\right)=\left(\mathbf{H}_{k,(rep,1),l,wr}I_{\{wr<k-l-1 \text{ or } a>0\}}+\mathbf{H}'_{k,(rep,1),l,wr}I_{\{wr=k-l-1 \text{ and } a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\dots,i_{l-a},1,0,\overset{wr}{\dots},0;\, j_1,\dots,j_l;1,wr,0\right)$$

Finally, for any number of warm standby failures,

$$\mathbf{D}_{lh}^{A_1,k}\left(i_1,\dots,i_h;\, j_1,\dots,j_l\right)=\sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}}\mathbf{D}_{lh}^{A_1,k,h-l-1+a}\left(i_1,\dots,i_h;\, j_1,\dots,j_l\right).$$

4. Measures

Several interesting reliability measures are worked out in this section. Firstly, the transient and stationary distribution is obtained in a well-structured way. The results have a similar structure to those developed in Ruiz-Castro et al. (2018), but in this case taking into account the new MMAP for the warm standby system. New measures are built, such as the expected time working on corrective repair from warm standby and the expected number of warm standby failures up to a certain time and in the stationary regime.

4.1. Transient and stationary distribution

The transition probability matrix associated to the system is given by $\mathbf{D}=\mathbf{D}^0+\mathbf{D}^{A_1}+\mathbf{D}^{A_2}+\mathbf{D}^{B_1}+\mathbf{D}^{B_2}+\mathbf{D}^{B_3}+\mathbf{D}^{C_1}+\mathbf{D}^{C_2}+\mathbf{D}^{FC_1}+\mathbf{D}^{FC_2}$. The transient distribution is calculated as $\mathbf{p}^{\nu}=\boldsymbol{\theta}\mathbf{D}^{\nu}$, where $\boldsymbol{\theta}$ is the initial distribution of the system and we denote as $\mathbf{p}^{\nu}_{\mathbf{E}_s^k}$ to the probability of occupying the macro-state $\mathbf{E}_s^k$ at time ν. It can be worked out from $\mathbf{p}^{\nu}$.

It is well-known that the stationary distribution, $\boldsymbol{\pi}$, is built solving the balance equations, $\boldsymbol{\pi}=\boldsymbol{\pi}\mathbf{D}$. It has been calculated by applying matrix analytic methods. The stationary distribution for the macro-states $\mathbf{E}^k$ and $\mathbf{E}_s^k$ is denoted by $\boldsymbol{\pi}_{\mathbf{E}^k}$ and $\boldsymbol{\pi}_{\mathbf{E}_s^k}$ respectively. The development to calculate the stationary distribution is given in Appendix D.

4.2. Reliability measures

The availability, reliability, mean times and mean number of events for the complex system are built in this section. These are calculated in transient and stationary regime. The availability is the probability that the system is operational at time ν. The reliability can be defined in different ways for this system. One of them is the time up to first time that the system is stopped (none operational unit) and other is the time up to first time that the all units of the system undergoes a non-repairable failures and therefore it is replaced by a new one. In both cases the time distribution is phase-type with representation $(\boldsymbol{\theta}',\mathbf{D}')$ and $(\boldsymbol{\theta},\mathbf{D}^*)$. The initial vector $\boldsymbol{\theta}'$ and $\mathbf{D}'$ are $\boldsymbol{\theta}$ and $\mathbf{D}$ restricted to the macro-states $\mathbf{E}_s^k$ for k = 1,…, K and s = 0,…,k–1, respectively. The matrix $\mathbf{D}^*$ is the matrix $\mathbf{D}$ with blocks $\mathbf{D}^{FC1,1}=\mathbf{D}^{FC2,1}=\mathbf{0}$.

The mean time in each macro-state up to a certain time has also been calculated. From this measure, the mean operational time and the expected time that the repairpersons are either idle or busy are worked out. These measures are given in Table 1.

	Transient regime (up to time ν)	**Stationary regime**
Availability	$A(\nu)=1-\sum_{k=1}^{K}\mathbf{p}_{E_k^k}^{\nu}\cdot\mathbf{e}$	$A=1-\sum_{k=1}^{K}\boldsymbol{\pi}_{E_k^k}\cdot\mathbf{e}$
Mean time in $\mathbf{E}_s^k$	$\psi_{k,s}(\nu)=\sum_{m=0}^{\nu}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{e}$	$\psi_{k,s}=\boldsymbol{\pi}_{E_s^k}\cdot\mathbf{e}$
Mean time in $\mathbf{E}^k$	$\psi_k(\nu)=\sum_{s=0}^{k}\psi_{k,s}(\nu)$	$\psi_k=\sum_{s=0}^{k}\psi_{k,s}$
Mean operational time	$\mu_{op}(\nu)=\sum_{k=1}^{K}\sum_{s=0}^{k-1}\psi_{k,s}(\nu)$	$\mu_{op}=\sum_{k=1}^{K}\sum_{s=0}^{k-1}\psi_{k,s}$
Mean idle time	$\mu_{idle}(\nu)=\sum_{k=1}^{K}\sum_{s=0}^{k-1}\left(R_k-\min\{R_k,s\}\right)\cdot\psi_{k,s}(\nu)$	$\mu_{idle\_s}=\sum_{k=1}^{K}\sum_{s=0}^{k-1}\left(R_k-\min\{R_k,s\}\right)\cdot\psi_{k,s}$
Mean busy time	$\mu_{busy}(\nu)=\sum_{k=1}^{K}\sum_{s=1}^{k}\min\{R_k,s\}\cdot\psi_{k,s}(\nu)$	$\mu_{busy\_s}=\sum_{k=1}^{K}\sum_{s=1}^{k}\min\{R_k,s\}\cdot\psi_{k,s}$

Table 1. Some Measures associated to the system

4.3. Expected time working on corrective repair from online, warm standby and preventive repair

Three different types of repair can be carried out by the repairpersons: corrective repair (units from warm standby and from online place) or preventive maintenance. The expected time that all repairpersons are working on each type up to time ν is given respectively by

$$\mu_{warmcorr}(\nu) = \sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(0),\ \mu_{onlinecorr}(\nu) = \sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(1)\ \text{ and}$$

$$\mu_{pm}(\nu) = \sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(2),$$

where $\mathbf{q}_s^k(0)$, $\mathbf{q}_s^k(1)\,\mathbf{q}_s^k(1)$ and $\mathbf{q}_s^k(2)$ are column vectors with the number of repairpersons working on corrective repair for warm standby units, corrective repair for online units and preventive maintenance respectively according to the macro-state $\mathbf{E}_s^k$.

The column vectors are

$$\mathbf{q}_s^k(0) = \begin{pmatrix} h_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(1)} z_1^{d_s^k(1)} z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ h_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(2)} z_1^{d_s^k(2)} z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(2)}} \\ \vdots \\ h_s^k\left(i = I_{\{s=K\}}2\cdot 3^{k-1} + I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

$$\mathbf{q}_s^k(1) = \begin{pmatrix} d_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(1)} z_1^{d_s^k(1)} z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ d_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(2)} z_1^{d_s^k(2)} z_2^{\min\{s,R_k\}-h_s^k(2)-d_s^k(2)}} \\ \vdots \\ d_s^k\left(i = I_{\{s=K\}}2\cdot 3^{k-1} + I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

and

$$\mathbf{q}_s^k(2) = \begin{pmatrix} g_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(1)} z_1^{d_s^k(1)} z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ g_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(2)} z_1^{d_s^k(2)} z_2^{\min\{s,R_k\}-h_s^k(2)-d_s^k(2)}} \\ \vdots \\ g_s^k\left(i = I_{\{s=K\}}2\cdot 3^{k-1} + I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}\, z_0^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

being

- $h_s^k\ i$ the i-th element of the vector $1 \le s < k$; $\mathbf{h}_s^k = \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{3^{\max\{s-R_k,0\}}}$

  with

$$\mathbf{h}_k^k = I_{\{k=R_k \neq K\}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \begin{pmatrix}1\\0\end{pmatrix}\right]$$
$$+ I_{\{K \neq k > R_k\}} \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}} \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}}$$

- $d_s^k\ i$ the i-th element of the vector $\mathbf{d}_s^k = \begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \mathbf{e}_{3^{\max\{s-R_k,0\}}}$ with

$$\mathbf{d}_k^k = I_{\{k=R_k \neq K\}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \odot \begin{pmatrix}0\\1\end{pmatrix}\right]$$
$$+ I_{\{K \neq k > R_k\}} \begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}} \begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}}$$

- $g_s^k\ i$ the i-th element of the vector $\mathbf{g}_s^k = \begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{3^{\max\{s-R_k,0\}}}$ with

$$\mathbf{g}_k^k = I_{\{k=R_k \neq K\}}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_2\right]$$
$$+ I_{\{K \neq k > R_k\}} \begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}} \begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}}$$

for $k = 1,\ldots, K$ and $s = 1,\ldots,k$, where the operator $\odot$ is defined as follows. Let **a** and **b** be column vectors with order n and m respectively, then $\mathbf{a} \odot \mathbf{b} = \mathbf{a} \otimes \mathbf{e}_m + \mathbf{e}_n \otimes \mathbf{b}$.

In the stationary case they are

$$\mu_{warmcorr} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(0),\ \mu_{onlinecorr} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(1),\ \mu_{pm} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(2).$$

4.4. Expected number of events

The expected number of events up to time ν is determined using the Markovian Arrival Process with Marked arrivals obtained in Section 3.5. If the event considered is denoted by Y then it is

$$\Lambda^{Y}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\mathbf{D}^{Y}\mathbf{e},$$

for $Y = A_1, A_2, B_1, B_2, B_3, C_1, C_2\ FC_1, FC_2$.

This value in stationary regime is $\Lambda^{Y}=\boldsymbol{\pi}\mathbf{D}^{Y}\mathbf{e}$.

4.4.1. Mean number of warm standby failures

The expected number of failures from warm standby units is given by

$$\Lambda^{w}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\mathbf{V},$$

where $\mathbf{V}$ is the column vector $\mathbf{V}=(\mathbf{V}^{K}, \mathbf{V}^{K-1}, \mathbf{V}^{K-2},\dots, \mathbf{V}^{2}, \mathbf{V}^{1})'$, where $\mathbf{V}^{k}=\left(\mathbf{V}_{0}^{k},\mathbf{V}_{1}^{k},\dots,\mathbf{V}_{k-2}^{k},\mathbf{0},\mathbf{0}\right)'$ for $k = 2,\dots, K$ and $\mathbf{V}^1 = (\mathbf{0}, \mathbf{0})'$. The column vector $\mathbf{V}_{l}^{k}$ is given by

$$\mathbf{V}_{l}^{k}=\sum_{Y=O,C_1,C_2,A_1,A_2,B_1,B_2}\ \sum_{h=l-\min\{l,R_k\}+1+I_{\{A_1,A_2,B_1,B_2\}}}^{k-1+I_{\{A_1,A_2,B_1,B_2\}}}\ \sum_{a=\max\left\{0,l-h+I_{\{A_1,A_2,B_1,B_2\}}\right\}}^{\min\left\{k-h-1+I_{\{A_1,A_2,B_1,B_2\}},\min\{l,R_k\}\right\}}\left(h-l+a-I_{\{A_1,A_2,B_1,B_2\}}\right)\mathbf{D}_{lh}^{Y,k,h-l+a-I_{\{A_1,A_2,B_1,B_2\}}}\cdot\mathbf{e},$$

for $l = 0, 1, \dots, k–2$.

In the stationary case it is given by $\Lambda^{w}=\boldsymbol{\pi}\mathbf{V}$.

5. Net profit vector and total net profit

To analyze the effectiveness of the model from an economic point of view, costs and rewards have been taken into account and from them a net profit vector associated to the state-space is built. The following values have been introduced:

B: Gross profit per unit of time if the system is operational
$\mathbf{c_0}$: expected cost per unit of time depending on the operational phase if the system is operational
$\mathbf{cr_0}$: expected cost per unit of time for a unit that failed from online depending on the repair phase

$\mathbf{cr}_1$: expected cost per unit of time for a unit that failed from warm standby depending on the repair phase
$\mathbf{cr}_2$: expected cost per unit of time for a unit that was observed with major damage depending on the preventive maintenance phase
H: fixed cost for each repairperson per unit of time
C: loss per unit of time while the system is not operational
fcr: fixed cost each time that the online unit undergoes a repairable failure from the online unit
fwr: fixed cost each time that the online unit undergoes a repairable failure from warm standby
fpm: fixed cost each time that the online unit undergoes a major inspection
fnu: cost for a new unit

5.1. Net profit vector

When the system occupies a state, a net profit value is produced. Costs and rewards from the online unit and the cost provoked by the repairpersons have been taken into account to build the net profit vector.

Online unit

If only the online unit is considered when the system visits the macro-state $\mathbf{E}_s^k$, a net reward for the phases of this macro-state is provoked as

$$\mathbf{nr}_s^k = \begin{cases} B\mathbf{e}_{ntd\varepsilon} - \mathbf{c_0} \otimes \mathbf{e}_{td\varepsilon} & ; \quad s = 0 \\ B\mathbf{e}_{ntd\varepsilon \cdot \sum_{i=1}^{3^s} z_o^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} - \begin{pmatrix} \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(1)} z_1^{d_s^k(1)} z_2^{g_s^k(1)}} \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(2)} z_1^{d_s^k(2)} z_2^{g_s^k(2)}} \\ \vdots \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(3^s)} z_1^{d_s^k(3^s)} z_2^{g_s^k(3^s)}} \end{pmatrix} & ; \quad s = 1, \ldots, k-1 \\ -C \cdot \mathbf{e}_{t \cdot \sum_{i=1}^{I_{\{s=K\}} 2 \cdot 3^{k-1} + I_{\{s \neq K\}} 3^k} z_0^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} & ; \quad s = k. \end{cases}$$

Then, if the state space is considered it is

$$\mathbf{nr} = \left(\mathbf{nr}_0^K{}', \mathbf{nr}_1^K{}', \ldots, \mathbf{nr}_K^K{}', \mathbf{nr}_0^K{}', \mathbf{nr}_1^{K-1}{}', \ldots, \mathbf{nr}_{K-1}^{K-1}{}', \ldots, \mathbf{nr}_0^1{}', \mathbf{nr}_1^1{}' \right)'.$$

Repair facility

If only the repair facility is considered when the system visits the macro-state $\mathbf{E}_s^k$, a cost vector for the phases of this macro-state, for s = 1,…, k, with (k, s) ≠ (K, K), is

$$\mathbf{nc}_s^k = \begin{pmatrix} \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_0 \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_1 \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_2 \\ \vdots \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_0 \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_1 \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_2 \end{pmatrix}, \ s \geq 1, \ \mathbf{nc}_0^k = \mathbf{0}.$$

$$\mathbf{nc}_K^K = \begin{pmatrix} \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_0 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_0 \odot \mathbf{cr}_1 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_1 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_1 \odot \mathbf{cr}_1 \\ \vdots \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_2 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I_{\{K>R_K\}}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{K}{\cdots} \odot \mathbf{cr}_2 \odot \mathbf{cr}_1 \end{pmatrix}.$$

Then, the cost vector associated to the state space due to repair is given by

$$\mathbf{nc} = \left(\mathbf{nc}_0^K{}', \mathbf{nc}_1^K{}', \ldots, \mathbf{nc}_K^K{}', \mathbf{nc}_0^K{}', \mathbf{nc}_1^{K-1}{}', \ldots, \mathbf{nc}_{K-1}^{K-1}{}', \ldots, \mathbf{nc}_0^1{}', \mathbf{nc}_1^1{}'\right)'.$$

Therefore, the net profit vector for the macro-state $\mathbf{E}_s^k$ is given by $\mathbf{c}_0^k = \mathbf{nr}_0^k$ and $\mathbf{c}_s^k = \mathbf{nr}_s^k - \mathbf{nc}_s^k$ for s =1,…,k. Then, if the macro-state $\mathbf{E}^k$ is considered, the net column profit vector is $\mathbf{c}^k = \left(\mathbf{c}_0^k, \ldots, \mathbf{c}_k^k\right)'$. From these, the global net column profit vector is built for the state-space,

$$\mathbf{c} = \mathbf{nr} - \mathbf{nc} = \begin{pmatrix} \mathbf{c}^K \\ \mathbf{c}^{K-1} \\ \vdots \\ \mathbf{c}^1 \end{pmatrix}.$$

5.2. Expected net profit, expected cost and total net profit

Rewards measures are worked out, in transient and stationary regime, to analyze the effectiveness of the system from an economic point of view.

Expected net profit from the online unit up to time ν

If the online unit is considered, the expected net profit up to time ν is

$$\Phi_w^\nu = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{nr}.$$

In stationary regime it is given by $\Phi_{w\_s} = \boldsymbol{\pi} \cdot \mathbf{nr}$.

Expected cost from corrective repair and preventive maintenance

The expected cost because of corrective repair from online, warm standby and preventive maintenance up to time ν is calculated. It is

$\Phi_{online\_cr}^\nu = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{online\_cr}$, $\Phi_{warm\_cr}^\nu = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{warm\_cr}$ and $\Phi_{pm}^\nu = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{pm}$ where

$\mathbf{mc}^{warm\_cr}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_1 = \mathbf{0}_{z_1}$, $\mathbf{cr}_2 = \mathbf{0}_{z_2}$; $\mathbf{mc}^{online\_cr}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_0 = \mathbf{0}_{z_0}$, $\mathbf{cr}_2 = \mathbf{0}_{z_2}$ and $\mathbf{mc}^{pm}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_0 = \mathbf{0}_{z_0}$ and $\mathbf{cr}_1 = \mathbf{0}_{z_1}$, being $\mathbf{0}_a$ a column vector of 0's with order a.

If the stationary regime is considered, then

$$\Phi_{online\_cr\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{online\_cr},\ \Phi_{warm\_cr\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{warm\_cr} \text{ and } \Phi_{pm\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{pm}.$$

Total net profit

If costs, fixed costs and profits are considered, the total net profit up to time ν is

$$\Phi^{\nu}=\Phi_{w}^{\nu}-\Phi_{online\_cr}^{\nu}-\Phi_{warm\_cr}^{\nu}-\Phi_{pm}^{\nu}-\left(1+\Lambda^{FC1}(\nu)+\Lambda^{FC2}(\nu)\right)\cdot K\cdot fnu$$
$$-\Lambda^{w}(\nu)\cdot fwr-\left(\Lambda^{A1}(\nu)+\Lambda^{A2}(\nu)\right)\cdot fcr$$
$$-\left(\Lambda^{B1}(\nu)+\Lambda^{B2}(\nu)+\Lambda^{B3}(\nu)\right)\cdot fpm-\left(\mu_{idle}+\mu_{busy}\right)\cdot H.$$

In the stationary case it is

$$\Phi_{s}=\Phi_{w\_s}-\Phi_{cr\_s}-\Phi_{pm\_s}-\left(1+\Lambda^{FC1}+\Lambda^{FC2}\right)\cdot K\cdot fnu$$
$$-\Lambda^{w}\cdot fwr-\left(\Lambda^{A1}+\Lambda^{A2}\right)\cdot fcr$$
$$-\left(\Lambda^{B1}+\Lambda^{B2}+\Lambda^{B3}\right)\cdot fpm-\left(\mu_{idle}+\mu_{busy}\right)\cdot H.$$

6. *A numerical example*

In this section, we consider a system similar to that described in Ruiz-Castro et al. (2018) but in which warm standby units are included. This kind of system is motivated by situations such as the following. A ship's power station consists of three generator sets, to ensure reliable electrical supply. During cruising and berthing, the power station normally employs one generator set for power supply, while another two remain in standby state. This constitutes a typical warm standby system in which any of the generating sets may fail, from different levels of degradation, and preventive maintenance may be necessary. For this reason, a warm standby system with three units is employed. Two questions are addressed to optimise the system; the first about whether preventive maintenance is profitable or not, and the second question about the number of repairpersons that would have to be deployed to optimise the profitability. The optimum number of repairpersons and the effectiveness of preventive maintenance is calculated in this complex numerical example. We show that the optimum system when warm standby units are included is different from the cold case.

Five internal performance levels for the online unit are assumed, in which the first three stages correspond to minor degradation and the last two, to major degradation. This online unit can undergo external shocks and inspections. The time distributions are all phase-type distributed and the representations are given in Table 2.

Internal operational time	External shock	Inspection time
$\boldsymbol{\alpha} = (1,0,0,0,0)$ $\mathbf{T} = \begin{pmatrix} 0.99 & 0.002 & 0 & 0 & 0 \\ 0 & 0.9 & 0.001 & 0 & 0 \\ 0 & 0 & 0.9 & 0.002 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \end{pmatrix}$ Mean time: 102.0201	$\boldsymbol{\gamma} = (1,0)$ $\mathbf{L} = \begin{pmatrix} 0.89 & 0.1 \\ 0.1 & 0.8 \end{pmatrix}$ Mean time: 25	$\boldsymbol{\eta} = (1,0)$ $\mathbf{M} = \begin{pmatrix} 0.85 & 0.1 \\ 0.45 & 0.4 \end{pmatrix}$ Mean time: 15.56

Table 2. Operational times embedded in the system.

The probability of undergoing a total extreme failure (non-repairable) after an external shock is equal to ω^0=0.05. If this failure does not occur, the internal performance level may be modified under the probability matrix

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.6 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 \end{pmatrix}.$$

Each time that an external shock takes place, cumulative external damage occurs. The number of external degradation levels is four, where the first two are minor and the rest are major. The following probability matrix governs these changes

$$\mathbf{D} = \begin{pmatrix} 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}.$$

Initially, the external degradation level is 1 (no external damage). Each warm standby is also subject to failure. Each one can fail at any time with a probability equal to p = 0.008.

When the online unit is broken or major damage is observed by inspection, it goes to the repair facility. The corrective repair times, for the cases from online and from warm standby place, and the preventive maintenance time distribution are shown in Table 3.

Corrective repair time distribution Warm standby unit	Corrective repair time distribution Online unit	Preventive maintenance time distribution
$\boldsymbol{\beta}^0 = (1,0)$ $\mathbf{S}_0 = \begin{pmatrix} 0.9 & 0.02 \\ 0 & 0.6 \end{pmatrix}$ Mean time: 10.5	$\boldsymbol{\beta}^1 = (1,0)$ $\mathbf{S}_1 = \begin{pmatrix} 0.91 & 0.01 \\ 0 & 0.8 \end{pmatrix}$ Mean time: 11.67	$\boldsymbol{\beta}^2 = (1,0)$ $\mathbf{S}_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{pmatrix}$ Mean time: 1.23

Table 3. Time distributions embedded in the repair facility

Analysis of the system according to the number of repairpersons

A comparative study is performed to analyse the behaviour of the systems with and without preventive maintenance by considering all possible combinations for the number of repairpersons. We denote as *i_j_k* to the system with *i*, *j*, *k* repairpersons when there are 1, 2, 3 units in the system respectively, where the number of repairpersons is always fewer or equal to the number of units. Six possible systems with preventive maintenance and six without are analyzed and compared.

The expected operational time (first row in figure) and the expected number of idle repairpersons (second row in figure) for all systems are shown in Figure 3. The first column of this figure is focused on the case with preventive maintenance and the second on without it. If the mean operational ratio is observed (first row) the optimum value for the system with preventive maintenance, in the stationary case, is reached for system 1_2_3 and it is equal to 0.9410. The optimum value is reached for system 1_2_3 and it is equal to 0.9292 for without preventive maintenance case.

The expected number of idle repairpersons per unit of time is also studied. If Figure 3 is observed (second row), this value is equal to 1.8023 for the case with preventive maintenance and 1.6940 for the case without both in the stationary regime. In both case it is reached for system 1_2_3.

Each time that the units of the system undergo an event, a cost occurs. The expected number of events in a period of time has been calculated for every system in transient regime. The Tables 4 and 5 show these values for the case up to 1500 units of time.

The last column of Table 5 shows the number of replacements up to time 1500. This fact is essential given that each new system has a cost. The minimum is reached for system 1_1_1 (one repairperson when there are 1, 2 or 3 units in the system).

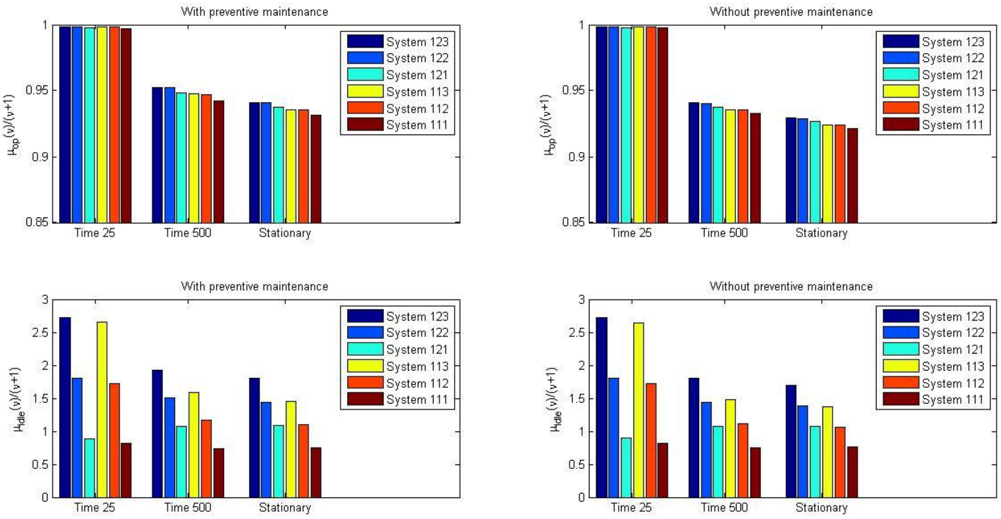

Figure 3. Expected operational time per unit of time (first row) and expected number of idle repairpersons per unit of time (second row)

	SYSTEM	$\Lambda^{A_1}(1500)$	$\Lambda^{A_2}(1500)$	$\Lambda^{B_1}(1500)$	$\Lambda^{B_2}(1500)$	$\Lambda^{B_3}(1500)$
Time 1500	1_2_3	21.5309 (22.2357)	0.0159 (0.0157)	0.0702	11.7203	0.2394
	1_2_2	21.5260 (22.2310)	0.0159 (0.0157)	0.0701	11.7174	0.2393
	1_2_1	21.4674 (22.1762)	0.0159 (0.0156)	0.0680	11.5083	0.2331
	1_1_3	21.4132 (22.1166)	0.0158 (0.0156)	0.0697	11.6528	0.2379
	1_1_2	21.4083 (22.1119)	0.0158 (0.0156)	0.0697	11.6499	0.2379
	1_1_1	21.3324 (22.0481)	0.0158 (0.0155)	0.0675	11.4354	0.2316

Table 4. Expected number of events up to time 1500 (no preventive maintenance in parenthesis)

	SYSTEM	$\Lambda^{w}(1500)$	$\Lambda^{C_1}(1500)$	$\Lambda^{C_2}(1500)$	$\Lambda^{FC_1}(1500)$ $+\Lambda^{FC_2}(1500)$
Time 1500	1_2_3	0.3397 (0.2751)	3.5492 (3.1521)	5.6844 (8.4947)	4.1467 (5.2923)
	1_2_2	0.3396 (0.2750)	3.5484 (3.1514)	5.6832 (8.4929)	4.1458 (5.2912)
	1_2_1	0.3301 (0.2698)	3.5343 (3.1440)	5.7003 (8.4729)	4.1469 (5.2785)
	1_1_3	0.3379 (0.2735)	3.5303 (3.1358)	5.6554 (8.4508)	4.1226 (5.2624)
	1_1_2	0.3378 (0.2735)	3.5295 (3.1351)	5.6541 (8.4490)	4.1216 (5.2613)
	1_1_1	0.3282 (0.2682)	3.5129 (3.1265)	5.6665 (8.4257)	4.1191 (5.2463)

Table 5. Expected number of events up to time 1500 (no preventive maintenance in parenthesis)

Including costs and rewards

An optimization from an economical point of view has been performed by including rewards and costs. The following costs and rewards have been considered,

B=100	Reward per unit of time while the system is operational.
$\mathbf{c}_0 = (10, 20, 30, 40, 50)'$	Operational cost per unit of time while the online unit is working.
$\mathbf{cr_0}$=(3, 3)'	Cost associated to a unit broken from warm standby that is being repaired per unit of time.
$\mathbf{cr_1}$ = (5, 5)'	Cost associated to a unit broken from online that is being repaired per unit of time.
$\mathbf{cr_2}$=(0.5, 0.5)'	Cost associated to a unit in preventive maintenance per unit of time.
fcr = 20	Fixed cost each time that a repairable failure takes place (online unit).
fwr = 5	Fixed cost each time that a repairable failure takes place (warm standby).
fpm = 1	Fixed cost due to one major inspection.
fnu = 200	Cost of a new unit.

The expected net reward has been worked out for every system and the most profitable one is achieved. Figure 4 and Table 6 show this value per unit of time.

If the stationary case is observed, the optimum expected net profit is observed for system 1_2_2 with preventive maintenance (1 repairperson when there is only 1 unit, 2 repairpersons when there are 2 units in the system and 2 repairpersons for the case 3 units in the system). The optimum expected net profit is equal to 72.6360 as it can see in Table 6.

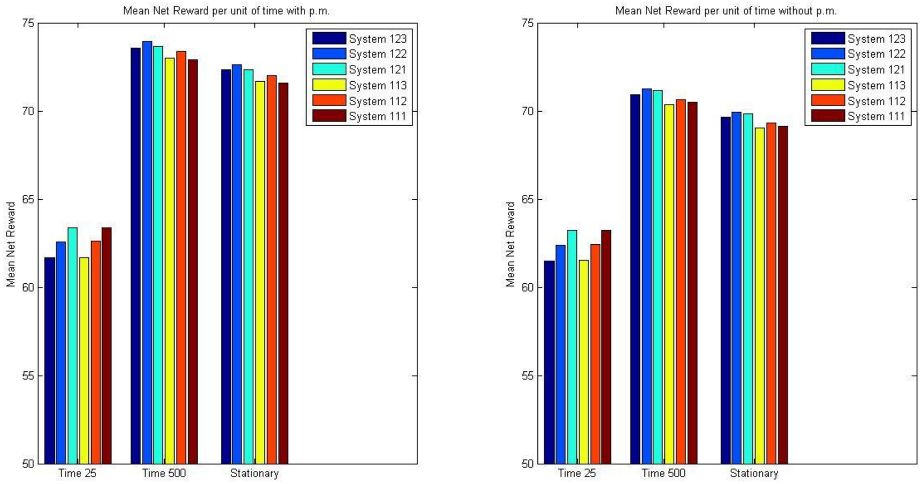

Figure 4. Expected net profit per unit of time up to time 25 and 500 units of time and the stationary case (with and without preventive maintenance)

	$\phi^{\nu}/(\nu+1)$					
System	1_2_3	1_2_2	1_2_1	1_1_3	1_1_2	1_1_1
$\nu = 25$	61.6543 (61.4718)	62.5604 (62.3677)	63.3604 (63.2103)	61.6905 (61.5142)	62.5965 (62.4102)	63.3758 (63.2470)
$\nu = 500$	73.5558 (70.9204)	73.9376 (71.2474)	73.6411 (71.1516)	72.9723 (70.3308)	73.3527 (70.6565)	72.9027 (70.4836)
Stationary	72.3181 (69.6454)	72.6360 (69.9214)	72.3478 (69.8170)	71.6840 (69.0346)	72.0000 (69.3091)	71.5751 (69.1337)

Table 6. Expected net profit per unit of time up to time 25 and 500 and the stationary case (without preventive maintenance in parenthesis)

7. Conclusions

A complex multi-state warm standby system subject to multiple events, with a variable number of repairpersons, is modelled. A Markovian Arrival Process with Marked arrivals is employed in this modelling process. The online and the warm standby units may undergo failures. Both repairable and non-repairable failures may take place, according to the internal behaviour of the online unit and the possible occurrence of external shocks. The system may experience the loss of units. Whenever a non-repairable failure occurs, the unit is discarded and the number of repairpersons may be modified. The system continues working as long as it contains at least one unit. The system is optimised according to this number. In addition, preventive maintenance is performed to establish an optimum system, according to the distribution of the number of repairpersons.

Multiple events (including failures) may occur simultaneously, affecting any warm standby unit and/or the online unit. Several performance and economic measures were obtained, algorithmically and computationally. A numerical application shows that the number of repairpersons at any time can be determined, and whether preventive maintenance is profitable, with respect to optimising the system.

The methodology used in this paper, in which phase type distributions and Markovian arrival processes play an essential role, makes it possible to successfully model complex systems, and thanks to the well-matched properties of these distributions and processes, the main results are obtained in a matrix-algorithmic form. Moreover, with the structure described, costs and rewards can be included. The results obtained can be interpreted straightforwardly and implemented without computational difficulties. In addition, the methodology applied to model the system proposed can also be used to analyse other types of redundant complex multi-state systems, such as k–out–of–n: G systems. Finally, other features, such as the vacation time taken by repairpersons, can be included in the system and the effects of this circumstance determined.

The model proposed can be applied to the analysis of real-world systems. As stated in the Introduction, this system can be applied in many fields, including civil and industrial engineering. Examples of a warm standby repairable system in practical use include the installation of a back-up server, or the provision of a reliable electrical supply, a fluid transfer system or a control system for an industrial-scale greenhouse. This control system consists of a warm standby repairable system that is available as an immediate replacement if the main CPU fails. During the standby time, because of the aging effect and the impact of the environment, many units (especially those with delicate instrumentation) will break down.

A major problem that can arise when phase type distributions and MMAPs are used to model real-life problems is that of estimating the parameters embedded in the model. Such estimation depends on the model in question, and diverse results have been obtained. Asmussen et al. (1996) developed an EM algorithm to estimate these parameters, and Buchholz et al. (2014) analysed this method in detail. This research group is currently working to estimate PH distributions and MMAPS in real-world problems (Acal et al., 2019; Pérez et al., 2019), determining the number of phases and parameters using the EM algorithm and various models, and performing pre-analyses in order to reduce the number of parameters to be addressed.

The authors of this paper are currently investigating several fields of research, one of which is the modelling of complex redundant systems, such as cold, warm and *k*-out-of-*n*: *G* systems. The number of possible events affecting the units and the features of the repair facility are addressed in detail. The ideas presented in this paper can be extended in several ways. For example, different types of replacement policies, when a minimal repair is required, might be included. Another possibility would be to consider a *k*-out-of-*n*: *G* system with an indeterminate and variable number of repairpersons in the repair facility with and without the loss of units. Thus, each time that a non-repairable failure occurs and the system is able to continue working, the number of repairpersons can be modified, which would enable us to optimise the model.

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Appendix A

Modelling the online unit

- *O*: No events:

$$\mathbf{H}_0 = \left[\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{I} + \mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{D}\left(1-\omega^0\right)\right]\otimes\mathbf{M}$$
$$+\left[\mathbf{U}_1\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{V}_1 + \mathbf{U}_1\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{D}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta$$

$$\mathbf{H}^{'}_{\text{mr}} = \left[\mathbf{U}_1\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{V}_2\mathbf{I} + \mathbf{U}_2\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{I}\right.$$
$$\left.+\mathbf{U}_1\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_2\mathbf{D}\left(1-\omega^0\right) + \mathbf{U}_2\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{D}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta$$

- A_1: Internal repairable failure (online unit)

$$\mathbf{H}_{\text{rep},1} = \left[\mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{e}\boldsymbol{\omega} + \mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon\eta.$$

$$\mathbf{H}^{'}_{\text{rep},1} = \left[\mathbf{T}_r^0\otimes\mathbf{L}\otimes\mathbf{e}_d + \mathbf{T}_r^0\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon.$$

- A_2: External repairable failure of the online unit

$$\mathbf{H}_{\text{rep},2} = \left[\mathbf{TW}^0\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon\eta.$$

$$\mathbf{H}'_{\text{rep},2} = \left[\mathbf{TW}^0 \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{De}\left(1-\omega^0\right) \right] \otimes \mathbf{e}_{\varepsilon}.$$

- *B*: Major revision
  - B_1: Major revision (only internal major damage)

$$\mathbf{H}_{\text{mr},1} = \left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{V}_1\mathbf{e}\boldsymbol{\omega} + \mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{V}_1\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right) \right] \otimes \mathbf{M}^0\eta$$

  - B_2: Major revision (only external cumulative damage)

$$\mathbf{H}_{\text{mr},2} = \left[\mathbf{U}_1\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{V}_2\mathbf{e}\boldsymbol{\omega} + \mathbf{U}_1\mathbf{TWe}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{V}_2\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right) \right] \otimes \mathbf{M}^0\eta$$

  - B_3: Major revision (both internal and external cumulative damage)

$$\mathbf{H}_{\text{mr},3} = \left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{V}_2\mathbf{e}\boldsymbol{\omega} + \mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{V}_2\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right) \right] \otimes \mathbf{M}^0\eta$$

- *C*: Non-repairable failure
  - C_1: Non-repairable internal failure:

$$\mathbf{H}_{\text{nrep},1} = \mathbf{T}^0_{nr}\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{e}\boldsymbol{\omega} \otimes \mathbf{e}\boldsymbol{\eta} + \mathbf{T}^0_{nr}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right) \otimes \mathbf{e}\eta$$

$$\mathbf{H}'_{\text{nrep},1} = \mathbf{T}^0_{nr} \otimes \mathbf{L} \otimes \mathbf{e} \otimes \mathbf{e} + \mathbf{T}^0_{nr} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{De}\left(1-\omega^0\right) \otimes \mathbf{e}$$

  - C_2: Non-repairable failure because of shock:

$$\mathbf{H}_{\text{nrep},2} = \mathbf{e}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \left(\mathbf{e}\boldsymbol{\omega}\omega^0 + \mathbf{D}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right) \otimes \mathbf{e}\eta$$

$$\mathbf{H}'_{\text{nrep},2} = \mathbf{e} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \left(\mathbf{e}\omega^0 + \mathbf{D}^0\left(1-\omega^0\right)\right) \otimes \mathbf{e}$$

Appendix B

Auxiliary functions for the repair facility

The output of the matrix function $C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right)$ is the transition probability for the units that are being repaired when the order of the units repaired is specified.

$$C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right) = \begin{cases} \mathbf{S}(1) \otimes \cdots \otimes \mathbf{S}\left(\min\{l,R_k\}\right) & ; \begin{array}{l} i_{s-\sum_{z=1}^{a} I_{\{k_z<s\}}} = j_s; s=1,\ldots,l; s \neq k_z, \forall z \\ i_{l-a+1} = 1,2;\ \text{if } mr = 1 \\ i_s = 0; s = l-a+mr+1,\ldots,l-a+mr+wr \end{array} \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

for $k \le K$, $l \ge 1$, $a \ge 1$, $b \ge 0$, where

$$S(h)=\begin{cases}\mathbf{S}^0_{j_h} & ; \quad \exists z\in\{1,\ldots,a\} \mid h=k_z\\ \mathbf{e}-\mathbf{S}^0_{j_h} & ; \quad h \text{ is the ordinal of the last } b \text{ units being repaired without ending}\\ \mathbf{S}_{j_h} & ; \quad \text{otherwise}\end{cases}.$$

If a = 0, then it is denoted as

$$C(k,l,a=0,b;i_1,\ldots,i_{l+mr+wr};j_1,\ldots,j_l)=$$

$$\begin{cases}S_{j_1}\otimes\ldots\otimes S_{j_{\min\{l,R_k\}-b}}\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}-b+1}}\right)\otimes\ldots\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}}}\right) & ; \quad \begin{array}{l} i_s=j_s;1\le s\le l\\ i_{l+1}=1,2 \text{ if } mr=1\\ i_s=0; s=l+mr+1,\ldots,l+mr+wr\end{array}\\ \mathbf{0} & ; \quad \text{otherwise}\end{cases}$$

If $a=\min\{l,R_k\}$, it is

$$C(k,l,a=\min\{l,R_k\},0;j_1,\ldots,j_l)=S^0_{j_1}\otimes\ldots\otimes S^0_{j_{\min\{l,R_k\}}}.$$

Given a and if the sequence of the units that are repaired is not specified, the transition probability function is

$$B(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l)$$
$$=\begin{cases}C(k,l,0,b;i_1,\ldots,i_{l+mr+wr};j_1,\ldots,j_l) & ; \quad a=0,b\ge 0,l>0\\ \sum_{k_1=1}^{\min\{l,R_k\}-a+1}\sum_{k_2=k_1+1}^{\min\{l,R_k\}-a+2}\cdots\sum_{k_a=k_{a-1}+1}^{\min\{l,R_k\}}C(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l) & ; \quad a>0, a\ne\min\{l,R_k\}\\ C(k,l,a,0;j_1,\ldots,j_l) & ; \quad a=\min\{l,R_k\},\\ 1 & ; \quad a=0,b=0,l=0\end{cases}$$

Appendix C

Blocks of the Markovian Arrival Process

$$\left(\mathbf{D}^{O},\mathbf{D}^{A_1},\mathbf{D}^{A_2},\mathbf{D}^{B_1},\mathbf{D}^{B_2},\mathbf{D}^{B_3},\mathbf{D}^{C_1},\mathbf{D}^{C_2},\mathbf{D}^{FC_1},\mathbf{D}^{FC_2}\right).$$

Each matrix block is composed again of matrix-blocks as follows.

$\mathbf{D}^{Y}=diag\left(\mathbf{D}^{Y,K},\mathbf{D}^{Y,K-1},\mathbf{D}^{Y,K-2},\ldots,\mathbf{D}^{Y,1}\right)$, for $Y = O, A_1, A_2, B_1, B_2$ and

$$\mathbf{D}^{Y}=\begin{pmatrix}\mathbf{0} & \mathbf{D}^{Y,K} & & & \\ & \mathbf{0} & \mathbf{D}^{Y,K-1} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mathbf{D}^{Y,2}\\ \mathbf{0} & & & & \mathbf{0}\end{pmatrix}, \text{ for } Y=C_1, C_2, \text{ and } \mathbf{D}^{Y}=\begin{pmatrix}\mathbf{0} & \ldots & \ldots & \mathbf{0}\\ \vdots & \ddots & & \vdots\\ \mathbf{0} & & \ddots & \vdots\\ \mathbf{D}^{Y,1} & \mathbf{0} & \ldots & \mathbf{0}\end{pmatrix},$$

for $Y=FC_1, FC2$.

- Matrix $\mathbf{D}^{Y,k}$ for $Y = O$, C_1, C_2, FC_1, FC_2

The elements of the matrix $\mathbf{D}^{Y,k}$ for $k = 1,...,K$ and $Y = O$, C_1, C_2, FC_1, FC_2, are

$$\mathbf{D}^{Y,k} = \begin{cases} \left(\mathbf{D}_{lh}^{Y,k}\right)_{l,h=0,\ldots,k} & ; \quad Y = O \\ \left(\mathbf{D}_{lh}^{Y,k}\right)_{\substack{l=0,\ldots,k \\ h=0,\ldots,k-1}} & ; \quad Y = C_1, C_2 \\ \left(\mathbf{D}_{lh}^{Y,1}\right)_{\substack{l=0,1 \\ h=0,\ldots,K}} & \quad Y = FC_1, FC_2; k = 1, \end{cases}$$

where $\mathbf{D}_{lh}^{O,k} = \mathbf{0}$ if $h = k$ for $l < k$ or $h < l - \min\{l, R_k\}$, $\mathbf{D}_{lh}^{C_i,k} = \mathbf{0}$ if $h < l - \min\{l, R_k\}$ or $l = k$ and $\mathbf{D}_{lh}^{FC_i,1} = \mathbf{0}$ for all l and h excepting when $l = h = 0$.

For $k = 1,\ldots, K$, and $type = 1, 2$,

$$\mathbf{D}_{00}^{Y,k} = \begin{cases} \mathbf{H}_{k,0,0,0} + I_{\{k=1\}}\mathbf{H}'_{k,mr,0,0} & ; \quad Y = O \\ \mathbf{H}_{k,(nrep,type),0,0} & ; \quad Y = C_{type} \text{ or } Y = FC_{type} \end{cases}.$$

For $l = 1,..., R_k$,

$$\mathbf{D}_{l,0}^{Y,k}\left(j_1,\ldots,j_l\right) = \begin{cases} \mathbf{H}_{k,0,l,0} \otimes E\left(k,l,l,0; j_1,\ldots,j_l; 0,0,0\right) & ; \quad Y = O \text{ and } l < k \\ \mathbf{H}_{k,(nrep,type),l,0} \otimes E\left(k,l,l,0; j_1,\ldots,j_l; 0,0,1\right) & ; \quad Y = C_{type} \text{ and } l < k \\ \boldsymbol{\zeta} \otimes E\left(k,l,l,0; j_1,\ldots,j_l; 0,0,0\right) & ; \quad Y = O \text{ and } l = k, \end{cases}$$

with $\boldsymbol{\zeta} = \alpha \otimes \left(\mathbf{L} + \mathbf{L}^0\boldsymbol{\gamma}\right) \otimes \boldsymbol{\eta} \otimes \boldsymbol{\omega}$.

For $wr = 1,\ldots, k-1$ with $k > 1$

$$\mathbf{D}_{0,wr}^{Y,k}\left(0,\overset{wr}{\ldots},0\right) =$$

$$\begin{cases} \left[\mathbf{H}_{k,0,0,wr} + I_{\{wr=k-1\}}\mathbf{H}'_{k,mr,0,wr}\right] \otimes E\left(k,0,0,0;0,\overset{wr}{\ldots},0;0,wr,0\right) & ; \quad Y = O \\ \left(\mathbf{H}_{k,(nrep,type),0,wr} I_{\{wr<k-1\}} + \mathbf{H}'_{k,(nrep,type),0,wr} I_{\{wr=k-1\}}\right) \otimes E\left(k,0,0,0;0,\overset{wr}{\ldots},0;0,wr,1\right) & ; \quad Y = C_{type}. \end{cases}$$

For $l = 1,\ldots, k-1$; $a = 0,\ldots, \min\{R_k, l\}$; $wr = 0,\ldots, k-l-1$ with $k > 1$ and $l + wr - a > 0$

$$\mathbf{D}^{Y,k,wr}_{l,l+wr-a}\left(i_1,\ldots,i_{l-a},0,\overset{wr}{\ldots},0;\, j_1,\ldots,j_l\right)=$$

$$\begin{cases} \left[\mathbf{H}_{k,0,l,wr}+I_{\{wr=k-l-1 \text{ and } a=0\}}\mathbf{H}'_{k,mr,l,wr}\right] & \\ \otimes E\left(k,l,a,0;i_1,\ldots,i_{l-a},0,\overset{wr}{\ldots},0;\, j_1,\ldots,j_l;0,wr,0\right) & ;\quad Y=O \\ \left(\mathbf{H}_{k,(nrep,type),l,wr}I_{\{wr<k-l-1 \text{ or } a>0\}}+\mathbf{H}'_{k,(nrep,type),l,wr}I_{\{wr=k-l-1 \text{ and } a=0\}}\right) & \\ \otimes E\left(k,l,a,b;i_1,\ldots,i_{l-a};\, j_1,\ldots,j_l;0,wr,1\right) & ;\quad Y=C_{type}. \end{cases}$$

Then,

$$\mathbf{D}^{Y,k}_{lh}\left(i_1,\ldots,i_h;\, j_1,\ldots,j_l\right)=\sum_{a=\max\{0,l-h\}}^{\min\{k-h-1,\min\{l,R_k\}\}}\mathbf{D}^{Y,k,h-l+a}_{lh}\left(i_1,\ldots,i_h;\, j_1,\ldots,j_l\right)\ .$$

For $a = 1,\ldots, \min\{R_k, k-1\}$,

$\mathbf{D}^{O,k}_{k,k-a}\left(i_1,\ldots,i_{k-a};\, j_1,\ldots,j_k\right)=\boldsymbol{\zeta}\otimes E\left(k,k,a,0;i_1,\ldots,i_{k-a};\, j_1,\ldots,j_k;0,0,0\right)$, with $k > 1$.

For $k = 1,\ldots, K$,

$\mathbf{D}^{O,k}_{k,k}\left(i_1,\ldots,i_k;\, j_1,\ldots,j_k\right)=\left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)\otimes E\left(k,k,0,0;i_1,\ldots,i_k;\, j_1,\ldots,j_k;0,0,0\right)$.

- *Matrix* $\mathbf{D}^{A_i,k}$

The elements of the matrix $\mathbf{D}^{A_i,k}$ for i=1, 2 and for k = 1,...,K are given by

$$\mathbf{D}^{A_i,k}=\left(\mathbf{D}^{A_i,k}_{lh}\right)_{l,h=0,\ldots,k},$$

where $\mathbf{D}^{A_i,k}_{lh}=\mathbf{0}$ if $h < l+1-\min\{l, R_k\}$ or $l = k$.

For *type* = 1, 2 and h=1, …, k then

$$\mathbf{D}^{A_{type},k}_{0h}\left(1,0,\overset{h-1}{\ldots},0\right)=\left(\mathbf{H}_{k,(rep,type),0,h-1}I_{\{h<k\}}+\mathbf{H}'_{k,(rep,type),0,h-1}I_{\{h=k\}}\right)$$
$$\otimes E\left(k,0,0,0;1,0,\overset{h-1}{\ldots},0;1,h-1,0\right).$$

For l=1,…,k–1; a=0,…,min{l, R_k}; wr = 0,…, k–l–1 with k > 1,

$$\mathbf{D}_{l,l+wr+1-a}^{A_{type},k,wr}\left(i_1,\ldots,i_{l-a},1,0,\overset{wr}{\ldots},0;\,j_1,\ldots,j_l\right)=\left(\mathbf{H}_{k,(rep,type),l,wr}I_{\{wr<k-l-1\text{ or }a>0\}}+\mathbf{H}'_{k,(rep,type),l,wr}I_{\{wr=k-l-1\text{ and }a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\ldots,i_{l-a},1,0,\overset{wr}{\ldots},0;\,j_1,\ldots,j_l;1,wr,0\right).$$

Then,

$$\mathbf{D}_{lh}^{A_{type},k}\left(i_1,\ldots,i_h;\,j_1,\ldots,j_l\right)=\sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}}\mathbf{D}_{lh}^{A_{type},k,h-l-1+a}\left(i_1,\ldots,i_h;\,j_1,\ldots,j_l\right)\;.$$

- *Matrix* $\mathbf{D}^{B_i,k}$

The elements of the matrix $\mathbf{D}^{B_i,k}$ for i = 1, 2, 3 and for k = 1,..., K are given by

$$\mathbf{D}^{B_i,k}=\left(\mathbf{D}_{lh}^{B_i,k}\right)_{l,h=0,\ldots,k},$$

where $\mathbf{D}_{lh}^{B_i,k}=\mathbf{0}$ if $h<l+1-\min\{l,R_k\}$ or $h=k$ or $l=k$.

For *type* = 1, 2, 3 then for h = 1, …, k–1

$$\mathbf{D}_{0h}^{B_{type},k}\left(2,0,\overset{h-1}{\ldots},0\right)=\mathbf{H}_{k,(mr,type),0,h-1}I_{\{k>1\}}\otimes E\left(k,0,0,0;2,0,\overset{h-1}{\ldots},0;1,h-1,0\right).$$

For l=1,…,k–1; a=0,…,min{l, R_k}; wr = 0,…, k–l–1 with k > 1,

$$\mathbf{D}_{l,l+wr+1-a}^{B_{type},k,wr}\left(i_1,\ldots,i_{l-a},2,0,\overset{wr}{\ldots},0;\,j_1,\ldots,j_l\right)$$
$$=\mathbf{H}_{k,(mr,type),l,wr}\otimes E\left(k,l,a,0;i_1,\ldots,i_{l-a},2,0,\overset{wr}{\ldots},0;\,j_1,\ldots,j_l;1,wr,0\right)$$

Then, $\mathbf{D}_{lh}^{B_{type},k}\left(i_1,\ldots,i_h;\,j_1,\ldots,j_l\right)=\sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}}\mathbf{D}_{lh}^{B_{type},k,h-l-1+a}\left(i_1,\ldots,i_h;\,j_1,\ldots,j_l\right)$

APPENDIX D

The stationary distribution, $\boldsymbol{\pi}=\left(\boldsymbol{\pi}_{\mathbf{E}^K},\boldsymbol{\pi}_{\mathbf{E}^{K-1}},\ldots,\boldsymbol{\pi}_{\mathbf{E}^1}\right)$, verifies $\boldsymbol{\pi}\mathbf{D}=\boldsymbol{\pi}$ and $\boldsymbol{\pi}\mathbf{e}=1$. Matrix-algebraic methods have been used to work out $\boldsymbol{\pi}_{\mathbf{E}^k}$, stationary distribution for $\mathbf{E}^k$. The transition probability matrices $\mathbf{D}_{k,k}$ and $\mathbf{D}_{k,k-1}$ corresponds to the transitions $\mathbf{E}^k\rightarrow\mathbf{E}^k$ and $\mathbf{E}^k\rightarrow\mathbf{E}^{k-1}$ respectively. The balance equations are

$$\begin{aligned}
\mathbf{D}_{k,k}&=\mathbf{D}^{O,k}+\mathbf{D}^{A_1,k}+\mathbf{D}^{A_2,k}+\mathbf{D}^{B_1,k}+\mathbf{D}^{B_2,k}+\mathbf{D}^{B_3,k} &&;\quad k=1,\ldots,K\\
\mathbf{D}_{k,k-1}&=\mathbf{D}^{C_1,k}+\mathbf{D}^{C_2,k} &&;\quad k=2,\ldots,K\\
\mathbf{D}_{1,K}&=\mathbf{D}^{FC_1,1}+\mathbf{D}^{FC_2,1}
\end{aligned}$$

The stationary distribution is calculated. This is equal to

$$\boldsymbol{\pi}_{\mathbf{E}^k} = \boldsymbol{\pi}_{\mathbf{E}^1}\mathbf{R}_{1,k}, \text{ for } k = 2,\ldots,K,$$

being

$$\mathbf{R}_{1,K} = \mathbf{D}_{1,K}\left(\mathbf{I}-\mathbf{D}_{K,K}\right)^{-1}$$

$$\mathbf{R}_{1,k} = \mathbf{R}_{1,k+1}\mathbf{D}_{k+1,k}\left(\mathbf{I}-\mathbf{D}_{k,k}\right)^{-1} \text{ for } k=2,\ldots,K-1.$$

The vector $\boldsymbol{\pi}_{\mathbf{E}^1}$ is

$$\boldsymbol{\pi}_{\mathbf{E}^1} = (1,\mathbf{0})\left(\left(\mathbf{I}+\sum_{k=2}^{K}\mathbf{R}_{1,k}\right)\mathbf{e}\middle|\left[\mathbf{D}_{1,1}+\mathbf{R}_{1,2}\mathbf{D}_{2,1}-\mathbf{I}\right]^{*}\right)^{-1},$$

where $\mathbf{A}^*$ denotes the matrix $\mathbf{A}$ without the first column.

References

Acal, C.; Ruiz-Castro, J.E.; Aguilera, A.M.; Jiménez-Molinos, F. and Roldán, J.B. (2019) Phase-type distributions for studying variability in resistive memories. *Journal of Computational and Applied Mathematics*, **345**, 23–32.

Alfa A.S. (2016) *Applied discrete-time queues*. Springer Science+Business Media New York.

Artalejo, J. R.: Gómez-Corral, A. and He, Q.M. (2010) Markovian arrivals in stochastic modelling: a survey and some new results. *Statistics and Operations Research Transactions*, **34**, 2, 101–144.

Asmussen, S.; Nerman, O. and Olsson, O. (1996) Fitting phase-type distributions via EM-algorithm. *Scandinavian Journal of Statistics*, **23**, 4, 419–441.

Barbu, V.S. and Limnios, N. (2008) Reliability of Semi-Markov Systems in Discrete Time: Modeling and Estimation. *In: Handbook of Performability Engineering*, 369–380. Springer, London.

Buchholz, P.; Kriege, J. and Felko, I. (2014) *Input modeling with phase-type distributions and Markov models. Theory and applications*. Springer Cham, Heidelberg New York Dordrecht London.

El-Neweihi, E. and Proschan, F. (1984) Degradable systems: a survey of multistate system theory. *Communication Statistics Theory Methods*, **13**, 405–432.

Goel, M. and Kumar, J. (2018). Stochastic analysis of a two –unit cold standby system with preventive maintenance and general distribution of all random variables. *International Journal of Statistics and Reliability Engineering*, **5**, 1, 10–21.

Georgiadis, S. and Limnios, N. (2014) Interval reliability for semi-Markov systems in discrete time. *Journal de la Société Française de Statistique*, **153**, 3, 152−166.

He, Q-M. (2014) *Fundamentals of Matrix-Analytic Methods*. Springer, New York.

Jia, H.; Ding, Y.; Peng, R. and Song, Y. (2017) Reliability evaluation for demand-based warm standby systems considering degradation process. *IEEE Transactions on Reliability*, **66**, 3, 795−805.

Levitin, G.; Finkelstein, M. and Dai, Y. (2017) Redundancy optimization for series-parallel phased mission systems exposed to random shocks. *Reliability Engineering and System Safety*, **167**, 554−560.

Levitin, G.; Xing, L.; Dai, Y. (2013) Optimal sequencing of warm standby elements. *Computers & Industrial Engineering*, **65**, 570−576.

Li, F.; Yin, D. and Hu, B. (2016) Analysis on reliability model for warm standby system with a repairman taking multiple vacations based on phase-type distribution. *In: Proceedings of the 2016 IEEE IEEM*, 1436−1442.

Li, Y.; Cui L. and Lin C. (2017) Modeling and analysis for multi-state systems with discrete-time Markov regime-switching. *Reliability Engineering and System Safety*, **166**, 41-49.

Lisnianski, A. and Frenkel, I. (2012). *Recent advances in system reliability: Signatures, multi-state systems and statistical inference* . Springer-Verlag London Limited .

Lisnianski, A.; Frenkel, I. and Ding, Y. (2010) Multi-state system reliability analysis and optimization for engineers and industrial managers. Springer-Verlag London.

Murchland, J. (1975) Fundamental concepts and relations for reliability analysis of multistate systems, In: Barlow, R.E., Fussell, J.B. and Singpurwalla, N. (eds) *Reliability and fault tree analysis*: *theoretical and applied aspects of system reliability. SIAM, Philadelphia*, pp. 581−618.

Nakagawa, T. (2005) *Maintenance Theory on Reliability*. Springer Series in Reliability Engineering series. Springer-Verlag London Limited.

Natvig, B. (1985) Multi-state coherent systems. In: Jonson N, Kotz S (eds) *Encyclopedia of statistical sciences*, **5**, 732–735. Wiley, New York.

Neuts, M.F. (1975) Probability Distributions of Phase Type. *In: Emeritus H Florin, editor*. *Liber Amicorum. Belgium:* Department of Mathematics, University of Louvain, 173–206.

Neuts, M.F. (1979) A versatile Markovian point process. *Journal of Applied Probability*, **16**, 764-779.

Neuts, M.F. (1981) *Matrix geometric solutions in stochastic models. An algorithmic approach,* Baltimore: John Hopkins University Press.

Osaki, S. and Asakura, A. (1970) A two-unit standby redundant system with repair and preventive maintenance. *Journal of Applied Probability*, **7**, 641-648.

Peng, R.; Xiao, H. and Liu, H. (2017) Reliability of multi-state systems with a performance sharing group of limited size. *Reliability Engineering and System Safety*, **166**, 164–170.

Pérez, E.; Maldonado, D.; Acal, C.; Ruiz-Castro, J.E.; Alonso, F.J. ; Aguilera, A.M.; Jiménez-Molinos, F.; Wenger, Ch. and J.B. Roldán (2019) Analysis of the statistics of device-to-device and cycle-to-cycle variability in TiN/Ti/Al:HfO2/TiN RRAMs. *Microelectronic Engineering* , **214**, 1, 104-109.

Qiu, Q.; Cui, L. and Dong.Q (2018) Preventive maintenance policy of single-unit systems based on shot-noise process. Quality and Reliability Engineering,..

Ruiz-Castro and Dawabsha, M. (2018) A discrete MMAP for analysing the behaviour of a multi-state complex dynamic system subject to multiple events. *Discrete event dynamic system*, DOI: 10.1007/s10626-018-0274-0

Ruiz-Castro, J.E.; Dawabsha, M. and Alonso, F.J. (2018) Discrete-timemarkovian arrival processes to model multi-state complex systems with loss of units and an indeterminate variable number of repairpersons. *Reliability Engineering and System Safety*, **174,** 114-127.

Ruiz-Castro, J. E and Li, Q.-L. (2011) Algorithm for a general discrete *k*-out-of-*n*: *G* system subject to several types of failure with an indefinite number of repairpersons. *European Journal of Operational Research*, **211**, 97–11.

Ruiz-Castro, J.E. (2013) A preventive maintenance policy for a standby system subject to internal failures and external shocks with loss of units. *International Journal of Systems Science*, **46**, 9, 1600–1613.

Ruiz-Castro, J.E. (2014) Preventive maintenance of a multi-state device subject to internal failure and damage due to external shocks. *IEEE Transactions on Reliability*, **63**, 2, 646-660.

Ruiz-Castro, J.E. (2016a) Complex multi-state systems modelled through Marked Markovian Arrival Processes. *European Journal of Operational Research,* **252**, 3,852-865.

Ruiz-Castro, J.E. (2016b) Markov counting and reward processes for analyzing the performance of a complex system subject to random inspections. *Reliability Engineering and System Safety*, vol. **145**, pp. 155–168.

Ruiz-Castro, J.E. (2018) A D-MMAP to model a complex multi-state system with loss of units. *In: Recent advances in Multi-state systems Reliability. Anatoly Lisnianski, Ilia Frenkel and Alex Karagrigoriou (Eds),* pp. 39−59. Springer.

Ruiz-Castro, J.E. and Dawabsha, M. (2019) A discrete MMAP for analysing the behaviour of a multi-state complex dynamic system subject to multiple events. *Discrete Event Dynamic Systems*, **29**, 1−29.

Ruiz-Castro, J.E. and Fernández-Villodre, G. (2012) A complex discrete warm standby system with loss of units. *European Journal of Operational Research*, **218**, 456-469.

Sadeghi, M and Roghanian, E. (2017) Reliability analysis of a warm standby repairable system with two cases of imperfect switching mechanism. *Scientia Iranica Transactions E: Industrial Engineering*, **24**, 2, 808−822.

Shatnawi, O. (2016) An Integrated Framework for Developing Discrete-Time Modelling in Software Reliability Engineering. *Quality and Reliability Engineering*, **32**, 8, 2925−2943.

Singh, J. (1989) A warm standby redundant system with common cause failures. *Reliability Engineering and System Safety*, **26**, 135−141.

Wang,W.; Xing, J. and Xie, M. (2018) Cold-standby redundancy allocation problem with degrading components. *International Journal of General Systems,* **44**, 876-888.

Yu, J.; Zhang, Pham, H. and Chen, T. (2018) Reliability modeling of multi-state degraded repairable systems and its applications to automotive systems. *Quality and Reliability Engineering International,* **34**, 459-474.

Zhai, Q.; Peng, R.; Xing, L. and Yang, J. (2015) Reliability of demand-based warm standby systems subject to fault level coverage. *Applied Stochastic models in Business and Industry*, **31**, 380−393.

Zhong, C.Q. and Jin, H. (2014) A novel optimal preventive maintenance policy for a cold standby system based on semi-Markov theory. *European Journal of Operational Research*, **232**, 405–411.