UNIVERSIDAD DE GRANADA

**Facultad de Ciencias de la Educación**

**Departamento de Didáctica de la Matemática**

DOCTORAL THESIS

# POSING INVERSE MODELING PROBLEMS FOR TASK ENRICHMENT IN A SECONDARY MATHEMATICS TEACHERS TRAINING PROGRAM

**VÍCTOR MARTÍNEZ-LUACES**

**Granada**

**February, 2021**

Editor: Universidad de Granada. Tesis Doctorales
Autor: Víctor Martínez Luaces
ISBN: 978-84-1306-868-8
URI: http://hdl.handle.net/10481/68580

# Acknowledgements

I would like to express my sincere gratitude to all the people who supported me in completing this Doctorate in Mathematics Education.

In particular, I want to thank María Esther Luaces and Marjorie Chaves, who always believed in the importance of this project. Their support for this endeavor has been very important along this process, which started five years ago with my Master studies.

In addition to the above, I wish to thank Marjorie Chaves for improving my English original version of the thesis.

Also, I want to express my gratitude to my supervisors, Dr. Luis Rico and Dr. José Antonio Fernández Plaza for the enthusiasm for this project, support and encouragement. Besides my supervisors, I would like to thank Dr. Juan Francisco Ruiz Hidalgo, who provided me relevant advice and assistance at the beginning of this thesis. All of them guided me wisely, with infinite patience, making it a true learning experience.

Finally, I want to thank professors Juan Francisco Ruiz Hidalgo, Antonio Moreno Verdejo and Luis Rico Romero, since without their collaboration in the fieldwork, this thesis would not have been possible.

## PRESENTATION

In this presentation the structure of the thesis is briefly explained. The thesis fieldwork was carried out with prospective teachers, focusing on the reformulation of given problems in an inverse way, in order to enrich tasks for secondary school students. As a consequence, in the first chapter, important mathematics education topics, such as tasks, problems, modeling, problem posing, problem solving and task enrichment –among other less frequent ones, like inverse problems– are introduced. Thus, the introduction finishes with the research questions and objectives.

Taking into account the above considerations, the different sections of the thesis theoretical framework, were chosen. The theoretical framework –which constitutes the second chapter of the thesis–, includes problem posing, inverse problems, mathematical modeling and task enrichment. Besides, it should be pointed out that all the results obtained in the fieldwork are analyzed using the Didactic Analysis, developed by Rico and collaborators (Rico & Fernández-Cano, 2013; Rico, Lupiáñez, & Molina, 2013; Rico & Ruiz-Hidalgo, 2018). In particular, our work focuses on three of the dimensions of Didactic Analysis: analysis of meanings, cognitive analysis, and instructional analysis. For this reason, Didactic Analysis is a fundamental part of the theoretical framework of the thesis, developed in the second chapter.

The third chapter is devoted to the methodological framework and it describes the characteristics of the sample and the instruments utilized to analyze the results. It is important to mention that these instruments used to analyze the productions of the subjects (i.e., the prospective teachers), have been transformed during the research, and these modifications are also described in this chapter.

Additionally, it is worth mentioning that the fieldwork was carried out in two well-differentiated parts: a pilot study that took place in 2017 and a definitive study, done in 2019. Between both experiments, there was a research design in which modifications were proposed, in order to correct or at least attenuate the difficulties observed during the pilot study; this is also part of the third chapter.

The fourth chapter is devoted to the description and analysis of the results obtained during the first part of the fieldwork, that is, the pilot study, done in 2017. In that opportunity, two direct problems were provided to the prospective teachers and they were asked to reformulate in an inverse form the second one. Furthermore, the participants were requested to design the tasks associated with their own reformulated problem and their responses are the inputs for the corresponding cognitive and instruction analysis.

In the fifth chapter, the results of the second fieldwork, designed throughout 2018 and carried out in 2019, are described and analyzed. In this second experience, the direct problems provided to the participants were the same as in the pilot study and the prospective teachers were asked to reformulate both in an inverse form. Once again, the participants were asked to propose the corresponding tasks, associated with their reformulation, so the productions included the cognitive and instructional analysis.

Finally, the sixth and last chapter is devoted to expound the conclusions of the research and the limitations of this work as well as the possible continuations of the research line.

**INDEX**

# Chapter 1.-INTRODUCTION

This document constitutes a research report on the development of professional skills, capacities and competencies, that is to say procedural contents knowledge on mathematical problems, reached by a group of prospective teachers who are being trained to be secondary school mathematics teachers. For this purpose, they follow a course about design, selection, evaluation and characterization of the school mathematics didactical tasks, within a training program. The foundations of this program were developed during the last years of the 20th century, focused on particular types of school mathematical problems and their relationships. These years coincide in time with the beginning of the Programme for International Student Assessment (PISA), promoted by the Organization for Economic Co-operation and Development (OECD), which are based on a different interpretation of school mathematics, understood as mathematical literacy. Even though our research is not based on the PISA assessment, the temporal coincidence with these studies can be observed in the foundation and conceptual framework of our work (OECD, 2004).

The course had a workshop format, based on instructional analysis of school tasks for writing new problems statements, characterized as inverse tasks of previous already known school problems, carrying out the direct and inverse problems solution and the analysis of the tasks involved. Task analysis is carried out within the framework of the didactic analysis and more specifically, the instructional analysis which provides the theoretical tools necessary to design, select and sequence tasks, ways of organizing the implementation in the classroom, their variables and complexity, as well as its cognitive and meaning aspects. It should be mentioned that the fieldwork corresponding to this research took place in several sessions throughout the 2016-2017 and 2018-2019 academic years– of the Master for Secondary School Mathematics Teachers, taught at the University of Granada (UGR).

This first chapter begins with a general analysis of school mathematical tasks and the notion of problem. Next, some of the most relevant previous experiences that have served as background to the current thesis are briefly commented.

Additionally, one of the distinctive elements of this work is the notion of inverse problem and in particular the inverse modeling problem. Both are introduced in section 1.3. Nevertheless, inverse problems themselves are not the final goal, but they are used for rich tasks proposals. For this reason, sections 1.4 and 1.5 are mainly devoted to problem posing and task enrichment.

Since the fieldwork of this study was carried out with prospective teachers who were studying the Master's Degree in Secondary School Teaching, then, the following section focuses on the teacher training courses offered by that institution.

It is important to recognize the difference between the mathematical knowledge and the mathematical knowledge that teachers need to effectively carry out their work. For this reason, these topics are exposed in section 1.7.

Finally, the last two sections (1.8 and 1.9) are devoted to the research questions and objectives of the thesis.

## 1.1 Mathematical tasks and problems

Mathematical problems constitute the central notion of this study and they are considered in this work from two different angles:

- Disciplinary: as the cultural and basic element of the mathematical activity, focused on its content, concepts and procedures, what makes mathematical knowledge grow. It is important to mention the distinction of Gowers. In his essay "The Two Cultures of Mathematics", Gowers (1999), classifies mathematicians by their work goal into two big groups: those whose "central objective is to solve problems" and those who are "more concerned in building and understanding theories", that's say the classical distinction between προβλεμτα –finding problems– θηορεματα –proving problems.
- Didactics of mathematics and Mathematics Education: as an object of teaching and learning, framed in a certain task, which contributes to the development of learning expectations as components in the short and medium term, with different functions and modes of organization, and which considers the structures, representations, meanings and modes of use, related to the educational level and specific students to which they are addressed.

Regarding these several approaches, our introduction summarizes the study background; identifies the didactical contents with respect to problems and tasks; characterizes the meanings of direct and inverse statements, particularly those related to modeling and applications, and establishes the techniques and tools for task analysis. We frame the object of this study within the third approach –i.e., the didactics of mathematics– focusing in how prospective teachers design tasks based on the inverse reformulation of an original direct modeling problem, and how they analyze those tasks by using categories and components for instruction content analysis and applying techniques and tools required by didactic analysis methodology (Rico, 2016; Moreno and Ramírez, 2016, Rico and Ruiz-Hidalgo, 2018).

Is worthy of remark that a problem can be described as a challenging or conflictive situation that proposes the achievement of a goal and makes necessary to find a way to answer it (Castro & Ruiz-Hidalgo, 2015). Then, in order to consider a given task as a problem, the subject who faces that must assume it as problematic.

The PISA assessment may be considered as based on a curricular framework since it takes a position and replies to the basic questions of a training plan: Why we teach mathematics? What mathematics to teach? and How to teach it?

PISA defines *mathematical literacy* "concerned with the capacity of students to analyze reason and communicate effectively as they pose, formulate, solve and interpret mathematical problems in a variety of situations involving a variety of mathematical concepts." Mathematical literacy is defined as "individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." (OECD, 2004, p. 37).

It should be noted that the characteristics of the PISA program support the argument that the tasks designed by prospective teachers must use various mathematical contents to solve everyday life situations and contribute to the development of medium and long-term expectations. In fact, the objective of PISA is to obtain information about the mastery of any community citizens when they use mathematical tools to respond to questions, work on tasks and solve real world problems. Regarding the role of problem solving in PISA, it is easy to observe a functional approach to school mathematics where the priority is not the content itself, but the phenomena organized by several integrated contents. PISA does not abandon the traditional organization of the mathematical content; on the contrary, it emphasizes how this content is used to solve contextualized problems in various situations.

According to Lupiáñez (2009) "school tasks are demands for actions that a teacher poses to the students, which can mobilize their knowledge on one or several specific mathematical topics, and which transform the specific objectives corresponding to this mathematical topic in terms of actions" (p. 60-61). These tasks involve the use of various structures, representations, and meanings or modes of use of mathematical concepts. Also, they contribute to the achievement of medium and long-term learning expectations; they have various functions and are characterized by their sequencing, complexity, meanings, authenticity, creativity and its potentiality for the organization of the classroom work.

Then, the planning of a school task, or any sequence of mathematical tasks, together with its organizational classroom work, as well as the design and implementation of materials and resources, are three well-known teaching organizers categories through which school mathematics contents and expectations are implemented.

When problems are framed in a certain school mathematical tasks, we will refer to them as school mathematical problems.

There are two central procedural contents related to the notion of problem: the *problem solving* components and strategies and the *problem posing* components and strategies. Problem posing is also part of school mathematical tasks, and holds a close relationship with problem solving, its creativity and its potentiality for the organization of the classroom work.

Although problem posing is usually considered as a supposed simple ability of students, in this study we assumed that problem posing is a competence of the future mathematics teacher, that is to say, it is required that prospective teachers study it and know how to reformulate an initial problem to give rise to new problems for students of a certain educational level. In particular, we propose and study statements inversion as a useful way to develop that competence. Invention of inverse statements from a given original statement: is shown as powerful strategy to enrich school mathematical problems and develop professional traits, capacities and skills for training mathematics teacher competences.

Therefore, in this study we focus on exploring and describing how prospective teachers design and characterize mathematical tasks corresponding to an inverse reformulation of an initial problem, thus contributing to its enrichment.

## 1.2 Preliminary experiences

Between years 1996 and 2002, an experience took place within six mathematics courses of the Faculty of Chemistry at the Universidad de la República, in Montevideo, Uruguay.

In these courses, an attempt was made to connect mathematics, through modeling and applications, with other disciplines taught at the aforementioned institution.

Those experiences started in 1996-1997 academic years, when modeling problems were incorporated, being almost all of them posed in a direct form. In the next academic years, since 1998-1999 to 2001-2002, courses inverse problems were also used in a systematic way.

Several years later, in 2013 a ten hours course-workshop was carried out at the University of Colima, Mexico, in the framework of an educational congress. In that context, a set of ten modeling problems was proposed to a group of prospective teachers and they were requested to reformulate them in an inverse form, aiming to enrich the problems for their use in secondary school mathematics courses. The participants reacted very positively to the course-workshop and this fact was reflected in their answers when they were asked to express their opinions by the corresponding satisfaction survey. This experience in Colima allowed obtaining a preliminary idea about developing a more complete research, similar to the one carried out in this thesis.

Our first international journal publication regarding inverse problems was released between both experiences (Martinez-Luaces, 2009), whereas other works that link inverse problems with mathematical modeling and teacher training appeared later (Martinez-Luaces, 2013, 2016).

All these experiences improved and broadened our knowledge about the advantages of inverse problem writing for teaching mathematics, and its important role in developing professional skills during professional teacher training. In particular, its potential for tasks enrichment was observed and as a result, a new research work emerged: the analysis of the connection between direct and inverse problems, as well as its relationship with mathematical modeling activities, in a line similar to the TFM (Final Master's Project) work (Martinez-Luaces, 2017).

Finally, it is important to remark that the aforementioned experiences took place in the same years in which the PISA international assessment results were being disseminated (OECD, 2004).

### 1.3 Direct and inverse problems: their relationship with modeling

As it was advanced, our growing interest was focused in school mathematics problems posing and invention, its meaning and structure, and its solving strategies and procedures. In scientific disciplines particularly in mathematics, it's easy to distinguish between two great different kinds of problems according to their statements: those posed in a direct way and those which statement is posed in an inverse form.

According to Groestch (1999, 2001), direct problems are those that provide the required information in order to execute a well-defined and stable procedure that leads to a single solution. Instead, inverse problems can be classified in two different types: causation and specification. In the causation problems the procedure is well-known and the question concerned the necessary data in order to obtain a certain result, while in the specification problems both data and result are given and the question is about which procedure can led to the desired result.

Consequently, inverse problems –which tend to be more interesting and more difficult to be solved– do not necessarily have a single solution and, when they do, usually do not have uniqueness (Bunge, 2006). Part of their difficulty arises from the fact that they require a certain regressive reasoning, which is not easy to teach and this may be a possible explanation that they have been almost ignored by traditional education proposals (Groestch, 1999, 2001; Martinez-Luaces, 2011). For instance, it can be mentioned that Kilpatrick (1987) proposed to change the conditions of a given problem in two different ways: add more or new conditions to the original problem then formulate a new demand; or remove conditions from the original problem then formulate a new demand. So, as it can be observed, in that approach the inversion is not included as a procedure for reformulating a given problem. A similar conclusion is obtained from the work of Fernandez-Plaza and Cañadas (2019),in fact, in their research with prospective Primary School teachers; none of the participants used inversion as a strategy for task enrichment. Besides, in our TFM (Martinez-Luaces, 2017), it was observed that this strategy is not common and usually, it does not appear spontaneously when working with prospective teachers.

Despite all the previous comments, some traditional problems clearly have an inverse structure, although this fact is not specifically remarked. This is observed, for instance, in several problems released by PISA (2012), such as the problem entitled “Apartment Purchase”.

Indeed, it proposes to measure the size of each room, calculate its area and then add all of them in order to obtain the total floor area of the apartment. The problem statement adds the following: "However, there is a more efficient method to estimate the total floor area where you only need to measure 4 lengths. Mark on the plan above the **four** lengths that are needed to estimate the total floor area of the apartment". Obviously the problem asks to measure four lengths that allow obtaining the same result and so, it is a causality inverse problem. Nevertheless, in order to choose the four lengths correctly it is necessary to know what to do with them, and then, implicitly it is also a specification problem.

As Groestch (1999, 2001) points out, the traditional curriculum in Mathematics is almost entirely dominated by direct statement' problems. Consequently, if we want to adapt the content and procedures taught in our courses, to include all kinds of problems –and not just only direct problems–, we must give inverse problems an important role. In addition, in the specific case of Mathematics, inverse problems provide a natural platform to investigate existence, uniqueness and stability of solutions, which are not so common and interesting in direct problems. Finally, this kind of problems helps to put mathematics closer to real-life contexts and future professional practices, since in the real world situations most of the problems are naturally posed in an inverse form.

Regarding the connections with real life, as Verschaffel et al. (2000, p. 119) observed, there exist "numerous examples in literature, from many parts of the world, of cases where students answered word problems apparently without regard for realistic considerations". Also, they added: "…we have considered a fundamental epistemological problem, namely how the abstract structures of mathematics relate to aspects of phenomena in the real world. The link between the two faces is modeling." (Verschaffel et al., 2000, p. 119).

As a consequence, modeling has an important role to play and then it is important to state the differences between "modeling" and "applications". According to Blum (2002), the term "modeling" is applicable when the process goes from the real world to mathematics whereas the term "applications" corresponds to the opposite direction, that is, from mathematics towards real life. Moreover, modeling refers especially to the process that takes place, while we use the term applications when the emphasis is on the object involved, particularly in those areas of the real world that are susceptible of a certain mathematical treatment (Blum, 2002).

If inverse problems and modeling activities are combined, we have the so-called "Inverse Modeling Problems", considered in previous works (Martinez-Luaces, 2009, 2013), which are even rarer in mathematics education. Exceptions to this fact are found by Liu (2003) and Yoon, Dreyfus and Thomas (2010). In these two cases, the researchers work with inverse problems connected with the real world, although the participants were students, not prospective teachers. Furthermore, the problems are given, so the participants were not requested to pose new problems related to the situations analyzed. Both articles have a certain connection with this thesis, but they differ with our work in many aspects.

Finally, it is important to remark that the proposal of inverse –or inverse modeling– problems constitutes a particular case of problem posing and for that reason, this topic will be considered in the next subsection.

### 1.4 Cross relationship: Problem solving and problem posing

It is important to note that in two works separated by almost 30 years, Kilpatrick (1987, 2016) shows that real-world problems that are susceptible to mathematical treatment are not like those posed in textbooks or those proposed by mathematics teachers, because they need an appropriate formulation. Likewise, a problem cannot be directly transferred from one person to another, since the receiver has to carry out a "reformulation" of it in order to give its "meaning", including new interpretations, implicit hypotheses, etc. This was observed several decades before by Pólya (1945), who pointed out that reformulations should be modified to yield more accessible problems statements: “We often have to try various modifications of the problem and we may arrive at a more successful trial by modifying an unsuccessful one. What we attain after various trials is very often ... a more accessible auxiliary problem.” (pp. 185–186).
It is well known that the solution of a problem usually requires several successive reformulations of the original statement, or sometimes to pose a similar problem, among other strategies. For instance, Duncker (1945) saw problem solving as productive reformulation when he commented: “It is therefore meaningful to say that what is really done in any solution of problems consists in formulating the problem more productively“ (pp. 8–9)

Kilpatrick (2016) commented these ideas expressing that "both Duncker and Pólya recognized that problem solvers need to take an active stance toward a problem, using the tool of reformulation to yield a solution" (p. 79).

As a consequence of these facts, it is easy to observe that problem solving and posing are not mutually exclusive strategies. On one hand, when posing a new problem, it is necessary to verify its potential solution, since it can be a conjecture, like the Goldbach conjecture which is one of the oldest and best-known unsolved problems in mathematics. It can be also a problem without solution, a problem with multiple solutions, or an ill-proposed problem, where the information is contradictory or inconsistent with the context, like in the example of the age of the captain, a famous nonsensical problem (Verschaffel, Greer, de Corte, 2000). In several of the previous cases, solving the problem is not possible. On the other hand, and more important, is that one of the heuristics of problem solving is to pose a similar problem or to make successive reformulations of the original one. This situation is well illustrated in the GPS problem analyzed by Kilpatrick (2016).

The publication of Pólya's book (1945) "How to Solve It" is generally considered as the origin of the importance attributed to problem solving in mathematics education. However, it was not until the mid-1970s that systematic research began in this field.

At the end of that decade, the orientation that considers problem solving as the central axis of mathematics education at compulsory levels – particularly in secondary school– gained strength. As a corollary to the above, the 1980 yearbook of NCTM (National Council of Teachers of

Mathematics, USA) was entirely devoted to problem solving teaching and learning and its curriculum development and treatment.

As Schoenfeld (2016) observed “In the 1978 draft program for the 1980 International Congress on Mathematics Education (ICME IV, Berkeley, California, 1980), only one session on problem solving was planned, and it was listed under ‘unusual aspects of the curriculum.’ Four years later, problem solving was one of the seven main themes of the next International Congress (ICME V, Adelaide, Australia)”. This is just an example of how situation towards problem solving changed dramatically in few years.

A special case, particularly important, is given by problem posing, which already appears initially in the works of Pólya (1945). This research line reappeared in the late 1980s with the work of Kilpatrick (1987) and was consolidated in the 1990s with the classic articles by English, (1997, 1998) Silver (1994, 1997) and Silver and Cai, 1996), among others.

It is possible to find in the literature, papers where problem posing is carried out working with prospective teachers. An example is the paper written by Leung and Silver (1997), where the authors examine the arithmetic problem-posing behaviors of prospective elementary school teachers. One more example is the paper of Işık et al. (Işık, Kar, Yalçın & Zehir, 2011), which tried to determine “prospective teachers’ problem posing skills appropriate to selecting, translating, comprehending and editing models and possible difficulties they could encounter during this process”. Another Turkish research group (Şengül & Katranci, 2015) carried out a study about free problem posing with prospective mathematics teachers, with the aim of analyzing the difficulties faced by the participants during the problem posing process. In Latin America, Felmer and Perdomo-Díaz (2016) conducted a research with a group of 30 novice Chilean mathematics teachers as problem solvers, although in this last case, no problem posing is reported as part of the research design.

At the University of Granada, the first published works regarding problem solving arose in the late 1980s and among them, the research study coordinated by Rico (1988) stands out as a relevant example.

Other works that deserve to be mentioned are the theses of Castro (1994) on verbal problems of multiplicative comparison and Fernández (1997), where graphic solution of verbal problems is one of the research subjects considered.

Finally, it should be mentioned that in some of our previous works we have inquired the solution of problems –and also the proposal of new ones– using statements' inversion as the main strategy, like in this study. In fact, it is important to underline that a possible way of enriching tasks is to reformulate traditional modeling problems, posing them inversely and increasing their educational value. This strategy has been put into practice on several opportunities in Latin America (Uruguay, 1996-2002; Chile, 2003; Argentina, 2009; Guatemala, 2010 and Mexico, 2013) and in Europe (France, 2016). Those mini-courses or course-workshops were attended by in-practice and prospective teachers.

In all those short training courses, the main purpose was to reformulate statements of given problems in an inverse form, for task enrichment. For that reason, we summarize rich tasks and task enrichment in the following section.

## 1.5 Rich tasks and task enrichment

One of the professional teacher skills is to select or design appropriate tasks for the intended purpose, and among them, those that best contribute to achieve the desired goals, so tasks need to be evaluated in terms of their didactical richness. Moreover, the Basic curriculum for Compulsory Secondary Education (ESO in Spanish) and Baccalaureate (BOE, 2015, p. 170), states that “The role of the teacher should be remarked, since he/she must be able to design tasks and/or learning situations that enable problem solving, the application of knowledge previously learned and the promotion of student activity.”

Among such tasks, we focus on those that include problem solving and problem posing. In accordance with that, Stacey et al (2015, p. 283), expressed their support to the “development of mathematics education away from looking at mathematical tasks as something that should be finalized with one right answer as quickly as possible towards looking at mathematical tasks as initiators for problem posing, problem solving, reasoning and communication”.

Regarding these facts, as we will note later, the inversion of an original problem is a problem posing strategy that deserves to be considered since it contributes to the problem enrichment.

Several papers remarked the importance of fostering prospective teachers’ creativity for the design of rich tasks. Indeed, the research in teacher training has paid special attention to the nature, role and use of tasks, both in specialized journals (Tzur, Sullivan & Zaslavsky, 2008; Watson and Mason, 2007) and also in specific books devoted to this theme (Zaslavsky & Sullivan, 2011).

For that reason, first of all it is important to analyze and characterize what a rich task is. Although there is not a general agreement, several authors considered this issue, particularly Grootenboer (2009), who described the key aspects of rich mathematical tasks. In a similar way, Clarke and Clarke (2002) proposed a list of characteristics for rich assessment tasks. Among those characteristics, we select a few ones that deserve to be highlighted:

- Different approaches and methods can be successfully utilized
- Each of them allow students to transfer knowledge from a well-known context to a new, less familiar one
- They provide several student responses, giving them an opportunity to show what they know about the mathematical content

These selected characteristics are particularly important because they illustrate very well how inverse problems are useful for task enrichment purposes. This is strongly connected with our work, which focus on prospective teachers and particularly with their strategies for problem posing and the design of rich tasks.

This purpose is also related to the work of Crespo and Sinclair (2008). In their paper, the authors explored the problem-posing behavior of elementary prospective teachers as a result of two interventions, designed to lead the improvement of problem posing and mathematically richer understandings of what makes a problem "good". Other authors like Lavy and Shriki (2007) reported some difficulties in their research in this area, since they observed that the prospective teachers tend to focus on common posed problems, being afraid of their inability to prove their findings. A similar observation was made by Abu-Elwan (1999) in his article about the development of mathematical problem posing strategies for prospective middle school teachers. Indeed, the researcher asked a group of prospective teachers to generate some good mathematics questions from a given life situation and to reformulate another mathematics problem from another one taken from a textbook. The author report that “there were bad questions and ill-formulated problems”. As it can be observed, it is not easy for prospective teachers, without a specific training in problem posing, to propose rich tasks.

Among many possibilities to reach this objective, the emphasis has been placed on the inversion of a given problem as an important strategy for task enrichment. This strategy is useful for the design of rich tasks and then it is a very important aspect to be considered in teacher training courses. Those courses are analyzed in the following section, focusing on the particular case of the UGR study.

### 1.6 Teacher training courses at UGR

Our fieldwork was carried out at University of Granada, which offer training programs for all the Masters in Secondary School Teaching and Learning, including the initial training for Secondary School Mathematics teachers. All those Masters programs consist of three content modules, being the first one a generic, common and mandatory module. In the particular case of Secondary School Mathematics teachers, the second module presents an offer of five optional courses – and two of them must be chosen– and the last one is a specific module, which has four subjects. Among them, the fourth one (Teaching and

Learning Mathematics) can be considered the most interesting for this work, since it was within this course where the fieldwork was carried out. In this subject, the mathematics curriculum is studied and particularly, the meaning of a concept, the learning of school mathematics, the teaching of mathematics and its evaluation are deeply analyzed. These contents establish well-defined connections with the four dimensions of the Didactic Analysis (Rico & Ruiz-Hidalgo, 2018). These dimensions, studied by the Didactic Analysis, within a functional approach supported by PISA, are listed below:

- Cultural/conceptual dimension
- Cognitive dimension
- Ethical/formative dimension
- Social dimension

Thus, when the curriculum is assumed as a way for the teacher work planning, it should describe certain content, objectives, methodology, and evaluation criteria.

This thesis work fundamentally focuses on the first three dimensions (cultural, cognitive and formative), which are directly connected with the analysis of the mathematical content, the cognitive and the instructional analysis, i.e., the first three components of the Didactic Analysis (Rico & Ruiz-Hidalgo, 2018).

Considering the aforementioned dimensions of the Didactic Analysis, our main interest is focused on teaching dimension –therefore, on Instruction Content Analysis–, that is to say teacher planning and implementation of mathematics teaching. Among other possibilities, modeling and problem solving are two fundamental strategies that prospective teachers should acquire to propose richer tasks for Secondary School courses, and teacher training courses should help their development.

In relation to the above, it should pointed out that two of the specific competences (SC) mentioned on the website of the Master[1], are the following:

> SC34. Transform the syllabus into activities and tasks.
>
> SC35. Acquire criteria for the selection and development of educational materials.

Moreover, the Master website includes the following objectives or learning outcomes:

- Transform the syllabus into activities and tasks.
- Acquire criteria for the selection and preparation of situations, activities, materials and educational resources, integrating them into didactic units and identifying their objectives, content, teaching methods and evaluation.
- Reflect on the development of teaching proposals in the classroom, analyzing specific didactic situations and proposing alternatives for improvement.

Lastly, in the section named "brief description of contents" the website includes:

[1]https://masteres.ugr.es/profesorado/docencia/plan-de-estudios

- Problem solving and problems as the center of the teaching-learning process.
- Models of mathematics teaching based on problem solving.
- Teach concepts and processes through problem solving.
- Curriculum design analysis: Reflection and analysis of the elements relevant for practice.

As a consequence of these facts, it is important to establish the difference between the mathematical content and the didactical mathematical content, both kinds of contents that teachers needed to know. These relevant aspects are analyzed in the next section.

### 1.7 The mathematical content and the didactic mathematical content

Three categories define the content-specific dimensions of Shulman's major categories of teacher knowledge (Shulman, 1987, p.8). On one hand, we find the mathematics content knowledge, which includes knowledge of the subject and its organizing structures. On the other hand, we have the curricular content, syllabus knowledge, "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p.10). Finally, the last and perhaps most influential of the three content-related categories is the didactic mathematical content.

Other mathematics education researchers followed Shulman’s ideas. In particular, Ball, Thames and Phelps (2008) continued this research line and defined “mathematical knowledge for teaching” as the mathematical knowledge about those three kinds of contents mentioned, that teachers need to carry out their work as teachers of mathematics. The latter is especially useful for that research, since it allows us to distinguish the prospective teacher’s reformulated problem from the didactic analysis carried out for the tasks associated with the corresponding proposal.

Also, it is important to remark that the didactic analysis allows developing and putting into practice the didactic knowledge of the content. Moreover, Rojas (2014), in her doctoral thesis analyzes the relationship connections between the didactic analysis and an alternative model of teacher knowledge developed by Carrillo and collaborators. In particular, in the second chapter of that thesis, the relationships between the components of the didactic analysis and the domains of the Mathematics Teacher’s Specialized Knowledge (MTSK) are deeply analyzed.

As a consequence, we consider the didactic analysis as an operational approach to the teacher's knowledge of the mathematical content and the didactic mathematical contents, required for teaching.

Both mathematical knowledge and didactical knowledge of the prospective teachers will appear in the research questions, which are developed in the next section.

## 1.8 Conjectures and research questions

Taking into account the experiences carried out in Latin America –and particularly the one that took place in Colima– and the final project of the Master, it seems reasonable to conjecture that prospective teachers are able to reformulate direct problems, turning them into richer inverse problems.

Consequently, the first conjecture is that prospective teachers are prepared to easily modify a certain kind of modeling problem, posed in a direct form, converting it in an inverse one, richer and more connected to the real world and therefore, more motivating for students.

Inverse problems are often ill-conditioned, which eventually represents an obstacle, however, it can also be seen as an advantage, since they allow us to study existence, uniqueness and stability issues, which are not so relevant in most direct problems. Therefore, a second conjecture that can be raised is that future teachers may use these potentialities to propose enriched mathematical tasks.

Besides, it is important to know the viewpoints of the prospective teachers concerned with the usefulness of inverse problems regarding classroom work, the students' motivation and the convenience or not of using them in their courses. Specifically, their opinions about the meanings and authenticity of the proposed tasks, as well as their elements and variables, are especially relevant.

In consequence, it is reasonable to ask about the potentiality of inverse mathematical modeling problems, focusing on their applicability in the classroom. Therefore, it is convenient to investigate whether future teachers consider these problems motivating or not and whether they see them as potentially useful their courses.

Lastly, it is appropriate to ask whether future teachers see themselves as able to transform direct modeling problems into richer inverse problems –with their corresponding tasks– to be used in secondary school courses.

Taking into account the previous comments, the following research questions arise:

RQ.1 How do prospective teachers use their mathematical knowledge when reformulating a direct modeling problem into an inverse problem that is coherent and adapted to the level of the students? What strengths and weaknesses are identified in their proposals?

RQ.2 How do prospective teachers use their didactic mathematical knowledge when designing meaningful tasks associated to the inverse problem, previously reformulated from a given direct problem? What strengths and weaknesses are identified in their task designs?

RQ.3 What are the strategies of future teachers for the reformulation of a given direct problem into an inverse one form, as a richer problem to be used in secondary school courses?

RQ.4 Which didactic characteristics can be described for the tasks designed by prospective teachers, associated with a certain inverse problem obtained through the reformulation of a given direct modeling problem?

### 1.9 Research objectives

Considering the aforementioned background, as well as possible conjectures and research questions, for this doctoral thesis the following general objectives are proposed:

OG.1 Identify and characterize the prospective teachers' strategies to pose inverse problems for secondary school courses, by reformulating a given direct problem.

OG.2 Study, analyze and characterize the prospective teachers' productions about the didactical analysis of tasks related to inverse problems, considering their relation to the original task based on the direct problem from where they come.

From these general objectives, the following specific objectives arise:

Specific objectives related to OG.1

O.1 Characterize the statements of the reformulations posed in an inverse form by the prospective teachers.

O.2 Characterize the complexity of the resolution process of the inverse problems proposed by the prospective teachers.

Specific objectives related to OG.2

O.3 Characterize the meanings of mathematical concepts that teachers use when they design tasks by reformulating direct problems.

O.4 Characterize intentional aspects (expectations, errors and cognitive demand) that appear in the prospective teachers' tasks related to reformulated problems.

O.5 Characterize the instructional components and elements, focused on the task variables that teachers use when they reformulate problems.

# Chapter 2. Theoretical framework

We articulate the didactic contents for the training of the mathematics teachers as professionals, according to the four dimensions of any mathematics curriculum as follows: conceptual contents, cognitive contents, instructional contents and evaluative contents (Rico, & Ruiz-Hidalgo, 2018).

Each of the aforementioned contents focuses its object of study on a modality or perspective on mathematics education matter, that is to say: its Meaning, Intentionality, Planning, and the Decision Making about its use.

Each dimension considers certain priority analysis categories with different criteria and utilities. In each case, the categories and concepts help to identify the components and themes that we will use to organize the specific proposals and documents to be studied. Thus, the mathematical contents are structured through themes, concepts and procedures, representation systems, contexts and modes of use. The cognitive contents are organized by expectations, limitations, and learning opportunities.

For this study we underline tasks and its sequences as the basic way to identify instructional contents, describe it in terms for classroom work's organization, materials and resources, and components. They will be identified in terms of task variables functions, its complexity, creativity, and characteristic types. This research do not consider contents and components of the evaluative curricular dimension. School instruction constitutes a basic dimension in the analysis of the didactic contents of any curricular proposal, required to carry out the processes of analysis and initial training of prospective mathematics teachers, as well as for the development and achievement of the professional competence, developed widely in the *Opportunity to Learn* (Cogan & Schmidt, 2014, pp. 207-220).

The experts establish three types of categories to deepen in didactic contents that are worked in this instructional dimension, among which we preferably highlight tasks and their sequencing, rules for school work planning and organization, and school materials and resources. In turn, each of these categories is described in terms of components through which the didactic scrutiny of the aforementioned categories is carried out (Rico & Ruiz-Hidalgo, 2018).

This chapter starts with a description of some structural components – notions, concepts and procedures– for the syllabus program at the University of Granada master' courses for prospective Secondary School Mathematics teachers (Rico, Fernández-Cano, Castro & Torralbo, 2008, pp. 203-211). In particular, in the first section, the corresponding specific subjects of the mathematics specialty are described, since it is essential for this work.

Within specific subjects we will focus on the so-called "Learning and teaching mathematics". This subject is based on the Didactic Analysis, which is the analysis methodology followed in this thesis and developed in the second section.

Instruction Analysis is the Didactic Analysis third dimension, focused on mathematical tasks and problem solving. It is strongly linked with the two previous dimensions, the Conceptual and the Cognitive ones, especially important for this work, due to the fundamental role that they play in the enrichment of mathematics tasks. Regarding the latter, third section of this chapter is entirely devoted to tasks enrichment.

In particular, problem posing and its invention are important strategies for mathematics education training about prospective teachers and at the same time are capital contents for tasks enrichment purposes. Due to these facts, the fourth section of this chapter is especially focused on problem posing competence.

It is worth highlighting the role of inverse problems as a strategy to pose new problems and contribute to their tasks enrichment. Throughout the fifth and last section, the characteristics of tasks enrichment that are generally met in the proposal of inverse problems will be analyzed.

In particular, from the research viewpoint, prospective teachers were asked to reformulate direct modeling problems following several and different strategies in an inverse form, such that the proposal can be considered as a rich task for secondary school students. For this reason, a subsection of section 2.5 is devoted to modeling and applications and inverse modeling problems.

The starting point for this chapter consists in describing several elements and components of the University of Granada's Mathematics Teachers Training Master Program, which we analyze in next section.

## 2.1 Syllabus program for mathematics teachers at the University of Granada

The training program structure for the Masters in Secondary School Teaching and Learning consists of three content modules[2].

[2]Web page: https://masteres.ugr.es/profesorado/docencia/plan-de-estudios

The first one is a generic, common and mandatory module, the second one presents an offer of five optional courses (from which two of them must be chosen) and the last one is a specific module, which will be described in detail here. In the case of mathematics teachers, this specific module consists of four subjects:

- Complements for disciplinary training
- Teaching innovation and educational research initiative (Part I)
- Teaching innovation and educational research initiative (Part II)
- Teaching and learning mathematics

The first subject treats several curricular topics of Geometry, Arithmetic, Mathematical Models (based on Difference Equations and Differential Equations), Mathematical Analysis and Complements of Probability and Statistics, all of them with a strong orientation to mathematics problem solving.

In the second subject, the action-research process is analyzed, including the design of a project as part of the theoretical-practical agenda which is carried out in the practical seminar of that course.

In the third –which continues the previous one– research and innovation in mathematics education are deepest worked and after that, resources needed for innovation and research are extensively considered. Finally, both the application in the classroom of the research and innovation results and a critical analysis of teaching practice are carried out.

The fourth subject is crucial for this work, since the participants were attending this course and the fieldwork was done during two of its sessions. For these reasons, it will be described here in more detail.

The structural lines of this subject's contents are developed in a manual coordinated by Rico and Moreno (2016). In the introduction of this book, one section refers to the syllabus of the subject "Learning and teaching mathematics", stating that "This manual establishes a program for the subject and proposes a work plan for the achievement of the competencies established for the Master, especially those skills that are linked to the aforementioned subject." And then, it adds that the book "...is based on the didactic analysis, a form of content analysis, sustained in mathematics didactic categories."

Regarding the manual's organization it says: "The chapters are organized in five blocks, attending to five central didactic notions, all of them based on the same framework of curricular theory" (Rico & Moreno, Eds. p. 25).

The first of those five blocks refers to the foundations, methods and didactic contents of school mathematics. The second block is devoted to the meanings of the syllabus school mathematical contents; the third one analyzes school mathematical learning; the fourth block is dedicated to the planning and teaching of mathematics, and the fifth one studies the assessment of learning and the derived decisions-making. As it can be observed, after the introductory first block, the other four blocks correspond to the curricular dimensions of the didactic analysis (Rico, 1999).

Hence, due to its importance, the didactic analysis constitutes the next section of this chapter.

## 2.2 Didactic Analysis

In this section, we present a summary of Didactic Analysis' functions considering: the structure, levels, dimensions, categories and components to perform and organize the didactic school mathematics content analysis and how it makes possible the design, implementation and evaluation of teaching and learning activities, corresponding to any specific mathematics subject (Rico, Lupiáñez & Molina, 2013; Rico & Moreno, 2016; Rico & Ruiz-Hidalgo, 2018).

This description gives rise to a cyclical structure, where the information obtained in a given analysis will be essential for a new implementation of the didactic analysis (Rico & Fernández Cano, 2013).

In the specific frame of this work, the didactic analysis has been used as a tool for the fieldwork design and the analysis of the productions of the prospective teachers. However, the evaluation of these productions has not been carried out, so in this theoretical framework we will focus on the first three stages: analysis of meanings, directed cognitive analysis, and instructional analysis.

In the following three sub-sections, these parts of didactic analysis are briefly described.

### 2.2.1 Mathematical Content Analysis

The mathematical content can be described, studied and analyzed according to different criteria, being its disciplinary organization one of the most important. Historic evolution of mathematics has taken place by using various external and internal branches of organization criteria. The external organization considers mathematics as a set of subjects with different purposes, although having common foundations and methods. The different branches are usually designated by their conventional names, being Arithmetic, Geometry, Algebra, Trigonometry, Calculus and Probability, etc., some of them included in the Secondary School Syllabus, that were clearly identified by the participants in this study (Rico, 2016).

The analysis of mathematical contents aims to describe and establish the different meanings of the mathematical concepts and structures involved in a certain teaching unit of content.

The notion of meaning is based on the semantic triangle, which constitutes an interpretation of Frege's ideas (Castro-Rodríguez, Pita-Panzatti, Rico & Gómez, 2016; Fernández-Plaza, Rico & Ruiz-Hidalgo, 2013; Martín-Fernández, Ruiz-Hidalgo & Rico, 2019). Following to this approach, the analysis of the mathematical content is organized into three categories: the representation systems, the senses or modes of use and the contents' structure.

Representation systems consider the different ways in which a certain mathematical content can be expressed through signs, symbols, graphics, relationships, rules, conventions, along with their translations into other concepts and conversions according to different procedures.

The sense considers the contexts and modes of use, including phenomena, contexts and situations that give meaning to the content.

Finally, the conceptual structure considers the relationships of the concepts and procedures involved in the content studied, attending to the mathematical structure of which they take part.

The starting point of content analysis is the revision of the topic in the curricular documents, with the aim of delimiting content focused. The content focused consist of specific groupings of concepts, procedures and relationships, which acquire special importance since they express, organize and summarize groupings of coherent contents. This information can be complemented by other documents that address the analysis of school mathematics (Rico et al., 2008).

In the methodological analysis, some of the previous topics, such as representation systems and context and phenomenological analysis, will be detailed.

### 2.2.2 Cognitive Analysis

Cognitive analysis lets teacher to carry out a detailed description and analysis of the problem for the learning orientation and understanding of a specific mathematics topic from a curricular and functional viewpoint (Lupiáñez, 2016).

Cognitive analysis is structured around the teacher expectations about what students may and must learn, what can interfere with that learning, and what allows them to learn, and also allows the teacher to observe whether the learning should effectively occurs or not. These three tools delimit three categories or curriculum organizers each of which structures and organizes cognitive analysis. Although each of these organizers provides an analysis' tool related to school learning, there are several other more specific finer aspects that should be considered. Organizers are named and described according to Lupiáñez (2009, p. 58):

- Learning expectations, which argue and organize what the teacher expects the students will learn, according to different level

- Learning limitations, which focus on the possible errors and difficulties that may arise in the learning process
- Learning opportunities that the teacher offers to the students.

The synthesis that results from this analysis in two or three levels shows a learning plan that based on the information obtained from the content analysis, takes into account the cognitive categories for those levels. All these conditions should guide the design and may help to select tasks adjusted to the contents and objectives considered.

Cognitive analysis is a central part of the professional competence of mathematics teacher to plan and implement didactic units, articulated through task sequencing, organize their implementation, and make decisions.

### 2.2.3 Instructional Analysis

Instructional analysis, in turn, focuses on categories to plan and implement the design of teaching units (Marín, 2013). We consider relevant for this dimension: the decisions to select and sequence tasks, the organization of students' work, the choice of materials and criteria for its use.

Regarding the previous statement, it is important to mention that Gómez (2007, pp. 78-79) limits the term school tasks to demands for action that the teacher gives to the students. Also, this author calls activities to everything that learners and teachers do related to a given school task.

Lupiáñez (2009) gives other interesting point of view when making a distinction between task and activity, linking it to learning expectations. This author considers that: "School tasks are demands for actions that a teacher poses to students, which can utilize their knowledge on a given mathematical topic and at same time, those tasks should implement the specific objectives of the mathematical topic in terms of actions. The tasks imply that a student should manifest his/her attitude and interest towards the work proposal and make explicit his/her knowledge of certain concepts and procedures and his/her mastery of certain capacities", and finally, he observes that "activities are the various responses of the students to the demands previously made; so, we refer to their activities in terms of the actions that take place when working on the proposed tasks" (pp. 60-61).

Lastly, Watson (2008) says that: “Individual access to mathematical concepts is structured through sequences of linked tasks –things learners do. The word ‘task’ has come to be associated with the ‘reform’ pole, and hence to mean something exploratory, extended, complex, open-ended. However, I use it to mean anything a learner is asked to do, or chooses to do. Listening to a teacher’s explanation, doing textbook questions, constructing a decahedron, designing packaging for tennis balls are all tasks. This means that lesson design is about sequencing tasks, and embedding tasks, rather than occasionally using tasks. This view of ‘task’ makes it possible to think through a lesson in terms of what the learner is expected to do, and hence to think about what and how learners might learn while doing these tasks” (p. 127).

Related to the previous comments, Moreno and Ramírez (2016) argue that the result of the analysis of instruction should be the design, justification and sequencing of tasks for every didactic unit corresponding to specific mathematical contents. Then, there exists a direct link with cognitive analysis since the selection of specific learning expectations marks the orientation of the tasks selected by the teacher. There is also a reciprocal relationship because at the same time, the analysis of a specific task can broaden or enrich the learning expectations.

So, mathematical tasks are central elements in both analyzes –cognitive and instructional– and constitute the organizational axis of the second one. Sometimes, just solving a task gives us information about its suitability for learning mathematics; while in other cases, it is necessary to deeper study the actions it promotes, the diversity of possible solutions or the complexity level. A simple modification of a certain statement can make it satisfy or not the requirements or the aims pursued by the teacher and so, it is important to have criteria to carry out and justify this modification.

### 2.2.4 Tasks variables

Instructional analysis is also based on three kinds of categories: Tasks and its sequences, Classroom work organization, and Materials and resources. In turn, each of these categories is articulated around different components, concepts or themes, which contribute to its analysis, essential in the planning process. Some of them may be identified as singular task variables, while others correspond to a set of tasks systematically organized.

These components respond to different facets in the design of didactic unit, for example Moreno and Ramírez (2016), listed:

- The adequacy of the tasks to the content and expectations
- The role of problem solving

- The use of materials and resources
- The sequencing chosen of tasks and sessions
- The attention to diversity from mathematics
- The organization of classroom management
- The learning proposed for evaluation.

Some of these facets will be analyzed in the chapter devoted to the research methodology.

According to the framework summarized, the elements and components we have selected as descriptors and utilized for a task's didactic analysis, have included the following:

- Previous knowledge
- Mathematics content activated by the task
- Challenge
- Task completion
- Event
- Question(s)
- Purpose
- Language
- Data
- Goal
- Formulation
- Material and resources
- Grouping
- Learning situation
- Timing

- Mathematical content
- Situation
- Complexity.

Descriptors for the analysis of the proposed tasks are briefly explained in Table 2.1. It is important to note that this table –where organizers and task components are mixed– was the format in which the description of the task variables was presented to the prospective teachers just in order to stimulate their activity. In fact, it was expected that this table could help them in order to compare and analyze the tasks proposed for both the original and the reformulated problem.

The descriptors are listed and briefly explained, based on the criteria developed by Moreno and Ramírez (2016, p. 244-251), who described the conditions for considering that a certain task is meaningful. They stated that the task should be based on students' prior knowledge, it allows activating other contents, it represents a challenge for the students and they can recognize when it has been done successfully.

**Table 2.1.** Task variables descriptors, from the components and elements of the Instructional Analysis organizers.

| Descriptors | Explanation of the descriptor |
|---|---|
| *Meanings* | |
| Prior knowledge | Refers to content that students already know; the task is based on concepts and p |
| Tasks activate mathematic content | Concepts and procedures that teacher wants to develop through the task work. |
| Challenge | This item asks if the task can be considered a challenge for the students and if the |
| Task completion: recognition/ justification | Students should be able to recognize if the task has been done successfully and explanations to decide if the given response completes the task or not. |
| *Authenticity* | |
| Event | The task refers to an event that happened before, or if it has a real chance of happ |
| Question | The question of the task can be considered as consistent with the expected questi |
| Purpose | The purpose of the task is consistent with the one that could be proposed in a rea |
| Language | The language appropriateness in which the task is expressed, |
| Data | This item is about the realistically of the given data. |
| *Task elements* | |
| Goal | This is about the learning expectation developed by the task. |
| Formulation | This item considers the way in which the task is presented (written text, oral, vid |
| Materials and resources | This is about the materials and resources needed to complete the task. |
| Grouping | About the ways of organizing the students when working on the given task. |
| Learning situation | The place or the physical situation where the task is carried out. |
| Timing | The adequacy of the timing for the work to be done in order to complete the task |
| *Task variables* | |
| Mathematical content | Quantity, space and shape, uncertainty and data, change and relationships. |
| Situation | Personal, educational/occupational, societal, scientific. |
| Complexity | Reproduction, connection and reflection. |

As it can be observed in Table 2.1, the participants were stimulated to give their opinions about significance, authenticity, elements of the task and task variables. A more complete explanation of the descriptors corresponding to those items, listed in Table 2.1, is given in the following paragraphs.

*Meanings*

Regarding the meanings aspect, four items have being considered:

- Prior knowledge: It refers to the content that the students already know and then, school tasks are based on these concepts and procedures that the students already possess.
- Mathematical content activated by the task: It refers to the concepts and procedures that the teacher wants to develop through the task work.

Regarding the two previous items, it is important to remark that in the case of Secondary School curriculum, the mathematical formal content is usually related to Geometry, Algebra, Trigonometry, Calculus and Probability. Then, it is expected that the concepts and procedures mentioned by the prospective teachers would be those corresponding to these branches of mathematics.

- Challenge: This item asks if the task can be considered a challenge for the students and if they are interested or not in it. This item is related to the learning opportunities since *conditions, specific demands and challenges* are the components of this organizer. So, this part of the task analysis is deeply related with the cognitive one.

- Task completion, recognition/justification: Students should be able to recognize if the task has been done successfully and also if their response is accurate, providing explanations to decide if the given response completes the task or not.

*Authenticity*

Regarding authenticity, five items have being considered:

- Event: The task refers to an event that happened before, or it has a real chance of happening. In other words, it can be considered as a realistic event.
- Question: The question of the task can be considered as consistent with the expected question in the real life. Once again, it means that the posed question is a realistic one.
- Purpose: The purpose of the task is consistent with the one that could be proposed in a real life situation, so, it is related to a realistic event.
- Language: It includes an appropriate terminology and also it has a sentence structure and extension that makes the situation easy to understand. In a few words, it means that the language in which the task is expressed is considered as appropriate.
- Data: The given numerical data can be considered as realistic.

*Elements of the task*

When considering the task elements, the first three items to be analyzed are: the goal of the task, its formulation and the materials and resources needed.

- Goal: This item is about the learning expectations developed by the task and they are chosen for the students of a specific educational level, on a certain mathematical topic, and it is important to persist in their achievement. These expectations can be structured at various levels of complexity and their common characteristic is that their achievement is accredited by the students' responses to the proposed tasks.
- Formulation: This item considers the way in which the task is presented. It can be presented as a written text, in an oral form, or using technology, like in a video or other formats.
- Materials and resources: This item is about the materials and resources needed to complete the task. It refers to the manipulative materials and the resources needed for the execution of the task. In general, in problems such as those proposed here, resources can be reduced to pencil and paper, but it may also require the use of graphic representation software, calculators or computer algebra systems. It is not so common, but some manipulative materials that allow a physical representation of the problem may also be suggested.

The other three items corresponding to the task variables are related to the classroom work organization. It is well known that there are different ways of organizing and managing the classes, in order to promote interactions and communication processes, according the conceived learning expectations. The following items consider different options for organizing the class for the proposed tasks.

- Grouping: about the ways of organizing the students when the working on the given task (individually, by pairs, groups, etc.).
- Learning situation: about the place or the physical situation where the task is carried out (at home, in class, etc.).
- Timing: about the timing for the work to be done in order to complete the task (15 minutes, one hour, etc.).

*Task variables*

Finally, for the task variables, the corresponding classification of the mathematical content, the situation and the complexity are based on those described by PISA (OECD, 2013, pp. 16-22).

- Mathematical content: There are considered four possibilities (change and relationships, space and shape, quantity, uncertainty and data), directly related to different mathematics branches.
- Situation: It refers to the context that is proposed for a given reformulated problem. These situations are personal (when they are related to daily activities of the students), educational/occupational (if they can be found in an educational situation or in a job), societal (if it appears in a community situation) and scientific (if it is an abstract situation related to scientific-technological activity).

- Complexity: It allows describing the difficulty of the task. Three degrees of difficulty are considered (Moreno & Ramírez, 2016), in an increasing order: Reproduction, which includes exercises that require repetition of the knowledge previously acquired; Connection, which considers tasks that need relating different representations of the same situation or linking different aspects to reach the solution; and Reflection, which includes tasks that involve a greater number of elements and require generalizations, explanations, more complex reasoning and results justification.

### 2.3 Task enrichment

As it has been mentioned, the instructional analysis focuses on the designing, selecting and sequencing of tasks for a certain didactic unit and for this purpose it is important to choose rich tasks and/or enrich other tasks previously proposed. Moreover, according to this idea, Lester and Cai (2016) stated "…teachers can develop worthwhile mathematical tasks by simply modifying problems from the textbooks" (p. 124).

Santos and Barmby (2010) observed in their paper about enrichment and engagement in mathematics "the question of what is meant by enrichment has been an ongoing question for researchers". For instance, Barbe (1960, pp. 199-206) says "an aura of vagueness and confusion seems to surround the term" and four decades after, Feng (2005) concludes in his article: "no overall consensus has yet been reached on the definition and nature of enrichment".

Due to these facts, rich tasks should be considered as a description more than an exact and precise definition of the term. For instance, Grootenboer (2009) described “the key aspects of rich mathematical tasks”, which include:

- Academic and intellectual quality.
- Extended engagement.
- Group work.
- Attention to the diversity, achieved through multiple solution pathways and entry points.
- Connectedness.
- Multi-representational.

Clarke and Clarke (2002) –after a big discussion with teachers– proposed a list containing the main characteristics of “rich assessment tasks”. Among those characteristics, we choose the following ones:

- Address several outcomes in only one task.
- Allow all the students to make “a start”.
- Different approaches and methods can be successfully utilized.
- Encourage students to reveal their own understanding of what they have learned.
- Allow students to make connections between the concepts they have learned.
- Authentically represents the forms in which mathematical knowledge and skills will be used in the future.
- Allow students to transfer knowledge from a well-known context to a new, less familiar one.

Although this is a list about “rich assessment tasks”, most of its items can be used to characterize “rich tasks”, in general, (independently of their assessment usefulness). Then, the previous characterizations cannot be considered as a definition, but they give an idea of what is expected when we talk about task enrichment.

In the following section, the connections between task enrichment and inverse problems will be analyzed.

## 2.4 School Mathematics Problems. Procedural knowledge.

### 2.4.1. What is a problem?

We review and develop some general ideas about mathematics problems that are in the core of this study.

Problems are the cornerstone of science. As Popper (1994) says:

"My whole view of scientific method may be summed up by saying that it consists of these four steps:

1. Selecting some open question - *problem* - perhaps by stumbling over it.
2. Trying to answer it by proposing a tentative solution by means of some *theory* or explanation.
3. Through the *critical discussion of our theories* our knowledge will grow by errors elimination. In this way theories will be understood and applied, problems will be solved, and new solutions will be reached.
4. New problems are always revealed by critical discussion of our best theories and solutions.

Or to put these four steps into four words "*problems - theories - criticisms - new problems*" (p. 158).

For both history of science and its philosophy, science is perceived as being essentially a problem-solving activity Laudan (1978, p.11).

### 2.4.2 Problems in PISA study

Recently, "PISA international comparative study starts with a concept of mathematical literacy that is concerned with the capacity of students to analyze, reason and communicate effectively as they pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts. (…) When thinking about what mathematics might mean for individuals, one must consider both the extent to which they possess mathematical knowledge and understanding, and the extent to which they can activate their mathematical competencies to solve problems they encounter in life (OECD, 2004, p.37). The curricular framework of the PISA study is based on the conviction that learning to mathematize is a basic objective for all students. The mathematical activity is specified in the mathematizing activity, which PISA identifies in the project with solving problems (OECD, 2003). As a consequence, mathematics teachers need to have a clear idea of what a (mathematical) problem is and (what it) means to solve a problem, in order to promote this competence in the students.

### 2.4.3 A common tradition and its boundaries

A problem can be described as a challenging or conflictive situation that proposes the achievement of a goal and makes it necessary to discover a way to solve it (Rico et al., 1988). So, to consider an open task as a problem, the subject who faces it must assume that it is an open and broad question, whose answer is unknown and that needs to be found.

Extended beliefs among the students take for granted that each problem has only one correct answer, that mathematics tasks should be worked on individually, that problems are solved quickly by the procedures previously studied in class, or cannot be solved, that the teacher knows the solution and the students may be able to get clues from him/her and finally, that the solution is not connected with the real world (Lester, 1983; Rico, Castro et cols., 1988, pp. 9-10).

One way to overcome this limited vision is to propose open problems where it is necessary to make decisions that change the final solution, or problems that may have several solutions.

Rico and collaborators (1988) stated that considering a certain mathematical task as a problem depends mainly on the challenge it proposes and the cognitive demand that the task poses to the students. Then, classify a task as a problem depends on several factors such as the type proposed, the school level in which it is proposed, the teacher's expectations about the learning that can be achieved, the difficulty of the task and the cognitive challenge posed to the students.

Depending on the challenge for the student, the tasks range from problems to exercises. As it was mentioned, a given task can be considered as a problem for one learner and it may be an exercise for another person (Rico et cols., 1988).

Castro and Ruiz-Hidalgo (2015, pp. 89-106), remark a relevant characteristic of mathematical problems: being an object of the didactic content. Therefore, they deserve to be studied and analyzed by the mathematics educators, taking into account their important role within the educational system.

## 2.5 Problems Reformulation and Problem Solving

Halmos is perhaps one of the most direct professional mathematicians in what concerns with his opinions on the role of problems in mathematics. He stated that: “The mathematician’s main reason for existence is to solve problems, and therefore, what mathematics really consists of problems and solutions” (Halmos, 1980 p. 519). His point of view includes posing good questions and not simply practicing on solving routine exercises. Particularly, he said that “One of the hardest parts of problem solving is to ask the right question, and the only way to learn to do so is practice” (Halmos, 1980, p. 524).

Duncker (1945) considered problem solving as strictly related to reformulation: “It is therefore meaningful to say that what is really done in any solution of problems consists in formulating the problem more productively” (pp. 8–9). Similarly, Pólya (1945) noted that we should try to modify our reformulation in order to obtain a more accessible problem: “We often have to try various modifications of a problem. We have to vary it, to restate it, to transform it again and again till we succeed eventually in finding something useful (…). What we attain after various trials is very often…a more accessible auxiliary problem” (pp. 185–186).

As a consequence, doing research is strongly related to problem solving and the last one is related to problem posing.

Also, it should be remarked that problem posing has a strong connection with task enrichment and it is considered as an important topic in mathematics education. Problem posing constitutes the core of the next subsection.

### 2.5.1 Problem posing and task enrichment

In this research area, there exists a traditional list of milestone papers, like those of Brown and Walter (2005, 2014), English, (1997, 1998), Kilpatrick (1987), Silver and Cai (1996), among others. These authors used the term “problem posing” for strategies to include both new problems formulation and reformulation of previously proposed problems. These activities were developed in different formats that can be more or less structured (English, 1997, 1998; Silver, 1994, 1997; Silver & Cai, 1996).

### 2.5.2 Strategies to solve problems from statements reformulation

One of the particular situations may happen when students pose a new problem as part of the solution of another one, which is more complex (Silver, Mamona-Downs, Leung & Kenney, 1996). This is an interesting situation that firstly appeared in the Polya’s research (1957), who proposes some possible strategies like approaching the problem in a different form or establishing some variants, for example, discarding one or more of its original conditions.

The reformulation of a problem not always is linked to problem solving. For instance, in some research papers problems are invented, starting from a certain experience or a given situation (Silver, 1994, 1997).

Another interesting option consists in combining the two previous approaches, that is, ask the students to solve a given problem, where the final question or a certain condition was changed, and so, it can be considered as a new problem (Silver, 1994).

A different approach can be found in Brown and Walter (2005, 2014). In order to raise new problems, these researchers propose a strategy so-called “What if not?” The idea consists in changing restrictions and/or conditions of the problem and then, creating a new one.

Furthermore, at UGR it is possible to find some interesting experiences about problem posing. For instance, Ortiz, Rico and Castro (2008) proposed to the participants of their study: "Suppose that a high school teacher needs to develop a didactic activity in order to show the usefulness of linear equations systems" and they required the subjects to describe or propose a real-world problem situation that meets that assignment. Additionally, they asked them to propose at least two questions, which answers require, modeling and using a graphic calculator. So, in their research, problem posing is combined with modeling and the use of technology.

### 2.5.3 Invention and structured problem statements

In the work of Stoyanova (1998, pp. 164-185) three possible cases in problem posing research can be identified: free, semi-structured and structured situations. In the case of the free situations, there are no restrictions for the problem posing activity. In the semi-structured situations, problem invention is based on some quantitative information and/or in any previous experience. In the structured situations, the original problem is reformulated or some condition is changed in order to obtain a new one.

It is important to mention that at UGR the participants of our research are asked to reformulate in an inverse form a given direct problem, so, this is an example of a structured situation, in the Stoyanova (1998) classification.

## 2.6 Inversion: a strategy to pose new problems and enrich tasks

In the previous section, different strategies for problem posing were analyzed, focusing on their potential for task enrichment. Another option that deserves to be considered involves using inverse problems, which give an interesting option to pose new problems, by reformulating a previous statement and then, contribute to task enrichment.

In a first approach, it can be mentioned that problems can be posed in a direct or an inverse form. According to Groestch (1999, 2001), *direct problems* are those where the required information is provided in order to follow a well-defined procedure leading to a single correct solution.

On the contrary, *inverse problems* are usually more difficult –and also more interesting– since they often do not have a solution and if they do, usually it is not the unique one. Moreover, these problems appear regularly in other subjects and in many professional careers. As Bunge (2006) observes, listing the symptoms of a certain disease is a simple problem posed directly, which can be done easily consulting specialized texts. On the contrary, being able to diagnose a disease in a patient, knowing only its symptoms, is a problem of greater difficulty that requires the work of an experienced medical doctor.

Bunge (2006) mentions other examples, such as the detective that has to solve a crime, studying victims, testimonies, etc., or simply in our daily dealings with other people, when we try to guess the intentions of the others based on their behavior. Finally, in Philosophy, says Bunge[3]

[3] https://grupobunge.wordpress.com/2006/07/20/119/

(2006): “The fact that almost all philosophers have ignored the peculiarities of inverse problems, raises this other inverse problem: that of guessing the reasons for this enormous oversight of philosophers”

For all the previous reasons, inverse problems are extremely important and they are deeply analyzed in the next subsections.

### 2.6.1 Inverse problems classification

We will see below a classification of inverse problems, complemented by some mathematics examples.

In our research, mathematical problems are considered in a simpler way than the one considered by Popper. In fact, following Groestch ideas (1999, 2001), and considering that the direct-inverse relationship between problem statements is symmetric, direct problem can be easily schematized. This schema is similar to the input-process-output model (IPO model), commonly taught in computing and information technology units and it was deeply discussed in a previous paper (Martinez-Luaces, Fernández-Plaza, Rico & Ruiz-Hidalgo, 2019).

An example of a direct problem takes place when two polynomials (dividend and divisor) are provided and the student must execute the corresponding division algorithm, thus obtaining two polynomials (quotient and remainder), which constitute the requested response.

Now, we analyze the situation when answers become to be questions. In contrast with direct problems, the creativity inherent in inverse problems can lead to them being unsolvable or having multiple solutions, which is why they often pose more difficult and interesting questions (Bunge, 2006).

Globally, inverse problems can be classified in two subgroups: causation and specification problems. In the causation problems the procedure is well-known and the question is about the data needed to get a certain result. An example of causation problem is the determination of the function $F(x)$ that by derivation allows obtaining a given function $f(x)$ (i.e., find a primitive function, or indefinite integral). The specification problems are those where both data and result are given and the question is about which procedure can led to the desired result. An example of specification problem happens when the student is asked to demonstrate a property. In this case, both the hypothesis and the thesis are well known (since they are part of the statement) and what is requested is the reasoning that allows arriving at the thesis (output), starting from the hypothesis (input).

Schemas for the causation and specification problems were provided and deeply discussed in a previous article (Martinez-Luaces, Fernández-Plaza, Rico & Ruiz-Hidalgo, 2019).

In the experimental sciences and real life problems, causation and specification problems are common and also very important. This fact was already discussed in previous papers and book chapters (Martinez-Luaces, 2013, 2016).

As a final reflection, it is worth mentioning that the direct-inverse relationship should not be considered as symmetrical. In fact, there exist problems that deserve the qualification of direct whereas others are inverse ones, and only when both lead to a well-established process there will be symmetry. This last situation usually happens in classic arithmetic problems.

### 2.6.2 Solving inverse problems

In his article about inverse problems, Bunge (2006) proposes some possible solving techniques. Firstly, he comments that: "Inverse problems are of great theoretical interest because they refer to the most difficult investigations in all fields..." and adds that "... solving an inverse problem involves synthesis or regressive reasoning, that is, going from conclusions to premises or from effects to causes".

These comments suggest that both teachers and researchers have an enormous margin for posing inverse, creative and interesting problems that allow the development of skills and capacities and also promote critical thinking.

Specifically, regarding the solving strategies for these problems, Bunge (2006) recommends the following:

- Convert the given problem into a different one that can be solved
- Analyze the entire family of associated direct problems, since any of its members may be the key to solving the inverse problem posed
- Propose different possible scenarios that can lead to the result that was finally obtained
- Propose and test several plausible hypotheses and analyze which one fits better with the facts observed

Just analyzing the previous list of strategies for inverse problems solving, it is possible to know-how of its didactic value. In effect, these problems lead us to propose different hypotheses, try several resolution mechanisms, etc., and finally, evaluate the results obtained. If we combine these problems with mathematical modeling (the so-called inverse modeling problems, that will be analyzed later in this chapter), then we obtain an excellent platform for exploration and multidisciplinary work.

Once again, Bunge (2006) comments that "inverse problems are so difficult and have been so discriminated that the first international congress on the subject was held as late as year 2002" and "the treaties on the subject can be counted on the fingers of one hand". This is quite surprising, since –according to Groestch (1999, 2001)– the first inverse problem related to mathematics has almost five centuries. In fact, that problem is attributed to Tartaglia, who studied a ballistics inverse problem in 1537. In mathematics education the situation is not very different, since these problems have been almost ignored in traditional courses (Groestch, 1999; Martinez-Luaces, 2011).

Based on the above and taking into account the educational potential of inverse problems it is obvious that studying these problems is a fundamental task for mathematics education research.

This thesis will didactical analyze inverse reformulations of a given problem, proposed by prospective teachers at their training courses. The selected problems for this purpose are analyzed in next chapter.

### 2.6.3 Inverse problems in Mathematics Education

Traditionally inverse problems have been underestimated by both Mathematics Education and by Mathematicians. Moreover, taking into account that many of the problems that we must face daily are inverse problems, this apparent forgetfulness is even more surprising and only in recent times seems to be lagging behind (Bunge, 2006).

Groestch (1999, 2001) has pointed out that direct problems have practically dominated theoretic traditional mathematics courses, although in a modern curriculum, inverse problems should have an important role. Among his arguments, he mentions that inverse problems are more suitable for exploring questions of existence and uniqueness, as well as the stability of solutions. On the other hand, these problems bring the courses closer to the situations that arise in real life.
In a previous paper (Martinez-Luaces, 2011) several examples were proposed in order to illustrate how the above can be applied at any educational level, from primary school to university.
Lastly, a simple direct problem of elementary arithmetic consists in adding two odd prime numbers to obtain–obviously– an even number. It is more interesting to analyze the corresponding inverse problem: is it always possible to decompose an even number as the sum of two prime numbers? This question gives rise to the so-called "Goldbach conjecture", which was postulated in 1742 and its solution remains elusive more than 250 years later.
A direct problem is definitely a problem which students may potentially find a way to solve it and get familiar with the processes and concepts involved. On the other hand, an inverse problem derived from a direct one leads to a new problem, with multiple or no solutions, and involves more complex processes and concepts or deeper understanding of process or concepts involved in the direct from of the problem. In fact, authentic problems may emerge from routine processes by convenient inverse modification. Finally the adjectives “direct” and “inverse” are relative and depend on the solver’s perspective, that is, when students

get familiar with the process of solution of inverse problems, these become into the direct form for further inverse modifications.

### 2.6.4 Inverse problems in modeling and applications

In modeling and applications of mathematics to other disciplines, the inverse problems can best be exploited from an educational point of view. Consequently, we will particularly study the situation in which mathematical modeling and inverse problems are combined, giving rise to the so-called *Inverse Modeling Problems*, considered in some previous works (Martinez-Luaces, 2009, 2013).

Before analyzing inverse modeling problems, it is important to clarify the similarities and differences between modeling and applications.

Mathematical modeling is an active area of Mathematics Education research that has had an important development, fundamentally since the 1980s decade. For instance, ICTMA (International Community of Teachers of Mathematical Modeling and Applications), affiliated to ICMI (International Commission on Mathematical Instruction), organizes its international congress in odd years. This congress has already taken place in countries such as Germany, Australia, Brazil, China, the United States, England and South Africa, among others. Moreover, the World Congress of Mathematical Education (ICME) has been organized twice by ICTMA researchers.

It is well-known that the vast majority of mathematics students do not want to be mathematics teachers, nor mathematicians (Varsavsky, Waldock, Harding, Bookman, Sheryn, Martinez-Luaces, 2011). Therefore –at least from a numerical point of view– nothing would be more important for mathematics education than what is usually called “mathematics as a service subject”. These arguments highlight the need of a motivating mathematics teaching, including abundant applications and modeling examples. Then, it is not surprising that almost all the modern mathematics textbooks include examples of applications and/or mathematical modeling and also establishing strong links with real-life problems (Abell & Braselton, 2016; Simmons, 2016).

Obviously, there exists an important connection between mathematical modeling, problem solving, and applications. Werner Blum (2002), in the ICMI Study Nº 14 Discussion Paper observed that the term "modeling" refers to the direction that goes from the real world to mathematics, while the word "applications" focuses on the opposite direction, i.e., from mathematics to real life.

Also, it should be observed that the term "modeling" focuses more on the process that takes place, whereas the term "applications" emphasizes the object involved, particularly in those areas of the real world to which a certain mathematical treatment is applied.

It is important to remark that in the discussion document already mentioned (Blum, 2002), the term “problem” is used in a broad sense, including abstract problems, or those that try to explain or describe real-world phenomena. This implies that a mathematical model can be developed even if there is no specific problem to be solved, being just a mathematical description of a certain phenomenon. Similarly, a certain “pure” math problem can be solved, without any modeling process associated. Then, problem solving can take place without modeling and modeling can also be done without involving the resolution of a specific problem.

The relationship between mathematical modeling and applications was outlined by a schema included in previous works (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2019, pp. 31-32).

A deeper and broader discussion of problem solving, mathematical modeling, and applications can be found in classical texts (Blum, 2002; Polya, 1979; Schoenfeld, 1985) and also in an earlier work (Bressoud, Ghedamsi, Martinez-Luaces, & Törner, 2016).

Now, if we focus on the relationship between inverse problems and modeling and applications, it should be noted that in most cases, inverse problems associated with applications are usually simpler than inverse modeling problems.

It is important to remark that all these problems –inverse modeling problems or inverse problems associated with a certain application– do not have a general methodology to solve them. Although it is possible apply some of the strategies suggested by Bunge, previously commented in this chapter.

# Chapter 3.- METHODOLOGY

This chapter describes the methodology implemented to achieve the research objectives stated in Chapter 1.

In the study of professional competencies of the mathematics teacher, there are multiple methods and techniques that may be used; both general techniques (interview, questionnaire, case study, among others), as well as some more specific ones. This is not a study of individuals since the work is proposed as a research about professional competencies developed by a teachers' group that pursues the University Master's Degree for training mathematics teachers in Secondary Education at the University of Granada. This group is interested in their mastery of the procedural contents on teaching and instruction, mainly their mastery of school mathematics problems and tasks, their enrichment, and its organization through teaching units.

To reach the objective (OG1), two different strategies were proposed:

- Firstly, a modeling task about the filling of a swimming pool was provided to the participants and they were asked to reformulate a problem from it with the aim of carry out a task enrichment for Secondary School courses using categories, components and elements from an instructional contents analysis.
- Secondly, a new problem about a sheep grazing in a square field was given to the prospective teachers; in this opportunity an inverse reformulation enrichment task was specially requested by means of the previous teaching instructional contents, of teaching

organization, and through specific resources and instructional materials.

Regarding the second general objective (OG2), in the first stage, a worksheet was provided for the participants, in order to perform the didactic analysis of the tasks associated with their own reformulation.
When the first problem reformulations' were analyzed working with the content analysis' categories, components, and descriptors for instructional contents, it was found that the statement' inversion does not constitute a spontaneous strategy for tasks enrichment proposed by the participants. Also, in the analysis of the second problem responses, it was observed that a large number of participants tended to imitate the given examples and an important number of reformulations were ill-posed problems or just trivial proposals. In other words, the quality of the responses was not as good as expected in order to obtain important conclusions regarding OG1. Moreover, considering the achievement of the second general objective, it was observed that future teachers gave their opinions concerning several items that attracted their attention, but leave aside many of the elements considered in Table 2.1.
For these reasons, a second stage of the research was proposed and several modifications were introduced to avoid or at least attenuate the inconveniences of the first stage, where both the problem reformulation strategies as well as the didactic knowledge of future teachers cannot be completely analyzed. In addition, the treatment of the data analysis was deeply modified, including a cluster analysis for the reformulations, in order to study better their richness.

We begin by characterizing the type of research carried out. This description –including other generalities– constitutes the first section of this chapter (Section 3.1).

The selected problems for the fieldwork design, i.e., the swimming pool and the sheep problems, are described in Section 3.2.

The fieldwork of this research was implemented in two different stages.

The third section of the chapter (Section 3.3) is devoted to the methodology corresponding to the first stage of the research. This section includes the criteria for participants' election, the design of the fieldwork, the description of the instruments used to collect the information, the choice and organization of the data, together with the criteria that deal with the analysis of the answers and productions of the prospective teachers. Regarding the analysis of the productions, the categories are explained in detail both for the analysis of the reformulated problems and the subsequent didactic analysis of the corresponding tasks.

In the last part of the chapter (section 3.4), the research' second stage is analyzed from a methodological viewpoint. For this purpose, once again, the participants' election, the fieldwork design and the categories used in the analysis of the productions are described. In this new study, the participants' reformulated problems are analyzed, codified and classified, in order to get a coherent feed for a cluster analysis. This cluster analysis was proposed with the aim of getting more information about the different strategies utilized by the prospective teachers in the reformulation of a given problem, as it was proposed in the OG1.

Finally, for the cognitive and the instructional analysis of the enriched tasks, the refined categories previously described at the first stage of the study were used.

### 3.1. Type of study and generalities

This research is an exploratory study, since it corresponds to a first systematic study related to the invention of problems by prospective Secondary Schoolteachers.

As antecedent of this research, a previous experience was developed at the University of Colima, Mexico, several years ago (Martinez-Luaces, 2013). In that opportunity–after working with prospective teachers in a ten hours mini-course– a survey was carried out in order to know their general opinion about the activities proposed. It should be remarked that the participants' productions were not deeply analyzed, so this work constitutes a first step for a systematic investigation within this field. Then, it can be said that this work is related to a previous one, carried out at the University of Colima (Martinez-Luaces, 2017), but it clearly differs in terms of experimental design and data treatment.

According to Hernández, Fernández, and Baptista (1991), exploratory studies are carried out to examine a research problem –which may be unknown or that has been scarcely studied– in order to become familiar with it.

Moreover, Dankhe (1986) mentions that exploratory studies rarely constitute an end in themselves, and they generally determine trends and potential relationships between variables, thus establishing a guide for further systematic and comprehensive research.

Returning to Hernández, Fernández, and Baptista (1991), these authors stated that the exploratory studies are carried out to examine an unknown or poorly studied research problem and this is exactly the situation with respect to this chosen problem when we begin to work on them. Indeed, there is little research that addresses inverse problems and mathematical modeling at the same time, as mentioned earlier in Chapter 2. Two exceptions to this fact are given by Liu's ICTMA 10 paper (Liu, 2003) and the article published in *Mathematics Education Research Journal* by Yoon, Dreyfus and Thomas (2010). In both cases, the authors work with inverse problems connected with the real world, although the participants are students, not prospective teachers. On the other hand, in both cases the mathematical model was provided to the students, so the modeling process did not take place and the participants neither were requested to pose new problems related to the situations analyzed. In other words, both works are slightly connected with this thesis, but they differ with our work in many aspects.

This work is also a descriptive study, since its main objective is to identify and document characteristics about how prospective teachers reformulate in an inverse form another problem, originally written as a direct one. Specifically, the study aims to describe and characterize –in qualitative terms– the participants' productions about rich tasks related to the inverse modeling problems that they previously reformulated from the given problem. Moreover, since the participants are prospective teachers, their reformulations should be part of an enrichment of mathematical tasks, to be used in Secondary School courses. For these reasons, they were asked to analyze the tasks they proposed, comparing both problems (direct and inverse) and the related tasks in terms of meanings, authenticity, elements and variables involved in the tasks.

The analysis of the productions of the aforementioned prospective teachers will be carried out for both to the first and the second stage of the research. For this purpose, a content analysis methodology is followed (Rico & Fernández-Cano, 2013). From a methodological view, content analysis is a rigorous procedure governed by systematic rules for the examination and verification of written data content (Cohen, Manion & Morrison, 2011, p. 563). This tool is briefly described in subsection 3.3.2 for the first stage and subsection 3.4.2 for the second stage of the research. It is important to remark that the analysis of the reformulated problems showed several particularities in each stage of the research and so, instruments utilized are not the same in both cases. Moreover, in the second stage it was decided to perform a cluster analysis to classify and characterize the kinds of inverse problems proposed by the participants, as well as their skills in proposing rich tasks for mathematics teaching. The categories and criteria of classification used for this purpose are described in section 3.4.4, as an important part of the content analysis.

Finally, it should be remarked that the content analysis is much more effective when itis complemented with the analysis of the enriched tasks. The categories utilized for the first research stage –described in subsection 3.3.4– are refined to obtain those that were used for the second research stage.

## 3.2. Selected problems for the fieldwork design

Rephrasing a given direct problem as inverse establishes an obvious connection with problem posing and task enrichment, considered in chapter 2.

For this exploratory study two problems were chosen:

- About the filling of a rectangular swimming pool with a variable depth;
- Referring to a sheep grazing in a rectangular field.

The statements of these problems are the following:

*The Volume of Water in a Swimming Pool*

A swimming pool is 3 m deep at the deepest part and 1 m deep at the shallowest end. The horizontal dimensions of the pool are 40 m by 20 m. Finally, if $h$ is the height of the water at the deepest end, the typical direct modeling problem consists of obtaining the volume of water, $V$, as a function of the height, $h$.

The corresponding diagram for this problem is shown in Figure 3.1.

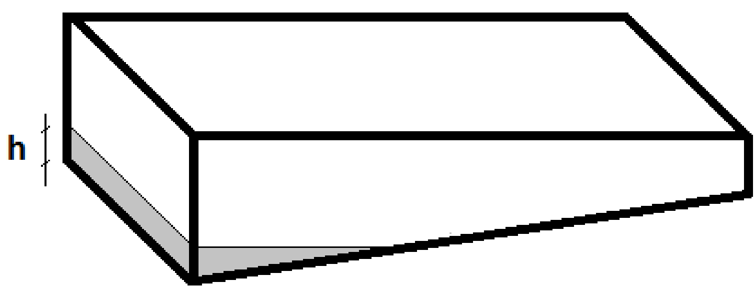



Figure 3.1. Diagram of the swimming pool.

Another diagram, showed in Figure 3.2, can be useful in solving this problem. In this diagram, the straight line that passes through the points $(0,-3)$ and $(40,-1)$ represents the bottom of the pool and the horizontal dashed line represents the level of the water given by the function,

$y = 3 - h$. The intersection point of these two lines is easily calculated and using simple mathematics the area of the grey triangle can be evaluated in Figure 3.2 and finally, the volume of water in the pool in Figure 3.1 can be obtained.

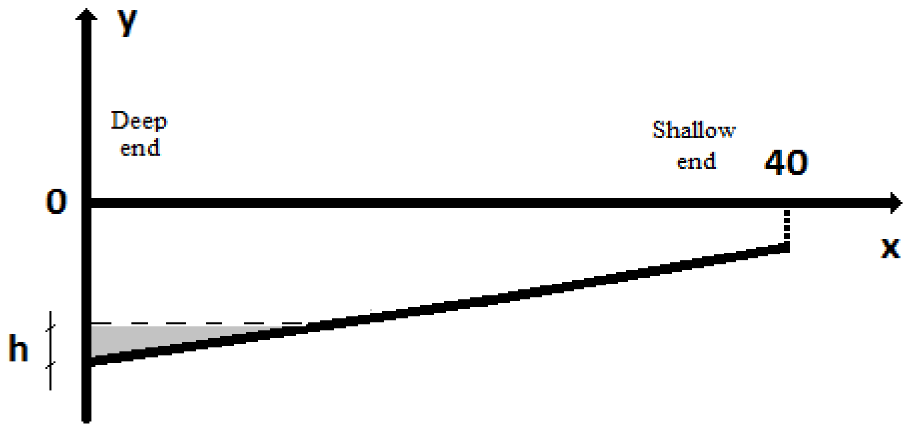



Figure 3.2. Another diagram for the swimming pool problem.

A more interesting problem –and more related to real-life– consist in obtaining the height as a function of time when the swimming pool is being filled with water at a flow rate of 0.8 cubic meters per minute. Particularly, a motivating inverse problem would be to calculate how much time is required to get a desired height at the deep end of the pool. In fact, this is the true real-life problem for both residential and commercial pool owners.

This is an example of a simple inverse modeling problem, which can be solved using only pre-calculus knowledge and skills.

*The Sheep Problem*

Let us consider a sheep that grazes in a square, being $L$ the side length. The sheep is tied with a rope of length $R$, at the point $(L/2\ ,\ 0)$ as can be observed in Figure 3.3.

In fact, in Figure 3.3, $A$ represents the area accessible for the sheep, $r = \frac{R}{L}$ is the ratio of the rope length to field side length and $f = \frac{A}{L^2}$ represents the fraction of the total area accessible for the animal. Obviously, $f$ is a function of the ratio $r$, which can be easily obtained by using known integration technique.

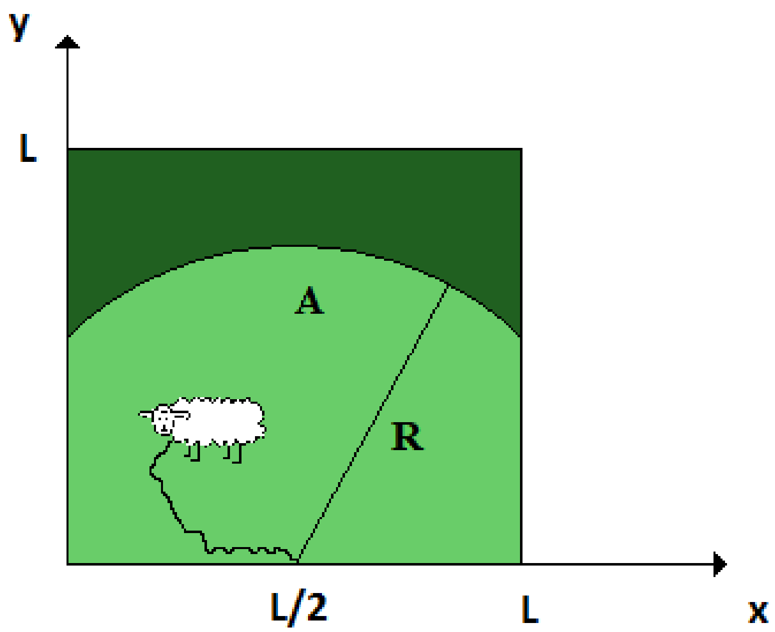



Figure 3.3. Grass area accessible to the sheep.

As mentioned, this typical direct problem appeared in a book chapter (Martinez-Luaces, 2016) and it asks the students to obtain the fraction

of area accessible for the sheep, $f$, that corresponds to a given value of the ratio $r$. Besides, in that book chapter it is observed that this direct problem can be solved just intersecting circles and squares.
Considering the intersections, four different cases are observed:

- In the first one, $r \in [0,1/2)$ and the sheep cannot reach the field lateral edges.
- In the second case, $r \in [1/2,1)$ and the sheep is able to reach the lateral edges, but not the upper one.
- If $r \in \left[1, \frac{\sqrt{5}}{2}\right)$, the sheep reaches the top edge of the field, but it is not able to graze in the whole field.
- In the last case, if $r \geq \frac{\sqrt{5}}{2}$, the sheep grazes without restrictions in the entire field.

Using integrals –among other possibilities that will be analyzed in next sections– this direct modeling problem can be solved. An obvious observation is that there exists a unique value of $A$, for every value of $r \geq 0$. Nevertheless, a more interesting question could be: for any given value $A$, does a corresponding value of $r$ exist? If so, is it the only one? In order to answer these questions, the continuity and strict growth of the function $f(r)$ must be studied, when $r \geq 0$, to ensure the necessary inversion of the function.

*A few comments about the richness of selected problems*

Both proposals –the swimming pool and the sheep problems– can be considered as intellectual and academic quality activities, for multiple reasons. Both problems allow group work and pay attention to the diversity, since both have multiple solution pathways and entry points. Particularly, the sheep problem can be solved by integrals or using just trigonometric concepts and procedures.

As we will see later, both problems produce engagement among the prospective teachers and allow using different representations and making connections with other mathematics topics and even with other subjects.

If we focus on the sheep problem, it should be remarked that it has multiple solutions (analytical, geometric or a mixture of both) and the possibility of using various representations, particularly the symbolic and the pictorial. This perfectly meets two of the Grootenboer (2009) criteria, since it attends the diversity and it allows several representations. Moreover, the task proposed to the participants (solving the direct problem, reformulating it in inverse and solving this new problem) allows obtaining several outcomes in only one task. Finally, since the proposed task includes an inverse reformulation proposed by the participant, it has a certain open character. The above covers two important aspects mentioned by Clarke and Clarke included in the list of Section 2.3.

## 3.3. First stage of the research

### 3.3.1. Participants of the first stage study

Considering the research characteristics, it was required to work with one or several groups of prospective Secondary School mathematics teachers. Taking into account the available options, it was decided to work with the students of both Groups, A and B, of the subject

“Teaching and Learning of Mathematics in Secondary School”, included in the University Master in Mathematics Teaching and Learning for Secondary School.

In the academic year 2016-2017 at the University of Granada –when the first stage of this research was carried out– Group A consisted of 33 students and Group B had 41 students, with regular class attendance. The fieldwork was possible due to the collaboration of the professors Rico (who was in charge of Group A) and Moreno (working with Group B).

### 3.3.2. Tools and application protocol in the first stage study

Among the techniques recently developed for mathematics teachers' education studies, some of the most relevant are focused on planning rich instructional tasks and problems, which contents satisfied curricular criteria linked to semantic contents, intentional expectations, tasks structure and coherence categories. In teachers training courses the semantic questionnaires (Klok, 2014; Matthewson, 2004) can be used to identify and assess the capacities of the prospective teachers when reformulating and enriching a school mathematical task through inversion and modeling, and when analyzing their didactic characteristics contents and variables.

For the empirical part of the study we worked with the participants in two time sessions, i.e., a training class where the participants were asked to solve a first modeling problem (the swimming pool problem) and a second session where they worked on a new problem (the sheep problem).

It should be mentioned that for the first problem an inverse reformulation was not requested, only to reformulate the given problem in order to make it richer. In fact, the participants were only asked to reformulate it with the aim of task enrichment whereas for the second, an inverse reformulation was specially requested. Neither for the direct problem, nor for its reformulation, a written solution was requested in the first stage of the research.

The first problem, about the filling time of a swimming pool (described in subsection 3.2), had already been proposed in Colima (Martinez-Luaces, 2013) and also it was included in a book chapter about inverse modeling problems (Martinez-Luaces, 2016). A few selected reformulations proposed by the participants were described in a book chapter published in New York, in 2019 (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2019).

Summarizing the main results, we should highlight that participants proposed several reformulations for the swimming pool problem, with the purpose of enriching the original task for Secondary Education courses and only a few ones were spontaneously posed as inverse problems.

On the other hand, the prospective teachers were provided with a spreadsheet to analyze the didactical components of the tasks associated to the reformulated problems. This worksheet is based on the manual coordinated by Rico and Moreno (2016, p. 96), where the process of didactic analysis is synthesized, in order to delimit the didactic content of a school mathematical topic in its four analysis dimensions. In our case, we are situated in the ethical-normative dimension, being the instructional analysis the corresponding analytical method. Therefore, the study's object consists in the planning and implementation of a mathematical teaching unit. For this purpose the organizers or categories of analysis are the tasks and sequences, the organization of the classroom work and the materials and resources. The prospective teachers are requested to give their viewpoints about these categories and so, they are expected to analyze the task variables, their complexity, creativity, organization, characteristics, types and uses, for both the direct and the inverse problem. As a final remark, it is expected that following this process of instruction analysis, it will be helpful to organize mathematics teaching in didactic units.

Taking into account the above comments, the characteristics and the descriptors utilized in the worksheet were selected. The corresponding details are described in other sections of this chapter.

When the swimming pool problem was given to the participants, the productions of both groups went to a first analysis and three of them were selected, since they had been spontaneously proposed in an inverse form (one of them came from group A and the other two corresponded to participants of group B). Then, in a second working session, taking these reformulations as the main examples, the participants were given a brief explaining talk about features of direct and inverse problems.

After analyzing the previous examples about inverse problems, a new direct problem –the sheep problem– was proposed to the prospective teachers (see Figure 3.4).

**PROBLEM: accessible area for a sheep in a square field**

Let us consider a sheep that grazes in a square, being $L$ the side length. The sheep is tied with a rope of length $R$, at the point $(L/2\ ,0)$ as can be observed in Figure 1.

In this figure, $A$ represents the area accessible for the sheep, $r = \frac{R}{L}$ is the ratio of the rope length to field side length and $f = \frac{A}{L^2}$ represents the fraction of the total area accessible for the animal. Note that both $f$ and $r$ are quotients, and so, they are dimensionless numbers since numerator and denominator have the same units.

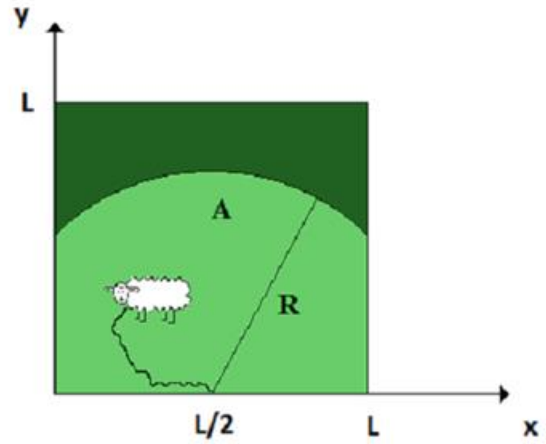



Figure 3.3.Grass area accessible to the sheep

A possible direct consists in asking the students to obtain the fraction of area accessible for the sheep, $f$, that corresponds to a given value of the ratio $r$, for instance: Which is the fraction of the total area which is accessible to the sheep if the rope is three-fourths the length of the side of the field?

Write a new problem that reformulates the initial statement in an inverse form. Explain which descriptors of the initial problem have been modified in your reformulation and justify in what extent the original problem was enriched.

Figure 3.4. Worksheet given to the participants, with the sheep direct problem (translation from the original one, in Spanish).

This new problem had already been proposed in Colima (Martinez-Luaces, 2013) and it has been published as a book chapter about inverse modeling problems (Martinez-Luaces, 2016), as happened with swimming pool problem. Unlike what happened with the swimming pool problem, in this case the participants were requested to reformulate the problem in an inverse way. However, they were not asked for a sketch of the solution, thus making it difficult to assess their difficulty properly.

Once again, the prospective teachers were requested to propose rich tasks for Secondary School courses, associated to their own reformulated problem specifically in a inverse form.

Finally, the participants were asked to make a comparison of the direct and the inverse problem –including the associated tasks– in terms of significance, authenticity, task elements and variables of the task. The worksheet (semantic questionnaire) provided to the prospective teachers for this purpose is shown in Figure 3.5.

Then, it can be noted that prospective teachers answered a semantic questionnaire divided into two parts: the reformulation of a problem and comparative analysis of the two proposals.

### 3.3.3. Data collected about the sheep problem

Due to thesis length reasons, results from the work with the swimming pool problem are not detailed, but some of them may be consulted in (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2019).

**Master's degree in Secondary School teaching. Specialization in mathematics.**

**Department of didactic of mathematics.Teaching and learning of mathematics.**

**Instructional analysis. Analysis of school mathematics tasks. Task enrichment.**

Comparison of problems' statements.

Indicate which characteristics you have taken into account in order to modify the task, specifically explaining the similarities and differences between the direct and your proposed inverse problem, considering your previous analysis.

| **Characteristics considered** | **Comments** |
|---|---|
| Meanings | |
| Authenticity | |
| Task elements | |
| Task variables | |

Figure 3.5.Worksheet for the comparison between direct and inverse problem (translation from the original one, in Spanish).

During the first stage, the participants were asked to submit a sheet with their inverse reformulation of the given problem and they filled a worksheet for statements' comparison (Figure 3.4). It is important to observe that only the inverse problem statement, without its corresponding solution, was requested in the first stage of the study.

Several remarks should be made about the total responses obtained, considering both groups A and B. Firstly, not all the responses corresponded to inverse reformulations of the given problem, so not all the productions have been taken into account for this study.

In contrast to the above, several prospective teachers proposed more than one reformulation, even more, a few of them proposed up to three inverse reformulations.

Moreover, it is worth mentioning that some participants in the first stage study worked individually whereas others worked with a classmate, so there is not a proposal for each prospective teacher, nor does each participant necessarily have a single answer.

For these reasons, the productions have been encoded as responses (R1, R2, etc.), rather than encoding by author. It should be mentioned that originally these productions had been coded as A1, A2, ... B1, B2, ... according to the group considered (A or B) and followed by a number that represents each of the participants.

With regard to the didactic analysis, the participants have responded on meanings, authenticity, elements of the task and task variables. Consequently, their values and opinions with respect to all categories of didactic analysis sometimes are obtained directly from the responses and in other opportunities are the result of the context and finally, in many of these categories there is not a precise opinion.

### 3.3.4. Content analysis categories

As already mentioned, in this study two semi-structured problem-solving tasks were proposed, according to one of the classifications above mentioned in the theoretical framework (Stoyanova, 1998).

In the first one, a direct modeling problem related to filling a swimming pool was proposed and future teachers were asked to reformulate it, focusing on task enrichment. This first experience with the students of the Master in Mathematics Teaching have served to three different purposes:

i) Acquire experience in reformulating problems,
ii) Carry out a comparative analysis of the tasks before and after its reformulation, and
iii) Provide examples of inverse reformulations.

Indeed, several of the reformulations –only three of them– were selected to introduce inverse modeling problems to prospective teachers.

For this purpose, it was considered that the context should be of interest, familiar to the students and that it should motivate them to pose inverse problems, which would allow the enrichment of the tasks proposed for Secondary School courses.

Moreover, both sessions were designed in such a way that the prospective teachers not only worked on problem posing, since they also carried out a comparative analysis of the proposed tasks, before and after reformulation.

The data analysis was performed following the content analysis method (Rico & Fernández-Cano, 2013), complemented with the analysis of the tasks.

As it has been mentioned, content analysis is a rigorous classification procedure, governed by specific categories and their components, which are determined by systematic rules and focused on its content inputs –both conceptual and procedural– and subjected to data analysis (Cohen, Manion & Morrison, 2011, p. 563). This tool has been used "in mathematics education as a method to establish and study the diversity of school meanings of the concepts and mathematical procedures that appear in a text" (Rico & Fernández-Cano, 2013, p. 11).

The stages to carry out a content analysis (Rico & Fernández-Cano, 2013) applied to prospective teachers' productions are:

- Define the categories.
- Induce the category system on the units of analysis, including each unit of analysis in a category.
- Relate the categories between them.
- Relate the content analysis process to the objective of the work.

Understanding a mathematical content involves interpreting and developing its concepts through a coherent mathematical system, which uses its structures and executes its procedures (Martín-Fernández et al., 2020).

We followed an existing content framework to describe school tasks, and interpret mathematical concepts meanings, which consists of three semantic categories: structural, representational and contextual (Bunge, 2008, pp. 24- 25; Rico, Martín-Fernández et al., 2016, p. 55). This semantic frame has also been conceived to study the content dimension of a curricular system.

As a consequence, in Mathematics Education (Rico & Fernández-Cano, 2013), the three categories considered to analyze the content and its components, are:

- Sense and phenomena, which considers the historical phenomena and modes of use, senses, moments at the communication components.
- Formal and cognitive contents, which considers structural components as definitions, procedures, and the cognitive types and the levels components of the studied contents.
- Representational, which includes the symbolic and graphic expressions and the sign systems used, and the processing rules, translations and conversions.

Taking into account all these elements, we consider the following categories for the analysis of conceptual and procedural elements related to the reformulated problems, proposed by the prospective teachers:

*Conceptual and procedural elements related to the reformulated problems*

- Concept/procedure involved: We classify the proposals according to a certain conceptual and procedural field.
- Structure of the problem: In this study we only considered inverse reformulations, however in some cases it is necessary to solve

first a direct problem, and then use the obtained result as the main input for an inverse problem.

- Input/output: In the reformulated problems, the supplied data (input) are not especially relevant, since they are not useful to discriminate among the different proposals. Regarding the required response, five variants were observed: the length of the rope, the side of the field, the stake position, the time for consuming all the grass available, and the speed of the sheep.
- Context variables: There are proposals that make changes in Geometry, while others include new variables that can be considered as external to the direct problem and finally, in several reformulations there are no changes in those aspects.

In the next chapter, these criteria will be used to analyze and exemplify the nine groups that had appeared in the first stage of this study.
It is important to mention that this classification in nine groups of problems is fundamentally based on their content aspects, not in cognitive and/or instruction aspects.

*Task analysis*

Task analysis requires special mention in this work since the content analysis can be helpful, but is most effective when completed with task analysis. Here, the prospective teachers described the tasks associated to their reformulations, how the students could work on these problems in the classroom, the objectives pursued, the meanings and concepts involved, etc. In order to analyze the prospective teachers' opinions it is convenient to make a choice of indicators for the tasks analysis. For this purpose, the guidelines that appear in Moreno and Ramírez (2016) were especially useful.

As it was mentioned, about the tasks associated to their own reformulation, the prospective teachers were asked about the following items: meanings, authenticity, elements of the task and task variables (see Figure 3.4).

It is important to remark that the participants were asked about four items, although the whole list of task variables descriptors (described in Section 2.2.4 and Table 2.1) provides more details about the participants' opinions.

Finally, as progress was made in the analysis of the productions of the prospective teachers, some of the previous variables were sub-divided in order to consider all the richness of the responses. At the end of this process, the final instrument was a worksheet with 21 columns, which was utilized for the analysis of the productions. The final version of the instrument for the first stage study is detailed in the next section of this chapter.

### 3.3.5 Instrument used to analyze the first stage outputs

The instrument used to ordered, organized and analyzed the productions of prospective teachers, generated a worksheet with a total of 21 columns, which are briefly described below:

- The responses were located in a first column –coded as R1, R2, etc.– where also another code indicating the group and student number (for example A1, A2, ..., B1, B2, etc.), was added in parentheses. It is important to mention that in some cases the same student proposed two and even three reformulations of the direct problem, and in this situation a lower case letter was added (for example, the answer R1 was given by the student A14a).

- Columns 2 and 3 –under the common title of “statement”– correspond to the type of statement (column 2) and change (column 3). In the first one, the type of statement is established (for example: inverse function, sequential problem, etc.) and in the second, it is specified what type of change is proposed (for example: change in the geometry of the field, in the position of the stake, etc.).
- Columns 4, 5, 6 and 7 –all of them under the common title of “analysis of meanings”–correspond to the mathematical content (column 4), systems of representation (column 5), meanings and modes of use (column 6) and situation (column 7). Regarding the mathematical content, the common answers are: areas and regions, integrals, etc. The representation systems usually include the symbolic and graphic expressions as the most typical options. In what concerns to senses and modes of use, the most common response is "space and form", although other singular responses also appear. Finally, if the situation is considered, the most common response is "educational/occupational", but this is not the only one that was obtained. All these responses are analyzed in more detail in the results chapter.
- The cognitive analysis covers the following five columns of the worksheet: cognitive content (column 8), learning expectations

(column 9), limitations (column 10) and learning opportunities, which was divided into two columns named "challenge" (column 11) and “comments” (column 12). In the column corresponding to the cognitive content, some prospective teachers’ responses start with a verb, for example: “to calculate areas using integrals” or “to control concepts”, among other options. Learning expectations and limitations did not undergo major changes. Learning opportunities was divided into two columns as many of the participants made general comments about the challenge (interesting challenge, greater challenge, etc.) and in many cases they added some additional comments in order to justify their response. Several long and interesting answers regarding learning opportunities justified dividing the column, to extract all its richness without losing relevant information.

- The instruction analysis was finally divided into eight columns (from 13 to 20) to avoid loss of information provided by the participants. The first of these columns was devoted to "language" (column 13), where the prospective teachers gave their opinions about clarity, simplicity, etc., of the language used in the problem statement. In the next column, named “authenticity” (column 14), the participants gave their opinions about if the proposal corresponds to a real situation or not. The third one, named “data” (column 15), is due to the participants’ comments about the data provided, for example, whether they are sufficient or not. In the “purpose” column (column 16), the participants expressed their opinions regarding the purpose of the task, for example: connecting ideas. The next column, “location and grouping” (column 17), could have been divided into two, nevertheless, since most of the answers were very brief (for example: “Group. In class”), they were kept as a single column. Column 18 “timing” also gave rise to very short responses, such as “one session”, or “half an hour”, among others. Column 19, about "complexity" has been previously analyzed and the same happens with the column 20, "materials and resources".

- The last column was used for observations. For instance, a participant proposed his/her reformulation of the given problem, without providing the corresponding comparative analysis and so, the corresponding observation was included in this last column.

### 3.4 The second study stage

During the development of the experimental design, in 2018, the need for a second experiment was observed, due to the deficiencies identified in the first stage.

In fact, when the swimming pool problem was proposed to the prospective teachers for free reformulation, during the first stage, it was observed that only a few participants proposed inverse problems. So, it was observed that inverse reformulation was not spontaneous in the vast majority of both groups (Martinez-Luaces, Fernández-Plaza & Rico, 2020). Moreover, when inversion was used as a strategy, the reformulated problems did not show much conceptual richness.

In the case of the sheep problem, an inverse reformulation was requested, after showing the participants some initial examples about both swimming pool and sheep problems. In this second part of the first stage, it was observed that the prospective teachers tend to imitate the given models, although some creative reformulations emerged. Also, many reformulations were classified as ill-posed problems or their solution was limited to applying a formula, trivializing the proposal. In other words, the quality of the responses was not as good as expected, in order to obtain important conclusions for the general objective OG1. Regarding the general objective OG2, a worksheet was provided for prospective teachers, in order to perform a didactic analysis of the tasks associated with their own reformulation. Regarding the achievement of this second general objective, it could be observed that prospective teachers gave their opinions concerning several items that attracted their attention, leaving aside many of the elements considered in Table 2.1.

For these reasons, a second stage of the research was proposed, including a new fieldwork designed to diminish the presence of direct reformulations and deep analyze inverse reformulations, using for this purpose some examples taken from other contexts, such that, possible imitations could be attenuated. In this second stage, the swimming pool problem served for training and familiarization with the inverse reformulation strategy (in fact, a preliminary study of the productions – not included in this thesis– was carried out for this problem).

Data derived from the sheep problem were analyzed here, since they better represent the knowledge and skills acquired by the participants in the experience. In addition, the sheep problem reformulations could be better described in terms of certain variables, which allowed classifying them by using appropriate quantitative techniques.

As it was mentioned, the new design showed some differences with the previous one and four of them were particularly important in terms of the experimental design:

- In this new research, the participants were asked to give a solution of the direct problem and after that, propose the corresponding reformulation in an inverse form.
- Before the proposal of the new task, other examples of inverse problems were discussed being none of them related to the sheep problem. This decision was made in order to avoid the adaptation or simple imitation of a previous procedure.
- The solution of the reformulated problem was requested to the participants. This means that the prospective teachers solved their own reformulation, or at least, gave a sketch of the solution. This request was proposed aiming to diminish the number of ill-posed problems.
- The participants were previously provided with a worksheet for the analysis of the direct and the inverse problem, which was accompanied by a miniature version of Table 2.1, which was added in order to give briefly explanations of the descriptors on which participants are asked to make comments. Then, it was expected that the participants could answer on all the aspects of Table 2.1, or at least, in those that most attract their attention.

This new experience with a different design produced a plethora of responses (that will be analyzed in the results section) and detailed in the following sub-sections.

### 3.4.1. Participants

The second stage of this research (during the academic year 2018-2019 at the University of Granada), considering the research characteristics and taking into account the available options, was carried out with two groups of prospective teachers. The first one, Group A, consisted of 32 students and the second one, Group B, had 33 students, with regular class attendance.

The fieldwork in this opportunity was possible due to the collaboration of the professors Moreno (Group A) and Ruiz-Hidalgo (Group B).

### 3.4.2. Tools and application protocol in the second stage of the research

As in the first stage study, the experimental part of this study was carried out in two time sessions, i.e., a training class where the participants were given a brief explaining talk about direct and inverse problems and also, some introductory examples were discussed. At the end of this first session, the inverse reformulation of the swimming pool problem was requested as homework, with the aim of enriching the original task, considering its future use in Secondary School courses.

According to the research design, the examples of inverse problems that were discussed during the first session were not related to the swimming pool problem.

Also, the prospective teachers were provided with a spreadsheet to analyze the tasks associated to their own proposals. The worksheet provided to the prospective teachers for this purpose was the same one that was used in the first stage study and it is shown in Figure 3.5.

In a second working session, the participants' reformulations of the swimming pool problem were analyzed in class, in a general discussion with the whole group. After that discussion session, the sheep problem was provided to the prospective teachers as homework. In this opportunity they were asked to solve the direct problem, to reformulate it in an inverse way and solve their own inverse reformulation.

Once again, the participants were requested to propose rich tasks for Secondary School courses, associated to the reformulated problem. As in the first stage of this research, the participants were asked to make a comparison of the direct and the inverse problem –including the associated tasks– in terms of meanings, authenticity, task elements and variables of the task (Figure 3.5).

It is important to mention that in this second stage, the participants were previously provided with a worksheet for the analysis of the direct problem (see Figure 3.6).

Figure 3.6 shows the English translation of the original worksheet in Spanish. In the fieldwork, this worksheet was accompanied by a miniature version of Table 2.1, which was added in order to give briefly explanations of the descriptors on which participants are asked to make comments.

Moreover, a similar worksheet was provided to the participants, for the analysis of the inverse problem, including the miniature version of Table 2.1. In fact, the only difference between both forms is in the heading, which in one case says "direct problem analysis" (like in Figure 3.6) and "inverse problem analysis" in the other one.

| **Master's degree in Secondary School teaching. Specialization in mathematics.<br>Department of didactic of mathematics. Teaching and learning of mathematics.<br>Academic year 2018-2019.<br>Instructional analysis. Analysis of school mathematics tasks. Task enrichment.**<br><br>Direct problem worksheet.<br>Indicate which characteristics and the descriptors within them, that have been modified when reformulating the original problem and add a brief explanation. | | |
|---|---|---|
| **Characteristics considered** | **Descriptors** | **Comments** |
| **Meanings** | Prior knowledge | |
| | Tasks that activate mathematic content | |
| | Challenge | |
| | Task completion: recognition / justification | |
| **Authenticity** | Event | |
| | Question | |
| | Purpose | |
| | Language | |
| | Data | |
| **Task elements** | Goal | |
| | Formulation | |
| | Materials and resources | |
| | Grouping | |
| | Learning situation | |
| | Timing | |
| **Task variables** | Mathematical content | |
| | Situation | |
| | Complexity | |

Figure 3.6. Worksheet for the direct problem analysis (translation from the original one, in Spanish)

All the worksheets used in the second stage –the direct and inverse problems worksheets and the comparative one– were based on the manual coordinated by Rico and Moreno (2016, p. 96), where the process of didactic analysis is synthesized, particularly in what corresponds to the section of the theoretical framework of the instruction analysis.

It is important to mention that during the fieldwork, some of the study participants made comments about the stress they experienced due to overwork related to the Master's subjects that they were studying at the same time. Due to this fact, the teachers in charge of the groups decided to request only the submission of the comparison worksheet, without requiring the direct and inverse problem analysis worksheets.

As a consequence, it can be observed that in Figure 3.5 only four items remained in the comparison worksheet (meanings, authenticity, elements and task variables), however, given the participants previous experiences with the other analysis forms, it was expected that they could answer on all the aspects of Table 2.1, or at least, in those that most attract their attention.

### 3.4.3. Data collected about the sheep problem in the second stage

During the second stage of the research, the participants were asked to submit a sheet with their solution of the given problem, posed in a direct form, an inverse reformulation of this problem, including a sketch of the solution, and the filled worksheet for statements’ comparison (Figure 3.5). As it was mentioned, other analysis forms were not finally requested.

In this second stage only a few responses were discarded because their reformulation was not posed in an inverse form and/or they corresponded to ill-posed problems. Then –as in the first stage– not all the productions have been taken into account for this study.

Also, several prospective teachers proposed more than one reformulation, even more, a few of them proposed up to three inverse reformulations.

Taking into account all these facts, there is not a proposal for each prospective teacher and there is not necessarily a single answer corresponding to each participant. Due to these reasons, the productions have been encoded as responses, rather than encoding by author.

Finally, with regard to the didactic analysis, the participants responded about the comparative analysis, as requested. Although a few of them submitted the analysis of the direct and/or the inverse problem, which were considered only as optional activities. Their responses were about meanings, authenticity, elements of the task and task variables and so, their opinions with respect to all categories of didactic analysis are sometimes obtained directly or in other cases by the context of the answer. As a consequence, in many of the categories of the didactic analysis there is not a concrete opinion.

### 3.4.4. Content analysis categories

*Cluster analysis of the problems: the classification and codification of productions*

The participants' productions, i.e., the inverse reformulations of the sheep problem, were classified taking into account the kind of inversion, the existence or not of changes in the geometry and the use of external variables, among other criteria.

In a first approach – as noted in subsection 2.5.1 – inverse problems can be classified into causation and specification problems. However, there are several ways of posing a causation problem, being the general inversion of the function, the inversion for a particular value and an interval inversion, just a few examples about how to do it. The same happens with specification problems, where the reformulation may ask about the interpretation of the parameters, or a graphical representation, or a different way to obtain the same results without using integrals. Once again, these are just a few possibilities for posing an inverse specification problem.

A similar situation happens with changes in geometry that include changes in the stake position and/or the shape of the field and the introduction of fences or other obstacles.

Taking into account the variables previously mentioned, a first classification allowed us to summarize the information about the participants and their reformulations in a table as showed in Figure 3.7.

| Group B Sheep Problem | Direct/ Inverse | Type of Inversion | Geometry and Dimensions | Other elements (fences, costs, etc.) | Difficulty | Solution | Inputs | Output | Comments |
|---|---|---|---|---|---|---|---|---|---|
| **PT1** | Inverse | Several pointwise inversions | Same Geometry No numbers | No | Greater | Incorrect and incomplete solution | f | r | Incomplete solutions for both direct and inverse problems |
| **PT2** | Inverse | Sketch of the region | Same Geometry No numbers | Change of context: cobblestone town area | Conceptual problem | From the integral the curve and the boundary terms are obtained | f ( R, L ) | Scketch | Gives f ( R, L ) as an integral |
| **PT3** | Inverse | Sketch of the region | Same Geometry No numbers | No | Conceptual problem | From the integral the curve and the boundary terms are obtained | f ( R, L ) | Scketch | Similar to PT2 but keep the original context |
| **PT4** | Inverse | Sketch of the region | Same Geometry No numbers | No | Conceptual problem | Gives the integral and the change of variable. Reverts the sustitution and obtains the region | A as an integral with corresponding change of variable | Scketch | Gives A as an integral after the change of variable |
| **PT5** | Inverse | Tends to ask for the sketch of the region | Same Geometry No numbers | Change of context: irrigation and fertilize part of the field | Conceptual problem | Gives f as a quotient of the integral and $L^2$ and from that obtains the region | Gives f as a quotient of the integral and $L^2$ | Scketch | Sketch is not requested explicitly, but it arises from the solution |

Figure 3.7. Example of first classification of proposals

Finally, in a second approach, the productions previously classified were codified in a Boolean format, using for this purpose the variables already described (see Figure 3.7). The Boolean results obtained were the inputs for the statistics software used for the Cluster Analysis.

We start explaining the criteria used to codify the productions in the Boolean spread-sheet exemplified in Figure 3.8.

These criteria were the following:

*Type of inversion*

All the selected proposals are inverse problems; some of them are causation problems, for example: what should be the length of the rope (cause) that makes the sheep access to 60% of the field (effect). In this case, particular values are provided (e.g., 60% of the field), so we should mark with "1" in the causation-point wise column. Let us suppose that a proposal asks to find $R$, such that the fraction of area accessible for the sheep is $f$, then, we should type "1" in the causation-general column since not particular values are given.

In the specification problems, cause and effect are known and what is asked is to specify something. If the output is a sketch that illustrates the situation, with particular or general values, then, "1" should be put in the specification-sketch column. If the output needs a certain process like finding a corner on the graph of the function or interpret the meaning of a certain parameter, we write "1" in the specification-process column.

*Inputs*

The inputs are interpreted as the data of the given problem that are provided to the student. They can be particular or general values, in the first case an example is: "it is known that the sheep accesses half of the land…" and for the second case: "Let $f$ , be the fraction of land where the sheep may graze..." It is also possible that the input is an integral, for example: "let $A$ be the area accessible to a sheep, which is expressed by the following integral…" Depending on the case considered, "1" should be written in values-particular, values-general or in the "integral" box.

*Outputs*

The outputs are the results of the problem that are requested. They can be particular values (for example: find the length of the rope, $R$, such that the sheep can accesses to a half the field), or general (for instance: find $R$ as a function of $f$ , which is the fraction of area where the sheep can graze). It is possible to see that the desired output is a sketch (for example: represent by a sketch the situation described above) or even a process (find the area accessible to the sheep by another method without using integrals and compare results). Depending on the case, we will mark "1" in values-particular, values-general, sketch or process.

| PT5 | PT4 | PT3 | PT2 | PT1 | | Code | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | pointwise | Causation | INVERSION |
| 0 | 0 | 0 | 0 | 0 | general | | |
| 1 | 1 | 1 | 1 | 0 | sketch | Specification | |
| 0 | 0 | 0 | 0 | 0 | process | | |
| 0 | 0 | 0 | 0 | 1 | particular | Values | INPUTS |
| 0 | 0 | 0 | 0 | 0 | general | | |
| 1 | 1 | 1 | 1 | 0 | | Integral | |
| 0 | 0 | 0 | 0 | 1 | particular | Values | OUTPUT |
| 0 | 0 | 0 | 0 | 0 | general | | |
| 1 | 1 | 1 | 1 | 0 | croquis | Other | |
| 0 | 0 | 0 | 0 | 0 | process | | |
| 0 | 0 | 0 | 0 | 0 | field | | CHANGE IN GEOMETRY |
| 0 | 0 | 0 | 0 | 0 | stake | | |
| 0 | 0 | 0 | 0 | 0 | additional | Variables | OTHER ELEMENTS |
| 1 | 0 | 0 | 1 | 0 | contextual | | |
| 0 | 0 | 0 | 0 | 0 | elementary | Geometry | SOLUTION |
| 0 | 0 | 0 | 0 | 0 | trigonometry | | |
| 1 | 1 | 1 | 1 | 0 | elemental | Integral | |
| 0 | 0 | 0 | 0 | 0 | theorems | | |
| 0 | 0 | 0 | 0 | 0 | | Numerical Methods | |
| 0 | 0 | 0 | 0 | 0 | algebraic manipulation | Other | |
| 0 | 0 | 0 | 0 | 0 | derivation | | |
| 0 | 0 | 0 | 0 | 0 | proportion | | |
| 0 | 0 | 0 | 0 | 0 | trivial | | DIFFICULTY |
| 1 | 1 | 1 | 1 | 0 | low | | |
| 0 | 0 | 0 | 0 | 0 | medium | | |
| 0 | 0 | 0 | 0 | 1 | high | | |

Figure 3.8. Table with results used for the Cluster Analysis

*Change in geometry*

There are some proposals that include changes in the geometry of the problem. One option is that they change the original square into a rectangle (in this case “1” goes to the column “field”) or may be that the field shape is the same, although the stake to which the sheep is tied is located in another point (in that case “1” goes to the "stake" box). Obviously if there are changes in both aspects, we will put “1” in each of the columns.

*Other elements*

Some prospective teachers add other variables, which can be additional variables unrelated to the original problem (costs, amount of fertilizer, etc.), or they may change the context of the problem (a soccer field, a bush fire in a certain region, etc.). In this last case, from a mathematical viewpoint the problem is basically the same, with a different context. In the first case we type “1” in the "variables-additional" box and in the second we write “1” in the "variables-contextual" column. If there are changes of both types, we should put “1” in each one.

*Solution*

The given solutions are quite varied and we could say that they can be classified according its mathematical contents into four subgroups: geometric, analytical, numerical, and others. Among the geometric solutions, there exists a first case where only elementary geometry is used (for example, the Pythagorean Theorem, or the formula for distance between two points), whereas others use trigonometric and inverse trigonometric functions (sine, cosine, inverse tangent, etc.). The analytical solutions can be classified in two groups: those that calculate integrals and occasionally use elementary results of this topic (change of variables, integration by parts, etc.) and those that use advanced theorems (like a result that under certain conditions links the area under the curve with the corresponding arc length). Another situation occurs when the sheep problem led to a non-linear equation which is solved by using numerical methods (Bisection, Secant, Newton-Raphson, etc.) with or without the help of technology (for instance, using Wolfram Alpha).

Lastly, there are solutions that cannot be classified in the previous groups and appear fundamentally in trivial reformulations, although not only in those cases. For example, there exist proposals that can be solved simply by finding roots of a quadratic equation or applying a well-known formula, without using geometry or integration techniques. Those solutions are considered as simple algebraic manipulations.

There are also some proposals that are solved by derivation, although it is important to confirm that the derivative is really useful; for instance, it is not necessary to use derivatives to find the maximum area accessible to the sheep, since the maximum does not occur at a stationary point. In those cases where derivatives are calculated for a useful purpose, are marked with “1” in the "derivation" column. Additionally, some proposals from an analytical point of view should be considered as direct problems (like finding an area by using integrals), nevertheless they are inverse problems of proportions (for example, the daily consumption of the sheep is given and it is requested

to determine for how long the animal may graze). In those cases “1” should be written in the "proportion" box and not in the integral one.
It should be mentioned that it is possible to have “1” in more than one box. For instance, if a prospective teacher finds the function by integrals, using a change of variables and then, the resulting nonlinear equation is solved by Bisection method, then “1” should be marked in the elementary-integral column and another “1” in the numerical methods box.

*Difficulty*

As a final obvious remark, there are proposals that show different degrees of difficulty. Firstly we have trivial problems, which can be reduced to calculate the distance between two points, or apply a given formula. Others may be easy because they involve a simple procedure, such as derivation, or the use of an intuitive result (for example, that the function $f$ increases with $r$), but at least they require some reflection and/or execute a certain procedure. Those cases are classified as low difficulty proposals. In contrast, a complicated integral of an irrational function, followed by a nonlinear equation in $R$, is considered as proposal of high difficulty. The proposals that are neither so difficult nor so simple will be marked with “1” on the medium difficulty box. In what concerns the solution, unlike other parts of the table, the different options are mutually exclusive. It is possible to see a solution that uses integrals and trigonometry, but it cannot be a trivial and medium difficulty proposal, both at the same time.

### 3.4.5 Cluster Analysis

As it was mentioned above, in a first classification the information about the participants and their reformulations was summarized (see Figure 3.6.), then the productions previously classified were codified in a Boolean format (as shown in Figure 3.7) and finally, they were used as inputs for the Cluster Analysis. For this reason, we finish this section explaining the main characteristics of Cluster Analysis.

Cluster analysis consists in grouping a set of objects in several clusters, such that the objects in a given group are more similar –in some sense– to each other, than to those objects included in other clusters.

Cluster analysis can be achieved by various algorithms that can differ significantly in their results. It should be mentioned that among the clustering algorithms there is no objectively a "correct" one. In fact, as Estivill-Castro (2002) mentioned, "clustering is in the eye of the beholder" and then, the most appropriate clustering algorithm often needs to be chosen experimentally and depends on the particular problem considered.

In clustering analysis, the connectivity models are especially important and in particular, the hierarchical clustering deserves to be considered. These models are built taking into account the distance connectivity and they are based on the idea that the objects are more related to nearby objects than to objects located farther away.

As it was mentioned, the connectivity models connect the different objects in order to form clusters based on their distance and so, as it can be expected, there are different options to be considered. In a first possibility, a cluster can be described by the maximum distance needed to connect parts of the cluster. At different distances, different clusters will be formed and they can be represented by a dendrogram, which explains where the common name “hierarchical clustering” comes from. These algorithms do not provide a single partitioning of the data set, but instead provide an extensive hierarchy of clusters that merge with each other at certain distances. Then, in a dendrogram, one axis is used to represent the distance at which the clusters merge, and the objects are placed along the other axis such that the clusters are not mixed.

As a consequence, connectivity-based clustering can be considered as a family of methods that differ by the way distances are computed. Then, the researcher needs to choice the distance function and the linkage criterion, since a given cluster has multiple objects, and so, there are multiple candidates to compute the distance. The most common choices are the single-linkage clustering (the minimum of object distances), the complete linkage clustering (the maximum of object distances) and the average linkage clustering (based on the arithmetic mean).

In our case, the cluster analysis was carried out considering the following characteristics:

- Software: IBM SPSS Statistics v.24 ® (Stehlik-Barry & Babinec, 2017).
- Metric used: Dice Similarity Coefficient.
- Linking method: Average linkage (between groups).

Finally, as it was mentioned, the metric used in this research was the Dice Similarity Coefficient (DSC). The DSC original formula –for discrete data– can be explained as follows: given two sets, X and Y, the

DSC is defined as $DSC = \frac{2|X \cap Y|}{|X| + |Y|}$ where $|X|$ and $|Y|$ are the cardinalities of the two sets, i.e. the number of elements in each set. Then, in a few words, the DSC index equals twice the number of elements common to both sets divided by the sum of the number of elements in each set.

### 4.4.6 Task analysis

As in the first stage, the prospective teachers were requested to describe the tasks associated to their own reformulations, analyze how the students could work on these problems in the classroom and make comments about the objectives pursued, the meanings and concepts involved, among other issues.

The task variables descriptors used in the second stage are those listed in Table 2.1 and also, they were described and explained in Section 2.2.4.

It is important to mention that, the final instrument utilized for the analysis of the productions in the first stage study –i.e., a worksheet with 21 columns– was once again utilized in the second stage of the research. Then, the final version of the instrument for both the first and the second stage of the research was the same and it corresponds to the one already described in subsection 3.3.5.

# Chapter 4. FIRST STAGE RESULTS

This section presents, describes and analyzes the results obtained from the reformulations of the "sheep problem" collected during the first stage of the research.

As a conclusion of the aforementioned analysis, we propose a classification into nine classes or groups, which in some cases were naturally divided into several subgroups.

The groups and their subgroups –with the corresponding examples– are deeply characterized in section 4.1.

Finally, section 4.2 is devoted to the didactic analysis of the tasks associated to every different reformulation, including general results (subsection 4.2.1), analysis of other semantic components of the meaning (subsection 4.2.2), cognitive analysis (subsection 4.2.3) and instructional analysis (subsection 4.2.4).

## 4.1. Analysis of the reformulations proposed by prospective teachers

In the analysis of the reformulations proposed by the prospective teachers, it was possible to identify, nine different groups, being some of them divisible into various subgroups:

- Group 1. Reformulations based on an inverse function, with the following modalities:
    - Subgroup 1a. Reformulation based on the inverse function, without other modifications

    - Subgroup 1b. Inverse function and change in the geometry of the field
    - Subgroup 1c. Inverse function and change of the stake position
    - Subgroup 1d. Inverse function and change of geometry and stake position
    - Subgroup 1e. Inverse function and field with obstacles.
- Group 2. Elementary problem, with several options and possibilities:
    - Subgroup 2a.Problems that can be reduced to a formula application
    - Subgroup 2b.Problems that require simple algebraic manipulations
    - Subgroup 2c.Simple reformulation problems, with the addition of other external variables
- Group 3. Inverse problem of stake location
- Group 4. Inverse problem about the length of the field
- Group 5. Optimization Problem
- Group 6. Sequential inverse problem
- Group 7. Incremental problem
- Group 8. Dynamic problem
- Group 9. Iso-surface problem

The number of proposals in every group or subgroup, only including inverse reformulations, is the following:

- Group 1 (inverse function) includes 22 proposals, which may be divided as follows: 16 proposals for subgroup1a, 2 for 1b and 1e and only 1 for subgroups 1c and 1d (it should be noted that the proposal included in 1c, can be also included in group 2).
- Group 2 (elementary problems) has 9 reformulations, the majority (7 proposals) of them included in subgroup 2a and only 1 proposal in each subgroup 2b and 2c.
- The other groups (Group 3 to Group 9) have a single reformulation each.

Also, it can be observed that only the first two groups showed more than one proposal. As a consequence, only a limited number of reformulations can be considered as truly creative.

Taking into account the described categories for the analysis of conceptual and procedural elements related to the reformulated problems, considered in the sub-section 3.3.4, it is possible to describe the characteristics of the different groups at the first stage of the research. This analysis includes the predominant characteristics of each group/subgroup (see Table 4.1). These features are illustrated in detail as follows:

| Group / subgroup | Concept/procedure | Structure | Output | Context variables |
|---|---|---|---|---|
| 1a | Integrals calculation | Inverse | Rope length | Same geometry |
| 1b | Integrals calculation | inverse | Rope length | Change of geometry |
| 1c | Integrals calculation | Inverse | Rope length | Same geometry |
| 1d | Integrals calculation | Inverse | Rope length | Change of geometry |
| 1e | Integrals calculation | Inverse | Rope length | Change of geometry |
| 2a | Trivial procedure | Inverse | Rope length | Same geometry |
| 2b | Trivial procedure | Inverse | Rope length | Same geometry |
| 2c | Trivial procedure | inverse | Rope length | Same geometry |
| 3 | Integrals calculation | Inverse | Stake position | Same geometry |
| 4 | Integrals calculation | Inverse | Side of the field | Change of geometry |
| 5 | Optimization | Inverse | Rope length | Same geometry |
| 6 | Integrals calculation and recurrence relations | Inverse | Rope length and time | Same geometry |
| 7 | Integrals calculation and increments | 2 stages: direct and inverse | Rope length | Same geometry |
| 8 | Arc length calculation | Inverse | Speed | Same geometry |
| 9 | Integrals calculation and surface invariance | 2 stages: direct and inverse | Rope length | Change of geometry |

Table 4.1.Characteristics at the first research stage.

Group 1. Reformulations based on the inverse function

A usual common reformulation consists in inverting the function involved in the direct problem, which would imply obtaining one or more values of $r$ for one or more values of $f$, or some equivalent version in which other changes may be or not added.

Depending on the aggregated changes –in addition to the reformulation as an inverse problem– five variants can be found:

- Reformulation based on the inverse function, without other modifications
- Inverse function and change in the geometry of the field
- Inverse function and change of the stake position
- Inverse function and change of geometry and stake position
- Inverse function and field with obstacles.

Some examples of these variants provided by the prospective teachers productions are shown and analyzed below.

Subgroup1a. Reformulations based on the inverse function, without other modifications

A group of 15 prospective teachers proposed an inversion of the original function, in different forms, for example, one of them asked "to find $r$ (relationship between the length of the string and the length of the field) such that $f = 3/4$ (i.e., the sheep may access to a fraction corresponding to $3/4$ of the total area)". Another participant opted for not asking about concrete values, proposing to "get the length of the rope $R$ knowing the value of $f$". It should be noted that in the latter case, the relationship $r$ is not requested; instead of it the length of the rope $R$ should be obtained. Other prospective teachers make more colloquial proposals, for instance one of them asks to determine "... how much the rope must measure so that the sheep can access 50% of the field".... Finally, equivalent versions can be found, but they are not well formulated from a dimensional point of view, such as one that requests "... the quotient $r$between the length of the rope and the side of the field, knowing that $A$ measures nine quarters of the side of the field ... " Evidently, when comparing a length with an area the result will depend on the units of measurement.

In a few words, all these proposals are variants of the classical problem of inverting the function presented in the direct problem.

Subgroup 1b. Inverse function with change in the geometry of the field

Three of the prospective teachers have proposed to invert the function $f\ (r)$, adding a change in the geometry of the field. An example of this variant is observed in a proposal in which the field acquires a trapezoidal shape, as Figure 4.1 shows. For this modified field, the prospective teacher proposes that: "If $A$ the area of the region is $300m^2$, what is the length of the rope?".

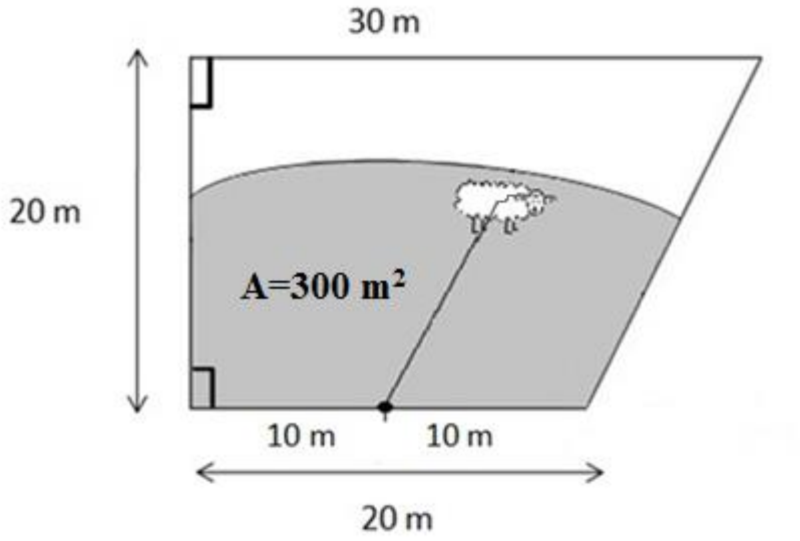



Figure 4.1. Trapezoidal field problem.

As it can be observed, the inversion of the function is added to a change in the geometry of the field, making slightly more difficult the resolution of the problem.

Subgroup 1c. Inverse function and change of the stake position

All the reformulations of this subgroup resulted in a trivialization of the problem. For this reason, they are included in Group 2: “The Elementary Problem”.

Subgroup 1d. Inverse function, change of geometry and stake position

Two of the prospective teachers proposed to invert the function $f(r)$, changing the geometry of the field and the position of the stake. For example, one of them proposed a rectangular field with the sheep tied at one of its vertices (see Figure 4.2). Specifically, the participant asks:

"if $f = 0.6$, how long is the rope?". Another reformulation with change of geometry and stake position proposed a field corresponding to an equilateral triangle with a side of $L/3$ and the sheep tied in a vertex.

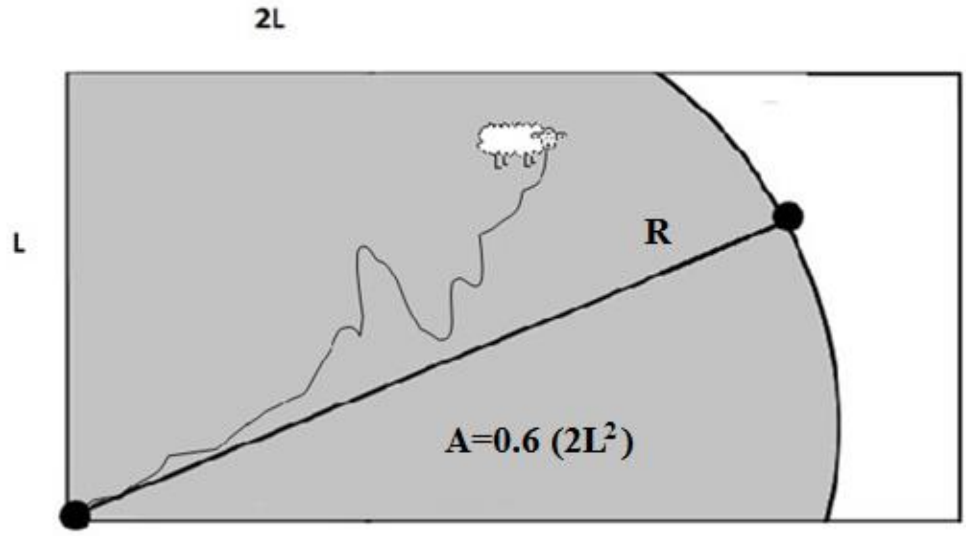



Figure 4.2. Rectangular field and stake in a vertex.

In this kind of problems, the change of geometry adds difficulty, somehow compensated by tying the sheep in a vertex, which facilitates the resolution.

Subgroup 1e. Inverse function and a field with obstacles

The prospective teachers proposed three inverse reformulations for which they have added obstacles to the possible displacement of the sheep. An example of this type of problem is a proposal where a fence was added as shown in Figure 4.3.

The reformulated problem asks about the fence length in order that the sheep can eat 60% of the grass eaten in part (a), being this part a direct problem in which the fraction of total area is requested, for $r = 3/4$, so the complete problem is much more complex. Indeed, first it is necessary to solve a direct problem and then, the obstacle makes the rope to take the form of a polygonal formed by two segments, with the consequent added difficulty.

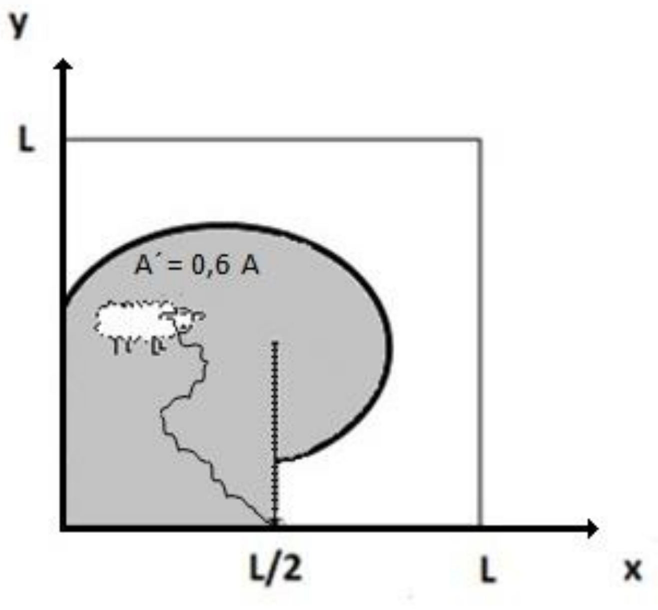



Figure 4.3. Field with a fence that adds an obstacle.

In other reformulations, different obstacles are added that make the pasture area much more difficult than a section of a circle and the rope adopts a polygonal shape with two or three segments.

Group 2. Elementary problem,

Some of the proposals can be considered as reformulations where the original problem has been trivialized such that it can be solved simply by applying a formula or by performing extremely simple algebraic manipulations. This group can be divided into three different sub-groups:

- Problems that can be reduced to a formula application
- Problems that require simple algebraic manipulations
- Simple reformulation problems, with the addition of other external variables

Next, examples of each of these variants are analyzed.

Subgroup 2a. Problems that can be reduced to apply a formula

In several analyzed productions it is proposed to calculate the minimum length of the rope so that the sheep may graze in the entire field. Obviously this problem can be reduced to find the distance between the point $(L/2, 0)$, where the sheep is tied, and $(L, L)$, which is the farthest point in the field. This reformulation reduces the problem to a simple application of the distance formula between two points, or possibly, it can be solved by Pythagorean Theorem. In any case, the resolution is immediate.

Subgroup 2b. Problems that require simple algebraic manipulations

In some cases the problem is trivialized –or at least greatly facilitated– since the resolution consists in proposing a simple equation, from which the variable requested in the reformulation can be obtained. An example of this variant appears in a reformulation where the sheep is tied at the point $(0,0)$, and the question is "what length, in relation to the side of the terrain, should the rope be so that the sheep can access to a fraction of $\pi/16$ of the total area of the field? "

To solve the problem, it is enough to observe that $\frac{1}{4}\pi R^2 = \frac{\pi}{16}L^2$ and from this equation it is easy to obtain that $\frac{R^2}{L^2} = \frac{4}{16} = \frac{1}{4} \Rightarrow \frac{R}{L} = \frac{1}{2}$ , so everything is reduced to posing the area of a quarter of a circle and then, obtain the requested fraction. In this reformulation, it is no longer necessary to propose integrals and everything is reduced to perform simple algebraic manipulations.

Subgroup 2c. Simple reformulations with the addition of other external variables

In one of the proposals two additional variables are added: the weight of the square meter of grass with 10cmheight and the daily consumption of a sheep of 40 kg weight (data that students should look for in the Internet). The question is: "How much should be $L$ such that the grass area will be sufficient for a day feeding of the sheep?"

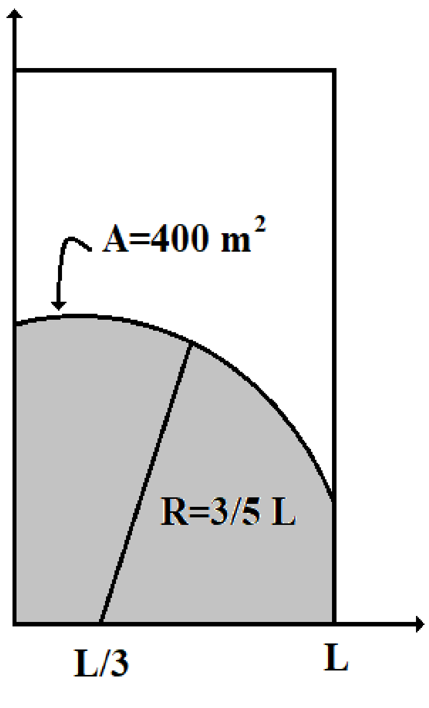
Since there is no other relevant information, everything seems to suggest that the sheep can graze in the entire field although in this case additional variables are added (weight of the grass per square meter and daily consumption of the sheep). Once again, the problem can be considered as trivialized, though only a little more elaboration is required to establish the proportionality between weight and area. Additionally, there is an extra task for the students that should search for data on Internet.

Group 3. Inverse problem of stake location

Another proposal consists in providing the area covered by the sheep and the length of the rope and it asks for the point from where the rope is tied. Due to the symmetry arguments, this inverse problem does not have a single solution and its proposal and resolution have a level of difficulty higher than those analyzed in the previous groups.

Group 4. Inverse problem about the length of the field

An interesting example proposes a rectangular field, being $L$ the base and $2L$ its height, and it also adds a change in the position of the stake, now located at the point$(L/3\,,0)$, as illustrated in Figure 4.4.




Fa

Figuee 4.4. Field with different geometry and new stake position.

The prospective teacher says: "the length of the rope is three fifths of the smallest side of the field" and also mentions that the "total area accessible to the sheep is 400 $m^2$" and the question is "which is the area of grass that is not accessible to the sheep?"

It is a different problem where the main goal consists in calculating the length of the smallest side of the field. Contrary to what it might seems, it is not only a problem about a change in geometry and stake position –as those described in subgroup 1.d– the relevant modification lies in which data are provided and what it is requested.

The problem combines a first part formulated in an inverse way, in which the area $A_r = L \times 2L$ of the field and the radius $R = \frac{3}{5}L$ are given, and the length of the land should be found. This first part is followed by a simple direct problem in which the area accessible to the sheep ($A = 400m^2$) is subtracted to the total area of the rectangle in order to obtain the requested output.

Group 5. Optimization Problem

In this reformulation, the prospective teacher proposes to tie two mountain goats in the opposite corners of a square field which side is $L$, as showed in Figure 4.5.

The proposal asks "to occupy the maximum possible area each, but without coinciding at any point" and it is known that "one of the areas has to be larger than the other". The participant asks about the length of each rope and how much area will be available for each goat.

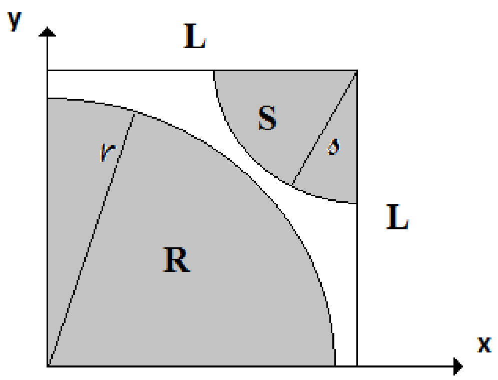



Figure 4.5. Problem of the mountain goats

Obviously, it is an optimization problem, which depends on two variables and the requested maximum does not exist, but there exists a supreme, not achievable for any pair of rope length values.

Group 6. Sequential inverse problem

In this case one of the prospective teachers proposed an interesting inverse problem in which the geometry of the field and the position of the stake are maintained, as can be seen in Figure 4.6.

In this reformulation the data corresponding to the length of the rope $R = \frac{L}{3}$ is provided and it is assumed that throughout a day, the sheep eats all the grass in the accessible area. The first question is about the length $R'$ which should have the rope so that the sheep can graze the same amount of grass. Then, the question is repeated for the third and for the fourth day, in order to finally ask after how many days the sheep will not find more the same amount of grass to graze?

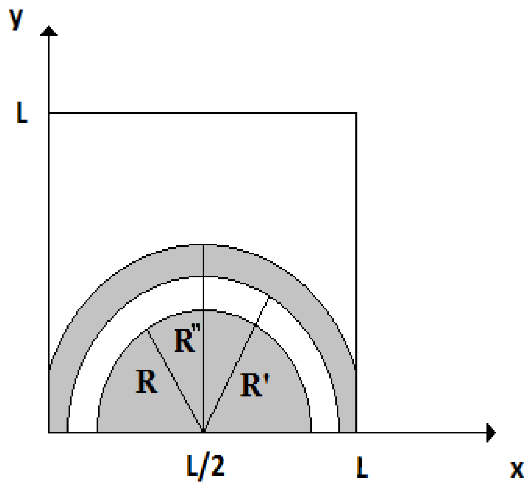



Figure 4.6. Sequential proposal for an inverse problem

As it can be observed, the order of the variables involved is inverted, the geometry and the position of the stake are maintained. Since it asks the same repeated question for different days, it can be considered as a sequential problem.

Group 7. Incremental problem

In this new proposal the prospective teacher does not present any diagram, so apparently the conditions of the original problem are maintained. It is stated that "the sheep may graze in a fraction of the field $f < 1$ and it is tied with a rope of length $R_f$" and asks "how long must the rope be lengthen so that the sheep may graze 10% more than what it can already graze?"
In this case – besides inverting the function – the main objective is to link the increase of the rope length to the increase of area and this fact

gives to this proposal a different feature comparing to others previously analyzed.

Group 8. Dynamic problem

In this reformulation there are two sheep tied to the same point, using ropes which length is $R = \frac{3}{4}L$ and both sheep are located at point P, as in Figure 4.7.

The first sheep, called $\alpha$ runs along the segment $PQ$ with a certain speed that is provided as a function of $L$ and the second sheep, called $\beta$ runs along the arch $PSQ$.

Several questions are posed in this reformulation, being the last one particularly interesting since the function is inverted. This question asks about the speed that the sheep $\beta$ must have to reach the point $Q$ at the same time as the sheep $\alpha$ arrives to it.

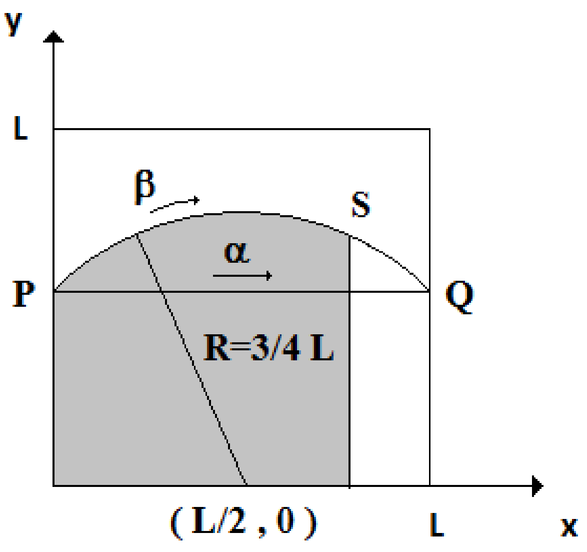



Figure 4.7. Dynamic problem with two sheep.

The problem can be considered inverse, since the arrival time is given and the speed is requested, but it totally changes the original problem that was a static problem unlike the proposed one, which is formulated dynamically.

Group 9. Iso-surface problem

In this last case an iso-surface problem is proposed. It considers for the first day, a sheep tied at the point $(L/2,0)$ with a rope that "measures three quarters of the length of the side of the field". Then, it is supposed that on the second day the shepherd "ties the sheep to a stake located at the point $(0,0)$ on the corner of the field" and the question is "how long should the rope be so that the accessible area is equivalent to the previous day?" Finally, a similar question is posed but now placing the stake in the center of the field.

It is a double iso-superficial problem where rope lengths are requested so that the grazing areas remain unchanged.

It is a double iso-superficial problem where rope lengths are requested so that the grazing areas remain unchanged.

### 4.2.- The didactic analysis of the tasks

#### 4.2.1.- General results of the didactic analysis

Firstly, it should be remarked that not all participants proposed inverse reformulations of the original given problem, being some of them direct proposals or ill-posed problems. In addition, some prospective teachers proposed more than one inverse reformulation, although they only filled one worksheet for the didactic analysis of the associated tasks. Taking into account these facts, for the didactic analysis only the responses of 29 participants were analyzed, since all of them proposed at least one correct inverse reformulation.

From this group of 29 prospective teachers, those who responded to the items of Table 2.1 are:

- Previous knowledge: 13 responses.
- Mathematics content activated by the task: 21 responses.
- Challenge: 22 responses.
- Task completion: 5 responses.
- Event: 22 responses.
- Question: 14 responses.
- Purpose: 7 responses.
- Language: 17 responses.
- Data: 15 responses.
- Goal: 23 responses
- Formulation: 13 responses
- Material and resources: 15 responses
- Grouping: 18 responses
- Learning situation: 10 responses
- Timing: 14 responses
- Mathematical content: 17 responses
- Situation: 21 responses
- Complexity: 27 responses.

Taking into account that the 75% of 29 is 21.75, then those who exceed 75% of responses (22 or more responses) are: Complexity (27), Goal (23), Event (22) and Challenge (22).

On the other hand, the 25% of 29 is 7.25, so those items that answer less than 25% of the participants are: Recognition / Justification (5) and Purpose (7).

It is not surprising to note that those that attract more the participants' attention (complexity, goal, whether the task is a challenge or not and whether the event is realistic or not) are four extremely important items. Perhaps much more surprising is that whether or not the purpose of the task is realistic, or whether the student recognizes that the task has been completed, does not seem to be of great concern to prospective teachers. In the next sub-sections, the tasks' didactic analysis, carried out by the participants, will be studied more in depth, including some interesting examples of the given responses.

### 4.2.2.- Analysis of other semantic components

Other semantic components of the meanings are the senses or modes of use. The results about these items are located in other columns of the instrument, which are important for the analysis of other aspects of the productions of prospective teachers. In particular, this subsection is devoted to the analysis of the senses meaning components.

The results obtained are the following:

- Regarding "senses and modes of use", almost 50% of the students who respond to this item affirm that it is a task about "space and form"; altogether 17 of 37 responses choose this interpretation. Another 25% (9 out of 37) choose the option "changes and relationships" and a little less than the previous ones (7 out of 37) respond that it is a "quantity" task. Finally, the remaining four (almost 10%) classifies it as "uncertainty and data".

- In what refers to the "situation", a large majority express themselves in favor of an "educational or work" context. Four prospective teachers consider the task as a "public" and only one participant classifies it as "personal". In addition to this, other responses mention the term "similar", but without clarifying what this classification implies.

#### 4.2.3 Cognitive analysis

The results of the cognitive analysis can be presented in four different items corresponding to learning expectations, limitations and opportunities, and a last one devoted to the participants' comments.
The results obtained are the following:

1.- With regard to learning expectations, taking into account the goals that prospective teachers declare, several groups can be described:

- Learning expectations specific to the problem, for instance: "find $r$ (ratio of lengths, $r = R/L$ , as in the direct problem) from the value of $f$ (ratio of areas, given as $f = A/L^2$, like in the direct problem) and the relationship between both magnitudes". Another example asks; "find the length of the string so that it does not reach more than half the grass area."

- Learning expectations specific to the topic in which the problem is framed. Some examples are: "apply Pythagorean Theorem, substitute and work with algebraic expressions"; "Learn to calculate area sections and operate with quotients"; "Calculate lengths given some areas"; "Calculate areas"; "Calculate areas and relate unknown parameters"; "Relate lengths and volumes"; "Relate the concepts of area and lengths" and "Relate variables and understand the difference between direct proportionality and functional relationship".
- Learning expectations which are transversals for teaching based on problem solving, as in the following answers: “understand the inversion of a problem and connect other mathematical contents”; "Consider various assumptions, reflect and draw a model and contrast results" and "find the initial conditions needed to obtain a pre-established final condition".
- Finally, there are some learning expectations, which can be considered as generic/imprecise, such as: "control area or region concepts" and also "functions, calculation of areas and proportionality coefficients".

2.- Regarding the analysis of learning limitations, there is practically no comment. A single prospective teacher mentions: "there is less chance of making errors in the reformulation". In general, the participants have not reflected on this topic.

3.- In relation to learning opportunities, four types of opinions can be distinguished about the challenge faced by students:

- There is a first group of opinions that we can consider as favorable, for example: “interesting”, “greater challenge”, or “more authentic challenge”, which constitute a majority of the responses.
- Three answers given by two prospective teachers express negative opinions in this regard, such as: “little interest” or “no interest for the student”.
- A case provides a neutral response, which is expressed in terms of “similar interest” without further explanation, since it is only a comparison with the direct problem that was provided to them.
- There are also three answers –given by two participants– that can be considered as generic/imprecise. In one of them the prospective teacher says "possible challenge" and in the other case the participant expresses: "interest of the challenge: 2", without clarifying which the corresponding scale is.

4.- Finally, with regard to the comments made by the participants on the previous item, in some cases they clarify the reasons for their response. A few examples of these comments are the following: "because of being an inverse problem"; "inverse problem and new contents"; "change of structure and isolating variables"; "consider various assumptions". Also, a couple of imprecise comments such as "depends on the approach" or "should give more freedom to the data” were observed.

### 4.2.4. Instructional analysis

This subsection concludes the didactic analysis of the prospective teachers' responses for the first stage of the research. For this purpose, the results obtained in the 12 answers received corresponding to the instruction analysis, are described below.

- In relation to *language*, all are expressed in a positive way, the most frequent being clear language (3 responses) and simple language (3 responses). Others comments such as "everyday language", "correct language", etc., showed favorable opinions.
- About *authenticity*, this item is clearly dominated by favorable options (14 in total) and a big number of them have expressed in comparative terms (higher authenticity: 5 responses), or simply say: it is a real problem (7 responses). On the other hand, there are 3 responses that can be considered negative (unlikely: 2 responses and not significant: 1 response), and the remaining 3 can be considered neutral (similar: 2 responses, possible problem: 1 response).
- Regarding the item *data*, several participants give their opinions about the quantity (sufficient data: 1 response, not all are given: 2 responses) and others express themselves about their quality (5 responses). In this last group, 3 are positive (realistic: 1 and easy to understand: 2), 1 is neutral (neither true nor improbable) and another is negative (non-concrete data). In other 2 answers, there is no comment about quality or quantity, although other aspects are addressed: one classifies the data as different from the original

problem and another says that data must be obtained by searching on the Internet.

- Regarding the *purpose*, almost all the answers invoke didactic reasons. Some of these responses refer to content (e.g., teaching proportions), while others address cognitive issues (e.g., the students may know when the problem is solved), or aspects of instruction (eg, enriching the task).Other similar answers aim to connect ideas, introduce content, etc. Finally, there was an answer that is limited to compare with the original problem, resulting that both have a similar purpose.
- In respect to the *location*, 8 answers proposed to carry out the task in the classroom and only one said that it could be done in class or at home.
- As regards *grouping*, opinions are quite divided: 11 propose that it should be an individual task, 7 recommend working in a group and 3 more propose working in pairs. In addition to the above, 4 answers say that the task can be done individually or working in pairs, without preferring one of the two options.
- In the column corresponding to the *timing*, 8 responses were expressed in comparative terms: 5 propose spending more time, 1 proposes less time and the other 2 suggest the same timing. All the other answers choose to quantify the duration of the task, the most common being: 30 minutes (7 responses), 1 full session (6 responses) and other minority options are also presented, such as 25 minutes (3 responses), 40-45 minutes (2 answers), etc. As a consequence, it can be observed that the results range from a minimum of 15 minutes to a maximum of one entire session.

- Regarding *complexity*, 16 prospective teachers limit themselves to give a comparative opinion, 15 of them saying that it is a task of greater complexity than the original one, while the other one says that it is similar. These answers do not indicate which is exactly is the level of complexity of the proposed task. Those who responded about the level of complexity can be divided into three groups: connection (15 responses), reflection (10 responses) and reproduction (only one response). In addition to the previous answers, one of the participants says that it is a task of "medium difficulty", a term that is not clarified by the prospective teacher.
- Finally, about *materials and resources*, once again several of the prospective teachers (7 answers) opted to compare saying that they are similar, without further explanation. Among those that recommend materials and resources, nine of them propose standard materials, such as pencil and paper (4 answers). A few propose not-so-traditional materials and resources: 3 propose adding software, 1 suggests using thread and thumbtack; and there is even a case that proposes a scale recreation on a field.

# Chapter 5. RESULTS OF THE SECOND STAGE OF THE RESEARCH

This chapter is organized into two sections. The first one (Section 5.1) consists of an analysis of the reformulations of the sheep problem –proposed by the prospective teachers– carried out through a cluster analysis. The second one (Section 5.2), develop an analysis of the tasks associated with that reformulation, since the prospective teachers were asked to give their opinions and value of the meanings, intentionality, authenticity, elements and task variables, works organization, materials and resources (Rico & Ruiz-Hidalgo, 2018, corresponding to its own proposal.

## 5.1 The analysis of the reformulations

As it was mentioned, the participants' reformulations were studied through a cluster analysis (see Section 3.4.4 and Section 3.4.5), which was carried out by using the table partially represented in Figure 3.8 as the input for the statistics software.

The nomenclature used is the following: the capital letters "PT" indicate that he or she is a prospective teacher. These letters are followed by two digits that correspond to the student number and the small letter refers to the first, second or third reformulation proposed by the participant, if applicable. For instance, PT23c means that the 23$^{rd}$ prospective teacher proposed, at least, three different reformulations and the one considered here is the third.

The first output of the clusters analysis was a dendrogram that showed 16 different clusters. This dendrogram is particularly puzzling due to the large number and the different sizes of the groups. Moreover, it should be noted that all the trivial reformulations are not grouped into a single cluster, as it would be reasonable to expect from a task enrichment viewpoint. Indeed, if the proposal results a very easy task or even more, a trivial exercise, its usefulness as an enriched task for secondary school students disappears.

For these reasons all the trivial proposals were removed from the input data for the statistical software. The result of these data renewed by that processing criteria was a dendrogram where nine different clusters can be easily identified, as Figure 5.1 shows. Eventually, some of these clusters can be subdivided into several subclusters, and also it is possible to follow the opposite way, considering superclusters. We discuss here all these possibilities.

If the dendrogram of Figure 5.1 is analyzed from an upper level, it is possible to identify three super-clusters.

- The first supercluster is related to procedural knowledge (skills) and it is formed by clusters Nº1, Nº2, Nº3 and Nº4, which are described in Subsection 5.1.1.
- The second one involves graphic representations and only includes cluster Nº5.
- The third supercluster is related to conceptual and deep knowledge (reasoning and strategies) and it is formed by clusters Nº6, Nº7, Nº8 and Nº9.

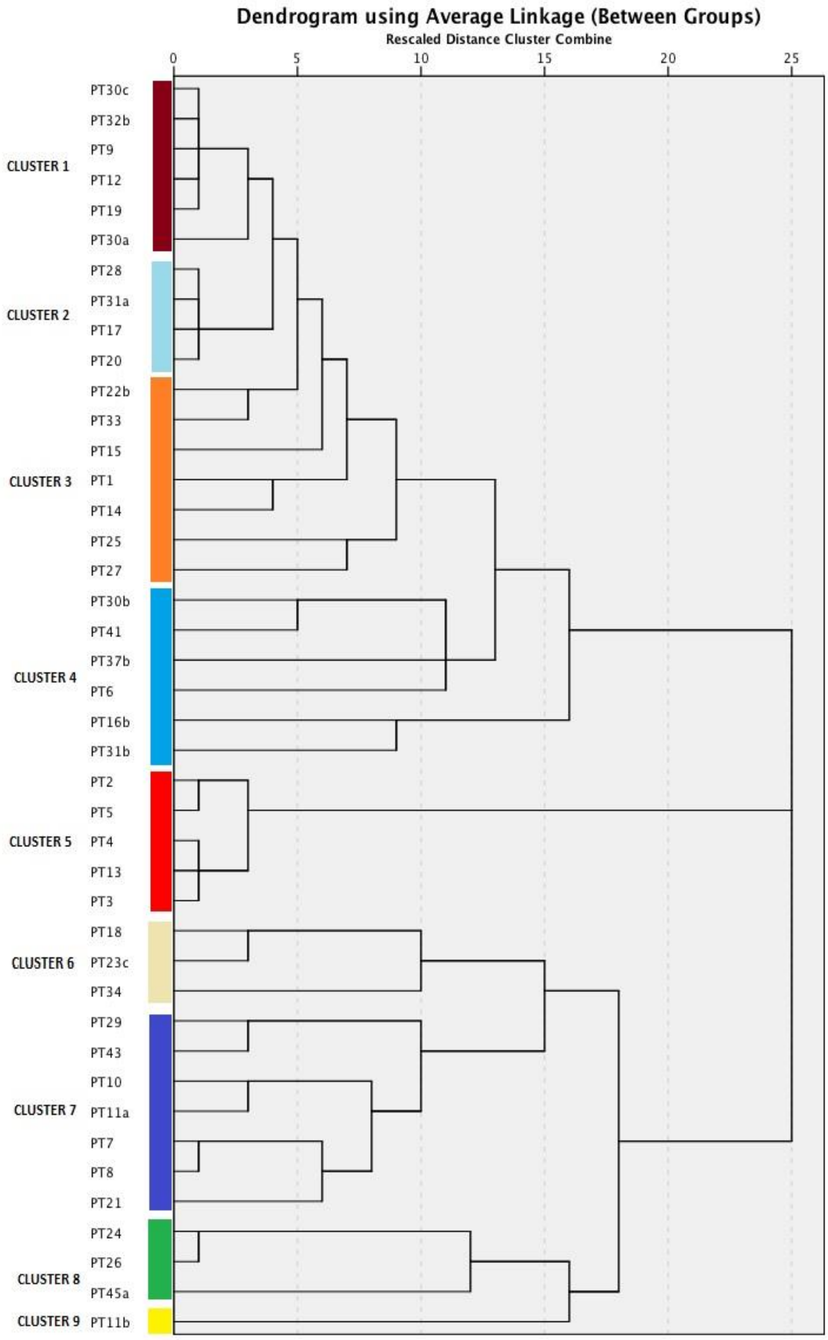


Figure 5.1. Dendrogram obtained with non trivial proposals

These three superclusters show a direct connection with reference, representation and meaning criteria, the semantic categories of the mathematical school content, analyzed by Rico and collaborators, based on Frege's ideas (Martín-Fernández, Ruiz-Hidalgo & Rico, 2019).

5.1.1 The first supercluster

The first supercluster is formed by clusters that have in common the use of procedural content knowledge. It can be exemplified by the production PT33, which proposes a rectangular field $\left[\frac{-L}{5},\frac{L}{5}\right]\times[0,L]$, where the sheep is tied at $(0,0)$. The corresponding sketch can be observed in Figure 5.2.

The author gives the following data: $r=\frac{R}{L}=\frac{3}{4}$ and the area accessible for the sheep is $A=433.36\ m^2$. The required output is the radius $R$.

For solving the problem, we consider the integral $A=2\int_0^{L/5}\sqrt{R^2-x^2}\,dx=433.36$, and if we use that $r=\frac{R}{L}=\frac{3}{4}\Rightarrow L=\frac{4}{3}R$, we obtain $A(R)=2\int_0^{4/15R}\sqrt{R^2-x^2}\,dx$. Finally if we put $A(R)=433.36$, it results a non-linear equation in $R$ and from this equation the requested value is obtained.

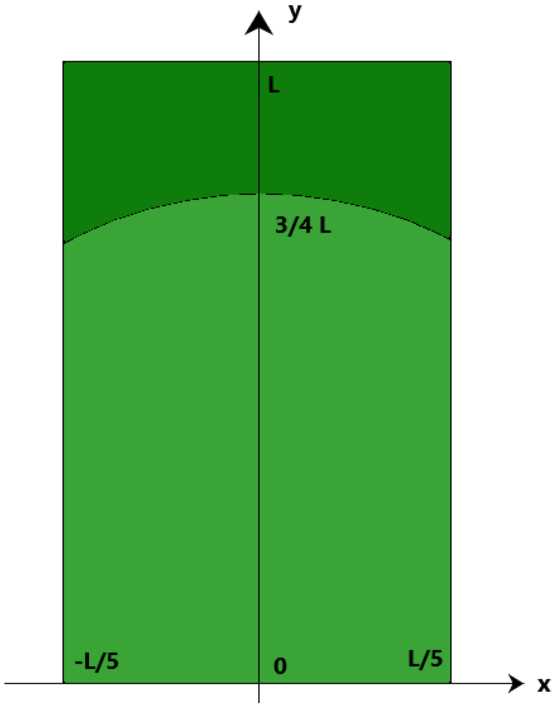



Figure 5.2. Geometry of the proposal PT33.

As it can be observed, this problem requires procedural knowledge to be solved.

This first supercluster can be divided into four clusters, which are described as follows:

*Cluster Nº 1: Pointwise inversions*

A first cluster includes the following productions: PT30c, PT32b, PT09, PT12, PT19 and PT30a. The proposal PT30c can be described in a few words as an inversion of the function where the radius $R$ is requested for an area fraction $f = 0.8$. The next one, PT32b, asked to obtain $R$ for an area fraction $A = 0.68L^2$. A new reformulation, PT09 gives

$L^2 = 36$ and $A = 25$, asking once more for the $R$ value that makes it possible. Other participant, PT12, gives the field length $L$ of $50m.$ and the area fraction $f$, expressed as a percentage, 70%, and he/she asks for the corresponding $R$ value.

Lastly, the proposal coded as PT19 gives $L = 4$ and $f = 7/8$ and again the $R$ value is requested.

These first five proposals (PT30c, PT32b, PT9, PT12 and PT19) can be considered as a sub-cluster and slightly separated we can find PT30a, which proposes a pointwise inversion although he/she adds a new variable: the sheep daily consumption.

Then, in all cases it is necessary to invert the function $f = f(R)$ with its corresponding variants, but only for a particular value of $f$ (given directly or indirectly by providing $A$ and $L$ or $L^2$).

*Cluster Nº 2: Numerical methods*

In Cluster Nº 2 there are four reformulations proposed by PT28, PT31a, PT17 and PT20. In the first one, the data are $L = 20$ and $f = 1/2$, and a sequence of approximated values is obtained using Microsoft Excel spreadsheet. The second proposal, coded as PT31a, gives $L = 20$ and $f = 0.877$, and the corresponding $R$ value is obtained by Bisection Method. In the third one $L^2 = 16$ and $A = 10$ are given and the corresponding inversion is carried out with the help of Wolfram Alpha. Finally, the last participant, PT20, proposed that $L^2 = 16$ and $A = 12$ and asks for the corresponding $R$ value, which is obtained as $R = 3.13219463\ldots$ and then, the number of digits of the response suggests that a numerical method not specified was used to get this result.

In this homogenous cluster, no sub-clusters can be observed. Also, it is easy to conclude that the common fact in all cases is that problems are solved using numerical methods and some technological support is necessary for this purpose.

*Cluster Nº 3: Pointwise inversion with variants, with informal solutions*

Cluster Nº 3 includes seven heterogeneous proposals: PT22b, PT33, PT15, PT01, PT14, PT25 and PT27. The first one, PT22b, changes the context and considers a handball field and a goal keeper that throws the ball, covering a certain area instead of the sheep grazing in a field and the solution is obtained by trial and error. In PT33, the participant proposes a rectangular field $\frac{2}{5}L \times L$ and gives $r = \frac{3}{4}$ and asks for the corresponding $R$ value, but after integrating he/she gets an area $A >> L^2$ and the inversion is not possible. In the third one,PT15, the sheep can reach between 60% and 70% of the entire field, the $r$ value is requested and a possible solution is obtained by trial and error. The fourth one, PT01, proposes to choose several values and invert the function (i.e., no particular values are given). It should be noted that it is impossible to get a general formula for this inversion, because of the transcendent function involved. Next participant, PT14, gives $A$ and asks for the corresponding $r$ value. In the solution the region is divided into three parts: two right triangles and a circular sector. Since this process gives inverse trigonometric functions and square roots, the resulting function $f$ is transcendent and the solution once again is delivered by trial and error. Next one, PT25, proposes another pointwise inversion giving $L = 20$ and $f = 0.09$ and asking for the corresponding $R$ value, although the problem is modified by putting a fence in the field and asking too for the point where the fence cuts the lateral edge of the fence. The last one, PT27, changes the context, which is about a bush fire without wind that evolves through concentric circles and the requested output is the standing point of the fire, knowing that $R = L$ and $f$ is 95.60%. This problem is solved as in other cases by a trial and error method.

This cluster can be divided in two parts, a sub-cluster formed by five proposals (PT22b, PT33, PT15, PT01 and PT14) and a second one where only PT25 and PT27 are included.

Both sub-clusters form a heterogeneous group with difficult problems; changes of context and most of them are solved approximately by trial and error, or simply remain unsolved.

*Cluster Nº 4: Additional conditions and heterogeneous solving approaches*

This cluster includes the following proposals: PT30b, PT41, PT37b,PT06, PT16b and PT31b. The first one, PT30b, gives $f = 0.8$ andincludes a new variable: the sheep daily consumption and he/she asks for the days that the sheep may graze in the field, so the integral is solved like in a direct problem, but the proportion is inversed. Similarly, PT41, gives $L = 30$ and $R = 20$ and the sheep daily consumption and he/she asks for the time that the sheep may graze in these conditions. Once again the integral is solved like in a direct problem and there is an inversion in the proportion. Next one, PT37b adds a new constraint $R \leq L$ and he/she asks for maximum value of $f$. In the solution, the participant tries to use derivatives to solve the problem. The prospective teacher PT06 gives $L = 1$, $f = 1/2$ and asks for the corresponding $R$ value, so, it seems to be a pointwise inversion, except for the solution, which avoid using integrals.In PT16b is $L = 2$, $R \geq 1$ and the problem changes because the angle $\theta$ of the circular sector is given, so, it can be solved without using integrals. Another participant, inPT31b includes a small building on one corner of the field and the dimensions of this obstacle for the sheep are given. The solution uses integrals and geometry to know if the sheep can reach or not the obstacle.

The whole group can be divided into three parts, a sub-cluster formed by PT30b and PT41, a second one that includes PT37b and PT06, and finally, a third one formed by PT16b and PT31b.

As a consequence of this fact, Cluster Nº4 can be considered as a heterogeneous group, although there are a few characteristics repeated in all the proposals. Indeed, in all of them there exist some additional conditions and the corresponding solutions include different approaches (geometry, integrals, proportions, etc.). These variants in the proposals and their solutions make them similar, and at same time, different than those of other clusters.

The second supercluster

The second supercluster, that only includes cluster Nº5, is based on representation systems. An example is given by the production PT04, which proposes a reformulation with the same geometry of the original problem and a different input. The author says that the area where the sheep may graze is given by the integral $A=\int_{\arcsin(-L/2R)}^{\arcsin(L/2R)} R^2\cos(\theta)\sqrt{1-\sin^2(\theta)}\,d\theta$, obtained after a given change of variables: $x=R\sin(\theta)+L/2$. The required output is a sketch of the region where the sheep may graze.

Being $x=R\sin(\theta)+L/2$, it follows by differentiation that $dx=R\cos(\theta)d\theta$, so, under the integral sign we obtain the following expression:

$$R^2\cos(\theta)\sqrt{1-\sin^2(\theta)}\,d\theta=\sqrt{R^2-R^2\sin^2(\theta)}\,R\cos(\theta)d\theta=$$
$$=\sqrt{R^2-(x-L/2)^2}\,dx$$

Finally, if we put $\theta=\arcsin(\pm L/2R)$, in the formula $x=R\sin(\theta)+L/2$ ,we obtain $x=R(\pm L/2R)+L/2=L/2\pm L/2$ and then the original integral is converted into:

$$A=\int_{\arcsin(-L/2R)}^{\arcsin(L/2R)} R^2\cos(\theta)\sqrt{1-\sin^2(\theta)}\,d\theta=\int_0^L\sqrt{R^2-(x-L/2)^2}\,dx$$ From this last integral, it can be obtained the requested region, which is sketched in Figure 5.3.

This supercluster is very homogeneous and contains only one cluster, which can be described as follows.

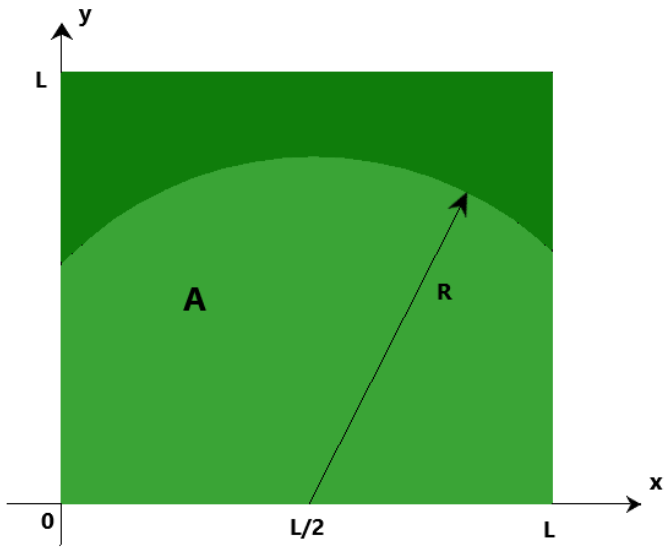



Figure 5.3. Sketch corresponding to the proposal coded as PT04.

*Cluster Nº 5: Sketches*

This cluster has the following participants: PT02, PT05, PT04, PT13 and PT03. The first one gives the integral corresponding to the area accessible to the sheep and asks for a sketch of the region. Something similar happens with PT05, who gives the integral divided by $L^2$ and –as in the previous case– a sketch of the region is the desired output. The participant PT04 also gives an integral and the main difference with the previous cases is that a change of variable was performed. Finally, the proposals coded asPT13 and PT03, give the integral corresponding to the area accessible to the sheep and ask for a sketch of the region.

It is possible to define a couple of sub-clusters the first formed by PT02 and PT05, while the second one includes PT04, PT13 and PT03. Nevertheless, we can conclude that the sketches are the desired outputs of all these proposals and this fact explains why they are all in the same cluster.

The third supercluster

The third supercluster's proposals are all about conceptual content knowledge. As an example, we consider the production coded as PT34, where the author maintains the geometry of the field and the stake position and asks for a criterion –in terms of the accessible area– that allows distinguishing among the possibilities illustrated in Figure 5.4.

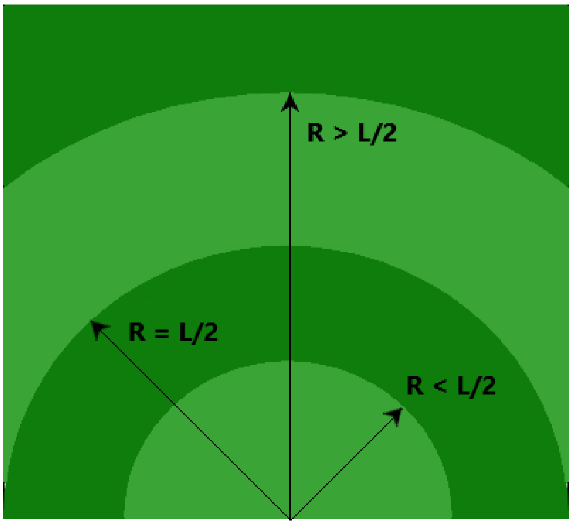



Figure 5.4. Different areas accessible to the sheep in the PT34 proposal.

Obviously, the corner point of the function $A(R)$ in the interval $0 \leq R \leq L$ is located at the point $R = \dfrac{L}{2}$ and in that case

$A=\frac{1}{2}\pi R^2=\frac{1}{2}\pi\left(\frac{L}{2}\right)^2=\frac{\pi L^2}{8}$. So, the requested criterion is very simple, if $A\le\frac{\pi L^2}{8}$, the accessible area is a semicircle; otherwise it is a region like the one sketched in Figure 5.3.

The solution only requires conceptual knowledge, since no analytical, geometrical or numerical procedures are necessary, and only a few simple algebraic manipulations were done.

The third supercluster is formed by four clusters which can be described as follows.

*Cluster Nº 6: Conceptual-heuristic*

This cluster has the following proposals: PT18, PT23c and PT34. The first one (PT18) and the last one (PT34) ask for a criterion to know for which value of $R$ the piecewise function $A(R)$ changes its functional expression. Finally, PT23c asks for the stake position and the rope length that are needed in order to get the possible greatest area for the sheep without touching the boundary of the field.

It can be considered that PT18 and PT23c form a sub-cluster and a little bit separated, it is possible to find the last one, PT34.

All these conceptual problems do not need integrals, numerical methods, etc., and can be solved using a heuristic technique followed by very simple calculations.

*Cluster Nº 7: Conceptual advanced problems*

This cluster includes seven reformulations: PT29, PT43, PT10, PT11a, PT07, PT08 and PT21. The first one, PT29, proposes a specification problem where appears the variable $r$, without any explanation and he/she asks for an interpretation of this variable. Next one, PT43, also proposes a specification problem where a function $f$ is given and the question is about if it includes all the possible cases –i.e., $0 \leq r < 1/2$, $1/2 \leq r < 1$, etc.– and if not, it is requested to complete it. The following two productions PT10 and PT11a give an integral function: $A(x)$ in the first case and $A(R)$ in the second and they ask for the curve involved, if it is possible to obtain it. The first one gets this curve by differentiation of the function $A(x)$ whereas the second one concludes that it is not possible to obtain the curve using differentiation since the given function is $A(R)$ instead of $A(x)$. The next two (PT07 and PT08), propose a specification problem where appear $f$ and $r$ –without explanation– and the solver must give meaning for both variables. The last prospective teacher PT21, proposed a specification problem where he/she asks for obtaining $f$ only using geometry and comparing the result with the one obtained by integrals. Of course the results are the same, so input and output are known and the solver must find a geometrical way to obtain the result.

The whole cluster shows three sub-clusters, being the first one formed by PT29 and PT43, the second one includes PT10 and PT11a and the proposals PT07 and PT08 form the third one, while PT21 remains separated.

As in the previous cluster, all of the proposals involve conceptual problems –most of them specification problems with a greater degree of difficulty– that cannot be solved heuristically.

*Cluster Nº 8: Unusual output problems*

This cluster includes the proposals: PT24, PT26 and PT45a. The first production, PT24, asks for a general inversion in $\Re^2$, i.e., a function $f$ with two variables should be inverted without giving any particular numbers. It is important to mention that in the solution given by the participant, some algebraic manipulations are wrong. In the second one, PT26, another general inversion is proposed –in this case with one variable– and also an intersection point with the lateral edge of the field is requested. In the participant solution the procedure is indicated, but the plan to solve it is not executed. The third one, PT45a, gives $L = 9$, a general $R$ value and the fertilizer rate consumption and he/she asks for the bottles of fertilizer needed for part of the field where the sheep grazes.

It is possible to observe a sub-cluster formed by PT24 and PT26, while PT45a remains separated.

Once again, the proposals included in this cluster are heterogeneous and the outputs of all these problems are unusual and not easy to obtain.

*Cluster Nº 9:Advanced theorems*

This last cluster is formed by only one proposal: PT11b. In this reformulation the rope is substituted by a fence which cost per meter is given and the author asks for the total cost of this substitution. He/she solves his/her own problem by using that $L = \frac{d}{dR}A(R)$, a formula that it is not true in all cases (see for instance (Dorff & Hall, 2003) or (Martinez-Luaces, Fernández-Plaza & Rico, 2019)). Fortunately, in this case the formula gives the correct result as it was shown in a previous paper (Martinez-Luaces, Fernández-Plaza & Rico, 2019), by using more advanced theorems like Leibnitz differentiation under the integral sign.

## 5.2 Didactic analysis of the tasks proposed by the prospective teachers

### 5.2.1 General results of the didactic analysis in the second stage.

As it was mentioned, the prospective teachers were asked to give their opinions about the tasks corresponding to their own inverse reformulation in comparison with those associated to the direct given problem. For this purpose, they were asked to write their points of view about meanings, authenticity, elements of the task and task variables.

In the responses concerning the items listed on Table 2.1 it is possible to observe some important changes. For instance, in the first stage only 29 responses were considered, whereas in the second stage that number was almost duplicated (being 54 responses), despite the fact that the groups of prospective teachers had similar sizes. It is important to note that this situation happened because the discarded responses –corresponding to direct reformulations, or ill-posed problems– were a minority in the second stage.

From this group of 54 prospective teachers, those who responded to the items of Table 2.1 are:

- Previous knowledge: 46 responses
- Mathematics content activated by the task: 49 responses
- Challenge: 43 responses
- Task completion: 37 responses
- Event: 50 responses
- Question: 32 responses
- Purpose: 34 responses
- Language: 40 responses
- Data: 34 responses
- Goal: 48 responses
- Formulation: 41 responses
- Material and resources: 35 responses
- Grouping: 42 responses
- Learning situation: 39 responses
- Timing: 45 responses
- Mathematical content: 50 responses
- Situation: 45 responses
- Complexity: 51 responses

Taking into account that the 75% of 54 is 40.5, , it turns out that this percentage is exceeded by 11 items: Previous knowledge (46), content activated by the task (49), challenge (43), event (50), goal (48), formulation (41), grouping (42), timing (45), mathematical content

(50), situation (45) and complexity (51), against only 7 that do not exceed the 75% of the responses. Moreover, the least answered is the item "Question", with 32 answers, which represent more than 59% of the total.

As it can be observed, all the most answered items in the first stage also attract the attention of more than 75% of the answers of the second stage. The main difference can be found in those items with a few answers in the first stage (under 25%), which in the second stage were answered in more than 60%.

It seems reasonable to suppose that this it is because a miniature of Table 2.1 was included in the worksheets, giving help and encouraging the prospective teachers to answer about all the items of that table.

The second stage of the research has achieved the objectives, since it was designed to reduce the number of direct or ill-posed reformulations and also encouraged participants to pay more attention to the elements of Table 2.1,

### 5.2.2. Results

As it was mentioned, the participants were asked to provide a form where the tasks associated to the original problem –posed in a direct way– should be compared with those corresponding to the inverse reformulation. In the cases where the prospective teacher proposed two or more inverse reformulations, it is also expected that all of them should be accompanied by the corresponding analysis of the corresponding tasks. Optionally, some participants also submitted their analysis of the direct problem and/or the inverse problem, in addition to the comparative study requested.

The results obtained in the second stage of the research, concerning the items of Table 2.1 are described as follows.

**Meanings**

With regard to meanings, the items to consider are prior knowledge, the mathematical content that activates the task, whether it constitutes a challenge or not, and the recognition of the task completion, which may be associated with a certain justification (Moreno & Ramírez, 2016, p. 244).

*Prior knowledge*

With reference to the prior knowledge, it is worth mentioning that some participants give their opinion on the direct problem (20 answers), about the inverse problem (19 answers) or related to both problems (15 answers), in addition to others that do not indicate to which problem they refer (19 responses). It should be noted that the same participant may give his opinions about the direct problem, the inverse one and the comparison between both problems, so he may be represented in two or more of the above groups.

Amid those who answered about the direct problem, Calculus topics, such as integrals (with and without change of variable), derivatives and functions, are mentioned in a whole group of 13 responses. Another 12 answers identify Geometry topics (plane areas, equation of the circumference, etc.), 4 mention Algebra topics (clear unknowns, solving equations, etc.) and only 2 answers mention trigonometry without indicating a specific topic of this area.

Among the opinions about the inverse problem, again Geometry (with 8 answers) and Calculus (7 answers) are among the most important, although in this case Algebra topics (7 answers) showed their importance and trigonometry topics are not mentioned. There are also 2 responses of another type, one of them is comparative ("similar content") and there exist another one about changes of representation system.

Concerning the participants who give their opinion explicitly about both problems, once again the majority identify Calculus (5 answers) and Geometry (4 answers) as the main areas, and a minority mentioned Algebra (2 answers) and Trigonometry (1 answer). There are also 3 other answers, mainly comparative.

Finally, among those who do not make explicit what problem they refer to, most of them identify topics of Calculus (16 responses) and Geometry (12 responses). On the other hand, fewer participants indicate Trigonometry topics (4 answers) and even fewer mention Algebra (2 answers). In this group there are 6 responses of another type, mostly comparative and there is also one response that emphasizes modeling.

*Mathematical content that activates the task*

With regard to the mathematical content activated by the task, 23 participants gave their opinion about the direct problem, 25 analyzed the inverse problem and only 9 gave their points of view on both problems. In addition to them, there is a group consisting of 26 opinions in which the referred problem is not indicated. As happened with the item prior knowledge, the same participant may give his opinions about several problems, therefore participating in at least two of the groups already mentioned.

Once again, comparing with the previous item –prior knowledge– it is important to observe that many responses are similar, answering basically the same in both items, using different words. This fact occurred in 21 responses, and so, it raises the question about whether prospective teachers always distinguish the differences between the two items.

On the other hand, there is a group consisting of 41 answers, where there are noticeable differences with the previous item, however, in the vast majority of them (25 answers), these differences are expressed in terms of mathematical content. A more interesting group –containing 29 responses– shows differences that do not refer to a certain mathematical content. Among these cases, those that mention modeling (5 answers) and visualization (4 answers) are the most important. Also, there is a group with three answers each that includes: problem solving, abstract thinking and reasoning, interpretation of results and making relations among the mathematical contents. The next group includes mathematizing and discussion of results (with 2 answers each) and finally (with one answer each), the following ones: approximation, inversion, manipulating unknown variables and reinforcing previous knowledge.

*Challenge*

Regarding whether future teachers consider the problems to be a challenge or not, something similar to what happened in the previous cases occurs. Indeed, there are 17 participants who give their opinion on the direct problem, 25 analyze the inverse problem, 16 give their points of view on both problems and finally 17 opinions do not indicate which problem they refer to. In a first approach to these numbers, it can be noted that there are many more opinions about the inverse problem, although, it should be noted that this situation follows from the analysis of the context in the answer itself, more than an explicit mention of this problem. Once again, the same participant may give opinions about several problems, participating in two or more groups of the aforementioned.

Regarding the opinions themselves, a large majority think that it constitutes a challenge for the students (37 answers) and only 12 answers correspond to the negative option. Obviously, there are also opinions that state that one of the problems constitutes a challenge while the other does not, and in those cases one is counted for the affirmative and one for the negative.

It is important to mention that there are 14 responses in which it is not clear whether the future teacher considers that the direct and/or the inverse problem constitute a challenge or not. The majority of these ambiguous answers make comments about what is necessary to do (for example: calculation of areas by integrals or decomposing the problem into simpler parts), but there is no definite opinion about if it is challenge for the student.

Finally, in some answers the character of the challenge is analyzed according to its interest, whereas the analysis is made in terms of the difficulty and both do not always coincide. For example, there is a response that says: “it could be a challenge because of the resolution, rather than its interest”, then, both aspects were considered. On the other hand, other participants only analyze the difficulty (for example: “not so challenging… less advanced operations”), or the interest (for example: “interest increases by specifying data and locating a booth”), but not both of them.

*Tasks Completion, Recognition & Justification.*

Regarding whether future teachers consider that students can recognize the end of the task –and eventually give a justification– something similar to what happened in the previous cases is observed. Indeed, there are 10 participants who give their opinion about the direct problem, 19 opinions are about the inverse problem, 12 give their points of view on both problems and finally there are 13 opinions that do not indicate which problem they refer to. Once again, among those who analyzed the inverse problem, some of them are detected by the context, rather than by an explicit response. Furthermore, as in previous cases, the same participant may give his opinions about two or more problems, then, being included in two or more previous groups.

Regarding the opinions themselves, 26 of them express (or imply) that the ending is clear for the students and in contrast, 18 opinions mention that the ending is ambiguous. It should be noted that in some opinions the ending of the direct problem is clear whereas the ending of the inverse one is ambiguous and other participants think that the situation is the opposite.

There are also 10 responses where the opinion of the prospective teacher is not clear because it is limited to comparing (for example: “similar”) or in other cases the participant says what should be done (for example: calculation of areas by integrals or geometry), without expressing whether or not this completion is clear for the student.

Finally, very few participants give an opinion about the justification or, in other words, the vast majority of future teachers only express themselves on the completion of the task independently of the justification of results.

**Authenticity**

With regard to authenticity, the items to consider are: event, question, purpose, language and data. Each of them is discussed below.

*Event*

Referring to the event, there are several participants who give their opinion about the direct problem (11 answers), others do it about the inverse problem (20 answers) or on both problems (20 answers), in addition to others that do not indicate which problem they refer to (23 answers). Once again, among those who write their opinions about the inverse problem, some of them were included due to the context of their response and not as a consequence of an explicit mention of the problem itself. As in previous cases, the same participant may have given several opinions and therefore may be represented in two or more of the aforementioned groups.

Among prospective teachers there is a high percentage of them which supports the opinion that it describes a realistic event (43 opinions). In fact they double those who think it is not a real event (21 responses). Finally, there is a group of 9 responses that do not say if the event can be considered as a realistic or non-realistic one. Some of them are comparative (for example: “it has not been modified”) whereas others suggest what could be done to improve it (for instance: adapting it to a dog in a garden). In all those cases they do not make it clear whether they consider the event realistic or not.

*Question*

With reference to the question, there are several participants who give their opinion about the direct problem (9 answers), whereas others analyze the inverse problem (12 answers) and other group write about both problems (10 answers). In addition, a large number of responses do not indicate to which problem they refer (19 responses). As in previous cases, some participants were classified by the context and several ones give more than one opinion, thus appearing in two or more of the aforementioned groups.

Among prospective teachers, there is a significant majority who think that the question is realistic (28 opinions) and a smaller number of participants think exactly the opposite (18 answers). Finally there is a small group (only 4 answers) that cannot be considered in favor of a realistic or unrealistic question. Some of them are comparative (for example: “similar”) and others make descriptive comments. For instance, a participant said: “in each problem something different is asked” and another one wrote: “a numerical question and another one to make relations between functions”, so, they do not clarify whether they consider realistic or not the asked question.

*Purpose*

With reference to purpose, almost half of the participants do not try to describe what problem they are referring to (20 responses). The rest of them (31 answers) mention the problem considered or it can be deduced from the context. Opinions are quite divided among prospective teachers, as many believe that the purpose is realistic (23 opinions) and a slightly smaller number of participants think the opposite (20 responses). Finally there is a rather smaller group (8 responses) that cannot be considered in favor of a realistic or unrealistic purpose. As always, in this last group, some of them are comparative (for example: “similar”), others say what to do (for example: “write the integral correctly”) and others are more critic (for instance: “there is no specific purpose”). Those examples and other similar cases do not clarify if they consider that the purpose is realistic or not.

*Language*

Regarding language, once again, almost half of the participants do not indicate which problem they are referring to (24 responses) and more than half of those who do it, give their opinion on both problems (15 responses). Now, considering the participants who give their opinion on a specific problem, only a few of them do so about the direct problem (4 answers) and a few more (9 answers) do the same about the inverse problem (or at least, this is the conclusion obtained by analyzing the context). As in previous cases, some participants contribute to more than one of the aforementioned groups.

Among prospective teachers, the vast majority of opinions consider that the language is realistic (41 opinions) and a much smaller number think that it is not (8 answers). Finally, there is a small group (only 3 answers) that cannot say clearly weather the language is realistic or not. As always, some of them are comparative (for example: "the same"), and in some cases a description of the language is made (for example: "with numbers and symbols"), without saying explicitly if language can be considered appropriate or not.

*Data*

Regarding the data, once again, many of the participants do not indicate what problem they are referring to (22 responses) and it does not emerge clearly from the context. Now, among the other participants, some of them give their opinions about the direct problem (10 answers), others about the inverse problem (11 answers) and the rest write about both problems (9 answers). As in the cases previously analyzed, there are some that appear in two or more groups.

Among prospective teachers, there are 28 opinions who consider that data are realistic and 17 answers express they are not.

Finally there is a smaller group (7 responses) which have no a favorable or unfavorable point of view with respect to the data. As always, some of them are comparative (for example: "more particular"), in some cases it is conditionally stated (for example: "authenticity that will depend on the values of the parameters") and in several cases, it is even denied the existence of data (in 3 responses) for at least one of the problems.

General results about authenticity.

As it can be observed, the event and the question are considered as realistic in the opinion of the majority of the participants. It seems that the purpose is not so clear for many prospective teachers and some of them think that it does not exist a purpose. Most of them believe that language and data are realistic, although a few participants even say that there are no data, at least in the case of the direct problem.

**Elements of the task**

In order to analyze the elements of the task, the items to consider are: goal, formulation, materials and resources, grouping, learning situation and timing, which are discussed below.

*Goal*

There are several prospective teachers that give their opinion about the direct problem (27 answers), several more do the same about the inverse problem (35 answers) and not so many write about both problems (11 answers), and finally, there are others that do not indicate which problem they refer to (16 responses). In some cases the participants make explicit what problem they write about and in other cases it arises from the context of the answer. It is important to remark that in some cases the participant gives his/her opinions regarding more than one problem and in that case he/she is represented in several of the aforementioned groups.

Among prospective teachers there is a very large majority in favor of the opinion that the goal is specific of the topic considered (like calculating integrals or plane areas), with a total number of 64 responses, which is more than double of all the other opinions combined (29 answers). Among the other responses, only 3 indicate that the goal should be considered as specific of the problem (for instance: obtain

$f(r)$ as a piecewise function), 8 responses identify transversal goals (like mathematizing and modeling) and another group of 7 responses propose generic goals (as "analysis and reflection").

Finally there is a group of 11 responses that cannot be considered as belonging to the previous groups. The vast majority are simply comparative (for example: "the same" or "without relevant changes") and others are very ambiguous (for example: "few possibilities for interpretation") and do not make it clear what the proposed goal would be.

*Formulation*

In this item, more than half of future teachers give their opinions without indicating what problem they refer to (23 responses). Among the other few who make explicit which problem they are analyzing, 4 give their opinion about the direct problem, another 4 do the same on the inverse problem and 9 participants write their opinions considering both problems. In a few cases it arises from the context of the answer to which problem they refer and, as always, some prospective teachers give their opinion about more than one problem and so, their answer is included in several of the aforementioned groups.

Among the prospective teachers there is a very large majority in favor of the opinion that the formulation is made in a written form (36 answers), followed in a descending order by graphic representation (17 answers) and almost the same for a pictorial formulation (15 responses). Other minority responses say oral/verbal (3 responses) and only one participant mention that in his/her opinion the formulation is symbolic.

It should be mentioned that there are 10 other responses that do not belong to the previous groups, since they only compare using terms like “similar”, “the same”, etc.

*Materials and resources*

In this item, almost half of prospective teachers give their opinions without indicating what problem they refer to (23 answers) and of the remaining 27 opinions, the vast majority (19 answers) analyzed both problems. Only 8 opinions consider individual problems: 2 about the direct problem and 6 about the inverse problem (which in some cases arises from the context). As always, there are prospective teachers who have opinions on more than one problem, so their answers contribute to more than one of the previous groups.

Among the participants there is a very large majority who identify traditional materials and resources, like pencil, paper, pen and notebook, among others (37 responses). Other minority options (6 responses) propose basic technological material, like electronic calculators. Another 6 answers propose the use of specific software (such as GeoGebra), or computer tools available on the Internet (such as Wolfram Alpha).

In addition to the above, there are 10 other responses that do not belong to the previous groups, all of them comparative except one that says that there exists a lack of resources in order to enrich the task.

*Grouping*

With regard to grouping, almost all the opinions are divided in two groups: those who give their opinions about both problems (26 answers) and those that do not indicate which problem they refer to (22 answers). Among the remaining 7 opinions, only 2 are about the problem direct and another 5 are about the inverse problem. In several of these cases, it is deduced from the context which problem they refer to and finally, some prospective teachers write their opinions on more than one problem.

Among the responses received, there is a very large majority in favor of an individual task (42 responses). Among the other 24 responses, 7 propose working in pairs, only 3 responses suggest working in small groups and another 4 responses recommend working on the task with the whole group with the aim of comparing the different solutions.

In addition to the above, there are 10 other responses that do not belong to the previous groups and all of them are comparative.

*Learning situation*

Regarding the learning situation, once again almost all opinions can be divided in two groups: those who do not indicate which problem they refer to (22 answers) and those who give their opinions about both problems (21 answers). All the other 6 remaining opinions are about the inverse problem and in many cases this fact is deduced from the context of the answer. In this item, there is no participant who gives an opinion that deserves to be included in more than one group at the same time.

Among the opinions received, there is a very large majority in favor of working in the classroom (35 responses), compared to 11 responses that propose working at home. It is worth mentioning that there are several that indicate both options, for example one participant says “in the classroom or at home" and he/she adds that it is "interesting to compare different solving methodologies in class".

It is important to mention that there are 12 responses that do not belong to the previous groups and almost all of them are comparative. There is one exception where it seems that the question was not understood, since he/she gives for answer "in the rural area".

*Timing*

When timing is considered –unlike what happened with other items – there is no a clear trend. Among the prospective teachers' opinions, 21 are about the direct problem, 23 are about the inverse problem, 14 write about both problems and 15 of the answers do not indicate which problem they refer to. Effectively, it is very common for participants to give their opinions about more than one problem, for instance, a typical response indicates a proposed timing for direct problem and another one for the inverse one. In some cases, the problem considered by the participant is deduced from the context.

Among the opinions received, there is a very large majority in favor of short time periods –less than one session– with a total number of 57 responses. Some of them even propose very short times, for example: "direct: 5 minutes, inverse: 10 minutes" (this is a response that appeared repeated a few times). Among the 17 remaining responses, only 4 suggest long time periods (one session or more) and even very long times, for instance: "1 or 2 sessions".

The 13 responses that do not belong to the previous groups are usually comparative (for example: "both the same"), and others do not belong to the previous groups due to their ambiguity (e.g., "it depends on the students' facility to solve integrals"), without proposing a concrete timing for the task.

General results about the elements of the task

Regarding this category, the participants tend to give a particular opinion about the goal and the timing and make more general comments in respect to other items. The goals more mentioned are those specific to the topic, which can be about Calculus (integrals, changes of variables, etc.) or Geometry (Pythagorean Theorem, equation of the circumference, etc.). The formulations are generally seen as written, including graphic or pictorial elements. Technology is proposed in a very limited way and the task is thought to be solved individually in the classroom. In general, short times are assigned for the task and this fact is quite surprising. It seems that for the pre-service teachers it is difficult to put themselves in the shoes of their future students.

## Task variables

Regarding the task variables, the items to consider are the mathematical content, the situation and the complexity. Each of these three items is analyzed below.

### *Mathematical content*

With reference to the mathematical content, the responses of the prospective teachers are mostly divided in two groups: those who give their opinion about both problems (30 responses) and those that do not indicate which problem they refer to (19 responses). Among the remaining 27 opinions, 12 are about the direct problem, whereas the other 15 are about the inverse one. As always, in many cases this classification arises from the context of the answer given and finally, in some cases the participant gives his/her opinion regarding more than one problem, being then represented in several of the aforementioned groups.

To analyze the contents, the framework is given by the PISA study, resulting in a very large majority in favor of both "Space and Form" (45 responses) and "Change and Relations" (38 responses). Among the minority responses, only 5 indicate that the task is about “Quantity” and just one participant mentions “Uncertainty and Data”. In many cases the classification was done by context, for example, a participant says "areas of plane figures, Pythagoras and relation between unknown variables", so his/her answer was included in "Space and Shape" and also in "Change and Relations."

Finally there is a group of 15 responses that cannot be considered as belonging to the previous groups. The vast majority are simply comparative (for example: “the same for both”) and others are very ambiguous (for example: “inverse problem: more content and more reasoning”), which do not make it clear what the mathematical content considered would be.

*Situation*

With reference to the situation, almost all the responses of the prospective teachers are divided between those who give their opinion on both problems (29 responses), or do not indicate which problem they refer to (25 responses). Among the remaining 3 responses, only one is about the direct problem and the other two are about the inverse problem. In this item, only in a few cases the participant gives his/her opinion regarding more than one problem.

Once again, the framework defined by PISA is used to classify, resulting in a significant majority in favor of "Educational/Labor" (32 responses) and "Personal" (22 responses). Among the minority options, nine indicate that it is a “Public” situation and seven consider it as “Scientific”. It should be mentioned that most of the answers to this item are clear enough, such that only a few of them needed to be classified by context.

Finally, there is a group of 10 responses that cannot be considered as belonging to the previous groups. Once again, most of them are simply comparative (for example: “both problems: the same”), while others are very ambiguous (for example: “both problems: the same mathematics” or “both problems: for Secondary School students”), not making it clear which is the considered situation.

*Complexity*

Regarding complexity, an important part of the prospective teachers give their opinion about the direct problem (24 answers) and the same happens with the inverse problem (29 answers), many of them classified by context. For instance, if the participant says “connection-reflection”, it is understood that the first refers to the direct problem and the second to the inverse problem, although it is not explicitly expressed. In addition, there is an important group that gives an opinion about both problems (26 answers) and finally, there are not so many responses that do not indicate which problem they refer to (8 answers). This last result is another consequence of the classification based on the context, previously described. In this item, many participants give their opinion about more than one problem.

Once again, the framework given by PISA is used for classifying the participants' productions. In this item the majority of the responses say "Connection" (42 responses), in the second place is "Reflection" (29 responses) and lastly "Reproduction" (6 answers). In this case, the vast majority of the answers are very clear and there are almost no cases in which the production must be classified by context.

Lastly, there is a group of 15 responses which do not belong to the previous groups, being comparative the vast majority of them. There are a few ambiguous answers (for example: "direct problem: clear a variable" or "inverse problem: more mechanical") that perhaps could be classified by context, but we chose to leave them in this group, since they do not say clearly which is the complexity of the task.

**Observations**

The observations column in the table was included just for indicating which forms were received from the prospective teachers. The results were: 52 comparison forms, 7 forms only for the analysis of the direct problem and 10 only for the inverse problem. Among the 10 forms corresponding to the inverse problem, in a couple of them, the problem analyzed was deduced by the context of the answers, since it was not made explicit.

# Chapter 6.- CONCLUSIONS

This last chapter presents the conclusions of the work corresponding to both the first and the second stage of the research. It begins analyzing the answers to the research questions posed in the first chapter and it discusses and reviews the achievement of the research objectives previously proposed.

It should be remarked that although in the fieldwork two different problems were proposed to the prospective teachers, only the second one –the sheep problem–was used to collect data and analyze the corresponding results. It is important to note that there are several factors that have an important influence in the creation of richer problems and some of them are related to the general research design, whereas others are more specific of the problem selected for the study.

After analyzing the productions, it was observed that prospective teachers have been particularly creative in their reformulations and they succeed in enriching the proposed tasks; however, an important group has opted for standard statements and in some cases, for the trivialization of the proposed problem.

Then, it is important to identify which elements tends to appear when reformulating the problem for the corresponding task enrichment are carried out in a more creative and effective way. This analysis is part of the Section 6.1, where the conclusions of the first and second stage of the research are developed. After that, in Section 6.2, the results of both research's stages are compared.

Finally, in the last two sections of this chapter, some of the limitations of the study (section 6.3) and possible perspectives for further investigation (Section 6.4) are discussed.

## 6.1. - Conclusions of the first and second stage of the research

In the first chapter of the study, the following general objectives were proposed:

- OG.1 Identify and characterize the prospective teachers' strategies to pose inverse problems for secondary school courses, by reformulating a given direct problem.
- OG.2 Study, analyze and characterize the prospective teachers' productions about the didactic analysis of tasks related to inverse problems, considering their relation to the original task based on the direct problem from where they come.

Firstly, regarding these general objectives, it should be considered the limitations of availability inherent to this type of study. In our case, the research was carried out working with two groups of prospective teachers of the Master's Degree in Secondary School Teaching at the University of Granada, and in both stages of the study (2017 and 2019) the fieldwork was organized in two sessions. In the first session, in 2017, the participants were asked to enrich a given problem (the swimming pool problem), without requesting that the reformulation should be presented in an inverse form. Anyway, a few prospective teachers spontaneously used this strategy and also, some of their productions were very creative. At same time, the participants were asked to analyze the components and elements that characterized their reformulations and compare them with those corresponding to the given problem.

In the second session of the first stage, a few selected inverse reformulations were commented and analyzed, using them to introduce inverse problems to the whole group. After this first experience, the participants were asked to reformulate a new problem –the sheep problem– in an inverse form and propose new tasks related to their own reformulation.

The work of the participants made it possible to obtain a significant number of productions, i.e., inverse reformulations with the corresponding tasks analysis.

In the first stage, nine different groups of inverse problems were identified, some of them with up to five variants within the same group. It is important to mention that, in this opportunity, the participants were not asked to solve their own proposal, making it difficult to know about the possible solution that the prospective teacher planned for the direct and/or the inverse problem. However, some keywords that emerged from the analysis of the productions helped to conjecture what kind of answers the pre-service teachers were expected.

From this descriptive first stage, a series of strategies to pose inverse problems were inferred, identified and characterized. In addition, Didactic Analysis was a useful tool to study and characterize the productions of the prospective teachers (i.e., the reformulated inverse problem and corresponding task analysis).

In the case of the second stage, it was not necessary to identify the solving strategies based on keywords that emerged from the analysis of the productions, since the research design was improved, requesting the solutions of both the direct and the inverse problem.

Consequently, in the second stage of the research it was possible to develop a more complete analysis of the reformulations, grouping them into clusters and even more, those clusters formed superclusters, based on the components of the semantic triangle.

In what concerns to the comparative didactic analysis of the tasks of both problems, it was also easier to extract their opinions in the second stage, since the participants were previously provided with a table (copy of Table 2.1) that helped them to consider in a more clear way the items needed to be taken into account.

Considering this is a first exploratory study –with the typical constraints about time and extension of the sample– it can be concluded that both general objectives have been achieved.

In addition to the above, the following specific objectives related to OG.1 were proposed:

- O.1. Characterize the statements of the reformulations posed in an inverse form by the prospective teachers.
- O.2. Characterize the complexity of the resolution process of the inverse problems proposed by the prospective teachers.

And the following specific objectives related to OG.2 were also proposed:

- O.3. Characterize the meanings of mathematical concepts that pre-service teachers use when they design tasks by reformulating direct problems.
- O.4. Characterize cognitive aspects (expectations, errors and cognitive demand) that appear in the prospective teachers' tasks related to reformulated problems.
- O.5. Characterize the instruction elements, focused on the task variables that teachers use when they reformulate problems.

As it was previously mentioned, a first classification of the proposals was done and this first approach was accurate to describe the productions received in the first stage. It was expected that repeating the experiment with a new population, more trained and with more structured work sessions, it could be observed other kinds of problems corresponding to a wider range of proposals. Clearly, the list was not closed after the first stage, since it only described the problems that have been proposed at that stage, which can be considered as mainly procedural problems.

When a new design of the experiment was carried out in 2019, other different problems appeared. In fact, the cluster analysis showed several new clusters and even more important, it appeared graphic representations and conceptual problems, which added two super-clusters that were not observed in the first stage of the study.

Also, for the definitive study a new instrument was developed (see Figure 3.7), including ten columns where the type of inversion, the difficulty and other items were analyzed. Moreover, all the productions were codified in a Boolean format (Figure 3.8) and they were classified into three super-clusters, which were subdivided to obtain nine clusters, which main characteristics and complexities were deeply described and analyzed.

Then, it can be considered that the specific objectives O.1 and O.2 have been achieved partially at first stage and even more at the second stage of the research.

Regarding the other three specific objectives (i.e., O.3, O.4 and O.5), a new analysis was carried out considering the prospective teachers' productions and instruments used for this purpose was improved. In fact, in the first stage, an original spreadsheet was built and after that, some columns were added, broken down and eliminated, until an advanced version of 21 columns was reached: a first one with the student/response number, then 20 analysis columns and a final column for observations.

In the 20 columns of the of the instrument advanced version, the proposed reformulation and its meanings are analyzed, being the majority of the columns devoted to cognitive analysis (5 columns) and instructional analysis (8 columns).

This instrument was improved in successive opportunities, arriving to a final version in the second stage with four columns for significance, five for authenticity, six for the task elements and three more columns for the variables of the task.

It is important to remark, regarding the didactic analysis of the tasks, that only 29 responses were considered in the first stage, while in the second stage that number increased to 54 responses. The reason of this improvement is that the discarded answers (direct reformulations, or poorly posed problems) were a minority in the second stage. Moreover, the least answered item received 32 responses (close to 60%) and in 11 items the75% was exceeded, whereas only 7 that did not exceed this percentage. These good results were possible because an explanation of the descriptors was included in the worksheets, motivating the prospective teachers answer regarding all the items. Therefore, the design of the second stage of the research has achieved the objectives proposed and the instrument has been validated in the second stage of the research, since the participants paid attention to all the items considered.

In what concerns to the results of the didactic analysis of the tasks, the most important ones in the first stage, were the following:

- Concerning the meanings, the responses show all the possible different options. Indeed, of the 37 responses, 17 chose "space and form", 9 "change and relationships", 7 use "quantity", and the remaining 4 mentioned "uncertainty and data". In addition, most of the participants place the task in the educational field.
- The cognitive aspects show a great variety of responses, showing four different groups about the goal of the task and another four groups with different components and categories about the learning opportunities.
- Regarding the categories of the instruction analysis, it should be mentioned that the majority has a favorable opinion about the task authenticity. On the other hand, when purposes and grouping are considered, there is no agreement and the opinions are varied. Lastly, there is a general agreement in considering that the complexity is greater than in the original task.

- It is worth mentioning that some of the pre-service teachers considered very important that the problem was reformulated in an inverse form. For instance, A14 coded student states that "The interest of the reformulation is because it is the inverse of the previous..." and student B14 says regarding complexity that "... increases when going from a direct problem to an indirect one." Others have a more critical position, such as the student B24who says "... it would not generate much interest ... the student may never face a problem like that."

In the second research stage, important conclusions are the following:

- Regarding the meanings, it should be noted that the participants tend to express their opinions about the differences between both problems (direct and inverse) more than about the similarities, and even more, they tend to focus their analysis on the inverse problem. In their comments about the prior knowledge, the dominant responses are Geometry and Calculus and with respect to the content that activates the task, 15 responses indicate several mathematical contents, whereas 29 participants mentioned different competencies without a clear trend. The participants generally consider that the task constitutes a challenge for Secondary School courses and its completion is considered as “clear” for the student, at least in the majority of the opinions.
- In what concerns with the authenticity, there is a general trend consisting in giving an opinion, which does not specify to what

problem it refers or they give an opinion about both problems at the same time, rather than giving an answer that refers to a specific problem. In general, the event and the question are realistic in the opinion of the majority, however the purpose is not so clear and there are even some participants who think that it does not exist. The language is adequate in the opinion of a very large majority and the data are generally seen as realistic, but there are also some opinions that even say that there are no data, at least for the case of the direct problem.

- With regard to the elements of the task, the participants tend to give a particular opinion about the goal and the timing and make more general comments about the other items. The most frequently goals indicated are those specific to the content matter, which can be about Calculus (integrals, changes of variables, etc.) or Geometry (Pythagorean Theorem, equation of the circumference, etc.). The formulations are seen as written, being in most cases graphic and/or pictorial. Technology is proposed in a very limited way and the tasks are thought to be solved individually in the classroom, adding in some cases a group session for comparing different solutions. Mostly, short and very short times are assigned for the tasks and this fact is quite surprising, since the participants had to solve the problem before giving their opinions and so, they have enough elements to know that the integrals were not trivial. Therefore, it seems that the pre-service teachers have no enough experience that allows them to put in the shoes of their future students.

- In accordance with what happened with the prior knowledge, the answers that choose "Space and Form" and "Change and Relationships" also prevail here, showing a clear connection with the Geometric and Analytical contents. The situation is identified firstly as “Educational/Labor” and “Personal” as the second option. The complexity is usually between “Connection” and “Reflection”, and a very common answer consists in saying that one problem is a connection task whereas the other needs reflection, depending on whether the prospective teacher proposed or not a more complex reformulation.
- The vast majority of the forms submitted corresponded to the comparison of both problems and only a few participants submitted others forms, considered as an optional, non-compulsory task.

Then, we can conclude that, taking limitations into account, the work complies with the specific objectives proposed for this research.

### 6.2.-First and second stage comparison

In the results section of the second stage, nine clusters were described when analyzing the productions of the prospective teachers in the fieldwork carried out in 2019 and the same number was obtained in the first stage in 2017. Nevertheless, only in the second stage, different proposals corresponding to three super-clusters were observed. Moreover, those super-clusters establish a direct connection with reference, representation and meaning, the so-called semantic triangle categories, adaptation of Frege's categories of mathematical school semantic contents, described by Rico and collaborators (Ruiz-Hidalgo &Rico, 2016).

As it happened with other research studies, even though the same materials were provided to all the participants, their attention was directed towards different representations and their analysis of the situation was based on different criteria. This fact is consistent with the research of Bautista et al. (2014), who observed the same situation and remarked the importance of the educational background, since it may contribute to shape the ideas of the pre-service mathematics' teachers. In our case, the participants come from different university careers (Mathematics, Engineering, Architecture, Sciences, etc.) and this may be help to give a possible explanation of the diversity observed in their proposals.

Although the number of clusters in the second stage was the same than in the first one (Martinez-Luaces, Fernández-Plaza& Rico, 2020), the groups were very different, being Cluster Nº 1, Cluster Nº 3 and Cluster Nº 4 the only ones that can be considered as strongly connected with those described in the first stage. Moreover, in that first stage there were no proposals about sketches, conceptual questions, specification problems or reformulated problems that require using advanced theorems.

These results obviously led us to the following questions:

- Why did both experiences obtain results so different?
- Do the design changes in the fieldwork help to justify the differences between the results?

- How did those changes in the fieldwork design affect the reformulations proposed by the prospective teacher's participants?

Before answering the first question, it is important to remark that even the sample was not the same; both populations can be considered as equivalents, chosen with the same availability criteria. In fact, in both cases we worked with prospective teachers that were regular students of the Master in Mathematics Teaching for Secondary School, the Group B professor was the same person and they worked in exactly the same problem (the sheep problem). Then, it can be concluded that the new fieldwork design was the only important difference between both experiences, so it is not difficult to conjecture that those changes in the design are the main reason in order to explain the new results.

As a consequence of the previous comments, a positive response to the second question seems reasonable and besides, it seems to be the only possible cause for the observed effect, i.e., the great differences in the prospective teachers' proposals in 2017 and 2019.

Then, if we assume that this is the most likely explanation, the third question should be analyzed. Let us start by considering Cluster Nº5 where five different proposals (PT02, PT05, PT04, PT13 and PT03) requested a sketch as the output, being an integral formula the corresponding input. This integral had not appeared if the direct problem would not have previously solved –or at least partially solved– in order to arrive to the corresponding definite integral. Another example comes from five proposals included in Cluster Nº7 (PT29, PT10, PT11a, PT07 and PT08), since in all of these problems a function is given –which can be $f(r)$, $A(x)$ or $A(R)$– and in order to obtain this function, at least a partial solution of the direct problem is needed. Consequently, these proposals, among others, would have not appeared, if the solution of the direct problem had not been requested. As a final remark, the analysis of PT21 in Cluster Nº7 and PT11b in Cluster Nº9, led to similar conclusions.

In the first stage the participants tried to imitate previous examples, related to the sheep problem. This comment agrees with Chapman (2012), who observed that the word problems that children tend to pose are variations of traditional ones, which can be found in textbooks. Even more, he stated that "since students grow up to become teachers, it is likely that prospective teachers maintain some of these issues that will then continue the cycle unless they are helped in appropriate ways".

When the participants were asked to reformulate the "sheep problem" at the first stage, they attempted to reproduce previous examples (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018). Then, their reformulations were based on the inversion of the function, changes in geometry and/or the inclusion of certain obstacles, among others possibilities.

The experience in 2019 was very different, since the previous examples showed to the prospective teachers were about other mathematics topics: arithmetic and geometric sequences, proportions, and obtaining unknown angles and/or edges of right triangles. Those simple examples did not allow the imitation of the given problems. At the same time, the subjects had to solve the original direct problem before posing their own reformulation and so, their experience was more connected with the direct problem solution than other inverse problems. This situation led them, in different directions, in terms of their inverse problems proposals. For instance, some of the participants gave a formula and requested an interpretation of a certain parameter, or asked to propose another solution without using integration techniques.

Also, some important differences in terms of use of external variables were observed. Those external variables can be chemical (amount of herbicide and fertilizer), physical (velocity or time), economical (the cost of a fence per unit of length) or even biological variables (kilograms of grass the sheep can eat per day). Those external variables widely appeared in 2017, although they were used only in a few cases in 2019.

It can be concluded that prospective teachers tend to propose their reformulated inverse problems, based on recent experiences. As a consequence, if those experiences consist in working with previous examples, they tend to imitate them. In the same way, if their only relevant experience consists in solving the original direct problem, then, they try to use the solution (or the process that led to it) as the main input for their proposal.

Finally, it is not easy to say that one of those experiences can be considered better than the other in terms of their proposals. It can be seen that in the first one, some characteristics were predominant, whereas in the second one, other characteristics were observed. As a consequence, the prospective teachers' proposals in both stages, more than antagonistic should be regarded as truly complementary.

## 6.3.- Work limitations

As already partially commented, the limitations of this study are related to the following four aspects: the participant's subjects, the sesions number devoted to the field work, certain instrument characteristics' utilized, and the mathematical content revealed in this work.

Regarding the first aspect mentioned, it was necessary to limit ourselves to describing the productions of two groups of prospective teachers, studying University Master's Degree for Secondary School Teachers at UGR. The experience has some points of contact with the work done at the University of Colima (Mexico) and also it has some important differences. Since in Colima they were not requested to write about their analysis of the reformulated problem and the corresponding tasks, it is not possible to make a full comparison. For obvious reasons, the results obtained cannot be generalized either, since for this purpose it would be necessary to repeat the research in other countries, with different educational contexts.

A second study limitation results from the scarcely sessions of the fieldwork. Indeed, due to time constraints, only two sessions took place in both stages and one of them was planned as an introductory class about task enrichment and problems reformulation. Only the last session was specifically used for inverse reformulations, at least in the first stage. As a consequence, even in the second stage of the research, the prospective teachers did not have many opportunities to adapt themselves to this type of task, thus being only partially trained to propose and discuss different reformulations. It can be conjectured that if the participants have more experience, perhaps other variants –and also more creative ones– could emerge, then leading to greater richness proposals.

Regarding the instrument used to collect the productions of the pre-service teachers at the first stage, it was evident that it would have been convenient to include a part for the participant's solution of his/her own reformulation. This modification –included in the second stage– diminished the number of ill-posed and trivial problems and it also decreased the number of non-inverse problems that were discarded for this study. Then, it can be concluded that if the prospective teachers solve their own reformulated problem, some of these inconveniences are less likely to occur.

On the other hand, in the worksheet provided for the task analysis, only the following items were explicitly included: meanings, authenticity, elements that make up the task and task variables. Then, it can be conjectured that a more extensive worksheet, where all the items of Table 2.1 appear explicitly, could have encouraged the participants to include all of them in their task analysis.

Finally, it can be observed that original problem mathematical contents' lead the participants towards three specific areas of mathematics: algebra, calculus and geometry. A different problem related to other branches of mathematics –also included in the Secondary School syllabus– could led the participants to propose different reformulations connected with other areas. For instance, trigonometry was practically absent in the productions of the prospective teachers and this situation could change if trigonometric functions are included in the given problem. The same happen with probability, where only one problem was proposed about this area, but it was discarded since it was not posed in an inverse form.

## 6.4.-Work to be followed with the research

Once this exploratory study is concluded, it can be considered that there are some aspects that could be modified and also improved in order to design further research projects in this topic. A first possibility consists on improving the tool as a semantic questionnaire, expanding the tasks to be considered, and the didactical questions to be posed (semantic, intentional and instructional).

A fourth assessment dimension will be pertinent when we reach and validate a semantic questionnaire, not only with a purpose of carrying out statistical tests obtained from a bigger sample size, but also for develop comparative results with other countries, similar experiences at least in other universities. This would produce more general results than the obtained in this study.

It would be also interesting to replicate the experience including more sessions in order to work on task enrichment activities and practicing the reformulation of a given problem in an inverse form. It would be convenient that the first sessions –more than one, if possible– take place with the teacher's accompaniment during the activity, in order to guide the work of the participants and to promote the discussion and reflection about the proposals within the whole group. It is worth mentioning that in this study the reformulation of inverse problems was left as homework. It can be expected that after working on these group activities –including the discussion of the proposals–, the prospective teachers would feel more comfortable and more prepared for the individual homework requested. Moreover, they would get used to performing task analysis considering more items than those considered by the participants in this study.

The above suggestions would allow obtaining more creative reformulations and, a better analysis of the corresponding tasks. In this regard, it should be noted that problem posing is a competence that involves thinking with original ideas, and surely many of the pre-service teachers are not used to work on this kind of tasks. It can be conjectured that having more time for practice skills and discuss strategies by groups of participants with more experience and confidence, it may appear new and more interesting results.

In summary, this research can be expanded to reach more general results; although it will be required to work with greater and diversified questions, didactically based on more open-wide population, and more sessions are necessary to prepare prospective teachers for this type of planning work..

## References


Abell, M. L., y Braselton, J. P. (2016). *Differential equations with Mathematica*. Massachusetts: Academic Press.

Abu-Elwan, R. (1999). The development of mathematical problem posing skills for prospective middle school teachers. In *Proceedings of the International conference on Mathematical Education into the 21st Century: Social challenges, Issues and approaches* (Vol. 2, pp. 1-8).

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of teacher education*, 59(5), 389-407.

Barbe, W. B. (1960). What is Enrichment? In J. F. Magary & J. R. Eichorn (Eds.), *The Exceptional Child: A Book of Readings* (pp. 199-206). New York: Holt, Rinehart and Winston Inc.

Blum, W. (2002). ICMI Study 14: Applications and modelling in mathematics education. Discussion document. *Educational studies in mathematics*, *51*(1),149-171.

Ministerio de Educación, Cultura y Deporte (2015), Real Decreto 1105/2014, de 26 de diciembre por el que se establece el currículo básico de la Educación Secundaria Obligatoria y del Bachillerato *BOE*, núm. 3. 169-546. Retrieved on February of 2021 from: https://www.boe.es/eli/es/rd/2014/12/26/1105/dof/spa/pdf

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and Learning of Calculus*. Heidelberg: Springer.

Brown S. I. & Walter M. I. (2005). *The art of problem posing*, 3rd ed. Mahwah, NJ: Lawrence Erlbaum Associates.

Brown S. I. & Walter M. I. (2014). *Problem posing. Reflections and Applications*. New York: Psychology Press.

Bunge, M. (2006). Filosofía y Ciencia. Problemas directos e inversos. Retrieved on February of 2021 from: http://grupobunge.wordpress.com/2006/07/20/119

Bunge, M. (2008). *Tratado de Filosofía. Semántica I, Sentido y referencia*. Barcelona, Spain: Gedisa Editorial.

Castro, E. (1994). *Niveles de comprensión en problemas verbales de comparación multiplicativa* (Doctoral Thesis, University of Granada).

Castro, E., & Ruiz-Hidalgo, J. F., (2015). Matemáticas y resolución de problemas. In P. Flores & L. Rico (Eds.), *Enseñanza y aprendizaje de las matemáticas en educación primaria* (pp. 89-106). Madrid: Pirámide.

Castro-Rodríguez, E., Pitta-Pantazi, D., Rico, L., & Gómez, P. (2016). Prospective teachers' understanding of themultiplicative part-whole relationship of fraction. *Educational Studies in Mathematics*, *92*, 129-146. https://doi.org/10.1007/s10649-015-9673-4

Clarke, D. & Clarke, B. (2002). Using rich assessment tasks in mathematics to engage students and inform teaching. Background paper for seminar for upper secondary teachers, Stockholm, September. Retrieved on February 2021 from http://ncm.gu.se/media/ncm/nollvision/assessment_paper_stockholm.doc

Cogan, L.S. & Schmidt, W. H. (2014). The concept of opportunity to learn (OTL) in international comparisons of education. In K. Stacey & R. Turner (Eds-),*Assessing Mathematical Literacy*. *The PISA experience*. Springer.

Cohen, L. Manion, L.,& Morrison, K. (2011). *Research Methods in Education*., London. UK: Routledge

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*(5), 395-415.

Dahnke, G. L. (1986). Investigación y Comunicación. In C. Fernández Collado (Ed.), *La Comunicación Humana: Ciencia Social* (pp. 385-454 ). Mexico D.F.: McGrawHill.

Dorff & Hall, 2003

Duncker, K. (1945). On problem-solving (L. S. Lees, Trans.). *Psychological Monographs, 58*(5), 1-113

English, L. (1997). The development of fifth-grade children's problem-posing abilities. *Educational Studies in Mathematics, 34*,183-217.

English, L. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*, 83-106.

Estivill-Castro, V. (2002). Why so many clustering algorithms: a position paper. *ACM SIGKDD explorations newsletter*, *4*(1), 65-75.

Felmer, P., & Perdomo-Díaz, J. (2016). Novice Chilean secondary mathematics teachers as problem solvers. In P. Felmer, E.

Pehkonen& J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 287-308). Cham : Springer..

Feng, W. Y. (2005). Conceptions of enrichment. Paper presented at the Cambridge Education Research Annual Conference (CamERA). England. April. University of Cambridge.

Fernández, F. (1998). *Evaluación de competencias en álgebra elemental a través de problemas verbales* (Doctoral Thesis, University of Granada).

Fernández-Plaza, J. A., Rico, L., & Ruiz-Hidalgo, J. F. (2013). Concept of Finite Limit of a Function at a Point: Meanings and Specific Terms. *International Journal of Mathematical Education in Science and Technology*, 44(5), 699-710. Doi: https://doi.org/10.1080/0020739X.2013.805887.

Fernández Plaza, J. A., & Cañadas, G. R. (2019). Modificación de tareas de porcentajes por futuros maestros de Educación Primaria. In S. Alonso, J. M. Romero, C. Rodríguez-Jiménez & J. M. Sola (Eds) *Investigación, Innovación docente y TIC. Nuevos horizontes educativos*. Madrid: Editorial Dykinson, S.L.

Gómez, P. (2007). *Desarrollo del conocimiento didáctico en un plan de formación inicial de profesores de matemáticas de secundaria*. (Doctoral Thesis, University of Granada).

Gowers, W. T. (2000). The two cultures of mathematics. In V. Arnold et al. (Eds) *Mathematics: Frontiers and Perspectives*, 65-78. International Mathematical Union.

Groestch, C. W. (1999). *Inverse problems: activities for undergraduates*. Washington D.C.: Mathematical Association of America.

Groetsch, C. W. (2001). Teaching-Inverse problems: The other two-thirds of the story. *Quaestiones Mathematicae, 24 (1), Supplement*, 89-94.

Grootenboer, P. (2009). Rich mathematical tasks in the Maths in the Kimberley (MITK) Project. *Crossing divides*, *1*, 696-699

Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly, 87* (7), 519–524.

Hernández, R., Fernández, C. & Baptista, P. (1991), *Metodología de la Investigación*. México: McGraw-Hill Interamericana

Işık, C., Kar, T., Yalçın, T., & Zehir, K. (2011). Prospective teachers' skills in problem posing with regard to different problem posing models. *Procedia-Social and Behavioral Sciences, 15*, 485-489.

Kilpatrick J. (1987). Problem formulating: Where do good problems come from? In A. Shoenfeld (Ed.) *Cognitive science and mathematics education*(pp. 123-148). New Jersey: Lawrence Erlbaum Associates.

Kilpatrick, J. (2016). Reformulating: Approaching mathematical problem solving as inquiry. InP. Felmer, E. Pehkonen & J. Kilpatrick (Eds), *Posing and solving mathematical problems* (pp. 69-81). Cham: Springer..

Klok, J.V., (2014). On the use of questionnaires in semantic fieldwork: A case study in modality. *Proceedings of language documentation and linguistic theory, 4*, 1-11.

Laudan, L. (1978). *Progress and its problems: towards a theory of scientific growth*. Berkeley and Los Angeles: University of California Press

Lavy, H. & Shriki A. (2007). Problem posing as a means for developing mathematicalknowledge of prospective teachers. En Woo, J. H., Lew, H. C., Park, K. S. y Seo, D.Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 129-136. Seoul: PME.

Lester, F. K. (1983). Trends and issues in mathematical problem-solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, 229-261. London: Academy Press.

Lester F. & Cai J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In P. Felmer., E. Pehkonen & J. Kilpatrick (Eds.) *Posing and solving mathematical problems* (pp. 117-135). Springer International Publishing.

Leung, S. S., & Silver, E. A. (1997).The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of prospective elementary school teachers. *Mathematics Education Research Journal*, 9(1), 5-24.

Liu, F. (2003). Teaching Inverse Problems in Undergraduate Level Mathematics, Modelling and Applied Mathematics Courses. In Y. Qi-Xiao, W. Blum, K. Houston & J. Qi-Yuan (Eds.), *Mathematical Modelling in Education and Culture* (pp. 165-172). Woodhead Publishing. https://doi.org/10.1533/9780857099556.4.165

Lupiáñez (2009) *Expectativas de aprendizaje y planificación curricular en un programa de formación inicial de profesores de* *matemáticas de secundaria*. (Doctoral Thesis, University of Granada).

Lupianez, J. L. (2016). Lo ordinario y lo extraordinario en el aula de Matemáticas. *Cuadernos de Investigación y Formación en Educación Matemática*, 253-268.

Marín, A. (2013). El análisis de instrucción: instrumento para la formación inicial de profesores de Secundaria. En L. Rico, J. L. Lupiáñez & M. Molina (Eds.), *Análisis Didáctico en Educación Matemática. Metodología de Investigación, Formación de profesores e Innovación curricular* (pp.103-120). Granada: Comares.

Martín-Fernández, E., Ruiz-Hidalgo, J. F. & Rico, L., (2016). Significado escolar de las razones trigonométricas elementales. *Enseñanza de las ciencias: revista de investigación y experiencias didácticas,34*(3), 51-71

Martín-Fernández, E., Ruiz-Hidalgo, J.F. & Rico, L., (2019). Meaning and understanding of school mathematical concepts by secondary students: The study of sine and cosine. *Eurasia Journal of Mathematics, Science and Technology Education*, *15* (12), 1-18.https://doi.org/10.29333/ejmste/110490

Martinez-Luaces, V., (2009). Modelling and inverse-modelling: experiences with O.D.E. linear systems in engineering courses. *International Journal of Mathematical Education in Science and Technology, 40 (2),* 259-268.

Martinez-Luaces, V. (2011). Problemas inversos: los casi olvidados de la Matemática Educativa. *Acta Latinoamericana de Matemática Educativa*, *24*, 439-447.

Martinez-Luaces, V. (2013). Inverse modelling problems in linear algebra undergraduate courses. *International Journal of Mathematical Education in Science and Technology, 44 (7),* 1056-1064.

Martinez-Luaces V. (2016). Inverse Modeling Problems and their Potential in Mathematics Education. In Vargas, M. (Ed.) *Teaching and Learning: Principles, Approaches and Impact Assessment* (pp. 151-185). New York: Nova Science Publishers.

Martinez-Luaces, V., (2017) *Estrategias de los profesores en formación para la invención de problemas inversos y enriquecimiento de tareas matemáticas destinadas a alumnos de enseñanza secundaria*. Master's Final Project Retrieved on February of 2021, from: https://fqm193.ugr.es/media/grupos/FQM193/cms/TFM_MARTINEZLUACES.pdf

Martinez-Luaces, V., Fernández-Plaza, J. A., Rico, L., & Ruiz-Hidalgo, J. F. (2019). Inverse reformulations of a modelling problem proposed by prospective teachers in Spain. *International Journal of Mathematical Education in Science and Technology*. DOI: 10.1080/0020739X.2019.1683773

Martinez-Luaces, V., Rico, L., Ruiz-Hidalgo, J. F & Fernández-Plaza, J. A. (2018). Inverse Modeling Problems and Task Enrichment in Teacher Training Courses. In Nata, R.V. (Ed.) *Progress in Education* (pp. 185-214) New York: Nova Science Publishers.

Martinez-Luaces, Fernández-Plaza & Rico, 2019 ijesim

Martinez-Luaces, V., Fernández-Plaza, J. A., & Rico, L. (2020). Inverse modeling problems in task enrichment for STEM courses. InK.

G. Fomunyam (Ed.), *Theorizing STEM Education in the 21st Century* (pp. 29-44). London: IntechOpen.

Matthewson, L. (2004). On the methodology of semantic fieldwork. *International journal of American linguistics*, *70*(4), 369-415.

Moreno, A. & Ramírez, R. (2016). Variables y funciones de las tareas matemáticas. In L. Rico& A. Moreno(Eds.), *Elementos de didáctica de la matemática para el profesor de Secundaria* (pp. 243-258). Madrid: Pirámide.

OECD (2003). *The PISA 2003 assessment framework. Mathematics, reading, science and problem solving knowledge and skills.*Retrieved on February of 2021, from: http://www.oecd.org/education/school/programmeforinternationalstudentassessmentpisa/33694881.pdf

OECD (2004) *Mathematical literacy*. Retrieved on February of 2021from: http://www.oecd.org/education/school/programmeforinternationalstudentassessmentpisa/33707192.pdf

OECD (2013).*PISA 2012 released mathematics items*. Retrieved on February of 2021, from: https://www.oecd.org/pisa/pisaproducts/pisa2012-2006-rel-items-maths-ENG.pdf

OECD (2013). *PISA 2015 draft mathematics framework*. Retrieved on February of 2021, from: http://www.oecd.org/pisa/pisaproducts/Draft%20PISA%202015%20Mathematics%20Framework%20.pdf

Ortíz, J., Rico, L., & Castro, E. (2008). La enseñanza del álgebra lineal utilizando modelización y calculadora gráfica: Un estudio con profesores en formación. *PNA, 2(4)*, 181-189.

Polya G. (1957). *How to Solve it? A New Aspect of Mathematical Method*. 2nd edition. Princeton, NJ: Princeton Science Library.

Popper, K.R. (1994). *The Myth of the Framework: In Defense of Science and Rationality*. London: Routledge.

Rico, L. (1988). *Didáctica activa para la resolución de problemas*. Granada. España: Grupo EGB de Granada

Rico, L. (1999) Development of Spanish Doctoral Studies in Didactic of Mathematics. In K. Hart & F. Hitt (Eds.),*Supervision of Higher Degrees in Mathematics Education. An International Perspective* (pp. 1- 26). México: CINVESTAV.

Rico, L. (2016). Matemáticas y análisis didáctico. In L. Rico&A. Moreno (Eds.), *Elementos de didáctica de la matemática para el profesor de Secundaria*. Madrid: Pirámide.

Rico, L., Almendros, A., Cobo, F., Casares, M., Casares, A., Castro Martínez, E., ..., Gutiérrez, J.,(1988). *Didáctica activa para la resolución de problemas. 6º nivel EGB Curso 86-87*. Departamento de didáctica de la matemática. Universidad de Granada.

Rico, L., Fernández-Cano, A., Castro, E.,& Torralbo, M. (2008). Postgraduate Study Program in Mathematics Education at the University of Granada (Spain). In R.E. Reys, & J. A. Dossey (Eds.): *U. S. Doctorates in Mathematics Education: Developing Stewards of the Discipline* (pp. 203-211). Providence, R.I.:American Mathematical Society, CBMS Issues in Mathematics Education.

Rico, L.& Fernández-Cano, A. (2013). Análisis didáctico y la metodología de investigación. In L. Rico, J. L. Lupiañez &M. Molina (Eds.) *Análisis didáctico en educación matemática. Metodología de investigación, formación de profesores e innovación curricular* (pp.1-22). Granada: Comares.

Rico, L., Lupiañez, J. L., & Molina, M. (Eds.) (2013). *Análisis Didáctico en Educación Matemática: metodología de investigación, formación de profesores e innovación curricular*. Granada: Editorial Comares.

Rico, L.& Moreno, A. (Eds.) (2016). *Elementos de didáctica de la matemática para el profesor de Secundaria*. Madrid: Pirámide.

Rico, L. & Ruiz-Hidalgo, J. F. (2018). Ideas to work for the curriculum change in school mathematics. In Y. Shimizu&R. Vithal (Eds.) *ICMI Study 24* (pp. 301-308).

Rojas, N., (2014). *Caracterización del conocimiento especializado del profesor de matemáticas: un estudio de casos*. (Doctoral Thesis, University of Granada).

Santos, S., & Barmby, P. (2010). Enrichment and engagement in mathematics. In M. Joubert & P. Andrews (Eds.) *Proceedings of the British Congress for Mathematics Education,* Vol. 30, No. 1, pp. 199-206.

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education*, *196*(2), 1-38.

Şengül, S., & Katranci, Y. (2015). Meta-cognitive aspects of solving indefinite integral problems. .*Procedia-Social and Behavioral Sciences*, *197*, 622-629.

Shulman, L.S.(1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*, 1-22.

Silver, E. A. (1994). On Mathematical Problem Posing. *For the Learning of Mathematics*, *14*(1), 19-28.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM-Zentrallblatt fur Didaktik der Mathematik*, *29*(3),75-80.

Silver, E. & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *27*(5),521-539.

Silver, E., Mamona-Downs, J., Leung, S., &Kenney, P. (1996). Posing mathematical problem: An exploratory study. *Journal for research in mathematics education, 27*(3), 293-309.

Simmons, G. F. (2016). *Differential equations with applications and historical notes*. Florida: CRC Press.

Stacey, K. et al.(2015). PISA's influence on thought and action in mathematics education. In K. Stacey & R. Turner (Eds.), *Assessing Mathematical Literacy* (pp. 275-306). Cham: Springer.

Stoyanova, E. (1998). Problem posing in mathematics classrooms. InA. McIntosh y N. Ellerton (Eds.) *Research in Mathematics Education: a contemporary perspective* (pp. 164-185). Perth, W.A.: MASTEC, Mathematics, Science and Technology Education Centre, Edith Cowan University.

Tzur, R., Zaslavsky, O., & Sullivan, P. (2008). Examining teachers' use of (non-routine) mathematical tasks in classrooms from three complementary perspectives: Teacher, teacher educator, researcher. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda, *Proceedings of the 32nd PME and 30th PME-NA Annual Meeting, Vol. 1*, pp. 121-123. Morelia, México: PME.

Varsavsky, C., Waldock, J., Harding, A., Bookman, J., Sheryn, L. & Martinez-Luaces, V. (2011). Undergraduate mathematics around the world. In J. Hannah & M. Thomas. (Eds.) *Proceedings of Volcanic Delta 2011* (pp. 399-410)**.** New Zeland

Verschaffel, L., Greer, B., & De Corte, E., (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger Publishers.

Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10*(4-6), 205-215.

Watson, A. (2008). How secondary teachers structure the subject matter of mathematics. En M. Joubert (Ed.), *Informal Proceedings of the British Society for Research into Learning Mathematics*, (Vol. 28, Nº 3, pp.126-131). London: British Society for Research into Learning Mathematics.

Yoon, C., Dreyfus, T., & Thomas, M. O. (2010). How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity. *Mathematics Education Research Journal*, *22*(2), 141-157.

Zaslavsky, O., & Sullivan, P. (2011).Setting the stage: A conceptual framework for examining and developing tasks for mathematics teacher education. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics* (pp. 1-19). Boston, MA: Springer.

## INTRODUCCIÓN

Este documento constituye un informe de investigación sobre el desarrollo de habilidades, capacidades y competencias profesionales, es decir, conocimientos de contenidos procedimentales sobre problemas matemáticos, alcanzado por un grupo de futuros docentes de matemáticas de secundaria. Para ello, los sujetos siguen un curso sobre diseño, selección, evaluación y caracterización de las tareas didácticas de la matemática escolar, dentro de un programa de formación. Los fundamentos de este programa se desarrollaron durante los últimos años del siglo XX, concentrándose en tipos particulares de problemas matemáticos escolares. Estos años coinciden en el tiempo en que se inició el Programa de Evaluación Internacional de Estudiantes (PISA, por su acrónimo en inglés), promovido por la Organización para la Cooperación y el Desarrollo Económicos (OCDE, por su acrónimo en inglés), que se basa en una interpretación diferente de las matemáticas escolares, entendidas como una competencia matemática. Si bien nuestra investigación no se basa en la evaluación PISA, la coincidencia temporal con estos estudios se puede observar en la base y en el marco conceptual de nuestro trabajo (OCDE, 2004).

El curso en el que se hizo el trabajo de campo, tuvo un formato de taller basado en el análisis de instrucción de las tareas escolares asociadas a la reformulación de problemas, caracterizados como inversos de los problemas originales, realizando además, la solución de ambos problemas (directo e inverso) y el análisis de las tareas involucradas. El análisis de tareas se lleva a cabo en el marco del análisis didáctico y más específicamente, el análisis de instrucción que proporciona las herramientas teóricas necesarias para diseñar, seleccionar y secuenciar tareas, formas de organizar su implementación en el aula, sus variables y complejidad, así como sus aspectos cognitivos y de significado.

Cabe mencionar que el trabajo de campo correspondiente a esta investigación se desarrolló en varias sesiones a lo largo de los cursos académicos 2016-2017 y 2018-2019 del Máster de Profesores de Matemáticas de Secundaria, impartido por la Universidad de Granada.

Este primer capítulo de esta memoria comienza con un análisis general de las tareas matemáticas escolares y la noción de problema. A continuación, se comentan brevemente algunas de las experiencias previas más relevantes que han servido de antecedentes de la tesis actual.

Adicionalmente, uno de los elementos distintivos de este trabajo es la noción de problema inverso y en particular el problema de modelación inverso. Ambos se presentan en la sección 1.3. Sin embargo, los problemas inversos en sí mismos no son el objetivo final, sino que se utilizan como herramienta para la proposición de tareas de mayor riqueza. Por esta razón, las secciones 1.4 y 1.5 están dedicadas principalmente a la formulación de problemas y al enriquecimiento de tareas.

Dado que el trabajo de campo de este estudio se realizó con futuros docentes que cursaban la Maestría en Docencia de Secundaria, entonces, el siguiente apartado se centra en los cursos de formación docente que ofrece dicha institución.

Es importante reconocer la diferencia entre el conocimiento matemático y el conocimiento matemático que los profesores necesitan para llevar a cabo su trabajo de manera eficaz. Por este motivo, estos temas se exponen en la sección 1.7.

Finalmente, en las dos últimas secciones del capítulo (secciones 1.8 y 1.9), se presentan las preguntas de investigación y los objetivos de la tesis.

Teniendo en cuenta las experiencias previas, que tuvieron lugar en distintos países de América Latina –y particularmente la que tuvo lugar en la Universidad de Colima, México– y el Trabajo Final del Máster (TFM), parece razonable conjeturar que los futuros profesores son capaces de reformular problemas directos, convirtiéndolos en problemas inversos de mayor riqueza.

En consecuencia, la primera conjetura es que los futuros profesores están preparados para modificar fácilmente un determinado tipo de problema de modelización, planteado de forma directa, convirtiéndolo en uno inverso, más rico y más conectado con el mundo real y por tanto, más motivador para los alumnos.

Los problemas inversos suelen estar mal condicionados, lo que eventualmente representa un obstáculo, sin embargo también puede ser una ventaja ya que permiten estudiar cuestiones de existencia, unicidad y estabilidad, que no suelen ser tan relevantes en la mayoría de los problemas directos. Por tanto, una segunda conjetura que puede plantearse consiste en que los futuros profesores pueden utilizar estas potencialidades para proponer tareas matemáticas enriquecidas.

Además, es importante conocer los puntos de vista de los futuros profesores preocupados por la utilidad de los problemas inversos en el trabajo en el aula, la motivación de los estudiantes y la conveniencia o no de utilizarlos en sus cursos. En concreto, sus opiniones sobre los significados y autenticidad de las tareas propuestas, así como sus elementos y variables, son especialmente relevantes. En consecuencia, es razonable preguntarse sobre la potencialidad de los problemas de modelado matemático inverso, centrándose en su aplicabilidad en el aula. Por lo tanto, es conveniente investigar si los futuros profesores consideran estos problemas motivadores o no y si los ven como potencialmente útiles en sus cursos. Por último, cabe preguntarse si los futuros profesores se ven capaces de transformar los problemas de modelización directos en problemas inversos más ricos –con sus correspondientes tareas– para ser utilizados en los cursos de secundaria.

Teniendo en cuenta los comentarios anteriores, surgen las siguientes preguntas de investigación:

P.I. 1 ¿Cómo utilizan los futuros profesores sus conocimientos matemáticos al reformular un problema de modelización directo, en un problema inverso que sea coherente y adaptado al nivel de los estudiantes? ¿Qué fortalezas y debilidades se identifican en sus propuestas?

P.I. 2 ¿Cómo utilizan los futuros profesores su conocimiento didáctico al diseñar tareas significativas asociadas al problema inverso, previamente reformulado a partir de un problema directo dado? ¿Qué fortalezas y debilidades se identifican en sus diseños de tareas?

P.I. 3 ¿Cuáles son las estrategias de los futuros profesores para la reformulación de un problema directo dado, en uno inverso de mayor riqueza, para su utilización en los cursos de la escuela secundaria?

P.I. 4 ¿Qué características didácticas pueden describirse para las tareas diseñadas por los futuros profesores, asociadas a un determinado problema inverso obtenido a través de la reformulación de un cierto problema directo de modelización?

## Objetivos de la investigación

Teniendo en cuenta los antecedentes ya mencionados, así como las posibles conjeturas y preguntas de investigación, para esta tesis doctoral se proponen los siguientes objetivos generales:

OG.1 Identificar y caracterizar las estrategias de los futuros profesores para plantear problemas inversos para los cursos de secundaria, mediante la reformulación de un problema directo dado.

OG.2 Estudiar, analizar y caracterizar las producciones de los futuros profesores sobre el análisis didáctico de tareas asociadas a los problemas inversos, considerando su relación con las tareas originales vinculadas al problema directo de donde proceden.

De estos objetivos generales surgen los siguientes objetivos específicos:

Objetivos específicos relacionados con OG.1

O.E. 1 Caracterizar los enunciados de las reformulaciones planteadas en forma inversa por los futuros profesores.

O.E. 2 Caracterizar la complejidad del proceso de resolución de los problemas inversos propuestos por los futuros profesores.

Objetivos específicos relacionados con OG.2

O.E. 3 Caracterizar los significados de los conceptos matemáticos que utilizan los profesores cuando diseñan tareas por reformulación de un problema directo dado.

O.E. 4 Caracterizar los aspectos cognitivos (expectativas, errores y demandas cognitivas), que aparecen en las tareas de los futuros profesores relacionadas con los problemas reformulados.

O.E. 5 Caracterizar los elementos de instrucción, enfocados en las variables de la tarea que utilizan los docentes cuando reformulan un problema dado.

## MARCO TEÓRICO

### La formación de futuros profesores y el análisis didáctico

Articulamos los contenidos didácticos para la formación de los docentes de matemáticas como profesionales, de acuerdo con las cuatro dimensiones de cualquier currículo de matemáticas de la siguiente manera: contenidos conceptuales, contenidos cognitivos, contenidos instruccionales y contenidos evaluativos (Rico, & Ruiz-Hidalgo, 2018). Cada uno de dichos contenidos centra su objeto de estudio en una cierta modalidad o perspectiva de la educación matemática, es decir: su significado; su intencionalidad; su planificación y la toma de decisiones sobre su empleo.

Cada dimensión considera ciertas categorías de análisis con diferentes criterios y utilidades. En cada caso, esas categorías y conceptos ayudan a identificar los componentes y temas que utilizaremos para organizar las propuestas y documentos específicos a estudiar. Así, los contenidos matemáticos se estructuran a través de temas, conceptos y procedimientos; sistemas de representación; contextos y modos de uso. Los contenidos cognitivos están organizados por expectativas; limitaciones y oportunidades de aprendizaje.

Para este estudio destacamos las tareas y las secuencias de tareas como la forma básica de identificar contenidos de la instrucción; describir dicho análisis en términos de organización del trabajo en el aula; materiales y recursos, y componentes. Se identificarán además sus de variables de tarea; su complejidad, creatividad y tipos característicos. Este estudio no considerará contenidos y componentes de la dimensión curricular evaluativa.

Cabe destacar que la instrucción escolar constituye una dimensión básica en el análisis de los contenidos didácticos de cualquier propuesta curricular, necesaria para llevar a cabo los procesos de análisis y formación inicial de los futuros profesores de matemáticas, así como para el desarrollo y logro de la competencia profesional, ampliamente desarrollada en la *oportunidad de aprender* (Cogan y Schmidt, 2014, pp. 207-220).

Los expertos establecen tres tipos de categorías para profundizar en los contenidos didácticos que se trabajan en esta dimensión instruccional, entre las que destacamos preferentemente las tareas y su secuenciación; las reglas para la planificación y organización del trabajo escolar; y los materiales y recursos escolares. A su vez, cada una de estas categorías se describe en términos de componentes a través de los cuales se realiza el escrutinio didáctico de las categorías antes mencionadas (Rico & Ruiz-Hidalgo, 2018).

Este capítulo parte de la descripción de algunos componentes estructurales –nociones, conceptos y procedimientos– del programa de estudios del Máster para Profesores de Matemáticas de Enseñanza Secundaria de la Universidad de Granada (Rico, Fernández-Cano, Castro & Torralbo, 2008, pp. 203-211). En particular, en la primera sección del capítulo, se describen las asignaturas específicas correspondientes de la especialidad matemática, ya que resulta fundamental para este trabajo.

Dentro de asignaturas específicas de dicho Máster nos centraremos en la denominada "Aprendizaje y enseñanza de las matemáticas". Esta asignatura se basa en el Análisis Didáctico, que además es la metodología de análisis seguida en esta tesis y desarrollada en el segundo apartado del capítulo.

El Análisis de Instrucción es la tercera dimensión del Análisis Didáctico, enfocado en tareas matemáticas y resolución de problemas. Está fuertemente ligada a las dos dimensiones anteriores, la Conceptual y la Cognitiva, especialmente importantes para este trabajo, debido al papel fundamental que juegan en el enriquecimiento de las tareas matemáticas. En referencia a esto último, cabe mencionar que la tercera sección de este capítulo está íntegramente dedicada al enriquecimiento de las tareas.

En particular, el planteamiento de problemas y su invención son estrategias importantes para la formación en educación matemática de futuros profesores y, al mismo tiempo, son contenidos fundamentales para el enriquecimiento de las tareas. Debido a estos hechos, la cuarta sección del capítulo se centrará especialmente en la competencia para la invención de problemas.

Es particularmente destacable el papel de los problemas inversos como una estrategia para el planteo de nuevos problemas y a su vez, para contribuir al enriquecimiento de las tareas asociadas. Por este motivo, a lo largo de la última sección del capítulo, se analizarán las características del enriquecimiento de tareas que generalmente tienen lugar en la proposición de problemas inversos.

**Los problemas inversos**

En una primera aproximación, se puede mencionar que los problemas se pueden plantear en forma directa o inversa. Según Groestch (1999, 2001), los problemas directos son aquellos en los que se proporciona la información necesaria para ejecutar un proceso bien definido que nos lleva a una única solución.

Globalmente, los problemas inversos se pueden clasificar en dos subgrupos, los problemas de causalidad y los de especificación. En los problemas de causalidad, el procedimiento es bien conocido y la pregunta es sobre los datos necesarios para obtener un resultado determinado. Un ejemplo de problema de causalidad es la determinación de la función $F(x)$ que por el proceso de derivación permite obtener una función dada $f(x)$ (es decir, encontrar una función primitiva, o integral indefinida).

Los problemas de especificación son aquellos en los que se dan tanto los datos como los resultados y la pregunta es sobre el procedimiento que puede, a partir de esos datos, llegar al resultado deseado. Un ejemplo de problema de especificación tiene lugar cuando se le pide al estudiante que demuestre una propiedad. En este caso, tanto la hipótesis y la tesis son bien conocidas (de hecho, forman parte del enunciado del problema) y lo que se solicita es el razonamiento que permita llegar a la tesis (resultado), partiendo de la hipótesis (dato).

Cabe mencionar que diversos esquemas, tanto para los problemas de causalidad como los de especificación, fueron discutidos en profundidad en un artículo anterior (Martinez-Luaces, Fernández-Plaza, Rico & Ruiz-Hidalgo, 2019).

Los problemas tradicionalmente inversos han sido subestimados tanto por la educación matemática como por los propios matemáticos. Además, teniendo en cuenta que muchos de los problemas que debemos afrontar a diario son problemas inversos, este aparente olvido es aún más sorprendente y solamente parece estar quedando atrás en tiempos bastante recientes (Bunge, 2006).

Groestch (1999, 2001) ha señalado que los problemas directos han dominado prácticamente los cursos tradicionales de matemáticas, aunque reconoce que en un currículo moderno los problemas inversos deberían tener un papel importante. Entre sus argumentos, menciona que los problemas inversos son más adecuados para explorar cuestiones de existencia y unicidad, así como la estabilidad de las soluciones. Por otro lado, estos problemas acercan los cursos a las situaciones que se presentan en la vida real y en la práctica profesional de diversas carreras.

En un trabajo anterior (Martínez-Luaces, 2011) se propusieron varios ejemplos que ilustran cómo se puede aplicar los problemas inversos en cualquier nivel educativo, desde la escuela primaria hasta la universidad.

Por último, un ejemplo muy sencillo de problema directo de aritmética elemental consiste en sumar dos números primos impares para obtener –obviamente– un número par. Sin embargo, resulta más interesante analizar el problema inverso correspondiente: ¿es siempre posible descomponer un número par como la suma de dos números primos? Esta pregunta da lugar a la llamada "conjetura de Goldbach", que fue postulada en 1742 y su solución sigue siendo esquiva más de 250 años después.

Un problema directo es definitivamente un problema que los estudiantes pueden potencialmente encontrar una manera de resolverlo y familiarizarse con los procesos y conceptos involucrados. Por otro lado, un problema inverso derivado de uno directo, conduce a un nuevo problema, con múltiples soluciones o insoluble, e involucra procesos y conceptos más complejos y una comprensión más profunda del proceso y de los conceptos involucrados en el problema directo original.

Finalmente, los adjetivos "directo" e "inverso" son relativos y dependen de la perspectiva del solucionador, es decir, cuando los estudiantes se familiarizan con el proceso de solución de problemas inversos, estos pueden pasar a ser problemas directos, susceptibles de posteriores modificaciones inversas.

## METODOLOGÍA

En este capítulo se describe la metodología implementada para lograr los objetivos de la investigación, previamente descritos en el Capítulo 1. Para alcanzar el primer objetivo (OG1), se propusieron dos estrategias diferentes en la primera etapa de esta investigación:

- En primer lugar, se proporcionó a los participantes un problema de modelización sobre el llenado de una piscina y se les solicitó que lo reformularan con el objetivo de enriquecer las tareas de los cursos de Secundaria. El objetivo principal de esta tarea fue explorar los conocimientos previos y las estrategias utilizadas por los profesores para enriquecer una tarea determinada, y si la inversión se daba o no de forma espontánea.
- En segundo lugar, se les dio a los futuros profesores otro problema sobre una oveja pastando en un campo cuadrado, y en dicha oportunidad se les solicitó especialmente una reformulación inversa. El objetivo principal de esta tarea fue obtener una reflexión más profunda sobre el enriquecimiento del problema dado, específicamente a través de su inversión, luego de un entrenamiento específico sobre la reformulación inversa de un problema dado.

En cuanto al segundo objetivo general (OG2), en la primera etapa se entregó una hoja de trabajo a los participantes, con el fin de realizar el análisis didáctico de las tareas asociadas a su propio problema reformulado.

Cuando se analizaron las reformulaciones del primer problema, se encontró que en general, la inversión no constituye una estrategia espontánea de enriquecimiento de tareas entre los futuros docentes que participaron del estudio. Asimismo, en el análisis de las respuestas del segundo problema, se observó que un gran número de profesores en formación tendía a imitar los ejemplos dados previamente, y un número importante de reformulaciones daban lugar a propuestas triviales o simplemente estaban mal planteadas. En otras palabras, la calidad de las respuestas no tuvo la riqueza que se esperaba para obtener conclusiones importantes sobre el OG1. Además, en lo que respecta a la consecución del segundo objetivo general (OG2), se observó que los futuros docentes expresaron sus opiniones sobre varios ítems que llamaron su atención, pero dejaron de lado muchos de los elementos considerados en la tabla 2.1.

Por estos motivos, se propuso una segunda etapa de investigación en la cual se introdujeron varias modificaciones en la formación impartida a los futuros docentes, para evitar o al menos atenuar los inconvenientes de la primera etapa, por ejemplo, se analizaron ejemplos de reformulación inversa en otros contextos distintos al del problema de la oveja. Por otra parte, se observó que los conocimientos didácticos de los futuros profesores no podían ser analizados profundamente debido a los formularios que entregaron estaban bastante incompletos. Además, al poder disponer en una segunda etapa de más datos y de mayor calidad, se modificó profundamente el tratamiento de los mismos, incluyendo un análisis cluster para las reformulaciones propuestas, con el fin de estudiar mejor la riqueza de las mismas.

Este capítulo referido al marco metodológico comienza caracterizando el tipo de investigación realizada como estudio exploratorio de tipo descriptivo. La fundamentación de esta caracterización –además de otras generalidades– constituye la primera sección del capítulo (Sección 3.1).

Los problemas seleccionados para el diseño del trabajo de campo, es decir, el problema de la piscina y el problema de la oveja, forman el núcleo de la siguiente sección del capítulo (Sección 3.2).

El trabajo de campo de esta investigación se implementó en dos etapas diferentes y la tercera sección del capítulo (Sección 3.3) está dedicada a la metodología correspondiente a la primera de estas dos etapas. En dicha sección se incluyen los criterios para la elección de la muestra, el diseño del trabajo de campo, la descripción de los instrumentos utilizados para la recogida de información, la elección y organización de los datos, junto con los criterios que abordan el análisis de las respuestas y producciones de los futuros profesores. En cuanto al análisis de las producciones, se explican en detalle las categorías tanto para el análisis de los problemas reformulados como para el posterior análisis didáctico de las tareas correspondientes.

En la última parte del capítulo (Sección 3.4) se analiza la segunda etapa de la investigación desde un punto de vista metodológico. Para ello, una vez más, se describe la elección de la muestra, el diseño del trabajo de campo y las categorías utilizadas en el análisis de las producciones. En este nuevo estudio, los problemas reformulados de los participantes se analizan, codifican y clasifican, con el fin de obtener una información coherente para un análisis de clusters. Dicho análisis se propuso con el objetivo de obtener más información sobre las diferentes estrategias utilizadas por los futuros profesores en la reformulación de un determinado problema, de acuerdo a lo propuesto por el O.G. 1.

Finalmente, para el análisis cognitivo e instruccional de las tareas enriquecidas, se utilizaron las mismas categorías, convenientemente refinadas, que se utilizaron en la primera etapa del estudio.

**Los problemas seleccionados para el trabajo de campo**

Para esta investigación se eligieron dos problemas: el primero sobre el llenado de una piscina rectangular de profundidad variable y el segundo referido a una oveja que se encuentra pastando en un campo rectangular. Los enunciados de estos problemas son los siguientes:

*El volumen de agua en una piscina*

Una piscina tiene 3 m de profundidad en la parte más profunda y 1 m de profundidad en el extremo menos profundo. Las dimensiones horizontales de la piscina son de 40 m por 20 m. Finalmente, si $h$ es la altura del agua en el extremo más profundo, el típico problema de modelado directo consiste en obtener el volumen de agua, $V$ ,en función de la altura, $h$.

El diagrama correspondiente a este problema se muestra en la Figura 3.1.

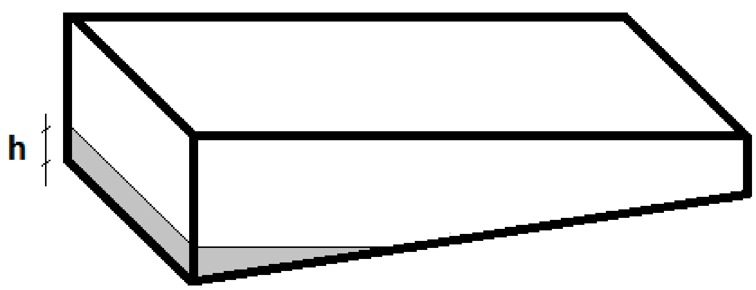



Figura 3.1. Diagrama de la piscina

Otro diagrama, que se muestra en la Figura 3.2, puede ser de utilidad para la resolución de este problema. En este segundo diagrama, la línea recta que pasa por los puntos $(0,-3)$ y $(40,-1)$ representa el fondo de la piscina y la línea punteada horizontal representa el nivel del agua, dado por la función $y = 3 - h$. El punto de intersección de estas dos líneas se puede calcular fácilmente y utilizando conocimientos elementales se puede hallar el área del triángulo sombreado en gris en la Figura 3.2. Finalmente, a partir de dicha área es fácil obtener el volumen de agua en la piscina de la Figura 3.1.

Un problema más interesante –y más relacionado con la vida real– consiste en obtener la altura del agua en el extremo más profundo en función del tiempo, cuando la piscina se está llenando de agua a un caudal de 0,8 metros cúbicos por minuto. En particular, un problema inverso motivador consiste en calcular cuánto tiempo se requiere para obtener la altura deseada en el extremo profundo de la piscina. De hecho, este es el verdadero problema de la vida real para los propietarios de piscinas residenciales y comerciales.

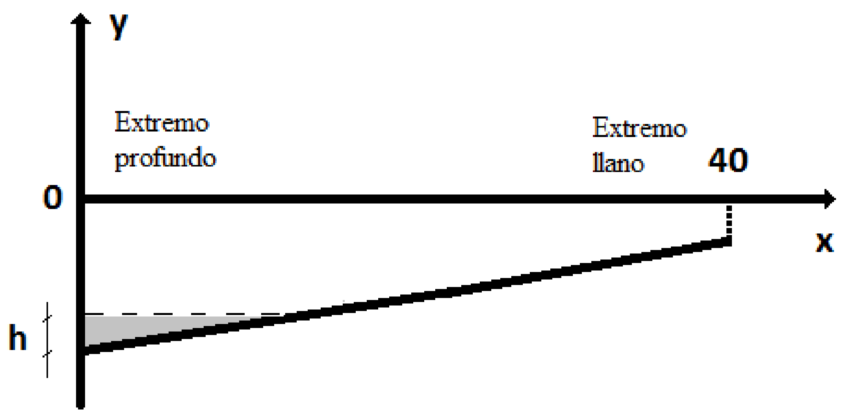



Figura 3.2. Otro diagrama para el problema de la piscina

Este es un ejemplo de un problema de modelización inverso, que se puede resolver utilizando solo conocimientos y habilidades propias del pre-cálculo.

El problema de la oveja

Consideremos una oveja que pasta en un cuadrado, siendo $L$ la longitud del lado del terreno. La oveja se ata con una cuerda de largo $R$, en el punto $(L/2\ ,\ 0)$, como se puede observar en la Figura 3.3.

En dicha figura, $A$ representa el área accesible para la oveja, $r = \frac{R}{L}$ es la relación entre la longitud de la cuerda y la longitud del lado del terreno y $f = \frac{A}{L^2}$ representa la fracción del área total que resulta accesible para el animal. Obviamente, $f$ es una función de la razón $r$, que puede obtenerse fácilmente utilizando técnicas de integración bien conocidas.

Es posible plantear un típico problema directo, solicitando el valor de $f$ para a un valor dado de la razón $r$. Dicho problema proviene de un capítulo de libro (Martínez-Luaces, 2016) en el que se observa que es posible resolver dicho problema simplemente por intersección de círculos y cuadrados.

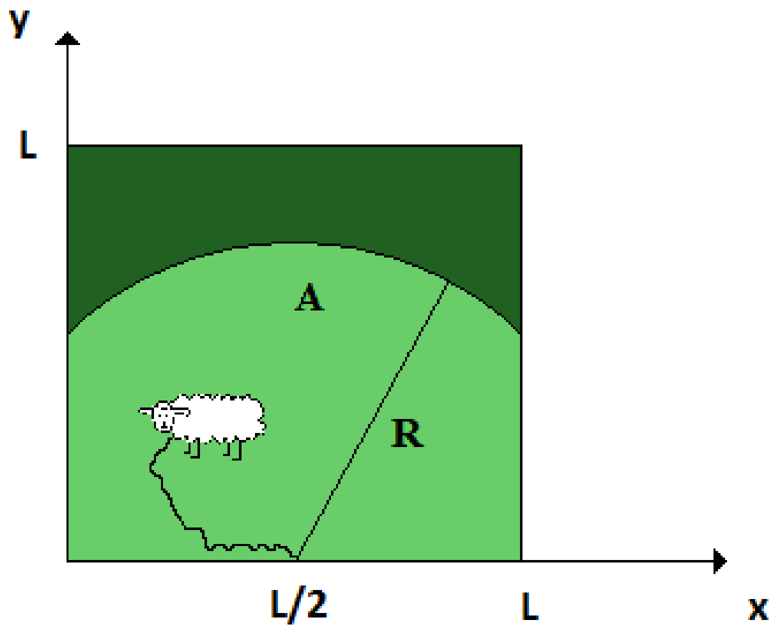



Figura 3.3 Zona de césped accesible a la oveja

Considerando las intersecciones, se pueden observar cuatro casos diferentes:

- En el primero $r \in [0,1/2)$ y la oveja no puede llegar a los bordes laterales del campo.
- En el segundo caso $r \in [1/2,1)$ y la oveja es capaz de llegar a los bordes laterales, pero no al superior.
- Si $r \in \left[1, \frac{\sqrt{5}}{2}\right)$, la oveja llega al borde superior del terreno, pero no puede pastar en todo la extensión.

- En el último caso, si $r \geq \frac{\sqrt{5}}{2}$, la oveja pasta sin restricciones en todo el campo.

Utilizando integrales –entre otras posibilidades que se analizarán en las próximas secciones– se puede resolver este problema de modelización directo.
Una observación obvia es que existe un valor único de $A$, para cada valor de $r \geq 0$. Sin embargo, una pregunta más interesante podría ser: para cualquier valor dado $A$, ¿existe siempre un valor correspondiente de $r$?, y si es así, ¿es el único posible? Para dar respuesta a estas preguntas, se debe estudiar la continuidad y crecimiento estricto de la función $f(r)$, cuando $r \geq 0$, de modo de asegurar la existencia de la función inversa.

*Algunos comentarios sobre la riqueza de problemas seleccionados*

Ambas propuestas – los problemas de la piscina y de la oveja– pueden considerarse tareas de calidad intelectual y académica, por múltiples motivos. Ambos problemas permiten trabajar el trabajo en grupo y prestan atención a la diversidad, ya que ambos tienen múltiples vías de solución y posibles puntos de entrada. En particular, el problema de la oveja se puede resolver mediante integrales o utilizando solamente conceptos y procedimientos trigonométricos.
Como veremos más adelante, ambos problemas son capaces de generar interés entre los futuros profesores, permiten el uso de diferentes representaciones y establecer conexiones entre diversas áreas de la matemática, así como la conexión con otras asignaturas.

Volviendo al problema de la oveja, cabe destacar que tiene múltiples soluciones (analíticas, trigonométricas, o una mezcla de ambas) y la permite la utilización de diversas representaciones, particularmente la simbólica y la pictórica. Esto cumple perfectamente con dos de los criterios de Grootenboer (2009), ya que atiende la diversidad y permite varias representaciones. Además, la tarea propuesta a los participantes (resolver el problema directo, reformularlo a la inversa y resolver este nuevo problema) permite obtener varios resultados en una sola tarea. Finalmente, dado que la tarea propuesta incluye una reformulación inversa propuesta por el participante, tiene cierto carácter abierto. Lo anterior cubre varios de los aspectos importantes mencionados por Clarke y Clarke incluidos en la lista de la sección correspondiente a la caracterización de tareas ricas.

## Primera etapa de la investigación

### Participantes del estudio

En la primera etapa de la investigación, los participantes fueron futuros profesores que cursaban la asignatura “Enseñanza y aprendizaje de las matemáticas“, del Máster de Profesorado de Educación Secundaria de la Universidad de Granada. En dicha primera etapa, 33 estudiantes formaban el grupo A y el grupo B contaba con 41 alumnos, con asistencia regular a los cursos. Para el trabajo de campo se contó con la invalorable colaboración de los profesores de los mismos: Rico y Moreno.

### Datos recopilados sobre el problema de la oveja

Por razones de espacio no se describen los resultados del trabajo con el problema de la piscina, pero algunos de ellos pueden consultarse en (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2019).

Durante la primera etapa, se pidió a los participantes que presentaran un formulario con su reformulación inversa del problema dado, complementado con una planilla para la comparación de ambos enunciados. Es importante observar que en la primera etapa de la investigación no se solicitó la resolución del problema inverso planteado.

Cabe hacer varias observaciones sobre el total de respuestas obtenidas en ambos grupos de futuros profesores. En primer lugar, no todas las respuestas obtenidas correspondieron a reformulaciones inversas del problema dado, por lo que no se han tenido en cuenta todas las producciones para este estudio. Por el contrario, hubo varios futuros profesores que propusieron más de una reformulación, de hecho hubo participantes que propusieron hasta tres reformulaciones inversas.

Además, cabe mencionar que algunos participantes en la primera etapa del estudio trabajaron individualmente mientras que otros trabajaron con un compañero, por lo que no existe una propuesta para cada futuro profesor, ni cada participante tiene necesariamente una única respuesta. Por estas razones, las producciones se han codificado como respuestas (R1, R2, etc.), en lugar de codificarse por autor. Cabe mencionar que originalmente estas producciones habían sido codificadas como A1, A2, ... B1, B2, ... según el grupo considerado (A o B) y seguidas de un número que representa a cada uno de los participantes.

En cuanto al análisis didáctico, los participantes han respondido sobre significados, autenticidad, elementos de la tarea y variables de la tarea.

En consecuencia, sus opiniones con respecto a cada una de las categorías de análisis didáctico, en ocasiones se pueden obtener directamente de las respuestas, mientras que en otros casos son el resultado del contexto. Finalmente, en muchas de estas categorías no existe una opinión concreta y precisa.

*Elementos conceptuales y procedimentales relacionados con los problemas reformulados*

- Concepto/procedimiento involucrado: Clasificamos las propuestas según un determinado campo conceptual y procedimental.
- Estructura del problema: En este estudio solo se consideraron reformulaciones inversas, sin embargo en algunos casos es necesario resolver primero un problema directo y luego utilizar el resultado obtenido como insumo para resolver el problema inverso.
- Datos y Respuesta solicitada: En los problemas reformulados, los datos suministrados no son especialmente relevantes, ya que no sirven para discriminar entre las distintas propuestas. En cambio, en cuanto a la respuesta requerida, se observaron cinco variantes: la longitud de la cuerda, el lado del campo, la posición de la estaca, el tiempo para consumir toda la hierba disponible y la velocidad de la oveja.
- Variables de contexto: Existen propuestas que hacen cambios en la Geometría, mientras que otras incluyen nuevas variables que pueden ser consideradas como externas al problema directo y finalmente, en varias reformulaciones no hay cambios en esos aspectos.

En el próximo capítulo, estos criterios se utilizarán para analizar y ejemplificar los nueve grupos que habían aparecido en la primera etapa de este estudio.

El instrumento utilizado para ordenar, organizar y analizar las producciones de los futuros profesores, dio lugar a una planilla con un total de 21 columnas, las cuales se describen brevemente a continuación:

- Las respuestas se ubicaron en una primera columna, codificadas como R1, R2, etc., donde también se agregó otro código que indica el grupo y número de estudiante (por ejemplo, A1, A2, ..., B1, B2, etc.) entre paréntesis. Es importante mencionar que en algunos casos el mismo alumno propuso dos e incluso tres reformulaciones distintas del problema dado, y en ese caso se agregó una letra minúscula (por ejemplo, la respuesta R1 la dio el alumno A14a).
- Las columnas 2 y 3 –bajo el título común de “reformulación” - corresponden al tipo de enunciado (columna 2) y el cambio realizado (columna 3). En el primero se establece el tipo de enunciado (por ejemplo: función inversa, problema secuencial, etc.) y en el segundo se especifica qué tipo de cambio se propone (por ejemplo: cambio en la geometría del campo, en la posición de la estaca, etc.).
- Las columnas 4, 5, 6 y 7 –todas ellas bajo el título común de “análisis de significados” - corresponden al contenido matemático (columna 4), sistemas de representación (columna 5), sentido y modos de uso (columna 6). ) y situación (columna 7).

En cuanto al contenido matemático, las respuestas comunes son: áreas y regiones, integrales, etc. Los sistemas de representación suelen incluir las expresiones simbólicas y gráficas como opciones más típicas. En lo que respecta a sentido y modos de uso, la respuesta más común es "espacio y forma", aunque también aparecen otras respuestas singulares. Finalmente, si se considera la situación, la respuesta más común es "educativa/ocupacional", pero esta no es la única que se obtuvo. Todas estas respuestas se analizan con más detalle en el capítulo de resultados.

- El análisis cognitivo ocupa las siguientes cinco columnas de la planilla: contenido cognitivo (columna 8), expectativas de aprendizaje (columna 9), limitaciones al aprendizaje (columna 10) y oportunidades de aprendizaje, que se dividió en dos columnas denominadas "reto" (columna 11) y “comentarios” (columna 12). En la columna correspondiente al contenido cognitivo, las respuestas de algunos futuros profesores comienzan con un verbo, por ejemplo: “calcular áreas mediante integrales” o “controlar conceptos”, entre otras opciones. Las expectativas y limitaciones de aprendizaje no sufrieron cambios importantes. Las oportunidades de aprendizaje se dividieron en dos columnas ya que muchos de los participantes hicieron comentarios generales sobre el reto (desafío interesante, mayor reto, etc.) y en muchos casos agregaron algunos comentarios adicionales para justificar su respuesta. Varias respuestas largas e interesantes sobre oportunidades de aprendizaje justificaron dividir la columna, para extraer toda su riqueza sin perder información relevante.

- El análisis de la instrucción finalmente se dividió en ocho columnas (de la 13 a la 20) para evitar la pérdida de información proporcionada por los participantes. La primera de estas columnas se dedicó al "lenguaje" (columna 13), donde los futuros profesores expresaron sus opiniones sobre la claridad, simplicidad, etc., del lenguaje utilizado en el planteamiento del problema. En la siguiente columna, denominada “autenticidad” (columna 14), los participantes expresaron sus opiniones sobre si la propuesta corresponde a una situación real o no. La tercera, denominada "datos" (columna 15), se dedicó a los comentarios de los participantes sobre los datos proporcionados, por ejemplo, si son suficientes o no. En la columna “propósito” (columna 16), los participantes expresaron sus opiniones sobre el propósito de la tarea, por ejemplo: conectar ideas. La siguiente columna, "situación de aprendizaje y agrupación" (columna 17), podría haberse dividido en dos, sin embargo, dado que la mayoría de las respuestas eran muy breves (por ejemplo: "Grupo. En clase"), se mantuvieron como una sola columna. La columna 18 “temporalización” también dio lugar a respuestas muy breves, como “una sesión”, o “media hora”, entre otras. La columna 19, sobre "complejidad", ha sido previamente analizada y lo mismo ocurre con la columna 20, "materiales y recursos".
- La última columna se utilizó para las observaciones. Por ejemplo, un participante propuso su reformulación del problema en cuestión, sin aportar el análisis comparativo correspondiente, por

lo que se incluyó la observación correspondiente en esta última columna.

## La segunda etapa del estudio

### Participantes del estudio

En la segunda etapa de la investigación, en el año 2019, los participantes fueron futuros docentes que cursaban la asignatura “Enseñanza y aprendizaje de las matemáticas“, del Máster de Profesorado de Educación Secundaria de la Universidad de Granada. En dicha etapa, 32 estudiantes integraban el grupo A y 33 alumnos formaban el grupo B. En ambos grupos, los futuros profesores asistieron regularmente a los cursos. Para el trabajo de campo se contó con la invalorable colaboración de los profesores de los mismos: Moreno y Ruiz-Hidalgo.

### Diseño de la segunda etapa.

Durante el desarrollo del diseño experimental, en 2018, se observó la necesidad de un segundo experimento, debido a las deficiencias identificadas en la primera etapa. De hecho, durante la primera etapa, cuando se propuso el problema de la piscina a los futuros profesores para que lo reformularan libremente, se observó que solo unos pocos participantes propusieron problemas inversos. Se puede entonces observar que la reformulación inversa no es espontánea para la mayoría de los futuros profesores (Martínez-Luaces, Fernández-Plaza & Rico,

2020) y aún en los casos en que se utilizó dicha estrategia, los problemas reformulados no mostraron mucha riqueza conceptual.

Por otra parte, se observó que los futuros profesores tendían a imitar los modelos dados, aunque también surgieron unas pocas reformulaciones creativas. Asimismo, se observó una cantidad importante de problemas mal planteados o que su solución se limitaba a aplicar una fórmula, trivializando la propuesta. En resumen, la calidad de las respuestas obtenidas no fue tan buena como se esperaba, a fin de obtener conclusiones importantes para el objetivo general OG1.

En cuanto al objetivo general OG2, se entregó una planilla a los futuros profesores, con el fin de realizar un análisis didáctico de las tareas asociadas a su propia reformulación. En cuanto al logro de este segundo objetivo general, se pudo observar que los futuros docentes opinaron sobre varios ítems que llamaron su atención, dejando de lado muchos otros que por algún motivo no eran tomados en cuenta.

Por estos motivos, se propuso una segunda etapa de la investigación, incluyendo un nuevo trabajo de campo diseñado para disminuir la presencia de reformulaciones directas y los problemas mal propuestos. También se analizaron algunos ejemplos de reformulaciones inversas de problemas tomados de otros contextos, de manera que no facilitar las posibles imitaciones.

En resumen, el nuevo diseño mostró algunas diferencias con el anterior y cuatro de ellas son particularmente importantes en cuanto al diseño experimental:

- En esta nueva investigación, se pidió a los participantes que dieran una solución al problema directo y luego, propusieran la reformulación inversa correspondiente.
- Antes de la propuesta del problema de la oveja, se discutieron otros ejemplos de problemas inversos, ninguno de los cuales estaba relacionado con dicho problema. Esta decisión se tomó para evitar la adaptación o simple imitación de un procedimiento anterior.
- Se solicitó a los participantes la solución del problema reformulado, o al menos, hicieran un boceto de la posible solución. Esto se propuso con el objetivo de disminuir el número de problemas mal planteados.
- Previamente se entregó a los participantes una planilla para el análisis del problema directo y otra para el inverso, las cuales fueron acompañadas de una tabla que explicaba brevemente los descriptores sobre los que se solicitaba la opinión a los participantes que. Con esta medida se esperaba que los participantes pudieran responder sobre todos los aspectos de la tabla mencionada, o al menos sobre aquellos que más les llaman la atención en el estudio comparativo de las tareas asociadas a ambos problemas.

Esta nueva experiencia con un diseño diferente produjo una cantidad mayor de respuestas que serán analizadas en la sección de resultados.

Los criterios utilizados para clasificar las reformulaciones fueron los siguientes:

*Tipo de inversión*

Todas las propuestas seleccionadas son problemas inversos; algunos de ellos son problemas de causalidad en los que se proporcionan valores particulares y otros son de tipo general ya que no se dan valores numéricos.
Entre los problemas de especificación los hay de croquis (si piden dibujar o interpretar un cierto esquema) y de proceso, como encontrar un punto anguloso o interpretar el significado de un determinado parámetro.

*Datos*

Los datos del problema que se proporcionan al estudiante, pueden ser valores particulares o generales y por último, también es posible que el dato dado sea una integral.

*Resultados*

Las resultados que se solicitan pueden ser valores particulares, o generales. También es posible que se solicite un croquis o incluso un proceso, por ejemplo, encontrar el área accesible a la oveja por otro método sin utilizar integrales y comparar resultados.

*Cambio de geometría*

Hay algunas propuestas que incluyen cambios en la geometría del problema. Una opción es que cambien el cuadrado original por una otra figura geométrica o puede que la forma del campo sea la misma, pero la estaca a la cual está atada la oveja se ubique en un punto diferente y finalmente, hay casos en que hay cambios en ambos aspectos.

*Otros elementos*

Algunos futuros profesores agregan otras variables, que pueden ser variables adicionales no relacionadas con el problema original (costos, cantidad de fertilizante, etc.), o pueden cambiar el contexto del problema (un campo de fútbol, un incendio forestal en una región determinada, etc.). En este último caso, desde un punto de vista matemático el problema es básicamente el mismo, con un contexto diferente. También puede haber cambios de los dos tipos antes descritos.

*Solución*

Las soluciones dadas son bastante variadas y podríamos decir que se pueden clasificar según su contenido matemático en cuatro subgrupos: geométricos, analíticos, numéricos y otros. Entre las soluciones geométricas, existe un primer caso donde solo se utiliza geometría elemental, mientras que otras utilizan funciones trigonométricas y trigonométricas inversas. Las soluciones analíticas se pueden clasificar en dos grupos: las que calculan integrales y ocasionalmente utilizan resultados elementales de dicho tema y las que utilizan teoremas avanzados, como el vincula la área bajo la curva con la longitud de arco correspondiente (bajo ciertas hipótesis especiales). En otros casos, el problema reformulado conduce a una ecuación no lineal que se resuelve utilizando métodos numéricos con o sin la ayuda de tecnología.

Por último, hay soluciones que no se pueden clasificar en los grupos anteriores y aparecen fundamentalmente en las reformulaciones triviales, aunque no solo en esos casos. Por ejemplo, también hay algunas propuestas que se resuelven por derivación, aunque en esos casos es importante confirmar si la derivada es realmente útil para la resolución del problema. Adicionalmente, algunas propuestas desde un punto de vista analítico deben ser consideradas como problemas directos (como encontrar un área mediante el uso de integrales), pero que sin embargo son problemas inversos de proporciones.

*Dificultad*

Como observación final obvia, hay propuestas que presentan diferentes grados de dificultad. En primer lugar tenemos problemas triviales, que se pueden reducir para calcular la distancia entre dos puntos, o aplicar una fórmula determinada. Otras pueden ser fáciles porque implican un procedimiento simple, como la derivación, o el uso de un resultado intuitivo, pero al menos requieren algo de reflexión y/o ejecutar un determinado procedimiento sencillo. Estos casos se clasifican como propuestas de baja dificultad. En cambio, una integral complicada, de una función irracional, seguida de una ecuación no lineal en $R$, se considera una propuesta de alta dificultad. Aquellas propuestas que no sean ni tan difíciles ni tan sencillas se marcarán con “1” en la casilla de dificultad media. En lo que respecta a la solución, a diferencia de otras partes del cuadro, las diferentes opciones se excluyen mutuamente. Es posible ver una solución que utilice integrales y trigonometría, pero no puede ser una propuesta trivial y de dificultad media al mismo tiempo.

Teniendo en cuenta los criterios anteriores, se puede hacer una clasificación Booleana donde se pone un “1” en cada casilla que corresponda y “0” en las restantes. Por último, esta planilla de ceros y unos se utilizó como datos de ingreso para la realización de un análisis cluster, cuyas características son las siguientes:

- Programa utilizado: IBM SPSS Statistics v.24 ® (Stehlik-Barry & Babinec, 2017).
- Métrica utilizada: Dice Similarity Coefficient.
- Método de agrupamiento: Average linkage (between groups).

Finalmente, en lo que refiere al análisis de las tareas, el instrumento fue el mismo que se utilizó en la primera etapa de la investigación.

## DISCUSIÓN Y CONCLUSIONES

En primer lugar se describen y analizan los resultados correspondientes a las reformulaciones del problema de la oveja durante la primera etapa del estudio.

### Análisis de las reformulaciones propuestas en la primera etapa

En el análisis de las reformulaciones propuestas por los futuros docentes, fue posible identificar nueve grupos diferentes, algunos de ellos divisibles en varios subgrupos. Específicamente, estos grupos son:

- Grupo 1. Reformulaciones basadas en la inversión de la función.
    - Subgrupo 1a. Reformulación basada en la función inversa, sin otras modificaciones
    - Subgrupo 1b. Función inversa y cambio en la geometría del campo.

- Subgrupo 1c. Función inversa y cambio de posición de la estaca.
- Subgrupo 1d. Función inversa y cambio de geometría y posición de estaca.
- Subgrupo 1e. Función inversa y campo con obstáculos.

- Grupo 2. Problema elemental
  - Subgrupo 2a. Problemas que se pueden reducir a la aplicación de una fórmula
  - Subgrupo 2b. Problemas que solo requieren manipulaciones algebraicas simples
  - Subgrupo 2c. Problemas de reformulación elementales, con el agragado de otras variables externas
- Grupo 3. Problema inverso de ubicación de la estaca
- Grupo 4. Problema inverso sobre la longitud del campo
- Grupo 5. Problema de optimización
- Grupo 6. Problema inverso secuencial
- Grupo 7. Problema incremental
- Grupo 8. Problema dinámico
- Grupo 9. Problema de iso-superficie

En el primer grupo se utilizan procedimientos del cálculo integral para obtener el área de pastura $A$, o la fracción de área $f$ en función de $R$ o de la razón $r$ y luego se trata simplemente de invertir la función, por ejemplo, dando la longitud de la cuerda que permite a la oveja pastar en la mitad del terreno. En las variantes 1b, 1d y 1e se cambia la geometría al modificar la forma del terreno y/o cambiar la posición de la estaca, o eventualmente agregar obstáculos, que hacen que la cuerda describa una poligonal. La propuesta del subgrupo 1c, termina dando origen a un problema trivial, por lo que se describe luego, en el grupo 2. En cuanto al número de reformulaciones, el grupo 1 incluye 22 propuestas, que se pueden dividir de la siguiente manera: 16 propuestas para el subgrupo 1a, 2 para el 1b y 1e y solo 1 para los subgrupos 1c y 1d.

Las reformulaciones del grupo 2, también piden la longitud de la cuerda, pero trivializan el problema, por ejemplo, solicitando $R$ para que la oveja pueda pastar en todo el campo. En dicho problema, basta aplicar la fórmula de la distancia entre dos puntos (subgrupo 2a), mientras que en otros casos hay que hacer un despeje sencillo de cierta variable (2b) o hacer una proporción sencilla (2c) con alguna variable externa (consumo de pasto, cantidad de fertilizante, etc.). Este grupo presenta 9 reformulaciones, estando la mayoría (7 propuestas) incluidas en el subgrupo 2a y solo 1 propuesta en cada subgrupo 2b y 2c.

A partir de este punto, todos los demás grupos (Grupo 3 a Grupo 9) tienen una sola reformulación en cada uno.

En la propuesta del grupo 3 se proporciona el área cubierta por la oveja y el largo de la cuerda y se pregunta en qué punto se debe atar la cuerda. Por las razones de simetría, este problema inverso no tiene solución única y su resolución tiene un mayor nivel de dificultad.

Un ejemplo interesante (grupo 4) propone un campo rectangular, siendo $A_r = L \times 2L$ y también agrega un cambio en la posición de la estaca, ahora ubicada en el punto $(L/3, 0)$ y además se informa que $R = \frac{3}{5}L$ y se da el área accesible a la oveja ($A = 400m^2$). En este caso lo que se pide es el valor de $L$, la longitud de la base del terreno.

La reformulación del grupo 5 propone 2 cabras montesas tales que ocupen la máxima superficie posible cada una, pero sin coincidir en ningún punto y se sabe además que una de las zonas tiene que ser más grande que la otra. El participante pregunta sobre la longitud de cada cuerda y el área disponible para cada cabra. Es un problema de optimización, con un supremo que no es alcanzable.

En esta reformulación del grupo 6 se da la longitud de la cuerda $R = \frac{L}{3}$ y se asume que el primer día, la oveja come toda la hierba de su zona accesible. La primera pregunta es sobre la longitud $R'$ que debe tener la cuerda para que la ovejas pueda consumir la misma cantidad de hierba. Luego, se repite la pregunta para el tercer día y para el cuarto día, para finalmente plantear ¿después de cuántos días la oveja no dispondrá de la misma cantidad de hierba para pastar? Es un problema secuencial de mayor dificultad, en el que se mantiene la geometría original.

En el problema del grupo 7 se plantea que la oveja puede pastar en una fracción del campo $f < 1$ y se la ata con una cuerda de longitud $R_f$ y se pregunta ¿cuánto más debe alargarse la cuerda para que la oveja pueda pastar un 10% más de lo pasta actualmente? Se trata por lo tanto de un problema incremental, con la misma geometría.

En el problema siguiente (grupo 8) se presenta un diagrama como en la Figura 4.1.

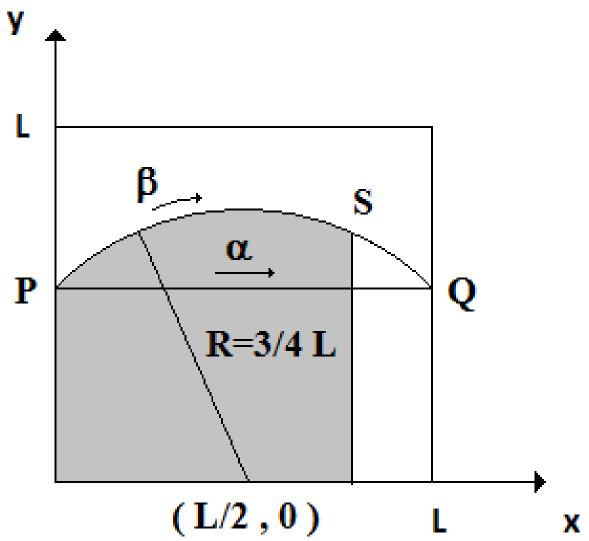



Figura 4.1. Problema dinámico con dos ovejas.

En este caso la cuerda mide $R = \frac{3}{4}L$ y la primera oveja ( $\alpha$ ) corre a lo largo del segmento PQ con una cierta velocidad dada en función de $L$, mientras que la segunda oveja ( $\beta$ ) corre a lo largo del arco de curva PSQ. El enunciado pregunta qué velocidad que debe tener la oveja $\beta$ para alcanzar el punto $Q$ al mismo tiempo que la oveja $\alpha$ .

En este último problema (grupo 9) la cuerda mide $R = \frac{3}{4}L$ en el primer día, mientras que en el segundo día el pastor ata la oveja en el punto $(0,0)$, en la esquina del campo. La pregunta es: ¿cuánto debe medir la cuerda para que el área sea la misma que la del día anterior? Finalmente, se plantea una pregunta similar pero esta vez, colocando la estaca en el centro del campo. Es un problema doble iso-superficial donde se solicitan longitudes de cuerda para que las áreas de pastoreo permanezcan invariantes.

**Análisis didáctico de las tareas asociadas a las reformulaciones en la primera etapa.**

En primer lugar se eliminaron las propuestas directas y/o los problemas mal planteados, pero además, algunos futuros profesores propusieron más de una reformulación inversa, aunque solo llenaron una planilla para el análisis didáctico de las tareas asociadas. Teniendo en cuenta lo anterior, para el análisis didáctico solo se consideraron las respuestas de 29 participantes, es decir, aquellos que propusieron al menos una reformulación inversa correcta.

De ese grupo de 29 futuros profesores, las respuestas obtenidas fueron:

- Conocimientos previos: 13 respuestas.
- Contenido matemático activado por la tarea: 21 respuestas.
- Reto: 22 respuestas.
- Finalización de la tarea, reconocimiento/justificación: 5 respuestas.
- Evento: 22 respuestas.
- Pregunta: 14 respuestas.
- Propósito: 7 respuestas.
- Lenguaje: 17 respuestas.
- Datos: 15 respuestas.
- Meta: 23 respuestas
- Formulación: 13 respuestas
- Materiales y recursos: 15 respuestas
- Agrupamiento: 18 respuestas
- Situación de aprendizaje: 10 respuestas
- Temporalización: 14 respuestas
- Contenido matemático: 17 respuestas

• Situación: 21 respuestas
• Complejidad: 27 respuestas.

Por lo tanto, los que superan el 75% de respuestas (22 o más respuestas) son: Complejidad (27), Meta (23), Evento (22) y Desafío (22), mientras que los ítems que responden menos del 25% de los participantes son: Reconocimiento/Justificación (5) y Propósito (7).
Los resultados obtenidos, en cuanto a otros componentes semánticos, fueron los siguientes:

• En lo vinculado a "sentidos y modos de uso", cerca de la mitad dicen que es "espacio y forma", casi una cuarta parte se decanta por "cambios y relaciones", algunos menos dicen "cantidad" y casi un 10% consideran que es de "incertidumbre y datos".
• En lo que se refiere a la "situación", una gran mayoría se expresa a favor de un contexto "educativo o laboral". Cuatro futuros profesores consideran la tarea como "pública" y solo un participante la clasifica como "personal".

El análisis cognitivo

Los resultados del análisis cognitivo se pueden dividir en expectativas, limitaciones y oportunidades de aprendizaje, siendo los resultados obtenidos, los que siguen:

1.- En cuanto a las expectativas de aprendizaje, teniendo en cuenta las metas declaradas por los futuros profesores, se pueden describir varios grupos:

• Expectativas de aprendizaje específicas del problema, por ejemplo: "Calcule la longitud de la cuerda de modo que no llegue a más de la mitad del área de césped".

• Expectativas de aprendizaje específicas del tema en el que se enmarca el problema, por ejemplo "aplicar el Teorema de Pitágoras", "Calcular áreas", etc.

• Expectativas de aprendizaje que son transversales para la enseñanza basada en la resolución de problemas, como por ejemplo "comprender la inversión de un problema y conectar con otros contenidos matemáticos", "Considerar varios supuestos, reflejar y dibujar un modelo y contrastar resultados", etc.

• Finalmente, existen algunas expectativas de aprendizaje que pueden considerarse genéricas/imprecisas, como: "conceptos de área o región de control" y también "funciones, cálculo de áreas y coeficientes de proporcionalidad".

2.- En cuanto al análisis de las limitaciones de aprendizaje, prácticamente no hay comentarios. Un solo participante menciona que "hay menos posibilidades de cometer errores en la reformulación". En general, no han reflexionado sobre este tema.

3.- En cuanto a las oportunidades de aprendizaje, se pueden distinguir cuatro tipos de opiniones sobre el reto que enfrentan los estudiantes:

• Existe un primer grupo de opiniones favorables, por ejemplo: "interesante", "mayor desafío", o "reto más auténtico", que constituyen la mayoría de las respuestas

• Tres respuestas dadas por dos futuros profesores expresan opiniones negativas al respecto, tales como: “poco interés” o “ningún interés por el alumno”.
• En un caso se da una respuesta neutra, que se expresa como “de interés similar” sin mayor explicación, por lo que es solamente comparativa.
• También hay tres respuestas que pueden considerarse genéricas/imprecisas, por ejemplo un futuro profesor dice "posible desafío" y en el otro caso el participante expresa: "interés del reto: 2", sin aclarar cuál es la escala correspondiente.

Análisis de instrucción

Los resultados obtenidos, correspondientes al análisis de la instrucción, fueron:

• En relación al lenguaje, todos se expresan de forma positiva, siendo las más frecuentes el lenguaje claro y el lenguaje sencillo.
• Sobre la autenticidad, este ítem está claramente dominado por opciones favorables (14 en total), hay 3 respuestas negativas y las 3 restantes pueden considerarse neutrales.
• Respecto a los datos, varios participantes opinan sobre su cantidad y otros se expresan sobre su calidad. En este último grupo, 3 son positivos, uno es negativo y otro es neutral.
• En cuanto al propósito, casi todas las respuestas invocan razones didácticas. Algunas de estas respuestas se refieren al contenido, otras abordan cuestiones cognitivas y algunas consideran aspectos de la instrucción.

- Respecto a la situación de aprendizaje, 8 respuestas proponen realizar la tarea en clase y solamente un participante dijo que podría ser en clase o en casa.
- En cuanto a la agrupación, las opiniones están bastante divididas: 11 proponen que sea una tarea individual, 7 recomiendan trabajar en grupo y 3 más proponen trabajar por parejas.
- En cuanto a la temporalización, 8 respuestas se expresaron en términos comparativos y todas las demás respuestas optan por cuantificar la duración de la tarea, siendo las más comunes: 30 minutos (7 respuestas), 1 sesión completa (6 respuestas) y también se presentan otras opciones minoritarias, como 25 minutos (3 respuestas), 40-45 minutos (2 respuestas), etc.
- En cuanto a la complejidad, 16 futuros docentes se limitan a dar una opinión comparativa, 15 de ellos señalando que es una tarea de mayor complejidad que la original, mientras que el otro dice que es similar. Entre los que respondieron sobre el nivel de complejidad, los resultados son: conexión (15 respuestas), reflexión (10 respuestas) y reproducción (solo una respuesta).
- Finalmente, en cuanto a materiales y recursos, una vez más varios de los futuros profesores (7 respuestas) optaron por comparar diciendo que son similares, sin mayor explicación. Entre los que recomiendan materiales y recursos, nueve de ellos proponen materiales estándar, como lápiz y papel (4 respuestas) y otros proponen materiales y recursos no tan tradicionales: 3 proponen agregar software, uno sugiere usar hilo y chincheta; e incluso hay un caso que propone una recreación a escala en un terreno.

**Análisis de las reformulaciones la segunda etapa de la investigación**

Las reformulaciones de los participantes se estudiaron a través de un análisis de clusters, utilizando como insumo la tabla Booleana, confeccionada a partir de los criterios considerados en el capítulo metodológico.

La nomenclatura utilizada es la siguiente: las letras mayúsculas “PT” (prospective teacher) indican que se trata de un futuro profesor. Estas letras van seguidas de dos dígitos que corresponden al número de alumno y la letra minúscula hace referencia a la primera, segunda o tercera reformulación propuesta por el participante, en su caso. Por ejemplo, PT23c significa que el 23º futuro docente propuso tres reformulaciones y la que se considera aquí es la tercera.

El primer resultado del análisis de clusters fue un dendrograma son 16 grupos diferentes, que resultó particularmente confuso debido al gran número y diferentes tamaños de los grupos. Además, las reformulaciones triviales no quedaron agrupadas en un solo cluster, como sería razonable esperar desde el punto de vista de las tareas. En efecto, si la propuesta resulta muy fácil o aún más, un ejercicio trivial, su utilidad como tarea enriquecida para los estudiantes de secundaria desaparece.

Por estas razones, todas las propuestas triviales se eliminaron de los datos de entrada para el software estadístico y el resultado consistió en un nuevo dendrograma donde se pueden identificar nueve grupos diferentes, como se muestra en la Figura 5.1. Eventualmente, algunos de estos clusters se pueden subdividir en varios subclusters, y también es posible seguir el camino opuesto, considerando superclusters. En la tesis se discuten todas estas posibilidades, pero en este resumen no se considerarán los subclusters.

Si el dendrograma de la Figura 5.1 se analiza desde un nivel superior, es posible identificar tres superclusters.

- El primer supercluster está relacionado con los conocimientos (habilidades) procedimentales y está integrado por los clusters Nº1, Nº2, Nº3 y Nº4.
- El segundo involucra representaciones gráficas y solo incluye el cluster Nº5.
- El tercer supercluster está relacionado con un conocimiento conceptual más profundo (razonamiento y estrategias) y está formado por los clusters Nº6, Nº7, Nº8 y Nº9.

Estos superclusters muestran una conexión directa con los criterios de referencia, representación y significado, las categorías semánticas del contenido de la escuela matemática, analizadas por Rico y colaboradores, a partir de las ideas de Frege (Ruiz-Hidalgo & Rico, 2019).

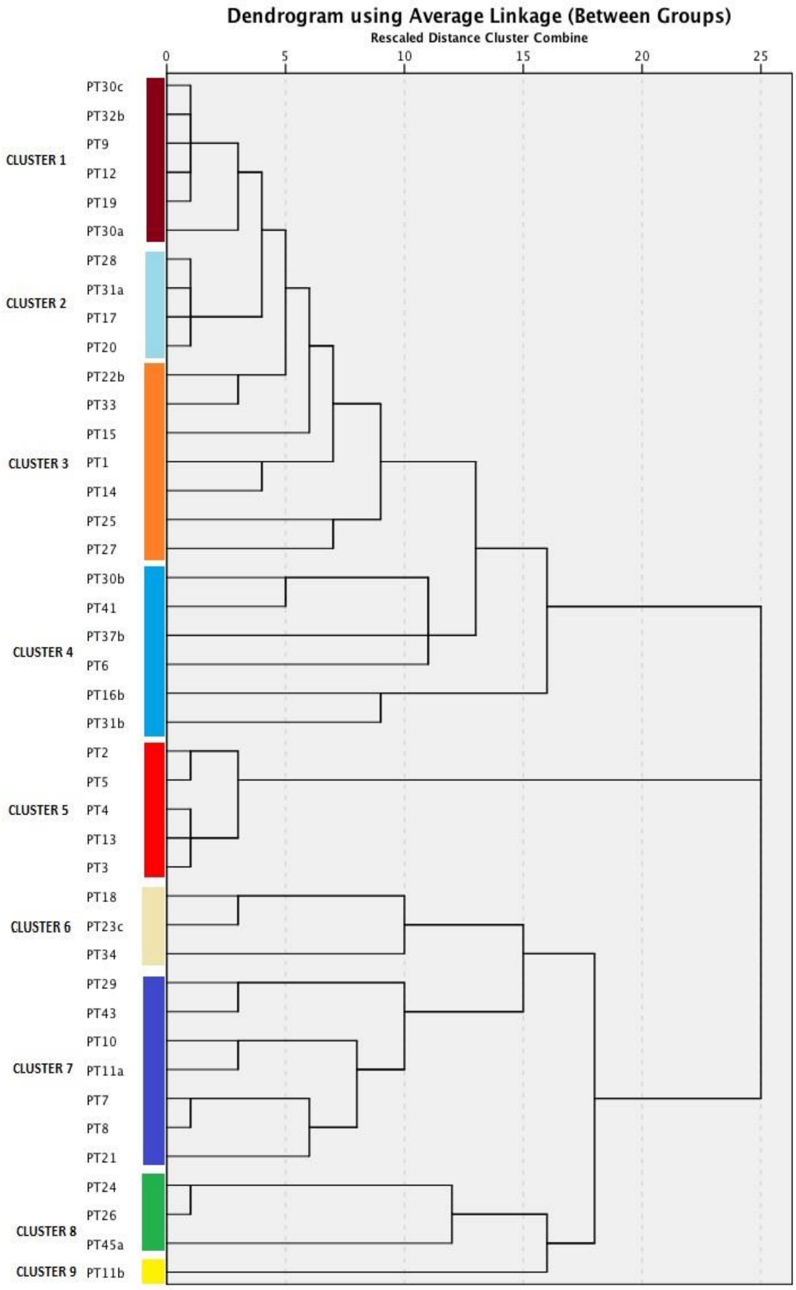



Figura 5.1. Dendrograma que se obtiene eliminando las propuestas triviales.

El primer supercluster

El primer supercluster está formado por clusters que tienen en común el uso del conocimiento de contenido procedimental. Puede ejemplificarse con la reformulación PT33, que propone un campo rectangular $\left[\frac{-L}{5},\frac{L}{5}\right]\times[0,L]$ con la oveja atada en $(0,0)$. El autor da los siguientes datos: $r=\frac{R}{L}=\frac{3}{4}$ y el área accesible a la oveja es $A=433.36\ m^2$. El enunciado pide el radio $R$ para que lo anterior sea posible.

En la resolución del problema, se plantea la integral, que se iguala a $A=433.36\ m^2$ y resulta una ecuación no lineal en $R$ de la cual se obtiene el valor solicitado.

Como se puede observar, este problema requiere de conocimientos procedimentales para ser resuelto.

Este primer supercluster se puede dividir en cuatro grupos:

*Cluster Nº 1: Inversiones puntuales.*

Un primer clúster incluye las siguientes producciones: PT30c, PT32b, PT09, PT12, PT19 y PT30a. En todas ellas es necesario invertir la función $f=f(R)$ (con algunas variantes), pero solo para un valor particular de $f$ (dado directa o indirectamente al conocer $A$ y $L$ o $L^2$).

*Cluster Nº 2: Métodos numéricos*

En el Cluster Nº 2 hay cuatro reformulaciones: PT28, PT31a, PT17 y PT20. En este grupo homogéneo, el hecho común en todos los casos es que los problemas se resuelven mediante métodos numéricos y para ello es necesario algún apoyo tecnológico.

*Cluster Nº 3: Inversión puntual con variantes, con soluciones informales*

Este grupo incluye siete propuestas: PT22b, PT33, PT15, PT01, PT14, PT25 y PT27.
Dichas propuestas forman un grupo heterogéneo con problemas difíciles; cambios de contexto (deportes, incendios forestales, etc.) y la mayoría de ellos se resuelven de manera aproximada por ensayo y error, o simplemente quedan sin solución por sus autores.

*Cluster Nº 4: Condiciones adicionales y métodos de resolución heterogéneos*

Este grupo incluye las siguientes propuestas: PT30b, PT41, PT37b, PT06, PT16b y PT31b. El mismo es bastante heterogéneo, aunque hay algunas características que se repiten en todas las propuestas. De hecho, en todos ellos existen algunas condiciones adicionales y las soluciones correspondientes incluyen diferentes enfoques (geometría, integrales, proporciones, etc.).

El segundo supercluster

El segundo supercluster, que solo incluye el cluster Nº5, se basa en sistemas de representación. Un ejemplo lo da la reformulación PT04, que mantiene la geometría del problema original y con datos diferentes. El autor dice que el área donde las oveja puede pastar viene dada por la integral $A = \int_{\arcsin(-L/2R)}^{\arcsin(L/2R)} R^2 \cos(\theta)\sqrt{1-\sin^2(\theta)}\, d\theta$, obtenida después de un cambio dado de variables: $x = R\sin(\theta) + L/2$. El resultado solicitado es un croquis de la región donde la oveja puede pastar. La resolución consiste en deshacer el cambio de variable y obtener: $A = \int_{\arcsin(-L/2R)}^{\arcsin(L/2R)} R^2 \cos(\theta)\sqrt{1-\sin^2(\theta)}\, d\theta = \int_0^L \sqrt{R^2 - (x - L/2)^2}\, dx$. A partir de esta última integral se puede obtener la región solicitada.

Este supercluster es muy homogéneo y contiene solo un grupo, que se puede describir de la siguiente manera.

*Cluster Nº 5: Croquis*

Este grupo tiene las siguientes propuestas: PT02, PT05, PT04, PT13 y PT03 y los croquis son los resultados solicitados de todas estas reformulaciones.

El tercer supercluster

Las propuestas del tercer supercluster tienen que ver con el contenido conceptual. Como ejemplo, consideramos la producción codificada como PT34, donde el autor mantiene la geometría del campo y la posición de la estaca y solicita un criterio –en términos del área accesible– que permita distinguir entre las posibilidades ilustradas en la Figura 5.2.

Obviamente el punto anguloso de la función $A(R)$ en el intervalo $0 \leq R \leq L$ se produce en el punto $R = \frac{L}{2}$ y en ese caso $A = \frac{1}{2}\pi R^2 = \frac{1}{2}\pi\left(\frac{L}{2}\right)^2 = \frac{\pi L^2}{8}$. Por lo tanto, el criterio solicitado es muy simple, si $A \leq \frac{\pi L^2}{8}$, el área accesible es un semicírculo y en otro caso es una región como la representada para $R > \frac{L}{2}$ en la Figura 5.2.

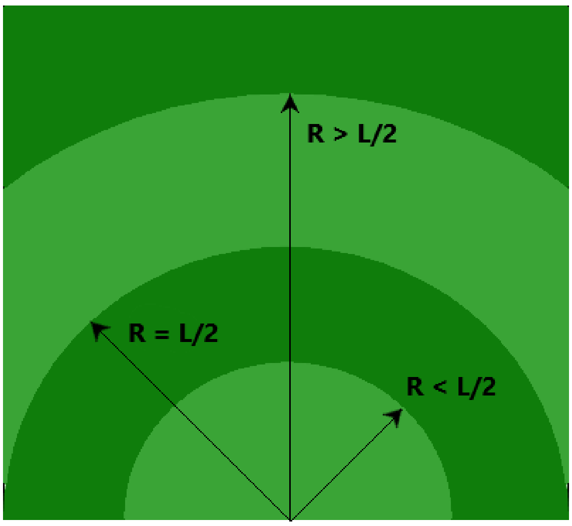



Figura 5.2. Áreas diferentes accesibles a la oveja en la propuesta PT34.

La solución solo requiere conocimientos conceptuales, ya que no son necesarios procedimientos analíticos, geométricos o numéricos, y solo se realizaron algunas manipulaciones algebraicas simples. Este tercer supercluster está formado por cuatro clusters que se pueden describir de la siguiente manera.

*Cluster Nº 6: Conceptual-heurístico*

Este clúster tiene las siguientes propuestas: PT18, PT23c y PT34. En todos estos problemas conceptuales no se necesitan integrales ni métodos numéricos y pueden resolverse en forma heurística, realizando cálculos muy sencillos.

*Cluster Nº 7: Problemas conceptuales avanzados*

Este grupo incluye siete reformulaciones: PT29, PT43, PT10, PT11a, PT07, PT08 y PT21. Por ejemplo, el último de ellos (PT21), propuso un problema de especificación donde pide obtener el mismo resultado del problema directo, pero solo usando geometría. Por supuesto, los datos y los resultados son los mismos (i.e., la entrada y la salida son conocidas) y lo que se busca es el procedimiento geométrico que provea el mismo resultado, sin utilizar integrales.

Como en el cluster anterior, todas las propuestas involucran problemas conceptuales –la mayoría de ellos problemas de especificación con mayor grado de dificultad– que no se pueden resolver heurísticamente.

*Cluster Nº 8: Problemas con solicitud de resultados inusuales*

Este clúster incluye las propuestas: PT24, PT26 y PT45a, donde por ejemplo la primera (PT24), solicita una inversión general en $\Re^2$, es decir, una función con dos variables que debe invertirse sin dar ningún valor particular. Las propuestas incluidas en este clúster son bastante heterogéneas y los resultados que se piden son un tanto inusuales y difíciles de obtener.

*Cluster Nº 9: Teoremas avanzados*

Este último clúster está formado por una única propuesta: PT11b, en la cual se sustituye la cuerda por una cerca cuyo costo por metro es conocido y el autor pregunta por el costo total de la misma. El futuro docente resuelve su propio problema utilizando que $L = \frac{d}{dR} A(R)$, es decir, una fórmula que no es válida en cualquier caso (Dorff & Hall, 2003). Afortunadamente, en este caso la fórmula da el resultado correcto, como se demostró en un artículo anterior (Martinez-Luaces, Fernández-Plaza & Rico, 2019), utilizando teoremas más avanzados como el de Leibnitz, sobre la diferenciación bajo el signo de integral.

## Análisis didáctico de las tareas propuestas por los futuros profesores en la segunda etapa

Resultados generales.

En lo que refiere al análisis didáctico de las tareas, en la primera etapa solo se consideraron 29 respuestas, mientras que en la segunda etapa ese número aumentó a 54 respuestas, a pesar de que los grupos de futuros profesores tenían tamaños similares en ambas experiencias. Esto sucedió porque las respuestas descartadas –correspondientes a reformulaciones directas, o problemas mal planteados– fueron minoritarias en la segunda etapa.

De los distintos ítems, el menos respondido fue "Pregunta", con 32 respuestas (cercano al 60 %) y el más respondido fue “Complejidad” con 51 respuestas. El 75% fue superado por 11 ítems: Conocimientos previos (46), contenidos activados por la tarea (49), reto (43), evento (50), propósito (48), formulación (41), agrupamiento (42), temporalización (45), contenido matemático (50), situación (45) y complejidad (51), frente a solo 7 que no superan dicho porcentaje. Parece razonable suponer que estos resultados se deben a que se incluyó una explicación de los descriptores en las planillas, lo que motivó a los futuros profesores a dar respuestas a todos los ítems. Por lo tanto, el diseño de la segunda etapa de la investigación, ha logrado los objetivos de reducir el número de reformulaciones directas o mal planteadas y que los participantes presten más atención a todos los ítems.

Significatividad

En cuanto a los significados, los ítems a considerar son el conocimiento previo, el contenido matemático que activa la tarea, constituya o no un desafío, y el reconocimiento de la finalización de la tarea, que puede estar asociado a una determinada justificación.

*Conocimientos previos*

Con referencia al conocimiento previo, cabe mencionar que algunos participantes opinan sobre el problema directo (20 respuestas) y/o sobre el problema inverso (19 respuestas) o sobre ambos problemas (15 respuestas), además de otros que no indican a qué problema se refieren (19 respuestas). En todos estos grupos, indican mayoritariamente temas de Cálculo y/o temas de Geometría, un número mucho menor señala temas de Álgebra y muy pocos mencionan la Trigonometría.

*Contenido matemático que activa la tarea*

Con respecto al contenido matemático activado por la tarea, 23 participantes dieron su opinión sobre el problema directo, 25 analizaron el problema inverso y solo 9 dieron su punto de vista sobre ambos problemas. Además de ellos, existe un grupo formado por 26 opiniones en las que no se indica el problema referido. Comparando con el ítem anterior –conocimientos previos– es importante observar que muchas respuestas son similares, respondiendo básicamente igual en ambos ítems, utilizando diferentes palabras.

Por otra parte, existe un grupo importante en el que se observan diferencias notorias con el ítem anterior, siendo en la gran mayoría de ellas, diferencias en términos de contenido matemático. Entre los que no se refieren a un determinado contenido matemático, los que mencionan modelización y visualización son los más importantes. Otras respuestas son: resolución de problemas, pensamiento y razonamiento abstracto, interpretación de resultados y en menor proporción: aproximación, inversión, manipulación de variables desconocidas y refuerzo de conocimientos previos.

*Reto*

En cuanto al reto, hay 17 participantes que opinan sobre el problema directo, 25 analizan el problema inverso, 16 dan sus puntos de vista sobre ambos problemas y finalmente 17 opiniones no indican a qué problema se refieren. En cuanto a las opiniones en sí mismas, una gran mayoría piensa que constituye un desafío para los estudiantes (37 respuestas) y solo 12 respuestas corresponden a la opción negativa. Obviamente, también hay opiniones que afirman que uno de los problemas constituye un reto, mientras que el otro no.

*Finalización de la tarea. Reconocimiento / Justificación.*

En este ítem hay 10 participantes que opinan sobre el problema directo, 19 opiniones son sobre el problema inverso, 12 dan sus puntos de vista sobre ambos problemas y finalmente hay 13 opiniones que no indican a qué problema se refieren.

En cuanto a las opiniones en sí, 26 de ellas expresan (o dan a entender) que el final es claro para los estudiantes y en cambio, 18 opiniones mencionan que el final es ambiguo. Cabe señalar que en algunas opiniones la finalización del problema directo es clara mientras que la finalización del inverso es ambigua y otros participantes piensan que la situación es al revés.

Autenticidad

En cuanto a la autenticidad, los ítems a considerar son: evento, pregunta, propósito, lenguaje y datos. Cada uno de ellos se analiza a continuación.

*Evento*

Con referencia al evento, hay varios participantes que opinan sobre el problema directo (11 respuestas), otros sobre el problema inverso (20

respuestas) o sobre ambos problemas (20 respuestas), además de otros que no indican a qué problema se refieren (23 respuestas).
Entre los futuros profesores hay una gran mayoría que opina que se trata de un evento realista y que de hecho duplican a quienes piensan que no lo es.

*Pregunta*

Con referencia a la pregunta, hay varios participantes que opinan sobre el problema directo (9 respuestas), mientras que otros analizan el problema inverso (12 respuestas) y otro grupo escribe sobre ambos problemas (10 respuestas). Además, una gran cantidad de respuestas no indican a qué problema se refieren (19 respuestas).
Entre los futuros profesores, hay una mayoría significativa que piensa que la pregunta es realista (28 opiniones) y un número menor de participantes piensa exactamente lo contrario (18 respuestas).

*Propósito*

Con referencia al propósito, casi la mitad de los participantes no indica a qué problema se refieren (20 respuestas). De los que mencionan el problema considerado (o se puede deducir del contexto), hay varios que hicieron comentarios sobre el problema directo (10 respuestas), algunos otros opinaron sobre el problema inverso (12 respuestas) y el resto (9 respuestas), escribió sobre ambos problemas.
Las opiniones están bastante divididas entre los futuros profesores, ya que muchos creen que el propósito es realista (23 opiniones) y un número ligeramente menor de participantes piensa lo contrario (20 respuestas). Finalmente, hay un grupo bastante más pequeño (8

respuestas) que no puede considerarse a favor de un propósito realista o poco realista.

*Lenguaje*

En cuanto al lenguaje, una vez más, casi la mitad de los participantes no indican a qué problema se refieren (24 respuestas) y de los que sí lo hacen, más de la mitad opinan sobre ambos problemas (15 respuestas). Entre los futuros profesores, la gran mayoría de opiniones considera que el lenguaje es adecuado (41 opiniones) y un número mucho menor opina que no lo es (8 respuestas).

*Datos*

En este ítem muchos de los participantes no señalan a qué problema se refieren (22 respuestas) y no surge claramente del contexto. Entre los demás, algunos opinan sobre el problema directo (10 respuestas), otros sobre el problema inverso (11 respuestas) y los demás escriben sobre ambos problemas (9 respuestas).
Entre los futuros profesores, hay una mayoría que considera que los datos son realistas (28 opiniones) y un número mucho menor (17 respuestas) expresa lo contrario.

Elementos de la tarea

Los ítems a considerar son: meta, formulación, materiales y recursos, agrupamiento, situación de aprendizaje y temporalización y cada uno de ellos se analiza a continuación.

*Meta*

Hay varios futuros profesores que dan su opinión sobre el problema directo (27 respuestas), otros lo hacen sobre el problema inverso (35 respuestas) y no muchos son los que escriben sobre ambos problemas (11 respuestas), y finalmente, hay otros que no indican a qué problema se refieren (16 respuestas).

Entre los futuros profesores hay una gran mayoría a favor de la opinión de que el objetivo es específico del tema considerado (como calcular integrales o áreas planas), con un total de 64 respuestas, que es más del doble de todas las demás opiniones juntas (29 respuestas).

*Formulación*

En este ítem, más de la mitad de los futuros profesores dan su opinión sin indicar a qué problema se refieren (23 respuestas). Se observa una gran mayoría a favor de la formulación escrita (36 respuestas), seguida en orden descendente de la representación gráfica (17 respuestas) y casi lo mismo para una formulación pictórica (15 respuestas). Hay además otras respuestas absolutamente minoritarias.

*Materiales y recursos*

En este ítem, casi la mitad de los futuros profesores dan su opinión sin indicar a qué problema se refieren (23 respuestas) y de las 27 opiniones restantes, la gran mayoría (19 respuestas) analiza ambos problemas.

Entre los participantes hay una gran mayoría que identifica materiales y recursos tradicionales, como lápiz, papel, bolígrafo y cuaderno, entre otros (37 respuestas), otros proponen material tecnológico básico, como calculadoras electrónicas (6 respuestas) y otras 6 respuestas proponen el uso de software específico (como GeoGebra) o herramientas informáticas disponibles en Internet (como Wolfram Alpha).

*Agrupamiento*

En este ítem prácticamente todas las opiniones se dividen en dos grupos: los que opinan sobre ambos problemas (26 respuestas) y las que no indican a qué problema se refieren (22 respuestas).

Entre las respuestas recibidas, hay una gran mayoría a favor de una tarea individual (42 respuestas). Entre las otras 24 respuestas, 7 proponen trabajar en parejas, solo 3 respuestas sugieren trabajar en pequeños grupos y otras 4 respuestas recomiendan trabajar en la tarea con todo el grupo con el objetivo de comparar las diferentes soluciones.

*Situación de aprendizaje*

En cuanto a la situación de aprendizaje, una vez más casi todas las opiniones se pueden dividir en dos grupos: las que no indican a qué problema se refieren (22 respuestas) y las que opinan sobre ambos problemas (21 respuestas).

Entre las opiniones recibidas, hay una gran mayoría a favor del trabajo en el aula (35 respuestas), frente a las 11 respuestas que proponen trabajar en casa.

*Temporalización*

Entre las opiniones de los futuros profesores, 21 son sobre el problema directo, 23 sobre el problema inverso, 14 escriben sobre ambos problemas y 15 de las respuestas no indican a qué problema se refieren.

De dichas opiniones hay una gran mayoría a favor de períodos breves, es decir, menos de una sesión, con un total de 57 respuestas y solo 4 sugieren períodos largos (una sesión o más) e incluso muy largos, por ejemplo: “1 o 2 sesiones”.

Variables de tarea

En cuanto a las variables de la tarea, los ítems a considerar son el contenido matemático, la situación y la complejidad. Cada uno de estos tres elementos se analiza a continuación.

*Contenido matemático*

En cuanto al contenido matemático, las respuestas de los futuros profesores se dividen mayoritariamente en dos grupos: los que opinan sobre ambos problemas (30 respuestas) y los que no indican a qué problema se refieren (19 respuestas). Entre las 27 opiniones restantes, 12 son sobre el problema directo, mientras que las otras 15 son sobre el inverso.
Los resultados muestran una gran mayoría a favor de "Espacio y forma" (45 respuestas) y "Cambio y relaciones" (38 respuestas). Entre las respuestas minoritarias, solo 5 indican que la tarea se trata de “Cantidad” y solo un participante menciona “Incertidumbre y datos”.

*Situación*

Con referencia a la situación, casi todas las respuestas de los futuros profesores se dividen entre quienes opinan sobre ambos problemas (29 respuestas), o no indican a qué problema se refieren (25 respuestas).

Resulta una mayoría significativa a favor de "Educativo/Laboral" (32 respuestas) y "Personal" (22 respuestas). Entre las opciones minoritarias, nueve indican que se trata de una situación “Pública” y siete la consideran “Científica”.

*Complejidad*

En cuanto a la complejidad, una parte importante de los futuros profesores dan su opinión sobre el problema directo (24 respuestas) y lo mismo ocurre con el problema inverso (29 respuestas), muchos de ellos clasificados por contexto. Además, existe un grupo importante que opina sobre ambos problemas (26 respuestas) y finalmente, no hay tantas respuestas que no indiquen a qué problema se refieren (8 respuestas).

En este ítem la mayoría de las respuestas se decantan por “Conexión” (42 respuestas), en segundo lugar está “Reflexión” (29 respuestas) y por último “Reproducción” (6 respuestas).

## Conclusiones de ambas etapas de investigación

En el primer capítulo del estudio se propusieron los objetivos generales OG1 y OG2. En primer lugar, con respecto a estos objetivos generales, se deben considerar las limitaciones de disponibilidad inherentes a este tipo de estudios. En nuestro caso, la investigación se realizó trabajando con dos grupos de futuros profesores del Máster en Docencia de Secundaria de la Universidad de Granada, y en ambas etapas del estudio (2017 y 2019) el trabajo de campo se organizó en dos sesiones. El trabajo de los participantes permitió obtener un número significativo de producciones, es decir, reformulaciones inversas con el correspondiente análisis de tareas.

En la primera etapa se identificaron nueve grupos diferentes de problemas inversos, algunos de ellos con hasta cinco variantes dentro de un mismo grupo. Es importante mencionar que en esa oportunidad, no se les pidió a los participantes que resolvieran su propia propuesta, lo que dificulta conocer la posible solución que el futuro docente planificó para el problema directo y/o inverso. Sin embargo, algunas palabras clave que surgieron del análisis de las producciones ayudaron a conjeturar qué tipo de respuestas eran las que esperaban los profesores en formación.

A partir de esta primera etapa descriptiva, se infirieron, identificaron y caracterizaron una serie de estrategias para plantear problemas inversos. Además, el Análisis Didáctico fue una herramienta útil para estudiar y caracterizar las producciones de los futuros profesores (es decir, el problema inverso reformulado y el análisis de tareas correspondiente).

En el caso de la segunda etapa, no fue necesario identificar las estrategias de resolución en base a palabras clave que surgieron del análisis de las producciones, ya que se mejoró el diseño de la investigación, solicitando las soluciones de ambos problemas.

En consecuencia, en esa segunda etapa de la investigación se logró realizar un análisis más completo de las reformulaciones, agrupándolas en clusters y más aún, esos clusters formaron superclusters, directamente relacionados con los elementos del triángulo semántico.

En lo que concierne al análisis didáctico comparativo de las tareas de ambos problemas, también fue más fácil extraer sus opiniones en la segunda etapa, ya que previamente se entregó a los participantes una tabla resumida con explicaciones de los descriptores, lo que les ayudó a considerar todos los ítems que debían tenerse en cuenta.

Considerando que se trata de un primer estudio exploratorio –con las típicas limitaciones de tiempo y extensión de la muestra– se puede concluir que se han alcanzado ambos objetivos generales.

Además de lo anterior, se propusieron los objetivos específicos O.1 y O.2, relacionados con el OG.1 y los objetivos específicos O.3, O.4 y O.5, relacionados con el OG.2.

Con respecto a dichos objetivos específicos, cabe mencionar que se realizó una primera clasificación de las propuestas, que fue acertado para describir las producciones recibidas en la primera etapa. Sin embargo, se esperaba que repitiendo el experimento con una nueva población, más capacitada y con sesiones de trabajo más estructuradas, se pudieran observar otro tipo de problemas correspondientes a un abanico más amplio de propuestas. Claramente, la lista no quedó cerrada después de la primera etapa, ya que solo aparecieron problemas de procedimiento.

Cuando se llevó a cabo un nuevo diseño del experimento en 2019, aparecieron otros problemas diferentes. De hecho, el análisis de clusters mostró varios grupos nuevos y aún más importante, aparecen representaciones gráficas y problemas conceptuales, en dos superclusters que no se observaron en la primera etapa de este estudio.

Asimismo, para el estudio definitivo se desarrolló un nuevo instrumento, que incluyó diez columnas donde se analizó el tipo de inversión, la dificultad y otros ítems, que fue el insumo que permitió que todas las producciones fueron codificadas en forma Booleana y así poder realizar el análisis de clusters. En este análisis se obtuvieron nueve clusters, cuyas principales características y complejidades fueron profundamente analizadas por lo que se puede considerar que los objetivos específicos O.1 y O.2 se han logrado parcialmente en la primera etapa y más aún en la segunda etapa de la investigación.

En cuanto a los otros tres objetivos específicos (O.3, O.4 y O.5), se realizó un nuevo análisis considerando las producciones de los futuros profesores y se mejoró el instrumento utilizado para tal fin. De hecho, en una primera etapa se construyó una hoja de cálculo original y luego se agregaron, desglosaron y eliminaron algunas columnas, hasta llegar a una versión avanzada de 21 columnas: una primera con el número alumno/respuesta, luego 20 análisis columnas y una columna final para las observaciones.

En las 20 columnas de la versión avanzada del instrumento se analiza la reformulación propuesta y sus significados, siendo la mayoría de las columnas dedicadas al análisis cognitivo (5 columnas) y análisis de instrucción (8 columnas).

Este instrumento fue mejorado en sucesivas oportunidades, llegando a una versión final en la segunda etapa con cuatro columnas de significación, cinco de autenticidad, seis de los elementos de la tarea y tres columnas más de las variables de la tarea.

Entre los resultados más importantes de la primera etapa, cabe mencionar los siguientes:

• En cuanto a los significados, de las 37 respuestas, 17 optan por "espacio y forma", 9 por "cambio y relaciones", 7 dicen "cantidad" y las 4 restantes mencionan "incertidumbre y datos". Además, la mayoría de los participantes sitúan la tarea en el ámbito educativo.

• Los aspectos cognitivos muestran una gran variedad de respuestas, mostrando cuatro grupos diferentes sobre el objetivo de la tarea y otros cuatro grupos con opiniones diferentes sobre las oportunidades de aprendizaje.
• En cuanto a las categorías del análisis de instrucción, cabe mencionar que la mayoría tiene una opinión favorable sobre la autenticidad de la tarea. Por otro lado, cuando se consideran “propósito” y “agrupamiento”, no hay acuerdo y las opiniones son variadas. Por último, existe un acuerdo generalizado en considerar que la complejidad es mayor que en la tarea original.
• Cabe mencionar que algunos de los docentes en formación consideraron muy importante que el problema se reformulara de forma inversa. Por ejemplo, el alumno A14 afirma que "El interés de la reformulación es porque es la inversa de la anterior ..." y el alumno B14 dice respecto a la complejidad que "... aumenta al pasar de un problema directo a uno indirecto. " Otros tienen una posición más crítica, como el estudiante B24 que dice "... no generaría mucho interés ... el estudiante nunca puede enfrentar un problema como ese".

Respecto a la segunda etapa de la investigación, las conclusiones más importantes son:

• En cuanto a los significados, cabe señalar que los participantes tienden a expresar su opinión sobre las diferencias entre ambos problemas (directos e inversos) más que sobre las similitudes, y más aún, tienden a centrar su análisis en el problema inverso. En sus comentarios sobre los conocimientos previos, las respuestas dominantes son Geometría y Cálculo y con respecto al contenido que activa la tarea, 15 respuestas indican varios contenidos matemáticos, mientras que 29 participantes mencionaron competencias diferentes sin una tendencia clara. Los participantes generalmente consideran que la tarea constituye un reto para los cursos de Bachillerato y su finalización se considera "clara" para el alumno, al menos en la mayoría de opiniones.

• En lo que se refiere a la autenticidad, existe una tendencia generalizada que consiste en opinar sin especificar a qué problema se hace referencia o dan una opinión sobre ambos problemas a la vez, en lugar de dar respuestas sobre un determinado problema. En principio, el evento y la pregunta son realistas en opinión de la mayoría, sin embargo el propósito no es tan claro e incluso hay algunos participantes que piensan que no existe. El lenguaje es adecuado en opinión de una gran mayoría y los datos generalmente se ven como realistas, pero también hay algunas opiniones que incluso dicen que no hay datos, al menos para el caso del problema directo.

• Con respecto a los elementos de la tarea, los participantes tienden a dar una opinión particular sobre "meta" y "temporalización" y hacen comentarios más generales sobre los otros ítems. Las metas más mencionadas son las específicas del tema, que pueden ser sobre Cálculo (integrales, cambios de variables, etc.) o Geometría (Teorema de Pitágoras, ecuación de la circunferencia, etc.). Las formulaciones se ven como escritas, siendo en la mayoría de los casos gráficas y/o pictóricas. La tecnología se propone de forma muy limitada y se piensa que la tarea se resolverá de forma individual y en el aula, añadiendo en algunos casos una sesión grupal para comparar diferentes soluciones. En general, se asignan tiempos cortos y muy cortos para la tarea y este hecho es bastante sorprendente, ya que los participantes debieron resolver el problema antes de dar su opinión y por lo tanto, contaban con elementos suficientes para saber que las integrales no eran triviales. En consecuencia, da la sensación de que los profesores en formación no tienen la experiencia suficiente que les permita ponerse en la situación de sus futuros alumnos.

• En concordancia con lo sucedido con los conocimientos previos, aquí también prevalecen las respuestas que eligen "Espacio y Forma" y "Cambio y Relaciones", mostrando una clara conexión con los contenidos Geométricos y Analíticos. La situación se identifica en primer lugar como "Educativa/Laboral" y "Personal" como segunda opción. La complejidad suele estar entre "Conexión" y "Reflexión", y una respuesta muy común consiste en decir que un problema es una tarea de conexión mientras que el otro necesita reflexión, dependiendo de si el futuro profesor propuso o no una reformulación más compleja.

• La gran mayoría de los formularios presentados correspondieron a la comparación de ambos problemas y solo unos pocos participantes presentaron otros formularios, considerados como una tarea opcional y no obligatoria.

Teniendo en cuenta lo anterior y las limitaciones del el trabajo, se puede concluir que el mismo cumple con los objetivos específicos propuestos para esta investigación.

Comparación de ambas etapas

En el apartado de resultados de la segunda etapa se describieron nueve clusters al analizar las producciones de los futuros docentes, es decir, el mismo número que los grupos descritos para la primera etapa. Sin embargo, solo en la segunda etapa, se observaron propuestas correspondientes a los tres superclusters. Dichos superclusters establecen una conexión directa con referencia, representación y sentido, el llamado triángulo semántico de un contenido matemático escolar, basado en ideas de Frege y desarrolladas por Rico y colaboradores (ver por ejemplo, Rico, 2016).

Al igual que sucedió con otras investigaciones, si bien se entregó el mismo material a todos los participantes, su atención se dirigió hacia diferentes representaciones y su análisis de la situación se basó en diferentes criterios. En nuestro caso, los participantes proceden de distintas disciplinas universitarias (Matemáticas, Ingeniería, Arquitectura, Ciencias, etc.) y esto puede ser una posible explicación de la diversidad observada en sus propuestas.

Si bien el número de clusters en la segunda etapa fue el mismo que en la primera (Martinez-Luaces, Fernández-Plaza & Rico, 2020), los grupos fueron muy diferentes, siendo los clusters 1, 3 y 4 los únicos que pueden considerarse fuertemente conectados con los descritos en la primera etapa. Además, en esa primera etapa no hubo propuestas sobre croquis, cuestiones conceptuales, problemas de especificación o que requieran el uso de teoremas avanzados.

Estos resultados nos llevan a formularnos las siguientes preguntas:

- ¿Por qué ambas experiencias obtuvieron resultados tan diferentes?
- ¿Fueron las modificaciones en el diseño del trabajo de campo las responsables de las diferencias en los resultados?
- ¿Cómo afectaron esos cambios en el diseño del trabajo de campo a las reformulaciones propuestas por los futuros profesores?

Es importante señalar que aunque la muestra no era la misma; ambas poblaciones pueden considerarse equivalentes y fueron elegidas con los mismos criterios de disponibilidad. Más aún, en ambos casos trabajamos con futuros profesores que cursaban el Máster en Profesorado de Matemática de Enseñanza Secundaria, el profesor del Grupo B era la misma persona y trabajaban exactamente en el mismo problema (el problema de la oveja). Entonces, se puede concluir que el nuevo diseño del trabajo de campo fue la única diferencia importante entre la primera y la segunda etapa de la investigación, por lo que no es difícil conjeturar que los cambios en el diseño fueron la razón principal de los diferentes resultados obtenidos.

Lo anterior da una explicación para la primera pregunta y una respuesta positiva a la segunda, eiendo el diseño la única causa posible del efecto observado, es decir, las grandes diferencias en las propuestas de los futuros profesores en 2017 y 2019.

Entonces, si asumimos que esta es la explicación más probable, se debe analizar la tercera pregunta. Empecemos por considerar el Cluster Nº5 donde cinco propuestas diferentes (PT02, PT05, PT04, PT13 y PT03) solicitaron un croquis como repuesta, siendo el dato principal, una fórmula integral dada. Esta integral no hubiera aparecido si el problema directo no se hubiera resuelto previamente –o al menos disponer de un esquema de resolución– para así llegar a la integral definida correspondiente. Otro ejemplo proviene de cinco propuestas incluidas en el Cluster Nº7 (PT29, PT10, PT11a, PT07 y PT08), ya que en todos estos problemas se da una función –que puede ser $f(r)$, $A(x)$ o $A(R)$– y para obtener esta función, una se necesita al menos una solución parcial del problema directo. En consecuencia, estas propuestas, entre otras, no habrían aparecido si no se hubiera solicitado la solución del problema directo. Como observación final, el análisis de PT21 en el Cluster Nº7 y PT11b en el Cluster Nº9, permite llegar a conclusiones similares.

En la primera etapa los participantes intentaron imitar ejemplos anteriores, relacionados con el problema de la oveja y en consecuencia, sus reformulaciones se basaron en la inversión de la función, cambios en la geometría y/o la inclusión de ciertos obstáculos, entre otras posibilidades.

Esta situación concuerda con lo observado por Chapman (2012), en cuanto a que los problemas verbales que suelen plantear los niños son variaciones de los tradicionales, que se pueden encontrar en los libros de texto. Más aún, afirmó que "dado que los estudiantes crecen para convertirse en maestros, es probable que los futuros maestros mantengan algunos de estos problemas, los que luego continuarán el ciclo a menos que se les ayude de un modo apropiado".

La experiencia en 2019 fue muy diferente, ya que los ejemplos previos mostrados a los futuros profesores trataban de otros temas matemáticos y no facilitaban la imitación de los problemas dados. Además, los sujetos tenían que resolver el problema dado antes de plantear su propia reformulación y, por tanto, su experiencia estaba más conectada con la solución del problema directo que otros problemas inversos. Esta situación los llevó, en distintas direcciones, en cuanto a sus propuestas de problemas inversos. Por ejemplo, algunos de los participantes dieron una fórmula y pidieron la interpretación de un determinado parámetro, solicitaron obtener la misma solución, pero sin utilizar técnicas de integración.

Además, se observaron algunas diferencias importantes en términos de uso de variables externas. Esas variables externas pueden ser químicas (cantidad de herbicida o fertilizante), físicas (velocidad o tiempo), económicas (el costo de una cerca por unidad de longitud) o incluso biológicas (kilogramos de pasto que las ovejas pueden comer por día). Esas variables externas aparecieron ampliamente en la primera etapa, pero fueron algo raras en la segunda.

Se puede concluir que los futuros profesores tienden a proponer sus reformulaciones inversos, basándose en sus experiencias recientes. En consecuencia, si esas experiencias consisten en trabajar con ejemplos anteriores, tienden a imitarlos, en cambio si su única experiencia relevante consiste en haber resuelto el problema directo original, entonces, intentan utilizar la solución (o el proceso que condujo a ella) como principal insumo para su propuesta.

## Limitaciones del trabajo

Las limitaciones de este estudio están relacionadas con los siguientes cuatro aspectos: los sujetos que participaron, el número de sesiones dedicadas al trabajo de campo, ciertas características de las planillas utilizadas y el contenido matemático vinculado a este trabajo.

En cuanto al primer aspecto mencionado, nos limitamos a describir las producciones de dos grupos de futuros profesores, cursando el Máster Universitario de Profesores de Educación Secundaria de la UGR. Por razones obvias, los resultados obtenidos no pueden generalizarse a cualquier contexto, ya que para ello sería necesario repetir la investigación en otros países, o al menos en otras universidades.

Una segunda limitación de este estudio se refiere al número de sesiones del trabajo de campo, que fueron solamente dos sesiones en cada etapa, de las cuales una de ellas se planeó como clase introductoria sobre el enriquecimiento de tareas y reformulación de problemas. De hecho, en la primera etapa solo la última sesión se utilizó específicamente para las reformulaciones inversas. Como consecuencia, incluso en la segunda etapa de la investigación, los futuros docentes no tuvieron muchas oportunidades de adaptarse a este tipo de tareas. Se puede conjeturar que si los participantes tuvieran más experiencia, quizás podrían surgir otras variantes, que lleven a propuestas de mayor riqueza.

En cuanto a las planillas utilizadas para recoger las producciones de los futuros docentes en cuanto al análisis didáctico de las tareas, solo se incluyeron explícitamente los siguientes ítems: significatividad, autenticidad, elementos que componen la tarea y variables de la tarea. Se puede conjeturar que una planilla más extensa, donde todos los ítems considerados en el trabajo aparecen explícitamente, podría haber animado a los participantes a opinar sobre todos ellos sin dejar de lado ninguno.

Finalmente, se puede observar que el contenido matemático del problema original conduce a los participantes hacia tres áreas específicas de las matemáticas: álgebra, cálculo y geometría. Un problema diferente relacionado con otras ramas de la matemática –también incluido en el temario de Educación Secundaria– podría llevar a los participantes a proponer distintas reformulaciones conectadas con otras áreas, como por ejemplo, la trigonometría.

## Líneas abiertas de la investigación

Una vez concluido este estudio exploratorio, se puede considerar que existen algunos aspectos que podrían modificarse y también mejorarse para poder diseñar nuevos proyectos de investigación en esta área. Una primera posibilidad consiste en ampliar la muestra, no solo con el propósito de realizar pruebas estadísticas con un tamaño muestral mayor, sino también para comparar resultados con experiencias similares en otros países, o al menos en otras universidades. Esto produciría resultados más generales que los obtenidos en este estudio.

Sería interesante repetir la experiencia incluyendo más sesiones para trabajar en actividades de enriquecimiento de tareas y practicar la reformulación de un problema dado de forma inversa. Sería conveniente que las primeras sesiones –más de una, si es posible– se realicen con el acompañamiento del profesor durante la actividad, con el fin de orientar el trabajo de los participantes y promover la discusión y reflexión sobre las propuestas dentro de todo el grupo. Se puede esperar que después de trabajar en estas actividades grupales, los futuros profesores se sientan más cómodos y más preparados para la tarea individual solicitada. Además, se acostumbrarían a realizar análisis de tareas considerando más ítems que los considerados por los participantes en este estudio.

En resumen, la investigación se puede ampliar y obtener resultados más generales, pero esto requiere trabajar con poblaciones mayores y más diversas y disponer de más sesiones para preparar a los futuros profesores para este tipo de trabajo creativo.