

TOWARDS A SYSTEMATIC METHOD OF IMPLICIT  
RENORMALIZATION: CHIRAL THEORIES AND  
HIGHER ORDERS

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## Resumen

Las predicciones teóricas de precisión en Física de Partículas, incluyendo las necesarias para comparar con experimentos en el Gran Colisionador de Hadrones (LHC), conllevan el cálculo de correcciones radiativas, es decir, correcciones de alto orden en una serie perturbativa. Estas correcciones cuánticas están plagadas de infinitos, siendo inevitable un proceso de renormalización para poder sacar información de ellas. Hay diferentes métodos de tratar la renormalización. La mayoría de ellos implican dos pasos: (i) regularización de las integrales, dando forma a su comportamiento divergente y (ii) substracción de las divergencias, usualmente mediante la incorporación de contratérminos en el Lagrangiano. El método más popular hoy en día consiste en regularizar las integrales mediante continuación analítica de la dimensión del espacio-tiempo, el método llamado Regularización Dimensional (DReg), y después usar substracción mínima (MS) para eliminar las partes singulares de la correspondiente expansión en serie de Laurent de la dimensión física. Las ventajas de este método son la relativa simplicidad de los cálculos y el hecho de que preserva invariancia *gauge*. Sin embargo, DReg tiene problemas en teorías quirales, lo que incluye al Modelo Estándar de interacciones electrodébiles y a teorías supersimétricas. La razón por la que presenta problemas con estas teorías recae en la continuación analítica de la matriz  $\gamma_5$  hacia dimensiones complejas. Por tanto es necesario crear variaciones de DReg que preserven la matriz  $\gamma_5$  común, como por ejemplo la regularización por reducción dimensional (DRed). Estas variaciones, sin embargo, también presentan inconsistencias o acaban con problemas similares a DReg con la matriz  $\gamma_5$ .

Otros métodos tradicionales de regularización o bien rompen la invariancia *gauge* o bien son demasiado complicados para usarse en situaciones realistas. Es por tanto interesante, desarrollar métodos alternativos en cuatro dimensiones que combinen la simplicidad con la preservación de las simetrías más importantes. Los métodos implícitos estudiados en esta tesis, renormalización diferencial restringida (CDR), renormalización implícita restringida (CIR) y renormalización cuatridimensional (FDR), son candidatos prometedores. Se ha demostrado que estos métodos respetan las identidades de Ward asociadas a la invariancia *gauge*. Los llamamos implícitos porque se saltan el paso (i) realizando la substracción sin necesidad de regularización. Esto puede ser bueno y malo al mismo tiempo: por una parte, los métodos son extraordinariamente simples y trabajan en cuatro dimensiones (u otra dimensión física de interés); pero por otro lado, debido a la ausencia de un regulador explícito, se debe tener mucho cuidado con el fin de evitar inconsistencias. Esto requiere el uso de reglas estrictas que limiten las operaciones que se puedan realizar con los integrandos e integrales. Además, con la ausencia de contratérminos explícitos, las substracciones en integrales de varios *loops*

pueden fácilmente romper propiedades sagradas de la Teoría Cuántica de Campos (QFT), como localidad o unitariedad. Esta tesis trata con estas sutilezas de los métodos implícitos. Nuestro objetivo es encontrar un proceso sistemático de renormalización implícita que automáticamente cumpla con las propiedades esenciales. Esto es necesario para realizar pruebas rigurosas o también para poder usar el método con confianza como alternativa a DReg para cálculos novedosos.

En la primera parte de la tesis, estudiamos el comportamiento de los métodos implícitos en teorías quirales. El hecho de que los métodos implícitos trabajen en dimensión fija, los hace buenos candidatos para la renormalización de teorías *gauge* quirales. Sin embargo, mostramos que, bajo leves suposiciones se da un conflicto inevitable entre la preservación de la invariancia gauge y la validez de identidades específicas de la dimensión que están relacionadas con las propiedades comunes de la matriz  $\gamma_5$ . Como consecuencia, los métodos implícitos que mencionamos anteriormente, presentan exactamente los mismos problemas con las teorías quirales que los métodos dimensionales. Las formulaciones originales de estos métodos llevan, de hecho, a inconsistencias en el trato del tensor antisimétrico  $\varepsilon$  y la matriz  $\gamma_5$ . Para remediarlo, añadimos unas reglas adicionales a estos métodos implícitos que eliminan cualquier ambigüedad en el resultado. De esta forma, pueden ser usados de forma segura en teorías *gauge* quirales. El precio a pagar es dejar a un lado las identidades de Fierz y algunas de las propiedades familiares de  $\gamma_5$ . Una vez bien definidas estas reglas, realizamos cálculos explícitos a un *loop* en teorías quirales con FDR para evaluar el comportamiento de estos métodos mejorados.

La segunda parte de la tesis trata con cálculos a varios *loops*. Nos concentramos en FDR. Como se ha mencionado anteriormente, en ausencia de contratérminos explícitos, la unitariedad y localidad de la teoría renormalizada no está garantizada a priori. De hecho, se descubrió que la formulación de FDR es inconsistente con la estructura de contratérminos de las teorías renormalizables. Esto fue solucionado en cálculos explícitos a dos *loops* incorporando al método una “sub-prescripción” que fuerza la consistencia al sub-integrar, es decir fuerza la correspondencia entre renormalizar primero el subdiagrama para después introducirlo en el diagrama completo y continuar la renormalización, con renormalizar todo el diagrama a la vez. Esto funciona en ejemplos conocidos, aun así, es una corrección a posteriori y no está claro que vaya a funcionar para cualquier diagrama de Feynman, especialmente a altos órdenes o en la presencia de singularidades infrarrojas.

Nuestra propuesta es imponer las propiedades esenciales organizando la renormalización de diagramas y subdiagramas de acuerdo con la *Forest Formula* de Zimmermann. Para hacer esto, definimos un operador de substracción en el contexto de FDR. Esto determina el

método. Analizamos diferentes definiciones y las probamos en cálculos explícitos a dos *loops*. Encontramos que todos los cálculos respetan las identidades de Ward. Sin embargo, algunas de las definiciones presentan problemas al reproducir los valores conocidos de la función beta del Grupo de Renormalización, lo que refleja que las amplitudes renormalizadas contienen partes no locales incorrectas. Finalmente, seleccionamos la definición más simple y comprobamos que lleva a las propiedades deseadas. De hecho, mostramos que la *Forest Formula* con el operador elegido genera automáticamente los mismos “*extra-extra*” términos que la “sub-prescripción” de FDR en todos los ejemplos que se han estudiado.

Los trabajos de ambos capítulos llevan, como principal resultado de esta tesis, a una definición precisa de un método sistemático e implícito que respeta la invariancia *gauge* y otras propiedades básicas de las teorías cuánticas quirales y no quirales, al menos hasta dos *loops*. Basado en nuestro análisis, creemos que el mismo método se puede usar también satisfactoriamente sin más perfeccionamientos, a cualquier orden. Sin embargo, una prueba rigurosa de este hecho va más allá del alcance de esta tesis.

# Abstract

Precise theoretical predictions in Particle Physics, including those required to compare with experimental results at the Large Hadron Collider, involve the calculation of radiative corrections, that is, higher order corrections in a perturbative expansion. These quantum corrections are plagued with infinities, and a non-trivial renormalization process is unavoidable to make sense of them. There are different methods to deal with renormalization. Most of them involve two steps: (i) regularization of the integrals, which gives a precise form to their divergent behaviour and (ii) subtraction of the divergences, typically by the addition of counterterms in the Lagrangian. The most popular method nowadays is to regularize the integrals by analytical continuation in the space-time dimension, the so-called dimensional regularization (DReg), and then use minimal subtraction (MS) to eliminate the singular part of the corresponding Laurent series at the physical dimension. The advantages of this method are the relative simplicity of the calculations and the fact that it preserves gauge invariance. However, DReg has problems in chiral theories, which include the Standard Model of electroweak interactions and supersymmetric theories. The reason lies in problems with the analytical continuation of the  $\gamma_5$  matrix into complex dimensions. Variations of DReg that preserve the standard  $\gamma_5$  matrix, such as regularization by dimensional reduction (DRed), have been proposed. These variations, however, either present inconsistencies or end up with similar problems with the  $\gamma_5$ .

Other traditional methods of regularization either break gauge invariance or are too complicated to be used in practice in realistic situations. It is therefore interesting to develop alternative methods in four dimensions that combine simplicity with the preservation of the relevant symmetries. The implicit methods studied in this thesis—constrained differential renormalization (CDR), constrained implicit renormalization (CIR) and four-dimensional renormalization (FDR)—are promising candidates. These methods have been shown to respect the Ward identities associated to gauge invariance. We call them implicit because they bypass step (i) by providing the necessary subtraction without any regularization. This is both a blessing and a curse: on the one hand, the methods are remarkably simple and work in four dimensions (or any other physical dimension of interest); on the other, in the absence of a regulator one must be extremely careful to avoid inconsistencies. This requires the use of strict rules that limit the operations that can be performed on integrands and integrals. Moreover, in the absence of explicit counterterms the subtractions in multi-loop integrals could easily break sacred properties of quantum field theory, such as locality or unitarity. This thesis deals with these subtleties of the implicit methods. Our aim is to find a systematic procedure of implicit renormalization that automatically fulfils the essential properties.

This is necessary for rigorous proofs and also if the method is to be used with confidence as an alternative to DReg in novel calculations.

In the first part of the thesis, we study the performance of implicit methods in chiral theories. The fact that implicit methods work at fixed dimension make them good candidates for the convenient renormalization of chiral gauge theories. However, we show that, under very mild assumptions (which hold in these methods) there is an unavoidable conflict between the preservation of gauge invariance and the validity of dimension-specific identities that are related to the standard properties of the  $\gamma_5$  matrix. As a consequence, the implicit methods of interest present exactly the same problems with chiral theories as the dimensional methods. The original formulations of these methods lead in fact to inconsistencies in dealing with the  $\varepsilon$  antisymmetric tensor and the  $\gamma_5$  matrix. To remediate this, we supplement the implicit methods with additional rules that render the results unambiguous. In this way, they can be safely used in chiral gauge theories. The price to pay is giving up standard Fierz identities and some of the familiar properties of the  $\gamma_5$ . We perform explicit one-loop calculations in chiral theories with FDR to assess the performance of the improved methods.

The second part of the thesis deals with multi-loop calculations. We concentrate on FDR. As mentioned above, in the absence of explicit counterterms the unitarity and locality of the renormalized theory is not guaranteed a priori. In fact, the naive formulation of FDR was soon found to be inconsistent with the counterterm structure of renormalizable theories. This was solved in explicit two-loop calculations by incorporating into the method a “*subprescription*” that enforces subintegration consistency. This works in known examples, but still, it is a correction a posteriori and it is not clear if it will work as such for all Feynman diagrams, especially at higher loops or in the presence of infrared singularities.

Our proposal is to enforce the essential properties by organizing the renormalization of diagrams and subdiagrams according to Zimmermann’s Forest Formula. To do this, we define a subtraction operator in the context of FDR. This determines the method. We analyze different definitions and test them in explicit two-loop calculations. They are all shown to respect the Ward identities. However, some of them present problems in reproducing the known beta functions, which reflect wrong non-local parts in the renormalized amplitudes. Finally, we select the simplest definition and check that it leads to all the desirable properties. In fact, we show that the Forest Formula with the selected operator automatically generates the same “*extra-extra*” terms as the FDR *subprescription* in all the examples we study.

The developments of the two chapters lead, as the main result of the thesis, to the precise definition of a systematic implicit method that respects gauge invariance and other basic properties of chiral and non-chiral quantum field theories, at least at the two-loop level.



Based on our analysis, we believe that the very same method can also be used successfully, without any further refinement, at arbitrary order. But a rigorous proof to all orders goes beyond the scope of this work.

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# Chapter 1

## Introduction

Nowadays, due to the emergence and improvement of particle colliders, precision Physics research at High Energy Physics framework has become so essential. Increasing colliders energy in order to look for New Physics beyond the Standard Model is a hard work and has engineering limitations. So finding another way of sensing New Physics turns necessary, and there is where precision acquires an important role. Detecting some small deviations between theory and experiment at available energy scales could be a signal of New Physics. A massive unknown particle that we cannot produce and detect in a collider due to its large mass, may virtually contribute to a lower energy process that we are able to measure. This contribution, which is inversely proportional to the particle mass, provokes a small discrepancy since it is not theoretically expected. Therefore, it is extremely important to extract measurable observable values from theory as well as from experiment with high precision.

In this way, Theoretical Physics task is to develop and improve mathematical methods which allow us to compute amplitudes of physical processes. These methods are framed at perturbative Quantum Field Theory (pQFT) where every fields interaction is described as an infinite series in fields coupling and  $\hbar$ , i.e. quantum effects. The lowest order is called tree level and it represents the more important contribution to the amplitude. The remaining are called radiative corrections because they mean a little rectification to the principal value. Furthermore, some physical process could be highly suppressed because of the lack of the tree level. Thus, the bigger the pQFT order we are able to reach while evaluating a physical amplitude, the higher the precision we get in the computation.

Nevertheless, beyond tree level, in an intermediate step of the physical process, particles without any momentum restriction arise forming the so-called loops. While computing the amplitude of that process, logically, a sum over all possible momenta of those particles needs

to be performed. That sum involves an integral of every free momenta, from  $-\infty$  to  $\infty$  in all space-time dimensions. And as expected, this integral could diverge. However, despite of those arisen infinities, real physical information related to the interaction process is hidden behind them and has to be extracted somehow.

In order to achieve that, a regularization procedure must be done. Regularization parametrizes the divergences in a regulator which can be isolated and later ejected from the theory by the introduction of local counterterms to the Lagrangian and reabsorbing them into the parameters of the theory, what is called a renormalization procedure. Throughout the years, multiple regularization schemes have been developed such as cut-off, Pauli-Villars, etc., as well as implicit renormalization methods where a regulator is not needed since renormalization occurs at the integrand level leaving the Lagrangian untouched. However, several of them does not respect shift or gauge invariance, unitarity, etc., that is important symmetries of the theory necessary for a consistent renormalization that fulfils the quantum action principle. That is why Dimensional Regularization (or some versions of it) has become the standard for QFT calculations since it respects all mentioned symmetries. It consists in performing the calculations in  $d$  space-time dimensions, and then taking the limit to four dimensions  $d = 4 + \varepsilon$  (or the target one  $n$ ), using  $\varepsilon \rightarrow 0$  as a regulator. In spite of its good properties, theories where dimension is relevant as in chiral or supersymmetric ones, it stops being consistent.

It is well known that regularization methods based on analytical continuation in a complex dimension  $d$  face some problems in the presence of  $\gamma_5$  matrices and completely antisymmetric  $\epsilon$  tensors.<sup>1</sup> The reason is that the usual properties of these objects in integer dimension  $n$  are not consistent with the treatment of Lorentz tensors in dimensional regularization (DReg) [2, 3]. Therefore, one has to give up some of these properties [4]. In particular, a consistently-defined  $\gamma_5$  that approaches the standard  $\gamma_5$  as  $d \rightarrow n$  cannot anticommute with the Dirac matrices in DReg [4, 5]<sup>2</sup> and there is no finite-dimensional complete set in Dirac space—which is an obstacle, for instance, for Fierz reorderings and supersymmetry. These complications are related to the fact that Lorentz covariants in complex dimension  $d$  are treated as formal objects, in which the indices do not take actual values. Even if quantities such as  $\{\gamma_5, \gamma_\mu\}$  are evanescent, i.e. they approach zero as  $d \rightarrow n$ , due to the poles at  $d = n$  in the loop integrals they leave a vestige in the renormalized functions after minimal subtraction

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<sup>1</sup>See [1] for a review. We follow the conventions in that reference only to some extent.

<sup>2</sup>An anticommuting  $\gamma_5$  is often employed in DReg, nevertheless. Although this may be safe for many calculations, as shown in [6], a well-defined renormalized theory requires a consistent unified treatment of arbitrary diagrams and amplitudes. In particular, this is necessary to prove important properties of the theory to all orders.

(MS).

A consistent set of rules in DReg to manipulate Lorentz tensors, including the  $\epsilon$  tensor and  $\gamma_5$  matrix, was proposed by Breitenlohner and Maison in [4]. They used the original definition of  $\gamma_5$  by 't Hooft and Veltman (tHV) [3]. Completed with these rules and MS (or  $\overline{\text{MS}}$ ), DReg provides a consistent renormalization scheme. However, besides genuine anomalies, spurious anomalies appear in some correlators of axial vector currents in QCD and chiral gauge theories, including the Standard Model [7]. These pose no fundamental problem, as it has been shown that they can be eliminated by an additional finite subtraction in a systematic way [8]. But such a correction represents a complication in explicit calculations. This is the main reason for looking for alternatives to the tHV definition of  $\gamma_5$ . Furthermore, DReg explicitly breaks supersymmetry, so it is not a convenient method in supersymmetric theories.

An alternative set of rules for Lorentz tensors and Dirac matrices, which define regularization by dimensional reduction (DRed), was proposed by Siegel in [9] with the purpose of preserving supersymmetry. In this case, the  $\epsilon$  tensor, the Dirac gamma matrices and the  $\gamma_5$  matrix are the original  $n$ -dimensional objects and thus the Dirac algebra is in principle performed in integer dimensions. The name of the method indicates that when these objects are contracted with tensors associated to the dimensionally-regularized integrals, they are projected into the formal  $d$ -dimensional Minkowski space. But due to this projection, the conflict between the  $n$ -dimensional relations and the  $d$ -dimensional Lorentz space reappears. In fact, Siegel himself showed in [10] that the set of rules in the original formulation is inconsistent. A consistent set of rules can be defined by, once again, giving up genuine  $n$ -dimensional relations that require giving explicit integer values to the Lorentz indices [11, 12, 13]. Unsurprisingly, this consistent version of DRed does not manifestly preserve supersymmetry. The four-dimensional helicity method (FDH) [14] is a variant of DRed that treats external vector fields as strictly four dimensional [1]. For our purposes we need not distinguish it from (consistent) DRed. On the other hand, we will often refer to the original, inconsistent version of DRed, for comparison with the methods we are interested in.

In view of the unavoidable difficulties of the dimensional methods when applied to chiral theories or chiral operators, it is reasonable to turn to methods defined in the fixed dimension of interest (often  $n = 4$ ). Indeed, none of the issues discussed above seems at first sight to be relevant when the original dimension is kept fixed. However, in this thesis we show that this expectation is too naïve. It turns out that the formal treatment of Lorentz tensors and Dirac space in certain efficient fixed-dimension methods shares many properties with the one in dimensional methods. As a consequence, the same consistency problems with  $\epsilon$  and  $\gamma_5$  are

found. Consistency can be recovered, once again, by giving up some  $n$ -dimensional relations. The resulting methods do not preserve supersymmetry.

Among mentioned implicit and fixed-dimension methods, we will focus on FDR due to its ability to work at the integrand level. The current definition of the method is order by order where new rules are introduced when a problem appears. Thus a systematic approach to the method is needed. In this way, Zimmermann's Forest Formula [39] ensures locality, unitarity and a systematic implementation of whatever renormalization method we choose. Therefore, Forest Formula could be a very helpful tool in FDR development.

The thesis is organized as follows. Chapter 2 is mostly composed by paper [15] also written by the author of this thesis, his supervisor and another collaborator. In section 2.1, we describe the treatment of Lorentz tensors in DReg/DRed and in implicit methods. We stress the fact that, in order to preserve basic properties of the integrals, the contraction of Lorentz indices cannot commute with renormalization. We also explain how this requirement is implemented in the different methods. In section 2.2 we show a consequence of it: some identities that are valid in standard  $n$ -dimensional spaces are spoiled by the renormalization process. Thus, using these identities leads to inconsistent results. In section 2.3 we study how the Dirac algebra is affected by these potential inconsistencies. We find that implicit methods have problems with the Dirac algebra in odd dimensions and with the  $\gamma_5$  matrix in even dimensions. These issues parallel the ones in dimensional methods. In section 2.4 we propose a well-defined procedure that avoids inconsistencies in implicit methods. This procedure is analogous to the consistent version of DRed. We discuss allowed simplifications within this scheme, including shortcuts that have already been used in FDR. In section 2.5 we show the FDR method as it was originally explained by its authors [20, 23]. We give simple examples of chiral calculations using FDR in  $n = 2$  and  $n = 4$  in section 2.6. Chapter 3 is a work in progress with the main FDR author Roberto Pittau (Full Professor at University of Granada) which is expected that could lead to a paper publication in the future. In section 3.1 we define Forest Formula and show how it works with several renormalization methods. In section 3.2 we systematize FDR thanks to Forest Formula. We present several conditions configurations to successfully achieve the systematization. In section 3.3 we introduce Renormalization Group Equation as a tool to analyze all FDR schemes we found in the previous section. Finally, in chapter 4 we perform some QED calculations trying to distinguish among the several FDR systematizations. We compute one and two-loop vacuum polarization in Spinorial and Scalar QED, and fermion self-energy in Spinorial QED in sections 4.1, 4.2 and 4.3 respectively. And we conclude in section 5. Appendix A contains all Feynman rules of the theories we have treated in this thesis, appendix B has all master integrals and needed



functions to evaluate one and two-loop integrals, and appendix C collects functions that appear in the evaluation of axial vertex in four dimensions.

## Chapter 2

# Lorentz algebra and chirality in implicit methods of renormalization

In explicit regularizations in fixed dimension  $n$ , such as a momentum ultraviolet cutoff or those based on a modification of the Lagrangian, the  $n$ -dimensional Lorentz and Dirac manipulations inside or outside loop integrals are well defined. The same is true at the renormalized level in physical renormalization schemes defined by renormalization conditions. However, when some *a priori* subtraction prescription is used (similar to MS in DReg), the commutation of the usage of  $n$ -dimensional identities with the subtraction of divergences needs to be checked. This is specially so in fixed-dimension methods that directly provide renormalized amplitudes without explicit counterterms. Here we are interested in methods of this kind with the potential of satisfying the quantum action principle [16], from which basic properties, such as Ward identities, follow to all orders. We will generically refer to them as *implicit methods*. Specifically, we study in detail three implicit methods: constrained differential regularization/renormalization (CDR)<sup>1</sup> [18], constrained implicit regularization/renormalization (CIReg) [19] and four-dimensional regularization/renormalization (FDR) [20]. These methods have been applied to one-loop and multi-loop calculations in QCD [21, 22, 23], the Standard Model [24, 25, 26], supersymmetric models [27, 28, 29] and supergravity [30], among other theories. In all these examples, gauge invariance and supersymmetry, when relevant, have been preserved. We will also make some comments about the four-dimensional formalism (FDF) [31] of FDH, which shares some features with FDR.

The first of these methods was originally defined in position space, as a gauge-invariant version of differential renormalization [32], but it works equally well in momentum space.

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<sup>1</sup>Not to be confused with “conventional dimensional regularization”.

Actually, its momentum-space version is implemented in `FormCalc` [33].<sup>2</sup> CDR gives renormalized expressions without any intermediate regularization, essentially by an implicit subtraction of local singularities (polynomial in external momenta, in agreement with Weinberg’s theorem). CIReg and FDR work in momentum space at the integrand level. Both methods use straightforward partial-fraction identities to isolate the ultraviolet divergences, with no external momenta in the denominators. The divergent parts are then ignored, that is, subtracted. Again, no regularization is necessary. One difference between them is that CIReg keeps physical masses in the divergent parts, while FDR does not, but these include an auxiliary mass  $\mu$ , which is introduced before the algebraic manipulations to avoid artificial infrared divergences and taken to zero at the end.<sup>3</sup> This scale is essential in FDR and will be very relevant in the discussion below. Let us stress that, notwithstanding its name, FDR can be used in the very same manner in any integer dimension. All three methods can in principle deal as well with genuine infrared divergences, but only FDR has been studied in detail in this context, both for virtual and real singularities [20, 25]. The equivalences in non-chiral theories and at the one-loop level of CDR, CIReg (in a massless scheme) and DRed have been established in [33, 35] and [36]. Concerning the preservation of unitarity and locality in multiloop calculations without counterterms, CDR and CIReg rely on Bogoliubov’s recursive renormalization and Zimmermann’s Forest Formula [37, 38, 39, 40, 34] which we will study in coming sections. At any rate, here we are concerned with the treatment of Lorentz tensors and Dirac matrices in these fixed-dimension methods, and one-loop examples will be sufficient to illustrate our main messages.

## 2.1 Lorentz tensors and index contraction

In dimensional methods, the contraction of Lorentz indices in a tensorial integral does not, in general, commute with regularization and renormalization. This comes from the simple fact that the trace of the  $d$ -dimensional metric tensor is  $g^\mu{}_\mu = d = n - \varepsilon \neq n$ . When it hits a pole  $1/\varepsilon$  in a divergent integral, the term linear in  $\varepsilon$  will give rise to a finite contribution, which is not subtracted in MS and survives when  $\varepsilon$  is taken to zero.

We show next that, actually, index contraction does not commute with renormalization in any implicit method that respects the quantum action principle and consistently

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<sup>2</sup>`FormCalc` has the option of working in  $d$  or 4 dimensions, that is, in DReg or CDR. The latter is more suitable for supersymmetric theories.

<sup>3</sup>CIReg can also be implemented without masses in divergent parts [34]. In this mass-independent scheme, and in all cases in massless theories, the scale  $\mu$  is introduced in CIReg as well, but only in denominators.

replaces each overall-divergent integral by a unique finite expression. CDR, CIReg and FDR belong to this class. The proof of the quantum action principle in perturbation theory relies on two non-trivial properties: invariance under shifts of the integration momenta and numerator-denominator consistency. The first property is related to translational invariance and guarantees independence of momentum rooting. The second one requires that the application of the kinetic operator to the propagator associated to some line in a Feynman graph is equivalent to pinching of that line, that is, its contraction to a point. This is necessary for a consistent treatment of the quadratic and interaction terms in perturbation theory [4]. These properties need not hold in arbitrary definitions of regularized or subtracted integrals.

Shift invariance can be related to the vanishing of total derivatives with respect to integration momenta:

$$0 = \left[ \int d^n k (f(k+p) - f(k)) \right]^R = p^\nu \left[ \int d^n k \frac{\partial}{\partial k_\nu} f(k) \right]^R + O(p^2) \quad (2.1.1)$$

Here,  $R$  indicates that the expression inside the corresponding brackets is renormalized, i.e. subtracted and with any possible regulator or auxiliary parameter removed (except for the unavoidable renormalization scale). We require that the operation  $[\cdot]^R$  be linear:

$$[aF + bG]^R = a[F]^R + b[G]^R \quad (2.1.2)$$

where  $a, b$  are numbers or external objects, such as external momenta. This holds in all the methods we study in this thesis. Consider the following two-dimensional integral:

$$\begin{aligned} f_{\mu\nu} &= \int d^2 k \frac{\partial}{\partial k_\mu} \frac{k_\nu}{k^2 - m^2} \\ &= \int d^2 k \left( \frac{g_{\mu\nu}}{k^2 - m^2} - 2 \frac{k_\mu k_\nu}{(k^2 - m^2)^2} \right) \end{aligned} \quad (2.1.3)$$

According to 2.1.1, shift invariance requires  $[f_{\mu\nu}]^R = 0$ , and thus, calling

$$I_{\mu\nu} = \int d^2 k \frac{k_\mu k_\nu}{(k^2 - m^2)^2} \quad (2.1.4)$$

we have

$$\begin{aligned}
[I_{\mu\nu}]^R &= \frac{1}{2}g_{\mu\nu} \left[ \int d^2k \frac{1}{k^2 - m^2} \right]^R \\
&= \frac{1}{2}g_{\mu\nu} \left( \left[ \int d^2k \frac{k^2}{(k^2 - m^2)^2} \right]^R - \left[ \int d^2k \frac{m^2}{(k^2 - m^2)^2} \right]^R \right) \\
&= \frac{1}{2}g_{\mu\nu} \left( [I^\alpha{}_\alpha]^R - i\pi \right)
\end{aligned} \tag{2.1.5}$$

That is, shift invariance forbids symmetric integration (in  $n$  dimensions). In the second line we have used numerator-denominator consistency,  $(k^2 - m^2)/(k^2 - m^2) = 1$ . This looks trivial in the formal equations above, but it is not so in methods that modify the propagators at intermediate steps of the calculation. In the third line we have assumed that integrals finite by power counting are not changed by renormalization. This assumption is essential in the definition of dimensional regularization and also in the definition of CDR, CIReg and FDR, as should already be clear from the brief explanations in the introduction. We can rewrite 2.1.5 as

$$g^{\mu\nu} [I_{\mu\nu}]^R = [g^{\mu\nu} I_{\mu\nu}]^R - i\pi \tag{2.1.6}$$

So, we see that *renormalization does not commute with index contraction if it commutes with shifts of integration momenta and respects numerator-denominator consistency*. This is in fact the origin of trace anomalies [41] and also of chiral anomalies, as we shall see. The same conclusion can be proven in arbitrary integer dimension  $n$  using similar arguments.

Let us now examine how the different renormalization methods we are discussing recover 2.1.5, and thus comply with 2.1.1. In the case of dimensional methods, we have

$$\begin{aligned}
[I_{\mu\nu}]^R &= \left[ \int d^d k \frac{k_\mu k_\nu}{(k^2 - m^2)^2} \right]^S \\
&= \left[ \int d^d k \frac{1}{d} g_{\mu\nu} \frac{k^2}{(k^2 - m^2)^2} \right]^S \\
&= \left[ \int d^d k \left( \frac{1}{2} + \frac{\varepsilon}{4} + O(\varepsilon^2) \right) g_{\mu\nu} \frac{k^2}{(k^2 - m^2)^2} \right]^S \\
&= \left[ \frac{1}{2} g_{\mu\nu} \int d^d k \frac{k^2}{(k^2 - m^2)^2} + \left( \frac{\varepsilon}{4} + O(\varepsilon^2) \right) g_{\mu\nu} \left( -2i\pi \frac{1}{\varepsilon} + O(\varepsilon^0) \right) \right]^S \\
&= \frac{1}{2} g_{\mu\nu} \left( [I^\alpha{}_\alpha]^R - i\pi \right)
\end{aligned} \tag{2.1.7}$$

in agreement with 2.1.5. Here,  $S$  indicates MS followed by  $\varepsilon \rightarrow 0$ . Note that before the  $S$

operation,  $g$  is the metric tensor in  $d$  formal dimensions, which satisfies  $g^\mu{}_\mu = d$ .

In CDR, the finite local terms in the renormalized value of the different overall-divergent tensor integrals are fixed by requiring compatibility with shift invariance and numerator-denominator consistency. Hence,  $[f_{\mu\nu}]^R = 0$  by construction and the extra local term in the tensor integral is fixed just as in equation 2.1.6.

CIReg has the advantage of working at the integrand level. Tensor integrands are expressed as simpler integrands plus total derivatives. Integrating the latter gives potential surface terms, which are dropped by definition. So, shift invariance is enforced by the very definition of the method. For instance, using the same relation as in 2.1.3,

$$\begin{aligned}
[I_{\mu\nu}]^R &= \left[ \int d^2k \left( \frac{1}{2} \frac{g_{\mu\nu}}{k^2 - m^2} - \frac{1}{2} \frac{\partial}{\partial k_\mu} \frac{k_\nu}{k^2 - m^2} \right) \right]^R \\
&= \frac{1}{2} g_{\mu\nu} \left[ \int d^2k \frac{1}{k^2 - m^2} \right]^R \\
&= \frac{1}{2} g_{\mu\nu} \left( [I^\alpha{}_\alpha]^R - i\pi \right)
\end{aligned} \tag{2.1.8}$$

We see that the same local terms as in CDR are found, but in this case there is a simple prescription to obtain them. Obviously  $[f_{\mu\nu}]^R = 0$  and 2.1.6 is satisfied. At this point, it is important to make the following observation. We can also write

$$[I^\alpha{}_\alpha]^R = \left[ \int d^2k \left( \frac{1}{k^2 - m^2} - \frac{1}{2} \frac{\partial}{\partial k_\alpha} \frac{k_\alpha}{k^2 - m^2} \right) \right]^R \tag{2.1.9}$$

Dropping the second term would contradict 2.1.8. Accordingly, CIReg does not drop this sort of surface term when the index in the total derivative is contracted with a loop momentum. Therefore, just as CDR, CIReg distinguishes by definition contracted and non-contracted Lorentz indices. Note that the vanishing of the second term on the right hand of 2.1.9 is not necessary for shift invariance: in 2.1.1 the index in the total derivative is contracted with the index in the (external) momentum shift, so it can never be contracted with the index of a loop momentum.

In FDR, which also works at the integrand level, the extra local terms necessary for shift invariance result automatically from the introduction of the scale  $\mu$ , together with some

additional prescriptions. In this method,

$$\begin{aligned}
[I_{\mu\nu}]^R &= \left[ \int d^2k \frac{k_\mu k_\nu}{(k^2 - \mu^2 - m^2)^2} \right]^S \\
&= \frac{1}{2} g_{\mu\nu} \left[ \int d^2k \frac{k^2}{(k^2 - \mu^2 - m^2)^2} \right]^S \\
&= \frac{1}{2} g_{\mu\nu} \left[ \int d^2k \frac{k^2 - \mu^2}{(k^2 - \mu^2 - m^2)^2} + \int d^2k \frac{\mu^2}{(k^2 - \mu^2 - m^2)^2} \right]^S \\
&= \frac{1}{2} g_{\mu\nu} \left( [I^\alpha_\alpha]^R + \left[ \int d^2k \frac{\mu^2}{(k^2 - \mu^2 - m^2)^2} \right]^S \right) \\
&= \frac{1}{2} g_{\mu\nu} \left( [I^\alpha_\alpha]^R - i\pi \right)
\end{aligned} \tag{2.1.10}$$

Several explanations are in order. The first step in FDR is the introduction of the scale  $\mu$ , as done in the first line of 2.1.10. The symbol  $[\cdot]^S$  in this case refers to the FDR subtractions, followed by the limit  $\mu \rightarrow 0$  (outside logarithms). In the second line, we have used the property of symmetric integration, which is allowed in this method *after* the scale  $\mu$  has been introduced. In the fourth line we have used the so-called *global prescription* of FDR, according to which the possible  $k^2$  in numerators inside  $[\cdot]^R$  should be also replaced by  $k^2 - \mu^2$ , just as in the denominators. As emphasized in [20], this is necessary to preserve numerator-denominator consistency. Finally, the integral in the second term of the fourth line of 2.1.10 is finite and goes to zero as  $\mu \rightarrow 0$ . However, a nonvanishing contribution is found as shown in the last line, because FDR performs an oversubtraction, treating this integral as divergent (for power counting,  $\mu$  is counted like an integration momentum). In the FDR language integrals of this kind are called *extra integrals*. They play the same role as the extra local terms in CDR, with the advantage that the necessary terms arise directly from a simple and universal prescription, formulated without reference to specific integrals. The result in 2.1.10 coincides with the one in the previous methods, as it should to guarantee  $[f_{\mu\nu}]^R = 0$  and, thereby, shift invariance.

Let us summarize this section. Just as in dimensional renormalization, the contraction of Lorentz indices does not commute with renormalization in the implicit methods we are considering, which respect invariance under shifts of the integration momenta and numerator-denominator consistency. In the latter methods,  $k^2$  and  $k_\mu k_\nu$  have to be treated in a different manner by hand. This requires writing the diagrams in some normal form that allows for a unique identification of tensors with contracted and uncontracted indices.

## 2.2 Relations in genuine integer dimension

Genuine  $n$ -dimensional identities (GnDI) spoil the uniqueness of the normal form and thus can lead to inconsistencies in implicit methods, which parallel the ones in the original version of DRed. By GnDI we mean equalities depending crucially on the fact that the Lorentz indices can take  $n$  different integer values. Consider the determinant

$$\text{Det}(\mu_1 \dots \mu_m; \nu_1 \dots \nu_m) \equiv \begin{vmatrix} g_{\mu_1 \nu_1} & g_{\mu_1 \nu_2} & \dots & g_{\mu_1 \nu_m} \\ g_{\mu_2 \nu_1} & g_{\mu_2 \nu_2} & \dots & g_{\mu_2 \nu_m} \\ \vdots & \vdots & & \vdots \\ g_{\mu_m \nu_1} & g_{\mu_m \nu_2} & \dots & g_{\mu_m \nu_m} \end{vmatrix} \quad (2.2.1)$$

In standard algebra, this object vanishes when  $m > n$ , since it is then unavoidable to have at least two identical rows, as the indices can take only  $n$  different values. However, this is not necessarily true when used inside  $[\cdot]^R$ , because contracted and uncontracted indices are treated differently if index contraction does not commute with renormalization. To show this more explicitly, let us consider the case with  $n = 2$  and  $m = 3$ . Requiring the determinant 2.2.1 to vanish we have

$$\begin{aligned} 0 &= [0]^R \\ &\stackrel{?}{=} [\text{Det}(\alpha\mu\nu; \beta\rho\sigma) p_{1\mu} p_{2\nu} p_{3\rho} p_{4\sigma} I_{\alpha\beta}]^R \\ &= (p_1 \cdot p_3 p_2 \cdot p_4 - p_1 \cdot p_4 p_2 \cdot p_3) [I_{\alpha\alpha}^R] - p_1^\mu p_3^\rho p_2 \cdot p_4 [I_{\rho\mu}]^R \\ &\quad + p_3^\rho p_2^\nu p_1 \cdot p_4 [I_{\rho\nu}]^R + p_4^\sigma p_1^\mu p_2 \cdot p_3 [I_{\sigma\mu}]^R - p_2^\nu p_4^\sigma p_1 \cdot p_3 [I_{\sigma\nu}]^R \end{aligned} \quad (2.2.2)$$

If we now use 2.1.5, we find

$$0 \stackrel{?}{=} i\pi(p_1 \cdot p_3 p_2 \cdot p_4 - p_1 \cdot p_4 p_2 \cdot p_3) \quad (2.2.3)$$

which is obviously not true for general  $p_i$ .

This simple example is sufficient to prove the main assertion of this thesis: *Using GnDI before renormalization can lead to inconsistencies in implicit methods.* The origin of this issue is the non-commutation of index contraction with renormalization. The difficulties with  $\gamma_5$ , discussed in the next section, are a direct consequence of it.

In dimensional methods, it is clear that the determinant 2.2.1 does not vanish if  $g$  is the  $d$ -dimensional metric, so obviously the second equality in 2.2.2 is invalid. However, an  $n$ -dimensional metric  $\bar{g}$  ( $\tilde{g}$ ) is introduced in DReg (DRed), with  $\bar{g}^\mu_\mu = \tilde{g}^\mu_\mu = n$ . The relation



between the  $n$ -dimensional and  $d$ -dimensional metrics is different in DReg and DRed:

$$g_\mu^\nu \bar{g}_{\nu\rho} = \bar{g}_{\mu\rho} \quad (\text{DReg}) \quad (2.2.4)$$

$$g_\mu^\nu \tilde{g}_{\nu\rho} = g_{\mu\rho} \quad (\text{DRed}) \quad (2.2.5)$$

Let us define

$$\overline{\text{Det}}(\mu_1 \dots \mu_m; \nu_1 \dots \nu_m) \equiv \begin{vmatrix} \bar{g}_{\mu_1\nu_1} & \bar{g}_{\mu_1\nu_2} & \cdots & \bar{g}_{\mu_1\nu_m} \\ \bar{g}_{\mu_2\nu_1} & \bar{g}_{\mu_2\nu_2} & \cdots & \bar{g}_{\mu_2\nu_m} \\ \vdots & \vdots & & \vdots \\ \bar{g}_{\mu_m\nu_1} & \bar{g}_{\mu_m\nu_2} & \cdots & \bar{g}_{\mu_m\nu_m} \end{vmatrix} \quad (2.2.6)$$

For  $n = 2$ , in DReg we have

$$\begin{aligned} 0 &= [0]^R \\ &= [\overline{\text{Det}}(\alpha\mu\nu; \beta\rho\sigma) p_{1\mu} p_{2\nu} p_{3\rho} p_{4\sigma} I_{\alpha\beta}]^R \\ &= (p_1 \cdot p_3 p_2 \cdot p_4 - p_1 \cdot p_4 p_2 \cdot p_3) [\bar{g}_{\alpha\beta} I_{\alpha\beta}]^R - p_1^\mu p_3^\rho p_2 \cdot p_4 [I_{\rho\mu}]^R \\ &\quad + p_3^\rho p_2^\nu p_1 \cdot p_4 [I_{\rho\nu}]^R + p_4^\sigma p_1^\mu p_2 \cdot p_3 [I_{\sigma\mu}]^R - p_2^\nu p_4^\sigma p_1 \cdot p_3 [I_{\sigma\nu}]^R \end{aligned} \quad (2.2.7)$$

This expression does vanish. The difference with 2.2.2 is that  $\bar{g}^{\mu\nu} k_\mu k_\nu \neq k^2$  if  $k$  is a  $d$ -dimensional vector. Then,

$$[\bar{g}^{\alpha\beta} I_{\alpha\beta}]^R = g^{\alpha\beta} [I_{\alpha\beta}]^R \quad (2.2.8)$$

Note that  $g$  is the same as  $\bar{g}$  outside  $[\cdot]^R$ . We see that the rules in DReg are perfectly consistent in our example:  $\text{Det}(\alpha\mu\nu; \beta\rho\sigma)$  does not vanish in  $d$  dimensions while  $\overline{\text{Det}}(\alpha\mu\nu; \beta\rho\sigma)$  can be safely set to zero in  $n = 2$ .

Things are very different in the original version of DRed. If we define  $\widetilde{\text{Det}}$  just as in 2.2.6 but with  $\bar{g} \rightarrow \tilde{g}$ , due to 2.2.5 and the fact that the integration momentum  $k$  is a  $d$ -dimensional vector (in the sense explained above), we find  $\tilde{g}^{\mu\nu} k_\mu k_\nu = k^2$ . Hence, we recover 2.2.2 and the inconsistency 2.2.3. The root of the problem in this case is apparent: the relation 2.2.5 projects  $n$ -dimensional objects into  $d$ -dimensions, which invalidates the GnDI used for the former.

Note that in this version of DRed, the inconsistencies arise at the regularized level, due to the incompatibility of the dimensional reduction rule 2.2.5 with GnDI. In implicit methods, the GnDI are also dangerous before the identification and distinction of the different tensors. But they can be safely used afterwards: in CDR, after the (non-trivial) trace-traceless decompositions; in CIReg, after rewriting tensor integrals and eliminating surface terms by

generalizations of 2.1.8; and in FDR, after the addition of  $\mu^2$  in numerators, according to the global prescription.

It will prove useful to mimic DReg and introduce in implicit methods a genuinely  $n$ -dimensional metric  $\bar{g}$ , with the properties<sup>4</sup>

$$\begin{aligned}\bar{g}_\mu{}^\nu \bar{g}_{\nu\rho} &= \bar{g}_\mu{}^\nu g_{\nu\rho} = \bar{g}_{\mu\rho} \quad (\text{implicit}) \\ \bar{g}^\mu{}_\mu &= n\end{aligned}\tag{2.2.9}$$

The distinguishing property of the metric  $\bar{g}$  with respect to  $g$  is that, by definition,

$$[\bar{g}^{\mu\nu} T_{\dots\mu\dots\nu\dots}]^R = g^{\mu\nu} [T_{\dots\mu\dots\nu\dots}]^R\tag{2.2.10}$$

for any tensor  $T$ . In general, 2.2.10 is different from  $[T_{\dots\mu\dots}^\mu\dots]^R$ . In other words, for renormalization purposes  $\bar{g}^{\mu\nu} k_\mu k_\nu = \bar{k}^2$  is not to be treated as  $k^2$  but as if the indices were not contracted. For instance, in FDR, no  $\mu^2$  is added to  $\bar{k}^2$ . (But once the  $\mu^2$  shifts have been performed, one can write  $\bar{k}^2 = k^2$ .) Because  $\bar{g}$  commutes with renormalization,  $\overline{\text{Det}}(\mu_1\mu_2\mu_3; \nu_1\nu_2\nu_3)$  vanishes for  $n = 2$ , just as in DReg. But importantly, in expressions such as 2.1.3, it is still the ordinary metric  $g$  of the formal  $n$ -dimensional space that appears. Otherwise, shift invariance or numerator-denominator consistency would be spoiled, as we have seen. If  $E$  is either the  $\epsilon$  tensor or an external tensor, then we can substitute at any moment one metric by the other one,

$$E_{\dots\mu\dots} g^{\mu\nu} = E_{\dots\mu\dots} \bar{g}^{\mu\nu}\tag{2.2.11}$$

since the metrics appearing here will never contract two internal momenta, as long as GnDI are not employed. We can also use  $\bar{g}$  in DRed, with the properties in 2.2.9 and 2.2.5, supplemented with

$$\bar{g}_\mu{}^\nu \tilde{g}_{\nu\rho} = \bar{g}_{\mu\rho} \quad (\text{DRed})\tag{2.2.12}$$

Actually, this is the key to the consistent formulation of DRed in [13]. In section 2.4 we will comment on the structure of the spaces with these different metrics.

The example we have examined may look quite contrived, but identities of this kind are often used to simplify expressions in the presence of completely antisymmetric tensors  $\epsilon_{\mu_1\dots\mu_n}$ . This object can be defined formally by its rank and its antisymmetric character. Note that the definition is dimension-specific: even if we do not assign values to the indices,  $\epsilon$  is only

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<sup>4</sup> $\bar{g}$  and  $g$  here play the same role as  $\bar{g}$  and  $g$ , respectively, in DReg, except for the fact that in the latter method  $g^\mu{}_\mu = d$ .

defined with  $n$  indices. The relations

$$\sum_{\pi \in S_{n+1}} \text{sign}(\pi) \epsilon_{\mu_{\pi(1)} \dots \mu_{\pi(n)}} g^{\mu_{\pi(n)} \nu} = 0 \quad (2.2.13)$$

and

$$\epsilon_{\mu_1 \dots \mu_n} \epsilon_{\nu_1 \dots \nu_n} = \text{Det}(\mu_1 \dots \mu_n; \nu_1 \dots \nu_n) \quad (2.2.14)$$

are GnDI. They can lead to inconsistencies when used inside  $[\cdot]^R$ . For instance, 2.2.13 in  $n = 2$  implies

$$\begin{aligned} 0 &\stackrel{?}{=} [(\epsilon_{\mu\nu} g_{\rho\alpha} - \epsilon_{\mu\rho} g_{\nu\alpha} + \epsilon_{\nu\rho} g_{\mu\alpha}) I^{\mu\alpha}]^R \\ &= i\pi \epsilon_{\nu\rho} \end{aligned} \quad (2.2.15)$$

where we have used 2.1.5. One might be tempted to avoid some ambiguities by defining the left-hand side of 2.2.14 by its right-hand side, and in this way eliminate products of two  $\epsilon$  tensors until one at most remains in a given factor. This definition is ill-defined. For instance, in a product  $\epsilon_{\mu_1 \nu_1} \epsilon_{\mu_2 \nu_2} \epsilon_{\mu_3 \nu_3} \epsilon_{\mu_4 \nu_4}$ , it is possible to apply 2.2.14 to three different pairs of pairs of  $\epsilon$  tensors. The result with each choice is formally different and can give rise to different index contractions. Hence, when multiplied by a divergent integral, the result after renormalization may depend on how the four  $\epsilon$  tensors have been paired. This is analogous to the DRed inconsistency pointed out in [10]. On the other hand, no inconsistencies arise in any of the methods from GnDI such as 2.2.13 or 2.2.14 when the metric  $\bar{g}$  is used instead of  $g$ .

## 2.3 Dirac algebra

The Dirac matrices  $\gamma_\mu$  transform as vectors under Lorentz transformations. In dimensional methods, they cannot have explicit  $n$ -dimensional form, since the Lorentz indices do not take explicit integer values. They are defined as a formal representation of the Clifford algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \mathbb{1} \quad (\text{DReg}) \quad (2.3.1)$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\tilde{g}_{\mu\nu} \mathbb{1} \quad (\text{DRed}) \quad (2.3.2)$$

Trace identities follow in each case from these definitions, the cyclicity of the trace (which we assume throughout the thesis) and the value of the trace of the identity, which in both methods can be taken to be  $\text{Tr} \mathbb{1} = n$ . Because of the projection rule 2.2.5, even if the Dirac

algebra looks  $n$ -dimensional in DRed, this can be effectively changed by contractions with the integration momenta. In fact, the relation  $\not{k} = k^2$  is necessary to preserve numerator-denominator consistency. Implicit methods also treat Lorentz tensors in a formal way, as we have seen, so the Dirac matrices are naturally defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1} \quad (\text{implicit}) \quad (2.3.3)$$

where of course  $g$  here is (formally)  $n$ -dimensional.

The formal treatment of the Dirac algebra in all of these methods suffers from a fundamental problem when  $n$  is odd. This can be checked most easily in DReg [42]. First, 2.3.1 and the cyclicity of the trace imply

$$d\text{Tr}(\gamma_\mu) = \text{Tr}(\gamma_\mu\gamma_\alpha\gamma^\alpha) = \text{Tr}(\gamma_\alpha\gamma_\mu\gamma^\alpha) = 2\text{Tr}(\gamma_\mu) - \text{Tr}(\gamma_\mu\gamma_\alpha\gamma^\alpha) = (2-d)\text{Tr}(\gamma_\mu) \quad (2.3.4)$$

Hence, unless  $d = 1$ ,  $\text{Tr}(\gamma_\mu) = 0$ . Similar manipulations for a product of an odd number  $m$  of Dirac matrices lead to

$$(d-m)\text{Tr}(\gamma_{\mu_1}\dots\gamma_{\mu_m}) = 0 \quad (2.3.5)$$

Therefore,  $\text{Tr}(\gamma_{\mu_1}\dots\gamma_{\mu_m}) = 0$  unless  $d = m$ . Analytical continuation in  $d$  then requires all these products to vanish identically for all  $d$ . But this is incompatible with the fact that the product of  $n$  Dirac matrices in  $n$  fixed odd dimensions is proportional to the  $\epsilon$  tensor, a property that should be recovered after renormalization. To solve this problem, one must break the  $d$ -dimensional Lorentz covariance of the Dirac algebra changing  $g$  by  $\bar{g}$  in 2.3.1, as proposed in [42]. This is consistent with 2.2.4 but compromises numerator-denominator consistency. On the other hand, even if the definition 2.3.2 is employed in DRed, the problem reappears when the indices in the initial trace are contracted with integration momenta, due to the projection rule 2.2.5.

Presented in this way, the inconsistency in odd dimensions looks like a specific problem of the analytical continuation in  $d$ . However, it turns out that implicit methods also treat the Dirac algebra inconsistently when the dimension  $n$  is odd. Let us show it for the case  $n = 3$ , for definiteness. In three dimensions,<sup>5</sup>

$$\text{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 2\epsilon_{\mu\nu\rho} \quad (2.3.6)$$

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<sup>5</sup>Lorentz covariance guarantees that this trace is proportional to the  $\epsilon$  tensor. The numerical factor can be determined by agreement with the usual algebraic result, for instance using the Pauli matrices as a representation of the 3-dimensional Dirac algebra.

Then, from 2.3.2 and the cyclicity of the trace,

$$\begin{aligned}
\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) &= -\text{Tr}(\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_\tau) + 2g_{\mu\nu} \text{Tr}(\gamma_\rho \gamma_\sigma \gamma_\tau) \\
&= \text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) + 2(g_{\mu\nu} \text{Tr}(\gamma_\rho \gamma_\sigma \gamma_\tau) - g_{\mu\rho} \text{Tr}(\gamma_\nu \gamma_\sigma \gamma_\tau) + g_{\mu\sigma} \text{Tr}(\gamma_\nu \gamma_\rho \gamma_\tau) \\
&\quad - g_{\mu\tau} \text{Tr}(\gamma_\nu \gamma_\rho \gamma_\sigma)) \\
&= \text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) + 4(g_{\mu\nu} \epsilon_{\rho\sigma\tau} - g_{\mu\rho} \epsilon_{\nu\sigma\tau} + g_{\mu\sigma} \epsilon_{\nu\rho\tau} - g_{\mu\tau} \epsilon_{\nu\rho\sigma}) \tag{2.3.7}
\end{aligned}$$

which in view of 2.2.13 looks fine at first sight. However, as we have seen in the previous section the combination of  $\epsilon$  tensors and deltas in the last line needs not vanish inside  $[\cdot]^R$  when two of the indices are contracted with the integration momenta of a divergent integral. Therefore, the result of the calculations can be ambiguous.

From now on, we will assume that the dimension  $n$  is even, unless otherwise indicated. One of the most important limitations of not being able to employ GnDI is the absence of a finite complete set in Dirac space. In ordinary  $n$ -dimensional space, the antisymmetric products

$$[\mu_1 \dots \mu_m] = \frac{1}{m!} \sum_{\pi \in S_m} \text{sign}(\pi) \gamma_{\mu_{\pi(1)}} \dots \gamma_{\mu_{\pi(m)}}, \quad m = 1, \dots, n \tag{2.3.8}$$

together with the identity  $\mathbf{1}$ , form a linearly independent complete set of the space of  $2^{n/2} \times 2^{n/2}$  complex matrices.<sup>6</sup> In the formal  $n$ -dimensional space, the Dirac matrices cannot be understood as matrices of any specific dimension, so completeness must be defined also in a formal way. As shown in [43], many useful relations can be proven using only formal manipulations. The matrices  $[\mu_1 \dots \mu_m]$  are orthogonal with respect to the trace bilinear form. Then, a string of Dirac gamma matrices

$$S_{\alpha_1 \dots \alpha_m} = \gamma_{\alpha_1} \dots \gamma_{\alpha_m} \tag{2.3.9}$$

can always be written as

$$S_{\alpha_1 \dots \alpha_m} = a_0^{\alpha_1 \dots \alpha_m} \mathbf{1} + a_\mu^{\alpha_1 \dots \alpha_m} [\mu] + \dots + a_{\mu_1 \dots \mu_m}^{\alpha_1 \dots \alpha_m} [\mu_1 \dots \mu_m] \tag{2.3.10}$$

with ( $n$ -independent) coefficients given by

$$a_{\mu_1 \dots \mu_k}^{\alpha_1 \dots \alpha_m} = \frac{1}{nm!} \text{Tr}(S_{\alpha_1 \dots \alpha_m} [\mu_k \dots \mu_1]) \tag{2.3.11}$$

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<sup>6</sup>We are discussing the case of even  $n$ . For odd  $n$ , the set  $\{[\mu_1 \dots \mu_m], m = 1, \dots, (n-1)/2\}$  is a complete set of  $2^{(n-1)/2} \times 2^{(n-1)/2}$  matrices.

Therefore,  $\mathcal{B} = \{\mathbf{1}, [\mu_1], [\mu_1\mu_2], \dots\}$  is a countable Hamel basis of the formal Dirac space, defined as the set of arbitrary linear combinations of strings of the form 2.3.9 (including the case with  $m = 0$ ,  $S = \mathbf{1}$ ). The main difference with a genuine  $n$ -dimensional space is that the objects  $[\mu_1 \dots \mu_m]$  do not vanish for  $m > n$ , so the space is infinite-dimensional. For instance, in formal  $n$ -dimensional space we have

$$S_{\mu\nu\rho} = g_{\mu\nu}\gamma_\rho - g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu + [\mu\nu\rho] \quad (2.3.12)$$

which is valid for any even  $n$ , including  $n = 2$ . Using the mentioned GnDI, in  $n = 2$  we could instead simplify this expression to

$$S_{\mu\nu\rho} = g_{\mu\nu}\gamma_\rho - g_{\mu\rho}\gamma_\nu + g_{\nu\rho}\gamma_\mu \quad (2.3.13)$$

But as stressed already many times, such simplifications are dangerous before renormalization.

The standard Fierz identities in  $n$  dimensions can be derived using the completeness of  $\{\mathbf{1}, [\mu_1], \dots, [\mu_1 \dots \mu_n]\}$ . Similarly, in the formal  $n$ -dimensional space one can derive Fierz identities from the completeness of  $\mathcal{B}$ . However, the Fierz reorderings in this case involve in general an infinite number of terms, just as in DReg [44], which makes them less useful. Moreover, the invariance under supersymmetry transformations of the action of supersymmetric theories relies on genuine  $n$  dimensional Fierz identities (and also on an anticommuting  $\gamma_5$ ). In fact, as shown in [11, 12, 13], the supersymmetry Ward identities are violated when relevant GnDI are not fulfilled.

In even dimensions, Weyl spinors can be defined from Dirac spinors by chiral projectors constructed with  $\gamma_5$ .<sup>7</sup> Several definitions of  $\gamma_5$  are in principle possible in the methods we are considering. First, it can be defined formally by the basic property

$$\{\gamma_5^{\text{AC}}, \gamma_\mu\} = 0 \quad (2.3.14)$$

where the label AC has been introduced to distinguish this definition from the one we favor below. This simple definition is consistent, as has been proven in [13] by explicit construction. Unfortunately, in all the methods we consider, it is incompatible with the correct  $n$ -dimensional value of odd-parity traces. This fact is well known in DReg [4, 5]. In  $n = 2$ ,

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<sup>7</sup>We call this object  $\gamma_5$  for any integer dimension  $n$ . Because in this thesis we never write Lorentz indices with explicit integer values, no confusion with  $\gamma_\mu$  should arise.

for example, after renormalization we would like to recover the standard value

$$\text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu) = -2\epsilon_{\mu\nu} \quad (2.3.15)$$

On the other hand, using 2.3.14 and the DReg rules in  $\text{Tr}(\gamma_5^{\text{AC}} \gamma_\mu \gamma_\nu \gamma_\rho \gamma^\rho)$ , it follows that

$$d(d-2)\text{Tr}(\gamma_5^{\text{AC}} \gamma_\mu \gamma_\nu) = 0 \quad (2.3.16)$$

which shows that  $\text{Tr}(\gamma_5^{\text{AC}} \gamma_\mu \gamma_\nu)$  vanishes identically and 2.3.15 cannot be recovered in the limit  $d \rightarrow 2$ . DRed faces the same situation when the free indices in the initial trace are contracted with integration momenta, due to the projection rule 2.2.5 [5]. Once again, this issue appears as well in implicit methods. Indeed, 2.3.14 and 2.3.15 imply

$$\text{Tr}(\gamma_5^{\text{AC}} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = \text{Tr}(\gamma_5^{\text{AC}} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) - 2\epsilon_{\nu\rho} g_{\mu\sigma} + 2\epsilon_{\mu\rho} g_{\nu\sigma} - 2\epsilon_{\mu\nu} g_{\rho\sigma} \quad (2.3.17)$$

Again, in spite of GnDI 2.2.13 and as shown in 2.2.15,  $-\epsilon_{\nu\rho} g_{\mu\sigma} + \epsilon_{\mu\rho} g_{\nu\sigma} - \epsilon_{\mu\nu} g_{\rho\sigma}$  can be nonzero inside  $[\cdot]^R$ , which then leads to a contradiction.<sup>8</sup> In the same way it can be shown that  $\text{Tr}(\gamma_5^{\text{AC}} \gamma_{\mu_1} \dots \gamma_{\mu_n}) = 0$  for any even  $n$ . This is certainly not what one would want in an  $n$ -dimensional method and it shows that the definition 2.3.14 does not provide a correct regularization of arbitrary diagrams in a chiral theory. Note that other traces with one  $\gamma_5^{\text{AC}}$  matrix also vanish, since they must be antisymmetric and there is no Lorentz-covariant completely antisymmetric tensor of rank  $m \neq n$ . This can be proven more explicitly by relating them to  $\text{Tr}(\gamma_5^{\text{AC}} \gamma_{\mu_1} \dots \gamma_{\mu_n})$ , and can be extended to traces with an odd number of  $\gamma_5^{\text{AC}}$  matrices if  $(\gamma_5^{\text{AC}})^2 = -\mathbb{1}$ , a property which is required to form chiral projectors.

This problem of  $\gamma_5^{\text{AC}}$  reappears in a more subtle form in open fermion lines. To see this, assume for a moment that  $\gamma_5^{\text{AC}}$  belongs to the formal Dirac space, as defined above. Then, using the completeness of  $\mathcal{B}$ , we would find

$$\begin{aligned} \gamma_5^{\text{AC}} &= \frac{1}{n} \text{Tr}(\gamma_5^{\text{AC}}) \mathbb{1} + \frac{1}{2n} \text{Tr}(\gamma_5^{\text{AC}} [\nu\mu]) [\mu\nu] + \frac{1}{24n} \text{Tr}(\gamma_5^{\text{AC}} [\sigma\rho\nu\mu]) [\mu\nu\rho\sigma] + \dots \\ &= 0 \end{aligned} \quad (2.3.18)$$

Therefore, if  $\gamma_5^{\text{AC}}$  is to be nontrivial, it cannot belong to the formal Dirac space.<sup>9</sup> But then, the eventual projection into the standard Dirac space of genuine  $n$ -dimensional space, which

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<sup>8</sup>This argument in even dimensions is almost identical to the one above in odd dimensions. Taking into account that the usual candidate for  $\gamma_5$  is proportional to the identity in odd dimensions, we see that the origin of the inconsistencies is essentially the same in odd and even dimensions.

<sup>9</sup>This is apparent in the explicit construction of [13].

is a subset of the former, will annihilate it. So, to recover standard Dirac strings with  $\gamma_5$  matrices, one needs to replace by hand  $\gamma_5^{\text{AC}}$  by  $\gamma_5$  after renormalization. It does not seem obvious to us that this *ad hoc* replacement in multiloop amplitudes will respect unitarity.

An alternative definition of  $\gamma_5$  is to generalize its explicit definition in genuine  $n$  dimensions in terms of the Dirac matrices:

$$\gamma_5 = \frac{1}{n!} \epsilon^{\mu_1 \dots \mu_n} \gamma_{\mu_1} \cdots \gamma_{\mu_n} \quad (2.3.19)$$

This is akin to the original tHV definition in DReg [3] and is the definition we will use in the following, unless otherwise indicated. Note that, even if we are not restricting the indices to have  $n$  different values, this object is  $n$ -dimensional in the sense that it contains  $n$  Dirac matrices. Furthermore, in view of 2.2.11, we can write 2.3.19 in the alternative form

$$\gamma_5 = \frac{1}{n!} \epsilon^{\mu_1 \dots \mu_n} \bar{\gamma}_{\mu_1} \cdots \bar{\gamma}_{\mu_n} \quad (2.3.20)$$

where  $\bar{\gamma}_\mu = \bar{g}_\mu{}^\nu \gamma_\nu$ . Like any other explicit definition, 2.3.19 does not introduce any consistency issues by itself. The non-trivial question is which familiar properties of the  $\gamma_5$  can be proven without using dangerous GnDI. The most important of these properties is the anticommutation with the Dirac matrices, but from the discussion above it is clear that this property cannot hold for the definition 2.3.19 in any of the methods we are discussing.<sup>10</sup> Indeed, for  $n = 2$ , for instance, 2.3.19 and 2.3.2 give

$$\text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) + \text{Tr}(\gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\sigma) = -4(g_{\mu\nu} \epsilon_{\rho\sigma} - g_{\mu\rho} \epsilon_{\nu\sigma} + g_{\mu\sigma} \epsilon_{\nu\rho}) \quad (2.3.21)$$

This expression vanishes when it accompanies finite integrals. However, using 2.2.15 we get

$$[(\text{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) + \text{Tr}(\gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\sigma)) I_{\mu\sigma}]^R = 4i\pi \epsilon_{\rho\nu} \quad (2.3.22)$$

In the same vein, let us point out that some of the explicit trace expressions of odd-parity products of Dirac matrices in the literature have been simplified with the help of the GnDI 2.2.13. To avoid inconsistencies, only the complete expressions derived from 2.3.19 and 2.3.2 or 2.3.3 should be used before renormalization. The nonvanishing anticommutator

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<sup>10</sup>The fact that  $\gamma_5$  does not anticommute with the Dirac matrices has already been observed in FDR [20] and CIReg [45, 46].



$\{\gamma_5, \gamma_\mu\}$  can be written in a simple form using  $\bar{g}$ . First, observe that in  $n = 2$ ,

$$\begin{aligned}
0 &= (\gamma_\mu \gamma_\nu \gamma_\rho)(\epsilon^{\mu\nu} \bar{g}_{\rho\alpha} - \epsilon^{\mu\rho} \bar{g}_{\nu\alpha} + \epsilon^{\nu\rho} \bar{g}_{\mu\alpha}) \\
&= -\epsilon^{\mu\rho} (\gamma_\mu \gamma_\rho \bar{\gamma}_\alpha + \bar{\gamma}_\alpha \gamma_\rho \gamma_\mu) \\
&= \{\gamma_5, \bar{\gamma}_\alpha\}
\end{aligned} \tag{2.3.23}$$

From this, similarly to DReg, we find

$$\{\gamma_5, \gamma_\alpha\} = 2\gamma_5 \hat{\gamma}_\alpha \tag{2.3.24}$$

where we have introduced the evanescent metric  $\hat{g} = g - \bar{g}$ , which has trace  $\hat{g}^\mu{}_\mu = 0$ , to write the evanescent matrix  $\hat{\gamma}_\mu = \hat{g}_\mu{}^\nu \gamma_\nu = \gamma_\mu - \bar{\gamma}_\mu$ , and used the fact that this matrix commutes with  $\gamma_5$ . Indeed, in  $n = 2$ ,

$$\begin{aligned}
[\gamma_5, \hat{\gamma}_\alpha] &= [\gamma_5, \gamma_\beta](g^\beta{}_\alpha - \bar{g}^\beta{}_\alpha) \\
&= \frac{1}{2} \epsilon^{\mu\nu} [\gamma_\mu \gamma_\nu, \gamma_\beta](g^\beta{}_\alpha - \bar{g}^\beta{}_\alpha) \\
&= -2\epsilon^{\beta\mu} \gamma_\mu (g_{\beta\alpha} - \bar{g}_{\beta\alpha}) \\
&= 0
\end{aligned} \tag{2.3.25}$$

due to 2.2.11. The proof of 2.3.24 can be generalized to arbitrary even  $n$ . Let us also note in passing the useful relations

$$\begin{aligned}
\{\bar{\gamma}_\mu, \bar{\gamma}_\nu\} &= \{\bar{\gamma}_\mu, \gamma_\nu\} = 2\bar{g}_{\mu\nu} \\
\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} &= \{\hat{\gamma}_\mu, \gamma_\nu\} = 2\hat{g}_{\mu\nu} \\
\{\bar{\gamma}_\mu, \hat{\gamma}_\nu\} &= 0
\end{aligned} \tag{2.3.26}$$

which follow from the definitions of the involved objects. Similarly,  $\bar{g}$  can be used to show that  $\gamma_5^2 = -\mathbf{1}$  in any even  $n$ . In  $n = 2$ , for example,

$$\begin{aligned}
\gamma_5 \gamma_5 &= \frac{1}{4} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \\
&= \frac{1}{4} (\bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - \bar{g}^{\mu\sigma} \bar{g}^{\nu\rho}) \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \\
&= -\mathbf{1}
\end{aligned} \tag{2.3.27}$$

In the second line we have used the GnDI 2.2.14, involving only the  $\epsilon$  tensors. In the last one, 2.3.2 and 2.2.9.

## 2.4 A consistent procedure in implicit fixed-dimension methods

In even dimension  $n$ , the inconsistencies of DRed can be avoided simply by forbidding the use of GnDI before renormalization, as proposed in [11, 12]. That is, the  $n$ -dimensional space to be used in a consistent version of DRed is not the genuine  $n$ -dimensional space (GnS), but a quasi- $n$ -dimensional space (QnS). Similarly to the case of quasi- $d$ -dimensional space (QdS) in DReg [47], QnS can be defined explicitly as an infinite-dimensional vector space endowed with a metric  $\tilde{g}$ , which satisfies  $\tilde{g}^\mu{}_\mu = n$  [13]. The relation with QdS is given by the direct-sum structure  $\text{QnS} = \text{QdS} \oplus \text{Q}\varepsilon\text{S}$ . Dirac matrices in the three spaces have been explicitly constructed in [13], following [47].

We propose here to define implicit methods in the same QnS. In this case, there is no need to embed QdS in it, so the setup is simpler. Moreover, the metric can be called  $g$  without confusion, in agreement with our notation thus far. Forbidding GnDI is actually not sufficient in fixed dimension, since the discrimination of Lorentz tensors is not automatic. As anticipated above, we need to specify some normal form of the expressions to uniquely identify the different tensor structures.<sup>11</sup> Following [4], we propose to simplify arbitrary Feynman diagrams with the following algorithm, which leads to a unique normal form:

- (i) All  $\gamma_5$  are substituted by their tHV definition 2.3.19.
- (ii) All Dirac matrices are removed from denominators.
- (iii) Dirac traces are computed using  $\text{Tr}AB = \text{Tr}BA$ , 2.3.2 and  $\text{Tr}\mathbf{1} = n$ .
- (iv) Products of Dirac gammas are decomposed into sums of antisymmetric combinations as in 2.3.10 and 2.3.11.
- (v) All possible contractions are performed, using  $g_\mu{}^\nu V_{\dots\nu\dots} \rightarrow V_{\dots\mu\dots}$  for arbitrary tensors  $V$ .
- (vi)  $g^\mu{}_\mu$  is replaced by  $n$ .

As we work in QnS from the start, GnDI cannot be applied. Indeed, if GnDI were allowed, the resulting expression would not have unique form, which could eventually translate into different renormalized results. There are however exceptions to this prohibition, which are discussed below. After performing the algebraic manipulations in steps (i–vi), the diagram

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<sup>11</sup>As a matter of fact, some standard form is also required in the dimensional methods to display explicitly all the  $d$  dependence and thus be able to apply  $\overline{\text{MS}}$  or  $\overline{\text{MS}}$  without ambiguities.

will be a sum of terms that contain  $\epsilon$  tensors, metrics with free indices, antisymmetric arrays of gamma functions, external momenta, possible background tensors and a tensor (multi-dimensional) integral  $T$ . In this way, the different integrals  $T$  that appear in a given diagram are determined. They are then to be renormalized as prescribed in the different methods. After this, there is no harm in using GnDI. In particular, they can and should be used after subtraction to simplify the final results. Note in particular that, because the final antisymmetric combinations of Dirac matrices  $[\alpha_1 \dots \alpha_m]$  are not touched by renormalization, only the combinations with  $m \leq n$  need to be included in the decomposition of step (iv).

Sometimes selected GnDI can be used to simplify expressions from the very beginning, as long as one is sure that they will not change the contractions of indices in the loop integrals  $T$ . One simple example in  $n = 2$  is using  $\epsilon^{\mu\nu}\epsilon_{\mu\nu} = 2$ . More generally, we can simplify the calculations significantly using the metric  $\bar{g}$ , defined above. The rules it obeys, 2.2.9, can be understood as the consequence of the structure  $\text{QnS} = \text{GnS} \oplus \text{X}$ , with  $g$ ,  $\bar{g}$  and  $\hat{g} = g - \bar{g}$  the metrics in QnS, GnS and the extra space X, respectively. Remember that the defining property of  $\bar{g}$  in implicit methods is that it commutes with renormalization. In expressions related to loop integrals, such as 2.1.3, or in the traces of Dirac matrices, it is still the ordinary metric  $g$  of QnS that appears, to comply with shift invariance and numerator-denominator consistency. The idea here is to allow for GnDI that involve *only*  $\bar{g}$ , the  $\epsilon$  tensor and external momenta or fields. Then,  $\bar{g}$  can appear as the result of these GnDI. Using such GnDI spoils the uniqueness of the normal form. However, the resulting expressions have the same renormalized value, thanks to 2.2.10. As a straightforward illustration in  $n = 2$ ,

$$\begin{aligned} [\epsilon^{\mu\nu}\epsilon_{\nu}{}^{\rho}I_{\mu\rho}]^R &= \epsilon^{\mu\nu}\epsilon_{\nu}{}^{\rho} [I_{\mu\rho}]^R \\ &= 2g^{\mu\rho} [I_{\mu\rho}]^R \\ &= [2\bar{g}^{\mu\rho}I_{\mu\rho}]^R \end{aligned} \tag{2.4.1}$$

In a next section we give simple examples that illustrate how the calculations can be simplified with the help of  $\bar{g}$  and related objects.

The same simplifications are valid also in the consistent version of DRed [13] with a tHV  $\gamma_5$ . The only difference is that in this method four different spaces are used, related by  $\text{QnS} = \text{QdS} \oplus \text{Q}\epsilon\text{S}$  and  $\text{QdS} = \text{GnS} \oplus \text{Q}(-\epsilon)\text{S}$ . Then, we can identify the extra space in fixed dimension with  $X = \text{Q}(-\epsilon)\text{S} \oplus \text{Q}\epsilon\text{S}$ . The relations between the metrics in 2.2.5 and 2.2.9 are those implied by this hierarchical structure, with  $\tilde{g}$ ,  $g$  and  $\bar{g}$  the metrics in QnS, QdS and GnS, respectively.

We have already pointed out that GnDI can be safely used after tensor identification.

Indeed, after that step,  $g$  behaves as  $\bar{g}$ . This is specially relevant to FDR, as in this method some useful shortcuts exist to identify tensors from the very start. As a salient example, in one-loop diagrams with fermion lines that do not have indices contracted with the ones in other fermion lines, it is easy to see that the correct  $\mu^2$  shifts can be obtained by shifting the integration momenta as  $1/\not{k} \rightarrow 1/(\not{k} \pm \mu)$ , with opposite signs for  $\not{k}$  separated by an even number of  $\gamma$  matrices and equal signs for those separated by an odd number of  $\gamma$  matrices. For this, it is important that terms with odd powers of  $\mu$  do not contribute after the limit  $\mu \rightarrow 0$ . We can easily generalize this rule to spinor chains that contain  $\gamma_5$  matrices: because, according to its definition 2.3.19,  $\gamma_5$  contains an even number of  $\gamma$  matrices in even dimension, the  $\gamma_5$  matrices should just be ignored in the determination of the signs. This approach allows, for instance, to use an anticommuting  $\gamma_5$  before evaluating Dirac traces. The results are unique and agree with the ones obtained from the normal form or with the  $\bar{g}$  formalism. When one Lorentz index is contracted between different fermion lines, a similar, more complicated rule can be found which gives the right  $\mu^2$  [48]. Modifications may also be necessary in diagrams that contain both Dirac traces and derivative interactions. To the best of our knowledge, no general prescription exists to treat any diagram in this way. A very similar idea is used in FDF. In this dimensional method, the necessary  $\mu^2$  are obtained from the extra-dimensional components of integration momenta and a set of selection rules for the extra-dimensional space (see also [49]). Then, GnDI are valid and  $\gamma_5$  anticommutes with the Dirac matrices. Comparing with the situation in FDR, it seems that in order to comply with the quantum action principle the method will require some refinements for multiloop calculations.

The consistent procedure for implicit methods in QnS can in principle be applied to multi-loop calculations. Let us sketch how the renormalization of a Feynman diagram could proceed. First, the diagram is expressed in its normal form, following the steps above. Allowed GnDI involving  $\bar{g}$  can be optionally used. Then, each tensor integral  $I$  is treated with Bogoliubov's recursive  $R$ -operation [37, 38] (or equivalently its solution, Zimmermann's Forest Formula [39]), in order to guarantee locality and unitarity of the renormalized theory. To do this, a subtraction operator, which selects the singular part of a primitively divergent (sub)graph  $\Gamma$  of  $I$ , can be defined without any explicit regularization as  $\mathcal{T}\Gamma = \Gamma - \mathcal{R}(\Gamma)$  [40]. Here,  $\mathcal{R}(\Gamma)$  is  $\Gamma$  with its (overall) divergence subtracted. Then,  $\mathcal{T}$  is applied according to Bogoliubov's formula.

This systematic method has been used in differential renormalization [40] and in CIReg [34], but only in non-derivative scalar theories, which have a simple tensor structure. In more complicated theories, it is essential to treat tensor integrals consistently. To do this, in calculating

$\mathcal{R}(\Gamma)$  for a tensor  $\Gamma \subset I$ , the Lorentz indices in  $\Gamma$  that are contracted with indices in  $I \setminus \Gamma$  should be treated as uncontracted free indices. This is a necessary condition to preserve invariance under shifts of the integration momenta in  $\Gamma$  that are proportional to the integration momenta in  $I \setminus \Gamma$ . We will not try to prove here that it is also a sufficient condition for shift invariance of the final renormalized multi-loop integrals. This issue has been addressed in particular examples in CIReg [50] and FDR [51]. We believe that the so-called *extra-extra integrals* that are introduced in FDR to impose sub-integration consistency are equivalent to the contribution of (sums of) forests with the tensor rule above. They are also related to the DRed contributions of  $\varepsilon$  scalars associated to virtual vector bosons, which renormalize independently.

Finally, we should stress that, even if implicit methods as treated in this section are consistent and preserve shift invariance and numerator-denominator consistency, some particular Ward identities based on GnDI may be broken. This is the origin of chiral anomalies and of the breakdown of supersymmetry. Also vectorial Ward identities associated to gauge invariance can be broken in the presence of the tHV  $\gamma_5$ , giving rise to spurious anomalies that must be eliminated with additional finite counterterms. We will give an example of this in the next section. In this regard, these methods are not better or worse than DReg.

## 2.5 FDR. An implicit method

In this section we will explain one of the implicit methods we mentioned before in order to put into practice in the next section all the rules we have discussed. To this purpose we have chosen FDR (Four-Dimensional Regularization/Renormalization). As a good implicit method, it does not need an UV regulator that forces the subsequent insertion of counterterms, and that is why the “R” in FDR stands for Regularization and Renormalization at the same time, because both are performed at once. This approach modifies directly the integrand with the help of partial fraction decomposition, and an “extra mass” that acts as a renormalization scale at the end of the calculation as well as becoming massless theories IR-safe. It also deals with algebra and integrals in fixed dimension  $n$  which allows the full use of numerical tools. As we said before, in spite of its name, it can be used in whatever dimension we want. Furthermore, the FDR integral is shift and gauge-invariant. So, all these mentioned properties make it a worthy method to be studied. The rules explained here are extracted from [20, 23].

### 2.5.1 Basic rules

One of its two bases is the introducing of something like a fictitious mass called  $\mu$ , which will be taken to zero after evaluating the integral. This mass is inserted following the so-called *global prescription*. It means that all squared loop momenta made by Feynman rules have to be shifted as  $q^2 \rightarrow q^2 - \mu^2$  even in the numerator. To simplify the notation, we define

$$\bar{q}^2 \equiv q^2 - \mu^2 \quad (2.5.1)$$

$$\bar{D} \equiv (q + p)^2 - m^2 - \mu^2 \quad (2.5.2)$$

The other base of the method is the partial fraction decomposition. And it is as simple as follows

$$\frac{1}{\bar{D}} = \frac{1}{\bar{q}^2} + \frac{m^2 - p^2 - 2q \cdot p}{\bar{q}^2 \bar{D}} \quad (2.5.3)$$

This trick can be repeated as many times as it is necessary because on the right hand side, the old denominator appears. Every time, the degree of divergence of the new terms are reduced and purely divergent fractions without any scale in the denominators are isolated. The rule is to stop expanding a term with scale when it is convergent. Obviously, we have to keep in mind the numerator to find out whether a term is convergent or not making use of power-counting. One important thing is that, as we see in (2.5.3), the denominators of divergent isolated fractions have to be powers of  $\bar{q}^2$  and not  $q^2$ . When all these divergent fractions are found, they are removed of the integration. This way the divergences vanish and the integral becomes finite.

Let us see these properties with two simply examples: one-loop photon and fermion self-energy diagrams in massless QED (fig. 2.5.1 and 2.5.2).

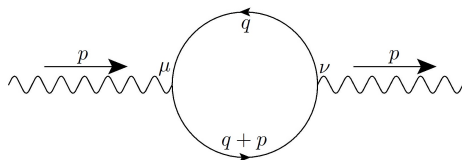


Figure 2.5.1: One-loop photon self-energy in massless QED.

According to QED Feynman rules fig. 2.5.1 can be written as

$$\Pi_{\mu\nu} = ie^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})]}{q^2 (q + p)^2} \quad (2.5.4)$$

Power-counting, we have a quadratically divergent integral so there is a difference of two powers of loop momentum between numerator and denominator.<sup>12</sup> So we expand the denominator as we have learnt.

$$\begin{aligned} \frac{1}{\bar{D}} &= \frac{1}{\bar{q}^2} - \frac{p^2 + 2q \cdot p}{\bar{q}^2 \bar{D}} = \frac{1}{\bar{q}^2} - \frac{p^2 + 2q \cdot p}{\bar{q}^4} + \frac{(p^2 + 2q \cdot p)^2}{\bar{q}^4 \bar{D}} = \\ &= \frac{1}{\bar{q}^2} - \frac{p^2 + 2q \cdot p}{\bar{q}^4} + \frac{4(q \cdot p)^2}{\bar{q}^6} + \frac{p^2(p^2 + 4q \cdot p)}{\bar{q}^4 \bar{D}} - \frac{4(q \cdot p)^2(p^2 + 2q \cdot p)}{\bar{q}^6 \bar{D}} \end{aligned} \quad (2.5.5)$$

where in this case  $\bar{D} \equiv (q + p)^2 - \mu^2$ . We had to expand  $\bar{D}$  three times to find the desired finite parts (last two in eq. 2.5.5). Before perform the integral, we cannot forget to implement *global prescription* also in the numerator. As we have fermions, we have to perform the trace to find  $q^2$  and then add it the “mass”.<sup>13</sup>

Now, we have our FDR integral ready to be evaluated.

$$\Pi_{\mu\nu} = ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} (\text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] + 4\mu^2 g_{\mu\nu}) \left[ \frac{p^2(p^2 + 4q \cdot p)}{\bar{q}^6 \bar{D}} - \frac{4(q \cdot p)^2(p^2 + 2q \cdot p)}{\bar{q}^8 \bar{D}} \right] \quad (2.5.6)$$

The term proportional to  $\mu^2$  in the numerator is called extra-integral. This is integrated along with the same denominator expansion as the original numerator because it must be considered as a loop momentum when power-counting. If we carry out the method as we did, first the denominator expansion and then the numerator shift, there will be not hesitation about this, but we have to be careful if we effectuate it reversed. Instead of using the finite parts of (2.5.5), we can also use the infinite ones and subtract them to the original denominator. Sometimes this integral is easier than the previous one, especially when performing extra-integrals because they always go to zero along with the original denominator. So we can write an alternative version of  $\Pi_{\mu\nu}$  as follows

$$\Pi_{\mu\nu} = ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} (\text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] + 4\mu^2 g_{\mu\nu}) \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{p^2 + 2q \cdot p}{\bar{q}^6} - \frac{4(q \cdot p)^2}{\bar{q}^8} \right] \quad (2.5.7)$$

Although this variant looks simpler, we recall that now every term (except for the extra-

<sup>12</sup>In fact, there is also a linearly divergent term, but in this example we can consider the whole integral quadratically divergent thanks to parity. Since every new arisen piece of the expansion is alternatively even or odd, a certain part of the numerator always can go together with a expansion which belongs to a degree of divergence incremented by one. Nevertheless, there is no other advantage than compactness.

<sup>13</sup>In spite of we have learnt that we can shift loop momenta with  $\pm\mu$  within the trace, the safest process to avoid inconsistencies is to find  $q^2$ , and then shift.

integral) is divergent. So we will have to use Dimensional Regularization to evaluate the integral, but renormalization will be not necessary because all the poles will vanish automatically thanks to the denominator expansion. Hence we have two possibilities to compute the physical amplitude and we will choose one of them depending on the situation: one purely four-dimensional (2.5.6), and one where Dimensional Regularization appears but only at its first stage and concerning only to integrals and not to Dirac algebra (2.5.7). Of course, both lead to the same result (2.5.8).

$$\Pi_{\mu\nu} = \frac{\alpha}{3\pi} \left[ \frac{5}{3} - \log \left( \frac{p^2}{\mu^2} \right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (2.5.8)$$

As we said before, the fictitious mass  $\mu$  disappears from the result besides logarithms where acts as a renormalization scale.

Let us show the procedure again but with another example (fig. 2.5.2).

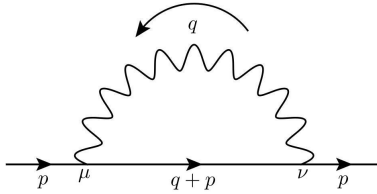


Figure 2.5.2: One-loop fermion self-energy in massless QED.

According to fig. 2.5.2 and QED Feynman rules:

$$\Sigma = -ie^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu (\not{q} + \not{p}) \gamma^\mu}{q^2 (q+p)^2} \quad (2.5.9)$$

Now we have a linear divergence<sup>14</sup>, so

$$\frac{1}{\bar{D}} = \frac{1}{\bar{q}^2} - \frac{p^2 + 2q \cdot p}{\bar{q}^2 \bar{D}} = \frac{1}{\bar{q}^2} - \frac{2q \cdot p}{\bar{q}^4} - \frac{p^2}{\bar{q}^2 \bar{D}} + \frac{2q \cdot p (p^2 + 2q \cdot p)}{\bar{q}^4 \bar{D}} \quad (2.5.10)$$

This graph has no possibilities of creating a squared loop momentum in the numerator, thus no more shifts need to be done. Therefore we have already reached the final outcome.

$$\Sigma = ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} [\gamma_\mu (\not{q} + \not{p}) \gamma^\mu] \left[ \frac{p^2}{\bar{q}^4 \bar{D}} - \frac{2q \cdot p (p^2 + 2q \cdot p)}{\bar{q}^6 \bar{D}} \right] \quad (2.5.11)$$

<sup>14</sup>Likewise, we can consider the whole integral linearly divergent because of parity.



Also, just like the other diagram, we can write the result using the infinities instead of the finite denominators.

$$\Sigma = -ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} [\gamma_\mu (\not{q} + \not{p}) \gamma^\mu] \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{2q \cdot p}{\bar{q}^6} \right] \quad (2.5.12)$$

As it is expected, both lead to the same result with  $\mu$  only working as a renormalization scale.

$$\Sigma = -\frac{\alpha}{4\pi} \not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (2.5.13)$$

## 2.5.2 Multi-loop rules

As we have seen with these examples, computing a diagram using FDR is so easy, because its rules are very simple. Nevertheless, when we move to two-loops and beyond things get harder. Fraction decomposition now has to take into account overall divergence as well as subdivergences, which complicates the expansion. Furthermore, in the numerator, besides squared loop momenta also their products need to be shifted according to *global prescription*. Moreover, a *subprescription* appears to ensure subintegration consistency, i.e. renormalizing the subdiagram and then inserting it back to the diagram and renormalizing again must lead to the same result as renormalizing the whole diagram at once. This *subprescription* consists, in addition to *global prescription*, in shifting subdivergent loop momenta only in the subdiagram and in the whole diagram as if the others were external and then subtracting the later to the former. Then this new part proportional to  $\mu^2$  must be integrated and taken its corresponding limit  $\mu \rightarrow 0$ , before continuing with the remaining integrals. The important point here is whether there is or not index contraction between the subdiagram and the outer part. If there is some index contraction this part will give some contribution, if not it will vanish. This new kind of integral is called extra-extra integral, because it is very similar to extra-integrals but with the new norm of integrating and taking its limit beforehand. So the numerator of a two-loop diagram must be modified as follows

$$\mathcal{N}(q_1, q_2) \rightarrow \bar{\mathcal{N}}(q_1^2 \rightarrow \bar{q}_1^2, q_2^2 \rightarrow \bar{q}_2^2, q_1 \cdot q_2 \rightarrow \overline{q_1 \cdot q_2}) + \sum_{i=1}^2 \left[ \hat{\mathcal{N}}_i(q_i^2 \rightarrow \bar{q}_i^2) - \bar{\mathcal{N}}_i(q_i^2 \rightarrow \bar{q}_i^2) \right] \quad (2.5.14)$$

where a bar symbolizes the normal shifting and a hat that those  $\mu$  have to be taken to zero before performing the external integral. Obviously, the sum only concerns to subdivergent subdiagrams.

Let us review all these new rules with a two-loop example (fig. 2.5.3). Using QED

Feynman rules:

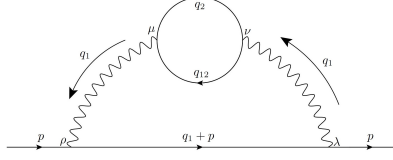


Figure 2.5.3: Two-loop fermion self-energy in massless QED.

$$\Sigma = e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu \text{Tr} [\gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12}]}{q_1^4 D_1 q_2^2 q_{12}^2} \quad (2.5.15)$$

where  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ . As we see in the graph, there are one subdivergence when  $q_2$  grows and  $q_1$  remains finite, and an overall divergence when both momenta get large.<sup>15</sup> We begin expanding the denominator source of the subdivergence.

$$\begin{aligned} \frac{1}{\bar{q}_{12}^2} &= \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2 \bar{q}_{12}^2} = \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^4} + \frac{(q_1^2 + 2q_1 \cdot q_2)^2}{\bar{q}_2^4 \bar{q}_{12}^2} = \\ &= \frac{1}{\bar{q}_2^2} - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^4} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^6} + \frac{q_1^2 (q_1^2 + 4q_1 \cdot q_2)}{\bar{q}_2^4 \bar{q}_{12}^2} - \frac{4(q_1 \cdot q_2)^2 (q_1^2 + 2q_1 \cdot q_2)}{\bar{q}_2^6 \bar{q}_{12}^2} \end{aligned} \quad (2.5.16)$$

And then we continue with the denominator with external momenta. We cannot expand  $q_{12}$  in favour of  $q_1$  or  $q_2$  when isolating the global divergence because both momenta get large. If there would be a term like  $D_2$  or  $D_{12}$ , we have had to expand them too, but we only have a  $D_1$  so this is the only term we must decompose in order to set apart the overall divergence.

$$\frac{1}{\bar{D}_1} = \frac{1}{\bar{q}_1^2} - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2 \bar{D}_1} = \frac{1}{\bar{q}_1^2} - \frac{2q_1 \cdot p}{\bar{q}_1^4} - \frac{p^2}{\bar{q}_1^2 \bar{D}_1} + \frac{2q_1 \cdot p (p^2 + 2q_1 \cdot p)}{\bar{q}_1^4 \bar{D}_1} \quad (2.5.17)$$

Before implementing *global prescription*, we have to contract all indices in the numerator and then we will be able to shift all squared momenta and their products. Let the numerator be

$$\begin{aligned} \mathcal{N} &= \gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu \text{Tr} [\gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12}] = \\ &= 4 \left[ \not{q}_2 (\not{q}_1 + \not{p}) \not{q}_{12} + \not{q}_{12} (\not{q}_1 + \not{p}) \not{q}_2 + 2q_2 \cdot q_{12} (\not{q}_1 + \not{p}) \right] = \end{aligned}$$

<sup>15</sup>Again, when we power-count searching for the overall divergence as well as for the subdivergence, we notice two different powers. Since the difference is one, we can consider the whole numerator having the same order because of parity.

$$= 8 \left[ q_{12} \cdot (q_1 + p) \not{q}_2 + q_2 \cdot (q_1 + p) \not{q}_{12} \right] \quad (2.5.18)$$

Once indices have been contracted, we can now shift squared momenta and their products as follows

$$q_i^2 \rightarrow \bar{q}_i^2 = q_i^2 - \mu^2 \quad (2.5.19)$$

$$q_i \cdot q_j = \frac{1}{2} (q_{ij}^2 - q_i^2 - q_j^2) \rightarrow \frac{1}{2} (\bar{q}_{ij}^2 - \bar{q}_i^2 - \bar{q}_j^2) = q_i \cdot q_j + \frac{\mu^2}{2} \quad (2.5.20)$$

So the numerator becomes

$$\bar{\mathcal{N}} = \mathcal{N} + 4\mu^2 \not{q}_1 \quad (2.5.21)$$

The last step is finding the *subprescription* we mentioned before. To do that, we must shift the subdiagram numerator without contracting its indices with the external ones and considering  $q_1$  an external momentum.

$$\mathcal{N} \supset \mathcal{N}_2 = \text{Tr} \left[ \gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12} \right] \quad (2.5.22)$$

$$\bar{\mathcal{N}}_2 = \mathcal{N}_2 + 4\mu^2 g_{\mu\nu} \quad (2.5.23)$$

And now we must shift the same loop momentum but after contracting all indices in the whole numerator. As there are not any  $q_2$  in the numerator after contraction, there is nothing to subtract. We remember that this extra-extra integral has to be evaluated and  $\mu$  taken to zero, before joining to the external part. Then, the result must be treated as a one-loop integral again. That is

$$\begin{aligned} & \int d^4 q_2 \frac{4\mu^2 g_{\mu\nu}}{q_2^2 q_{12}^2} \xrightarrow{\text{FDR}} \\ \xrightarrow{\text{FDR}} \lim_{\mu \rightarrow 0} \int d^4 q_2 (4\mu^2 g_{\mu\nu}) & \left[ \frac{1}{\bar{q}_2^2 \bar{q}_{12}^2} - \frac{1}{\bar{q}_2^4} + \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^6} - \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^8} \right] = -\frac{2}{3} i\pi^2 q_1^2 g_{\mu\nu} \end{aligned} \quad (2.5.24)$$

Now, we introduce this result in the left over integral and repeat the process.

$$\begin{aligned} & \int d^4 q_1 \frac{\gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu (-\frac{2}{3} i\pi^2 q_1^2 g_{\mu\nu})}{q_1^4 D_1} \xrightarrow{\text{FDR}} \\ \xrightarrow{\text{FDR}} \lim_{\mu \rightarrow 0} \int d^4 q_1 & \left[ \frac{4}{3} i\pi^2 (\not{q}_1 + \not{p}) \right] \left[ \frac{1}{\bar{q}_1^2 D_1} - \frac{1}{\bar{q}_1^4} + \frac{2q_1 \cdot p}{\bar{q}_1^6} \right] \end{aligned} \quad (2.5.25)$$

So

$$EEI = -\frac{\alpha^2}{24\pi^2} \not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (2.5.26)$$

And finally, we get the renormalized integrand of the two-loop fermion self-energy.

$$\begin{aligned} \Sigma &= e^4 \lim_{\mu \rightarrow 0} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \left\{ \gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu \text{Tr} \left[ \gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12} \right] + 4\mu^2 \not{q}_1 \right\} \times \\ &\times \left[ \frac{1}{\bar{q}_1^4 \bar{D}_1} - \frac{1}{\bar{q}_1^6} + \frac{2q_1 \cdot p}{\bar{q}_1^8} \right] \left[ \frac{1}{\bar{q}_2^2 \bar{q}_{12}^2} - \frac{1}{\bar{q}_2^4} + \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^6} - \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^8} \right] + EEI \end{aligned} \quad (2.5.27)$$

We can now confirm what we said before: the procedure becomes more complicated depending on the number of subdivergences. However, we are now ready for using the method under whatever circumstances.

Beyond one loop many kinds of  $\mu$  could appear. A  $\mu$  generated by a certain squared momentum or another are different when power-counting because we remember they have to be considered to possess the same loop momentum power as their generators. If we first split the distinct pieces of the numerator and link them with their respective denominators expansion, and only then perform the shifting, we will not have any problem with  $\mu$ . However, we can use subindices to remember where they come if necessary. Also we can place for example a hat on a  $\mu$  that has to be integrated before to remember its nature. But if all is well organized, any distinction will be necessary.

## 2.6 Chiral calculations

We will present simple off-shell calculations for non-exceptional momenta, such that no infrared divergences can arise.

### 2.6.1 Vector and axial currents in two dimensions

Let us consider a free massless Dirac fermion in space of dimension  $n = 2$ , with Lagrangian

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi \quad (2.6.1)$$

This Lagrangian is invariant under global vector (V) and axial (A) transformations. The corresponding, classically conserved Noether currents are

$$j_\mu = \bar{\psi} \gamma_\mu \psi \quad (2.6.2)$$

$$j_\mu^5 = \bar{\psi} \gamma_\mu \gamma_5 \psi \quad (2.6.3)$$

respectively.<sup>16</sup> We want to calculate the correlation functions of two of these currents. The three distinct possibilities are  $\Pi_{\mu\nu}(p) = \langle j_\mu(p)j_\nu(-p) \rangle$ ,  $\Pi_{\mu\nu}^5(p) = \langle j_\mu(p)j_\nu^5(-p) \rangle$  and  $\Pi_{\mu\nu}^{55}(p) = \langle j_\mu^5(p)j_\nu^5(-p) \rangle$ . The classical Ward identities are

$$p^\mu \Pi_{\mu\nu}(p) = p^\nu \Pi_{\mu\nu}(p) = 0 \quad (2.6.4)$$

$$p^\mu \Pi_{\mu\nu}^5(p) = 0 \quad (2.6.5)$$

$$p^\nu \Pi_{\mu\nu}^5(p) = 0 \quad (2.6.6)$$

$$p^\mu \Pi_{\mu\nu}^{55}(p) = p^\nu \Pi_{\mu\nu}^{55}(p) = 0 \quad (2.6.7)$$

A useful GnDI in  $n = 2$  is  $\gamma_\mu \gamma_5 = \epsilon_\mu^\alpha \gamma_\alpha$ . This can be proven, for instance, using the complete set in GnS Dirac space. The correlation functions can be calculated exactly at one loop. Before doing it, we can anticipate the form of the correlators. In fact, the previous GnDI implies  $j_\mu^5 = \epsilon_{\mu\alpha} j_\alpha$ , so the three correlators are algebraically related:

$$\Pi_{\mu\nu}^5(p) = \epsilon_\mu^\alpha \Pi_{\mu\alpha} \quad (2.6.8)$$

$$\Pi_{\mu\nu}^{55}(p) = g_{\mu\nu} \Pi_\alpha^\alpha - \Pi_{\nu\mu} \quad (2.6.9)$$

In the second of these equations we have also used the GnDI 2.2.14 for  $n = 2$ . From this, we can easily conclude that the Ward identities (2.6.4–2.6.7) cannot be satisfied simultaneously. Indeed, dimensional analysis and the fact that the longitudinal piece is finite imply

$$\Pi_{\mu\nu}(p) = X \left( \frac{p_\mu p_\nu}{p^2} - a g_{\mu\nu} \right) \quad (2.6.10)$$

where both  $X$  and  $a$  are numbers.  $X$  is fixed by the result of a finite integral, while  $a$  is regularization dependent and can be modified with a local finite counterterm. In order to fulfill 2.6.4, we need  $a = 1$ . Then, we see that 2.6.5 is also satisfied but 2.6.6 and 2.6.7 are not. Instead, we have the anomalous identities

$$p^\nu \Pi_{\mu\nu}^5(p) = X \epsilon_{\mu\nu} p^\nu \quad (2.6.11)$$

$$p^\mu \Pi_{\mu\nu}^{55}(p) = -X p_\nu \quad (2.6.12)$$

It should be noted that all the GnDI we have employed involve external tensors only. Therefore, we expect that these results hold in consistent regularization and renormalization schemes that respect 2.6.4, including the method proposed in a previous section.

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<sup>16</sup>Because this current will always be an external operator in our calculations, nothing would change should we write instead  $j_\mu^5 = -\bar{\psi} \gamma_5 \gamma_\mu \psi$  or the average of these two definitions.

Let us now check this by explicit computation. We will use FDR for definiteness and because it allows us to compare with the rule that allows to identify the tensor integrals a priori, before computing the trace. We have checked that all the results are identical in CDR and CIReg and also in consistent DRed and FDF. Because no  $g^\alpha_\alpha$  arises from the Dirac matrices, the results in DReg are identical as well in these examples. The only contributing diagram to the VV correlator gives

$$\Pi_{\mu\nu}(p) = - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p}} \gamma_\nu \frac{1}{\not{k}} \right) \right]^R \quad (2.6.13)$$

Performing the trace, we find

$$\Pi_{\mu\nu}(p) = - [4B_{\mu\nu}(p) - 2g_{\mu\nu}B^\alpha_\alpha(p)]^R \quad (2.6.14)$$

where

$$B_{\alpha\beta}(p) = \int \frac{d^2k}{4\pi^2} \frac{(k-p)_\alpha k_\beta}{(k-p)^2 k^2} \quad (2.6.15)$$

Note that 2.6.14 is written in normal form. In FDR, we have

$$\begin{aligned} [B_{\alpha\beta}(p)]^R &= \left[ \int \frac{d^2k}{4\pi^2} \frac{(k-p)_\alpha k_\beta}{[(k-p)^2 - \mu^2][k^2 - \mu^2]} \right]^S \\ &= -\frac{i}{4\pi} \left\{ g_{\alpha\beta} \left( 1 - \frac{1}{2} \log \frac{p^2}{\mu^2} \right) - \frac{p_\alpha p_\beta}{p^2} \right\}, \end{aligned} \quad (2.6.16)$$

whereas

$$\begin{aligned} [B^\alpha_\alpha(p)]^R &= \left[ \int \frac{d^2k}{4\pi^2} \frac{(k-p)_\alpha k^\alpha - \mu^2}{[(k-p)^2 - \mu^2][k^2 - \mu^2]} \right]^S \\ &= g^{\alpha\beta} [B_{\alpha\beta}(p)]^R + \frac{i}{4\pi} \\ &= \frac{i}{4\pi} \log \frac{p^2}{\mu^2} \end{aligned} \quad (2.6.17)$$

The extra local term in the second equality comes, just as in 2.1.10, from the oversubtracted integral proportional to  $\mu^2$ , which is added to the numerator in the first line, according to the global prescription. Combining everything, we find

$$\Pi_{\mu\nu}(p) = -\frac{i}{\pi} \left( \frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \quad (2.6.18)$$

which agrees with 2.6.10 with  $X = -i/\pi$  and  $a = 1$ . As expected in a method that respects shift invariance and numerator-denominator consistency, the vector Ward identity 2.6.4 is satisfied. The very same result is recovered if we directly write

$$\Pi_{\mu\nu}(p) = - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p} + \mu} \gamma_\nu \frac{1}{\not{k} + \mu} \right) \right]^S \quad (2.6.19)$$

as the same  $\mu^2$  term appears after the trace is evaluated.

Let us next compute the VA correlator:

$$\begin{aligned} \Pi_{\mu\nu}^5(p) &= - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p}} \gamma_\nu \gamma_5 \frac{1}{\not{k}} \right) \right]^R \\ &= - \left[ \text{Tr} (\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_5 \gamma_\beta) B^{\alpha\beta} \right]^R \end{aligned} \quad (2.6.20)$$

To evaluate the trace without ambiguities, we simply use the definition of  $\gamma_5$  2.3.19. Then, refraining from using 2.2.13, we have

$$\text{Tr} (\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_5 \gamma_\beta) = 2 (-\epsilon_{\beta\nu} g_{\alpha\mu} + \epsilon_{\mu\nu} g_{\alpha\beta} - \epsilon_{\alpha\nu} g_{\beta\mu} + \epsilon_{\beta\alpha} g_{\mu\nu} - \epsilon_{\mu\alpha} g_{\beta\nu} - \epsilon_{\beta\mu} g_{\alpha\nu}) \quad (2.6.21)$$

from which the normal form is readily obtained. Note that only the second term on the right-hand side of 2.6.21 gives rise to  $B_{\alpha\alpha}$ , with contracted indices. Using 2.6.16 and 2.6.17, we get

$$\Pi_{\mu\nu}^5(p) = -\frac{i}{\pi} \epsilon_{\nu\alpha} \left( \frac{p_\mu p_\alpha}{p^2} - g_{\mu\alpha} \right) \quad (2.6.22)$$

which agrees with 2.6.8. The vector Ward identity 2.6.5 and the anomalous axial one 2.6.11, with  $X = -i/\pi$ , follow. Observe that a different result, with the anomaly in the  $\mu$  index, would have been obtained had we anticommutated the  $\gamma_5$  with  $1/\not{k}$ . In fact, we can directly evaluate the left-hand side of 2.6.11:

$$\begin{aligned} p^\nu \Pi_{\mu\nu}^5(p) &= - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p}} (\not{p} - \not{k} + \not{k}) \gamma_5 \frac{1}{\not{k}} \right) \right]^R \\ &= 0 - 2 \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p}} \hat{k} \gamma_5 \frac{1}{\not{k}} \right) \right]^R \end{aligned} \quad (2.6.23)$$

where the non-vanishing, evanescent term comes from the anticommutator  $\{\hat{k}, \gamma_5\}$ , see 2.3.24. Using the relation

$$\hat{k} \not{k} = k^2 - \bar{k}^2 = \mu^2 \quad (2.6.24)$$

an extra integral appears which gives the result 2.6.11.

Again, the same result can be obtained writing

$$\Pi_{\mu\nu}^5(p) = - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \frac{1}{\not{k} - \not{p} + \mu} \gamma_\nu \gamma_5 \frac{1}{\not{k} + \mu} \right) \right]^S \quad (2.6.25)$$

As explained in the previous section, the presence of  $\gamma_5$  should be obviated in assigning the relative signs of the  $\mu$  shifts. After writing 2.6.25, GnDI are allowed, and in particular we can anticommute  $\gamma_5$  with the Dirac matrices. The origin of the anomaly can then be tracked to the extra integral arising from

$$\{\gamma_5, \not{k} - \mu\} = -2\mu\gamma_5 \quad (2.6.26)$$

which is closely related to 2.3.24.

Finally, let us calculate the AA correlator,

$$\begin{aligned} \Pi_{\mu\nu}^{55}(p) &= - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \gamma_5 \frac{1}{\not{k} - \not{p}} \gamma_\nu \gamma_5 \frac{1}{\not{k}} \right) \right]^R \\ &= - \left[ \text{Tr} (\gamma_\mu \gamma_5 \gamma_\alpha \gamma_\nu \gamma_5 \gamma_\beta) B^{\alpha\beta} \right]^R \end{aligned} \quad (2.6.27)$$

First note that if we used  $\gamma_5^{\text{AC}}$ , we would immediately find  $\Pi_{\mu\nu}^{55} = -\Pi_{\mu\nu}$ , at odds with 2.6.9. But in our method we should not anticommute before the  $\mu$  shift. The consistent result is obtained by using the definition 2.3.19 for the two  $\gamma_5$ . Then we need to evaluate a trace with eight Dirac matrices, contract with  $B_{\alpha\beta}$  and use 2.6.16 and 2.6.17. The computation is not difficult and gives the expected result, 2.6.9. A faster procedure is to make use of 2.3.24 and  $\gamma_5^2 = -1$  to write

$$\text{Tr} (\gamma_\mu \gamma_5 (\not{k} - \not{p}) \gamma_\nu \gamma_5 \not{k}) = -\text{Tr} (\gamma_\mu (\not{k} - \not{p}) \gamma_\nu \not{k} + 2\gamma_\mu \hat{k} \gamma_\nu \not{k}). \quad (2.6.28)$$

From this and 2.6.24 we easily obtain

$$\begin{aligned} \Pi_{\mu\nu}^{55}(p) &= -\Pi_{\mu\nu} + 4g_{\mu\nu} \left[ \int \frac{d^2k}{4\pi^2} \frac{\mu^2}{(k^2 - \mu^2)^2} \right]^S \\ &= -\Pi_{\mu\nu} + \frac{i}{\pi} g_{\mu\nu} \\ &= \frac{i}{\pi} \frac{p_\mu p_\nu}{p^2} \end{aligned} \quad (2.6.29)$$



Once again, the same extra integral and therefore the same result are obtained by shifting the denominators with the prescribed signs,

$$\Pi_{\mu\nu}^{55}(p) = - \left[ \int \frac{d^2k}{4\pi^2} \text{Tr} \left( \gamma_\mu \gamma_5 \frac{1}{\not{k} - \not{p} + \mu} \gamma_\nu \gamma_5 \frac{1}{\not{k} + \mu} \right) \right]^S \quad (2.6.30)$$

After this shift, which automatically performs the correct tensor identification, all the standard properties of  $\gamma_5$  can be safely employed to simplify the calculation. Note that the very same procedure is followed in FDF.

The situation in  $n = 4$  is completely analogous, except for the fact that in that case the VA correlator studied here vanishes and the axial anomaly manifests itself in the familiar triangular diagrams. These have been calculated in DReg [3], consistent DRed [52], CDR [17, 53], FDR [20], CIReg [45] and FDF [54]. These calculations show that, as long as no GnDI is used before tensor identification, the vector Ward identities are automatically preserved and the anomaly is localized in the axial current.

## 2.6.2 Axial vertex Ward identity in four dimensions

As an example with an open fermion chain, we consider the correlation function  $\Gamma_\mu^5(p_1, p_2) = \langle j_\mu^5(p_1 + p_2) \bar{\psi}(-p_1) \psi(-p_2) \rangle_{\text{1PI}}$  (with the Legendre transform applied only to the elementary fields) in four-dimensional<sup>17</sup> massless QED, that is,

$$\mathcal{L} = -\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{2} (\partial_\mu \mathcal{A}_\nu)^2 + \bar{\psi} i \not{D} \psi \quad (2.6.31)$$

with  $\mathcal{D}_\mu = \partial_\mu + ieA_\mu$ . As manifest in 4, we work in the 't Hooft-Feynman gauge. There is no anomaly associated to this correlator, i.e. the theory can be renormalized in such a way that the Ward identity

$$(p_1 + p_2)^\mu \Gamma_\mu^5(p_1, p_2) = e (\gamma_5 \Sigma(p_1) - \Sigma(p_2) \gamma_5) \quad (2.6.32)$$

is satisfied, with  $\Sigma(p) = \langle \bar{\psi}(p) \psi(-p) \rangle_{\text{1PI}}$ . However, it is known that this identity is not satisfied in DReg with the tHV definition of  $\gamma_5$  [3]. The reason is that the GnDI  $\not{p} \gamma_5 = (\not{k} + \not{p}) \gamma_5 - \gamma_5 \not{k}$ , which is needed in the combinatorial proof, does not hold for a non-anticommuting  $\gamma_5$ . The Ward identity can be recovered by adding a finite gauge-invariant counterterm. This is a necessity if the axial symmetry is gauged.

<sup>17</sup>The corresponding diagrams in  $n = 2$  are finite by power counting and have no ambiguities.

It is clear that the Ward identity 2.6.32 will also be violated in the consistent versions of DRed and implicit methods that employ the  $\gamma_5$  definition in 2.3.19. Let us check this explicitly by one-loop calculations. Again, we use FDR for definiteness, but exactly the same results are found in CDR, CIReg and also in consistent DRed and FDF in  $\overline{\text{MS}}$ . The results in DReg are quantitatively different in this case.  $\Sigma$  and  $\Gamma_\mu$  in the following are understood to be the one-loop contributions to the corresponding correlation functions.

The fermion self-energy is given at one loop by

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{k}} \gamma^\alpha \frac{1}{(k-p)^2} \quad (2.6.33)$$

It has no potential ambiguity of the kind we are discussing. The result in FDR is easily found to be

$$\Sigma(p) = -\frac{e^2}{(4\pi)^2} \not{p} \left( 2 - \log \frac{p^2}{\mu^2} \right) \quad (2.6.34)$$

Let us now compute the axial vertex  $\Gamma_\mu^5$ , which at one loop is given by

$$\begin{aligned} \Gamma_\mu^5(p_1, p_2) &= -ie^3 \left[ \int \frac{d^4k}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{k} - \not{p}_2} \gamma_\mu \gamma_5 \frac{1}{\not{k} + \not{p}_1} \gamma_\alpha \frac{1}{k^2} \right]^R \\ &= -ie^3 [S_{\alpha\beta\mu 5g\alpha} C_{\beta\kappa}(p_1, p_2)]^R \end{aligned} \quad (2.6.35)$$

with

$$C_{\alpha\beta}(p_1, p_2) = \int \frac{d^4k}{(2\pi)^4} \frac{(k-p_2)_\alpha (k+p_1)_\beta}{k^2 (k-p_2)^2 (k+p_1)^2} \quad (2.6.36)$$

Substituting  $\gamma_5$  by its definition 2.3.19,

$$S_{\alpha\beta\mu 5\kappa\alpha} = \frac{1}{4!} \epsilon^{\nu\rho\sigma\tau} S_{\alpha\beta\mu\nu\rho\sigma\tau\kappa\alpha} \quad (2.6.37)$$

Next decompose  $S_{\alpha\beta\mu\nu\rho\sigma\tau g\alpha}$  as in 2.3.10. Since the index  $\alpha$  is contracted, there are contributions proportional to  $[\mu_1 \dots \mu_m]$  with  $m = 1, 3, 5, 7$ . As pointed out before, these combinations can be factored out of  $[.]^R$ , so the ones with  $m = 5, 7$  can be directly set to zero, as in four genuine dimensions. Then, we contract indices with the resulting metrics and use the CIReg

results

$$\begin{aligned}
[C_{\alpha\beta}(p_1, p_2)]^R &= \frac{i}{(4\pi)^2} \left\{ \frac{g_{\alpha\beta}}{4} \left[ 3 - p_2^2 \xi_{0,1} - p_1^2 \xi_{1,0} - \log \frac{(p_1 + p_2)^2}{\mu^2} \right] \right. \\
&\quad + [(\xi_{0,2} - \xi_{0,1})(p_1)_\alpha (p_1)_\beta - \xi_{1,1}(p_1)_\alpha (p_2)_\beta + (p_1 \rightleftharpoons p_2, \xi_{m,n} \rightleftharpoons \xi_{n,m})] \\
&\quad \left. - (p_1)_\beta (p_2)_\alpha (\xi_{0,0} - \xi_{0,1} - \xi_{1,0}) \right\} \quad (2.6.38)
\end{aligned}$$

$$[C^\alpha{}_\alpha(p_1, p_2)]^R = \frac{i}{(4\pi)^2} \left[ 2 - \frac{(p_1 + p_2)^2}{2} \xi_{0,0} - \frac{1}{2} \log \frac{p_2^2}{\mu^2} - \frac{1}{2} \log \frac{p_1^2}{\mu^2} \right] \quad (2.6.39)$$

which in this massless case (and also in the massive case in the mass-independent version of CIReg) exactly coincide with the FDR ones. The functions  $\xi_{n,m} \equiv \xi_{n,m}(p_2, p_1)$  are defined in the appendix. Importantly, the last integral includes the shift  $k^2 \rightarrow k^2 - \mu^2$  in the numerator. The final result is

$$\begin{aligned}
\Gamma_\mu^5(p_1, p_2) &= \frac{e^3}{(4\pi)^2} \left[ \gamma^\mu \gamma_5 \left[ 3 - (p_1 + p_2)^2 \xi_{0,0} + p_2^2 \xi_{0,1} + p_1^2 \xi_{1,0} - \log \frac{p_1^2}{\mu^2} - \log \frac{p_2^2}{\mu^2} \right. \right. \\
&\quad \left. \left. + \log \frac{(p_1 + p_2)^2}{\mu^2} \right] + 2 \left\{ p_2^\mu \gamma_5 [p_1^\mu (2\xi_{1,1} - \xi_{0,1} - \xi_{1,0} - \xi_{0,0}) + 2p_2^\mu (\xi_{0,1} - \xi_{0,2})] \right. \right. \\
&\quad \left. \left. + (p_1 \rightleftharpoons p_2, \xi_{m,n} \rightleftharpoons \xi_{n,m}) \right\} - 2 (\xi_{0,0} + \xi_{0,1} + \xi_{1,0}) \epsilon_{g\mu\alpha\beta} p_2^\alpha p_1^\beta \gamma^g \right] \quad (2.6.40)
\end{aligned}$$

An equivalent procedure that simplifies the Dirac algebra is to anticommute the  $\gamma_5$  to the right, using 2.3.24. This leads to

$$\Gamma_\mu^5(p_1, p_2) = -ie^3 [(2\bar{g}^\rho{}_\kappa - g^\rho{}_\kappa) S_{\alpha\beta\mu\rho\alpha} C^{\beta\kappa}(p_1, p_2)]^R \quad (2.6.41)$$

Decomposing  $S_{\alpha\beta\mu\rho\alpha}$  and using the rules 2.2.9 and 2.2.10, we find again 2.6.40. Even more easily, the same result can be found fixing the  $\mu$  terms from the very beginning with the same rule used above,

$$\Gamma_\mu^5(p_1, p_2) = -ie^3 \left[ \int \frac{d^4 k}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{k} - \not{p}_2 + \mu} \gamma_\mu \gamma_5 \frac{1}{\not{k} + \not{p}_1 + \mu} \gamma^\alpha \frac{1}{k^2} \right]^S \quad (2.6.42)$$

After this,  $\gamma_5$  can be safely anticommuted with the Dirac matrices (and commuted with  $\mu$ ). Let us note again that this same prescription is used in FDF, so the result will be identical in that method. Even if the last procedure looks simpler, it should be noted that it is less universal than the other ones, as we have pointed out in the previous section.

The result 2.6.40 does not satisfy the Ward identity 2.6.32. Instead, using the relations

in the appendix we find

$$(p_1 + p_2)^\mu \Gamma_\mu(p_1, p_2) = e (\gamma_5 \Sigma(p_1) - \Sigma(p_2) \gamma_5) + 2 \frac{e^3}{(4\pi)^2} (\not{p}_1 + \not{p}_2) \gamma_5 \quad (2.6.43)$$

To isolate the origin of the extra local term, we can compute the left-hand side of 2.6.43 directly. For instance, using the expression in 2.6.42,

$$\begin{aligned} & (p_1 + p_2)^\mu \Gamma_\mu(p_1, p_2) \\ &= -ie^3 \left[ \int \frac{d^4 k}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{k} - \not{p}_2 + \mu} (\not{p}_1 + \not{k} + \mu + \not{p}_2 - \not{k} - \mu) \gamma_5 \frac{1}{\not{k} + \not{p}_1 + \mu} \gamma^\alpha \frac{1}{k^2} \right]^S \\ &= e (\gamma_5 \Sigma(p_1) - \Sigma(p_2) \gamma_5) - ie^3 \left[ \int \frac{d^4 k}{(2\pi)^4} \gamma_\alpha \frac{1}{\not{k} - \not{p}_2 + \mu} (2\mu \gamma_5) \frac{1}{\not{k} + \not{p}_1 + \mu} \gamma^\alpha \frac{1}{k^2} \right]^S \end{aligned} \quad (2.6.44)$$

It can be checked that the extra integral above gives the extra local term on the right-hand side of 2.6.43. The axial symmetry can be restored by canceling this term with a finite counterterm proportional to  $\bar{\psi} \not{B} \gamma_5 \psi$ , where  $B_\mu$  is a source coupled to  $j_\mu^5$ .

Our results are consistent with the ones in [54], where  $\Gamma_\mu$  is calculated for massive on-shell fermions in FDH with a tHV  $\gamma_5$  and FDF, which give the same result, and in FDH with  $\gamma_5^{\text{AC}}$ , which differs by a local term. In the context of dimensional methods, it has been observed that identity 2.6.32 and similar Ward identities can be preserved by moving all  $\gamma_5$  to one end of open fermion lines before regularization and renormalization [55]. The reason is that, by doing this, the  $\gamma_5$  does not interfere with the necessary identity in the combinatorial proof. This is not quite the same as using  $\gamma_5^{\text{AC}}$ , as the  $\gamma_5$  matrices are not allowed to be anticommutated to an arbitrary position. This trick works equally well in implicit methods and it has actually been advocated in FDR [20]. Observe, nevertheless, that this procedure goes beyond the basic idea in these methods of substituting the bare expressions, in the form obtained from the Feynman rules, by their renormalized value. A previous non-trivial manipulation is performed. Then, one needs to check that this does not interfere with unitarity or with the quantum action principle in multiloop calculations.

# Chapter 3

## Forest renormalization in FDR

In this chapter, we will define Forest Formula which systematizes the implementation of any renormalization method as well as ensures locality. Later, we will apply it to FDR in an attempt to improve its order by order definition, and see the different assumptions and conditions we can choose to implement that method, leading us to several renormalization schemes all based on FDR. Finally, we will introduce Renormalization Group Equation as a useful tool to analyze the different versions of FDR.

### 3.1 Forest Formula definition

The renormalization procedure as we know today, i.e. the recursive process we do to eliminate divergences from inside to outside of the diagrams, was first developed by Bogoliubov and Parasiuk in 1957 [37], with corrections due to Hepp in 1966 [38]. The idea was very simple. If we have a Feynman diagram  $\Gamma$ , we obtain its renormalized value  $\mathcal{R}(\Gamma)$  as

$$\mathcal{R}(\Gamma) = \Gamma + \mathcal{S}(\Gamma) \tag{3.1.1}$$

where  $\mathcal{S}(\Gamma)$  is the subtraction part (the sum of all counterterms).

Suppose a one-particle-irreducible diagram (1PI) without subdivergences, so the only possible divergence is the overall one. This means all integration momenta get huge at the same time. As it does not have subdivergences, its renormalized value is simply

$$\mathcal{R}(\Gamma) = \Gamma - \mathcal{T}\Gamma \tag{3.1.2}$$

This  $\mathcal{T}$  is an operator which extracts the divergence of the diagram regardless of the renormalization scheme we chose. For example, if we want to use Dimensional Regularization,  $\mathcal{T}$  expands  $\Gamma$  around  $d = n$  and keeps the pole part. But if we like Pauli-Villars regularization,  $\mathcal{T}$  will manage to put the extra mass in the photon propagators and then subtract them to the original diagram. And of course, if the diagram is finite  $\mathcal{T}\Gamma = 0$ .

In a general situation, the diagram under consideration may have subdivergences, i.e. there is at least one subdiagram inside that its loop momenta get large while the external momenta remain finite. And perhaps that subdiagram has also subdivergences, etc. As every subdivergence of a diagram is the global divergence of a subdiagram, the only thing we have to do to renormalize a diagram with subdivergences is to apply (3.1.2) recursively from inside to outside.

To do this we define  $\overline{\mathcal{R}}(\Gamma)$  as  $\Gamma$  with its subdivergences subtracted, thus the only possible divergence in the diagram is the overall one. So its renormalization is given by

$$\mathcal{R}(\Gamma) = \overline{\mathcal{R}}(\Gamma) - \mathcal{T}\overline{\mathcal{R}}(\Gamma) \quad (3.1.3)$$

where

$$\overline{\mathcal{R}}(\Gamma) = \Gamma - \sum_{\gamma \subsetneq \Gamma} \mathcal{T}\overline{\mathcal{R}}(\gamma) \quad (3.1.4)$$

As we said before, this procedure is recursive. The diagram with renormalized subdivergences is the diagram itself subtracting the infinite parts of all its subdiagrams ( $\gamma \subsetneq \Gamma$ ), but only if they don't have any more subdivergences, and that is why  $\overline{\mathcal{R}}(\gamma)$  has a bar on it. If they had any divergent subdiagram, we would extract them before doing the same process. Of course if  $\gamma$  does not diverge  $\mathcal{T}\overline{\mathcal{R}}(\gamma) = 0$ .

The operator  $\mathcal{T}$  satisfies the next trivial properties:

$$\mathcal{T}(\mathcal{T}\gamma) = \mathcal{T}\gamma \quad (3.1.5)$$

$$\gamma_1 \cap \gamma_2 = \emptyset \iff \mathcal{T}(\gamma_1 \cup \gamma_2) = (\mathcal{T}\gamma_1)(\mathcal{T}\gamma_2) \quad (3.1.6)$$

The first one means the infinite part of a infinite part is itself as it should be, and the second one denotes that if two diagrams are disconnected, then the operator works on them separately.

In 1969, Zimmermann [39] solved this recursion creating the so called Forest Formula. It tells us how to renormalize a diagram using forests. A forest  $f$  of a given diagram  $\Gamma$ , is a set of divergent 1PI subdiagrams so that they are non-overlapping, i.e. they can be either disjoint

( $\gamma_i \cap \gamma_j = \emptyset$ ) or nested ( $\gamma_i \subsetneq \gamma_j, \gamma_i \supsetneq \gamma_j$ ). The forests that do not contain the full diagram are called normal forests, and they are responsible for subtracting the subdivergences. In order to remove the overall divergence too, we add the full diagram to all normal forests. These are named full forests. And the set of all forests of a diagram  $\Gamma$  is called  $\mathcal{F}(\Gamma)$ . Once clarified all the concepts we can finally introduce the Forest Formula.

$$\mathcal{R}(\Gamma) = \sum_{f \in \mathcal{F}(\Gamma)} \prod_{\gamma \in f} (-\mathcal{T}_\gamma) \Gamma \quad (3.1.7)$$

where  $-\mathcal{T}_\gamma \Gamma$  means the replacement  $\gamma \rightarrow -\mathcal{T}\gamma$  inside  $\Gamma$ , and  $\mathcal{T}$  is the same operator we defined beforehand. We can see that (3.1.7) is simply going over all forests and then replacing the divergent 1PI subdiagrams inside each forest. To work properly, we obviously need a forest which contains the empty set, that is a forest with no replacements that will result in the diagram  $\Gamma$  itself.

Now we are ready to see an example in order to practice the identifying of forests. Consider a three-loop self-energy diagram in a  $\phi^3$  theory in  $d = 6$  dimensions (fig. 3.1.1).

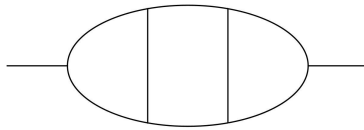


Figure 3.1.1: Three-loop self-energy diagram in  $\phi^3$  theory.

In fig. 3.1.2, we depict every forest separately with a dashed box surrounding its subdiagrams  $\gamma_i$ . Each box means that its inside is affected by  $\mathcal{T}$ .





### 3.1.1 Forest Formula at work

Once we have learnt how to choose the corresponding forests within a diagram, now we are going to show how  $\mathcal{T}$  implements the renormalization method we want. To do that we will use again the one-loop photon and fermion self-energy diagrams in massless QED (fig. 2.5.1 and 2.5.2) as examples.

In the first place, we will study the vacuum polarization. Since it is a one-loop diagram, we only have one normal forest (the diagram itself) and one full forest (the diagram in a box) (fig. 3.1.3). Moreover, concerning the divergences, the only possibility is an overall divergence. So

$$\mathcal{R}(\Pi_{\mu\nu}) = \Pi_{\mu\nu} - \mathcal{T}\Pi_{\mu\nu} \quad (3.1.8)$$

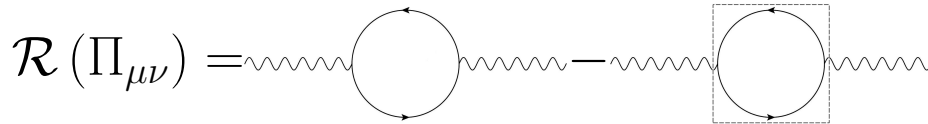


Figure 3.1.3: Renormalization of the photon self-energy diagram at one loop.

To compute  $\mathcal{T}\Pi_{\mu\nu}$ , we first have to decide what renormalization method we are going to use. To prove the flexibility of Forest Formula, we will show several ones. Some will require to do the integral and some will work at the integrand level, but the role of  $\mathcal{T}$  will always be the same.

For example, in Dimensional Regularization with minimal subtraction, after performing the integral in  $d$  dimensions and expanding the result around  $d = 4$ ,  $\mathcal{T}$  will pick the part proportional to  $1/\varepsilon$ . So, if evaluating (2.5.4) in  $d$  dimensions, the result is

$$\Pi_{\mu\nu} = \frac{\alpha}{3\pi} \left[ \frac{1}{\varepsilon} + \frac{5}{3} - \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (3.1.9)$$

then

$$\mathcal{T}\Pi_{\mu\nu} = \frac{\alpha}{3\pi\varepsilon} (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (3.1.10)$$

and thus

$$\mathcal{R}(\Pi_{\mu\nu}) = \frac{\alpha}{3\pi} \left[ \frac{5}{3} - \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (3.1.11)$$

As an example of a method that does not demand to do the integral before  $\mathcal{T}$  can act, we have the BPHZ method. This belongs to the class of implicit methods because it does

not use any regulator to renormalize the diagram and subtractions are applied directly to the integrand. In this approach,  $\mathcal{T}$  takes the external momenta series of the propagators and then keeps the terms with the same and less momenta power than the degree of divergence. Making use of power-counting, we see (2.5.4) has a quadratic divergence, so we have to Taylor expand the denominator up to order two (3.1.12).

$$f(p) = f(0) + \partial_\mu f(p) \Big|_{p=0} p^\mu + \frac{1}{2} \partial_\mu \partial_\nu f(p) \Big|_{p=0} p^\mu p^\nu + \mathcal{O}(p^3) \quad (3.1.12)$$

Now,  $\mathcal{T}$  keeps those terms instead of the original denominator.

$$\mathcal{T}\Pi_{\mu\nu} = ie^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] \left[ \frac{1}{q^4} - \frac{2q \cdot p + p^2}{q^6} + \frac{4(q \cdot p)^2}{q^8} \right] \quad (3.1.13)$$

$$\mathcal{R}(\Pi_{\mu\nu}) = ie^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] \left[ \frac{1}{q^2 (q+p)^2} - \frac{1}{q^4} + \frac{2q \cdot p + p^2}{q^6} - \frac{4(q \cdot p)^2}{q^8} \right] \quad (3.1.14)$$

Finally, we have a completely finite integral that we can perform in four dimensions. In fact, this method needs fermions to be massive in order to renormalize the diagram properly. If we put the masses in the propagators and in the trace, evaluate the integral and then take the limit of zero mass, we will get the same result as in (3.1.11).

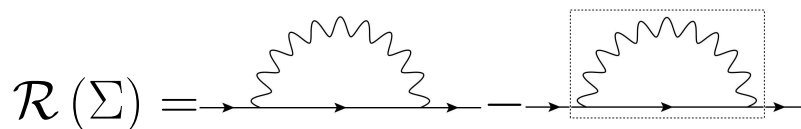


Figure 3.1.4: Renormalization of the fermion self-energy diagram at one loop.

Secondly, let us examine the fermion self-energy. This graph is also one-loop, hence it has only a normal and a full forest (fig. 3.1.4) to remove the overall divergence. Therefore its renormalized value is just

$$\mathcal{R}(\Sigma) = \Sigma - \mathcal{T}\Sigma \quad (3.1.15)$$

Here, there will be a slight difference between Dimensional Regularization and Dimensional Reduction, though  $\mathcal{T}\Sigma$  will remain the same, if we use minimal subtraction. Evaluating (2.5.9):

$$\Sigma(\text{DReg}) = -\frac{\alpha}{4\pi} \not{p} \left[ \frac{1}{\varepsilon} + 1 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (3.1.16)$$

$$\Sigma (\text{DRed}) = -\frac{\alpha}{4\pi}\not{p} \left[ \frac{1}{\varepsilon} + 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (3.1.17)$$

so

$$\mathcal{T}\Sigma = -\frac{\alpha}{4\pi\varepsilon}\not{p} \quad (3.1.18)$$

and then

$$\mathcal{R}(\Sigma) (\text{DReg}) = -\frac{\alpha}{4\pi}\not{p} \left[ 1 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (3.1.19)$$

$$\mathcal{R}(\Sigma) (\text{DRed}) = -\frac{\alpha}{4\pi}\not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (3.1.20)$$

Another method we can easily implement in this diagram is Pauli-Villars. In this case,  $\mathcal{T}$  will make the photon massive. Hence if

$$\Sigma = -ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu (\not{q} + \not{p}) \gamma^\mu}{q^2 (q+p)^2} \quad (3.1.21)$$

then

$$\mathcal{T}\Sigma = -ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu (\not{q} + \not{p}) \gamma^\mu}{(q^2 - \Lambda^2) (q+p)^2} \quad (3.1.22)$$

and

$$\begin{aligned} \Sigma - \mathcal{T}\Sigma &= ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{\Lambda^2 \gamma_\mu (\not{q} + \not{p}) \gamma^\mu}{q^2 (q^2 - \Lambda^2) (q+p)^2} = \\ &= -\frac{\alpha}{4\pi}\not{p} \left[ \frac{\Lambda^2}{p^2} + \frac{(p^2 - \Lambda^2)^2}{p^4} \log \left( -\frac{p^2 - \Lambda^2}{\Lambda^2} \right) - \log \left( -\frac{p^2}{\Lambda^2} \right) \right] \end{aligned} \quad (3.1.23)$$

If we take the limit  $\Lambda \rightarrow \infty$ , we will recover the physical massless photon. In fact,  $\Lambda$  is still infinite, so an implicit subtraction of  $\log(\mu^2/\Lambda^2)$  is assumed, where  $\mu$  is the renormalization constant.

$$\mathcal{R}(\Sigma) = -\frac{\alpha}{4\pi}\not{p} \left[ \frac{3}{2} - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (3.1.24)$$

As we see, this graph has multiple results depending on the renormalization scheme we choose, though the logarithmic behaviour remains unchanged which is the relevant aspect.

The main question here is why we would use Forest Formula with our chosen renormalization method instead of employing the method itself as we have been doing so far. The answer is that with a good definition of  $\mathcal{T}$ , Forest Formula automates the search for subdivergences and their renormalization, and ensures a correct structure of local counterterms providing  $\mathcal{T}$  is local. In fact, once  $\mathcal{T}$  is defined and forests are identified, Forest Formula will tell us step by step how to renormalize a diagram regardless of its size or its number of loops.

This makes it very powerful and attractive. In the next section, we will choose an implicit method to be improved by all these properties we are discussing in order to show the power of Forest Formula more explicitly.

## 3.2 FDR systematization

In section 2.5, we introduced FDR (Four-Dimensional Regularization/Renormalization) as an implicit method which modifies directly the integrand. Next, we are going to apply what we have learnt about Forest Formula to this method. FDR has so many rules and operations that makes its implementation not trivial. Thus we need some definitions to achieve it.

Every diagram, whatever its size is, is made up of a numerator and a denominator. So consider a general one-loop diagram in any fixed dimension where  $q$  is the loop momentum

$$\Gamma = \int d^n q \mathcal{N}(q) \mathcal{D}(q) \quad (3.2.1)$$

In spite of one of the advantages of using Forest Formula is that it automates the renormalization procedure, we have to shift the squared loop momenta in the denominator by hand before applying Forest Formula. That is

$$\mathcal{D}(q^2) \rightarrow \mathcal{D}(\bar{q}^2) \quad (3.2.2)$$

Henceforth we will omit this step and assume every graph is prepared for Forest Formula performing this shift and placing the limit  $\mu \rightarrow 0$  before its expression.

Our goal now is to define a good  $\mathcal{T}$  acting on the integrand so that it can properly lead us to FDR method as we know, and can reproduce all its results. Nevertheless, this election is not trivial. Let us review all definitions and configurations we have considered and their peculiarities and consequences, and also their comparison with the original FDR.

### 3.2.1 Minimal operator

As we said before, we can always consider that a diagram is formed by a numerator and denominator. So knowing FDR rules<sup>1</sup>, the first  $\mathcal{T}$  definition which comes to mind could be

$$\mathcal{T}(\mathcal{N}\mathcal{D}) = [\mathcal{N} + \mathcal{X}(\mathcal{N})] \mathcal{V}(\mathcal{D}) \quad (3.2.3)$$

---

<sup>1</sup>See section 2.5.

where we have defined other two linear operators acting on numerator and denominator separately.  $\mathcal{V}(\mathcal{D})$  expands the denominator using fraction decomposition and picks the vacuum part, and  $\mathcal{X}(\mathcal{N})$  shifts the numerator and picks the extra part proportional to  $\mu^2$ . The first thing we must wonder is if eq. (3.1.5) holds for this operator. So let us verify it.

$$\mathcal{T}(\mathcal{T}(\mathcal{N}\mathcal{D})) = [\mathcal{N} + \mathcal{X}(\mathcal{N})]\mathcal{V}(\mathcal{D}) + [\mathcal{X}(\mathcal{N}) + \mathcal{X}(\mathcal{X}(\mathcal{N}))]\mathcal{V}(\mathcal{V}(\mathcal{D})) \quad (3.2.4)$$

By definition, picking up the vacua of the denominator, it is clear that

$$\mathcal{V}(\mathcal{V}(\mathcal{D})) = \mathcal{V}(\mathcal{D}) \quad (3.2.5)$$

so to recover the  $\mathcal{T}$  idempotence relation, the only possibility is

$$\mathcal{X}(\mathcal{X}(\mathcal{N})) = -\mathcal{X}(\mathcal{N}) \quad (3.2.6)$$

what suggests that  $\mu^2$  must be treated as a loop momentum concerning to  $\mathcal{X}(\mathcal{N})$ .

Since a one-loop diagram has one normal forest (the diagram itself) and one full forest, according to eq. (3.1.7) we get

$$\mathcal{R}(\mathcal{N}\mathcal{D}) = \mathcal{N}[\mathcal{D} - \mathcal{V}(\mathcal{D})] - \mathcal{X}(\mathcal{N})\mathcal{V}(\mathcal{D}) \quad (3.2.7)$$

And obviously, if we perform the integral and take the limit, we get to the final result.

$$\mathcal{R}(\Gamma) = \lim_{\mu \rightarrow 0} \int d^n q \{ \mathcal{N}[\mathcal{D} - \mathcal{V}(\mathcal{D})] - \mathcal{X}(\mathcal{N})\mathcal{V}(\mathcal{D}) \} \quad (3.2.8)$$

It is important to recall that a numerator may have several parts with different powers of loop momentum. In this case, every piece needs a different denominator decomposition, so in the final result an implicit sum over all of them is assumed. Sometimes, especially when the difference between parts is one power, they can be joined under the same vacua because parity takes the misplaced fragment to zero.

As we have not made use of any explicit example to get to eq. (3.2.8), it will be valid for whatever one-loop diagram. Actually, it still works for any diagram in any theory, the only condition is not to have subdivergences.

Let us put this expression in context with fig. 2.5.1 and 2.5.2 again. Using eq. (3.2.8)

they are respectively

$$\mathcal{R}(\Pi_{\mu\nu}) = ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} \left\{ \text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{p^2 + 2q \cdot p}{\bar{q}^6} - \frac{4(q \cdot p)^2}{\bar{q}^8} \right] \right. \\ \left. - 4\mu^2 g_{\mu\nu} \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{p^2 + 2q \cdot p}{\bar{q}^6} - \frac{4(q \cdot p)^2}{\bar{q}^8} \right] \right\} \quad (3.2.9)$$

$$\mathcal{R}(\Sigma) = -ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} [\gamma_\mu (\not{q} + \not{p}) \gamma^\mu] \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{2q \cdot p}{\bar{q}^6} \right] \quad (3.2.10)$$

If we compare the results above with (2.5.7) and (2.5.12) (expressions we got using pure FDR), we see that photon self-energy is slightly different. This difference is exactly

$$\lim_{\mu \rightarrow 0} \int d^n q \mathcal{X}(\mathcal{N}) \mathcal{D} \xrightarrow{\Pi_{\mu\nu}} 4ig_{\mu\nu} \lim_{\mu \rightarrow 0} \int d^4 q \frac{\mu^2}{\bar{q}^2 \bar{D}} = 0 \quad (3.2.11)$$

In fact, this term on the left hand side will always be null for any diagram with only global divergence due to its structure. Furthermore, whatever term proportional to  $\mu^2$  integrated along with the original denominator of an any-loop diagram will always be zero after taking the limit. Actually, this fact was part of the motivation for the definition of the operator  $\mathcal{T}$ . In addition, thanks to that, equations (2.5.7) and (3.2.9) lead to the same result we showed in (2.5.8). Obviously, fermion self-energy shares the same integrand expression because no  $\mu$  can arise in its numerator. So we have verified that at least for a one-loop case, the chosen  $\mathcal{T}$  reproduces the FDR result despite of some differences between renormalized integrands.

The evident question now is whether this correspondence also follows beyond one-loop or not. So let us work out two-loop formal expressions. Consider a  $\phi^3$  theory in six dimensions but only to be able to depict some graphs and better understand how the method works, though the final expressions will be valid for any two-loop diagram. In fig. 3.2.1, we see the three possible self-energy diagrams for a  $\phi^3$  theory. Figures 3.2.1a and 3.2.1b are equivalent, so we will choose the first for the study. They have one subdivergence while the other have two. To start the analysis, first we must depict the forests as we have learnt previously. We will start with the case of one subdivergence.

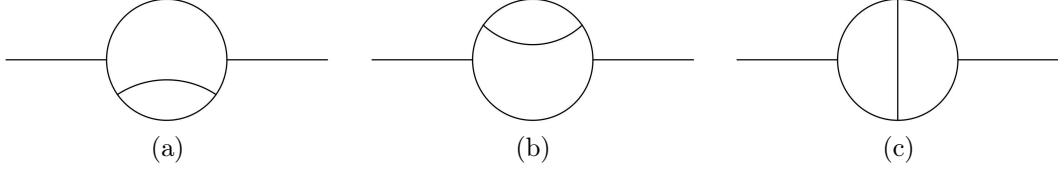


Figure 3.2.1: Two-loop self-energy diagrams in  $\phi^3$ .

$$\begin{aligned}
 \mathcal{R}(\Gamma_1) = & \text{---} \text{(a)} \text{---} - \text{---} \text{(b)} \text{---} \\
 & - \text{---} \text{(c)} \text{---} + \text{---} \text{(d)} \text{---}
 \end{aligned}$$

Figure 3.2.2: Necessary forests for the renormalization of fig. 3.2.1a.

Before writing down the expressions of each diagram, we must add two new rules to our method in order to generalize it properly beyond one loop. The reason for the first one is just practical. It simply consists in calling all loop momenta as  $q_i$ , such that operators  $\mathcal{V}$  and  $\mathcal{X}$  or even  $\mathcal{T}$  can carry a subindex or subindices according to all loop momenta inside a box.<sup>2</sup> This is very important not to forget what diagram part they are acting on once all forests are summed. This way, in a two-loop diagram for instance,  $\mathcal{V}_{ij}$  is identified with the global vacuum, and  $\mathcal{V}_i$  with subvacua. Likewise, we associate  $\mathcal{X}_{ij}$  to *global prescription*, and somehow  $\mathcal{X}_i$  to *subprescription*.

The second rule lies in that parts proportional to  $\mu$  generated by a particular box have to be integrated and then that  $\mu$  taken to zero, before joining them to the outside of the box. This rule is motivated, indeed, by the *subprescription* we mentioned when explaining FDR original method. In principle, Forest Formula does not suggest anything about it. Thus, we introduce it in order to reproduce FDR as much as possible.

<sup>2</sup>We remember that a box symbolizes that its inside is affected by the replacement  $\gamma \rightarrow \mathcal{T}\gamma$ , so that  $\mathcal{T}$  cannot see anything outside the box.

So now, let us write the expressions for the forests in fig. 3.2.2. We will consider that this graph have  $q_1$  and  $q_2$  as loop momenta, where the later will be the loop momentum of the subdiagram. Moreover, we will suppose a general numerator and denominator because the aim of this section does not need to write them explicitly, so that pictures are only a guide. We will work just with the integrands always keeping in mind  $\Gamma = \int d^6 q_1 \int d^6 q_2 \bar{\Gamma}(q_1, q_2)$ . Hence using eq. (3.2.3), we obtain

$$\bar{\Gamma}_1^{(a)} = \mathcal{N} \mathcal{D} \quad (3.2.12)$$

$$\bar{\Gamma}_1^{(b)} = [\mathcal{N} + \mathcal{X}_2(\mathcal{N})] \mathcal{V}_2(\mathcal{D}) \quad (3.2.13)$$

$$\bar{\Gamma}_1^{(c)} = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})] \mathcal{V}_{12}(\mathcal{D}) \quad (3.2.14)$$

$$\bar{\Gamma}_1^{(d)} = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})] \mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) + \mathcal{T}_{12}(\mathcal{X}_2(\mathcal{N}) \mathcal{V}_2(\mathcal{D})) \quad (3.2.15)$$

If we sum and reorganize them a bit

$$\mathcal{R}(\bar{\Gamma}_1) = \mathcal{N} [\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] - \mathcal{X}_{12}(\mathcal{N}) \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D})) - \mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N}) \mathcal{V}_2(\mathcal{D})) \quad (3.2.16)$$

where we have used the linearity of the operators, and we have respected the fact that a  $\mu$  generated by an inner box must be integrated before continuing with other external operations. And that is why we simply placed a  $\mathcal{T}$  to the part with already a  $\mu$  and did not write explicitly  $\mathcal{V}$  and  $\mathcal{X}$ . This expression is completely valid for any two-loop diagram with one subdivergence (changing subindices conveniently).

Here again, we can expect several differences with the example we studied with FDR in section 2.5 (fig. 2.5.3). If we examine eq. (3.2.16), we see the numerator along with a finite denominator exactly as in eq. (2.5.27). However, the remainder is not the same. Concerning to *global prescription*,  $\mathcal{X}_{12}(\mathcal{N})$  should also go with the complete finite denominator as in the original FDR result. Nevertheless, one can wonder if as well as in the one-loop case, the missing piece goes to zero so that both expressions lead to the same outcome. In fact, it happens at least in the diagram we are comparing with.

$$\lim_{\mu \rightarrow 0} \int d^n q_i \int d^n q_j \mathcal{X}_{ij}(\mathcal{N}) [\mathcal{D} - \mathcal{V}_j(\mathcal{D})] \xrightarrow{\Sigma} -4 \lim_{\mu \rightarrow 0} \int d^4 q_1 \int d^4 q_2 \frac{\mu^2 \not{q}_1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^4} = 0 \quad (3.2.17)$$

Unfortunately, this is just a coincidence and we cannot ensure it follows for whatever two-loop diagram. Actually, it does not. For example, in fig. 4.2.5 (a diagram we will study later),



this piece is divergent and non-local.

$$\lim_{\mu \rightarrow 0} \int d^n q_i \int d^n q_j \mathcal{X}_{ij}(\mathcal{N}) [\mathcal{D} - \mathcal{V}_j(\mathcal{D})] \xrightarrow{\hat{\Pi}_{\mu\nu}^{(a)}} \frac{\alpha^2}{288\pi^2} \left[ \frac{1}{\varepsilon} + \frac{8}{3} - \log\left(-\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (3.2.18)$$

This term is divergent, so it is evident that it is necessary to properly renormalize the diagram, since original FDR contains it and lead to a finite result. Moreover, despite of its appearance of a non-local counterterm, it is essential to compensate the non-locality introduced when shifting just the denominators.

On the other hand, we have the term  $\mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N}) \mathcal{V}_2(\mathcal{D}))$ . It is almost the extra-extra integral that appears in the original FDR method, except for the needed subtraction which does not arise here. Again, the diagram of fig. 2.5.3, is not affected by this fact since its subtraction is null. Nevertheless, we will find a serious problem related to this point in other diagrams that we will study later.

Therefore, besides the lucky diagram of fig. 2.5.3, it is clear that the suggested operator  $\mathcal{T}$  does not reproduce FDR. Actually, we have shown that it does not renormalize diagrams neither, since one of the missing pieces contains a pole.

### 3.2.2 Next to minimal operator

Once studied the previous  $\mathcal{T}$ , we should propose a new one that solves at least the problem with the divergences we found with the former  $\mathcal{T}$ , but not too different because some of the arisen structures were correct. A good election could be

$$\mathcal{T}(\mathcal{N}\mathcal{D}) = \mathcal{N}\mathcal{V}(\mathcal{D}) + \mathcal{X}(\mathcal{N})[\mathcal{V}(\mathcal{D}) - \mathcal{D}] \quad (3.2.19)$$

It is seemingly non-local but, as we said in the previous section, it compensates the non-locality introduced by shifting the denominators. Again, we must test this  $\mathcal{T}$  definition using eq. (3.1.5).

$$\begin{aligned} \mathcal{T}(\mathcal{T}(\mathcal{N}\mathcal{D})) &= \mathcal{N}\mathcal{V}(\mathcal{V}(\mathcal{D})) + \mathcal{X}(\mathcal{N})[\mathcal{V}(\mathcal{V}(\mathcal{D})) - \mathcal{V}(\mathcal{D})] + \\ &+ \mathcal{X}(\mathcal{N})\mathcal{V}(\mathcal{V}(\mathcal{D}) - \mathcal{D}) + \mathcal{X}(\mathcal{X}(\mathcal{N}))[\mathcal{V}(\mathcal{V}(\mathcal{D}) - \mathcal{D}) - (\mathcal{V}(\mathcal{D}) - \mathcal{D})] \end{aligned} \quad (3.2.20)$$

$$\begin{aligned} \mathcal{T}(\mathcal{T}(\mathcal{N}\mathcal{D})) &= [\mathcal{N} + 2\mathcal{X}(\mathcal{N}) + \mathcal{X}(\mathcal{X}(\mathcal{N}))]\mathcal{V}(\mathcal{V}(\mathcal{D})) \\ &- 2[\mathcal{X}(\mathcal{N}) + \mathcal{X}(\mathcal{X}(\mathcal{N}))]\mathcal{V}(\mathcal{D}) + \mathcal{X}(\mathcal{X}(\mathcal{N}))\mathcal{D} \end{aligned} \quad (3.2.21)$$

and if

$$\mathcal{V}(\mathcal{V}(\mathcal{D})) = \mathcal{V}(\mathcal{D}) \quad (3.2.22)$$

then

$$\mathcal{T}(\mathcal{T}(\mathcal{N}\mathcal{D})) = \mathcal{N}\mathcal{V}(\mathcal{D}) - \mathcal{X}(\mathcal{X}(\mathcal{N}))[\mathcal{V}(\mathcal{D}) - \mathcal{D}] \quad (3.2.23)$$

and finally we find the same idempotence relation for  $\mathcal{X}(\mathcal{N})$  as before.

$$\mathcal{X}(\mathcal{X}(\mathcal{N})) = -\mathcal{X}(\mathcal{N}) \quad (3.2.24)$$

So using this  $\mathcal{T}$  definition, for a one-loop diagram we have

$$\mathcal{R}(\Gamma) = \lim_{\mu \rightarrow 0} \int d^n q [\mathcal{N} + \mathcal{X}(\mathcal{N})][\mathcal{D} - \mathcal{V}(\mathcal{D})] \quad (3.2.25)$$

making final result more compact. Now, it exactly shares the same structure as eq. (2.5.7). Although we saw that for a one-loop diagram, the term that we have inserted now is null, it will be very important beyond one loop.

So let us write again the explicit expressions of the forests depicted in fig. 3.2.2 but using the current  $\mathcal{T}$ .

$$\bar{\Gamma}_1^{(a)} = \mathcal{N}\mathcal{D} \quad (3.2.26)$$

$$\bar{\Gamma}_1^{(b)} = \mathcal{N}\mathcal{V}_2(\mathcal{D}) + \mathcal{X}_2(\mathcal{N})[\mathcal{V}_2(\mathcal{D}) - \mathcal{D}] \quad (3.2.27)$$

$$\bar{\Gamma}_1^{(c)} = \mathcal{N}\mathcal{V}_{12}(\mathcal{D}) + \mathcal{X}_{12}(\mathcal{N})[\mathcal{V}_{12}(\mathcal{D}) - \mathcal{D}] \quad (3.2.28)$$

$$\bar{\Gamma}_1^{(d)} = \mathcal{N}\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) + \mathcal{X}_{12}(\mathcal{N})[\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) - \mathcal{V}_2(\mathcal{D})] + \mathcal{T}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{V}_2(\mathcal{D}) - \mathcal{D}]) \quad (3.2.29)$$

and summing them all

$$\mathcal{R}(\bar{\Gamma}_1) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] + \mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \quad (3.2.30)$$

As in the one-loop case, we have got a more compact and absolutely factorized expression, formed by the numerator shifted by *global prescription* along with the completely finite denominator. Then, out of the main structure, we find the same term than beforehand, since the new piece is always null. In spite of we have already shown that it does not lead to the same expression as FDR, we will continue studying this operator and leave that discussion to later sections, since the difference is finite. Furthermore, eq. (3.2.30) is such intuitive that no operator nor forests will be needed to remember it, and we can even directly foresee what would be the renormalized expression of a three-loop diagram (or more) with nested

subdivergences, i.e. where forests only can have nested boxes because of the topology we have used to reach to the result. Obviously, if some forest with more than one box at the same level existed, the configuration of the renormalized value would change, and all the process would be necessary. So for a three-loop diagram with just nested subdivergences, we can directly predict that

$$\begin{aligned} \mathcal{R}(\mathcal{N}\mathcal{D}) &= [\mathcal{N} + \mathcal{X}_{ijk}(\mathcal{N})][\mathcal{D} - \mathcal{V}_i(\mathcal{D}) - \mathcal{V}_{ij}(\mathcal{D} - \mathcal{V}_i(\mathcal{D})) - \mathcal{V}_{ijk}(\mathcal{D} - \mathcal{V}_i(\mathcal{D}) - \mathcal{V}_{ij}(\mathcal{D} - \mathcal{V}_i(\mathcal{D})))] + \\ &+ \mathcal{R}_{ijk}(\mathcal{X}_{ij}(\mathcal{N})[\mathcal{D} - \mathcal{V}_i(\mathcal{D}) - \mathcal{V}_{ij}(\mathcal{D} - \mathcal{V}_i(\mathcal{D}))]) + \mathcal{R}_{ijk}(\mathcal{R}_{ij}(\mathcal{X}_i(\mathcal{N})[\mathcal{D} - \mathcal{V}_i(\mathcal{D})])) \end{aligned} \quad (3.2.31)$$

where  $i, j, k$  are the loop momenta subindices from inside to outside respectively. Therefore, the formal renormalized expression of any-loop diagram which fulfils these features can be deduced this way.

Next, the turn is for the two-loop self-energy diagram with two subdivergences in  $\phi^3$  (fig. 3.2.1c). In order to make easier the formal expression deduction, we will depict and write the corresponding forests. So seeing fig. 3.2.3, supposing  $q_1$  is related to left loop and  $q_2$  to right loop, and obviously using eq. (3.2.19)

Figure 3.2.3: Necessary forests for the renormalization of fig. 3.2.1c.

$$\bar{\Gamma}_2^{(a)} = \mathcal{N}\mathcal{D} \quad (3.2.32)$$

$$\bar{\Gamma}_2^{(b)} = \mathcal{N}\mathcal{V}_1(\mathcal{D}) + \mathcal{X}_1(\mathcal{N})[\mathcal{V}_1(\mathcal{D}) - \mathcal{D}] \quad (3.2.33)$$

$$\bar{\Gamma}_2^{(c)} = \mathcal{N}\mathcal{V}_2(\mathcal{D}) + \mathcal{X}_2(\mathcal{N})[\mathcal{V}_2(\mathcal{D}) - \mathcal{D}] \quad (3.2.34)$$

$$\bar{\Gamma}_2^{(d)} = \mathcal{N}\mathcal{V}_{12}(\mathcal{D}) + \mathcal{X}_{12}(\mathcal{N})[\mathcal{V}_{12}(\mathcal{D}) - \mathcal{D}] \quad (3.2.35)$$

$$\bar{\Gamma}_2^{(e)} = \mathcal{N}\mathcal{V}_{12}(\mathcal{V}_1(\mathcal{D})) + \mathcal{X}_{12}(\mathcal{N})[\mathcal{V}_{12}(\mathcal{V}_1(\mathcal{D})) - \mathcal{V}_1(\mathcal{D})] + \mathcal{T}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{V}_1(\mathcal{D}) - \mathcal{D}]) \quad (3.2.36)$$

$$\bar{\Gamma}_2^{(f)} = \mathcal{N}\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) + \mathcal{X}_{12}(\mathcal{N})[\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) - \mathcal{V}_2(\mathcal{D})] + \mathcal{T}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{V}_2(\mathcal{D}) - \mathcal{D}]) \quad (3.2.37)$$

And finally

$$\begin{aligned} \mathcal{R}(\bar{\Gamma}_2) &= [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}))] + \\ &+ \mathcal{R}_{12}(\mathcal{X}_1(\mathcal{N})[\mathcal{D} - \mathcal{V}_1(\mathcal{D})]) + \mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \end{aligned} \quad (3.2.38)$$

As we see again, we obtained a very simple and predictable structure: the shifted numerator together with the finite denominator, and this nearly extra-extra for each subdivergence. So with this equation, we can expect the formal expression of whatever any-loop diagram whose subdivergences are arranged into two groups, i.e. its forests can have only two sets of nested subdivergences, due to the topology of the diagram we have utilized to deduce the formal expression. Actually, studying both equations (3.2.30 and 3.2.38), we could predict any situation. But if we want to ensure the result, depicting forests, applying  $\mathcal{T}$  and summing them all is the safest procedure.<sup>3</sup>

Henceforth, we will use this operator definition since it leads to a finite result. However, in the following sections, we will try several assumptions and configurations that could change the final result beyond one loop.

### 3.2.3 Simplest configuration

Although the previous definition of  $\mathcal{T}$  leads to a finite expression and the main structure is exactly the same as the expected by FDR, the so-called *extra-extra integrals* are a bit different, so FDR has not been entirely reproduced. In principle, the deviation is always finite by definition and even for some diagrams vanishes. A constant difference in renormalization is not a real problem, since it is an usual thing that occurs when working with several renormalization schemes. In this way, it could be considered that FDR has been successfully systematized. The problem is that a logarithmic dependence can arise due to our extra-extra piece configuration. It would make the difference between original FDR and our deduction non-local what would change the diagram behaviour.

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<sup>3</sup>We remember that in all formal expressions we wrote, a sum over all parts with different degree of divergence is assumed.

Making an attempt to avoid this, we could, for example, remove the rule that distinguishes between  $\mu$  generated by inner or outer boxes. This means that all  $\mu$  should be taken to zero after all integrals have been performed. Therefore, there is no reason to place a  $\mathcal{T}$  before  $\mathcal{X}_2(\mathcal{N})$  when acting with a big box on a small one. Now, explicit  $\mathcal{V}$  and  $\mathcal{X}$  should be use instead. So using the same  $\mathcal{T}$  as before, this respectively transforms equations (3.2.30) and (3.2.38) into

$$\mathcal{R}(\bar{\Gamma}_1) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N}) + \mathcal{X}_2(\mathcal{N}) + \mathcal{X}_{12}(\mathcal{X}_2(\mathcal{N}))][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \quad (3.2.39)$$

$$\begin{aligned} \mathcal{R}(\bar{\Gamma}_2) = & [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}))] + \\ & + [\mathcal{X}_1(\mathcal{N}) + \mathcal{X}_{12}(\mathcal{X}_1(\mathcal{N}))][\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_1(\mathcal{D}))] + \\ & + [\mathcal{X}_2(\mathcal{N}) + \mathcal{X}_{12}(\mathcal{X}_2(\mathcal{N}))][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \end{aligned} \quad (3.2.40)$$

Beforehand, when we were distinguishing between  $\mu$ , it was impossible that a  $\mathcal{X}$  could act on an existing  $\mu$  because it must have been taken to zero before any outer operator could act on it. So, every time a  $\mathcal{X}$  acted on a numerator, no  $\mu$  could exist. Nevertheless, now this rule has been removed what means that the way  $\mathcal{X}$  acts on  $\mu$  needs to be define. The answer is in eq. (3.2.24) which has not been necessary until now. This idempotence relation suggests that  $\mu$  should be treated as a loop momentum just like in the power-counting procedure.

$$\mathcal{X}(\mu^2) = -\mu^2 \quad (3.2.41)$$

In any case, idempotence relation allows us to simplify equations (3.2.39) and (3.2.40).

$$\mathcal{R}(\bar{\Gamma}_1) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \quad (3.2.42)$$

$$\mathcal{R}(\bar{\Gamma}_2) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}))] \quad (3.2.43)$$

leaving just the main structure and deleting any track of *extra-extra integrals*. We already know that these equations lead to a finite renormalization but some diagrams may require the missing extra-extra parts in order to keep unchanged its logarithmic behaviour.

### 3.2.4 Complete configuration

Studying the way *extra-extra integrals* are introduced in FDR, we realize that it cannot be exactly replicated by Forest Formula if we continue with the same assumptions. In eq. (2.5.14)

we learnt that *extra-extra integrals* are made by a  $\mu$ -part generated by the subdiagram numerator without any index contraction with the external part, minus a  $\mu$ -part also generated by the subdiagram loop momentum but after contracting all numerator indices. In addition, both  $\mu$ -parts have to be integrated and taken to zero before performing the outer integral. Both conditions cannot be achieved by Forest Formula at once, since the only box which can obtain information of the external part of the diagram is the big one, but, by definition, its  $\mu$  has to be taken to zero after all integrals have been performed. Moreover, Forest Formula is so strict, its operator  $\mathcal{T}$  always has to act the same way on whatever diagram or subdiagram, so we cannot expect that bigger boxes behave differently to smaller ones. Therefore, it is clear, that it will be impossible to get the missing part to be subtracted. Unless, we give Forest Formula the needed information.

Let us consider that before giving the diagram to Forest Formula, besides shifting the denominator we perform the *global prescription* in the numerator. Thus the diagram without boxes will be

$$\mathcal{N}\mathcal{D} \rightarrow [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})]\mathcal{D} \quad (3.2.44)$$

Furthermore, we need the same last  $\mathcal{T}$  but restoring the rule that keeps  $\mu$  hierarchy, i.e. the same conditions as in subsection (3.2.2). And now if we repeat all the process for a one subdivergence we get

$$\begin{aligned} \mathcal{R}(\bar{\Gamma}_1) = & [\mathcal{N} + 2\mathcal{X}_{12}(\mathcal{N}) + \mathcal{X}_2(\mathcal{N}) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})) + \mathcal{X}_{12}(\mathcal{X}_{12}(\mathcal{N})) + \\ & + \mathcal{X}_{12}(\mathcal{X}_2(\mathcal{N})) + \mathcal{X}_{12}(\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})))][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \end{aligned} \quad (3.2.45)$$

where all numerator pieces have been put together for compactness and convenience, but always keeping in mind that a  $\mu$  generated by  $\mathcal{X}_2$  has to be taken to zero before any other operator can act on it. Using again the idempotence relation, it is clear that

$$\mathcal{X}_{12}(\mathcal{X}_{12}(\mathcal{N})) = -\mathcal{X}_{12}(\mathcal{N}) \quad (3.2.46)$$

Now we wonder what the value of  $\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))$  is, and precisely it is just the piece we were looking for. Consider the next numerator

$$\mathcal{N}_1 = q_1^2 + q_2^2 \quad (3.2.47)$$

Applying  $\mathcal{X}_2$

$$\mathcal{X}_2(\mathcal{N}_1) = -\mu_2^2 \quad (3.2.48)$$

and now  $\mathcal{X}_{12}$

$$\mathcal{X}_{12}(\mathcal{N}_1) = -\mu_1^2 - \mu_2^2 \quad (3.2.49)$$

and last

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_1)) = \mu_2^2 \quad (3.2.50)$$

The result is just  $-\mathcal{X}_2(\mathcal{N})$  but obtained with information of all diagram. In this case

$$\mathcal{X}_2(\mathcal{N}_1) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_1)) = 0 \quad (3.2.51)$$

just as we need in the diagrams where there is no index contraction between subdiagram and the outer part. Now consider the following numerator

$$\mathcal{N}_2 = \gamma^\mu \not{q}_1 \gamma^\nu [2q_{2\mu} q_{2\nu} - q_2^2 g_{\mu\nu}] \quad (3.2.52)$$

where the part inside brackets belongs to the subdiagram. Now there are external indices contracted with inner ones. Let us compute the interesting  $\mathcal{X}$  expressions.

$$\mathcal{X}_2(\mathcal{N}_2) = -2\mu_2^2 \not{q}_1 \quad (3.2.53)$$

Before applying  $\mathcal{X}_{12}$  we have to contract all indices.

$$\mathcal{N}_2 = 2\not{q}_2 \not{q}_1 \not{q}_2 + 2q_2^2 \not{q}_1 = 4q_1 \cdot q_2 \not{q}_2 \quad (3.2.54)$$

Now we can continue.

$$\mathcal{X}_{12}(\mathcal{N}_2) = 2\mu^2 \not{q}_2 \quad (3.2.55)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_2)) = 0 \quad (3.2.56)$$

since there is no  $\mu^2$  generated by a  $q_2^2$  in  $\mathcal{X}_{12}(\mathcal{N}_2)$ . So in this case

$$\mathcal{X}_2(\mathcal{N}_2) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_2)) = -2\mu_2^2 \not{q}_1 \quad (3.2.57)$$

We see that now Forest Formula can distinguish among diagrams with and without index contraction between the subdiagrams and the external part. Thus reordering eq. (3.2.45)

$$\begin{aligned} \mathcal{R}(\bar{\Gamma}_1) &= [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})] [\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] + \\ &+ \mathcal{R}_{12}([\mathcal{X}_2(\mathcal{N}) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))] [\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \end{aligned} \quad (3.2.58)$$

and analogously

$$\begin{aligned} \mathcal{R}(\bar{\Gamma}_2) &= [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})] [\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_1(\mathcal{D}) - \mathcal{V}_2(\mathcal{D}))] + \\ &+ \mathcal{R}_{12}([\mathcal{X}_1(\mathcal{N}) + \mathcal{X}_1(\mathcal{X}_{12}(\mathcal{N}))] [\mathcal{D} - \mathcal{V}_1(\mathcal{D})]) + \mathcal{R}_{12}([\mathcal{X}_2(\mathcal{N}) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))] [\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \end{aligned} \quad (3.2.59)$$

So the exactly FDR procedure has been replicated. This is not necessary the correct way, we are just testing the method.

### 3.2.5 Subintegration

Analyzing the original *extra-extra integrals* in FDR, we realize that they were introduced as a trick to get a subintegration consistency. Manipulating the whole numerator, evaluating the overlapped integrals and taking  $\mu$  to zero after that, spoils the subintegration consistency for some diagrams. That is, this process does not give the same result as computing the subdiagram renormalized value, then inserting it to the outer part and renormalizing again. So, *extra-extra integrals* were introduced to get a complete equivalence between the two ways.

Let us put this fact in context with fig. 3.2.1a again. If we renormalize the subdiagram first we have

$$\mathcal{R}(\Gamma_1) = \lim_{\mu \rightarrow 0} \int d^n q_1 \mathcal{R} \left( \lim_{\mu \rightarrow 0} \int d^n q_2 [\mathcal{N} + \mathcal{X}_2(\mathcal{N})] [\mathcal{D} - \mathcal{V}_2(\mathcal{D})] \right) \quad (3.2.60)$$

but if we perform the integrals and limits at the end

$$\mathcal{R}(\Gamma_1) = \lim_{\mu \rightarrow 0} \int d^n q_1 \int d^n q_2 [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})] [\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \quad (3.2.61)$$

The first difference is clear, the term

$$\lim_{\mu \rightarrow 0} \int d^n q_1 \mathcal{R} \left( \lim_{\mu \rightarrow 0} \int d^n q_2 \mathcal{X}_2(\mathcal{N}) [\mathcal{D} - \mathcal{V}_2(\mathcal{D})] \right) \quad (3.2.62)$$

does not appear in the second equation, so it is reasonable that it must be added to (3.2.61) in order to recover the subintegration consistency. Precisely, it is exactly the same piece that FDR adds and the one we found in eq. (3.2.30) with Forest Formula. If we first renormalize the subdiagram, then  $\mathcal{X}_{12}$  cannot find any  $q_2^2$  because they have already been integrated out. So all  $\mu$  found by  $\mathcal{X}_{12}$  in eq. (3.2.61) must be added to (3.2.60). And now both ways of renormalizing are equivalent. As we will use only one of them at once, we will have to add



one part and subtract the other, accordingly. That is why an extra-extra integral is made by a sum and a subtraction.

Once better understood the nature of *extra-extra integrals*, we can affirm that Forest Formula with the second  $\mathcal{T}$  we defined (eq. 3.2.19) leads to the same result by two ways. The first consists in renormalizing the subdiagram first but without shifting the numerator before giving it to Forest Formula, while the second lies in preparing the numerator with *global prescription* and then using Forest Formula as usual.

### 3.3 Renormalization Group Equation

As we have shown in the previous sections, we have found several renormalizations that lead to a finite result. However, distinct logarithmic behaviour is expected making their differences non-local. This means that maybe some renormalization could be wrong. In this way, Renormalization Group Equation (RGE) could help us to discard some of them.

A particular renormalization scheme is a reordering of the perturbative expansion and a change in the scheme is compensated by changes in the parameters of the theory. All these changes can be parametrized by the energy scale or renormalization constant  $\mu$ . Consider that the relation between an unrenormalized correlator and a renormalized one in a particular renormalization scheme is

$$\Gamma_R = Z(R) \Gamma \quad (3.3.1)$$

As the unrenormalized one cannot depend on  $\mu$

$$\frac{d}{d\mu} \Gamma = 0 \quad (3.3.2)$$

so

$$\frac{d}{d\mu} [Z^{-1}(R) \Gamma_R] = 0 \quad (3.3.3)$$

If we perform the total derivative, i.e. differentiating all parameters in QED (Quantum Electrodynamics) with respect to  $\mu$ , we find that whatever renormalized correlation function in QED must fulfil the next relation [58].

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(\alpha, \xi) \frac{\partial}{\partial \alpha} + \gamma_m(\alpha, \xi) m \frac{\partial}{\partial m} + \beta_\xi(\alpha, \xi) \frac{\partial}{\partial \xi} - n_A \gamma_A(\alpha, \xi) - n_f \gamma_f(\alpha, \xi) \right] \Gamma_R^{(n_A, n_f)} = 0 \quad (3.3.4)$$

where  $\mu$  is the renormalization constant or energy scale,  $\alpha$  the coupling constant,  $\xi$  the gauge,

$m$  the mass,  $n_A$  and  $n_f$  the number of external photon and fermion fields respectively in the renormalized correlator  $\Gamma_R^{(n_A, n_f)}$ ,  $\beta$ 's the beta function which represents the dependence of the coupling constant or gauge on energy scale, and  $\gamma$ 's the anomalous dimensions. This relation is the so-called RGE what we mentioned above and it was discovered by Callan and Symanzik [59, 60, 61].

Thanks to that relation, if we compute some two-loop diagrams in QED, we can use RGE to analyze the several FDR renormalizations that we have found with Forest Formula. Moreover, in QED, the following well-known relation holds and it will be useful to our analysis.

$$\beta = 2\alpha\gamma_A \tag{3.3.5}$$

That is, in Spinorial QED as well as in Scalar QED, the beta function is two times the anomalous dimension of the photon field to all orders in  $\alpha$ . Furthermore, if we expand the  $\beta$  function with respect the constant coupling  $\alpha$

$$\beta(\alpha) = \beta^{(2)}\alpha^2 + \beta^{(3)}\alpha^3 + \dots \tag{3.3.6}$$

the first two coefficients of the  $\beta$  function are scheme-independent in Spinorial QED as well as in Scalar QED. Such values can be useful to our work. In Spinorial QED [62] we have

$$\beta^{(2)} = \frac{2}{3\pi} \tag{3.3.7}$$

$$\beta^{(3)} = \frac{1}{2\pi^2} \tag{3.3.8}$$

and in Scalar QED [63]

$$\beta^{(2)} = \frac{1}{6\pi} \tag{3.3.9}$$

$$\beta^{(3)} = \frac{1}{2\pi^2} \tag{3.3.10}$$

# Chapter 4

## QED Calculations

In this section, we are going to compare among some procedures we talked about in the last section. To do this, we will compute some two-loop examples. All of them will be in the framework of QED (Quantum Electrodynamics) with massless fermions, but off-shell to avoid infrared divergences, and in the 't Hooft-Feynman gauge ( $\xi = 1$ ). We will begin with vacuum polarization diagrams in usual Spinorial QED and Scalar QED (where fermions are swapped by scalars) followed by fermion self-energy.

$$\mathcal{L}_{QED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + \bar{\psi}(i\mathcal{D})\psi \quad (4.0.1)$$

$$\mathcal{L}_{SQED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + |\mathcal{D}_\mu\phi|^2 \quad (4.0.2)$$

While evaluating the mentioned diagrams, we will always follow the same process. We will first compute the main structure of the formal expressions we found as in equations (3.2.42) and (3.2.43). Then it will be the turn of the controversial *extra-extra integrals*. We will compute them using equations (3.2.30) and (3.2.38); and (3.2.58) and (3.2.59) to see the difference. To clarify, we will call these different methods as  $\mathcal{R}^{(1)}$ ,  $\mathcal{R}^{(2)}$  and  $\mathcal{R}^{(3)}$  respectively. That is

$$\mathcal{R}^{(1)}(\bar{\Gamma}_1) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] \quad (4.0.3)$$

$$\mathcal{R}^{(2)}(\bar{\Gamma}_1) = [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] + \mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \quad (4.0.4)$$

$$\begin{aligned} \mathcal{R}^{(3)}(\bar{\Gamma}_1) &= [\mathcal{N} + \mathcal{X}_{12}(\mathcal{N})][\mathcal{D} - \mathcal{V}_2(\mathcal{D}) - \mathcal{V}_{12}(\mathcal{D} - \mathcal{V}_2(\mathcal{D}))] + \\ &+ \mathcal{R}_{12}([\mathcal{X}_2(\mathcal{N}) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))][\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \end{aligned} \quad (4.0.5)$$

Moreover, to facilitate the comparison, we define

$$\Delta_1 \mathcal{R}(\bar{\Gamma}_1) \equiv \mathcal{R}^{(2)}(\bar{\Gamma}_1) - \mathcal{R}^{(1)}(\bar{\Gamma}_1) \equiv \mathcal{R}_{12}(\mathcal{X}_2(\mathcal{N})[\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \quad (4.0.6)$$

$$\Delta_2 \mathcal{R}(\bar{\Gamma}_1) \equiv \mathcal{R}^{(3)}(\bar{\Gamma}_1) - \mathcal{R}^{(1)}(\bar{\Gamma}_1) \equiv \mathcal{R}_{12}([\mathcal{X}_2(\mathcal{N}) + \mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))][\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \quad (4.0.7)$$

$$\Delta_3 \mathcal{R}(\bar{\Gamma}_1) \equiv \mathcal{R}^{(2)}(\bar{\Gamma}_1) - \mathcal{R}^{(3)}(\bar{\Gamma}_1) \equiv -\mathcal{R}_{12}([\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}))][\mathcal{D} - \mathcal{V}_2(\mathcal{D})]) \quad (4.0.8)$$

and also the corresponding expressions for the case with two subdivergences.

The procedure to get the main structure consists, as we know, in shifting the numerator by *global prescription*, and then finding the corresponding vacua by using fraction decomposition and power-counting. Only then, we can evaluate the integrals and take the limit  $\mu \rightarrow 0$ . To get *extra-extra integrals*, we shift and fraction decompose only the subdiagram, evaluate its integral and take its  $\mu$  to zero. Next the remaining one-loop integral must be renormalize as usual.

All Feynman rules and needed integrals as well as other useful relations will be in the appendices.

## 4.1 Vacuum polarization

We will start this example section with the study of vacuum polarization. The Lagrangian interaction part that leads to the diagrams we want to study is

$$\mathcal{L}_{QED} \supset -e\bar{\psi}\mathcal{A}\psi \quad (4.1.1)$$

where  $e$  is the fermion electric charge, that we will replace for the fine structure constant, the usual coupling constant in QED.

$$\alpha = 4\pi e^2 \quad (4.1.2)$$

### 4.1.1 One loop

For the completeness of this section, we will begin with the one-loop graph (fig. 4.1.1).

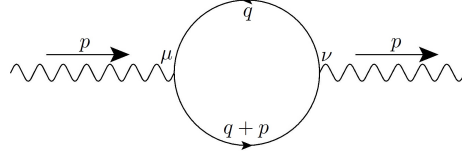


Figure 4.1.1: One-loop photon self-energy in QED.

According to QED Feynman rules

$$\Pi_{\mu\nu} = ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{\text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})]}{q^2 (q+p)^2} \quad (4.1.3)$$

Power-counting, we can put all numerator together with

$$\mathcal{V}(\mathcal{D}) = \frac{1}{\bar{q}^4} \left[ 1 - \frac{p^2 + 2q \cdot p}{\bar{q}^2} + \frac{4(q \cdot p)^2}{\bar{q}^4} \right] \quad (4.1.4)$$

Now, we must extract the extra part from the numerator. Performing the trace we easily find

$$\mathcal{X}(\mathcal{N}) = 4\mu^2 g_{\mu\nu} \quad (4.1.5)$$

So the final expression is

$$\mathcal{R}(\Pi_{\mu\nu}) = ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4q}{(2\pi)^4} (\text{Tr} [\gamma_\mu \not{q} \gamma_\nu (\not{q} + \not{p})] + 4\mu^2 g_{\mu\nu}) \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{p^2 + 2q \cdot p}{\bar{q}^6} - \frac{4(q \cdot p)^2}{\bar{q}^8} \right] \quad (4.1.6)$$

The last step now is evaluating the integral.

$$\mathcal{R}(\Pi_{\mu\nu}) = \frac{\alpha}{3\pi} \left[ \frac{5}{3} - \log \left( \frac{p^2}{\mu^2} \right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.1.7)$$

## 4.1.2 Two loops

When we move to two loops, there are three possible diagrams as we see in fig. 4.1.2, however two first are equivalent so we can just study one of them and double the result.

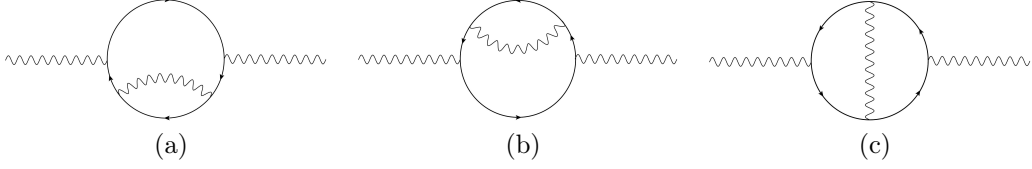


Figure 4.1.2: Two-loop photon self-energy diagrams in QED.

#### 4.1.2.1 $\Pi_{\mu\nu}^{(a+b)}$

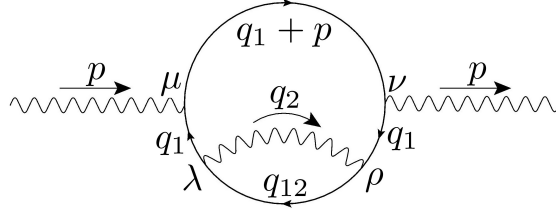


Figure 4.1.3: Momenta description of fig. 4.1.2a.

As we said before, figures 4.1.2a and 4.1.2b are equivalent. So, for example, fig. 4.1.2a has been chosen to be studied. Reading fig. 4.1.3 and using QED Feynman rules we have

$$\Pi_{\mu\nu}^{(a+b)} = 2e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\lambda \not{q}_{12} \gamma^\lambda \not{q}_1 \gamma_\nu (\not{q}_1 + \not{p}) \right]}{q_1^4 D_1 q_2^2 q_{12}^2} \quad (4.1.8)$$

where we recall  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ .

This diagram has one subdivergence and it shares topology with 3.2.1a, so their depicted forests will be the same. Therefore, our task now is to find the explicit expressions of all necessary pieces, in order to renormalize the graph. Obviously, the numerator and denominator are

$$\mathcal{N}_{\mu\nu} = 2\text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\lambda \not{q}_{12} \gamma^\lambda \not{q}_1 \gamma_\nu (\not{q}_1 + \not{p}) \right] \quad (4.1.9)$$

$$\mathcal{D} = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^2 \bar{q}_{12}^2} \quad (4.1.10)$$

Power-counting, we notice that the difference between degrees of divergence among parts related to subdivergence as well as to overall divergence is one. So we can compact all numerator under the same vacua and subvacua. Therefore, making use of fraction decomposition we easily get

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^4} \left[ 1 - \frac{2q_1 \cdot q_2}{\bar{q}_2^2} \right] \quad (4.1.11)$$

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.1.12)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \left[ 1 - \frac{2q_1 \cdot q_2}{\bar{q}_2^2} \right] \quad (4.1.13)$$

And now, we can shift the numerator to find extra-extra parts. As  $q_2^2$  cannot be made up in the numerator, we directly have

$$\mathcal{X}_2(\mathcal{N}) = 0 \quad (4.1.14)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})) = 0 \quad (4.1.15)$$

Thus all renormalizations will concur.

$$\begin{aligned} \mathcal{R}^{(1)}(\Pi_{\mu\nu}^{(a+b)}) &= -\frac{\alpha^2}{36\pi^2} \left\{ \left[ 33.43 - 17 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -45.62 + 20 \log\left(\frac{p^2}{\mu^2}\right) - 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.1.16)$$

$$\Delta_1 \mathcal{R}(\Pi_{\mu\nu}^{(a+b)}) = \Delta_2 \mathcal{R}(\Pi_{\mu\nu}^{(a+b)}) = \Delta_3 \mathcal{R}(\Pi_{\mu\nu}^{(a+b)}) = 0 \quad (4.1.17)$$

#### 4.1.2.2 $\Pi_{\mu\nu}^{(c)}$

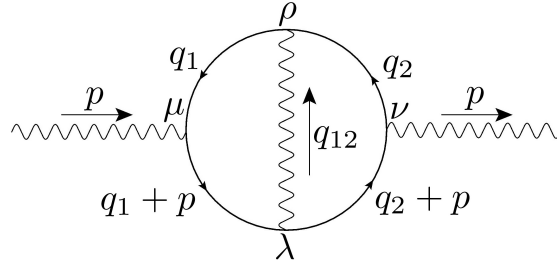


Figure 4.1.4: Momenta description of fig. 4.1.2c.

Let us evaluate the third graph. Reading fig. 4.1.4 and using QED Feynman rules we have

$$\Pi_{\mu\nu}^{(c)} = e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\lambda \not{q}_2 \gamma_\nu (\not{q}_2 + \not{p}) \gamma^\lambda (\not{q}_1 + \not{p}) \right]}{q_1^2 D_1 q_2^2 D_2 q_{12}^2} \quad (4.1.18)$$

where this time  $q_{12} \equiv q_1 - q_2$  and  $D_i \equiv (q_i + p)^2$ .

This graph has two coupled subdivergences thus its computing will be harder. So now let us find the pieces. Concerning subdivergences, we can put all numerator together.

$$\mathcal{V}_1(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{D}_2} \quad (4.1.19)$$

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{D}_1 \bar{q}_2^6} \quad (4.1.20)$$

However, regarding overall divergence, the piece with two external momenta in the trace must be separated because the difference between its degree of divergence and the leader is two. Indeed, we can split the numerator into three pieces but we will split into two for compactness.

$$\mathcal{N}_{\mu\nu}^{(1)} = \text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\lambda \not{q}_2 \gamma_\nu \left\{ \not{q}_2 \gamma^\lambda (\not{q}_1 + \not{p}) + \not{p} \gamma^\lambda \not{q}_1 \right\} \right] \quad (4.1.21)$$

$$\mathcal{N}_{\mu\nu}^{(2)} = \text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\lambda \not{q}_2 \gamma_\nu \not{p} \gamma^\lambda \not{p} \right] \quad (4.1.22)$$

So their corresponding vacua are

$$\mathcal{V}_{12}^{(1)}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{q}_2^4 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} - \frac{p^2 + 2q_2 \cdot p}{\bar{q}_2^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} + \frac{4(q_2 \cdot p)^2}{\bar{q}_2^4} + \frac{4(q_1 \cdot p)(q_2 \cdot p)}{\bar{q}_1^2 \bar{q}_2^2} \right] \quad (4.1.23)$$

$$\mathcal{V}_{12}^{(2)}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{q}_2^4 \bar{q}_{12}^2} \quad (4.1.24)$$

$$\mathcal{V}_{12}^{(1)}(\mathcal{V}_1(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{p^2 + 2q_2 \cdot p}{\bar{q}_2^2} + \frac{4(q_2 \cdot p)^2}{\bar{q}_2^4} \right] \quad (4.1.25)$$

$$\mathcal{V}_{12}^{(2)}(\mathcal{V}_1(\mathcal{D})) = \frac{1}{\bar{q}_1^6} \frac{1}{\bar{q}_2^4} \quad (4.1.26)$$

$$\mathcal{V}_{12}^{(1)}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^4 \bar{q}_2^6} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.1.27)$$

$$\mathcal{V}_{12}^{(2)}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^4} \frac{1}{\bar{q}_2^6} \quad (4.1.28)$$

And last step is to extract extra-extra parts from the numerator.

$$\mathcal{X}_1(\mathcal{N}_{\mu\nu}) = -2\mu^2 \text{Tr} \left[ \gamma_\mu \not{q}_2 \gamma_\nu (\not{q}_2 + \not{p}) \right] \quad (4.1.29)$$

$$\mathcal{X}_2(\mathcal{N}_{\mu\nu}) = -2\mu^2 \text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\nu (\not{q}_1 + \not{p}) \right] \quad (4.1.30)$$



It is easy to show that

$$\mathcal{X}_1(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = 2\mu^2 \text{Tr} \left[ \gamma_\mu \not{q}_2 \gamma_\nu (\not{q}_2 + \not{p}) \right] \quad (4.1.31)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = 2\mu^2 \text{Tr} \left[ \gamma_\mu \not{q}_1 \gamma_\nu (\not{q}_1 + \not{p}) \right] \quad (4.1.32)$$

due to the lack of index contraction between subdiagrams and outside. Thus this time, first and third renormalizations will have no differences.

$$\begin{aligned} \mathcal{R}^{(1)}(\Pi_{\mu\nu}^{(c)}) &= \frac{\alpha^2}{36\pi^2} \left\{ \left[ 25.25 - 26 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -37.44 + 29 \log\left(\frac{p^2}{\mu^2}\right) - 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.1.33)$$

$$\begin{aligned} \mathcal{R}^{(2)}(\Pi_{\mu\nu}^{(c)}) &= \frac{\alpha^2}{36\pi^2} \left\{ \left[ 15.25 - 20 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -27.44 + 23 \log\left(\frac{p^2}{\mu^2}\right) - 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.1.34)$$

$$\Delta_1 \mathcal{R}(\Pi_{\mu\nu}^{(c)}) = \Delta_3 \mathcal{R}(\Pi_{\mu\nu}^{(c)}) = -\frac{\alpha^2}{18\pi^2} \left[ 5 - 3 \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.1.35)$$

$$\Delta_2 \mathcal{R}(\Pi_{\mu\nu}^{(c)}) = 0 \quad (4.1.36)$$

### 4.1.2.3 Sum

We cannot forget that photon must be transverse in its propagation, so the sum of all diagrams should accomplish Ward Identities. Thus let us sum them all.

$$\mathcal{R}^{(1)}(\Pi_{\mu\nu}) = -\frac{\alpha^2}{12\pi^2} \left[ 2.73 + 3 \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.1.37)$$

$$\mathcal{R}^{(2)}(\Pi_{\mu\nu}) = -\frac{\alpha^2}{12\pi^2} \left[ 6.06 + \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.1.38)$$

$$\Delta_1 \mathcal{R}(\Pi_{\mu\nu}) = \Delta_3 \mathcal{R}(\Pi_{\mu\nu}) = -\frac{\alpha^2}{18\pi^2} \left[ 5 - 3 \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.1.39)$$

$$\Delta_2 \mathcal{R}(\Pi_{\mu\nu}) = 0 \quad (4.1.40)$$

Besides squared logarithm has vanished, as expected

$$p^\mu \mathcal{R}(\Pi_{\mu\nu}) = p^\nu \mathcal{R}(\Pi_{\mu\nu}) = 0 \quad (4.1.41)$$

### 4.1.3 RGE analysis

Gauge invariance and finiteness are a very good test that renormalization has been successful. However, our results have distinct logarithmic behaviour as expected, i.e. their difference are non-local. So, as we advanced in a previous section, perhaps the Renormalization Group Equation (3.3.4) give us some useful information.

$$\left[ 2\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha, \xi) \frac{\partial}{\partial \alpha} + \beta_\xi(\alpha, \xi) \frac{\partial}{\partial \xi} - 2\gamma_A(\alpha, \xi) \right] \Pi_{\mu\nu} = 0 \quad (4.1.42)$$

where the derivative with respect to the energy scale has been put in a more convenient form, mass has been removed because of the massless nature of our work, and  $n_A = 2$  since we have computed two-point functions. Before starting the analysis we should know the behaviour of every component in terms of  $\alpha$ , given that different  $\alpha$  orders cannot mix.

$$\beta(\alpha) = \beta^{(2)}\alpha^2 + \beta^{(3)}\alpha^3 + \dots \quad (4.1.43)$$

$$\beta_\xi(\alpha) = \beta_\xi^{(1)}\alpha + \beta_\xi^{(2)}\alpha^2 + \dots \quad (4.1.44)$$

$$\gamma_A(\alpha) = \gamma_A^{(1)}\alpha + \gamma_A^{(2)}\alpha^2 + \dots \quad (4.1.45)$$

$$\Pi_{\mu\nu} = (p_\mu p_\nu - p^2 g_{\mu\nu}) [\Pi^{(0)} + \alpha \Pi^{(1)} + \alpha^2 \Pi^{(2)} + \dots] + p_\mu p_\nu \Pi_{\text{long}}^{(0)} \quad (4.1.46)$$

since the photon propagator is not completely transverse due to the gauge parameter  $\xi$ . We summarize here for convenience all needed vacuum polarization results before inserting them to RGE. The order zero parts are extracted from the photon propagator inverse.

$$\Pi^{(0)} = 1 \quad (4.1.47)$$

$$\Pi_{\text{long}}^{(0)} = -\frac{1}{\xi} \quad (4.1.48)$$

$$\Pi^{(1)} = -\frac{1}{3\pi} \log\left(\frac{p^2}{\mu^2}\right) + \mathcal{C} \quad (4.1.49)$$

$$\Pi_1^{(2)} = -\frac{1}{4\pi^2} \log\left(\frac{p^2}{\mu^2}\right) + \mathcal{C}_1 \quad (4.1.50)$$

$$\Pi_2^{(2)} = -\frac{1}{12\pi^2} \log\left(\frac{p^2}{\mu^2}\right) + \mathcal{C}_2 \quad (4.1.51)$$

where, in principle, we have two possibilities for  $\Pi^{(2)}$  as we know.

If we insert the longitudinal part into (4.1.42), we get the simply relation to all orders

$$\beta_\xi = -2\gamma_A \quad (4.1.52)$$

and also due to eq. (3.3.5)

$$\beta = -\beta_\xi \alpha \quad (4.1.53)$$

To study the transverse part, we have to go order by order in  $\alpha$ . We do not find any contribution in  $\mathcal{O}(\alpha^0)$ , thus we start for  $\mathcal{O}(\alpha)$ .

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(1)} - 2\gamma_A^{(1)} \Pi^{(0)} = 0 \quad (4.1.54)$$

$$\gamma_A^{(1)} = \mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(1)} \quad (4.1.55)$$

so

$$\gamma_A^{(1)} = \frac{1}{3\pi} \quad (4.1.56)$$

In QED, the first two coefficients of the beta function are scheme-independent, so we can use their known values to compare with our results. So using eq. (3.3.5) and the known first beta function coefficient (3.3.7), we can conclude that the result (4.1.56) agrees with the known values. This means that FDR has successfully renormalize the one-loop graph.

Now, we continue with  $\mathcal{O}(\alpha^2)$ , the interesting one.

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} + \beta^{(2)} \Pi^{(1)} - 2\gamma_A^{(1)} \Pi^{(1)} - 2\gamma_A^{(2)} \Pi^{(0)} = 0 \quad (4.1.57)$$

Thanks to eq. (4.1.42), we can simplify as follows.

$$\gamma_A^{(2)} = \mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} \quad (4.1.58)$$

As  $\gamma_A^{(2)}$  does not have any logarithmic dependence, it means that  $\Pi^{(2)}$  cannot contain any squared logarithm (or beyond). Although we have found squared logarithms in our computations, they cancelled out when we summed them all as we can see in (4.1.50) and (4.1.51). Using again (3.3.5) and the known value for the second beta function coefficient (3.3.8), we

deduce that any good renormalization of the two-loop vacuum polarization in Spinorial QED, must fulfil

$$\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} = \frac{1}{4\pi^2} \quad (4.1.59)$$

so the renormalization scheme  $\mathcal{R}^{(2)}$  is discarded. Unfortunately, we cannot distinguish from  $\mathcal{R}^{(1)}$  and  $\mathcal{R}^{(3)}$ , since they concord in this calculation.

## 4.2 Scalar vacuum polarization

Now it is the turn for vacuum polarization but in Scalar QED, i.e. replacing fermions with scalars. The interaction Lagrangian we need is

$$\mathcal{L}_{SQED} \supset -ig [\phi^* (\partial_\mu \phi) - (\partial_\mu \phi)^* \phi] \mathcal{A}^\mu + g^2 \mathcal{A}_\mu \mathcal{A}^\mu |\phi|^2 \quad (4.2.1)$$

where  $g$  is the coupling constant of the scalars and photon fields, that no necessarily has to agree with the fermion electric charge that we used in Spinorial QED. Nevertheless, we will replace it for a “fine structure constant” a hat will be placed on it to distinguish it from usual QED.

$$\hat{\alpha} = 4\pi g^2 \quad (4.2.2)$$

This time we notice that besides a triple vertex, a quadruple one has arisen, which will cause the emergence of more diagrams than in usual QED.

To avoid confusion between Scalar and usual Spinorial QED, we will place a hat over the scalar symbol.

### 4.2.1 One loop

We will start with the one-loop case. This one-loop self-energy is composed of two graphs: one with the same topology as usual QED (fig. 4.2.1a) and one with a quadruple vertex (fig. 4.2.1b).

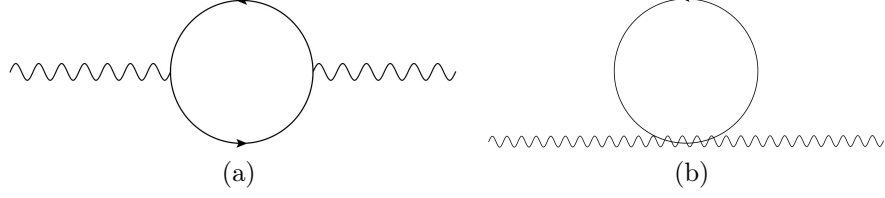


Figure 4.2.1: One-loop photon self-energy graphs in Scalar QED.

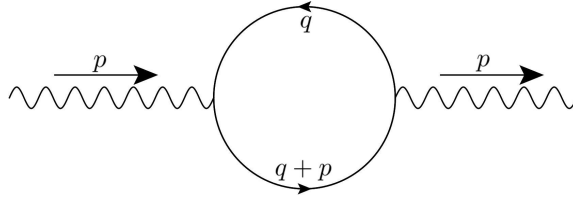


Figure 4.2.2: Momenta description of fig. 4.2.1a.

Let us begin computing fig. 4.2.2. According to Scalar QED Feynman rules

$$\hat{\Pi}_{\mu\nu}^{(a)} = -ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{(2q+p)_\mu (2q+p)_\nu}{q^2 (q+p)^2} \quad (4.2.3)$$

Power-counting, we find several degrees of divergence in the numerator, so we have to split it up. It will be enough to distinguish between two parts always thanks to parity.

$$\mathcal{N}_{\mu\nu}^{(1)} = 4q_\mu q_\nu + 2(q_\mu p_\nu + p_\mu q_\nu) \quad (4.2.4)$$

$$\mathcal{N}_{\mu\nu}^{(2)} = p_\mu p_\nu \quad (4.2.5)$$

and their corresponding vacua are respectively

$$\mathcal{V}^{(1)}(\mathcal{D}) = \frac{1}{\bar{q}^4} \left[ 1 - \frac{p^2 + 2q \cdot p}{\bar{q}^2} + \frac{4(q \cdot p)^2}{\bar{q}^4} \right] \quad (4.2.6)$$

$$\mathcal{V}^{(2)}(\mathcal{D}) = \frac{1}{\bar{q}^4} \quad (4.2.7)$$

As there is no  $q^2$  in the numerator, we can write the final expression.

$$\mathcal{R}(\hat{\Pi}_{\mu\nu}^{(a)}) = -ig^2 \lim_{\mu \rightarrow 0} \int \frac{d^4q}{(2\pi)^4} \{ [4q_\mu q_\nu + 2(q_\mu p_\nu + p_\mu q_\nu)] \times$$

$$\times \left\{ \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{p^2 + 2q \cdot p}{\bar{q}^6} - \frac{4(q \cdot p)^2}{\bar{q}^8} \right] + p_\mu p_\nu \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} \right] \right\} \quad (4.2.8)$$

The remaining task now is evaluating the integral.

$$\mathcal{R} \left( \hat{\Pi}_{\mu\nu}^{(a)} \right) = \frac{\hat{\alpha}}{12\pi} \left[ \frac{8}{3} - \log \left( \frac{p^2}{\mu^2} \right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.2.9)$$

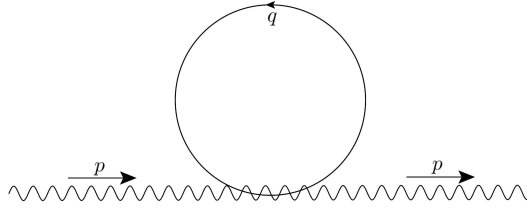


Figure 4.2.3: Momenta description of fig. 4.2.1b.

Now the turn is for fig.4.2.3. Using Feynman rules

$$\hat{\Pi}_{\mu\nu}^{(b)} = 2ig^2 g_{\mu\nu} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \quad (4.2.10)$$

This is a scaleless and purely divergent integral. So

$$\mathcal{R} \left( \hat{\Pi}_{\mu\nu}^{(b)} \right) = 0 \quad (4.2.11)$$

Therefore, renormalized value of the one-loop photon self-energy is directly the result of fig. 4.2.2, since the another one does not contribute.

$$\mathcal{R} \left( \hat{\Pi}_{\mu\nu} \right) = \frac{\hat{\alpha}}{12\pi} \left[ \frac{8}{3} - \log \left( \frac{p^2}{\mu^2} \right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.2.12)$$

We can see that it is transverse as expected, i.e. it is gauge invariant and respects Ward Identities.

$$p^\mu \mathcal{R} \left( \hat{\Pi}_{\mu\nu} \right) = p^\nu \mathcal{R} \left( \hat{\Pi}_{\mu\nu} \right) = 0 \quad (4.2.13)$$

## 4.2.2 Two loops

In the two-loop case, the appearance of a quadruple vertex produces a total of twelve diagrams (fig. 4.2.4). Indeed, there is no need to study them all because some are equivalent as we

will see.

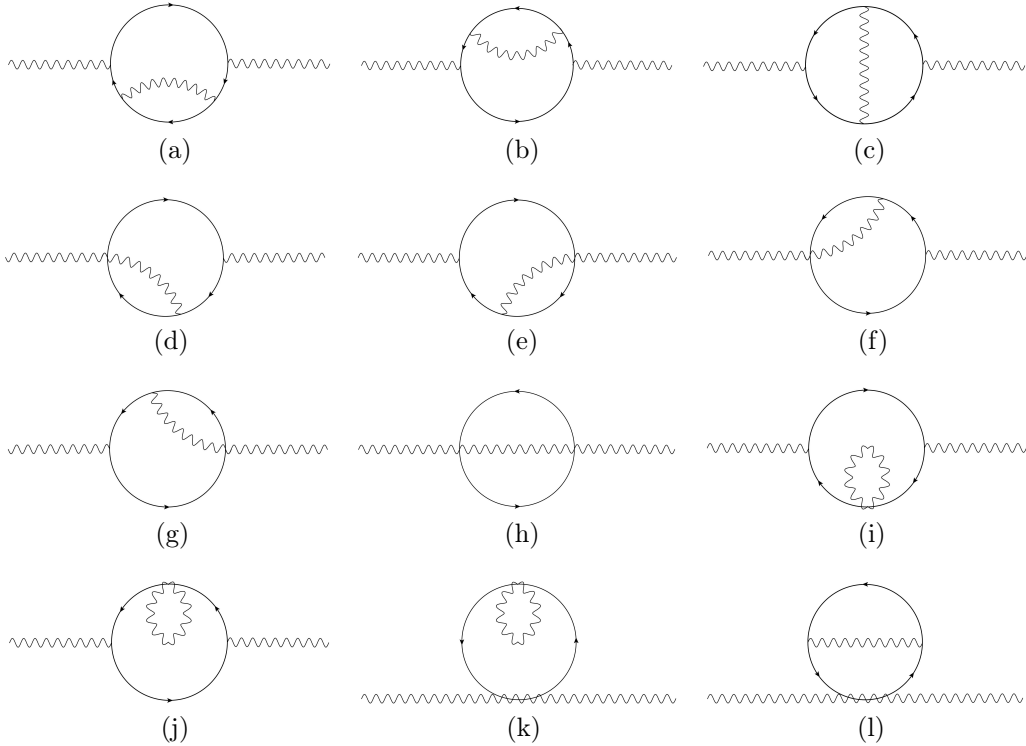


Figure 4.2.4: Two-loop photon self-energy diagrams in Scalar QED.

#### 4.2.2.1 $\hat{\Pi}_{\mu\nu}^{(a+b)}$

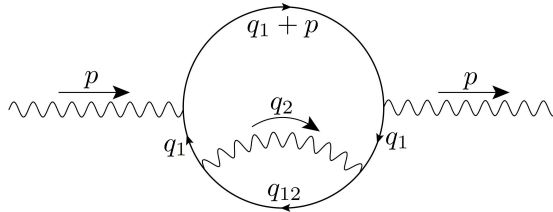


Figure 4.2.5: Momenta description of fig. 4.2.4a.

Let us begin with the diagrams which shares topology with usual QED. Figures 4.2.4a and 4.2.4b are equivalent, so we will study the first one for example and double the result. So now using Scalar QED Feynman rules and reading fig. 4.2.5

$$\hat{\Pi}_{\mu\nu}^{(a+b)} = -2g^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{(2q_1 + p)_\mu (2q_1 + p)_\nu (2q_1 + q_2)^2}{q_1^4 D_1 q_2^2 q_{12}^2} \quad (4.2.14)$$

where  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ .

This diagram has only one subdivergence. When searching for subvacua as well as global vacua, we have to split the numerator into pieces. The ones with one or two  $q_2$  can go with

$$\mathcal{V}_2^{(1)}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^4} \left[ 1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^4} \right] \quad (4.2.15)$$

while the one with no  $q_2$  must be integrated with

$$\mathcal{V}_2^{(2)}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^4} \quad (4.2.16)$$

Parts with four or three loop momenta go with

$$\mathcal{V}_{12}^{(1)}(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.17)$$

$$\mathcal{V}_{12}^{(1)}(\mathcal{V}_2^{(1)}(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \left[ 1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^4} \right] \quad (4.2.18)$$

$$\mathcal{V}_{12}^{(1)}(\mathcal{V}_2^{(2)}(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.19)$$

and parts with two loop momenta must go with

$$\mathcal{V}_{12}^{(2)}(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2} \quad (4.2.20)$$

$$\mathcal{V}_{12}^{(2)}(\mathcal{V}_2^{(1)}(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^4} \right] \quad (4.2.21)$$

$$\mathcal{V}_{12}^{(2)}(\mathcal{V}_2^{(2)}(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \quad (4.2.22)$$

Finally, subdiagram extra part is so easy to extract.

$$\mathcal{X}_2(\mathcal{N}_{\mu\nu}) = 2\mu^2 (2q_1 + p)_\mu (2q_1 + p)_\nu \quad (4.2.23)$$

and since subdiagram has not indices

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = -2\mu^2 (2q_1 + p)_\mu (2q_1 + p)_\nu \quad (4.2.24)$$



so the first and third renormalizations will agree. Now let us compute them.

$$\begin{aligned} \mathcal{R}^{(1)}\left(\hat{\Pi}_{\mu\nu}^{(a+b)}\right) &= \frac{\hat{\alpha}^2}{72\pi^2} \left\{ \left[ 44.24 - 20 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -56.43 + 23 \log\left(\frac{p^2}{\mu^2}\right) - 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.2.25)$$

$$\begin{aligned} \mathcal{R}^{(2)}\left(\hat{\Pi}_{\mu\nu}^{(a+b)}\right) &= \frac{\hat{\alpha}^2}{144\pi^2} \left\{ \left[ 91.15 - 41 \log\left(\frac{p^2}{\mu^2}\right) + 6 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -115.52 + 47 \log\left(\frac{p^2}{\mu^2}\right) - 6 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.2.26)$$

$$\Delta_1 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(a+b)}\right) = \Delta_3 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(a+b)}\right) = \frac{\hat{\alpha}^2}{432\pi^2} \left[ 8 - 3 \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.2.27)$$

$$\Delta_2 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(a+b)}\right) = 0 \quad (4.2.28)$$

#### 4.2.2.2 $\hat{\Pi}_{\mu\nu}^{(c)}$

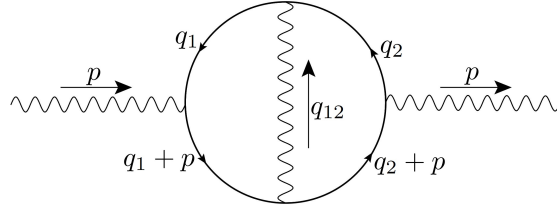


Figure 4.2.6: Momenta description of fig. 4.2.4c.

Next, the remaining diagram which shares topology with usual Spinorial QED: the one with two coupled subdivergences. Using Scalar QED Feynman rules and reading fig. 4.2.6

$$\hat{\Pi}_{\mu\nu}^{(c)} = -g^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{(2q_1 + p)_\mu (2q_2 + p)_\nu (q_1 + q_2) \cdot (q_1 + q_2 + 2p)}{q_1^2 D_1 q_2^2 D_2 q_{12}^2} \quad (4.2.29)$$

where in this occasion  $q_{12} \equiv q_1 - q_2$  and  $D_i \equiv (q_i + p)^2$ .

The pieces in the numerator with three  $q_1^2$  or  $q_2^2$  must be integrated with

$$\mathcal{V}_1(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{D}_2} \left[ 1 + \frac{2q_1 \cdot (q_2 - p)}{\bar{q}_1^2} \right] \quad (4.2.30)$$

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{D}_1 \bar{q}_2^6} \left[ 1 + \frac{2q_2 \cdot (q_1 - p)}{\bar{q}_2^2} \right] \quad (4.2.31)$$

respectively, and the remaining do not have any subdivergence. And regarding overall divergence, parts with four or three loop momenta go with the following vacua and the remaining are globally finite.

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{q}_2^4 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} - \frac{p^2 + 2q_2 \cdot p}{\bar{q}_2^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} + \frac{4(q_2 \cdot p)^2}{\bar{q}_2^4} + \frac{4(q_1 \cdot p)(q_2 \cdot p)}{\bar{q}_1^2 \bar{q}_2^2} \right] \quad (4.2.32)$$

$$\mathcal{V}_{12}(\mathcal{V}_1(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 + \frac{2q_1 \cdot (q_2 - p)}{\bar{q}_1^2} \right] \left[ 1 - \frac{p^2 + 2q_2 \cdot p}{\bar{q}_2^2} + \frac{4(q_2 \cdot p)^2}{\bar{q}_2^4} \right] \quad (4.2.33)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^4 \bar{q}_2^6} \left[ 1 + \frac{2q_2 \cdot (q_1 - p)}{\bar{q}_2^2} \right] \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.34)$$

To finish, we extract from the subdiagrams the extra parts.

$$\mathcal{X}_1(\mathcal{N}_{\mu\nu}) = \mu^2 (2q_1 + p)_\mu (2q_2 + p)_\nu \quad (4.2.35)$$

$$\mathcal{X}_2(\mathcal{N}_{\mu\nu}) = \mu^2 (2q_1 + p)_\mu (2q_2 + p)_\nu \quad (4.2.36)$$

and again, due to the lack of index contraction

$$\mathcal{X}_1(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = \mu^2 (2q_1 + p)_\mu (2q_2 + p)_\nu \quad (4.2.37)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = \mu^2 (2q_1 + p)_\mu (2q_2 + p)_\nu \quad (4.2.38)$$

thus the first and third renormalizations will coincide.

$$\begin{aligned} \mathcal{R}^{(1)}(\hat{\Pi}_{\mu\nu}^{(c)}) &= \frac{\hat{\alpha}^2}{432\pi^2} \left\{ \left[ 148.72 - 87 \log\left(\frac{p^2}{\mu^2}\right) + 9 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -294.97 + 123 \log\left(\frac{p^2}{\mu^2}\right) - 9 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.2.39)$$

$$\mathcal{R}^{(2)}(\hat{\Pi}_{\mu\nu}^{(c)}) = \frac{\hat{\alpha}^2}{144\pi^2} \left\{ \left[ 46.91 - 28 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right.$$

$$+ \left[ -95.66 + 40 \log \left( \frac{p^2}{\mu^2} \right) - 3 \log^2 \left( \frac{p^2}{\mu^2} \right) \right] p^2 g_{\mu\nu} \} \quad (4.2.40)$$

$$\Delta_1 \mathcal{R} \left( \hat{\Pi}_{\mu\nu}^{(c)} \right) = \Delta_3 \mathcal{R} \left( \hat{\Pi}_{\mu\nu}^{(c)} \right) = -\frac{\hat{\alpha}^2}{432\pi^2} \left[ 8 - 3 \log \left( \frac{p^2}{\mu^2} \right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.2.41)$$

$$\Delta_2 \mathcal{R} \left( \hat{\Pi}_{\mu\nu}^{(c)} \right) = 0 \quad (4.2.42)$$

### 4.2.2.3 $\hat{\Pi}_{\mu\nu}^{(d+e+f+g)}$

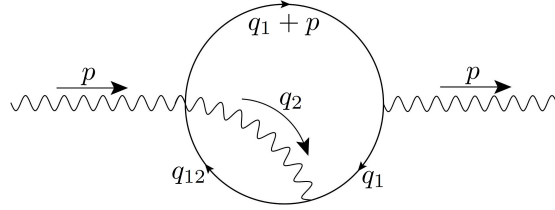


Figure 4.2.7: Momenta description of fig. 4.2.4d.

This graph is the first one with no equivalence in Spinorial QED due to the emergence of a quadruple vertex. Here, we find an equivalence among four diagrams, all of them with one subdivergence. That is, fig. 4.2.4d and 4.2.4f are completely equivalent, and fig. 4.2.4e and 4.2.4g too. And to relate each couple an exchange of Lorentz indices must be performed. So we can choose for example the first one and by using Scalar QED Feynman rules on fig. 4.2.7 we have

$$\hat{\Pi}_{\mu\nu}^{(d+e+f+g)} = 4g^4 (g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda}) \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{(2q_1 + q_2)^\lambda (2q_1 + p)^\rho}{q_1^2 D_1 q_2^2 q_{12}^2} \quad (4.2.43)$$

where  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ .

In this diagram is not necessary to split the numerator while searching for the subvacua as well as global vacua. Therefore the only needed expressions are

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{D}_1 \bar{q}_2^4} \left[ 1 - \frac{2q_1 \cdot q_2}{\bar{q}_2^2} \right] \quad (4.2.44)$$

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.45)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^4 \bar{q}_2^4} \left[ 1 - \frac{2q_1 \cdot q_2}{\bar{q}_2^2} \right] \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.46)$$

As there are not any squared loop momenta in the numerator, any extra or extra-extra part can be generated.

$$\mathcal{X}_{12}(\mathcal{N}_{\mu\nu}) = 0 \quad (4.2.47)$$

$$\mathcal{X}_2(\mathcal{N}_{\mu\nu}) = 0 \quad (4.2.48)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}_{\mu\nu})) = 0 \quad (4.2.49)$$

so inevitably the three renormalizations will concur.

$$\begin{aligned} \mathcal{R}^{(1)}\left(\hat{\Pi}_{\mu\nu}^{(d+e+f+g)}\right) &= -\frac{\hat{\alpha}^2}{48\pi^2} \left\{ \left[ 7.68 - 11 \log\left(\frac{p^2}{\mu^2}\right) + 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p_\mu p_\nu + \right. \\ &\quad \left. + \left[ -56.43 + 23 \log\left(\frac{p^2}{\mu^2}\right) - 3 \log^2\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \right\} \end{aligned} \quad (4.2.50)$$

$$\Delta_1 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(d+e+f+g)}\right) = \Delta_2 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(d+e+f+g)}\right) = \Delta_3 \mathcal{R}\left(\hat{\Pi}_{\mu\nu}^{(d+e+f+g)}\right) = 0 \quad (4.2.51)$$

#### 4.2.2.4 $\hat{\Pi}_{\mu\nu}^{(h)}$

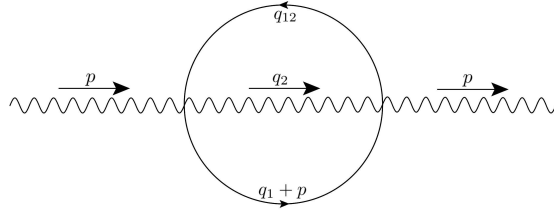


Figure 4.2.8: Momenta description of fig. 4.2.4h.

Using Scalar QED Feynman rules on fig. 4.2.8 we have

$$\hat{\Pi}_{\mu\nu}^{(h)} = -4g^4 g_{\mu\nu} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{D_1 q_2^2 q_{12}^2} \quad (4.2.52)$$

where  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ .

In principle, as we see in fig. 4.2.8 as well as in eq. (4.2.52), this graph seems to have two subdivergences. However, the subvacuum corresponding to  $q_1$  is completely divergent with no possibility of being expanded again when searching for overall divergence. Therefore, it

has to be subtracted entirely and the final result will be as if the graph had one subdivergence instead of two.

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{D_1 \bar{q}_2^4} \quad (4.2.53)$$

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.54)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^2 \bar{q}_2^4} \left[ 1 - \frac{p^2 + 2q_1 \cdot p}{\bar{q}_1^2} + \frac{4(q_1 \cdot p)^2}{\bar{q}_1^4} \right] \quad (4.2.55)$$

There are not any squared loop momenta in the numerator so

$$\mathcal{X}_{12}(\mathcal{N}^{\mu\nu}) = 0 \quad (4.2.56)$$

$$\mathcal{X}_2(\mathcal{N}^{\mu\nu}) = 0 \quad (4.2.57)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N}^{\mu\nu})) = 0 \quad (4.2.58)$$

and again the three renormalizations will be identical.

$$\mathcal{R}^{(1)}(\hat{\Pi}_{\mu\nu}^{(h)}) = -\frac{\hat{\alpha}^2}{8\pi^2} \left[ 4.06 + \log\left(\frac{p^2}{\mu^2}\right) \right] p^2 g_{\mu\nu} \quad (4.2.59)$$

$$\Delta_1 \mathcal{R}(\hat{\Pi}_{\mu\nu}^{(h)}) = \Delta_2 \mathcal{R}(\hat{\Pi}_{\mu\nu}^{(h)}) = \Delta_3 \mathcal{R}(\hat{\Pi}_{\mu\nu}^{(h)}) = 0 \quad (4.2.60)$$

#### 4.2.2.5 $\hat{\Pi}_{\mu\nu}^{(i+j)}$

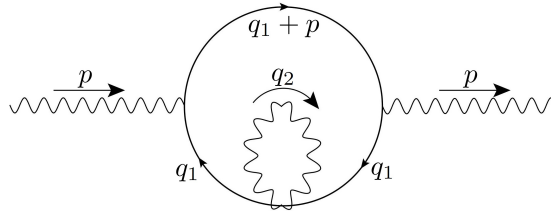


Figure 4.2.9: Momenta description of fig. 4.2.4i.

Figures 4.2.4i and 4.2.4j are equivalent so we will choose for example the first and double the result. Using Scalar QED Feynman rules on fig. 4.2.9 we have

$$\hat{\Pi}_{\mu\nu}^{(i+j)} = 16g^4 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{(2q_1 + p)_\mu (2q_1 + p)_\nu}{q_1^4 D_1 q_2^2} \quad (4.2.61)$$

where  $D_1 \equiv (q_1 + p)^2$ .

As we see, this graph has one subdivergence. If we write the subvacuum

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^2} \quad (4.2.62)$$

we realize that it is equal to the original denominator, so

$$\mathcal{V}_2(\mathcal{D}) = \mathcal{D} \implies \mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \mathcal{V}_{12}(\mathcal{D}) \quad (4.2.63)$$

Therefore, according to formal expressions given by Forest Formula, this graph vanishes in all schemes.

$$\mathcal{R}^{(1)}(\hat{\Pi}_{\mu\nu}^{(i+j)}) = \mathcal{R}^{(2)}(\hat{\Pi}_{\mu\nu}^{(i+j)}) = \mathcal{R}^{(3)}(\hat{\Pi}_{\mu\nu}^{(i+j)}) = 0 \quad (4.2.64)$$

We could have also noticed that this integral is disconnected because of the non-existence of  $\bar{q}_{12}^2$  in the denominator, and have treated them separately. This way, we quickly see that the  $q_2$  integral must be subtracted itself because it is purely divergent.

#### 4.2.2.6 $\hat{\Pi}_{\mu\nu}^{(k)}$

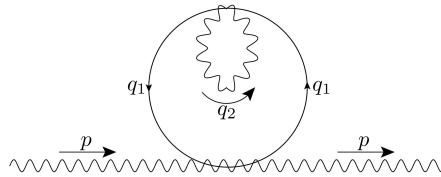


Figure 4.2.10: Momenta description of fig. 4.2.4k.

Using Scalar QED Feynman rules on fig. 4.2.10 we have

$$\hat{\Pi}_{\mu\nu}^{(k)} = -16g^4 g_{\mu\nu} \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{\bar{q}_1^4 \bar{q}_2^2} \quad (4.2.65)$$

This diagram has only  $\mu$  as scale, so the result will be proportional to it. And after

taking the limit  $\mu \rightarrow 0$  we will inevitably find that

$$\mathcal{R}^{(1)}\left(\hat{\Pi}_{\mu\nu}^{(k)}\right) = \mathcal{R}^{(2)}\left(\hat{\Pi}_{\mu\nu}^{(k)}\right) = \mathcal{R}^{(3)}\left(\hat{\Pi}_{\mu\nu}^{(k)}\right) = 0 \quad (4.2.66)$$

which has sense because the integral is purely divergent.

#### 4.2.2.7 $\hat{\Pi}_{\mu\nu}^{(l)}$

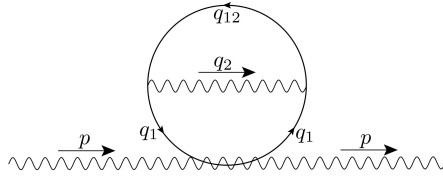


Figure 4.2.11: Momenta description of fig. 4.2.4l.

Using Scalar QED Feynman rules on fig. 4.2.11 we have

$$\hat{\Pi}_{\mu\nu}^{(l)} = 2g^4 g_{\mu\nu} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{(2q_1 + q_2)^2}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} \quad (4.2.67)$$

Again, this diagram has only  $\mu$  as scale, so

$$\mathcal{R}^{(1)}\left(\hat{\Pi}_{\mu\nu}^{(l)}\right) = \mathcal{R}^{(2)}\left(\hat{\Pi}_{\mu\nu}^{(l)}\right) = \mathcal{R}^{(3)}\left(\hat{\Pi}_{\mu\nu}^{(l)}\right) = 0 \quad (4.2.68)$$

#### 4.2.2.8 Sum

In the end, despite of beginning with twelve diagrams, only seven topologies needed to be studied of which just four gave real contribution. And summing them all we get

$$\mathcal{R}^{(1)}\left(\hat{\Pi}_{\mu\nu}\right) = \frac{\hat{\alpha}^2}{4\pi^2} \left[ 3.19 - \log\left(\frac{p^2}{\mu^2}\right) \right] (p_\mu p_\nu - p^2 g_{\mu\nu}) \quad (4.2.69)$$

$$\Delta_1 \mathcal{R}(\Pi_{\mu\nu}) = \Delta_2 \mathcal{R}(\Pi_{\mu\nu}) = \Delta_3 \mathcal{R}(\Pi_{\mu\nu}) = 0 \quad (4.2.70)$$

Curiously, notwithstanding that graphs of figures 4.2.4a, 4.2.4b and 4.2.4c have differences among renormalizations, the sum of them does not. We can easily verify it, noticing that eq. (4.2.27) and (4.2.41) are opposite.

And of course, transversality and gauge invariance have been recovered, so Ward Identities are fulfilled.

$$p^\mu \mathcal{R} \left( \hat{\Pi}_{\mu\nu} \right) = p^\nu \mathcal{R} \left( \hat{\Pi}_{\mu\nu} \right) = 0 \quad (4.2.71)$$

### 4.2.3 RGE analysis

In this case, RGE analysis will not give more information about renormalizations because they all agree. Nevertheless, it is convenient to perform the study to see if it is consistent with the result we got. So using (3.3.4)

$$\left[ 2\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha, \xi) \frac{\partial}{\partial \alpha} + \beta_\xi(\alpha, \xi) \frac{\partial}{\partial \xi} - 2\gamma_A(\alpha, \xi) \right] \Pi_{\mu\nu} = 0 \quad (4.2.72)$$

Obviously, all  $\alpha$  expansions we did in spinorial case hold. So we directly summarize here the diagrams results.

$$\Pi^{(0)} = 1 \quad (4.2.73)$$

$$\Pi_{\text{longitudinal}}^{(0)} = -\frac{1}{\xi} \quad (4.2.74)$$

$$\Pi^{(1)} = -\frac{1}{12\pi} \log \left( \frac{p^2}{\mu^2} \right) + \mathcal{C} \quad (4.2.75)$$

$$\Pi^{(2)} = -\frac{1}{4\pi^2} \log \left( \frac{p^2}{\mu^2} \right) + \mathcal{C}' \quad (4.2.76)$$

where, this time, we only have one  $\Pi^{(2)}$ .

If we insert the longitudinal part into (4.2.72), we get the same relations as beforehand.

$$\beta_\xi = -2\gamma_A \quad (4.2.77)$$

$$\beta = -\beta_\xi \alpha \quad (4.2.78)$$

The relation for  $\mathcal{O}(\alpha)$  is also the same.

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(1)} - 2\gamma_A^{(1)} \Pi^{(0)} = 0 \quad (4.2.79)$$

$$\gamma_A^{(1)} = \mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(1)} \quad (4.2.80)$$



though this time we find a different value

$$\gamma_A^{(1)} = \frac{1}{12\pi} \quad (4.2.81)$$

In Scalar QED, the eq. (3.3.5) still holds and also the two first beta function coefficients are scheme-independent. So using the known value for the first coefficient (3.3.9), we can say that the result (4.2.81) agrees with the known values. This means that FDR has successfully renormalize the one-loop graph.

We continue with  $\mathcal{O}(\alpha^2)$ .

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} + \beta^{(2)} \Pi^{(1)} - 2\gamma_A^{(1)} \Pi^{(1)} - 2\gamma_A^{(2)} \Pi^{(0)} = 0 \quad (4.2.82)$$

And again, thanks to eq. (4.2.72), we also get the same relation as in Spinorial QED.

$$\gamma_A^{(2)} = \mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} \quad (4.2.83)$$

Thus in Scalar QED  $\Pi^{(2)}$  cannot contain any squared logarithm (or beyond) as well as in Spinorial QED. This fact also agrees with our result. Using again (3.3.5) and the known value of the second beta function coefficient which actually coincides with the spinorial one (3.3.10), we find that the same relation as beforehand must hold.

$$\mu^2 \frac{\partial}{\partial \mu^2} \Pi^{(2)} = \frac{1}{4\pi^2} \quad (4.2.84)$$

so all renormalizations scheme we have performed agree with this result. Therefore, as we said before, this analysis does not give us new information but at least we found that all renormalizations are consistent.

### 4.3 Fermion self-energy

We end this chapter with the study of fermion self-energy in Spinorial QED. Again, the interaction Lagrangian part that we are interested in is

$$\mathcal{L}_{QED} \supset -e\bar{\psi}\mathcal{A}\psi \quad (4.3.1)$$

where we will replace  $e$  for the fine structure constant  $\alpha$ .

### 4.3.1 One loop

We will begin evaluating the one-loop fermion self-energy (fig. 4.3.1), though we have already worked with it previously.

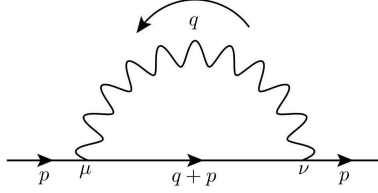


Figure 4.3.1: One-loop fermion self-energy in QED.

However, this time we will compute this graph in a general gauge for discussion purposes. According to Feynman rules

$$\Sigma = -ie^2 \int \frac{d^4 q}{(2\pi)^4} \left[ \frac{\gamma_\mu (\not{q} + \not{p}) \gamma^\mu}{q^2 (q+p)^2} + (\xi - 1) \frac{\not{q} (\not{q} + \not{p}) \not{q}}{q^4 (q+p)^2} \right] \quad (4.3.2)$$

The part in the 't Hooft-Feynman gauge ( $\xi = 1$ ) and the new one must be respectively integrated with

$$\mathcal{V}^{(1)}(\mathcal{D}) = \frac{1}{\bar{q}^4} \left[ 1 - \frac{2q \cdot p}{\bar{q}^6} \right] \quad (4.3.3)$$

$$\mathcal{V}^{(2)}(\mathcal{D}) = \frac{1}{\bar{q}^6} \left[ 1 - \frac{2q \cdot p}{\bar{q}^8} \right] \quad (4.3.4)$$

And performing the shift in the numerator, we get to the final expression.

$$\begin{aligned} \mathcal{R}(\Sigma) = & -ie^2 \lim_{\mu \rightarrow 0} \int \frac{d^4 q}{(2\pi)^4} \left\{ [\gamma_\mu (\not{q} + \not{p}) \gamma^\mu] \left[ \frac{1}{\bar{q}^2 \bar{D}} - \frac{1}{\bar{q}^4} + \frac{2q \cdot p}{\bar{q}^6} \right] + \right. \\ & \left. + (\xi - 1) [\not{q} (\not{q} + \not{p}) \not{q} - \mu^2 (\not{q} - \not{p})] \left[ \frac{1}{\bar{q}^4 \bar{D}} - \frac{1}{\bar{q}^6} + \frac{2q \cdot p}{\bar{q}^8} \right] \right\} \quad (4.3.5) \end{aligned}$$

and performing the integral

$$\mathcal{R}(\Sigma) = -\frac{\alpha}{4\pi} \not{p} \left[ 1 + \xi - \xi \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.6)$$

where if  $\xi = 1$ , we recover the result we have computed beforehand.

### 4.3.2 Two loops

We find three different two-loop fermion self-energy as we see in fig. 4.3.2.

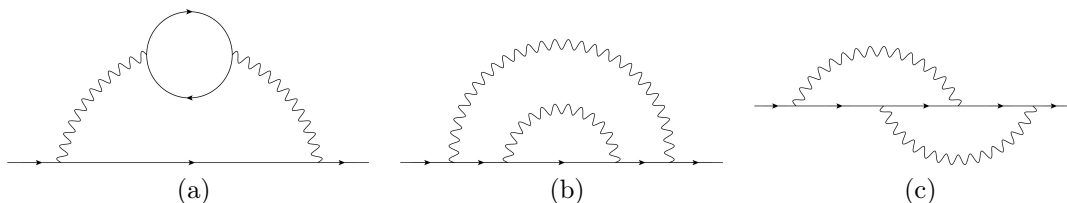


Figure 4.3.2: Two-loop fermion self-energy diagrams in QED.

#### 4.3.2.1 $\Sigma^{(a)}$

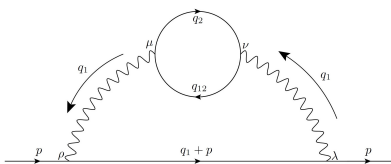


Figure 4.3.3: Momenta description of fig. 4.3.2a.

We will begin with fig. 4.3.3 which we used to explain FDR procedure when we have a two-loop graph. This time we will make use of Forest Formula to see or not the emergence of the same terms. Reading fig. 4.3.3 and using QED Feynman rules we have

$$\Sigma^{(a)} = e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu \text{Tr} [\gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12}]}{q_1^4 D_1 q_2^2 q_{12}^2} \quad (4.3.7)$$

where  $q_{12} \equiv q_1 + q_2$  and  $D_1 \equiv (q_1 + p)^2$ .

First of all, we notice that it has one subdivergence, so we will use one subdivergence formal expressions. Thus now we only have to find out all pieces.

$$\mathcal{N} = \gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu \text{Tr} [\gamma_\mu \not{q}_2 \gamma_\nu \not{q}_{12}] \quad (4.3.8)$$

$$\mathcal{D} = \frac{1}{q_1^4 D_1 q_2^2 q_{12}^2} \quad (4.3.9)$$

Once we have identified numerator and denominator, we can figure out all vacua by fraction decomposing and power-counting. All parts can be together because of parity and in order

to get the most compactness.

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{D}_1 \bar{q}_2^4} \left[ 1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^4} \right] \quad (4.3.10)$$

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{2q_1 \cdot p}{\bar{q}_1^2} \right] \quad (4.3.11)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{2q_1 \cdot p}{\bar{q}_1^2} \right] \left[ 1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2} + \frac{4(q_1 \cdot q_2)^2}{\bar{q}_2^4} \right] \quad (4.3.12)$$

In fact, since  $\mathcal{N}$  and  $\mathcal{X}_{12}(\mathcal{N})$  always appear together as a sum, usually it will be no necessary to know the exact form of  $\mathcal{X}_{12}(\mathcal{N})$  when renormalizing a particular diagram. Just placing a bar over squared loop momenta in  $\mathcal{N}$  as in eq. (2.5.19) and (2.5.20) is enough. Moreover, this facilitates cancellations between numerator and denominator. Once having all pieces, we can perform the integrals making use of the appendix.

Now it is the turn of extra-extra parts. Shifting only  $q_2^2$  without index contraction we have

$$\mathcal{X}_2(\mathcal{N}) = -8\mu^2 (\not{q}_1 + \not{p}) \quad (4.3.13)$$

and it has to be integrated along with the already obtained  $\mathcal{V}_2(\mathcal{D})$ . The process have been shown in section 2.5. It is important to recall that after generating  $\mu$ , we can contract all indices without any problem. Now, we contract all indices to see the numerator appearance.

$$\mathcal{N} = 8 \left[ q_{12} \cdot (q_1 + p) \not{q}_2 + q_2 \cdot (q_1 + p) \not{q}_{12} \right] \quad (4.3.14)$$

As there is no  $q_2^2$  in the numerator after contraction

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})) = 0 \quad (4.3.15)$$

thus  $\mathcal{R}^{(2)}(\Sigma^{(a)})$  and  $\mathcal{R}^{(3)}(\Sigma^{(a)})$  will agree. So the final results are

$$\mathcal{R}^{(1)}(\Sigma^{(a)}) = \frac{\alpha^2}{6\pi^2} \not{p} \left[ 2.80 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.16)$$

$$\mathcal{R}^{(2)}(\Sigma^{(a)}) = \frac{\alpha^2}{8\pi^2} \not{p} \left[ 3.06 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.17)$$

$$\Delta_1 \mathcal{R}(\Sigma^{(a)}) = \Delta_2 \mathcal{R}(\Sigma^{(a)}) = -\frac{\alpha^2}{24\pi^2} \not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.18)$$

$$\Delta_3 \mathcal{R}(\Sigma^{(a)}) = 0 \quad (4.3.19)$$

### 4.3.2.2 $\Sigma^{(b)}$

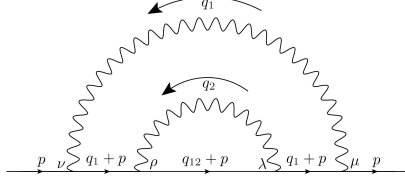


Figure 4.3.4: Momenta description of fig. 4.3.2b.

We follow now with fig. 4.3.4. Let us write its expression by QED Feynman rules.

$$\Sigma^{(b)} = -e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\gamma_\mu (\not{q}_1 + \not{p}) \gamma_\nu (\not{q}_{12} + \not{p}) \gamma^\nu (\not{q}_1 + \not{p}) \gamma^\mu}{q_1^2 D_1^2 q_2^2 D_{12}} \quad (4.3.20)$$

where  $q_{12} \equiv q_1 + q_2$ ,  $D_1 \equiv (q_1 + p)^2$  and  $D_{12} \equiv (q_{12} + p)^2$ .

This diagram has also one subdivergence so we will use the corresponding expressions. Now we must search for vacua. In spite of numerator several parts, we can put it all under the same subvacuum.

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{D}_1^2 \bar{q}_2^4} \left[ 1 - \frac{2(q_1 + p) \cdot q_2}{\bar{q}_2^2} \right] \quad (4.3.21)$$

Nevertheless, we have to split the numerator when finding global vacua. Parts with three and two loop momenta can go with the following infinities. The remaining are already globally finite.

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{q}_{12}^2} \left[ 1 - \frac{4q_1 \cdot p}{\bar{q}_1^2} - \frac{2q_{12} \cdot p}{\bar{q}_{12}^2} \right] \quad (4.3.22)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{4q_1 \cdot p}{\bar{q}_1^2} \right] \left[ 1 - \frac{2(q_1 + p) \cdot q_2}{\bar{q}_2^2} \right] \quad (4.3.23)$$

And reading the numerator, we realize there is no possibility to make a  $q_2^2$  so

$$\mathcal{X}_2(\mathcal{N}) = 0 \quad (4.3.24)$$

and therefore

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})) = 0 \quad (4.3.25)$$

thus all renormalizations will be the same.

$$\mathcal{R}^{(1)}(\Sigma^{(b)}) = \frac{\alpha^2}{32\pi^2} \not{p} \left[ 8.28 - 5 \log\left(\frac{p^2}{\mu^2}\right) + \log^2\left(\frac{p^2}{\mu^2}\right) \right] \quad (4.3.26)$$

$$\Delta_1 \mathcal{R}(\Sigma^{(b)}) = \Delta_2 \mathcal{R}(\Sigma^{(b)}) = \Delta_3 \mathcal{R}(\Sigma^{(b)}) = 0 \quad (4.3.27)$$

### 4.3.2.3 $\Sigma^{(c)}$

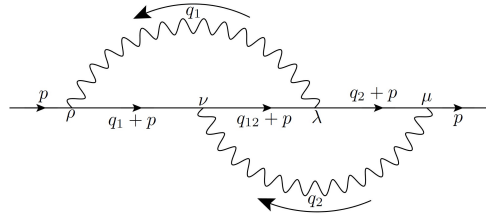


Figure 4.3.5: Momenta description of fig. 4.3.2c.

And finally, the last graph which takes part in the two-loop fermion self-energy (fig. 4.3.5). So by Feynman rules we have

$$\Sigma^{(c)} = -e^4 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\gamma_\mu (\not{q}_2 + \not{p}) \gamma_\nu (\not{q}_{12} + \not{p}) \gamma^\mu (\not{q}_1 + \not{p}) \gamma^\nu}{q_1^2 D_1 q_2^2 D_2 D_{12}} \quad (4.3.28)$$

where  $q_{12} \equiv q_1 + q_2$ ,  $D_i \equiv (q_i + p)^2$  and  $D_{12} \equiv (q_{12} + p)^2$ .

This diagram is more complicated than former ones because it has two subdivergences. Concerning to them, only numerator parts with two  $q_1$  or two  $q_2$  need to be regulated with

$$\mathcal{V}_1(\mathcal{D}) = \frac{1}{\bar{q}_1^6 \bar{q}_2^2 \bar{D}_2} \quad (4.3.29)$$

$$\mathcal{V}_2(\mathcal{D}) = \frac{1}{\bar{q}_1^2 \bar{q}_2^6 \bar{D}_1} \quad (4.3.30)$$

respectively. The other parts are not subdivergent. Likewise, only parts with three or two loop momenta must be integrated under

$$\mathcal{V}_{12}(\mathcal{D}) = \frac{1}{\bar{q}_1^4 \bar{q}_2^4 \bar{q}_{12}^2} \left[ 1 - \frac{2q_1 \cdot p}{\bar{q}_1^2} - \frac{2q_2 \cdot p}{\bar{q}_2^2} - \frac{2q_{12} \cdot p}{\bar{q}_{12}^2} \right] \quad (4.3.31)$$

$$\mathcal{V}_{12}(\mathcal{V}_1(\mathcal{D})) = \frac{1}{\bar{q}_1^6 \bar{q}_2^4} \left[ 1 - \frac{2q_2 \cdot p}{\bar{q}_2^2} \right] \quad (4.3.32)$$

$$\mathcal{V}_{12}(\mathcal{V}_2(\mathcal{D})) = \frac{1}{\bar{q}_1^4 \bar{q}_2^6} \left[ 1 - \frac{2q_1 \cdot p}{\bar{q}_1^2} \right] \quad (4.3.33)$$

because the remaining are globally finite.

Finally, the turn for *extra-extra integrals*. We easily find

$$\mathcal{X}_1(\mathcal{N}) = -4\mu^2 \left( \not{q}_2 + \not{p} \right) \quad (4.3.34)$$

$$\mathcal{X}_2(\mathcal{N}) = -4\mu^2 \left( \not{q}_1 + \not{p} \right) \quad (4.3.35)$$

Now, contracting all indices in the numerator

$$\mathcal{N} = 8(q_1 + p) \cdot (q_2 + p) \left( \not{q}_{12} + \not{p} \right) \quad (4.3.36)$$

Since we cannot generate<sup>1</sup> any  $\mu^2$  from a  $q_1^2$  or  $q_2^2$

$$\mathcal{X}_1(\mathcal{X}_{12}(\mathcal{N})) = 0 \quad (4.3.37)$$

$$\mathcal{X}_2(\mathcal{X}_{12}(\mathcal{N})) = 0 \quad (4.3.38)$$

so the second and third renormalization will coincide. Let us compute them and their differences.

$$\mathcal{R}^{(1)}(\Sigma^{(c)}) = -\frac{\alpha^2}{16\pi^2} \not{p} \left[ 8.28 - 5 \log \left( \frac{p^2}{\mu^2} \right) + \log^2 \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.39)$$

$$\mathcal{R}^{(2)}(\Sigma^{(c)}) = -\frac{\alpha^2}{16\pi^2} \not{p} \left[ 4.28 - 3 \log \left( \frac{p^2}{\mu^2} \right) + \log^2 \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.40)$$

$$\Delta_1 \mathcal{R}(\Sigma^{(c)}) = \Delta_2 \mathcal{R}(\Sigma^{(c)}) = \frac{\alpha^2}{8\pi^2} \not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.41)$$

$$\Delta_3 \mathcal{R}(\Sigma^{(c)}) = 0 \quad (4.3.42)$$

#### 4.3.2.4 Sum

Once computed all diagrams we can sum them to get the renormalized value of the two-loop fermion self-energy, and the differences between renormalizations.

$$\mathcal{R}^{(1)}(\Sigma) = \frac{\alpha^2}{96\pi^2} \not{p} \left[ 19.91 - \log \left( \frac{p^2}{\mu^2} \right) - 3 \log^2 \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.43)$$

---

<sup>1</sup> $\mu^2$  can be extracted from  $q_1 \cdot q_2$  but this only concerns to  $\mathcal{X}_{12}(\mathcal{N})$ .

$$\mathcal{R}^{(2)}(\Sigma) = \frac{\alpha^2}{32\pi^2} \not{p} \left[ 11.97 - 3 \log \left( \frac{p^2}{\mu^2} \right) - \log^2 \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.44)$$

$$\Delta_1 \mathcal{R}(\Sigma) = \Delta_2 \mathcal{R}(\Sigma) = \frac{\alpha^2}{12\pi^2} \not{p} \left[ 2 - \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.45)$$

$$\Delta_3 \mathcal{R}(\Sigma) = 0 \quad (4.3.46)$$

As we can deduce, there is no distinction between third and second renormalization since  $\mathcal{X}_i(\mathcal{X}_{ij}(\mathcal{N}))$  vanishes for all diagrams. Concerning to first renormalization, we see that its logarithmic behaviour is different to the other ones (except for the squared one).

### 4.3.3 RGE analysis

Let us write RGE for fermion self-energy, in order to try to discard some of the renormalization schemes.

$$\left[ 2\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha, \xi) \frac{\partial}{\partial \alpha} + \beta_\xi(\alpha, \xi) \frac{\partial}{\partial \xi} - 2\gamma_f(\alpha, \xi) \right] \Sigma = 0 \quad (4.3.47)$$

where this time,  $\gamma_f$  appears instead of  $\gamma_A$ , as this graph has two external fermions. We already know all functions dependence on  $\alpha$ , but let us remember the results for  $\Sigma$ .

$$\Sigma = -\not{p} \left[ \Sigma^{(0)} + \alpha \Sigma^{(1)} + \alpha^2 \Sigma^{(2)} + \dots \right] \quad (4.3.48)$$

$$\Sigma^{(0)} = 1 \quad (4.3.49)$$

$$\Sigma^{(1)} = \frac{1}{4\pi} \left[ 1 + \xi - \xi \log \left( \frac{p^2}{\mu^2} \right) \right] \quad (4.3.50)$$

$$\Sigma_1^{(2)} = \frac{1}{32\pi^2} \left[ \frac{1}{3} \log \left( \frac{p^2}{\mu^2} \right) + \log^2 \left( \frac{p^2}{\mu^2} \right) \right] + \mathcal{C}_1 \quad (4.3.51)$$

$$\Sigma_2^{(2)} = \frac{1}{32\pi^2} \left[ 3 \log \left( \frac{p^2}{\mu^2} \right) + \log^2 \left( \frac{p^2}{\mu^2} \right) \right] + \mathcal{C}_2 \quad (4.3.52)$$

Now, we can understand why we computed the one-loop self-energy in a general gauge. As the two-loop ones are in the 't Hooft-Feynman gauge, after differentiating with respect to  $\xi$ , we must set it to  $\xi = 1$ .

For  $\mathcal{O}(\alpha)$  we find

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Sigma^{(1)} - 2\gamma_f^{(1)} \Sigma^{(0)} = 0 \quad (4.3.53)$$

$$\gamma_f^{(1)} = \mu^2 \frac{\partial}{\partial \mu^2} \Sigma^{(1)} \quad (4.3.54)$$



and for  $\xi = 1$

$$\gamma_f^{(1)} = \frac{1}{4\pi} \quad (4.3.55)$$

Unfortunately,  $\gamma_f$  is not related to  $\beta$  or to another function as  $\gamma_A$ , and also it is scheme-dependant, so we cannot ensure whether this value is correct or not.

Let us write the  $\mathcal{O}(\alpha^2)$ .

$$2\mu^2 \frac{\partial}{\partial \mu^2} \Sigma^{(2)} + \beta^{(2)} \Sigma^{(1)} + \beta_\xi^{(1)} \frac{\partial}{\partial \xi} \Sigma^{(1)} - 2\gamma_f^{(1)} \Sigma^{(1)} - 2\gamma_f^{(2)} \Sigma^{(0)} = 0 \quad (4.3.56)$$

In this equation there are terms proportional to a logarithm and terms which are not. Thanks to that, we can find two different relations from the equation above.

$$\left[ -\frac{1}{8\pi^2} - \frac{1}{4\pi} \beta^{(2)} - \frac{1}{4\pi} \beta_\xi^{(1)} + \frac{1}{2\pi} \gamma_f^{(1)} \right] \log \left( \frac{p^2}{\mu^2} \right) = 0 \quad (4.3.57)$$

Using eq. (4.1.53) we find again

$$\gamma_f^{(1)} = \frac{1}{4\pi} \quad (4.3.58)$$

This could mean that the one-loop self-energy and the squared logarithm coefficient of the two-loop self-energy are consistent with a well-done renormalization. Nevertheless, this coefficient coincides in all renormalizations we have made and one-loop does not make any distinction neither, so we cannot discard any one yet. Now, we write the  $\mathcal{O}(\alpha^2)$  constant part.

$$-\frac{1}{48\pi^2} + \frac{1}{2\pi} \beta^{(2)} + \frac{1}{4\pi} \beta_\xi^{(1)} - \frac{1}{\pi} \gamma_f^{(1)} - 2\gamma_f^{(2)} = 0 \quad (4.3.59)$$

$$-\frac{3}{16\pi^2} + \frac{1}{2\pi} \beta^{(2)} + \frac{1}{4\pi} \beta_\xi^{(1)} - \frac{1}{\pi} \gamma_f^{(1)} - 2\gamma_f^{(2)} = 0 \quad (4.3.60)$$

where we have used  $\Sigma_1^{(2)}$  and  $\Sigma_2^{(2)}$  respectively. Substituting the values we have

$$\gamma_f^{(2)} = -\frac{5}{96\pi^2} \quad (4.3.61)$$

$$\gamma_f^{(2)} = -\frac{13}{96\pi^2} \quad (4.3.62)$$

In contrast to the case of vacuum polarization, here we cannot conclude anything. We already knew that  $\gamma_f^{(2)}$  is scheme-dependant so in principle the logarithm coefficient depends on scheme too. One clear prove is that while calculating  $\gamma_f^{(2)}$ , we had to use the constant value of  $\Sigma^{(1)}$  which we explicitly showed in section (3.1.1) that depends on scheme.

Therefore, thanks to vacuum polarization we were able to discard  $\mathcal{R}^{(2)}$ , but after fermion

self-energy analysis we cannot still distinguish between  $\mathcal{R}^{(1)}$  and  $\mathcal{R}^{(3)}$ .

# Chapter 5

## Conclusions

In the last decade, there has been a renewed interest in alternative methods to perform perturbative calculations in quantum field theory (see [1] for a recent review). This has been motivated by the increasing complexity of the computations required to match the sensitivity of present and future experiments and by the development of new techniques for on-shell scattering amplitudes, based on unitarity and analyticity. The most efficient methods are either variations of dimensional regularization or implicit methods in fixed dimension, which act directly on the bare integrals, often at the integrand level, and do not need to keep track of counterterms. Besides other possible advantages, the latter could be expected to handle more easily chiral theories, such as the Standard Model, since the concept of chirality is dimension specific. In this thesis we have examined this issue in implicit fixed-dimension methods such as CDR, CIREg and FDR. We have shown that, somewhat counterintuitively, the difficulties one has to address in these methods are very similar to the ones in dimensional methods. They can be dealt with in a similar manner.

The origin of these difficulties is the fact that contraction of Lorentz indices does not commute with renormalization in these implicit methods. We have observed that this is actually required to preserve both shift invariance and numerator-denominator consistency, which are the crucial ingredients in the perturbative proof of the quantum action principle. The latter leads to Ward identities of local and global symmetries in the renormalized theory. But it turns out that this non-commutation property is incompatible with certain identities, specific to the ordinary  $n$ -dimensional space in which a given theory is defined. Hence, a naïve usage of these identities may lead to inconsistencies. The situation is similar to the one in dimensional methods. And a way out is also to simply avoid using these identities before renormalization. This statement can be made more formal by defining the theory in an

infinite dimensional space QnS, which only shares a few features with the real  $n$ -dimensional space.

Working in QnS is necessary for consistency, but it brings about some complications in theories with fermions. First, it turns out that the standard Dirac algebra cannot be preserved in odd dimensions. Second, there is no finite complete set in Dirac space and hence the standard Fierz identities do not hold. One consequence of this is that these methods break supersymmetry. And third, we have argued that it is impossible to define a unique  $\gamma_5$  matrix that anticommutes with the Dirac matrices and reduces to the standard  $\gamma_5$  after renormalization (or in finite expressions). We have then proposed to use the standard explicit definition with the antisymmetric  $\epsilon$  tensor in terms of the Dirac matrices. This is similar to the 't Hooft-Veltman definition in dimensional regularization and has the same consequences. Axial anomalies are reproduced, but in addition some spurious anomalies emerge, which should be removed *a posteriori* by local counterterms, added by hand. This is equivalent to the direct use of an anticommuting  $\gamma_5$ , when allowed [6].

In the implicit methods, it is also necessary to discriminate between different tensor structures. To avoid ambiguities in this discrimination, we have proposed a systematic renormalization procedure, in which the expressions to be renormalized are first put in a certain normal form, using only relations valid in QnS. We have also suggested some simplifications that make use of the decomposition  $\text{QnS} = \text{GnS} \oplus X$ , where GnS is the genuine  $n$ -dimensional space. The advantage of introducing this direct-sum structure is that it allows to use standard identities in GnS at some steps of the calculations.

In the context of chiral theories, we have also reconsidered shortcuts that exist at one loop and in simple higher-loop diagrams in FDR, which allow to discriminate the tensor structures from the very beginning and obtain the same results in a more direct way. A generalization of these shortcuts to arbitrary diagrams would allow to reduce the computational cost of heavy calculations. We think that the ideas in FDF can be helpful in this regard.

In addition, we have treated chiral theories in a formalism with Dirac spinors and chiral projectors. It would be interesting to see how our findings are translated to calculations with Weyl spinors and in superspace [56].

On the other hand, concerning the systematization of FDR, we have seen that distinct Forest Formula operator definitions and configurations lead to different FDR schemes. We have focused on three of them, which give a finite result. The discrepancies among them are non-local in some amplitudes, which suggest that some of these schemes might be inconsistent. In order to test them, we have computed two-loop calculations in massless QED and in

massless Scalar QED. In Spinorial QED we have found that for the vacuum polarization the first and third schemes agree, while for the fermion self-energy, the second and third one do. For the vacuum polarization in Scalar QED all schemes agree after summing all graphs, even if they do not for individual graphs. The Ward identities of gauge invariance hold in all cases. We have shown that the renormalised amplitudes obey the Renormalization Group Equation. We found agreement with the known values of the photon anomalous dimension, except in the case of the second scheme for Spinorial QED. This shows that the single logarithm in this scheme is not compatible with multiplicative renormalization. For the fermion self-energy the anomalous dimension at two loops is scheme-dependent, and it can be adjusted in such a way that all three schemes respect the Renormalization Group Equation.

Therefore, we have been able to discard scheme two, while scheme one and three respect gauge invariance and the Renormalization Group. Nevertheless, only the third scheme reproduce the results of the standard FDR *subprescription*. The logarithmic discrepancies seem to indicate that the discarded divergences cannot be absorbed into local counterterms in the first scheme. Therefore, we favour the third scheme, which reproduces the standard FDR results. Even if we have shown why this is the case, it remains to justify its validity in a more rigorous way and to extend it to all orders.

Finally, we observe that we have discarded the simplest operator definition because it did not give finite results at two loops. However, this conclusion is dependent on the order of limits and should be checked more carefully.

# Appendix A

## Lagrangians. Feynman rules

In this appendix we review the lagrangians and their derived Feynman rules of Spinorial and Scalar QED.

### A.1 Spinorial QED

The Spinorial QED Lagrangian is

$$\mathcal{L}_{QED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + \bar{\psi}(i\mathcal{D} - m)\psi \quad (\text{A.1.1})$$

where

$$\mathcal{D}_\mu = \partial_\mu + ie\mathcal{A}_\mu \quad (\text{A.1.2})$$

so

$$\mathcal{L}_{QED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + \bar{\psi}(i\mathcal{D} - m)\psi - e\bar{\psi}\mathcal{A}\psi \quad (\text{A.1.3})$$

where now the interaction among fields appears explicit.

$$\begin{aligned}
& \text{---}\xrightarrow{p}\text{---} = \frac{i}{\not{p} - m + i\varepsilon} \\
& \text{(a) Fermion propagator.} \\
& \text{~~~~~}\xrightarrow{q}\text{~~~~~} = -\frac{i}{q^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right] \\
& \text{(b) Photon propagator.} \\
& \begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} = -ie\gamma_\mu \\
& \text{(c) Fermion-Photon vertex.}
\end{aligned}$$

Figure A.1.1: Spinorial QED Feynman rules.

In fig. A.1.1 we see the Feynman rules generated by the Lagrangian (A.1.3). However, in this thesis we have always worked with massless fermions ( $m = 0$ ) and in the 't Hooft-Feynman gauge ( $\xi = 1$ ).<sup>1</sup> The symbol  $e$  represents the coupling constant between fermion and photon fields, i.e. the electron charge. After computing a diagram, we have replaced it with the fine structure constant.

$$\alpha = 4\pi e^2 \tag{A.1.4}$$

Moreover, to clarify, we have evaluated fermion and photon self-energy respectively as  $-i\Sigma$  and  $i\Pi_{\mu\nu}$ .

## A.2 Scalar QED

We present the Scalar QED Lagrangian with a complex scalar field.

$$\mathcal{L}_{SQED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + |\mathcal{D}_\mu\phi|^2 - m^2|\phi|^2 \tag{A.2.1}$$

---

<sup>1</sup>Except for one-loop fermion self-energy where we worked in a general gauge as required by Renormalization Group.

where a term like  $\lambda |\phi|^4$  has been no necessary for our purposes. If we introduce the covariant derivative, we explicitly see the fields interaction.

$$\mathcal{L}_{SQED} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu\mathcal{A}^\mu)^2 + |\partial_\mu\phi|^2 - m^2|\phi|^2 - ig[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi]\mathcal{A}^\mu + g^2\mathcal{A}_\mu\mathcal{A}^\mu|\phi|^2 \quad (\text{A.2.2})$$

Here, we see new kind of vertices. First, there is one triple vertex but made up of derivatives, which means the appearance of momenta and their directions in Feynman rules. On the other hand, there is a quadruple vertex which will lead to new graph topologies.

In fig. A.2.1 we see Scalar QED Feynman rules. Nevertheless, our work is massless ( $m = 0$ ), in the 't Hooft-Feynman gauge ( $\xi = 1$ ), and due to nature of our studied diagrams, only the first triple vertex has arisen. And after all, we replace  $g$  for an other “fine structure constant” that we will distinguish it placing a hat as well as in  $i\hat{\Pi}_{\mu\nu}$ .

$$\hat{\alpha} = 4\pi g^2 \quad (\text{A.2.3})$$



$$\text{---}\overrightarrow{p}\text{---} = \frac{i}{p^2 - m^2 + i\varepsilon}$$

(a) Scalar propagator.

$$\text{~~~~~}\overrightarrow{q}\text{~~~~~} = -\frac{i}{q^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right]$$

(b) Photon propagator.

$$\text{---}\overrightarrow{p_2}\text{---} \text{---}\overrightarrow{p_1}\text{---} \text{~~~~~}\mu = -ie (p_1 + p_2)_\mu$$

(c) Triple Scalar-Photon vertex 1.

$$\text{---}\overrightarrow{p_2}\text{---} \text{---}\overrightarrow{p_1}\text{---} \text{~~~~~}\mu = ie (p_1 + p_2)_\mu$$

(d) Triple Scalar-Photon vertex 2.

$$\text{---}\overrightarrow{p_1}\text{---} \text{---}\overrightarrow{p_2}\text{---} \text{~~~~~}\mu = -ie (p_1 - p_2)_\mu$$

(e) Triple Scalar-Photon vertex 3.

$$\text{---}\overrightarrow{p_1}\text{---} \text{---}\overrightarrow{p_2}\text{---} \text{~~~~~}\mu = ie (p_1 - p_2)_\mu$$

(f) Triple Scalar-Photon vertex 4.

$$\text{---}\text{---} \text{~~~~~}\mu \text{~~~~~}\nu = 2ig^{\mu\nu}$$

(g) Quadruple Scalar-Photon vertex.

Figure A.2.1: Scalar QED Feynman rules.

# Appendix B

## FDR Integrals

Here we explain the procedure we have followed to compute FDR integrals and useful relations.

### B.1 Dirac Algebra

Although a FDR Integral can be also understood as a difference of two  $d$ -dimensional ones, whatever way we choose, Dirac algebra can always be performed in the target fixed dimension ( $n$ ). Since if one  $\varepsilon$  hits  $1/\varepsilon$  in one side, it will also hit the same in the other side and the finite parts they could produce will vanish. Here we present several useful relations.

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{B.1.1}$$

where  $g_{\mu\nu}$  is the Minkowski metric  $\text{diag}(+, -, -, -)$ .

$$\gamma^\mu \gamma_\mu = n \tag{B.1.2}$$

$$\gamma^\mu \gamma_\nu \gamma_\mu = -(n-2) \gamma_\nu \tag{B.1.3}$$

$$\gamma^\mu \gamma_\nu \gamma_\lambda \gamma_\mu = 4g_{\nu\lambda} + (n-4) \gamma_\nu \gamma_\lambda \tag{B.1.4}$$

$$\gamma^\mu \gamma_\nu \gamma_\lambda \gamma_\rho \gamma_\mu = -2\gamma_\rho \gamma_\lambda \gamma_\nu - (n-4) \gamma_\nu \gamma_\lambda \gamma_\rho \tag{B.1.5}$$

$$\gamma_\mu \gamma_\nu \gamma_\lambda + \gamma_\lambda \gamma_\nu \gamma_\mu = 2(g_{\nu\lambda} \gamma_\mu - g_{\mu\lambda} \gamma_\nu + g_{\mu\nu} \gamma_\lambda) \tag{B.1.6}$$

## B.2 One-loop integrals

We write now some help and shortcuts to evaluate one-loop integrals. If we choose the way where a FDR integral is expressed as some  $d$ -dimensional ones, it is important to perform tensor reduction and integrals in  $d$  dimensions.

When the denominator has a dependence on external momentum, usual Passarino-Veltman relations could be useful.

$$\mathcal{A}^\mu = \frac{q^\mu}{\mathcal{D}(q, p)} = \mathcal{C}q^\mu \quad (\text{B.2.1})$$

$$\mathcal{C} = \frac{1}{p^2} p_\mu \mathcal{A}^\mu \quad (\text{B.2.2})$$

$$\mathcal{B}^{\mu\nu} = \frac{q^\mu q^\nu}{\mathcal{D}(q, p)} = \mathcal{C}_1 p^\mu p^\nu + \mathcal{C}_2 p^2 g_{\mu\nu} \quad (\text{B.2.3})$$

$$\mathcal{C}_1 = -\frac{1}{(d-1)p^2} \left( g_{\mu\nu} - d \frac{p_\mu p_\nu}{p^2} \right) \mathcal{B}^{\mu\nu} \quad (\text{B.2.4})$$

$$\mathcal{C}_2 = \frac{1}{(d-1)p^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \mathcal{B}^{\mu\nu} \quad (\text{B.2.5})$$

If the integrals are vacua ones, the following symmetric integration relations will help.

$$q^\mu q^\nu \rightarrow \frac{1}{d} q^2 g^{\mu\nu} \quad (\text{B.2.6})$$

$$q^\mu q^\nu q^\lambda q^\rho \rightarrow \frac{1}{d(d+2)} q^4 (g^{\mu\nu} g^{\lambda\rho} + g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}) \quad (\text{B.2.7})$$

Then, we use Feynman parameters to convert denominator into one term.

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma(\sum_{i=1}^n \alpha_i)}{\prod_{i=1}^n \Gamma(\alpha_i)} \left[ \prod_{i=1}^n \int_0^1 dx_i \right] \frac{\prod_{i=1}^n x_i^{\alpha_i-1}}{[\sum_{i=1}^n x_i A_i]^{\sum_{i=1}^n \alpha_i}} \delta\left(1 - \sum_{i=1}^n x_i\right) \quad (\text{B.2.8})$$

where  $\Gamma(x)$  is the gamma function, and  $\delta(x)$  the Dirac delta. Now the integral can be performed using the following equation.

$$\int d^d q \frac{(q^2)^\alpha}{(q^2 - M^2)^\beta} = (-1)^{\alpha-\beta} i\pi^{d/2} \frac{\Gamma(\alpha + d/2) \Gamma(\beta - \alpha - d/2)}{\Gamma(d/2) \Gamma(\beta)} (M^2)^{d/2 + \alpha - \beta} \quad (\text{B.2.9})$$

If there is no loop momentum in the numerator, the following master integrals without the

necessity of Feynman parameters could be useful.

$$\int d^d q \frac{1}{(q^2)^\alpha [(q+p)^2]^\beta} = i^{d+1} \pi^{d/2} \frac{\Gamma(\alpha + \beta - d/2) \Gamma(d/2 - \alpha) \Gamma(d/2 - \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(d - \alpha - \beta)} (p^2)^{d/2 - \alpha - \beta} \quad (\text{B.2.10})$$

$$\begin{aligned} & \int d^d q \frac{1}{(q^2 - m_1^2)^\alpha (q^2 - m_2^2)^\beta} = \\ & = (-1)^{-(\alpha+\beta)} i \pi^{d/2} \frac{\Gamma(\alpha + \beta - d/2)}{\Gamma(\alpha + \beta)} {}_2F_1 \left( \alpha, \alpha + \beta - d/2; \alpha + \beta; 1 - \frac{m_2^2}{m_1^2} \right) \end{aligned} \quad (\text{B.2.11})$$

where  ${}_2F_1(a, b; c; z)$  is the Gauss hypergeometric function.

The last step now is to evaluate Feynman parameters integrals if they exist, sum all pieces and take the limit  $d \rightarrow n$  by  $d = n + \varepsilon \rightarrow n$  if necessary. And finally take  $\mu \rightarrow 0$ , since we are evaluating a FDR integral.

## B.3 Two-loop integrals

Two-loop integrals are far more complicated than one-loop ones. First, we have to distinguish between real two-loop integrals, where a term  $(q_1 + q_2)^\alpha$  appears in the denominator making the integrals coupled; and disconnected two-loop integrals which can be separately evaluated by previous one-loop techniques, or in particular cases the following master integral can help.

$$\begin{aligned} & \int d^d q_1 \int d^d q_2 \frac{1}{(q_1^2 - m_1^2)^{\alpha_1} (q_2^2 - m_2^2)^{\alpha_2}} = \\ & = (-1)^{1 - \alpha_1 - \alpha_2} \pi^d \frac{\Gamma(\alpha_1 - d/2) \Gamma(\alpha_2 - d/2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} (m_1^2)^{d/2 - \alpha_1} (m_2^2)^{d/2 - \alpha_2} \end{aligned} \quad (\text{B.3.1})$$

### B.3.1 Integrals with external momentum

Regarding real two-loop integrals, first we must write them in a standard form in order to use a big set of master integrals that we will show next. The first step is to use Feynman parameters to combine all propagators with the same loop momentum ( $q_1, q_2$  or  $q_{12} \equiv q_1 + q_2$ ) in such a way that finally we get

$$\mathcal{I}^{\mu\nu} = \int d^d q_1 \int d^d q_2 \frac{f^{\mu\nu}(q_1, q_2)}{(q_1^2 - m_1^2)^{\alpha_1} (q_2^2 - m_2^2)^{\alpha_1} [(q_{12} + p)^2 - m_3^2]^{\alpha_3}} \quad (\text{B.3.2})$$

obviously, with Feynman parameters implicit dependence. Fortunately, only tensor reduction with two Lorentz indices within a non-vacuum integral has been necessary in this thesis. And it can be done as follows. First, we make the following replacement.

$$q_i^\mu = q_{i\perp}^\mu + \frac{q_i \cdot p}{p^2} p^\mu \quad (\text{B.3.3})$$

and then

$$q_{i\perp}^\mu f(q_1 \cdot p, q_2 \cdot p) \rightarrow 0 \quad (\text{B.3.4})$$

$$q_i^\mu f(q_1 \cdot p, q_2 \cdot p) \rightarrow \frac{p^\mu}{p^2} q_i \cdot p f(q_1 \cdot p, q_2 \cdot p) \quad (\text{B.3.5})$$

$$q_{i\perp}^\mu q_{i\perp}^\nu f(q_1 \cdot p, q_2 \cdot p) \rightarrow \frac{1}{d-1} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left[ q_i^2 - \frac{(q_i \cdot p)^2}{p^2} \right] f(q_1 \cdot p, q_2 \cdot p) \quad (\text{B.3.6})$$

$$q_{1\perp}^\mu q_{2\perp}^\nu f(q_1 \cdot p, q_2 \cdot p) \rightarrow \frac{1}{d-1} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left[ q_1 \cdot q_2 - \frac{(q_1 \cdot p)(q_2 \cdot p)}{p^2} \right] f(q_1 \cdot p, q_2 \cdot p) \quad (\text{B.3.7})$$

After all these substitutions, reordering, making cancellations between numerator and denominator when it is possible and performing Wick rotation, we should have several integrals in the following form.

$$\mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab} \equiv \int d^d q_1 \int d^d q_2 \frac{(q_1 \cdot p)^a (q_2 \cdot p)^b}{(q_1^2 + m_1^2)^{\alpha_1} (q_2^2 + m_2^2)^{\alpha_2} [(q_{12} + p)^2 + m_3^2]^{\alpha_3}} \quad (\text{B.3.8})$$

where  $\alpha_i \geq 1$ , and  $a, b \geq 0$ . We can increase any  $\alpha_i$  by differentiating with respect the corresponding mass. For example

$$\mathcal{P}_{\alpha_1+1 \alpha_2 \alpha_3}^{ab} = -\frac{1}{\alpha_1} \frac{\partial}{\partial m_1^2} \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab} \quad (\text{B.3.9})$$

This is the more useful relation concerning to our work, but we can obtain others by differentiating with respect external momentum to increase  $a$  or  $b$ .

$$\mathcal{P}_{\alpha_1+1 \alpha_2 \alpha_3}^{a+1b} = \frac{1}{2\alpha_1} \left[ 2p^2 \frac{\partial}{\partial p^2} - (a+b) \right] \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab} + \frac{ap^2}{2\alpha_1} \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{a-1b} \quad (\text{B.3.10})$$

$$\mathcal{P}_{\alpha_1 \alpha_2+1 \alpha_3}^{ab+1} = \frac{1}{2\alpha_2} \left[ 2p^2 \frac{\partial}{\partial p^2} - (a+b) \right] \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab} + \frac{bp^2}{2\alpha_2} \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab-1} \quad (\text{B.3.11})$$

$$\left[ 2p^2 \frac{\partial}{\partial p^2} - (a+b) \right] \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{ab} = -2\alpha_3 \left[ p^2 \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3+1}^{ab} + \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3+1}^{a+1b} + \mathcal{P}_{\alpha_1 \alpha_2 \alpha_3+1}^{ab+1} \right] \quad (\text{B.3.12})$$

taking into a count that a  $\mathcal{P}$  with  $a = -1$  or  $b = -1$  vanishes. We will get all needed integrals from  $\mathcal{P}_{211}^{ab}$  because they have simpler integral representations than  $\mathcal{P}_{111}^{ab}$ . To get these we will have to use the next relation.

$$\begin{aligned} \mathcal{P}_{\alpha_1\alpha_2\alpha_3}^{ab} = & -\frac{2}{2d-2(\alpha_1+\alpha_2+\alpha_3)+a+b} \left\{ \alpha_1 m_1^2 \mathcal{P}_{\alpha_1+1\alpha_2\alpha_3}^{ab} + \alpha_2 m_2^2 \mathcal{P}_{\alpha_1\alpha_2+1\alpha_3}^{ab} + \right. \\ & \left. + \alpha_3 m_3^2 \mathcal{P}_{\alpha_1\alpha_2\alpha_3+1}^{ab} - \frac{1}{2} \left[ 2p^2 \frac{\partial}{\partial p^2} - (a+b) \right] \mathcal{P}_{\alpha_1\alpha_2\alpha_3}^{ab} \right\} \end{aligned} \quad (\text{B.3.13})$$

Now we present several  $\mathcal{P}_{\alpha_1\alpha_2\alpha_3}^{ab}$  built by other functions that we will introduce later. A complete list can be consulted in [67], since here we only show the necessary ones to evaluate our diagrams.

$$\mathcal{P}_{211}^{00} = \mathcal{H}_1 \quad (\text{B.3.14})$$

$$\mathcal{P}_{211}^{10} = -\mathcal{H}_2 - p^2 \mathcal{H}_1 \quad (\text{B.3.15})$$

$$\mathcal{P}_{211}^{01} = -\mathcal{H}_3 \quad (\text{B.3.16})$$

$$\mathcal{P}_{211}^{20} = \mathcal{H}_4 + \frac{p^2}{d} \left\{ [(d-1)p^2 - m_1^2] \mathcal{H}_1 + 2(d-1) \mathcal{H}_2 + \mathcal{P}_{111}^{00} \right\} \quad (\text{B.3.17})$$

$$\begin{aligned} \mathcal{P}_{211}^{11} = & \mathcal{H}_5 + p^2 \mathcal{H}_3 + \frac{p^2}{2d} \left\{ (m_1^2 + m_2^2 - m_3^3 + p^2) \mathcal{H}_1 + \right. \\ & \left. + 2\mathcal{H}_2 - \mathcal{P}_{111}^{00} + T_2(m_1^2) [T_1(m_2^2) - T_1(m_3^3)] \right\} \end{aligned} \quad (\text{B.3.18})$$

$$\mathcal{P}_{211}^{02} = \mathcal{H}_6 + \frac{p^2}{d} [-m_2^2 \mathcal{H}_1 + T_2(m_1^2) T_1(m_3^2)] \quad (\text{B.3.19})$$

$$\begin{aligned} \mathcal{P}_{111}^{00} = & -\frac{1}{d-3} [(m_1^2 + p^2) \mathcal{H}_1 + \mathcal{H}_2 + \\ & + m_2^2 \mathcal{H}_1 (m_1 \rightleftharpoons m_2) + m_3^3 \mathcal{H}_1 (m_1 \rightarrow m_3, m_2 \rightarrow m_1, m_3 \rightarrow m_2)] \end{aligned} \quad (\text{B.3.20})$$

where  $T_i$  are the Euclidean one-loop tadpole integrals.

$$T_1(m^2) = -\pi^{2+\varepsilon/2} (m^2)^{\varepsilon/2} \Gamma\left(-\frac{\varepsilon}{2}\right) \frac{2m^2}{2+\varepsilon} \quad (\text{B.3.21})$$

$$T_2(m^2) = \pi^{2+\varepsilon/2} (m^2)^{\varepsilon/2} \Gamma\left(-\frac{\varepsilon}{2}\right) \quad (\text{B.3.22})$$

Next, we show  $\mathcal{H}_i$  with  $d = 4 + \varepsilon$ .

$$\mathcal{H}_1 = \pi^4 \left[ \frac{2}{\varepsilon^2} - \frac{1}{\varepsilon} + \frac{2}{\varepsilon} \log\left(\frac{m_1^2}{\mu^2}\right) - \frac{1}{2} + \frac{\pi^2}{12} - \log\left(\frac{m_1^2}{\mu^2}\right) + \log^2\left(\frac{m_1^2}{\mu^2}\right) + h_1 \right] \quad (\text{B.3.23})$$

$$\mathcal{H}_2 = \pi^4 p^2 \left[ -\frac{2}{\varepsilon^2} + \frac{1}{2\varepsilon} - \frac{2}{\varepsilon} \log \left( \frac{m_1^2}{\mu^2} \right) + \frac{13}{8} - \frac{\pi^2}{12} + \frac{1}{2} \log \left( \frac{m_1^2}{\mu^2} \right) - \log^2 \left( \frac{m_1^2}{\mu^2} \right) - h_2 \right] \quad (\text{B.3.24})$$

$$\mathcal{H}_3 = \pi^4 p^2 \left[ \frac{1}{\varepsilon^2} - \frac{1}{4\varepsilon} + \frac{1}{\varepsilon} \log \left( \frac{m_1^2}{\mu^2} \right) - \frac{13}{16} + \frac{\pi^2}{24} - \frac{1}{4} \log \left( \frac{m_1^2}{\mu^2} \right) + \frac{1}{2} \log^2 \left( \frac{m_1^2}{\mu^2} \right) + h_3 \right] \quad (\text{B.3.25})$$

$$\mathcal{H}_4 = \pi^4 p^4 \left[ \frac{3}{2\varepsilon^2} + \frac{3}{2\varepsilon} \log \left( \frac{m_1^2}{\mu^2} \right) - \frac{175}{96} + \frac{\pi^2}{16} + \frac{3}{4} \log^2 \left( \frac{m_1^2}{\mu^2} \right) + \frac{3}{4} h_4 \right] \quad (\text{B.3.26})$$

$$\mathcal{H}_5 = \pi^4 p^4 \left[ -\frac{3}{4\varepsilon^2} - \frac{3}{4\varepsilon} \log \left( \frac{m_1^2}{\mu^2} \right) + \frac{175}{192} - \frac{\pi^2}{32} - \frac{3}{8} \log^2 \left( \frac{m_1^2}{\mu^2} \right) - \frac{3}{4} h_5 \right] \quad (\text{B.3.27})$$

$$\mathcal{H}_6 = \pi^4 p^4 \left[ \frac{1}{2\varepsilon^2} - \frac{1}{24\varepsilon} + \frac{1}{2\varepsilon} \log \left( \frac{m_1^2}{\mu^2} \right) - \frac{19}{32} + \frac{\pi^2}{48} - \frac{1}{24} \log \left( \frac{m_1^2}{\mu^2} \right) + \frac{1}{4} \log^2 \left( \frac{m_1^2}{\mu^2} \right) + \frac{3}{4} h_6 \right] \quad (\text{B.3.28})$$

where every time a logarithm appears,  $\gamma_E + \log \pi$  has been omitted. Now a new set of finite functions  $h_i$  have arisen.

$$h_1(m_1, m_2, m_3; p^2) = \int_0^1 dz g(z) \quad (\text{B.3.29})$$

$$h_2(m_1, m_2, m_3; p^2) = \int_0^1 dz [g(z) + f_1(z)] \quad (\text{B.3.30})$$

$$h_3(m_1, m_2, m_3; p^2) = \int_0^1 dz [g(z) + f_1(z)] (1-z) \quad (\text{B.3.31})$$

$$h_4(m_1, m_2, m_3; p^2) = \int_0^1 dz [g(z) + f_1(z) + f_2(z)] \quad (\text{B.3.32})$$

$$h_5(m_1, m_2, m_3; p^2) = \int_0^1 dz [g(z) + f_1(z) + f_2(z)] (1-z) \quad (\text{B.3.33})$$

$$h_6(m_1, m_2, m_3; p^2) = \int_0^1 dz [g(z) + f_1(z) + f_2(z)] (1-z)^2 \quad (\text{B.3.34})$$

And these functions are built by the following ones.

$$g(m_1, m_2, m_3; p^2; z) = \text{Li}_2 \left( \frac{1}{1-y_1} \right) + \text{Li}_2 \left( \frac{1}{1-y_2} \right) + y_1 \log \left( \frac{y_1}{y_1-1} \right) + y_2 \log \left( \frac{y_2}{y_2-1} \right) \quad (\text{B.3.35})$$

$$f_1(m_1, m_2, m_3; p^2; z) = \frac{1}{2} \left[ -\frac{1-\lambda^2}{\kappa^2} + y_1^2 \log \left( \frac{y_1}{y_1-1} \right) + y_2^2 \log \left( \frac{y_2}{y_2-1} \right) \right] \quad (\text{B.3.36})$$

$$f_2(m_1, m_2, m_3; p^2; z) = \frac{1}{3} \left[ -\frac{2}{\kappa^2} - \frac{1-\lambda^2}{\kappa^2} - \left( \frac{1-\lambda^2}{\kappa^2} \right)^2 + y_1^3 \log \left( \frac{y_1}{y_1-1} \right) + y_2^3 \log \left( \frac{y_2}{y_2-1} \right) \right] \quad (\text{B.3.37})$$

where  $\text{Li}_2$  is the dilogarithm defined by

$$\text{Li}_2(x) = - \int_0^x dt \frac{1}{t} \log(1-t) \quad (\text{B.3.38})$$

and

$$y_{1,2} = \frac{1 + \kappa^2 - \lambda^2 \pm \sqrt{\Delta}}{2\kappa^2} \quad (\text{B.3.39})$$

$$\Delta = (1 + \kappa^2 - \lambda^2)^2 + 4\kappa^2\lambda^2 - 4i\kappa^2\eta \quad (\text{B.3.40})$$

$$\lambda^2 = \frac{c_1 z + c_2(1-z)}{z(1-z)} \quad (\text{B.3.41})$$

$$c_1 = \frac{m_2^2}{m_1^2} \quad (\text{B.3.42})$$

$$c_2 = \frac{m_3^2}{m_1^2} \quad (\text{B.3.43})$$

$$\kappa^2 = \frac{p^2}{m_1^2} \quad (\text{B.3.44})$$

$$\eta \rightarrow 0 \quad (\text{B.3.45})$$

Some of these integrals may have to be integrated numerically along with Feynman parameters. And, obviously, the last step is to take the limit  $\mu \rightarrow 0$ .

### B.3.2 Vacua integrals

If there is no external momentum in the denominator, the previous procedure simplifies. First, we can use symmetric integration to perform tensor reduction.

$$q_i^\mu q_j^\nu \rightarrow \frac{1}{d} q_i \cdot q_j g^{\mu\nu} \quad (\text{B.3.46})$$

$$q_i^\mu q_i^\nu q_i^\lambda q_j^\rho \rightarrow \frac{1}{d(d+2)} q_i^2 (q_i \cdot q_j) (g^{\mu\nu} g^{\lambda\rho} + g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda}) \quad (\text{B.3.47})$$

$$q_i^\mu q_i^\nu q_j^\lambda q_j^\rho \rightarrow \frac{1}{3d(d+2)} [q_i^2 q_j^2 (5g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda}) - 2(q_i \cdot q_j)^2 (g^{\mu\nu} g^{\lambda\rho} - 2g^{\mu\lambda} g^{\nu\rho} - 2g^{\mu\rho} g^{\nu\lambda})] \quad (\text{B.3.48})$$

After performing that, no loop momenta can be in the numerator, masses can only be  $\mu$ , and also  $p = 0$ , thus only  $\mathcal{P}_{\alpha_1\alpha_2\alpha_3}^{00}$  will contribute, no logarithms of masses can appear, and now



$g(z)$  can be integrated analytically [68]. As only  $h_1$  is necessary, an analytical integration of  $f_i(z)$  is not needed.

$$\begin{aligned}
h_1(m_1, m_2, m_3; 0) = & 1 - \frac{1}{2} \log c_1 \log c_2 - \frac{c_1 + c_2 - 1}{\sqrt{\tilde{\Delta}}} \left[ \text{Li}_2\left(-\frac{u_2}{v_1}\right) + \text{Li}_2\left(-\frac{v_2}{u_1}\right) + \right. \\
& \left. + \frac{1}{4} \log^2\left(\frac{u_2}{v_1}\right) + \frac{1}{4} \log^2\left(\frac{v_2}{u_1}\right) + \frac{1}{4} \log^2\left(\frac{u_1}{v_1}\right) - \frac{1}{4} \log\left(\frac{u_2}{v_2}\right) + \frac{\pi^2}{6} \right] \quad (\text{B.3.49})
\end{aligned}$$

where

$$u_{1,2} = \frac{1}{2} \left( 1 + c_2 - c_1 \pm \sqrt{\tilde{\Delta}} \right) \quad (\text{B.3.50})$$

$$v_{1,2} = \frac{1}{2} \left( 1 - c_2 + c_1 \pm \sqrt{\tilde{\Delta}} \right) \quad (\text{B.3.51})$$

$$\tilde{\Delta} = 1 - 2(c_1 + c_2) + (c_1 - c_2)^2 \quad (\text{B.3.52})$$

We know that in vacua integrals  $m_1 = m_2 = m_3 = \mu$  so all above expressions can simplify even more. Nevertheless, derivatives of  $h_1$  with respect to  $m_i$  are required to get some  $\mathcal{P}_{\alpha_1 \alpha_2 \alpha_3}^{00}$ , so it is important that all different masses appear symbolically. And of course, after reordering all expressions, the limit  $\mu \rightarrow 0$  has to be taken.

# Appendix C

## Three-point functions

In this appendix, we collect the finite three-points functions used in the evaluation of the axial vertex in four dimensions. We define the functions  $\xi_{nm} \equiv \xi_{nm}(p_2, p_1)$  as

$$\xi_{nm}(p_2, p_1) = \int_0^1 dz \int_0^{1-z} dy \frac{z^n y^m}{Q(y, z)} \quad (.0.1)$$

with

$$Q(y, z) = [\mu^2 - p_2^2 y(1-y) - p_1^2 z(1-z) - 2(p_2 \cdot p_1)yz] \quad (.0.2)$$

and notice that these functions have the property  $\xi_{nm}(p_2, p_1) = \xi_{mn}(p_1, p_2)$ . Using integration by parts [57], the relations below follow

$$p_1^2 \xi_{11} - (p_2 \cdot p_1) \xi_{02} = \frac{1}{2} \left[ -\frac{1}{2} \log \frac{(p_1 + p_2)^2}{\mu^2} + \frac{1}{2} \log \frac{p_2^2}{\mu^2} + p_1^2 \xi_{01} \right] \quad (.0.3)$$

$$p_2^2 \xi_{11} - (p_2 \cdot p_1) \xi_{20} = \frac{1}{2} \left[ -\frac{1}{2} \log \frac{(p_1 + p_2)^2}{\mu^2} + \frac{1}{2} \log \frac{p_1^2}{\mu^2} + p_2^2 \xi_{10} \right] \quad (.0.4)$$

$$p_1^2 \xi_{10} - (p_2 \cdot p_1) \xi_{01} = \frac{1}{2} \left[ -\log \frac{(p_1 + p_2)^2}{\mu^2} + \log \frac{p_2^2}{\mu^2} + p_1^2 \xi_{00} \right] \quad (.0.5)$$

$$p_2^2 \xi_{01} - (p_2 \cdot p_1) \xi_{10} = \frac{1}{2} \left[ -\log \frac{(p_1 + p_2)^2}{\mu^2} + \log \frac{p_1^2}{\mu^2} + p_2^2 \xi_{00} \right] \quad (.0.6)$$

$$p_1^2 \xi_{20} - (p_2 \cdot p_1) \xi_{11} = \frac{1}{4} [-1 + p_2^2 \xi_{01} + 3p_1^2 \xi_{10}] \quad (.0.7)$$

$$p_2^2 \xi_{02} - (p_2 \cdot p_1) \xi_{11} = \frac{1}{4} [-1 + p_1^2 \xi_{10} + 3p_2^2 \xi_{01}] \quad (.0.8)$$

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