Constant mean curvature spheres in homogeneous three-manifolds

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Abstract We prove that two spheres of the same constant mean curvature in an arbitrary homogeneous three-manifold only differ by an ambient isometry, and we determine the values of the mean curvature for which such spheres exist.

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This gives a complete classification of immersed constant mean curvature spheres in three-dimensional homogeneous manifolds.

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Contents

1 Introduction

In this manuscript we solve the fundamental problem of classifying constant mean curvature spheres in an arbitrary homogeneous three-manifold, where by a *sphere*, we mean a closed immersed surface of genus zero:

Theorem 1.1 *Let M be a Riemannian homogeneous three-manifold. Then, any two spheres in M of the same constant mean curvature differ by an isometry of M. Moreover, if X denotes the Riemannian universal cover of M, and $\mathrm{Ch}(X)$ denotes the Cheeger constant of X, then:*

1. *If X is not diffeomorphic to $\mathbb{R}^3$, then, for every $H \in \mathbb{R}$, there exists a sphere of constant mean curvature H in M.*

2. *If X is diffeomorphic to $\mathbb{R}^3$, then the values $H \in \mathbb{R}$ for which there exists a sphere of constant mean curvature H in M are exactly those with $|H| > \mathrm{Ch}(X)/2$.*

Our study of constant mean curvature spheres in homogeneous three-manifolds provides a natural parameterization of their moduli space and fundamental information about their geometry and symmetry, as explained in the next theorem.

Theorem 1.2 *Let M be a Riemannian homogeneous three-manifold, X denote its Riemannian universal cover and let S_H be a sphere of constant mean curvature $H \in \mathbb{R}$ in M. Then:*

1. *S_H is* maximally symmetric*; that is, there exists a point $p \in M$, which we call the* center of symmetry *of S_H, with the property that any isometry Φ of M that fixes p also satisfies $\Phi(S_H) = S_H$. In particular, constant mean curvature spheres are totally umbilical if M has constant sectional curvature, and are spheres of revolution if M is rotationally symmetric.*
2. *If $H = 0$ and X is a Riemannian product $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where $\mathbb{S}^2(\kappa)$ is a sphere of constant curvature $\kappa > 0$, then it is well-known that S_H is totally geodesic, stable and has nullity 1 for its stability operator. Otherwise, S_H has index 1 and nullity 3 for its stability operator, and it is* Alexandrov embedded*; that is, the immersion of S_H into M extends to an isometric immersion $F: B \to M$ of a Riemannian topological 3-ball B, such that $\partial B = S_H$ is mean convex, i.e. the mean curvature of ∂B with respect to its inward pointing normal vector is nonnegative.*

Moreover, if $\mathcal{M}_M(p)$ denotes the space of spheres of nonnegative constant mean curvature in M that have a base point $p \in M$ as a center of symmetry, then the map $S_H \in \mathcal{M}_M(p) \mapsto H \in \mathbb{R}$ that assigns to each sphere S_H its mean curvature is a homeomorphism between: (i) $\mathcal{M}_M(p)$ and $[0, \infty)$ if X is not diffeomorphic to $\mathbb{R}^3$, or (ii) $\mathcal{M}_M(p)$ and $(\mathrm{Ch}(X)/2, \infty)$ if X is diffeomorphic to $\mathbb{R}^3$. This homeomorphism is real analytic except at $H = 0$ when $X = \mathbb{S}^2(\kappa) \times \mathbb{R}$.

Theorems 1.1 and 1.2 can be generalized to the case where M is a complete, locally homogeneous three-manifold; see Sect. 3.3.

Remark 1.3 The Cheeger constant $\mathrm{Ch}(Y)$ of a Riemannian manifold Y with infinite volume is defined as the infimum of the quotients $\frac{\mathrm{Area}(\partial D)}{\mathrm{Volume}(D)}$ where D is any smooth compact domain in Y. When $Y = X$ is a homogeneous manifold diffeomorphic to $\mathbb{R}^3$, the value of $\mathrm{Ch}(X)$ is explicitly known, except when X is isometric to the universal cover of the special linear group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ endowed with a left invariant metric with a three-dimensional isometry group; see Section 2.2 in [27]. It is a problem of independent interest to determine the

exact value of Ch(X) in such spaces $X = (\widetilde{\mathrm{SL}}(2, \mathbb{R}), \langle, \rangle)$, which is known to be positive; see [27] and item 5 of Lemma 4.9 of the present paper for this and other related partial results.

The classification of constant mean curvature spheres is an old problem. In the nineteenth century, Jellet [18] proved that any constant mean curvature sphere in $\mathbb{R}^3$ that is star-shaped with respect to some point is a round sphere. In 1951, Hopf [16] introduced a holomorphic quadratic differential for any constant mean curvature surface in $\mathbb{R}^3$ and then used it to prove that constant mean curvature spheres in $\mathbb{R}^3$ are round. His proof also worked in the other simply-connected three-dimensional spaces of constant sectional curvature, which shows that these spheres are, again, totally umbilical (hence, they are the boundary spheres of geodesic balls of the space); see e.g. [6,7].

In 2004, Abresch and Rosenberg [1,2] proved that any constant mean curvature sphere in the product spaces $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where $\mathbb{S}^2(\kappa)$ (resp. $\mathbb{H}^2(\kappa)$) denotes the two-dimensional sphere (resp. the hyperbolic plane) of constant Gaussian curvature $\kappa \neq 0$, and more generally, in any simply connected Riemannian homogeneous three-manifold with an isometry group of dimension four, is a rotational sphere. Their theorem settled an old problem posed by Hsiang and Hsiang [17], and reduced the classification of constant mean curvature spheres in these homogeneous spaces to an ODE analysis. For their proof, Abresch and Rosenberg introduced a *perturbed* Hopf differential, which turned out to be holomorphic for constant mean curvature surfaces in these spaces. The Abresch–Rosenberg theorem was the starting point for the development of the theory of constant mean curvature surfaces in rotationally symmetric homogeneous three-manifolds; see e.g., [9,11] for a survey on the beginnings of this theory.

It is important to observe here that a generic homogeneous three-manifold M has an isometry group of dimension three, and that the techniques used by Abresch and Rosenberg in the rotationally symmetric case do not work to classify constant mean curvature spheres in such an M.

In 2013, Daniel and Mira [10] introduced a new method for studying constant mean curvature spheres in the homogeneous manifold Sol$_3$ with its usual Thurston geometry. Using this method, they classified constant mean curvature spheres in Sol$_3$ for values H of the mean curvature greater than $\frac{1}{\sqrt{3}}$, and reduced the general classification problem to the obtention of area estimates for the family of spheres of constant mean curvature greater than any given positive number. These crucial area estimates were subsequently proved by Meeks [23], which completed the classification of constant mean curvature spheres in Sol$_3$: *For any $H > 0$ there is a unique constant mean curvature sphere S_H in* Sol$_3$ *with mean curvature H; moreover, S_H is maximally symmetric, embedded, and has index one* [10,23].

In [26] we extended the Daniel–Mira theory in [10] to arbitrary simply connected homogeneous three-manifolds X not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$. However, the area estimates problem in this general setting cannot be solved following Meeks' approach in Sol_3, since the proof by Meeks uses in an essential way special properties of Sol_3 that are not shared by a general X.

In order to solve the area estimates problem in any homogeneous three-manifold, the authors have developed in previous works an extensive theoretical background for the study of constant mean curvature surfaces in homogeneous three-manifolds, see [24–28]. Specifically, in [28] there is a detailed presentation of the geometry of *metric Lie groups*, i.e., simply connected homogeneous three-manifolds given by a Lie group endowed with a left invariant metric. In [26,28] we described the basic theory of constant mean curvature surfaces in metric Lie groups. Of special importance for our study here is [26], where we extended the Daniel–Mira theory [10] to arbitrary metric Lie groups X, and we proved Theorems 1.1 and 1.2 in the case that M is a homogeneous three-sphere. In [24] we studied the geometry of spheres in metric Lie groups whose left invariant Gauss maps are diffeomorphisms. In [27] we proved that any metric Lie group X diffeomorphic to $\mathbb{R}^3$ admits a foliation by leaves of constant mean curvature of value $\text{Ch}(X)/2$. In [25] we established uniform radius estimates for certain stable minimal surfaces in metric Lie groups. All these works are used in the present manuscript to solve the area estimates problem.

The main results of this paper, Theorems 1.1 and 1.2, can be reduced to demonstrating Theorem 1.5 below; this reduction is explained in Sect. 3. To state Theorem 1.5, we need the notion of entire Killing graph given in the next definition, where X is a simply connected homogeneous three-manifold.

Definition 1.4 Given a nowhere zero Killing field K on X, we say that an immersed surface $\Sigma \subset X$ is a *Killing graph* with respect to K if whenever Σ intersects an integral curve of K, this intersection is transversal and consists of a single point. If additionally Σ intersects every integral curve of K, then we say that Σ is an *entire* Killing graph.

By classification, any homogeneous manifold X diffeomorphic to $\mathbb{R}^3$ is isometric to a metric Lie group. In this way, any nonzero vector field K on X that is right invariant for such Lie group structure is a nowhere-zero Killing field on X. We note that any integral curve of K is a properly embedded curve in X diffeomorphic to $\mathbb{R}$.

Theorem 1.5 *Let X be a homogeneous manifold diffeomorphic to $\mathbb{R}^3$, and let S_n be a sequence of constant mean curvature spheres in X with $\text{Area}(S_n) \to \infty$. Then, there exist isometries Φ_n of X and compact domains $\Omega_n \subset \Phi_n(S_n)$ with the property that a subsequence of the Ω_n converges uniformly on compact subsets of X to an entire Killing graph with respect to some nowhere zero*

Killing vector field on X, which in fact is right invariant with respect to some Lie group structure on X.

One should note that Theorem 1.5 is well-known if the isometry group of X has dimension 6 (equivalently, X has constant curvature) or dimension 4. For instance, if X has constant curvature $c \leq 0$, constant mean curvature spheres are totally umbilical by Hopf's theorem, and converge in the sense of Theorem 1.5 to planes (if $c = 0$) or horospheres (if $c < 0$). If X has an isometry group of dimension four, then constant mean curvature spheres are rotational by the Abresch–Rosenberg theorem, and they converge in the sense of Theorem 1.5 to entire graphs of constant mean curvature in X; this follows from classification theorems of rotational CMC surfaces in these spaces by Abresch and Rosenberg [1], Figueroa et al. [12] and Torralbo [42].

The uniqueness statement in Theorem 1.1 relies on the following uniqueness property, whose proof uses in a fundamental way the Poincaré-Hopf index theorem: *two spheres Σ_1, Σ_2 of the same constant mean curvature in a metric Lie group X are congruent provided that the left invariant Gauss map of one of them is a diffeomorphism* (see Definition 3.3 for the notion of left invariant Gauss map). This property was proved in [26, Theorem 4.1(2)], and it also follows directly from the general Gálvez-Mira uniqueness theory for immersed spheres in three-manifolds in [13]. A consequence of Theorem 1.5 above will be the existence, in the case that X is diffeomorphic to $\mathbb{R}^3$ and for every $H > \text{Ch}(X)/2$, of a sphere with constant mean curvature H whose left invariant Gauss map is a diffeomorphism. This existence result and the uniqueness property explained above imply the uniqueness statement of Theorem 1.1.

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2 Organization of the paper

We next explain the organization of the paper and outline the strategy of the proof of the main theorems. In our study in [26] of the space of constant mean curvature spheres of index one in metric Lie groups X, we proved that when X is diffeomorphic to $\mathbb{S}^3$, Theorems 1.1 and 1.2 hold for $M = X$. In that same paper we proved that if X is a metric Lie group not diffeomorphic to $\mathbb{S}^3$ (and hence, diffeomorphic to $\mathbb{R}^3$), then there exists a constant $h_0(X) \geq 0$ such that: (i) for any $H > h_0(X)$ there is an index-one sphere S_H of constant mean curvature H in X; (ii) any sphere of constant mean curvature $H > h_0(X)$ is a left translation of S_H, and (iii) if $H_n \searrow h_0(X)$, then the areas of the spheres S_{H_n} diverge to ∞ as $n \to \infty$; furthermore, the spheres S_{H_n} bound isometrically immersed Riemannian three-balls $\widetilde{f}_n \colon B_n \to X$ with the positive mean curvature H_n; in particular, S_{H_n} is Alexandrov-embedded.

In Sect. 3 we explain how Theorems 1.1 and 1.2 can be deduced from Theorem 1.5 and from the result stated in the last paragraph.

Section 4 is an introductory section; in it we prove some basic properties of constant mean curvature surfaces $f \colon \Sigma \looparrowright X$ in a metric Lie group X that are *invariant*, i.e., such that $f(\Sigma)$ is everywhere tangent to a nonzero right invariant Killing vector field K_Σ on X (equivalently, $f(\Sigma)$ is invariant under the flow of K_Σ), and we explain in more detail the geometry of metric Lie groups diffeomorphic to $\mathbb{R}^3$. For this analysis, we divide these metric Lie groups into two categories: *metric semidirect products* and those of the form $(\widetilde{\mathrm{SL}}(2, \mathbb{R}), \langle\,,\,\rangle)$, where $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the universal cover of the special linear group $\mathrm{SL}(2, \mathbb{R})$ and $\langle\,,\,\rangle$ is a left invariant metric; here the metric semidirect products X under consideration are the semidirect product of a normal subgroup $\mathbb{R}^2$ with $\mathbb{R}$ and we refer to the cosets in X of $\mathbb{R}^2$ as being *horizontal planes*.

In Sect. 5 we prove that if $\{S_n\}_n$ is a sequence of index-one spheres with constant mean curvatures of values $H_n \searrow h_0(X)$ in a metric Lie group X diffeomorphic to $\mathbb{R}^3$, then, after taking limits of an appropriately chosen subsequence of left translations of these spheres, there exists an invariant surface $f \colon \Sigma \looparrowright X$ of constant mean curvature $h_0(X)$ that is a limit of compact domains of the spheres S_n. Moreover, f is complete, stable, Σ has the topology of a plane or a cylinder, and if $f(\Sigma)$ is not a coset of a two-dimensional subgroup of X, then the closure $\overline{\gamma_f}$ of its left invariant Gauss map image γ_f in $\mathbb{S}^2$ (see Definition 3.3 for this notion) has the structure of a one-dimensional lamination. Another key property proved in Sect. 5 is that whenever $f(\Sigma)$ is tangent to some coset of a two-dimensional subgroup of X, then $f(\Sigma)$ lies in one of the two closed *half-spaces* bounded by this coset; see Corollary 5.9.

In Sect. 6 we prove in Theorem 6.1 that the invariant limit surface $f(\Sigma)$ can be chosen so that its left invariant Gauss map image $\gamma_f \subset \mathbb{S}^2$ is a point or a closed embedded regular curve. In the case γ_f is a point, Theorem 1.5 follows easily from the fact that $f(\Sigma)$ is a coset of a two-dimensional subgroup of X. In the case that γ_f is a closed curve, one can choose f so that one of the following mutually exclusive possibilities occurs:

1. Σ is diffeomorphic to an annulus.
2. Σ is diffeomorphic to a plane, and $f(\Sigma)$ is an immersed annulus in X.
3. Σ is diffeomorphic to a plane and there exists an element $a \in X$ such that the left translation by a in X leaves $f(\Sigma)$ invariant, but this left translation does not lie in the 1-parameter subgroup of isometries generated by the nonzero right invariant Killing vector field K_Σ that is everywhere tangent to $f(\Sigma)$.

In Sect. 7 we prove Theorem 1.5 in the case that X is a metric semidirect product. For that, it suffices to prove that $f(\Sigma)$ is an entire Killing graph for some right invariant Killing vector field V in X. This is proved as follows.

First, we show that if case 3 above holds, then the Killing field K_Σ which is everywhere tangent to $f(\Sigma)$ is horizontal with respect to the semidirect product structure of X, and we prove that $f(\Sigma)$ is an entire graph with respect to any other horizontal Killing field V linearly independent from K_Σ. So, it suffices to rule out cases 1 and 2 above. We prove that case 2 is impossible by constructing in that situation geodesic balls of a certain fixed radius $R^* > 0$ in the abstract Riemannian three-balls B_n that the Alexandrov-embedded index-one spheres S_n bound, and whose volumes tend to infinity as $n \to \infty$. This unbounded volume result eventually provides a contradiction with Bishop's theorem. Finally, we show that case 1 is impossible by constructing an abstract three-dimensional cylinder bounded by Σ that submerses isometrically into X with boundary $f(\Sigma)$, and then proving that a certain CMC flux of Σ in this abstract cylinder is different from zero. This gives a contradiction with the homological invariance of the CMC flux and the fact that $f(\Sigma)$ is a limit of the (homologically trivial) Alexandrov-embedded constant mean curvature spheres S_n.

In Sect. 8 we prove Theorem 1.5 in the remaining case where X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. The arguments and the basic strategy of the proof in this situation follow closely those from the previous Sect. 7. However, several of these arguments are by necessity different from those in Sect. 7, as many geometric properties of metric semidirect products do not have analogous counterparts in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

The paper finishes with an Appendix that can be read independently from the rest of the manuscript. In it, we prove a general nonvanishing result for the CMC flux that is used in Sects. 7 and 8 for ruling out case 1 above, as previously explained.

3 Reduction of Theorems 1.1 and 1.2 to Theorem 1.5

In order to show that Theorems 1.1 and 1.2 are implied by Theorem 1.5, we will prove the next two assertions in Sects. 3.1 and 3.2, respectively.

Assertion 3.1 *Assume that Theorem 1.5 holds. Then Theorems 1.1 and 1.2 hold for the particular case that the homogeneous three-manifold M is simply connected.*

Assertion 3.2 *Theorems 1.1 and 1.2 hold provided they hold for the particular case that the homogeneous three-manifold M is simply connected.*

In what follows, M will denote a homogeneous three-manifold and X will denote its universal Riemannian cover; if M is simply connected, we will identify $M = X$. By an H-*surface* (or H-*sphere*) in M or X we will mean an immersed surface (resp. sphere) of constant mean curvature of value $H \in \mathbb{R}$ immersed in M or X.

The *stability operator* of an H-surface $f : \Sigma \hookrightarrow M$ is the Schrödinger operator $L = \Delta + q$, where Δ stands for the Laplacian with respect to the induced metric in Σ, and q is the smooth function given by

$$q = |\sigma|^2 + \mathrm{Ric}(N),$$

where $|\sigma|^2$ denotes the squared norm of the second fundamental form of f, and N is the unit normal vector field to f. The immersion f is said to be *stable* if $-L$ is a nonnegative operator. When Σ is closed, the *index* of f is defined as the number of negative eigenvalues of $-L$, and the *nullity* of f is the dimension of the kernel of L. We say that a function $u \in C^\infty(\Sigma)$ is a *Jacobi function* if $Lu = 0$. Functions of the type $u = \langle N, F \rangle$ where F is a Killing vector field on the ambient space M are always Jacobi functions on Σ.

The classification of simply connected homogeneous Riemannian three-manifolds is well-known, and we explain next some of its basic features. For details about what follows, see e.g. Milnor [34], Scott [39] or Sekigawa [40]; see also the notes [28] by two of the authors, which will be used extensively along this paper, and which contain proofs of the results that we detail next.

A simply connected homogeneous three-manifold X has an isometry group $I(X)$ of dimension $n = 6, 4$ or 3. If $n = 6$, M is a space of constant curvature $c \in \mathbb{R}$. If $n = 4$, M has rotational symmetry, and is isometric to one of the so-called $\mathbb{E}(\kappa, \tau)$-spaces; see e.g. [8] for a description of their geometries. All these $\mathbb{E}(\kappa, \tau)$ spaces are fibrations over a 2-dimensional manifold $\mathbb{M}^2(\kappa)$ of constant curvature κ, and the constant τ denotes the bundle curvature of the fibration. At last, in the generic case that $I(X) = 3$, the homogeneous manifold X is isometric to a metric Lie group, with a finite isotropy group at any point.

From a topological viewpoint, a simply connected homogeneous three-manifold X is diffeomorphic to $\mathbb{R}^3$, to $\mathbb{S}^3$ or to $\mathbb{S}^2 \times \mathbb{R}$. In the $\mathbb{S}^2 \times \mathbb{R}$ case, X is actually isometric to the product space $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for some $\kappa > 0$. In all other cases, X is isometric to a metric Lie group. If X is diffeomorphic to $\mathbb{S}^3$, the underlying Lie group structure is $\mathrm{SU}(2)$. If X is diffeomorphic to $\mathbb{R}^3$, then X is isometric to either a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in \mathcal{M}_2(\mathbb{R})$ equipped with its canonical metric (see Sect. 4.2) or to the universal cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group with some left invariant metric (see Sect. 4.3). In particular, every simply connected homogeneous Riemannian three-manifold not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ is isometric to a Lie group endowed with a left invariant metric.

3.1 Proof of Assertion 3.1

Let $M = X$ denote a simply connected homogeneous three-manifold. When X is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, the statements contained in Theorems 1.1 and 1.2

follow from the Abresch–Rosenberg theorem that states that constant mean curvature spheres in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ are rotational spheres, and from an ODE analysis of the profile curves of these spheres. More specifically, an explicit expression of the rotational H-spheres in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ can be found in Pedrosa and Ritoré [36, Lemma 1.3], or in [1]. From this expression it follows that if $H \neq 0$, then each of these rotational H-spheres is embedded, and bounds a unique compact subdomain in X which is a mean convex ball that is invariant under all the isometries of X that fix the mid point in the ball of the revolution axis of the sphere (this point can be defined as the *center of symmetry* of the sphere). If $H = 0$, the corresponding rotational sphere is a slice $\mathbb{S}^2(\kappa) \times \{t_0\}$, that is trivially stable, and hence has index zero and nullity one. Moreover, any point of $\mathbb{S}^2(\kappa) \times \{t_0\}$ can be defined as the center of symmetry of the sphere in this case, in the sense that any isometry of $\mathbb{S}^2(\kappa) \times \mathbb{R}$ that fixes the point must leave invariant $\mathbb{S}^2(\kappa) \times \{t_0\}$. That the rotational H-spheres in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for $H \neq 0$ have index one and nullity three is shown in Souam [41, proof of Theorem 2.2]. Also, the rotational H-spheres for $H > 0$ with a fixed center of symmetry p_0 converge as $H \to 0$ to a double cover of the slice $\mathbb{S}^2(\kappa) \times \{t_0\}$ that contains p_0. So, from all this information and the Abresch–Rosenberg theorem, it follows that Theorems 1.1 and 1.2 hold when X is isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

When X is diffeomorphic to $\mathbb{S}^3$, the statements in Theorems 1.1 and 1.2 were proven in our previous work [26].

So, from now on in this Sect. 3.1, X will denote a homogeneous manifold diffeomorphic to $\mathbb{R}^3$. As explained above, X is isometric to a metric Lie group. In the sequel, X will be regarded as a metric Lie group, and e will denote its identity element. Given $x \in X$, we will denote by $l_x, r_x \colon X \to X$ the left and right translations by x, respectively given by $l_x(y) = xy, r_x(y) = yx$ for all $y \in X$. Thus, l_x is an isometry of X for every $x \in X$.

Definition 3.3 Given an immersed oriented surface $f \colon \Sigma \hookrightarrow X$ with unit normal vector field $N \colon \Sigma \to TX$ (here TX refers to the tangent bundle of X), we define the *left invariant Gauss map* of f to be the map $G \colon \Sigma \to \mathbb{S}^2 \subset T_eX$ that assigns to each $p \in \Sigma$, the unit tangent vector $G(p)$ to X at the identity element e given by $(dl_{f(p)})_e(G(p)) = N_p$.

The following proposition follows from a rearrangement of ideas taken from [26], but for the sake of clarity, we will include a proof here.

Proposition 3.4 *Let X be any metric Lie group X diffeomorphic to $\mathbb{R}^3$. Then, there exists a number $h_0(X) \geq 0$ such that:*

(1) For every $|H| > h_0(X)$, there exists an H-sphere S_H in X with index one and nullity three for its stability operator. Furthermore, S_H is Alexandrov embedded, i.e., S_H is the boundary of a (unique) mean convex immersed ball $\widetilde{f}_n \colon B_n \to X$ in X.

(2) *If Σ is an H-sphere in X for some $|H| > h_0(X)$, then Σ is a left translation of S_H.*

(3) *The left invariant Gauss map $G : S_H \rightarrow \mathbb{S}^2$ of S_H is an orientation-preserving diffeomorphism.*

(4) *Each S_H lies inside a real-analytic family $\{\Sigma_{H'} \mid H' \in (H - \varepsilon, H + \varepsilon)\}$ of index-one spheres in X for some $\varepsilon > 0$, where $\Sigma_H = S_H$ and $\Sigma_{H'}$ has constant mean curvature of value H'; note that, by item 2 above, each $\Sigma_{H'}$ is a left translation of $S_{H'}$. In particular, there exists a unique component $\mathcal{C}$ of the space of index-one spheres with constant mean curvature in X such that $S_H \in \mathcal{C}$ for all $H > h_0(X)$.*

(5) *For every $H_0 > h_0(X)$ there exists some positive constant $C = C(X, H_0)$ such that if $|H| \in (h_0(X), H_0]$, then the norm of the second fundamental form of S_H is at most C.*

(6) *For every $H_0 > h_0(X)$ there exists some positive constant $D = D(X, H_0)$ such that if $|H| > H_0$, then the area of S_H is at most D.*

(7) *If S_n is a sequence of constant mean curvature spheres in X whose mean curvatures $H_n > h_0(X)$ satisfy $H_n \rightarrow h_0(X)$, then $\mathrm{Area}(S_n) \rightarrow \infty$.*

Proof After a change of orientation if necessary, it clearly suffices to prove all statements for the case where $H \geq 0$. By [26, item 3 of Theorem 4.1], there exists a unique component $\mathcal{C}$ of the space of index-one spheres with constant mean curvature in X such that the values of the mean curvatures of the spheres in $\mathcal{C}$ are not bounded from above. As X is diffeomorphic to $\mathbb{R}^3$, then [26, item 6 of Theorem 4.1] ensures that the map $\mathcal{H} : \mathcal{C} \rightarrow \mathbb{R}$ that assigns to each sphere in $\mathcal{C}$ its mean curvature is not surjective. By item 5 in the same theorem, there exists $h_0(X) \geq 0$ such that for every $H > h_0(X)$, there is an index-one sphere $S_H \in \mathcal{C}$, and the following property holds:

(7)′ The areas of any sequence $\{S_{H_n}\}_n \subset \mathcal{C}$ with $H \rightarrow h_0(X)$ satisfy $\lim_{n \rightarrow \infty} \mathrm{Area}(S_{H_n}) = \infty$.

Since the sphere S_H has index one, then Cheng [5, Theorem 3.4] gives that the nullity of S_H is three. Alexandrov embeddedness of S_H follows from [26, Corollary 4.4]. Now item 1 of Proposition 3.4 is proved.

Once this item 1 has been proved, items 2, 3, 4, 5 of Proposition 3.4 are proved in items 1, 2, 3, 4 of [26, Theorem 4.1]. Item 6 of Proposition 3.4 is a direct consequence of the existence, uniqueness and analyticity properties of the spheres S_H stated in items 1, 2, 4, and of the fact also proved in [26] that, for H large enough, the spheres S_H bound small isoperimetric regions in X. Finally, item 7 of Proposition 3.4 follows from property (7)′ above and from the uniqueness given by the already proven item 2 of Proposition 3.4. Now the proof is complete. □

A key notion in what follows is the *critical mean curvature of X*.

Definition 3.5 Let X be a homogeneous manifold diffeomorphic to $\mathbb{R}^3$, let $\mathcal{A}(X)$ be the collection of all closed, immersed surfaces in X, and given a surface $\Sigma \in \mathcal{A}(X)$, let $|H_\Sigma|: \Sigma \to [0, \infty)$ stand for the absolute mean curvature function of Σ. The *critical mean curvature* of X is defined as

$$H(X) = \inf \left\{ \max_\Sigma |H_\Sigma| \; : \; \Sigma \in \mathcal{A}(X) \right\}.$$

Item 2 of Theorem 1.4 in [27] gives that $\mathrm{Ch}(X) = 2H(X)$. By definition of $H(X)$, every compact H-surface in X satisfies $|H| \geq H(X)$. Observe that Proposition 3.4 gives that for every $H > h_0(X)$ there is an H-sphere S_H in X. Thus,

$$h_0(X) \geq H(X) = \mathrm{Ch}(X)/2. \tag{3.1}$$

In order to prove Assertion 3.1, **in the remainder of this section we will assume that Theorem 1.5 holds.** Actually, we will just assume that Theorem 1.5 holds for any sequence S_n of spheres in X satisfying the conditions of item 7 of Proposition 3.4.

We will prove next that $h_0(X) = H(X)$. If S_n are constant mean curvature spheres satisfying item 7 of Proposition 3.4, then by Theorem 1.5 we obtain the existence of a surface Σ_0 in X of constant mean curvature $h_0(X)$ that is an entire Killing graph with respect to some nonzero right invariant Killing field K. Let $\Gamma = \{\Gamma(t) \mid t \in \mathbb{R}\}$ be the 1-parameter subgroup of X given by $\Gamma(0) = e$, $\Gamma'(0) = K(e)$. As left translations are isometries of X, it follows that $\{l_a(\Sigma_0) \mid a \in \Gamma\}$ defines a foliation of X by congruent surfaces of constant mean curvature $h_0(X)$, and this foliation is topologically a product foliation. A standard application of the mean curvature comparison principle shows then that there are no closed surfaces Σ in X with $\max_\Sigma |H| < h_0(X)$ (otherwise, we left translate Σ_0 until Σ is contained in the region of X on the mean convex side of Σ_0, and then start left translating Σ_0 towards its mean convex side until it reaches a first contact point with Σ; this provides a contradiction with the mean curvature comparison principle). Therefore, we conclude that $h_0(X) = H(X)$ and that, by item 1 of Proposition 3.4, the values $H \in \mathbb{R}$ for which there exists a sphere of constant mean curvature H in X are exactly those with $|H| > H(X)$. This fact together with item 2 of Proposition 3.4 proves item 2 of Theorem 1.1, and thus it completes the proof of Theorem 1.1 in the case X is diffeomorphic to $\mathbb{R}^3$ (assuming that Theorem 1.5 holds).

Regarding the proof of Theorem 1.2 when X is diffeomorphic to $\mathbb{R}^3$, similar arguments as the one in the preceding paragraph prove item 2 of Theorem 1.2. To conclude the proof of Theorem 1.2 (and hence of Assertion 3.1) we need to show that all spheres S_H in X are maximally symmetric, something that we will prove next. We note that the '*Moreover*' part in the statement of Theorem 1.2

follows directly from the existence of a center of symmetry of each sphere S_H, and the analyticity properties in item 4 of Proposition 3.4.

In the case that the isometry group $I(X)$ of X is of dimension 6, all constant mean curvature spheres in X are totally umbilical; in particular they are maximally symmetric, and the definition of the center of symmetry of the sphere is clear.

When $I(X)$ has dimension 4, it follows from the Abresch–Rosenberg theorem that constant mean curvature spheres in X are rotational, and thus they can be explicitly classified. In particular, if X is diffeomorphic to $\mathbb{R}^3$, these spheres are embedded and intersect their rotation axis orthogonally at exactly two points. In this way, we may define their center of symmetry as the midpoint of these two intersection points along the rotation axis. It can be checked that, with this definition, any constant mean curvature sphere in such space X is invariant with respect to any ambient isometry that fixes its center of symmetry, and in particular they are maximally symmetric.

So, in the remainder of this section we deal with the case that X is a metric Lie group diffeomorphic to $\mathbb{R}^3$ whose isometry group $I(X)$ has dimension 3. Let $\mathrm{Stab}_e(X)$ denote the group of isometries ψ of X with $\psi(e) = e$, and let $\mathrm{Stab}_e^+(X)$ denote the index-two subgroup of orientation-preserving isometries in $\mathrm{Stab}_e(X)$. The next proposition gives some basic properties of the elements of $\mathrm{Stab}_e(X)$ that will be needed for proving the maximal symmetry of H-spheres in X. Its proof follows from the analysis of metric Lie groups in [28].

Proposition 3.6 *Assume that $I(X)$ has dimension three. Then, there exists a 1-parameter subgroup Γ of X such that:*

(1) There exists an isometry $\phi \in \mathrm{Stab}_e^+(X)$ of order two in X that is a group automorphism of X, and whose fixed point set is Γ, i.e., $\Gamma = \{x \in X : \phi(x) = x\}$.

(2) Every $\psi \in \mathrm{Stab}_e(X)$ leaves Γ invariant, in the sense that $\psi(\Gamma) = \Gamma$.

Moreover, if there exists another 1-parameter subgroup $\widehat{\Gamma} \neq \Gamma$ of X that satisfies the previous two properties, then $\phi(\widehat{\Gamma}) = \widehat{\Gamma}$, where ϕ is given with respect to Γ by item 1 above.

Proof First suppose that the underlying Lie group structure Y of X is not unimodular[1]. Hence Y is isomorphic to some semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$ with $\mathrm{trace}(A) \neq 0$ and $\det(A) \neq 0$ since the dimension of $I(X) = 3$; see Sect. 4.2 for this notion of nonunimodular metric Lie group. Then [28, item 4 of Proposition 2.21] gives that $\mathrm{Stab}_e^+(X) = \{1_X, \phi\}$ where $\phi(x, y, z) = (-x, -y, z)$. Therefore, item 1 of Proposition 3.6 holds with the choice $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$ and the above ϕ. Regarding item 2, we

---

[1] A Lie group with Lie algebra $\mathfrak{g}$ is called unimodular if for all $V \in \mathfrak{g}$, the Lie algebra endomorphism $\mathrm{ad}_V : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_V(W) = [V, W]$ has trace zero.

divide the argument into two cases in this nonunimodular case for Y. Given $b \in \mathbb{R} - \{0\}$, consider the matrix

$$A(b) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}. \tag{3.2}$$

(A1) If X is not isomorphic and homothetic to any $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ with A given by (3.2) with $b \neq -1$, then [28, item 3 of Proposition 2.24] gives that $\text{Stab}_e(X) = \text{Stab}_e^+(X)$.

(A2) If X is isomorphic and homothetic to $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ for some $b \neq -1, 0$, then [28, item (3a) of Proposition 2.24] implies that then $\text{Stab}_e(X) - \text{Stab}_e^+(X) = \{\psi_1, \psi_2\}$ where $\psi_1(x, y, z) = (-x, y, z)$ and $\psi_2(x, y, z) = (x, -y, z)$. In the case that $b = 0$, then $\det(A(b)) = 0$, which is not the case presently under consideration as the isometry group of X for $b = 0$ has dimension four.

In both cases (A1) and (A2), item 2 of Proposition 3.6 holds with Γ being the z-axis. The moreover part of Proposition 3.6 also holds, because item 1 only holds for $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

Assume next that Y is unimodular. Since $\dim(I(X)) = 3$ and Y is unimodular, then there exists a frame of left invariant vector fields E_1, E_2, E_3 on Y that are eigenfields of the Ricci tensor of X, independently of the left invariant metric on X, see [28, Section 2.6]. By the proof of Proposition 2.21 of [28], for each $i = 1, 2, 3$ there exists an orientation-preserving automorphism $\phi_i \colon X \to X$ whose differential satisfies $(\phi_i)_*(E_i) = E_i$ and $(\phi_i)_*(E_j) = -E_j$ whenever $j \neq i$. Furthermore, ϕ is an isometry of every left invariant metric on Y (in particular, of the metric of X). Also, item 1 of Proposition 2.24 in [28] gives that $\text{Stab}_e^+(X)$ is the dihedral group $\{1_X, \phi_1, \phi_2, \phi_3\}$. Therefore, item 1 of Proposition 3.6 holds with any of the choices $\phi = \phi_i$ and $\Gamma = \Gamma_i$, for $i = 1, 2, 3$, where Γ_i is the 1-parameter subgroup of X generated by $(E_i)_e$. Regarding item 2, we again divide the argument into two cases in this unimodular case for Y.

(A1)$'$ If X is not isomorphic and homothetic to $\mathbb{R}^2 \rtimes_{A(-1)} \mathbb{R}$ (recall that $\mathbb{R}^2 \rtimes_{A(-1)} \mathbb{R}$ is Sol_3 with its standard metric), then [28, item 3 of Proposition 2.24] ensures that $\text{Stab}_e(X) = \text{Stab}_e^+(X)$. Thus item 2 of Proposition 3.6 holds for every choice of the form $\Gamma = \Gamma_i$ and $\phi = \phi_i$ with $i = 1, 2, 3$. Once here, and since it is immediate that $\phi_i(\Gamma_j) = \Gamma_j$ for any $i, j \in \{1, 2, 3\}$, we conclude that the moreover part of Proposition 3.6 holds in this case.

(A2)$'$ If X is isomorphic and homothetic to $\mathbb{R}^2 \rtimes_{A(-1)} \mathbb{R}$, then [28, item 3b of Proposition 2.24] gives that $\text{Stab}_e(X) - \text{Stab}_e^+(X) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$,

where

$$\psi_1(x, y, z) = (-x, y, z), \qquad \psi_2(x, y, z) = (x, -y, z),$$
$$\psi_3(x, y, z) = (y, -x, -z), \quad \psi_4(x, y, z) = (-y, x, -z).$$

Therefore, item 2 of Proposition 3.6 only holds for the choice $\Gamma = \{(0, 0, z) \mid z \in \mathbb{R}\}$ and $\phi(x, y, z) = (-x, -y, z)$ (in particular, the moreover part of Proposition 3.6 also holds).

Now the proof is complete. □

We are now ready to prove the existence of the center of symmetry of any H-sphere in X, in the remaining case that $I(X)$ has dimension three. Let $f_H : S_H \hookrightarrow X$ denote an H-sphere, with left invariant Gauss map $G : S_H \to \mathbb{S}^2$. By item 3 of Proposition 3.4 and $h_0(X) = H(X)$, we know G is an orientation preserving diffeomorphism. Without loss of generality we can assume that $f_H(q_H) = e$, where $q_H \in S_H$ is the unique point such that $G(q_H) = v := \Gamma'(0)$ and $\Gamma : \mathbb{R} \to X$ is an arc length parameterization of the 1-parameter subgroup of X that appears in Proposition 3.6. Let $\phi \in \text{Stab}_e^+(X)$ be an isometry associated to Γ in the conditions of item 1 of Proposition 3.6. Clearly $d\phi_e(v) = v$, hence $f_H(S_H)$ and $(\phi \circ f_H)(S_H)$ are two H-spheres in X passing through e with the same left invariant Gauss map image at e. By the uniqueness of H-spheres up to left translations in item 2 of Proposition 3.4, we conclude that $f_H(S_H) = (\phi \circ f_H)(S_H)$. Let q_H^* be the unique point in S_H such that $G(q_H^*) = -v$. Since $d\phi_e(-v) = -v$, we see that the Gauss map image of $\phi \circ f_H : S_H \hookrightarrow X$ at q_H^* is also $-v$. Since $f_H(S_H) = (\phi \circ f_H)(S_H)$, this implies (by uniqueness of the point q_H^*) that $f_H(q_H^*)$ is a fixed point of ϕ, which by item 1 of Proposition 3.6 shows that $f_H(q_H^*) \in \Gamma$.

Let $x_H \in \Gamma$ be the midpoint of the connected arc of Γ that connects e with $f_H(q_H^*)$, and define $\widehat{f}_H := l_{x_H^{-1}} \circ f_H : S_H \hookrightarrow X$. As the family $\{f_H : S_H \hookrightarrow X : H > H(X)\}$ is real analytic with respect to H (by items 2 and 4 of Proposition 3.4), then the family $\{\widehat{f}_H : S_H \hookrightarrow X \mid H > H(X)\}$ is also real analytic in terms of H. Also, note that the points $p_H := \widehat{f}_H(q_H)$ and $p_H^* := \widehat{f}_H(q_H^*)$ both lie in Γ, and are equidistant from e along Γ. This property and item 2 of Proposition 3.6 imply that every $\psi \in \text{Stab}_e(X)$ satisfies $\psi(\{p_H, p_H^*\}) = \{p_H, p_H^*\}$, and hence either $\psi(p_H) = p_H$ and $\psi(p_H^*) = p_H^*$, or alternatively $\psi(p_H) = p_H^*$ and $\psi(p_H^*) = p_H$.

Take $\psi \in \text{Stab}_e(X)$. We claim that ψ leaves $\widehat{f}_H(S_H)$ invariant. To see this, we will distinguish four cases.

(B1) Suppose that ψ preserves orientation on X and satisfies $\psi(p_H) = p_H$, $\psi(p_H^*) = p_H^*$. The first condition implies that $(\psi \circ \widehat{f}_H)(S_H)$ is an H-sphere in X. So, by items 2 and 3 of Proposition 3.4 we have $l_a(\widehat{f}_H(S_H)) = (\psi \circ \widehat{f}_H)(S_H)$ for some $a \in X$, and that $l_a(p_H)$ corresponds to the unique point of $(\psi \circ \widehat{f}_H)(S_H)$ where its Gauss map takes

the value v. But now, from the second condition we obtain that ψ restricts to Γ as the identity map, which implies that the value of the left invariant Gauss map image of $(\psi \circ \widehat{f}_H)(S_H)$ at p_H is v. Thus, $l_a(p_H) = p_H$, hence $a = e$ and $\widehat{f}_H(S_H) = (\psi \circ \widehat{f}_H)(S_H)$, that is, ψ leaves $\widehat{f}_H(S_H)$ invariant.

(B2) Suppose that ψ reverses orientation on X and satisfies $\psi(p_H) = p_H^*$, $\psi(p_H^*) = p_H$. The first condition implies that after changing the orientation of $(\psi \circ \widehat{f}_H)(S_H)$, this last surface is an H-sphere in X. Since by the second condition ψ restricts to Γ as minus the identity map, then the left invariant Gauss map image of $(\psi \circ \widehat{f}_H)(S_H)$ (with the reversed orientation) at p_H is v. Now we deduce as in case (B1) that ψ leaves $\widehat{f}_H(S_H)$ invariant.

(B3) Suppose that ψ reverses orientation on X and satisfies $\psi(p_H) = p_H$, $\psi(p_H^*) = p_H^*$. As in case (B2), we change the orientation on $(\psi \circ \widehat{f}_H)(S_H)$ so it becomes an H-sphere in X. Item 2 of Proposition 3.4 then gives that there exists $a \in X$ such that $(\psi \circ \widehat{f}_H)(S_H) = l_a(\widehat{f}_H(S_H))$. As $p_H^* \in \widehat{f}_H(S_H) \cap \Gamma$ and ψ fixes Γ pointwise, then $p_H^* \in (\psi \circ \widehat{f}_H)(S_H)$. Moreover, the value of the left invariant Gauss map image of $(\psi \circ \widehat{f}_H)(S_H)$ (with the reversed orientation) at p_H^* is v. Since the only point of $l_a(\widehat{f}_H(S_H))$ where its left invariant Gauss map image (with the original orientation) takes the value v is $l_a(p_H)$, we deduce that $l_a(p_H) = p_H^*$. Similarly, $p_H = l_a(p_H^*)$. This is only possible if $a = a^{-1}$, which, in our setting that X is diffeomorphic to $\mathbb{R}^3$, implies that $a = e$ (note that this does not happen when X is diffeomorphic to the three-sphere, because then X is isomorphic to SU(2), which has one element of order 2). The claim then holds in this case.

(B4) Finally, assume that ψ preserves the orientation on X and satisfies $\psi(p_H) = p_H^*$, $\psi(p_H^*) = p_H$. As in case (B1), $(\psi \circ \widehat{f}_H)(S_H)$ is an H-sphere in X and so there exists $a \in X$ such that $(\psi \circ \widehat{f}_H)(S_H) = l_a(\widehat{f}_H(S_H))$. As $p_H \in \widehat{f}_H(S_H)$, then $p_H^* = \psi(p_H) \in (\psi \circ \widehat{f}_H)(S_H)$. Since ψ restricts to Γ as minus the identity map, then the value of the left invariant Gauss map image of $(\psi \circ \widehat{f}_H)(S_H)$ at p_H^* is v. Since the only point of $l_a(\widehat{f}_H(S_H))$ where its left invariant Gauss map image takes the value v is $l_a(p_H)$, we deduce that $l_a(p_H) = p_H^*$. Now we finish as in case (B3).

To sum up, we have proved that any $\psi \in \text{Stab}_e(X)$ leaves $\widehat{f}_H(S_H)$ invariant; this proves that the H-sphere $\widehat{f}_H(S_H)$ is maximally symmetric with respect to the identity element e, in the sense of item 1 of Theorem 1.2. So, we may define e to be the *center of symmetry* of $\widehat{f}_H(S_H)$. We next show that the above definition of center of symmetry does not depend on the choice of the 1-parameter subgroup Γ of X that appears in Proposition 3.6. Suppose $\widehat{\Gamma}$ is another 1-parameter subgroup of X satisfying the conclusions of

Proposition 3.6. As we explained in the paragraph just after Proposition 3.6, $\widehat{\Gamma}$ determines two points $\widehat{q}, \widehat{q}^* \in S_H$ with $\widehat{f}_H(\widehat{q}), \widehat{f}_H(\widehat{q}^*) \in \widehat{\Gamma}$, such that $G(\widehat{q}) = \widehat{\Gamma}'(0)$, $G(\widehat{q}^*) = -\widehat{\Gamma}'(0)$, where G is the left invariant Gauss map of $\widehat{f}_H$. To show that the above definition of center of symmetry of $\widehat{f}_H(S_H)$ does not depend on Γ, we must prove that the mid point of the arc of $\widehat{\Gamma}$ between $\widehat{f}_H(\widehat{q})$ an $\widehat{f}_H(\widehat{q}^*)$ is e. To do this, let $\phi \in \mathrm{Stab}_e^+(X)$ be an order-two isometry that satisfies item 1 of Proposition 3.6 with respect to Γ. By Proposition 3.6, $\phi(\widehat{\Gamma}) = \widehat{\Gamma}$. As ϕ leaves $\widehat{f}_H(S_H)$ invariant and G is a diffeomorphism, then $\phi(\{\widehat{f}_H(\widehat{q}), \widehat{f}_H(\widehat{q}^*)\}) = \{\widehat{f}_H(\widehat{q}), \widehat{f}_H(\widehat{q}^*)\}$. Since the set of fixed points of ϕ is Γ and $\widehat{f}_H(\widehat{q}), \widehat{f}_H(\widehat{q}^*) \notin \Gamma$ (because two different 1-parameter subgroups in X only intersect at e, here we are using again that X is diffeomorphic to $\mathbb{R}^3$), then $\phi(\widehat{f}_H(\widehat{q})) = \widehat{f}_H(\widehat{q}^*)$. As ϕ is an isometry with $\phi(e) = e$ and $\phi(\widehat{\Gamma}) = \widehat{\Gamma}$, we deduce that the lengths of the arcs of $\widehat{\Gamma}$ that join e to $\widehat{f}_H(\widehat{q})$, and e to $\widehat{f}_H(\widehat{q}^*)$, coincide. This implies that the mid point of the arc of $\widehat{\Gamma}$ between $\widehat{f}_H(\widehat{q}))$ an $\widehat{f}_H(\widehat{q}^*)$ is e, as desired. As every H-sphere in X is a left translation of $\widehat{f}_H(S_H)$, we conclude that all H-spheres in X are maximally symmetric with respect to some point $p \in X$, which is obtained by the corresponding left translation of the identity element e, and thus can be defined as the center of symmetry of the sphere. This completes the proof of Theorem 1.2 (assuming that Theorem 1.5 holds) for the case that X is diffeomorphic to $\mathbb{R}^3$. Thus, the proof of Assertion 3.1 is complete.

The next remark gives a direct definition of the center of symmetry of any H-sphere in X:

Remark 3.7 (Definition of the center of symmetry). Let $f : S_H \looparrowright X$ be an H-sphere in a homogeneous manifold X diffeomorphic to $\mathbb{R}^3$ with an isometry group of dimension three. Let $q, q^* \in S_H$ be given by $G(q) = v$, $G(q^*) = -v$, where $G : S_H \to \mathbb{S}^2$ denotes the left invariant Gauss map of S_H and $v := \Gamma'(0)$, with $\Gamma(t)$ a 1-parameter subgroup satisfying the conditions in Proposition 3.6. Then, by our previous discussion, the left coset $\alpha := l_{f(q)}(\Gamma)$ passes through both $f(q)$ and $f(q^*)$, and we define the *center of symmetry* of S_H as the midpoint $p \in \alpha$ of the subarc of α that joins $f(q)$ with $f(q^*)$.

It is worth mentioning that this definition gives a nonambiguous definition of the center of symmetry of an H-sphere; however, when X has an isometry group of dimension three and is nonunimodular, there are many points $p \in X$ besides this center of symmetry such that the H-sphere is invariant under all isometries of X that fix p. Nevertheless, in order to make sense of the analyticity properties of the family of constant mean curvature spheres with a fixed center of symmetry (see Theorem 1.2), we need to make the definition of center of symmetry nonambiguous.

3.2 Proof of Assertion 3.2

Let M be a homogeneous three-manifold with universal covering space $\Pi\colon X \to M$. To prove Assertion 3.2, **we will assume in this Sect. 3.2 that Theorems 1.1 and 1.2 hold for X.**

We first prove Theorem 1.1 in M. Since every constant mean curvature sphere $f\colon S \looparrowright M$ is the projection via Π of some lift $\widetilde{f}\colon S \looparrowright X$ of f with the same mean curvature, then clearly the first and second items in Theorem 1.1 hold in M, since they are true by hypothesis in X. In order to prove the uniqueness statement in Theorem 1.1, let $f_1\colon S_1 \looparrowright M$, $f_2\colon S_2 \looparrowright M$ be two spheres with the same constant mean curvature in M. For $i = 1, 2$, choose $\widetilde{f_i}$ respective lifts of f_i, let $\widetilde{p_i}$ be the centers of symmetry of $\widetilde{f_i}$, denote $p_i = \Pi(\widetilde{p_i})$, and let $I\colon M \to M$ be an isometry with $I(p_1) = p_2$. Let $\widetilde{I}\colon X \to X$ be the lift of I that takes $\widetilde{p_1}$ to $\widetilde{p_2}$. It follows that the H-spheres $\widetilde{I} \circ \widetilde{f_1}$ and $\widetilde{f_2}$ have the same center of symmetry in X and so these immersions have the same images. In particular, it follows that $I \circ f_1$ and f_2 also have the same images, which completes the proof of Theorem 1.1 in M.

We next prove Theorem 1.2 in M. Suppose $f\colon S \looparrowright M$ is an oriented H-sphere, and let $\widetilde{f}\colon S \looparrowright X$ denote some lift of f. Since the stability operators of f and $\widetilde{f}$ are the same, the index and nullity of f and $\widetilde{f}$ agree. Also, note that if the immersion $\widetilde{f}$ extends to an isometric immersion $F\colon B \to X$ of a mean convex Riemannian three-ball B into X, then $\Pi \circ F$ is an isometric immersion of B into M that extends f. These two trivial observations prove that item 2 of Theorem 1.2 holds in M.

If M is covered by $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $f(S)$ is totally geodesic, then any point of the image sphere satisfies the property of being a center of symmetry of f. This observation follows by the classification of homogeneous three-manifolds covered by $\mathbb{S}^2(\kappa) \times \mathbb{R}$ as being those spaces isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, $\mathbb{S}^2(\kappa) \times \mathbb{S}^1(R)$, $\mathbb{P}^2(\kappa) \times \mathbb{R}$ or $\mathbb{P}^2(\kappa) \times \mathbb{S}^1(R)$, where $\mathbb{S}^1(R)$ is a circle of circumference R for some $R > 0$ and $\mathbb{P}^2(\kappa)$ is the projective plane of constant Gaussian curvature $\kappa > 0$.

Suppose now that M is not covered by $\mathbb{S}^2(\kappa) \times \mathbb{R}$ with $f(S)$ being totally geodesic. In order to complete the proof of item 1 of Theorem 1.2, we next define the center of symmetry of f in M as $p := \Pi(\widetilde{p})$, where $\widetilde{p}$ is the center of symmetry of a lift $\widetilde{f}$ in X. Since the center of symmetry of the lift $\widetilde{f}$ is uniquely defined by $\widetilde{f}$ and any other lift is equal to the composition of $\widetilde{f}$ with an isometry $\sigma\colon X \to X$ that is a covering transformation, then by uniqueness of the center of symmetry in X, σ maps the center of symmetry of $\widetilde{f}$ to the center of symmetry of the composed oriented immersion $\sigma \circ \widetilde{f}$; hence p is independent of the choice of the lift of f to X. We now show that $f(S)$ is invariant under all isometries of M that fix p. Let $I\colon M \to M$ be any such isometry, and let $\widetilde{I}\colon X \to X$ be the lift of I such that $\widetilde{I}(\widetilde{p}) = \widetilde{p}$. As $\widetilde{p}$ is

a center of symmetry of $\widetilde{f}$, then, $\widetilde{I}$ induces an isometry of $\widetilde{f}\colon S \hookrightarrow X$. It follows that I induces an isometry of the H-sphere f, which proves item 1 of Theorem 1.2 in M. Once here, the homeomorphism and analyticity properties in the last statement of Theorem 1.2 follow trivially from the validity of the corresponding statement in X. This completes the proof of Theorem 1.2 in M, and thus also the proof of Assertion 3.2.

3.3 Constant mean curvature spheres in complete locally homogeneous three-manifolds

As a consequence of Theorems 1.1 and 1.2 and of the discussion in Sects. 3.1 and 3.2 we can provide a description of the space of constant mean curvature spheres in complete, locally homogeneous three-manifolds. Specifically, let Y be a complete, connected, locally homogeneous Riemannian three-manifold. Then, the universal Riemannian covering space $\Pi\colon X \to Y$ is a simply connected homogeneous three-manifold. Let $\text{Ch}(X)$ be the Cheeger constant of X. Remark 3.7, the proof of Proposition 3.6 and the results on the existence and uniqueness of the center of symmetry for any H-sphere in X imply that associated to any H-sphere S_H in Y with $H > 0$, there is a unique point $p(S_H) \in Y$ that is the image by Π of the center of symmetry of any lift of S_H to X; that $p(S_H)$ does not depend on the lift of S_H can be proved following the arguments used in Sect. 3.2. The following properties of spheres with nonzero constant mean curvature in Y are then easy to check, and provide a parameterization of the space of such constant mean curvature spheres.

1. For any $H > \text{Ch}(X)/2$ and any $p_0 \in Y$ there exists a unique H-sphere S_H in Y with $p(S_H) = p_0$.
2. If $\text{Ch}(X) > 0$, there are no H-spheres in Y for $0 < H \leq \text{Ch}(X)/2$.
3. The space $\mathcal{M}_H(Y)$ of all immersed H-spheres in Y for a given $H > \text{Ch}(X)/2$ is naturally diffeomorphic to Y (by the map that sends every S_H into $p(S_H)$), where H-spheres in Y with distinct images are considered to be different elements in $\mathcal{M}_H(Y)$.
4. Any H-sphere in Y for $H > 0$ has index one and nullity three for its stability operator.
5. Any H-sphere S_H in Y for $H > 0$ is invariant under every isometry of Y that fixes $p(S_H)$.
6. Given any two H-spheres $S_H^0, S_H^1 \in \mathcal{M}_H(Y)$ for some $H > 0$, there exist an isometry $\phi\colon S_H^0 \to S_H^1$ that also preserves the second fundamental forms of S_H^0 and S_H^1 at corresponding points. Moreover, given any path $\alpha\colon [0, 1] \to Y$ joining $p(S_H^0)$ to $p(S_H^1)$, there is an associated family of isometric H-spheres $S_H(t)$, $t \in [0, 1]$, in Y with $S_H^0 = S_H(0)$ and $S_H^1 = S_H(1)$, such that $p(S_H(t)) = \alpha(t)$ for every t.

We observe that item 6 above is the natural generalization to complete locally homogeneous three-manifolds of the uniqueness result in Theorem 1.1 that any two spheres of the same constant mean curvature in a homogeneous three-manifold differ by an ambient isometry.

4 Background on constant mean curvature surface theory in metric Lie groups

In this section we will collect some basic material about metric Lie groups diffeomorphic to $\mathbb{R}^3$ and immersed surfaces of constant mean curvature in these spaces, that will be needed in later sections. For the purposes of this paper, it is worth dividing metric Lie groups X diffeomorphic to $\mathbb{R}^3$ into two classes depending on whether the underlying group structure of X is that of a semidirect product, or it is the one of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, that can be naturally identified as the universal cover of the group $\mathrm{PSL}(2, \mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane.

We divide this section into three parts: in Sect. 4.1 we describe some geometric aspects of constant mean curvature surfaces in a metric Lie group X that are invariant under the flow of a nonzero right invariant vector field on X. Sections 4.2 and 4.3 describe the ambient geometry and particular aspects of surface theory in a metric Lie group isomorphic to a semidirect product or to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, respectively. We will use the contents of Sect. 4.1 in Sect. 5, but Sects. 4.2 and 4.3 will not be needed until Sect. 6. The basic references for this material are Milnor [34], the notes [28] and the paper [26].

4.1 Invariant constant mean curvature surfaces

Given a unit vector $v \in T_e X$ in the Lie algebra of X, we can extend v by left translation to a left-invariant unit vector field V globally defined on X. By using right translations in X in a similar way, we can extend v to a right-invariant vector field K in X. The vector field K will not be, in general, of unit length anymore, but it is a nowhere-zero Killing field on X.

By [28, Corollary 3.17], left cosets of two-dimensional subgroups of X are characterized by their left invariant Gauss map being constant, and they all have constant mean curvature with absolute value at most $H(X)$. As every right coset Λx of a two-dimensional subgroup Λ of X is the left coset of a conjugate subgroup of Λ (namely $\Lambda x = x \Lambda_1$ where $\Lambda_1 = x^{-1} \Lambda x$), then we deduce:

(C) The left invariant Gauss map of a left or right coset of a two-dimensional subgroup of X is constant.

The next proposition is a reformulation of Corollary 3.8 in [26]. We remark that Corollary 3.8 in [26] was written in terms of the H-potential of X, a concept that we will not introduce here, but one can easily translate that formulation into the one below by using [28, Corollary 3.21].

Proposition 4.1 *Let $f \colon \Sigma \hookrightarrow X$ be a complete immersed H-surface in X and let $G \colon \Sigma \to \mathbb{S}^2 \subset T_e X$ be its left invariant Gauss map. Assume that there are no two-dimensional subgroups in X with mean curvature H whose (constant) left invariant Gauss map lies in the Gauss map image $G(\Sigma)$. Then:*

(1) The differential dG of G has rank at most 1 everywhere on Σ if and only if f is invariant under the flow of a nonzero right invariant vector field K on X.

(2) If f is invariant under the flow of a nonzero right invariant vector field of X, then $rank(dG) = 1$ everywhere on Σ, and $G(\Sigma)$ is a regular curve in $\mathbb{S}^2$.

Remark 4.2 If K is a nonzero right invariant vector field on a metric Lie group X diffeomorphic to $\mathbb{R}^3$, then each integral curve of K is diffeomorphic to $\mathbb{R}$ and the quotient space X/K of integral curves of K inherits a two-dimensional differentiable structure that makes the natural projection $\Pi_K \colon X \to X/K$ a submersion with $\ker[d(\Pi_K)_x] = \mathrm{Span}\{K(x)\}$ for all $x \in X$. Therefore, when $f \colon \Sigma \hookrightarrow X$ satisfies the hypothesis in item 2 of Proposition 4.1, we can consider $f(\Sigma)$ as the surface in X obtained by pulling back via Π_{K_Σ} an immersed curve β contained in X/K_Σ. Moreover, after identifying in the standard way X/K_Σ with any entire Killing graph $S_0 \subset X$ with respect to K_Σ, the invariant surface $f \colon \Sigma \hookrightarrow X$ can be parameterized locally (and even globally if Σ is simply connected) as

$$f(s, t) = l_{\Gamma(t)}(\beta(s)), \qquad (s, t) \in I \times \mathbb{R},$$

where $\beta(s) \colon I \subset \mathbb{R} \to X/K_\Sigma \equiv S_0$ is the immersed curve above, and $\Gamma = \Gamma(t)$ is the 1-parameter subgroup of X given by $\Gamma'(0) = K_\Sigma(e)$.

Let $f \colon \Sigma \hookrightarrow X$ be an immersed H-surface in X whose left invariant Gauss map $G \colon \Sigma \to \mathbb{S}^2$ has rank one at every point. It is shown in [26, proof of Corollary 3.8] that around any $z_0 \in \Sigma$ there exist conformal parameters (s, t) on Σ such that G does not depend on t, so we can write $G = G(s)$. With respect to these coordinates, the Gauss map $G(s)$ satisfies a second order autonomous ODE of the form

$$G'' = \Psi(G, G') \tag{4.1}$$

where $\Psi : \mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ is real analytic; this follows directly from the ODE (4.3) in [26], which is of the form

$$\widehat{g}'' = A(\widehat{g})(\widehat{g}')^2 + B(\widehat{g})|\widehat{g}'|^2, \tag{4.2}$$

where A, B are real analytic. Here, $\widehat{g} = \pi \circ G$, where $\pi : \mathbb{S}^2 \to \overline{\mathbb{C}}$ is a suitable stereographic projection with respect to an orthonormal basis of the Lie algebra of X. The special form of this ODE has the following trivial consequence:

(D) If $\widehat{g}(s)$ is a solution of (4.2), then $\widehat{g}(\delta_1 s + \delta_2)$, $\delta_1 \neq 0$, is also a solution of the same ODE. Consequently, the same property holds for a solution $G(s)$ of (4.1).

Lemma 4.3 *Let $f_i \colon \Sigma_i \looparrowright X$, $i = 1, 2$ denote two complete immersed H-surfaces in X whose Gauss map images $\gamma_i := G_i(\Sigma_i)$ are regular curves in $\mathbb{S}^2$. Assume that there exist points $p_i \in \Sigma_i$ with $f_1(p_1) = f_2(p_2)$ and $G_1(p_1) = G_2(p_2) =: v \in \mathbb{S}^2$, so that γ_1, γ_2 intersect tangentially at v. Then $f_1(\Sigma_1) = f_2(\Sigma_2)$.*

Proof By the previous comments, we can view f_1, f_2 locally around p_1, p_2 as conformal immersions $f_i(s, t)$ into X, and G_1, G_2 as regular parameterized curves $G_i(s) \colon J_i \to \mathbb{S}^2$, both of them satisfying the ODE (4.1) on an open interval $J_i \subset \mathbb{R}$. The conditions in the statement of the lemma imply that $G_1(s_1) = G_2(s_2)$ and $(G_1)'(s_1) = \pm (G_2)'(s_2)$ for some $s_i \in J_i$. But once here, property (D) above and the uniqueness of solution to the Cauchy problem for ODEs imply that $G_1(s) = G_2(\delta_1 s + \delta_2)$ for adequate constants $\delta_1 \neq 0$ and δ_2. In particular, the two Gauss maps coincide after conformal reparameterization of f_1 or f_2. Recall now that, by [26, Theorem 3.7], the Gauss map of a conformally parameterized H-surface in X determines the surface up to left translation in X. Since $f_1(p_1) = f_2(p_2)$, this left translation is trivial in our case, and so we conclude that $f_1(\Sigma_1) = f_2(\Sigma_2)$. □

4.2 Metric semidirect products

Given a real 2×2 matrix $A \in \mathcal{M}_2(\mathbb{R})$, the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the Lie group $(\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, *)$ endowed with the group operation $(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A} \mathbf{p}_2, z_1 + z_2)$; here $e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$ denotes the usual exponentiation of a matrix $B \in \mathcal{M}_2(\mathbb{R})$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}. \tag{4.3}$$

Then, a left invariant frame $\{E_1, E_2, E_3\}$ of X is given by

$$E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y,$$

$$E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z. \tag{4.4}$$

Observe that $\{E_1, E_2, E_3\}$ is the left invariant extension of the canonical basis $(\partial_x)_e, (\partial_y)_e, (\partial_z)_e$ of the tangent space $T_e X$ at the identity element $e = (0, 0, 0)$. The right invariant extensions on X of the same vectors of $T_e X$ define the frame F_1, F_2, F_3 where

$$F_1 = \partial_x, \quad F_2 = \partial_y,$$
$$F_3(x, y, z) = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z. \tag{4.5}$$

In terms of A, the Lie bracket relations are:

$$[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2.$$

Definition 4.4 We define the *canonical left invariant metric* on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ to be that one for which the left invariant frame $\{E_1, E_2, E_3\}$ given by (4.4) is orthonormal. Equivalently, it is the left invariant extension to $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ of the inner product on $T_e X$ that makes $(\partial_x)_e, (\partial_y)_e, (\partial_z)_e$ an orthonormal basis.

Some basic properties of the canonical left invariant metric $\langle, \rangle$ on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are the following ones.

(E1) The vector fields F_1, F_2, F_3 are Killing.
(E2) The mean curvature of each leaf of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ with respect to the unit normal vector field E_3 is the constant $H = \text{trace}(A)/2$. All the leaves of the foliation $\mathcal{F}$ are intrinsically flat, and have critical constant mean curvature. Moreover, the Gauss map image of each leaf of $\mathcal{F}$ with respect to the orientation given by E_3 (resp. $-E_3$) is the North (resp. South) pole of $\mathbb{S}^2$ with respect to the left invariant frame $\{E_1, E_2, E_3\}$.
(E3) The change (4.4) from the orthonormal basis $\{E_1, E_2, E_3\}$ to the basis $\{\partial_x, \partial_y, \partial_z\}$ produces the following expression of $\langle, \rangle$ in the x, y, z coordinates of X:

$$\begin{aligned}
\langle, \rangle = {}& \left[a_{11}(-z)^2 + a_{21}(-z)^2\right] dx^2 \\
& + \left[a_{12}(-z)^2 + a_{22}(-z)^2\right] dy^2 + dz^2 \\
& + \left[a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)\right] \\
& (dx \otimes dy + dy \otimes dx).
\end{aligned} \tag{4.6}$$

(E4) The π-rotation about any of the integral curves of $\partial_z = E_3$ (vertical lines in the (x, y, z)-coordinates) is an order-two orientation preserving isometry.

In what follows, we will use the standard notation $\mathbb{E}(\kappa, \tau)$ (as in [8]) for a simply connected homogeneous three-manifold X with an isometry group of dimension 4; see Sect. 3. As in our situation X is diffeomorphic to $\mathbb{R}^3$, then by classification $\kappa \leq 0$ and X is isometric to $\mathbb{H}^2(\kappa) \times \mathbb{R}$ (if $\tau = 0, \kappa < 0$), to Heisenberg space Nil_3 (if $\tau \neq 0, \kappa = 0$), or to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ with a rotationally symmetric left invariant metric if $\tau \neq 0, \kappa < 0$.

Lemma 4.5 *If A is a singular matrix, then $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is isometric to $\mathbb{R}^3$ or to an $\mathbb{E}(\kappa, \tau)$-space with $\kappa \leq 0$ and $\tau \in \mathbb{R}$.*

Proof If $A = 0$, then X is the standard $\mathbb{R}^3$. If $A \neq 0$ and trace$(A) = 0$, then [28, Theorem 2.15] gives that either $\det(A) \neq 0$ (which contradicts our hypothesis) or X is isometric to the Heisenberg space Nil_3 (recall that Nil_3 admits a 1-parameter family of homogeneous metrics, all of which are homothetic). Finally, if trace$(A) \neq 0$ then X is isometric to some $\mathbb{E}(\kappa, \tau)$ with $\kappa < 0$ and $\tau \in \mathbb{R}$ as explained in [28, Section 2.8]. □

Remark 4.6 Let us explain the relation between the basic geometric properties of $\mathbb{E}(\kappa, \tau)$ when viewed as a fibration $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$, and when viewed as a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, with A singular. When trace$(A) \neq 0$ (with A singular) we have, as explained in the proof of the above lemma, that $\kappa < 0$. In that situation, one can check that the unit left invariant vector field E_2 in (4.4) is actually equal to the Killing field $F_2 = \partial_y$ in (4.5). Thus, E_2 is actually the unit Killing vertical field associated to the canonical fibration of $\mathbb{E}(\kappa, \tau)$; see the proof of [28, Theorem 2.15]. With this, the family of horizontal planes $\mathbb{R}^2 \rtimes_A \{z\}$, $z \in \mathbb{R}$, in the semidirect product model corresponds to a family of parallel horocylinders of $\mathbb{E}(\kappa, \tau)$ in the fibration model.

In the case that trace$(A) = 0$ (with A singular, and non-zero), X is isometric to $\mathbb{E}(0, \tau) \equiv \mathrm{Nil}_3$, for some $\tau \neq 0$. This time, the vector field E_1 in (4.4) corresponds to the unit Killing field associated to the fibration model of X, and the horizontal planes $\mathbb{R}^2 \rtimes_A \{z\}$, $z \in \mathbb{R}$, correspond to a family of vertical, parallel minimal planes in the usual $\mathbb{E}(\kappa, \tau)$ model of Heisenberg space.

A simple consequence of Eqs. (4.5) and (4.6) is that every horizontal right invariant vector field (i.e., every linear combination of ∂_x, ∂_y with constant coefficients) is bounded in $\mathbb{R}^2 \rtimes_A [z_1, z_2]$, for all $z_1, z_2 \in \mathbb{R}$ with $z_1 \leq z_2$. The following lemma guarantees the existence of at least one nonzero horizontal right invariant vector field in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ which is bounded in a horizontal halfspace.

Lemma 4.7 *Let X be a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, where $A \in \mathcal{M}_2(\mathbb{R})$. Then, there exists a nonzero horizontal right invariant vector field which is bounded in $\mathbb{R}^2 \rtimes_A [0, \infty)$.*

Proof If $\text{trace}(A) \neq 0$, the lemma follows from [25, item 2 of Proposition 6.1]. If $\text{trace}(A) = 0$, then either $A = 0$ and $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is the standard Euclidean space, for which the lemma is obvious, or else $A \neq 0$ and [28, item 2 of Theorem 2.15] implies that there are three possible cases for A up to rescaling the metric. We will prove the lemma by studying each case separately.

- $A = \begin{pmatrix} 0 & -c \\ 1/c & 0 \end{pmatrix}$ for some $1 \le c < \infty$. This case corresponds to the group structure of $\widetilde{\text{E}}(2)$, the universal cover of the group of orientation-preserving rigid motions of the Euclidean plane. A direct computation using (4.5) and (4.6) gives $|F_1|^2 = \cos^2 z + c^{-2} \sin^2 z$ and $|F_2|^2 = \cos^2 z + c^2 \sin^2 z$, both bounded.

- $A = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$ for some $1 \le c < \infty$. In this case, X is isomorphic to Sol_3. A similar reasoning as in the preceding case gives that the basis of horizontal right invariant vector fields $\widehat{F}_1 = -c\partial_x + \partial_y$, $\widehat{F}_2 = c\partial_x + \partial_y$ satisfies $|\widehat{F}_1| = \sqrt{1 + c^2} e^z$, $|\widehat{F}_2| = \sqrt{1 + c^2} e^{-z}$.

- $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and X is isomorphic to the Heisenberg space Nil_3. Using again (4.5) and (4.6) we have that $|F_1| = 1$.

$\square$

4.3 Metric Lie groups isomorphic to $\widetilde{\text{SL}}(2, \mathbb{R})$

The Lie group $\widetilde{\text{SL}}(2, \mathbb{R})$ is the universal cover of the special linear group $\text{SL}(2, \mathbb{R}) = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det A = 1\}$, and of the projective special linear group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I_2\}$. The Lie algebra of any of the groups $\widetilde{\text{SL}}(2, \mathbb{R})$, $\text{SL}(2, \mathbb{R})$, and $\text{PSL}(2, \mathbb{R})$ is $\mathfrak{sl}(2, \mathbb{R}) = \{B \in \mathcal{M}_2(\mathbb{R}) \mid \text{trace}(B) = 0\}$. It is worth recalling that $\text{PSL}(2, \mathbb{R})$ has several isomorphic models, that we will use in the sequel:

(F1) The group of orientation-preserving isometries of the hyperbolic plane,

$$z \in \mathbb{H}^2 \equiv (\mathbb{R}^2)^+ \mapsto \frac{az + b}{cz + d} \in (\mathbb{R}^2)^+ \qquad (a, b, c, d \in \mathbb{R}, \ ad - bc = 1).$$

(F2) The group of conformal automorphisms of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$; these automorphisms are the Möbius transformations of the type $\phi(z) = \xi \frac{z+a}{1+\bar{a}z}$, with $a \in \mathbb{D}$ and $|\xi| = 1$. The universal cover map $\theta \mapsto \xi := e^{i\theta} : \mathbb{R} \to \mathbb{S}^1$ gives $\widetilde{\text{SL}}(2, \mathbb{R})$ a structure of an $\mathbb{R}$-bundle over $\mathbb{D}$.

(F3) The unit tangent bundle of $\mathbb{H}^2$. This interpretation of $\text{PSL}(2, \mathbb{R})$ as an $\mathbb{S}^1$-bundle over $\mathbb{H}^2$ (and of $\widetilde{\text{SL}}(2, \mathbb{R})$ as an $\mathbb{R}$-bundle over $\mathbb{H}^2$) produces a submersion from $\widetilde{\text{SL}}(2, \mathbb{R})$ to $\mathbb{H}^2$.

There are three types of 1-parameter subgroups of $\widetilde{SL}(2, \mathbb{R})$:

- *Elliptic subgroups.* Elements of any of these subgroups correspond to liftings to $\widetilde{SL}(2, \mathbb{R})$ of rotations around any fixed point in $\mathbb{H}^2$. These groups of rotations fix no points in the boundary at infinity $\partial_\infty \mathbb{H}^2$. Any two elliptic 1-parameter subgroups are conjugate.
- *Hyperbolic subgroups.* These subgroups correspond to liftings to $\widetilde{SL}(2, \mathbb{R})$ of translations along any fixed geodesic in $\mathbb{H}^2$. In the Poincaré disk model of $\mathbb{H}^2$, the hyperbolic subgroup associated to a geodesic Γ fixes the two points at infinity given by the end points of Γ. As in the elliptic case, any two 1-parameter hyperbolic subgroups are conjugate.
- *Parabolic subgroups.* They correspond to liftings to $\widetilde{SL}(2, \mathbb{R})$ of rotations about any fixed point $\theta \in \partial_\infty \mathbb{H}^2$. Any of these rotation subgroups only fixes the point θ at infinity, and leaves invariant the 1-parameter family of horocycles based at θ. As in the previous cases, 1-parameter parabolic subgroups are all conjugate by elliptic rotations of the Poincaré disk.

The *character* of a 1-parameter subgroup Γ of $\widetilde{SL}(2, \mathbb{R})$ refers to the property of Γ being elliptic, parabolic or hyperbolic. The character is invariant by conjugation, i.e., if Γ is a 1-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ and we define $\Gamma_a := a\Gamma a^{-1} = l_a(r_{a^{-1}}(\Gamma))$ for some $a \in \widetilde{SL}(2, \mathbb{R})$, then Γ_a has the same character as Γ.

Nonzero right invariant vector fields K are in one-to-one correspondence with 1-parameter subgroups Γ and with tangent vectors at the identity element e of $\widetilde{SL}(2, \mathbb{R})$ by the formula $K_e = \Gamma'(0)$. We will say that a nonzero right invariant vector field K on $\widetilde{SL}(2, \mathbb{R})$ is elliptic (resp. hyperbolic, parabolic) when the related 1-parameter subgroup Γ in $\widetilde{SL}(2, \mathbb{R})$ is elliptic (resp. hyperbolic, parabolic).

The matrices in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ given by

$$(E_1)_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (E_2)_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (E_3)_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.7)$$

define a left invariant frame $\{E_1, E_2, E_3\}$ on $\widetilde{SL}(2, \mathbb{R})$ with the property that $[E_1, E_2] = -2E_3$, $[E_2, E_3] = 2E_1$, $[E_3, E_1] = 2E_2$. The fields E_1, E_2, E_3 are respectively generated by the 1-parameter subgroups of $SL(2, \mathbb{R})$ given by

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R}, \quad (4.8)$$

where the first two are hyperbolic subgroups and the third one is an elliptic subgroup.

The center Z of $\widetilde{SL}(2, \mathbb{R})$ is an infinite cyclic subgroup (hence isomorphic to $\mathbb{Z}$) whose elements correspond to liftings to $\widetilde{SL}(2, \mathbb{R})$ of rotations by angles that are multiples of 2π; if $\Lambda\colon \widetilde{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ denotes the universal covering map, then $Z = \Lambda^{-1}(e)$ where e is the identity element of $\mathrm{PSL}(2, \mathbb{R})$. In this way, Z is contained in the integral curve $\Gamma_3 \subset \widetilde{SL}(2, \mathbb{R})$ of the left invariant vector field E_3 given by (4.8) (indeed, Z is contained in every elliptic subgroup of $\widetilde{SL}(2, \mathbb{R})$).

By declaring the left invariant frame $\{E_1, E_2, E_3\}$ to be orthonormal, we define a left invariant metric $\langle, \rangle$ on $\widetilde{SL}(2, \mathbb{R})$ such that the metric Lie group $(\widetilde{SL}(2, \mathbb{R}), \langle, \rangle)$ is isometric to the $\mathbb{E}(\kappa, \tau)$-space with base curvature $\kappa = -4$ and bundle curvature $\tau^2 = 1$. Let

$$\Pi_0\colon \mathbb{E}(-4, 1) \to \mathbb{H}^2(-4) \tag{4.9}$$

be the associated Riemannian submersion onto the hyperbolic plane endowed with the metric of constant curvature -4 (this submersion can be naturally identified with any of the submersions described in properties (F2) and (F3) above). Let $\phi\colon (\widetilde{SL}(2, \mathbb{R}), \langle, \rangle) \to \mathbb{E}(-4, 1)$ be a Riemannian isometry. Then, the image by ϕ of every integral curve of E_3 is a Π_0-vertical geodesic. The composition

$$\Pi = \Pi_0 \circ \phi\colon \widetilde{SL}(2, \mathbb{R}) \to \mathbb{H}^2(-4) \tag{4.10}$$

defines a submersion which is Riemannian with respect to the particular left invariant metric $\langle, \rangle$ defined above.

The next result follows from [28, Section 2.7] and describes the geometry of each 1-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ in terms of the coordinates of its velocity vector at the identity element e with respect to the basis $\{E_1, E_2, E_3\}$.

Lemma 4.8 *Consider a vector $\Gamma'(0) \in T_e\widetilde{SL}(2, \mathbb{R})$ with coordinates (a, b, c) with respect to the basis given in (4.7), and let Γ denote the 1-parameter subgroup generated by $\Gamma'(0)$. Then:*

1. *Γ is elliptic if and only if $a^2 + b^2 < c^2$. If $a = b = 0$, then Γ is the lift to $\widetilde{SL}(2, \mathbb{R})$ of the elliptic subgroup of rotations of $\mathbb{H}^2(-4)$ around the point $\Pi(e)$. If $0 < a^2 + b^2 < c^2$, then $\Pi(\Gamma)$ is a constant geodesic curvature circle passing through $\Pi(e)$ and completely contained in $\mathbb{H}^2(-4)$.*
2. *Γ is hyperbolic if and only if $a^2 + b^2 > c^2$. In this case, $\Pi(\Gamma)$ is a constant geodesic curvature arc passing through $\Pi(e)$ with two end points in the boundary at infinity of $\mathbb{H}^2(-4)$.*
3. *Γ is parabolic if and only if $a^2 + b^2 = c^2$. In this case, then $\Pi(\Gamma)$ is a horocycle in $\mathbb{H}^2(-4)$.*

Every left invariant metric in $\widetilde{SL}(2, \mathbb{R})$ can be obtained by choosing numbers $\lambda_1, \lambda_2, \lambda_3 > 0$ and declaring the length of the left invariant vector field E_i to

be λ_i, 1, 2, 3, while keeping them orthogonal; see [27, Proposition 4.2]. For the remainder of this section, we will suppose that the metric Lie group X is isometric and isomorphic to $\widetilde{SL}(2, \mathbb{R})$ endowed with the left invariant metric given by an (arbitrary but fixed) choice of $\lambda_i > 0$, $i = 1, 2, 3$.

Lemma 4.9 *Let $\gamma \subset \mathbb{H}^2(-4)$ be a geodesic. Denote by $\Sigma(\gamma) = \Pi^{-1}(\gamma)$ the surface in $\widetilde{SL}(2, \mathbb{R})$ obtained by pulling back γ through Π. Then:*

(1) If $\Gamma \subset \widetilde{SL}(2, \mathbb{R})$ is an integral curve of E_3, then $\phi(\Gamma)$ is an integral curve of the vertical vector field on $\mathbb{E}(-4, 1)$ that generates the kernel of the differential of Π_0. Furthermore, if $R^\Gamma \colon \mathbb{E}(-4, 1) \to \mathbb{E}(-4, 1)$ denotes the rotation of angle π around $\phi(\Gamma)$ (i.e., the lifting through Π_0 of the rotation $\psi_{\Pi(\Gamma)}$ by angle π around the point $\Pi(\Gamma) \in \mathbb{H}^2(-4)$), then the map $R_\Gamma = \phi^{-1} \circ R^\Gamma \circ \phi$ is an order-two, orientation-preserving isometry of X that fixes Γ.

(2) $\Sigma(\gamma)$ is ruled by geodesics of X which are integral curves of E_3, and each integral curve Γ of E_3 contained in $\Sigma(\gamma)$ is the fixed point set of the isometry $R_\Gamma \colon X \to X$, which restricts to an order-two, orientation-reversing isometry on $\Sigma(\gamma)$.

(3) $\Sigma(\gamma)$ is a stable minimal surface in X.

(4) Given a curve $\gamma_r \subset \mathbb{H}^2(-4)$ at constant distance $r > 0$ from γ, the surface $\Sigma(\gamma_r) = \Pi^{-1}(\gamma_r)$ has mean curvature vector of length bounded away from zero (not necessarily constant[2]) and pointing towards $\Sigma(\gamma)$.

(5) The critical mean curvature of X (see Definition 3.5) satisfies $H(X) \geq \delta$, where δ is any positive lower bound for the mean curvature function of $\Sigma(\gamma_r)$ for any given $r > 0$.

Proof Item 1 is well-known and follows from the fact that E_3 is a principal Ricci curvature direction at every point in X; see [28, Proposition 2.21] for details.

To prove item 2, consider the integral curve $\Gamma \subset \widetilde{SL}(2, \mathbb{R})$ of E_3 that passes through a given point $p \in \Sigma(\gamma)$. As $\Pi(\Gamma) = \Pi(p) \in \gamma$, then Γ is entirely contained in $\Sigma(\gamma)$. Since $R^\Gamma \colon \mathbb{E}(-4, 1) \to \mathbb{E}(-4, 1)$ projects through Π_0 to the rotation $\psi_{\Pi(\Gamma)} \colon \mathbb{H}^2(-4) \to \mathbb{H}^2(-4)$ of angle π around $\Pi(\Gamma) \in \gamma$, and $\psi_{\Pi(\Gamma)}$ leaves invariant the geodesic γ, then $R_\Gamma(\Sigma(\gamma)) = \Sigma(\gamma)$. This proves that item 2 holds.

Minimality in item 3 follows since the mean curvature vector of $\Sigma(\gamma)$ maps to its negative under the differential of R_Γ. $\Sigma(\gamma)$ is stable because it admits a positive Jacobi function (namely, $\langle K, N \rangle$ where K is the right invariant vector field on $\widetilde{SL}(2, \mathbb{R})$ generated by the hyperbolic translations along a geodesic orthogonal to γ in $\mathbb{H}^2(-4)$, and N is a unit normal field to $\Sigma(\gamma)$).

---

[2] The mean curvature of $\Pi_0^{-1}(\gamma_r) \subset \mathbb{E}(-4, 1)$ is constant equal to half of the geodesic curvature of γ_r in $\mathbb{H}^2(-4)$.

To prove item 4, take a curve γ_r at distance $r > 0$ from a geodesic $\gamma \subset \mathbb{H}^2(-4)$. Given a point $q \in \gamma_r$, there exists a geodesic $\widehat{\gamma} \subset \mathbb{H}^2(-4)$ passing through q which is tangent to γ_r at q, and $\widehat{\gamma}$ lies entirely on the side of γ_r that does not contain γ. By item 3, the surface $\Sigma(\widehat{\gamma}) = \Pi^{-1}(\widehat{\gamma})$ is minimal. Since $\Sigma(\widehat{\gamma})$ is tangent to $\Sigma(\gamma_r)$ and lies at one side of $\Sigma(\gamma_r)$, then the usual mean curvature comparison principle implies that the mean curvature of $\Sigma(\gamma_r)$ with respect to the unit normal vector field that points towards $\Sigma(\gamma)$, is nonnegative along $\Gamma = \Pi^{-1}(\{q\})$. In fact, the mean curvature of $\Sigma(\gamma_r)$ is strictly positive along Γ as we explain next: otherwise the second fundamental forms II of $\Sigma(\gamma_r)$ and $\widehat{II}$ of $\Sigma(\widehat{\gamma})$ would have the same trace and the same value in the direction given by the tangent vector Γ' to Γ; choosing Γ' as the first vector of an orthonormal basis $\mathcal{B}$ of the common tangent plane of both surfaces at any point of Γ, the matrix of the difference $II - \widehat{II}$ with respect to $\mathcal{B}$ is symmetric with zero diagonal entries; therefore, $II - \widehat{II}$ has opposite eigenvalues $a, -a$. If $a \neq 0$, then we contradict that $\Sigma(\widehat{\gamma})$ lies at one side of $\Sigma(\gamma_r)$; hence $II = \widehat{II}$ along Γ, which in turn implies that γ_r would have a point of second order contact with $\widehat{\gamma}$, which is false. Therefore, the mean curvature function of $\Sigma(\gamma_r)$ is strictly positive.

To see that the mean curvature function of $\Sigma(\gamma_r)$ is bounded away from zero, consider the 1-parameter subgroup G of hyperbolic translations of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ associated to the translations of $\mathbb{H}^2(-4)$ along γ. Then, $\Sigma(\gamma_r)$ is invariant under every element in G. Also, $\Sigma(\gamma_r)$ is invariant under left translation by every element in the center of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ (recall that elements in this center correspond to liftings to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of rotations by multiples of 2π around a point in $\mathbb{H}^2(-4)$). Hence $\Sigma(\gamma_r)$ can be considered to be a doubly periodic surface. This property, together with the positivity of the mean curvature function of $\Sigma(\gamma_r)$, imply that item 4 holds.

Finally, item 5 is a direct consequence of item 4 and of the mean curvature comparison principle by the following argument. Let $\Lambda \subset X$ be any closed surface and take $r > 0$. Since the mean convex side of $\Sigma(\gamma_r)$ contains arbitrarily large balls of X in its interior, then after a left translation of Λ, we can assume that Λ lies on the mean convex side of $\Sigma(\gamma_r)$. After further continuous left translations inside the mean convex side of $\Sigma(\gamma_r)$ applied to Λ, we can assume that Λ is also tangent to $\Sigma(\gamma_r)$ at some common point p. By the mean curvature comparison principle, the absolute mean curvature function of Λ at p is at least as large as any lower positive bound δ of the positive mean curvature function of $\Sigma(\gamma_r)$. Hence, $H(X) \geq \delta$, which completes the proof. $\square$

Lemma 4.10 *The push-forward by any left translation in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of a right invariant elliptic (resp. hyperbolic, parabolic) vector field is again right invariant elliptic (resp. hyperbolic, parabolic) vector field. Furthermore, given a nonzero elliptic or parabolic vector field F on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the inner product of*

F and the left invariant vector field E_3 given by (4.7)*, has constant nonzero sign on* $\widetilde{SL}(2, \mathbb{R})$*, independently of the left invariant metric on* $\widetilde{SL}(2, \mathbb{R})$*.*

Proof The fact that the push-forward by a left translation of a right invariant vector field is again right invariant holds in any Lie group, and is well-known.

Consider now $F_a := (l_a)_*(F)$, where F is a right invariant vector field and $a \in \widetilde{SL}(2, \mathbb{R})$. If Γ denotes the 1-parameter subgroup with $\Gamma'(0) = F(e)$, then the corresponding 1-parameter subgroup Γ_a that generates the right invariant vector field F_a is given by $\Gamma_a = a\Gamma a^{-1} = l_a(r_{a^{-1}}(\Gamma))$. In particular, Γ_a and Γ are conjugate subgroups, and so they have the same character. Thus, F_a and F also have the same character, as claimed.

Finally, observe that from the paragraph just before the statement of Lemma 4.9 we deduce that the sign of the inner product of a pair of tangent vectors $v, w \in T_x\widetilde{SL}(2, \mathbb{R})$ at any $x \in X$ does not depend on the left invariant metric on $\widetilde{SL}(2, \mathbb{R})$. Therefore, in order to prove the last sentence of the lemma, it suffices to choose a nonzero elliptic or parabolic vector field F and prove that the function $\langle F, E_3 \rangle \colon \widetilde{SL}(2, \mathbb{R}) \to \mathbb{R}$ has no zeros, where $\langle, \rangle$ is the metric defined just before (4.9).

Given $x \in \widetilde{SL}(2, \mathbb{R})$, let F^x be the vector field on $\widetilde{SL}(2, \mathbb{R})$ defined by $(l_x)_*(F^x) = F$. By the first sentence of Lemma 4.10, F^x is a nonzero right invariant vector field of the same character as F (i.e., elliptic or parabolic), and

$$\langle F, E_3 \rangle(x) = \langle (dl_x)_e(F_e^x), (E_3)_x \rangle = \langle F_e^x, (E_3)_e \rangle = \langle F^x, E_3 \rangle(e),$$

where we have used that E_3 is left invariant and that l_x is an isometry of $\langle, \rangle$. As F^x is elliptic or parabolic, by Lemma 4.8, the coordinates (a, b, c) of F_e^x satisfy $a^2 + b^2 \leq c^2$, which implies that $\langle F, E_3 \rangle(x)$ cannot vanish, as desired. Now the proof of Lemma 4.10 is complete. □

We will finish this section with some considerations about two-dimensional subgroups of $\widetilde{SL}(2, \mathbb{R})$ and their constant left invariant Gauss map images.

Given $\theta \in \partial_\infty \mathbb{H}^2$, let $\mathbb{H}_\theta^2 \subset \widetilde{SL}(2, \mathbb{R})$ be the set of liftings to $\widetilde{SL}(2, \mathbb{R})$ of the orientation-preserving isometries of $\mathbb{H}^2$ that fix θ. Elements in $\mathbb{H}_\theta^2$ are of two types: liftings to $\widetilde{SL}(2, \mathbb{R})$ of hyperbolic translations along geodesics one of whose end points is θ, and liftings to $\widetilde{SL}(2, \mathbb{R})$ of parabolic rotations around θ. $\mathbb{H}_\theta^2$ is a noncommutative, two-dimensional simply connected subgroup of $\widetilde{SL}(2, \mathbb{R})$, and $\mathbb{H}_\theta^2$ contains a unique 1-parameter parabolic subgroup, namely the liftings to $\widetilde{SL}(2, \mathbb{R})$ of parabolic rotations around θ. Every two-dimensional subgroup of $\widetilde{SL}(2, \mathbb{R})$ is of the form $\mathbb{H}_\theta^2$ for some $\theta \in \partial_\infty \mathbb{H}^2$, see the paragraph around equation (2.30) in [28] for more details.

Given a hyperbolic 1-parameter subgroup Γ of $\widetilde{SL}(2, \mathbb{R})$, by Lemma 4.8, the image of Γ through the map Π defined in (4.10) is an arc in $\mathbb{H}^2(-4)$ of
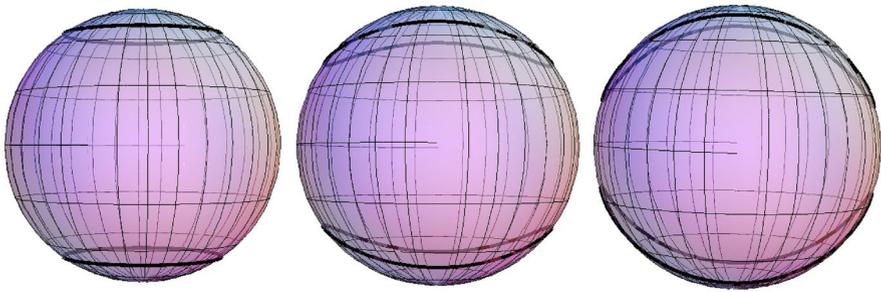
Fig. 1 Gauss map image of the family of two-dimensional subgroups $\mathbb{H}^2_\theta$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, with three different left invariant metrics. Left: $\lambda_1 = \lambda_2 = 2, \lambda_3 = 1$ ($\mathbb{E}(\kappa, \tau)$-metric). Center: $\lambda_1 = 0.5, \lambda_2 = 2, \lambda_3 = 1$. Right: $\lambda_1 = 0.1, \lambda_2 = 4, \lambda_3 = 1$

constant geodesic curvature that has two extrema $\theta_1, \theta_2 \in \partial_\infty \mathbb{H}^2$. These two end-point values θ_1, θ_2 determine the two two-dimensional subgroups $\mathbb{H}^2_{\theta_1}$, $\mathbb{H}^2_{\theta_2}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ that contain Γ.

The following result describes the image under the left invariant Gauss map of the circle family $\{\mathbb{H}^2_\theta \mid \theta \in \partial_\infty \mathbb{H}^2\}$ (recall that the left invariant Gauss map depends on the left invariant metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, and thus, it depends on the numbers $\lambda_1, \lambda_2, \lambda_3 > 0$ chosen just before Lemma 4.9).

Lemma 4.11 *The constant value $G_\theta \in \mathbb{S}^2$ of the left invariant Gauss map of $\mathbb{H}^2_\theta$ is (up to a sign coming from a change of orientation):*

$$
G_\theta = \frac{\pm 1}{\sqrt{\lambda_2\lambda_3 \sin^2 \theta + \lambda_1\lambda_3 \cos^2 \theta + \lambda_1\lambda_2}} \left(-\sqrt{\lambda_2\lambda_3} \sin \theta, \sqrt{\lambda_1\lambda_3} \cos \theta, -\sqrt{\lambda_1\lambda_2} \right), \tag{4.11}
$$

in coordinates with respect to the orthonormal frame $\{\lambda_i^{-1/2}(E_i)_e \mid i = 1, 2, 3\}$ (see Eq. (4.7)). The set $\{\pm G_\theta \mid \theta \in \partial_\infty \mathbb{H}^2\}$ is a pair of antipodal, simple closed curves $\Upsilon \cup (-\Upsilon) \subset \mathbb{S}^2$, and it is invariant under the π-rotations in $\mathbb{S}^2$ in the directions of $(E_1)_e, (E_2)_e, (E_3)_e$, see Fig. 1.

Proof Let $v \in T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ be a tangent vector with coordinates (a, b, c) with respect to $\{(E_1)_e, (E_2)_e, (E_3)_e\}$. By Lemma 4.8, the tangent vector $v^P \in T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with coordinates $(a, b, c) = (-\sin \theta, \cos \theta, 1)$ (resp. $v^H \in T_e\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with $(a, b, c) = (\cos \theta, \sin \theta, 0)$) produces a parabolic (resp. hyperbolic) 1-parameter subgroup Γ^P (resp. Γ^H) of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Furthermore, the projection of Γ^P (resp. of Γ^H) through the map Π defined in (4.10) is the horocycle that passes through the origin of the Poincaré disk and through the point at infinity $e^{i\theta} \in \partial_\infty \mathbb{H}^2$ (resp. the segment with extrema $\pm e^{i\theta}$). It

follows that both subgroups Γ^P, Γ^H are contained in the two-dimensional subgroup $\mathbb{H}_\theta^2$ of $\widetilde{SL}(2, \mathbb{R})$.

The coordinates of v^P, v^H with respect to the frame $\{\lambda_i^{-1/2}(E_i)_e \mid i = 1, 2, 3\}$ are respectively $(-\sqrt{\lambda_1} \sin \theta, \sqrt{\lambda_2} \cos \theta, \sqrt{\lambda_3})$ and $(\sqrt{\lambda_1} \cos \theta, \sqrt{\lambda_2} \sin \theta, 0)$. After normalizing the cross product of these 3-tuples, we deduce that the (constant) value of the left invariant Gauss map of $\mathbb{H}_\theta^2$ in the left invariant metric on $\widetilde{SL}(2, \mathbb{R})$ determined by $\lambda_1, \lambda_2, \lambda_3 > 0$ is the one given by Eq. (4.11). The remaining conclusions of the lemma are direct consequences of (4.11). □

5 The geometry of invariant limit surfaces of index-one spheres

In what follows, we will denote by $\mathcal{M}_X^1$ the moduli space of all index-one constant mean curvature spheres in a metric Lie group X diffeomorphic to $\mathbb{R}^3$. Inside $\mathcal{M}_X^1$ we have the component $\mathcal{C}$ described in item 4 of Proposition 3.4. We will also use the number $h_0(X) \geq 0$ defined in Proposition 3.4.

Definition 5.1 We say that a complete, noncompact, connected H-surface $f: \Sigma \looparrowright X$ is a *limit surface* of $\mathcal{C}$ with base point $p \in \Sigma$ (also called a *pointed limit immersion* and denoted by $f: (\Sigma, p) \looparrowright (X, e)$) if $e = f(p)$ and there exists a sequence $\{\widehat{f}_n: S_n \looparrowright X\}_n \subset \mathcal{C}$, compact domains $\Omega_n \subset S_n$ and points $p_n \in \Omega_n$ such that the following two conditions hold:

(G1) f is a limit as $n \to \infty$ of the immersions $f_n = l_{\widehat{f}_n(p_n)^{-1}} \circ (\widehat{f}_n|_{\Omega_n}): \Omega_n \looparrowright X$ obtained by left translating $\widehat{f}_n|_{\Omega_n}$ by the inverse of $\widehat{f}_n(p_n)$ in X (hence $f_n(p_n) = e$). Here, the convergence is the uniform convergence in the C^k-topology for every $k \geq 1$, when we view the surfaces as local graphs in the normal bundle of the limit immersion.

(G2) The area of f_n is greater than n, for all $n \in \mathbb{N}$.

Remark 5.2 (1) By item 6 of Proposition 3.4, any such pointed limit immersion has constant mean curvature equal to $h_0(X)$. Moreover, by [26, Lemma 5.2], the space of pointed limit immersions of $\mathcal{C}$ is nonempty.

(2) If, with the notation in the last definition, q_n is a point in Ω_n for each $n \in \mathbb{N}$ and the distance in S_n from q_n to p_n is bounded independently of n, then for each n the pointed immersions $l_{\widehat{f}_n(p_n)^{-1}} \circ (\widehat{f}_n|_{\Omega_n}): (\Omega_n, p_n) \looparrowright (X, e)$ and $l_{\widehat{f}_n(q_n)^{-1}} \circ (\widehat{f}_n|_{\Omega_n}): (\Omega_n, q_n) \looparrowright (X, e)$ differ only in the left translation in X by the element $x_n := \widehat{f}_n(q_n)^{-1} \widehat{f}_n(p_n)$. Observe that the distance in X from x_n to e equals the distance on X from $\widehat{f}_n(p_n)$ to $\widehat{f}_n(q_n)$, which is at most $d_{S_n}(p_n, q_n)$. In particular, if $d_{S_n}(p_n, q_n) \to 0$ as $n \to \infty$, we deduce that the $l_{\widehat{f}_n(q_n)^{-1}} \circ (\widehat{f}_n|_{\Omega_n}): (\Omega_n, q_n) \looparrowright (X, e)$ converge (after passing to a subsequence) to the same pointed limit immersion as the $l_{\widehat{f}_n(p_n)^{-1}} \circ (\widehat{f}_n|_{\Omega_n}): (\Omega_n, p_n) \looparrowright (X, e)$.

Frequently in this paper, we will apply the Transversality Lemma below. This is a basic property of immersed spheres in metric Lie groups proved by the authors in [24, Lemma 3.1], and that we formulate next adapted to our context.

Let Λ be a two-dimensional subgroup of X. Then the set $\mathcal{F} = \{l_a(\Lambda) = a\Lambda \mid a \in X\}$ of left cosets of Λ is a foliation of X. The quotient space X/Λ is diffeomorphic to $\mathbb{R}$. Moreover, every element of $\mathcal{F}$ has the same constant Gauss map, say $g_0 \in \mathbb{S}^2$, by property (C). With these considerations in mind, we have:

Lemma 5.3 (Transversality Lemma, [24]). *Let $f : S \hookrightarrow X$ be an immersed sphere in X whose left invariant Gauss map is a diffeomorphism. Then, with the notation above, $f(S)$ lies in the closed topological slab $[L_a, L_b]$ of X determined by two leafs L_a, L_b of $\mathcal{F}$, so that:*

1. *Each of the two leaves L_a, L_b intersects $f(S)$ at a single point. At these two points, the Gauss map of f is equal to $\pm g_0$.*
2. *Each leaf L of $\mathcal{F}$ in the interior of $[L_a, L_b]$ intersects $f(S)$ transversely along a closed, connected, immersed curve. Moreover, $\alpha := f^{-1}(L)$ is a closed, embedded curve in S.*

Proposition 5.4 *Let $f : (\Sigma, p) \hookrightarrow (X, e)$ be a pointed limit immersion of $\mathcal{C}$, and let $G : \Sigma \to \mathbb{S}^2 \subset T_e X$ be its left invariant Gauss map. Then, the rank of the differential dG is a constant $k \in \{0, 1, 2\}$ on Σ. Furthermore:*

(1) *If $\mathrm{rank}(dG) = 0$, then f is injective and $f(\Sigma)$ is a two-dimensional subgroup of X.*
(2) *If $\mathrm{rank}(dG) = 1$, then there exists a unique (up to scaling) right invariant vector field on X which is everywhere tangent to $f(\Sigma)$.*

Proof Suppose first that dG has rank zero at some point $q \in \Sigma$. By [28, equation (3.9) and Corollary 3.21], there exists a two-dimensional subgroup Λ of X which has the same mean curvature as f and (constant) left invariant Gauss map of value $G(q) \in \mathbb{S}^2$. We are going to prove next that $f(\Sigma) = l_{f(q)}(\Lambda) = f(q)\Lambda$.

Consider the foliation $\mathcal{F} = \{l_a(\Lambda) = a\Lambda \mid a \in X\}$ of left cosets of Λ. Then, we may write the leaves of $\mathcal{F}$ as the level sets of a Morse function without critical points $h : X \to \mathbb{R}$, so that $\{h = 0\} = f(q)\Lambda$. Also, we can consider local coordinates (x_1, x_2, x_3) in X around $f(q)$ with $x_3 := h$, so that $f(q) = (0, 0, 0)$ in these coordinates. By taking a small neighborhood V of $f(q)$ in $f(\Sigma)$, we may view V as a graph $x_3 = \varphi(x_1, x_2)$ over a small neighborhood U of the origin in the (x_1, x_2)-plane.

Arguing by contradiction, assume that $f(\Sigma)$ and $f(q)\Lambda$ do not coincide in a neighborhood of $f(q)$. Then, φ is not identically zero around the origin. Since the graphs $x_3 = \varphi(x_1, x_2)$ and $x_3 = 0$ have the same constant mean

curvature in X (and thus define solutions to the same quasilinear elliptic PDE), then $\varphi - 0 = \varphi$ satisfies a second order linear homogeneous elliptic equation with smooth coefficients. So, by Bers' theorem [4], we have $\varphi(x_1, x_2) = p_k(x_1, x_2) + o(\sqrt{x_1^2 + x_2^2})^k$, where $p_k(x_1, x_2)$ is a homogeneous polynomial of the form $p_k = h_k \circ \Phi$, with h_k a harmonic homogeneous polynomial of degree $k \geq 2$ in $\mathbb{R}^2$, and Φ a linear transformation of the (x_1, x_2)-coordinates. In particular, the vector field $D\varphi := (\varphi_{x_1}, \varphi_{x_2})$ on U has at the origin an isolated zero of negative index.

As f is a pointed limit immersion of $\mathcal{C}$, there exist a sequence $\{\widehat{f}_n \colon S_n \looparrowright X\}_n \subset \mathcal{C}$, compact domains $\Omega_n \subset S_n$ and points $p_n \in \Omega_n$ such that the conditions (G1) and (G2) hold. For n large, let $V(n) \subset \widehat{f}_n(\Omega_n)$ be subdomains that are expressed as small normal graphs over V and converge to V as $n \to \infty$. Thus, the $V(n)$ can be seen as graphs $x_3 = \varphi_n(x_1, x_2)$ over U, making U smaller if necessary. Since the φ_n converge to φ, we have that for n large enough, the winding number of each $D\varphi_n := ((\varphi_n)_{x_1}, (\varphi_n)_{x_2})$ along a sufficiently small circle around the origin is negative. Thus, by degree theory, for n large enough we have $D\varphi_n = (0, 0)$ at some point $z_n \in U$, what means that the graph $x_3 = \varphi_n(x_1, x_2)$ intersects tangentially some leaf $x_3 = a_n$ of $\mathcal{F}$ at (z_n, a_n). Note that, by the Transversality Lemma, the sphere $\widehat{f}_n(S_n)$ is tangent to the foliation $\mathcal{F}$ at exactly two points; in particular, φ_n has then a local extremum at z_n and no other critical points in U except for, possibly, another local extremum. This contradicts the fact that the winding number of $D\varphi_n$ is negative around the origin, by Poincaré-Hopf. Thus, $f(\Sigma) = f(q)\Lambda$, and dG has rank zero everywhere.

Suppose now that the rank of dG is constant one on Σ. By item 1 of Proposition 4.1, $f(\Sigma)$ is invariant under the flow of a nonzero right invariant vector field K on X. If $\widetilde{K}$ is a right invariant vector field on X linearly independent from K, and $\widetilde{K}$ is everywhere tangent to $f(\Sigma)$, then the Lie bracket $[K, \widetilde{K}]$ is also everywhere tangent to $f(\Sigma)$. This implies that $K, \widetilde{K}$ generate an integrable two-dimensional subalgebra of the algebra of right invariant vector fields on X, whose integral submanifold passing through e is a two-dimensional subgroup Λ of X. By uniqueness of integral submanifolds of an integrable distribution, we have $f(\Sigma) = \Lambda$, which contradicts that the rank of dG is constant one on Σ. Hence item 2 of the proposition is proved.

We next prove that if dG has rank one at a some point $q \in \Sigma$, then the rank of dG is one everywhere on Σ, which together with the previous paragraphs implies the constancy of the rank of dG in any case and finishes the proof of Proposition 5.4. Suppose then that $\text{rank}(dG_q) = 1$ at some $q \in \Sigma$. By the argument in Step 1 of the proof of [26, Theorem 4.1], there is a nonzero right invariant vector field V on X that has a contact at $f(q)$ with $f(\Sigma)$ of order at least two. Suppose for the moment that V is everywhere tangent to $f(\Sigma)$. In this case, the differential of G has rank at most one at every point of Σ by

item 1 of Proposition 4.1. But by the previous arguments of this proof, the rank of dG cannot be zero at any point of Σ, and so, dG has constant rank one.

To finish the proof, we assume that V is not everywhere tangent to $f(\Sigma)$ and we will obtain a contradiction. This contradiction will follow from the existence of two disjoint compact domains in the index-one H_n-spheres $\widehat{f}_n\colon S_n \looparrowright X$ that give rise to the limit immersion f (in the sense of conditions (G1) and (G2) above), such that each of these compact domains is unstable. To create these compact subdomains, we proceed as follows. Consider the Jacobi function $J = \langle V, N \rangle$ on Σ, where N stands for the unit normal vector field along f. By [5, Theorem 2.5], in a small compact neighborhood E_q of q in Σ, the set $J^{-1}(0)$ has the appearance of a set of k embedded arcs $\alpha_1, \ldots, \alpha_k$ crossing at equal angles at q with very small geodesic curvatures (their geodesic curvatures all vanish at the common point q), where $k \geq 2$ is the degree of vanishing of J at q; see Fig. 2. Let $E(n) \subset \Omega_n$ be compact disks that converge to E_q as $n \to \infty$, where $\Omega_n \subset S_n$ is defined in condition (G1) above. As S_n has index one, any Jacobi function on S_n is a second eigenfunction of its stability operator. Hence, it follows by Courant's nodal domain theorem that any Jacobi function on S_n has at most two nodal domains; see e.g. [10] for a similar argument. Since S_n is simply connected, we conclude then that the zero set of the Jacobi function $J_n = \langle V, N_n \rangle$ (here N_n is the unit normal vector of $\widehat{f}_n$) is a regular, connected, analytic Jordan curve that decomposes S_n into two nodal domains $D_1(n)$, $D_2(n)$, each of them diffeomorphic to a disk, and the zero sets $J_n^{-1}(0) \cap \Omega_n$ converge as $n \to \infty$ to the zero set $J^{-1}(0)$ of J. Then one of the two nodal domains, say $D_1(n)$, intersects $E(n)$ in a connected set, see Fig. 2.

Let $U_1, U_2 \subset \Sigma$ be two small compact disjoint geodesic disks of radius $\varepsilon_0 > 0$ centered at points $q_1 \in \alpha_1, q_2 \in \alpha_2$ such that $U_1 \cup U_2 \subset E_q$ and $(U_1 \cup U_2) \cap \alpha_j = \emptyset$ for all $j \neq 1, 2$. Let $q_1(n), q_2(n) \in J_n^{-1}(0) \cap E(n)$ be points in the zero set of J_n that converge to q_1, q_2, respectively, and let $U_1(n), U_2(n)$ be ε_0-disks in $E(n)$ centered at the points $q_1(n), q_2(n)$ that converge to U_1, U_2 as $n \to \infty$. Choose a compact embedded short arc $\beta_n \subset D_1(n) - [U_1(n) \cup U_2(n)]$ such that

(H1) β_n joins points in different components of $E(n) \cap \partial D_1(n)$, and β_n separates $U_1(n) \cap D_1(n)$ from $U_2(n) \cap D_1(n)$ in $D_1(n)$.

(H2) $\{x \in S_n \mid d_{S_n}(x, \beta_n) \leq \frac{1}{n}\}_n$ converges to $\{q\}$ as $n \to \infty$, the length of β_n is less than $1/n$ for each n and $\mathrm{dist}_{S_n}(\beta_n, U_1(n) \cup U_2(n)) > \delta$ for some small $\delta > 0$ independent of n, where dist_{S_n} denotes intrinsic distance in S_n.

Note that β_n decomposes $D_1(n)$ into two disk components $D_1(n)^+$, $D_1(n)^-$, each one being disjoint from exactly one of the disks $U_1(n), U_2(n)$. Without loss of generality, we can assume that $U_1(n) \cap D_1(n)^- = \emptyset$ (and thus, $U_2(n) \cap$
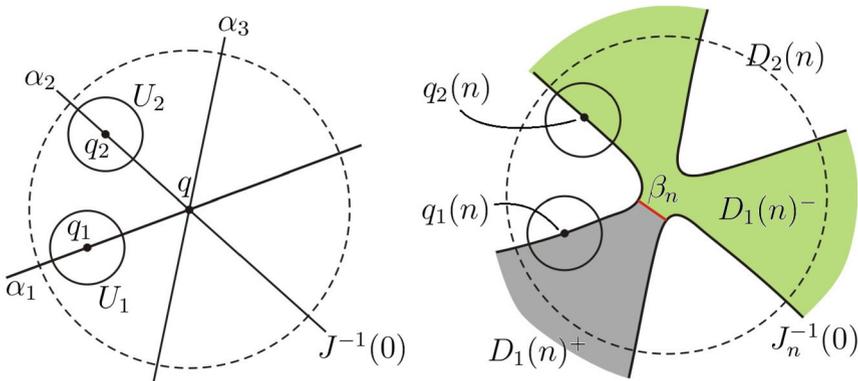
Fig. 2 Left: In a small neighborhood E_q (represented as a dotted disk) of the point $q \in \Sigma$, the nodal lines of the Jacobi function J form an equiangular system of (almost geodesic) arcs crossing at q. Right: The nodal lines of the related Jacobi functions J_n in a (dotted) disk $E(n) \subset S_n$ converging to E_q. $J_n^{-1}(0)$ divides S_n into two disk regions $D_1(n)$, $D_2(n)$, and the short arc β_n divides $D_1(n)$ in $D_1(n)^+$, $D_1(n)^-$. The two small disks centered at $q_1(n)$, $q_2(n)$ are $U_1(n)$, $U_2(n)$

$D_1(n)^+ = \emptyset$), see Fig. 2. Let $\phi_n \colon S_n \to [0, 1]$ be a piecewise smooth cut-off function with the following properties:

(I1) $\phi_n(x) = 0$ if $\mathrm{dist}_{S_n}(x, \beta_n) \leq \frac{1}{3n}$.
(I2) $\phi_n(x) = 1$ if $\mathrm{dist}_{S_n}(x, \beta_n) \geq \frac{1}{2n}$.
(I3) $|\nabla \phi_n| \leq nC$ in S_n, where $C > 0$ is a universal constant.

Note that $\phi_n J_n$ is a piecewise smooth function that vanishes along $\partial D_1(n)^+$.

We claim that

$$\int_{D_1(n)^+} \left(|\nabla(\phi_n J_n)|^2 - P_n \phi_n^2 J_n^2 \right) \to 0 \quad \text{as } n \to \infty, \tag{5.1}$$

where $\mathcal{L}_n = \Delta + P_n$ is the stability operator of S_n, i.e., $P_n = |\sigma_n|^2 + \mathrm{Ric}(N_n)$, $|\sigma_n|$ is the norm of the second fundamental form of S_n and $\mathrm{Ric}(N_n)$ denotes the Ricci curvature of X in the direction of the unit normal to S_n. To prove (5.1), observe that since $\mathcal{L}_n J_n = 0$ on S_n, then

$$|\nabla J_n|^2 - P_n J_n^2 = |\nabla J_n|^2 + J_n \Delta J_n = \mathrm{div}(J_n \nabla J_n). \tag{5.2}$$

Multiplying by ϕ_n^2 in (5.2) and integrating on $D_1(n)^+$ we find

$$\int_{D_1(n)^+} \left(\phi_n^2 |\nabla J_n|^2 - P_n \phi_n^2 J_n^2 \right) = \int_{D_1(n)^+} \phi_n^2 \mathrm{div}(J_n \nabla J_n). \tag{5.3}$$

On the other hand, $\operatorname{div}(\phi_n^2 J_n \nabla J_n) = \frac{1}{2}\langle \nabla(\phi_n^2), \nabla(J_n^2)\rangle + \phi_n^2 \operatorname{div}(J_n \nabla J_n)$, hence the Divergence Theorem and the fact that $\phi_n J_n = 0$ along $\partial D_1(n)^+$ give that

$$\int_{D_1(n)^+} \phi_n^2 \operatorname{div}(J_n \nabla J_n) = -\frac{1}{2}\int_{D_1(n)^+} \langle \nabla(\phi_n^2), \nabla(J_n^2)\rangle. \tag{5.4}$$

Equations (5.3), (5.4) together with (I1), (I2) and (I3) imply that

$$\int_{D_1(n)^+} \left(|\nabla(\phi_n J_n)|^2 - P_n \phi_n^2 J_n^2\right)$$

$$= \int_{D_1(n)^+} J_n^2 |\nabla \phi_n|^2$$

$$== \int_{\{\frac{1}{3n} < d_{S_n}(x, \beta_n) < \frac{1}{2n}\} \cap D_1(n)^+} J_n^2(x)|\nabla \phi_n|^2(x)$$

$$\leq C^2 n^2 \cdot \operatorname{Area}(\{\tfrac{1}{3n} < d_{S_n}(x, \beta_n) < \tfrac{1}{2n}\}) \cdot \max_{\{\frac{1}{3n} < d_{S_n}(x, \beta_n) < \frac{1}{2n}\}} (J_n^2(x)).$$

As $\{\widehat{f}_n|_{\Omega_n}\}_n$ is convergent, we conclude that $\operatorname{Area}(\{\tfrac{1}{3n} < d_{S_n}(x, \beta_n) < \tfrac{1}{2n}\})$ can be made smaller than C_1/n^2 for n large, for some $C_1 > 0$ independent of n. Therefore, the last right-hand-side is bounded from above by

$$C^2 C_1 \max_{\{\frac{1}{3n} < d_{S_n}(x, \beta_n) < \frac{1}{2n}\}} (J_n^2(x)).$$

which tends to zero as $n \to \infty$ by the first property in (H2) above, because $J(q) = 0$. This proves our claim (5.1).

Next we will define a test function for the stability operator on the domain $D_1(n)^+ \cup U_1(n)$.

To start, let us consider for $\varepsilon_0 > 0$ sufficiently small, local coordinates (x, y) around q_1 in Σ, so that:

- (x, y) are defined on $\mathbb{D}(\varepsilon) = \{(x, y) : x^2 + y^2 \leq \varepsilon\}$ for some $\varepsilon \in (0, \varepsilon_0]$, q_1 corresponds in these coordinates to $(0, 0)$ and $\mathbb{D}(\varepsilon)$ corresponds to a compact neighborhood $V_1 \subset U_1$ of q_1.
- $\alpha_1 \cap V_1$ corresponds to the arc $\mathbb{D}(\varepsilon) \cap \{x = 0\}$.
- The curves $\{y = \text{constant}\}$ are geodesic arcs orthogonal to $\{x = 0\} \subset \alpha_1$.

By noting the convergence of the disks $U_1(n)$ to U_1 as $n \to \infty$, it is clear that we can choose coordinates $(x_{(n)}, y_{(n)})$ in S_n around $q_1(n)$ for each n large enough on domains $V_1(n)$ converging to V_1, given by a slight deformation of the coordinates (x, y), and with similar properties. More specifically, for $n \geq n_0(\varepsilon)$ large enough there exist local coordinates around $q_1(n)$ in S_n, that will be denoted for simplicity also by (x, y), so that:
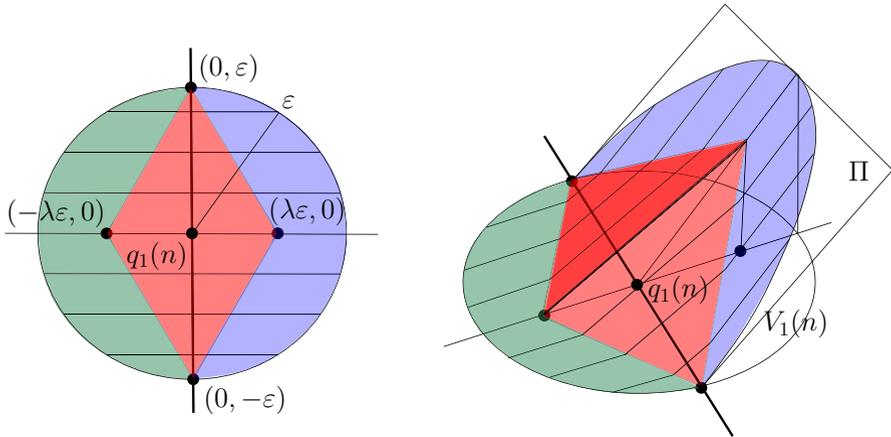
Fig. 3 Left: The compact neighborhood $V_1(n) \equiv \{x^2 + y^2 \leq \varepsilon^2\}$ of $q_1(n) = (0,0)$ intersects $\{x \geq 0\}$ in $D_1(n)^+ \cap V_1(n)$. $V_1(n)$ contains the rhombus $R(n)$ (in red color). Right: The function $v_1(n)$ is zero in the green region, coincides with J_n in the blue region and linearly interpolates these boundary values on $\partial R(n)$ as a function of x in the interior of $R(n)$ (we have represented J_n as a linear graph whose slope at $q_1(n)$ is that of the tilted plane Π) (color figure online)

- (x, y) are defined on $\mathbb{D}(\varepsilon) = \{(x,y) : x^2 + y^2 \leq \varepsilon^2\}$ for some $\varepsilon \in (0, \varepsilon_0]$, $q_1(n)$ corresponds in these coordinates to $(0, 0)$ and $\mathbb{D}(\varepsilon)$ corresponds to a compact neighborhood $V_1(n) \subset U_1(n)$ of $q_1(n)$.
- $D_1(n)^+ \cap V_1(n)$ corresponds to $\mathbb{D}(\varepsilon) \cap \{x \geq 0\}$.
- The curves $\{y = \text{constant}\}$ are geodesic arcs orthogonal to $\{x = 0\} \subset \partial D_1(n)^+$.

Given $\lambda \in (0, 1)$, consider the *rhombus* $R(n) = R(\varepsilon, \lambda, n) \subset \mathbb{D}(\varepsilon)$ that corresponds to the convex hull of the points $(\pm \lambda \varepsilon, 0)$, $(0, \pm \varepsilon)$ in these coordinates, see Fig. 3.

We define a function $v_1(n) \colon \mathbb{D}(\varepsilon) \to \mathbb{R}$ in these local coordinates as a function of x, y as follows:

(J1) $v_1(n) = 0$ in $[D_2(n) \cap V_1(n)] - R(n)$.
(J2) $v_1(n) = J_n$ in $[D_1(n)^+ \cap V_1(n)] - R(n)$.
(J3) $v_1(n)$ linearly interpolates the values of $J_n|_{D_1(n)^+ \cap \partial R(n)}$ and the value zero on $\partial R(n) \cap D_2(n)$ along the geodesics $\{y = \text{constant}\}$.

For n large, $v_1(n)$ can be extended to a piecewise smooth map on S_n (also denoted by $v_1(n)$), with the following properties for λ, ε sufficiently small:

(J1)$'$ $v_1(n) = 0$ in $S_n - [D_1(n)^+ \cup R(n)]$,
(J2)$'$ $v_1(n) = \phi_n J_n$ in $D_1(n)^+ - R(n)$,
(J3)$'$ $\displaystyle \max_{R(n)} v_1(n)^2 \leq \max_{D_1(n)^+ \cap \partial R(n)} J_n^2 = \max_{D_1(n)^+ \cap R(n)} J_n^2$.
(J4)$'$ $\displaystyle \max_{R(n)} |\nabla v_1(n)| \leq \frac{3}{5} \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|$.

Observe that (J1)′, (J2)′ respectively follow from (J1), (J2) and the fact that $\phi_n = 1$ in $U_1(n)$ for n large. We explain next why properties (J3)′ and (J4)′ hold for $v_1(n)$.

To start, note that, in terms of the (x, y) coordinates in $V_1(n)$, the Jacobi function J_n satisfies $J_n(0, y) = 0$ and $|\nabla J_n|(0, y) \neq 0$ for every $y \in [-\varepsilon, \varepsilon]$. Thus, for ε small enough, J_n can be arbitrarily well approximated in $\mathbb{D}(\varepsilon)$ by a linear function of the type ax for some $a \neq 0$. This shows that the equality in (J3)′ holds for ε, λ small enough. The inequality in (J3)′ is immediate from the definition of $v_1(n)$.

Similarly, by the previous comments and the definition by interpolation of $v_1(n)$ on $R(n)$, we see that $v_1(n)$ is arbitrarily well-approximated in $R(n) \subset \mathbb{D}(\varepsilon)$ for ε small enough by the piecewise linear function $\frac{a}{2}(x - \lambda|y| + \lambda\varepsilon)$, where $a \neq 0$ is the previous constant associated to J_n. Thus, the (almost constant) norm of the gradient of $v_1(n)$ in $R(n) \subset \mathbb{D}(\varepsilon)$ is approximately $\sqrt{1 + \lambda^2}/2$ times the (almost constant) norm of the gradient of J_n in $\mathbb{D}(\varepsilon)$. By taking λ small enough, we obtain condition (J4)′.

Observe that $v_1(n)$ vanishes along the boundary of $D_1(n)^+ \cup R(n)$. We will prove that, for ε, λ sufficiently small, there is $n_0 = n_0(\varepsilon, \lambda)$ such that, whenever $n \geq n_0$, the following inequality holds:

$$\int_{D_1(n)^+ \cup R(n)} \left(|\nabla v_1(n)|^2 - P_n v_1(n)^2\right) < 0. \tag{5.5}$$

Assuming (5.5) holds, we conclude that $D_1(n)^+ \cup R(n)$ is a strictly unstable domain in S_n, and thus, $D_1(n)^+ \cup U_1(n)$ is strictly unstable as well. Arguing in a similar way we also conclude that $D_1(n)^- \cup U_2(n)$ is strictly unstable. As $D_1(n)^+ \cup U_1(n)$, $D_1(n)^- \cup U_2(n)$ have disjoint interiors, we contradict that the index of S_n is one. Hence it only remains to prove (5.5) in order to find the desired contradiction in the case that the differential dG has rank one at some point but the right invariant Killing vector field V is not everywhere tangent to Σ. We next show (5.5).

Since $v_1(n) = \phi_n J_n$ in $D_1(n)^+ - R(n)$, then

$$\int_{D_1(n)^+ \cup R(n)} \left(|\nabla v_1(n)|^2 - P_n v_1(n)^2\right)$$
$$= \int_{D_1(n)^+ - R(n)} \left(|\nabla(\phi_n J_n)|^2 - P_n \phi_n^2 J_n^2\right) \tag{5.6}$$
$$+ \int_{R(n)} \left(|\nabla v_1(n)|^2 - P_n v_1(n)^2\right).$$

Using property (I2) above, we split the first integral in the right-hand-side of (5.6) for n large as

$$\int_{D_1(n)^+} \left(|\nabla(\phi_n J_n)|^2 - P_n \phi_n^2 J_n^2\right) - \int_{D_1(n)^+ \cap R(n)} \left(|\nabla J_n|^2 - P_n J_n^2\right). \quad (5.7)$$

By the previous claim in Eq. (5.1), the first integral in (5.7) tends to zero as $n \to \infty$. Hence, to prove (5.5) we just need to show that if ε, λ are sufficiently small and n is large enough, then

$$\int_{R(n)} \left(|\nabla v_1(n)|^2 - P_n v_1(n)^2\right) - \int_{D_1(n)^+ \cap R(n)} \left(|\nabla J_n|^2 - P_n J_n^2\right)$$
$$\leq C(\varepsilon) \quad (5.8)$$

for some constant $C(\varepsilon, \lambda) < 0$.

First note that for ε, λ sufficiently small and n large enough (once ε, λ are fixed), we have by (J3)$'$

$$\left| -\int_{R(n)} P_n v_1(n)^2 + \int_{D_1(n)^+ \cap R(n)} P_n J_n^2 \right|$$
$$\leq 2 \max_{R(n)} |P_n| \cdot \max_{D_1(n)^+ \cap R(n)} (J_n^2) \cdot \text{Area}[R(n)] \quad (5.9)$$
$$\leq 2 \max_{R(n)} |P_n| \cdot \varepsilon^2 \max_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[R(n)].$$

Observe that the potential P_n converges smoothly to the corresponding potential $P = |\sigma|^2 + \text{Ric}(N)$ for the limit surface Σ. Therefore, $\max_{R(n)} |P_n|$ can be supposed to be less than some $\mu > 0$ independent of ε, λ. This implies that

$$2 \max_{R(n)} |P_n| \cdot \varepsilon^2 \max_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[R(n)]$$
$$\overset{(A)}{\leq} 4\mu \cdot \varepsilon^2 \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[R(n)], \quad (5.10)$$

where in (A) we have used that the sequences of numbers

$$\{ \max_{D_1(n)^+ \cap V_1(n)} |\nabla J_n| \}_n, \quad \{ \min_{D_1(n)^+ \cap V_1(n)} |\nabla J_n| \}_n$$

converge to the same positive limit $|(\nabla J)(q_1)|$ when we make $\varepsilon > 0$ decrease to zero, and take n large enough with respect to each such choice of ε.

On the other hand,

$$\int_{R(n)} |\nabla v_1(n)|^2 - \int_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2$$

$$\leq \max_{R(n)} |\nabla v_1(n)|^2 \cdot \text{Area}[R(n)] - \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[D_1(n)^+ \cap R(n)]$$

$$\overset{(B)}{\leq} \max_{R(n)} |\nabla v_1(n)|^2 \cdot \text{Area}[R(n)] - \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \frac{2}{5} \text{Area}[R(n)],$$

where (B) holds for ε small and n large enough; here, we are using that by taking ε small enough, the metric space structure on S_n induced by $\widehat{f_n}$ can be assumed to be arbitrarily close to the flat one of the (x, y)-coordinates. By property (J4)$'$, we obtain from the above inequality for ε, λ small and n large that

$$\int_{R(n)} |\nabla v_1(n)|^2 - \int_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2$$

$$\leq \left(\frac{9}{25} - \frac{2}{5}\right) \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[R(n)]. \qquad (5.11)$$

Hence, (5.9), (5.10) and (5.11) give that

$$\int_{R(n)} \left(|\nabla v_1(n)|^2 - P_n v_1(n)^2\right) - \int_{D_1(n)^+ \cap R(n)} \left(|\nabla J_n|^2 - P_n J_n^2\right)$$

$$\leq \left(4\mu \varepsilon^2 - \frac{1}{25}\right) \min_{D_1(n)^+ \cap R(n)} |\nabla J_n|^2 \cdot \text{Area}[R(n)],$$

which implies directly that inequality (5.8) holds for ε, λ small enough and n large. It then follows that for n large, there exist two disjoint unstable regions on S_n which contradicts that S_n has index one. This completes the proof of the proposition. $\qquad \square$

Definition 5.5 Suppose $f' : (\Sigma', p') \looparrowright (X, e)$ is a pointed limit immersion of $\mathcal{C}$. We define $\Delta(f')$ as the set of pointed immersions $f : (\Sigma, p) \looparrowright (X, e)$ where Σ is a complete, noncompact connected surface, $p \in \Sigma$, $f(p) = e$ and f is obtained as a limit of f' under an (intrinsically) divergent sequence of left translations. In other words, there exist compact domains $\Omega'_n \subset \Sigma'$ and points $q_n \in \Omega'_n$ diverging to infinity in Σ' such that the sequence of left translated immersions $\{(l_{f'(q_n)^{-1}} \circ f')|_{\Omega'_n}\}_n$ converges on compact sets of Σ to f as $n \to \infty$.

Proposition 5.6 *Let $f' : (\Sigma', p') \looparrowright (X, e)$ be a pointed limit immersion of $\mathcal{C}$. Then, the space $\Delta(f')$ is nonempty and every $[f : (\Sigma, p) \looparrowright (X, e)] \in \Delta(f')$ satisfies:*

(1) f is a limit surface of $\mathcal{C}$, thus of constant mean curvature $h_0(X)$.
(2) f is stable.
(3) There exists a nonzero right invariant vector field on X which is everywhere tangent to $f(\Sigma)$.

(4) $f(\Sigma)$ is topologically an immersed plane or annulus in X.

(5) Σ is diffeomorphic to a plane or an annulus.

Moreover, if $f(\Sigma)$ is not a two-dimensional subgroup of X, then the left invariant Gauss map image $G(\Sigma)$ of f is a regular curve in $\mathbb{S}^2$.

Proof The fact that $\Delta(f')$ is not empty and items 1, 2 of the proposition follow directly from the main statement of [26, Corollary 5.4]. Item 3 of the proposition was proved in item 1 of [26, Corollary 5.4], under the extra assumption (K1) below (as in our explanation previous to Proposition 4.1, assumption (K1) was stated in [26, Corollary 5.4] in terms of the H-potential, so one must use [28, Corollary 3.21] to reformulate it in the following manner):

(K1) No left translation of a two-dimensional subgroup in X with constant mean curvature $h_0(X)$ is tangent to $f(\Sigma)$ at some point.

Therefore, item 3 of the proposition will be proved if we demonstrate that

(K2) If $f(\Sigma)$ is not a two-dimensional subgroup, then property (K1) holds.

We next prove (K2). Arguing by contradiction, suppose that $f(\Sigma)$ is not a two-dimensional subgroup and there exist a two-dimensional subgroup Λ of X with mean curvature $h_0(X)$, and points $x \in X$, $y \in f(\Sigma) \cap (x\Lambda)$ such that $f(\Sigma)$ and $x\Lambda = l_x(\Lambda)$ are tangent at y. Therefore, $f(\Sigma)$, $x\Lambda$ are different surfaces tangent at y with the same unit normal at this point and the same constant mean curvature. Since f is a limit surface of $\mathcal{C}$ by the already proven item 1 of this proposition, we can easily adapt the arguments in the first part of the proof of Proposition 5.4 to deduce that $f(\Sigma) = x\Lambda$, which is impossible since $e \in f(\Sigma)$, and so $f(\Sigma) = \Lambda$, a subgroup. Therefore, property (K2) holds and the proof of item 3 of Proposition 5.6 is complete.

Once item 3 is proved, the proofs of items 4, 5 of the proposition are the same as the proofs of the related items 2, 3 of [26, Corollary 5.4], respectively.

It remains to prove the 'Moreover part' of the proposition, so assume that $f(\Sigma)$ is not a two-dimensional subgroup. By properties (K1), (K2) above, we conclude that there are no two-dimensional subgroups in X with mean curvature $h_0(X)$, whose (constant) left invariant Gauss map lies in $G(\Sigma)$. In this setting, the already proven item 3 of this proposition implies that we can apply Proposition 4.1 to deduce that $f(\Sigma)$ is a regular curve. This completes the proof. □

The next technical lemma will be used in the proof of Corollary 5.9.

Lemma 5.7 (Unique Limit Surface Lemma). *Let $f_1 \colon (\Sigma_1, p) \looparrowright (X, e)$ be a pointed limit immersion of $\mathcal{C}$, with associated pointed immersions $\widehat{f}_n \colon (S_n, p_n) \looparrowright (X, e)$, $S_n \in \mathcal{C}$, so that each H_n-sphere S_n contains a compact subdomain Ω_n^1 with $p_n \in \Omega_n^1$ and $\widehat{f}_n|_{\Omega_n^1}$ converges to f_1 as $n \to \infty$,*

and $\mathrm{Area}(\widehat{f}_n|_{\Omega_n^1}) > n$. Suppose that the rank of the differential dG^1 of the left invariant Gauss map G^1 of f_1 is one. Let G_n be the left invariant Gauss map of $\widehat{f}_n$ and let $q_n \in S_n$ be the unique point with $G_n(q_n) = v = G^1(p)$. Then, after choosing a subsequence, there exist compact domains $\Omega_n^2 \subset S_n$ with $q_n \in \Omega_n^2$ such that:

(1) The sequence of pointed H_n-immersions $(l_{\widehat{f}_n(q_n)^{-1}} \circ \widehat{f}_n)|_{\Omega_n^2} : (\Omega_n^2, q_n) \looparrowright (X, e)$ obtained by left translating $\widehat{f}_n|_{\Omega_n^2}$ by the inverse of $\widehat{f}_n(q_n)$ in X, converges to a pointed immersion $f_2 : (\Sigma_2, q) \looparrowright (X, e)$ that is a limit of $\mathcal{C}$, and which has unit normal vector v at $q \in \Sigma_2$.

(2) The immersion f_2 has the same image as f_1.

Remark 5.8 Lemma 5.7 remains true if we drop the hypothesis that the rank of dG^1 is one, but we will not need this more general version in this paper.

Proof The existence of the compact subdomains $\Omega_n^2 \subset S_n$ (after passing to a subsequence) and item 1 follow from standard arguments in elliptic theory, as we have uniform curvature estimates for the H_n-immersions $l_{\widehat{f}_n(q_n)^{-1}} \circ \widehat{f}_n : (S_n, q_n) \looparrowright (X, e)$ by item 5 of Proposition 3.4. By construction, the $h_0(X)$-immersions f_1, f_2 have an oriented contact of order at least one at e. By Proposition 5.4, the differentials of the respective Gauss maps G^1 and G^2 of f_1 and f_2 have constant ranks, which are possibly different. We now distinguish cases depending on the value of $\mathrm{rank}(dG^2)$.

(L1) Suppose $\mathrm{rank}(dG^2) = 0$. In this case, item 1 of Proposition 5.4 gives that f_2 is injective and $f_2(\Sigma_2)$ is a two-dimensional subgroup of X. Since $f_1(\Sigma)$ is not a two-dimensional subgroup (because $\mathrm{rank}(dG^1) = 1$), and both $f_1(\Sigma_1), f_2(\Sigma_2)$ are tangent at e with the same constant mean curvature, we can apply the arguments in the first part of the proof of Proposition 5.4 (see also the proof of property (K2) above) to find a contradiction. Therefore, this case cannot occur.

(L2) Suppose $\mathrm{rank}(dG^2) = 1$. In this case, $G^1(\Sigma_1)$ and $G^2(\Sigma_2)$ are analytic immersed curves α_1, α_2 in $\mathbb{S}^2$ passing through $v \in \mathbb{S}^2$. If α_1, α_2 intersect tangentially at v, then by Lemma 4.3 we have $f_1(\Sigma_1) = f_2(\Sigma_2)$ and so the lemma holds. So, assume next that α_1, α_2 are transverse at v. Thus, given $\varepsilon > 0$ small enough there is some $n_0 = n_0(\varepsilon)$ such that, if $n \geq n_0$, the images through G_n of the intrinsic balls $B_{S_n}(p_n, \varepsilon), B_{S_n}(q_n, \varepsilon)$ centered at p_n and q_n of radius ε, must intersect in an open nonempty set of $\mathbb{S}^2$ near v. As G_n is a diffeomorphism, $B_{S_n}(p_n, \varepsilon) \cap B_{S_n}(q_n, \varepsilon) \neq \emptyset$, which implies that $d_{S_n}(p_n, q_n) < 2\varepsilon$. Therefore, $d_{S_n}(p_n, q_n) \to 0$ as $n \to \infty$, and this shows by item 2 of Remark 5.2 that $f_1(\Sigma_1) = f_2(\Sigma_2)$. This proves Lemma 5.7 in this case.

(L3) Suppose $\mathrm{rank}(dG^2) = 2$. By the Inverse Function Theorem applied to G^2 around q, we can find $\varepsilon > 0$ small such that if B_ε denotes the intrinsic

ball in Σ_2 of radius ε centered at q, then $G^2|_{B_\varepsilon}$ is a diffeomorphism onto its image. Choosing $\varepsilon > 0$ small enough, we can assume that for n large, $G^2(B_\varepsilon) \subset G_n(B_{S_n}(q_n, 2\varepsilon))$, where $B_{S_n}(q_n, 2\varepsilon)$ is the intrinsic ball of radius 2ε around q_n in S_n. As $G_n(p_n) \to v$, we have $G_n(p_n) \in G_n(B_{S_n}(q_n, 2\varepsilon))$, and as G_n is a diffeomorphism, this implies that $p_n \in B_{S_n}(q_n, 2\varepsilon)$ for n large enough. As ε is arbitrarily small, we conclude that $d_{S_n}(p_n, q_n) \to 0$ as $n \to \infty$. This implies that the limit surfaces $f_1(\Sigma_1)$, $f_2(\Sigma_2)$ are the same by item 2 of Remark 5.2, so in particular $\mathrm{rank}(dG^2) = 1$, a contradiction. Therefore, this case does not occur, and this completes the proof of Lemma 5.7.

□

As a consequence of Lemma 5.7 and the Transversality Lemma (Lemma 5.3), we have the following corollary.

Corollary 5.9 *Suppose* $f : (\Sigma, p) \looparrowright (X, e) \in \Delta(f')$, *where* $f' : (\Sigma', p') \looparrowright (X, e)$ *is a limit immersion of* $\mathcal{C}$. *If* $f(\Sigma)$ *is tangent at some point to a left or right coset of a two-dimensional subgroup of* X, *then* $f(\Sigma)$ *is contained in one of the two closed complements of this coset in* X.

Proof Suppose the image immersed surface $f(\Sigma)$ is tangent to a left or right coset E of some two-dimensional subgroup of X at a point $f(y) \in f(\Sigma) \cap E$, where $y \in \Sigma$. By the discussion just before property (C) in Sect. 4.1, we can assume that $E = f(y)\Lambda$ for some two-dimensional subgroup Λ of X. After a left translation of f by $f(y)^{-1}$, we may assume that $p = y$, and hence $f(\Sigma)$ is tangent to Λ at e.

If $f(\Sigma)$ is a two-dimensional subgroup of X, then $f(\Sigma) = \Lambda$ and there is nothing to prove. Hence in the sequel we will assume that $f(\Sigma)$ is not a two-dimensional subgroup of X. In this case, as $f \in \Delta(f')$ and f' is a limit immersion of $\mathcal{C}$, then Proposition 5.6 gives that $G(\Sigma)$ is a regular curve in $\mathbb{S}^2$, where $G \colon \Sigma \to \mathbb{S}^2 \subset T_e X$ is the left invariant Gauss map of f. In particular, $\mathrm{rank}(dG) = 1$.

Applying Lemma 5.7 to $f_1 = f$, we conclude that $f(\Sigma)$ coincides with the image set $f_2(\Sigma_2)$ of the limit $f_2 \colon (\Sigma_2, p_2) \looparrowright (X, e)$ of a sequence of pointed immersions $f_n^2 \colon (\Omega_n^2, q_n) \looparrowright (X, e)$, where each Ω_n^2 is a compact subdomain of an H_n-sphere S_n in $\mathcal{C}$ and such that the left invariant Gauss map of f_n^2 at $q_n \in \Omega_n^2$ is $v = G(p)$. By the Transversality Lemma, the spheres $f_n^2(S_n)$ all lie on one side of Λ. Thus, $f(\Sigma) = f_2(\Sigma_2) = \lim_{n \to \infty} f_n^2(\Omega_n^2)$ must lie on one side of Λ. This completes the proof of the corollary. □

Lemma 5.10 *Suppose* $f : (\Sigma, p) \looparrowright (X, e) \in \Delta(f')$, *where* $f' : (\Sigma', p') \looparrowright (X, e)$ *is a limit immersion for* $\mathcal{C}$. *For each* $q \in \Sigma$, *let* $f_q = l_{f(q)}^{-1} \circ f : (\Sigma, q) \looparrowright (X, e)$ *denote the related pointed immersion from* (Σ, q) *to* (X, e) *obtained*

from f after a change of base point and left translating by $f(q)^{-1}$. Consider the set $\Delta(f)$ of limits of f under a divergent sequence left translations, in the sense of Definition 5.5. Then:

(1) $\Delta(f)$ is a subset of $\Delta(f')$.

Assume that $\Delta(f)$ contains no elements with constant left invariant Gauss map. Then:

(2) The image $\gamma_f = G(\Sigma)$ of the left invariant Gauss map $G\colon \Sigma \to \mathbb{S}^2 \subset T_e X$ of f is a complete embedded regular curve in $\mathbb{S}^2$. Moreover, its closure $\overline{\gamma_f} \subset \mathbb{S}^2$ admits the structure of a lamination of $\mathbb{S}^2$, whose leaves correspond to the Gauss map images $\gamma_{\widehat{f}}$ of elements $\widehat{f} \in \Delta(f)$. In particular, if $\widehat{f} \in \Delta(f)$, then $\Delta(\widehat{f}) \subset \Delta(f)$ and so, $\overline{\gamma_{\widehat{f}}}$ is a sublamination of $\overline{\gamma_f}$.

(3) There is a uniform upper bound on the absolute geodesic curvature of all the leaves of $\overline{\gamma_f}$.

(4) There exists $\widehat{f} \in \Delta(f)$ such that $\overline{\gamma_{\widehat{f}}}$ contains no proper sublaminations. Furthermore, one of the following two possibilities holds:

(a) If $\gamma_{\widehat{f}}$ is a closed curve, then the lamination $\overline{\gamma_{\widehat{f}}}$ contains a single leaf.

(b) If $\gamma_{\widehat{f}}$ is not a simple closed curve, then the lamination $\overline{\gamma_{\widehat{f}}}$ has uncountably many leaves and $\gamma_{\widehat{f}}$ has the following recurrency property: given any compact arc I of $\gamma_{\widehat{f}}$, there exists a sequence of intrinsically divergent, pairwise disjoint arcs in $\gamma_{\widehat{f}}$ that converge to I in the C^1-topology.

Proof Item 1 follows from a standard diagonal argument.

In the sequel, we will assume that $\Delta(f)$ contains no elements with constant left invariant Gauss map. In particular, $f(\Sigma)$ is not a two-dimensional subgroup of X.

We next prove item 2. By Proposition 5.6, there exists a nonzero, right invariant vector field on X which is everywhere tangent to $f(\Sigma)$ and the Gauss map image γ_f is a regular curve in $\mathbb{S}^2$. Next we show that the curve γ_f is complete, or equivalently, it has no end points in $\mathbb{S}^2$. If such an end point $x \in \mathbb{S}^2$ of γ_f exists, then we can consider a divergent sequence $q_n \in \Sigma$ and compact geodesic disks $D_n \subset \Sigma$ around q_n of fixed radius such that $G(D_n) \to \{x\}$. After passing to a subsequence, there exist compact subdomains $\Omega_n \subseteq \Sigma$ with $q_n \in \Omega_n$ such that the restrictions $f_{q_n}|_{\Omega_n}$ converge to an element $\widetilde{f} \in \Delta(f)$ (see the first paragraph in the proof of Lemma 5.7 for a similar argument). By construction, the Gauss map of $\widetilde{f}$ is constantly equal to x around e. So, by analyticity, $\widetilde{f}$ has constant left invariant Gauss map, which contradicts our hypothesis. Therefore, γ_f is complete.

By the arguments in the proof of Lemma 4.3, γ_f has no tangential self-intersections. Transversal self-intersections of γ_f can also be ruled out by using a straightforward modification of the arguments in case (L2) of the proof of

Lemma 5.7. Therefore, the curve γ_f is embedded. Also, note that the surface $f(\Sigma)$ can be viewed locally as the graph of a solution to a quasilinear elliptic PDE, and recall that f has uniformly bounded second fundamental form. By standard elliptic estimates, this provides a priori C^3 estimates for $f(\Sigma)$, and this shows in particular that γ_f has bounded geometry. These properties for γ_f imply that its closure $\overline{\gamma_f}$ has the structure of a lamination of $\mathbb{S}^2$, all whose leaves have bounded geometry as well. Hence item 3 of the lemma is proved.

If $\widehat{f} \in \Delta(f)$, then the arguments above apply to $\widehat{f}$ to give that the Gauss map image $\gamma_{\widehat{f}}$ of $\widehat{f}$ is a complete embedded regular curve in $\mathbb{S}^2$. By definition, there exists a sequence $\{q'_n\}_n \subset \Sigma$ such that the immersions $f_{q'_n}$ converge smoothly to $\widehat{f}$, and thus the corresponding Gauss map images $\gamma_{f_{q'_n}}$ converge to $\gamma_{\widehat{f}}$ in $\mathbb{S}^2$. But clearly $\gamma_{f_{q'_n}} = \gamma_f$ as sets, from where we deduce that the leaves of the lamination $\overline{\gamma_f}$ correspond to the Gauss map images of elements of $\Delta(f)$. This proves item 2 of the lemma.

We finish by proving item 4. Consider the set $\mathcal{S}$ of sublaminations of the lamination $\overline{\gamma_f}$, which is partially ordered by the inclusion. We want to apply Zorn's lemma to $\mathcal{S}$ in order to find a minimal element in $\mathcal{S}$, i.e., a sublamination of $\overline{\gamma_f}$ with no proper sublaminations. To do this, we must check that every totally ordered subset $\mathcal{S}_1$ of $\mathcal{S}$ has a lower bound. This is clear provided that the intersection of all sublaminations in $\mathcal{S}_1$ is nonempty. Since $\mathcal{S}_1$ is a collection of closed sets of the compact topological space $\mathbb{S}^2$ and $\mathcal{S}_1$ clearly satisfies the finite intersection property, then the intersection of all sublaminations in $\mathcal{S}_1$ is nonempty. By Zorn's lemma, there exists a sublamination $\mathcal{L}$ of $\overline{\gamma_f}$ with no proper sublaminations. Take a leaf Γ of $\mathcal{L}$. As Γ is a leaf of $\overline{\gamma_f}$, then item 2 implies that there exists an immersion $\widehat{f} \in \Delta(f)$ whose Gauss map image $\gamma_{\widehat{f}}$ is equal to Γ. Therefore, $\overline{\gamma_{\widehat{f}}}$ is a sublamination of $\mathcal{L}$, and by minimality of $\mathcal{L}$ in $\mathcal{S}$ implies that $\overline{\gamma_{\widehat{f}}} = \mathcal{L}$. This proves the first sentence of item 4 of the lemma.

In order to prove items 4a and 4b, it is worth adapting some known facts about laminations to our setting. Take a sublamination $\overline{\gamma_{\widehat{f}}}$ of $\overline{\gamma_f}$ with no proper sublaminations, corresponding to an element $\widehat{f} \in \Delta(f)$. A point $x \in \overline{\gamma_{\widehat{f}}}$ is called a *limit point of* $\overline{\gamma_{\widehat{f}}}$ if x is the limit in $\mathbb{S}^2$ of an intrinsically divergent sequence $\{x_n\}_n \subset \gamma_{\widehat{f}}$. If $x \in \overline{\gamma_{\widehat{f}}}$ is a limit point of $\overline{\gamma_{\widehat{f}}}$, then the leaf component L of the lamination $\overline{\gamma_{\widehat{f}}}$ that contains x consists entirely of limit points of $\overline{\gamma_{\widehat{f}}}$ (and L is called a *limit leaf*). The set $\mathrm{Lim}(\gamma_{\widehat{f}})$ of limit points of $\overline{\gamma_{\widehat{f}}}$ is a (closed) sublamination of $\overline{\gamma_{\widehat{f}}}$, possibly empty.

If $\gamma_{\widehat{f}}$ is a closed curve, then clearly $\mathrm{Lim}(\gamma_{\widehat{f}})$ is empty and so, $\overline{\gamma_{\widehat{f}}}$ consists of the single leaf $\gamma_{\widehat{f}}$, which is item 4a of the lemma. Next suppose that $\gamma_{\widehat{f}}$ is not a simple closed curve. Thus, $\mathrm{Lim}(\gamma_{\widehat{f}}) \neq \emptyset$. Since $\overline{\gamma_{\widehat{f}}}$ contains no proper sublaminations, then $\mathrm{Lim}(\gamma_{\widehat{f}}) = \overline{\gamma_{\widehat{f}}}$. We next show that $\overline{\gamma_{\widehat{f}}}$ contains an uncountable number of leaves. Consider a small compact arc $\alpha \subset \mathbb{S}^2$ cutting $\overline{\gamma_{\widehat{f}}}$ transversally. Since complete one-manifolds are second countable in

their intrinsic topology (each leaf of the lamination $\overline{\gamma_{\widehat{f}}}$ has this property), to prove that $\overline{\gamma_{\widehat{f}}}$ has an uncountable number of leaves, as a lamination has a local product structure it suffices to prove that $W = \alpha \cap \overline{\gamma_{\widehat{f}}}$ is uncountable. We can consider W to be a complete metric space (note that α is a compact arc in $\mathbb{S}^2$ with its usual topology, hence α has a natural structure of a complete metric space, and W is a closed subset of α). W has no isolated points, since $\mathrm{Lim}(\gamma_{\widehat{f}}) = \overline{\gamma_{\widehat{f}}}$. In this setting, the uncountability of W is a consequence of the following well-known elementary application of the Baire category theorem: *any complete metric space without isolated points is uncountable.*

Finally, the recurrency property in item 4b of the lemma follows easily from the fact that $\gamma_{\widehat{f}} \subset \overline{\gamma_{\widehat{f}}} = \mathrm{Lim}(\gamma_{\widehat{f}})$. Now Lemma 5.10 is proved. □

For the remainder of this section we will use the notation introduced in Lemma 5.10, and denote by $f : (\Sigma, p) \hookrightarrow (X, e)$ an element of $\Delta(f')$ such that $\Delta(f)$ **contains no elements with constant left invariant Gauss map**. In particular, $f(\Sigma)$ is not a two-dimensional subgroup of X.

By Proposition 5.6, γ_f is a regular curve in $\mathbb{S}^2$ and thus, the right invariant vector field K_Σ given by item 2 of Proposition 5.4 is unique up to scaling. Once we pick K_Σ and an orientation on X, the curve $\gamma_f \subset \mathbb{S}^2$ has a natural orientation as follows.

Definition 5.11 Suppose that $q \in \Sigma$ and $N(q) \in T_{f(q)}X$ is the unit normal vector to Σ at q. Consider a short arc β in Σ transverse to the integral curves of the vector field $\widetilde{K}_\Sigma$ induced by K_Σ on Σ, parameterized so that $\beta(0) = q$ and $\{df_q(\beta'(0)), K_\Sigma(f(q)), N(q)\}$ is a positively oriented basis for $T_{f(q)}X$. Now, the orientation on γ_f is defined to be the one given by $G \circ \beta$ where G is the Gauss map of f.

Lemma 5.12 *Assume that γ_f is not a simple closed curve in $\mathbb{S}^2$. Then, there exists a compact interval $\sigma \subset \gamma_f$ with $G(p) \in \sigma$ (here G is the left invariant Gauss map of f and p is the base point of f) and there exists a sequence of pairwise disjoint arcs $\sigma_n \subset \gamma_f$ that are small normal graphs over σ, which converge to σ in the C^1-topology and, as graphs oriented by γ_f, satisfy that their induced orientations are opposite of the orientation of σ, see Fig. 4.*

Proof By item 4b of Lemma 5.10, given a small arc $\sigma \subset \gamma_f$ containing $G(p)$, there exists a sequence of pairwise disjoint intervals $\sigma_n \subset \gamma_f$ which converge C^1 in $\mathbb{S}^2$ to σ. After extracting a subsequence, we can assume that all the arcs σ_n lie on the same side of σ; more specifically, we may assume that there exists a one-sided neighborhood U of σ diffeomorphic to a square $[0, 1] \times [0, 1]$, so that σ corresponds to $\{0\} \times [0, 1]$ and for each $n \in \mathbb{N}$, σ_n corresponds to $\{1/n\} \times [0, 1]$ under this diffeomorphism. Also note that the lamination structure on the closure of γ_f ensures that we can take U so that

Fig. 4 If γ_f is not a simple closed curve, then it contains compact arcs σ_n converging to a compact arc σ, with the orientations of the σ_n being opposite to the one of σ
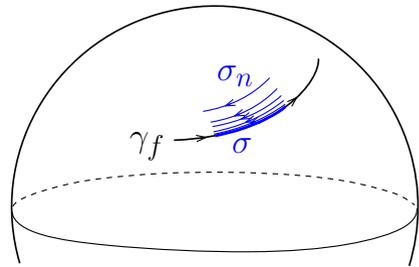
Fig. 5 The future of I' in γ_f must enter D by crossing U pointing down, a contradiction
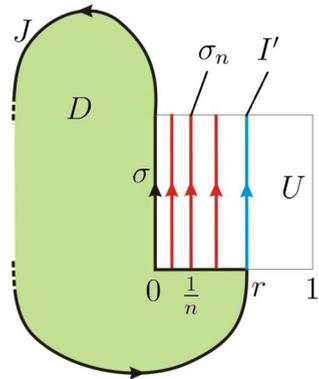
every component of $\gamma_f \cap U$ is a small normal graph over σ. Therefore, in the model of U as $[0, 1] \times [0, 1]$, $\gamma_f \cap U$ can be represented as $A \times [0, 1]$ where $A \subset [0, 1]$ is a compact infinite subset, $1 \in A$ and 0 is an accumulation point of A.

Without loss of generality, we can also assume that the orientation on σ induced by the one of γ_f is pointing upwards in this model of σ as $\{0\} \times [0, 1]$. If Lemma 5.12 fails, then we have that after choosing U small enough, every component of $\gamma_f \cap U \equiv A \times [0, 1]$ is oriented by pointing upwards. We can assume that γ_f is parameterized by the real line $\mathbb{R}$, and its orientation is the induced one by this parameterization. Then, each of the segments σ, σ_n corresponds respectively to a closed interval I, $I_n \subset \mathbb{R}$ by this parameterization, and these intervals form a pairwise disjoint collection. Each point or closed interval J in $\mathbb{R}$ defines a "future" (the component of $\mathbb{R} - J$ that limits to $+\infty$) and a "past" (the component of $\mathbb{R} - J$ that limits to $-\infty$), and these definitions can be translated to γ_f via the parameterization. After passing to a subsequence, we can assume that one of the two following possibilities holds:

(M1) I_n is contained in the future of I, for all $n \in \mathbb{N}$.
(M2) I_n is contained in the past of I, for all $n \in \mathbb{N}$.

If case (M1) holds, then consider the end point $t_1 = \max(I)$ and the first $t_2 > t_1$ whose image by the parameterization lies in the closed topological

square $U \subset \mathbb{S}^2$; the image of the interval $(t_1, t_2) \subset \mathbb{R}$ is an open Jordan arc J in $\mathbb{S}^2 - U$ with endpoints in ∂U corresponding to the points $(0, 1)$ and $(r, 0)$ for a certain $r \in (0, 1]$, in the model of U as $[0, 1] \times [0, 1]$, see Fig. 5. By definition, $J \cap U = \emptyset$ and $(r, 0)$ lies in the boundary of an interval component I' of $\gamma_f \cap U$, which in our square model is represented by $\{r\} \times [0, 1]$. Then, $\alpha := \sigma \cup J \cup ([0, r] \times \{0\})$ defines a Jordan curve in $\mathbb{S}^2$. Let $D \subset \mathbb{S}^2$ be the topological disk bounded by α whose interior is disjoint from the square U. Consider the embedded open arc $(I')^+$ of γ_f given by the future of I'. Note that $(I')^+$ must eventually intersect the interior of D (to see this, observe that for n sufficiently large, the segment $\sigma_n \equiv \{1/n\} \times [0, 1]$ lies in $(I')^+$ and intersects ∂D transversely at its initial point). Since γ_f has no self-intersections, we deduce that $(I')^+$ must cross ∂D along the arc $(0, r) \times \{0\}$. Moreover, at the points of $(I')^+ \cap \partial D$ along the arc $(0, r) \times \{0\}$ where $(I')^+$ enters D, the induced orientation of $(I')^+$ points down, which contradicts our assumption that every component of $\gamma_f \cap U \equiv A \times [0, 1]$ is oriented by pointing up. If possibility (M2) above occurs, then a similar argument leads to a contradiction. This finishes the proof of Lemma 5.12. $\qquad\square$

Lemma 5.13 *Suppose that γ_f is not a closed curve. Let $\sigma, \sigma_n \subset \gamma_f, n \in \mathbb{N}$, be the arcs given by Lemma 5.12, and assume that $G(p) \in \sigma$ (here p is the base point of f and G is its left invariant Gauss map). Let $D, D_n \subset \Sigma$ be given by $D := G^{-1}(\sigma), D_n := G^{-1}(\sigma_n)$. Take points $p_n \in D_n$ so that $\{G(p_n)\}_n$ converges to $G(p)$. Then, there exist diffeomorphisms $\Phi_n : D \to D_n$ such that the immersions $l_{f(p_n)^{-1}} \circ f \circ \Phi_n$ converge to f uniformly on compact sets of D.*

As a consequence, the pointed immersions $l_{f(p_n)^{-1}} \circ f : (\Sigma, p_n) \looparrowright (X, e)$ converge uniformly on compact sets to $f : (\Sigma, p) \looparrowright (X, e)$.

Proof As γ_f is not a closed curve, Σ cannot be diffeomorphic to an annulus, and so by Proposition 5.6 it is simply connected. Thus, by the arguments described after Proposition 4.1 and the uniformization theorem, we can parameterize f as a conformal immersion $f : \mathcal{U} \subset \mathbb{C} \looparrowright X$ where $\mathcal{U}$ is either $\mathbb{C}$ or a vertical strip $\{s + it \mid a_1 < s < a_2\}$, and the left invariant Gauss map $G : \mathcal{U} \to \mathbb{S}^2$ of f depends solely on s. From now on, we will identify Σ with $\mathcal{U}$ and assume that the imaginary part of the base point $p \in \mathcal{U}$ of f is zero. We will also consider the stereographically projected Gauss map $g := \pi \circ G : \mathcal{U} \to \overline{\mathbb{C}}$, where $\pi : \mathbb{S}^2 \to \overline{\mathbb{C}}$ is the stereographic projection from the South pole of $\mathbb{S}^2$ defined in terms of a previously chosen left invariant orthonormal frame $\{E_1, E_2, E_3\}$ of X (see the paragraph before [26, Definition 3.2]). By item 2 of Lemma 5.10, $s \mapsto g(s)$ is a complete embedded regular curve in $\overline{\mathbb{C}}$. Moreover, after choosing an adequate left invariant frame $\{E_1, E_2, E_3\}$, [26, Theorem 3.7] implies that $g(s)$ is a solution of a second

order analytic autonomous ODE, see (4.2). In particular, $g(s)$ satisfies property (D), introduced just before Lemma 4.3.

Write $\sigma = g([b_1, b_2])$, and $\sigma_n = g(I_n)$, where $[b_1, b_2]$ and each I_n are compact arcs in (a_1, a_2); note that p can be then identified with some point in $[b_1, b_2]$. Let $\Phi_n(z) = \delta_n z + \mu_n$, where $\delta_n, \mu_n \in \mathbb{R}$ satisfy that $\Phi_n([b_1, b_2]) = I_n$. Then, we can consider the sequence of maps $g_n := g \circ \Phi_n : [b_1, b_2] \to \overline{\mathbb{C}}$. It follows from property (D) above and from the C^1 convergence of the arcs σ_n to σ given by Lemma 5.12 that the following properties hold:

(N1) Each g_n is a solution of the ODE (4.2). In fact, g_n is the Gauss map of $f \circ \Phi_n$ restricted to $[b_1, b_2] \times \mathbb{R} \subset \mathbb{C}$.
(N2) $\{g_n\}_n \to g$ uniformly on $[b_1, b_2]$ in the C^1-topology.

Let $t_n \in [b_1, b_2]$ such that $g_n(t_n)$ is the value of the left invariant Gauss map of f at p_n. By hypothesis on the points p_n, we have that $\{t_n\}_n \to p \in [b_1, b_2]$.

We next translate property (N2) above into the desired convergence of $l_{f(p_n)^{-1}} \circ f \circ \Phi_n$ to f. To do this, we observe that given $z \in \mathcal{U}$, we have

$$f_z(z) = \sum_{i=1}^{3} A_i(z)(E_i)_{f(z)}, \qquad (5.12)$$

where $A_1, A_2, A_3 : \mathcal{U} \to \mathbb{C}$ are given in terms of g by [26, equation (3.4)], and $\{E_1, E_2, E_3\}$ is the orthonormal basis of left invariant vector fields on X that we prescribed previously. Note that the immersion $f \circ \Phi_n$ satisfies the same complex linear differential equation (5.12) with respect to g_n. Using properties (N1), (N2) above and the smooth dependence of the solution of an ODE with respect to initial conditions, we deduce that $l_{f(p_n)^{-1}} \circ f \circ \Phi_n$ converges uniformly on compact sets of $[b_1, b_2] \times \mathbb{R}$ to f.

Finally we prove the last sentence in the statement of Lemma 5.13. Now consider the pointed immersions $l_{f(p_n)^{-1}} \circ f : (\Sigma, p_n) \looparrowright (X, e)$, which have uniform curvature estimates by item 5 of Proposition 3.4. By elliptic theory, these immersions converge (up to subsequence) to some other complete pointed immersion of constant mean curvature. Now, the uniform convergence property that we obtained in the previous paragraph, and the unique continuation principle for surfaces of constant mean curvature (see e.g., Aronszajn [3]) concludes the proof of Lemma 5.13. □

6 The existence of periodic invariant limit surfaces

In this section we will apply the results of Sect. 5 to prove Theorem 6.1 below, as well as its consequence, Corollary 6.7. In later sections we will also quote some of the steps in the proof of Theorem 6.1.

We will use the following notation throughout this section. X is a metric Lie group diffeomorphic to $\mathbb{R}^3$; $h_0(X) \geq 0$, $\mathcal{C}$ are respectively the number and the component of the space $\mathcal{M}_X^1$ of index-one spheres with constant mean curvature in X that appear in Proposition 3.4. $f : (\Sigma, p) \looparrowright (X, e)$ is a pointed limit immersion of $\mathcal{C}$ produced by Proposition 5.6; in particular, f is stable with constant mean curvature $h_0(X)$, and there exists a nonzero, right invariant vector field K_Σ on X which is everywhere tangent to $f(\Sigma)$. $\Gamma_\Sigma \subset X$ will denote the 1-parameter subgroup of X generated by K_Σ.

Observe that if the space $\Delta(f)$ defined in Lemma 5.10 contains an element $f_1 : (\Sigma_1, p_1) \looparrowright (X, e)$ with constant left invariant Gauss map (equivalently, $f_1(\Sigma_1)$ is a two-dimensional subgroup of X), then f_1 is injective and $f_1(\Sigma_1)$ is an entire Killing graph in X with respect to any nonzero right invariant vector field in X transverse to $f_1(\Sigma_1)$. In particular, Theorem 1.5 holds trivially in this case (note that by item 1 of Proposition 5.6, f_1 is a limit surface of $\mathcal{C}$, and so it has the convergence properties in the statement of Theorem 1.5). Therefore, **in the sequel we will assume that no element in $\Delta(f)$ has constant left invariant Gauss map;** in particular, $f(\Sigma)$ is not a two-dimensional subgroup of X.

By Proposition 5.6, Σ is either simply connected (with $f(\Sigma)$ being a plane or an annulus) or Σ is diffeomorphic to an annulus (so $f(\Sigma)$ is an annulus as well). By Lemma 5.10, the left invariant Gauss map image $\gamma_f = G(\Sigma)$ of f is an embedded regular curve in $\mathbb{S}^2 \subset T_e X$, and we can choose f so that the closure $\overline{\gamma_f}$ is a lamination of $\mathbb{S}^2$ with no proper sublaminations.

Theorem 6.1 *In the above situation, γ_f is a closed curve in $\mathbb{S}^2$.*

Proof As γ_f is not a point, then we can consider on γ_f the orientation given by Definition 5.11. We will divide the proof of the theorem into three assertions for the sake of clarity.

Assertion 6.2 *γ_f is a closed curve if one of the following two conditions hold:*

(1) X is isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some matrix $A \in \mathcal{M}_2(\mathbb{R})$ and $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$ where F_1, F_2 are given by Eq. (4.5).
(2) X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and K_Σ is either elliptic or parabolic.

Proof of Assertion 6.2 Arguing by contradiction, suppose that γ_f is not a simple closed curve. Let σ and $\{\sigma_n\}_n$ be the arcs given in Lemma 5.12, where $q := G(p)$ lies in σ. Let p_n be points of Σ such that the sequence $G(p_n) \in \sigma_n$ converges to q as $n \to \infty$. By Lemma 5.13, the pointed immersions $l_{f(p_n)^{-1}} \circ f : (\Sigma, p_n) \looparrowright (X, e)$ converge uniformly on compact sets to $f : (\Sigma, p) \looparrowright (X, e)$. As K_Σ is everywhere tangent to $f(\Sigma)$, then the nonzero vector field $(l_{f(p_n)^{-1}})_*(K_\Sigma)$ is everywhere tangent to $(l_{f(p_n)^{-1}} \circ f)(\Sigma)$. Observe that $(l_{f(p_n)^{-1}})_*(K_\Sigma)$ is right invariant by the first paragraph of the

proof of Lemma 4.10. After normalizing, we deduce that the sequence of right invariant vector fields

$$K(n) = \frac{1}{|[(l_{f(p_n)^{-1}})_*(K_\Sigma)](e)|}(l_{f(p_n)^{-1}})_*(K_\Sigma) \tag{6.1}$$

converges to a nonzero right invariant vector field K_1 on X which is everywhere tangent to $f(\Sigma)$. Since $f(\Sigma)$ is not a two-dimensional subgroup of X, then K_1 must be a (constant) nonzero multiple of K_Σ. As the orientations of the arcs σ_n are all opposite to the orientation of the arc σ, by definition of the orientation of γ_f, we deduce that K_1 is a *negative* multiple of K_Σ.

We now check that the last property cannot happen under the hypotheses of Assertion 6.2. First suppose that we are in case 1 of this assertion. Since $K_\Sigma \notin \text{Span}\{F_1, F_2\}$, then we may assume that

$$K_\Sigma = F_3 + c_1\partial_x + c_2\partial_y$$

for some constants $c_1, c_2 \in \mathbb{R}$; here F_3 is given by (4.5). Plugging (4.5) into this last equation we get $K_\Sigma = \partial_z + h_1(x, y)\partial_x + h_2(x, y)\partial_y$ for some functions h_1, h_2. Since ∂_z is left invariant and $(l_a)_*$ preserves $\text{Span}\{\partial_x, \partial_y\}$ for every $a \in X$, then we conclude that $K(n)$ cannot converge to a negative multiple of K_Σ, which is the desired contradiction if case 1 holds.

In case 2, we can use similar arguments to find a contradiction, using the last sentence of Lemma 4.10. This finishes the proof of Assertion 6.2. □

Assertion 6.3 *If $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ and $K_\Sigma \in \text{Span}\{F_1, F_2\}$, then γ_f is a closed curve. Moreover, if γ_f does not pass through the North or South pole, then $f(\Sigma)$ is an entire Killing graph with respect to any horizontal right invariant vector field linearly independent from K_Σ.*

Proof of Assertion 6.3 As $K_\Sigma \in \text{Span}\{F_1, F_2\}$, Eq. (4.5) implies that K_Σ is a linear combination $\lambda\partial_x + \mu\partial_y$ for some $\lambda, \mu \in \mathbb{R}$. Since $f(\Sigma)$ is foliated by integral curves of K_Σ, then $f(\Sigma)$ is ruled by straight lines in the direction of $\lambda\partial_x + \mu\partial_y$. There are two cases to consider:

(O1) The angles between the immersed surface $f(\Sigma)$ and the planes $\mathbb{R}^2 \rtimes_A \{z\}$, $z \in \mathbb{R}$, are bounded away from zero. Equivalently, $\overline{\gamma_f}$ does not contain the North or South poles of $\mathbb{S}^2$.

(O2) There exists a sequence $\{q_n\}_n \subset \Sigma$ such that the angle between $f(\Sigma)$ and $\mathbb{R}^2 \rtimes_A \{z(f(q_n))\}$ at $f(q_n)$ tends to zero as $n \to \infty$. Equivalently, the North pole or the South pole in $\mathbb{S}^2$ is in $\overline{\gamma_f}$.

Suppose case (O1) holds. Note that the length of the pullback $\nabla(z \circ f)$ by f of the tangential component of ∂_z to Σ is bounded away from zero. Without loss of generality, we can assume that $K_\Sigma = \partial_x$ (this is just a linear change

of the coordinates x, y, which does not affect to our hypotheses). Hence, the image through f of the integral curve of $\nabla(z \circ f)$ passing through the base point p is a curve in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ that passes through the origin and intersects every horizontal plane $\mathbb{R}^2 \rtimes_A \{t\}$ at a single point $Q(t) = (x(t), y(t), t)$. Furthermore, the intersection of $f(\Sigma)$ with $\mathbb{R}^2 \rtimes_A \{t\}$ is the straight line passing through $Q(t)$ in the direction of ∂_x. Hence, $f(\Sigma)$ is an entire Killing graph with respect to any horizontal right invariant vector field $V = a\partial_x + b\partial_y$ linearly independent from ∂_x.

We now prove that γ_f is a simple closed curve by using arguments that are similar to the ones in the proof of the first item of Assertion 6.2. Consider the cross product $K_\Sigma \times N$, which defines a nowhere zero tangent vector field to $f(\Sigma)$ (here N is the unit normal of f). Thus, the tangent vector field W to Σ given by pulling back $K_\Sigma \times N$ through f is never tangent to the foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$. By connectedness of Σ, the function $\langle W, \nabla(z \circ f)\rangle$ has constant (nonzero) sign on Σ. Observe that this sign does not change when we compose f with a left translation in X and do the same process with the translated surface. Assume γ_f is not a simple closed curve and consider for each $n \in \mathbb{N}$ the right invariant vector field $K(n)$ given by (6.1), which in this case lies in $\mathrm{Span}\{\partial_x, \partial_y\}$ (because $(l_a)_*$ preserves $\mathrm{Span}\{\partial_x, \partial_y\}$ for every $a \in X$). As before, the $K(n)$ converge to a negative multiple of K_Σ, and so the vector fields W_n obtained by pulling back $K(n) \times N_n$ through $l_{f(p_n)^{-1}} \circ f$ converge to a negative multiple of W (here, N_n denotes the unit normal of $l_{f(p_n)^{-1}} \circ f$). This contradicts the previously explained invariance of the sign of the function $\langle W, \nabla(z \circ f)\rangle$ under left translations. Hence, γ_f is a simple closed curve, what proves Assertion 6.3 if case (O1) holds.

In order to analyze case (O2), we will prove the following general property to be used later on.

Claim 6.4 (Slab Property in $\mathbb{R}^2 \rtimes_A \mathbb{R}$). *Suppose* $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ *and* $K_\Sigma \in \mathrm{Span}\{F_1, F_2\}$. *Let* $\widetilde{K}_\Sigma = f^*(K_\Sigma)$ *be the Killing field on* Σ *induced from the right invariant vector field* K_Σ. *If there exists a sequence* $\{q_n\}_n \subset \Sigma$ *such that the angle between* $f(\Sigma)$ *and* $\mathbb{R}^2 \rtimes_A \{z(f(q_n))\}$ *at* $f(q_n)$ *tends to zero as* $n \to \infty$ *(i.e., case (O2) holds), then:*

(1) $f(\Sigma)$ is contained in a smallest horizontal slab $\mathbb{R}^2 \rtimes_A [a, b]$ with $a \leq 0 \leq b$, $a \neq b$, and $f(\Sigma)$ is tangent to both $\mathbb{R}^2 \rtimes_A \{a\}$ and $\mathbb{R}^2 \rtimes_A \{b\}$.
(2) If $q \in \Sigma$ satisfies that $f(\Sigma)$ is tangent to $\mathbb{R}^2 \rtimes_A \{z(f(q))\}$, then $z(f(q)) \in \{a, b\}$.
(3) If Σ is simply connected, then the points q_n can be taken to lie on distinct integral curves of $\widetilde{K}_\Sigma$ such that

$$z(f(q_n)) = \begin{cases} a & \text{if } n \text{ is odd,} \\ b & \text{if } n \text{ is even.} \end{cases}$$

(4) γ_f is a simple closed curve that passes through the North and South poles of $\mathbb{S}^2$.

Proof of Claim 6.4 By the uniform curvature estimates in item 5 of Proposition 3.4, after passing to a subsequence, the sequence of left translated pointed immersions $l_{f(q_n)^{-1}} \circ f \colon (\Sigma, p) \looparrowright (X, e)$ converges to a pointed immersion $[\widehat{f} \colon (\widehat{\Sigma}, \widehat{p}) \looparrowright (X, e)] \in \Delta(f)$. Since the angle between $f(\Sigma)$ and $\mathbb{R}^2 \rtimes_A \{z(f(q_n))\}$ at $f(q_n)$ is not bounded away from zero, then $\widehat{f}(\widehat{\Sigma})$ is tangent to $\mathbb{R}^2 \rtimes_A \{0\}$ at $e = (0, 0, 0)$. By Corollary 5.9, $\widehat{f}(\widehat{\Sigma})$ is contained in $\mathbb{R}^2 \rtimes_A (-\infty, 0]$ or $\mathbb{R}^2 \rtimes_A [0, \infty)$. From now on, we will assume that $\widehat{f}(\widehat{\Sigma}) \subset \mathbb{R}^2 \rtimes_A [0, \infty)$ (this does not affect the arguments that follow).

Note that $\widehat{f}$ is invariant under a nonzero right invariant vector field $K_{\widehat{\Sigma}}$, which is the limit as $n \to \infty$ of the right invariant vector fields $K(n)$ given by Eq. (6.1) after replacing p_n by q_n; in particular, both $K(n)$ and $K_{\widehat{\Sigma}}$ lie in $\mathrm{Span}\{\partial_x, \partial_y\}$. Without loss of generality, we can assume that $K_{\widehat{\Sigma}} = \partial_x$.

Therefore, $\widehat{f}(\widehat{\Sigma})$ is ruled by straight lines in the direction of ∂_x. If $\widehat{f}(\widehat{\Sigma}) = \mathbb{R}^2 \rtimes_A \{0\}$, then $\widehat{f}$ has constant left invariant Gauss map, which contradicts our assumption stated before Theorem 6.1. Since $\widehat{f}(\widehat{\Sigma})$ is analytic and $\widehat{f}(\widehat{\Sigma}) \neq \mathbb{R}^2 \rtimes_A \{0\}$, then there exists $\varepsilon > 0$ such that if $B_{\widehat{\Sigma}}(\widehat{p}, \varepsilon)$ denotes the intrinsic closed disk in $\widehat{\Sigma}$ centered at the base point $\widehat{p}$ with radius ε, then $D = \widehat{f}(B_{\widehat{\Sigma}}(\widehat{p}, \varepsilon))$ is a small graphical disk over its vertical projection to $\mathbb{R}^2 \rtimes_A \{0\}$, D is foliated by line segments in the direction of ∂_x, exactly one of these line segments lies in the x-axis, and all the other ones lie in $\mathbb{R}^2 \rtimes_A (0, \infty)$.

As $l_{f(q_n)^{-1}} \circ f$ converges to $\widehat{f}$, then the sequence of disks $D_n = (l_{f(q_n)^{-1}} \circ f)(B_\Sigma(q_n, \varepsilon))$ converges smoothly to D, where $B_\Sigma(q_n, \varepsilon)$ stands for the intrinsic closed disk in Σ centered at q_n with radius ε. Consider the integral curves of $K(n)$ (which are horizontal straight lines not necessarily parallel to the x-axis) which intersect D_n. For each n, this set of parallel lines forms a smooth surface D'_n contained in $(l_{f(q_n)^{-1}} \circ f)(\Sigma)$. Observe that D'_n contains D_n, D'_n is foliated by complete parallel horizontal lines, $\partial D'_n$ consists of two horizontal lines, and the D'_n converge smoothly on compact sets as $n \to \infty$ to the ruled surface $D' \subset \widehat{f}(\widehat{\Sigma})$ consisting of the integral curves of ∂_x that intersect D; note that D' contains the x-axis, D' is foliated by complete lines parallel to the x-axis, and its boundary $\partial D'$ consists of two lines that both lie in $\mathbb{R}^2 \rtimes_A (0, \infty)$, i.e., above the lowest straight line of D'. In particular, for n large, the boundary straight lines of D'_n lie strictly above the lowest straight line in D'_n. Therefore, along this lowest straight line in D'_n, the tangent plane to this surface is horizontal. Hence, $f(\Sigma)$ is also tangent to some horizontal plane and, by Corollary 5.9, we see that $f(\Sigma)$ lies above this plane.

After possibly replacing f by a left translation of it and changing the base point p, we may assume that $f(\Sigma)$ is tangent to the plane $P_0 = \mathbb{R}^2 \rtimes_A \{0\}$ at $f(p) = (0, 0, 0)$ and it lies above P_0.

By analyticity of f, we can choose an open strip $\mathcal{S} \subset \Sigma$ containing p and such that:

- $\mathcal{S}$ is invariant under the flow of $\widetilde{K}_\Sigma$.
- $\nabla(z \circ f)$ only vanishes in $\mathcal{S}$ along the integral curve β_1 of $\widetilde{K}_\Sigma$ that passes through p.
- $z \circ f$ attains the same value at the two boundary components of $\mathcal{S}$.

Let $C_1 \subset \Sigma$ be one of the (at most two) components of $\Sigma - \mathcal{S}$, and let P' be the horizontal plane that contains $f(\partial C_1)$. We claim that the angle that $f(C_1)$ makes with the foliation by horizontal planes is not bounded away from zero: otherwise there exists a sequence of points $r_n \in \Sigma$ such that $|(z \circ f)(r_n)| \to \infty$ as $n \to \infty$. After left translating f by $f(r_n)^{-1}$ and extracting a subsequence, we find a new limit $f' \in \Delta(f)$ such that its image makes angles bounded away from zero with the foliation of horizontal planes $\{\mathbb{R}^2 \rtimes_A \{t\} \mid t \in \mathbb{R}\}$. Since γ_f is a leaf of the lamination $\overline{\gamma_f}$ and $\overline{\gamma_f} = \overline{\gamma_{f'}}$ (recall that we were assuming that $\overline{\gamma_f}$ has no proper sublaminations), then item 2 of Lemma 5.10 implies that f is also a limit of f' under left translations. This clearly contradicts that $f(\Sigma)$ is tangent to P_0. Hence, our claim is proved.

By repeating the above arguments, there is an integral curve $\beta_2 \subset C_1$ of $\widetilde{K}_\Sigma$ such that $\nabla(z \circ f)$ vanishes along β_2 and being the closest such integral curve to β_1 in C_1; let $b := (z \circ f)(\beta_2) > 0$. As before, Corollary 5.9 implies that $f(\Sigma)$ is contained on one side of the plane $P_b = \mathbb{R}^2 \rtimes_A \{b\}$; by connectedness of $f(\Sigma)$, this side must be $\mathbb{R}^2 \rtimes_A (-\infty, b]$, since $f(\Sigma)$ intersects $\mathbb{R}^2 \rtimes_A \{0\}$. This implies that the immersed surface $f(\Sigma)$ is contained in the slab between the planes P_0 and P_b; also, the strip $Y_1 \subset \Sigma$ between β_1 and β_2 has an embedded image contained in that slab (because $z \circ f$ has no critical points in the interior of Y_1). Repeating these arguments in the domain $C_2 := C_1 - Y_1$, one finds an integral curve β_3 of $\widetilde{K}_\Sigma$ in C_2 closest to β_2 where $z \circ f$ has a critical value; call $c := (z \circ f)(\beta_3)$. Note that $0 \le c < b$, since $f(\Sigma)$ lies between P_0 and P_b and is not equal to P_b. But also, $f(\Sigma)$ lies on one side of the corresponding horizontal plane P_c. Again by connectedness of $f(\Sigma)$, and since $f(\Sigma)$ intersects both P_0 and P_b, we must have $c = 0$. Continuing inductively with this argument, in the case that Σ is simply connected, one produces the desired sequence of points $\{q_n\}_{n \in \mathbb{N}}$, where $q_n \in \Gamma_n$ and for n odd (resp. even) one has the related integral curves β_n contained in $\mathbb{R}^2 \rtimes_A \{0\}$ (resp. $\mathbb{R}^2 \rtimes_A \{b\}$). The remainder of the statements listed in Claim 6.4 follow immediately from this discussion. □

Finally, Assertion 6.3 follows from the Slab Property (Claim 6.4) and our previous discussion of case (O1). □

In the last step of the proof of Theorem 6.1 we will study the case in which X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and K_Σ is hyperbolic, or equivalently, the 1-parameter subgroup Γ_Σ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by K_Σ is hyperbolic. Recall

from Sect. 4.3 that the image of Γ_Σ through the projection $\Pi\colon \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathbb{H}^2$ defined by (4.10) is an arc of constant geodesic curvature (possibly zero) in the standard metric of $\mathbb{H}^2$, that passes through the origin of the Poincaré disk and its two extrema $\theta_1, \theta_2 \in \partial_\infty \mathbb{H}^2$ determine the two two-dimensional subgroups $\mathbb{H}^2_{\theta_1}, \mathbb{H}^2_{\theta_2}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ that contain Γ_Σ. We will also use in the next statement the pair of antipodal, simple closed curves $\Upsilon \cup (-\Upsilon) \subset \mathbb{S}^2$ defined in Lemma 4.11.

Assertion 6.5 *If X is isomorphic to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and K_Σ is hyperbolic, then γ_f is a simple closed curve. Moreover, if γ_f does not intersect $\Upsilon \cup (-\Upsilon)$, and $\mathbb{H}$ is either $\mathbb{H}^2_{\theta_1}$ or $\mathbb{H}^2_{\theta_2}$ (where K_Σ is tangent to both $\mathbb{H}^2_{\theta_1}$ and $\mathbb{H}^2_{\theta_2}$), then $f(\Sigma)$ is an entire Killing graph with respect to the parabolic right invariant vector field which is tangent at e to $\mathbb{H}$.*

Proof of Assertion 6.5 We denote by Γ_3 the 1-parameter elliptic subgroup of X defined by $\Gamma_3'(0) = (E_3)_e$ (see Eq. 4.7)). Let $\mathbb{H}$ be one of the subgroups $\mathbb{H}^2_{\theta_1}$ or $\mathbb{H}^2_{\theta_2}$. Let $\mathcal{F} = \{r_t(\mathbb{H}) = \mathbb{H}\,t \mid t \in \widetilde{\mathrm{SL}}(2,\mathbb{R})\}$ be the foliation of X whose leaves are the right cosets of $\mathbb{H}$. Since Γ_3 intersects $\mathbb{H}$ only at the identity element, it is easily seen that every element of X lies in a unique right coset of $\mathbb{H}$ of the form $\mathbb{H}\,t$, for some $t \in \Gamma_3$.

Consider the oriented distance function $\tilde{d}\colon X \to \mathbb{R}$ to the closed set $\mathbb{H}$, which is positive on $\mathbb{H}\,t$, for $t > 0$, where we have naturally parameterized Γ_3 by $\mathbb{R}$. As each right coset of a codimension one connected subgroup of a metric Lie group is at a constant distance from the subgroup (see [28, Lemma 3.9]) and Γ_3 is isomorphic to $\mathbb{R}$, then the level sets of $\tilde{d}$ are the leaves of $\mathcal{F}$. Hence, we have an induced function $d\colon \mathcal{F} \to \mathbb{R}$ so that the absolute value of d at a given element $\mathbb{H}\,t$ of $\mathcal{F}$ is the constant distance from $\mathbb{H}$ to $\mathbb{H}\,t$; thus $\mathcal{F}$ can be parameterized by Γ_3, i.e., $\mathcal{F} = \{\mathbb{H}\,t \mid t \in \Gamma_3\}$, and we may assume that $\langle \nabla \tilde{d}, E_3 \rangle > 0$, where here ∇ stands for gradient in X.

The approach to prove Assertion 6.5 is to follow, when possible, the arguments in the proof of Assertion 6.3 by exchanging the former foliation $\{\mathbb{R}^2 \rtimes_A \{t\} \mid t \in \mathbb{R}\}$ by the current one $\mathcal{F}$. Notice that the ambient function z in Assertion 6.3 is now replaced by the oriented distance function $\tilde{d}\colon X \to \mathbb{R}$. For the proof, it is also important to take into account the following list of elementary observations:

(P1) Each of the right cosets $\mathbb{H}\,t$ with $t \in \mathbb{R} \equiv \Gamma_3$ can be written as $\mathbb{H}\,t = t(t^{-1}\,\mathbb{H}\,t)$, therefore $\mathbb{H}\,t$ is a left coset of a conjugate two-dimensional subgroup of $\mathbb{H}$; see Sect. 4.1. If $t \in \Gamma_3$ lies in the center of X (which is isomorphic to $\mathbb{Z}$), then $\mathbb{H}\,t = t\,\mathbb{H}$. In this sense, $\mathcal{F}$ is a *periodic* foliation invariant under the left action of $\mathbb{Z} = \mathrm{center}(X) \subset \Gamma_3$.

(P2) Limits of f after left translations by the inverses of points on the image surface $f(\Sigma)$ give rise to limit surfaces $\widehat{f}(\widehat{\Sigma})$ that are invariant under some other *hyperbolic* right invariant vector field and such that $\gamma_{\widehat{f}}$ is a

leaf of $\overline{\gamma_f}$. Otherwise, Lemma 4.10 implies that $\widehat{f}(\widehat{\Sigma})$ would be tangent to a right invariant parabolic vector field (because the limit of a normalized sequence of hyperbolic right invariant vector fields is either hyperbolic or parabolic). As the set of parabolic vector fields normalized to have length 1 at e is compact, then every element in $\Delta(\widehat{f})$ is tangent to a parabolic right invariant vector field. As γ_f is a leaf of the lamination $\overline{\gamma_f} = \overline{\gamma_{\widehat{f}}}$ (here we are using $\overline{\gamma_f}$ has no proper sublaminations), then f is itself a limit of $\widehat{f}$, i.e., $f \in \Delta(\widehat{f})$, and so f is tangent to a right invariant parabolic vector field. This is a contradiction since K_Σ is hyperbolic and $f(\Sigma)$ is not a two-dimensional subgroup of X.

(P3) Given $a \in X$, the family $a\,\mathcal{F} = \{a\,L = l_a(L) \mid L \in \mathcal{F}\}$ is a foliation of X by the right cosets of some conjugate subgroup of $\mathbb{H}$.

(P4) As the foliation $\mathcal{F}$ is periodic in the sense of item (P1), then for every sequence $\{a_n\}_n \subset X$, the sequence of foliations $\{a_n\,\mathcal{F}\}_{n\in\mathbb{N}}$ defined as in (P3) has a convergent subsequence to another foliation of X by the right cosets of some conjugate subgroup of $\mathbb{H}$.

(P5) Let Γ^P be the unique 1-parameter parabolic subgroup contained in $\mathbb{H}$. As $\Gamma_\Sigma \mathbb{H} = \mathbb{H}$ and $\Gamma^P \mathbb{H} = \mathbb{H}$, we deduce that the left actions of Γ_Σ, Γ^P on X leave invariant each of the right cosets of $\mathbb{H}$. In particular, K_Σ and the parabolic right invariant vector field P on X generated by Γ^P are everywhere tangent to each of the leaves of $\mathcal{F}$.

(P6) Given a leaf $\mathbb{H}\,t$ of $\mathcal{F}$, each integral curve of K_Σ restricted to $\mathbb{H}\,t$ intersects each integral curve of P restricted to $\mathbb{H}\,t$ in exactly one point.

To prove Assertion 6.5, let us start by assuming that the Gaussian image γ_f is at a positive distance from the set $\Upsilon \cup (-\Upsilon)$. In particular, the angle between $f(\Sigma)$ and the leaves of $\mathcal{F}$ is bounded away from zero. Using observations (P5) and (P6) above, it follows from simple modifications of the arguments in the proof of case (O1) of Assertion 6.3 that in these conditions, $f(\Sigma)$ is an entire Killing graph with respect to P, and that γ_f is closed. Details are left to the reader.

Claim 6.6 (Slab Property in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$). *Suppose $X = \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and K_Σ is hyperbolic. Let $\widetilde{K}_\Sigma = f^*(K_\Sigma)$ be the Killing field on Σ induced by the right invariant vector field K_Σ. If there exists a sequence $\{q_n\}_n \subset \Sigma$ such that the angle between $f(\Sigma)$ and the leaf of $\mathcal{F}$ passing through $f(q_n)$ tends to zero as $n \to \infty$, then:*

(1) $f(\Sigma)$ is contained in a smallest topological slab bounded by two different leaves L_a, L_b of $\mathcal{F}$, and $f(\Sigma)$ is tangent to both leaves.

(2) If $f(\Sigma)$ is tangent to a leaf L of $\mathcal{F}$, then $L = L_a$ or $L = L_b$.

(3) γ_f is a simple closed curve that intersects both Υ and $-\Upsilon$.

Proof of Claim 6.6 Consider for each $n \in \mathbb{N}$ the left translated pointed immersion $f_n := l_{f(q_n)^{-1}} \circ f : (\Sigma, q_n) \hookrightarrow (X, e)$, which is everywhere tangent to

the hyperbolic right invariant vector field $K_n := (l_{f(q_n)^{-1}})_*(K_\Sigma)$; see observation (P2). Note that K_n is everywhere tangent to the foliation by right cosets $\mathcal{F}_n := \{\mathbb{H}_n \, t \mid t \in \mathbb{R}\}$, where $\mathbb{H}_n$ is the two-dimensional subgroup of X given by $\mathbb{H}_n = f(q_n)^{-1} \, \mathbb{H} \, f(q_n)$. After extracting a subsequence, the f_n converge on compact sets to a pointed limit immersion $\widehat{f} \colon (\widehat{\Sigma}, \widehat{p}) \hookrightarrow (X, e)$ and by observation (P4), $\mathcal{F}_n$ converges to some foliation $\widehat{\mathcal{F}}$ of X by the right cosets of some two-dimensional subgroup $\widehat{\mathbb{H}}$. Observe that by hypothesis, the angle that $f_n(\Sigma)$ makes with $\mathcal{F}_n$ at $f_n(q_n) = e$ tends to zero as $n \to \infty$. Thus, $\widehat{f}(\widehat{\Sigma})$ is tangent to $\widehat{\mathbb{H}}$ at $\widehat{f}(\widehat{p}) = e$. In particular, the left invariant Gauss map of $\widehat{f}$ at $\widehat{p}$ lies in $\Upsilon \cup (-\Upsilon)$. Furthermore, $\widehat{f}$ is everywhere tangent to some right invariant hyperbolic Killing field $\widehat{K}$, by observation (P2) ($\widehat{K}$ is the limit of appropriate rescalings of the K_n after extracting a subsequence, as we did in (6.1)). In particular, $\widehat{K}$ is everywhere tangent to $\widehat{\mathbb{H}}$, which implies that $\widehat{K}$ is generated by a 1-parameter subgroup of $\widehat{\mathbb{H}}$. Let $\widehat{d} \colon X \to \mathbb{R}$ stand for the oriented distance function associated to the foliation $\widehat{\mathcal{F}}$. Note that $\widehat{d}$ is constant along each of the integral curves of $\widehat{K}$. By Corollary 5.9, $\widehat{f}(\widehat{\Sigma})$ is contained in one side of $\widehat{\mathbb{H}}$, i.e., the restriction of $\widehat{d}$ to $\widehat{f}(\widehat{\Sigma})$ is nonpositive or nonnegative. Moreover, arguing as in the proof of Claim 6.4 there exists a small compact topological disk $\widehat{D} = \widehat{f}\big(B_{\widehat{\Sigma}}(\widehat{p}, \varepsilon)\big)$ which is foliated by compact arcs of integral curves of $\widehat{K}$, such that:

(Q) Exactly one of these compact arcs of integral curves of $\widehat{K}$ in $\widehat{D}$ lies in the zero set of $\widehat{d}$, while all the other ones lie in $\widehat{d}^{-1}(0, \infty)$ (or in $\widehat{d}^{-1}(-\infty, 0)$).

Since f_n converges to $\widehat{f}$, then the sequence of disks $D_n := f_n(B_\Sigma(q_n, \varepsilon))$ converges smoothly to $\widehat{D}$. As $\{\mathbb{H}_n\}_n \to \widehat{\mathbb{H}}$, then we can assume that the distance function $d_n \colon \widetilde{SL}(2, \mathbb{R}) \to \mathbb{R}$ associated to the foliation $\mathcal{F}_n$ converges on compact sets of $\widetilde{SL}(2, \mathbb{R})$ to $\widehat{d}$. As K_n is everywhere tangent to the leaves of $\mathcal{F}_n$, we conclude that d_n is constant along the integral curves of K_n. Furthermore, item (Q) and the previous convergence properties ensure that for n large enough, D_n contains an integral curve segment of K_n along which d_n has a local extremum. Hence, there are points $r_n \in B_\Sigma(q_n, \varepsilon)$ such that f_n is tangent at $f_n(r_n)$ to some leaf of $\mathcal{F}_n$. Therefore, f is tangent at $f(r_n)$ to some leaf L_a of $\mathcal{F}$. By Corollary 5.9, we deduce that $f(\Sigma)$ lies on one side of L_a. Once here, and using similar ideas, we can adapt the last four paragraphs of the proof of Claim 6.4 to our $\widetilde{SL}(2, \mathbb{R})$ setting, and conclude that $f(\Sigma)$ is tangent to another leaf L_b of $\mathcal{F}$, and lies in the topological slab of $\widetilde{SL}(2, \mathbb{R})$ bounded by $L_a \cup L_b$. This proves item 1 of Claim 6.6, and item 2 is a direct consequence of item 1, Corollary 5.9 and the connectedness of Σ that we leave to the reader.

As for item 3 of Claim 6.6, since the leaves of $\mathcal{F}$ are minimal, the mean curvature comparison principle applied to $f(\Sigma)$ and L_a, L_b implies that the mean curvature vector of $f(\Sigma)$ points towards the interior of the previously defined topological slab at the tangency points between $f(\Sigma)$ and $L_a \cup L_b$.

From here, a straightforward continuity argument applied to the normal field to $\mathcal{F}$ gives that γ_f intersects both Υ and $-\Upsilon$. If Σ is an annulus, then γ_f is clearly a closed curve. Otherwise, Σ is simply connected and $f(\Sigma)$ is tangent to L_a along at least two (in fact, infinitely many) distinct integral curves of $\widetilde{K}_\Sigma = f^*(K_\Sigma)$. This implies that γ_f takes the same value at these integral curves of $\widetilde{K}_\Sigma$. As γ is embedded, we now conclude that γ_f is a closed curve. $\square$

Assertion 6.5 follows from the Slab Property (Claim 6.6) and our discussion in the paragraph just before Claim 6.6. $\square$

We now complete the proof of Theorem 6.1. Since X is not isomorphic to SU(2), it is either isomorphic to a semidirect product or it is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. By Assertions 6.2, 6.3 and 6.5, γ_f is a simple closed curve, which completes the proof of Theorem 6.1. $\square$

Corollary 6.7 *Let $f : (\Sigma, p) \looparrowright (X, e)$ be a pointed limit immersion that satisfies the hypotheses stated just before Theorem 6.1.*

(1) If Σ is diffeomorphic to an annulus, then it has linear area growth.
(2) If Σ is simply connected, then one of the two following possibilities holds:
(a) f factors through a pointed immersion $\widehat{f} : (\widehat{\Sigma}, \widehat{p}) \looparrowright (X, e)$ of an annulus. In this case, Σ has quadratic area growth and the constant mean curvature immersion $\widehat{f}$ is stable.
(b) f is periodic, in the sense that there exists an element $a_\Sigma \in X - \Gamma_\Sigma$ such that the left translation by a_Σ leaves $f(\Sigma)$ invariant.

Proof Recall that $f(\Sigma)$ is foliated by integral curves of K_Σ. Since X is a simply connected Lie group diffeomorphic to $\mathbb{R}^3$, then the integral curves of K_Σ are proper Jordan arcs. Let $\widetilde{K}_\Sigma = f^*(K_\Sigma)$ denote the Killing field induced on Σ by K_Σ after pulling back K_Σ through f.

Assume that Σ is diffeomorphic to an annulus and we will show that item 1 holds. Consider the natural projection $\Pi_{\Gamma_\Sigma} : X \to X/\Gamma_\Sigma$, where X/Γ_Σ denotes the quotient surface whose points are the integral curves of K_Σ in X. Since Σ is an annulus, then $(\Pi_{\Gamma_\Sigma} \circ f)(\Sigma)$ is an immersed closed curve $\beta \subset X/\Gamma_\Sigma$ and $f : \Sigma \to \beta$ is a trivial $\mathbb{R}$-bundle over β, whose fibers are the integral curves of $\widetilde{K}_\Sigma$. Take a global section of this trivial bundle. The image set of this global section is an embedded closed curve $\alpha \subset \Sigma$ which intersects transversely each of the integral curves of $\widetilde{K}_\Sigma$. Since $\widetilde{K}_\Sigma$ is a Killing vector field, then the image $\psi_t(\alpha)$ of α through the flow $\{\psi_t \mid t \in \mathbb{R}\}$ by isometries of Σ associated to $\widetilde{K}_\Sigma$ produces a foliation of Σ by curves isometric to α, so in particular all of these curves $\psi_t(\alpha)$ have the same length. From here it is straightforward to show that Σ has linear area growth, so item 1 is proved.

Next suppose that Σ is simply connected. Consider the quotient space $\Sigma/\widetilde{K}_\Sigma$ of integral curves of $\widetilde{K}_\Sigma$. Thus, $\Sigma/\widetilde{K}_\Sigma$ is a 1-dimensional manifold

diffeomorphic to $\mathbb{R}$, and there is a natural projection $\Pi\colon \Sigma \to \Sigma/\widetilde{K}_\Sigma$. We can identify $\Sigma/\widetilde{K}_\Sigma = \mathbb{R}$, so that $\Pi(p) = 0$ for the base point p of Σ satisfying $f(p) = e$. Let $\widehat{G}\colon \mathbb{R} = \Sigma/\widetilde{K}_\Sigma \to \gamma_f \subset \mathbb{S}^2$ be the mapping such that $G = \widehat{G} \circ \Pi$, where G stands for the left invariant Gauss map of f. Since γ_f is a simple closed regular curve by Theorem 6.1, it is easy to check that the map $\widehat{G}$ is the universal cover of γ_f. Let $\tau\colon \Sigma/\widetilde{K}_\Sigma \to \Sigma/\widetilde{K}_\Sigma$ be one of the two generators of the group of automorphisms of the covering $\widehat{G}\colon \Sigma/\widetilde{K}_\Sigma \to \gamma_f \subset \mathbb{S}^2$. Since the left invariant Gauss map determines the surface up to left translations by [26, Theorem 3.7], given a point $q \in \Pi^{-1}(\tau(0))$, the left translation $l_{f(q)}\colon X \to X$ induces a nontrivial isometry $\widetilde{l}_q\colon \Sigma \to \Sigma$ that is a lift of τ to Σ, i.e., $\tau \circ \Pi = \Pi \circ \widetilde{l}_q$, and satisfies $\widetilde{l}_q(p) = q$. Note that $\widetilde{l}_q$ acts freely on Σ and the quotient $\widehat{\Sigma} = \Sigma/\widetilde{l}_q$ is an annulus. We now have two possibilities.

(R1) $f(\Pi^{-1}(0)) = f(\Pi^{-1}(\tau(0)))$, in which case f factors through the quotient annulus $\widehat{\Sigma}$ to an immersion $\widehat{f}\colon \widehat{\Sigma} \looparrowright X$.

(R2) $f(\Pi^{-1}(0)) \neq f(\Pi^{-1}(\tau(0)))$, in which case the integral curves $f(\Pi^{-1}(0))$, $f(\Pi^{-1}(\tau(0)))$ of K_Σ are disjoint.

In any of the two possibilities above, the isometry group $\mathrm{Iso}(\Sigma)$ of the induced metric on Σ by f contains an $\mathbb{R}$-type subgroup (which corresponds to left translations by any element in Γ_Σ) and a $\mathbb{Z}$-type subgroup (which corresponds to the subgroup generated by $\widetilde{l}_q$), such that these two subgroups generate a subgroup Λ of $\mathrm{Iso}(\Sigma)$, although Λ is not necessarily isomorphic to $\mathbb{R} \times \mathbb{Z}$, because elements in the $\mathbb{R}$-type subgroup do not necessarily commute with those in the $\mathbb{Z}$-subgroup; also note that Σ would have at most quadratic area growth provided that Λ is isomorphic to $\mathbb{R} \times \mathbb{Z}$.

If case (R1) holds, then the quotient metric on $\widehat{\Sigma}$ (which is the induced metric by $\widehat{f}$) has linear area growth, and so Σ has quadratic area growth. Since f is stable and the cover $\Sigma \to \widehat{\Sigma}$ is cyclic, then the quotient immersion $\widehat{f}\colon \widehat{\Sigma} \looparrowright X$ is also stable (see for instance [29, Proposition 2.5]), which proves that item 2a in the statement of Corollary 6.7 holds. Finally, if case (R2) holds, then we define $a_\Sigma = f(q)$, which lies in $X - \Gamma_\Sigma$ since $f(p) = e \in \Gamma_\Sigma$ and the integral curves $f(\Pi^{-1}(0))$, $f(\Pi^{-1}(\tau(0)))$ of K_Σ are disjoint. Then, the left translation by a_Σ leaves $f(\Sigma)$ invariant and we have item 2b of the corollary. Now the proof is complete. □

7 Proof of Theorem 1.5 when X is a semidirect product

In this section we will prove Theorem 1.5 in the case that X is a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric.

Let S_n be a sequence of constant mean curvature spheres in X with $\mathrm{Area}(S_n) > n$ for all n. In order to prove Theorem 1.5 in X it suffices to consider the case where all the spheres S_n lie in the connected component $\mathcal{C}$ of

the space of index-one spheres in X with $H > h_0(X)$ given in Proposition 3.4. Indeed, if Theorem 1.5 holds for any such sequence of spheres $(S_n)_n \subset \mathcal{C}$, the discussion after Definition 3.5 ensures that $h_0(X) = H(X)$, and hence any H-sphere Σ in X satisfies that $H > h_0(X)$, and so Theorem 1.5 holds in full generality.

So, assuming these conditions (in particular, that $S_n \in \mathcal{C}$ for every n), Proposition 5.6, Theorem 6.1 and Corollary 6.7 imply that there exists a pointed limit immersion $f : (\Sigma, p) \looparrowright (X, e)$ of the spheres S_n with the following properties:

(S1) f is complete, has constant mean curvature $h_0(X)$, it is stable and its left invariant Gauss map image γ_f is either a point or a regular, simple closed curve in $\mathbb{S}^2 \subset T_e X$.

(S2) f is everywhere tangent to a nonzero, right invariant Killing vector field K_Σ in X. Moreover, exactly one of the following three situations happens:

(S2.1) Σ is diffeomorphic to an annulus, and it has linear area growth.

(S2.2) Σ is simply connected and $f : \Sigma \looparrowright X$ factors through an immersion $\widehat{f} : \widehat{\Sigma} \looparrowright X$ of an annulus. In this case, Σ has quadratic area growth and $\widehat{f}$ is a stable immersion with constant mean curvature.

(S2.3) There exists $a \in X - \Gamma_\Sigma$ such that the left translation by a leaves $f(\Sigma)$ invariant. Here, Γ_Σ is the 1-parameter subgroup of X that generates K_Σ.

Remark 7.1 In case (S2.2), we have that $f : \Sigma \looparrowright X$ is an infinite covering of the topological annulus $f(\Sigma)$. In case (S2.1), Σ is a topological annulus itself, but the immersion $f : \Sigma \looparrowright X$ can be a finite covering of the topological annulus $f(\Sigma)$.

Recall that if γ_f is a point, then $f(\Sigma)$ is a two-dimensional subgroup of X. In particular $f(\Sigma)$ is an entire Killing graph with respect to some right invariant vector field in X, and so Theorem 1.5 holds. Hence, **from now on we will assume that** $f(\Sigma)$ **is not a two-dimensional subgroup**; in particular, γ_f is a regular Jordan curve in $\mathbb{S}^2$, and K_Σ is unique (up to scaling) among right invariant vector fields on X everywhere tangent to $f(\Sigma)$.

First assume that the matrix A is singular. By Lemma 4.5, X is isometric to $\mathbb{R}^3$ or to some $\mathbb{E}(\kappa, \tau)$ space with $\kappa \leq 0$. So, X has isometry group of dimension 6 or 4 and, as explained at the end of the introduction, Theorem 1.5 holds for X.

So, from now on, we will assume that the matrix A is regular. By the previous discussion, the next result directly implies that Theorem 1.5 holds for any metric semidirect product X, as desired.

Theorem 7.2 *In the above conditions, K_Σ is horizontal, i.e., $K_\Sigma \in \text{Span}\{F_1, F_2\}$, and $f(\Sigma)$ is an entire Killing graph in X with respect to any nonzero right*

invariant Killing vector field $V \in \text{Span}\{F_1, F_2\}$ that is linearly independent from K_Σ (with the notation of (4.5)).

Observe that if Theorem 7.2 holds, then case (S2.3) occurs for f (because in cases (S2.1) and (S2.2) the image surface $f(\Sigma)$ is an immersed annulus in X, which contradicts that $f(\Sigma)$ is an entire graph).

We will divide the proof of Theorem 7.2 into four steps. In Sect. 7.1 we will prove that Theorem 7.2 holds when $K_\Sigma \in \text{Span}\{F_1, F_2\}$; thus, Theorem 7.2 will be proved provided that we find a contradiction whenever K_Σ is not horizontal. Assuming that $K_\Sigma \notin \text{Span}\{F_1, F_2\}$, in Sects. 7.2, 7.3, 7.4 we will respectively show that cases (S2.3), (S2.2), (S2.1) above cannot occur. This will complete the proof of Theorem 7.2 (and thus of Theorem 1.5 for X a metric semidirect product).

7.1 Proof of Theorem 7.2 when K_Σ is horizontal

Lemma 7.3 *If $K_\Sigma \in \text{Span}\{F_1, F_2\}$, then f is an entire Killing graph with respect to any right invariant vector field $V \in \text{Span}\{F_1, F_2\}$ that is linearly independent from K_Σ.*

Proof In this case, $K_\Sigma = \lambda \partial_x + \mu \partial_y$ for some $\lambda, \mu \in \mathbb{R}$. Since $f(\Sigma)$ is foliated by integral curves of K_Σ, then $f(\Sigma)$ is ruled by straight lines in the direction of K_Σ. By Assertion 6.3, we conclude that the lemma will hold provided that γ_f does not pass through the North or South pole of $\mathbb{S}^2$. Arguing by contradiction, assume that γ_f passes through the North or South pole of $\mathbb{S}^2$. Then, f is in the conditions of Claim 6.4, which ensures that $f(\Sigma)$ is contained in a smallest slab-type region $\mathbb{R}^2 \rtimes_A [a, b]$ with $a \leq 0 \leq b$, $a \neq b$, $f(\Sigma)$ intersects each of the planes $\mathbb{R}^2 \rtimes_A \{a\}$, $\mathbb{R}^2 \rtimes_A \{b\}$ in nonempty sets whose components are integral curves of K_Σ, and $f(\Sigma)$ transversely intersects every intermediate plane $\mathbb{R}^2 \rtimes_A \{z\}$ with $a < z < b$. Let Y be the flat Riemannian product $\mathbb{R}^2 \times [a, b]$. Thus, we can consider $f \colon \Sigma \looparrowright Y$ to be a complete flat immersion, which is ruled by straight lines parallel to K_Σ. In particular, Σ has quadratic area growth with respect to the pullback metric of Y through f. Since the coefficients of the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with respect to the (x, y, z)-coordinates only depend on z by Eq. (4.6), we conclude that the identity map $\text{Id} \colon Y \to \mathbb{R}^2 \rtimes_A [a, b]$ is a quasi-isometry. From here, we deduce that Σ has at most quadratic area growth with respect to the pullback metric of X through f.

Let V be any horizontal right invariant vector field on X that is linearly independent from K_Σ. As the coefficients of the canonical metric are bounded in every horizontal slab of finite width by Eq. (4.6), then V is bounded in $\mathbb{R}^2 \rtimes_A [a, b]$. Denoting by N the unit normal vector to f, we deduce that $J := \langle N, V \rangle$ is a bounded Jacobi function on Σ that changes sign (changes of

sign of J occur at points of $f^{-1}(\mathbb{R}^2 \rtimes_A \{a, b\})$). As $f : \Sigma \hookrightarrow X$ is a complete stable $h_0(X)$-surface with at most quadratic area growth, then the conformal structure of Σ is parabolic, and Corollary 1 in Manzano et al. [21] implies that J cannot change sign, which is a contradiction. This contradiction completes the proof of the lemma. □

7.2 Case (S2.3) is impossible when K_Σ is not horizontal

Assume that $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$. The next lemma reduces this case to the specific nonhorizontal, right invariant vector field $K_\Sigma = F_3$ given by (4.5), which is generated by the 1-parameter subgroup of X corresponding to the z-axis. In proving the next lemma, we will be making use of the assumption that A is a regular matrix, as discussed just before the statement of Theorem 7.2.

Lemma 7.4 *If $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$, then after a horizontal left translation of f and a scaling of K_Σ, the following statements hold:*

(1) $K_\Sigma = F_3$.
(2) Case (S2.3) does not happen.

Proof Since $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$, then after scaling and using (4.3) and (4.5),

$$K_\Sigma = F_3 + s\partial_x + t\partial_y = (ax + by + s)\partial_x + (cx + dy + t)\partial_y + \partial_z$$
$$\equiv \begin{pmatrix} A\mathbf{p} + \mathbf{q} \\ 1 \end{pmatrix},$$

for some $s, t \in \mathbb{R}$, where $\mathbf{p} = (x, y)$, $\mathbf{q} = (s, t)$. Given $\mathbf{P}_0 = (\mathbf{p}_0, 0) \in \mathbb{R}^2 \rtimes_A \{0\}$, the right invariant vector field $(l_{\mathbf{P}_0})_*(K_\Sigma)$ is everywhere tangent to the left translation of f by $\mathbf{P}_0$. Hence, to prove item 1 of the lemma it suffices to find $\mathbf{p}_0 \in \mathbb{R}^2$ such that $(l_{\mathbf{P}_0})_*(K_\Sigma) = F_3$. By direct computation using the group operation in $\mathbb{R}^2 \rtimes_A \mathbb{R}$, see Sect. 4.2, we have that $l_{\mathbf{P}_0}(\mathbf{p}, z) = (\mathbf{p}_0 + \mathbf{p}, z)$ is an Euclidean translation, hence its differential is the identity after identifying the tangent spaces of X at both points $(\mathbf{p}, z)$, $(\mathbf{p}_0 + \mathbf{p}, z)$ with $\mathbb{R}^3$ in the basis $\partial_x, \partial_y, \partial_z$. Therefore, the value of $(l_{\mathbf{P}_0})_*(K_\Sigma)$ at $l_{\mathbf{P}_0}(\mathbf{p}, z)$ is given by

$$(dl_{\mathbf{P}_0})_{\binom{\mathbf{p}}{z}} \begin{pmatrix} A\mathbf{p} + \mathbf{q} \\ 1 \end{pmatrix} = \begin{pmatrix} A\mathbf{p} + \mathbf{q} \\ 1 \end{pmatrix},$$

while

$$(F_3)_{l_{\mathbf{P}_0}\binom{\mathbf{p}}{z}} = \begin{pmatrix} A(\mathbf{p}_0 + \mathbf{p}) \\ 1 \end{pmatrix}; \tag{7.1}$$

hence we only need to define $\mathbf{p}_0$ such that $A\mathbf{p}_0 = \mathbf{q}$, which can be done since A is regular. This proves the first statement in the lemma.

To prove the second assertion, assume that case (S2.3) holds with $K_\Sigma = F_3$. Hence, there exists $a \in X - \Gamma_\Sigma = X - \{z\text{-axis}\}$ such that the left translation by a leaves $f(\Sigma)$ invariant. Writing $a = (\mathbf{p}_0, z_0) \in \mathbb{R}^2 \rtimes_A \mathbb{R}$ as the product $(\mathbf{p}_0, 0) * (\mathbf{0}, z_0)$ and using that $(\mathbf{0}, z_0) \in \Gamma_\Sigma$, we conclude that a can be chosen so that $z_0 = 0$. As l_a leaves $f(\Sigma)$ invariant and F_3 is unique up to rescaling among right invariant vector fields which are nonzero and everywhere tangent to $f(\Sigma)$, then $(l_a)_*(F_3) = \lambda F_3$ for some $\lambda \in \mathbb{R}$. But the value at $l_a(\mathbf{p}, z) = (\mathbf{p}_0 + \mathbf{p}, z)$ of $(l_a)_*(F_3)$ is

$$(dl_a)_{\binom{\mathbf{p}}{z}} \begin{pmatrix} A\mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} A\mathbf{p} \\ 1 \end{pmatrix},$$

while $(F_3)_{l_a(\binom{\mathbf{p}}{z})}$ is given by (7.1). Since A is a regular matrix, then we deduce that $(l_a)_*(F_3)$ cannot be a multiple of F_3. This completes the proof. $\qquad \square$

7.3 Case (S2.2) is impossible when K_Σ is not horizontal

Assume that $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$ and that case (S2.2) above holds. By item 1 of Lemma 7.4, we can assume after a horizontal left translation that $K_\Sigma = F_3$. This of course may produce a change of base point for f, of the form $f(p) := Q \in \mathbb{R}^2 \rtimes_A \{0\}$. Recall from the beginning of Sect. 7 that $f : (\Sigma, p) \looparrowright (X, e)$ is obtained as a limit of pointed limit immersions $f_n : (S_n, p_n) \looparrowright (X, e)$ of spheres $S_n \in \mathcal{C}$. Applying to the f_n the same horizontal left translation that we applied to f in order to have $K_\Sigma = F_3$, we have $f_n(p_n) = Q$ for all n.

By Proposition 3.4, each sphere $S_n \in \mathcal{C}$ is Alexandrov embedded. In this way, for each $n \in \mathbb{N}$ there exists a three-dimensional Riemannian manifold (B_n, g_n) which is topologically a closed ball and a Riemannian submersion $\widetilde{f}_n : (B_n, g_n) \to X$ such that $\partial B_n = S_n$, $\widetilde{f}_n|_{S_n} = f_n$ and ∂B_n is mean convex.

Lemma 7.5 *In the conditions above, there is some $R^* > 0$ and points $q_n \in S_n$ such that the volumes of the Riemannian metric balls $B_{g_n}(q_n, R^*)$ in (B_n, g_n) of radius R^* centered at q_n tend to ∞ as $n \to \infty$.*

Proof As explained in Remark 4.2, the surface $f(\Sigma)$ can be obtained by pulling back via Π_{K_Σ} a closed immersed curve α contained in X/K_Σ. Since in our case $K = K_\Sigma = F_3$ and every integral curve of F_3 intersects any horizontal plane $\mathbb{R}^2 \rtimes_A \{z_0\}$ transversely at a single point, we can identify X/K_Σ with $\mathbb{R}^2 \rtimes_A \{0\}$.

Consider the vertical geodesic of X

$$\Gamma = \{\Gamma(t) = (0, 0, t) \mid t \in \mathbb{R}\}. \tag{7.2}$$

Given $r > 0$, let $\mathcal{W}(\Gamma, r)$ denote the solid metric cylinder of radius r around Γ, i.e., the set of points of X whose distance to Γ is at most r. Since $f(\Sigma)$ is an

annulus invariant under the flow of F_3, and since all cylinders $\mathcal{W}(\Gamma, r)$ are also invariant under this flow, it is clear that $f(\Sigma)$ is contained in a solid metric cylinder $\mathcal{W}_0 := \mathcal{W}(\Gamma, r_0)$ for some $r_0 > 0$, such that $\partial \mathcal{W}_0$ is at a positive distance from $f(\Sigma)$.

We next recall the following independent a priori radius estimate for compact stable minimal surfaces with boundary in $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, obtained by the authors in [25, Theorem 1.1]: there exists some $R = R(r_0) > 0$ associated to the solid metric cylinder $\mathcal{W}_0 \subset X$, with the following property:

($\star$) Let M be any compact, stable minimal surface in X whose boundary is contained inside $\mathcal{W}_0$. Then, the intrinsic radius[3] of M is less than R.

In particular, there are no complete stable minimal surfaces contained in $\mathcal{W}_0$. In our current setting, this has the consequence that $h_0(X) > 0$. Indeed, recall that the surface $f(\Sigma)$ is complete, stable, it has constant mean curvature equal to $h_0(X)$, and it is contained in $\mathcal{W}_0$. Thus, it cannot be minimal, by ($\star$); therefore, $h_0(X) > 0$.

Let us also consider the positive number $\mu_0 = \mu_0(r_0)$ defined in the following way. Observe that $\mathcal{W}_0$ is foliated by integral curves of F_3. We define μ_0 as the maximum length of all the arcs of integral curves of F_3 that are contained in the compact piece of $\mathcal{W}_0$ that lies in the slab determined by the planes $\mathbb{R}^2 \rtimes_A \{R+2\}$ and $\mathbb{R}^2 \rtimes_A \{-R-2\}$; note that μ_0 exists since $\mathcal{W}_0 \cap [\mathbb{R}^2 \rtimes_A \{0\}]$ is compact.

As Σ is simply connected and $f : \Sigma \hookrightarrow X$ is invariant under the flow of F_3, then, by the discussion in Remark 4.2, we may parameterize f as

$$f(s, t) = l_{\Gamma(t)}(\alpha(s)), \quad (s, t) \in \mathbb{R}^2, \tag{7.3}$$

where $\alpha = \alpha(s) : \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ is a curve parameterized by arc length in $\mathbb{R}^2 \rtimes_A \{0\}$, and so that under the natural identification $\Sigma \equiv \{(s, t) \in \mathbb{R}^2\}$, the base point $p \in \Sigma$ for f corresponds to $(0, 0)$. In particular, $f(0, 0) = \alpha(0) = Q$. Note that since we are in the conditions of case (S2.2), the image $\alpha(\mathbb{R})$ is an immersed closed curve and $\alpha(s)$ is L-periodic, where L is the length of $\alpha(\mathbb{R})$.

For each $s_0 \in \mathbb{R}$ we may define numbers $c^-(s_0) < 0, c^+(s_0) > 0$ such that

$$A(s_0) := \{l_{\Gamma(t)}(\alpha(s_0)) \mid c^-(s_0) \leq t \leq c^+(s_0)\} \tag{7.4}$$

is the arc of the integral curve of F_3 that passes through $\alpha(s_0)$ whose endpoints lie in $\mathbb{R}^2 \rtimes_A \{R+2\}$ and $\mathbb{R}^2 \rtimes_A \{-R-2\}$. Note that $A(s_0) \subset f(\Sigma)$, and that the length of $A(s_0)$ is at most μ_0.

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[3] The radius of a compact Riemannian manifold M with boundary is the maximum distance of points in M to its boundary ∂M.
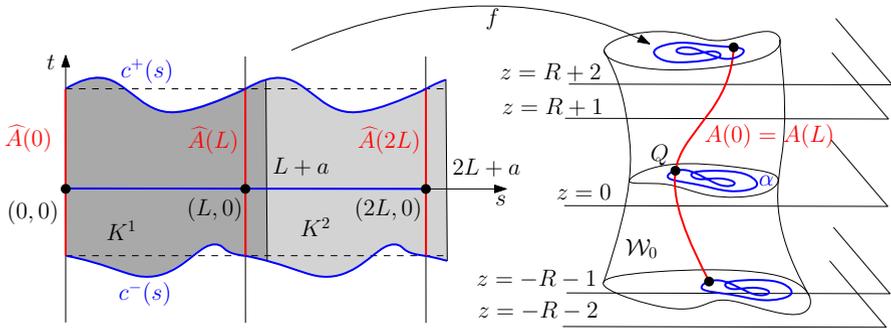
Fig. 6 Left: the compact regions $K^1 \subset K^2$ in the (s, t)-plane. The vertical segments $\widehat{A}(0)$, $\widehat{A}(L)$, $\widehat{A}(2L)$ apply through f into the compact arc $A(0)$, which is part of the integral curve of F_3 passing through Q. Each K^j wraps j times through f around its image, due to the L-periodicity of f in the s-variable

Take some $a \in (0, L)$ and consider for each $j \in \mathbb{N}$ the compact region of Σ (viewed as $\mathbb{R}^2$ with coordinates (s, t)) given by

$$K^j = \{(s, t) \mid 0 \leq s \leq jL + a, \ c^-(s) \leq t \leq c^+(s)\}, \qquad (7.5)$$

see Fig. 6.

Since f is a limit surface of the f_n, it follows that for n large enough, there exist compact simply connected domains $K_n^j \subset S_n$ such that $f_n(K_n^j) \subset \mathcal{W}_0$ and the sequence $\{f_n|_{K_n^j}\}_n$ converges to $f|_{K^j}$ uniformly as $n \to \infty$. For $\varepsilon > 0$ sufficiently small and less than the injectivity radius of Σ, denote by $\mathcal{U}^k$ the closed geodesic disk in Σ of radius ε centered at $(kL, 0)$ with $k \in \{1, \ldots, j\}$. After choosing $\varepsilon > 0$ sufficiently small, we may assume that $\{\mathcal{U}^1, \ldots, \mathcal{U}^j\}$ forms a pairwise disjoint collection, and the intrinsic distance between $\mathcal{U}^i, \mathcal{U}^k$ for $i, k \in \{1, \ldots, j\}, i \neq k$, is greater than 2δ for some $\delta > 0$ independent of j. Also, there exist closed geodesic disks of radius ε, $\mathcal{U}_n^k \subset K_n^j$, such that $\{f_n|_{\mathcal{U}_n^k}\}_n$ converges to $f|_{\mathcal{U}^k}$ uniformly as $n \to \infty$. Observe that for each j, there exists $n_0(j) \in \mathbb{N}$ such that if $n \geq n_0(j)$, the related compact disks $\mathcal{U}_n^k \subset K_n^j$ satisfy that the intrinsic distance between $\mathcal{U}_n^i$ and $\mathcal{U}_n^k$ inside K_n^j is greater than δ for every $i, k \in \{1, \ldots, j\}, i \neq k$. It is important to notice that δ does not depend on j, since $f(s, t)$ is L-periodic with respect to s.

Let I denote a small compact segment in $\mathbb{R}^2 \rtimes_A \{0\}$ centered at Q and transversal to $f(\Sigma)$ at Q. Let $\alpha_n \subset S_n$ denote the set $f_n^{-1}(\mathbb{R}^2 \rtimes_A \{0\})$. Since f is a limit of the f_n and $f(\Sigma)$ intersects $\mathbb{R}^2 \rtimes_A \{0\}$ transversely, then by the Transversality Lemma (Lemma 5.3), α_n is a closed simple curve in S_n, for n large enough; to see this, note that $\mathbb{R}^2 \rtimes_A \{0\}$ is a two-dimensional subgroup of X. As $f(s, t)$ is L-periodic in the variable s, the previous convergence properties imply that for $n = n(j)$ large enough there exist j points

$q_n^1, \ldots, q_n^j \in \alpha_n \cap K_n^j$ such that, for every $k = 1, \ldots, j$, we have $q_n^k \in \mathcal{U}_n^k$ and $f_n(q_n^k) \in I$. In particular, we note for later use the following property:

(★★) For n large enough, the intrinsic compact metric disks $D_{S_n}(q_n^k, \delta/2)$ in S_n of radius $\delta/2$ and center q_n^k, $k = 1, \ldots, j$, are pairwise disjoint.

Let J_n^k be the compact connected arc of α_n contained in K_n^j whose endpoints are q_n^1 and q_n^k, for each $k \in \{1, \ldots, j\}$. Let W_n be the vector field on S_n obtained by pulling back via f_n the tangent part to $f_n(S_n)$ of the right invariant vector field $K_\Sigma = F_3$; clearly, W_n has no zeros on $K_n^j \subset S_n$ for n large enough, since K_Σ has no zeros, is everywhere tangent to $f(\Sigma)$ and $\{f_n|_{K_n^j}\}_n$ converges to $f|_{K^j}$ uniformly as $n \to \infty$. Moreover, again for n large enough, the integral curves of W_n that start at any point of any of the arcs J_n^k satisfy that their images by f_n intersect both planes $\mathbb{R}^2 \rtimes_A \{-R-1\}$ and $\mathbb{R}^2 \rtimes_A \{R+1\}$. In particular, we can define the compact disks $D_n^k \subset S_n$ obtained by letting the arc J_n^k flow under W_n between the sets $f_n^{-1}(\mathbb{R}^2 \rtimes_A \{-R-1\})$ and $f_n^{-1}(\mathbb{R}^2 \rtimes_A \{R+1\})$ of S_n. In fact, $D_n^k \subset K_n^j$ for all $k \in \{1, \ldots, j\}$ and for n large.

Let $\gamma_n^k := \partial D_n^k$, which is a Jordan curve in S_n. Observe that γ_n^k can be written as a union $A_1^n \cup A_2^n \cup B_1^n \cup B_2^n$, where:

(T1) A_1^n, A_2^n are compact arcs of integral curves of W_n that pass through q_n^1 and q_n^k, respectively. In particular, the endpoints of both $f_n(A_1^n)$ and $f_n(A_2^n)$ lie on the planes $\mathbb{R}^2 \rtimes_A \{-R-1\}$, $\mathbb{R}^2 \rtimes_A \{R+1\}$. Moreover, since $f_n(A_1^n)$ and $f_n(A_2^n)$ converge uniformly as $n \to \infty$ to proper subarcs of the compact arc $A(0)$ defined by equation (7.4), then the lengths of A_1^n and A_2^n are smaller than the constant μ_0 for n large enough.

(T2) The arc $f_n(B_1^n)$ (resp. $f_n(B_2^n)$) lies in the plane $\mathbb{R}^2 \rtimes_A \{-R-1\}$ (resp. $\mathbb{R}^2 \rtimes_A \{R+1\}$).

Recall that (B_n, g_n) is the abstract Riemannian three-ball that S_n bounds, and $\widetilde{f}_n \colon (B_n, g_n) \to X$ is a Riemannian submersion with mean convex boundary $\widetilde{f}_n|_{\partial B_n} = f_n$. In particular, S_n is a good barrier to solve Plateau problems in B_n in the sense of Meeks and Yau [33]. More precisely, for each $k \in \{1, \ldots, j\}$ (and n large enough), there exists a compact stable embedded minimal disk $M_n^k \subset B_n$ of least area, with $\partial M_n^k = \gamma_n^k$. Let us stress here that M_n^k is embedded in the abstract ball B_n; in contrast, the image $\widetilde{f}(M_n^k) \subset X$ of M_n^k via the Riemannian submersion $\widetilde{f}$ might be non-embedded in X. Furthermore, also by [33], M_n^k is an immersion transverse to ∂B_n at the regular points of its boundary γ_n^k (this transversality property does not hold at the vertices of γ_n^k). In particular, M_n^k is an immersion transverse to ∂B_n at the interior points of the arcs A_1^n and A_2^n. Therefore, if we denote by $\Lambda_n \subset B_n$ the embedded surface $\{q \in B_n \mid \widetilde{f}_n(q) \in \mathbb{R}^2 \rtimes_A \{0\}\}$ and $\beta_n^k := M_n^k \cap \Lambda_n$, then $\beta_n^k \cap \partial M_n^k = \{q_n^1, q_n^k\}$, and the intersection of M_n^k with Λ_n is clearly transverse at q_n^1, q_n^k since the arcs A_1^n, A_2^n are transverse to Λ_n. Moreover, up to a small perturbation of Λ_n in B_n
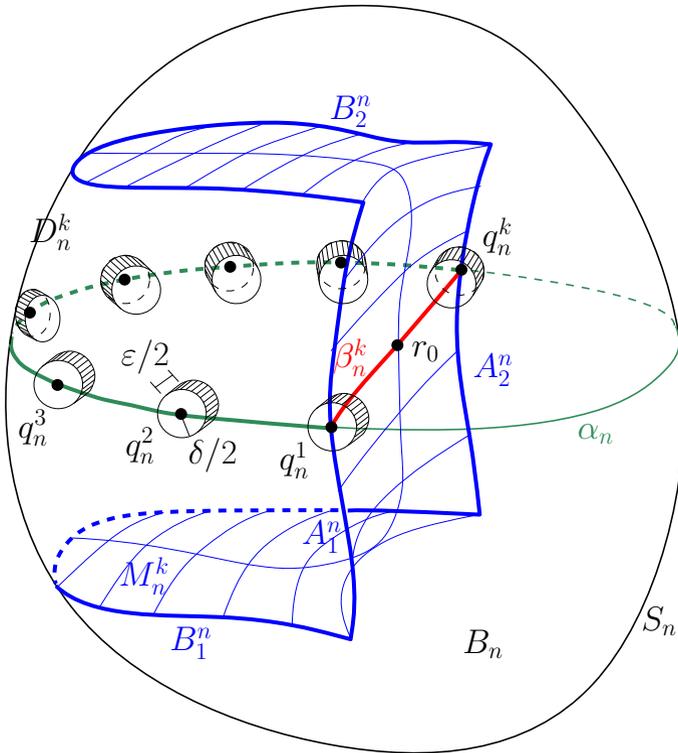
Fig. 7 The least area disk M_n^k inside the abstract Riemannian ball B_n has the same boundary γ_n^k as the disk $D_n^k \subset S_n = \partial B_n$. γ_n^k consists of four consecutive arcs A_1^n, B_1^n, A_2^n, B_2^n satisfying properties (T1), (T2)

that fixes its boundary, we may assume that the intersection of M_n^k with Λ_n is transverse. Therefore, β_n^k is a (possibly disconnected) compact 1-dimensional manifold with boundary, and a connected component of β_n^k is a compact arc in B_n joining q_n^1 with q_n^k. We will keep denoting this compact arc by β_n^k, see Fig. 7.

Take $r \in \beta_n^k$. We next give a lower bound of $d_X(\widetilde{f}_n(r), f_n(B_1^n \cup B_2^n))$, the distance in X from $\widetilde{f}_n(r)$ to $f_n(B_1^n \cup B_2^n)$. Since $f_n(B_1^n \cup B_2^n)$ is contained in the union of planes $\mathbb{R}^2 \rtimes_A \{\pm(R+1)\}$, then $d_X(\widetilde{f}_n(r), f_n(B_1^n \cup B_2^n)) \geq d_X(\widetilde{f}_n(r), \mathbb{R}^2 \rtimes_A \{\pm R_1\})$. As $\widetilde{f}_n(r)$ lies at an arbitrarily small distance of $\mathbb{R}^2 \rtimes_A \{0\}$ and $d_X(\mathbb{R}^2 \rtimes_A \{0\}, \mathbb{R}^2 \rtimes_A \{\pm(R+1)\}) = R+1$ (this follows, for instance, from [28, Lemma 3.9], since $\mathbb{R}^2 \rtimes_A \{\pm(R+1)\}$ are right cosets of $\mathbb{R}^2 \rtimes_A \{0\}$), then $d_X(\widetilde{f}_n(r), \mathbb{R}^2 \rtimes_A \{\pm(R+1)\})$ can be taken arbitrarily close to $R+1$. In particular,

$$d_X(\widetilde{f}_n(r), f_n(B_1^n \cup B_2^n)) > R, \quad \text{for all } r \in \beta_n^k. \tag{7.6}$$

As $\widetilde{f}_n|_{M_n^k} \colon M_n^k \hookrightarrow X$ is by construction a compact immersed stable minimal disk in X whose boundary is contained in the solid metric cylinder $\mathcal{W}_0$, we deduce from the radius estimate $(\star)$ above that for any $r \in \beta_n^k$ the distance $d_{M_n^k}(r, \gamma_n^k)$ in M_n^k from r to $\gamma_n^k = \partial M_n^k$ is not greater than R. Note that $d_{M_n^k}(r, B_1^n \cup B_2^n) \geq d_X(\widetilde{f}_n(r), f_n(B_1^n \cup B_2^n))$, which by (7.6) is greater than R. Therefore,

$$d_{M_n^k}(r, \gamma_n^k) = d_{M_n^k}(r, A_1^n \cup A_2^n), \quad \text{for all } r \in \beta_n^k. \tag{7.7}$$

Moreover, since $q_n^1 \in A_1^n$ and $q_n^k \in A_2^n$, we deduce by the continuity and connectedness of the arc β_n^k that there exists some *midpoint* $r_0 \in \beta_n^k$ such that $d_{M_n^k}(r_0, A_1^n) = d_{M_n^k}(r_0, A_2^n)$. Thus, we can estimate the distance in B_n from r_0 to A_i^n, $i = 1, 2$, by

$$
\begin{aligned}
d_{B_n}(r_0, A_i^n) \;&\leq\; d_{M_n^k}(r_0, A_i^n) \\
&=\; d_{M_n^k}(r_0, A_1^n \cup A_2^n) \quad (r_0 \text{ is a midpoint}) \\
&=\; d_{M_n^k}(r_0, \gamma_n^k) \qquad \text{(by Eq. (7.7))} \\
&\leq\; R.
\end{aligned}
\tag{7.8}
$$

Therefore,

$$
\begin{aligned}
d_{B_n}(q_n^1, q_n^k) \;&\leq\; d_{B_n}(q_n^1, r_0) + d_{B_n}(r_0, q_n^k) && \text{(triangle inequality)} \\
&\leq\; \mu_0 + d_{B_n}(r_0, A_1^n) + \mu_0 + d_{B_n}(r_0, A_2^n) && \text{(triangle inequality and (T1))} \\
&\leq\; 2\mu_0 + 2R. && \text{(inequality (7.8))}
\end{aligned}
\tag{7.9}
$$

Recall that, by item 5 of Proposition 3.4, the norms of the second fundamental forms of the spheres S_n are uniformly bounded by some constant $C > 0$. Also, note that since we have already proved that $h_0(X) > 0$ in this case, the values H_n of the mean curvatures of S_n are uniformly bounded away from zero (since $H_n > h_0(X)$ for every n). Hence, by [30], there exists some $\varepsilon_1 > 0$ smaller than the injectivity radius of B_n such that $S_n = \partial B_n$ has a regular neighborhood $\mathcal{V}_n^{\varepsilon_1}$ in B_n of fixed size ε_1 (independent of n). In other words, we have a diffeomorphism

$$\Phi_n \colon S_n \times [0, \varepsilon_1) \to \mathcal{V}_n^{\varepsilon_1}, \quad \Phi(x, t) = \exp_x^n(t N_n(x))$$

where N_n is the inward pointing unit normal in (B_n, g_n) of the mean convex sphere $S_n = \partial B_n$ and $\exp^n$ denotes the exponential map in (B_n, g_n).

By Property $(\star\star)$ above, for n large enough, the solid cylinders $C_n^1, \ldots, C_n^j$ in $\mathcal{V}_n^{\varepsilon_1}$ given by the image under Φ_n of the compact cylinders $D_{S_n}(q_n^k, \delta/2) \times [0, \varepsilon_1/2]$ are mutually disjoint, see Fig. 7. Also, note that the volume of each C_n^k in (B_n, g_n) is greater than some $V > 0$, independent of n, j and k (again this follows from the uniform bound of the second fundamental forms of the

S_n, as well as from the fact ε_1 does not depend on n). In particular, for n large enough, the total volume in (B_n, g_n) of the (disjoint) union $C_n^1 \cup \cdots \cup C_n^j$ is at least jV. Note that this volume can be made arbitrarily large, as $j \in \mathbb{N}$ was chosen fixed but arbitrary.

Also, by the triangle inequality we have for any $x \in C_n^k$, $k \in \{1, \ldots, j\}$, that $d_{B_n}(q_n^k, x) \leq \frac{\varepsilon_1 + \delta}{2}$. This inequality and (7.9) imply that

$$d_{B_n}(q_n^1, x) \leq d_{B_n}(q_n^1, q_n^k) + d_{B_n}(q_n^k, x) \leq 2(\mu_0 + R) + \frac{\varepsilon_1 + \delta}{2},$$

from where we deduce that $C_n^1 \cup \cdots \cup C_n^j$ is contained in the Riemannian ball of (B_n, g_n) centered at q_n^1 and of radius $R^* := 2(\mu_0 + R) + \frac{\varepsilon_1 + \delta}{2}$.

In particular, we have proved the following: given a fixed but arbitrary $j \in \mathbb{N}$, then for all $n \in \mathbb{N}$ large enough, the metric ball $B_{g_n}(q_n^1, R^*)$ in the Riemannian manifold (B_n, g_n) centered at $q_n^1 \in \partial B_n$ of radius R^* has volume at least jV, where $V, R^* > 0$ do not depend on j, n. This clearly proves Lemma 7.5. □

We will next obtain a contradiction with Lemma 7.5, that will prove that case (S2.2) is impossible when $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$, which was the objective of the present Sect. 7.3. First, observe that as the second fundamental forms of the f_n are uniformly bounded, there exists $\delta > 0$ such that, for all n, the following properties hold:

(U1) (B_n, g_n) can be extended to a compact Riemannian three-manifold with boundary $(\widetilde{B}_n, \widetilde{g}_n)$, in the sense that B_n is a compact subset of $\widetilde{B}_n$ and $\widetilde{g}_n|_{B_n} = g_n$.

(U2) The map $(x, t) \in S_n \times [-\delta, 0] \to \exp_x^n(t N_n(x)) \in \widetilde{B}_n - \mathrm{Int}(B_n)$ is a diffeomorphism, where $\exp^n$ denotes the exponential map of $(\widetilde{B}_n, \widetilde{g}_n)$ and N_n is the unit normal vector to S_n at x with respect to $\widetilde{g}_n$ such that $H_n N_n$ is the mean curvature vector of S_n.

(U3) The metrics $\widetilde{g}_n$ are uniformly bounded in the C^2-topology.

Let $\varphi \colon (-\delta, 0] \to (0, \infty)$ be a smooth positive function such that $\varphi = 1$ in $[-\delta/3, 0]$ and $\varphi(t) = \frac{1}{t+\delta}$ in $(-\delta, -2\delta/3]$. We define the complete Riemannian metric $\widehat{g}_n$ on $\widetilde{B}_n - \partial \widetilde{B}_n$ by

$$\widehat{g}_n = \begin{cases} g_n & \text{in } B_n, \\ \varphi^2 \widetilde{g}_n & \text{in } S_n \times (-\delta, 0] \end{cases}$$

(we are identifying $S_n \times [-\delta, 0]$ with $\widetilde{B}_n - \mathrm{Int}(B_n)$ through the diffeomorphism that appears in property (U2)). Observe that

$$\widehat{g}_n = \frac{1}{(t+\delta)^2} \widetilde{g}_n \quad \text{in} \quad S_n \times (-\delta, -2\delta/3]. \tag{7.10}$$

As $\widetilde{g}_n|_{\partial \widetilde{B}_n}$ is a Riemannian metric on the closed surface $\partial \widetilde{B}_n$, then (7.10) implies that given $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) \in (2\delta/3, \delta)$ such that the restriction of $\widehat{g}_n$ to $S_n \times (-\delta, -\delta_1]$ is ε-close in the C^2-topology to the hyperbolic metric of constant sectional curvature -1. In particular, the Ricci curvature of $\widehat{g}_n$ satisfies

$$\left| \mathrm{Ric}_{\widehat{g}_n} + 2 \right|_{L^\infty(S_n \times (-\delta, -\delta_1])} < b(\varepsilon), \tag{7.11}$$

where $b(\varepsilon)$ can be made arbitrarily small for ε small.

As the coefficients of the Ricci tensor of $\varphi^2 \widetilde{g}_n$ are smooth expressions of the coefficients of $\widetilde{g}_n$, of those of the Ricci tensor of $\widetilde{g}_n$ and of $\varphi, \varphi', \varphi''$, then property (U3) and the smoothness of φ imply that the Ricci curvature of $\varphi^2 \widetilde{g}_n$ satisfies for all n:

$$\mathrm{Ric}_{\varphi^2 \widetilde{g}_n} \text{ is uniformly bounded in } S_n \times [-\delta_1, 0]. \tag{7.12}$$

As g_n is locally homogeneous, (7.11) and (7.12) imply that there exist $c < 0$ independent of n such that $\mathrm{Ric}_{\widehat{g}_n} \geq 2c$ in $\widetilde{B}_n - \partial \widetilde{B}_n$ for all n. Since $(\widetilde{B}_n - \partial \widetilde{B}_n, \widehat{g}_n)$ is a complete Riemannian manifold, then Bishop's comparison theorem gives the following upper bound for the volume of the metric ball $B_{\widehat{g}_n}(q_n, R^*)$ in $(\widetilde{B}_n - \partial \widetilde{B}_n, \widehat{g}_n)$ of radius $R^* > 0$ centered at q_n:

$$\mathrm{Vol}\left(B_{\widehat{g}_n}(p_n, R^*) \right) \leq V(c, R^*),$$

where $V(c, R^*)$ denotes the volume of any metric ball of radius R^* in the three-dimensional space form of constant sectional curvature c. This is a contradiction with Lemma 7.5, as desired.

7.4 Case (S2.1) is impossible when K_Σ is not horizontal

Assume that case (S2.1) holds and $K_\Sigma \notin \mathrm{Span}\{F_1, F_2\}$. We start with the same normalizations as in the previous Sect. 7.3. So, $K_\Sigma = F_3$, and $f : (\Sigma, p) \looparrowright (X, Q)$ (here $Q \in \mathbb{R}^2 \rtimes_A \{0\}$) is obtained as a limit of pointed immersions $f_n : (S_n, p_n) \looparrowright (X, Q)$ of spheres $S_n \in \mathcal{C}$. Also, we will consider the compact Riemannian balls (B_n, g_n) with $\partial B_n = S_n$, and Riemannian submersions $\widetilde{f}_n : (B_n, g_n) \to X$ such that ∂B_n is mean convex and $\widetilde{f}_n|_{\partial B_n} = f_n$. Let $\alpha_n := f_n^{-1}(\mathbb{R}^2 \rtimes_A \{0\}) \subset S_n$, which is a simple closed curve in S_n for n large enough, by the Transversality Lemma (Lemma 5.3). Also, let K_Σ^n be the Killing field on B_n that is induced by K_Σ via the pullback by the submersion $\widetilde{f}_n$.

Lemma 7.6 *In the conditions above,* $\alpha_\infty := f^{-1}(\mathbb{R}^2 \rtimes_A \{0\}) \subset \Sigma$ *is a simple closed curve in* Σ, *and the sequence* $\{f_n|_{\alpha_n}\}_n$ *converges uniformly to the immersion* $f|_{\alpha_\infty}$.

Proof Recall that in this case (S2.1), Σ is an annulus. Also, recall that the space X/K_Σ whose points are the integral curves of $K_\Sigma = F_3$ can be identified with $\mathbb{R}^2 \rtimes_A \{0\}$; see the beginning of the proof of Lemma 7.5. In particular, by Remark 4.2, $f(\Sigma)$ is the immersed annulus obtained by letting the immersed closed curve $\beta := f(\Sigma) \cap (\mathbb{R}^2 \rtimes_A \{0\})$ flow under F_3. Also, by the argument in the proof of item 1 of Corollary 6.7, we have that $\alpha_\infty := f^{-1}(\mathbb{R}^2 \rtimes_A \{0\})$ is an embedded closed curve in Σ that generates the homology group of the cylinder Σ. Note that $f|_{\alpha_\infty} : \alpha_\infty \to \mathbb{R}^2 \rtimes_A \{0\}$ is not an embedding, and it could actually be a finite covering of the closed immersed curve $\beta = f(\alpha_\infty)$.

Finally, the convergence property of $\{f_n|_{\alpha_n}\}_n$ to $f|_{\alpha_\infty}$ follows from the convergence of the pointed immersions $f_n : (S_n, p_n) \looparrowright (X, Q)$ to $f : (\Sigma, p) \looparrowright (X, Q)$, and the compactness of α_∞ in Σ. $\square$

By Theorems 1 and 2 in [33], α_n is the boundary of a smooth, embedded least-area disk $M(n)$ in the abstract Riemannian 3-ball B_n and this disk is transverse to $S_n = \partial B_n$ along its boundary α_n. Note that, even though $M(n)$ is embedded in B_n, $\tilde{f}_n(M(n))$ might be non-embedded in X. Also, note that the intrinsic radii of the disks $M(n)$ are bounded by a universal constant $R > 0$ by Theorem 1.1 in [25], since their compact image boundary curves $f_n|_{\partial M(n)} = f_n|_{\alpha_n}$ converge uniformly as $n \to \infty$ to $f|_{\alpha_\infty}$, and $f(\alpha_\infty)$ is contained in a solid metric cylinder around $\Gamma = \{(0, 0, t) \mid t \in \mathbb{R}\}$ (so the same holds for $f_n(\partial M(n))$ for all $n \in \mathbb{N}$, for a slightly larger solid metric cylinder around Γ).

Notice that each of the immersed curves α_n is the boundary curve of the intrinsically flat "horizontal" disk in $D_n \subset B_n$. Since the lengths of the curves α_n are uniformly bounded, then the areas of the disks D_n are also uniformly bounded by the classical isoperimetric inequality for flat disks. In particular, the areas of the least-area disks $M(n)$ are also uniformly bounded. It follows that a subsequence of the immersed stable minimal disks $\tilde{f}_n(M(n)) \subset X$, which lie locally in the mean convex side of $f_n(S_n)$ near $f_n(\alpha_n)$, converge uniformly on compact sets to an immersed stable minimal disk $M(\infty)$ in X with boundary $f(\alpha_\infty)$. We claim that $M(n)$ lies in the piecewise smooth, closed upper half-ball $B_n^+ = \tilde{f}_n^{-1}(\mathbb{R}^2 \rtimes_A [0, \infty))$ determined by the disk $\tilde{f}_n^{-1}(\mathbb{R}^2 \rtimes_A \{0\}) \subset B_n$. To see this, first recall that for every $z_0 \in \mathbb{R}$, the halfspace of the form $\mathbb{R}^2 \rtimes_A [z_0, \infty)$ is mean convex in X, and the constant mean curvature of its boundary is equal to $H(X) \geq 0$; recall that if $\text{tr}(A) = 0$, then $H(X) = 0$ and so both half-spaces limited by $\mathbb{R}^2 \rtimes_A \{z_0\}$ are mean convex. Since $f_n(\alpha_n) \subset \mathbb{R}^2 \rtimes_A \{0\}$ is the boundary of the immersed minimal disk $f_n(M(n))$, then the maximum principle (or the mean curvature comparison

principle) applied to $M(n)$ and to the smooth surfaces $\widetilde{f}_n^{-1}(\mathbb{R}^2 \rtimes_A \{z\})$, $z \in \mathbb{R}$, implies that $M(n) \subset B_n^+$ as desired.

We next claim that there exists an $\eta \in (0, \pi/4)$ such that the angle that $M(n)$ makes with S_n along α_n is greater than η for all $n \in \mathbb{N}$ large enough. Assume that the claim does not hold; in this case, $M(\infty)$ is tangent to $f(\Sigma)$ at some boundary point $Q' \in \partial M(\infty)$. As $M(\infty)$ lies locally in the mean convex side of $f(\Sigma)$ along $f(\alpha_\infty)$, then the Hopf boundary maximum principle implies that $f(\Sigma)$ coincides with $M(\infty)$ nearby Q'. This is a contradiction since we know that $h_0(X) > 0$ in this case, as it was explained just after property ($\star$) in the previous section. Thus, the claim at the beginning of this paragraph holds.

Let $N_{M(n)}$ denote the unit normal vector field to $M(n)$ in (B_n, g_n). We claim that for n sufficiently large, $g_n(N_{M(n)}, K_\Sigma^n)$ has no zeros along α_n. Arguing by contradiction, suppose that after passing to a subsequence, K_Σ^n is tangent to $M(n)$ at some point $q_n \in \alpha_n$ for all n. Then $K_\Sigma^n(q_n)$ and the unit tangent vector w_n to α_n at q_n generate the tangent space $T_{q_n} M(n)$. By the last paragraph, the angle between $T_{q_n} M(n)$ and $T_{q_n} S_n$ is greater than η. Since K_Σ is everywhere tangent to $f(\Sigma)$, then the angle that $K_\Sigma^n(q_n)$ makes with $T_{q_n} S_n$ tends to zero as $n \to \infty$. Since $K_\Sigma^n(q_n)$ is tangent to $M(n)$ at q_n, then the angle that $K_\Sigma^n(q_n)$ makes with w_n becomes arbitrarily small if n is large enough (note that $w_n \in T_{q_n} S_n \cap T_{q_n} M(n)$). Applying the differential of $\widetilde{f}_n$ at q_n, we deduce that the angle between $(K_\Sigma)_{f_n(q_n)} = (F_3)_{f_n(q_n)}$ and $(df_n)_{q_n}(w_n)$ tends to zero as $n \to \infty$. Since $(df_n)_{q_n}(w_n)$ is a nonzero horizontal vector, this only can occur provided that $(F_3)_{f_n(q_n)}$ becomes horizontal as $n \to \infty$, or equivalently, the ratio between the component of $(F_3)_{f_n(q_n)}$ in the direction of ∂_z and its horizontal component tends to zero. By Eq. (4.5), this only can occur if $f_n(q_n)$ diverges in $\mathbb{R}^2 \rtimes_A \{0\}$. But this contradicts the fact that $f(\alpha_\infty)$ lies at a finite distance from the origin. Hence the claim follows.

Therefore, after replacing by a subsequence and choosing $N_{M(n)}$ so that the Jacobi function $J_n = g_n(N_{M(n)}, K_\Sigma^n)$ is positive along α_n, we deduce from the stability of the minimal disk $M(n)$ that J_n is positive on $M(n)$.

Consider the Jacobi function $J_\infty = \langle N_{M(\infty)}, K_\Sigma \rangle$ defined on the stable minimal disk $M(\infty)$. Observe that J_∞ does not vanish at any point of $\partial M(\infty) = f(\alpha_\infty)$ (otherwise $M(\infty)$ would become tangent to $f(\Sigma)$ at some boundary point of $M(\infty)$, which we have seen that contradicts the Hopf boundary maximum principle). Once we know that J_∞ does not vanish at any point of $\partial M(\infty)$, then $J_\infty|_{\partial M(\infty)}$ is positive (because $J_n|_{\alpha_n}$ is positive for n large), and thus, $J_\infty > 0$ on $M(\infty)$ because $M(\infty)$ is stable.

Consider the mapping $T_n \colon M(n) \times \mathbb{R} \to X$ given by

$$T_n(q, t) = l_{\Gamma(t)}(\widetilde{f}_n(q)), \tag{7.13}$$

where $\Gamma(t)$ is given by (7.2). As J_n is positive on $M(n)$, T_n is a proper submersion of $M(n) \times \mathbb{R}$ into X (note that the projection $(x, t) \mapsto t$ is proper). Our next goal is to take limits of these immersions T_n. To do this, we will use conformal parameters for $M(n)$ and $M(\infty)$ and a standard three-point condition argument.

Specifically, let $\overline{\mathbb{D}}$ be the closed unit disk in $\mathbb{R}^2$ and parameterize each $M(n)$ by a conformal diffeomorphism $\phi_n \colon \overline{\mathbb{D}} \to M(n)$ in such a way that the points $(1, 0), (0, 1), (-1, 0)$ of $\partial \mathbb{D}$ have as images by $f_n \circ \phi_n$ three ordered points w_1^n, w_2^n, w_3^n in $f_n(\alpha_n)$. Similarly, let $\widetilde{f}_\infty \colon \overline{\mathbb{D}} \hookrightarrow X$ be the analogous conformal parametrization of $M(\infty)$ with respect to three ordered points ξ_1, ξ_2, ξ_3 in $f(\alpha_\infty)$. Since $\{f_n|_{\alpha_n}\}_n$ converges uniformly to $f|_{\alpha_\infty}$ by Lemma 7.6, we can choose the points w_i^n so that they converge to ξ_i, for $i = 1, 2, 3$. With this choice, and taking in mind the a priori radius estimates, the uniform area estimates and the boundary convergence of the minimal disks $D(n)$, we can use a classical argument via the Courant-Lebesgue theorem to ensure that the conformal immersions $\widetilde{f}_n \circ \phi_n \colon \overline{\mathbb{D}} \hookrightarrow X$ converge smoothly up to a subsequence to the conformal immersion $\widetilde{f}_\infty \colon \overline{\mathbb{D}} \hookrightarrow X$. Details of this standard compactness argument can be found for instance in [32, pages 423–424], see also [35].

Note that as $n \to \infty$, the proper immersions $T_n \circ (\phi_n \times 1_\mathbb{R}) \colon \overline{\mathbb{D}} \times \mathbb{R} \to X$ converge to the submersion $T_\infty \colon \overline{\mathbb{D}} \times \mathbb{R} \to X$ given by

$$T_\infty(x, t) = l_{\Gamma(t)}(\widetilde{f}_\infty(x)), \tag{7.14}$$

which again is a proper submersion since $(x, t) \in \overline{\mathbb{D}} \times \mathbb{R} \to \mathbb{R} \mapsto t \in \mathbb{R}$ is proper (because J_∞ is positive on $M(\infty)$). Consider the codimension-one foliation of $\overline{\mathbb{D}} \times \mathbb{R}$ given by the family of compact disks

$$\mathcal{F} = \{D_t = T_\infty^{-1}(\mathbb{R}^2 \rtimes_A \{t\}) \mid t \in \mathbb{R}\}. \tag{7.15}$$

Let K_1 be the nowhere zero, smooth vector field on $\overline{\mathbb{D}} \times \mathbb{R}$ obtained after pulling back K_Σ through T_∞. Clearly, each of the integral curves of K_1 in $\overline{\mathbb{D}} \times \mathbb{R}$ intersects the disk D_0 in a single point. Thus, we can view $\overline{\mathbb{D}} \times \mathbb{R}$ in a natural manner as being the topological product $W := D_0 \times \mathbb{R}$, and we endow this three-manifold with boundary with the pulled-back metric by T_∞. We will find the desired contradiction as an application of Theorem 9.1 below, as we explain next.

Observe that the boundary $\partial W = (\partial D_0) \times \mathbb{R}$ of W satisfies

$$\begin{aligned} T_\infty(\partial W) &= \{l_{\Gamma(t)}(T_\infty(x)) \mid x \in \partial D_0, \ t \in \mathbb{R}\} \\ &= \{l_{\Gamma(t)}(f(x)) \mid x \in \alpha_\infty, \ t \in \mathbb{R}\} = f(\Sigma). \end{aligned} \tag{7.16}$$

Recall that $f(\Sigma)$ has constant mean curvature $h_0(X) > 0$ in X with respect to the unit normal field obtained as the limit of the inward pointing unit normal fields to the spheres S_n; see the paragraph after property $(\star)$ in Sect. 7.3. Also note that the inward pointing unit normal field on ∂W has positive inner product with the inner conormal to $M(\infty)$ by the Hopf boundary maximum principle, and that we can view $M(\infty)$ as an embedded minimal disk in W. Since $M(\infty)$ can be viewed as the limit of the disks $M(n)$, each of which lies in the mean convex ball B_n bounded by S_n, then we conclude that ∂W has constant mean curvature $h_0(X)$ with respect to the inward pointing normal field to its boundary (thus, W has mean convex boundary of constant mean curvature).

The vector field K_1 is clearly a Killing field on W endowed with the pulled-back metric by T_∞. Note that K_1 arises from the proper action of $\Gamma = \Gamma(t)$, which we can view as a 1-parameter subgroup of isometries of W.

We claim that there exists a nowhere zero Killing field K_2 in W which is tangent to the foliation $\mathcal{F}$ defined in (7.15), bounded on the end of W given by the end representative $D_0 \times [0, \infty)$ and such that the closure in W of every integral curve of K_2 in $\mathrm{Int}(W)$ is a compact arc with end points in ∂W. To see this, first note that by Lemma 4.7 there exists a nonzero horizontal right invariant vector field V in X that is bounded in the halfspace $\mathbb{R}^2 \rtimes_A [0, \infty)$. Let K_2 be the Killing field in W defined as the pullback by T_∞ of V. Since V is everywhere horizontal and bounded in $\mathbb{R}^2 \rtimes_A [0, \infty)$, then K_2 is tangent to $\mathcal{F}$ and bounded on the end representative $T_\infty^{-1}(\mathbb{R}^2 \rtimes_A [0, \infty))$ of W. Clearly, both noncompact domains $D_0 \times [0, \infty)$, $T_\infty^{-1}(\mathbb{R}^2 \rtimes_A [0, \infty))$ represent the same end of the solid cylinder W. Finally, since T_∞ is proper, then the closure of every integral curve of K_2 in $\mathrm{Int}(W)$ is a compact arc in a disk D_t for some $t \in \mathbb{R}$, with end points in ∂W. Thus our claim in this paragraph is proved.

By Theorem 9.1, the CMC flux (see Definition 9.2)

$$\mathrm{Flux}(\partial W, \partial D_0 \times \{0\}, K_1) \neq 0.$$

We will arrive to a contradiction by showing that the above CMC flux is actually zero. Note that α_n is homologous to zero in S_n, and so, the homological invariance of the CMC flux gives that $\mathrm{Flux}(\partial B_n, \alpha_n, K_\Sigma^n) = 0$ for all n. But this sequence of numbers converges as $n \to \infty$ to $\mathrm{Flux}(\partial W, \partial D_0 \times \{0\}, K_1)$, which gives the desired contradiction.

8 Proof of Theorem 1.5 when X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$

In this section we will prove Theorem 1.5 when X is a metric Lie group isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. As explained at the beginning of Sect. 7, to prove Theorem 1.5 it suffices to prove that if S_n is a sequence of constant mean

curvature spheres in X that lie in the connected component $\mathcal{C}$ of the space of index-one spheres in X with $H > h_0(X)$ defined in Proposition 3.4, and for which $\text{Area}(S_n) \to \infty$ as $n \to \infty$, then there exists a pointed limit immersion $f : (\Sigma, p) \looparrowright (X, e)$ of this sequence such that $f(\Sigma)$ is an entire Killing graph in X with respect to a right invariant Killing vector field in X. Also by the discussion at the beginning of Sect. 7, we can also assume that there exists a pointed limit immersion $f : (\Sigma, p) \looparrowright (X, e)$ of the spheres S_n such that properties (S1), (S2) hold. Observe that properties (S1), (S2) and the three mutually exclusive possibilities (S2.1), (S2.3) and (S2.3) make sense in this case when X is isomorphic to $\widetilde{\text{SL}}(2, \mathbb{R})$.

By item 5 of Lemma 4.9, $H(X) > 0$. Therefore, by (3.1) we have $h_0(X) > 0$, which implies that no immersions in $\Delta(f)$ have constant left invariant Gauss map (because two-dimensional subgroups of X are minimal, see Corollary 3.17 in [28]). In particular, $f(\Sigma)$ is not a two-dimensional subgroup of X and the right invariant vector field K_Σ to which $f(\Sigma)$ is everywhere tangent is unique up to scaling.

Theorem 1.5 when X is isomorphic to $\widetilde{\text{SL}}(2, \mathbb{R})$ will follow directly from the next result.

Theorem 8.1 *In the above conditions, $f(\Sigma)$ is an entire Killing graph with respect to a nonzero right invariant vector field on X. Moreover, f satisfies the periodicity condition (S2.3) stated at the beginning of Sect. 7.*

We next explain the strategy of the proof of Theorem 8.1. Firstly observe that the second sentence of Theorem 8.1 is a consequence of the first one (by the same argument given in the paragraph just after the statement of Theorem 7.2; note that this argument did not use that X was a semidirect product and remains valid in the present case that X is isomorphic to $\widetilde{\text{SL}}(2, \mathbb{R})$). So we only need to prove that $f(\Sigma)$ is an entire Killing graph with respect to some nonzero right invariant vector field. The proof of this property will be divided into four steps, depending on the character of the right invariant vector field K_Σ (elliptic, hyperbolic or parabolic), and on the situations (S2.1), (S2.2) and (S2.3). In Sect. 8.1 we will show that K_Σ cannot be elliptic. In Sect. 8.2 we will prove that Theorem 8.1 holds if K_Σ is parabolic. In Sect. 8.3 we will prove that Theorem 8.1 holds if K_Σ is hyperbolic and case (S2.3) occurs. In Sects. 8.4 and 8.5 we will respectively show that cases (S2.2) and (S2.1) are impossible when K_Σ is hyperbolic. This discussion exhausts all possible cases, and thus proves Theorem 8.1, and therefore Theorem 1.5 when X is isomorphic to $\widetilde{\text{SL}}(2, \mathbb{R})$.

8.1 K_Σ cannot be elliptic

Lemma 8.2 K_Σ *is not generated by an elliptic 1-parameter subgroup of* $\widetilde{SL}(2, \mathbb{R})$.

Proof Let Γ^E be any elliptic 1-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ and let K^E be the right invariant, elliptic vector field on $\widetilde{SL}(2, \mathbb{R})$ generated by the left action of Γ^E on $\widetilde{SL}(2, \mathbb{R})$. Each elliptic 1-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ consists of the set of liftings to $\widetilde{SL}(2, \mathbb{R})$ of all rotations of $\mathbb{H}^2$ around a given point; see Sect. 4.3. If $a \in \widetilde{SL}(2, \mathbb{R})$ satisfies $(l_a)_*(K^E) = \lambda K^E$ for some $\lambda \in \mathbb{R} - \{0\}$, then $a\Gamma^E a^{-1} = \Gamma^E$, see e.g., the proof of Lemma 4.10. Once here, it is well-known that on an elliptic subgroup this is only possible if $a \in \Gamma^E$.

Suppose now that K_Σ is generated by the left action on $\widetilde{SL}(2, \mathbb{R})$ of an elliptic 1-parameter subgroup $\Gamma_\Sigma = \Gamma^E$ of $\widetilde{SL}(2, \mathbb{R})$ and we will obtain a contradiction. If case (S2.3) holds, then there exists $a \in X - \Gamma_\Sigma$ such that $l_a(f(\Sigma)) = f(\Sigma)$. As the right invariant vector field $(l_a)_*(K_\Sigma)$ is everywhere tangent to $l_a(f(\Sigma))$ and K_Σ is unique up to scaling among nonzero right invariant vector fields that are everywhere tangent to $f(\Sigma)$, then we conclude that $(l_a)_*(K_\Sigma) = \lambda K_\Sigma$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Thus, the first paragraph in this proof ensures that $a \in \Gamma^E$, a contradiction. Therefore, case (S2.3) cannot hold.

In both cases (S2.1) and (S2.2), Remark 4.2 implies that $f(\Sigma)$ is obtained by the flow under the elliptic right invariant vector field K_Σ of an immersed closed curve contained in $\widetilde{SL}(2, \mathbb{R})/K_\Sigma$. In particular, since K_Σ is Killing, the distance in X of $f(\Sigma)$ to Γ^E is bounded, or equivalently, $f(\Sigma)$ is contained in a solid metric cylinder in X of fixed radius $r > 0$ centered along Γ^E,

$$\mathcal{W}(\Gamma^E, r) = \{x \in X \mid d_X(x, \Gamma^E) \leq r\},$$

where d_X denotes distance in X. As every elliptic 1-parameter subgroup of X contains the center Z of $\widetilde{SL}(2, \mathbb{R})$, then so does Γ^E, from where we deduce that $\mathcal{W}(\Gamma^E, r)$ is invariant under every left translation in Z. Also, it is clear that every right invariant vector field of $\widetilde{SL}(2, \mathbb{R})$ descends to the quotient group $\widetilde{SL}(2, \mathbb{R})/Z$ given by the action on $\widetilde{SL}(2, \mathbb{R})$ of the group of left translations by elements of Z. As $\mathcal{W}(\Gamma^E, r)/Z$ is compact, the restriction to $\mathcal{W}(\Gamma^E, r)$ of every right invariant vector field in $\widetilde{SL}(2, \mathbb{R})$ is bounded in $\mathcal{W}(\Gamma^E, r)$.

Let V be any nonzero right invariant vector field in X tangent to $f(\Sigma)$ at $f(p) = e$ and linearly independent from K_Σ. As V is not everywhere tangent to $f(\Sigma)$ and V is bounded in $\mathcal{W}(\Gamma^E, r)$, then the Jacobi function $J = \langle V, N \rangle$ is not identically zero, vanishes at p and it is bounded on Σ (here N is the unit normal vector field to f). Note that J changes sign on Σ by the maximum principle. As $f : \Sigma \mapsto X$ is a complete stable $h_0(X)$-surface with at most quadratic area growth, the existence of the function J contradicts [21,

Corollary 1] (see the last paragraph of the proof of Lemma 7.3 for a similar argument). This contradiction proves the lemma. □

8.2 Proof of Theorem 8.1 when K_Σ is parabolic

In order to prove Theorem 8.1 if K_Σ is parabolic, we will use the following auxiliary result:

Lemma 8.3 *Let Γ^P, Γ^H be two 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ such that Γ^P is parabolic, Γ^H is hyperbolic and they generate a two-dimensional subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Let K^P, K^H denote the right invariant vector fields on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by Γ^P, Γ^H. Then, there exists a function $h \in C^\infty(\widetilde{\mathrm{SL}}(2, \mathbb{R}))$ such that the vector field $K^H - h\,K^P$ satisfies:*

(1) $K^H - h\,K^P$ is bounded in $X = (\widetilde{\mathrm{SL}}(2, \mathbb{R}), \langle, \rangle)$ (regardless of the left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$).

(2) If the isometry group of X has dimension four, then $K^H - h\,K^P$ has constant length.

Proof First note that if we decompose K^H as $K^H = h\,K^P + V$ for some $h \in C^\infty(\widetilde{\mathrm{SL}}(2, \mathbb{R}))$ and some smooth vector field V on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, then the boundedness of V in $X = (\widetilde{\mathrm{SL}}(2, \mathbb{R}), \langle, \rangle)$ does not depend on the left invariant metric $\langle, \rangle$ on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, because all such metrics are quasi-isometric by the identity map in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Also, note that the particular bound for V on each specific X might depend on the metric $\langle, \rangle$. Thus, it suffices to prove item 2 of the lemma.

We next prove item 2. Let $\mathbb{H}^2 = \mathbb{H}^2_\theta$ be the 2-dimensional subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ that contains Γ^P, Γ^H (see the description of $\mathbb{H}^2_\theta$ before Lemma 4.11). Consider the Lie group given by the semidirect product $\mathbb{R} \rtimes_{(1)} \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$, endowed with the group operation

$$(x, y) * (x', y') = (x + e^y x', y + y'). \tag{8.1}$$

Let $\Phi_1 \colon \mathbb{H}^2 \to \mathbb{R} \rtimes_{(1)} \mathbb{R}$ be a group isomorphism, which exists since both $\mathbb{H}^2, \mathbb{R} \rtimes_{(1)} \mathbb{R}$ are noncommutative Lie groups of dimension two. As Φ_1 is an isomorphism, the push-forward vector fields $\widetilde{K}^P = (\Phi_1)_*(K^P)$, $\widetilde{K}^H = (\Phi_1)_*(K^H)$ are nowhere zero and right invariant on $\mathbb{R} \rtimes_{(1)} \mathbb{R}$. Furthermore, as the unique parabolic one-dimensional subgroup of $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ is $\{(x, 0) \mid x \in \mathbb{R}\}$, then we conclude that there exists $\lambda \in \mathbb{R} - \{0\}$ such that $\widetilde{K}^P = \lambda\, \partial_x$. Since $\{\partial_x, x\, \partial_x + \partial_y\}$ is a basis of right invariant vector fields on $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ and $\lambda \neq 0$, there exist $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$\widetilde{K}^H = \mu_1 \widetilde{K}^P + \mu_2 (x\, \partial_x + \partial_y) = \left(\mu_1 + \frac{\mu_2}{\lambda} x\right) \widetilde{K}^P + \mu_2 \partial_y. \tag{8.2}$$

Consider the left invariant metric $\langle, \rangle$ on $\mathbb{R} \rtimes_{(1)} \mathbb{R}$ that makes Φ_1 an isometry (recall that $\mathbb{H}^2$ has the induced metric from $\widetilde{\mathrm{SL}}(2, \mathbb{R})$). As ∂_y is left invariant on $\mathbb{R} \rtimes_{(1)} \mathbb{R}$, then the length of ∂_y with respect to $\langle, \rangle$ is constant (nonzero). Hence, Eq. (8.2) implies that the length of $\widetilde{K}^H - \widetilde{h} \, \widetilde{K}^P$ with respect to $\langle, \rangle$ is constant, where $\widetilde{h} = \mu_1 + \frac{\mu_2}{\lambda} x \in C^\infty(\mathbb{R} \rtimes_{(1)} \mathbb{R})$. This implies that if we call $h = \widetilde{h} \circ \Phi_1 \in C^\infty(\mathbb{H}^2)$, then the vector field $(K^H)|_{\mathbb{H}^2} - h \, (K^P)|_{\mathbb{H}^2}$ has constant length on $\mathbb{H}^2$.

Parameterize by $\{c(t) \mid t \in \mathbb{R}\}$ the 1-parameter subgroup Γ_3 of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by E_3. As the family $\{r_{c(t)}(\mathbb{H}^2) \mid t \in \mathbb{R}\}$ of right cosets of $\mathbb{H}^2$ is a foliation of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, we can extend h to a smooth function, also called $h \in C^\infty(\widetilde{\mathrm{SL}}(2, \mathbb{R}))$, so that h is constant along the orbits of the right action by elements in Γ_3. Suppose now that the left invariant metric on X has isometry group of dimension four. Thus, the right translations $r_{c(t)}$ by elements in Γ_3 are isometries of the ambient metric (because $\Gamma_3'(0) = (E_3)_e$ given by (4.7) and E_3 is a left invariant Killing vector field of every left invariant metric $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ whose isometry group has dimension four). This property and the right invariance of K^H, K^P imply that the length of $K^H - hK^P$ is constant along the orbits of the right action by elements of Γ_3. As $(K^H)|_{\mathbb{H}^2} - h \, (K^P)|_{\mathbb{H}^2}$ has constant length on $\mathbb{H}^2$, we conclude that $K^H - h \, K^P$ has constant length on X and item 2 of Lemma 8.3 is proved. $\qquad \square$

The next lemma implies that Theorem 8.1 holds in the case that K_Σ is parabolic, which is the objective of the present Sect. 8.2.

Lemma 8.4 *Suppose that K_Σ is parabolic. Let $\mathbb{H}^2_\theta$ be the two-dimensional subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ that contains the 1-parameter subgroup Γ_Σ generated by K_Σ. Let Γ^H be any hyperbolic 1-parameter subgroup of $\mathbb{H}^2_\theta$ and K^H the nonzero right invariant vector field on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by Γ^H. Then, $f(\Sigma)$ is a smooth entire Killing K^H-graph in $X = (\widetilde{\mathrm{SL}}(2, \mathbb{R}), \langle, \rangle)$.*

Proof Consider the foliation $\mathcal{F}$ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by right cosets of $\mathbb{H}^2_\theta$. As Γ^H (resp. Γ_Σ) is a 1-parameter subgroup of $\mathbb{H}^2_\theta$, then the integral curves of K^H (resp. K_Σ) are entirely contained in right cosets of $\mathbb{H}^2_\theta$; see property (P5) just before Claim 6.6. Since $f(\Sigma)$ is ruled by integral curves of K_Σ, we have that if $f(\Sigma)$ intersects a leaf $r_x(\mathbb{H}^2_\theta)$ of $\mathcal{F}$ at a point x, then the integral curve $r_x(\Gamma_\Sigma)$ of K_Σ passing through x is also contained in $f(\Sigma) \cap r_x(\mathbb{H}^2_\theta)$, and the angle that $f(\Sigma)$ makes with $r_x(\mathbb{H}^2_\theta)$ along $r_x(\Gamma_\Sigma)$ is constant, since left translations by elements of Γ_Σ are isometries of X.

Fix one of the leaves of $\mathcal{F}$, say $r_x(\mathbb{H}^2_\theta)$, $x \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$. Since each of the integral curves of K^H contained in $r_x(\mathbb{H}^2_\theta)$ intersects exactly once the unique integral curve $r_x(\Gamma_\Sigma)$ of K_Σ contained in $r_x(\mathbb{H}^2_\theta)$, then it follows that Lemma 8.4 will hold provided that we show that $f(\Sigma)$ intersects each of the right cosets $r_x(\mathbb{H}^2_\theta)$ transversely.

By straightforward modifications of the arguments in Assertion 6.5 (note that in that assertion, K_Σ was assumed to be *hyperbolic* while we are now assuming that K_Σ is parabolic; nevertheless the arguments in Assertion 6.5 still hold since they will be applied to the foliation by right cosets of $\mathbb{H}^2_\theta$, which are tangent to both K_Σ and K^H), we deduce that one of the two following possibilities holds for f:

(V1) $f(\Sigma)$ makes angles with the leaves of $\mathcal{F}$ that are bounded away from zero.

(V2) $f(\Sigma)$ is contained in a smallest slab-type region between two different leaves $L := r_x(\mathbb{H}^2_\theta)$, $L' := r_y(\mathbb{H}^2_\theta)$ of $\mathcal{F}$, for some $x, y \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$, and $f(\Sigma)$ is tangent to L (resp. L') along an integral curve Γ_x (resp. Γ_y) of K_Σ (in other words, the conclusions of Claim 6.6 hold).

A straightforward modification of the arguments in the paragraph just before Claim 6.6 implies that if case (V1) holds, then Lemma 8.4 also holds. Therefore, we will assume that case (V2) holds and we will find a contradiction.

Consider the Jacobi function $J = \langle K^H, N \rangle$ on Σ, where N stands for the unit normal vector to f. As K_Σ is parabolic and everywhere tangent to $f(\Sigma)$, then Lemma 8.3 applied to $K^P := K_\Sigma$ and K^H ensures that J is bounded on Σ. Also, note that J vanishes along the curves in $f^{-1}(L \cup L')$, which exist by condition (V2). Since each nodal domain $\Omega \subset \Sigma$ of J is a smooth infinite strip invariant under a 1-parameter group of isometries (the ones generated by the flow of K_Σ), then Ω has linear area growth. Hence, the closure $\overline{\Omega}$ of Ω is a parabolic Riemannian surface with boundary, in the sense that there exists a sequence $\{\varphi_j\}_j \subset C_0^\infty(\overline{\Omega})$ such that $0 \leq \varphi_j \leq 1$, the $\varphi_j^{-1}(1)$ form an increasing exhaustion of $\overline{\Omega}$ and $\int_\Omega |\nabla \varphi_j|^2 \leq \frac{1}{j}$ for each j. As f is stable, there exists $v \in C^\infty(\Sigma)$ such that $v > 0$ and $Lv = 0$ in Σ, where L is the Jacobi operator of f. Since J is bounded in Σ and changes sign, then we have a contradiction with Lemma 8.5 below. This contradiction proves the lemma. □

Lemma 8.5 *Suppose Σ is a complete Riemannian surface without boundary. Suppose that $L = \Delta + P$ is a Schrödinger operator with potential $P \in C^\infty(\Sigma)$. Let $u, v \in C^\infty(\Sigma)$ satisfying $Lu = Lv = 0$, with u not identically zero and $v > 0$ in Σ. If $\Omega \subset \Sigma$ is a nodal domain of u such that $\overline{\Omega}$ is parabolic and u is bounded in Ω, then u is a nonzero multiple of v and $\Omega = \Sigma$.*

Proof The arguments in the proof of Theorem 1 in [21] are still valid in our current setting and we leave the details to the reader. □

8.3 Proof of Theorem 8.1 when K_Σ is hyperbolic and case (S2.3) holds

Lemma 8.6 *Suppose that K_Σ is generated by a 1-parameter hyperbolic subgroup Γ^H of $\widetilde{SL}(2, \mathbb{R})$, and that case (S2.3) holds. Let $\mathbb{H}_\theta^2$ be one of the two two-dimensional subgroups of $\widetilde{SL}(2, \mathbb{R})$ containing Γ^H. Let Γ^P, K^P be the parabolic subgroup of $\mathbb{H}_\theta^2$ and the right invariant vector field on $\widetilde{SL}(2, \mathbb{R})$ generated by Γ^P, respectively. Then, $f(\Sigma)$ is a smooth entire Killing K^P-graph.*

Proof We argue in a similar way as in the proof of Lemma 8.4. Consider the foliation $\mathcal{F} = \{r_t(\mathbb{H}_\theta^2) \mid t \in \Gamma_3\}$ of $\widetilde{SL}(2, \mathbb{R})$ by right cosets of $\mathbb{H}_\theta^2$, where Γ_3 is the elliptic 1-parameter subgroup associated to E_3 given by Eq. (4.7). Recall that the leaves of $\mathcal{F}$ are equidistant surfaces to $\mathbb{H}_\theta^2$.

Suppose first that the angles that $f(\Sigma)$ makes with $\mathcal{F}$ are bounded away from zero. Then, $f(\Sigma)$ is an entire Killing graph with respect to K^P as explained just before Claim 6.6, and so Lemma 8.3 holds in this case. Otherwise, we can apply the Slab Property in $\widetilde{SL}(2, \mathbb{R})$ (Claim 6.6) to conclude that $f(\Sigma)$ is contained in a smallest slab-type region S between two different leaves L, L' of $\mathcal{F}$. As we are in the conditions of case (S2.3), there is an element $a \in \widetilde{SL}(2, \mathbb{R}) - \Gamma^H$ such that $l_a(f(\Sigma)) = f(\Sigma)$. As $f(\Sigma)$ is everywhere tangent to the right invariant vector field K_Σ and $f(\Sigma)$ is not a two-dimensional subgroup of $\widetilde{SL}(2, \mathbb{R})$, then there exists $\lambda \in \mathbb{R} - \{0\}$ such that $(l_a)_*(K_\Sigma) = \lambda K_\Sigma$. Observe that the 1-parameter subgroup of $\widetilde{SL}(2, \mathbb{R})$ generated by $(l_a)_*(K_\Sigma)$ is $a\Gamma_\Sigma a^{-1} = (l_a \circ r_{a^{-1}})(\Gamma_\Sigma)$ (see e.g., the proof of Lemma 4.10), hence this last 1-parameter subgroup must be a reparameterization of Γ_Σ. By Lemma 8.7 below, either a^2 lies in the center Z of $\widetilde{SL}(2, \mathbb{R})$, or a is the product of an element $h \in \Gamma^H$ with an element $b \in Z$. In the first case, $a^2 = e$ since otherwise the left translation by a large power a^{2k} of a^2 sends the topological slab S into another topological slab disjoint from S, which contradicts that $f(\Sigma)$ is invariant under the left translation by a^{2k} and $f(\Sigma)$ is contained in S. But $a^2 = e$ implies $a = e$ (because Z is an infinite cyclic subgroup), which contradicts our hypothesis that $a \in \widetilde{SL}(2, \mathbb{R}) - \Gamma^H$. Therefore, $a = bh$ with $h \in \Gamma^H$ and $b \in Z$. Observe that $b \neq e$ since $a \notin \Gamma^H$. Therefore, a large power $a^n = b^n h^n$ of a sends the topological slab S into another topological slab disjoint from S and we find a contradiction as before. This proves Lemma 8.6. $\qquad\square$

Lemma 8.7 *Suppose $a \in \widetilde{SL}(2, \mathbb{R})$ satisfies $a\Gamma^H a^{-1} = \Gamma^H$ for some hyperbolic 1-parameter subgroup Γ^H of $\widetilde{SL}(2, \mathbb{R})$. Let $\mathbf{P}\colon \widetilde{SL}(2, \mathbb{R}) \to PSL(2, \mathbb{R}) = \widetilde{SL}(2, \mathbb{R})/Z$ be the quotient homomorphism where Z is the center of $\widetilde{SL}(2, \mathbb{R})$. Then $\mathbf{P}(a) \in \mathbf{P}(\Gamma^H)$ or $a^2 \in Z$.*

Proof We identify $\mathbf{P}(\Gamma^H)$ with the set of hyperbolic translations of the Poincaré disk $\mathbb{D}$ along a geodesic γ. Since $\mathbf{P}(a)\mathbf{P}(\Gamma^H)\mathbf{P}(a^{-1}) = \mathbf{P}(\Gamma^H)$, we conclude that $\mathbf{P}(a)$ lies in the normalizer subgroup of $\mathbf{P}(\Gamma^H)$ in $PSL(2, \mathbb{R})$. In

this case, it is well-known that $\mathbf{P}(a) \in \mathbf{P}(\Gamma^H)$ or $\mathbf{P}(a)$ is an elliptic isometry of order two around some point in γ. In the second case, $\mathbf{P}(a)^2$ is the identity transformation of $\mathbb{D}$, or equivalently a^2 lies in the kernel of $\mathbf{P}$ which is Z. $\qquad \square$

8.4 Case (S2.2) is impossible if K_Σ is hyperbolic

The proof that $f(\Sigma)$ cannot be in the conditions of case (S2.2) if K_Σ is hyperbolic will be an adaptation to the $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ setting of our arguments in Sect. 7.3. In order to do this, we prove first a radius estimate (Lemma 8.9) for compact stable minimal surfaces with boundary contained in some special regions of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, similar in spirit to Theorem 1.1 in [25]. To start, we next associate to every hyperbolic 1-parameter subgroup Γ^H of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ a cylindrical-type neighborhood of Γ^H in X, which depends on the left invariant metric and on a number $r > 0$. We will call this neighborhood of Γ^H an *infinite box*.

Definition 8.8 Let Γ^H be a hyperbolic 1-parameter subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and let $\gamma \subset \mathbb{H}^2(-4)$ be the geodesic with the same extrema as the circle arc $\Pi(\Gamma^H) \subset \mathbb{H}^2(-4)$ (recall that Π was defined in (4.10)). Note that $\Pi(\Gamma^H)$ is at constant hyperbolic distance r_0 from γ, for some $r_0 \geq 0$. Given $r > r_0$, let $\gamma_r^+, \gamma_r^- \subset \mathbb{H}^2(-4)$ be the two arcs at distance r from $\Pi(\Gamma^H)$ and let $P^+(r) = \Pi^{-1}(\gamma_r^+)$, $P^-(r) = \Pi^{-1}(\gamma_r^-)$ be the topological planes inside $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ obtained after lifting these arcs through Π. Let $\mathbb{H} = \mathbb{H}_\theta^2$ be one of the two two-dimensional subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ that contain Γ^H, and call $\mathbb{H} a_r^+, \mathbb{H} a_r^-$ the right cosets of $\mathbb{H}$ at distance r from $\mathbb{H}$ (here a_r^+, a_r^- are some elements in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$). Observe that each of the surfaces $P^+(r)$, $P^-(r)$ intersects transversely both $\mathbb{H} a_r^+$ and $\mathbb{H} a_r^-$, along integral curves of the hyperbolic right invariant vector field K^H generated by the left action of Γ^H on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, see Fig. 8.

With these ingredients, we define the *infinite box* $\mathcal{B}(\Gamma^H, r)$ to be the closure of the component of $\widetilde{\mathrm{SL}}(2, \mathbb{R}) - [P^+(r) \cup P^-(r) \cup (\mathbb{H} a_r^+) \cup (\mathbb{H} a_r^-)]$ that has portions of all four surfaces in its boundary. Observe that Γ^H is contained in the interior of $\mathcal{B}(\Gamma^H, r)$.

Lemma 8.9 *Consider an infinite box $\mathcal{B}(\Gamma^H, r)$ as before. Then:*

(1) $\mathcal{B}(\Gamma^H, r)$ is invariant under the left translations by elements in Γ^H.
(2) Every compact, immersed minimal surface $M \subset X$ whose boundary lies in $\mathcal{B}(\Gamma^H, r)$ satisfies that $M \subset \mathcal{B}(\Gamma^H, r)$.
(3) There exists a constant $R(r) > 0$ such that for every compact, stable, immersed minimal surface $h \colon M \looparrowright X$ with $h(\partial M) \subset \mathcal{B}(\Gamma^H, r)$, the intrinsic radius of M is less than $R(r)$.

Proof Item 1 follows from the definitions of the surfaces in the boundary of $\mathcal{B}(\Gamma^H, r)$, all of which are invariant under left translations by elements in Γ^H.
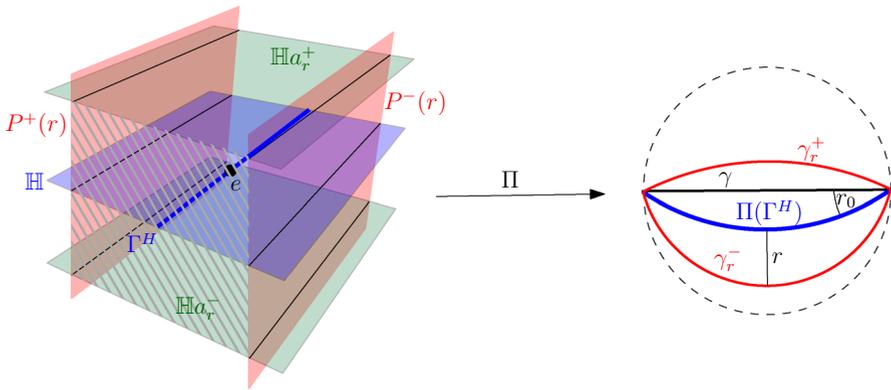
Fig. 8 Schematic representation of an infinite box $\mathcal{B}(\Gamma^H, r)$ in $\widetilde{SL}(2, \mathbb{R})$, which is the closed, non-compact three-dimensional region invariant under the flow of the hyperbolic 1-parameter subgroup Γ^H (the blue curve on the left) trapped between the following four topological planes: the two ones represented by red vertical planes are the liftings $P^+(r) = \Pi^{-1}(\gamma_r^+)$, $P^-(r) = \Pi^{-1}(\gamma_r^-)$ under the canonical projection Π of the arcs $\gamma_r^+, \gamma_r^- \subset \mathbb{H}^2(-4)$ at distance r of $\Pi(\Gamma^H)$, and the other two represented by green horizontal planes on the left are the two right cosets $\mathbb{H}\, a_r^+, \mathbb{H}\, a_r^-$ of $\mathbb{H}$ at distance r from $\mathbb{H}$, where $\mathbb{H}$ is one of the two two-dimensional subgroups of $\widetilde{SL}(2, \mathbb{R})$ that contain Γ^H (represented by a blue horizontal plane in the left) (color figure online)

Item 2 is a direct consequence of the maximum principle and the following facts:

- For every $r' \geq r$, $\mathcal{B}(\Gamma^H, r')$ has piecewise smooth mean convex boundary (this follows from item 4 of Lemma 4.9 and the fact that every right coset $\mathbb{H}\, a$ is a left translation of some two-dimensional subgroup of X, hence minimal), and the inner angles between adjacent smooth surfaces in this boundary are less than π. Furthermore, $\mathcal{B}(\Gamma^H, r) \subset \mathcal{B}(\Gamma^H, r')$.
- $\bigcup_{r' > r} \mathcal{B}(\Gamma^H, r') = X$.

Suppose item 3 of the lemma fails to hold. Then, there exists a sequence of compact, stable, immersed minimal surfaces $h_n : M_n \looparrowright X$ such that $h_n(\partial M_n) \subset \mathcal{B}(\Gamma^H, r)$ and points $p_n \in M_n$ such that the intrinsic distances from p_n to the boundary of M_n satisfy $d_{M_n}(p_n, \partial M_n) > n$ for all $n \in \mathbb{N}$. Next we show that after appropriate left translations and passing to a subsequence, we can take limits of the h_n. Let β be the geodesic in $\mathbb{H}^2(-4)$ that passes through $\Pi(e)$ and is orthogonal to γ. Note that each integral curve in $\widetilde{SL}(2, \mathbb{R})$ of the hyperbolic right invariant vector field K^H generated by Γ^H intersects the surface $\Pi^{-1}(\beta)$ exactly once; in particular, the integral curve $\Gamma^H h_n(p_n) = r_{h_n(p_n)}(\Gamma^H)$ of K^H passing through the point $h_n(p_n)$ intersects the compact surface $Y = \Pi^{-1}(\beta) \cap \mathcal{B}(\Gamma^H, r)$ just once (to see this, observe that by the already proven items 1, 2 of this lemma, $\mathcal{B}(\Gamma^H, r)$ is invariant under the left action of Γ^H and $h_n(M_n) \subset \mathcal{B}(\Gamma^H, r)$). Hence, after replacing h_n by

a left translation by some element in Γ^H, we will have that $h_n(p_n) \in Y$ and $h_n(M_n)$ still lies in $\mathcal{B}(\Gamma^H, r)$. As Y is compact, after choosing a subsequence we may assume that $h_n(p_n)$ converges to a point $q_\infty \in Y$.

By curvature estimates for stable minimal surfaces (Schoen [38], Ros [37]), the immersions h_n restricted to the intrinsic balls

$$B_{M_n}(p_n, n/2) = \{x \in M_n \mid d_{M_n}(p_n, x) < \tfrac{n}{2}\}$$

have uniformly bounded second fundamental forms. Hence, after choosing a subsequence, we obtain a complete minimal immersion $h_\infty \colon M_\infty \looparrowright \mathcal{B}(\Gamma^H, r)$ with bounded second fundamental form that is a limit of the restriction of the h_n to compact domains in M_n, and such that $h_\infty(p_\infty) = q_\infty$ for some point $p_\infty \in M_\infty$. Consider the set

$$\mathcal{M} = \overline{\{a\, h_\infty(M_\infty) = l_a\, (h_\infty(M_\infty)) \mid a \in \Gamma^H\}} \subset \mathcal{B}(\Gamma^H, r),$$

where for any $A \subset X$, $\overline{A}$ is the closure of A. Since $h_\infty(M_\infty)$ has bounded second fundamental form, then given any point $q \in \mathcal{M}$, there exists a compact, embedded minimal disk $D(q) \subset \mathcal{M}$ with $q \in \mathrm{Int}(D(q))$; specifically, the disk $D(q)$ is a limit of embedded geodesic disks of fixed small geodesic radius centered at points q_n in some left translate $l_{a_n}(h_\infty(M_\infty))$ such that $\lim_{n \to \infty} q_n = q$, where a_n is some element in Γ^H. Since $\mathcal{M} \cap Y$ is a compact set, there exists a largest positive $r' \leq r$ such that the set

$$W = [\mathbb{H}\, a_{r'}^+ \cup \mathbb{H}\, a_{r'}^-] \cap \mathcal{M} \cap Y$$

is nonempty. Note that the Γ^H-invariance of $[\mathbb{H}\, a_{r'}^+ \cup \mathbb{H}\, a_{r'}^-] \cap \mathcal{M}$ implies that r' is also the largest positive number such that $[\mathbb{H}\, a_{r'}^+ \cup \mathbb{H}\, a_{r'}^-] \cap \mathcal{M}$ is nonempty. Let $q \in W$. Then by the maximum principle for minimal surfaces, the set $\mathbb{H}\, a_{r'}^+ \cup \mathbb{H}\, a_{r'}^-$ must contain the aforementioned embedded minimal disk $D(q)$. But in this case, $\mathcal{M}$ would have to contain one of the surfaces $\mathbb{H}\, a_{r'}^+$, $\mathbb{H}\, a_{r'}^-$ which is false since neither of these surfaces is entirely contained in $\mathcal{B}(\Gamma^H, r)$. This contradiction proves the lemma. □

With Lemma 8.9 at hand, we can now rule out case (S2.2).

Proposition 8.10 *If K_Σ is hyperbolic, then case* (S2.2) *does not occur for f.*

Proof We will adapt the arguments of Sect. 7.3 to our current setting, focusing on the differences with the case of X being a semidirect product. Suppose that $K_\Sigma = K^H$ is hyperbolic and that case (S2.2) holds. Recall that $f \colon (\Sigma, p) \looparrowright (X, e)$ is a complete stable pointed immersion with constant mean curvature $h_0(X)$, that is obtained as a limit of pointed immersions $f_n \colon (S_n, p_n) \looparrowright (X, e)$

of spheres $S_n \in \mathcal{C}$. Since each sphere $S_n \in \mathcal{C}$ is Alexandrov embedded, for each $n \in \mathbb{N}$ there exists a three-dimensional Riemannian manifold (B_n, g_n) which is topologically a closed ball, together with a Riemannian submersion $\widetilde{f}_n \colon (B_n, g_n) \to X$ such that $\partial B_n = S_n$, $\widetilde{f}_n|_{S_n} = f_n$ and ∂B_n is mean convex. The key step in the proof of Proposition 8.10 will be proving that Lemma 7.5 holds in our situation. Observe that if we admit the validity of Lemma 7.5 in our current setting, then all the arguments in Sect. 7.4 after the proof of Lemma 7.5 remain valid in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and the proof of Proposition 8.10 will be finished. Therefore, we will only concentrate on the proof of the statement of Lemma 7.5 in our current $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-setting.

Consider the hyperbolic 1-parameter subgroup Γ^H of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by K_Σ. Since $f(\Sigma)$ is an annulus invariant under the flow of K^Σ, we can find $r_0 > 0$ such that $f(\Sigma)$ is contained in an infinite box $\mathcal{B}(\Gamma^H, r_0)$ and $\partial\mathcal{B}(\Gamma^H, r_0)$ is at a positive distance from $f(\Sigma)$; in what follows, $\mathcal{B}(\Gamma^H, r_0)$ will play the role of the solid cylinder $\mathcal{W}(\Gamma, r_0)$ of the original proof of Lemma 7.5 for metric semidirect products.

By item 3 of Lemma 8.9, there exists some $R = R(r_0) > 0$ with the following property:

$(\star)'$ If M is any compact, stable immersed minimal surface in X whose boundary ∂M is contained in $\mathcal{B}(\Gamma^H, r_0)$, then the intrinsic radius of M is less than R.

The role of the horizontal planes in the proof of the semidirect product case of Lemma 7.5 will be now played by the leaves of the foliation $\mathcal{F}' = \{P_t = l_t(\Pi^{-1}(\beta)) \mid t \in \Gamma^H \equiv \mathbb{R}\}$, where β is the geodesic in $\mathbb{H}^2(-4)$ that passes through $\Pi(e)$ and is orthogonal to γ, where both γ and β were previously defined in the proof of Lemma 8.9. Notice that $\mathcal{F}'$ is also the foliation of X whose projection by Π to $\mathbb{H}(-4)$ is the foliation of $\mathbb{H}(-4)$ by the set of geodesics orthogonal to γ, and the leaves of $\mathcal{F}'$ are minimal topological planes by Lemma 4.9.

As Σ is simply connected and $f \colon \Sigma \hookrightarrow X$ is invariant under the flow of K_Σ, then by the discussion in Remark 4.2 we may parameterize f as in Eq. (7.3) where Γ is replaced by Γ^H, and $\alpha = \alpha(s) \colon \mathbb{R} \to P_0$ is an immersed L-periodic curve parameterized by arc length in the plane P_0, so that $f(0, 0) = \alpha(0) = e$, $L > 0$ being the length of $\alpha(\mathbb{R})$.

Observe that $\mathcal{B}(\Gamma^H, r_0)$ is foliated by integral curves of K^H. As $\mathcal{B}(\Gamma^H, r_0) \cap P_0$ is compact, there exists the maximum length μ_0 of all the arcs of integral curves of K^H that are contained in the compact piece of $\mathcal{B}(\Gamma^H, r_0)$ that lies in the topological slab of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ determined by the leaves P_{-R-2} and P_{R+2} of $\mathcal{F}'$. Given $s_0 \in \mathbb{R}$, the definition (7.4) of the compact arc $A(s_0) \subset f(\Sigma)$ of the integral curve of Γ^H that passes through $\alpha(s_0)$ whose endpoints lie in P_{-R-2} and P_{R+2}, remains valid after replacing Γ by Γ^H, as well as the property that the length of $A(s_0)$ is at most μ_0.

For each $j \in \mathbb{N}$, consider the compact region $K^j \subset \Sigma$ defined by Eq. (7.5). Repeating the arguments in the proof of Lemma 7.5 verbatim, each K^j produces compact simply connected domains $K_n^j \subset S_n$ for n large, such that $f_n(K_n^j) \subset \mathcal{B}(\Gamma^H, r_0)$ and the sequence $\{f_n|_{K_n^j}\}_n$ converges to $f|_{K^j}$ uniformly as $n \to \infty$, and we can also find for each $k \in \{1, \ldots, j\}$ a small compact disk $\mathcal{U}^k$ centered at $(kL, 0)$ in K^j and related compact disks $\mathcal{U}_n^k \subset K_n^j$ such that $\{f_n|_{\mathcal{U}_n^k}\}_n$ converges to $f|_{\mathcal{U}^k}$ uniformly as $n \to \infty$. Furthermore, the $\mathcal{U}_n^k$ satisfy that the intrinsic distance between $\mathcal{U}_n^i$ and $\mathcal{U}_n^k$ inside K_n^j is greater than δ for some $\delta > 0$ independent of n, j, k. Notice however, that in the present setting, while the plane P_0 intersects $f(\Sigma)$ transversely along the closed immersed curve $\alpha(\mathbb{R})$, we no longer know that $f_n^{-1}(P_0)$ is a simple closed curve (since P_0 is not a two dimensional subgroup). Instead, for any fixed j and for n sufficiently large, we have that $f_n^{-1}(P_0) \cap K_n^j$ is a connected compact embedded arc which intersects each of the disks $\mathcal{U}_n^k \subset K_n^j$, where $k \in \{1, \ldots, j\}$. Let I denote a small compact segment in P_0 centered at $\alpha(0) = e$ and transversal to $f(\Sigma)$ at this point. The L-periodicity of $f(s, t)$ in the variable s and the previous convergence properties allow to find for $n = n(j)$ large enough, j points $q_n^k \in \mathcal{U}_n^k$ with $f_n(q_n^k) \in I$, for every $k = 1, \ldots, j$. In particular, property (⋆⋆) in the proof of Lemma 7.5 holds.

Let J_n^k be the compact connected arc of $f_n^{-1}(P_0) \cap K_n^j$ whose endpoints are q_n^1 and q_n^k, for each $k \in \{1, \ldots, j\}$. Pulling back via f_n the tangent part to $f_n(S_n)$ of K_Σ, we obtain a nowhere zero vector field W_n on K_n^j for n large enough. Letting the arc J_n^k flow under W_n between the sets $f_n^{-1}(P_{-R-1})$ and $f_n^{-1}(P_{R+1})$ we obtain a compact disk $D_n^k \subset K_n^j$ which is foliated by compact arcs of integral curves of W_n, each of which joins points in $(\partial D_n^k) \cap f_n^{-1}(P_{-R-1})$ and $(\partial D_n^k) \cap f_n^{-1}(P_{R+1})$, for n large. Then, the Jordan curve $\gamma_n^k := \partial D_n^k$, can be decomposed as $\gamma_n^k = A_1^n \cup A_2^n \cup B_1^n \cup B_2^n$, where properties (T1), (T2) hold after replacing the planes $\mathbb{R}^2 \rtimes_A \{\pm(R+1)\}$ by the topological planes $P_{\pm(R+1)}$.

We next solve for $k \in \{1, \ldots, j\}$ fixed (and n large enough) the Plateau problem with boundary γ_n^k in (B_n, g_n), finding a compact stable embedded minimal disk $M_n^k \subset B_n$ of least area, with $\partial M_n^k = \gamma_n^k$. Furthermore, M_n^k is an immersion transverse to ∂B_n at the interior points of A_1^n and A_2^n. Let Λ_n be the embedded surface $\{q \in B_n \mid \tilde{f}_n(q) \in P_0\}$ and $\beta_n^k := M_n^k \cap \Lambda_n$. Following the same reasoning as in the proof of Lemma 7.5 we deduce that after a small perturbation of Λ_n in B_n that fixes its boundary, we may assume that β_n^k contains a compact arc joining q_n^1 with q_n^k along which M_n^k intersects Λ_n transversally. Equations (7.6), (7.7), (7.8) and (7.9) hold in our current situation with the only changes of $\mathbb{R}^2 \rtimes_A \{\pm(R+1)\}$ by $P_{\pm(R+1)}$, $\mathbb{R}^2 \rtimes_A \{0\}$ by P_0 and Property (⋆) by Property (⋆)′. Finally, the arguments in the last five

paragraphs of the proof Lemma 7.5 remain valid without changes, by noticing that in our $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-setting, the mean curvature $h_0(X)$ of f is positive as explained just before the statement of Theorem 8.1. This finishes the sketch of proof of Lemma 7.5 when X is isomorphic to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and K_Σ is hyperbolic, and also finishes the proof of Proposition 8.10. □

8.5 Case (S2.1) is impossible if K_Σ is hyperbolic

Proposition 8.11 *If K_Σ is hyperbolic, then $f(\Sigma)$ cannot be in the conditions of case* (S2.1).

Proof Again the proof of this proposition is basically an adaptation of the contents of Sect. 7.4 to the setting of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. As we did when proving Proposition 8.10, we will only focus on the differences with the case of X being a semidirect product, and we will also use part of the notation established in the proof of Proposition 8.10.

Suppose that case (S2.1) holds and K_Σ is hyperbolic, with associated 1-parameter subgroup $\Gamma^H \colon \mathbb{R} \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$. We use the same notation $f \colon (\Sigma, p) \looparrowright (X, e)$, $f_n \colon (S_n, p_n) \looparrowright (X, e)$ and $\tilde{f}_n \colon B_n \to X$ as in the first paragraph of the proof of Proposition 8.10. The desired contradiction will come from application of a flux argument (Theorem 9.1) that we explain next.

Let $\mathbb{H} = \mathbb{H}^2_\theta$ be one of the two two-dimensional subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ that contain Γ^H. Associated to $\mathbb{H}$ and to Γ^H, we have the infinite boxes $\mathcal{B}(\Gamma^H, r)$ for all $r > 0$. Since $f(\Sigma)$ is an annulus invariant under the flow of K_Σ, we can find $r_0 > 0$ such that $f(\Sigma)$ is contained in $\mathcal{B}(\Gamma^H, r_0)$ and $\partial \mathcal{B}(\Gamma^H, r_0)$ is at a positive distance from $f(\Sigma)$. Consider the related foliation $\mathcal{F}' = \{P_t \mid t \in \Gamma^H \equiv \mathbb{R}\}$ by topological minimal planes defined in the proof of Proposition 8.10.

For n sufficiently large, consider the simple closed curve component α_n of $f_n^{-1}(P_0) \subset S_n$ containing p_n and let K_Σ^n be the Killing vector field on B_n obtained by pulling-back K_Σ through $\tilde{f}_n$. As we are assuming that case (S2.1) holds, Σ is an annulus and $f(\Sigma)$ is the immersed annulus obtained by letting the immersed closed curve $f(\Sigma) \cap P_0$ flow under K_Σ. Let $\alpha_\infty = f^{-1}(P_0)$, which is a simple closed curve in Σ such that the sequence $\{f_n|_{\alpha_n}\}_n$ converges uniformly to $f|_{\alpha_\infty}$.

As $\mathcal{F}'$ is a foliation by minimal planes, then the maximum principle implies that the smooth, embedded least-area disk $M(n)$ in the abstract Riemannian 3-ball B_n obtained by solving the Plateau problem in B_n with boundary α_n, satisfies that $\tilde{f}_n$ restricts to $M(n)$ producing an immersion of $M(n)$ into P_0. By the convergence of the sequence $\{f_n|_{\alpha_n}\}_n$, it follows that the areas of the minimal disks $\tilde{f}_n(M(n))$ are uniformly bounded. Furthermore, as the mean curvatures of the S_n are positive, then $M(n)$ is transverse to ∂B_n along α_n.

The same argument as in the third paragraph after the proof of Lemma 7.6 proves that there exists an $\eta \in (0, \pi/4)$ such that the angle that $M(n)$ makes with S_n along α_n is greater than η for all $n \in \mathbb{N}$ large enough, after replacement of the intrinsic radius estimate Theorem 1.1 in [25] by item 3 of Lemma 8.9, changing Γ by Γ^H, and noticing that $h_0(X) \geq H(X) > 0$.

The next step consists of proving that if $N_{M(n)}$ stands for the unit normal field to $M(n)$ in (B_n, g_n), then for n sufficiently large the function $J_n = g_n(N_{M(n)}, K_\Sigma^n): M(n) \to \mathbb{R}$ has no zeros along α_n. The proof of the same property in Sect. 7.4 does not work now, since it uses Eq. (4.5) which is only valid in a semidirect product. Instead, we will argue in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ as follows. Observe that

(W) K_Σ is everywhere transversal to P_0.

Now, if J_n vanishes at some point $q_n \in \alpha_n$, then $K_\Sigma^n(q_n)$ is tangent to $M(n)$ at q_n. Applying the differential of $\widetilde{f}_n$ at q_n, we deduce that $K_\Sigma(f_n(q_n))$ is tangent to P_0 at $f_n(q_n)$, which contradicts (W). Therefore, J_n has no zeros along α_n for all $n \in \mathbb{N}$.

Once we know that J_n has no zeros along α_n for all n, we can follow the arguments in Sect. 7.4 with the following changes (we are using the same notation as in Sect. 7.4):

- Replace Γ by Γ^H in Eqs. (7.13) and (7.14). In this way, we can obtain a submersion $T_\infty: \overline{\mathbb{D}} \times \mathbb{R} \to X$.
- Replace $\mathbb{R}^2 \rtimes_A \{t\}$ in Eq. (7.15) by the horocylinder $\Pi^{-1}(c_t)$ where c_t is the horocircle in $\mathbb{H}^2(-4)$ whose end point at $\partial_\infty \mathbb{H}^2$ is θ defined by $\mathbb{H} = \mathbb{H}_\theta^2$, such that c_t passes through $\Pi(t)$, $t \in \Gamma^H$. That is, define $D_t = T_\infty^{-1}(\Pi^{-1}(c_t))$, and then define the foliation $\mathcal{F}$ as in Eq. (7.15).
- Define K_1 as the nowhere zero, smooth vector field on $\overline{\mathbb{D}} \times \mathbb{R}$ obtained after pulling K_Σ by the submersion T_∞. Observe that each of the integral curves of K_1 in $\overline{\mathbb{D}} \times \mathbb{R}$ intersects D_0 in a single point.
- Define K_2 as the nowhere zero vector field on $W := D_0 \times \mathbb{R}$ which is the pullback by T_∞ of a parabolic right invariant vector field V on X associated to $\mathbb{H}$ (V is defined up to a multiplicative nonzero constant). Observe that K_2 is tangent to the foliation $\mathcal{F}$ and that once we endow W with the pulled-back metric by T_∞, K_2 is a Killing field. We claim that K_2 is bounded on one of the two ends of W: to do this, first note that to prove this boundedness property, it suffices to assume that X has isometry group of dimension four (as every left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is quasi-isometric to any fixed left invariant metric on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$). Also notice that $\Pi^{-1}(c_0) \cap \mathbb{H}$ is the 1-parameter parabolic subgroup of $\mathbb{H}$, and that the restriction of V to $\Pi^{-1}(A_0) \cap \mathbb{H}$ is bounded, where A_0 is the open horodisk in $\mathbb{H}^2$ bounded by c_0 (in fact, if we identify $\mathbb{H}$ with the semidirect product $\mathbb{R} \rtimes_{(1)} \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$ as explained after (8.1), then V

identifies with ∂_x up to a multiplicative constant, $\Pi^{-1}(A_0) \cap \mathbb{H}$ is the open halfspace $\mathbb{R} \rtimes_{(1)} (0, \infty)$ and ∂_x is bounded in $\mathbb{R} \rtimes_{(1)} (0, \infty)$). As the right translations by elements in the elliptic subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ generated by the vector $(E_3)_e$ given by Eq. (4.7) are isometries of any $\mathbb{E}(\kappa, \tau)$-metric in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and V is right invariant, we conclude that V is bounded in $\Pi^{-1}(A_0)$. Therefore, our claim follows.

The above changes allow us to apply Theorem 9.1 to find the same contradiction with CMC flux as in Sect. 7.4. This contradiction finishes the proof of Proposition 8.11. □

9 Appendix: A nonvanishing result for the CMC flux

Let W be a Riemannian $(n + 1)$-manifold and $M \subset W$ a two-sided n-hypersurface with constant mean curvature $H \in \mathbb{R}$ (throughout this appendix we define the mean curvature of M to be the trace of its second fundamental form divided by n). The CMC flux of M associated to a Killing vector field K on W and to a piecewise smooth $(n - 1)$-chain α in M which is the boundary of a piecewise smooth two-sided n-chain β in W, is the real number

$$\mathrm{Flux}(M, \alpha, K) = \int_\alpha \langle K, \eta \rangle + nH \int_\beta \langle K, N \rangle, \tag{9.1}$$

where η is a unit conormal vector to M along α and N is a unit normal vector field along β. In order for (9.1) not to depend on orientation choices, certain compatibility between H, η and N must be satisfied, see a more precise definition in Definition 9.2. A key property of $\mathrm{Flux}(M, \alpha, K)$ is that it does not depend on the homology class of α in M.

The goal of this section is to prove the following general result, in which the CMC flux of an n-hypersurface $M = \partial \Sigma \times \mathbb{R}$ with respect to a certain Killing vector field $K = K_1$ is nonzero. Theorem 9.1 has been used in Sects. 7.4 and 8.5.

Theorem 9.1 *Let Σ be a smooth, compact oriented n-manifold with nonempty boundary, $n \leq 6$. Let W be $\Sigma \times \mathbb{R}$ equipped with a Riemannian metric such that:*

(1) *The boundary $\partial \Sigma \times \mathbb{R}$ of W is mean convex, with constant mean curvature $h_0 \geq 0$.*
(2) *There exists a nowhere zero Killing field K_1 of W arising from a proper action[4] of an $\mathbb{R}$-subgroup of the isometry group of W.*

Suppose that there exists a Killing field K_2 in W which is tangent to the foliation $\{\Sigma \times \{s\} \mid s \in \mathbb{R}\}$, bounded on either $\Sigma \times [0, \infty)$ or on $\Sigma \times (-\infty, 0]$ and such that every integral curve of K_2 that intersects $\mathrm{Int}(W)$ is a compact arc with boundary in ∂W. Then:

$$\mathrm{Flux}(\partial \Sigma \times \mathbb{R}, \partial \Sigma \times \{0\}, K_1) \neq 0.$$

Definition 9.2 *(CMC flux).* Suppose M is a hypersurface with constant mean curvature $H \in \mathbb{R}$ in an oriented Riemannian $(n + 1)$-manifold X, and let K be a Killing field on X. For each $[\alpha] \in H_{n-1}(M)$ in the kernel of the induced inclusion homomorphism $i_*\colon H_{n-1}(M) \to H_{n-1}(X)$, consider two homologous piecewise smooth $(n - 1)$-cycles $\alpha, \alpha_1 \subset M$ representing $[\alpha]$ and let β, β_1 be piecewise smooth n-chains in X with boundaries $\partial \beta = \alpha$, $\partial \beta_1 = \alpha_1$ (which exist since $[\alpha] \in \ker(i_*)$). We assume that if $M(\alpha, \alpha_1)$ is the n-chain in M whose boundary is $\alpha_1 - \alpha$, then $\beta - \beta_1 - M(\alpha, \alpha_1)$ is the boundary of an $(n + 1)$-chain Ω in X. Applying the Divergence Theorem to K in Ω, we obtain

$$0 = -\int_\beta \langle K, N \rangle + \int_{\beta_1} \langle K, N_1 \rangle - \int_{M(\alpha,\alpha_1)} \langle K, N^M \rangle, \qquad (9.2)$$

where $-N$, N_1 are unit normal vector fields to β, β_1 (defined almost everywhere) pointing outward Ω, and N^M is the unit normal vector field to M pointing inward Ω (see Fig. 9). By the Gauss equation, the divergence in M of the tangential part K^T of K is given by $\mathrm{div}^M(K^T) = nH\langle K, N^M \rangle$, where H is the mean curvature of M with respect to N^M. Applying the Divergence Theorem in $M(\alpha, \alpha_1)$,

$$nH \int_{M(\alpha,\alpha_1)} \langle K, N^M \rangle = \int_\alpha \langle K, \eta \rangle - \int_{\alpha_1} \langle K, \eta_1 \rangle, \qquad (9.3)$$

---

[4] This means that if we denote by $\{\phi_s \mid s \in \mathbb{R}\}$ the 1-parameter subgroup of isometries of W corresponding to the flow of the Killing field K_1, then for each $x \in W$ the map $s \in \mathbb{R} \mapsto \phi_s(x)$ is proper.

Fig. 9 Interpretation of the CMC flux homological invariant
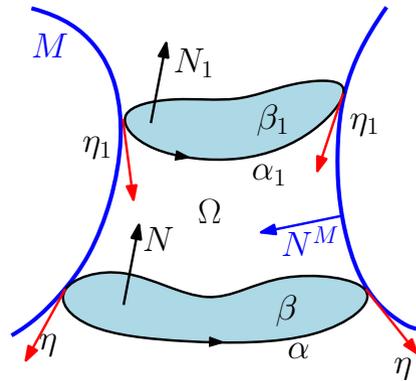
where $\eta, -\eta_1$ are the unit conormal vector fields to M along α, α_1 pointing outward $M(\alpha, \alpha_1)$, which are defined almost everywhere on α, α_1. Equations (9.2), (9.3) show that the real number

$$\text{Flux}(M, \alpha, K) = \int_\alpha \langle K, \eta \rangle + nH \int_\beta \langle K, N \rangle \qquad (9.4)$$

is independent of the choice of the representative α in a given homology class $[\alpha] \in \ker(i_*) \subset H_{n-1}(M)$ and of the n-chain β as above. This number is usually called the *CMC flux* of M along $[\alpha]$ associated to K. Observe that the mean curvature of M is computed with respect to the unit normal vector that, along α, points to the same closed component of $M \cup \beta$ as N, and that $\langle N, \eta \rangle \leq 0$ along α. The above argument shows the homological invariance of the CMC flux; we refer the reader to the papers [15,19,20,22] for applications of this CMC flux.

Proof of Theorem 9.1 Without loss of generality, we will assume that the Killing field K_2 is bounded on the end $\Sigma \times (-\infty, 0]$ of W. Let $\{\phi_s \mid s \in \mathbb{R}\}$ be the 1-parameter subgroup of isometries of X corresponding to the flow of the Killing field K_1. Since the action $(s, x) \in \mathbb{R} \times W \mapsto \phi_s(x)$ is proper, then after possibly changing K_1 by $-K_1$, we may suppose that $\phi_s(\Sigma \times \{0\})$ lies eventually in the upper end $\Sigma \times [0, \infty)$ of W as $s \to +\infty$. Consider the quotient map $\Pi: W \to W/G = \widehat{W}$, where $G = \{\phi_s \mid s \in 2\pi\mathbb{Z}\}$ (note that the discrete group of isometries G acts proper discontinuously on W). We will denote by $\mathbb{S}^1$ the related 1-parameter subgroup of isometries of $\widehat{W}$.

Let $\widehat{\Gamma}$ be an area-minimizing oriented $(n-1)$-dimensional hypersurface in $\partial\widehat{W}$ that represents $[\partial\Sigma] \in H_{n-1}(\partial\widehat{W})$ ($\widehat{\Gamma}$ is an area-minimizing integral current, which is smooth in this dimension and it exists since $\partial\widehat{W}$ is a closed Riemannian manifold). By cut-and-paste type arguments and the maximum

principle for minimal hypersurfaces, any two distinct solutions to this area-minimization problem in $\partial \widehat{W}$ are disjoint or equal. It follows that $\partial \widehat{W}$ is foliated by the area-minimizing oriented $(n-1)$-dimensional submanifolds $\phi_s(\widehat{\Gamma})$, $\phi_s \in \mathbb{S}^1$. For homological reasons, $\widehat{\Gamma}$ has a lift through Π which is a compact $(n-1)$-dimensional hypersurface Γ in ∂W that is area-minimizing in ∂W, and when considered to lie in W, it is the boundary of an integral n-chain. By the properness of the $\mathbb{R}$-action giving rise to K_1 and the compactness of Γ, we deduce that K_1 cannot be everywhere tangent to Γ. On the other hand, since for $s \neq 0$ it holds $\phi_s(\Gamma) \cap \Gamma = \emptyset$, then the Jacobi function $J_\Gamma = \langle K_1, N_\Gamma \rangle \colon \Gamma \to \mathbb{R}$, is nonnegative on Γ (after choosing an appropriate orientation of Γ, or equivalently a unit normal N_Γ along Γ, i.e., N_Γ is tangent to ∂W, orthogonal to Γ and unitary); note that J_Γ is positive at the points of Γ where K_1 is not tangent to Γ. The maximum principle (see e.g., [29, Assertion 2.2]) implies that $J_\Gamma > 0$ on Γ; in particular, K_1 is transverse to Γ and each integral curve of K_1 intersects Γ in a single point.

Let $\Lambda \subset W$ be an area-minimizing hypersurface with boundary Γ, which in this dimension is smooth by standard interior and boundary regularity results for minimizing integral varifolds; see for example, Hardt and Simon [14] for boundary regularity results and also see [31,33] for related barrier arguments. By the mean curvature comparison principle, Λ is never tangent to ∂W along Γ. Since $J_\Gamma > 0$ by the previous paragraph, then we can choose a unit normal vector N_Λ to Λ such that $\langle (N_\Lambda)|_{\partial \Gamma}, N_\Gamma \rangle$ is positive along Γ. As K_1 is tangent to ∂W, $(N_\Lambda)|_\Gamma$ is orthogonal to Γ and both $\langle K_1, N_\Gamma \rangle$, $\langle (N_\Lambda)|_{\partial \Gamma}, N_\Gamma \rangle$ are positive along Γ, then we conclude that $\langle K_1, N_\Lambda \rangle > 0$ along Γ. As Λ is stable and $\langle K_1, N_\Lambda \rangle > 0$ along $\partial \Lambda$, then $\langle K_1, N_\Lambda \rangle$ is positive on Λ and so, Λ intersects each of the integral curves of K_1 transversely in a single point. Therefore, the set $\mathcal{F} = \{\phi_s(\Lambda) \mid s \in \mathbb{R}\}$ is a smooth minimal foliation of W, where each leaf $\phi_s(\Lambda)$ of this foliation intersects every integral curve of K_1 transversely in a single point.

We will now deform the minimal surface Λ by a 1-parameter family of surfaces with constant mean curvature and the same boundary as Λ. By the "blowing a bubble" technique described in [19], for $\delta > 0$ sufficiently small there exists a unique 1-parameter family of hypersurfaces $t \in [0, \delta) \mapsto \Lambda(t) \subset W$ depending smoothly on t such that for all $t \in [0, \delta)$:

(X1) $\Lambda(t)$ is the graph of a smooth function $f_t \colon \Lambda \to \mathbb{R}$ (in the direction of K_1), with $\Lambda(0) = \Lambda$ and $f_0 = 0$.

(X2) $\Lambda(t)$ has constant mean curvature $-t$ with respect to the unit normal vector field $N_{\Lambda(t)}$ to $\Lambda(t)$ that satisfies $\langle K_1, N_{\Lambda(t)} \rangle > 0$.

(X3) $f_t = 0$ on $\partial \Lambda$.

(X4) $f_t(x) > f_{t'}(x)$ whenever $t > t'$ and $x \in \mathrm{Int}(\Lambda)$.

We now study the maximal half-open interval $[0, T_0)$ of t-values in which the family $\Lambda(t)$ can be defined (here $T_0 \in (0, \infty]$).

First consider the case in which the constant mean curvature h_0 of ∂W satisfies $h_0 < T_0$. Then, there exists a hypersurface $\Lambda(t_1) \subset X$ with constant mean curvature $-t_1$, which is the graph of a function $f_{t_1} : \Lambda \to \mathbb{R}$ such that $f_{t_1} = 0$ on $\partial \Lambda$ and $t_1 > h_0$. Since $\partial \Lambda(0) = \partial \Lambda(t_1)$, then the CMC flux of $\Lambda(t_1)$ along $[\partial \Lambda(t_1)]$ associated to K_1 is given by

$$\text{Flux}(\Lambda(t_1), \partial \Lambda(t_1), K_1) = \int_{\partial \Lambda(0)} \langle K_1, \eta_{\Lambda(t_1)} \rangle + nt_1 \int_{\Lambda(0)} \langle K_1, N_{\Lambda(0)} \rangle,$$

where $\eta_{\Lambda(t_1)}$ is the outward pointing unit conormal vector field to $\Lambda(t_1)$ along $\partial \Lambda(t_1)$ and $N_{\Lambda(0)}$ is the unit normal vector field to $\Lambda(0)$ pointing towards the compact region of W bounded by $\Lambda(0)$ and $\Lambda(t_1)$. Note that

$$\text{Flux}(\Lambda(t_1), \partial \Lambda(t_1), K_1) = 0$$

by the homological invariance of the CMC flux ($\partial \Lambda(t_1)$ is homologically trivial in $\Lambda(t_1)$). On the other hand, along $\partial \Lambda(0)$ the inequality

$$\langle K_1, \eta_{(\partial W)^+} \rangle \leq \langle K_1, \eta_{\Lambda(t_1)} \rangle \tag{9.5}$$

holds, where $\eta_{(\partial W)^+}$ denotes the outward pointing unit conormal field to the portion $(\partial W)^+$ of ∂W which lies above $\partial \Lambda(0)$. This last inequality together with $h_0 < t_1$ imply that

$$\begin{aligned}
\text{Flux}\left((\partial W)^+, \partial \Lambda(0), K_1\right) &= \int_{\partial \Lambda(0)} \langle K_1, \eta_{(\partial W)^+} \rangle + nh_0 \int_{\Lambda(0)} \langle K_1, N_{\Lambda(0)} \rangle \\
&< \int_{\partial \Lambda(0)} \langle K_1, \eta_{(\partial W)^+} \rangle + nt_1 \int_{\Lambda(0)} \langle K_1, N_{\Lambda(0)} \rangle \\
&\overset{(9.5)}{\leq} \text{Flux}(\Lambda(t_1), \partial \Lambda(t_1), K_1) = 0.
\end{aligned}$$

Finally, the homological invariance of the CMC flux gives

$$\begin{aligned}
\text{Flux}\left(\partial W, \partial \Sigma \times \{0\}, K_1\right) &= \text{Flux}\left((\partial W)^+, \partial \Sigma \times \{0\}, K_1\right) \\
&= \text{Flux}\left((\partial W)^+, \partial \Lambda(0), K_1\right) < 0,
\end{aligned}$$

which completes the proof of the theorem in the case $h_0 < T_0$.

Assume in the sequel that $T_0 \leq h_0$ and we will find a contradiction, which will finish the proof of the theorem. We first prove the following property.

Assertion 9.3 $\cup_{t \in [0, T_0)} \Lambda(t)$ *is not contained in any compact region of W. In particular, for every $n \in \mathbb{N}$ there exists a $t(n) \in [0, T_0)$ such that $\Lambda(t)$ contains points at distance at least n from $\Lambda(0)$ whenever $t \geq t(n)$.*

Proof of the assertion. Suppose the assertion does not hold. Then, the constant mean curvature hypersurfaces $\{\Lambda(t) \mid t \in [0, T_0)\}$ have uniform height estimates (as K_1-graphs). By the "blowing a bubble" technique in [19], these height estimates imply that the $\{\Lambda(t) \mid t \in [0, T_0)\}$ satisfy uniform curvature estimates, i.e., they have *uniform* bounds on the length of their second fundamental forms; these uniform curvature estimates can also be seen to hold since $\Lambda(t)$ minimizes the functional $\text{Volume}_n + nt \text{Volume}_{n+1}$ with respect to compact hypersurfaces in W with boundary $\partial \Lambda$ that are homologous to Λ relative to $\partial \Lambda$, where Volume_k is the volume of any k-chain. From here, standard compactness results show that the top boundary component of $\cup_{t \in [0, T_0)} \Lambda(t)$ is a hypersurface $\Lambda(T_0)$ with constant mean curvature $-T_0$.

Note that with respect to the unit normal vector field $N_{\Lambda(T_0)}$ to $\Lambda(T_0)$ that is the limit of the $N_{\Lambda(t)}$ as $t \nearrow T_0$, the function $\langle K_1, N_{\Lambda(T_0)} \rangle$ is nonnegative, and it is positive at some point by previous arguments. Since $\langle K_1, N_{\Lambda(T_0)} \rangle$ is a Jacobi function on $\Lambda(T_0)$, then the maximum principle (see e.g., [29, Assertion 2.2]) implies that $\langle K_1, N_{\Lambda(T_0)} \rangle > 0$ at $\text{Int}(\Lambda(T_0))$. In particular, $\Lambda(T_0)$ is the K_1-graph of a continuous function $f_{T_0} \colon \Lambda \to \mathbb{R}$ that is smooth at $\text{Int}(\Lambda(T_0))$. As $\Lambda(T_0)$ is not contained in ∂W, then the mean curvature comparison principle and the Hopf boundary maximum principle imply that the gradient of f_{T_0} is bounded as we approach $\partial \Lambda(T_0)$; hence, $\langle K_1, N_{\Lambda(T_0)} \rangle$ is positive along $\partial \Lambda(T_0)$. It follows from the maximum principle that $\langle K_1, N_{\Lambda(T_0)} \rangle$ is positive in $\Lambda(T_0)$ and thus, f_{T_0} is smooth in Λ. Therefore, items (X1)-(X4) hold for $t = T_0$. In this situation, the strict stability of $\Lambda(T_0)$ implies that the family of hypersurfaces $\{\Lambda(t) \mid t \in [0, T_0]\}$ can be extended to a related family of hypersurfaces defined for t-values in a larger interval $[0, T_0 + \varepsilon)$, $\varepsilon > 0$, which contradicts the definition of the value T_0. This contradiction proves the assertion. $\qquad\square$

For each $t \in [0, T_0)$ fixed, consider the related foliation of W

$$\mathcal{F}(t) = \{\phi_s(\Lambda(t)) \mid s \in \mathbb{R}\},$$

all whose leaves have constant mean curvature $-t$, see Fig. 10 left. Let

$$L(t) = \phi_{s(t)}(\Lambda(t))$$

be the unique leaf of $\mathcal{F}(t)$ such that $L(t)$ intersects $\Lambda(0)$ and lies on the lower side of $\Lambda(0)$. Assertion 9.3 implies that $s(t) \to -\infty$ as $t \to T_0$. Let $L(T_0)$ be the limit of the hypersurfaces $L(t)$ as $t \nearrow T_0$. Such a limit exists and it is a complete graphical hypersurface with constant mean curvature $-T_0$ by the stability of $L(t)$ for all $t < T_0$, and the same arguments as in the proof of Assertion 9.3. Let Λ' be the open subdomain of Λ over which $L(T_0)$ is a K_1-graph. Clearly, $L(T_0)$ intersects $\Lambda(0)$ and lies below $\Lambda(0)$. Define $t_0 \in \mathbb{R}$
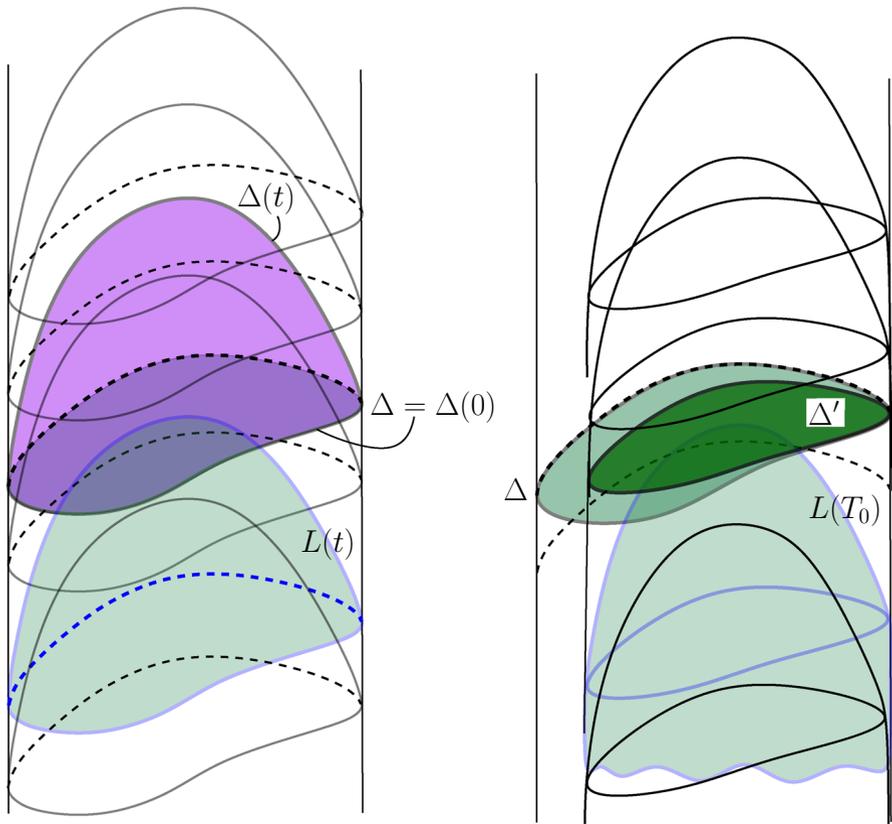
Fig. 10 Left: After blowing a bubble $\Lambda(t)$ with constant mean curvature $-t \in (-T_0, 0]$ (which is K_1-graphical over $\Lambda = \Lambda(0)$), we translate it with $\{\phi_s\}_{s \in \mathbb{R}}$ to produce a foliation $\mathcal{F}(t)$, and we highlight the highest graphical leaf $L(t)$ of $\mathcal{F}(t)$ lying below $\Lambda(0)$ with $L(t) \cap \Lambda(0) \neq \emptyset$. Right: Since the $\Lambda(t)$ with $t \to T_0^-$ do not lie in a fixed compact set, we produce a noncompact limit $L(T_0) = \lim_{t \to T_0^-} L(t)$ which is a graph over a domain Λ' of Λ

such that $L(T_0)$ lies in $\Sigma \times (-\infty, t_0]$ and t_0 is the smallest value with this property. Observe that $L(T_0)$ intersects $\Sigma \times \{t_0\}$ tangentially at some interior point of $\Sigma \times \{t_0\}$.

Assertion 9.4 $L(T_0)$ *has linear volume growth. In particular, $L(T_0)$ is parabolic.*

Proof of the assertion. A straightforward limit argument shows that $L(T_0)$ minimizes the functional $\text{Volume}_n + nT_0\text{Volume}_{n+1}$ on compact subsets of $\text{Int}(W)$. This property implies that $L(T_0)$ has bounded second fundamental form. Therefore, the gradient of the graphing function $h: \Lambda' \to \mathbb{R}$ that produces $L(T_0)$ as a K_1-graph becomes unbounded as we approach any point of the boundary of Λ' (otherwise we could enlarge Λ'). Therefore the asser-

tion holds provided that we check that the total $(n-1)$-dimensional volume function of the submanifold $\Gamma_s := L(T_0) \cap \phi_s(\Lambda(0))$ of $L(T_0)$ is bounded as $s \to -\infty$. To see this, for s large ($s \ll -1$) let us denote by $\Omega(s)$ the portion of $L(T_0)$ lying above $\phi_s(\Lambda(0))$, and let η_s be the outward pointing unit conormal vector field of $\Omega(s)$ along its boundary Γ_s. Note that for every $\varepsilon > 0$ small, there exists $s_0(\varepsilon) \ll -1$ such that for each $s \le s_0$,

$$\langle K_1, \eta_s \rangle < -\varepsilon \quad \text{along } \Gamma_s, \tag{9.6}$$

for some $\varepsilon > 0$ independent of s. On the other hand,

$$\text{Flux}(\Omega(s), \Gamma_s, K_1) = \int_{\Gamma_s} \langle K_1, \eta_s \rangle + n T_0 \int_{\phi_s(\Lambda_s)} \langle K_1, N_{\phi_s(\Lambda_s)} \rangle, \tag{9.7}$$

where Λ_s is the subdomain of Λ' over which $\Omega(s)$ is a K_1-graph, and $N_{\phi_s(\Lambda_s)} = (\phi_s)_*(N_{\Lambda(0)})$. By homological invariance of the flux, we have $\text{Flux}(\Omega(s), \Gamma_s, K_1) = 0$. As the second integral in the right-hand-side of (9.7) is bounded above in absolute value by $|K_1|_\infty$ times the n-dimensional volume of $\Lambda(0)$ (here $|K_1|_\infty = \max_\Lambda |K_1|$, which exists since Λ is compact and K_1 is generated by $\{\phi_s \mid s \in \mathbb{R}\}$), then the first integral in the right-hand-side of (9.7) is bounded independently of $s \to -\infty$. This property together with (9.6) ensure that the $(n-1)$-dimensional volume of Γ_s is bounded as $s \to -\infty$. Now the assertion is proved. □

Recall that by assumption, K_2 is bounded in $\Sigma \times (-\infty, 0]$. Define $u = \langle K_2, N_{L(T_0)} \rangle \colon L(T_0) \to \mathbb{R}$, where $N_{L(T_0)}$ denotes a unit normal vector field along $L(T_0)$. u is a bounded Jacobi function on $L(T_0)$, which vanishes at each of the points of the nonempty set $L(T_0) \cap (\Sigma \times \{t_0\})$. Since every integral curve of K_2 that intersects $L(T_0)$ is a compact interval with its boundary in ∂W, then K_2 is not everywhere tangent to $L(T_0)$, which implies u is not identically zero; by the maximum principle, u must change sign in a neighborhood of any of the points in $L(T_0) \cap (\Sigma \times \{t_0\})$. As $L(T_0)$ is complete, stable, and is parabolic by Assertion 9.4, we conclude that every bounded nonzero Jacobi function on $L(T_0)$ has constant sign [21, Corollary 1], which is a contradiction. This contradiction finishes the proof of Theorem 9.1. □

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