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On Incomplete Fuzzy and Multiplicative Preference Relations In Multi-Person Decision Making

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Abstract

Rapid changes in the business environment such us the globalization as well as the increasing necessity to make crucial decisions involving a huge range of alternatives in short period of time or even in real time have made that computerized group decision support systems become very useful tools. However in the majority of the cases the panel of experts cannot provide all the information about their preferences due to different reasons such as lack of knowledge, time etc. Therefore different approaches have been presented to deal with the missing preferences in group decision making contexts. In this paper we review and analyse the state-of-the-art research efforts carried out on this topic for incomplete fuzzy preference relations and multiplicative preference relations.

Keywords: Group decision making, GDM, incomplete information, fuzzy preference relations (APR), multiplicative preference relations (MPR)

1. Introduction

Group decision making (GDM) consist of multiple individual interacting to choose the best option between all the available ones. Each decision maker (expert) may have his/her own opinions and background, which enables them to approach the problem from different perspectives, but they share a common interest in achieving agreement on selecting the most suitable option.

In these systems experts have to express their preferences by means of a set of evaluations over a set of alternatives using different representation formats. In real world situations the expert panel is composed of diverse specialists with very different backgrounds and expertise, therefore sometimes and expert might not possess a precise or sufficient level of knowledge of part of the problem and as a consequence he/she might not give all the information that is required. Indeed, this could be due to different causes such as a high number of alternatives and limited time, experts not having

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enough knowledge of a part of the problem or are unable to discriminate the degree up to which an option is better than another, or even when conflict in a comparison situation appears which happens when each alternative outperforms the other one on some criterion and imposes a trade-off. In these sense Deparis et al. have carried out recently and empirical study in[1] which tests the hypothesis that increasing the intensity of conflict in a multicriteria comparison increases the likelihood that DMs consider two alternatives as incomparable expressing therefore incomplete preferences. Results show that depending on whether the participants are allowed to express incomplete preferences or not, attribute spread has different effects: a large attribute spread increases the frequency of incomparability statements, when available, while it increases the use of indifference statements when only indifference and preference answers are permitted.

In all these situations experts provide incomplete preference relations, that is, preference relations with some of their values missing or unknown. An extreme case happens when an expert does not provide any information about a particular alternative. This situations are called in literature *total ignorance* or simply *ignorance* situations.

The key issue in these situations is how the decision making algorithms should deal with the missing information.In the literature we can three main approaches to deal with missing judgements[2]: i. deletion, ii. using incomplete preference relations without carrying out any estimation process, iii. Carrying out a completion process prior to the aggregation.

According to the first approach the objects which contain missing values are deleted. It is also possible that attributes or fields containing many missing values are ignored. The main disadvantage of this approach is that the elimination of useful information in the data which could lead to serious biases[2]. The second one consists on using the incomplete preferences provided by the expert to reach the decision without estimating the missing values. Finally, the majority of the models in the literature follows the third approach that carry out completion methods to estimate the missing preferences. Some of these approaches use the information provided by the other experts together with aggregation procedures[3] requiring therefore several experts to estimate the missing values of a particular one and they do not take into account the differences between the experts preferences. Therefore this approach could lead to estimate missing values not naturally compatible with the opinion of the expert. Hence the majority of the estimation techniques uses only the information provided by the expert who provides the incomplete preference relation. In this paper we focus on the foundations and developments in estimation of missing additive and multiplicative preferences in GDM. Finally, several current trends and prospects about the topic are introduced.

The remainder of the paper is set out as follows: In section 2 we review the most relevant concepts in GDM including the definition of the additive fuzzy preference relation (APR), the multiplicative preference relation (MPR), and the concept of incomplete preference relation. Section 3 presents the main strategies in the literature to deal with missing judgements in the context of GDM for APR and MPR. Whereas in section 4 the approaches to deal with total ignorance situations are explained. Finally in section 5 our concluding remarks are pointed out among with some future work.

2. Frameworks for GDM with incomplete information

In the context of GDM the objective is choosing the best alternative(s) among a finite set, $X = \{x_1, \cdots, x_n\}$, $(n \geq 2)$. The alternatives will be classified from the best to worst, using the information known according to a set of m experts, i.e., $E = \{e_1, \cdots, e_m\}$, $(m \geq 2)$. Let $w = (w_1, w_2, \cdots, w_n)$ be the vector of priority, where w_i reflects the importance degree of the alternative x_i. All the $w_i (i = 1, 2, \cdots n)$ are greater than zero and sum to one, that is

$$w_i > 0,\ i = 1, 2, \cdots, n\ \sum_{i=1}^{n} w_i = 1 \tag{1}$$

Modelling how each expert $e_k \in E$ express his/her preferences is a key factor. Pair comparison of alternatives is usually used in many models since they integrate processes linked to some degree of credibility of preference of one alternative over another.According to the Millet comparatives study on different alternative preference elicitation methods[4], pairwise comparison methods are more accurate than non-pairwise methods. This is due to the fact that focusing exclusively on two alternatives at a time facilitates experts when expressing their preferences.

Preference relations are one of the most common formats to represent the information provided by the experts in the decision making context. A preference relation is a special function which can be defined as follows:

Definition 1. [5]. *A preference relation P on the set X is characterized by a function* $\mu_p : X \times X \to D$*, where D is the domain of representation of preference degrees provided by the decision maker for each pair of alternatives.*

When cardinality of X is small, the preference relation may be conveniently represented by an $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_p(x_i, x_j) \forall i, j \in \{1, \cdots, n\}$

According to the nature of the information expressed for every pair of alternatives, many different representation formats can be used to express preferences. Xu presents in [6] a survey of preference relations. However in this paper we are going to focus in Multiplicative Preference relations (MPR) and in Additive Fuzzy Preference relations (APR).

2.1. Additive Preference Relation

The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy or [0,1]-valued preference relation, $P = (p_{ij})$ [7], referred to as additive preference relation (APR) in this paper:

Definition 2 (Additive Preference Relation (APR)). *An APR P on a finite set of alternatives X is characterised by a membership function* $\mu_P : X \times X \longrightarrow [0, 1]$, $\mu_P(x_i, x_j) = p_{ij}$*, verifying* $p_{ij} + p_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}$.

The following interpretation is assumed:

- $p_{ij} > 0.5$ indicates that the expert prefers the alternative x_i to the alternative x_j, with $p_{ij} = 1$ being the maximum degree of preference for x_i over x_j;
- $p_{ij} = 0.5$ represents indifference between x_i and x_j.

An APR can be seen as a particular case of a (weakly) complete fuzzy preference relation [8], i.e. a fuzzy preference relation satisfying $p_{ij} + p_{ji} \geq 1 \ \forall i, j$.

2.2. Multiplicative Preference Relation

The measuring of the intensity of preferences can be done using a ratio scale instead, with the most widely ratio scale used being the interval $D = [1/9, 9]$ [9].

Definition 3 (Multiplicative Preference Relation (MPR)). *A MPR A on a finite set of alternatives X is characterised by a membership function* $\mu_A : X \times X \longrightarrow [1/9, 9]$, $\mu_A(x_i, x_j) = a_{ij}$*, verifying* $a_{ij} \cdot a_{ji} = 1 \ \forall i, j \in \{1, \ldots, n\}$.

The following interpretation is assumed: x_i is a_{ij} times as good as x_j, and in particular:

- $a_{ij} = 1$ indicates indifference between x_i and x_j;
- $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j;

In [10], it was proved that multiplicative and additive preference relations are isomorphic:

Proposition 1. *Suppose that we have a set of alternatives,* $X = \{x_1, \ldots, x_n\}$*, and associated with it a MPR* $A = (a_{ij})$*, with* $a_{ij} \in [1/9, 9]$ *and* $a_{ij} \cdot a_{ji} = 1, \forall i, j$*. Then the corresponding APR,* $P = (p_{ij})$*, associated to A, with* $p_{ij} \in [0, 1]$ *and* $p_{ij} + p_{ji} = 1, \forall i, j$*, is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2}\left(1 + \log_9 a_{ij}\right) \tag{2}$$

The above transformation function is bijective and, therefore, allows to transpose concepts that have been defined for APRs to MPRs, and vice-versa.

3. Processes to estimate missing judgements in GDM

It is often assumed in theoretical approaches to GDM that all the experts are able to provide preference degrees between any pair of possible alternatives, which means that complete PRs are assumed. However this is not always possible because of time pressure, lack of knowledge, decision maker's limited expertise on the field dealt with, or incapacity to quantify the degree of preference of one alternative over another. Thus, an expert might decide not to guess the preference values in doubt to maintain the consistency of the values already provided. To model these situations the concept of incomplete PR was introduced in [11]. In this section we analyse the main techniques in the literature to deal with incomplete information in decision making when the experts express their judgements by means of APR and MPR. We should remark at this point that the algorithms developed for one of them can be directly applied to the other one using the transformation function between (reciprocal) MPR with values in the interval scales $[1/9, 9]$ and reciprocal APR with values between $[0, 1]$ pointed out in [10].

The techniques to complete an incomplete APR and/or MPR can be widely divided into two main groups depending on the approach used to obtain the missing preferences:

1. Iterative approaches
2. Optimisation approaches

3.1. Iterative approaches

These approaches seek to fill the missing preferences in an incomplete PR following a repetitive procedure in which the missing values are calculated using known ones. We can highlight two main iterative approaches to estimate incomplete APRs and MPRs : additive consistency based approaches [12,5,13,14,15] and its generalisation approach based on the use of uninorm operators [16] and multiplicative consistency based approaches [17].

1. **Additive consistency based approaches:** The main additive consistency based method is due to Herrera-Viedma et al. [12], which consists of an iterative procedure to estimate missing preference values followed by a choice process of the solution alternative. Given an unknown preference value p_{ij} $(i \neq j)$ the iterative procedure starts by using intermediate alternatives, x_k, to create indirect chains of known preference values, (p_{ik}, p_{kj}), that will be used to derive, using the additive consistency property. Notice that the cases when an incomplete APR cannot be successfully completed are reduced to those cases when no preference values involving a particular alternative are known, which means that a whole row or column of the APR is completely missing.

   In [5], an extension to deal with MPR, IVPR, and LPR is presented. The original approach by Herrera-Viedma et al. has been taken forward by many authors to tackle different research problems with incomplete APRs. Notable examples can be found in [18,13,14,15].

   Due to the fact that additive consistency property does not generalise the concept of transitivity of crisp preferences, in [19] it is shown that, under a set of conditions, consistency of APR can be characterised by representable uninorms. Therefore in [16], Herrera-Viedma et al's iterative method is adapted to implement the modelling of consistency of preferences using a self-dual almost continuous uninorm operator. Since Tanino's multiplicative transitivity property is an example of such type of uninorms [16,19], this approach to deal with incomplete information in APRs is more general than the above one.

2. **Multiplicative consistency based approaches:** The most relevant method is due to Xu [17]. In this method, each individual incomplete APR is completed using the multiplicative consistency property, followed by their aggregation into a collective preference relation. Based on the deviations between the collective and individuals APRs, the decision makers interact to increase the level of consensus.

3.2. Optimisation and linear programming based methods

The two optimisation approaches to deal with incomplete PRs are analysed next:

1. **Optimisation methods to estimate missing preference values.** These approaches aim to estimate the missing preference values by maximizing the consistency and/or the consensus of the experts' preferences.The most relevant of these methods are due to Fedrizzi and Giove [20] and Zhang et al. [21]

   (a) In Fedrizzi and Giove [20] it is proposed a function that measures the global additive inconsistency of the incomplete APR, and in which the missing preference values are the variables. Under this approach, the stationary vector that minimises the global inconsistency function is taken as the estimated values for the unknown preference values. Obviously, these estimated values are the most consistent with respect to the known preference values.
   A comparison between Fedrizzi and Giove's method and Herrera-Viedma et al.'s in [12] is found in [22]. Both methods are driven by the additive consistency property. Concluding that both methods, as originally presented, provide the same set of solutions for independent sets of missing comparisons but not for dependent missing comparisons. This comparative study also shows that a modification of Herrera-Viedma et al.'s coincides with Fedrizzi and Giove's method. However, the main difference between both resides in their successful application in reconstructing an incomplete APR. Fedrizzi and Giove's method performs worse than Herrera-Viedma et al.'s method for a large number of alternatives. As mentioned before, Herrera-Viedma et al.'s method fails, as well as Fedrizzi and Giove's method, to complete an incomplete APR only when no preference values are known for at least one of the alternatives. Therefore,. it was concluded that both methods are complementary, rather than antagonistic, in their application, and as such, a new policy for reconstructing incomplete APRs that makes use of both methods was proposed.

   (b) Zhang et al. [21] propose a model for incomplete APR $F = (f_{ij})_{n\times n}$ that aims to calculate a complete fuzzy preference relation $F' = (f'_{ij})_{n\times n}$ with $f'_{ij} = f_{ij}$ for non-null entries of F maximising the consistency level proposed by Herrera-Viedma et al. [12]. To increase the individual consistency they propose a linear optimisation method that minimises the Manhattan distance between the provided preference relation and the completed consistent based one.

2. **Methods where priority weights are directly computed:** These methods aim at ranking the alternatives using directly the incomplete APR, and therefore no completion process is needed. They are based on Saaty's assumption for MPR that there is an exact functional relation between the preference values and the priority vector. Two main approaches are used to develop indirect completion models based on the computation of the priority vector: linear based methods [23,24,25,26,27,28], and least square optimisation based methods [29,26,30,31].

   (a) Linear based methods

      i. Harker extends in [23] the eigenvector approach proposed by Saaty [9] for non-negative quasi reciprocal matrices in order to apply it to the case of incomplete APRs.

      ii. Xu [24] presents a method based on a system of equations to determine the priority vector of an incomplete APR, by replacing a missing preference value p_{ij} with the following priority weighting vector relation: $\frac{w_i}{w_i + w_j}$. The main advantage of this procedure is that if there exists a unique solution to this system of equations, then the obtained solution is used to rank the alternatives and to select the most desirable one; otherwise, it requires the experts to provide more evaluation information until the unique priority vector can be obtained.

      iii. Xu and Chen propose in [25] a completion method based on the additive transitivity property that requires solving a linear system of equations for ranking alternatives. Later, this proposal was proved in [32] and [26] that the relation of the PR and the elements of the priority weight vector $r_{ij} = 0.5(w_i - w_j + 1)$ postulated in Xu and Chen's methods does not always hold, resulting, in some cases, in ambiguous priority vectors. To overcome this drawback, Xu proposed to use the following auxiliary additive transitivity based ,PR $R' = (r'_{ij})_{n\times n}$, to estimate the missing preferences values [26]:

$$\begin{aligned} r'_{ij} &= r_{ij} \text{ , if } r_{ij} \text{ is known;} \\ r'_{ij} &= \frac{n-1}{2}(w_i - w_j) + \frac{1}{2} \text{ , otherwise.} \end{aligned} \tag{3}$$

iv. In[27], Xu proposes two goal programming models for obtaining the priority vector of an incomplete APR, and extends these models to obtain the collective priority vector.

v. In[28], a parametric goal programming model based on the consistency property for MPR, to obtain the weighted priority vector, is proposed. This model makes use of a dissimilarity function between the ideal case, when the preferences are consistent and there is unanimous consensus among experts, $I^k = \left(\frac{w_i}{w_j}\right)$, and the provided incomplete MPR, M^k. The objective function corresponds to a compromise criterion constructed as a convex combination of the two extreme criteria: to minimise the weighted sum of expert deviations and to minimise the largest weighted deviation. In this model, the relative residual aggregation is modelled by a parameter α that is used to control the importance given to the most discrepant expert.

(b) Least square based methods

i. Gong presented in[29] a multiplicative consistency based least-square model for APRs aiming at maximising the consensus among the experts by minimising the following error function:

$$\min g(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{d_{ij}} (r_{ijl} w_j - r_{jil} w_i)^2 \tag{4}$$

$$\text{s.t.} \sum_{i=1}^{n} w_i = 1, \ , w_i > 0, \ i \in n \tag{5}$$

where d_{ij} stands for the number of experts who have provided a preference between the alternatives x_i and x_j. A similar approach that allows for the following three formats of incomplete preference relations: MPRs, APRs, and LPRs, is proposed in[33]. A variant that uses a logarithmic least squares instead is proposed in[31].

ii. An approach based on additive consistency property is found in[30]. This approach is based on the solving of the following optimisation problem:

$$\min D' = \sum_{(i,j)\in E} (\frac{1}{2}(w_i - w_j + 1) - r_{ij})^2 \tag{6}$$

where $E = \{(i, j\}|r_{ij}$ is known and $i \neq j\}$.

In[34] a comparative study of seven different methods for reconstructing incomplete APR in terms of the consistency of the resulting complete preference relation is presented. They compare 4 methods for MPR and three for FPR using both, consistent and highly inconsistent preference relations. Finally they compare the numerical results in terms of the consistency Ratio, introduced by Saaty in[9], of the reconstructed preference relation.The best results are obtained by using the optimization methods where the missing entries are directly computed as is the case of the algorithm in[20], followed by the methods where the priority weights w_i are first computed. The two least square calculation approaches are the ones which presents worse performance.

4. Processes dealing with ignorance situations in GDM

The procedures exposed previously are not succesfull in situations when some experts do not provide any information about a particular alternative, which is known as ignorance situations. Alonso et al.[35] developed several strategies to deal with ignorance situations in the context of GDM with APRs. These strategies can be broadly classified as social strategies and individual strategiesdepending on whether the information provided by other experts is used to estimate the missing values.

4.1. Individual strategies

The proposed individual strategies can be divided in two main steps:

1. Setting some particular seed values to provide some initial information to the estimation procedure to be able to compute the other missing values. The selection of the seed values can be accomplished following different methodologies such us choosing indifference seed values, or choosing proximity seed values. In this second case the seed values are obtained from the preference values given to similar alternatives. This is possible if some extra information or properties about alternatives, which strongly suggest that the ignored alternative is similar to another one, are known. This strategy could be useful in some decision making problems where the alternatives to be evaluated are goods with similar characteristics (similar models).

2. Estimating the rest of the missing values using the consistency based procedure proposed in [12].

4.2. Social strategies

Social strategies are based on the use of the information provided by the set of experts. The authors present three main approaches in this case:

1. The first social strategy uses consensus preference values of the collective PR, computed by aggregating all the experts' individual PRs. The main advantage of this approach is that it improves the consensus of the set of experts making their opinions close to each other.

2. The second strategy uses only the consensus preference values provided by those experts nearest to the expert whose PR is incomplete. This strategy is aimed to narrow the differences between the expert with an ignored alternative and those who have a similar opinion about the rest of alternatives.

3. The third approach integrates the previous two by taking into account both information from the collective preference relation and from the nearest experts. This strategy encompasses the advantages of the previous two social strategies since the estimated information not only helps in the consensus process but also tries to keep a high consistency level in the individual experts' PR. Therefore it is considered by the authors of the proposal as the best strategy to deal with ignorance situations in GDM.

5. Conclusion and future work

In this contribution we have reviewed the main completion approaches in the literature to deal with missing information for APR and MPR including total ignorance situations. The majority of the analysed approaches in this contribution takes advantage of the additive and or the multiplicative properties to estimate the missing values from the known ones and they can be broadly classified as iterative approaches and optimization approaches. One the one hand the iterative ones seek to fill the missing preferences following a repetitive procedure in which the missing values are calculated using known ones. On the other hand, the optimisation approaches carry out the completion process by maximizing the consistency and/or the consensus of the experts' preferences using a wide range of mathematical procedures such as goal programming and least square minimization. Finally we should point out that there are a wide variety of methodologies in the literature to deal with missing APR and MPR. However few techniques have been investigated to enable the expression of some kind of imprecision in the experts' judgements. Two very promising types of PRs that are being recently and widely used in decision making are type-2 fuzzy PR and hesitant preference relations. Hence, effective methods to estimate the missing information when working with these types of PRs are worthwhile to be investigated in the future.

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