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KINDERGARTEN AND FIRST-GRADE STUDENTS' UNDERSTANDINGS AND REPRESENTATIONS OF ARITHMETIC PROPERTIES

We present a study that explores Kindergarten and first-grade students' understandings and representations of arithmetic properties. Sixteen students participated in a classroom teaching experiment designed to explore children's algebraic understandings, including their understandings and symbolic representations of three arithmetic properties: additive identity, additive inverse, and commutativity. We characterized students' understandings in terms of Skemp's framework of understandings: rules without reason (instrumental) and knowing what to do and why (relational). Then, following Vergnaud, we analyzed the types of additive relationships (transformation, comparison, or combination) and representations used by students. Our findings show that students' understandings developed in sophistication over time. We observed the least sophisticated understandings for the commutative property, particularly among Kindergarten students who exhibited instrumental understandings even after instruction.

Key Words: Young Children, Arithmetic Understandings, Representations, Arithmetic Properties

From a very young age, children develop understandings about arithmetic—or the study of numbers, operations, and the properties of operations. Children develop these understandings in everyday settings (see Nunes, Schliemann, & Carraher, 1993), prior to and regardless of the start of formal schooling. However, the start of schooling seeks to formalize their understandings, build on their knowledge of numbers, of how numbers relate to each other, of operations on numbers, and of the properties of those operations. In our study, after four weeks of participation in a classroom teaching experiment (CTE), a first grade student expressed the following understanding of additive identity, or the property that “anytime you add zero to a number, you get the number back:”

B1_2: Zero is nothing, so if you ever see a number with zero, it’s just going to be the same number.
Interviewer: Okay, great. Do you think that’s true for— You said whenever you see a number plus zero. So are you saying that’s true for any number, or for just some numbers?
B1_2: Any.
Interviewer: Any number? Okay, great. Could you represent that idea using a letter, a variable? Could you write an equation that shows me what you’ve said so well in words? You can just put it down here if you want. Can you write an equation that would describe that idea to your friend?
B1_2: Mmm—
Interviewer: Tell me what you wrote. Explain it to me.
B1_2: Y plus zero equals Y.

In this paper, we report on a study in which we explored Kindergarten and first grade students’ understandings and representations of properties of arithmetic operations, such as additive identity. The importantce of understanding the relationship between mathematics education and early childhood education is underscored because it can help illuminate our knowledge about young children’s broader learning and development (Saracho & Spodek, 2009).

Within early mathematics education, the study of early elementary students’ algebraic reasoning, or early algebra, has drawn considerable attention in recent decades (Cai & Howson, 2013). Within early algebra, generalized arithmetic has been viewed as a way to introduce algebraic reasoning while simultaneously deepening children’s understanding of number and operations through arithmetic tasks with which young children are familiar (Carpenter, Franke, & Levi, 2003; Kaput, 2008). More specifically, the arithmetic properties of additive identity, additive inverse, and commutativity (see Table 1) are considered relevant to teaching early algebraic thinking because they can foster arithmetic generalization (Blanton, Levi, Crites, & Dougherty, 2011) and contribute to understanding the nature of number, to computacional fluency, and to the ability to solve problems (Ching & Nunes, 2017). At the same time, understanding the general nature of properties may allow students to delve more deeply into the meaning of operations and connect arithmetic with algebraic reasoning (Schifter, 2009). Further research is needed, however, to fully understand the relationship between arithmetic and algebraic reasoning in the early grades and its effect on older students' ability to learn algebra (Warren, Trigueros, & Ursini, 2016).

Table 1
Arithmetic Properties of Numbers and Operations (adapted from Author, 2008; Blanton, 2008, p. 15)

Arithmetic property	Natural language expression	Symbolic expression	Example
Additive Identity	Anytime you add zero to a number, you get the number back.	$a + 0 = a$	$3+0=3$
Additive Inverse	If you subtract a number from itself, you get zero.	$a - a = 0$	$3-3=0$
Commutative Property of Addition	You can add two numbers in any order and get the same result.	$a + b = b + a$	$3+5=5+3$

Some of the main aspects of quality early mathematics instruction are students' mathematical understandings and representations (Cerezci, 2020). While research provides evidence that students have the potential to recognize, generalize, and represent properties, we have found no prior studies regarding the relationship between the types of understandings students exhibit and the types of representations they use when generalizing properties. As Baroody, Torbeyns, and Verschaaffel (2009) state, "our understanding of when and how an accurate, reliable, and general undersantanding of these arithmetic principles emerges and develops is particulary incomplete" (p. 6).

In this paper we address the following research questions:

(1) What kinds of understandings and representations of the arithmetic properties of *additive identity*, *additive inverse*, and *commutativity* are exhibited by Kindergarten and first-grade students who participate in a CTE that supports relational understandings (Skemp, 2006) and symbolic representations?

(2) What, if any, connections are there between children's understandings and their representations?

As young children develop, they are also developing many fundamental concepts that can be built upon by adults in their later schooling (Seo, 2003). It is essential to explore children's understanding of fundamental concepts of arithmetic (Charlesworth & Leali, 2012), which are later applied to more advanced concepts. However, prior studies have provided contradictory results regarding Kindergarten and first grade children's understandings of arithmetic properties. For example, in terms of the commutative property, Bermejo and Rodríguez's (1993) study showed that primary school children seemed to understand commutativity only with small sets while preschoolers could appreciate commutativity in larger sets. This study's goal is to explore the relationship between algebraic understandings and representations for the three fundamental properties of arithmetic, comparing Kindergarten and first grade students. Most research on children's arithmetic concepts is based on one concept at a time, limiting the conclusions that can be made about how children's conceptual knowledge of arithmetic develops (Robinson, Dubé, & Beatch, 2017).

Arithmetic properties in the early grades

In this study, our aim was to explore students' understandings and representations of arithmetic properties and relationships between these. Three such properties are explored in this study, expressed in ways that are deemed appropriate for young children (see Table 1). In their own study on this topic, Carpenter and his co-authors found that third- to fifth-graders exhibited sufficient understanding of *additive identity* and *inverse* to represent their generalizations using natural language (Carpenter et al., 2003). Elsewhere, Pang and Kim (2018) found that as early as third grade, students recognize and express the *additive inverse* and *commutativity* properties using algebraic symbols. Further, Blanton, Stephens, Knuth, Gardiner, Isler and Kim (2015) document that third-grade children recognize the underlying structure of arithmetic properties, and use them to justify their arguments.

In earlier grades, researchers found that after instruction focused on arithmetic properties, first- and second-graders were able to represent the properties symbolically with letters (Carpenter et al., 2003; Carpenter & Levi, 2000; Carpenter, Levi, Berman, & Pligge, 2005). The first and second graders also generalized *additive identity* in both addition and subtraction, while attributing to zero the meaning of a number "that doesn't change [the initial quantity]." They generalized *additive inverse*, in turn, by comparing quantities and used variable notation to represent the property. In terms of *commutativity*, they recognized it when represented symbolically, but did not articulate a general explanation (Carpenter & Levi, 2000).

While children at these ages were observed to recognize *commutativity*, it is unclear whether they were exhibiting understanding of the property itself or only understanding that switching the placement of the two quantities did not change the total (Schifter, 2009). Other studies, for instance, have found that students focus on the order of the operands without taking into account whether the operation involved is subtraction or addition (Bastable & Schifter, 2017; McGowen & Tall, 2010). Bastable and Schifter (2017) also found that children who might readily use *commutativity* to solve problems with small numbers might doubt whether the property holds for all numbers. In a study with 5 to 6-year-old students, Ching and Nunes (2017) observed that knowledge of *commutativity* seems to develop from thinking in the context of specific quantities to thinking about more abstract symbols. Bermejo and Rodríguez (1993) found that the differences between preschoolers and first- and second-grade students were significant in tasks that involved the *commutative property*. These prior studies suggest that students' understanding of properties is associated to the meanings they attribute to additive situations; for example, several different studies showed that children performed better on *commutativity* tasks that were presented in the format of Combine problems (Ching & Nunes, 2017). In this study, we will explore students' understandings of the three properties, focusing on additive structures as framed in our theoretical framework. Gathering this kind of data and information is crucial as we look to make informed instructional decisions and is an integral part of most early childhood programs (Snow & Van Hermel, 2008).

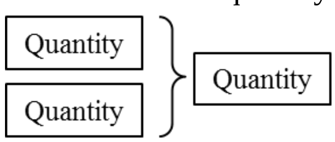
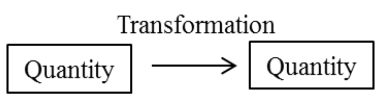
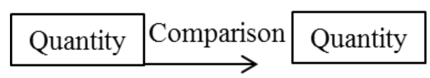
THEORETICAL FRAMEWORK

The focus of this study is on *generalized arithmetic*, which we take here to involve generalizing arithmetic relationships, including properties of number and operation, and "reasoning about the structure of arithmetic expressions rather than their computational value" (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015, p. 43). Within a generalized arithmetic framework, students are encouraged to perceive and represent underlying structures, such as the arithmetic properties, and justify and reason based on the generalizations they recognize (Kaput, 2008). Blanton et al. (2011) deem the ability to build on arithmetic properties to generalize arithmetic operations to be an "essential understanding" of early algebraic reasoning.

We used Skemp's (2006) terminology here to analyze understandings, drawing a distinction between relational understandings that imply "knowing both what to do and why" (p. 89) and instrumental understandings that imply that a student knows a rule or procedure and has the ability to use it, but "without reason" (p. 89).

In this study, we further refined *relational understandings of arithmetic properties* using Vergnaud's (1996) description of additive structures as the suite of situations that can be generated from six basic relationships. The first three of these relationships are combination, transformation, and comparison (see Table 2). The other three relationships are described from the first three: combinations of transformations, transformation of a relationship, and combinations of relationships.

Table 2
First Three Basic Relationships in Additive Situations (Vergnaud, 1996).

I Combination	II Transformation	III Comparison
Combination of two quantities into a third quantity	Transformation of an initial quantity into a final quantity	Comparison of two quantities
		
Mary has two red pencils and three green pencils. How many pencils does she have in total?	*Mary has eight candies and she is given five more. What is her total number of candies?*	*Mary has six cookies and John has three more than she does. How many cookies does John have?*

In the basic relationship of combination, there are two static quantities that are combined to form a third total quantity. In the basic relationship of transformation there are three different moments: there is an initial quantity, there is a transformation of that quantity, and there is a final quantity. In the basic relationship of comparison, a comparison links two different quantities.

Given findings from previous studies that understanding a concept is associated with the various ways in which it can be represented (Rico, 2009), the study reported in this paper also analyzed the kinds of representations used by students. Kaput (2008) defined generalization and its representation in a growing system of conventional symbols as a core aspect of algebraic reasoning. Generalizations may be represented in non-conventional forms, however, such as natural language (Radford, 2011). Ureña, Ramírez, and Molina (2019) distinguish among numerical, verbal, generic, and symbolic representations for generalizations of functions.

In the fourth category, symbolic representations, students will use algebraic expressions and equations to represent a generalization. Students' use of symbolic representations are facilitated by the kind of instruction they experience and the kinds of tasks they work on. Relational understanding need not necessarily precede the introduction of symbols as notation, for meanings and symbols may co-emerge (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015).

METHODOLOGY

Participants

The research was conducted in two elementary schools in the northeastern United States. School A was in a district with 20 % low-income families and 6 % English Language Learners, and School B was in a different district with 45 % low-income families and 27 % English Language Learners. Each Kindergarten and first-grade class selected in each school had around 22 students, for a total study population of approximately 88 students.

Data collection

Classroom teaching experiment

The CTE (Steffe & Thompson, 2000) was carried out in the four classrooms over a 7-week period and included 14 30-minute lessons (i.e., two per week) led by the project researchers. The lessons were video-recorded and the research team met once a week to discuss interim findings with the goal of making any necessary revisions to the upcoming lessons.

Interviews

The classroom teachers identified four students in each class with low, medium, or high ability to work with numbers, count, add, subtract, and express themselves orally. At least one student from each performance group was chosen from each of the four classes. During the CTE, five individual interviews were held with each of the 16 students chosen (80 interviews in total). The pre- and post-clinical interviews conducted prior to and after the CTE consisted of the same set of questions. Because the aim was to assess students' initial and final knowledge of arithmetic properties, these interviews were conducted with no interviewer follow-up to the students' responses. The three teaching experiment interviews (Steffe & Thompson, 2000) were held before (but after the pre-clinical interview), during, and at the end (but before the post-clinical interview) of the CTE (see Figure 1).

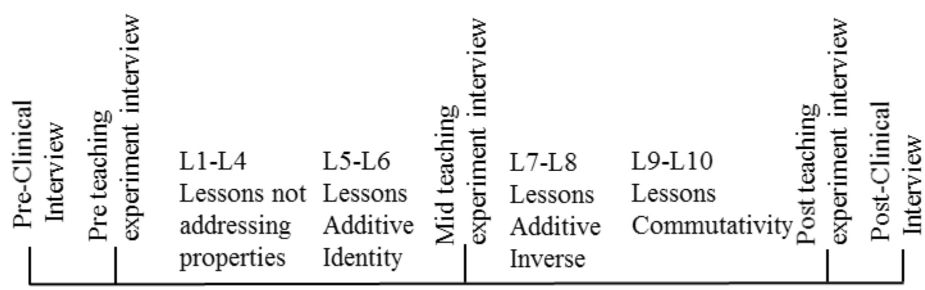


Figure 1: Chronology of CTE and interviews.

The interviewer's role was similar to that of the teacher researcher during the lessons (encouraging argumentation and the use of letters), while the tasks involved were similar to those implemented in the lessons in an effort to capture children's thought processes. The interviews were carried out by a project researcher and observed by a second researcher whenever possible.

Lessons on arithmetic properties

During the first four lessons in the CTE, L1-L4, we did not focus on arithmetic properties. The focus was on equations, the equal sign, and equivalence, and we introduced the use of letters to represent indeterminate quantities. As shown in Figure 1, during the CTE, each arithmetic property was the focus of two lessons that followed the same overall structure and focused on algebraic reasoning (Kaput, 2008). The first lesson for each property had two goals. The first goal was to analyze information to describe a conjecture about the property and represent it in words. The modus operandi consisted of describing an initial situation in which the property was introduced for a given value (e.g., "Charlotte's birthday is coming soon. One day, she got 5 birthday cards in the mail. The next day, she didn't get any cards. How many cards did she get all together? Draw a picture that shows your thinking. Can you write an equation that shows how you got your answer?"). We collected examples from students expressing the properties (e.g., 3+2 = 2+3) and used these examples to generate, with the students, a list of equations representing the property. The students were asked if they noticed an underlying structure in the equations and how they would represent the generalization they identified. The second goal of the first lesson for each property was to identify the values for which the premise was true. A game was set up in which the students were asked to match flash cards (i.e., "_=9+0) showing open equations to other cards containing the solutions. After identifying the values, the students were again asked to describe a conjecture about the property and to identify the numbers for which they deemed it to be true. The second lesson for each property came back to the idea of conjectures and their validity for any number; the goal of these lessons was to represent the generalization using variables (e.g., "Katie started writing an equation on her paper that her teacher was writing on the board. She didn't get to finish. The following is what she wrote: _____ + 0 =_ . What number(s) could Katie have put in the missing places of the equation to make the equation true?"). The final part of the lesson was devoted to identifying the generalization (i.e., property) underlying the different computational tasks.

The tasks used in the interviews were similar to the ones introduced during the lessons: while the specifics and values changed, the types of equations were similar.

Data analysis

For our analysis of the data, we used videotaped recordings of student interviews and students' written work produced during the interviews. We analyzed all transcripts and written work produced by students for each of the interviews, focusing on both their *understandings* and their *representations*. The categories defined for systematic analysis of students' understandings and representations are presented below. We used each student's complete answer to each one of the interview questions as a unit of analysis. The first author of this paper coded each one of the students' responses. The second author confirmed all coding with the first author and any disagreements were discussed until consensus was reached.

Categories for analyzing understandings

The categories listed below to characterize students' understandings of arithmetic properties were drawn from Skemp's (2006) description that distinguishes between *relational* and *instrumental understandings* and Vergnaud's (1996) classification of additive structures into combination, transformation, and comparison.

- *Instrumental understanding (P).* Students exhibited understandings based on learned rules without describing the generality of the property. In these cases, students mentioned or alluded to learned rules or memorized facts.

- *Relational understanding with combination (I).* Students described the property using a combination type relationship, i.e., by combining two initial quantities to obtain a third (e.g., “they just switched it around. They are both equal the same thing”). This type of understanding was observed only in *commutativity*, where the students would combine two quantities to obtain a final quantity noting that the order in which the quantities are combined doesn’t matter. In *additive identity* and *inverse* the initial quantity was compared or transformed, but not combined with another quantity.
- *Relational understanding with transformation (II).* Students described the property by transforming an initial quantity to obtain a final quantity. In these cases, additions and subtractions are interpreted as actions that add or subtract elements from an initial quantity (e.g., “you put zero that means you don´t take anything away or you don’t add anything”).
- *Relational understanding with comparison (III).* Students described the property drawing on a comparison between two quantities. This kind of understanding was only observed for *additive inverse*, where students alluded to the relationship between two quantities (e.g., “they have the same number, that means that both of them are going to be zero”).

Some responses were coded as emergent when a student articulated a justification that was incomplete or imprecise.

A prior coding scheme for analyzing representations

In this study, we analyzed representations using the previously developed categories of numerical, verbal, generic, and symbolic, drawn from Ureña et al. (2019):

- *Numerical.* Students identified the property and found the underlying structure in a series of numbers, but were unable to describe the general rule using algebraic symbolism. In both *additive inverse* and *commutativity* they restricted the rule to a certain number set.

- *Verbal.* Students generalized the property in natural language alluding to indeterminate quantities, but did not use algebraic expressions. That is, the students used natural language and not algebraic symbolism to articulate the general rule even when the questions they were asked about properties included algebraic expressions.
- *Generic.* Students generalized verbally or numerically, describing the general rule using generic examples in which specific quantities were used as examples that were meant to represent several values at the same time.
- *Symbolic.* Students generalized using algebraic expressions (letters to represent any value, equations, and so on).

RESULTS

We first show findings for the types of understandings exhibited by students for each one of the properties. Following this, we present the types of representations used by students as well as the relationship between understandings and representations. We conclude with an overview of the results and an analysis of the properties as a whole.

Students' understandings of arithmetic properties

The findings for students' understandings of the properties are presented in Table 3.

Table 3

Students' Understandings and Representations of the Arithmetic Properties in the CTE Interviews

Student	Pre-clinical interview			Pre interview	Mid-interview			Post-interview		Post-clinical interview		
	AP	IP	CP	AP	AP	IP	CP	IP	CP	AP	IP	CP
AK_1	NO	NO	P:V	NO	P:S	NO	P:V	II*:G	P:V	P:V	P:N	P:V
AK_2	II:G	NO	NO	II:G	II:G	II + III*:G	P:V	III*:G	P:V	II:G	NO	P:V
AK_3	II:V	II:V	I*:G	II:V	II	II + III:G	I*:G	II:G	I*:G	II:G	II:V	P:V
AK_4	II:V	NO	NO	II:G	II:S	III*:V	P:S	III*:V	P:V	II:V	II:V	NO
BK_1	NO	NO	NO	II:G	II:V	III*:G	NO	P:V	NO	II:V	II:V	I*
BK_2	II:V	P + II*:V	NO	II*:G	II*:G	II*:V	NO	II:V	P:V	II:V	II:V	P:V
BK_3	II:V	II:V	NO	II:V	II+S	III:S	NO	III:S	P:V	II:V	II:V	P:V
BK_4	P:V	P:V	NO	P:G	P:G	P:G	NO	P:V	NO	P:V	P:V	NO

Table 3

Students' Understandings and Representations of the Arithmetic Properties in the CTE Interviews

Student	Pre-clinical interview			Pre interview	Mid-interview			Post-interview		Post-clinical interview		
	AP	IP	CP	AP	AP	IP	CP	IP	CP	AP	IP	CP
A1_1	P:V	II:G	P:N	P + II:G	II:V	II + III:S	P:S	II:S	P:S	P:V	II:G	P:V
A1_2	II:V	II:V	II:V	II:S	P + II:S	II:S	P + I:S	II + III:S	P:S	P + II:V	III:V	I:V
A1_3	P:G	NO	I*:V	P:S	P + II:V	III:G	NO	P + II:S	NO	P:G	II:V	I:V
A1_4	II*:V	III*:V	P:V	II:S	P:S	III:S	NO	II:S	P:S	P+II:G	II:V	P:V
B1_1	II:G	P:V	P:V	II:S	P:S	II:S	P + I:S	II:S	P:S	II:V	II:V	P:V
B1_2	II:G	II:G	I:G	II:G	II:S	II:G	I:S	II:S	I:S	II:V	II:G	I:G
B1_3	P:V	III:V	P:V	II:S	II:S	III + II:S	P:G	III + II:S	NO	II:V	II:V	P:V
B1_4	II:V	III:G	P:V	II:V	II:G	III:S	I:G	III + II:S	I	II:V	III:G	I:V

Note: NO: no response; N/A: not applicable; I: combination; II: transformation; III: comparison; P: instrumental; (*)=emergent; N: numerical; V: Verbal; G: Generic; S: Symbolic
AP: Additive Identity Property; IP: Additive Inverse Property; CP: Commutativity Property

Students' understandings of additive identity

With the exception of two Kindergarten students, all students exhibited a transformation relational understanding in at least one of the five interviews carried out during the CTE (see Table 3); eight of the 16 did so in all the CTE interviews. The *additive identity* property was perceived as a transformation (II) in which zero meant "absence of change." In their explanations, students described the rationale for the property (B1_1*:* "zero's nothing, so when you add it, it's still the same amount and number; The zero doesn't add anything. It doesn't get any larger or any smaller"). Although the question focused on addition, the students recognized that the *identity* property was also applicable to subtraction (BK_2*:* "doesn't mean you take anything away or add nothing; you put zero that means you don't take anything away or you don't add anything. You just put the number you tried but equal it"). The property was closely associated with the meaning students attributed to zero as the absence of quantity and the smallest number they knew (B1_1*:* "zero its the lowest number in the numbers, so it doesn't give you anything"). Associating the number with the quantity represented, they even asserted that zero is not a number (B1_1*:* "The zero is no number; it's always a number only doesn't have no [sic] number; it looks like a zero so it can make numbers not go there; the zero doesn't count"). Given that they regarded zero as a "different" number from all others, when they were asked whether the property was valid for "any number," A1_4, AK_3, and B1_1 denied its applicability to zero (AK_3*:* "Any number in the world, except zero"). One first-grader exhibited a transformation type of relational understanding, even alluding to the unicity of identity (zero is the only number that when added to another results in the number itself, unchanged), realizing that if the initial and final states were the same, the transformation had to be zero (A1_1*:* "if you want to get the same number adding and subtracting, you have to use a zero").

Table 3 shows that, even prior to the lessons on additive identity, 15 of the 16 students (except AK_1) identified the property and 12 exhibited a relational understanding. Only two Kindergarten students (AK_1 and BK_4) did not exhibit a relational understanding at any time, instead reciting the rule they had learned (BK_4: "it equals back to a big number"). No substantial differences were observed between Kindergarten and first grade students' recognition of additive identity, with all but the two Kindergarten students mentioned above exhibiting a relational understanding and attributing the absence of transformation to the zero in the underlying operation.

Students' understandings of additive inverse

Except for the same two Kindergartners (AK_1 and BK_4) who, as noted above, understood *additive identity* only instrumentally, all the other students exhibited a relational understanding of *additive inverse* in at least one of the CTE interviews (see Table 3); seven of them do so in all CTE interviews. Relational understandings based on transformations also prevailed in all the interviews (33 of the 64 records in Table 3 for this property), where *additive* *inverse* was understood as a transformation of the initial quantity (A1_2*:* "because if you have a certain number and then you take away all of them away it's just zero"). Two first-graders (A1_1 and A1_2) exhibited a transformation type of relational understanding that suggested the unicity of the *additive inverse* property as the sole number that when subtracted from the initial number yields zero (A1_1*:* "you're subtracting the same number from itself…. if you want to get zero, you had to subtract the same number from itself"). Although children of these ages may be less familiar with subtraction, which is required to understand this property, they did not seem to encounter any greater difficulty with *additive inverse* than with *additive identity*. Kindergarten students associated subtraction with "taking away" from the initial quantity and obtaining a smaller number (AK_3*:* "minus means take away; it will get a lower number than you already have"). Two Kindergarten and six first-grade students based their relational understandings on comparisons. They found zero to be the result of having no elements left over after matching the quantity representing the number and its inverse (BK_3*:* "Cause there's 15 friends and 15 juice boxes. If each kid gets one, there's no more, cause there's the same number of juice boxes and people"). Three Kindergartners exhibited emergent understandings of this property, alluding to the comparison of quantities but without expressing it generally (BK_1: "took away all of his cards, so he doesn't have any more"), although in a later interview in the CTE the same students alluded to transformations of the initial quantity, an indication of relational understanding.

Table 3 shows that in the mid-interview, prior to the lesson on *additive inverse*, the property was understood by all students except one Kindergarten student (AK_1). This Kindergarten student did not allude to, generalize, or represent the property accurately. Nine students (all the first-graders and two Kindergartners, AK_3 and BK_3) exhibited relational understandings. While the two Kindergarten students (AK_1 and BK_4) showed only instrumental understandings in the form of learned rules throughout the CTE interviews, all the others achieved relational understandings at some point during the CTE interviews.

Students' understandings of commutativity

Students had difficulties explaining the commutative property's generality in natural language, and instrumental understandings of commutativity prevailed (30 of the 64 records in Table 8 for this property); B1_3*:* "even though they are in different places, they are all the same numbers that they were in before, so that they will have the same equations"). Of course, it is also possible that students did understand the what and why of the commutative property (i.e., relational understanding) but that they simply did not have the natural language resources to explain it. Some students also provided evidence that they did not take into account the operation, stating for example that the expression *a*-*b* is the same as *b*-*a* (e.g., A1_2*:* "just the same number sentence, just the numbers are switched in order").

Students' relational understandings were associated with considering that the additions *a*+*b* and *b*+*a* constitute the same quantities combined in different ways to obtain the same final quantity. Of the five first-graders who provided evidence of relational understanding, only one student (B1_2) did so in all CTE interviews. Their interpretation entailed either a comparison between the two quantities or the transformation of one into the other. In the combination understandings, the students focused on the combination of quantities *a* and *b*: B1_4*:* "just switching around. Put this one first and that last, it equals the same thing" (carrying out the operation to check his answer). We only observed one case (A1_2) of a transformation understanding, associated with the transformation from *a*+*b* to *b*+*a*, in which the student alluded to the quantity that would need to be added to one to obtain the other (A1_2*:* "It's just like turning the number around but you're not adding or subtracting any, so it's just going to stay the same").

Table 3 shows that six students (four Kindergarten and two first grade) provided no evidence that they understood the property in the mid-interview prior to the commutativity lesson. Despite instruction, none of the Kindergarten students exhibited a relational understanding of commutativity. Instead, they showed only a rules-based instrumental understanding or no understanding of the property at all by the final interview of the CTE. One Kindergarten student (BK_4) showed no indication of understanding the property in any of the interviews and another Kindergarten student (AK_4), who had described it instrumentally in the mid-interview, claimed not to understand it in the post-clinical interview. Substantial differences between Kindergarten and first-grade students' understandings were observed in connection with commutativity. Six Kindergartners did not understand it in the pre-clinical interview and none exhibited a relational understanding during any of the CTE interviews, while one did not understand the property during any of the CTE interviews. While we observed instrumental understandings among three first-graders (A1_1, A1_4, and B1_3), the other five exhibited relational understandings based primarily on considering the quantities *a* and *b* and *b* and *a* to be equivalent but combined in a different order.

Representations of arithmetic properties

Verbal representations prevailed in students' answers (80 of the 192 records in Table 3). Perhaps because all questions were asked using natural language, it makes sense that natural language was the most frequently used representation when they were asked to explain the arithmetic properties. We observed sophistication among the first-grade students' natural language expressions (e.g., B1_1*:* "It stays the same. Zero is nothing and when you put zero with a number, it just stays the same it doesn't matter what number- what number it's subtracted or added to, it just stays the same" for *additive identity*). Students in first grade also recited mnemonic rules, which we assume were learned from their classroom teachers, in their verbal explanations of the *additive identity* property (A1_2*:* “going through zero and coming out the other side”). When students used generic representations, they used examples of equations involving both large and small numbers, indicating their recognition of the property and generic use of numbers (B1_4*:* “zero is nothing, but if you, it there’s any number before, like hundred” [the student uses the example of 100+0=100 to explain the property when asked about 8+0=8]).

All first-grade and four Kindergarten (AK_1, AK_3, AK_4, and BK_3) students used symbols to represent some of the properties correctly or not, including the use of letters (*J*+0=*J*; 0+*D*=*D*; 0=*K*-*K*) with the additive operation on both sides of the equal sign. One first grade student (B1_2) expressed the commutative property with a system of two equations (*s*+*r*=*a* and *r*+*s*=*a*), showing that she understood that the two combinations were the same because the same letters were involved, albeit in a different order. One of the Kindergarten students (BK_2) represented the *additive identity* property symbolically in two steps, first writing *J*+0=*E* and then replacing *E* with *J*.

We did not observe major differences across properties in terms of the kinds of representations used, although symbolic representations were used less frequently for commutativity compared to the two other properties. All first-grade students represented at least one property symbolically, providing evidence of these students’ potential to work with that kind of representation. In contrast, four Kindergarten students (AK_2, BK1, BK_2, and BK_4) never used symbolic representations, despite exhibiting relational understandings. The other four Kindergarten students used symbolic representations at least once.

Relationships between understandings and representations

The data show that neither instrumental nor relational understandings were associated with a single type of representation. Both instrumental and the various types of relational understandings were represented verbally, generically, or symbolically across different properties, apparently ruling out relationships between the two.

Symbolic representations did not necessarily entail relational understandings, or vice-versa. For instance, one student who used symbolic representation to represent the *additive identity* property (a+0=0) explained the property in a way that indicated an instrumental understanding (AK_4*:* "I don't remember. It came through zero"). Similarly, another student who provided evidence of a relational understanding and even identified unicity (A1_1*:* "if you want to get the same number adding and subtracting, you have to use a zero"), represented the property with generic examples, and not with symbolic representation. Cases such as these make a general relationship between relational understandings of properties and symbolic representations of properties doubtful. It seems that students do not *need* algebraic symbols to represent a property relationally and that they may rely on the simplest representation possible to articulate relational understandings of properties.

However, in contrast, we did find that numerical representations were only associated with instrumental understandings. The two students (AK_1 and A1_1) who who used a numerical representation only exhibited instrumental understandings based on unexplained rules. In this case, the use of a numerical representation where the student is not yet able to describe a general rule for the properties implies a less sophisticated understanding of the property, relying on learned or memorized rules (i.e., an instrumental understanding). Table 4 provides us with information regarding the kinds of representations associated with relational understandings across the three properties. In Table 4, for each student who provided evidence of a relational understanding for a given property, we also looked at their most sophisticated representation for this understanding. We considered the symbolic representation the most sophisticated and the numerical representation to be the least sophisticated. It is noteworthy that of the 14 students who exhibited a relational understanding of *additive identity*, nine used symbolic representation at some time while five of them did so verbally or using numbers in a generic way.

Table 4
Number of Students Exhibiting Relational Understandings of Properties by Type of Representation

	Relational and numerical	Relational and verbal/generic	Relational and symbolic
Additive identity	0	5	9
Additive inverse	0	5	9
Commutativity	0	1	4

Table 4 shows that no student whose most sophisticated representation was numerical provided evidence of a relational understanding during any of the CTE interviews. Across the three properties, however, verbal or generic representations were enough to show a relational understanding. Moreover, a majority of the students exhibiting relational understandings also used symbolic representations.

Relational understandings and symbolic representations in all three properties were observed in three first-grade students (A1_2, B1_1, and B1_2). The least sophisticated performance was observed for a Kindergarten student who exhibited neither relational understandings nor symbolic representations across any of the tasks.

CONCLUSIONS

This study explored the relationship between Kindergarten and first-grade students' understandings and representations of the arithmetic properties of *additive identity*, *additive* *inverse*, and *commutativity*. The five- and six-year-olds interviewed in this study were able to generalize arithmetic relationships and reason algebraically, as reported in previous research (e.g., Blanton et al., 2011). The present findings provide new data on students' relational understandings (Skemp, 2006) of arithmetic properties and their use of symbols to represent generalizations (Kaput, 2008).

Relational understandings in first grade and Kindergarten

Students showed relational understandings of *additive identity*, generalizing the property involved (i.e., adding zero) as the absence of transformation of the initial quantity. Even prior to the lessons on *additive identity*, 12 of the 16 students exhibited a relational understanding of this property. This finding is consistent with the results of an earlier CTE with first- and second-graders (Carpenter & Levi, 2000) in which students generalized *additive identity* and its unicity while attributing to zero the meaning as a number "that doesn't change." The relational understanding associated with understanding addition as a transformation was the most frequent understanding among students for *additive inverse*, where the "–*a*" was interpreted as a transformation on the initial quantity to obtain zero in $a\text{-}a=0$. We also observed relational understandings based on the comparison of quantities for *additive inverse* (Carpenter & Levi, 2000), where students alluded to the equal magnitude of both the initial quantity and the quantity that was subtracted. With the exception of two Kindergartners, all students exhibited relational understandings for *additive inverse* and *identity*, an indication of students' potential, even in Kindergarten, to generalize the arithmetic operations involved, even though the students were less familiar with subtraction (*additive inverse*) than with addition (*additive identity*).

In contrast, we observed a lower rate of relational understandings for *commutativity*, particularly in Kindergarten where, although it seemed to be emerging for some students, we observed no evidence of generalization of this property. This greater difficulty was also observed among three first-grade students, who exhibited only instrumental understandings of the property, explaining it with rules learned in their regular instruction outside of the CTE in which they prioritized the order of the addends (e.g., “turn around fact”), a finding consistent with reports of studies with older children (Bastable & Schifter, 2017; Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015). A main contribution of this study are the different results for each of the three properties. Relational understandings were more prevalent for the *additive identity* and *additive inverse* properties, both for Kindergarten and first grade students. For these properties, students applied the basic relationships associated with transformation and comparison. However, the understanding of *commutativity* in Kindergarten as well as three first grade students was instrumental. The five first-grade students who exhibited relational understandings of the *commutative property* did so by applying the basic relationship associated with combining the quantities in the additive operations $a+b$ and $b+a$ (Russell et al., 2011; Schifter, 2009).

Representing generalization

Most of the Kindergarten and first-grade students in this study represented generalization using natural language (Radford, 2011). This kind of representation prevailed across all three properties studied. This might be expected, given that even though the tasks did involve numerical operations, equations, and letters, when students were asked to explain the properties, the tasks were presented using natural language as well. While first graders’ understandings were more sophisticated, Kindergarten students were also able to verbalize in ways similar to older students (Carpenter et al., 2003).

One prominent finding was that all the first-grade students and half of the Kindergartners represented some of the properties symbolically. This, as noted for higher grades in connection with early algebra instruction, provided evidence of their ability to express properties algebraically (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015; Pang & Kim, 2018). As in other studies with first- and second-grade students, the children in this study generalized the properties symbolically after exploring cases with small and large numbers (Carpenter et al., 2003; Carpenter & Levi, 2000; Carpenter et al., 2005). Nonetheless, two Kindergarten students did find it difficult to write the numbers and use the signs correctly, using very rudimentary mathematical notations.

Relationship between understandings and representations

We only observed numerical representations sporadically (one in Kindergarten and one in first grade) and, as reported by Bastable and Schifter (2017), only when students deemed a property to be valid for small numbers while doubting its applicability to all numbers. Students who used numerical representations seemed limited in their ability to understand properties relationally. Although they could describe a rule for a small set of numbers, they were unable to understand the general validity involved.

Our study provides main contributions in terms of the relationship between understandings and representations. The first, noted above, is that students who only used numerical representations also only exhibited instrumental understandings. In contrast, students who exhibited relational understandings also represented properties verbally (verbal) or through representations that used numbers generically (generic). Students who provided evidence of relational understandings used symbolic representations much more frequently than other types of representations. The second contribution is that in spite of these findings, our results show that relational understandings and symbolic representations seem to be independent of each other. However, a majority of the students exhibiting relational understandings also used symbolic representations.

Implications for teaching

We did not observe any clear differences between the two grades in connection with *additive identity*. The gap was slightly wider for *additive inverse*, which was initially emergent among Kindergartners. However, by the final interview of the CTE even most Kindergarten students exhibited relational understandings for this property. Significant differences were found between Kindergarten and first grade students' understandings of *commutativity*, however. Kindergartners found relational understandings of *commutativity* to be a challenge, due primarily to the presence of two variables and more complex additive operations. Such differences indicate that different approaches may be needed to teach these three properties. The approaches for *additive identity* and *inverse* that we took in the CTEs seem to have supported children's generalization of these properties. However, for the *commutative property*, our results indicate that instruction should emphasize comparison of the quantities obtained after combining the elements; for instance, through visualizing the comparison of two combined quantities with manipulatives. In addition, examples with equations representing the *commutative property* that include both small and large numbers should be used to help students recognize the validity of this property for any two numbers. Our results indicate that Kindergarten and first grade students have the potential to develop relational understandings and to symbolically represent arithmetic properties. This has important implications for the design of innovative learning and teaching environments at these grade levels. For instance, the design of learning environments could include situations known to children, the use of manipulative materials, and small numbers. Starting there, larger numbers could be included, encouraging generalizations and the inclusion of the use of letters to represent them.

The use of mnemonic rules, “without reason,” seems to be associated with students’ instrumental understandings, which may persist in higher grades (Russell et al., 2011). Some students, while sometimes exhibiting relational understandings, continued to use the learned rules to explain properties. This study emphasizes the need for teaching practices that foster relational understandings. Teaching that simply focuses on memorization or mnemonic rules without reason, which might seem more easily accesible to students, can become significant challenges for their learning. Prior studies have shown that relational understandings emerge along with the meanings built by early-grade students (Stephens, Ellis, Blanton, & Brizuela, 2017). With a view to develop teaching approaches that could favor relational understandings, future research could look to identify the tasks that most effectively facilitate students’ relational understandings and use of symbolic representations.

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