Patterns in Partial Differential Equations Arising from Fluid Mechanics

Presented by

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A mis padres, Paqui y Carlos,
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Dissertation Summary

This dissertation is centered around the existence of time–periodic solutions for Hamiltonian models that arise from Fluid Mechanics. In the first part, we explore relative equilibria taking the form of rigid motion (pure rotations or translations) in the plane with uniform and non uniform distributions for standard models like the incompressible Euler equations or the generalized quasi–geostrophic equation. In the second part, we focus on a similar study for the 3D quasi–geostrophic system. The study of this model shows a remarkable diversity compared to the 2D models due to the existence of a large set of stationary solutions or the variety of the associated spectral problems. In the last part, we show some works in progress of this dissertation, and also some conclusions and perspectives.

In what follows, we briefly explain the contents of this thesis and the works contained in it.

- Chapter 1 deals with a general introduction to the above mentioned models, the contribution of this dissertation and related literature. It is divided in two sections: two–dimensional Euler equations and three–dimensional quasi–geostrophic system.

- Chapter 2 is devoted to the work [67], which is a collaboration with my thesis advisors T. HMIDI and J. SOLER. This work is currently accepted for publication in *Archive for Rational Mechanics and Analysis*. There, we focus on the existence of non uniform rotating solutions for the 2D Euler equations, which are compactly supported in bounded domains. The main idea is the bifurcation from stationary radial solutions. The system reduces to two coupled nonlinear equations for the shape of the support and the density inside it. We will deeply analyze the bifurcation diagram around a quadratic profile, i.e. $(A|x|^2+B)\mathbf{1}_{\mathbb{D}}$, by using Crandall–Rabinowitz theorem and also refined properties of hypergeometric functions.

- Chapter 3 refers to the work [65], which is published in *Nonlinearity*. This chapter aims to provide a robust model for the well–known phenomenon of Kármán Vortex Street arising in nonlinear transport equations. The first theoretical attempt to model this pattern was given by VON KÁRMÁN [89, 90] using a system of point vortices. The author considered two parallel staggered rows of Dirac masses, with opposite strength, that translate at the same speed. Following the numerical simulations of SAFFMAN and SCHATZMAN [134], we propose to study this phenomenon in a more realistic way using two infinite arrows of vortex patches. Hence, by desingularizating the system of point vortices, we are able to rigorously prove these numerical observations

- Chapter 4 is the content of [66], which is a collaboration with my thesis advisor T. HMIDI and with J. MATEU, and is currently submitted for publication. It aims to study time periodic solutions for the 3D inviscid quasi–geostrophic model. We show the existence of non trivial rotating patches by suitable perturbation of stationary solutions given by generic revolution shapes around the vertical axis. The construction of those special solutions is

achieved through bifurcation theory. In general, the spectral problem is very delicate and strongly depends on the shape of the initial stationary solutions. Restricting ourselves to a particular class of revolution shapes and exploiting the particular structure of our model, we are able to implement the bifurcation at the largest eigenvalue of a family of 1D Fredholm type operators.

- Chapter 5 is devoted to explain some works in progress of this dissertation. Some conclusions of the above mentioned works together with some new perspective and future works are also given at the end of this chapter.
- Finally, Appendices A, B and C collect some necessary results about bifurcation theory, potential theory and special functions.

Resumen en castellano

Esta tesis se centra en la existencia de soluciones periódicas en tiempo de modelos hamiltonianos que surgen en Mecánica de Fluidos. En la primera parte, exploraremos en el plano soluciones con movimiento rigido (rotaciones puras o translaciones) con distribucion uniforme o no uniforme para modelos como las ecuaciones de Euler incompresibles o el modelo quasi–geostrófico generalizado. En la segunda parte, nos centraremos en un estudio similar para el sistema quasi–geostrófico tridimensional. El estudio de este modelo muestra una gran riqueza comparado con los modelos bidimensionales, esto es debido al conjunto de soluciones estacionarias y también a la gran diversidad de problemas espectrales asociados. En la última parte, mostramos varios trabajos en desarrollo de esta tesis junto con algunas conclusiones y perspectivas.

A continuación, explicaremos brevemente los contenidos de la tesis.

- En el primer capítulo presentamos el estado del arte acerca de los principales temas tratados en esta tesis y otros temas relacionados. Está dividido en dos secciones: las Ecuaciones de Euler bidimensionales y el sistema quasi–geostrófico tridimensional.

- El capítulo 2 está enfocado al trabajo [67], el cual es una colaboración con mis supervisores de tesis T. HMIDI y J. SOLER. Este trabajo está actualmente aceptado para publicación en *Archive for Rational Mechanics and Analysis*. En este capítulo, nos centramos en la existencia de soluciones no uniformes, con soporte compacto en dominios acotados, que rotan en las Ecuaciones de Euler bidimensionales. La principal idea es la bifurcación desde soluciones radiales (las cuales son estacionarias). El sistema está compuesto de dos ecuaciones acopladas no lineales para la forma del soporte y para la densidad dentro de él. Analizaremos profundamente el diagrama de bifurcación alrededor de perfiles cuadráticos, esto es $(A|x|^2 + B)\mathbf{1}_{\mathbb{D}}$, usando el teorema de Crandall–Rabinowitz y propiedades refinadas de funciones hipergeométricas.

- El tercer capítulo se centra en el trabajo [65], el cual está publicado en *Nonlinearity*. Este capítulo propone un modelo para el fenómeno conocido como Kármán Vortex Street que aparece en ecuaciones de transporte no lineales. Los primeros intentos teóricos para entender este modelo fueron los de VON KÁRMÁN [89, 90] mediante un sistema de puntos de vorticidad. El autor consideró dos calles paralelas de masa de Dirac, con fuerza opuesta, que se transladan a velocidad constante. Siguiendo las simulaciones numéricas de SAFFMAN y SCHATZMAN [134], proponemos estudiar este fenómeno de una forma más realística considerando dos calles infinitas de *parches de vorticidad* (vortex patches). Mediante la desingularización del modelo de puntos de vorticidad propuesto por VON KÁRMÁN, somos capaces de demostrar rigurosamente las simulaciones numéricas propuestas por SAFFMAN y SCHATZMAN.

- El capítulo 4 incluye el trabajo [66], el cual es una colaboración con mi supervisor de tesis T. HMIDI y con J. MATEU; está actualmente sometido a publicación. Se centra en el estudio de soluciones periódicas para el modelo tridimensional quasi–geostrófico sin viscosidad. Mostramos la existencia de *parches* que rotan, los cuales son una perturbación de *parches* estacionarios que son superficies de revolución alrededor del eje vertical. La construcción de estas soluciones especiales se consigue mediante teoría de bifurcación. En general, el problema espectral es muy delicado y depende fuertemente de la solución estacionaria inicial. Restringiéndonos a una clase de superficies de revolución y explotando la forma particular de nuestro modelo, somos capaces de implementar la bifurcación a partir del autovalor más grande de una familia de operadores tipo Fredholm en una dimensión.

- El capítulo 5 explica algunos trabajos en proceso de esta tesis. Algunas conclusiones y perspectivas de trabajo son también dadas a final de este capítulo.
- Finalmente, en los apéndices A, B y C damos algunos resultados necesarios sobre teoría de bifurcación, teoría del potencial y funciones especiales.

Résumé en Français

Cette thèse est consacrée à l'émergence de solutions périodiques en temps pour des modèles hamiltoniens issus de la mécanique des fluides. Dans la première partie, nous explorons dans le plan les solutions en mouvement rigide (rotation ou tranlation pures) avec des distributions uniformes ou non pour des modèles standards comme les équations d'Euler incompressibles ou l'équation de surface quasi–géostrophique généralisée. Dans la deuxième partie, nous menons une étude analogue pour le système quasi–géostrophique en 3D. L'étude de ce modèle montre une remarquable richesse par rapport aux modèles 2D que ce soit par rapport l'ensemble des solutions stationnaires ou la diversité des problèmes spectraux associés. Dans la dernière partie, nous discutons quelques travaux en cours de cette thèse.

Dans la suite, nous allons expliquer brièvement le contenu de cette thèse et les travaux qu'elle contient.

- Le chapitre 1 traite d'une introduction générale aux modèles mentionnés ci-dessus, de l'apport de cette thèse et de la littérature associée. Il est divisé en deux sections: les équations d'Euler bidimensionnelles et le système quasi–géostrophique tridimensionnel.

- Le chapitre 2 est consacré au travail [67], qui est une collaboration avec mes directeurs de thèse T. HMIDI et J. SOLER. Ce travail est actuellement accepté pour publication dans *Archive for Rational Mechanics and Analysis*. IL s'agit de montrer l'existence de solutions inhomogènes en rotation uniforme pour les équations d'Euler 2D, avec un support compact. L'idée principale consiste mettre en place les techniques de bifurcation partir des solutions radiales stationnaires. Le système se ramène à deux équations non linéaires couplées liant la forme du support et la densité du tourbillon à l'intérieur. Nous avons examiné en détail le diagramme de bifurcation autour de la solution stationnaire $(A|x|^2+B)\mathbf{1}_{\mathbb{D}}$ avec une distribution quadratique. Ceci repose en partie sur le théorème de Crandall-Rabinowitz, combiné avec des propriétés fines des fonctions hypergéométriques.

- Le chapitre 3 porte sur le travail [65], qui est publié dans *Nonlinearity*. L'objectif principal est de fournir une description rigoureuse de l'existence des allées de von Kármán pour divers modèles de transport nonlinéaires. Signalons que la première tentative théorique pour modéliser ces structures a été élaborée par VON KÁRMÁN [89, 90] dans le cadre du système des points vortex. L'auteur a considéré deux allées parallèles de masse de Dirac, avec des circulations opposées, et qui sont animées d'un mouvement de translation uniforme. Des simulations numériques obtenues par SAFFMAN et SCHATZMAN [134], montrent que ce type de structures persiste également pour des tourbillon plus réalistes de type poches de tourbillon fortement concentrées. Nous proposons de démontrer rigoureusement ces obsrvations numériques en procédant par une désingularisation du modèle des points vortex.

- Le chapitre 4 concerne le travail [66], qui est une collaboration avec mon directeur de thèse T. HMIDI et J. MATEU, et soumis pour publication. Il vise à étudier l'existence des solutions périodiques en temps pour le modèle quasi–géostrophique non visqueux 3D. Nous montrons l'existence de poches en rotation uniforme en perturbant adéquatement des solutions stationnaires données par des formes de révolution régulières autour de l'axe vertical. La construction de ces solutions spéciales est réalisée grâce à la théorie de la bifurcation. En général, le problème spectral sous-jacent est très délicat et dépend fortement de la forme des solutions stationnaires initiales. Cependant en exploitant la structure particulière de notre modèle, nous réussissons à valider la bifurcation à partir des grandes valeurs propres d'une famille discrète d'opétaeurs 1D de type Fredholm.

- Le chapitre 5 est consacré à quelques travaux en cours de cette thèse. Certaines concluſiones des travaux mentionnés ci-dessus ainsi que de nouvelles perspectives et travaux futurs sont également données à la fin de ce chapitre.

- Enfin, les annexes A, B et C collectent quelques résultats nécessaires sur la théorie de la bifurcation, la théorie du potentiel et les fonctions spéciales.

Contents

Chapter 1

Introduction

Contents

Understanding the hydrodynamic turbulence remains the greatest challenge at the interface of mathematics and other sciences, and has been of great interest even for artists. In 1932, SIR HORACE LAMB expressed his interest in turbulence: *"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."*

LEONARDO DA VINCI, also called "Master of Water", revealed his preoccupations with the power of water: *"If humans could not control water, they could nonetheless work with it"*. See Figure 1.1 for "Studies of Water", one of his scientific works about fluid flows. Later, VINCENT VAN GOGH documented his observation of turbulent flows in its well–known piece of art "The Starry Night", see Figure 1.2. There, light and clouds flow in turbulent swirls on the night sky. Recently, a group of scientists took digital pictures of the painting and calculated the relative probability that two pixels at a certain distance have the same luminance. They found that same patters are repeated at different spatial scales. Remarkably, paintings from his own turbulent period show luminance with a scaling similar to that of the mathematical theory of turbulence.

Informally speaking, turbulence refers to a chaotic and disordered behavior in the fluid which seems only to be predicted statistically or in an averaged sense, rather than exactly. We refer as *fluids* both gases and liquids, provided that the gas is not too thin. Both obey very similar mathematical laws. In a turbulent state, fluid flows become unstable and fluctuating. This state is associated with a large number of highly mixing vortices that interact on a wide range of temporal and spatial scales.

In 1557, LEONARD EULER proposed the so called Euler equations which are ones of the first partial differential equations in the literature. They describe the dynamics of an inviscid fluid and consist of a coupled system for the continuity equation of the density and the

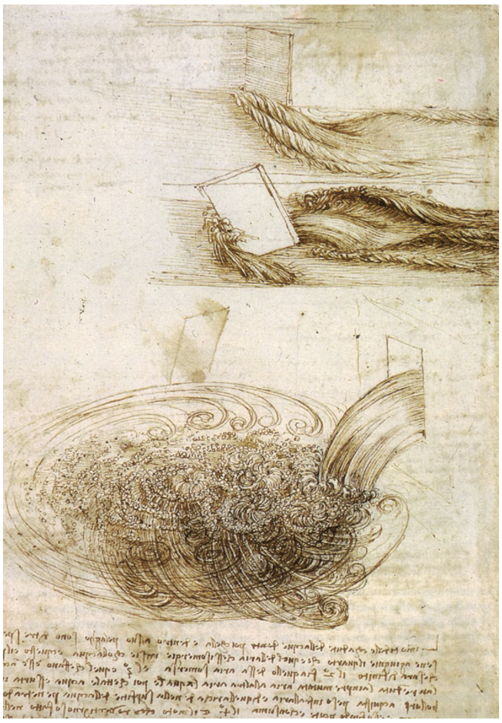

Figure 1.1: “Studies of Water” by DA VINCI

Figure 1.2: “Starry Night” by VAN GOGH

balance equation of the linear momentum. In this thesis, we will be interested in the case of two–dimensional perfect fluids, that is, ideal homogeneous incompressible fuids lying in the plane $\mathbb{R}^2$. By an ideal fluid we mean when the only forces are normal to the boundaries and their strength by surface unit is determined by the pressure p of the fluid. We refer to [4] for the derivation of the Euler equations and their viscous counterpart (i.e., the Navier–Stokes equations).

Specifically, the question of global in time well posedness or blow up in three dimensions is a extremely hard open problem with relevant implications in sciences. As a consequence, important mathematics have been emerged when trying to solve related questions. Indeed, the analogue version of such problem for the Navier–Stokes equations is one of the problems of the millennium by Clay Institut of Mathematics. Its resolution entails a 1 million dollards prize to the discoverer, as a recognition for a strong advance in mathematics.

The Euler equations can be reformulated in terms of the vorticity of the fluid, which represents the tendency of the fluid to rotate, and the velocity of the fluid. In the two–dimensional case, it becomes a pure transport equation for the vorticity and whose associated velocity can be recovered from a singular integral operator. Vortices are important components of the fluid flows. We observe the formation and dissipation of vortices in a very wide range of the fluid, and then their formulation and dynamics is an important area of Fluid Mechanics. We can think of different categories of vortices. On the one hand, we find a point vortex which is an “infinite” vorticity distribution concentrated on a point (meaning a Dirac delta on a point) and it is characterized by its vortex strength. A vortex sheet consists in an “infinite” vorticity concentrated along a line, that is, a Dirac delta on a line. In this case, it is not only characterized by its circulation but also by the line shape. On the other hand, a vortex patch has a finite and uniform distribution of vorticity inside some domain of the plane, that is characterized by the shape of the boundary. These definitions have their counterparts in the three–dimensional case that we will mention later.

Along this thesis, we will be interested in special patters of weak solutions of fluid models. In Chapter 2 and Chapter 3 we focus on the 2D Euler equations. In the first one, we show the existence of non uniform rotating solutions which are compactly supported in a bounded domain. Whereas, in the second one, we study the very special structure of the von Kármán Vortex Street in the Euler equations. Chapter 4 deals with the 3D quasi–geostrophic system, where uniformly rotating weak solutions of patch form are found. In Chapter 5, we present some works in progress of this dissertation. Some conclusions of the previous works are also given in this chapter together with some new perspectives. Finally, we give some necessary results on bifurcation theory, potential theory and special functions in Appendices A, B and C.

1.1 Two–dimensional Euler equations

Wild weak solutions of the incompressible Euler equations can capture the essence of turbulence, even in the two–dimensional case. The importance of finding significant patterns of weak solutions of the Euler equations is, therefore, essential for a deep knowledge about the complex nature of turbulent fluids.

The evolution of an homogeneous incompressible ideal fluid without viscosity in $\mathbb{R}^d$, with $d \geq 2$, is described by the Euler equations

$$\begin{cases} v_t + v \cdot \nabla v = -\nabla p, & \text{in } [0,+\infty) \times \mathbb{R}^d, \\ \operatorname{div} v = 0, & \text{in } [0,+\infty) \times \mathbb{R}^d, \\ v(0,x) = v_0(x), & \text{with } x \in \mathbb{R}^d. \end{cases} \tag{1.1.1}$$

Here, the unknowns are the velocity field $v = (v^1, \ldots, v^d)$ that depends on $(t,x) \in [0,+\infty) \times \mathbb{R}^d$ and the pressure $p : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$. Since the fluid is taken to be homogeneous, we have normalized the density to one for simplicity. Then, the continuity equation (the conservation of mass) is trivial and we can ignore it. The first equation in (1.1.1) agrees with the conservation of linear momentum, which is derived from Newton's second law assuming that the fluid is ideal. Finally, the second equation in (1.1.1) describes the incompressibility condition. See [4] for a derivation of these equations.

Regarding the well–posedness of the system, KATO and PONCE [92] addressed the local existence and uniqueness of solution in the Sobolev space H^s, for any $s > \frac{d}{2} + 1$. Later, this result has been extended to other spaces such as Hölder or Besov spaces. We refer for instance to [29, 36]. The question of global existence of solution is still an open problem except for the two dimensional case, where the global well–posed is achieved in H^s, for $s > 2$, see [145].

Along this thesis we shall focus mainly on the two–dimensional case. Actually, we will pay special attention to the evolution of vorticity, which represents the tendency of the fluid to rotate. Vorticity is described through

$$\omega := \nabla^\perp \cdot v = \partial_1 v_2 - \partial_2 v_1.$$

We set here $(x_1,x_2)^\perp = (-x_2,x_1)$. Indeed, applying such operator $\nabla^\perp \cdot$ to (1.1.1) we observe that the pressure term disappears from the dynamics and we arrive at the following system

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ \omega = \nabla^\perp \cdot v, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ \operatorname{div} v = 0, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ \omega(0,x) = \omega_0(x), & \text{with } x \in \mathbb{R}^2. \end{cases}$$

The first equation in the above system is a pure transport equation for the vorticity ω. In fact, looking at the L^p norm of the solution and using the incompressibility condition, the following conservation law is achieved

$$||\omega(t)||_{L^p} = ||\omega_0||_{L^p},$$

for any $t \geq 0$ and $p \in [1, +\infty]$. Such property is crucial in order to construct global classical solutions of Kato type.

Notice that we can face the second and third equation in (1.1.1) dealing with a div $-$ curl problem by virtue of a function ψ such that $v = \nabla^\perp \psi$. In this setting, such function is often called the stream function. Therefore, applying the $\nabla^\perp$ operator above defined we arrive at the elliptic equation

$$\Delta \psi = \omega. \tag{1.1.2}$$

Indeed, it can be solved by means of G the fundamental solution to the Δ operator, that is,

$$\psi(t, x) = (G * \omega)(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| \omega(t, y) dy.$$

Hence, the velocity field can be recovered via the vorticity in the following way

$$v(t, x) = \nabla^\perp \psi(t, x) = (K * \omega)(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(t, y) dy.$$

This is the so called Biot–Savart law that links the velocity field with the vorticity. Such ideas yield an equivalent system to (1.1.1) given by

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ v = K * \omega, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ \omega(t = 0, x) = \omega_0(x), & \text{with } x \in \mathbb{R}^2. \end{cases} \tag{1.1.3}$$

In the case that the initial data belong to $L^1 \cap L^\infty$, global existence and uniqueness of solution is stated by YUDOVICH in [147]. One of the main consequences of Yudovich's theorem is that although the velocity field is not Lipschitz continuous (but log–Lipschitz), the trajectories are well–defined and the vorticity is transported along the trajectories. That is, let X be the flow associated to v, i.e.,

$$\frac{\partial X(t, x)}{\partial t} = v(t, X(t, x)), \quad X(0, x) = x \in \mathbb{R}^2, \tag{1.1.4}$$

for any $t \geq 0$. Hence denoting $X_t^{-1} = X(t, \cdot)^{-1}$ we have

$$\omega(t, x) = \omega_0(X_t^{-1}(x)),$$

for any $t \geq 0$ and $x \in \mathbb{R}^2$.

Along this introductory part, we review some recent studies on Euler equations and the main contributions of this dissertation on this field. First, Section 1.1.1 contains an outline of the main works on special solutions: stationary, rotating and translating solutions. Second, we focus on vortex patches in Section 1.1.2, and more specifically, V–states (rotating vortex patches). That is the main motivation behind the works developed in Chapter 2 and 3. Finally, those works and related literature will be briefly introduced in Sections 1.1.3 and 1.1.4.

1.1.1 Special solutions

In this section, we show some recent results on special solutions. First, we will introduce the stationary solutions from which later we will look for periodic solutions around them. Later, we will see that in the search of rigid motions (pure rotations or translations), the Euler system can be simplified to an equation involving only the initial data.

Therefore, the search of stationary solutions reduces to studying the following equations

$$v \cdot \nabla \omega = 0, \quad \text{and} \quad v = K * \omega. \tag{1.1.5}$$

By using the structure of the velocity field via Biot–Savart law, it is easy to check that every radial vorticity defines a stationary solution. Hence, we find a large family of stationary solutions from which we can look for periodic solutions around them. That is a key observation that will support later our search for non uniform (and non radial) rotating vortices. See Section 1.1.3 or Chapter 2. Moreover, by using the stream function, (1.1.5) amounts to

$$\nabla^{\perp}\psi \cdot \nabla \Delta \psi = 0. \tag{1.1.6}$$

In particular, if we assume $\Delta\psi = F(\psi)$, for some scalar function F, then the equation (1.1.6) is automatically satisfied, see [105].

In the literature, there are several works about the study of stationary solutions. On the one hand, LUO and SHVYDKOY [104] looked for solutions in polar coordinates taking the form $\psi(r,\theta) = r^{\lambda}\psi(\theta)$, for some $\lambda > 0$. They found that if $0 < \lambda < \frac{1}{2}$, then only trivial solutions, in the sense of parallel shear and rotational flows, are found. Outside that range of λ, new solutions with hyperbolic, parabolic and elliptic structure of the streamlines appear.

Later, CHOFFRUT and ŠVERÁK [39] showed analogies of some finite-dimensional models in the infinite-dimensional setting of Euler's equations. Moreover, under some non–degeneracy assumptions, they proved a local one–to–one correspondence between steady–states and co–adjoint orbits. Then, CHOFFRUT and SZÉKELYHIDI [40] found, using an h-principle [59], that there is an abundant set of weak, bounded stationary solutions in the neighborhood of any smooth stationary solution. In [73, 74, 72], HAMEL and NADIRASHVILI studied the Euler equations in the plane or in some domains assuming tangential boundary conditions. If such a flow has no stagnation point in the domain or at infinity, in the sense that the infimum of its norm over the domain is positive, then it inherits the geometric properties of the domain, for some simple classes of domains. In particular, in an infinite two–dimensional strip, the same authors proved that a steady flow is parallel to the boundary of the domain. Moreover, they proved that the stationary solutions to the Euler equations in the full space (with no stagnation point at infinity) are shear flows.

Other approaches to study stationary solutions have been proposed through the study of the characteristic trajectories (1.1.4) associated with stationary velocities, in connection with the elliptic equation $\Delta\psi = \omega = F(\psi)$, i.e. (1.1.2). In this context, NADIRASHVILI in [113] studied the geometry (curvature) of streamlines of smooth stationary solutions. KISELEV and ŠVERÁK [97] constructed an example of initial data in the disc such that the corresponding solutions for the 2D Euler equations exhibit double exponential growth in the gradient of vorticity. That is related to an example of the singular stationary solution provided by BAHOURI and CHEMIN [11] with lack of Lipschitz regularity in the velocity field. Recently, GÓMEZ–SERRANO, PARK, SHI and YAO [69] proved that any smooth compactly supported non–negative stationary vorticity must be radial. Other recent important results are due to GAVRILOV [68] and CONSTANTIN, LA and VICOL [41], where they obtained very interesting examples of smooth compactly supported stationary solutions for the 3D Euler equations, based on Grad–Shafranov equations.

Furthermore, the Euler system is a Hamiltonian system that develops various interesting behaviors at different levels, which are in the center of intensive research activities. Lots of studies have been devoted to the existence and stability of relative equilibria (in general, translating and rotating steady–state solutions). We point out that despite the complexity of the motion and the deformation process that the vorticity undergoes, some special vortices subsist without any deformation and keep their shape during the motion. These fascinating and intriguing structures illustrate somehow the emergence of order from disordered motion.

Indeed, one can study the existence of rotating solutions of the Euler equations (1.1.3), that is,

$$\omega(t,x)=\omega_0(e^{-i\Omega t}x),$$

for some constant angular velocity $\Omega\in\mathbb{R}$. Inserting this ansatz in (1.1.3), we arrive at the equivalent system

$$R(\Omega,\omega_0):=(v_0(x)-\Omega x^{\perp})\cdot\nabla\omega_0(x)=0, \tag{1.1.7}$$

for any $x\in\mathbb{R}^2$. Here $v_0=v(0,x)$ is the initial velocity and is given in terms of the initial vorticity ω_0 via the Biot–Savart law. Note that the time dependence has disappeared due to the special form of the solution.

A priori, there is an infinite family of solutions to (1.1.7) given by every radial function. That is, given any radial function ω_0, one finds

$$R(\Omega,\omega_0)(x)=0,$$

for any $x\in\mathbb{R}^2$ and $\Omega\in\mathbb{R}$. In the special case that the support of ω_0 is a smooth bounded domain D, that is,

$$\omega_0(x)=q(x)\mathbf{1}_D(x),$$

for some smooth function q, hence (1.1.7) agrees with the coupled equations

$$R_1(\Omega,q)(x):=(v_0(x)-\Omega x^{\perp})\cdot\nabla q(x)=0,\quad x\in D, \tag{1.1.8}$$

$$R_2(\Omega,q)(x):=q(x)(v_0(x)-\Omega x^{\perp})\cdot\vec{n}(x)=0,\quad x\in\partial D, \tag{1.1.9}$$

where $\vec{n}$ is a unit normal vector to the boundary ∂D.

A solution is called a vortex patch if ω_0 is constant in D. In such a case the above system reduces only to one equation given by (1.1.9), for any $x\in\partial D$. In Section 1.1.2 we will discuss some recent results on V–states, that are the rotating vortex patches. Such results study the equation (1.1.9) via bifurcation techniques around the stationary circular patch. However, the case of non uniform rotating solutions is a more complicate problem since now one needs to solve the full coupled system (1.1.8) and (1.1.9), and not only (1.1.9) as for uniform patches. One of the main works of this thesis is to consider the existence of non uniform solutions close to some radial profiles by using bifurcation arguments. This problem will be softly discussed in Section 1.1.3 and the full description will be analyzed later in Chapter 2.

Another class of special interest is given by translating motions, that is,

$$\omega(t,x)=\omega_0(x-Vt),$$

for some constant speed $V\in\mathbb{R}^2$. In this case, the Euler equations (1.1.3) can be written as

$$T(\Omega,\omega_0):=(v_0(x)-V)\cdot\nabla\omega_0(x)=0, \tag{1.1.10}$$

for any $x\in\mathbb{R}^2$. This equation will be carefully studied in Chapter 3 in order to find some special structures in the Euler equations with periodic spatial patters. More precisely we shall investigate the so called Kármán Vortex Street that consists in two rows of point vortices (or vortex patches with small area) that translate along the space at constant speed. Such models for the point vortices and vortex patches will be introduced also in Section 1.1.4 in this introduction.

1.1.2 V–states: rotating vortex patches

The main goal of this section is to explore some special features of the vortex patch motion. We shall write down the contour dynamics equations and investigate later the rotating patches, which are called by V–states. Later, we will explain some results about the existence of such solutions using bifurcation techniques. This is the main motivation of this thesis in order to look for non uniform rotating solutions in Chapter 2, Kármán Vortex Street structures in Chapter 3 or the existence of V–states in the 3D quasi–geostrophic system in Chapter 4.

Consider an initial vorticity ω_0 in the vortex patch form, that is,

$$\omega_0(x) = \mathbf{1}_D(x), \quad x \in \mathbb{R}^2.$$

According to Yudovich theorem, this structure is preserved in time and one has

$$\omega(t,x) = \mathbf{1}_{D_t}(x), \quad x \in \mathbb{R}^2,$$

for some domain D_t with $D_0 = D$. Note that D_t is nothing but the evolution of D_0 by the flow, that is $D_t = X(t, D_0)$. By using the equation of the characteristic trajectories (1.1.4), one arrives to the well known contour dynamics equation:

$$\frac{dz(t,s)}{dt} = -\frac{1}{2\pi}\int_0^{2\pi} \log|z(t,s)-z(t,\tau)|\partial_\tau z(t,\tau)d\tau, \tag{1.1.11}$$

where $s \in \mathbb{T} \mapsto z(t,s)$ is a parametrization of ∂D_t. Hence, the vortex patch problem is reduced to the resolution of the above nonlocal differential equation. The global in time regularity persistence of the boundary with $\mathscr{C}^{1,\alpha}$-regularity is delicate and was first shown by CHEMIN [35], and then via different techniques by BERTOZZI and CONSTANTIN [17] and SERFATI [137].

The dynamics of the boundary is in general complex and hard to tackle its evolution. However few examples with a full description are known in the literature. Note that there is a trivial solution to (1.1.11) given by the circular patch (this is due to its radial symmetry), which is known as the Rankine vortex. Later, KIRCHHOFF [96] discovered that a vorticity uniformly distributed inside an elliptic shape rotates about its center with constant angular velocity. Further uniformly rotating m–fold patches with lower symmetries generalizing Kirchhoff ellipses were discovered numerically by DEEM and ZABUSKY [49]. We understand by a m–fold symmetric domain if it is invariant by the dihedral group D_m. See Figure 1.3 for some examples.

Having this kind of V–states solutions in mind, BURBEA [20] designed a rigorous approach to generate them close to a Rankine vortex through complex analytical tools and bifurcation theory. Later this idea was improved and extended to different directions: regularity of the boundary, various topologies, effects of the boundary conditions, and different nonlinear transport equations. For the first subject, the regularity of the contour was analyzed in [25, 26, 84]. There, it was proved that close to the unit disc the boundary of the rotating patches are not only $\mathscr{C}^\infty$ but also analytic. Regarding the second point, similar results with richer structures have been obtained for doubly connected patches [51, 81]. The existence of small loops in the bifurcation diagram has been achieved very recently in [85]. For disconnected patches, the existence of co–rotating and counter–rotating vortex pairs was discussed in [83]. We mention that the approach of BURBEA is so robust that partial results have been extended to different models such as the generalized surface quasi-geostrophic equations [26, 76] or shallow-water quasi-geostrophic equations [56], but the computations turn out to be much more involved in those cases.

Let us explain the main idea of BURBEA for the existence of uniformly rotating vortex patches with m–fold symmetry. Assume that

$$\omega_0(x) = \mathbf{1}_D(x), \quad x \in \mathbb{R}^2,$$

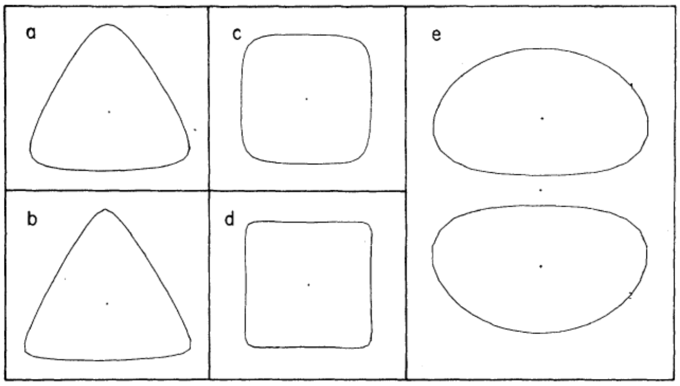

Figure 1.3: Numerical existence of V–States by DEEM and ZABUSKY [49]

for a simply–connected bounded domain D. Inserting such initial data in the equation for the rotating solutions (1.1.7), we arrive at (1.1.9), that is,

$$(v_0(x) - \Omega x^{\perp}) \cdot \vec{n}(x) = 0, \quad x \in \partial D, \tag{1.1.12}$$

where $\vec{n}$ is a unit normal vector to ∂D. Recall that v_0 is given in terms of ω_0 via the Biot–Savart law, that is,

$$v_0(x) = \frac{1}{2\pi}\int_D \frac{(x-y)^{\perp}}{|x-y|^2}dA(y) = \frac{i}{2\pi}\int_D \frac{dA(y)}{\overline{x-y}},$$

where we show in the second term its expression in the complex sense. Next, we can parametrize D by its conformal mapping

$$\Phi : \mathbb{D} \to D,$$

where $\mathbb{D}$ is the unit disc. Straightforward computations using complex notation implies that the rotating equation (1.1.12) is equivalent to

$$F(\Omega,\phi)(w) := \operatorname{Im}\left[\left(\Omega\overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{|\Phi'(y)|^2}{\overline{\Phi(w)-\Phi(y)}}dA(y)\right)\Phi'(w)w\right] = 0, \quad w \in \mathbb{T}, \tag{1.1.13}$$

where

$$\Phi = \mathrm{Id} + \phi.$$

We have a trivial solution to (1.1.13) coming from $\phi = 0$, that is $\Phi = \mathrm{Id}$. Indeed, in that case we obtain the circular patch and then $F(\Omega, 0) = 0$, for any $\Omega \in \mathbb{R}$. For that reason we have normalized the conformal map to work with ϕ which is the perturbation of the trivial solution (the circular patch).

The equation (1.1.13) is equivalent to the formulation used by BURBEA in [20]. The key idea of BURBEA was looking for nontrivial solutions around the trivial solution (in this case $\phi = 0$) to (1.1.13) via bifurcation arguments. Thus, one needs that the linearized operator around the trivial solution is a Fredholm operator of zero index (see Appendix A). By working with singular integrals and using Appendix B, one finds

$$\partial_\phi F(\Omega,0)h(w) = \sum_{n\geq 1} a_n \sin(n\theta)\left\{n\Omega - \frac{n-1}{2}\right\},$$

where $w = e^{i\theta}$ and we have decomposed h as

$$h(w) = \sum_{n\geq 1} a_n w^{n+1}.$$

Then, fixing $\Omega_n = \frac{n-1}{2n}$ we can check that the kernel of $\partial_\phi F(\Omega_n, 0)$ is one dimensional in some appropriate function spaces.

Using these ideas, BURBEA found the existence of families of V–states that are m–fold symmetric, for $m \geq 2$. That proof was later revisited by HMIDI, MATEU and VERDERA in [84]. The main result reads as follows.

Theorem 1.1.1. *[84, Theorem 1] Given $0 < \alpha < 1$ and $m = 2, 3, \ldots$, there exists a curve of m–fold rotating vortex patches with boundary of class $\mathscr{C}^{1,\alpha}$ bifurcating from the disc at $\Omega_m = \frac{m-1}{2m}$.*

In [84], the authors showed that the bifurcated patches near the Rankine vortices are in fact $\mathscr{C}^\infty$ and convex. Notice that the case $m = 1$ corresponds to translations of the circular patch.

We point out that the countable curves bifurcate at the points $\Omega_m \in (0, \frac{1}{2})$ and numerical simulations conjectures that the associated angular velocities still lie in the same range. In [64], FRAENKEL proved, using the moving plane method, that in the case that $\Omega = 0$ the only solution is the trivial one, i.e., the Rankine vortex. Later, HMIDI in [80] adapted this proof to check that in the case $\Omega = \frac{1}{2}$ or $\Omega \leq 0$ (under some suitable geometric constraints) the only rotating patch is the trivial one. Recently, GÓMEZ–SERRANO, PARK, SHI and YAO did an important progress in this topic and proved the same result for the range $\Omega \notin (0, \frac{1}{2})$ following a new approach based on variational arguments and steiner symmetrization, see [69].

1.1.3 Non uniform rotating vortices

Motivated by the previous section, here we will investigate whether there are periodic solutions around radial functions (which are stationary solutions). In particular, this section aims to overview some recent studies on rotating non uniform solutions which are compactly supported in a simply–connected bounded domain. First, we will introduce the work of CASTRO, CÓRDOBA and GÓMEZ–SERRANO in [27] about the desingularization of the V–states. Later, we present one of the main works of this thesis which is the bifurcation from quadratic radial profiles that are far from the patches, we refer to Chapter 2 for more details.

Assume that we have an initial vorticity of the type

$$\omega_0(x) = q(x)\mathbf{1}_D(x),$$

where $q : D \to \mathbb{R}$ is a smooth profile and D is a simply–connected bounded domain. That solution rotates at a constant angular velocity $\Omega \in \mathbb{R}$, that is,

$$\omega(t, x) = \omega_0(e^{-i\Omega t}x),$$

if and only if

$$R_1(\Omega, q) = (v_0(x) - \Omega x^\perp) \cdot \nabla q(x) = 0, \quad x \in D,$$
$$R_2(\Omega, q) = q(x)(v_0(x) - \Omega x^\perp) \cdot \vec{n}(x) = 0, \quad x \in \partial D.$$

Such functionals R_1 and R_2 were introduced in (1.1.8) and (1.1.9). Recall that v_0 is obtained through Biot–Savart law and it has the following expression

$$v_0(x) = \frac{1}{2\pi}\int_D \frac{(x-y)^\perp}{|x-y|^2} q(y)dA(y) = \frac{i}{2\pi}\int_D \frac{q(y)}{\overline{x-y}} dA(y).$$

Given any radial function q_0, one trivially has that $R_1(\Omega, q_0) = R_2(\Omega, q_0) = 0$ for any $\Omega \in \mathbb{R}$. Hence, one could try to implement Crandall–Rabinowitz theorem and find branches of solutions around such initial radial profile. However, a priori one can not apply bifurcation arguments to R_1 due to its bad spectral properties.

Namely, in order to apply Crandall–Rabinowitz theorem (or some generalization of the theorem), one needs that the dimension of the kernel and codimension of the range of the linearized operator around the trivial solution (in this case, q_0) are finite. However, the linearized operator $\mathcal{L}$ of R_1 presents some technical problems: its kernel contains every radial function and it is smoothing in the radial component. Indeed, it has the expression

$$\mathcal{L}(h) = \left(\frac{v_\theta}{r} - \Omega\right)\partial_\theta h + K(h)\cdot\nabla q_0, \quad K(h)(x) = \frac{1}{2\pi}\int_D \frac{(x-y)^\perp}{|x-y|^2}h(y)dA(y).$$

The loss of information in the radial direction can not be compensated by the operator K which is compact. Hence, using standard function spaces (say Hölder spaces), the codimension of the range will not be finite. Moreover, one has that if h_0 is radial then $\mathcal{L}(h_0) = 0$, which gives us that the kernel is of infinite dimension: it contains every radial function. Hence, one must change such an equation by restricting to some particular solutions in order to apply bifurcation arguments.

To the best of our knowledge, there are two ways to tackle that problem in the literature. The first one is due to CASTRO, CÓRDOBA and GÓMEZ–SERRANO in [27]. They found the existence of $\mathscr{C}^2$ rotating vortices with m–fold symmetry, for any $m \geq 2$. The proof is based on the desingularization and bifurcation from the Burbea patches of Section 1.1.2.

They considered the level sets of ω_0 that can be parametrized by

$$x(\alpha,\rho) = r(\alpha,\rho)(\cos(\alpha), \sin(\alpha)),$$

for a scalar function $r(\alpha,\rho)$, $\alpha \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$. Since we are looking for rotating solutions, the level sets $z(\alpha,\rho,t)$ rotate with constant angular velocity

$$z(\alpha,\rho,t) = e^{i\Omega t}x(\alpha,\rho). \tag{1.1.14}$$

Moreover, by definition one has

$$\omega(z(\alpha,\rho,t),t) = f(\rho),$$

for some scalar function f. By using the Euler equations and the above equation one obtains

$$(-v_0(z(\alpha,\rho,t),t) + \partial_t z(\alpha,\rho,t))\cdot(\partial_\alpha z)^\perp(\alpha,\rho,t)\frac{f'(\rho)}{(\partial_\alpha z)^\perp\cdot(\partial_\rho z)(\alpha,\rho,t)} = 0,$$

where

$$v_0(z(\alpha,\rho,t),t) = \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}\log|z(\alpha,\rho,t) - z(\alpha',\rho',t)|f'(\rho')\partial_\alpha z(\alpha',\rho',t)d\alpha' d\rho'.$$

Assuming now (1.1.14), one arrives at the equivalent equation

$$\begin{aligned} F_f(r,\Omega)(\alpha,\rho) =& \Omega r(\alpha,\rho)\partial_\alpha r(\alpha,\rho) \\ &+ \frac{r(\alpha,\rho)}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}(\partial_\rho f)(\rho')\log(|x(\alpha,\rho)-x(\alpha',\rho')|)\cos(\alpha-\alpha')(\partial_\alpha r)(\alpha',\rho')d\alpha' d\rho' \\ &- \frac{\partial_\alpha r(\alpha,\rho)}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}(\partial_\rho f)(\rho')\log(|x(\alpha,\rho)-x(\alpha',\rho')|)\cos(\alpha-\alpha')r(\alpha',\rho')d\alpha' d\rho' \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}(\partial_\rho f)(\rho')\log(|x(\alpha,\rho)-x(\alpha',\rho')|)\sin(\alpha-\alpha')\\
&\times(r(\alpha,\rho)r(\alpha',\rho')+(\partial_\alpha r)(\alpha,\rho)(\partial_\alpha r)(\alpha',\rho'))d\alpha' d\rho'\\
=&0.
\end{aligned}
$$

Using $\text{supp}(\partial_\rho f)\subset(1-a,1+a)$, then such equation must be verified for $\alpha\in\mathbb{T}$ and $\rho\in(1-a,1+a)$. The parameter a is used as a bifurcation parameter in [27].

Note that the above functional strictly depends on the choice of the function f. In [27], they chose f such that $f^a(\rho)=H\left(\frac{\rho-1}{a}\right)$, where

$$
H(\rho):=\begin{cases}1, & \rho\in(-\infty,-1],\\ 1+\int_{-1}^{\rho}\phi(\rho')d\rho', & \rho\in(-1,1),\\ 0 & \rho\in[1,\infty),\end{cases}\tag{1.1.15}
$$

for some function $\phi\in\mathscr{C}_c^3((-1,1))$ with $\int_{-1}^{1}\phi(\rho')d\rho'=-1$.

Hence, the main result of [27] reads as follows.

Theorem 1.1.2. *[27, Corollary 2.2] There exist global rotating solutions for the 2D Euler vorticity equation with $\mathscr{C}^2$–regularity with compact support, with m–fold symmetry for any integer $m\geq 2$.*

On the other hand, there is another way to tackle the problem presented by $R_1(\Omega,q)$ about its spectral properties, and thus to find non uniform rotating solutions supported on a simply–connected bounded domain. The idea is to restrict our class of solutions and then change the equation $R_1(\Omega,q)=0$ into a more particular one. This is one of the main problems of this dissertation presented in Chapter 2. The main difference from the work done in [27] is that now the bifurcated solutions are far away the patches.

More precisely, we will describe our solutions with a conformal map $\Phi:\mathbb{D}\to D$ from the unit disc $\mathbb{D}$ into a simply–connected bounded domain D and with a real function $f:\overline{\mathbb{D}}\to\mathbb{R}$ which denotes the density profile. That is, we will look for rotating solutions with initial data

$$
\omega_0(x)=(f\circ\Phi^{-1})(x)\mathbf{1}_{\Phi(\mathbb{D})}(x),\quad x\in\mathbb{R}^2.
$$

Moreover, we will consider that f is a perturbation of a radial function f_0 and Φ is a perturbation of the identity map in the following sense

$$
f=f_0+g,\quad\text{and}\quad\Phi=\text{Id}+\phi.
$$

Then, using those variables the equation $R_2(\Omega,q)=0$ agrees with

$$
F(\Omega,g,\phi)(w):=\text{Im}\left[\left(\Omega\,\overline{\Phi(w)}-\frac{1}{2\pi}\int_{\mathbb{D}}\frac{f(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2dA(y)\right)\Phi'(w)w\right]=0,\quad w\in\mathbb{T},\tag{1.1.16}
$$

and we will refer to it as the boundary equation.

Let us now proceed to change the first equation $R_1(\Omega,q)=0$. The strategy is to look for solutions such that

$$
\nabla(f\circ\Phi^{-1})(x)=\mu(\Omega,(f\circ\Phi^{-1})(x))(v(s)-\Omega x^\perp)^\perp,\tag{1.1.17}
$$

for any $x\in D$ and for some scalar function μ. Notice that any solution to (1.1.17) is a solution of the initial density equation (1.1.9). However, the reverse is not in general true. The scalar function μ will be fixed in such a way that the radial profile f_0, around which we look for non trivial solutions, is also a solution to (1.1.17) for any angular velocity.

We can integrate the above equation (1.1.17) arriving to the equivalent form

$$G(\Omega, g, \phi)(z) := \mathcal{M}(\Omega, f(x)) + \frac{1}{2\pi}\int_{\mathbb{D}} \log|\Phi(z)-\Phi(y)|f(y)|\Phi'(y)|^2 dA(y) - \frac{1}{2}\Omega|\Phi(z)|^2 = 0, \tag{1.1.18}$$

for any $z \in \mathbb{D}$. As for μ, the function $\mathcal{M}$ is fixed in order to have that $G(\Omega, 0, 0) = 0$ for any $\Omega \in \mathbb{R}$. Hence, different initial profiles f_0 give us different equations.

In Chapter 2 of this dissertation we focus on the bifurcation around quadratic profiles. That is, we study the particular case

$$f_0(r) = Ar^2 + B, \tag{1.1.19}$$

where $A > 0$ and $B \in \mathbb{R}$. For that choice of f_0, the function $\mathcal{M}$ is given by

$$\mathcal{M}(\Omega, s) = \frac{4\Omega - B}{8A}s - \frac{1}{16A}s^2 + \frac{3B^2 + A^2 + 4AB - 8\Omega B}{16A}.$$

The first point that must be verified is that there are no trivial solutions (in the sense of radial solutions) around the quadratic profile $f_0(r) = Ar^2 + B$. This is proved in Chapter 2, in particular Proposition 2.4.3, that states

Proposition 1.1.3. *Let x_0 be the unique root of the Gauss hypergeometric function $F(1-\sqrt{2}, 1+\sqrt{2}; 1; x)$, and $\Omega_0 := \frac{B}{2} + \frac{A}{4x_0}$. Moreover, let $G_{\text{rad}}(\Omega, g)$ the functional $G(\Omega, g, 0)$ restricted to radial functions g, and let $f_0(r) = Ar^2 + B$ be the quadratic profile, with $A \in \mathbb{R}^\star, B \in \mathbb{R}$. Then, there exists $\varepsilon > 0$ such that*

$$G_{\text{rad}}(\Omega, g) = 0 \Longleftrightarrow g = 0,$$

for any $(\Omega, g) \in I \times B(0, \varepsilon)$ and any bounded interval I, with $I \cap \left(\left[\frac{B}{2}, \frac{B}{2} + \frac{A}{4}\right] \cup \{\Omega_0\}\right) = \emptyset$.

This rules out the possibility of getting radial solutions near the trivial one.

Define now the set $\mathcal{S}_{\text{sing}}$ as

$$\mathcal{S}_{\text{sing}} := \Big\{\Omega \, : \, \partial_\phi F(\Omega, 0, 0) \text{ is not an isomorphism}\Big\},$$

where in the particular case of the quadratic profile it takes the form

$$\mathcal{S}_{\text{sing}} = \left\{\frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad n \in \mathbb{N}^\star \cup \{+\infty\}\right\}. \tag{1.1.20}$$

Then, the Implicit Function theorem can be applied to the boundary equation (1.1.16) outside the singular set $\mathcal{S}_{\text{sing}}$ obtaining that $\phi = \mathcal{N}(\Omega, g)$, for some smooth function $\mathcal{N}$, see Proposition 2.3.3. Hence the density equation (1.1.18) can be reduced to

$$\widehat{G}(\Omega, g)(z) := G(\Omega, g, \mathcal{N}(\Omega, g)).$$

The idea now is to apply Crandall–Rabinowitz theorem to $\widehat{G}$ and try to bifurcate from Ω that belong to the dispersion set:

$$\mathcal{S}_{\text{disp}} := \Big\{\Omega \, : \, \text{Ker}\, D_g\widehat{G}(\Omega, 0) \neq \{0\}\Big\}. \tag{1.1.21}$$

Then, apart from analyzing the spectral properties of $\widehat{G}$ (that is, dimension of the kernel and codimension of the range of the linearized operator), one must verify that the singular and dispersion sets are well–separated.

The linearized operator can be related to a Volterra type integro–differential equation, that can be written in terms of Gauss hypergeometric functions. In this way, we find explicit special regimes on A and B such that there exist nontrivial solutions around the quadratic profile. A (simplified) version of our main theorem can be found in Theorem 2.1.1 and reads as follows.

Theorem 1.1.4. *Let $A > 0$, $B \in \mathbb{R}$ and m a positive integer. Then the following results hold true.*

1. *If $A+B < 0$, then there is $m_0 \in \mathbb{N}$ (depending only on A and B) such that for any $m \geq m_0$, there exists a branch of non radial rotating solutions with $m-$fold symmetry for the Euler equation, bifurcating from the radial solution* (1.1.19) *at some given $\Omega_m > \frac{A+2B}{4}$.*
2. *If $B > A$, then for any integer $m \in \left[1, \frac{B}{A} + \frac{1}{8}\right]$ or $m \in \left[1, \frac{2B}{A} - \frac{9}{2}\right]$ there exists a branch of non radial rotating solutions with $m-$fold symmetry for the Euler equation, bifurcating from the radial solution* (1.1.19) *at some given $0 \leq \Omega_m < \frac{B}{2}$. However, there is no solutions to* (1.1.18) *close to the quadratic profile, for any symmetry $m \geq \frac{2B}{A} + 2$.*
3. *If $B > 0$ or $B \leq -\frac{A}{1+\epsilon}$ for some $0,0581 < \epsilon < 1$, then there exists a branch of non radial $1-$fold symmetric rotating solutions for the Euler equation, bifurcating from the radial solution* (1.1.19) *at $\Omega_1 = 0$.*
4. *If $-\frac{A}{2} < B < 0$ and $\Omega \notin \mathcal{S}_{\text{sing}}$, then there is no solutions to* (1.1.18) *close to the quadratic profile.*
5. *In the frame of the rotating vortices constructed in* $(\mathbf{1}), (\mathbf{2})$ *and* $(\mathbf{3})$*, the particle trajectories inside their supports are concentric periodic orbits around the origin.*

From the transformation $(A, B, \Omega) \mapsto (-A, -B, -\Omega)$ and the homogeneity of the equation we can recover the case $A < 0$ excluded in the above theorem.

The main difficulty of the previous theorem is the spectral study. As we mentioned before, one must analyze first the roots of (1.1.18), which strongly depends on the choice of r_0. Let us briefly explain why we restricted ourselves to the particular profile $r_0(r) = Ar^2 + B$. In order to find nontrivial roots of (1.1.18) around the trivial profile, one needs that the kernel of the linearized operator around it is not trivial. Moreover, we can restate such kernel as a second order differential equation, which will depend on the choice of r_0. A priori, for general profiles we are not able to find an explicit solution of such equation but we do for some particular ones: quadratic or, more generally, polynomial profiles. This is the step where we crucially need to fix the initial profile, that we set as $r_0(r) = Ar^2 + B$. However, in general, one could try to solve the differential equation with more general profiles, like Gaussian ones, and to perform the same analysis. This is an open problem and we refer to Section 5.4 for more details.

In our case, we can write the linearized operator of $\widehat{G}$ around the trivial solution as

$$\partial_g \widehat{G}(\Omega, 0)h(z) = \left[\frac{1}{8}\left\{\frac{4}{A}\left(\Omega - \frac{B}{2}\right) - |z|^2\right\}\text{Id} + \mathcal{K}\right]h(z),$$

where $\mathcal{K}$ is a compact operator. In the case that $\frac{4}{A}\left(\Omega - \frac{B}{2}\right) \notin [0,1]$, we are able to get that $\partial_g \widehat{G}(\Omega, 0)$ is Fredholm of zero index. However, in the opposite case the linearized operator is injective but the range is not closed. A more refined analysis and some coercivity estimates imply also the injectivity of the nonlinear functional $\widehat{G}$ getting the negative results of Theorem 1.1.4.

Moreover, after solving the mentioned second order differential equation via Gauss Hypergeometric functions, we can relate the kernel equation with finding the roots of an algebraic scalar equation depending on the angular velocity Ω and the m–fold symmetry. More precisely, we find the following identity

$$\dim \operatorname{Ker} \partial_g \widehat{G}(\Omega, 0) = \operatorname{Card} \mathcal{A}_{\hat{\Omega}}.$$

The set $\mathcal{A}_{\hat{\Omega}}$ is defined as

$$\mathcal{A}_{\hat{\Omega}} := \left\{ m \in \mathbb{N}^{\star}, \quad \zeta_m(\hat{\Omega}) = 0 \right\},$$

with $\frac{1}{\hat{\Omega}} := \frac{4}{A}\left(\Omega - \frac{B}{2}\right)$, and

$$\zeta_m(\hat{\Omega}) := F_m(\hat{\Omega}) \left[1 - \hat{\Omega} + \frac{A + 2B}{A(m+1)} \hat{\Omega} \right] + \int_0^1 F_m(\tau\hat{\Omega})\tau^m \left[-1 + 2\hat{\Omega}\tau \right] d\tau, \quad \hat{\Omega} \in (-\infty, 1].$$

The function F_m is a Gauss Hypergeometric function where the parameters depend on m:

$$F_m(z) := F(a_m, b_m; m+1; z), \quad a_m = \frac{m - \sqrt{m^2 + 8}}{2}, \ b_m = \frac{m + \sqrt{m^2 + 8}}{2}.$$

See Appendix C for more details about Hypergeometric functions.

Therefore, the kernel study reduces to finding roots of ζ_m, which depends on the parameters A and B. These are the elements within the dispersion set (1.1.21) defined above and different regions of such parameters give us the different scenarios described in Theorem 1.1.4. Moreover, once we are able to apply Crandall–Rabinowitz theorem to (1.1.18), one must come back to the boundary equation (1.1.16) via the Implicit Function theorem mentioned previously. Hence, we need that the chosen angular velocity $\Omega \in \mathcal{S}_{\text{disp}}$ does not belong to the singular set (1.1.20). To prove that we use a deep asymptotic analysis on $\Omega \in \mathcal{S}_{\text{disp}}$ depending on m and also for the points of the singular set $\mathcal{S}_{\text{sing}}$ in order to show that they are well–separated.

Finally, let us remark the existence of 1–fold branches in Theorem 1.1.4–(3) which survive even in regions where no other symmetry is allowed. Moreover, it occurs from $\Omega = 0$ and we could find stationary solutions there. Note that the recent work of GÓMEZ SERRANO, PARK, SHI and YAO [69] implies that if the solution has a fixed sign and is stationary then it must be radial. However, we have a region in Theorem 1.1.4–(3) where the profile changes sign and non radial stationary solutions may live in this branch. See Section 5.4 for a discussion about this topic.

1.1.4 Kármán Vortex Street

The von Kármán Vortex Street is a particular spatial periodic pattern observed in the wake of a two–dimensional bluff body placed in a uniform stream at certain velocities. It can be observed in atmospheric flows about island or in aeronautical systems, see [89, 90, 127, 134, 135, 136, 139]. We refer to Figure 1.4 for the observation of this pattern close to Juan Fernandez Chilean Islands.

The more classical model was given by VON KÁRMÁN in [89, 90]. The author considered the vortices that are modeled as two infinite rows of point vortices which can be staggered with respect to each other. VON KÁRMÁN found that the vortex street is only stable for a particular geometric configuration of the vortices, and unstable for all others. In such a model, the vortices are assumed to have already been formed, and their dynamics is studied in a inviscid flow.

However, the model given by VON KÁRMÁN uses point vortices which are singular solutions of the incompressible Euler equations. Indeed, these structures arise in the context of the Euler equations in the works of SAFFMAN and SCHATZMAN [134, 135, 136]. They considered

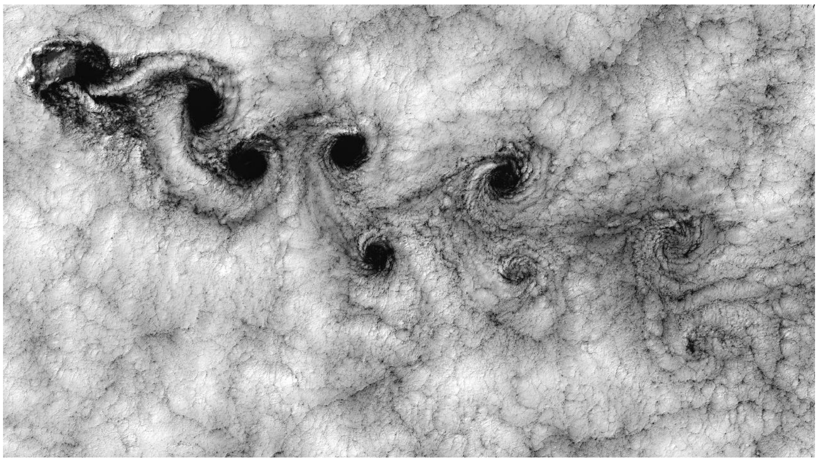

Figure 1.4: Kármán Vortex Street close to Juan Fernandez Chilean Islands. Source: Wikipedia.

two rows of vortices with finite area, i.e., vortex patches of small area. They found numerically the existence of this kind of solutions that translate at a constant speed. Moreover, they identified the stability region where the vortex patch street is stable, and this region shrinks to a point when the vortex patches shrink to point vortices. Following these ideas, we focus on the analytical existence of the Kármán vortex Street in the Euler equations (and in other incompressible models). This will be developed in Chapter 3.

First of all, let us introduce the N–vortex problem. Consider that the vorticity is formed by N points situated in the plane with positions $z_k(t)$, that is,

$$\omega_0(x) = \sum_{k=0}^{N} \Gamma_k \delta(x - z_k(0)),$$

where Γ_k is the strength of $z_k(t)$ for any $t \geq 0$. The Kelvin circulation theorem (see for instance [133]) implies that

$$\omega(t, x) = \sum_{k=0}^{N} \Gamma_k \delta(x - z_k(t)).$$

If we neglect the induced effect of a point on itself, the evolution of each point z_m is given by

$$\Gamma_m \frac{d}{dt} z_m(t) = \nabla^{\perp} H(z_m),$$

where H is the Hamiltonian

$$H(z_1, \ldots, z_m) = -\frac{1}{4\pi} \sum_{k \neq j} \Gamma_m \Gamma_k \log(|z_m - z_k|).$$

This system has some particular relative equilibria that translate or rotate without deformation.

The more simplest example is the two point vortices. Actually, given two initial point vortices $z_1(0)$ and $z_2(0)$, with strengths Γ_1 and Γ_2, then the time evolution consists in a rotation or a translation depending on the circulations. Indeed, if $\Gamma_1 + \Gamma_2 = 0$, then $z_m(t) = z_m(0) + V_0 t$, for some constant speed V_0 that is related to the initial points. In the opposite case that $\Gamma_1 + \Gamma_2 \neq 0$, then we find a rotation evolution in the sense that $z_m(t) = e^{i\Omega t} z_m(0)$ (where we are assuming that the center of masses of $z_1(0)$ and $z_2(0)$ is the origin), for some constant angular velocity Ω. For more details we refer to Proposition 3.2.1 in Chapter 3.

The point vortex model can be seen as a limit of highly concentrated vortex patches with finite area. Indeed, if D is a bounded domain then the family

$$\omega_{0,\varepsilon}(x) = \frac{1}{\varepsilon^2}\mathbf{1}_{D_\varepsilon}, \quad \text{with} \quad D_\varepsilon = \varepsilon D,$$

converges weakly as $\varepsilon \to 0$ to the Dirac measure centered at zero. Hence, we can desingularize some configurations of point vortices to find solutions of the Euler equations consisting in a family of vortex patches. This is the aim of HMIDI and MATEU in [82] where they desingularized two point vortices finding a pair of vortex patches that rotate or translate in time. Let us now explain the basic idea of this work and how to generate Kármán Vortex Street structures in Chapter 3.

Consider an initial vorticity composed of two point vortices with opposite circulations

$$\omega_0(x) = \delta_{z_1(0)}(x) - \delta_{z_2(0)}(x). \tag{1.1.22}$$

As it is mentioned before, its time evolution is given by a translating pair of vortices at constant speed V_0. The goal is to find two simply–connected bounded domains D_1 and D_2 such that the rescaled vorticity

$$\omega_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\mathbf{1}_{\varepsilon D_1 + z_1(0)}(x) - \frac{1}{\pi\varepsilon^2}\mathbf{1}_{\varepsilon D_2 + z_2(0)}(x), \tag{1.1.23}$$

evolves under the same law of the point vortices and translate uniformly with some constant speed V for some ε ranging in some interval $(0, \varepsilon_0)$. Consider without loss of generality that the initial point vortices are located in the real axis in the following way: $z_1(0) = 0$ and $z_2(0) = 2d$, for some $d > 1$. In order to find D_1 and D_2 such that the evolution of (1.1.23) is a translation of constant speed V, that is

$$\omega(t, x) = \omega_0(x - Vt),$$

then it must verify the equation for translating solutions (1.1.10) given in Section 1.1.1 by

$$T(\Omega, \omega_0) = (v_0(x) - V) \cdot \nabla\omega_0(x) = 0, \tag{1.1.24}$$

for any $x \in \mathbb{R}^2$. By using the expression of (1.1.23) in the preceding equation one finds that it agrees with

$$(v_0(x) - V) \cdot \vec{n}(x) = 0, \quad x \in (\varepsilon\partial D_1) \cup (\varepsilon\partial D_2 + 2d), \tag{1.1.25}$$

where $\vec{n}$ is an unit normal vector to the boundary. In order to reduce the unknowns of the equation, we assume some relation between D_1 and D_2, and analyzing the symmetries of the equation one finds that the suitable one is $D_2 = -D_1$. Moreover, it implies that if (1.1.25) is true for any $x \in (\varepsilon\partial D_1)$, then it holds also for $x \in (\varepsilon\partial D_2 + 2d)$. Hence, it is enough to analyze (1.1.25) only for $x \in (\varepsilon\partial D_1)$.

By virtue of the ideas developed in Section 1.1.2, one can rewrite such equation by using a conformal map Φ from the unit disc $\mathbb{D}$ into D_1. However, here the conformal map Φ must depend on ε in order to overcome the singularity given by such parameter. Hence, such map will take the form

$$\Phi = \text{Id} + \varepsilon f. \tag{1.1.26}$$

Notice that the rescaled of Φ in terms of ε strongly depends on the singularity of the velocity field. In order to adapt these techniques to other equations (such as the generalized surface quasi–geostrophic equation) one must use a different scaling.

Finally, one finds that $T(\Omega, \omega_0) = 0$ is equivalent to

$$F(\varepsilon, V, f)(w) := \text{Re}\left[\left\{\overline{I}(\varepsilon, f)(w) - \overline{V}\right\} w\Phi'(w)\right] = 0,$$

for any $w \in \mathbb{T}$, where

$$I(\varepsilon, f)(w) = \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(w)} - \overline{\Phi(\xi)}}{\Phi(w) - \Phi(\xi)}\Phi'(\xi)d\xi} + \frac{1}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(\xi)}}{\varepsilon(\Phi(w) - \Phi(\xi)) + ih}\Phi'(\xi)d\xi}.$$

Then, the initial problem is transformed into looking nontrivial roots of the equation $F(\varepsilon, V, f)$. Analyzing such equation, HMIDI and MATEU were able in [82] to establish the following result.

Theorem 1.1.5. *[82, Main Theorem] Let*

$$\omega_{0,\varepsilon}(x) = \frac{1}{\varepsilon^2}\mathbf{1}_{\varepsilon D_1}(x) + \delta\frac{1}{\varepsilon^2}\mathbf{1}_{-\varepsilon D_1 + 2d}(x), \tag{1.1.27}$$

for $d > 1$, and where $\delta \in \{\pm 1\}$. Then, there exists $\varepsilon_0 > 0$ such that the following results hold true.

- *The case $\delta = 1$. For any $\varepsilon \in (0, \varepsilon_0]$, there exists a strictly convex domain D_1^ε at least of class $\mathscr{C}^1$ such that $\omega_{0,\varepsilon}$ in* (1.1.27) *generates a corotating vortex pair for the Euler equations.*
- *The case $\delta = -1$. For any $\varepsilon \in (0, \varepsilon_0]$, there exists a strictly convex domain D_1^ε at least of class $\mathscr{C}^1$ such that $\omega_{0,\varepsilon}$ in* (1.1.27) *generates a counter–rotating vortex pair for the Euler equations.*

They refer to corotating and counter rotating vortex pairs to rotating or translating vortex pairs, respectively. The idea of the proof is the implementation of the infinite dimensional Implicit Function theorem to F around the initial solution $F(0, V_0, 0) = 0$ (in the case of $\delta = -1$), and more details will be given in the following when analyzing the desingularization of the Kármán Vortex Street model.

Coming back to the Kármán Vortex Street structures, consider the model proposed by VON KÁRMÁN. Then, take a uniformly distributed arrow of points, with the same circulations and located in the horizontal axis, i.e., $(kl, 0)$, with $l > 0$ and $k \in \mathbb{Z}$. The second arrow is similar to the preceding one and contains an infinite number of points, with opposite strength, which will be parallel to the preceding one and with arbitrary stagger: $(a + kl, -h)$ with $a \in \mathbb{R}$ and $h \neq 0$. We refer to Figure 1.5 for a better understanding of the localization of the points. Hence, we consider the following distribution:

$$\omega_0(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x), \tag{1.1.28}$$

where $a \in \mathbb{R}$, $l > 0$ and $h \neq 0$. The evolution of each point is given by

$$\frac{d}{dt}z_m(t) = \sum_{m\neq k\in\mathbb{Z}} \frac{(z_m(t) - z_k(t))^\perp}{|z_m(t) - z_k(t)|^2} - \sum_{k\in\mathbb{Z}} \frac{(z_m(t) - \tilde{z}_k(t))^\perp}{|z_m(t) - \tilde{z}_k(t)|^2},$$
$$\frac{d}{dt}\tilde{z}_m(t) = \sum_{k\in\mathbb{Z}} \frac{(\tilde{z}_m(t) - z_k(t))^\perp}{|\tilde{z}_m(t) - z_k(t)|^2} - \sum_{m\neq k\in\mathbb{Z}} \frac{(\tilde{z}_m(t) - \tilde{z}_k(t))^\perp}{|\tilde{z}_m(t) - \tilde{z}_k(t)|^2},$$

with initial conditions $z_m(0) = ml$ and $\tilde{z}_m(0) = a + ml - ih$, for $m \in \mathbb{Z}$. It can be checked that all the points translate uniformly at the same speed. In the case $a = 0$ or $a = \frac{l}{2}$, the translation is parallel to the real axis and the speed can be expressed by elementary functions:

$$V_0 = \frac{1}{2l}\coth\left(\frac{\pi h}{l}\right), \quad \text{for } a = 0,$$

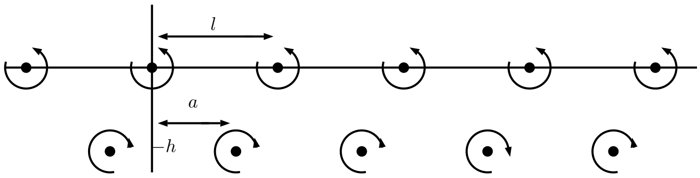

Figure 1.5: Kármán Vortex Street located at the points $(kl, 0)$ and $(a+kl, -h)$, with $l > 0$, $a \in \mathbb{R}$, $h \neq 0$, and $k \in \mathbb{Z}$.

$$V_0 = \frac{1}{2l}\tanh\left(\frac{\pi h}{l}\right), \quad \text{for } a = \frac{l}{2}.$$

Following the ideas of HMIDI and MATEU in [82], let us define

$$\omega_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_1+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D_1+a-ih+kl}(x), \tag{1.1.29}$$

for $l > 0$, $a = 0$ or $a = \frac{l}{2}$, and $\varepsilon > 0$, and for some simply–connected bounded domain D_1. In the limit when $\varepsilon \to 0$, we find in the weak formulation the point vortex street (1.1.28).

Similarly to the vortex pairs, we need to analyze (1.1.24) in the case that ω_0 is given by (1.1.29). To tackle such equation, we parametrize again the domain D_1 via a conformal map from $\mathbb{D}$ into the domain taking the form

$$\Phi = i\left(\text{Id} + \varepsilon f\right). \tag{1.1.30}$$

Notice that the conformal map used for the desingularization of the vortex pairs (1.1.26) is different to (1.1.30) and this is due to the direction of translation. Indeed, the vortex pairs presented previously are located in the real axis and then the translation is along the y–axis. However, here the translation is horizontal. Thus, for the well–posed of the equation it forces us to have a different symmetry for the domain. Recall that we need to introduce ε in the definition of the conformal map in order to work with the singularity given by such parameter.

Hence, inserting (1.1.29) into (1.1.24), we get the equivalent equation

$$F_{KVS}(\varepsilon, f, V)(w) := \text{Re}\left[\left\{\overline{I_{KVS}(\varepsilon, f)(w)} - \overline{V}\right\} w\Phi'(w)\right] = 0, \quad w \in \mathbb{T}, \tag{1.1.31}$$

where

$$I_{KVS}(\varepsilon, f)(w) = v_{0,\varepsilon}(\varepsilon\Phi(w)),$$

and $v_{0,\varepsilon}$ is the velocity field associated to (1.1.29). Hence I_{KVS} is a priori defined via two infinite sums. However, by performing suitable integral computations in the periodic setting we find some useful compact expression for the velocity field amounting to

$$\begin{aligned} I_{KVS}(\varepsilon, f)(w) := &-\frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w) - \Phi(\xi)))}{l}\right)\right| \Phi'(\xi)\, d\xi \\ &- \frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l}\right)\right| \Phi'(\xi)\, d\xi. \end{aligned} \tag{1.1.32}$$

The main result in Chapter 3 concerns the existence of nontrivial solutions (ε, V, Φ) to the equation $F_{KVS}(\varepsilon, V, f) = 0$. The main statement can be found in Theorem 3.3.9 and reads as follows.

Theorem 1.1.6. *Let $h, l \in \mathbb{R}$, with $h \neq 0$ and $l > 0$, and $a = 0$ or $a = \frac{l}{2}$. Then, there exist D^ε such that*

$$\omega_0(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D^\varepsilon + kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D^\varepsilon + a - ih + kl}(x), \tag{1.1.33}$$

defines a horizontal translating solution to the Euler equations, with constant speed, for any $\varepsilon \in (0, \varepsilon_0)$ and small enough $\varepsilon_0 > 0$. Moreover, D^ε is at least $\mathscr{C}^1$.

In the proof of the previous theorem, the cornerstone is the choice of the appropriate function spaces for the perturbation f. Define the following spaces

$$X_\alpha = \left\{ f \in \mathscr{C}^{1,\alpha}(\mathbb{T}), \quad f(w) = \sum_{n\geq 1} a_n w^{-n},\ a_n \in \mathbb{R} \right\},$$

$$\tilde{Y}_\alpha = \left\{ f \in \mathscr{C}^{0,\alpha}(\mathbb{T}), \quad f(e^{i\theta}) = \sum_{n\geq 1} a_n \sin(n\theta),\ a_n \in \mathbb{R} \right\},$$

$$Y_\alpha = \left\{ f \in \mathscr{C}^{0,\alpha}(\mathbb{T}), \quad f(e^{i\theta}) = \sum_{n\geq 2} a_n \sin(n\theta),\ a_n \in \mathbb{R} \right\}.$$

By using some potential theory arguments on singular integrals, we are able to achieve that F_{KVS} is well–defined from $\mathbb{R} \times X_\alpha \times \mathbb{R}$ to $\tilde{Y}_\alpha$, for $\alpha \in (0,1)$. However, the linearized operator around the trivial solution, that is $\partial_f F(0,0,V_0)$, is not an isomorphism in these function spaces, which is needed to implement the infinite dimensional Implicit Function theorem. Indeed, the linearized operator takes the form

$$\partial_f F_{KVS}(0,0,V_0)h(w) = -\frac{1}{2\pi}\mathrm{Im}\left[h'(w)\right],$$

which is an isomorphism from X_α to Y_α, for $\alpha \in (0,1)$. In order to overcome this difficulty and get the persistence of the nonlinear functional F_{KVS} from $\mathbb{R} \times X_\alpha \times \mathbb{R}$ to Y_α one needs to fix V in terms of ε and f. That is achieved by the following nonlinear relation

$$V(\varepsilon, f) = \frac{\int_{\mathbb{T}} \overline{I(\varepsilon, f)} w\Phi'(w)(1-\overline{w}^2)dw}{\int_{\mathbb{T}} w\Phi'(w)(1-\overline{w}^2)dw},$$

and we can check that $V(\varepsilon, f) \in \mathbb{R}$ for $(\varepsilon, f) \in \mathbb{R} \times X_\alpha$. The idea now is the implementation of the Implicit Function theorem to $\tilde{F}_{KVS} : \mathbb{R} \times X_\alpha \to Y_\alpha$ defined by

$$\tilde{F}_{KVS}(\varepsilon, f) = F_{KVS}(\varepsilon, f, V(\varepsilon, f)).$$

Finally, note that the construction given in the previous theorem is flexible and can also be extended to more general incompressible models such as the generalized quasi–geostrophic or the shallow water quasi–geostrophic systems. There, one needs to adapt the choice of the scaling of the conformal map for every case. Thus, we find Kármán Vortex Street structures (in the sense of vortex patches) for those equations. This point will be discussed in detail later in Chapter 3. Moreover, we shall investigate possisble extensions around this topic. Indeed, following the same ideas we can desingularize other special configurations of point vortices. For instance, point vortices with same circulation located at the vertex of a regular polygon rotate uniformly around the center of masses and the desingularization to the Euler equations may happen. We refer to Section 5.2 for more details.

1.2 The 3D quasi–geostrophic model

Geophysical fluids are characterized by the strong influence of the Earth's rotation about its own axis (this gives rise to the Coriolis force) and the stratification, which refers to the in–homogenization of the density. The core of Chapter 4 deals with quasi–geostrophic equations, which describe large scale motion of fluids around the geostrophic and hydrostatic balance. On the one hand, the geostrophic balance refers to a balance between the advected term and the pressure. On the other hand, hydrostatic balance is the balance between the gravitational force and the pressure. Both geostrophic and hydrostatic balance need small Rossby and Froude numbers, which will be defined later.

In the next section, we formally derive this system from the inhomogeneous Euler equations via the Boussinesq approximation. At the end, we will introduce the main contribution of this thesis in this topic. That is, the existence of periodic rotating solutions in the 3D quasi–geostrophic system.

1.2.1 Formal derivation from primitive equations

The derivation of the quasi–geostrophic system comes out from the inhomogeneous Euler equations by taking into account the stratification effects and the Earth's rotation: these are denoted by the primitive equations. We refer for a formal derivation of the quasi–geostrophic system to [14, 33, 34, 37, 52, 118], and for a rigorous justification to [87]. Although we consider here inviscid fluid, let us remark that there is the analogue viscous quasi–geostrophic system by using the Navier–Stokes equations, see the work of CHEMIN [37].

The inhomogeneous Euler equations read as

$$\rho(u_t + u\cdot\nabla u + f_0(-u_2, u_1, 0)) = -\nabla p - \rho(0,0,g), \tag{1.2.1}$$

$$\rho_t + u\cdot\nabla\rho = 0, \tag{1.2.2}$$

$$\nabla\cdot u = 0. \tag{1.2.3}$$

Here ρ is the scalar density, $u = (u_1, u_2, u_3)$ is a vector depending on (t, x, y, z), and p is the pressure. The term f_0 is linked to the speed of Earth's rotation and is written as $f_0 = 2\Omega\sin(\theta)$, where Ω stands for the angular velocity of the Earth and θ its latitude. Let us forget in this section about initial conditions and boundary terms since our main goal is to formally derive the quasi–geostrophic system.

The next step is to introduce the Boussinesq approximation (see for instance [47]). That approximation is a way to study non isothermal flows instead of studying the full Euler equations. It states that the density variation is only important in the buoyancy term, meaning the term $\rho(0,0,g)$, and it can be neglected in the rest of the equations. Under the Boussinesq aproximation, (1.2.1)–(1.2.3) reads as

$$u_t + u\cdot\nabla u + f_0(-u_2, u_1, 0) = -\nabla p - \rho(0,0,g), \tag{1.2.4}$$

$$\nabla\cdot u = 0. \tag{1.2.5}$$

We write the conservation of mass (1.2.2) in terms of the density ϱ such that

$$\rho(t,x,y,z) = \overline{\varrho}(z) + \varrho(t,x,y,z),$$

where $\overline{\varrho}$ is the (known) background density profile. Hence ϱ represents the fluctuation from $\overline{\varrho}$. Then, using (1.2.2) we find that ϱ satisfies

$$\varrho_t + u\cdot\nabla\varrho + u_3\overline{\varrho}_z = 0. \tag{1.2.6}$$

Coming back to (1.2.4), decompose the pressure in the following terms

$$p(t,x,y,z) = p_0(z) + p'(t,x,y,z),$$

and assume that we are in the hydrostatic balance $p_0'(z) = -g\overline{\varrho}(z)$. In this way, omitting the prime of p', we find that (1.2.4) reads as

$$u_t + u\cdot\nabla u + f_0(-u_2,u_1,0) = -\nabla p - \varrho(0,0,g). \tag{1.2.7}$$

We can adimensionalize the system by taking L the typical horizontal length, U the typical horizontal speed and H the typical depth, see [14, 52] for more details. That is, we do the following change of variables

$$\begin{gathered} x = Lx', \quad y = Ly', \quad z = Hz', \quad t = \frac{L}{U}t', \\ u_1 = Uu_1', \quad u_2 = Uu_2', \quad u_3 = \frac{UH}{L}u_3', \\ \overline{\varrho} = P\overline{\varrho}', \quad \varrho = \frac{f_0UL}{gH}\varrho', \quad p = f_0ULp'. \end{gathered}$$

Let us define now the next three numbers:

- Aspect ratio:
$$\sigma = \frac{H}{L}.$$
- Rossby number:
$$\varepsilon = \frac{U}{Lf_0}.$$
- Froude number:
$$\mathrm{Fr} = \frac{U}{\sqrt{gD}}.$$

In what follows we assume that the Froude number is of order ε and we will consider ε to be small. Note that small Rossby number implies a strong effect of the Earth's rotation. Here the aspect ratio does not play an important role and let us take $\sigma = 1$. Moreover, let us remark that

$$\frac{f_0UL}{gH} = P\varepsilon.$$

Hence, the above equations (leaving off primes) can be written as

$$\varepsilon(u_t + u\cdot\nabla u) + (-u_2,u_1,0) = -\nabla p - \varrho(0,0,1), \tag{1.2.8}$$

$$\nabla\cdot u = 0, \tag{1.2.9}$$

$$\varepsilon(\varrho_t + u\cdot\nabla\varrho) + u_3\overline{\varrho}_z = 0. \tag{1.2.10}$$

Formally, assume that we have the following expansions for u_i around ε:

$$u_i = u_i^0 + \varepsilon u_i^1 + \dots,$$

and similarly for the density and the pressure terms. Let us describe the singular limit when ε tends to 0 in a formal way. The first two equations of (1.2.8) together with (1.2.10) at order zero reads as

$$u_2^0 = \partial_x p^0, \quad u_1^0 = -\partial_y p^0, \quad \text{and} \quad u_3^0 = 0,$$

which is known as the geostrophic balance. The hydrostatic balance comes from the third equation of (1.2.8) achieving

$$\varrho^0 = -\partial_z p^0.$$

Note also that $\partial_x u_1^0 + \partial_y u_2^0 = 0$ from (1.2.9). Define now $\xi^0 := \partial_x u_2^0 - \partial_y u_1^0$, then one finds the following equation for ξ^0:

$$d_0\xi := \partial_t \xi^0 + u_1^0 \partial_x \xi^0 + u_2^0 \partial_y \xi^0 = -\partial_x u_2^1 - \partial_y u_1^1.$$

Using (1.2.9) at the first order we find that $\partial_x u_1^1 + \partial_y u_2^1 + \partial_z u_3^1 = 0$. It implies

$$d_0\xi = \partial_z u_3^1.$$

Define the stratification frequency (also called buoyancy or Brunt–Väisälä frequency) $N^2 = -1/\partial_z\overline{\varrho}$. Hence

$$\partial_z u_3^1 = d_0 \partial_z \left(\frac{1}{N^2} \varrho^0 \right),$$

by using that $d_0 N^2 = 0$. Then, one finally gets

$$d_0 \left(\partial_x u_2^0 - \partial_y u_1^0 - \partial_z \left(\frac{1}{N^2}\varrho^0\right) \right) = 0. \tag{1.2.11}$$

We define then the potential vorticity q as

$$q = \partial_x u_2^0 - \partial_y u_1^0 - \partial_z\left(\frac{1}{N^2}\varrho^0\right).$$

Notice that taking $\psi = p^0$, one easily finds that

$$q = \Delta_h \psi + \partial_z\left(\frac{1}{N^2}\partial_z \psi\right),$$

where $\Delta_h = \partial_x^2 + \partial_y^2$.

In the following section we analyze the very special case that $N^2 = 1$ (this is the case of the works [37, 87]). In this case, one has that

$$q = \Delta\psi,$$

where now Δ is the laplacian operator in three dimensions, and

$$(u_1^0, u_2^0) = (-\partial_y \psi, \partial_x \psi).$$

The structure of the system for $N^2 = 1$ is similar to the 2D Euler equations and hence we can ensure the persistence of the vortex patches. However, the persistence of the geometrical structures of the patches for $N^2 \neq 1$ has been also studied, first with the work of DUTRIFOY [58], and later by CHARVE [32].

1.2.2 Rotating patches around the vertical axis

The main contribution of this thesis to the field of quasi–geostrophic motion is to analyze the existence of 3D patches uniformly rotating around the vertical axis motivated by the V–states described in Section 1.1.2. From now on we assume that the buoyancy frequency N^2 is constant and $N^2 = 1$. Hence, we aim to study the following system

$$\begin{cases} \partial_t q + u\partial_1 q + v\partial_2 q = 0, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ \Delta\psi = q, \\ u = -\partial_2\psi, \ v = \partial_1\psi, \\ q(t=0,x) = q_0(x). \end{cases} \tag{1.2.12}$$

The stream function ψ is given by

$$\psi(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(t,y)}{|x-y|} dA(y),$$

where dA denotes the usual Lebesgue measure. Then, the velocity field takes the form

$$(u,v)(t,x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x_1-y_1, x_2-y_2)^\perp}{|x-y|^3} q(t,y)dA(y).$$

From the evolution equation we recover that the potential vorticity is transported along the trajectories. Considering an initial datum in the patch form $q_0 = \mathbf{1}_D$, where D is a bounded domain in $\mathbb{R}^3$, then

$$q(t,\cdot) = \mathbf{1}_{D_t},$$

where D_t is the image of D by the flow. Similarly to the 2D case, as for Euler equations, we can implement the contour dynamics equations. Namely, for any sufficiently smooth parametrization γ_t of the boundary ∂D_t one has that

$$(\partial_t\gamma_t - U(t,\gamma_t)) \cdot n(\gamma_t) = 0,$$

where $U = (u_1, u_2, 0)$ and $n(\gamma_t)$ is a unit normal vector to the boundary at the point γ_t.

Compared to the 2D equations that we have seen before, the quasi–geostrophic equations enjoy richer structures and offer several new perspectives. Actually, at the level of stationary solutions, those in the patch form are more abundant here than in the planar case. Indeed, any domain with a revolution shape about the $z-$axis generates a stationary solution. The analogous to Kirchhoff ellipses still surprisingly survive in the 3D case. In [108], MEACHAM showed that a standing ellipsoid of arbitrary semi–axis lengths a, b and c rotates uniformly about the $z-$axis with the angular velocity

$$\Omega = \mu\frac{\lambda^{-1}R_D(\mu^2,\lambda,\lambda^{-1}) - \lambda R_D(\mu^2,\lambda^{-1},\lambda)}{3(\lambda^{-1}-\lambda)},$$

where $\lambda = \frac{a}{b}$ is the horizontal aspect ratio, $\mu := \frac{c}{\sqrt{ab}}$ the vertical aspect ratio and R_D denotes the elliptic integral of second order

$$R_D(x,y,z) := \frac{3}{2}\int_0^{+\infty} \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}.$$

For more details about the stability of those ellipsoids we refer to works of DRISTCHEL, SCOTT, RENAUD, MCKIVER [54, 55, 57].

The main objective of Chapter 4 is to show the existence of non trivial rotating patches by suitable perturbation of stationary solutions given by generic revolution shapes around the vertical axis. Indeed, we look for solutions of the type

$$q(t,x) = q_0(e^{-i\Omega t}x_h, x_3), \quad q_0 = \mathbf{1}_D, \quad x_h = (x_1,x_2). \tag{1.2.13}$$

The initial domain is chosen so that it can be parametrized in the following way

$$D = \left\{ (re^{i\theta}, \cos(\phi)) \,:\, 0 \le r \le r(\phi,\theta), 0 \le \theta \le 2\pi, 0 \le \phi \le \pi \right\}, \tag{1.2.14}$$

where the shape is sufficiently close to a revolution shape domain, meaning that

$$r(\phi,\theta) = r_0(\phi) + f(\phi,\theta),$$

for a small non–axisymmetric perturbation $f(\phi,\theta)$ of the generatrix curve $r_0(\phi)$. Moreover, we assume the following Dirichlet boundary conditions

$$r_0(0) = r_0(\pi) = f(0,\theta) = f(\pi,\theta) = 0,$$

so that the poles $(0,0,\pm 1)$ remain unchanged. Inserting the ansatz (1.2.13) in the contour dynamics equation and using the parametrization of the domain given by (1.2.14), we get the following equivalent formulation

$$F(\Omega, f)(\phi,\theta) := \psi_0(r(\phi,\theta)e^{i\theta}, \cos(\phi)) - \frac{\Omega}{2} r^2(\phi,\theta) - m(\Omega, f)(\phi) = 0, \tag{1.2.15}$$

for any $(\phi,\theta) \in [0,\pi] \times [0,2\pi]$, with

$$m(\Omega, f)(\phi) := \frac{1}{2\pi} \int_0^{2\pi} \left\{ \psi_0(r(\phi,\theta)e^{i\theta}, \cos(\phi)) - \frac{\Omega}{2} r^2(\phi,\theta) \right\} d\theta,$$

where ψ_0 stands for the stream function associated to q_0. Moreover, from the structure of the stream function ψ_0 we achieve that $F(\Omega, 0) = 0$ for any angular velocity $\Omega \in \mathbb{R}$.

The main tool to find periodic solutions around the stationary shapes is the bifurcation theory, and in particular, finding nontrivial solutions (Ω, f) to (1.2.15). However when implementing such program several hard difficulties emerge at the linear and nonlinear levels. In particular, the spectral problem here is very delicate and strongly depends on the shape of the initial stationary solutions (that is, r_0). Surprisingly, in this case we are finally able to bifurcate from a quite large family of initial domains with mild regularity conditions. Consequently, we do not need to restrict to a specific initial shape, as opposed to the non uniform rotating patches in the Euler equations in Section 1.1.3, where we had to restrict to specific initial vorticity configurations, namely, quadratic profiles.

The main result in Chapter 4 reads as

Theorem 1.2.1. *Assume that r_0 satisfies*

(H1) $r_0 \in \mathscr{C}^2([0,\pi])$.

(H2) *There exists $C > 0$ such that*

$$\forall\, \phi \in [0,\pi], \quad C^{-1} \sin\phi \le r_0(\phi) \le C \sin(\phi).$$

(H3) *r_0 is symmetric with respect to $\phi = \frac{\pi}{2}$, i.e., $r_0\left(\frac{\pi}{2} - \phi\right) = r_0\left(\frac{\pi}{2} + \phi\right)$, for any $\phi \in [0, \frac{\pi}{2}]$.*

Then for any $m \ge 2$, there exists a curve of non trivial rotating solutions with m-fold symmetry to the equation (1.2.12) *bifurcating from the trivial revolution shape associated to r_0 at some angular velocity Ω_m.*

We precise by m–fold shape symmetric shape of $\mathbb{R}^3$ we mean a surface whose horizontal sections are m–fold symmetric as for the 2D case.

There are many difficulties when trying to apply bifurcation theory to (1.2.15), and the main one is related to the choice of the function spaces. First, notice the following simplification in (1.2.15):

$$\begin{aligned} r^2(\phi,\theta) - \frac{1}{2\pi}\int_0^{2\pi} r^2(\phi,\theta)d\theta =& (r_0(\phi)+f(\phi,\theta))^2 - \frac{1}{2\pi}\int_0^{2\pi}(r_0(\phi)+f(\phi,\theta)^2 d\theta \\ =& 2r_0(\phi)f(\phi,\theta)+f(\phi,\theta)^2 - \frac{1}{2\pi}\int_0^{2\pi}(2r_0(\phi)f(\phi,\theta)+f(\phi,\theta)^2)d\theta. \end{aligned}$$

Then, we observe that the nonlinear functional (1.2.15) can be written as $F = -\Omega\, r_0\,\mathrm{Id} + \mathscr{F}$, for an appropriate functional $\mathscr{F}$. Note that r_0 is vanishing at the boundary $\{0,\pi\}$, and thus we need to include this degeneracy in the function spaces. Specifically, in order to avoid this problem we can rescale F as follows

$$\tilde{F}(\Omega,f) := \frac{F(\Omega,f)}{r_0}, \tag{1.2.16}$$

and work with $\tilde{F}$ instead of F. By doing so we can write $\tilde{F} = -\Omega\,\mathrm{Id} + \tilde{\mathscr{F}}$, that is a better suited operator. Our bifucation method then is divided into two main parts: the spectral study and the regularity study, that we briefly discuss in the sequel.

On the one hand, from the point of view of the spectral study, we shall look at the linearized operator and show that it is a Fredholm operator of zero index. Take $h(\phi,\theta) = \sum_{n\geq 1} h_n(\phi)\cos(n\theta)$, then

$$\partial_f \tilde{F}(\Omega,0)h(\phi,\theta) = \sum_{n\geq 1}\cos(n\theta)\left[\{\mathcal{F}_1(1)(\phi)-\Omega\}\,h_n(\phi) - \mathcal{F}_n(h_n)(\phi)\right], \tag{1.2.17}$$

where

$$\mathcal{F}_n(h_n)(\phi) := \frac{1}{4\pi r_0(\phi)}\int_0^{\pi}\int_0^{2\pi}\mathcal{H}_n(\phi,\varphi,\eta)h(\varphi)d\eta d\varphi, \tag{1.2.18}$$

$$\mathcal{H}_n(\phi,\varphi,\eta) := \frac{\sin(\varphi)r_0(\varphi)\cos(n\eta)}{((r_0(\phi)-r_0(\varphi))^2 + 2r_0(\phi)r_0(\varphi)(1-\cos(\eta)) + (\cos(\phi)-\cos(\varphi))^2)^{\frac{1}{2}}}. \tag{1.2.19}$$

Observe then that the linearized operator contains both a local and nonlocal part. Moreover, the function ν_Ω defined as

$$\nu_\Omega(\phi) := \mathcal{F}_1(1)(\phi) - \Omega,$$

is crucial in order to have a Fredholm operator. Defining $\kappa := \inf_{\phi\in(0,\pi)}\mathcal{F}_1(1)(\phi)$, we get that ν_Ω is strictly positive when $\Omega \in (-\infty,\kappa)$ and thus we will restrict to angular velocities lying in such interval.

Furthermore, after computing explicitly the integral in η, we achieve that the linearized operator can be expressed in terms of Gauss Hypergeometric functions as follows:

$$\partial_f \tilde{F}(\Omega,0)h(\phi,\theta) = \nu_\Omega(\phi)\sum_{n\geq 1}\cos(n\theta)\mathcal{L}_n(h_n)(\phi),$$

where

$$\mathcal{L}_n(h_n)(\phi) := h_n(\phi) - \mathcal{K}_n^{\Omega}(h_n)(\phi),$$

$$\mathcal{K}_n^\Omega(h) := \int_0^\pi \frac{H_n(\phi,\varphi)}{\sin(\varphi)\nu_\Omega(\phi)\nu_\Omega(\varphi)r^2(\varphi)} h(\varphi)d\mu_\Omega(\varphi),$$
$$H_n(\phi,\varphi) := \frac{2^{2n-1}\left(\frac{1}{2}\right)_n^2}{(2n)!} \frac{\sin(\varphi)r_0^{n-1}(\phi)r_0^{n+1}(\varphi)}{[R(\phi,\varphi)]^{n+\frac{1}{2}}} F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right).$$

Moreover ν_Ω can be recovered as

$$\nu_\Omega(\phi) = \int_0^\pi H_1(\phi,\varphi)d\varphi - \Omega,$$

and we set

$$R(\phi,\varphi) := (r_0(\phi)+r_0(\varphi))^2 + (\cos(\phi)-\cos(\varphi))^2.$$

Here F_n denotes the Gauss Hypergeometric function

$$F_n(x) := F\left(n+\frac{1}{2}, n+\frac{1}{2}, 2n+1, x\right), \quad x \in [0,1),$$

and we refer to Appendix C for more information about these special functions. In addition, the measure $d\mu_\Omega$ has the following expression

$$d\mu_\Omega(\varphi) := \sin(\varphi)r_0^2(\varphi)\nu_\Omega(\varphi)d\varphi.$$

The goal of such new measure is to symmetrize the linearized operator by working over the new Hilbert spaces determined by the weighted Lebesgue spaces: $L^2_{\mu_\Omega}(0,\pi)$. We emphasize that our hypothesis that $\Omega \in (-\infty,\kappa)$ guarantees that μ_Ω is a signed measure and the weighted space is well defined.

Bearing all the above notation in mind, the kernel equation reduces to study whether 1 is an eigenvalue of $\mathcal{K}_n^\Omega$. In fact, we prove that $\mathcal{K}_n^\Omega : L^2_{\mu_\Omega} \to L^2_{\mu_\Omega}$ is a compact self–adjoint integral operator, and more precisely, it is of Hilbert–Schmidt type. A careful spectral study allows us to determine that that the largest eigenvalue $\lambda_n(\Omega)$ is simple and monotone. Moreover, there exists sequence $\Omega_n \to \kappa$ such that $\lambda_n(\Omega_n) = 1$, which is crucial in order to have that the kernel of the linearized operator is one dimensional.

On the other hand, from the point of view of regularity, at this moment the preceding weighted spaces are too weak to get the persistence and regularity of the nonlinear functional F on those spaces. In order to solve that, our candidate will be the Hölder space $\mathscr{C}^{1,\alpha}$ with Dirichlet boundary conditions, and $\alpha \in (0,1)$. First, we need to check that the above eigenfunctions of $\mathcal{K}_n^\Omega$ belong to this new space, which a priori is not trivial and is equivalent to show such regularity for h satisfying $\mathcal{F}_n(h) = \nu_\Omega h$. By using a bootstrap argument starting at $h \in L^2_{\mu_\Omega}$, we are able to achieve that $h \in \mathscr{C}^{1,\alpha}$ and fulfills the Dirichlet conditions, for any $n \geq 2$. Here it appears the restriction $m \geq 2$ in Theorem 1.2.1. Note also that the integral kernel of $\mathcal{F}_n$, that is (1.2.19), is singular inside the interval $(0,2\pi)$ but also on the boundary since r_0 is vanishing and thus one strongly needs the Dirichlet conditions for h. In particular, this creates some problems in order to guarantee that $\nu_\Omega = \mathcal{F}_n(1) - \Omega$ belongs to $\mathscr{C}^{1,\alpha}$ since the singularity at the boundary can not be compensated with any function inside the integral. This will be solved with a more delicate analysis using the Gauss Hypergeometric functions. Secondly, the persistence of the nonlinear functional can not be achieved by standard potential theory arguments. Note that the Euclidean kernel of the stream function ψ_0 is deformated by the cylindrical coordinates amounting to new singularities at the boundary and a more refined analysis is needed.

Finally, let us mention that the prototypes of domains satisfying the hypotheses **(H1)**–**(H3)** are the sphere, agreeing with $r_0 = \sin(\phi)$, or ellipsoids with same x and y axes, that is,

$r_0 = A\sin(\phi)$. For these particular shapes, the associated stream function is very well–known and thus some of the previous computations can be simplified, we refer to Section 4.6.1 for more details. Indeed, hypothesis **(H2)** means that the initial domain can be located between two ellipsoids and thus gives us some regularity on the domain. However, this is a technical assumption needed for the persistence of the nonlinear functional but not for the spectral study. Hence, the bifurcation of singular shapes not verifying **(H2)** may occur using some appropriate weaker arguments. This will be discussed in Section 5.4.

Chapter 2

Non uniform rotating vortices and periodic orbits for the two–dimensional Euler equations

This chapter is the subject of the following publication:
C. García, T. Hmidi, J. Soler, *Non uniform rotating vortices and periodic orbits for the two–dimensional Euler equations*, arXiv:1807.10017.
Accepted for publication in *Archive for Rational Mechanics and Analysis.*

Contents

2.1 Introduction

The search for Euler and Navier–Stokes solutions is a classical problem of permanent relevance that seeks to understand the complexity and dynamics of certain singular structures in Fluid Mechanics. Only a few solutions are known without much information about their dynamics.

We will focus on the two-dimensional Euler equations that can be written in the velocity–vorticity formulation as follows

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ v = K * \omega, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ \omega(t = 0, x) = \omega_0(x), & \text{with } x \in \mathbb{R}^2. \end{cases}$$

The second equation links the velocity to the vorticity through the so called Biot–Savart law, where $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$.

This chapter focuses on the existence of non uniform rotating solutions for the Euler equations, which are motivated by the rotating patches described in Section 1.1.2. It should be noted that the particularity of the rotating patches is that the dynamics is reduced to the motion of a finite number of curves in the complex plane, and therefore the implementation of the bifurcation is straightforward. However, the construction of smooth rotating vortices is much more intricate due to the size of the kernel of the linearized operator, which is in general infinite dimensional because it contains at least every radial function. Some strategies have been elaborated in order to capture some non trivial rotating smooth solutions. The first result amounts to CASTRO, CÓRDOBA and GÓMEZ-SERRANO [28] who established for the SQG and Euler equations the existence of 3-fold smooth rotating vortices using a reformulation of the equation through the level sets of the vorticity. However the spectral study turns to be highly complex and they use computer–assisted proofs to check the suitable spectral properties. In a recent paper [27] the same authors removed the computer assistance part and proved the existence of $\mathscr{C}^2$ rotating vorticity with m-fold symmetry, for any $m \geq 2$. The proof relies on the desingularization and bifurcation from the vortex patch problem. We point out that the profile of the vorticity is constant outside a very thin region where the transition occurs, and the thickness of this region serves as a bifurcation parameter. Remark that different variational arguments were developed in [21, 70].

The main objective of this chapter is to construct a systematic scheme which turns to be relevant to detect non trivial rotating vortices with non uniform densities, far from the patches but close to some known radial profiles. Actually, we are looking for compactly supported rotating vortices in the form

$$\omega(t, x) = \omega_0(e^{-it\Omega}x), \quad \omega_0 = (f \circ \Phi^{-1})\mathbf{1}_D, \quad \forall x \in \mathbb{R}^2, \tag{2.1.1}$$

where Ω is the angular velocity, $\mathbf{1}_D$ is the characteristic function of a smooth simply connected domain D, the real function $f : \overline{\mathbb{D}} \to \mathbb{R}$ denotes the density profile and $\Phi : \mathbb{D} \to D$ is a conformal mapping from the unit disc $\mathbb{D}$ into D. It is a known fact that an initial vorticity ω_0 with velocity v_0 generates a rotating solution, with constant angular velocity Ω, if and only if

$$(v_0(x) - \Omega x^\perp) \cdot \nabla \omega_0(x) = 0, \quad \forall x \in \mathbb{R}^2, \tag{2.1.2}$$

where $(x_1, x_2)^\perp = (-x_2, x_1)$. Thus the ansatz (2.1.1) is a solution of Euler equations (1.1.3) if and only if the following equations

$$(v(x) - \Omega x^\perp) \cdot \nabla (f \circ \Phi^{-1})(x) = 0, \quad \text{in} \quad D, \tag{2.1.3}$$
$$(v(x) - \Omega x^\perp) \cdot (f \circ \Phi^{-1})(x) \vec{n}(x) = 0, \quad \text{on} \quad \partial D, \tag{2.1.4}$$

are simultaneously satisfied, where $\vec{n}$ is the upward unit normal vector to the boundary ∂D. Regarding its relationship with the issue of finding vortex patches, the problem presented here exhibits a greater complexity. While a rotating vortex patch solution can be described by the boundary equation (2.1.4), here we also need to work with the corresponding coupled density equation (2.1.3). One major problem that one should face in order to make the bifurcation argument useful is related to the size of the kernel of the linearized operator which is in general infinite-dimensional. In the vortex patch framework we overcome this difficulty using the contour dynamics equation and by imposing a suitable symmetry on the V-states: they should be invariant by the dihedral group D_m. In this manner we guarantee that the linearized operator becomes a Fredholm operator with zero index. In the current context, we note that all smooth radial functions belong to the kernel. One possible strategy that one could implement is to filter those non desirable functions from the structure of the function spaces by removing the mode zero. However, this attempt fails because the space will not be stable by the nonlinearity especially for the density equation (2.1.3): the frequency zero can be obtained from a resonant regime, for example the square of a non vanishing function on the disc generates always the zero mode. Even though, if we assume that we were able to solve this technical problem by some special fine tricks, a second but more delicate one arises with the formulation (2.1.2). The linearized operator around any radial solution is not of Fredholm type: it is smoothing in the radial component. In fact, if ω_0 is radial, then the linearized operator associated with the nonlinear map

$$F(\omega)(x) = (v(x) - \Omega x^\perp) \cdot \nabla \omega(x),$$

is given in polar coordinates by

$$\mathcal{L}(h) = \left(\frac{v_\theta^0}{r} - \Omega \right) \partial_\theta h + K(h) \cdot \nabla \omega_0, \quad K(h)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} h(y) dy.$$

The loss of information in the radial direction can not be compensated by the operator K which is compact. This means that when using standard function spaces, the range of the linearized operator will be of infinite codimension. This discussion illustrates the limitation of working directly with the model (2.1.2). Thus, we should first proceed with reformulating differently the equation (2.1.2) in order to avoid the preceding technical problems and capture non radial solutions by a bifurcation argument. We point out that the main obstacle comes from the density equation (2.1.3) and the elementary key observation is that a solution to this equation means that the density is constant along the level sets of the relative stream function. This can be guaranteed if one looks for solutions to the restricted problem,

$$G(\Omega, f, \Phi)(z) := \mathcal{M}\big(\Omega, f(z)\big) + \frac{1}{2\pi} \int_D \log |\Phi(z) - \Phi(y)| f(y) | |\Phi'(y)|^2 dy - \frac{1}{2} \Omega |\Phi(z)|^2 = 0, \tag{2.1.5}$$

for every $z \in \mathbb{D}$, and for some suitable real function $\mathcal{M}$. The free function $\mathcal{M}$ can be fixed so that the radial profile is a solution. For instance, as it will be shown in Section 2.4, for the radial profile

$$f_0(r) = Ar^2 + B, \tag{2.1.6}$$

we get the explicit form

$$\mathcal{M}(\Omega, s) = \frac{4\Omega - B}{8A}s - \frac{1}{16A}s^2 + \frac{3B^2 + A^2 + 4AB - 8\Omega B}{16A}.$$

Moreover, with this reformulation, we can ensure that no other radial solution can be captured around the radial profile except for a singular value, see Proposition 2.4.3.

Before stating our result we need to introduce the following set, which is nothing but the singular set introduced later, see (2.1.9), in the case of the quadratic profile,

$$\mathcal{S}_{\text{sing}} = \left\{ \frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad n \in \mathbb{N}^\star \cup \{+\infty\} \right\}.$$

The main result of this chapter concerning the quadratic profile is the following.

Theorem 2.1.1. *Let $A > 0$, $B \in \mathbb{R}$ and m a positive integer. Then the following results hold true.*

1. *If $A+B < 0$, then there is $m_0 \in \mathbb{N}$ (depending only on A and B) such that for any $m \geq m_0$, there exists a branch of non radial rotating solutions with $m-$fold symmetry for the Euler equation, bifurcating from the radial solution* (2.1.6) *at some given $\Omega_m > \frac{A+2B}{4}$.*
2. *If $B > A$, then for any integer $m \in \left[1, \frac{B}{A} + \frac{1}{8}\right]$ or $m \in \left[1, \frac{2B}{A} - \frac{9}{2}\right]$ there exists a branch of non radial rotating solutions with $m-$fold symmetry for the Euler equation, bifurcating from the radial solution* (2.1.6) *at some given $0 \leq \Omega_m < \frac{B}{2}$. However, there is no solutions to* (2.1.5) *close to the quadratic profile, for any symmetry $m \geq \frac{2B}{A} + 2$.*
3. *If $B > 0$ or $B \leq -\frac{A}{1+\epsilon}$ for some $0,0581 < \epsilon < 1$, then there exists a branch of non radial $1-$fold symmetry rotating solutions for the Euler equation, bifurcating from the radial solution* (2.1.6) *at $\Omega_1 = 0$.*
4. *If $-\frac{A}{2} < B < 0$ and $\Omega \notin \mathcal{S}_{\text{sing}}$, then there is no solutions to* (2.1.5) *close to the quadratic profile.*
5. *In the frame of the rotating vortices constructed in $(\mathbf{1})$, $(\mathbf{2})$ and $(\mathbf{3})$, the particle trajectories inside their supports are concentric periodic orbits around the origin.*

This theorem will be fully detailed in Theorem 2.5.6, Theorem 2.7.3 and Theorem 2.8.2. Before giving some details about the main ideas of the proofs, we wish to draw some useful comments:

- The upcoming Theorem 2.8.2 states that the orbits associated with (2.1.3) are periodic with smooth period, and at any time the flow is invariant by a rotation of angle $\frac{2\pi}{m}$. Moreover, it generates a group of diffeomorphisms of the closed unit disc.
- The V-states constructed in the above theorem have the form $f \circ \Phi^{-1}\mathbf{1}_D$. Also, it is proved that the density f is $\mathscr{C}^{1,\alpha}(\mathbb{D})$ and the boundary ∂D is $\mathscr{C}^{2,\alpha}$ with $\alpha \in (0,1)$. We believe that by implementing the techniques used in [26] it could be shown that the density and the domain are analytic. An indication supporting this intuition is provided by the generator of the kernel associated with the density equation, see (2.5.34), which is analytic up to the boundary. The dynamics of the 1–fold symmetric V-states is rich and very interesting.

The branch can survive even in the region where no other symmetry is allowed. Since the bifurcation occurs from $\Omega_1 = 0$, it is not clear from our result whether or not the branch contains stationary solutions. However, we know that this branch is not given by a pure translation of the radial solution. This follows from the structure of the function space describing the conformal mapping regularity: there we kill the invariance by translation by removing the frequency zero, for more details see Section 2.2.2 and Theorem 2.7.3. It should be noted that in the context of vortex patches the bifurcation from the disc or the annulus occurs only with symmetry $m \geq 2$ and never with the symmetry 1. The only examples that we know in the literature about the emergence of the symmetry one is the bifurcation from Kirchhoff ellipses [26, 82] or the presence of the boundary effects [53]. Interesting discussion about stationary solutions for active scalar equations can be found in the recent paper of GÓMEZ-SERRANO,PARK,SHI and YAU [69].

- From the homogeneity of Euler equations the transformation $(A, B, \Omega) \mapsto (-A, -B, -\Omega)$ leads to the same class of solutions in Theorem 2.1.1. This observation allows including in the main theorem the case $A < 0$.

- The assumptions on A and B seen in Theorem 2.1.1 $-$ (1) $-$ (2) about the bifurcation cases imply that the radial profile f_0 is not changing the sign in the unit disc. However in the point (3) the profile can change the sign.

- The bifurcation with $m-$fold symmetry, $m \geq 1$, when $B \in (-A, -\frac{A}{2})$ is not well understood. We only know that we can obtain a branch of 1-fold symmetric solutions bifurcating from $\Omega_1 = 0$ for $B \in (-A, -\frac{A}{1+\epsilon})$ for some $\epsilon \in (0, 1)$, nothing is known for other symmetries. We expect that similarly to the result of Theorem 2.1.1-(2), they do exist but only for lower frequencies, and the bifurcation curves are rarefied when B approaches $-\frac{A}{2}$.

- Let us remark the existence of solutions with lower m-fold symmetry coming from the second point of the above Theorem. Fixing A, the number of allowed symmetries increases when $|B|$ increases. We guess that there is a smooth curve when passing from one symmetry to another one, see Fig.1.

Let us briefly outline the general strategy we follow to prove the main result and that could be implemented for more general profiles. Using the conformal mapping Φ we can translate the equations (2.1.3)–(2.1.4) into the disc $\mathbb{D}$ and its boundary $\mathbb{T}$. Equations (2.1.3)–(2.1.4) depend functionally on the parameters (Ω, f, Φ), so that we can write them as

$$\begin{cases} G(\Omega, f, \Phi)(z) = & 0, \quad \forall z \in \mathbb{D}, \\ F(\Omega, f, \Phi)(w) = & 0, \quad \forall w \in \mathbb{T}, \end{cases} \tag{2.1.7}$$

with

$$F(\Omega, f, \Phi)(w) := \mathrm{Im}\left[\left(\Omega\,\overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)}|\Phi'(y)|^2 dA(y)\right)\Phi'(w)w\right] = 0, \quad \forall w \in \mathbb{T},$$

where the functional G is described in (2.1.5). The aim is to parametrize the solutions in (Ω, f, Φ) close to some initial radial solution $(\Omega, f_0, \mathrm{Id})$, with f_0 being a radial profile and Id the identity map. Then, we will deal with the unknowns g and ϕ defined by

$$f = f_0 + g, \quad \Phi = \mathrm{Id} + \phi. \tag{2.1.8}$$

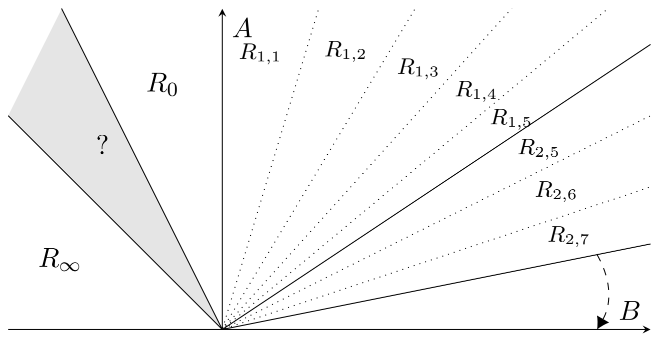

Figure 2.1: This diagram shows the different bifurcation regimes given in Theorem 2.1.1, with $A > 0$. In the case $B > 0$, we can find only a finite number of eigenvalues Ω_m for which it is possible to obtain a branch of non radial m–fold symmetric solutions of the Euler equation. Here, $m \geq 1$ increases as B does. The region $R_{i,j}$ admits solutions with m–fold symmetry for $1 \leq m \leq i$. In addition, solutions with m–fold symmetry for $m > j$ are not found. The transition between $m = i$ and $m = j$ is not known. Notice that in the region R_∞ the bifurcation occurs with an infinite countable family of eigevalues. However, the bifurcation is not possible in the region R_0 but the transition between R_0 and R_∞ is not well-understood due to some spectral problem concerning the linearized operator. We only know the existence of 1–fold symmetric solutions in a small region.

Thus, the equations in (2.1.7) are parametrized in the form $G(\Omega, g, \phi)(z) = 0$ and $F(\Omega, g, \phi)(w) = 0$, where $G(\Omega, 0, 0)(z) = 0$ and $F(\Omega, 0, 0)(w) = 0$. The idea is to start by solving the boundary equation, which would reduce a variable through a mapping $(\Omega, f) \mapsto \Phi = \mathcal{N}(\Omega, f)$, i.e. to prove that under some restrictions $F(\Omega, g, \phi) = 0$ is equivalent to $\phi = \mathcal{N}(\Omega, g)$. However, the argument stumbles when we realize that this can only be done outside a set of singular values

$$\mathcal{S}_{\text{sing}} := \Big\{\Omega \,:\, \partial_\phi F(\Omega, 0, 0) \text{ is not an isomorphism}\Big\}, \tag{2.1.9}$$

for which the Implicit Function Theorem can be applied. Then, we prove that there exists an open interval I for Ω such that $\overline{I} \subset \mathbb{R}\backslash\mathcal{S}_{\text{sing}}$ and $\mathcal{N}$ is well–defined in appropriated spaces, which will be subspaces of Hölder–continuous functions. Under the hypothesis that $\Omega \in I$, the problem of finding solutions of (2.1.7) is reduced to solve

$$\widehat{G}(\Omega, g)(z) := G(\Omega, g, \mathcal{N}(\Omega, g))(z) = 0, \quad \forall z \in \mathbb{D}. \tag{2.1.10}$$

In order to find time dependent non radial rotating solutions to (1.1.3) we use the procedure developed in [20] that suggests the bifurcation theory as a tool to generate solutions from a stationary one via the Crandall–Rabinowitz Theorem. The values Ω that could lead to the bifurcation to non trivial solutions are located in the dispersion set

$$\mathcal{S}_{\text{disp}} := \Big\{\Omega \,:\, \operatorname{Ker} D_g\widehat{G}(\Omega, 0) \neq \{0\}\Big\}. \tag{2.1.11}$$

The problem then consists in verifying that the singular (2.1.9) and dispersion (2.1.11) sets are well-separated, for a correct definition of the interval I. Achieving this objective together with the analysis of the dimension properties of the kernel and the codimension of the range of

$D_g\widehat{G}(\Omega,0)$, as well as verifying the transversality property requires a complex and precise spectral and asymptotic analysis. Although our discussion is quite general, we focus our attention on the special case of quadratic profiles (2.1.6). In this case we obtain a compact representation of the dispersion set. Indeed, as we shall see in Section 2.5, the resolution of the kernel equation leads to a Volterra type integro-differential equation that one may solve through transforming it into an ordinary differential equation of second order with polynomial coefficients. Surprisingly, the new equation can be solved explicitly through variation of the constant and is connected to Gauss hypergeometric functions. The structure of the dispersion set is very subtle and appears to be very sensitive to the parameters A and B. Our analysis allows us to highlight some special regimes on A and B, see Proposition 2.6.6 and Proposition 2.6.7.

Let us emphasize that the techniques developed in the quadratic profile are robust and could be extended to other profiles. In this direction, we first provide in Section 2.4 the explicit expression of the function $\mathcal{M}$ when the density admits a polynomial or Gaussian distribution. In general, the explicit resolution of the kernel equations may turn out to be a very challenging problem. Second, we will notice in Remark 2.5.9 that when $f_0 = Ar^{2m}+B$ with $m\in\mathbb{N}^\star$, explicit formulas are expected through some elementary transformations and the kernel elements are linked also to hypergeometric equations.

In Section 2.8 we shall be concerned with the proof of the point (5) of Theorem 2.1.1 concerning the planar trajectories of the particles located inside the support of the rotating vortices. We analyze the properties of periodicity and symmetries of the solutions via the study of the associated dynamical Hamiltonian structure in Eulerian coordinates, which was highlighted by ARNOLD [10]. This Hamiltonian nature of the Euler equations has been the idea behind the study of conservation laws in the hydrodynamics of an ideal fluid [10, 109, 112, 115], as well as in a certain sense to justify Boltzmanns principle from classical mechanics [146].

We shall give in Theorem 2.8.2 a precise statement and prove that close to the quadratic profile all the trajectories are periodic orbits located inside the support of the V-states, enclosing a simply connected domain containing the origin, and are symmetric with respect to the real axis. In addition, every orbit is invariant by a rotation of angle $\frac{2\pi}{m}$, as it has been proved for the branch of bifurcated solutions, where the parameter m is determined by the spectral properties. The periodicity of the orbits follows from the Hamiltonian structure of the autonomous dynamical system,

$$\partial_t\Psi(t,z)=W(\Omega,f,\Phi)(\Psi(t,z)),\quad \Psi(0,z)=z\in\overline{\mathbb{D}}, \tag{2.1.12}$$

where

$$W(\Omega,f,\Phi)(z)=\left(\frac{i}{2\pi}\int_{\mathbb{D}}\frac{f(y)}{\overline{\Phi(z)}-\overline{\Phi(y)}}|\Phi'(y)|^2dy-i\Omega\Phi(z)\right)\overline{\Phi'(z)},\quad z\in\mathbb{D}.$$

Notice that W is nothing but the pull-back of the vector filed $v(x)-\Omega x^\perp$ by the conformal mapping Φ. This vector field remains Hamiltonian and is tangential to the boundary $\mathbb{T}$. Moreover, we check that close to the radial profile, it has only one critical point located at the origin which must be a center. As a consequence, the trajectories near the origin are organized through periodic orbits. Since the trajectories are located in the level sets of the energy functional given by the relative stream function, then using simple arguments we show the limit cycles are excluded and thus all the trajectories are periodic enclosing the origin which is the only fixed point, which, together with the trajectories defined above, is a way of solving the hyperbolic system (2.1.3). This allows to define the period map $z\in\mathbb{D}\mapsto T_z$, whose regularity will be at the same level as the profiles. As a by-product we find the following equivalent reformulation of the density equation

$$f(z)-\frac{1}{T_z}\int_0^{T_z} f(\Psi(\tau,z))d\tau=0,\quad \forall z\in\overline{\mathbb{D}}. \tag{2.1.13}$$

We finally comment on three recent approaches to the analysis of rotating solutions. The first one concerns rotating vortex patches. In [77], HASSANIA, MASMOUDI and WHEELER construct continuous curves of rotating vortex patch solutions, where the minimum along the interface of the angular fluid velocity in the rotating frame becomes arbitrarily small, which agrees with the conjecture about singular limiting patches with 90° corners [24, 116]. In the second contribution [27], it was studied the existence of smooth rotating vortices desingularized from a vortex patch, as it was mentioned before. The techniques are based on the analysis of the level sets of the vorticity of a global rotating solution. Since the level sets $z(\alpha,\rho,t)$ rotate with constant angular velocity, they satisfy $\omega(z(\alpha,\rho,t)),t)=f(\rho)$. Thus, in [27] is studied the problem of bifurcating it for some specific choice of f. In a broad sense, this result connects with that developed in this chapter about the study of orbits and their periodicity. Finally, BEDROSSIAN, COTI ZELATI and VICOL in [15] analyzed the incompressible 2D Euler equations linearized around a radially symmetric, strictly monotone decreasing vorticity distribution. For sufficiently regular data, inviscid damping of the θ-dependent radial and angular velocity fields is proved. In this case, the vorticity weakly converges back to radial symmetry as $t\to\infty$, a phenomenon known as vortex axisymetrization. Also they show that the θ-dependent angular Fourier modes in the vorticity are ejected from the origin as $t\to\infty$, resulting in faster inviscid damping rates than those possible with passive scalar evolution (vorticity depletion).

The results and techniques presented in this chapter are powerful enough to be extended to other situations and equations such as SQG equations, co–rotating time–dependent solutions, solutions depending only on one variable,...

2.2 Preliminaries and statement of the problem

The aim of this section is to formulate the equations governing general rotating solutions of the Euler equations. We will also set down some of the tools that we use throughout the chapter such as the functional setting or some properties about the extension of Cauchy integrals.

2.2.1 Equation for rotating vortices

Let us begin with the equations for compactly supported rotating solutions (2.1.3)–(2.1.4) and assume that f is not vanishing on the boundary. In the opposite case, the equation (2.1.4) degenerates and becomes trivial, which implies that we just have one equation to analyze. Thus, (2.1.4) becomes

$$(v(x)-\Omega x^\perp)\cdot\vec{n}(x)=0,\quad \text{on}\quad \partial D. \tag{2.2.1}$$

We will rewrite these equations in the unit disc through the use of the conformal map $\Phi:\mathbb{D}\to D$. Note that from now on and for the sake of simplicity we will identify the Euclidean and the complex planes. Then, we write the velocity field as

$$v(x)=\frac{i}{2\pi}\int_D\frac{(f\circ\Phi^{-1})(y)}{\overline{x}-\overline{y}}dA(y),\quad \forall x\in\mathbb{C},$$

where dA refers to the planar Lebesgue measure, getting

$$v(\Phi(z)) = \frac{i}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}}|\Phi'(y)|^2 dA(y), \quad \forall z \in \mathbb{D}.$$

Using the conformal parametrization $\theta \in [0, 2\pi] \mapsto \Phi(e^{i\theta})$ of ∂D, we find that a normal vector to the boundary is given by $\vec{n}(\Phi(w)) = w\Phi'(w)$, with $w \in \mathbb{T}$. In order to deal with (2.1.3), we need to transform carefully the term $\nabla(f \circ \Phi^{-1})(\Phi(z))$ coming from the density equation. Recall that for any complex function $\varphi : \mathbb{C} \to \mathbb{C}$ of class $\mathscr{C}^1$ seen as a function of $\mathbb{R}^2$, we can define

$$\partial_{\overline{z}}\varphi(z) := \frac{1}{2}\left(\partial_1\varphi(z) + i\partial_2\varphi(z)\right), \quad \text{and} \quad \partial_z\varphi(z) := \frac{1}{2}\left(\partial_1\varphi(z) - i\partial_2\varphi(z)\right),$$

which are known in the literature as Wirtinger derivatives. Let us state some of their basic properties:

$$\overline{\partial_z\varphi} = \partial_{\overline{z}}\overline{\varphi}, \quad \overline{\partial_{\overline{z}}\varphi} = \partial_z\overline{\varphi}.$$

Given two complex functions $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}$ of class $\mathscr{C}^1$ in the Euclidean coordinates, the chain rule comes as follows

$$\begin{aligned}
\partial_z(\varphi_1 \circ \varphi_2) &= (\partial_z\varphi_1 \circ \varphi_2)\,\partial_z\varphi_2 + (\partial_{\overline{z}}\varphi_1 \circ \varphi_2)\,\partial_z\overline{\varphi_2},\\
\partial_{\overline{z}}(\varphi_1 \circ \varphi_2) &= (\partial_z\varphi_1 \circ \varphi_2)\,\partial_{\overline{z}}\varphi_2 + (\partial_{\overline{z}}\varphi_1 \circ \varphi_2)\,\partial_{\overline{z}}\overline{\varphi_2}.
\end{aligned}$$

Moreover, since Φ is a conformal map, one has that $\partial_{\overline{z}}\Phi = 0$. Identifying the gradient with the operator $2\partial_{\overline{z}}$ leads to

$$\nabla(f \circ \Phi^{-1}) = 2\partial_{\overline{z}}(f \circ \Phi^{-1}).$$

From straightforward computations using the holomorphic structure of Φ^{-1}, combined with the previous properties of the Wirtinger derivatives, we get

$$\begin{aligned}
\partial_{\overline{x}}(f \circ \Phi^{-1})(x) =& (\partial_x f)(\Phi^{-1}(x))\partial_{\overline{x}}\Phi^{-1}(x) + (\partial_{\overline{x}} f)(\Phi^{-1}(x))\partial_{\overline{x}}\overline{\Phi^{-1}(x)}\\
=& (\partial_{\overline{x}} f)(\Phi^{-1}(x))\overline{(\Phi^{-1})'(x)},
\end{aligned}$$

where the prime notation $'$ for Φ^{-1} denotes the complex derivative in the holomorphic case. Using that $(\Phi^{-1} \circ \Phi)(z) = z$ and differentiating it we obtain

$$(\Phi^{-1})'(\Phi(z)) = \frac{\overline{\Phi'(z)}}{|\Phi'(z)|^2},$$

which implies

$$\partial_{\overline{z}}(f \circ \Phi^{-1})(\Phi(z)) = \frac{\partial_{\overline{z}} f(z)\Phi'(z)}{|\Phi'(z)|^2}. \tag{2.2.2}$$

Putting everything together in (2.1.3)-(2.2.1) we find the following equivalent expression

$$\begin{aligned}
W(\Omega, f, \Phi) \cdot \nabla f &= 0, \quad \text{in} \quad \mathbb{D}, & (2.2.3)\\
W(\Omega, f, \Phi) \cdot \vec{n} &= 0, \quad \text{on} \quad \mathbb{T}, & (2.2.4)
\end{aligned}$$

where $\vec{n}$ stands for a unit normal vector to $\mathbb{T}$, and W is given by

$$W(\Omega, f, \Phi)(z) = \left(\frac{i}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}}|\Phi'(y)|^2 dy - i\Omega\Phi(z)\right)\overline{\Phi'(z)}. \tag{2.2.5}$$

Then, the vector field $W(\Omega, f, \Phi)$ is incompressible. This fact is a consequence of the lemma below. Given a vector field $X : \mathbb{C} \to \mathbb{C}$ of class $\mathscr{C}^1$ in the Euclidean variables, let us associate the divergence operator with the Wirtinger derivatives as follows

$$\text{div}X(z) = 2\text{Re}\left[\partial_z X(z)\right]. \tag{2.2.6}$$

Lemma 2.2.1. *Given $X : D_1 \to \mathbb{C}$ an incompressible vector field, $\Phi : D_2 \to D_1$ a conformal map, where $D_1, D_2 \subset \mathbb{C}$, then $(X \circ \Phi)\overline{\Phi'} : D_2 \to \mathbb{C}$ is incompressible.*

Proof. Using (2.2.6) we have that $\text{Re}\left[\partial_x X(x)\right] = 0$, for any $x \in D_1$.

The properties of the Wirtinger derivatives lead to

$$\begin{aligned}\partial_z \left[X(\Phi(z))\overline{\Phi'(z)}\right] =&\partial_z \left[X(\Phi(z))\right]\overline{\Phi'(z)} + X(\Phi(z))\partial_z\overline{\Phi'(z)}\\ =&(\partial_z X)(\Phi(z))\Phi'(z)\overline{\Phi'(z)} + (\partial_{\overline{z}} X)(\Phi(z))\partial_z\overline{\Phi(z)}\,\overline{\Phi'(z)}\\ &+ X(\Phi(z))\partial_z\overline{\Phi'(z)}\\ =&(\partial_z X)(\Phi(z))|\Phi'(z)|^2, \quad \forall z \in D_2.\end{aligned}$$

Hence, we have that

$$\text{Re}\left[\partial_z\left(X(\Phi(z))\overline{\Phi'(z)}\right)\right] = |\Phi'(z)|^2\text{Re}\left[(\partial_z X)(\Phi(z))\right] = 0, \quad \forall z \in D_2,$$

and $(X \circ \Phi)\overline{\Phi'}$ is incompressible. □

Let us remark that the equation associated to the V-states in [84] is nothing but the boundary equation (2.2.4). In [84], V-states close to a trivial solution are obtained by means of a perturbation of the domain via a conformal mapping. Since we are perturbing also the initial density, we must analyze one more equation: the density equation (2.2.3). In order to apply the Crandall–Rabinowitz Theorem we will not deal with (2.2.3) because it seems not to be suitable when studying the linearized operator. Hence, we will reformulate this equation in Section 2.4. Moreover, we will provide an alternative way of writing (2.2.3) in Section 2.8 to understand the behavior of the orbits of the dynamical system associated to it.

2.2.2 Function spaces

The right choice of the function spaces will be crucial in order to construct non radial rotating solutions different to the vortex patches.

Before going into further details we introduce the classical Hölder spaces in the unit disc $\mathbb{D}$. Let us denote $\mathscr{C}^{0,\alpha}(\mathbb{D})$ as the set of continuous functions such that

$$\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} := \|f\|_{L^\infty(\mathbb{D})} + \sup_{z_1 \neq z_2 \in \mathbb{D}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < +\infty,$$

for any $\alpha \in (0,1)$. By $\mathscr{C}^{k,\alpha}(\mathbb{D})$, with $k \in \mathbb{N}$, we denote the $\mathscr{C}^k$ functions whose k-order derivative lies in $\mathscr{C}^{0,\alpha}(\mathbb{D})$. Recall the Lipschitz space with the semi-norm defined as

$$\|f\|_{\text{Lip}(\mathbb{D})} := \sup_{z_1 \neq z_2 \in \mathbb{D}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}.$$

Similarly, we define the Hölder spaces $\mathscr{C}^{k,\alpha}(\mathbb{T})$ in the unit circle $\mathbb{T}$. Let us supplement these spaces with additional symmetry structures:

$$\mathscr{C}^{k,\alpha}_s(\mathbb{D}) := \left\{g : \mathbb{D} \to \mathbb{R} \in \mathscr{C}^{k,\alpha}(\mathbb{D}), \quad g(re^{i\theta}) = \sum_{n \geq 0} g_n(r)\cos(n\theta),\ g_n \in \mathbb{R},\ \forall z = re^{i\theta} \in \mathbb{D}\right\},$$

$$\mathscr{C}_a^{k,\alpha}(\mathbb{T}) := \Big\{\rho : \mathbb{T} \to \mathbb{R} \in \mathscr{C}^{k,\alpha}(\mathbb{T}), \quad \rho(e^{i\theta}) = \sum_{n\geq 1} \rho_n \sin(n\theta), \ \forall w = e^{i\theta} \in \mathbb{T}\Big\}. \tag{2.2.7}$$

These spaces are equipped with the usual norm $\|\cdot\|_{\mathscr{C}^{k,\alpha}}$. One can easily check that if the functions $g \in \mathscr{C}_s^{k,\alpha}(\mathbb{D})$ and $\rho \in \mathscr{C}_a^{k,\alpha}(\mathbb{T})$, then they satisfy the following properties

$$g(\overline{z}) = g(z), \quad \rho(\overline{w}) = -\rho(w), \quad \forall z \in \mathbb{D}, \forall\, w \in \mathbb{T}. \tag{2.2.8}$$

The space $\mathscr{C}_s^{k,\alpha}(\mathbb{D})$ will contain the perturbations of the initial radial density. The condition on g means that this perturbation is invariant by reflexion on the real axis. Let us remark that we introduce also a radial perturbation coming from the frequency $n = 0$, this fact will be a key point in the bifurcation argument.

The second kind of function spaces is $\mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{D})$, which is the set of holomorphic functions ϕ in $\mathbb{D}$ belonging to $\mathscr{C}^{k,\alpha}(\mathbb{D})$ and satisfying

$$\phi(0) = 0, \quad \phi'(0) = 0 \quad \text{and} \quad \overline{\phi(z)} = \phi(\overline{z}), \quad \forall z \in \overline{\mathbb{D}}.$$

With these properties, the function ϕ admits the following expansion

$$\phi(z) = z \sum_{n\geq 1} a_n z^n, \quad a_n \in \mathbb{R}. \tag{2.2.9}$$

Thus, we have

$$\mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{T}) := \Big\{\phi \in \mathscr{C}^{k,\alpha}(\mathbb{T}), \quad \phi(w) = w \sum_{n\geq 1} a_n w^n, \ a_n \in \mathbb{R}, \forall w \in \mathbb{T}\Big\}.$$

Notice that if $\Phi := \mathrm{Id} + \phi$ is conformal then $\Phi(\mathbb{D})$ is a simply connected domain, symmetric with respect to the real axis and whose boundary is $\mathscr{C}^{k,\alpha}$. The space $\mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{D})$ is a closed subspace of $\mathscr{C}^{k,\alpha}(\mathbb{D})$ equipped with the same norm, so it is complete. In the bifurcation argument, we will perturb also the initial domain $\mathbb{D}$ via a conformal map that will lie in this space.

The last condition on ϕ, given by (2.2.9), together with the symmetry condition for the density (2.2.8), means that we are looking for rotating initial data which admit at least one axis of symmetry. For the rotating patch problem this is the minimal requirement that we should impose and up to now we do not know whether such structures without any prescribed symmetry could exist.

Now, we introduce the following trace problem concerning the extension of Cauchy integrals, which is a classical result in complex analysis and potential theory. It is directly linked to [124, Proposition 3.4] and [132, Theorem 2.2]) and for the convenience of the reader we give a proof.

Lemma 2.2.2. *Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Denote by $\mathcal{C} : \mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{T}) \to \mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{D})$ the linear map defined by*

$$\phi(w) = w \sum_{n\in\mathbb{N}} a_n w^n, \quad \forall w \in \mathbb{T} \Longrightarrow \mathcal{C}\,(\phi) = \sum_{n\in\mathbb{N}} a_n z^n, \quad \forall z \in \overline{\mathbb{D}}.$$

Then, $\mathcal{C}\,(\phi)$ is well-defined and continuous.

Proof. First, it is a simple matter to check that the map $\mathcal{C}$ is well-defined. Thus, it remains to check the continuity. We recall from [132, Theorem 2.2]) the following estimates on the modulus of continuity

$$\sup_{\substack{z_1,z_2\in\mathbb{D}\\ |z_1-z_2|\leq\delta}} |\mathcal{C}(\phi)(z_1) - \mathcal{C}(\phi)(z_2)| \leq 3 \sup_{\substack{w_1,w_2\in\mathbb{T}\\ |w_1-w_2|\leq\delta}} |\phi(w_1) - \phi(w_2)|, \tag{2.2.10}$$

for any $\delta < \frac{\pi}{2}$ and for any continuous function $\mathcal{C}(\phi)$ in $\overline{\mathbb{D}}$, analytic in $\mathbb{D}$ and having trace function ϕ on the unit circle. Therefore, given $z_1, z_2 \in \mathbb{D}$ with $|z_1 - z_2| \leq 1$, we obtain

$$\begin{aligned}\frac{|\mathcal{C}(\phi)(z_1) - \mathcal{C}(\phi)(z_2)|}{|z_1 - z_2|^\alpha} \leq & 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq |z_1 - z_2|}} \frac{|\phi(w_1) - \phi(w_2)|}{|z_1 - z_2|^\alpha} \\ \leq & 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha} \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq |z_1 - z_2|}} \frac{|w_1 - w_2|^\alpha}{|z_1 - z_2|^\alpha} \\ \leq & 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha}.\end{aligned}$$

Now, let $z \in \mathbb{D}$ and $w \in \mathbb{T}$ such that $|z - w| \leq 1$, then we also get from (2.2.10)

$$|\mathcal{C}(\phi)(z)| \leq |\phi(w)| + 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} |\phi(w_1) - \phi(w_2)| \leq |\phi(w)| + 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha},$$

which implies that

$$\|\mathcal{C}(\phi)\|_{L^\infty(\overline{\mathbb{D}})} \leq 3\|\phi\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}.$$

Combining the preceding estimates, we deduce that

$$\|\mathcal{C}(\phi)\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \leq 6\|\phi\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}.$$

Note that this estimate can be extended to higher derivatives and thus we obtain

$$\|\mathcal{C}(\phi)\|_{\mathscr{C}^{k,\alpha}(\mathbb{D})} \leq 6\|\phi\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})},$$

which completes the proof. □

2.3 Boundary equation

This section focuses on studying the second equation (2.2.4) concerning the boundary equation and prove that we can parametrize the solutions in (Ω, f, Φ) close to the initial radial solution (f_0, Id), with f_0 being a radial profile, through a mapping $(\Omega, f) \mapsto \Phi = \mathcal{N}(\Omega, f)$. We will deal with the unknowns g and ϕ defined by

$$f = f_0 + g, \quad \Phi = \mathrm{Id} + \phi.$$

Equation (2.2.4) can be written in the following way

$$F(\Omega, g, \phi)(w) := \mathrm{Im}\left[\left(\Omega\, \overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y)\right)\Phi'(w) w\right] = 0, \qquad (2.3.1)$$

for any $w \in \mathbb{T}$. Notice that from this formulation we can retrieve the fact that

$$F(\Omega, 0, 0) = 0, \quad \forall\, \Omega \in \mathbb{R},$$

which is compatible with the fact that any radial initial data leads to a stationary solution of the Euler equations. Indeed, this identity follows from Proposition B.0.8 which implies that

$$\frac{1}{2\pi}\int_{\mathbb{D}} \frac{f_0(y)}{w - y} dA(y) = \overline{w}\int_0^1 s f_0(s) ds, \quad \forall\, w \in \mathbb{T}.$$

The idea to solve the nonlinear equation (2.3.1) is to apply the Implicit Function Theorem. Define the open balls

$$\begin{cases} B_{\mathscr{C}_s^{k,\alpha}}(g_0,\varepsilon) &= \Big\{ g\in \mathscr{C}_s^{k,\alpha}(\mathbb{D}) \quad \text{s.t.} \quad \|g-g_0\|_{k,\alpha}<\varepsilon\Big\}, \\ B_{\mathscr{H}\mathscr{C}^{k,\alpha}}(\phi_0,\varepsilon) &= \Big\{ \phi\in \mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{D}) \quad \text{s.t.} \quad \|\phi-\phi_0\|_{k,\alpha}<\varepsilon\Big\}, \end{cases} \tag{2.3.2}$$

for $\varepsilon>0$, $k\in\mathbb{N}$, $\alpha\in(0,1)$, $g_0\in\mathscr{C}_s^{k,\alpha}(\mathbb{D})$ and $\phi_0\in\mathscr{C}^{k,\alpha}(\mathbb{D})$. The first result concerns the well–definition and regularity of the functional F introduced in (2.3.1).

Proposition 2.3.1. *Let $\varepsilon\in(0,1)$, then $F:\mathbb{R}\times B_{\mathscr{C}_s^{1,\alpha}}(0,\varepsilon)\times B_{\mathscr{H}\mathscr{C}^{2,\alpha}}(0,\varepsilon)\longrightarrow \mathscr{C}_a^{1,\alpha}(\mathbb{T})$ is well-defined and of class $\mathscr{C}^1$.*

Remark 2.3.2. *If $\phi\in B_{\mathscr{H}\mathscr{C}^{k,\alpha}}(0,\varepsilon)$ with $\varepsilon<1$ and $k\in\mathbb{N}^\star$, then $\Phi=\text{Id}+\phi$ is conformal and bi-Lipschitz.*

Proof. Let us show that $F\in\mathscr{C}^{1,\alpha}(\mathbb{T})$. Since $\overline{\Phi},\Phi'\in\mathscr{C}^{1,\alpha}(\mathbb{T})$ it remains to study the integral term. This is a consequence of Lemma B.0.5 in Appendix B, which yields $F\in\mathscr{C}^{1,\alpha}(\mathbb{T})$.

Let us turn to the persistence of the symmetry. According to (2.2.8) one has to check that

$$F(\Omega,g,\phi)(\overline{w})=-F(\Omega,g,\phi)(w),\quad \forall\, w\in\mathbb{T}. \tag{2.3.3}$$

Using the symmetry properties of the density and the conformal mapping we write

$$\begin{aligned} F(\Omega,g,\phi)(\overline{w}) =&\text{Im}\left[\left(\Omega\overline{\Phi(\overline{w})}-\frac{1}{2\pi}\int_{\mathbb{D}}\frac{f(y)}{\Phi(\overline{w})-\Phi(y)}|\Phi'(y)|^2dA(y)\right)\Phi'(\overline{w})\overline{w}\right]\\ =&\text{Im}\left[\left(\Omega\Phi(w)-\frac{1}{2\pi}\int_{\mathbb{D}}\frac{f(y)}{\overline{\Phi(w)}-\Phi(y)}|\Phi'(y)|^2dA(y)\right)\overline{\Phi'(w)}\overline{w}\right]\\ =&\text{Im}\left[\left(\Omega\Phi(w)-\frac{1}{2\pi}\overline{\int_{\mathbb{D}}\frac{f(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2dA(y)}\right)\overline{\Phi'(w)}\overline{w}\right]. \end{aligned}$$

Therefore, we get (2.3.3). This concludes that $F(\Omega,g,\phi)$ is in $\mathscr{C}_a^{1,\alpha}(\mathbb{T})$. Notice that the dependance with respect to Ω is smooth and we will focus on the Gâteaux derivatives of F with respect to g and ϕ. Straightforward computations lead to

$$\begin{aligned} D_gF(\Omega,g,\phi)h(w) =& -\text{Im}\left[\frac{w\Phi'(w)}{2\pi}\int_{\mathbb{D}}\frac{h(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2dA(y)\right],\\ D_\phi F(\Omega,g,\phi)k(w) =&\text{Im}\Bigg[\Omega\overline{k(w)}\Phi'(w)w+\Omega\overline{\Phi(w)}k'(w)w\\ &-\frac{wk'(w)}{2\pi}\int_{\mathbb{D}}\frac{f(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2dA(y)\\ &+\frac{w\Phi'(w)}{2\pi}\int_{\mathbb{D}}\frac{k(w)-k(y)}{(\Phi(w)-\Phi(y))^2}f(y)|\Phi'(y)|^2dA(y)\\ &-\frac{w\Phi'(w)}{\pi}\int_{\mathbb{D}}\frac{f(y)}{\Phi(w)-\Phi(y)}\text{Re}\left[\overline{\Phi'(y)}k'(y)\right]dA(y)\Bigg]. \end{aligned} \tag{2.3.4}$$

Let us use the operator $\mathscr{F}[\Phi]$ defined in (B.0.11). Although in Lemma B.0.5 $\mathscr{F}[\Phi]$ is defined in $\mathbb{D}$ we can extend it up to the boundary $\overline{\mathbb{D}}$ getting the same result. Hence, all the above expressions can be written through this operator as

$$\begin{aligned}
D_gF(\Omega,g,\phi)h(w) =& -\operatorname{Im}\left[\frac{w\Phi'(w)}{2\pi}\mathscr{F}[\Phi](h)(w)\right],\\
D_\phi F(\Omega,g,\phi)k(w) =& \operatorname{Im}\Bigg[\Omega\overline{k(w)}\Phi'(w)w+\Omega\overline{\Phi(w)}k'(w)w-\frac{wk'(w)}{2\pi}\mathscr{F}[\Phi](f)(w)\\
&+\frac{w\Phi'(w)}{2\pi}\int_{\mathbb{D}}\frac{k(w)-k(y)}{(\Phi(w)-\Phi(y))^2}f(y)|\Phi'(y)|^2dA(y)\\
&-\frac{w\Phi'(w)}{\pi}\mathscr{F}[\Phi]\left(\frac{\operatorname{Re}\left[\overline{\Phi'(\cdot)}k'(\cdot)\right]}{|\Phi'(\cdot)|^2}\right)(w)\Bigg].
\end{aligned}$$

Since $\frac{\operatorname{Re}\left[\overline{\Phi'(\cdot)}k'(\cdot)\right]}{|\Phi'(\cdot)|^2}, \Phi', \overline{\Phi}, k' \in \mathscr{C}^{1,\alpha}(\mathbb{D})$ and are continuous with respect to Φ, Lemma B.0.5 entails that all the terms except the integral one lie in $\mathscr{C}^{1,\alpha}(\mathbb{D})$ and they are continuous with respect to Φ. The continuity with respect to f comes also from the same result. Note that although our unknowns are (g,ϕ), studying the continuity with respect to (g,ϕ) is equivalent to doing it with respect to (f,Φ). We shall now focus our attention on the integral term by splitting it as follows

$$\begin{aligned}
\int_{\mathbb{D}}\frac{k(w)-k(y)}{(\Phi(w)-\Phi(y))^2}f(y)|\Phi'(y)|^2dA(y) =& \int_{\mathbb{D}}\frac{(k(w)-k(y))(f(y)-f(w))}{(\Phi(w)-\Phi(y))^2}|\Phi'(y)|^2dA(y)\\
&+f(w)\int_{\mathbb{D}}\frac{k(w)-k(y)}{(\Phi(w)-\Phi(y))^2}|\Phi'(y)|^2dA(y)\\
=:& \mathscr{J}_1[\Phi]f(z)+f(w)\mathscr{J}_2[\Phi].
\end{aligned}$$

First, we deal with $\mathscr{J}_1[\Phi]$. Clearly

$$|\mathscr{J}_1[\Phi](w)|\leq C\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})},$$

and we define

$$\begin{aligned}
K(w,y) :=& \nabla_w\frac{(k(w)-k(y))(f(y)-f(w))}{(\Phi(w)-\Phi(y))^2}\\
=& -\frac{(k(w)-k(y))}{(\Phi(w)-\Phi(y))^2}\nabla_w f(w)+(f(y)-f(w))\nabla_w\frac{(k(w)-k(y))}{(\Phi(w)-\Phi(y))^2}\\
=:& -\nabla_w f(w)K_1(w,y)+K_2(w,y).
\end{aligned}$$

Using the same argument as in (B.0.13), we can check that K_1 and K_2 verify both the hypotheses of Lemma B.0.4. This implies that $\mathscr{J}_1[\Phi]$ lies in $\mathscr{C}^{1,\alpha}(\mathbb{D})$. Taking two conformal maps Φ_1 and Φ_2 and estimating $\mathscr{J}_1[\Phi_1]-\mathscr{J}_1[\Phi_2]$, we find integrals similar to those treated in Lemma B.0.4.

Concerning the second integral $\mathscr{J}_2[\Phi]$, which seems to be more singular, we use the Cauchy–Pompeiu's formula (B.0.10) to find

$$\int_{\mathbb{D}}\frac{k(w)-k(y)}{\Phi(w)-\Phi(y)}|\Phi'(y)|^2dA(y)=\frac{1}{2i}\int_{\mathbb{T}}\frac{k(w)-k(\xi)}{\Phi(w)-\Phi(\xi)}\overline{\Phi(\xi)}\Phi'(\xi)d\xi.$$

Differentiating it, we deduce

$$\int_{\mathbb{D}} \frac{k(w)-k(y)}{(\Phi(w)-\Phi(y))^2}|\Phi'(y)|^2 dA(y) = \frac{k'(w)\overline{\Phi(w)}\pi}{2\Phi'(w)} + \frac{1}{2i}\int_{\mathbb{T}} \frac{k(\xi)-k(w)}{(\Phi(\xi)-\Phi(w))^2}\overline{\Phi(\xi)}\Phi'(\xi)d\xi$$
$$=: \frac{k'(w)\overline{\Phi(w)}\pi}{2\Phi'(w)} + \frac{1}{2i}\mathscr{T}[\Phi](w).$$

The first term is in $\mathscr{C}^{1,\alpha}(\mathbb{T})$ and is clearly continuous with respect to Φ. Integration by parts in the second term $\mathscr{T}[\Phi]$ leads to

$$\mathscr{T}[\Phi](w) = -\int_{\mathbb{T}} \frac{k'(\xi)\overline{\Phi(\xi)}}{\Phi(\xi)-\Phi(w)}d\xi + \int_{\mathbb{T}} \frac{k(\xi)-k(w)}{\Phi(\xi)-\Phi(w)}\overline{\xi}^2\overline{\Phi'(\xi)}d\xi.$$

Differentiating and integrating it by parts again one obtain the following expression

$$\begin{aligned}
\mathscr{T}[\Phi]'(w) =& \Phi'(w)\int_{\mathbb{T}} \frac{\partial_\xi\left(\frac{k'(\xi)\overline{\Phi(\xi)}}{\Phi'(\xi)}\right)}{\Phi(\xi)-\Phi(w)}d\xi - \Phi'(w)\int_{\mathbb{T}} \frac{\partial_\xi\left(\frac{(k(\xi)-k(w))\overline{\xi}^2\overline{\Phi'(\xi)}}{\Phi'(\xi)}\right)}{\Phi(\xi)-\Phi(w)}d\xi\\
=& \Phi'(w)\int_{\mathbb{T}} \frac{\partial_\xi\left(\frac{k'(\xi)\overline{\Phi(\xi)}}{\Phi'(\xi)}\right)}{\Phi(\xi)-\Phi(w)}d\xi - \Phi'(w)\int_{\mathbb{T}} \frac{\frac{k'(\xi)\overline{\xi}^2\overline{\Phi'(\xi)}}{\Phi'(\xi)^2}}{\Phi(\xi)-\Phi(w)}\Phi'(\xi)d\xi\\
&- \Phi'(w)\int_{\mathbb{T}} \frac{(k(\xi)-k(w))\partial_\xi\left(\frac{\overline{\xi}^2\overline{\Phi'(\xi)}}{\Phi'(\xi)}\right)}{\Phi(\xi)-\Phi(w)}d\xi\\
=& \Phi'(w)\mathscr{I}[\Phi]\left(\frac{\partial\left(\frac{k'(\cdot)\overline{\Phi(\cdot)}}{\Phi'(\cdot)}\right)}{\Phi'(\cdot)}\right)(w) - \Phi'(w)\mathscr{I}[\Phi]\left(\frac{k'(\cdot)\overline{(\cdot)}^2\overline{\Phi'(\cdot)}}{\Phi'(\cdot)^2}\right)(w)\\
&- \Phi'(w)\mathscr{I}[\Phi]\left(k(\cdot)\partial\left(\frac{\overline{\cdot}^2\overline{\Phi'(\cdot)}}{\Phi'(\cdot)}\right)\right)(w) + \Phi'(w)k(w)\mathscr{I}[\Phi]\left(\partial\frac{\overline{(\cdot)}^2\overline{\Phi'(\cdot)}}{\Phi'(\cdot)}\right)(w),
\end{aligned}$$

where $\mathscr{I}[\Phi]$ is the operator defined in (B.0.20). Since the functions

$$\frac{\partial\left(\frac{k'(\cdot)\overline{\Phi(\cdot)}}{\Phi'(\cdot)}\right)}{\Phi'(\cdot)}, \quad \frac{k'(\cdot)\overline{(\cdot)}^2\overline{\Phi'(\cdot)}}{\Phi'(\cdot)^2}, \quad k, \quad \partial\left(\frac{\overline{(\cdot)}^2\overline{\Phi'(\cdot)}}{\Phi'(\cdot)}\right) \in \mathscr{C}^{0,\alpha}(\mathbb{D}),$$

and are continuous with respect to Φ, we can use Lemma B.0.7. Hence, these terms lie in $\mathscr{C}^{0,\alpha}(\mathbb{D})$. Moreover, the same argument gives us the continuity with respect to Φ. To conclude, we use the fact that the Gâteaux derivatives are continuous with respect to (g,ϕ) and so they are in fact Fréchet derivatives. □

The next task is to implement the Implicit Function Theorem in order to solve the boundary equation (2.3.1) through a two-parameters curve solutions in infinite-dimensional spaces. Given a radial function $f_0 \in \mathscr{C}^{1,\alpha}(\mathbb{D})$, we associate to it the singular set

$$\mathcal{S}_{\text{sing}} := \left\{\widehat{\Omega}_n := \int_0^1 sf_0(s)ds - \frac{n+1}{n}\int_0^1 s^{2n+1}f_0(s)ds, \quad \forall n \in \mathbb{N}^\star \cup \{+\infty\}\right\}. \tag{2.3.5}$$

This terminology will be later justified in the proof of the next proposition. Actually, this set corresponds to the location of the points Ω where the partial linearized operator $\partial_\phi F(\Omega,0,0)$ is not invertible. Let us establish the following result.

Proposition 2.3.3. *Let $f_0 : \overline{\mathbb{D}} \to \mathbb{R}$ be a radial function in $\mathscr{C}^{1,\alpha}(\mathbb{D})$. Let I be an open interval such that $\overline{I} \subset \mathbb{R}\backslash\mathcal{S}_{\text{sing}}$. Then, there exists $\varepsilon > 0$ and a $\mathscr{C}^1$ function*

$$\mathcal{N} : I \times B_{\mathscr{C}_s^{1,\alpha}}(0,\varepsilon) \longrightarrow B_{\mathscr{H}\mathscr{C}^{2,\alpha}}(0,\varepsilon),$$

with the following property:

$$F(\Omega, g, \phi) = 0 \Longleftrightarrow \phi = \mathcal{N}(\Omega, g),$$

for any $(\Omega, g, \phi) \in I \times B_{\mathscr{C}_s^{1,\alpha}}(0,\varepsilon) \times B_{\mathscr{H}\mathscr{C}^{2,\alpha}}(0,\varepsilon)$. In addition, we obtain the identity

$$D_g\mathcal{N}(\Omega, 0)h(z) = z\sum_{n\geq 1} A_n z^n,$$

for any $h \in \mathscr{C}_s^{1,\alpha}(\mathbb{D})$, with $h(re^{i\theta}) = \sum_{n\geq 0} h_n(r)\cos(n\theta)$, and

$$A_n = \frac{\displaystyle\int_0^1 s^{n+1}h_n(s)ds}{2n\big(\widehat{\Omega}_n - \Omega\big)}, \tag{2.3.6}$$

for any $n \geq 1$ where $\widehat{\Omega}_n$ is defined in (2.3.5)*. Moreover, we have*

$$\|\mathcal{N}(\Omega, 0)h\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})} \leq C\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}. \tag{2.3.7}$$

Remark 2.3.4. *From the definition of the function space $\mathscr{C}_s^{1,\alpha}(\mathbb{D})$ we are adding also a radial perturbation of the initial radial part given by the first mode $n = 0$. However, from the expression of $D_g\mathcal{N}(\Omega, 0)h(z)$ the first frequency disappears and the sum starts at $n = 1$. This is an expected fact because $(\Omega, g, 0)$ is a solution of $F(\Omega, g, \phi)$ for any radial smooth function g. This means that $\mathcal{N}(\Omega, g) = 0$, and hence $D_g\mathcal{N}(\Omega, 0)h$ is vanishing when h is radial.*

Proof. Applying the Implicit Function Theorem consists in checking that

$$D_\phi F(\Omega, 0, 0) : \mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D}) \longrightarrow \mathscr{C}_a^{1,\alpha}(\mathbb{T}),$$

is an isomorphism. A combination of (2.3.4) with Proposition B.0.8 allow us to compute explicitly the differential of $F(\Omega, g, \phi)$ on the initial solution. In fact, let $w \in \mathbb{D} \mapsto k(w) = w\sum_{n\geq 1} a_n w^n$ be a holomorphic function in $\mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$, then

$$\begin{aligned}
D_\phi F(\Omega, 0, 0)k(w) =& \text{Im}\Bigg[\Omega\overline{k(w)}w + \Omega\overline{w}k'(w)w - \frac{wk'(w)}{2\pi}\int_{\mathbb{D}}\frac{f_0(y)}{w-y}dA(y) \\
&+ \frac{w}{2\pi}\int_{\mathbb{D}}\frac{k(w)-k(y)}{(w-y)^2}f_0(y)dA(y) - \frac{w}{\pi}\int_{\mathbb{D}}\frac{f_0(y)}{w-y}\text{Re}[k'(y)]dA(y)\Bigg] \\
=& \sum_{n\geq 1} a_n\,\text{Im}\Bigg[\Omega\overline{w}^n + \Omega(n+1)w^n - (n+1)w^n\int_0^1 sf_0(s)ds \\
&+ w^n\int_0^1 sf_0(s)ds - (n+1)\overline{w}^n\int_0^1 s^{2n+1}f_0(s)ds\Bigg] \\
=& \sum_{n\geq 1} a_n n\left\{\Omega - \int_0^1 sf_0(s)ds + \frac{n+1}{n}\int_0^1 s^{2n+1}f_0(s)ds\right\}\sin(n\theta).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}D_gF(\Omega,0,0)h(w)=&-\operatorname{Im}\left[\frac{w}{2\pi}\int_{\mathbb{D}}\frac{h(y)}{w-y}dA(y)\right]\\=&-\frac{\pi}{2\pi}\sum_{n\geq 1}\operatorname{Im}\left[w\overline{w}^{n+1}\int_0^1 s^{n+1}h_n(s)ds\right]\\=&\frac{1}{2}\sum_{n\geq 1}\int_0^1 s^{n+1}h_n(s)ds\sin(n\theta),\end{aligned}\tag{2.3.8}$$

where

$$z\mapsto k(z)=\sum_{n\geq 1}a_nz^{n+1}\in\mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})\quad\text{and}\quad z\mapsto h(z)=\sum_{n\geq 0}h_n(r)\cos(n\theta)\in\mathscr{C}^{1,\alpha}_s(\mathbb{D}),$$

are given as in (2.2.9) and (2.2.7), respectively. Then, we have that $D_\phi F(\Omega,0,0)$ is one–to–one linear mapping and is continuous according to Proposition 2.3.1, for any $\Omega\in\overline{I}$. Using the Banach Theorem it suffices to check that this mapping is onto. Notice that at the formal level the inverse operator can be easily computed from the expression of $D_\phi F(\Omega,0,0)$ and it is given by

$$D_\phi F(\Omega,0,0)^{-1}\rho(z)=z\sum_{n\geq 1}\frac{\rho_n}{n\big(\Omega-\widehat{\Omega}_n\big)}z^n,\tag{2.3.9}$$

for any $\rho(e^{i\theta})=\sum_{n\geq 1}\rho_n\sin(n\theta)$. Thus the problem reduces to check that $D_\phi F(\Omega,0,0)^{-1}\rho\in\mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$. First, we will prove that this function is holomorphic inside the unit disc $\mathbb{D}$. For this purpose we use that

$$\rho_n=\frac{1}{\pi}\int_0^{2\pi}\rho(e^{i\theta})\sin(n\theta)d\theta.\tag{2.3.10}$$

Since $\rho\in L^\infty(\mathbb{T})$, we obtain that the coefficients sequences $(\rho_n)\in\ell^\infty$. Using the facts that $\lim_{n\to\infty}\widehat{\Omega}_n=\widehat{\Omega}_\infty$ and that Ω is far away from the singular set, then we deduce that the Fourier coefficients of $D_\phi F(\Omega,0,0)^{-1}\rho$ are bounded. Consequentely, this function is holomorphic inside the unit disc. It remains to check that this function belongs to $\mathscr{C}^{2,\alpha}(\mathbb{D})$. By virtue of Lemma 2.2.2 it is enough to check that the restriction on the boundary belongs to $\mathscr{C}^{2,\alpha}(\mathbb{T})$. First, we must notice that if $\rho\in\mathscr{C}^{1,\alpha}(\mathbb{T})$, then

$$w\mapsto\rho_+(w):=\sum_{n\geq 1}\rho_nw^n\in\mathscr{C}^{1,\alpha}(\mathbb{T}).$$

For this purpose, let us write ρ in the form

$$\rho(w)=\frac{1}{2i}\sum_{n\in\mathbb{Z}}\rho_nw^n,\quad\text{with}\quad\rho_{-n}=-\rho_n.$$

Hence, ρ_+ is nothing but the Szegö projection of ρ, which is continuous on $\mathscr{C}^{1,\alpha}(\mathbb{T})$. Note that this latter property is based upon the fact that ρ_+ can be expressed from ρ through the Cauchy integral operator

$$\rho_+(w)=\frac{1}{\pi}\int_{\mathbb{T}}\frac{\rho(\xi)}{\xi-w}d\xi,$$

and one may use $T(1)-$Theorem of Wittmann for Hölder spaces, see for instance [144, Theorem 2.1] and [84, page 10] or Lemma B.0.7. Secondly, we will prove that $D_\phi F(\Omega,0,0)^{-1}\rho\in\mathscr{C}^{2,\alpha}(\mathbb{T})$. We define

$$D_\phi F(\Omega,0,0)^{-1}\rho(w):=wq(w).$$

Let us show that q is bounded. Using (2.3.10) and integration by parts, we have

$$|\rho_n|\leq 2\frac{\|\rho'\|_{L^\infty(\mathbb{T})}}{n},$$

which implies that q is bounded. To prove higher regularity, we write q as a convolution

$$q=\rho_+\star K_1,$$

where

$$K_1(w)=\sum_{n\geq 1}\frac{w^n}{n\big(\Omega-\widehat{\Omega}_n\big)}.$$

Since $\rho_+\in\mathscr{C}^{1,\alpha}$, we just need to check that $K_1\in L^1$. To do that, we use the Parseval's identity, which provides that $K_1\in L^2(\mathbb{T})$:

$$\|K_1\|_2^2=\sum_{n\geq 1}\frac{1}{n^2\big(\Omega-\widehat{\Omega}_n\big)^2}\leq C\sum_{n\geq 1}\frac{1}{n^2}<+\infty,$$

where C is a constant connected to the distance between Ω and the singular set $\mathcal{S}_{sing}$ defined in (2.3.5). To study its derivative, let us write it as

$$q'(w)=\overline{w}\sum_{n\geq 1}\frac{\rho_n}{\Omega-\widehat{\Omega}_n}w^n=\overline{w}\left[\sum_{n\geq 1}\frac{\rho_n}{\beta}w^n+\sum_{n\geq 1}\rho_n\left(\frac{1}{\beta+u_n}-\frac{1}{\beta}\right)w^n\right]=:\overline{w}\left[\frac{1}{\beta}\rho_+(w)+S\right],$$

where

$$\beta=\Omega-\widehat{\Omega}_\infty,\quad u_n=\frac{n+1}{n}\int_0^1 s^{2n+1}f_0(s)ds.$$

From the foregoing discussion, we have seen that $\rho_+\in\mathscr{C}^{1,\alpha}(\mathbb{T})$. As to the term S, it can be written in convolution form

$$S=\rho_+\star K_2,$$

with

$$K_2(w)=-\sum_{n\geq 1}\frac{u_n}{(\beta+u_n)\beta}w^n.$$

Since $\rho_+\in\mathscr{C}^{1,\alpha}(\mathbb{T})$, then we just need to check that $K_2\in L^1(\mathbb{T})$ to conclude. Using Parseval's identity we have that $K_2\in L^2$ because

$$\begin{aligned}\|K_2\|_2^2=\frac{1}{\beta^2}\sum_{n\geq 1}\frac{u_n^2}{(\beta+u_n)^2}\leq & C\sum_{n\geq 1}\frac{(n+1)^2}{n^2}\left(\int_0^1 s^{2n+1}f_0(s)ds\right)^2\\ \leq & C\sum_{n\geq 1}\frac{(n+1)^2}{n^2(2n+1)^2}<+\infty.\end{aligned}$$

This achieves that $D_\phi F(\Omega,0,0)^{-1}\rho\in\mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$ and consequently the linearized operator $D_\phi F(\Omega,0,0)$ is an isomorphism. Hence, the Implicit Function Theorem can be used and it ensures the existence of a $\mathscr{C}^1$–function $\mathcal{N}$ such that

$$F(\Omega,g,\phi)=0\Longleftrightarrow\phi=\mathcal{N}(\Omega,g),$$

for any $(\Omega, g, \phi) \in I \times B_{\mathscr{C}_s^{1,\alpha}}(0,\varepsilon) \times B_{\mathscr{H}\mathscr{C}^{2,\alpha}}(0,\varepsilon)$. Differentiating with respect to g, we obtain

$$D_g F(\Omega, g, \mathcal{N}(\Omega, g)) = \partial_g F(\Omega, g, \phi) + \partial_\phi F(\Omega, g, \phi) \circ \partial_g \mathcal{N}(\Omega, g) = 0,$$

which yields

$$\partial_g \mathcal{N}(\Omega, 0) h(z) = -\partial_\phi F(\Omega, 0, 0)^{-1} \circ \partial_g F(\Omega, 0, 0) h(z).$$

Then, using (2.3.9) and (2.3.8), straightforward computations show that

$$D_g \mathcal{N}(\Omega, 0) h(z) = -z \sum_{n \geq 1} \frac{\int_0^1 s^{n+1} h_n(s) ds}{2n\big(\Omega - \widehat{\Omega}_n\big)} z^n.$$

This concludes the proof of the announced result. □

2.4 Density equation

This section aims at studying the density equation (2.2.3) in order to get non radial rotating solutions via the Crandall–Rabinowitz Theorem. We will reformulate it in a more convenient way since we are not able to use the original expression (2.2.3) due to the structural defect on its linearized operator as it has been pointed out previously. We must have in mind that under suitable assumptions, the conformal map is recovered from the angular velocity Ω and the density function via Proposition 2.3.3.

2.4.1 Reformulation of the density equation

Taking an initial data in the form (2.1.1) and noting that if the density f is fixed close to f_0 and Ω does not lie in the singular set $\mathcal{S}_{\text{sing}}$, then the conformal mapping is uniquely determined as a consequence of Proposition 2.3.3. Now, we turn to the analysis of the first equation of (2.1.3) that we intend to solve for a restricted class of initial densities. The strategy to implement it is to look for solutions satisfying the specific equation

$$\begin{aligned} \nabla(f \circ \Phi^{-1})(x) =& \mu\left(\Omega, (f \circ \Phi^{-1})(x)\right)(v(x) - \Omega x^\perp)^\perp \\ =& \mu\left(\Omega, (f \circ \Phi^{-1})(x)\right)(v^\perp(x) + \Omega x), \quad \forall x \in D, \end{aligned} \tag{2.4.1}$$

for some scalar function μ. One can easily check that any solution of (2.4.1) is a solution of the initial density equation (2.1.3) but the reversed is not in general true. Remark that from this latter equation we are looking for particular solutions due to the precise dependence of the scalar function μ with respect to f. The scalar function μ must be fixed in such a way that the radial profile f_0, around which we look for non trivial solutions, is also a solution of (2.4.1). Therefore, for any initial radial profile candidate to be bifurcated, we will obtain a different density equation. Notice also that it is not necessary in general to impose to μ to be well–defined on $\mathbb{R}$ but just on some open interval containing the image of $\overline{\mathbb{D}}$ by f_0.

Now, let us show how to construct concretely the function μ. By virtue of Proposition 2.3.3, the associated conformal map to any radial profile is the identity map. Therefore, it is obvious that a smooth radial profile f_0 is a solution of (2.4.1) if and only if

$$\begin{aligned} f_0'(r)\frac{z}{r} =& \mu(\Omega, f_0(z)) \left[-\frac{1}{2\pi}\int_{\mathbb{D}} \frac{z-y}{|z-y|^2} f_0(y) dA(y) + \Omega z\right] \\ =& \mu(\Omega, f_0(z)) \left[-\frac{1}{2\pi}\overline{\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y)} + \Omega z\right] \end{aligned}$$

$$=\mu(\Omega, f_0(z))\left[-\frac{1}{r^2}\int_0^r sf_0(s)ds+\Omega\right]z, \quad \forall z\in\mathbb{D},\, r=|z|,$$

where we have used the explicit computations given in Proposition B.0.8. Thus, we infer that the function μ must satisfy the compatibility condition

$$\mu(\Omega, f_0(r))=\frac{1}{r}\frac{f_0'(r)}{\Omega-\frac{1}{r^2}\int_0^r sf_0(s)ds}, \quad \forall\, r\in(0,1]. \tag{2.4.2}$$

We emphasize that not for all radial profiles f_0 we can find a function μ such that f_0 satisfies (2.4.2) In fact, we can violate this equation by working with non monotonic profiles. Taking f_0 verifying (2.4.2), let us go through the above procedure and see how to reformulate the density equation. Consider the function

$$\mathcal{M}_{f_0}(\Omega, \mathtt{t})=\int_{t_0}^{\mathtt{t}}\frac{1}{\mu(\Omega,s)}ds, \tag{2.4.3}$$

for some $t_0\in\mathbb{R}$. We use the subscript f_0 in order to stress that the above function depends on the choice of the initial profile f_0. This rigidity is very relevant in our study and enables us to include the structure of the solution into the formulation. By this way, we expect to remove the pathological behavior of the old formulation and to prepare the problem for the bifurcation arguments. From the expression of the velocity field, it is obvious that

$$v^{\perp}(x)+\Omega x=-\nabla\left(\frac{1}{2\pi}\int_D \log|x-y|(f\circ\Phi^{-1})(y)dA(y)-\frac{1}{2}\Omega|x|^2\right).$$

Since D is a simply connected domain, then integrating (2.4.1) yields to the equivalent form

$$\mathcal{M}_{f_0}\big(\Omega,(f\circ\Phi^{-1})(x)\big)+\frac{1}{2\pi}\int_D\log|x-y|(f\circ\Phi^{-1})(y)dA(y)-\frac{1}{2}\Omega|x|^2=\lambda, \quad \forall x\in D,$$

for some constant λ. Using a change of variable through the conformal map $\Phi:\mathbb{D}\to D$, we obtain the equivalent formulation in the unit disc

$$\mathcal{M}_{f_0}(\Omega, f(z))+\frac{1}{2\pi}\int_{\mathbb{D}}\log|\Phi(z)-\Phi(y)|f(y)|\Phi'(y)|^2dA(y)-\frac{1}{2}\Omega|\Phi(z)|^2=\lambda, \quad \forall z\in\mathbb{D}. \tag{2.4.4}$$

It remains to fix the constant λ by using that the initial radial profile should be a solution of (2.4.4). Thus the last integral identity in Proposition B.0.8 entails that

$$\lambda=\mathcal{M}_{f_0}(f_0(r))-\int_r^1\frac{1}{\tau}\int_0^\tau sf_0(s)dsd\tau-\frac{1}{2}\Omega r^2. \tag{2.4.5}$$

Notice that λ does not depend on r since f_0 verifies (2.4.2). Then, we finally arrive at the following reformulation for the density equation

$$G_{f_0}(\Omega, g,\phi)(z):=\mathcal{M}_{f_0}(\Omega, f(z))+\frac{1}{2\pi}\int_{\mathbb{D}}\log|\Phi(z)-\Phi(y)|f(y)|\Phi'(y)|^2dA(y)-\frac{\Omega}{2}|\Phi(z)|^2-\lambda=0,$$

for any $z\in\mathbb{D}$. The above expression yields

$$G_{f_0}(\Omega,0,0)=0, \quad \forall\Omega\in\mathbb{R}.$$

Thanks to Proposition 2.3.3, the conformal mapping is parametrized outside the singular set by Ω and g and thus the equation for the density becomes

$$\widehat{G}_{f_0}(\Omega, g) := G_{f_0}(\Omega, g, \mathcal{N}(\Omega, g)) = 0. \tag{2.4.6}$$

Next, let us analyze the constraint (2.4.2), for some particular examples. Since we are looking for smooth solutions, it is convenient to deal with smooth radial profiles. Then, one stands

$$f_0(r) = \widehat{f_0}(r^2),$$

and thus (2.4.2) becomes

$$\mu(\Omega, \widehat{f_0}(r)) = \frac{4r\widehat{f_0}'(r)}{2\Omega r - \int_0^r \widehat{f_0}(s)ds}, \quad \forall\, r \in (0,1]. \tag{2.4.7}$$

At this stage, there are two ways to proceed. The first one is to start with $\widehat{f_0}$ and reconstruct μ, and the second one is to impose μ and solve the nonlinear differential equation on $\widehat{f_0}$. This last approach is implicit and more delicate to implement. Therefore, let us proceed with the first approach and apply it to some special examples.

Quadratic profiles

The first example is the quadratic profile of the type

$$f_0(r) = Ar^2 + B,$$

where $A, B \in \mathbb{R}$. In this case, $\widehat{f_0}(r) = Ar + B$ and thus (2.4.7) agrees with

$$\mu(\Omega, \widehat{f_0}(r)) = \frac{4Ar}{2\Omega r - \frac{Ar}{2} - Br} = \frac{8A}{4\Omega - B - \widehat{f_0}(r)}, \quad \forall r \in (0,1].$$

Then, we find

$$\mu(\Omega, \mathtt{t}) = \frac{8A}{4\Omega - B - \mathtt{t}}, \tag{2.4.8}$$

which implies from (2.4.3) that

$$\mathcal{M}_{f_0}(\Omega, \mathtt{t}) = \frac{4\Omega - B}{8A}\mathtt{t} - \frac{1}{16A}\mathtt{t}^2.$$

Thus, using (2.4.5), we deduce that

$$\begin{aligned} \lambda &= \frac{4\Omega - B}{8A} f_0(r) - \frac{1}{16A} f_0(r)^2 - \int_r^1 \frac{1}{\tau}\int_0^\tau s f_0(s) ds d\tau - \frac{\Omega r^2}{2} \\ &= \frac{8\Omega B - 3B^2 - A^2 - 4AB}{16A}. \end{aligned} \tag{2.4.9}$$

As we have mentioned before, the conformal mapping is determined by Ω and g and so the last equation takes the form (2.4.6). The subscript f_0 will be omitted when we refer to this equation with the quadratic profile if there is no confusion.

Let us remark some comparison to the vortex patch problem. The case $A = 0$ agrees with a vortex patch of the type $f_0(r) = B$. It was mentioned before that the boundary equation studied in Section 2.3 is the one studied in [84] when analyzing the vortex patch problem. Here we have one more equation in $\mathbb{D}$ given by the density equation. This amounts to look for solutions of the type

$$\omega_0(x) = (B+g)\left(\Phi^{-1}(x)\right)\,\mathbf{1}_{\Phi(\mathbb{D})}(x),$$

which implies that the initial vorticity B of the vortex patch is perturbed by a function that could not to be constant. However, using (2.4.1) and evaluating in $f_0(r) = B$, for any $r \in [0,1]$, one gets that for this case $\mu \equiv 0$. Then if you perturb with g the equation to be studied is

$$\nabla((f_0+g)\circ\Phi^{-1})(x) = 0, \quad \forall x \in \Phi(\mathbb{D}).$$

Using the conformal map Φ and changing the variables we arrive at

$$\nabla((f_0+g)\circ\Phi^{-1})(\Phi(z)) = 0, \quad \forall z \in \mathbb{D}.$$

By virtue of (2.2.2), the above equation leads to

$$\nabla(f_0+g)(z) = 0,$$

which gives us that g must be a constant. Hence, using our approach we get that starting with a vortex patch we just can obtain another vortex patch solution.

Polynomial profiles

The second example is to consider a general polynomial profile of the type

$$\widehat{f_0}(r) = Ar^m + B, \quad m \in \mathbb{N}^\star, \quad A \geq 0, \quad B \in \mathbb{R}.$$

From (2.4.7) we obtain that

$$\mu(\Omega, \mathtt{t}) = 4m(m+1)A^{\frac{1}{m}} \frac{(\mathtt{t}-B)^{\frac{m-1}{m}}}{2\Omega(m+1) - mB - \mathtt{t}}, \quad \forall \mathtt{t} \geq B.$$

Consequently, we find that

$$\mathcal{M}_{f_0}(\Omega, \mathtt{t}) = \frac{1}{4m(m+1)A^{\frac{1}{m}}}\left[(2\Omega(m+1) - mB)\int_B^{\mathtt{t}} (s-B)^{\frac{1-m}{m}} ds - \int_B^{\mathtt{t}} s(s-B)^{\frac{1-m}{m}} ds\right].$$

Remark that for $m = 1$ we recover the previous quadratic profiles. The discussion developed later about the quadratic profile can be also extended to this polynomial profile as we shall comment in detail in Remark 2.5.9.

Gaussian profiles

Another example which is relevant is given by the Gaussian distribution,

$$\widehat{f_0}(r) = e^{Ar}, \quad A \in \mathbb{R}^\star.$$

Inserting $\widehat{f_0}$ into (2.4.7) one obtains that

$$\mu(\Omega, \widehat{f_0}(r)) = \frac{4A^2 r}{2\Omega Ar - e^{Ar} + 1},$$

and thus

$$\mu(\Omega, \mathtt{t}) = \frac{4A\ln\mathtt{t}}{1 - \mathtt{t} + 2\Omega\ln\mathtt{t}}.$$

Then the formula (2.4.3) allows us to get

$$\mathcal{M}_{f_0}(\Omega, \mathtt{t}) = \frac{1}{4A}\left[2\Omega\mathtt{t} + \int_1^{\mathtt{t}} \frac{1-s}{\ln s} ds\right].$$

2.4.2 Functional regularity

In this section, we will be interested in the regularity of the functional $\widehat{G}$ obtained in (2.4.6) for a the quadratic profile. Notice that in the case of the quadratic profile (2.1.6), the singular set (2.3.5) becomes

$$\mathcal{S}_{\text{sing}} = \left\{ \widehat{\Omega}_n := \frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad \forall n \in \mathbb{N}^\star \cup \{+\infty\} \right\}. \tag{2.4.10}$$

Proposition 2.4.1. *Let f_0 be the quadratic profile given by* (2.1.6) *and I be an open interval with $\overline{I} \subset \mathbb{R} \backslash \mathcal{S}_{\text{sing}}$. Then, there exists $\varepsilon > 0$ such that*

$$\widehat{G} : I \times B_{\mathscr{C}_s^{1,\alpha}(\mathbb{D})}(0, \varepsilon) \to \mathscr{C}_s^{1,\alpha}(\mathbb{D}),$$

is well–defined and of class $\mathscr{C}^1$, where $\widehat{G}$ is defined in (2.4.6) *and $B_{\mathscr{C}_s^{1,\alpha}(\mathbb{D})}(0, \varepsilon)$ in* (2.3.2).

Proof. Let us show that $\widehat{G}(\Omega, g) \in \mathscr{C}^{1,\alpha}(\mathbb{D})$. Clearly, $\mathcal{M}(\Omega, f)$ is polynomial in f and by the algebra structure of Hölder spaces we deduce that $\mathcal{M}(\Omega, f) \in \mathscr{C}^{1,\alpha}(\mathbb{D})$. Since $\Phi \in \mathscr{C}^{2,\alpha}(\mathbb{D})$, then the only term which deserves attention is the integral one. It is clear that

$$\int_{\mathbb{D}} \log |\Phi(\cdot) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) \in \mathscr{C}^0(\mathbb{D}).$$

To estimate its derivative, we note that

$$\nabla_z \log |\Phi(z) - \Phi(y)| = \frac{(\Phi(z) - \Phi(y)) \overline{\Phi'(z)}}{|\Phi(z) - \Phi(y)|^2} = \frac{\overline{\Phi'(z)}}{\overline{\Phi(z)} - \overline{\Phi(y)}},$$

which implies that

$$\begin{aligned}\nabla_z \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) =& \overline{\Phi'(z)} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}} |\Phi'(y)| dA(y) \\ =& \overline{\Phi'(z)}\, \overline{\mathscr{F}[\Phi](f)(z)},\end{aligned}$$

where the operator $\mathscr{F}[\Phi]$ is defined in (B.0.11). Thus, we can use Lemma B.0.5 obtaining that $\mathscr{F}[\Phi]$ belongs to $\mathscr{C}^{1,\alpha}(\mathbb{D})$. Since $\Phi' \in \mathscr{C}^{1,\alpha}(\mathbb{D})$ we deduce that the integral term of $\widehat{G}(\Omega, g)$ lies in the space $\mathscr{C}^{2,\alpha}(\mathbb{D})$ and is continuous with respect to (f, Φ).

Let us check the symmetry property. Take g and ϕ satisfying $g(\overline{z}) = g(z)$ and $\phi(\overline{z}) = \overline{\phi(z)}$. It is a simple matter to verify that

$$\mathcal{M}(\Omega, f(\overline{z})) = \mathcal{M}(\Omega, f(z)) \quad \text{and} \quad |\Phi(\overline{z})|^2 = |\Phi(z)|^2, \quad \forall\, z \in \overline{\mathbb{D}}.$$

For the Newtonian potential, the change of variables $y \mapsto \overline{y}$ leads to

$$\begin{aligned}\frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(\overline{z}) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) =& \frac{1}{2\pi} \int_{\mathbb{D}} \log |\overline{\Phi(z)} - \Phi(\overline{y})| f(\overline{y}) |\Phi'(\overline{y})|^2 dA(y) \\ =& \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y).\end{aligned}$$

Let us turn to the computations of the Gâteaux derivatives, that can be computed as

$$D_g \widehat{G}(\Omega, g) h(z) = \partial_g G(\Omega, g, \phi) h(z) + \partial_\phi G(\Omega, g, \phi) \circ \partial_g \mathcal{N}(\Omega, g) h(z).$$

By virtue of Proposition 2.3.3, it is known that $\mathcal{N}$ is $\mathscr{C}^1$, which implies that $\partial_g \mathcal{N}(\Omega, g)$ is continuous. Gâteaux derivatives are given by

$$\begin{aligned} D_g G(\Omega, g, \phi) h(z) &= D_g \mathcal{M}(\Omega, f(z)) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| h(y) |\Phi'(y)|^2 dA(y) \\ &= \frac{4\Omega - B}{8A} h(z) - \frac{1}{8A} f(z) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| h(y) |\Phi'(y)|^2 dA(y), \end{aligned}$$

and

$$\begin{aligned} D_\phi G(\Omega, g, \phi) k(z) =& \frac{\mathrm{Re}}{2\pi} \int_{\mathbb{D}} \frac{k(z) - k(y)}{\Phi(z) - \Phi(y)} f(y) |\Phi'(y)|^2 dA(y) - \Omega \mathrm{Re} \left[\overline{\Phi}(z) k(z) \right] \\ &+ \frac{1}{\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) \mathrm{Re} \left[\overline{\Phi'(y)} k'(y) \right] dA(y). \end{aligned}$$

We focus our attention on the integral terms. The operator $\mathscr{F}[\Phi]$ in (B.0.11) allows us to write

$$\int_{\mathbb{D}} \frac{k(\cdot) - k(y)}{\Phi(\cdot) - \Phi(y)} f(y) |\Phi'(y)|^2 dA(y) = k(z) \mathscr{F}[\Phi](f) - \mathscr{F}[\Phi](kf)(z).$$

Thus, Lemma B.0.5 concludes that this term lies in $\mathscr{C}^{1,\alpha}(\mathbb{D})$ and is continuous with respect to Φ. For the other terms involving the logarithm, we can compute its gradient as before; for instance

$$\begin{aligned} &\nabla_z \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) \mathrm{Re} \left[\overline{\Phi'(y)} k'(y) \right] dA(y) \\ &= \overline{\Phi'(z)} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}} \mathrm{Re} \left[\overline{\Phi'(y)} k'(y) \right] dA(y) = \overline{\Phi'(z)}\, \overline{\mathscr{F}[\Phi]} \left(\frac{\mathrm{Re} \left[\overline{\Phi'(\cdot)} k'(\cdot) \right]}{|\Phi'(\cdot)|^2} \right) (z). \end{aligned}$$

Since $\frac{\mathrm{Re}\left[\overline{\Phi}'(\cdot) k'(\cdot)\right]}{|\Phi'(\cdot)|^2} \in \mathscr{C}^{1,\alpha}(\mathbb{D})$ and is continuous with respect to Φ, Lemma B.0.5 concludes that this term lies in $\mathscr{C}^{1,\alpha}(\mathbb{D})$ and is continuous with respect to Φ. For the other terms involving a logarithmic part, the same procedure can be done. Trivially, both $D_g G(\Omega, g, \phi)$ and $D_\phi G(\Omega, g, \phi)$ are continuous with respect to g. We have obtained that the Gateaux derivatives are continuous with respect to (g, ϕ) and hence they are Fréchet derivatives. □

Remark 2.4.2. *Although we have done the previous discussion for the quadratic profile, the same argument may be applied for any radial profile f_0. Note that the only difference with the quadratic profile is that the function $\mathcal{M}_{f_0}$ and the constant λ_{f_0} will depend on f_0. Hence, we just have to study the regularity of function $\mathcal{M}_{f_0}$ in order to give a similar result.*

2.4.3 Radial solutions

The main goal of this section is the resolution of Equation (2.4.6) in the class of radial functions but in a small neighborhood of the quadratic profiles (2.1.6). We establish that except for one singular value for Ω, no radial solutions different from f_0 may be found around it. This discussion is essential in order to ensure that with the new reformulation we avoid the main defect of the old one (2.1.2): the kernel is infinite-dimensional and contains radial solutions. As it was observed before, Proposition 2.3.3 gives us that the associated conformal mapping of any radial function is the identity map, and therefore (2.4.6) becomes

$$\widehat{G}(\Omega, f - f_0)(z) = \frac{4\Omega - B}{8A} f(|z|) - \frac{1}{16A} f^2(|z|) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| f(|y|) dA(y) - \frac{\Omega |z|^2}{2} - \lambda = 0,$$

for any $z \in \mathbb{D}$ where λ is given by (2.4.9). Thus the last integral identity of Proposition B.0.8 gives

$$\frac{4\Omega - B}{8A} f(r) - \frac{1}{16A} f^2(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s f(s) ds d\tau - \frac{1}{2}\Omega r^2 - \lambda = 0, \quad \forall r \in [0,1].$$

Introduce the function $G_{\text{rad}} : \mathbb{R} \times \mathscr{C}([0,1];\mathbb{R}) \to \mathscr{C}([0,1];\mathbb{R})$, defined by

$$G_{\text{rad}}(\Omega, f)(r) = \frac{4\Omega - B}{8A} f(r) - \frac{1}{16A} f^2(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s f(s) ds d\tau - \frac{\Omega r^2}{2} - \lambda, \quad \forall r \in [0,1].$$

It is obvious that G_{rad} is well–defined and furthermore it satisfies

$$G_{\text{rad}}(\Omega, f_0) = 0, \quad \forall\, \Omega \in \mathbb{R}. \tag{2.4.11}$$

Through this chapter, it will be more convenient to work with the variable x instead of Ω defined as

$$\frac{1}{x} = \frac{4}{A}\left(\Omega - \frac{B}{2}\right). \tag{2.4.12}$$

Before stating our result, some properties of the hypergeometric function $x \in (-1,1) \mapsto F(1-\sqrt{2}, 1+\sqrt{2}; 1; x)$, are needed. A brief account on some useful properties of Gauss hypergeometric functions will be discussed later in the Appendix C. In view of (C.0.8) we obtain the identity

$$F(1-\sqrt{2}, 1+\sqrt{2}; 1; x) = \frac{1}{1-x} F(-\sqrt{2}, \sqrt{2}; 1; x). \tag{2.4.13}$$

According to Appendix C, we have $F(1-\sqrt{2}, 1+\sqrt{2}; 1; 0) = 1$, and it diverges to $-\infty$ at 1. This implies that there is at least one root in $(0,1)$. Combined with the fact that its derivative is negative according to (C.0.4), we may show that this root is unique. Denote this zero by $x_0 \in (0,1)$ and set

$$\Omega_0 := \frac{B}{2} + \frac{A}{4x_0}. \tag{2.4.14}$$

Setting the ball

$$B(f_0, \varepsilon) = \{ f \in \mathscr{C}([0,1];\mathbb{R}), \quad \|f - f_0\|_{L^\infty} \le \varepsilon \},$$

for any $\varepsilon > 0$, the first result can be stated as follows.

Proposition 2.4.3. *Let f_0 be the quadratic profile* (2.1.6), *with $A \in \mathbb{R}^\star, B \in \mathbb{R}$ and I be any bounded interval with $I \cap \left(\left[\frac{B}{2}, \frac{B}{2} + \frac{A}{4}\right] \cup \{\Omega_0\}\right) = \varnothing$. Then, there exists $\varepsilon > 0$ such that*

$$G_{\text{rad}}(\Omega, f) = 0 \Longleftrightarrow f = f_0,$$

for any $(\Omega, f) \in I \times B(f_0, \varepsilon)$.

Proof. We remark that G_{rad} is a $\mathscr{C}^1$ function on (Ω, f). The idea is to apply the Implicit Function Theorem to deduce the result. By differentiation with respect to f, one gets that

$$\begin{aligned} D_f G_{\text{rad}}(\Omega, f_0) h(r) =& \frac{4\Omega - B}{8A} h(r) - \frac{1}{8A} f_0(r) h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h(s) ds d\tau \\ =& \frac{4\Omega - Ar^2 - 2B}{8A} h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h(s) ds d\tau, \end{aligned} \tag{2.4.15}$$

for any $h \in \mathscr{C}([0,1];\mathbb{R})$. Now we shall look for the kernel of this operator, which consists of elements h solving a Volterra integro-differential equation of the type

$$\frac{4\Omega - Ar^2 - 2B}{8A} h(r) - \int_r^1 \frac{1}{\tau}\int_0^\tau sh(s)dsd\tau = 0, \quad \forall r \in [0,1].$$

The assumption $\Omega \notin \left[\frac{B}{2}, \frac{B}{2} + \frac{A}{4}\right]$ implies that $r \in [0,1] \mapsto \frac{4\Omega - Ar^2 - 2B}{8A}$ is not vanishing and smooth. Thus from the regularization of the integral, one can check that any element of the kernel is actually $\mathscr{C}^\infty$. Our purpose is to derive a differential equation by differentiating successively this integral equation. With the notation (2.4.12) the kernel equation can be written in the form

$$\mathcal{L}h(r) := \left(\frac{1}{x} - r^2\right) h(r) - 8\int_r^1 \frac{1}{\tau}\int_0^\tau sh(s)dsd\tau = 0, \quad \forall r \in [0,1].$$

Remark that the assumptions on Ω can be translated into x, as $x \in (-\infty, 1)$ and $x \neq 0$. Differentiating the function $\mathcal{L}h$ yields

$$(\mathcal{L}h)'(r) = \left(\frac{1}{x} - r^2\right) h'(r) - 2rh(r) + \frac{8}{r}\int_0^r sh(s)ds.$$

Multiplying by r and differentiating again we deduce that

$$\frac{[r(\mathcal{L}h)'(r)]'}{r} = \left(\frac{1}{x} - r^2\right) h''(r) + \left(\frac{1}{x_0} - 5r^2\right)\frac{h'(r)}{r} + 4h(r) = 0, \quad \forall r \in (0,1). \tag{2.4.16}$$

In order to solve the above equation we look for solutions in the form

$$h(r) = \rho(x\, r^2).$$

This ansatz can be justified *a posteriori* by evoking the uniqueness principle for ODEs. Doing the change of variables $y = x\, r^2$, we transform the preceding equation to

$$y(1-y)\rho''(y) + (1-3y)\rho'(y) + \rho(y) = 0.$$

Appendix C leads to assure that the only bounded solutions close to zero to this hypergeometric equation are given by

$$\rho(y) = \gamma\ F(1+\sqrt{2}, 1-\sqrt{2}; 1; y), \quad \forall \gamma \in \mathbb{R},$$

and thus

$$h(r) = \gamma\ F(1+\sqrt{2}, 1-\sqrt{2}; 1; x\, r^2), \quad \forall \gamma \in \mathbb{R}. \tag{2.4.17}$$

It is important to note that from the integral representation (C.0.2) of hypergeometric functions, we can extend the above solution to $x \in (-\infty, 1)$. Coming back to the equation (2.4.16) and integrating two times, we obtain two real numbers $\alpha, \beta \in \mathbb{R}$ such that

$$\mathcal{L}h(r) = \alpha \ln r + \beta, \quad \forall r \in (0,1].$$

Since $\mathcal{L}h \in \mathscr{C}([0,1],\mathbb{R})$, we obtain that $\alpha = 0$ and thus $\mathcal{L}h(r) = \beta$. By definition one has $\mathcal{L}h(1) = \left(\frac{1}{x} - 1\right) h(1)$. The fact that $x \neq 1$ implies that $\mathcal{L}h = 0$ if and only if $h(1) = 0$. According to (2.4.17), this condition is equivalent to $\gamma F(1-\sqrt{2}, 1+\sqrt{2}; 1; x) = 0$. It follows that the kernel is trivial ($\gamma = 0$) if and only if $x \neq x_0$, with x_0 being the only zero of $F(1-\sqrt{2}, 1+\sqrt{2}; 1; \cdot)$.

However, for $x = x_0$ the kernel is one-dimensional and is generated by this hypergeometric function. Those claims will be made more rigorous in what follows.

• **Case** $x \neq x_0$. As we have mentioned before, the kernel is trivial and it remains to check that $\mathcal{L}$ is an isomorphism. With this aim, it suffices to prove that $\mathcal{L}$ is a Fredholm operator of zero index. First, we can split $\mathcal{L}$ as follows

$$\mathcal{L} := \mathcal{L}_0 + \mathcal{K}, \quad \mathcal{L}_0 := \left(\frac{1}{x} - r^2\right)\mathrm{Id} \quad \text{and} \quad \mathcal{K}\,h(r) := -8\int_r^1 \frac{1}{\tau}\int_0^\tau sh(s)dsd\tau.$$

Second, it is obvious that $\mathcal{L}_0 : \mathscr{C}([0,1];\mathbb{R}) \to \mathscr{C}([0,1];\mathbb{R})$ is an isomorphism, it is a Fredholm operator of zero index. Now, since the Fredholm operators with given index are stable by compact perturbation (for more details, see Appendix A), then to check that $\mathcal{L}$ has zero index it is enough to establish that

$$\mathcal{K} : \mathscr{C}([0,1];\mathbb{R}) \to \mathscr{C}([0,1];\mathbb{R}),$$

is compact. One can easily obtain that for $h \in \mathscr{C}([0,1];\mathbb{R})$ the function $\mathcal{K}h$ belongs to $\mathscr{C}^1([0,1];\mathbb{R})$. Furthermore, by change of variables

$$(\mathcal{K}h)'(r) = r\int_0^1 sh(rs)ds \quad \text{and} \quad (\mathcal{K}h)'(0) = 0, \quad \forall r \in (0,1],$$

which implies that

$$\|\mathcal{K}h\|_{\mathscr{C}^1} \leq C\|h\|_{L^\infty}.$$

Since the embedding $\mathscr{C}^1([0,1];\mathbb{R}) \hookrightarrow \mathscr{C}([0,1];\mathbb{R})$ is compact, we find that $\mathcal{K}$ is a compact operator. Finally, we get that $\mathcal{L}$ is an isomorphism. This ensures that $D_f G_{\text{rad}}(\Omega, f_0)$ is an isomorphism, and therefore the Implicit Function Theorem together with (2.4.11) allow us to deduce that the only solutions of $G(\Omega, f) = 0$ in $I \times B(f_0, \varepsilon)$ are given by the trivial ones $\{(\Omega, f_0), \Omega \in I\}$.

• **Case** $x = x_0$. In this special case the kernel of $\mathcal{L}$ is one–dimensional and is generated by

$$\mathrm{Ker}\mathcal{L} = \langle F(1-\sqrt{2}, 1+\sqrt{2}; 1; x\,(\cdot)^2)\rangle. \tag{2.4.18}$$

This case will be deeply discussed below in Proposition 2.4.4. □

Let us focus on the case $\Omega = \Omega_0$. From (2.4.18), the kernel of the linearized operator is one–dimensional, and we will be able to implement the Crandall–Rabinowitz Theorem. Our result reads as follows.

Proposition 2.4.4. *Let f_0 be the quadratic profile* (2.1.6) *with $A \in \mathbb{R}^\star$, $B \in \mathbb{R}$ and fix Ω_0 as in* (2.4.14). *Then, there exists an open neighborhood U of (Ω_0, f_0) in $\mathbb{R} \times \mathscr{C}([0,1];\mathbb{R})$ and a continuous curve $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi) \in U$ with $a > 0$ such that*

$$G_{\text{rad}}(\Omega_\xi, f_\xi) = 0, \quad \forall \xi \in (-a, a).$$

Proof. We must check that the hypotheses of the Crandall–Rabinowitz Theorem are achived. It is clear that

$$G_{\text{rad}}(\Omega, f_0) = 0, \quad \forall\, \Omega \in \mathbb{R}.$$

It is not difficult to show that the mapping $(\Omega, f) \mapsto G_{\text{rad}}(\Omega, f)$ is $\mathscr{C}^1$. In addition, we have seen in the foregoing discussion that $D_f G_{\text{rad}}(\Omega_0, f_0)$ is a Fredholm operator with zero index and its kernel is one–dimensional. Therefore to apply the bifurcation arguments it remains just to check the transversality condition in the Crandall-Rabinowitz Theorem. Having this in mind, we should first find a practical characterization for the range of the linearized operator. We note that an element $d \in \mathscr{C}([0,1];\mathbb{R})$ belongs to the range of $D_f G_{\text{rad}}(\Omega_0, f_0)$ if the equation

$$\frac{\frac{1}{x_0}-r^2}{8}h(r)-\int_r^1\frac{1}{\tau}\int_0^\tau sh(s)dsd\tau=d(r),\quad \forall r\in[0,1], \tag{2.4.19}$$

admits a solution h in $\mathscr{C}([0,1];\mathbb{R})$, where x_0 is given by (2.4.14). Consider the auxiliary function H

$$H(r):=\int_r^1\frac{1}{\tau}\int_0^\tau sh(s)dsd\tau,\quad \forall\, r\in[0,1].$$

Then H belongs to $\mathscr{C}^1([0,1];\mathbb{R})$ and it satisfies the boundary condition

$$H(1)=0\quad\text{and}\quad H'(0)=0. \tag{2.4.20}$$

In order to write down an ordinary differential equation for H, let us define the linear operator

$$\mathcal{L}h(r):=\int_r^1\frac{1}{\tau}\int_0^\tau sh(s)dsd\tau,\quad \forall r\in[0,1],$$

for any $h\in\mathscr{C}([0,1];\mathbb{R})$. Then, we derive successively,

$$(\mathcal{L}h)'(r)=-\frac{1}{r}\int_0^r sh(s)ds,$$

and

$$r(\mathcal{L}h)''(r)+(\mathcal{L}h)'(r)=-rh(r).$$

Applying this identity to H one arrives at

$$rH''(r)+H'(r)=-\frac{8rx_0}{1-x_0r^2}\left[H(r)+d(r)\right].$$

Thus, H solves the second order differential equation

$$r(1-x_0r^2)H''(r)+(1-x_0r^2)H'(r)+8rx_0H(r)=-8rx_0d(r), \tag{2.4.21}$$

supplemented with boundary conditions (2.4.20). The argument is to come back to the original equation (2.4.19) and show that the candidate

$$h(r):=\frac{8x_0}{1-x_0r^2}\left[H(r)+d(r)\right],$$

is actually a solution to this equation. Then, we need to check that $\mathcal{L}h=H$. By setting $\mathscr{H}:=\mathcal{L}h-H$, we deduce that

$$r(\mathscr{H})''(r)+(\mathscr{H})'(r)=0,$$

with the boundary conditions $\mathscr{H}(1)=0$ and $\mathscr{H}'(0)=0$, which come from (2.4.20). The solution of this differential equation is

$$\mathscr{H}(r)=\lambda_0+\lambda_1\ln r,\quad \forall\lambda_0,\lambda_1\in\mathbb{R}$$

Since $\mathcal{L}h$ and H belong to $\mathscr{C}([0,1];\mathbb{R})$, then necessarily $\lambda_1 = 0$, and from the boundary condition we find $\lambda_0 = 0$. This implies that $\mathcal{L}h = H$, and it shows finally that solving (2.4.19) is equivalent to solving (2.4.21). Now, let us focus on the resolution of (2.4.21). For this purpose, we proceed by finding a particular solution for the homogeneous equation and use later the method of variation of constants. Looking for a solution to the homogeneous equation in the form $\mathcal{H}_0(r) = \rho(x_0r^2)$, and using the variable $y = x_0r^2$, one arrives at

$$(1-y)y\rho''(y) + (1-y)\rho'(y) + 2\rho(y) = 0.$$

This is a hypergeometric equation, and one solution is given by $y \mapsto F(-\sqrt{2},\sqrt{2};1;y)$. Thus, a particular solution to the homogeneous equation is $\mathcal{H}_0 : r \in [0,1] \mapsto F(-\sqrt{2},\sqrt{2};1;x_0r^2)$. Then, the general solutions for (2.4.21) are given through the formula

$$H(r) = \mathcal{H}_0(r)\left[K_2 + \int_\delta^r \frac{1}{\tau\mathcal{H}_0^2(\tau)}\left[K_1 - 8x_0\int_0^\tau \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)ds\right]d\tau\right], \quad \forall r \in [0,1],$$

where K_1, K_2 are real constants and $\delta \in (0,1)$ is any given number. Since $\mathcal{H}_0$ is smooth on the interval $[0,1]$, with $\mathcal{H}_0(0) = 1$, one can check that H admits a singular term close to zero taking the form $K_1 \ln r$. This forces K_1 to vanish because H is continuous up to the origin. Therefore, we infer that

$$H(r) = \mathcal{H}_0(r)\left[K_2 - 8x_0\int_\delta^r \frac{1}{\tau\mathcal{H}_0^2(\tau)}\int_0^\tau \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)dsd\tau\right], \quad \forall r \in [0,1].$$

The last integral term is convergent at the origin and one may take $\delta = 0$. This implies that

$$H(r) = \mathcal{H}_0(r)\left[K_2 - 8x_0\int_0^r \frac{1}{\tau\mathcal{H}_0^2(\tau)}\int_0^\tau \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)dsd\tau\right], \quad \forall r \in [0,1].$$

From this expression, we deduce the second condition of (2.4.20). For the first condition, $H(1) = 0$, we first note from (2.4.13) that $F(-\sqrt{2},\sqrt{2};1;x_0) = 0$. Then, we can compute the limit at $r = 1$ via l'Hôpital's rule leading to

$$H(1) = -8x_0\lim_{r\to 1^-}\mathcal{H}_0(r)\int_0^r \frac{1}{\tau\mathcal{H}_0^2(\tau)}\int_0^\tau \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)dsd\tau = 8x_0\frac{\displaystyle\int_0^1 \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)ds}{\mathcal{H}_0'(1)}.$$

From the expression of $\mathcal{H}_0$ and (C.0.4) we recover that $\mathcal{H}_0'(1) = -4x_0F(1-\sqrt{2},1+\sqrt{2};2;x_0)$. We point out that this quantity is not vanishing. This can be proved by differentiating the relation (2.4.13), which implies that

$$F(1-\sqrt{2},1+\sqrt{2};2;x_0) = (x_0-1)F(2-\sqrt{2},2+\sqrt{2};2;x_0),$$

and the latter term is not vanishing from the definition of hypergeometric functions. It follows that the condition $H(1) = 0$ is equivalent to

$$\int_0^1 \frac{s\mathcal{H}_0(s)}{1-x_0s^2}d(s)ds = 0. \tag{2.4.22}$$

This characterizes the elements of the range of the linearized operator. Now, we are in a position to check the transversality condition. According to the expression (2.4.15), one gets by differentiating with respect to Ω that

$$D_{\Omega,f}G_{\text{rad}}(\Omega_0,f_0)h(r) = \frac{1}{2A}h(r).$$

Recall from (2.4.18) and the relation (2.4.13) that the kernel is generated by $r \in [0,1] \mapsto \frac{\mathscr{H}_0(r)}{1-x_0r^2}$. Hence, from (2.4.22) the transversality assumption in the Crandall–Rabinowitz Theorem becomes

$$\int_0^1 \frac{s\mathscr{H}_0^2(s)}{(1-x_0s^2)^2}ds \neq 0,$$

which is trivially satisfied, and concludes the announced result. □

2.5 Linearized operator for the density equation

This section is devoted to the study of the linearized operator of the density equation (2.4.6). First, we will compute it with a general f_0 and provide a suitable formula in the case of quadratic profiles. Second, we shall prove that the linearized operator is a Fredholm operator of index zero because it takes the form of a compact perturbation of an invertible operator. More details about Fredholm operators can be found in Appendix A. Later we will focus our attention on the algebraic structure of the kernel and the range and give explicit expressions by using hypergeometric functions. We point out that the kernel description is done through the resolution of a Volterra integro-differential equation.

2.5.1 General formula and Fredholm index of the linearized operator

Let f_0 be an arbitrary smooth radial function satisfying (2.4.2) and let us compute the linearized operator of the functional $\widehat{G}_{f_0}$ given by (2.4.6). First, using Proposition 2.3.3 one gets

$$D_g\mathcal{N}(\Omega,0)h(z) = z\sum_{n\geq 1} A_n z^n \quad =: \quad k(z), \tag{2.5.1}$$

where A_n is given in (2.3.6) and $h \in \mathscr{C}_s^{1,\alpha}(\mathbb{D})$. Therefore, differentiating with respect to g yields

$$\begin{aligned} D_g\widehat{G}_{f_0}(\Omega,0)h =& \partial_g G_{f_0}(\Omega,0,0)h + \partial_\phi G_{f_0}(\Omega,0,0)\partial_g\mathcal{N}h(\Omega,0) \\ =& \partial_g G_{f_0}(\Omega,0,0)h + \partial_\phi G_{f_0}(\Omega,0,0)k. \end{aligned}$$

Using the Fréchet derivatives from Proposition 2.4.1, we have that

$$\begin{aligned} D_g\widehat{G}_{f_0}(\Omega,0)h(z) =& D_g\mathcal{M}_{f_0}(\Omega,f_0(z))h(z) + \frac{1}{2\pi}\int_{\mathbb{D}} \log|z-y|h(y)dA(y) - \Omega\text{Re}[\overline{z}k(z)] \\ &+ \frac{1}{2\pi}\int_{\mathbb{D}} \text{Re}\left[\frac{k(z)-k(y)}{z-y}\right] f_0(y)dA(y) \\ &+ \frac{1}{\pi}\int_{\mathbb{D}} \log|z-y| f_0(y)\text{Re}[k'(y)]dA(y). \end{aligned}$$

From the definition of $\mathcal{M}_{f_0}$ in (2.4.3) we infer that

$$D_g\mathcal{M}_{f_0}(\Omega,f_0(z))h(z) = \frac{h(z)}{\mu(\Omega,f_0(|z|))}, \quad \forall\, z \in \mathbb{D}.$$

Putting together the preceding formulas we obtain

$$D_g\widehat{G}_{f_0}(\Omega,0)h(z) = \frac{h(z)}{\mu(\Omega,f_0(r))} + \frac{1}{2\pi}\int_{\mathbb{D}} \log|z-y|h(y)dA(y) - \Omega\text{Re}[\overline{z}k(z)]$$

$$+\frac{1}{2\pi}\int_{\mathbb{D}}\text{Re}\left[\frac{k(z)-k(y)}{z-y}\right]f_0(y)dA(y)$$
$$+\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)\text{Re}[k'(y)]dA(y), \tag{2.5.2}$$

where μ is given by the compatibility condition (2.4.2). Next, we shall rewrite the linearized operator for the quadratic profile, and we will omit the subscript f_0 for the sake of simplicity. Taking μ as in (2.4.8), we get

$$\begin{aligned}D_g\widehat{G}(\Omega,0)h(z)=&\frac{1}{8}\left(\frac{1}{x}-r^2\right)h(z)+\frac{1}{2\pi}\int_{\mathbb{D}}\log|z-y|h(y)dA(y)-\Omega\text{Re}[\overline{z}k(z)]\\&+\frac{1}{2\pi}\int_{\mathbb{D}}\text{Re}\left[\frac{k(z)-k(y)}{z-y}\right]f_0(y)dA(y)\\&+\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)\text{Re}[k'(y)]dA(y),\end{aligned} \tag{2.5.3}$$

where x is given by (2.4.12).

In the following result we show that the linearized operator associated to a quadratic profile is a Fredholm operator with index zero. Similar result may be obtained in the general case imposing suitable conditions on the profiles.

Proposition 2.5.1. *Let f_0 be the profile* (2.1.6)*, with $A \in \mathbb{R}^\star$ and $B \in \mathbb{R}$. Assume that $\Omega \notin \left[\frac{B}{2},\frac{B}{2}+\frac{A}{4}\right]\cup\mathcal{S}_{\text{sing}}$. Then, $D_g\widehat{G}(\Omega,0):\mathscr{C}_s^{1,\alpha}(\mathbb{D})\to\mathscr{C}_s^{1,\alpha}(\mathbb{D})$ is a Fredholm operator with zero index.*

Proof. Using (2.5.3) we have

$$\begin{aligned}D_g\widehat{G}(\Omega,0)h(z)=&\frac{1}{8}\left(\frac{1}{x}-r^2\right)h(z)+\frac{1}{2\pi}\int_{\mathbb{D}}\log|z-y|h(y)dA(y)-\Omega\text{Re}[\overline{z}k(z)]\\&+\frac{1}{2\pi}\int_{\mathbb{D}}\text{Re}\left[\frac{k(z)-k(y)}{z-y}\right]f_0(y)dA(y)\\&+\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)\text{Re}[k'(y)]dA(y)\\=:&\left[\frac{1}{8}\left(\frac{1}{x}-r^2\right)\text{Id}+\mathcal{K}\right]h(z),\end{aligned}$$

where k is related to h through (2.5.1). The assumption on Ω entails that the smooth function $z\in\overline{\mathbb{D}}\mapsto\frac{1}{x}-|z|^2$ is not vanishing on the closed unit disc. Then the operator

$$\frac{1}{8}\left(\frac{1}{x}-|z|^2\right)\text{Id}:\mathscr{C}_s^{1,\alpha}(\mathbb{D})\to\mathscr{C}_s^{1,\alpha}(\mathbb{D}),$$

is an isomorphism. Hence, it is a Fredholm operator with zero index. To check that $\mathcal{L}$ is also a Fredholm operator with zero index, it suffices to prove that the operator $\mathcal{K}:\mathscr{C}_s^{1,\alpha}(\mathbb{D})\to\mathscr{C}_s^{1,\alpha}(\mathbb{D})$ is compact. To do that, we will prove that $\mathcal{K}:\mathscr{C}^{1,\alpha}(\mathbb{D})\to\mathscr{C}^{1,\gamma}(\mathbb{D})$, for any $\gamma\in(0,1)$. We split $\mathcal{K}$ as follows

$$\mathcal{K}h=\sum_{j=1}^{4}\mathcal{K}_jh,$$

with

$$\mathcal{K}_1h=-\Omega\text{Re}[\overline{z}k(z)],\quad \mathcal{K}_2h(z)=\frac{1}{2\pi}\int_{\mathbb{D}}\log|z-y|h(y)dA(y),$$

$$\mathcal{K}_3h(z)=\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)\mathrm{Re}[k'(y)]dA(y),\quad \mathcal{K}_4h=\frac{1}{2\pi}\int_{\mathbb{D}}\mathrm{Re}\left[\frac{k(z)-k(y)}{z-y}\right]f_0(y)dA(y).$$

The estimate of the first term $\mathcal{K}_1h$ follows from (2.5.1) and (2.3.7), leading to

$$\|\mathcal{K}_1h\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq C\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}. \tag{2.5.4}$$

Concerning the term $\mathcal{K}_2$ we note that

$$\|\mathcal{K}_2h\|_{L^\infty(\mathbb{D})}\leq C\|h\|_{L^\infty(\mathbb{D})},$$

and differentiating it, we obtain

$$\nabla_z\mathcal{K}_2h(z)=\frac{1}{2\pi}\int_{\mathbb{D}}\frac{h(y)}{\overline{z}-\overline{y}}dA(y).$$

Lemma B.0.4 yields

$$\|\nabla_z\mathcal{K}_2h\|_{\mathscr{C}^{0,\gamma}(\mathbb{D})}\leq C\|h\|_{L^\infty(\mathbb{D})},$$

for any $\gamma\in(0,1)$, and thus

$$\|\mathcal{K}_2h\|_{\mathscr{C}^{1,\gamma}(\mathbb{D})}\leq C\|h\|_{L^\infty(\mathbb{D})}. \tag{2.5.5}$$

The estimate of $\mathcal{K}_3$ is similar to that of $\mathcal{K}_2$, and using (2.5.5) and (2.3.7) we find that

$$\|\mathcal{K}_3h\|_{\mathscr{C}^{1,\gamma}(\mathbb{D})}\leq C\|f_0\|_{L^\infty(\mathbb{D})}\|k'\|_{L^\infty(\mathbb{D})}\leq C\|f_0\|_{L^\infty(\mathbb{D})}\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}. \tag{2.5.6}$$

Setting

$$K(z,y)=\frac{k(z)-k(y)}{z-y},\quad \forall z\neq y,$$

it is obvious that $|K(z,y)|\leq\|k'\|_{L^\infty(\mathbb{D})}$. Therefore, we have

$$\|\mathcal{K}_4h\|_{L^\infty}\leq C\|f_0\|_{L^\infty}\|k'\|_{L^\infty(\mathbb{D})}.$$

Moreover, by differentiation we find

$$\nabla_z\mathcal{K}_4h(z)=\frac{1}{2\pi}\mathrm{Re}\int_{\mathbb{D}}\nabla_zK(z,y)f_0(y)dA(y).$$

Straightforward computations show that

$$\begin{aligned}|\nabla_zK(z,y)|\leq&C\|k'\|_{L^\infty(\mathbb{D})}|z-y|^{-1},\\ |\nabla_zK(z_1,y)-\nabla_zK(z_2,y)|\leq&C|z_1-z_2|\left[\frac{\|k''\|_{L^\infty(\mathbb{D})}}{|z_1-y|}+\frac{\|k'\|_{L^\infty(\mathbb{D})}}{|z_1-y||z_2-y|}\right].\end{aligned}$$

Thus, hypotheses (B.0.8) are satisfied and we can use Lemma B.0.4 and (2.3.7) to find

$$\|\nabla_z\mathcal{K}_4h\|_{\mathscr{C}^{0,\gamma}(\mathbb{D})}\leq C\|f_0\|_{L^\infty}\|k\|_{\mathscr{C}^2(\mathbb{D})}\leq C\|f_0\|_{L^\infty}\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})},$$

obtaining

$$\|\mathcal{K}_4h\|_{\mathscr{C}^{1,\gamma}(\mathbb{D})}\leq C\|f_0\|_{L^\infty(\mathbb{D})}\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}. \tag{2.5.7}$$

Combining the estimates (2.5.4),(2.5.5),(2.5.6) and (2.5.7), we deduce

$$\|\mathcal{K}h\|_{\mathscr{C}^{1,\gamma}(\mathbb{D})}\leq C\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})},$$

which concludes the proof. □

To end this subsection, we give a more explicit form of the linearized operator. Coming back to the general expression in (2.5.2) and using Proposition B.0.8 we get that

$$\begin{aligned}D_g\widehat{G}_{f_0}(\Omega,0)h(z) =& \sum_{n\geq 1}\cos(n\theta)\left[\frac{h_n(r)}{\mu(\Omega,f_0(r))}-\frac{r}{n}\left(A_nG_n(r)+\frac{1}{2r^{n+1}}H_n(r)\right)\right]\\&+\frac{h_0(r)}{\mu(\Omega,f_0(r))}-\int_r^1\frac{1}{\tau}\int_0^\tau sh_0(s)dsd\tau,\end{aligned}$$

for

$$re^{i\theta}\mapsto h(re^{i\theta})=\sum_{n\in\mathbb{N}}h_n(r)\cos(n\theta)\in\mathscr{C}^{1,\alpha}_s(\mathbb{D}),$$

where

$$\begin{aligned}G_n(r) :=& n\Omega r^{n+1}+r^{n-1}\int_0^1 sf_0(s)ds-(n+1)r^{n-1}\int_0^r sf_0(s)ds+\frac{n+1}{r^{n+1}}\int_0^r s^{2n+1}f_0(s)ds,\\H_n(r) :=& r^{2n}\int_r^1\frac{1}{s^{n-1}}h_n(s)ds+\int_0^r s^{n+1}h_n(s)ds,\end{aligned}$$

for any $n\geq 1$. The value of A_n is given by (2.3.6) and recall that it was derived from the expression $\partial_g\mathcal{N}(\Omega,0)$ when studying the boundary equation. Moreover, there is another useful expression for A_n coming from the value of $G_n(1)$

$$G_n(1)=n\left[\Omega-\int_0^1 sf_0(s)ds+\frac{n+1}{n}\int_0^1 s^{2n+1}f_0(s)ds\right]=n\left(\Omega-\widehat{\Omega}_n\right).$$

Those preceding identities agrees with

$$A_n=-\frac{H_n(1)}{2G_n(1)},\quad\forall n\geq 1.$$

In the special case of f_0 being a quadratic profiles of the type (2.1.6), straightforward computations imply that

$$\begin{aligned}D_g\widehat{G}(\Omega,0)h(z) =& \sum_{n\geq 1}\cos(n\theta)\left[\frac{\frac{1}{x}-r^2}{8}h_n(r)-\frac{r}{n}\left(A_nG_n(r)+\frac{1}{2r^{n+1}}H_n(r)\right)\right]\\&+\frac{\frac{1}{x}-r^2}{8}h_0(r)-\int_r^1\frac{1}{\tau}\int_0^\tau sh_0(s)dsd\tau,\end{aligned}\tag{2.5.8}$$

with

$$G_n(r) = -\frac{An(n+1)}{4(n+2)}r^{n-1}P_n(r^2),\tag{2.5.9}$$

$$H_n(r) = r^{2n}\int_r^1\frac{1}{s^{n-1}}h_n(s)ds+\int_0^r s^{n+1}h_n(s)ds,\tag{2.5.10}$$

$$P_n(r) = r^2-\frac{n+2}{n+1}\frac{1}{x}r-\frac{A+2B}{A}\frac{n+2}{n(n+1)},\tag{2.5.11}$$

$$G_n(1) = -\frac{An(n+1)}{4(n+2)}P_n(1)=n\left[\frac{A}{4}\left(\frac{1}{x}-1\right)+\frac{A(n+1)}{2n(n+2)}+\frac{B}{2n}\right],\tag{2.5.12}$$

$$A_n = -\frac{H_n(1)}{2G_n(1)}=\frac{H_n(1)}{2n\left\{\widehat{\Omega}_n-\Omega\right\}}.\tag{2.5.13}$$

Remark that $G_n(1) \neq 0$ since we are assuming that $\Omega \notin \mathcal{S}_{\text{sing}}$, the singular set defined in (2.4.10).

From now on we will work only with the quadratic profiles. Similar study could be implemented with general profiles but the analysis may turn out to be very difficult because the spectral study is intimately related to the distribution of the selected profile.

2.5.2 Kernel structure and negative results

The current objective is to conduct a precise study for the kernel structure of the linearized operator (2.5.8). We must identify the master equation describing the dispersion relation. As a by-product we connect the dimension of the kernel to the number of roots of the master equation. We shall distinguish in this study between the regular case corresponding to $x \in (-\infty, 1)$ and the singular case associated to $x > 1$. For this latter case we prove that the equation (2.4.10) has no solution close to the trivial one.

Regular case

Let us start with a preliminary result devoted to the explicit resolution of a second order differential equation with polynomial coefficients taking the form

$$(1 - xr^2)rF''(r) - (1 - xr^2)(2n - 1)F'(r) + 8rxF(r) = g(r), \quad \forall r \in [0, 1], \tag{2.5.14}$$

This will be applied later to the study of the kernel and the range. Before stating our result we need to introduce some functions

$$F_n(r) = F(a_n, b_n; c_n; r), \quad a_n = \frac{n - \sqrt{n^2 + 8}}{2}, \quad b_n = \frac{n + \sqrt{n^2 + 8}}{2}, \quad c_n = n + 1. \tag{2.5.15}$$

where $r \in [0, 1) \mapsto F(a, b; c; r)$ denotes the Gauss hypergeometric function defined in (C.0.1).

Lemma 2.5.2. *Let $n \geq 1$ be an integer, $x \in (-\infty, 1)$ and $g \in \mathscr{C}([0, 1]; \mathbb{R})$. Then, the general continuous solutions of equation* (2.5.14) *supplemented with the initial condition $F(0) = 0$, are given by a one–parameter curve*

$$F(r) = r^{2n} F_n(xr^2) \left[\frac{F(1)}{F_n(x)} - \frac{x^{n-1}}{4} \int_{xr^2}^{x} \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^{\tau} \frac{F_n(s)}{1 - s} \left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right].$$

Proof. Consider the auxiliary function $F(r) = \mathcal{F}(xr^2)$ and set $y = xr^2$. Note that $y \in [0, x]$ when $x > 0$ and $y \in [x, 0]$ if $x < 0$, then $r = \left(\frac{y}{x}\right)^{\frac{1}{2}}$ in both cases. Hence, the equation governing this new function is

$$(1 - y)y\mathcal{F}''(y) - (n - 1)(1 - y)\mathcal{F}'(y) + 2\mathcal{F}(y) = \frac{1}{4x} \left(\frac{x}{y}\right)^{\frac{1}{2}} g\left(\left(\frac{y}{x}\right)^{\frac{1}{2}}\right), \tag{2.5.16}$$

with the boundary condition $\mathcal{F}(0) = 0$. The strategy to be followed consists in solving the homogeneous equation and using later the method of variation of constants. The homogeneous problem is given by

$$(1 - y)y\mathcal{F}_0''(y) - (n - 1)(1 - y)\mathcal{F}_0'(y) + 2\mathcal{F}_0(y) = 0.$$

Comparing it with the general differential equation (C.0.16), we obviously find that $\mathcal{F}_0$ satisfies a hypergeometric equation with the parameters

$$a = \frac{-n - \sqrt{n^2 + 8}}{2}, \quad b = \frac{-n + \sqrt{n^2 + 8}}{2}, \quad c = 1 - n.$$

The general theory of hypergeometric functions gives us that this differential equation is degenerate because c is a negative integer, see discussion in Appendix C. However, we still get two independent solutions generating the class of solutions to this differential equation: one is smooth and the second is singular and contains a logarithmic singularity at the origin. The smooth one is given by

$$y \in (-\infty, 1) \mapsto y^{1-c} F\left(1+a-c, 1+b-c, 2-c, y\right).$$

With the special parameters (2.5.15), it becomes

$$y \in (-\infty, 1) \mapsto y^n F\left(a_n, b_n; c_n; y\right) = y^n F_n(y).$$

It is important to note that, by Taylor expansion, the hypergeometric function initially defined in the unit disc $\mathbb{D}$ admits an analytic continuation in the complex plane cut along the real axis from 1 to $+\infty$. This comes from the integral representation (C.0.2).

Next, we use the method of variation of constants with the smooth homogeneous solution and set

$$\mathcal{F}_0 : (-\infty, 1) \in \mathbb{D} \mapsto y^n F_n(y). \tag{2.5.17}$$

We wish to mention that when using the method of variation of constants with the smooth solution we also find the trace of the singular solution. As we will notice in the next step, this singular part will not contribute for the full inhomogeneous problem due to the required regularity and the boundary condition $F(0) = 0$. Now, we solve the equation (2.5.16) by looking for solutions in the form $\mathcal{F}(y) = \mathcal{F}_0(y)K(y)$. By setting $\mathcal{K} := K'$, one has that

$$\mathcal{K}'(y) + \left[2\frac{\mathcal{F}_0'(y)}{\mathcal{F}_0(y)} - \frac{n-1}{y}\right]\mathcal{K}(y) = \frac{1}{4x}\frac{\left(\frac{x}{y}\right)^{\frac{1}{2}} g\left(\left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{y(1-y)\mathcal{F}_0(y)},$$

which can be integrated in the following way

$$\mathcal{K}(y) = \frac{y^{n-1}}{\mathcal{F}_0^2(y)}\left\{K_1 + \frac{1}{4x}\int_0^y \frac{\mathcal{F}_0(s)}{s^n(1-s)}\left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right)ds\right\},$$

where K_1 is a constant. Thus integrating successively we find K and $\mathcal{F}$ and from the expression of $\mathcal{F}_0$ we deduce

$$\mathcal{F}(y) = y^n F_n(y)\left[K_2 - \int_y^{\operatorname{sign} y} \frac{1}{\tau^{n+1}F_n^2(\tau)}\left\{K_1 + \frac{1}{4x}\int_0^\tau \frac{F_n(s)}{1-s}\left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right)ds\right\}d\tau\right],$$

for any $y \in (-\infty, 1)$, with K_2 a constant and where $sign$ is the sign function. From straightforward computations using integration by parts we get

$$\mathcal{F}(0) = -K_1 \lim_{y\to 0} y^n F_n(y)\int_y^{\operatorname{sign} y}\frac{1}{\tau^{n+1}F_n^2(\tau)}d\tau = -\frac{K_1}{n}.$$

Combined with the initial condition $\mathcal{F}(0) = 0$ we obtain $K_1 = 0$. Coming back to the original function $F(r) = \mathcal{F}(xr^2)$, we obtain

$$F(r) = x^n r^{2n} F_n(xr^2)\left[K_2 - \frac{1}{4x}\int_{xr^2}^{\operatorname{sign} x}\frac{1}{\tau^{n+1}F_n^2(\tau)}\int_0^\tau \frac{F_n(s)}{1-s}\left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right)ds d\tau\right].$$

The constant K_2 can be computed by evaluating the preceding expression at $r = 1$. We finally get

$$F(r) = r^{2n} F_n(xr^2) \left[\frac{F(1)}{F_n(x)} - \frac{x^{n-1}}{4} \int_{xr^2}^{x} \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \sqrt{x/s}\, g\left(\sqrt{s/x}\right) ds d\tau \right], \quad (2.5.18)$$

for $r \in [0,1]$. Observe from the integral representation (C.0.2) that the function F_n does not vanish on $(-\infty, 1)$ for $n \geq 1$. Hence, (2.5.18) is well-defined and F is $\mathscr{C}^\infty$ in $[0,1]$ when $x < 1$. □

The next goal is to give the kernel structure of the linearized operator $D_g\widehat{G}(\Omega, 0)$. We emphasize that according to Proposition 2.5.1, this is a Fredholm operator of zero index, which implies in particular that its kernel is finite–dimensional. Before that, we introduce the singular set for x connected to the singular set of Ω through the relations (2.4.12) and (2.4.10)

$$\widehat{\mathcal{S}}_{\text{sing}} = \left\{ \widehat{x}_n = \frac{A}{4\left(\widehat{\Omega}_n - \frac{B}{2}\right)}, \quad \widehat{\Omega}_n \in \mathcal{S}_{\text{sing}} \right\}. \quad (2.5.19)$$

For any $n \geq 1$, consider the following sequences of functions

$$\zeta_n(x) := F_n(x) \left[1 - x + \frac{A+2B}{A(n+1)} x \right] + \int_0^1 F_n(\tau x) \tau^n \left[-1 + 2x\tau \right] d\tau, \quad \forall x \in (-\infty, 1], \quad (2.5.20)$$

where F_n has been introduced in (2.5.15). Then we prove the following.

Proposition 2.5.3. *Let $A \in \mathbb{R}^\star$, $B \in \mathbb{R}$ and $x \in (-\infty, 1) \backslash \{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$, with* (2.4.12) *and* (2.4.14). *Define the set*

$$\mathcal{A}_x := \Big\{ n \in \mathbb{N}^\star, \quad \zeta_n(x) = 0 \Big\}. \quad (2.5.21)$$

Then, the kernel of $D_g\widehat{G}(\Omega, 0)$ is finite–dimensional and generated by the $\mathscr{C}^\infty$ functions $\{\hbar_n,\ n \in \mathcal{A}_x\}$, with $\hbar_n : z \in \overline{\mathbb{D}} \mapsto \text{Re}\left[\mathscr{G}_n(|z|^2) z^n\right]$ and

$$\mathscr{G}_n(t) = \frac{1}{1-xt} \left[-\frac{P_n(t)}{P_n(1)} + \frac{F_n(xt)}{F_n(x)} - \frac{2xF_n(xt)}{P_n(1)} \int_t^1 \frac{1}{\tau^{n+1} F_n^2(x\tau)} \int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds d\tau \right].$$

As a consequence, dim Ker $D_g\widehat{G}(\Omega, 0) =$ Card $\mathcal{A}_x$. *The functions P_n and F_n are defined in* (2.5.11) *and* (2.5.15), *respectively.*

Remark 2.5.4. *Notice that the set $\mathcal{A}_x$ can be empty; in that case the kernel of $D_g\widehat{G}(\Omega, 0)$ is trivial. Otherwise, the set $\mathcal{A}_x$ is finite.*

Proof. To analyze the kernel structure, we return to (2.5.8) and solve the equations keeping in mind the relations (2.5.13). Thus we should solve

$$\begin{aligned} h_n(r) + \frac{4r}{n\left(r^2 - \frac{1}{x}\right)} \left[-\frac{H_n(1)}{G_n(1)} G_n(r) + \frac{H_n(r)}{r^{n+1}} \right] = 0, \quad \forall r \in [0,1], \quad \forall n \in \mathbb{N}^\star, \\ \frac{\frac{1}{x} - r^2}{8} h_0(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau = 0, \quad \forall r \in [0,1], \end{aligned} \quad (2.5.22)$$

where the functions involved in the last expressions are given in (2.5.9)-(2.5.13). The term $(r^2 - \frac{1}{x})$ is not vanishing from the assumptions on x. Note that the last equation for $n = 0$ has been already studied in Proposition 2.4.3, which implies that if $x \neq x_0$, then the zero function is the only solution. Hence, let us focus on the case $n \geq 1$ and solve the associated equation. To deal with this equation we write down a differential equation for H_n and use Lemma 2.5.2. Firstly, we define the linear operator

$$\mathscr{L}h(r) := r^{2n}\int_r^1 \frac{1}{s^{n-1}}h(s)ds + \int_0^r s^{n+1}h(s)ds, \tag{2.5.23}$$

for any $h \in \mathscr{C}([0,1];\mathbb{R})$. Then, by differentiation we obtain

$$(\mathscr{L}h)'(r) = 2nr^{2n-1}\int_r^1 \frac{1}{s^{n-1}}h(s)ds = \frac{2n}{r}\left[\mathscr{L}h(r) - \int_0^r s^{n+1}h(s)ds\right].$$

It is important to precise at this stage that $\mathscr{L}h$ satisfies the boundary conditions

$$(\mathscr{L}h)(0) = (\mathscr{L}h)'(1) = 0. \tag{2.5.24}$$

Indeed, the second condition is obvious and to get the first one we use that h is bounded:

$$|\mathscr{L}h(r)| \leq \|h\|_{L^\infty}\left(\frac{|r^{2n} - r^{2+n}|}{|n-2|} + \frac{r^{n+2}}{n+2}\right),$$

for any $n \neq 2$. In the case $n = 2$ we have

$$|\mathscr{L}h(r)| \leq \|h\|_{L^\infty}\left(r^4|\ln r| + \frac{r^4}{4}\right),$$

and for $n = 1$ it is clearly verified. Differentiating again we obtain

$$\frac{1}{2n}\left[r(\mathscr{L}h)'(r)\right]' - (\mathscr{L}h)'(r) = -r^{n+1}h(r). \tag{2.5.25}$$

Since $H_n = \mathscr{L}h_n$, one has

$$\frac{1}{2n}\left[rH_n'(r)\right]' - H_n'(r) = -r^{n+1}h_n(r). \tag{2.5.26}$$

Using Equation (2.5.22), we deduce that H_n satisfies the following differential equation

$$(1-xr^2)rH_n''(r) - (1-xr^2)(2n-1)H_n'(r) + 8rxH_n(r) = 8\frac{H_n(1)x}{G_n(1)}r^{n+2}G_n(r), \tag{2.5.27}$$

complemented with the boundary conditions (2.5.24). Let us show how to recover the full solutions of (2.5.22) from this equation. Assume that we have constructed all the solutions H_n of (2.5.27), with the boundary conditions (2.5.24). Then, to obtain the solutions of (2.5.22), we should check the compatibility condition $\mathscr{L}h_n = H_n$, by setting

$$h_n(r) := \frac{4r}{n\left(r^2 - \frac{1}{x}\right)}\left[H_n(1)\frac{G_n(r)}{G_n(1)} - \frac{H_n(r)}{r^{n+1}}\right]. \tag{2.5.28}$$

Combining (2.5.25) and (2.5.26), we deduce that $\mathscr{H} := \mathscr{L}h_n - H_n$ satisfies

$$\frac{1}{2n}\left[r\,\mathscr{H}'(r)\right]' - \mathscr{H}'(r) = 0.$$

By solving this differential equation we obtain the existence of two real constants λ_0 and λ_1 such that $\mathcal{H}(r) = \lambda_0+\lambda_1 r^{2n}$, for any $r \in [0,1]$. Since both $\mathscr{L}h_n$ and H_n satisfy (2.5.24), then $\mathcal{H}$ satisfies also these conditions. Hence, we find $\mathcal{H}(r) = 0$, for any $r \in [0,1]$, and this concludes that h_n, given by (2.5.28), is a solution of (2.5.22). We emphasize that h_n satisfies the compatibility condition

$$H_n(1) = \int_0^1 r^{n+1}h_n(r)dr.$$

Indeed, integrating (2.5.26) from 0 to 1, we obtain

$$\frac{H_n'(1)}{2n} - (H_n(1) - H_n(0)) = -\int_0^1 r^{n+1}h_n(r)dr.$$

Thus, if H_n satisfies the boundary conditions (2.5.24), then the compatibility condition is automatically verified. Now, let us come back to the resolution of (2.5.27). Since $H_n(0) = 0$, then one can apply Lemma 2.5.2 with

$$g(r) = 8\frac{H_n(1)x}{G_n(1)}r^{n+2}G_n(r). \tag{2.5.29}$$

Therefore, we obtain after a change of variables that

$$\begin{aligned}
H_n(r) =& r^{2n}F_n(xr^2)H_n(1)\left[\frac{1}{F_n(x)}\right.\\
&\left.-\frac{2x^n}{G_n(1)}\int_{xr^2}^{x}\frac{1}{\tau^{n+1}F_n^2(\tau)}\int_0^\tau \frac{F_n(s)}{1-s}\left(\frac{s}{x}\right)^{\frac{n+1}{2}}G_n\left(\left(\frac{s}{x}\right)^{\frac12}\right)dsd\tau\right]\\
=& H_n(1)r^{2n}F_n(xr^2)\left[\frac{1}{F_n(x)}\right.\\
&\left.-\frac{8x}{G_n(1)}\int_r^1\frac{1}{\tau^{2n+1}F_n^2(x\tau^2)}\int_0^\tau \frac{s^{n+2}F_n(xs^2)}{1-xs^2}G_n(s)dsd\tau\right].
\end{aligned} \tag{2.5.30}$$

It remains to check the second initial condition: $H_n'(1) = 0$. From straightforward computations using (2.5.9) and (2.5.12) we find that

$$\begin{aligned}
H_n'(1) =& \frac{H_n(1)}{F_n(x)}\left[2nF_n(x) + 2xF_n'(x)\right] + \frac{8xH_n(1)}{G_n(1)F_n(x)}\int_0^1 \frac{s^{n+2}F_n(xs^2)}{1-xs^2}G_n(s)ds\\
=& 2\frac{H_n(1)}{F_n(x)}\left[\varphi_n(x) - \frac{An(n+1)x}{(n+2)G_n(1)}\int_0^1\frac{s^{2n+1}F_n(xs^2)}{1-xs^2}P_n(s^2)ds\right]\\
=& \frac{2nH_n(1)}{F_n(x)G_n(1)}\Psi_n(x),
\end{aligned} \tag{2.5.31}$$

where

$$\Psi_n(x) := \varphi_n(x)\left[\frac{A}{4}\left(\frac{1}{x}-1\right) + \frac{A}{2}\frac{n+1}{n^2+2n} + \frac{B}{2n}\right] - \frac{A(n+1)x}{n+2}\int_0^1 \frac{s^{2n+1}F_n\left(xs^2\right)}{1-xs^2}P_n(s^2)ds,$$

and

$$\varphi_n(x) := nF_n(x) + xF_n'(x). \tag{2.5.32}$$

Note that $G_n(1) \neq 0$ because $\Omega \notin \mathcal{S}_{\text{sing}}$. Let us link Ψ_n to the function ζ_n. Recall from (2.5.16) and (C.0.16) that

$$x(1-x)F_n'' + (n+1)(1-x)F_n'(x) + 2F_n = 0.$$

Differentiating the function $\varphi_n(x)$ and using the differential equation for F_n we realize that

$$\begin{aligned}\varphi_n'(x) =&(n+1)F_n'(x)+xF_n''(x)\\ =&\frac{1}{1-x}\left[(1-x)(n+1)F_n'(x)+(1-x)xF_n''(x)\right]=-\frac{2F_n(x)}{1-x}.\end{aligned}$$

The change of variables $xs^2\mapsto\tau$ in the integral term yields

$$\int_0^1\frac{s^{2n+1}F_n\left(xs^2\right)}{1-xs^2}P_n(s^2)ds=\frac{1}{2x^{n+1}}\int_0^x\frac{\tau^nF_n(\tau)}{1-\tau}P_n\left(\frac{\tau}{x}\right)d\tau.$$

Therefore we get

$$\Psi_n\left(x\right)=\varphi_n(x)\left[\frac{A}{4}\left(\frac{1}{x}-1\right)+\frac{A}{2}\frac{n+1}{n^2+2n}+\frac{B}{2n}\right]+\frac{A(n+1)}{4(n+2)x^n}\int_0^x\varphi_n'(\tau)\tau^nP_n\left(\frac{\tau}{x}\right)d\tau.$$

From Definition (2.5.11) we find the identity

$$\int_0^x\varphi_n'(\tau)\tau^nP_n\left(\frac{\tau}{x}\right)d\tau=\frac{1}{x^2}\int_0^x\varphi_n'(\tau)\tau^n\left[\tau^2-\tau\frac{n+2}{n+1}-\frac{A+2B}{A}\frac{n+2}{n(n+1)}x^2\right]d\tau.$$

Integrating by parts, we deduce

$$\begin{aligned}\frac{A(n+1)}{4(n+2)x^n}\int_0^x\varphi_n'(\tau)\tau^nP_n\left(\frac{\tau}{x}\right)d\tau=&\frac{A}{4}\varphi_n(x)\left(\frac{n+1}{n+2}-\frac{1}{x}-\frac{A+2B}{An}\right)\\ &-\frac{A(n+1)}{4x^{n+2}}\int_0^x\varphi_n(\tau)\tau^{n-1}\left(\tau^2-\tau-\frac{A+2B}{A(n+1)}x^2\right)d\tau.\end{aligned}$$

By virtue of the following identity

$$\frac{A}{4}\left(\frac{1}{x}-1\right)+\frac{A}{2}\frac{n+1}{n^2+2n}+\frac{B}{2n}+\frac{A}{4}\left(\frac{n+1}{n+2}-\frac{1}{x}-\frac{A+2B}{An}\right)=0,$$

the boundary term in the integral is canceled with the first part of Ψ_n. Thus

$$\Psi_n(x)=-\frac{A(n+1)}{4x^{n+2}}\int_0^x\varphi_n(\tau)\tau^{n-1}\left(\tau^2-\tau-\frac{A+2B}{A(n+1)}x^2\right)d\tau,$$

where, after a change of variables in the integral term, we get that

$$\Psi_n(x)=\frac{A(n+1)}{4x}\int_0^1\varphi_n(\tau x)\tau^{n-1}\left(-x\tau^2+\tau+\frac{A+2B}{A(n+1)}x\right)d\tau.$$

Setting

$$\zeta_n(x)=\int_0^1\varphi_n(\tau x)\tau^{n-1}\left(-x\tau^2+\tau+\frac{A+2B}{A(n+1)}x\right)d\tau,$$

we find the relation

$$\Psi_n(x)=\frac{A(n+1)}{4x}\zeta_n(x).$$

Observe first that the zeroes of Ψ_n and ζ_n are the same. Coming back to (2.5.32) and integrating by parts we get the equivalent form

$$\zeta_n(x)=F_n(x)\left(1-x+\frac{A+2B}{A(n+1)}x\right)+\int_0^1F_n(\tau x)\tau^n\left(-1+2x\tau\right)d\tau.$$

This gives (2.5.20). According to (2.5.31), the constraint $H_n'(1) = 0$ is equivalent to $H_n(1) = 0$ or $\zeta_n(x) = 0$. In the first case, we get from (2.5.30) that $H_n \equiv 0$ and inserting this into (2.5.28) we find $h_n(r) = 0$, for any $r \in [0,1]$. Thus, for $n \notin \mathcal{A}_x$, where $\mathcal{A}_x$ is given by (2.5.21), we obtain that there is only one solution for the kernel equation, which is the trivial one. As to the second condition $\zeta_n(x) = 0$, which agrees with $n \in \mathcal{A}_x$, one gets from (2.5.30) and (2.5.28) that the kernel of $D_g\widehat{G}(\Omega, 0)$ restricted to the level frequency n is generated by

$$\hbar_n(re^{i\theta}) = h_n^\star(r)\cos(n\theta),$$

with

$$h_n^\star(r) = \frac{1}{1-xr^2}\left[-\frac{rG_n(r)}{G_n(1)} + \frac{r^n F_n(xr^2)}{F_n(x)} - 8xr^n F_n(xr^2)\int_r^1 \frac{\displaystyle\int_0^\tau \frac{s^{n+2}F_n(xs^2)}{1-xs^2}\frac{G_n(s)}{G_n(1)}ds}{\tau^{2n+1}F_n^2(x\tau^2)}d\tau\right].$$

The fact that $h_n = \frac{4x}{n}H_n(1)\, h_n^\star$ together with $H_n(1) = \displaystyle\int_0^1 s^{n+1}h_n(s)ds$ imply

$$\int_0^1 s^{n+1}h_n^\star(s)ds = \frac{n}{4x}. \tag{2.5.33}$$

Using

$$\frac{rG_n(r)}{G_n(1)} = r^n\frac{P_n(r^2)}{P_n(1)},$$

and a suitable change of variables allows getting the formula,

$$\begin{aligned}\int_r^1 \frac{\displaystyle\int_0^\tau \frac{s^{n+2}F_n(xs^2)}{1-xs^2}\frac{G_n(s)}{G_n(1)}ds}{\tau^{2n+1}F_n^2(x\tau^2)}d\tau &= \frac{1}{P_n(1)}\int_r^1 \frac{\displaystyle\int_0^\tau \frac{s^{2n+1}F_n(xs^2)}{1-xs^2}P_n(s^2)ds}{\tau^{2n+1}F_n^2(x\tau^2)}d\tau\\ &= \frac{1}{4P_n(1)}\int_{r^2}^1 \frac{\displaystyle\int_0^\tau \frac{s^n F_n(xs)}{1-xs}P_n(s)ds}{\tau^{n+1}F_n^2(x\tau)}d\tau.\end{aligned}$$

We have

$$h_n^\star(r) = \frac{r^n}{1-xr^2}\left[-\frac{P_n(r^2)}{P_n(1)} + \frac{F_n(xr^2)}{F_n(x)} - \frac{2xF_n(xr^2)}{P_n(1)}\int_{r^2}^1 \frac{1}{\tau^{n+1}F_n^2(x\tau)}\int_0^\tau \frac{s^nF_n(xs)}{1-xs}P_n(s)dsd\tau\right].$$

Setting

$$\mathscr{G}_n(\mathtt{t}) = \frac{1}{1-x\mathtt{t}}\left[-\frac{P_n(\mathtt{t})}{P_n(1)} + \frac{F_n(x\mathtt{t})}{F_n(x)} - \frac{2xF_n(x\mathtt{t})}{P_n(1)}\int_{\mathtt{t}}^1 \frac{1}{\tau^{n+1}F_n^2(x\tau)}\int_0^\tau \frac{s^nF_n(xs)}{1-xs}P_n(s)dsd\tau\right],$$

we deduce that

$$\hbar_n(z) = \mathscr{G}_n(|z|^2)|z|^n\cos(n\theta) = \mathrm{Re}\left[\mathscr{G}_n(|z|^2)z^n\right], \quad \forall\, z \in \overline{\mathbb{D}}. \tag{2.5.34}$$

We intend to check that $\hbar_n$ belongs to $\mathscr{C}^\infty(\overline{\mathbb{D}};\mathbb{R})$. To get this it is enough to verify that $\mathscr{G}$ belongs to $\mathscr{C}^\infty([0,1];\mathbb{R})$. Since $x \in (-\infty, 1)$, then $\tau \in [0,1] \mapsto F_n(x\tau)$ is in $\mathscr{C}^\infty([0,1];\mathbb{R})$. The change of variables $s = \tau\theta$ implies

$$\int_{\mathtt{t}}^1 \frac{\displaystyle\int_0^\tau \frac{s^nF_n(xs)}{1-xs}P_n(s)ds}{\tau^{n+1}F_n^2(x\tau)}d\tau = \int_{\mathtt{t}}^1 \frac{\displaystyle\int_0^1 \frac{\theta^nF_n(x\tau\theta)P_n(\tau\theta)}{1-x\tau\theta}d\theta}{F_n^2(x\tau)}d\tau.$$

Since F_n does not vanish on $(-\infty, 1)$, then the mapping $\tau \in [0,1] \mapsto \frac{1}{F_n(x\tau)}$ belongs to $\mathscr{C}^\infty([0,1];\mathbb{R})$. It suffices to observe that the integral function is $\mathscr{C}^\infty$ on $[0,1]$. Then, we have an independent element of the kernel given by $h_n^\star(\cdot)$, for any $n \in \mathcal{A}_x$. This concludes the announced result. □

Remark 2.5.5. *The hypergeometric function F_n for $n=1$ can be computed as $F_1(r) = F(a_1, b_1; c_1; r) = 1-r$. Hence, the function* (2.5.20) *becomes*

$$\begin{aligned}\zeta_1(x) =& F_1(x)\left[1-x+\frac{A+2B}{2A}x\right] + \int_0^1 F_1(\tau x)\tau\left[-1+2x\tau\right]d\tau\\ =&(1-x)\left[1-x+\frac{A+2B}{2A}x\right] + \int_0^1 (1-\tau x)\tau\left[-1+2x\tau\right]d\tau = (1-x)\left(\frac{B}{A}x+\frac{1}{2}\right).\end{aligned}$$

The root $x=1$ is not allowed since $x \notin \widehat{\mathcal{S}}_{\text{sing}}$. Therefore, the unique root is $x = -\frac{A}{2B}$. Coming back to Ω using (2.4.12)*, one has that $\Omega = 0$.*

Singular case

The singular case $x \in (1, +\infty)$ is studied in this section. Notice that from (2.4.12), we obtain

$$\frac{B}{2} < \Omega < \frac{B}{2} + \frac{A}{4}. \tag{2.5.35}$$

It is worthy to point out that this case is degenerate because the leading terms of the equations of the linearized operator (2.5.22) vanish inside the unit disc. To understand this operator one should deal with a second differential equation of hypergeometric type with a singularity. Thus, the first difficulty amounts to solving those equations across the singularity and invert the operator. This can be done in a straightforward way getting that the operator is injective with an explicit representation of its formal inverse. However, it is not an isomorphism and undergoes a loss of regularity in the Hölder class. Despite this bad behavior, one would expect at least the persistence of the injectivity for the nonlinear problem. This problem appears in different contexts, for instance in the inverse backscattering problem [138]. The idea to overcome this difficulty is to prove two key ingredients. The first one concerns the coercivity of the linearized operator with a quantified loss in the Hölder class. The second point is to use the Taylor expansion and to establish a soft estimate for the reminder combined with an interpolation argument. This argument leads to the following result.

Theorem 2.5.6. *Let $0<\alpha<1$, $A>0$ and $B \in \mathbb{R}$ such that $\frac{B}{A} \notin \left[-1, -\frac{1}{2}\right]$. Assume that Ω satisfies* (2.5.35) *and $\Omega \notin \mathcal{S}_{\text{sing}}$, where this latter set is defined in* (2.4.10)*. Then, there exists a small neighborhood V of the origin in $\mathscr{C}_s^{2,\alpha}(\mathbb{D})$ such that the nonlinear equation* (2.4.6) *has no solution in V, except the origin. Notice that in the case $B < -A$, the condition $\Omega \notin \mathcal{S}_{\text{sing}}$ follows automatically from* (2.5.35).

The proof of this theorem will be given at the end of this section. Before we should develop some tools. Let us start with solving the kernel equations, for this reason we introduce some auxiliary functions. Set

$$\widehat{F}_n(x) = F(-a_n, b_n; b_n - a_n + 1; x), \tag{2.5.36}$$

and define the functions

$$\mathscr{F}_{K_1,K_2}(y) = \begin{cases} y^n F_n(y) \left[\frac{K_1}{F_n(1)} - \displaystyle\int_y^1 \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}_x(\frac{s}{x}) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \right], & y \in [0,1], \\ y^{a_n} \widehat{F}_n\left(\frac{1}{y}\right) \left[\frac{K_1}{\widehat{F}_n(1)} + \displaystyle\int_{\frac{1}{y}}^1 \frac{K_2 + \int_{\frac{1}{x}}^\tau \frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s} \mathcal{R}_x(\frac{1}{sx}) ds}{\tau^{n+1-2a_n} \widehat{F}_n^2(\tau)} d\tau \right], & y \geq 1, \end{cases} \tag{2.5.37}$$

and

$$\widehat{\mathscr{F}}_{K_1,K_2}(y) = \begin{cases} y(1-y) \left[K_1 + \displaystyle\int_0^y \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x}) ds}{\tau^2 (1-\tau)^2} d\tau \right], & y \in [0,1], \\ \frac{1}{y} \widehat{F}_1\left(\frac{1}{y}\right) \left[\frac{1}{3} \displaystyle\int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) ds + \displaystyle\int_{\frac{1}{y}}^1 \frac{K_2 + \int_{\frac{1}{x}}^\tau \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}_x(\frac{1}{sx}) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau \right], & y \geq 1, \end{cases}$$

with

$$\mathcal{R}_x(y) = \frac{1}{4x} y^{-\frac{1}{2}} g\big(y^{\frac{1}{2}}\big), \tag{2.5.38}$$

where g is the source term in (2.5.14). Our first result reads as follows.

Lemma 2.5.7. *Let $n \geq 2$ be an integer, $x \in (1, +\infty)$ and $g \in \mathscr{C}([0,1]; \mathbb{R})$. Then, the continuous solutions in $[0, +\infty)$ to the equation* (2.5.14)*, such that $F(0) = 0$, are given by the two–parameters curve*

$$r \in [0,1] \mapsto F(r) = \mathscr{F}_{K_1,K_2}(xr^2), \quad K_1, K_2 \in \mathbb{R}.$$

Moreover, if $g \in \mathscr{C}^\mu([0,1]; \mathbb{R})$, for some $\mu > 0$, then the above solutions are $\mathscr{C}^1$ on $[0,1]$ if and only if the following conditions hold true:

$$\mathcal{R}_x\left(\frac{1}{x}\right) = 0 \tag{2.5.39}$$

and

$$K_1 = 0, \quad \text{and} \quad K_2 = -\frac{\widehat{F}_n(1)}{F_n(1)} \int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}_x\left(\frac{s}{x}\right) ds - \int_{\frac{1}{x}}^1 \frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s} \mathcal{R}_x(\frac{1}{sx}) ds. \tag{2.5.40}$$

Let $n = 1$ and $g \in \mathscr{C}([0,1]; \mathbb{R})$. Then the continuous solutions to (2.5.14)*, with $F(0) = 0$, are given by the two-parameters curve*

$$r \in [0,1] \mapsto F(r) = \widehat{\mathscr{F}}_{K_1,K_2}(xr^2), \quad K_1, K_2 \in \mathbb{R}.$$

If $g \in \mathscr{C}^\mu([0,1]; \mathbb{R})$, for some $\mu > 0$, then this solution is $\mathscr{C}^1$ if and only if

$$\mathcal{R}_x\left(\frac{1}{x}\right) = \mathcal{R}_x(0) = \int_0^{\frac{1}{x}} \mathcal{R}_x(\tau) \, d\tau = 0, \tag{2.5.41}$$

and, K_1 and K_2 satisfy

$$K_2 = -3\left(K_1 + \int_0^1 \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x}) ds}{\tau^2(1-\tau)^2} d\tau \right) - \int_{\frac{1}{x}}^{\frac{1}{y}} \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}_x\left(\frac{1}{sx}\right) ds. \tag{2.5.42}$$

Moreover, if $F'(1)=0$, we have the additional constraint

$$K_2\left(-\frac{1}{x^2}\left[\widehat{F}_1\left(\frac{1}{x}\right)+\frac{1}{x}\widehat{F}_1'\left(\frac{1}{x}\right)\right]\int_{\frac{1}{x}}^1\frac{d\tau}{\tau^4\widehat{F}_1^2(\tau)}+\frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)}\right)$$
$$=\frac{1}{x^2}\left[\widehat{F}_1\left(\frac{1}{x}\right)+\frac{1}{x}\widehat{F}_1'\left(\frac{1}{x}\right)\right]\int_{\frac{1}{x}}^1\frac{\int_{\frac{1}{x}}^\tau\frac{s\widehat{F}_1(s)}{1-s}\mathcal{R}_x(\frac{1}{sx})ds}{\tau^4\widehat{F}_1^2(\tau)}d\tau. \tag{2.5.43}$$

Proof. We proceed as in the proof of Lemma 2.5.2. The resolution of the equation (2.5.16), in the interval $[0,1]$, is exactly the same and we find

$$\mathcal{F}(y)=\mathscr{F}_{K_1,K_2}(y),\quad \forall y\in[0,1].$$

In the interval $[1,x]$, we first solve the homogeneous equation associated to (2.5.16). By virtue of Appendix C, one gets two independent solutions, one of them is described by

$$\mathcal{F}_0(y)=y^{a_n}\widehat{F}_n\left(\frac{1}{y}\right),\quad \forall y\in[1,x].$$

Using the method of variation of constants, we obtain that the general solutions to the equation (2.5.16) in this interval take the form

$$\mathcal{F}(y)=\mathscr{F}_{K_1,K_2}(y),\quad \forall y\in[1,x].$$

The continuity of $\mathcal{F}$ in the interval $[0,x]$ follows from the fact that the integrals in y appearing in the right-hand side of (2.5.37) vanish when y goes to 1 and the constant K_1 is the same in both sides. Therefore, we get that $r\in[0,1]\mapsto F(r)=\mathcal{F}(xr^2)$ is continuous.

Let us now select in this class those solutions who are $\mathscr{C}^1$. Notice that the solutions $\mathcal{F}$ are $\mathscr{C}^1$ in $[0,x]\backslash\{1\}$. So it remains to study the derivatives from the left and the right sides of $y=1$. Since F_n and $\widehat{F}_n$ have no derivatives on the left at 1 and verify

$$|F_n'(y)|\sim C\ln(1-y)\quad\text{and}\quad|\widehat{F}_n'(y)|\sim C\ln(1-y),$$

for any $y\in[0,1)$, see (C.0.6), then the first members of (2.5.37) have no derivatives at 1. This implies necessary that $K_1=0$. Moreover, one gets

$$\mathscr{F}_{K_1,K_2}'(1^-)=\frac{\int_0^1\frac{F_n(s)}{1-s}\mathcal{R}_x\left(\frac{s}{x}\right)ds}{F_n(1)}. \tag{2.5.44}$$

Since $F_n(1)>0$ and $\mathcal{R}_x$ is Hölder continuous near 1, then the convergence of this integral is equivalent to the condition $\mathcal{R}_x\left(\frac{1}{x}\right)=0$. Now, using (2.5.37), we deduce that the right derivative of $\mathscr{F}_{K_1,K_2}$ at 1 is given by

$$\mathscr{F}_{K_1,K_2}'(1^+)=-\frac{1}{\widehat{F}_n(1)}\left(K_2+\int_{\frac{1}{x}}^1\frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s}\mathcal{R}_x(\frac{1}{sx})ds\right). \tag{2.5.45}$$

Combining (2.5.44) with (2.5.45), we deduce that F admits a derivative at 1 if and only if

$$K_2=-\frac{\widehat{F}_n(1)}{F_n(1)}\int_0^1\frac{F_n(s)}{1-s}\mathcal{R}_x\left(\frac{s}{x}\right)ds-\int_{\frac{1}{x}}^1\frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s}\mathcal{R}_x(\frac{1}{sx})ds$$

Thus, the $\mathscr{C}^1-$solution to (2.5.37) is given by

$$\mathcal{F}(y) = y^n F_n(y) \int_1^y \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}_x\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \, \mathbf{1}_{[0,1]}(y)$$

$$-y^{a_n} \widehat{F}_n\left(\frac{1}{y}\right) \int_{\frac{1}{y}}^1 \frac{\frac{\widehat{F}_n(1)}{F_n(1)} \int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}_x\left(\frac{s}{x}\right) ds + \int_\tau^1 \frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s} \mathcal{R}_x\left(\frac{1}{sx}\right) ds}{\tau^{n+1-2a_n} \widehat{F}_n^2(\tau)} d\tau \, \mathbf{1}_{[1,\infty)}(y),$$

and, therefore, the solution to (2.5.14) takes the form

$$F(r) = \mathcal{F}(xr^2). \tag{2.5.46}$$

This implies in particular that there is only one $\mathscr{C}^1$ solution to (2.5.14) and satisfies $F(x^{-\frac{1}{2}}) = 0$. The case $n = 1$ is very special since $F_1(x) = 1 - x$ and hence F_1 vanishes at 1. As in the previous discussion, the solution of (2.5.16) in $[0, 1]$ is given by

$$\mathcal{F}(y) = y(1-y)\left[K_1 + \int_0^y \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x}) ds}{\tau^2(1-\tau)^2} d\tau\right],$$

whereas for $y \in [1, \infty)$ the solution reads as

$$\mathcal{F}(y) = \frac{1}{y} \widehat{F}_1\left(\frac{1}{y}\right)\left[K_3 + \int_{\frac{1}{y}}^1 \frac{K_2 + \int_{\frac{1}{x}}^\tau \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}_x(\frac{1}{sx}) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau\right],$$

where K_1, K_2 and K_3 are constants. We can check that the continuity of $\mathcal{F}$ at 1 is satisfied if and only if

$$\begin{aligned} K_3 =& \frac{1}{\hat{F}_1(1)} \int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) ds \\ =& \frac{1}{3} \int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) ds, \end{aligned}$$

by using in the last line the explicit expression of $\hat{F}_1(1)$ coming from (C.0.5):

$$\hat{F}_1(1) = F(1, 2; 4; 1) = \frac{\Gamma(4)\Gamma(1)}{\Gamma(3)\Gamma(2)} = 3.$$

Let us deal with the derivative, given by

$$\mathcal{F}'(y) = \begin{cases} (1-2y)\left[K_1 + \int_0^y \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x}) ds}{\tau^2(1-\tau)^2} d\tau\right] + \frac{1}{y(1-y)} \int_0^y \mathcal{R}_x\left(\frac{s}{x}\right) ds, & y \in [0,1], \\ -\frac{1}{y^2}\left[\widehat{F}_1\left(\frac{1}{y}\right) + \frac{1}{y}\widehat{F}_1'\left(\frac{1}{y}\right)\right]\Bigg[\frac{1}{3}\int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) ds & \\ +\int_{\frac{1}{y}}^1 \frac{K_2 + \int_{\frac{1}{x}}^\tau \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}_x(\frac{1}{sx}) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau\Bigg] + \frac{y}{\widehat{F}_1\left(\frac{1}{y}\right)}\left[K_2 + \int_{\frac{1}{x}}^{\frac{1}{y}} \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}_x\left(\frac{1}{sx}\right) ds\right], & y \geq 1. \end{cases}$$

The convergence in 0 and 1 of the first part comes from $\mathcal{R}_x(0) = \mathcal{R}_x\left(\frac{1}{x}\right) = \int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) d\tau = 0$. In which case, one gets

$$\mathcal{F}'(1^-) = -K_1 - \int_0^1 \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x}) ds}{\tau^2(1-\tau)^2} d\tau.$$

For the second part of $\mathcal{F}$, one needs $\int_0^1 \mathcal{R}_x\left(\frac{s}{x}\right) d\tau = 0$ since $\widehat{F}'$ is singular at 1 as it was mentioned before. Then,

$$\mathcal{F}'(1^+) = \frac{1}{3}\left[K_2 + \int_{\frac{1}{x}}^{\frac{1}{y}} \frac{s\widehat{F}_1(s)}{1-s}\mathcal{R}_x\left(\frac{1}{sx}\right) ds\right].$$

Clearly, we have the constraint

$$K_2 = -3\left(K_1 + \int_0^1 \frac{\int_0^\tau \mathcal{R}_x(\frac{s}{x})ds}{\tau^2(1-\tau)^2} d\tau\right) - \int_{\frac{1}{x}}^{\frac{1}{y}} \frac{s\widehat{F}_1(s)}{1-s}\mathcal{R}_x\left(\frac{1}{sx}\right) ds, \tag{2.5.47}$$

in order to obtain $\mathscr{C}^1$ solutions. If, in addition, $F'(1) = 0$, which agrees with $\mathcal{F}'(x) = 0$, we obtain the following additional equation for K_2:

$$\begin{aligned} K_2 &\left(-\frac{1}{x^2}\left[\widehat{F}_1\left(\frac{1}{x}\right) + \frac{1}{x}\widehat{F}_1'\left(\frac{1}{x}\right)\right]\int_{\frac{1}{x}}^1 \frac{d\tau}{\tau^4\widehat{F}_1^2(\tau)} + \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)}\right) \\ &= \frac{1}{x^2}\left[\widehat{F}_1\left(\frac{1}{x}\right) + \frac{1}{x}\widehat{F}_1'\left(\frac{1}{x}\right)\right]\int_{\frac{1}{x}}^1 \frac{\int_{\frac{1}{x}}^\tau \frac{s\widehat{F}_1(s)}{1-s}\mathcal{R}_x(\frac{1}{sx})ds}{\tau^4\widehat{F}_1^2(\tau)} d\tau. \end{aligned}$$

We claim that

$$-\frac{1}{x^2}\left[\widehat{F}_1\left(\frac{1}{x}\right) + \frac{1}{x}\widehat{F}_1'\left(\frac{1}{x}\right)\right]\int_{\frac{1}{x}}^1 \frac{d\tau}{\tau^4\widehat{F}_1^2(\tau)} + \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \neq 0, \tag{2.5.48}$$

for any $x > 1$, and then we can obtain the exact value of K_2 and then K_1 via the relation (2.5.47). Hence, it remains to check (2.5.48). Note that $\widehat{F}_1(z) = F(1,2;4;z)$, for $z \in (0,1)$, which is a positive and increasing function. Take $z = 1/x$ and hence, (2.5.48) is equivalent to

$$q(z) := z^3\widehat{F}_1(z)\left[\widehat{F}_1(z) + z\widehat{F}_1'(z)\right]\int_z^1 \frac{d\tau}{\tau^4\widehat{F}_1^2(\tau)} \neq 1, \tag{2.5.49}$$

for any $z \in (0,1)$. Note that q is positive and we will prove that $q(z) < 1$, for any $z \in [0,1]$. It is clear that $q(1) = 0$ and

$$\lim_{z\to 0} z^3\int_z^1 \frac{d\tau}{\tau^4\widehat{F}_1^2(\tau)} = \frac{1}{3\widehat{F}_1^2(0)} = \frac{1}{3}.$$

Therefore

$$\lim_{z\to 0} q(z) = \frac{1}{3}.$$

Moreover, since $z \in (0,1) \mapsto \frac{1}{\widehat{F}_1^2(z)}$ is decreasing then

$$\int_z^1 \frac{d\tau}{z^4\widehat{F}_1^2(\tau)} \leq \frac{1}{\widehat{F}_1^2(z)}\int_z^1 \frac{d\tau}{\tau^4} = \frac{1}{3\widehat{F}_1^2(z)}\frac{1-z^3}{z^3},$$

which implies

$$\begin{aligned} q(z) &\leq \frac{1}{3\widehat{F}_1(z)}\left[\widehat{F}_1(z) + z\widehat{F}_1'(z)\right](1-z)(z^2+z+1) \\ &\leq \left[\frac{1-z^3}{3} + \frac{z(z^2+z+1)}{3\widehat{F}_1(z)}(1-z)\widehat{F}_1'(z)\right]. \end{aligned} \tag{2.5.50}$$

From the integral representation for hypergeometric functions (C.0.2) one achieves that

$$
\begin{aligned}
(1-z)\widehat{F}_1'(z) &= 6(1-z)\int_0^1 \frac{x^2(1-x)}{(1-zx)^2}dx\\
&\leq \frac{8}{9},
\end{aligned}
$$

where we have used that $\sup\limits_{x\in[0,1]} x^2(1-x)=\frac{4}{27}$. Combining the last inequality with $1\leq \widehat{F}_1(z)$, we deduce from (2.5.50)

$$
\begin{aligned}
q(z) &\leq \left[\frac{1-z^3}{3}+\frac{8z(z^2+z+1)}{27}\right]\\
&\leq \frac{-z^3+8z^2+8z+9}{27}\\
&\leq \frac{26}{27},
\end{aligned}
$$

for any $z\in[0,1]$. Consequently we get $q(z)<1$ and the proof of (2.5.49) is completed. □

Proposition 2.5.8. *Let $A>0$ and $B\in\mathbb{R}$ such that $\frac{B}{A}\notin\left[-1,-\frac{1}{2}\right]$. Let Ω satisfy* (2.5.35) *with $\Omega\notin\mathcal{S}_{\text{sing}}$, where the last set is defined in* (2.4.10)*. Then, the following holds true:*

1. *The kernel of $D_g\widehat{G}(\Omega,0)$ is trivial in $\mathscr{C}^1(\overline{\mathbb{D}})$.*
2. *Let $h\in\mathscr{C}^0([0,1])$ and $d\in\mathscr{C}^1([0,1])$ such that*

$$
D_g\widehat{G}(\Omega,0)h=d,\quad h(re^{i\theta})=\sum_{n\in\mathbb{N}}h_n(r)\cos(n\theta),\quad d(re^{i\theta})=\sum_{n\in\mathbb{N}}d_n(r)\cos(n\theta),
$$

then, there exists an absolute constant $C>0$ such that

$$
\|h_n\|_{\mathscr{C}^0([0,1])}\leq C\|d_n\|_{\mathscr{C}^1([0,1])},\quad \forall\, n\in\mathbb{N}.
$$

3. *Coercivity with loss of derivative: for any $\alpha\in(0,1)$, there exists $C>0$ such that*

$$
\|h\|_{\mathscr{C}^0(\mathbb{D})}\leq C\|D_g\widehat{G}(\Omega,0)h\|_{\mathscr{C}_s^{2,\alpha}(\mathbb{D})}.
$$

Proof. (**1**) First, note that $x\neq x_0$, where x_0 is defined in (2.4.14), because $x_0\in(0,1)$. Then, Proposition 2.4.3 implies that the last equation in (2.5.22) admits only the zero function as a solution. We will check how the condition (2.5.39) gives us that there are no nontrivial $\mathscr{C}^1$-solutions for $n=1$. This can be done easily with the explicit expression of g given in (2.5.29) for $n=1$. Since

$$
G_1(r)=-2\left[r^4-\frac{3}{2x}r^2-\frac{3}{2}\left(1-\frac{1}{x}\right)\right],
$$

one obtains that

$$
g(r)=-16H_1(1)xr^3\left[r^4-\frac{3}{2x}r^2-\frac{3}{2}\left(1-\frac{1}{x}\right)\right].
$$

Then, condition (2.5.39) is equivalent to

$$
3x^2-3x+1=0\quad\text{or}\quad H_1(1)=0.
$$

Since $x > 1$, one has $H_1(1) = 0$ and $g \equiv 0$. Then, $\mathcal{R} \equiv 0$. By Lemma 2.5.7, one has that the solution of (2.5.27) with $H_1(0) = 0$ and $H_1'(1) = 0$ is the trivial one: $H_1(r) \equiv 0$. Coming back to (2.5.28), we deduce that $h_1 \equiv 0$. Therefore, the only $\mathscr{C}^1$-solution is the zero function, which implies finally that the kernel is trivial.

Let us now deal with $n \geq 2$. Applying Lemma 2.5.7 to the equation (2.5.27) with (2.5.38) we get that this equation admits a $\mathscr{C}^1$ solution if and only if

$$H_n(1)G_n(x^{-\frac{1}{2}}) = 0. \tag{2.5.51}$$

Using the expression (2.5.9), we find that

$$G_n(x^{-\frac{1}{2}}) = -\frac{An(n+1)}{4(n+2)}x^{\frac{1-n}{2}}P_n\left(x^{-1}\right),$$

and from (2.5.11) one has

$$P_n\left(x^{-1}\right) = -\frac{1}{n+1}\left(\frac{1}{x^2} + \frac{A+2B}{A}\frac{n+2}{n}\right). \tag{2.5.52}$$

With the assumptions $\frac{A+2B}{A} \notin [-1,0]$ and $x > 1$ one gets

$$P_n\left(x^{-1}\right) \neq 0, \quad \forall n \geq 2,$$

obtaining from (2.5.51) that

$$H_n(1) = 0, \quad \forall n \geq 2.$$

Coming back to (2.5.27) we find that the source term is vanishing everywhere. Now, from Lemma 2.5.7 and (2.5.37) we infer that

$$H_n(r) = 0, \quad \forall n \geq 2, r \in [0,1].$$

Inserting this into (2.5.22), we obtain $h_n \equiv 0$ for any $n \in \mathbb{N}^\star$. Finally, this implies that the vanishing function is the only element of the kernel.

(**2**) To get this result we should derive *a priori estimates* for the solutions to the equation

$$D_g\widehat{G}(\Omega,0)h = d.$$

The pre-image equation is equivalent to solve the infinite-dimensional system

$$\begin{aligned} \frac{\frac{1}{x}-r^2}{8}h_n(r) - \frac{r}{n}\left[A_nG_n(r) + \frac{1}{2r^{n+1}}H_n(r)\right] &= d_n(r), \quad \forall n \geq 1, \\ \frac{\frac{1}{x}-r^2}{8}h_0(r) - \int_r^1 \frac{1}{\tau}\int_0^\tau sh_0(s)dsd\tau &= d_0(r), \end{aligned} \tag{2.5.53}$$

where we use the notations of (2.5.9), (2.5.10) and (2.5.13). We first analyze the case of large n, for which we can apply the contraction principle and get the announced estimates. Later, we will deal with low frequencies, which is more delicate and requires the integral representation (2.5.46). Let us first work with large values of n. Observe that the first equation of (2.5.53) can be transformed into

$$h_n(r) = \frac{8}{\frac{1}{x}-r^2}\left[\frac{A_n}{n}rG_n(r) + \frac{H_n(r)}{2nr^n} + d_n(r)\right]. \tag{2.5.54}$$

The estimate of A_n, defined by (2.5.10) and (2.5.13), can be done as follows

$$|A_n| \le \frac{\int_0^1 s^{n+1}|h_n(s)|ds}{2n|\widehat{\Omega}_n - \Omega|}.$$

Keeping in mind the relation (2.4.12) and the assumption $x \notin \widehat{\mathcal{S}}_{\text{sing}}$, we obtain

$$\sup_{n\ge 1}\frac{1}{|\widehat{\Omega}_n - \Omega|} < +\infty,$$

and, on the other hand, it is obvious that

$$\int_0^1 s^{n+1}|h_n(s)|ds \le \frac{\|h_n\|_{\mathscr{C}^0([0,1])}}{n+2}.$$

Combining the preceding estimates, we find

$$|A_n| \le \frac{C}{n^2}\|h_n\|_{\mathscr{C}^0([0,1])}, \tag{2.5.55}$$

for some constant C independent of n. Let us remark that according to the equation (2.5.54), one should get the compatibility condition

$$\frac{A_n}{n}x^{-\frac{1}{2}}G_n(x^{-\frac{1}{2}}) + \frac{H_n(x^{-\frac{1}{2}})}{2nx^{-\frac{n}{2}}} + d_n(x^{-\frac{1}{2}}) = 0.$$

Hence, applying the Mean Value Theorem, combined with (2.5.55), we obtain

$$\begin{aligned}\|h_n\|_{\mathscr{C}^0([0,1])} \quad \le \quad & C\frac{\|h_n\|_{\mathscr{C}^0([0,1])}}{n^3}\left\|(rG_n(r))'\right\|_{\mathscr{C}^0([0,1])} \\ & + \frac{1}{2n}\left\|\left(\frac{H_n(r)}{r^n}\right)'\right\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])},\end{aligned} \tag{2.5.56}$$

where in this inequality C may depend on x but not on n. Now, it is straightforward to check that

$$\left|(rG_n(r))'\right| \le Cn^2, \quad \forall\, r \in [0,1], \tag{2.5.57}$$

and second

$$\left(\frac{H_n(r)}{r^n}\right)' = nr^{n-1}\int_r^1 \frac{h_n(s)}{s^{n-1}}ds - \frac{n}{r^{n+1}}\int_0^r s^{n+1}h_n(s)ds.$$

From this latter identity, we infer that

$$\left|\left(\frac{H_n(r)}{r^n}\right)'\right| \quad \le \quad \left(\frac{n}{n-2} + \frac{n}{n+2}\right)\|h_n\|_{\mathscr{C}^0([0,1])} \le C\|h_n\|_{\mathscr{C}^0([0,1])}, \quad \forall\, r \in [0,1],$$

for any $n \ge 3$. Consequently, we get

$$\left\|\left(\frac{H_n(r)}{r^n}\right)'\right\|_{\mathscr{C}^0([0,1])} \le C\|h_n\|_{\mathscr{C}^0([0,1])}. \tag{2.5.58}$$

Plugging (2.5.58) and (2.5.57) into (2.5.56), we find

$$\|h_n\|_{\mathscr{C}^0([0,1])} \le \frac{C}{n}\|h_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])}.$$

Hence, choosing n_0 large enough we deduce that

$$\|h_n\|_{\mathscr{C}^0([0,1])} \leq \|d_n'\|_{\mathscr{C}^0([0,1])}, \quad \forall\, n \geq n_0.$$

Next, we deal with the cases $1 \leq n \leq n_0$. The preceding argument fails and to invert the operator we recover explicitly the solution h_n from d_n according to the integral representation (2.5.46). For this purpose we will proceed as in the range study in Subsection 2.5.3. By virtue of (2.5.72), we find that H_n satisfies an equation of the type (2.5.14). Thus, using (2.5.46), we deduce

$$\begin{aligned} H_n(r) =& x^n r^{2n} F_n(xr^2) \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{\tau^{n+1}F_n^2(\tau)} d\tau\, \mathbf{1}_{[0,x^{-\frac{1}{2}}]}(r) + x^{a_n} r^{2a_n} \widehat{F}_n\left(\frac{1}{xr^2}\right) \\ &\times \int_{\frac{1}{xr^2}}^1 \frac{\frac{\widehat{F}_n(1)}{F_n(1)}\int_0^1 \frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds - \int_\tau^1 \frac{s^{n-a_n-1}\widehat{F}_n(s)}{1-s}\mathcal{R}(\frac{1}{sx})ds}{\tau^{n+1-2a_n}\widehat{F}_n^2(\tau)} d\tau\, \mathbf{1}_{(x^{-\frac{1}{2}},1]}(r), \end{aligned} \tag{2.5.59}$$

for $n \geq 2$, and

$$\begin{aligned} H_1(r) =& xr^2(1-xr^2)\left[K_1 - \int_{xr^2}^0 \frac{\int_0^\tau \mathcal{R}(\frac{s}{x})ds}{\tau^2(1-\tau)^2} d\tau\right] \mathbf{1}_{[0,x^{-\frac{1}{2}}]}(r) \\ &+ \frac{1}{xr^2}\widehat{F}_1\left(\frac{1}{xr^2}\right)\left[\frac{1}{3}\int_0^1 \mathcal{R}\left(\frac{s}{x}\right)ds + \int_{\frac{1}{xr^2}}^1 \frac{K_2 - \int_\tau^1 \frac{s\widehat{F}_1(s)}{1-s}\mathcal{R}(\frac{1}{sx})ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau\right] \mathbf{1}_{(x^{-\frac{1}{2}},1]}(r), \end{aligned}$$

where K_1 and K_2 are given in (2.5.42)-(2.5.43). From (2.5.38), we get

$$\mathcal{R}(y) = \frac{1}{4x}\left[AA_n \frac{n(n+1)}{n+2} y^n P_n(y) - 4xny^{\frac{n}{2}} d_n(y^{\frac{1}{2}})\right],$$

where P_n is defined in (2.5.11).

Let us relate A_n with d_n. This can be obtained from the constraint $\mathcal{R}(\frac{1}{x}) = 0$, which implies

$$A_n = \frac{(n+2)x^{\frac{n}{2}+1} d_n(x^{-\frac{1}{2}})}{A(n+1)P_n(x^{-1})}.$$

Let us remark that this relation is different from (2.5.55), which is not useful for low frequencies. Consequently, using (2.5.52) we infer that

$$|A_n| \leq C\|d_n\|_{\mathscr{C}^0([0,1])}, \quad \forall n \in [1, n_0].$$

Concerning $\mathcal{R}$, we can get successively

$$|\mathcal{R}(y)| \leq C\left[|A_n|y^n + y^{\frac{n}{2}}\left|d_n(y^{\frac{1}{2}})\right|\right], \quad \forall\, y \in [0,1], \tag{2.5.60}$$

which implies that

$$||\mathcal{R}||_{\mathscr{C}^0([0,1])} \leq C\left[|A_n| + ||d_n||_{\mathscr{C}^0([0,1])}\right] \leq C||d_n||_{\mathscr{C}^0([0,1])},$$

for any $n \geq 1$. In the case $n \geq 2$, we also obtain that

$$|\mathcal{R}'(y)| \leq C\left[|A_n|y^{n-1} + y^{\frac{n}{2}-1}\left(\left|d_n(y^{\frac{1}{2}})\right| + \left|d_n'(y^{\frac{1}{2}})\right|\right)\right], \quad \forall\, y \in [0,1],$$

which amounts to

$$\|\mathcal{R}'\|_{\mathscr{C}^0([0,1])} \leq C\left(\|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])}\right), \quad \forall n \geq 2. \tag{2.5.61}$$

Note that this last estimate can not be used for $n = 1$ since $\mathcal{R}$ is only Hölder continuous. Then, we can find in this case that

$$||\mathcal{R}||_{\mathscr{C}^{0,\gamma}([0,1])} \leq C||d_n||_{\mathscr{C}^{0,\gamma}([0,1])},$$

for $\mu = \min\left(\frac{1}{2}, \alpha\right)$.

Let us begin with $n \geq 2$. Using the boundedness property of F_n, which we shall see later in Lemma 2.6.4, combined with an integration by parts imply

$$\begin{aligned}\left|\int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{\tau^{n+1}F_n^2(\tau)}d\tau\right| \leq & C\left|\int_1^{xr^2} \frac{\int_0^\tau \frac{|\mathcal{R}(s/x)|}{1-s}ds}{\tau^{n+1}}d\tau\right| \\ \leq & C\int_0^1 \frac{\left|\mathcal{R}(\frac{s}{x})\right|}{1-s}ds + Cr^{-2n}\int_0^{xr^2} \frac{\left|\mathcal{R}(\frac{s}{x})\right|}{1-s}ds \\ & + \left|\int_1^{xr^2} \frac{\tau^{-n}\left|\mathcal{R}(\frac{\tau}{x})\right|}{1-\tau}d\tau\right|. \end{aligned} \tag{2.5.62}$$

We discuss first the case $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$. From the compatibility assumption (2.5.39), we recall that $\mathcal{R}(\frac{1}{x}) = 0$, and therefore we deduce from (2.5.60) and (2.5.61) that

$$\begin{aligned}\int_0^1 \frac{\left|\mathcal{R}(\frac{s}{x})\right|}{1-s}ds + r^{-2n}\int_0^{xr^2} \frac{\left|\mathcal{R}(\frac{s}{x})\right|}{1-s}ds \leq & \int_0^1 \frac{\left|\mathcal{R}(\frac{s}{x})\right|}{1-s}ds + Cr^{-2n}\int_0^{xr^2}\left|\mathcal{R}\left(\frac{s}{x}\right)\right|ds \\ \leq & C\left(|A_n| + \|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])} + r^{-n+2}\|d_n\|_{\mathscr{C}^0([0,1])}\right),\end{aligned}$$

for $2 \leq n \leq n_0$. In a similar way to the last integral term of (2.5.62), we split it as follows using the estimates (2.5.60) and (2.5.61)

$$\begin{aligned}\int_{xr^2}^1 \frac{\tau^{-n}\left|\mathcal{R}(\frac{\tau}{x})\right|}{1-\tau}d\tau = & \int_{\frac{1}{2}}^1 \frac{\tau^{-n}\left|\mathcal{R}(\frac{\tau}{x})\right|}{1-\tau}d\tau + \int_{xr^2}^{\frac{1}{2}} \frac{\tau^{-n}\left|\mathcal{R}(\frac{\tau}{x})\right|}{1-\tau}d\tau \\ \leq & C\left(|A_n| + \|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])}\right) + C\int_{xr^2}^{\frac{1}{2}} \tau^{-n}\left|\mathcal{R}\left(\frac{\tau}{x}\right)\right|d\tau \\ \leq & C\left(|A_n| + \|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])} + Cr^{-n+2}\|d_n\|_{\mathscr{C}^0([0,1])}\right).\end{aligned}$$

Putting together the preceding estimates we get that

$$\left|\int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{\tau^{n+1}F_n^2(\tau)}d\tau\right| \leq C\left(|A_n| + \|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])} + Cr^{-n+2}\|d_n\|_{\mathscr{C}^0([0,1])}\right),$$

for $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$. Plugging this into (2.5.59) yields

$$|H_n(r)| \leq C\left(|A_n| + \|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])}\right)r^{2n} + Cr^{n+2}\|d_n\|_{\mathscr{C}^0([0,1])}, \tag{2.5.63}$$

for $r\in[0,\frac12 x^{-\frac12}]$ and $2\leq n\leq n_0$. Now, we wish to estimate the derivative of $\frac{H_n(r)}{r^n}$. Coming back to (2.5.59), we deduce from elementary computations that

$$H_n'(r)=H_n(r)\left(\frac{2n}{r}+\frac{2xrF_n'(xr^2)}{F_n(xr^2)}\right)+\frac{2}{rF_n(xr^2)}\int_0^{xr^2}\frac{F_n(s)}{1-s}\mathcal{R}\left(\frac{s}{x}\right)ds,$$

for $r\in[0,\frac12 x^{-\frac12}]$. By (2.5.63), we get

$$\left|H_n(r)\left(\frac{2n}{r}+\frac{2xrF_n'(xr^2)}{F_n(xr^2)}\right)\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right)r^{2n-1}+Cr^{n+1}\|d_n\|_{\mathscr{C}^0([0,1])}.$$

Concerning the integral term, it suffices to apply (2.5.60) in order to get

$$\frac{2}{rF_n(xr^2)}\int_0^{xr^2}\frac{F_n(s)}{1-s}\left|\mathcal{R}\left(\frac{s}{x}\right)\right|ds\leq\frac{C}{r}\int_0^{xr^2}\left|\mathcal{R}\left(\frac{s}{x}\right)\right|ds\leq C\left(|A_n|r^{2n+1}+r^{n+1}\|d_n\|_{\mathscr{C}^0([0,1])}\right).$$

Hence, combining the preceding estimates leads to

$$\left|H_n'(r)\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right)r^{2n-1}+Cr^{n+1}\|d_n\|_{\mathscr{C}^0([0,1])}. \tag{2.5.64}$$

This estimate together with (2.5.63) allows getting

$$\left|\left(\frac{H_n(r)}{r^n}\right)'\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right),\quad\forall\, r\in\left[0,\frac12 x^{-\frac12}\right],$$

for $2\leq n\leq n_0$. The case $n=1$ can be done using similar ideas since we only have the singularity at 0 in this interval. Note that K_1 and K_2 can be estimated in terms of $\mathcal{R}$ having

$$|K_1|,|K_2|\leq||\mathcal{R}||_{\mathscr{C}^{0,\gamma}([0,1])}.$$

Let us now move to the intermediate case $x\in[\frac12 x^{-\frac12},x^{-\frac12}]$. Then, there is no singularity in this range except for $r=x^{-\frac12}$ due to the logarithmic behavior of F_n' close to this point. This logarithmic divergence can be controlled from the smallness of the integral term in H_n. Let us show the idea. When we differentiate H_n, we obtain one term of the type

$$2x^{n+1}r^{2n+1}F_n'(xr^2)\int_1^{xr^2}\frac{\int_0^\tau\frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{\tau^{n+1}F_n^2(\tau)}d\tau,$$

where we notice the logarithmic singularity coming from F_n' at 1. However, one has

$$\begin{aligned}\left|2x^{n+1}r^{2n+1}F_n'(xr^2)\int_1^{xr^2}\frac{\int_0^\tau\frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{\tau^{n+1}F_n^2(\tau)}d\tau\right|&\leq C\left|F_n'(xr^2)(1-xr^2)\frac{\int_0^{xr^2}\frac{F_n(s)}{1-s}\mathcal{R}(\frac{s}{x})ds}{(xr^2)^{n+1}F_n^2(xr^2)}\right|\\&\leq C\left(||\mathcal{R}||_{\mathscr{C}^0([0,1])}+||\mathcal{R}'||_{\mathscr{C}^0([0,1])}\right)\\&\leq C\left(||d_n||_{\mathscr{C}^0([0,1])}+||d_n'||_{\mathscr{C}^0([0,1])}\right).\end{aligned}$$

Therefore, after straightforward efforts on (2.5.59) using (2.5.61), it implies that

$$\left|H_n'(r)\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right), \tag{2.5.65}$$

for $2\leq n\leq n_0$. Hence, we obtain

$$\left|\left(\frac{H_n(r)}{r^n}\right)'\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right),\quad \forall\, r\in\left[\frac{1}{2}x^{-\frac12},x^{-\frac12}\right],$$

for $2\leq n\leq n_0$. In this interval, the case $n=1$ is different. This is because we have not singularity coming from the hypergeometric function but we do have it coming from the integral. Hence, some more manipulations are needed. In this case, H_1 reads as

$$H_1(r)=xr^2(1-xr^2)\left[K_1+\int_0^{\frac12}\frac{\int_0^\tau\mathcal{R}(\frac{s}{x})ds}{\tau^2(1-\tau)^2}d\tau+\int_{\frac12}^{xr^2}\frac{\int_0^\tau\mathcal{R}(\frac{s}{x})ds}{\tau^2(1-\tau)^2}d\tau\right].$$

The first integral term can be treated as in the previous computations in the interval $[0,\frac12 x^{-\frac12}]$. Let us focus on the singular integral term. By a change of variables, one has

$$\begin{aligned}\left|\int_{\frac12}^{xr^2}\frac{\int_0^\tau\mathcal{R}(\frac{s}{x})ds}{\tau^2(1-\tau)^2}d\tau\right|&=\left|\frac{\int_0^{xr^2}\mathcal{R}\left(\frac{s}{x}\right)ds}{x^2r^4(1-xr^2)}-8\int_0^{\frac12}\mathcal{R}\left(\frac{s}{x}\right)ds-\int_{\frac12}^{xr^2}\frac{\tau\mathcal{R}(\frac{\tau}{x})-2\int_0^\tau\mathcal{R}(\frac{s}{x})ds}{\tau^3(1-\tau)}d\tau\right|\\&\leq C\left(||\mathcal{R}||_{\mathscr{C}^0([0,1])}+||\mathcal{R}||_{\mathscr{C}^{0,\gamma}([0,1])}\right),\end{aligned}$$

where we have used (2.5.39)-(2.5.41). Then, we obtain

$$|H_1(r)|\leq C(1-xr^2)||\mathcal{R}||_{\mathscr{C}^{0,\gamma}([0,1])}.$$

Similar arguments can be done to find that

$$\left|\left(\frac{H_1(r)}{r}\right)'\right|\leq C||d_n||_{\mathscr{C}^{0,\gamma}([0,1])},$$

for any $r\in[\frac12 x^{-\frac12},x^{-\frac12}]$.

It remains to establish similar results for the case $r\in[x^{-\frac12},1]$. With this aim we use the second integral in (2.5.59). Notice that the only singular point is $r=x^{-\frac12}$ due to the logarithmic singularity of the hypergeometric function $\widehat{F}_n'$ defined in (2.5.36). One can check that this function is strictly increasing, positive and satisfies

$$1\leq\widehat{F}_n(r)\leq\sup_{n\in\mathbb{N}}\widehat{F}_n(1)<+\infty,\quad \forall r\in[0,1].$$

As in the previous interval, the smallness of the integral term controls this singularity. The same happens for the case $n=1$. Hence, we get

$$\left|\left(\frac{H_n(r)}{r^n}\right)'\right|\leq C\left(|A_n|+\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right),\quad \forall\, r\in[x^{-\frac12},1],$$

for $1\leq n\leq n_0$. Therefore, in all the cases we have

$$\left|\left(\frac{H_n(r)}{r^n}\right)'\right|\leq C\left(\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right),\quad \forall\, r\in[0,1],\tag{2.5.66}$$

for $1\leq n\leq n_0$. Applying the Mean Value Theorem to (2.5.54), and using (2.5.66) and (2.5.57), allow us to obtain

$$\|h_n\|_{\mathscr{C}^0([0,1])}\leq C\left(\|d_n\|_{\mathscr{C}^0([0,1])}+\|d_n'\|_{\mathscr{C}^0([0,1])}\right),\quad \forall\, n\in[1,n_0].$$

Similar arguments can be done in order to deal with the equation for $n = 0$. Note that the resolution of this equation is similar to the work done in Proposition 2.4.4.

Then, combining all the estimates, it yields

$$\|h_n\|_{\mathscr{C}^0([0,1])} \leq C\left(\|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])}\right), \quad \forall\, n \in \mathbb{N}.$$

This achieves the proof of the announced result.

(**3**) Let us recall the formula for the Fourier coefficients

$$d_n(r) = \frac{1}{\pi}\int_0^{2\pi} d(r\cos\theta, r\sin\theta)\cos(n\theta)d\theta.$$

We can prove that

$$\|d_n\|_{\mathscr{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}}\|d\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})},$$

for $n \geq 1$. This can be done integrating by parts as

$$d_n(r) = -\frac{r}{n\pi}\int_0^{2\pi} \nabla d(r\cos\theta, r\sin\theta)\cdot(-\sin\theta, \cos\theta)\sin(n\theta)d\theta,$$

and writing it as

$$\begin{aligned}
d_n(r) =& \frac{r}{n\pi}\int_0^{2\pi} \nabla d\left(r\cos\left(\theta+\frac{\pi}{n}\right), r\sin\left(\theta+\frac{\pi}{n}\right)\right)\cdot\left(-\sin\left(\theta+\frac{\pi}{n}\right), \cos\left(\theta+\frac{\pi}{n}\right)\right)\sin(n\theta)d\theta\\
=&\frac{r}{2n\pi}\int_0^{2\pi}\left[\nabla d\left(r\cos\left(\theta+\frac{\pi}{n}\right), r\sin\left(\theta+\frac{\pi}{n}\right)\right)\cdot\left(-\sin\left(\theta+\frac{\pi}{n}\right), \cos\left(\theta+\frac{\pi}{n}\right)\right)\right.\\
&\left. - \nabla d\left(r\cos\theta, r\sin\theta\right)\cdot(-\sin\theta, \cos\theta)\right]\sin(n\theta)d\theta.
\end{aligned}$$

Consequently,

$$|d_n(r)| \leq \frac{C}{n^{1+\alpha}}||d||_{\mathscr{C}^{1,\alpha}(\mathbb{D})}.$$

With similar arguments, one achieves that

$$\|d_n\|_{\mathscr{C}^0([0,1])} + \|d_n'\|_{\mathscr{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}}\|d\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})},$$

and then

$$\|h_n\|_{\mathscr{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}}\|d\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}.$$

Therefore, we obtain

$$\|h\|_{\mathscr{C}^0(\mathbb{D})} \leq \sum_{n\in\mathbb{N}}\|h_n\|_{\mathscr{C}^0([0,1])} \leq C\|d\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})},$$

which completes the proof. □

The next target is to provide the proof of Theorem 2.5.6.

Proof of Theorem 2.5.6. In a small neighborhood of the origin we have the following decomposition through Taylor expansion at the second order

$$\widehat{G}(\Omega,h)=D_g\widehat{G}(\Omega,0)(h)+\frac{1}{2}D^2_{g,g}\widehat{G}(\Omega,0)(h,h)+\mathscr{R}_2(\Omega,h),$$

where $\mathscr{R}_2(\Omega,h)$ is the remainder term, which verifies

$$\|\mathscr{R}_2(\Omega,h)\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq\frac{1}{6}\|D^3_{g,g,g}\widehat{G}(\Omega,g)(h,h,h)\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})},$$

where $g=th$ for some $t\in(0,1)$. We intend to show the following,

$$\left\|\frac{1}{2}D^2_{g,g}\widehat{G}(\Omega,0)(h,h)+\mathscr{R}_2(\Omega,h)\right\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq C\|h\|^{\delta_1}_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\|h\|^{\delta_2}_{\mathscr{C}^{0}(\mathbb{D})},\tag{2.5.67}$$

for some $\delta_1>0$ and $\delta_2\geq 1$, getting the last bound will be crucial in our argument. First, let us deal with the second derivative of $\widehat{G}$. Straightforward computations, similar to what was done in Proposition 2.4.1, lead to

$$\begin{aligned}
D^2_{g,g}\widehat{G}(\Omega,0)(h,h)(z)=&-\frac{h(z)^2}{8A}+\frac{\text{Re}}{\pi}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(z)-\partial_g\phi(\Omega,0)(h)(y)}{z-y}h(y)dA(y)\\
&+\frac{2}{\pi}\int_{\mathbb{D}}\log|z-y|h(y)\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\\
&-\frac{\text{Re}}{2\pi}\int_{\mathbb{D}}\frac{(\partial_g\phi(\Omega,0)(h)(z)-\partial_g\phi(\Omega,0)(h)(y))^2}{(z-y)^2}f_0(y)dA(y)\\
&+\frac{2\text{Re}}{\pi}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(z)-\partial_g\phi(\Omega,0)(h)(y)}{z-y}f_0(y)\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\\
&-\Omega|\partial_g\phi(\Omega,0)(h)(z)|^2+\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)|\partial_g\phi(\Omega,0)(h)'(y)|^2dA(y)\\
&-\Omega\text{Re}\left[\partial^2_{g,g}\phi(\Omega,0)(h,h)(z)\right]\\
&+\frac{\text{Re}}{2\pi}\int_{\mathbb{D}}\frac{\partial^2_{g,g}\phi(\Omega,0)(h,h)(z)-\partial^2_{g,g}\phi(\Omega,0)(h,h)(y)}{z-y}f_0(y)dA(y)\\
&+\frac{1}{\pi}\int_{\mathbb{D}}\log|z-y|f_0(y)\text{Re}\left[\partial^2_{g,g}\phi(\Omega,0)(h,h)'(y)\right]dA(y).
\end{aligned}$$

Recall the relation between $\partial_g\phi(\Omega,0)(h)$ and h

$$\partial_g\phi(\Omega,0)(h)(z)=z\sum_{n\geq 1}A_nz^n,\quad A_n=\frac{\displaystyle\int_0^1 s^{n+1}h_n(s)ds}{2n\big(\widehat{\Omega}_n-\Omega\big)}.$$

By Proposition 2.3.3, one has that

$$\|\partial_g\phi(\Omega,0)h\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq C\|h\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}.$$

We claim that one can reach the limit case,

$$\|\partial_g\phi(\Omega,0)h\|_{\mathscr{C}^{1}(\mathbb{D})}\leq ||h||_{\mathscr{C}^{0}(\mathbb{D})}.\tag{2.5.68}$$

The last estimate can be done using Proposition 2.2.2 in order to work in $\mathbb{T}$ as follows

$$\begin{aligned}\left|\partial_g\phi(\Omega,0)h'(e^{i\theta})\right| &= \left|\sum_{n\geq 1}\int_0^1 s^{n+1}h_n(s)dse^{in\theta}\right|\\ &= \frac{1}{2}\left|\sum_{n\geq 1}\int_0^{2\pi}\int_0^1 s^{n+1}h(se^{i\theta'})\cos(n\theta')e^{in\theta}dsd\theta'\right|\\ &= \frac{1}{4}\left|\sum_{n\geq 1}\int_0^{2\pi}\int_0^1 s^{n+1}h(se^{i\theta'})\big(e^{in\theta'}+e^{-in\theta'}\big)e^{in\theta}dsd\theta'\right|\\ &= \frac{1}{4}\left|\sum_{n\geq 1}\int_0^{2\pi}\int_0^1 s^{n+1}h(se^{i\theta'})\left(e^{in(\theta+\theta')}+e^{in(\theta-\theta')}\right)dsd\theta'\right|.\end{aligned}$$

Using Fubini we deduce that

$$\begin{aligned}\left|\partial_g\phi(\Omega,0)h'(e^{i\theta})\right| =&\frac{1}{4}\left|\int_0^{2\pi}\int_0^1 s^2h(se^{i\theta'})\left(\frac{e^{i(\theta+\theta')}}{1-se^{i(\theta+\theta')}}+\frac{e^{i(\theta-\theta')}}{1-se^{i(\theta-\theta')}}\right)dsd\theta'\right|\\ \leq& C||h||_{\mathscr{C}^0(\mathbb{D})}.\end{aligned}$$

The latter estimate follows from the convergence of the double integral

$$\int_0^{2\pi}\int_0^1\frac{dsd\theta}{|1-se^{i\theta}|}<\infty.$$

The next step is to deal with $\partial^2_{g,g}\phi(\Omega,0)(h,h)$. By differentiating it, similarly to the proof of in Proposition 2.3.3, we obtain

$$\begin{aligned}\partial^2_{g,g}\phi(\Omega,0)(h,h) =& -\frac{1}{2}\partial_\phi F(\Omega,0,0)^{-1}\left[\partial^2_{g,g}F(\Omega,0,0)(h,h)+2\partial^2_{g,\phi}F(\Omega,0,0)(h,\partial_g\phi(\Omega,0)h)\right.\\ &\left.+\partial^2_{\phi,\phi}F(\Omega,0,0)(\partial_g\phi(\Omega,0)h,\partial_g\phi(\Omega,0)h)\right]\\ =& -\frac{1}{2}\sum_{n\geq 1}\frac{\rho_n}{n(\Omega-\widehat{\Omega}_n)}z^{n+1},\end{aligned}$$

where $\rho(w)=\sum_{n\geq 1}\rho_n\sin(n\theta)$ and

$$\begin{aligned}\rho(w) =&\partial^2_{g,g}F(\Omega,0,0)(h,h)(w)+2\partial^2_{g,\phi}F(\Omega,0,0)(h,\partial_g\phi(\Omega,0)h)(w)\\ &+\partial^2_{\phi,\phi}F(\Omega,0,0)(\partial_g\phi(\Omega,0)h,\partial_g\phi(\Omega,0)h)(w)\\ =&\mathrm{Im}\Bigg[-\frac{w\partial_g\phi(\Omega,0)(h)'(w)}{\pi}\int_{\mathbb{D}}\frac{h(y)}{w-y}dA(y)\\ &+\frac{w}{\pi}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(w)-\partial_g\phi(\Omega,0)(h)(y)}{(w-y)^2}h(y)dA(y)\\ &-\frac{2w}{\pi}\int_{\mathbb{D}}\frac{h(y)}{w-y}\mathrm{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\;+2\Omega\,\overline{\partial_g\phi(\Omega,0)(h)(w)}\partial_g\phi(\Omega,0)(h)'(w)w\\ &+\frac{w\partial_g\phi(\Omega,0)(h)'(w)}{\pi}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(w)-\partial_g\phi(\Omega,0)(h)(y)}{(w-y)^2}f_0(y)dA(y)\end{aligned}$$

$$
\begin{aligned}
&-\frac{2w\partial_g\phi(\Omega,0)(h)'(w)}{\pi}\int_{\mathbb{D}}\frac{f_0(y)}{w-y}\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\\
&-\frac{w}{\pi}\int_{\mathbb{D}}\frac{\left[\partial_g\phi(\Omega,0)(h)(w)-\partial_g\phi(\Omega,0)(h)(y)\right]^2}{(w-y)^3}f_0(y)dA(y)\\
&+\frac{2w}{\pi}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(w)-\partial_g\phi(\Omega,0)(h)(y)}{(w-y)^2}f_0(y)\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\\
&-\frac{w}{\pi}\int_{\mathbb{D}}\frac{f_0(y)}{w-y}|\partial_g\phi(\Omega,0)(h)'(y)|^2dA(y)\Bigg].
\end{aligned}
$$

By Proposition 2.3.7 and due to the fact that $\partial^2_{g,g}\phi(\Omega,0)$ can be seen as a convolution operator, one has that

$$||\partial^2_{g,g}\phi(\Omega,0)||_{\mathscr{C}^{k,\alpha}(\mathbb{D})}\leq C||\rho||_{\mathscr{C}^{k,\alpha}(\mathbb{T})}.$$

Moreover, we claim that

$$||\rho||_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\leq C||h||^{\sigma_1}_{\mathscr{C}^{2,\alpha}(\mathbb{D})}||h||^{\sigma_2}_{\mathscr{C}^{0}(\mathbb{D})},\quad k=0,1,2, \tag{2.5.69}$$

with $\sigma_2\geq 1$. First, we use the interpolation inequalities for Hölder spaces

$$||h||_{\mathscr{C}^{k,\alpha}(\mathbb{D})}\leq C||h||^{\beta}_{\mathscr{C}^{k_1,\alpha_1}(\mathbb{D})}||h||^{1-\beta}_{\mathscr{C}^{k_2,\alpha_2}(\mathbb{D})}, \tag{2.5.70}$$

for k, k_1 and k_2 non negative integers, $0\leq\alpha_1,\alpha_2\leq 1$ and

$$k+\alpha=\beta(k_1+\alpha_1)+(1-\beta)(k_2+\alpha_2),$$

where $\beta\in(0,1)$. The proof of the interpolation inequality can be found in [78]. In order to get the announced results, we would need to use some classical results in Potential Theory dealing with the Newtonian potential and the Beurling transform, see Appendix B or for instance [62, 99, 106, 111]. Now, let us show the idea behind (2.5.69). To estimate the first term of ρ, we combine (2.5.68) with the law products in Hölder spaces, as follows,

$$
\begin{aligned}
\left\|\partial_g\phi(\Omega,0)(h)'(\cdot)\int_{\mathbb{D}}\frac{h(y)}{(\cdot)-y}dA(y)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}&\leq C\left\|\partial_g\phi(\Omega,0)(h)'(\cdot)\right\|_{\mathscr{C}^0(\mathbb{T})}\left\|\int_{\mathbb{D}}\frac{h(y)}{(\cdot)-y}dA(y)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\\
&\quad+C\left\|\partial_g\phi(\Omega,0)(h)'(\cdot)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\left\|\int_{\mathbb{D}}\frac{h(y)}{(\cdot)-y}dA(y)\right\|_{\mathscr{C}^0(\mathbb{T})}\\
&\leq C\left(\|h\|_{\mathscr{C}^0(\mathbb{D})}\|h\|_{\mathscr{C}^{k,\alpha}(\mathbb{D})}+\|h\|_{\mathscr{C}^{k,\alpha}(\mathbb{D})}\|h\|_{\mathscr{C}^0(\mathbb{D})}\right).
\end{aligned}
$$

Then, (2.5.69) is satisfied for the first term. Let us deal with the second term of ρ. For $k=0$, one has

$$
\begin{aligned}
&\left\|\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(\cdot)-\partial_g\phi(\Omega,0)(h)(y)}{((\cdot)-y)^2}h(y)dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\\
&\quad\leq C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\,\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\\
&\qquad\qquad+C\left\|\text{p.v.}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(y)\,h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\\
&\quad\leq C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\left\|\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^0(\mathbb{T})}+C\left\|\partial_g\phi(\Omega,0)(h)\,h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}
\end{aligned}
$$

$$
\begin{aligned}
&+C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\right\|_{\mathscr{C}^0(\mathbb{T})}\left\|\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\\
\leq& C\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}.
\end{aligned}
$$

For $k=1,2$, we would need the use of the interpolation inequalities:

$$
\begin{aligned}
&\Big\|\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(\cdot)-\partial_g\phi(\Omega,0)(h)(y)}{((\cdot)-y)^2}h(y)dA(y)\Big\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\\
\leq& C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\,\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\\
&+C\left\|\text{p.v.}\int_{\mathbb{D}}\frac{\partial_g\phi(\Omega,0)(h)(y)\,h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\\
\leq& C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\left\|\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^0(\mathbb{T})}\\
&+C\left\|\partial_g\phi(\Omega,0)(h)(\cdot)\right\|_{\mathscr{C}^0(\mathbb{T})}\left\|\text{p.v.}\int_{\mathbb{D}}\frac{h(y)}{((\cdot)-y)^2}dA(y)\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}+C\left\|\partial_g\phi(\Omega,0)(h)\,h\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{T})}\\
\leq& \left\|h\right\|_{\mathscr{C}^{k-1,\alpha}(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}+C\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^{k,\alpha}(\mathbb{D})}.
\end{aligned}
$$

It remains to use the interpolation inequalities in order to conclude. For the case $k=1$, we need to use (2.5.70) in order to get

$$
\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\leq C\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}^{2-\frac{2\alpha}{2+\alpha}}\left\|h\right\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}^{\frac{2\alpha}{2+\alpha}}.
$$

We use again (2.5.70) for $k=2$:

$$
\left\|h\right\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\leq C\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}^{\frac{3}{2+\alpha}}\left\|h\right\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}^{\frac{1+2\alpha}{2+\alpha}}.
$$

Note that in every case, the exponent of the $\mathscr{C}^0$-norm is bigger than 1. As to the remaining terms of ρ, we develop similar estimates with the same order of difficulties leading to the announced inequality (2.5.69).

Once we have these preliminaries estimates, we can check that (2.5.67) holds true. For example, let us illustrate the basic idea to implement through the second term.

$$
\left\|\int_{\mathbb{D}}\log|(\cdot)-y|h(y)\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\right\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq C\left\|h\partial_g\phi(\Omega,0)(h)'\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}.
$$

Using the classical law products and (2.5.68) we find

$$
\begin{aligned}
\left\|\int_{\mathbb{D}}\log|(\cdot)-y|h(y)\text{Re}\left[\partial_g\phi(\Omega,0)(h)'(y)\right]dA(y)\right\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq& C\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}\left\|\partial_g\phi(\Omega,0)(h)'\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\\
&+C\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\left\|\partial_g\phi(\Omega,0)(h)'\right\|_{\mathscr{C}^0(\mathbb{D})}\\
\leq& C\left\|h\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\left\|h\right\|_{\mathscr{C}^0(\mathbb{D})}.
\end{aligned}
$$

The other terms can be estimated in a similar way, achieving (2.5.67). The same arguments applied to the remainder term lead to

$$
\|\mathscr{R}_2(\Omega,h)\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\leq C\|h\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}^{\delta_1}\|h\|_{\mathscr{C}^0(\mathbb{D})}^{\delta_2}.
$$

The computations are long here but the analysis is straightforward.

Let us see how to achieve the argument. Assume that h is a zero to $\widehat{G}$ in a small neighborhood of the origin, then

$$D_g\widehat{G}(\Omega,0)h=-\frac{1}{2}D^2_{g,g}\widehat{G}(\Omega,0)(h,h)-\mathscr{R}_2(\Omega,h).$$

Applying Lemma 2.5.8–(3) we deduce that

$$\begin{aligned}\|h\|_{\mathscr{C}^0(\mathbb{D})}&\leq C\|D_g\widehat{G}(\Omega,0)h\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\\&\leq C\|\frac{1}{2}D^2_{g,g}\widehat{G}(\Omega,0)(h,h)+\mathscr{R}_2(\Omega,h)\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\\&\leq C\|h\|^{\delta_1}_{\mathscr{C}^{2,\alpha}}\|h\|^{\delta_2}_{\mathscr{C}^0(\mathbb{D})},\end{aligned}$$

Consequently, if $\|h\|^{\delta_1}_{\mathscr{C}^{2,\alpha}(\mathbb{D})}<C^{-1}$, then necessary $\|h\|_{\mathscr{C}^0(\mathbb{D})}=0$ since $\delta_2\geq 1$. Therefore, we deduce that there is only the trivial solution in this ball. □

Remark 2.5.9. *The quadratic profiles are particular cases of the polynomial profiles studied in Section 2.4.1; $f_0(r)=Ar^{2m}+B$. Here, we briefly show how to develop this case. Studying the kernel for this case is equivalent to study the equations*

$$\begin{aligned}h_n(r)+\frac{r}{n}\frac{m(2m+2)r^{2m-2}}{\left(r^{2m}-\frac{1}{x_m}\right)}\left[-\frac{H_n(1)}{G_n(1)}G_n(r)+\frac{H_n(r)}{r^{n+1}}\right]&=0,\quad\forall r\in[0,1],\quad\forall n\in\mathbb{N}^\star,\\\frac{\frac{1}{x_m}-r^2}{8}h_0(r)-\int_\tau^1\frac{1}{\tau}\int_0^\tau sh_0(s)dsd\tau&=0,\quad\forall r\in[0,1],\end{aligned}$$

where the functions H_n and G_n are defined in (2.5.9)-(2.5.10) *and*

$$\frac{1}{x_m}=\frac{2m+2}{A}\left(\Omega-\frac{B}{2}\right).$$

Thus H_n verifies the following equation

$$\begin{aligned}r(1-x_mr^{2m})H_n''(r)-&(2n-1)(1-x_mr^{2m})H_n'(r)\\&+2m(2m+2)r^{2m-1}x_mH_n(r)=2m(2m+2)x_mr^{n+2m}\frac{H_n(1)}{G_n(1)}G_n(r).\end{aligned}$$

Using the change of variables $y=x_mr^{2m}$ and setting $H_n(r)=\mathcal{F}(x_mr^{2m})$, one has that

$$\begin{aligned}y(1-y)\mathcal{F}''(y)+&\frac{m-n}{m}(1-y)\mathcal{F}'(y)+\frac{m+1}{m}\mathcal{F}(y)\\&=\frac{m+1}{m}\left(\frac{y}{x_m}\right)^{\frac{n+1}{2m}}\frac{H_n(1)}{G_n(1)}G_n\left(\left(\frac{y}{x_m}\right)^{\frac{1}{2m}}\right).\end{aligned}$$

The homogeneous equation of the last differential equation can be solved in terms of hypergeometric functions as it was done in the quadratic profile. Then, similar arguments can be applied to this case.

2.5.3 Range structure

Here, we provide an algebraic description of the range. This will be useful when studying the transversality assumption of the Crandall-Rabinowitz Theorem. Our result reads as follows.

Proposition 2.5.10. *Let $A \in \mathbb{R}^{\star}$, $B \in \mathbb{R}$ and x_0 be given by* (2.4.14). *Let $x \in (-\infty, 1)\backslash\{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$, where the set $\widehat{\mathcal{S}}_{\text{sing}}$ is defined in* (2.5.19). *Then*

$$\operatorname{Im} D_g\widehat{G}(\Omega,0)=\left\{d\in\mathscr{C}_s^{1,\alpha}(\mathbb{D}):\quad \int_{\mathbb{D}} d(z)\mathcal{K}_n(z)dz=0,\ n\in\mathcal{A}_x\right\},$$

where

$$\mathcal{K}_n(z)=\operatorname{Re}\left[\frac{F_n(x|z|^2)}{1-x|z|^2}z^n\right],\quad \Omega=\frac{B}{2}+\frac{A}{4x},$$

and the set $\mathcal{A}_x$ is defined by (2.5.21).

Proof. In order to describe the range of the $D_g\widehat{G}(\Omega,0):\mathscr{C}_s^{1,\alpha}(\mathbb{D})\to\mathscr{C}_s^{1,\alpha}(\mathbb{D})$, we should solve the equation

$$D_g\widehat{G}(\Omega,0)h=d,\quad h(re^{i\theta})=\sum_{n\geq 0}h_n(r)\cos(n\theta),\quad d(re^{i\theta})=\sum_{n\geq 0}d_n(r)\cos(n\theta).$$

From the structure of the linearized operator seen in (2.5.8), this problem is equivalent to

$$\begin{aligned}\frac{\frac{1}{x}-r^2}{8}h_n(r)-\frac{r}{n}\left[A_nG_n(r)+\frac{1}{2r^{n+1}}H_n(r)\right]&=d_n(r),\quad \forall n\geq 1\\ \frac{\frac{1}{x}-r^2}{8}h_0(r)-\int_r^1\frac{1}{\tau}\int_0^\tau sh_0(s)dsd\tau&=d_0(r),\end{aligned}\tag{2.5.71}$$

where the functions involved in the last expressions are defined in (2.5.9)-(2.5.13). By Proposition 2.4.3, the case $n=0$ can be analyzed through the Inverse Function Theorem getting a unique solution. Let us focus on the case $n\geq 1$ and proceed as in the preceding study for the kernel. We use the linear operator defined in (2.5.23),

$$\mathscr{L}h=r^{2n}\int_r^1\frac{1}{s^{n-1}}h(s)ds+\int_0^r s^{n+1}h(s)ds,$$

for any $h\in\mathscr{C}([0,1];\mathbb{R})$, which satisfies the boundary conditions in (2.5.24) and

$$\frac{1}{2n}(r(\mathscr{L}h)'(r))'-(\mathscr{L}h)'(r)=-r^{n+1}h(r).$$

Taking $H_n:=\mathscr{L}h_n$ and using (2.5.71) we find that H_n solves

$$\begin{aligned}(1-xr^2)rH_n''(r)-(1-xr^2)(2n-1)H_n'(r)+8rxH_n(r)&\\ =-16A_nxr^{n+2}G_n(r)-16xnr^{n+1}d_n(r),&\end{aligned}\tag{2.5.72}$$

complemented with the boundary conditions $H_n(0)=H_n'(1)=0$. This differential equation is equivalent to (2.5.71). Once we have a solution of the differential equation (2.5.72) we have to verify that $\mathscr{L}h_n=H_n$, where

$$h_n(r):=\frac{8x}{1-xr^2}\left[d_n(r)+\frac{A_nr}{n}G_n(r)+\frac{1}{2nr^n}H_n(r)\right].$$

Denote $\mathscr{H} := \mathscr{L}h_n - H_n$, then it satisfies

$$\frac{1}{2n}\left[r\,\mathscr{H}'(r)\right]' - \mathscr{H}'(r) = 0.$$

From the boundary conditions one obtains that $\mathscr{H} = 0$ and thus $\mathscr{L}h_n = H_n$. Now, since $H_n(0) = 0$, Lemma 2.5.14 can be applied with

$$g(r) = -16A_n x r^{n+2} G_n(r) - 16xnr^{n+1} d_n(r).$$

Thus, the solutions are given by

$$\begin{aligned} H_n(r) =& r^{2n}F_n(xr^2)\left[\frac{H_n(1)}{F_n(x)} + 4A_n x^n \int_{xr^2}^{x} \frac{1}{\tau^{n+1}F_n^2(\tau)}\int_0^\tau \frac{F_n(s)}{1-s}\left(\frac{s}{x}\right)^{\frac{n+1}{2}} G_n\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right. \\ &\left. + 4nx^n \int_{xr^2}^{x} \frac{1}{\tau^{n+1}F_n^2(\tau)}\int_0^\tau \frac{F_n(s)}{1-s}\left(\frac{s}{x}\right)^{\frac{n}{2}} d_n\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau\right]. \end{aligned}$$

A change of variables combined with (2.5.13) yield

$$\begin{aligned} H_n(r) =& H_n(1)r^{2n}F_n(xr^2)\left[\frac{1}{F_n(x)} - \frac{8x}{G_n(1)}\int_r^1 \frac{1}{\tau^{2n+1}F_n^2(x\tau^2)}\int_0^\tau \frac{s^{n+2}F_n(xs^2)}{1-xs^2}G_n(s) ds d\tau\right] \\ &+ 16nxr^{2n}F_n(xr^2)\int_r^1 \frac{1}{\tau^{2n+1}F_n^2(x\tau^2)}\int_0^\tau \frac{s^{n+1}F_n(xs^2)}{1-xs^2} d_n(s) ds d\tau. \end{aligned}$$

Note that when $d_n \equiv 0$, the function H_n agrees with the one obtained for the kernel. It remains to check the boundary conditions. Clearly $H_n(0) = 0$, then we focus on proving $H_n'(1) = 0$. Following the computations leading to (2.5.31), we obtain

$$H_n'(1) = \frac{2nH_n(1)}{F_n(x)G_n(1)}\Psi_n(x) - \frac{16nx}{F_n(x)}\int_0^1 \frac{s^{n+1}F_n(xs^2)}{1-xs^2} d_n(s) ds.$$

We will distinguish two cases. In the first one $\mathcal{A}_x$ is empty. Then, by virtue of Proposition 2.5.3 and Proposition 2.5.1, we obtain that $D_g\widehat{G}(\Omega, 0)$ is an isomorphism. Otherwise, we have $\Psi_n(x) = 0$, for some $n \in \mathbb{N}^\star$, and the boundary condition is equivalent to

$$\int_0^1 \frac{r^{n+1}F_n(xr^2)}{1-xr^2} d_n(r) dr = 0. \tag{2.5.73}$$

Define $z \in \overline{\mathbb{D}} \mapsto \mathscr{K}_n(z) = \text{Re}\left[\frac{F_n(x|z|^2)}{1-x|z|^2} z^n\right]$, and consider the linear form $T_{\mathscr{K}_n} : \mathscr{C}_s^{1,\alpha}(\mathbb{D}) \mapsto \mathbb{R}$ given by

$$T_{\mathscr{K}_n} d := \int_{\mathbb{D}} d(y)\mathscr{K}_n(y) dA(y).$$

Condition (2.5.73) leads to $T_{\mathscr{K}_n} d = 0$, which follows from

$$\int_0^{2\pi} d(re^{i\theta})\cos(n\theta) d\theta = \pi d_n(r).$$

Since $\mathscr{K}_n$ belongs to $\mathscr{C}^\infty(\overline{\mathbb{D}}; \mathbb{R})$, we deduce that $T_{\mathscr{K}_n}$ is continuous. Thus, Ker $T_{\mathscr{K}_n}$ is closed and of co-dimension one. In addition, from the preceding analysis one has that

$$\text{Im}\, D_g\widehat{G}(\Omega, 0) \subseteq \left\{ d \in \mathscr{C}_s^{1,\alpha}(\mathbb{D}) \, : \, \int_{\mathbb{D}} d(z)\mathscr{K}_n(z) dz = 0, \; n \in \mathcal{A}_x \right\} \subseteq \bigcap_{n \in \mathcal{A}_x} \text{Ker}\, T_{\mathscr{K}_n}.$$

The elements of the family $\{\mathcal{K}_n : n \in \mathcal{A}_x\}$ are independent, and thus $\bigcap_{n\in\mathcal{A}_x} \text{Ker } T_{\mathcal{K}_n}$ is closed and of co–dimension card$\mathcal{A}_x$. As a consequence of Proposition 2.5.3, Ker $D_g\widehat{G}(\Omega, 0)$ is of dimension card$\mathcal{A}_x$. Using Proposition 2.5.1, $D_g\widehat{G}(\Omega, 0)$ is a Fredholm operator of index zero. Consequently, Im $D_g\widehat{G}(\Omega, 0)$ is of co–dimension card$\mathcal{A}_x$, and thus

$$\text{Im } D_g\widehat{G}(\Omega, 0) = \bigcap_{n\in\mathcal{A}_x} \text{Ker } T_{\mathcal{K}_n}.$$

This achieves the proof of the announced result. □

2.6 Spectral study

The aim of this section is to study some qualitative properties of the roots of the spectral function (2.5.20) that will be needed when we apply bifurcation arguments. For instance, to identify the eigenvalues and explore the kernel structure of the linearized operator, we should carefully analyze the existence and uniqueness of roots x_n of (2.5.20) at each frequency level n and study their monotonicity. This part is highly technical and requires cautious manipulations on hypergeometric functions and their asymptotics with respect to n. Notice that for some special regime in A and B, the monotonicity turns to be very intricate and it is only established for higher frequencies through refined expansions of the eigenvalues x_n with respect to n. Another problem that one should face is connected to the separation between the eigenvalues set and the singular set associated to (2.3.5). It seems that the two sequences admit the same leading term and the separation is obtained at the second asymptotics level, which requires much more efforts because the sequence $\{x_n\}$ converges to 1, which is a singular point for the hypergeometric function involved in (2.5.20). Recall that n and m are non negative integers.

2.6.1 Reformulations of the dispersion equation

In what follows, we intend to write down various formulations for the dispersion equation (2.5.20) describing the set (2.1.11). This set is given by the zeroes of (2.5.20) and the elements of this set are called "eigenvalues". As we shall notice, the study of some qualitative behavior of the zeroes will be much more tractable through the use of different representations connected to some specific algebraic structure of the hypergeometric equations. Recall the use of the notation $F_n(x) = F(a_n, b_n; c_n; x)$, where the coefficients are given by (2.5.15). The Kummer quadratic transformations introduced in Appendix C leads to the following result:

Lemma 2.6.1. *The following identities hold true:*

$$\zeta_n(x) = \left[1 + x\left(\frac{A+2B}{A(n+1)} - 1\right)\right] F(a_n, b_n; n+1; x) + \frac{2x-1}{n+1} F(a_n, b_n; n+2; x) - \frac{2x}{(n+1)(n+2)} F(a_n, b_n; n+3; x), \tag{2.6.1}$$

$$\zeta_n(x) = \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) + \frac{n-(n+1)x}{n+1} F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)} F(a_n, b_n; n+3; x), \tag{2.6.2}$$

for any $n \in \mathbb{N}$ and $x \in (-\infty, 1)$, where we have used the notations (2.5.15).

Proof. Let us begin with (2.6.1). The integral term in (2.5.20) can be written as follows

$$\int_0^1 F_n(\tau x)\tau^n\left[-1+2x\tau\right]d\tau = (2x-1)\int_0^1 F_n(\tau x)\tau^n d\tau - 2x\int_0^1 F_n(\tau x)\tau^n(1-\tau)d\tau.$$

This leads to

$$\begin{aligned}\zeta_n(x) =& F_n(x)\left[1-x+\frac{A+2B}{A(n+1)}x\right]+(2x-1)\int_0^1 F_n(\tau x)\tau^n d\tau\\ &-2x\int_0^1 F_n(\tau x)\tau^n(1-\tau)d\tau.\end{aligned} \tag{2.6.3}$$

We use (C.0.10) in order to get successively

$$\int_0^1 F(a_n,b_n;n+1;\tau x)\tau^n d\tau = \frac{F(a_n,b_n;n+2;x)}{n+1}$$

and

$$\int_0^1 F(a_n,b_n;n+1;\tau x)\tau^n\left[1-\tau\right]d\tau = \frac{F(a_n,b_n;n+3;x)}{(n+1)(n+2)},$$

for any $x\in(-\infty,1)$. Taking into account these identities, we can rewrite (2.6.3) as (2.6.1).

In order to obtain (2.6.2), we use (C.0.9) with $a=a_n,\ b=b_n$ and $c=n+1$, which yields

$$\begin{aligned}F(a_n,b_n;n+1)(x-1) =& \frac{(n+3)x-(n+1)}{n+1}F(a_n,b_n;n+2;x)\\ &+\frac{(a_n-(n+2))((n+2)-b_n)x}{(n+1)(n+2)}F(a_n,b_n;n+3;x)\\ =& \frac{(n+3)x-(n+1)}{n+1}F(a_n,b_n;n+2;x)\\ &-\frac{(2n+2)x}{(n+1)(n+2)}F(a_n,b_n;n+3;x),\end{aligned}$$

where we have taken into account the identities $a_n+b_n=n$ and $a_nb_n=-2$. By virtue of the first assertion of this lemma we obtain

$$\begin{aligned}\zeta_n(x) =& \frac{A+2B}{A(n+1)}xF(a_n,b_n;n+1;x)\\ &+\frac{(n+1)-(n+3)x}{n+1}F(a_n,b_n;n+2;x)+\frac{(2n+2)x}{(n+1)(n+2)}F(a_n,b_n;n+3;x)\\ &+\frac{2x-1}{n+1}F(a_n,b_n;n+2;x)-\frac{2x}{(n+1)(n+2)}F(a_n,b_n;n+3;x)\\ =& \frac{A+2B}{A(n+1)}xF(a_n,b_n;n+1;x)\\ &+\frac{n-(n+1)x}{n+1}F(a_n,b_n;n+2;x)+\frac{2nx}{(n+1)(n+2)}F(a_n,b_n;n+3;x).\end{aligned}$$

This achieves the proof of the second identity (2.6.2).

Let us also remark that using (C.0.7) we can deduce another useful equivalent expression for ζ_n

$$\zeta_n(x) = I_n^1(x)F(a_n,b_n;n+1;x)+I_n^2(x)F(a_n+1,b_n;n+2;x)+I_n^3(x)F(a_n,b_n;n+3;x),$$

where

$$\begin{aligned}
I_n^1(x) :=& \frac{n-a_n}{n+1-a_n} + x\left(\frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1-a_n}\right),\\
I_n^2(x) :=& -\frac{a_n(2x-1)}{(n+1)(n+1-a_n)},\\
I_n^3(x) :=& -\frac{2x}{(n+1)(n+2)}.
\end{aligned}$$

□

2.6.2 Qualitative properties of hypergeometric functions

The main task of this section is to provide suitable properties about the analytic continuation of the mapping $(n,x) \mapsto F(a_n, b_n; c_n; x)$ and some partial monotonicity behavior. First, applying the integral representation (C.0.2) with the special coefficients (2.5.15), we find

$$F(a_n, b_n; c_n; x) = \frac{\Gamma(n+1)}{\Gamma(n-a_n)\Gamma(1+a_n)} \int_0^1 \tau^{n-a_n-1}(1-\tau)^{a_n}(1-\tau x)^{-a_n} d\tau, \tag{2.6.4}$$

for $x \in (-\infty, 1]$. Notice that, due to (C.0.5) we can evaluate it at 1, obtaining

$$F(a_n, b_n; c_n; 1) = \frac{\Gamma(n+1)}{\Gamma(n-a_n+1)\Gamma(1+a_n)}, \tag{2.6.5}$$

for any $n \geq 2$, where we have used the identity $\Gamma(x+1) = x\Gamma(x)$. We observe that the representation (2.6.4) fails for the case $n = 1$ because $a_1 = -1$. This does not matter since as we have already mentioned in Remark 2.5.5, the case $n = 1$ is explicit and the study of ζ_1 can be done by hand. It is a well-known fact that the Gamma function can be extended analytically to $\mathbb{C}\backslash\{0, -1, -2, \dots\}$. Therefore, the map $n \in \mathbb{N}^\star\backslash\{1\} \mapsto F(a_n, b_n; c_n; 1)$ admits a $\mathscr{C}^\infty$- extension given by

$$\mathcal{F} : \mathtt{t} \in]1, +\infty[\mapsto \frac{\Gamma(\mathtt{t}+1)}{\Gamma(\mathtt{t}-a_\mathtt{t}+1)\Gamma(1+a_\mathtt{t})},$$

with $a_\mathtt{t} = -\frac{4}{\mathtt{t}+\sqrt{\mathtt{t}^2+8}}$.
The first result that we should discuss concerns some useful asymptotic behaviors for $\mathtt{t} \mapsto \mathcal{F}(\mathtt{t})$.

Lemma 2.6.2. *The following properties are satisfied:*

1. *Let $\mathtt{t} \geq 1$, then the function $x \in (-\infty, 1] \mapsto F(a_\mathtt{t}, b_\mathtt{t}; c_\mathtt{t}; x)$ is positive and strictly decreasing.*
2. *For large $\mathtt{t} \gg 1$, we have*

$$\mathcal{F}(\mathtt{t}) = 1 - 2\frac{\ln \mathtt{t}}{\mathtt{t}} - \frac{2\gamma}{\mathtt{t}} + O\left(\frac{\ln^2 \mathtt{t}}{\mathtt{t}^2}\right)$$

and

$$\mathcal{F}'(\mathtt{t}) = 2\frac{\ln \mathtt{t}}{\mathtt{t}^2} + \frac{2(\gamma-1)}{\mathtt{t}^2} + O\left(\frac{\ln^2 \mathtt{t}}{\mathtt{t}^3}\right),$$

where γ is the Euler constant. In particular, we have the asymptotics

$$F(a_n, b_n; c_n; 1) = 1 - 2\frac{\ln n}{n} - \frac{2\gamma}{n} + O\left(\frac{\ln^2 n}{n^2}\right)$$

and

$$\frac{d}{dn}F(a_n, b_n; c_n; 1) = 2\frac{\ln n}{n^2} + \frac{2(\gamma - 1)}{n^2} + O\left(\frac{\ln^2 n}{n^3}\right),$$

for large n.

Proof. (**1**) The case $\mathtt{t} = 1$ follows obviously from the explicit expression given in Remark 2.5.5. Now let us consider $\mathtt{t} > 1$. According to (C.0.4), we can differentiate F with respect to x:

$$F'(a_\mathtt{t}, b_\mathtt{t}; c_\mathtt{t}; x) = \frac{a_\mathtt{t} b_\mathtt{t}}{c_\mathtt{t}} F(a_\mathtt{t} + 1, b_\mathtt{t} + 1; c_\mathtt{t} + 1; x), \quad \forall x \in (-\infty, 1).$$

Using the integral representation (C.0.2) and the positivity of Gamma function, one has that for any $c > b > 0$

$$F(a, b; c, x) > 0, \quad \forall x \in (-\infty, 1). \tag{2.6.6}$$

Since $a_\mathtt{t} \in (-1, 0), b_\mathtt{t}, c_\mathtt{t} > 0$, then we deduce that $F(a_\mathtt{t} + 1, b_\mathtt{t} + 1; c_\mathtt{t} + 1; x) > 0$ and thus

$$F'(a_\mathtt{t}, b_\mathtt{t}; c_\mathtt{t}; x) < 0, \quad \forall x \in (-\infty, 1).$$

This implies that $x \mapsto F(a_\mathtt{t}, b_\mathtt{t}; c_\mathtt{t}; x)$ is strictly decreasing and together with (2.6.5) we obtain

$$F(a, b; c, x) \geq F(a_\mathtt{t}, b_\mathtt{t}; c_\mathtt{t}, 1) > 0, \quad \forall x \in (-\infty, 1).$$

(**2**) The following asymptotic behavior

$$\frac{\Gamma(\mathtt{t} + \alpha)}{\Gamma(\mathtt{t} + \beta)} = \sum_{n \in \mathbb{N}} C_n(\alpha - \beta, \beta) \mathtt{t}^{\alpha - \beta - n},$$

holds as $\mathtt{t} \to +\infty$ by using [140, Identity 12], where the coefficients $C_n(\alpha - \beta, \beta)$ can be obtained recursively and are polynomials on the variables α, β. In addition, the first coefficients can be calculated explicitly

$$C_0(\alpha - \beta, \beta) = 1 \quad \text{and} \quad C_1(\alpha - \beta, \beta) = \frac{1}{2}(\alpha - \beta)(\alpha + \beta - 1).$$

Taking $\alpha = 1$ and $\beta = 1 - a_\mathtt{t}$ we deduce

$$\frac{\Gamma(\mathtt{t} + 1)}{\Gamma(\mathtt{t} + 1 - a_\mathtt{t})} = \mathtt{t}^{a_\mathtt{t}} + \frac{1}{2} a_\mathtt{t} (1 - a_\mathtt{t}) \mathtt{t}^{a_\mathtt{t} - 1} + O\left(\frac{1}{\mathtt{t}^2}\right) = \mathtt{t}^{a_\mathtt{t}} + O\left(\frac{1}{\mathtt{t}^2}\right),$$

where we have used that $a_\mathtt{t} \sim -\frac{2}{\mathtt{t}}$. From the following expansion

$$\mathtt{t}^{a_\mathtt{t}} = e^{a_\mathtt{t} \ln \mathtt{t}} = 1 - 2\frac{\ln \mathtt{t}}{\mathtt{t}} + O\left(\frac{\ln^2 \mathtt{t}}{\mathtt{t}^2}\right),$$

we get

$$\frac{\Gamma(\mathtt{t} + 1)}{\Gamma(\mathtt{t} + 1 - a_\mathtt{t})} = 1 - 2\frac{\ln \mathtt{t}}{\mathtt{t}} + O\left(\frac{\ln^2 \mathtt{t}}{\mathtt{t}^2}\right). \tag{2.6.7}$$

Using again Taylor expansion, we find $\Gamma(1 + a_\mathtt{t}) = 1 + a_\mathtt{t}\Gamma'(1) + O\left(a_\mathtt{t}^2\right)$. Therefore, combining this with $\Gamma'(1) = -\gamma$ and $a_\mathtt{t} \sim -\frac{2}{\mathtt{t}}$ yields

$$\Gamma(1 + a_\mathtt{t}) = 1 + \frac{2\gamma}{\mathtt{t}} + O\left(\frac{1}{\mathtt{t}^2}\right).$$

Consequently, it follows that $\mathcal{F}$ admits the following asymptotic behavior at infinity

$$\mathcal{F}(\mathtt{t}) = \frac{1 - 2\frac{\ln \mathtt{t}}{\mathtt{t}} + O(\mathtt{t}^{-2}\ln^2 \mathtt{t})}{1 + \frac{2\gamma}{\mathtt{t}} + O(\mathtt{t}^{-2})} = 1 - 2\frac{\ln \mathtt{t}}{\mathtt{t}} - 2\frac{\gamma}{\mathtt{t}} + O\left(\frac{\ln^2 \mathtt{t}}{\mathtt{t}^2}\right).$$

Since Γ is real analytic, we have that $\mathcal{F}$ is also real analytic and one may deduce the asymptotics at $+\infty$ of the derivative $\mathcal{F}'$ through the differentiation term by term the asymptotics of $\mathcal{F}$. Thus, we obtain the second expansion in assertion (2). □

Our next purpose is to provide some useful estimates for $F(a_n, b_n; c_n; x)$ and its partial derivatives. More precisely, we state the following result.

Lemma 2.6.3. *With the notations* (2.5.15)*, the following assertions hold true.*

1. *The sequence $n \in [1, +\infty) \mapsto F(a_n, b_n; c_n; x)$ is strictly increasing, for any $x \in (0, 1]$, and strictly decreasing, for any $x \in (-\infty, 0)$.*
2. *Given $n \geq 1$ we have*
$$|\partial_n F_n(x)| \leq \frac{-2xF_n(x)}{(n+1)^2},$$
for any $x \in (-\infty, 0]$.
3. *There exists $C > 0$ such that*
$$|\partial_x F(a_n, b_n; c_n; x)| \leq C + C|\ln(1-x)|,$$
for any $x \in [0, 1]$, and $n \geq 2$.
4. *There exists $C > 0$ such that*
$$|\partial_n F(a_n, b_n; c_n; x)| \leq C\frac{\ln n}{n^2},$$
for any $x \in [0, 1]$, and $n \geq 2$.
5. *There exists $C > 0$ such that*
$$|\partial_{xx} F(a_n, b_n; c_n; x)| \leq \frac{C}{1-x},$$
for any $x \in [0, 1]$ and $n \geq 2$.

Proof. **(1)** Recall that F_n solves the equation

$$x(1-x)F_n''(x) + (n+1)(1-x)F_n'(x) + 2F_n(x) = 0,$$

with $F_n(0) = 1$ and $F_n'(0) = \frac{a_n b_n}{c_n} = \frac{-2}{n+1}$. As we have mentioned in the beginning of this section the dependence with respect to n is smooth, here we use n as a continuous parameter instead of $\mathtt{t}$. Then, differentiating with respect to n we get

$$x(1-x)(\partial_n F_n)''(x) + (n+1)(1-x)(\partial_n F_n)'(x) + 2(\partial_n F_n) = -(1-x)F_n'(x),$$

with $(\partial_n F_n)(0) = 0$ and $(\partial_n F_n)'(0) = \frac{2}{(n+1)^2}$. We can explicitly solve the last differential equation by using the variation of the constant and keeping in mind that $x \mapsto F_n(x)$ is a homogeneous solution. Thus, we obtain

$$\partial_n F_n(x) = F_n(x)\left[K_2 - \int_x^1 \frac{1}{F_n^2(\tau)\tau^{n+1}}\left(K_1 - \int_0^\tau F_n'(s)F_n(s)s^n ds\right)d\tau\right],$$

where the constant K_1 must be zero to remove the singularity at 0, in a similar way to the proof of Lemma 2.5.2. Since $(\partial_n F_n)(0) = 0$, we deduce that

$$K_2 = -\int_0^1 \frac{1}{F_n^2(\tau)\tau^{n+1}} \int_0^\tau F_n'(s)F_n(s)s^n ds\, d\tau,$$

and then

$$\partial_n F_n(x) = -F_n(x) \int_0^x \frac{1}{F_n^2(\tau)\tau^{n+1}} \int_0^\tau F_n'(s)F_n(s)s^n ds d\tau. \tag{2.6.8}$$

The change of variables $s = \tau\theta$ leads to

$$\partial_n F_n(x) = -F_n(x) \int_0^x \frac{1}{F_n^2(\tau)} \int_0^1 F_n'(\tau\theta)F_n(\tau\theta)\theta^n d\theta d\tau. \tag{2.6.9}$$

Hence, it is clear that $\partial_n F_n(x) > 0$, for $x \in [0, 1)$, using Lemma 2.6.2-(1) . In the case $x \in (-\infty, 0]$ we similarly get $\partial_n F_n(x) < 0$. Let us observe that the compatibility condition $(\partial_n F_n)'(0) = \frac{2}{(n+1)^2}$ can be directly checked from the preceding representation. Indeed, one has

$$(\partial_n F_n)'(0) = \lim_{x\to 0} \frac{\partial_n F_n(x)}{x} = -\lim_{x\to 0^+} \frac{F_n'(0)\int_0^1 \theta^n d\theta}{F_n(0)} = \frac{2}{(n+1)^2}.$$

(2) First, notice that

$$F_n'(x) = -\frac{2}{n+1} F(a_n + 1, b_n + 1, c_n + 1, x),$$

and from (2.6.6) we deduce that

$$|F_n'(x)| \leq \frac{2}{n+1} F(a_n + 1, b_n + 1, c_n + 1, x).$$

Now studying the variation of $x \in (-\infty, 1) \mapsto F(a_n + 1, b_n + 1, c_n + 1, x)$ by means of the integral representation (C.0.2), we can show that it is strictly increasing and positive, which implies in turn that

$$0 < F(a_n + 1, b_n + 1, c_n + 1, x) \leq F(a_n + 1, b_n + 1, c_n + 1, 0) = 1, \quad \forall x \in (-\infty, 0].$$

This allows us to get,

$$0 \leq -F_n'(x) \leq \frac{2}{n+1}, \quad \forall x \in (-\infty, 0]. \tag{2.6.10}$$

Lemma 2.6.2-(1) implies in particular that

$$F_n(x) \geq 1, \quad \forall x \in (-\infty, 1),$$

and coming back to (2.6.9) we find

$$|\partial_n F_n(x)| \leq \frac{2F_n(x)}{n+1} \int_x^0 \frac{1}{F_n^2(\tau)} \int_0^1 F_n(\tau\theta)\theta^n d\theta d\tau \leq \frac{2F_n(x)}{(n+1)^2} \int_x^0 \frac{d\tau}{F_n(\tau)} \leq \frac{-2xF_n(x)}{(n+1)^2},$$

for $x \in (-\infty, 0]$. This achieves the proof of the announced inequality.

(3) From previous computations we have

$$\partial_x F(a_n, b_n; c_n; x) = \frac{-2}{n+1} F(1 + a_n, n + 1 - a_n; n + 2; x),$$

which admits the integral representation

$$\begin{aligned}\partial_x F(a_n, b_n; c_n; x) =& \frac{-2\Gamma(n+2)}{(n+1)\Gamma(n+1-a_n)\Gamma(1+a_n)} \int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}(1-\tau x)^{-1-a_n} d\tau \\ =& \frac{-2(n+2)}{n+1}\mathcal{F}(n) \int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}(1-\tau x)^{-1-a_n} d\tau \\ =& \frac{-2(n+2)}{n+1}\mathcal{F}(n) J_n(x), \end{aligned} \tag{2.6.11}$$

where

$$J_n(x) := \int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}(1-\tau x)^{-1-a_n} d\tau.$$

Using

$$\sup_{n\geq 1} \frac{2(n+2)}{n+1} \leq 3,$$

and the first assertion of Lemma 2.6.2, we have $\mathcal{F}(n) \in [0,1]$, and

$$\sup_{n\geq 1} \frac{2(n+2)\mathcal{F}(n)}{n+1} \leq 3. \tag{2.6.12}$$

Consequently,

$$|\partial_x F(a_n, b_n; c_n; x)| \leq 3J_n. \tag{2.6.13}$$

To estimate J_n we simply write

$$J_n \leq \int_0^1 (1-\tau)^{a_n} \Big(1 - \frac{\tau}{2}\Big)^{-1-a_n} d\tau \leq C \int_0^1 (1-\tau)^{a_n} d\tau \leq \frac{C}{1+a_n} \leq C,$$

for some constant C independent of $n \geq 2$, and for $x \in \big[0, \frac{1}{2}\big]$. In the case $x \in \big[\frac{1}{2}, 1\big)$, making the change of variable

$$\tau = 1 - \frac{1-x}{x}\tau', \tag{2.6.14}$$

and denoting the new variable again by τ, we obtain

$$\begin{aligned} J_n \leq& x^{-a_n-1} \int_0^{\frac{x}{1-x}} \tau^{a_n}(1+\tau)^{-1-a_n} d\tau \\ \leq& x^{-a_n-1} \int_0^1 \tau^{a_n}(1+\tau)^{-1-a_n} d\tau + x^{-a_n-1} \int_1^{\frac{x}{1-x}} \tau^{a_n}(1+\tau)^{-1-a_n} d\tau \\ \leq& C + C \int_1^{\frac{x}{1-x}} \Big(\frac{1+\tau}{\tau}\Big)^{-a_n} (1+\tau)^{-1} d\tau \leq C + C|\ln(1-x)|, \end{aligned}$$

which achieves the proof.

(4) According to (2.6.8) and Lemma 2.6.2$-(1)$ we may write

$$\begin{aligned} |\partial_n F_n(x)| \leq& F_n(x) \int_0^x \frac{1}{F_n^2(\tau)\tau^{n+1}} \int_0^\tau |F_n'(s)| F_n(s) s^n ds d\tau \\ \leq& \frac{1}{F_n^2(1)} \int_0^1 \tau^{-n-1} \int_0^\tau |F_n'(s) s^n ds d\tau, \end{aligned} \tag{2.6.15}$$

for $x \in [0,1]$ and $n \geq 2$. Using (2.6.13), the definition of J_n and the fact that $0 < -a_n < 1$, we obtain

$$|F_n'(x)| \leq 3\int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}(1-\tau x)^{-1-a_n}d\tau \leq \frac{3}{1-x}\int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}d\tau.$$

Now, recall the classical result on the Beta function B defined as follows

$$\int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}d\tau = \mathrm{B}(n+1-a_n, 1+a_n) = \frac{\Gamma(n+1-a_n)\Gamma(1+a_n)}{\Gamma(n+2)}, \tag{2.6.16}$$

which implies in view of (2.6.5) that

$$\int_0^1 \tau^{n-a_n}(1-\tau)^{a_n}d\tau = \frac{1}{(n+1)F_n(1)}.$$

Consequently

$$|F_n'(x)| \leq \frac{3}{(1-x)(n+1)F_n(1)}.$$

Inserting this inequality into (2.6.15) we deduce that

$$|\partial_n F_n(x)| \leq \frac{1}{(n+1)F_n^3(1)}\int_0^1 \tau^{-n-1}\int_0^\tau \frac{s^n}{1-s}dsd\tau,$$

and integrating by parts we find

$$\begin{aligned}\int_0^1 \tau^{-n-1}\int_0^\tau \frac{s^n}{1-s}dsd\tau &= \left[\frac{1-\tau^{-n}}{n}\int_0^\tau \frac{s^n}{1-s}ds\right]_0^1 + \frac{1}{n}\int_0^1 \frac{s^n-1}{s-1}ds\\ &= \frac{1}{n}\int_0^1 \sum_{k=0}^{n-1} s^k ds = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

Thus, it follows from the classical inequality $\sum_{k=1}^n \frac{1}{k} \leq 1 + \ln n$ that

$$|\partial_n F_n(x)| \leq \frac{1+\ln n}{n^2 F_n^3(1)},$$

for $n \geq 2$. Since $F_n(1) > 0$ and converges to 1, as n goes to ∞, then one can find an absolute constant $C > 0$ such that

$$|\partial_n F_n(x)| \leq C\frac{\ln n}{n^2},$$

for any $n \geq 2$, which achieves the proof of the estimate.

(5) Differentiating the integral representation (2.6.11) again with respect to x we obtain

$$\partial_{xx}F(a_n, b_n; c_n; x) = \frac{(a_n+1)a_n(n-a_n)(n+2)}{n+1}\mathcal{F}(n)\widehat{J}_n(x),$$

with

$$\widehat{J}_n(x) := \int_0^1 \tau^{n-a_n+1}(1-\tau)^{a_n}(1-\tau x)^{-2-a_n}d\tau.$$

According to (2.6.12) one finds that

$$\left|\partial_{xx}F(a_n, b_n; c_n; x)\right| \leq 4\widehat{J}_n(x). \tag{2.6.17}$$

The procedure for estimating $\widehat{J}_n$ matches the one given for J_n in the previous assertion. Indeed, we have the uniform bound $\widehat{J}_n(x) \leq C$, for $x \in \left[0, \frac{1}{2}\right]$. Now, the change of variables (2.6.14) leads to

$$\widehat{J}_n(x) \leq \int_0^1 (1-\tau)^{a_n}(1-\tau x)^{-2-a_n} d\tau \leq \frac{x^{-a_n-1}}{1-x}\int_0^{\frac{x}{1-x}} \tau^{a_n}(1+\tau)^{-2-a_n} d\tau \leq \frac{C}{1-x},$$

where we have used the bounds $0 < -a_n < \frac{2}{1+\sqrt{3}} < 1$, which are verified for any $n \geq 2$ and $x \in \left[\frac{1}{2}, 1\right)$. Inserting this estimate into (2.6.17) we obtain the announced inequality. □

Next we shall prove the following.

Lemma 2.6.4. *There exists $C > 0$ such that*

$$|F(a_n, b_n; n+1; x) - 1| \leq C\frac{\ln n}{n}, \tag{2.6.18}$$

$$1 \leq F(a_n+1, b_n; n+2; x) \leq Cn, \tag{2.6.19}$$

$$|F(a_n, b_n; n+3; x) - 1| \leq C\frac{\ln n}{n}, \tag{2.6.20}$$

for any $n \geq 2$ and any $x \in [0,1]$.

Proof. The estimate (2.6.18) follows easily from the second assertion of Lemma 2.6.3, combined with the monotonicity of F_n and Lemma 2.6.2. Indeed,

$$|F(a_n, b_n; c_n; 0) - F(a_n, b_n; c_n; x)| \leq F(a_n, b_n; c_n; 0) - F(a_n, b_n; c_n; 1) \leq C\frac{\ln n}{n}.$$

In the case (2.6.19), applying similar arguments as in the first assertion of Lemma 2.6.2, we conclude that the function $x \in [0,1] \to F(a_n+1, b_n; n+2; x)$ is positive and strictly increasing. Hence,

$$1 \leq F(a_n+1, b_n; n+2; x) \leq F(a_n+1, b_n; n+2; 1).$$

Combining (C.0.5) and (2.6.7) we obtain the estimate:

$$1 \leq F(a_n+1, b_n; n+2; x) \leq F(a_n+1, b_n; n+2; 1) \leq \frac{\Gamma(n+2)}{\Gamma(n+1-a_n)\Gamma(2+a_n)} \leq Cn.$$

To check (2.6.20), we use the first assertion of Lemma 2.6.2,

$$0 \leq 1 - F(a_n, b_n; n+3; x) \leq 1 - F(a_n, b_n; n+3; 1).$$

Moreover, by virtue of (C.0.5) one has

$$F(a_n, b_n; n+3; 1) = \frac{\Gamma(n+3)}{\Gamma(n+3-a_n)}\frac{\Gamma(3)}{\Gamma(3+a_n)}.$$

As a consequence of (2.6.7) and $a_n \sim -\frac{2}{n}$, we obtain

$$\frac{\Gamma(n+3)}{\Gamma(n+3-a_n)} = 1 - 2\frac{\ln n}{n} + O\left(\frac{1}{n}\right), \quad \text{and} \quad \frac{\Gamma(3)}{\Gamma(3+a_n)} = 1 + O\left(\frac{1}{n}\right).$$

Therefore, the following asymptotic expansion

$$F(a_n, b_n; n+3; 1) = 1 - 2\frac{\ln n}{n} + O\left(\frac{1}{n}\right)$$

holds and the estimate follows easily. □

Another useful property deals with the behavior of the hypergeometric function with respect to the third variable c.

Lemma 2.6.5. *Let $n \geq 1$, then the mapping $c \in (b_n, \infty) \mapsto F(a_n, b_n; c; x)$ is strictly increasing for $x \in (0,1)$ and strictly decreasing for $x \in (-\infty, 0)$.*

Proof. First, we check the case $n = 1$ that comes by

$$F(a_1, b_1; c; x) = 1 - \frac{2}{c}x.$$

Let $n > 1$ and recall that the hypergeometric function $F(a_n, b_n; c; x)$ solves the differential equation

$$x(1-x)\partial^2_{xx}F(a_n, b_n; c; x) + [c-(n+1)x]\partial_x F(a_n, b_n; c; x) + 2F(a_n, b_n; c; x) = 0,$$

with $F(a_n, b_n; c; 0) = 1$ and $\partial_x F(a_n, b_n; c; 0) = -\frac{2}{c}$. Hence by differentiation it is easy to check that $\partial_c F(a_n, b_n; c; x)$ solves

$$\begin{aligned} x(1-x)\partial^2_{xx}(\partial_c F(a_n, b_n; c; x)) + [c-(n+1)x]\partial_x(\partial_c F(a_n, b_n; c; x)) + 2(\partial_c F(a_n, b_n; c; x)) \\ = -\partial_x F(a_n, b_n; c; x), \end{aligned}$$

with initial conditions

$$\partial_c F(a_n, b_n; c; 0) = 0, \quad \text{and} \quad \partial_x(\partial_c F(a_n, b_n; c; 0)) = \frac{2}{c^2}.$$

Note that a homogeneous solution of the last differential equation is $F(a_n, b_n; c; x)$. By the variation of constant method one can look for the full solution to the differential equation in the form

$$\partial_c F(a_n, b_n; c; x) = K(x)F(a_n, b_n; c; x),$$

and from straightforward computations we find that $T = K'$ solves the first order differential equation,

$$T'(x) + \left[2\frac{\partial_x F(a_n, b_n; c; x)}{F(a_n, b_n; c; x)} + \frac{c-(n+1)x}{x(1-x)}\right] T(x) = -\frac{\partial_x F(a_n, b_n; c; x)}{x(1-x)F(a_n, b_n; c; x)}.$$

The general solution to this latter equation is given by

$$T(x) = \frac{(1-x)^{c-(n+1)}}{F(a_n, b_n; c; x)^2|x|^c}\left[K_1 - \int_0^x \frac{|s|^c s^{-1}F(a_n, b_n; c; s)\partial_s F(a_n, b_n; c; s)}{(1-s)^{c-n}}ds\right],$$

for $x \in (-\infty, 1)$ where $K_1 \in \mathbb{R}$ is a real constant. Thus

$$\begin{aligned} \partial_c F(a_n, b_n; c; x) = F(a_n, b_n; c; x)\Bigg[K_2 + \\ \int_{x_0}^x \frac{(1-\tau)^{c-(n+1)}}{F(a_n, b_n; c; \tau)^2|\tau|^c}\left\{K_1 - \int_0^\tau \frac{|s|^c s^{-1}F(a_n, b_n; c; s)\partial_s F(a_n, b_n; c; s)}{(1-s)^{c-n}}ds\right\}d\tau\Bigg], \end{aligned}$$

where $K_1, K_2 \in \mathbb{R}$ and $x_0 \in (-1,1)$. Since $\partial_c F(a_n, b_n; c; x)$ is not singular at $x = 0$, we get that $K_1 = 0$. Then, changing the constant K_2 one can take $x_0 = 0$ getting

$$\partial_c F(a_n, b_n; c; x) = F(a_n, b_n; c; x)\Bigg[K_2 -$$

$$\int_0^x \frac{(1-\tau)^{c-(n+1)}}{F(a_n,b_n;c;\tau)^2|\tau|^c}\int_0^\tau \frac{|s|^c s^{-1}F(a_n,b_n;c;s)\partial_s F(a_n,b_n;c;s)}{(1-s)^{c-n}}dsd\tau\Bigg].$$

The initial condition $\partial_c F(a_n,b_n;c;0)=0$ implies that $K_2=0$ and hence

$$\begin{aligned}\partial_c F(a_n,b_n;c;x) &= -F(a_n,b_n;c;x)\times\\ &\int_0^x (1-\tau)^{c-(n+1)}\int_0^\tau \frac{|s|^c s^{-1}F(a_n,b_n;c;s)\partial_s F(a_n,b_n;c;s)}{|\tau|^c F(a_n,b_n;c;\tau)^2(1-s)^{c-n}}dsd\tau.\end{aligned}$$

Similarly to the proof of Lemma 2.6.2$-(1)$ one may obtain that

$$F(a_n,b_n;c;x)>0 \quad \text{and} \quad \partial_x F(a_n,b_n;c;x)<0,$$

for any $c>b_n$ and $x\in(-\infty,1)$. This entails that $\partial_c F(a_n,b_n;c;x)$ is positive for $x\in(0,1)$, and negative when $x\in(-\infty,0)$, which concludes the proof. □

2.6.3 Eigenvalues

The existence of eigenvalues, that are the elements of the dispersion set defined in (2.1.11), is connected to the problem of studying the roots of the equation introduced in (2.5.20). Here, we will develop different cases illustrating strong discrepancy on the structure of the dispersion set. Assuming $A>0$ and $B<-A$, we find that the dispersion set is infinite. However, for the case $A>0$ and $B\geq -\frac{A}{2}$, the dispersion set is finite. Notice that the transient regime corresponding to $-A\leq B\leq -\frac{A}{2}$ is not covered by the current study and turns to be more complicate due to the complex structure of the spectral function (2.5.20).
Let us begin with studying the cases

$$A>0, \qquad A+B<0, \tag{2.6.21}$$
$$A>0, \qquad A+2B\leq 0, \tag{2.6.22}$$
$$A>0, \qquad A+2B\geq 0. \tag{2.6.23}$$

Our first main result reads as follows.

Proposition 2.6.6. *The following assertions hold true:*

1. *Given A,B satisfying* (2.6.21), *there exist $n_0\in\mathbb{N}^\star$, depending only on A and B, and a unique root $x_n\in(0,1)$ of* (2.5.20), *i.e. $\zeta_n(x_n)=0$, for any $n\geq n_0$. In addition,*
$$x_n\in\left(0,1+\frac{A+B}{An}\right),$$
and the sequence $n\in[n_0,+\infty)\mapsto x_n$ is strictly increasing.
2. *Given A,B satisfying the weak condition* (2.6.22) *and $n\in\mathbb{N}^\star$, then ζ_n has no solution in $(-\infty,0]$.*
3. *Given A,B satisfying* (2.6.23), *then ζ_n has no solution in $[0,1]$, for $n\in\mathbb{N}^\star$.*

Proof. (**1**) The expression of the spectral equation (2.6.1) agrees with

$$\zeta_n(x)=I_n^1(x)F(a_n,b_n;n+1;x)+I_n^2(x)F(a_n+1,b_n;n+2;x)+I_n^3(x)F(a_n,b_n;n+3;x),$$

where I_n^1, I_n^2 and I_n^3 are defined in Lemma 2.6.1. From this expression we get $\zeta_n(0) = \frac{n}{n+1}$. To find a solution in $(0,1)$ we shall apply the Intermediate Value Theorem, and for this purpose we need to check that $\zeta_n(1) < 0$. Applying (C.0.5) we get

$$\begin{aligned}\zeta_n(1) = &\left[\frac{1}{n+1-a_n} + \frac{A+2B}{A(n+1)}\right] \frac{\Gamma(n+1)}{\Gamma(n+1-a_n)\Gamma(1+a_n)} \\ &- \frac{a_n \Gamma(n+1)}{\Gamma(n+2-a_n)\Gamma(2+a_n)} - \frac{4\Gamma(n)}{\Gamma(n+3-a_n)\Gamma(3+a_n)}.\end{aligned}$$

Using the following expansion for large $n \gg 1$

$$\frac{1}{n+1-a_n} = \frac{1}{n+1} + O\left(\frac{1}{n^3}\right),$$

and (2.6.7), we find

$$\zeta_n(1) = 2\frac{A+B}{n+1} + O\left(\frac{\ln n}{n^2}\right).$$

Thus, under the hypothesis (2.6.21), there exists $n_0 \in \mathbb{N}^\star$, depending on A, B, such that

$$\zeta_n(1) < 0, \quad \forall n \geq n_0.$$

This proves the existence of at least one solution $x_n \in (0,1)$ to the equation $\zeta_n(x_n) = 0$, for any $n \geq n_0$. The next objective is to localize this root and show that $x_n \in \left(0, 1 + \frac{A+B}{An}\right)$. For this goal it suffices to verify that

$$\zeta_n(1-\varepsilon) < 0, \quad \forall \varepsilon \in \left(0, -\frac{A+B}{An}\right).$$

Let us begin with the first term $I_n^1(x)$ in the expression (2.6.1) which implies that

$$I_n^1(1-\varepsilon) = \left[\frac{1-2a_n}{n+1-a_n} + \frac{A+2B}{A(n+1)}\right] - \varepsilon\left[\frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1-a_n}\right].$$

Now, it is straightforward to check the following asymptotic expansions,

$$\frac{1-2a_n}{n+1-a_n} + \frac{A+2B}{A(n+1)} = 2\frac{A+B}{A(n+1)} - \frac{(2n+1)a_n}{(n+1)(n+1-a_n)} = 2\frac{A+B}{A(n+1)} + O\left(\frac{1}{n^2}\right),$$

and

$$\begin{aligned}-\varepsilon\left[\frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1-a_n}\right] =& -\varepsilon\left[-1 + \frac{A+2B}{A(n+1)} + \frac{2-2a_n}{n+1-a_n}\right] \\ \leq& \varepsilon + O\left(\frac{1}{n^2}\right) \leq -\frac{A+B}{An} + O\left(\frac{1}{n^2}\right),\end{aligned}$$

Therefore, we obtain

$$I_n^1(1-\varepsilon) \leq \frac{A+B}{An} + O\left(\frac{1}{n^2}\right).$$

Thanks to (2.6.18), we deduce

$$|1 - F(a_n, b_n; n+1; 1-\varepsilon)| \leq C\frac{\ln n}{n},$$

which yields in turn

$$I_n^1(1-\varepsilon)F(a_n,b_n;n+1;1-\varepsilon)\leq \frac{A+B}{A(n+1)}+O\left(\frac{\ln n}{n^2}\right). \tag{2.6.24}$$

Next, we will deal with the second term I_n^2 of (2.6.1). Directly from (2.6.19) we get

$$|I_n^2(x)|F(a_n+1,b_n;n+2;x)=\frac{|a_n(2x-1)|}{(n+1)(n+1-a_n)}F(a_n+1,b_n;n+2;x)\leq \frac{C}{n^2}. \tag{2.6.25}$$

Similarly, the estimate (2.6.20) implies that

$$|I_n^3(x)|F(a_n,b_n;n+3;x)=\frac{2xF(a_n,b_n;n+3;x)}{(n+1)(n+2)}\leq \frac{C}{n^2}. \tag{2.6.26}$$

Inserting (2.6.24), (2.6.25) and (2.6.26) into the expression of ζ_n we find

$$\zeta_n(1-\varepsilon)\leq \frac{A+B}{A(n+1)}+O\left(\frac{\ln n}{n^2}\right),$$

for any $\varepsilon\in\big(0,-\frac{A+B}{An}\big)$. From this we deduce the existence of n_0 depending on A and B such that

$$\zeta_n(1-\varepsilon)\leq \frac{A+B}{2A(n+1)}<0,$$

for any $n\geq n_0$. Then ζ_n has no zero in $(1+\frac{A+B}{An},1)$ and this achieves the proof of the first result.

Next we shall prove that x_n is the only zero of ζ_n in $(0,1)$. For this purpose it appears to be more convenient to use the expression for ζ_n given by (2.5.20). Let us differentiate ζ_n with respect to x as follows

$$\begin{aligned}\partial_x\zeta_n(x)=&F_n'(x)\left[1-x+\frac{A+2B}{A(n+1)}x\right]+F_n(x)\left[-1+\frac{A+2B}{A(n+1)}\right]\\&+\int_0^1F_n'(\tau x)\tau^{n+1}\left[-1+2x\tau\right]d\tau+2\int_0^1F_n(\tau x)\tau^{n+1}d\tau.\end{aligned}$$

From Lemma 2.6.2$-(1)$, we recall that $F_n>0$ and $F_n'<0$. Hence for $A+2B<0$ and $x\in(0,1)$ we get

$$\partial_x\zeta_n(x)\leq F_n'(x)\frac{A+2B}{A(n+1)}x-F_n(x)+\int_0^1F_n'(\tau x)\tau^{n+1}\left[-1+2x\tau\right]d\tau+2\int_0^1F_n(\tau x)\tau^{n+1}d\tau.$$

Applying the third assertion of Lemma 2.6.3 we find

$$|F_n'(x)|\leq C\ln n,\quad \forall n\geq 2,$$

for any $x\in(0,1+\frac{A+B}{An})$ and with C a constant depending only on A and B. It follows that

$$\partial_x\zeta_n(x)\leq C\frac{|A+2B|}{A}\frac{\ln n}{n}-F_n(x)+C\frac{\ln n}{n}+\frac{C}{n}\leq -1+C\frac{2A+2|B|}{A}\frac{\ln n}{n}+(1-F_n(x)),$$

for $x\in\big[0,1+\frac{A+B}{An}\big)$, which implies according to (2.6.18) that

$$\partial_x\zeta_n(x)\leq -1+C\frac{A+|B|}{A}\frac{\ln n}{n}.$$

Hence, there exists n_0 such that

$$\partial_x \zeta_n(x) \le -\frac{1}{2}, \quad \forall x \in \left[0, 1+\frac{A+B}{An}\right), \quad \forall n \ge n_0.$$

Thus, the function $x \in \left[0, 1+\frac{A+B}{An}\right) \mapsto \zeta_n(x)$ is strictly decreasing and admits only one zero that we have denoted by x_n.

It remains to show that $n \in [n_0, +\infty) \mapsto x_n$ is strictly increasing, which implies in particular that

$$\zeta_m(x_n) \neq 0, \quad \forall n \neq m \ge n_0. \tag{2.6.27}$$

For this aim, it suffices to show that the mapping $n \in [n_0, +\infty) \mapsto \zeta_n(x)$ is strictly increasing, for any $x \in (0,1)$. Setting

$$F_n(x) = 1 + \rho_n(x), \tag{2.6.28}$$

we can write

$$\begin{aligned}\zeta_n(x) \quad &= \quad \frac{n}{n+1} - \frac{n}{n+2}x + \frac{A+2B}{A(n+1)}x + \rho_n(x)\left[1 - x + \frac{A+2B}{A(n+1)}x\right] \\ &+ \int_0^1 \rho_n(\tau x)\tau^n \left[-1+2x\tau\right] d\tau =: \frac{n}{n+1} - \frac{n}{n+2}x + \frac{A+2B}{A(n+1)}x + R_n(x). \end{aligned} \tag{2.6.29}$$

Since F_n is analytic with respect to its parameters and we can think in n as a continuous parameter, $n \mapsto \zeta_n(x)$ is also analytic. Therefore, differentiating with respect to n, we deduce that

$$\partial_n \zeta_n(x) = \frac{1}{(n+1)^2} - 2\frac{x}{(n+2)^2} - \frac{A+2B}{A(n+1)^2}x + \partial_n R_n(x).$$

Consequently,

$$\partial_n \zeta_n(x) \ge \frac{1}{(n+1)^2}\left[1 - 2x - \frac{A+2B}{A}x\right] + \partial_n R_n(x) \ge \frac{1}{(n+1)^2}\left[1 - \frac{3A+2B}{A}x\right] + \partial_n R_n(x).$$

We use the following trivial bound

$$1 - \frac{3A+2B}{A}x \ge \min(1, \kappa), \quad \forall x \in [0,1],$$

where $\kappa = -2\frac{A+B}{A}$ is strictly positive due to the assumptions (2.6.21). Therefore, we can rewrite the bound for $\partial_n \zeta_n(x)$ as follows

$$\partial_n \zeta_n(x) \ge \frac{\min(1,\kappa)}{(n+1)^2} + \partial_n R_n(x). \tag{2.6.30}$$

To estimate $\partial_n R_n(x)$ we shall differentiate (2.6.29) with respect to n,

$$\begin{aligned}\partial_n R_n(x) =& \partial_n F_n(x)(1-x) + \partial_n F_n(x)\frac{A+2B}{A(n+1)}x - \rho_n(x)\frac{A+2B}{A(n+1)^2}x \\ &+ \int_0^1 (\partial_n F_n(\tau x))\tau^n\left[-1+2x\tau\right] d\tau + \int_0^1 \rho_n(\tau x)\tau^n \ln\tau \left[-1+2x\tau\right] d\tau.\end{aligned}$$

From (2.6.18) and Lemma 2.6.3, we deduce

$$\partial_n R_n(x) \ge \partial_n F_n(x)\frac{A+2B}{A(n+1)}x - \rho_n(x)\frac{A+2B}{A(n+1)^2}x$$

$$+\int_0^1 \partial_n F_n(\tau x)\,\tau^n\,[-1+2x\tau]\,d\tau+\int_0^1 \rho_n(\tau x)\tau^n\ln\tau\,[-1+2x\tau]\,d\tau,$$

and

$$\left|\partial_n F_n(x)\frac{A+2B}{A(n+1)}x-\rho_n(x)\frac{A+2B}{A(n+1)^2}x\right|\leq C\frac{|A+2B|}{A}\frac{\ln n}{n^3}.$$

Observe that the first integral can be bounded as follows

$$\left|\int_0^1 \partial_n F_n(\tau x)\,\tau^n\,[-1+2x\tau]\,d\tau\right|\leq C\frac{\ln n}{n^2}\int_0^1\tau^n|-1+2x\tau|d\tau\leq C\frac{\ln n}{n^3},$$

while for the second one we have

$$\left|\int_0^1 \rho_n(\tau x)\tau^n\ln\tau\,[-1+2x\tau]\,d\tau\right|\leq C\frac{\ln n}{n}\int_0^1\tau^n|\ln\tau|d\tau\leq C\frac{\ln n}{n^3},$$

where we have used (2.6.28) and Lemma 2.6.4. Plugging these estimates into (2.6.30), we find

$$\partial_n\zeta_n(x)\geq\frac{\min(1,\kappa)}{(n+1)^2}-C\frac{|A+2B|}{A}\frac{\ln(n+1)}{(n+1)^3},\quad\forall x\in[0,1].$$

Then, there exists n_0 depending only on A,B such that

$$\partial_n\zeta_n(x)\geq\frac{\min(1,\kappa)}{2(n+1)^2},$$

for any $n\geq n_0$ and any $x\in[0,1]$. This implies that $n\in[n_0,+\infty[\mapsto\zeta_n(x)$ is strictly increasing and thus (2.6.27) holds.

(**2**) From (2.6.2) one has

$$\begin{aligned}\zeta_n(x)=&\frac{A+2B}{A(n+1)}xF(a_n,b_n;n+1;x)-\frac{x}{n+1}F(a_n,b_n;n+2;x)\\&+\frac{n(1-x)}{n+1}F(a_n,b_n;n+2;x)+\frac{2nx}{(n+1)(n+2)}F(a_n,b_n;n+3;x).\end{aligned}$$

Remark that the involved hypergeometric functions are strictly positive, which implies that

$$\zeta_n(x)>\frac{n}{n+1}\left((1-x)F(a_n,b_n;n+2;x)+\frac{2x}{n+2}F(a_n,b_n;n+3;x)\right).$$

To get the announced result, it is enough to check that

$$(1-x)F(a_n,b_n;n+2;x)+\frac{2x}{(n+2)}F(a_n,b_n;n+3;x)\geq 1,\quad\forall x\in(-\infty,0),$$

which follows from Lemma 2.6.5:

$$\begin{aligned}(1-x)F(a_n,b_n;n+2;x)+&\frac{2x}{(n+2)}F(a_n,b_n;n+3;x)\\&\geq F(a_n,b_n;n+2;x)\left(1-\frac{nx}{n+2}\right)\geq F(a_n,b_n;n+2;x)\geq 1,\end{aligned}$$

for any $x\leq 0$. Thus, $\zeta_n(x)>0$ for any $x\in(-\infty,0)$ and this concludes the proof.

(**3**) Let us use the expression of ζ_n given in (2.6.2) obtaining

$$\zeta_n(x) = \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) + \frac{n(1-x)}{n+1} F(a_n, b_n; n+2; x) - \frac{x}{n+1} F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)} F(a_n, b_n; n+3; x).$$

Then

$$\zeta_n(x) > \frac{x}{n+1}\left(-F(a_n, b_n; n+2; x) + \frac{2n}{n+2} F(a_n, b_n; n+3; x)\right),$$

for any $x \in [0,1)$. From Lemma 2.6.5 we deduce

$$-F(a_n, b_n; n+2; x) + \frac{2n}{(n+2)} F(a_n, b_n; n+3; x) \geq \frac{n-2}{n+2} F(a_n, b_n; n+2; x) \geq 0,$$

for any $n \geq 2$ and $x \in (0,1)$. This implies that $\zeta_n(x) > 0$, for $x \in (0,1)$ and $n \geq 2$. The case $n = 1$ can be checked directly by the explicit expression stated in Remark 2.5.5. □

In the following result, we investigate more the case (2.6.23). We mention that according to Proposition 2.6.6$-(3)$ there are no eigenvalues in $(0,1)$. Thus, it remains to explore the region $(-\infty, 0)$ and study whether one can find eigenvalues there. Our result reads as follows.

Proposition 2.6.7. *Let $n \geq 2$ and $A, B \in \mathbb{R}$ satisfying* (2.6.23). *Then, the following assertions hold true:*

1. *If $n \leq \frac{B}{A} + \frac{1}{8}$, there exists a unique $x_n \in (-1,0)$ such that $\zeta_n(x_n) = 0$.*
2. *If $n \leq \frac{2B}{A}$, there exists a unique $x_n \in (-\infty, 0)$ such that $\zeta_n(x_n) = 0$, with*

$$\frac{1}{1 - \frac{A+2B}{A(n+1)}} < x_n < 0. \tag{2.6.31}$$

In addition, the map $x \in (-\infty, 0] \mapsto \zeta_n(x)$ is strictly increasing.

3. *If $n \geq \frac{B}{A} + 1$, then ζ_n has no solution in $[-1, 0]$.*
4. *If $n \geq \frac{2B}{A} + 2$, then ζ_n has no solution in $(-\infty, 0]$.*

Proof. **(1)** Thanks to (2.5.20) we have that $\zeta_n(0) = \frac{n}{n+1} > 0$. So to apply the Intermediate Value Theorem and prove that ζ_n admits a solution in $[-1, 0]$ it suffices to guarantee that $\zeta_n(-1) < 0$. Now coming back to (2.5.20) and using that $x \in (-1,1) \mapsto F_n(x)$ is strictly decreasing we get

$$\begin{aligned}\zeta_n(-1) =& F_n(-1)\left(2 - \frac{A+2B}{A(n+1)}\right) - \int_0^1 F_n(-\tau)\tau^n\big(1+2\tau\big)d\tau \\ <& F_n(-1)\left(2 - \frac{A+2B}{A(n+1)}\right) - \int_0^1 \tau^n\big(1+2\tau\big)d\tau \\ <& F_n(-1)\left(2 - \frac{A+2B}{A(n+1)}\right) - \frac{3n+4}{(n+1)(n+2)}, \quad \forall x \in [-1, 0).\end{aligned}$$

Consequently, to get $\zeta_n(-1) < 0$ we impose the condition

$$2 - \frac{A+2B}{A(n+1)} \leq \frac{3n+4}{(n+1)(n+2)F_n(-1)}. \tag{2.6.32}$$

Coming back to the integral representation, one gets

$$F_n(-1) \leq F_n(0)2^{-a_n} \leq 2, \tag{2.6.33}$$

due to the fact $a_n \in (-1, 0)$. In addition, it is easy to check that

$$\frac{3n+4}{n+2} \geq \frac{5}{2}, \quad \forall n \geq 2,$$

and the assumption (2.6.32) is satisfied if

$$2 - \frac{A+2B}{A(n+1)} \leq \frac{5}{4(n+1)}$$

holds, or equivalently, if

$$2 \leq n \leq \frac{B}{A} + \frac{1}{8}. \tag{2.6.34}$$

In conclusion, under the assumption (2.6.34), the function ζ_n admits a solution $x_n \in (-1, 0)$. Now, we localize this zero. Since F_n is strictly positive in $[-1, 1]$, then the second term in (2.5.20) is always strictly negative. Let us analyze the sign of the first term

$$F_n(x)\left[1 - x + \frac{A+2B}{A(n+1)}x\right],$$

which has a unique root

$$x_c = \frac{1}{1 - \frac{A+2B}{A(n+1)}}. \tag{2.6.35}$$

This root belongs to $(-\infty, 0)$ if and only if $n < \frac{2B}{A}$, which follows automatically from (2.6.34). Moreover the mapping $x \mapsto 1 - x + \frac{A+2B}{A(n+1)}x$ will be strictly increasing. Hence, if $x_c \leq -1$, then $x_n > x_c$. So let us assume that $x_c \in (-1, 0)$, then

$$F_n(x)\left[1 - x + \frac{A+2B}{A(n+1)}x\right] < 0, \quad \forall x \in [-1, x_c],$$

which implies that

$$\zeta_n(x) < 0, \quad \forall x \in [-1, x_c].$$

Therefore, the solution x_n must belong to $(x_c, 0)$, and equivalently

$$\frac{1}{1 - \frac{A+2B}{A(n+1)}} < x_n < 0.$$

The uniqueness of this solutions comes directly from the second assertion.

(**2**) As in the previous argument we have that $\zeta_n(0) = \frac{n}{n+1} > 0$ and the idea is to apply also the Intermediate Value Theorem. We intend to find the asymptotic behavior of ζ_n for x going to $-\infty$. We first find an asymptotic behavior of F_n. For this purpose we use the identity (C.0.3), which implies that

$$F_n(x) = (1-x)^{-a_n} F\left(a_n, a_n + 1, n + 1, \frac{x}{x-1}\right), \quad \forall x \leq 0.$$

Setting

$$\varphi_n(y) := F\left(a_n, a_n+1, n+1, y\right), \quad \forall y \in [0,1], \tag{2.6.36}$$

we obtain

$$\varphi_n'(y) = \frac{a_n(1+a_n)}{n+1} F\left(a_n+1, a_n+2, n+2, y\right), \quad \forall y \in [0,1)$$

By a monotonicity argument we deduce that

$$|\varphi_n'(y)| \leq C|F\left(a_n+1, a_n+2, n+2, y\right)| \leq C|F\left(a_n+1, a_n+2, n+2, 1\right)| \leq C_n, \quad \forall\, y \in [0,1],$$

with C_n a constant depending on n. However, the dependence with respect to n does not matter because we are interested in the asymptotics for large negative x but for a fixed n. Then, let us drop n from the subscript of the constant C_n. Applying the Mean Value Theorem we get

$$|\varphi_n(y) - \varphi_n(1)| \leq C(1-y), \quad \forall\, y \in [0,1].$$

Combining this estimate with (2.6.36) and $a_n \in (-1,0)$, we obtain

$$|F_n(x) - (1-x)^{-a_n}\varphi_n(1)| \leq C(1-x)^{-a_n-1} \leq C, \quad \forall x \leq 0,$$

which implies in turn that

$$|F_n(x) - (1-x)^{-a_n}\varphi_n(1)| \leq C, \quad \forall x \leq -1. \tag{2.6.37}$$

Consequently, we deduce that

$$\begin{aligned}\int_0^1 F_n(\tau x)\tau^n\big(-1+2x\tau\big)d\tau \sim & 2x(-x)^{-a_n}\varphi_n(1)\int_0^1 \tau^{-a_n+n+1}d\tau \\ \sim & \frac{2}{n+2-a_n}x(-x)^{-a_n}\varphi_n(1), \quad \forall -x \gg 1.\end{aligned}$$

Coming back to (2.5.20) and using once again (2.6.37) we get the asymptotic behavior

$$\begin{aligned}\zeta_n(x) \sim & \varphi_n(1)(-x)^{-a_n}\left(1-x+\frac{A+2B}{A(n+1)}x+\frac{2}{n+2-a_n}x\right) \\ \sim & \varphi_n(1)\left(\frac{A+2B}{A(n+1)}-1+\frac{2}{n+2-a_n}\right)(-x)^{-a_n}x, \quad \forall -x \gg 1.\end{aligned}$$

The condition

$$n \leq \frac{2B}{A}, \tag{2.6.38}$$

implies that

$$\frac{A+2B}{A(n+1)} - 1 + \frac{2}{n+2-a_n} > 0.$$

Since $\varphi_n(1) > 0$, we deduce that

$$\lim_{x\to-\infty}\zeta_n(x) = -\infty.$$

Therefore we deduce from the Intermediate Value Theorem that under the assumption (2.6.38), the function ζ_n admits a solution $x_n \in (-\infty, 0)$. Moreover, by the previous proof we get (2.6.31).

It remains to prove the uniqueness of this solution. For this goal we check that the mapping $x \in (-\infty,0) \mapsto \zeta_n(x)$ is strictly increasing when $n \in \left[1, \frac{2B}{A}\right]$. Differentiating ζ_n with respect to x yields

$$\zeta_n'(x) = F_n'(x)\left[1-x+\frac{A+2B}{A(n+1)}x\right] + F_n(x)\left[\frac{A+2B}{A(n+1)}-1\right]$$

$$+\int_0^1 F_n'(\tau x)\tau^{n+1}(-1+2x\tau)d\tau + 2\int_0^1 F_n(\tau x)\tau^{n+1}d\tau.$$

From Lemma 2.6.2-(1) we infer that $F_n'(x) < 0$, for $x \in (-\infty, 0)$, and therefore we get

$$\zeta_n'(x) > 0, \quad \forall\, x \in (-\infty, x_c).$$

Let $x \in (x_c, 0)$, then by a monotonicity argument we get

$$0 \leq 1 - x + \frac{A+2B}{A(n+1)}x \leq 1,$$

and thus

$$F_n'(x)\left[1 - x + \frac{A+2B}{A(n+1)}x\right] + 2\int_0^1 F_n(\tau x)\tau^{n+1}d\tau \quad \geq \quad F_n'(0) + \frac{2}{n+2} \geq -\frac{2}{(n+1)(n+2)},$$

by using that $F_n'(0) = -\frac{2}{n+1}$ and that $F_n'(x)$ is decreasing and negative in $(-\infty, 1)$. From the assumption (2.6.38) and the positivity of F_n we get

$$F_n(x)\left(\frac{A+2B}{A(n+1)} - 1\right) \geq 0, \quad \forall x \in (-\infty, 0].$$

Therefore, putting together the preceding estimates we deduce that

$$\begin{aligned}\zeta_n'(x) >& -\frac{2}{(n+1)(n+2)} + \int_0^1 F_n'(\tau x)\tau^{n+1}(-1+2x\tau)d\tau \\ >& -\frac{2}{(n+1)(n+2)} - \int_0^1 F_n'(\tau x)\tau^{n+1}d\tau, \quad \forall x \in (-\infty, 0).\end{aligned}$$

At this stage it suffices to make appeal to (2.6.10) in order to obtain

$$-\int_0^1 F_n'(\tau x)\tau^{n+1}d\tau \geq \frac{2}{(n+1)(n+2)}, \quad \forall x \in (-\infty, 0],$$

from which it follows

$$\zeta_n'(x) > 0, \quad \forall x \in (-\infty, 0],$$

which implies that ζ_n is strictly increasing in $(-\infty, 0]$, and thus x_n is the only solution in this interval.

(**3**) Using the definition of ζ_n in (2.5.20) and the monotonicity of F_n, one has

$$\zeta_n(x) > F_n(x)\left[1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1}\right]. \tag{2.6.39}$$

Now it is easy to check that

$$1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1} \geq \min\left\{\frac{n}{n+1}, \frac{2n-1}{n+1} - \frac{A+2B}{A(n+1)}\right\},$$

for any $x \in [-1, 0]$. This claim can be derived from the fact that the left-hand-side term is polynomial in x with degree one. Consequently, if we assume

$$n \geq 1 + \frac{B}{A},$$

we get $\frac{2n-1}{n+1} - \frac{A+2B}{A(n+1)} \geq 0$, and therefore (2.6.39) implies

$$\zeta_n(x) > 0, \quad \forall x \in [-1, 0].$$

Then, ζ_n has no solution in $[-1, 0]$.

(4) Using the expression of ζ_n in (2.5.20) and the monotonicity of F_n, one has

$$\begin{aligned}\zeta_n(x) >& F_n(x)\left[1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1}\right]\\ \geq& F_n(x)\left[\frac{n}{n+1} - x\left(\frac{n-1}{n+1} - \frac{A+2B}{A(n+1)}\right)\right], \quad \forall\, x \in (-\infty, 0).\end{aligned} \tag{2.6.40}$$

The assumption

$$n \geq 2 + \frac{2B}{A},$$

yields $\frac{n-1}{n+1} - \frac{A+2B}{A(n+1)} \geq 0$, and therefore (2.6.40) implies

$$\zeta_n(x) > 0, \quad \forall x \in (-\infty, 0].$$

Thus, ζ_n has no solution in $(-\infty, 0]$. □

In the next task we discuss the localization of the zeroes of ζ_n and, in particular, we improve the lower bound (2.6.31). Notice that $B > 0$ in order to get solutions of ζ_n in $(-\infty, 0]$ in the case $A > 0$ and $n \geq 2$, by using Proposition 2.6.6 and Proposition 2.6.7-(4). Our result reads as follows.

Proposition 2.6.8. *Let $A, B > 0$ and $n \geq 2$. If $x_n \in (-\infty, 0)$ is any solution of ζ_n, then the following properties are satisfied:*

1. *We have $\mathscr{P}_n(x_n) < 0$, with $\mathscr{P}_n(x) = \dfrac{n}{n+1} + x\left[-\dfrac{n}{n+2} + \dfrac{A+2B}{A(n+1)}\right]$.*
2. *If $x_n \in (-1, 0)$, then $x_\star := -\dfrac{2n+1}{2(n+1)} \dfrac{1}{\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2}} < x_n$.*
3. *We always have $x_n < -\dfrac{A}{2B}$.*

Proof. (**1**) Since $F_n(\tau x) < F_n(x)$, for any $\tau \in [0, 1)$ and $x \in (-\infty, 0)$, then we deduce from the expression (2.5.20) that

$$\zeta_n(x) > F_n(x)\Big[1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+2}\Big] = F_n(x)\mathscr{P}_n(x). \tag{2.6.41}$$

As F_n is strictly positive in $(-\infty, 1)$, then $\mathscr{P}_n(x_n) < 0$, for any root of ζ_n.

(**2**) Recall from Proposition 2.6.7–(3) that if ζ_n admits a solution in $(-1, 0)$ with $n \geq 2$ then necessary $2 \leq n \leq 1 + \frac{B}{A}$. This implies that $n \leq \frac{2B}{A}$ and hence the mapping $x \mapsto 1 - x + \frac{A+2B}{A(n+1)}x$ is increasing. Combined with the definition of (2.6.35) and (2.6.31) we deduce that

$$1 - x + \frac{A+2B}{A(n+1)}x \geq 0, \quad \forall x \in (x_c, 0)$$

and $x_n \in (x_c, 0)$. Using the monotonicity of F_n combined with the bound (2.6.33) we find from (2.5.20)

$$\zeta_n(x) < 2\left[1 - x + \frac{A+2B}{A(n+1)}x\right] - \frac{1}{n+1} + \frac{2x}{n+2}, \quad \forall x \in (x_c, 0).$$

Evaluating at any root x_n we obtain

$$-\frac{2n+1}{n+1} < 2x_n\left[\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2}\right].$$

Keeping in mind that $n \leq \frac{2B}{A}$, we get $\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2} > 0$, and therefore we find the announced lower bound for x_n.

(**3**) In a similar way to the upper bound for x_n, we turn to (2.6.41), and evaluate this inequality at $-\frac{A}{2B}$. Then we find

$$\zeta_n\left(-\frac{A}{2B}\right) > F_n\left(-\frac{A}{2B}\right)\mathscr{P}_n\left(-\frac{A}{2B}\right) > \mathscr{P}_n\left(-\frac{A}{2B}\right).$$

Explicit computations yield

$$\mathscr{P}_n\left(-\frac{A}{2B}\right) = \frac{n-1}{n+1} + \frac{A}{2B}\Big[\frac{n}{n+2} - \frac{1}{n+1}\Big]$$

Since $\frac{A}{2B} > 0$ and $n \geq 2$, then we infer that $\mathscr{P}_n\left(-\frac{A}{2B}\right) > 0$ and $\zeta_n\left(-\frac{A}{2B}\right) > 0$. Now we recall from Proposition 2.6.7-(2) that $x \in (-\infty, 0) \mapsto \zeta_n(x)$ is strictly increasing. Thus combined this property with the preceding one we deduce that

$$x_n < -\frac{A}{2B},$$

which achieves the proof. □

Notice that from Proposition 2.6.7-(4) when $B \leq 0$, the function ζ_n has no solution in $(-\infty, 0]$ for any $n \geq 2$. Moreover, in the case that $0 < B \leq \frac{A}{4}$, Proposition 2.6.7-(4) and Proposition 2.6.8-(1) give us again that ζ_n has no solution in $(-\infty, 0]$ for any $n \geq 2$. Combining these facts with Proposition 2.6.6–(3), we immediately get the following result.

Corollary 2.6.9. *Let $A > 0$ and B satisfying*

$$-\frac{A}{2} \leq B \leq \frac{A}{4}.$$

Then, the function ζ_n has no solution in $(-\infty, 1]$ for any $n \geq 2$. However, the function ζ_1 admits the solution $x_1 = -\frac{A}{2B}$. Notice that this latter solution belongs to $(-\infty, 1)$ if and only if $B \notin \left[-\frac{A}{2}, 0\right]$.

In the next result, we study the case when $x_1 \in \left(0, \frac{1+\epsilon}{2}\right]$ for some $0 < \epsilon < 1$, showing that there is no intersection with other eigenvalues.

Proposition 2.6.10. *Let $A > 0$. There exists $\epsilon \in (0,1)$ such that if $B \leq -\frac{A}{1+\epsilon}$, then $\zeta_n(x_1) \neq 0$ for any $n \geq 2$ and $x_1 = -\frac{A}{2B}$, with $\epsilon \approx 0,0581$.*

Proof. From (2.5.20), one has

$$\zeta_n(x_1) = F_n(x_1)\frac{n}{n+1}(1-x_1) + \int_0^1 F_n(x_1\tau)\tau^n\left[-1+2x_1\tau\right]d\tau.$$

By the integral representation of F_n given in (C.0.2), we obtain that

$$F_n(x) > \frac{\Gamma(n+1)}{\Gamma(n-a_n)\Gamma(a_n+1)}\int_0^1 \mathrm{t}^{n-a_n-1}(1-\mathrm{t})^{a_n}d\mathrm{t}(1-x)^{-a_n} = (1-x)^{-a_n},$$

for any $x \in (0,1)$, using the Beta function (2.6.16). By the monotonicity of $F_n(x)$ with respect to x and the above estimate, we find that

$$(1-x_1)^{-a_n} < F_n(x_1) < 1,$$

for any $n \geq 2$ and $x_1 \in (0,1)$, which agrees with the hypothesis on A and B. Hence, we have that

$$\zeta_n(x_1) > \frac{n}{n+1}(1-x_1)^{1-a_n} + 2x_1\frac{(1-x_1)^{-a_n}}{n+2} - \frac{1}{n+1}.$$

The above expression is increasing with respect to n, which implies that

$$\zeta_n(x_1) > \frac{2}{3}(1-x_1)^{1-a_2} + \frac{x_1}{2}(1-x_1)^{-a_2} - \frac{1}{3}.$$

Since $a_2 = 1-\sqrt{3}$, we get

$$\zeta_n(x_1) > \frac{2}{3}(1-x_1)^{\sqrt{3}} + \frac{x_1}{2}(1-x_1)^{\sqrt{3}-1} - \frac{1}{3} =: \mathcal{P}(x_1).$$

The function $\mathcal{P}$ decreases in $(0,1)$ and admits a unique root $\overline{x}$ whose approximate value is given by $\overline{x} = 0,52907$. Hence, $\mathcal{P}(x_1) \geq 0$ for $x_1 \in (0,\overline{x}]$, and consequently we get

$$\zeta_n(x_1) > 0,$$

for $B \leq -\frac{A}{2\overline{x}}$, achieving the announced result. □

We finish this section by the following result concerning the monotonicity of the eigenvalues.

Proposition 2.6.11. *Let $A > 0$ and $2B > A$. Then, the following assertions hold true:*

1. *Let $x \in (-\infty, 0)$, then $n \in \left[1, \frac{2B}{A}\right] \mapsto \zeta_n(x)$ is strictly increasing. In addition, we have*

$$\Big\{x \in (-\infty,0],\, \zeta_n(x) = 0\Big\} \cap \Big\{x \in (-\infty,0],\, \zeta_m(x) = 0\Big\} = \emptyset,$$

   for any $n \neq m \in \left[1, \frac{2B}{A}\right]$, and each set contains at most one element.

2. *The sequence $n \in \left[1, \frac{B}{A} + \frac{1}{8}\right] \mapsto x_n$ is strictly decreasing, where the $\{x_n\}$ are constructed in Proposition 2.6.7.*

3. *If $m \in \left[1, \frac{2B}{A} - 2\right]$, then*

$$\zeta_n(x_m) \neq 0, \quad \forall n \in \mathbb{N}^\star\backslash\{m\}.$$

Proof. (**1**) We shall prove that the mapping $n \in \left[1, \frac{2B}{A}\right] \mapsto \zeta_n(x)$ is strictly increasing for fixed $x \in (-\infty, 0]$ and $2B \geq A$. Differentiating (2.5.20) with respect to n we get

$$\begin{aligned}\partial_n\zeta_n(x) =&\partial_n F_n(x)\left[1-x+\frac{A+2B}{A(n+1)}x\right]+\int_0^1 \partial_n F_n(\tau x)\tau^n\left[-1+2x\tau\right]d\tau\\&-F_n(x)\frac{A+2B}{A(n+1)^2}x+\int_0^1 F_n(\tau x)\tau^n\ln\tau\left[-1+2x\tau\right]d\tau.\end{aligned}$$

Using Lemma 2.6.3-(1) and the positivity of F_n, we deduce that

$$\int_0^1 \partial_n F_n(\tau x)\tau^n\left[-1+2x\tau\right]d\tau>0,\quad \int_0^1 F_n(\tau x)\tau^n\ln\tau\left[-1+2x\tau\right]d\tau>0,\quad \forall x\in(-\infty,0).$$

Due to the assumption $n \in \left[1, \frac{2B}{A}\right]$, we have that $x_c < 0$, where x_c is defined in (2.6.35). If $x \in (-\infty, x_c]$, we find that

$$1-x+\frac{A+2B}{A(n+1)}x\leq 0,$$

which implies

$$\partial_n\zeta_n(x)>0,\quad \forall x\in(-1,x_c),\forall n\geq 1.$$

We obtain

$$0<1-x+\frac{A+2B}{A(n+1)}x<1, \tag{2.6.42}$$

for $x \in (x_c, 0)$, which yields in view of Lemma 2.6.3-(2)

$$\begin{aligned}\partial_n\zeta_n(x) >&\partial_n F_n(x)\left[1-x+\frac{A+2B}{A(n+1)}x\right]-F_n(x)\frac{A+2B}{A(n+1)^2}x\\>&\partial_n F_n(x)-F_n(x)\frac{A+2B}{A(n+1)^2}x>\frac{-2xF_n(x)}{(n+1)^2}\left[-1+\frac{A+2B}{2A}\right].\end{aligned}$$

Taking into account $2B \geq A$, one gets $\partial_n\zeta_n(x) > 0, \forall\, x \in (-\infty, 0]$. It remains to discuss the case $x_c \leq -1$. Remark that the estimate (2.6.42) is satisfied for any $x \in (-\infty, 0)$, and then the foregoing inequality holds, and one gets finally

$$\partial_n\zeta_n(x)>0,\quad \forall\, x\in[-1,0],\forall n\in\left[1,\frac{2B}{A}\right].$$

Consequently, we deduce that the mapping $n \in \left[1, \frac{2B}{A}\right] \mapsto \zeta_n(x)$ is strictly increasing for any $x \in (-\infty, 0)$. This implies in particular that the functions ζ_n and ζ_m have no common zero in $(-\infty, 0)$ for $n \neq m \in \left[1, \frac{2B}{A}\right]$.

(**2**) This follows by combining that $x \in (-\infty, 0) \mapsto \zeta_n(x)$ and $n \in \left[1, \frac{2B}{A}\right] \mapsto \zeta_n(x)$ are strictly increasing, proved in Proposition 2.6.7$-(1)$ and Proposition 2.6.11$-(1)$.

(**3**) By the last assertions, this is clear for $n \leq \frac{2B}{A}$ and it is also true for $n \geq \frac{2B}{A} + 2$, since ζ_n has not roots in $(-\infty, 1)$, by Proposition 2.6.6 and Proposition 2.6.7. Then, let us study the case $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$. First, using (2.6.31), we get that x_m, which is a solution of $\zeta_m = 0$ with $m \leq \frac{2B}{A} - 2$, verifies

$$x_m\geq-\frac{2B-A}{2A}. \tag{2.6.43}$$

Now, the strategy is to show that $\mathscr{P}_n(x_m) > 0$ for $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$, and then Proposition 2.6.8 will imply that $\zeta_n(x_m) \neq 0$. By definition, we have that

$$\mathscr{P}_n(x_m) = \frac{n}{n+1} + x_m \left[-\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \right].$$

If $-\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \leq 0$, then $\mathscr{P}_n(x_m) > 0$. Otherwise, we use (2.6.43) getting

$$\begin{aligned}\mathscr{P}_n(x_m) >& \frac{n}{n+1} - \frac{2B-A}{2A} \left[-\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \right] \\ =& \frac{1}{2(n+1)(n+2)} \left[n^2 \left(1 + \frac{2B}{A} \right) + n \left(4 + \frac{2B}{A} - 4\frac{B^2}{A^2} \right) + 2 \left(1 - 4\frac{B^2}{A^2} \right) \right].\end{aligned}$$

Straightforward computations yield that the above parabola is increasing in $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$. Evaluating at $n = \frac{2B}{A}$ in the parabola, we find

$$\mathscr{P}_n(x_m) > \frac{8\frac{B}{A} + 2}{2(n+1)(n+2)} > 0.$$

□

2.6.4 Asymptotic expansion of the eigenvalues

When solving the boundary equation in Proposition 2.3.3, one requires that the angular velocity is located outside the singular set (2.3.5). Consequently, in order to apply the bifurcation argument for the density equation we should check that the eigenvalues $\{x_n\}$ constructed in Proposition 2.6.6 do not intersect the singular set. This problem sounds to be very technical and in the case $A + B < 0$, where we know that the dispersion set is infinite, we reduce the problem to studying the asymptotic behavior of each sequence. Let us start with a preliminary result.

Lemma 2.6.12. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(-1, 1)$ such that $x_n = 1 - \frac{\kappa}{n} + o\left(\frac{1}{n}\right)$, for some strictly positive number κ. Then the following asymptotics*

$$F(a_n + 1, b_n; n + 2; x_n) = n\left(\eta + o(1)\right),$$

holds, with $\eta = \kappa \displaystyle\int_0^{+\infty} \frac{\tau e^{-\kappa\tau}}{1+\tau} d\tau.$

Proof. The integral representation of hypergeometric functions (C.0.2) allows us to write

$$F(a_n + 1, b_n; n + 2; x_n) = \frac{\Gamma(n+2)}{\Gamma(n-a_n)\Gamma(2+a_n)} \ell_n = \frac{n(n+1)\Gamma(n)}{\Gamma(n-a_n)\Gamma(2+a_n)} \ell_n,$$

where

$$\ell_n := \int_0^1 \tau^{n-a_n-1}(1-\tau)^{1+a_n}(1-\tau x_n)^{-1-a_n} d\tau.$$

Set $\varepsilon_n = \frac{1-x_n}{x_n}$, making the change of variables $\tau = 1 - \varepsilon_n\tau'$, and keeping the same notation τ to the new variable, we obtain

$$\ell_n = x_n^{-a_n-1}\varepsilon_n \int_0^{\frac{1}{\varepsilon_n}} (1-\varepsilon_n\tau)^{n-a_n-1} \tau^{1+a_n}(1+\tau)^{-a_n-1} d\tau.$$

From the first order expansion of x_n, one has the pointwise convergence

$$\lim_{n\to+\infty}(1-\varepsilon_n\tau)^{n-a_n-1}=e^{-\kappa\tau},$$

for any $\tau>0$. Since the sequence $n\mapsto\left(1-\frac{\kappa}{n}\right)^n$ is increasing, the Lebesgue Theorem leads to

$$\lim_{n\to+\infty}\int_0^{\frac{1}{\varepsilon_n}}(1-\varepsilon_n\tau)^{n-a_n-1}\tau^{1+a_n}(1+\tau)^{-a_n-1}d\tau=\int_0^{+\infty}e^{-\kappa\tau}\frac{\tau}{1+\tau}d\tau.$$

Therefore, we obtain the equivalence $\ell_n\sim\frac{\eta}{n}$. Combining the previous estimates with (2.6.7) we find the announced estimate. □

The next objective is to give the asymptotic expansion of the eigenvalues.

Proposition 2.6.13. *Let A and B be such that* (2.6.21) *holds. Then, the sequence $\{x_n, n\geq n_0\}$, constructed in Proposition* 2.6.6, *admits the following asymptotic behavior*

$$x_n=1-\frac{\kappa}{n}+\frac{c_\kappa}{n^2}+o\left(\frac{1}{n^2}\right),$$

where

$$\kappa=-2\frac{A+B}{A},\quad \textit{and}\quad c_\kappa=\kappa^2-2+2\int_0^{+\infty}\frac{e^{-\kappa\tau}}{(1+\tau)^2}d\tau.$$

Proof. First we will check that

$$x_n=1-\frac{\kappa}{n}+o\left(\frac{1}{n}\right). \tag{2.6.44}$$

Recall that $x_n\to1$ and write

$$x_n=1-\beta_n. \tag{2.6.45}$$

Clearly $\beta_n\to0$ and we intend to give an equivalent. From (2.6.1), we know that x_n satisfies the equation

$$\begin{aligned}\zeta_n(x_n)=&I_n^1(x_n)F(a_n,b_n;n+1;x_n)+I_n^2(x_n)F(a_n+1,b_n;n+2;x_n)\\&+I_n^3(x_n)F(a_n,b_n;n+3;x_n)\\=&0,\end{aligned} \tag{2.6.46}$$

with

$$I_n^1(x_n)=\frac{n-a_n}{n+1-a_n}-x_n\left(\frac{1+\kappa}{n+1}+\frac{n-1-a_n}{n+1-a_n}\right),$$

$$I_n^2(x_n)=-\frac{a_n(2x_n-1)}{(n+1)(n+1-a_n)},\quad\text{and}\quad I_n^3(x_n)=-\frac{2x_n}{(n+1)(n+2)}.$$

By virtue of Lemma 2.6.4 one can write (2.6.46) as

$$I_n^1(x_n)=I_n^2(x_n)O(n)+I_n^1(x_n)(1-F(a_n,b_n;n+1;x_n)-I_n^3(x_n)F(a_n,b_n;n+3;x_n). \tag{2.6.47}$$

Using (2.6.45) one gets that

$$I_n^1(x_n)=-\frac{\kappa}{n+1-a_n}+\frac{(1+\kappa)a_n}{(n+1)(n+1-a_n)}+\beta_n\frac{1+\kappa}{n+1}+\beta_n\frac{n-1-a_n}{n+1-a_n}.$$

Since $a_n \sim -\frac{2}{n}$ then we get successively

$$I_n^1(x_n) = -\frac{\kappa}{n+1-a_n} + \frac{(1+\kappa)a_n}{(n+1)(n+1-a_n)} + \beta_n \frac{1+\kappa}{n+1} + \beta_n \frac{n-1-a_n}{n+1-a_n}$$
$$= -\frac{\kappa}{n} + \beta_n \frac{1+\kappa}{n} + \beta_n \frac{n-1}{n+1-a_n} + O(1/n^2). \tag{2.6.48}$$

and

$$I_n^2(x_n) = -\frac{a_n}{(n+1)(n+1-a_n)} + \frac{2a_n\beta_n}{(n+1)(n+1-a_n)} = O\left(1/n^3\right). \tag{2.6.49}$$

In addition, $I_n^3(x_n)$ agrees with

$$I_n^3(x_n) = -\frac{2}{(n+1)(n+2)} + \frac{2\beta_n}{(n+1)(n+2)}$$
$$= O(1/n^2). \tag{2.6.50}$$

Then, inserting (2.6.48)–(2.6.49)–(2.6.50) into (2.6.47) and using Lemma 2.6.4 for the right hand side, we obtain

$$-\frac{\kappa}{n} + \beta_n \frac{1+\kappa}{n} + \beta_n \frac{n-1}{n+1-a_n} = O\left(\frac{\ln n}{n^2}\right). \tag{2.6.51}$$

Thus, we find

$$\beta_n \left[1 + O\left(\frac{1}{n}\right)\right] = \frac{\kappa}{n} + O\left(\frac{\ln n}{n^2}\right), \tag{2.6.52}$$

which achieves a the announced result (2.6.44). At this stage we can write x_n in the form

$$x_n = 1 - \frac{\kappa}{n} - u_n, \quad u_n = o\left(\frac{1}{n}\right). \tag{2.6.53}$$

Inserting (2.6.53) into (2.6.48), (2.6.49) and (2.6.50) we easily get

$$I_n^1(x_n) = -\frac{\kappa(1-\kappa)}{(n+1)^2} + u_n + o(u_n) + O\left(\frac{1}{n^3}\right)$$

and

$$I_n^2 = \frac{2}{(n+1)^3} + o\left(\frac{1}{n^3}\right), \quad \text{and} \quad I_n^3 = \frac{-2}{(n+1)^2} + O\left(\frac{1}{n^3}\right),$$

where we have used $a_n \sim -\frac{2}{n}$. By virtue of the above estimates, Lemma 2.6.4 and Lemma 2.6.12, the expansion of u_n reads as

$$u_n = \frac{\kappa(1-\kappa) + 2 - 2\eta}{(n+1)^2} + o\left(\frac{1}{n^2}\right) = \frac{\kappa - \kappa^2 + 2 - 2\eta}{n^2} + o\left(\frac{1}{n^2}\right).$$

Using $\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$, we find

$$x_n = 1 - \frac{\kappa}{n} + \frac{\kappa^2 - 2 + 2\eta}{n^2} + o\left(\frac{1}{n^2}\right).$$

The final expression holds as a consequence of the following integration by parts

$$\eta = 1 - \kappa \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1+\tau} d\tau = \int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau.$$

□

2.6.5 Separation of the singular and dispersion sets

In Section 2.2.4 we have established some conditions in order to solve the boundary equation. If we want to apply Proposition 2.3.3, we must verify that Ω does not lie in the singular set $\mathcal{S}_{\text{sing}}$ given in (2.4.10). Moreover, from the last analysis, we have checked that the dispersion set $\mathcal{S}$, defined in (2.1.11), contains different sets depending on the assumptions on A and B. We will prove the following results.

Proposition 2.6.14. *Let A and B such that* (2.6.21) *holds. Denote by*

$$\Omega_n := \frac{B}{2} + \frac{A}{4x_n},$$

where the sequence $(x_n)_{n\geq n_0}$ has been defined in Proposition 2.6.6. *If n_0 is large enough depending on A and B, then*

$$\Omega_n \neq \widehat{\Omega}_{np}, \qquad \forall\, n \geq n_0,\ \forall p \in \mathbb{N}^\star,$$

where $\widehat{\Omega}_{np}$ belongs to the set $\mathcal{S}_{sing}$ introduced in (2.4.10). *Moreover, there exists $\kappa_c > 0$ such that for any $\kappa > \kappa_c$ we find $n_0 \in \mathbb{N}$ such that*

$$\Omega_n \neq \widehat{\Omega}_m, \qquad \forall\, m \geq n \geq n_0.$$

The number $\kappa_c \in (0, 2)$ is the unique solution of the equation

$$\kappa_c - 2\int_0^{+\infty} \frac{e^{-\kappa_c \tau}}{(1+\tau)^2} d\tau = 0.$$

Proof. It is a simple matter to have

$$\frac{4}{A}\left(\widehat{\Omega}_n - \frac{B}{2}\right) = 1 + \frac{\kappa}{n} + \frac{2}{n^2} + O\left(\frac{1}{n^3}\right).$$

Setting $\widehat{x}_n = \frac{1}{\frac{4}{A}\left(\widehat{\Omega}_n - \frac{B}{2}\right)}$, we obtain

$$\widehat{x}_n = 1 - \frac{\kappa}{n} + \frac{\kappa^2 - 2}{n^2} + O\left(\frac{1}{n^3}\right). \tag{2.6.54}$$

Thus, condition $\Omega_n \neq \widehat{\Omega}_m$ is equivalent to $x_n \neq \widehat{x}_m$. According to Proposition 2.6.13, we have

$$x_n = 1 - \frac{\kappa}{n} + \frac{c_\kappa}{n^2} + o\left(\frac{1}{n^2}\right),$$

for $n \geq n_0$, where

$$c_\kappa = \kappa^2 - 2 + 2\int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau.$$

This implies that $x_n \neq \widehat{x}_n$ for large n. Moreover,

$$\widehat{x}_{np} = 1 - \frac{\kappa}{np} + \frac{\kappa^2 - 2}{n^2 p^2} + O\left(\frac{1}{n^3}\right) = 1 - \frac{\kappa}{np} + O\left(\frac{1}{n^2}\right),$$

for any $p \in \mathbb{N}^\star$ and $O\left(\frac{1}{n^2}\right)$ being uniform on p. Therefore, we get that $\widehat{x}_{np} > x_n$, for any $p \geq 2$, with $n \geq n_0$ and n_0 large enough. Consequently, we deduce that $x_n \neq \widehat{x}_{np}$, for any $n \geq n_0$, and $p \in \mathbb{N}^\star$. To establish the second assertion, we will use an asymptotic expansion for $\widehat{x}_{n+1}$. Thanks to (2.6.54) we can write

$$\widehat{x}_{n+1} = 1 - \frac{\kappa}{n} + \frac{\kappa^2 + \kappa - 2}{n^2} + O\left(\frac{1}{n^3}\right).$$

Then, since $(\widehat{x}_n)_{n \geq n_0}$ is strictly increasing and $\widehat{x}_n \neq x_n$, to prove $\widehat{x}_m \neq x_n$ for any $m \geq n$ it suffices just to check that $\widehat{x}_{n+1} > x_n$, which leads to

$$g(\kappa) := \kappa - 2\int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau > 0.$$

Since g is strictly increasing on $[0, +\infty)$ and satisfies $g(0) = -2$ and $g(2) > 0$, there exists only one solution $\kappa_c \in (0, 2)$ for the equation $g(\kappa) = 0$. This concludes the proof. $\square$

The next task is to discuss the separation problem when the dispersion set is finite.

Proposition 2.6.15. *Let $A > 0, B \in \mathbb{R}$ and $\widehat{\mathcal{S}}_{sing}$ being the set defined in* (2.5.19). *Then the following assertions hold true:*

1. *If $B \notin [-\frac{A}{2}, -\frac{A}{4}]$, then $x_1 = -\frac{A}{2B} \notin \widehat{\mathcal{S}}_{sing}$.*
2. *If $B > A$, then the sequence $m \in \left[2, \frac{2B}{A}\right] \mapsto x_m$ defined in Proposition* 2.6.7 *satisfies*

$$x_m \neq \widehat{x}_{nm}, \quad \forall n \geq 1.$$

Proof. Recall from the definition of the set $\widehat{\mathcal{S}}_{sing}$ given in (2.5.19) that

$$\frac{1}{\widehat{x}_n} = 1 - \frac{2(n+1)}{n(n+2)} - \frac{2B}{An},$$

where we have used (2.4.10) and (2.4.12). Notice that when A and B are positive then $n \mapsto \frac{1}{\widehat{x}_n}$ is strictly increasing.

(**1**) Let us prove that

$$-\frac{2B}{A} \neq 1 - \frac{2(n+1)}{n(n+2)} - \frac{2B}{An}, \quad \forall n \geq 1. \tag{2.6.55}$$

Note that for $n = 1$ this constraint is always satisfied since we get

$$-\frac{1}{3} - \frac{2B}{A} \neq -\frac{2B}{A}.$$

Thus (2.6.55) is equivalent to

$$\frac{n^2 - 2}{n^2 + n - 2} \neq -\frac{2B}{A}, \quad \forall n \geq 2.$$

One can easily check that left part is strictly increasing on $[2, +\infty)$ and so

$$\frac{1}{2} \leq \frac{n^2 - 2}{n^2 + n - 2} \leq 1, \quad \forall n \geq 2.$$

Consequently, if $-\frac{2B}{A} \notin [\frac{1}{2}, 1]$ then the condition (2.6.55) is satisfied and this ensures the first point.

(**2**) From the expression of $\mathscr{P}_n$ given in Proposition 2.6.8, we may write

$$\mathscr{P}_n(x_m) = \frac{n}{n+1}\left(1 - \frac{x_m}{\widehat{x}_n}\right),$$

and $\mathscr{P}_m(x_m) < 0$, which implies $\frac{1}{\widehat{x}_m} < \frac{1}{x_m}$. By the monotonicity of $\frac{1}{\widehat{x}_n}$, it is enough to prove

$$\frac{1}{x_m} < \frac{1}{\widehat{x}_{2m}}, \tag{2.6.56}$$

in order to conclude. According to (2.6.31), we have $\frac{1}{x_m} < 1 - \frac{A+2B}{A(m+1)}$. To obtain (2.6.56), it is enough to establish

$$1 - \frac{A+2B}{A(m+1)} \leq \frac{1}{\widehat{x}_{2m}} = 1 - \frac{2m+1}{2m(m+1)} - \frac{B}{Am},$$

which is equivalent to

$$\frac{A+2B}{A(m+1)} \geq \frac{2m+1}{2m(m+1)} + \frac{B}{Am}.$$

This latter one agrees with

$$\frac{B}{A} \geq \frac{1}{2(m-1)},$$

which holds true since $\frac{B}{A} \geq 1$. □

2.6.6 Transversal property

This section is devoted to the transversality assumption concerning the fourth hypothesis of the Crandall–Rabinowitz Theorem A.0.3. We shall reformulate an equivalent tractable statement, where the problem reduces to check the non-vanishing of a suitable integral. However, it is slightly hard to check this property for all the eigenvalues. We give positive results for higher frequencies using the asymptotics, which have been developed in the preceding sections for some special regimes on A and B. The first result in this direction is summarized as follows.

Proposition 2.6.16. *Let $A > 0$ and $B \in \mathbb{R}$, $x \in (-\infty, 1)\backslash\{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$ and $n \in \mathcal{A}_x$ where all the elements involved can be found in* (2.4.12) (2.4.14)-(2.5.19) *and* (2.5.21). *Let*

$$re^{i\theta} \in \overline{\mathbb{D}} \mapsto \hbar_n(re^{i\theta}) = h_n^\star(r)\cos(n\theta) \in \text{Ker } D_g\widehat{G}(\Omega, 0),$$

Then

$$D_{\Omega,g}\widehat{G}(\Omega, 0)\hbar_n \notin \text{Im } D_g\widehat{G}(\Omega, 0)$$

if and only if $h_n^\star$ satisfies

$$\int_0^1 \frac{s^{n+1}F_n(xs^2)}{1-xs^2}\left[\frac{h_n^\star(s)}{2A} - A_n s^{n+2} + A_n\frac{sG_n(s)}{G_n(1)}\right] ds \neq 0, \tag{2.6.57}$$

where

$$A_n = -\frac{\int_0^1 s^{n+1}h_n^\star(s)ds}{2G_n(1)},$$

and G_n is defined by (2.5.9).

Proof. Differentiating the expression of the linearized operator (2.5.8) with respect to Ω we obtain

$$D_{\Omega,g}\widehat{G}(\Omega,0)h(re^{i\theta})=\sum_{n\geq 1}\cos(n\theta)\left[\frac{1}{2A}h_n(r)-\frac{r}{n}\Big((\partial_\Omega A_n)G_n(r)+A_n\,\partial_\Omega G_n(r)\Big)\right]+\frac{1}{2A}h_0(r).$$

Differentiating the identity (2.5.13) with respect to Ω yields

$$\partial_\Omega A_n=\frac{H_n(1)}{2n(\Omega-\widehat{\Omega}_n)^2}=-\frac{nA_n}{G_n(1)}.$$

Similarly, we get from (2.5.9), (2.5.11) and the relation (2.4.12) between x and Ω that

$$\partial_\Omega G_n(r)=\frac{An}{4}r^{n+1}\partial_\Omega\left(\frac{1}{x}\right)=nr^{n+1}.$$

Putting together the preceding identities we find

$$D_{\Omega,g}\widehat{G}(\Omega,0)h(re^{i\theta})=\sum_{n\geq 1}\cos(n\theta)\left[\frac{1}{2A}h_n(r)+A_n\frac{rG_n(r)}{G_n(1)}-A_nr^{n+2}\right]+\frac{1}{2A}h_0(r).$$

Evaluating this formula at $\hbar_n$ yields

$$D_{\Omega,g}\widehat{G}(\Omega,0)\hbar_n(re^{i\theta})=\left[\frac{h_n^\star(r)}{2A}-A_nr^{n+2}+A_n\frac{rG_n(r)}{G_n(1)}\right]\cos(n\theta),$$

where A_n is related to $h_n^\star$ via (2.3.6). Now applying Proposition 2.5.10 we obtain that this element does not belong to $\mathrm{Im}D_g\widehat{G}(\Omega,0)$ if and only if the function

$$r\in[0,1]\mapsto d_n^\star(r):=\frac{h_n^\star(r)}{2A}-A_nr^{n+2}+A_n\frac{rG_n(r)}{G_n(1)}$$

verifies

$$\int_0^1\frac{s^{n+1}F_n(xs^2)}{1-xs^2}d_n^\star(s)ds\neq 0,$$

which gives the announced result. □

The next goal is to check the condition (2.6.57) for large n in the regime (2.6.21). We need first to rearrange the function $d_n^\star$ defined above and use the explicit expression of $\hbar_n$ given in (2.5.34). From (2.5.33) we get

$$A_n=-\frac{n}{8xG_n(1)}.$$

Then, multiplying $d_n^\star$ by $\frac{1}{A_n}$, we obtain

$$\frac{1}{A_n}\left[\frac{h_n^\star(r)}{2A}-A_nr^{n+2}+A_n\frac{rG_n(r)}{G_n(1)}\right]=\left[\frac{h_n^\star(r)}{2AA_n}-r^{n+2}+\frac{rG_n(r)}{G_n(1)}\right]=:r^n\mathcal{H}(r^2),$$

where the function $\mathcal{H}$ takes the form

$$\mathcal{H}(\mathtt{t})=\frac{4xG_n(1)}{An(1-x\mathtt{t})}\left[\frac{P_n(\mathtt{t})}{P_n(1)}-\frac{F_n(x\mathtt{t})}{F_n(x)}+\frac{2xF_n(x\mathtt{t})}{P_n(1)}\int_{\mathtt{t}}^1\frac{1}{\tau^{n+1}F_n^2(x\tau)}\int_0^\tau\frac{s^nF_n(xs)}{1-xs}P_n(s)dsd\tau\right]$$
$$-\mathtt{t}+\frac{P_n(\mathtt{t})}{P_n(1)}.$$

With the change of variables $s\rightsquigarrow s^2$ in the integral, condition (2.6.57) is equivalent to

$$\int_0^1\frac{s^nF_n(xs)}{1-xs}\mathcal{H}(s)ds\neq 0.\tag{2.6.58}$$

Regime $A > 0$ and $A + B < 0$

Here, we study the transversality assumption in the regime (2.6.21). Notice that the existence of infinite countable set of eigenvalues has been already established in Proposition 2.6.6. However, due to the complex structure of the integrand in (2.6.58) it appears quite difficult to check the non-vanishing of the integral for a given frequency. Thus, we have to overcome this difficulty using an asymptotic behavior of the integral and checking by this way the transversality only for high frequencies. More precisely, we prove the following result.

Proposition 2.6.17. *Let A and B satisfying* (2.6.21) *and $\{x_n\}$ the sequence constructed in Proposition* 2.6.6 $-$ (1). *Then there exists $n_0 \in \mathbb{N}$ such that*

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - x_n s}\mathcal{H}(s)ds = -\frac{n}{2}\big(1 + o(1)\big), \quad \forall n \geq n_0.$$

Proof. We proceed with studying the asymptotic behavior of the above integral for large frequencies n. We write $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$, with

$$\begin{aligned}
\mathcal{H}_1(\mathtt{t}) =& \frac{4x_n G_n(1)}{An(1 - x_n\mathtt{t})}\left[\frac{P_n(\mathtt{t})}{P_n(1)} - \frac{F_n(x_n\mathtt{t})}{F_n(x_n)}\right],\\
\mathcal{H}_2(\mathtt{t}) =& \frac{4x_n G_n(1)}{An(1 - x_n\mathtt{t})}\frac{2x_n F_n(x_n\mathtt{t})}{P_n(1)}\int_{\mathtt{t}}^1 \frac{1}{\tau^{n+1}F_n^2(x_n\tau)}\int_0^\tau \frac{s^n F_n(x_n s)}{1 - x_n s}P_n(s)dsd\tau,\\
\mathcal{H}_3(\mathtt{t}) =& -\mathtt{t} + \frac{P_n(\mathtt{t})}{P_n(1)}.
\end{aligned}$$

Let us start with the function $\mathcal{H}_1$. Proposition 2.6.13 leads to

$$\frac{1}{x_n} = 1 + \frac{\kappa}{n} - \frac{c_\kappa - \kappa^2}{n^2} + O\left(\frac{1}{n^3}\right),$$

which, together with the expression of $G_n(1)$ given in (2.5.12), imply

$$G_n(1) \sim -\frac{A}{4n}\left(c_\kappa - \kappa^2 + 2\right). \tag{2.6.59}$$

Recall that the inequality $F_n(1) \leq F_n(x_n\mathtt{t}) \leq 1$ holds for any $\mathtt{t} \in [0, 1]$, and thus Lemma 2.6.2 gives that

$$\frac{1}{2} \leq F_n(1), \quad \forall n \geq n_0.$$

Hence, $\mathcal{H}_1$ can be bounded as follows

$$|\mathcal{H}_1(\mathtt{t})| \leq \frac{C}{n^2(1 - x_n\mathtt{t})}\left[\frac{|P_n(\mathtt{t})|}{|P_n(1)|} + 1\right], \quad \forall \mathtt{t} \in [0, 1],$$

with C a constant depending in A and B. From the definition of P_n in (2.5.11) we may obtain

$$|P_n(\mathtt{t})| \leq C\left[1 - x_n\mathtt{t} + \frac{1}{n}\right], \quad \forall \mathtt{t} \in [0, 1]. \tag{2.6.60}$$

Moreover, plugging (2.6.59) into (2.5.12) we deduce that

$$P_n(1) \sim -\frac{4}{An}G_n(1) \sim \frac{c_\kappa - \kappa^2 + 2}{n^2}. \tag{2.6.61}$$

Putting everything together one gets

$$|\mathcal{H}_1(\mathtt{t})| \le C\left[\frac{1}{n(1-x_n\mathtt{t})}+1\right], \quad \forall\, \mathtt{t}\in[0,1],$$

from which we infer that

$$\left|\int_0^1 \frac{s^nF_n(x_ns)}{1-x_ns}\mathcal{H}_1(s)ds\right| \le \frac{C}{n}\int_0^1\frac{s^n}{(1-x_ns)^2}ds + C\int_0^1\frac{s^n}{1-x_ns}ds =: C\left(\frac{I_1}{n}+I_2\right).$$

To estimate the second integral we use the change of variables $s = 1-\varepsilon_n\tau$, with $\varepsilon_n = \frac{1-x_n}{x_n}$, leading to the asymptotic behavior

$$I_2 = \frac{1}{x_n}\int_0^{\frac{1}{\varepsilon_n}}\frac{(1-\varepsilon_n\tau)^n}{1+\tau}d\tau \sim \int_0^{\frac{n}{\kappa}}\frac{(1-\frac{\kappa\tau}{n})^n}{1+\tau}\sim\int_0^{+\infty}\frac{e^{-\kappa\tau}}{1+\tau}d\tau, \tag{2.6.62}$$

where we have used the expansion of x_n given by Proposition 2.6.13. As to the first integral we just have to integrate by parts and use the previous computations,

$$I_1 = \frac{1}{x_n}\frac{1}{1-x_n} - \frac{n}{x_n}\int_0^1\frac{s^{n-1}}{1-x_ns}ds = \left(\frac{1}{\kappa}-\int_0^\infty\frac{e^{-\kappa\tau}}{1+\tau}d\tau\right)n + o(n),$$

and consequently,

$$\sup_{n\ge n_0}\Big|\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}\mathcal{H}_1(s)ds\Big| < +\infty. \tag{2.6.63}$$

The estimate $\mathcal{H}_2$ it is straightforward. Indeed, from (2.5.12) we may write

$$|\mathcal{H}_2(\mathtt{t})| \le \frac{C}{1-x_n\mathtt{t}}\int_{\mathtt{t}}^1\frac{1}{\tau^{n+1}}\int_0^\tau\frac{s^n}{1-x_ns}|P_n(s)|dsd\tau. \tag{2.6.64}$$

Using once again (2.6.60) and Proposition 2.6.13 we get

$$\begin{aligned}\int_0^\tau\frac{s^n}{1-x_ns}|P_n(s)|ds \le& \frac{C}{n}\int_0^\tau\frac{s^n}{1-x_ns}ds + C\int_0^\tau s^nds\\ \le& \frac{C}{n(1-x_n)}\frac{\tau^{n+1}}{n+1}+C\frac{\tau^{n+1}}{n+1}\le\frac{C}{n}\tau^{n+1}, \quad \forall\tau\in[0,1].\end{aligned}$$

Hence, we deduce that

$$\int_{\mathtt{t}}^1\frac{1}{\tau^{n+1}}\int_0^\tau\frac{s^n}{1-x_ns}|P_n(s)|ds \le C\frac{1-\mathtt{t}}{n}.$$

Since the function $\mathtt{t}\in[0,1]\mapsto\frac{1-\mathtt{t}}{1-x_n\mathtt{t}}$ is strictly decreasing , then we have

$$0\le\frac{1-\mathtt{t}}{1-x_n\mathtt{t}}\le 1, \quad \forall \mathtt{t}\in[0,1].$$

Consequently, inserting the preceding two estimates into (2.6.64), we obtain

$$|\mathcal{H}_2(t)|\le\frac{C}{n}, \quad \forall\mathtt{t}\in[0,1].$$

Thus we infer

$$\Big|\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}\mathcal{H}_2(s)ds\Big| \le \qquad \frac{C}{n}\int_0^1\frac{s^n}{1-x_ns}ds,$$

which implies

$$\sup_{n\geq n_0}\Big|\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}\mathcal{H}_2(s)ds\Big|\leq\frac{C}{n}, \tag{2.6.65}$$

where we have used (2.6.62). It remains to estimate the integral term associated with $\mathcal{H}_3$ which takes the form

$$\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}\mathcal{H}_3(s)ds=-\int_0^1\frac{s^{n+1}F_n(x_ns)}{1-x_ns}ds+\frac{1}{P_n(1)}\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}P_n(s)ds.$$

Similarly to (2.6.62), one has

$$\sup_{n\geq n_0}\int_0^1\frac{s^{n+1}F_n(x_ns)}{1-x_ns}ds<+\infty.$$

To finish we just have to deal with the second integral term. Observe from (2.5.11) that P_n is a monic polynomial of degree two, and thus from Taylor formula one gets

$$P_n(\mathtt{t})=P_n\left(\frac{1}{x_n}\right)+\left(\mathtt{t}-\frac{1}{x_n}\right)P_n'\left(\frac{1}{x_n}\right)+\left(\mathtt{t}-\frac{1}{x_n}\right)^2.$$

It is easy to check the following behaviors

$$P_n\left(\frac{1}{x_n}\right)\sim\frac{\kappa}{n},\quad P_n'\left(\frac{1}{x_n}\right)\sim 1.$$

Hence, we obtain

$$\begin{aligned}\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}P_n(s)ds=&P\left(\frac{1}{x_n}\right)\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}ds-\frac{1}{x_n}P_n'\left(\frac{1}{x_n}\right)\int_0^1 s^nF_n(x_ns)ds\\&+\frac{1}{x_n^2}\int_0^1 s^nF_n(x_ns)(1-x_ns)ds.\end{aligned}$$

Concerning the last term we use the asymtotics of x_n, leading to

$$\frac{1}{x_n^2}\int_0^1 s^nF_n(x_ns)(1-x_ns)ds\leq C\int_0^1 s^n(1-x_ns)ds\leq C\left(\frac{1}{n+1}-\frac{x_n}{n+2}\right)\leq\frac{C}{n^2}.$$

For the first and second terms we use the estimate $|F_n(x_nt)-1|<C\frac{\ln n}{n}$ coming from Lemma 2.6.4. Hence, we find

$$-\frac{1}{x_n}P_n'\left(\frac{1}{x_n}\right)\int_0^1 s^nF_n(x_ns)ds=-\frac{1}{n}\big(1+o(1)\big).$$

Again from (2.6.62) we find

$$\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}ds=\int_0^1\frac{s^n}{1-x_ns}ds+o(1)=\int_0^{+\infty}\frac{e^{-\kappa\tau}}{1+\tau}d\tau+o(1).$$

Putting together the preceding estimates, we obtain

$$\int_0^1\frac{s^nF_n(x_ns)}{1-x_ns}P_n(s)ds=\frac{\kappa\displaystyle\int_0^{+\infty}\frac{e^{-\kappa\tau}}{1+\tau}d\tau-1}{n}+o\left(\frac{1}{n}\right),$$

and combining this estimate with (2.6.61), we infer

$$\int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s}\mathcal{H}_3(s)ds = \frac{\kappa\displaystyle\int_0^{+\infty}\frac{e^{-\kappa\tau}}{1+\tau}d\tau - 1}{c_\kappa - \kappa^2+2}n + o(n).$$

Using the explicit expression of above constants defined in Proposition 2.6.13, we get that

$$\frac{\kappa\displaystyle\int_0^{+\infty}\frac{e^{-\kappa\tau}}{1+\tau}d\tau - 1}{c_\kappa - \kappa^2+2} = -\frac{1}{2},$$

and therefore

$$\int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s}\mathcal{H}_3(s)ds = -\frac{n}{2} + o(n).$$

Combining this estimate with (2.6.65) and (2.6.63), we deduce that

$$\int_0^1 \frac{s^n F_n(x_n s)}{1-sx_n}\mathcal{H}(s)ds = -\frac{n}{2} + o(n),$$

which achieves the proof of the announced result. □

Regime $B > A > 0$

In this special regime there is only a finite number of eigenvalues that can be indexed by a decreasing sequence, see Proposition 2.6.11-(1). In what follows we shall prove that the transversality assumption is always satisfied without any additional constraint on the parameters. More precisely, we prove the following result.

Proposition 2.6.18. *Let $B > A > 0$. Then, the transversal property* (2.6.57) *holds, for every subsequence $\Big\{x_n; n \in [2, \mathscr{N}_{A,B}]\Big\}$ defined in Proposition* 2.6.7, *where*

$$\mathscr{N}_{A,B} := \max\left(\frac{B}{A}+\frac{1}{8}, \frac{2B}{A}-\frac{9}{2}\right).$$

Proof. Let us start with the case $n \in \big[2, \frac{B}{A}+\frac{1}{8}\big]$. Using the expression of P_n introduced in (2.5.11) one has

$$x_n P_n(1) = x_n\left[1 - \frac{A+2B}{A}\frac{n+2}{n(n+1)}\right] - \frac{n+2}{n+1}.$$

Moreover, from the definition of $\mathscr{P}$ seen in Proposition 2.6.8, we get

$$\begin{aligned}\mathscr{P}_n(x_n) =& \frac{n}{n+1} + x_n\left[-\frac{n}{n+2} + \frac{A+2B}{A(n+1)}\right]\\ =& \frac{n}{n+2}\left(\frac{n+2}{n+1} + x_n\left[-1+\frac{(A+2B)(n+2)}{An(n+1)}\right]\right) = -\frac{n}{n+2}x_n P_n(1) < 0.\end{aligned}$$

This implies that $P_n(1) < 0$. Since $\mathtt{t} \in [0,1] \mapsto P_n(\mathtt{t})$ is strictly increasing with $x_n < 0$, we deduce that

$$P_n(\mathtt{t}) < 0, \quad \forall \mathtt{t} \in [0,1]. \tag{2.6.66}$$

Therefore, we get from (2.5.12) that $G_n(1) > 0$. Let us study every term involved in (2.6.58) by using the decomposition $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$ of Proposition 2.6.17. From the preceding properties, it is clear that $\mathcal{H}_2(\mathtt{t}) > 0$, for $\mathtt{t} \in (0,1)$. In addition, we also have that $\mathtt{t} \in [0,1] \mapsto -\mathtt{t} + \frac{P_n(\mathtt{t})}{P_n(1)}$ is strictly decreasing and thus

$$\mathcal{H}_3(\mathtt{t}) = -\mathtt{t} + \frac{P_n(\mathtt{t})}{P_n(1)} \geq \mathcal{H}_3(1) = 0, \quad \forall \mathtt{t} \in [0,1].$$

Concerning $\mathcal{H}_1$ we first note that the mapping $\mathtt{t} \in [0,1] \mapsto \frac{P_n(\mathtt{t})}{P_n(1)} - \frac{F_n(x_n\mathtt{t})}{F_n(x_n)}$ is strictly decreasing which follows from (2.6.66) and the fact that F_n is decreasing and P_n is increasing. Thus,

$$\frac{P_n(\mathtt{t})}{P_n(1)} - \frac{F_n(x_n\mathtt{t})}{F_n(x_n)} \geq \frac{P_n(1)}{P_n(1)} - \frac{F_n(x_n)}{F_n(x_n)} = 0, \quad \forall \mathtt{t} \in [0,1].$$

Combining (2.6.66) with (2.5.12), we deduce

$$\mathcal{H}_1(\mathtt{t}) = \frac{4x_nG_n(1)}{An(1-x_n\mathtt{t})}\left[\frac{P_n(\mathtt{t})}{P_n(1)} - \frac{F_n(x_n\mathtt{t})}{F_n(x_n)}\right] < 0, \quad \mathtt{t} \in (0,1).$$

We continue our analysis assuming that

$$\left|\frac{4x_nG_n(1)}{An(1-x_n\mathtt{t})}\right| \leq 1, \quad \forall\, \mathtt{t} \in [0,1), \tag{2.6.67}$$

holds, we see how to conclude with. Since $\mathcal{H}_2$ is always positive then $\mathcal{H}$ will be strictly positive if one can show that $\mathcal{H}_1(\mathtt{t}) + \mathcal{H}_3(\mathtt{t}) > 0$, for any $\mathtt{t} \in (0,1)$. With (2.6.67) in mind, one gets

$$\mathcal{H}_1(\mathtt{t}) + \mathcal{H}_3(\mathtt{t}) \geq -\left[\frac{P_n(\mathtt{t})}{P_n(1)} - \frac{F_n(x_n\mathtt{t})}{F_n(x_n)}\right] - \mathtt{t} + \frac{P_n(\mathtt{t})}{P_n(1)} = \frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - \mathtt{t}, \quad \forall \mathtt{t} \in [0,1].$$

Computing the derivatives of the function in the right-hand side term, we find

$$\partial_{\mathtt{t}}\left(\frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - \mathtt{t}\right) = \frac{x_nF_n'(x_n\mathtt{t})}{F_n(x_n)} - 1, \quad \partial^2_{\mathtt{tt}}\left(\frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - \mathtt{t}\right) = \frac{x_n^2F_n''(x_n\mathtt{t})}{F_n(x_n)} < 0.$$

The latter fact implies that the first derivative is decreasing, and thus

$$\partial_{\mathtt{t}}\left(\frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - \mathtt{t}\right) \leq \frac{x_nF_n'(0)}{F_n(x_n)} - 1 \leq \frac{-2x_n}{(n+1)F_n(x_n)} - 1 \leq \frac{2}{n+1} - 1 < 0, \quad \forall \mathtt{t} \in [0,1],$$

where we have used Lemma 2.6.3$-$(1). Therefore, we conclude that the mapping $\mathtt{t} \in [0,1] \mapsto \frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - t$ decreases and, since it vanishes at $\mathtt{t} = 1$, we get

$$\mathcal{H}_1(\mathtt{t}) + \mathcal{H}_3(\mathtt{t}) \geq \frac{F_n(x_n\mathtt{t})}{F_n(x_n)} - \mathtt{t} > 0, \quad \forall \mathtt{t} \in [0,1).$$

This implies that $\mathcal{H}(\mathtt{t}) > 0$ for any $\mathtt{t} \in [0,1)$ and hence the transversality assumption (2.6.58) is satisfied. Let us now turn to the proof of (2.6.67) and observe that from (2.5.12)

$$\begin{aligned}\left|\frac{4x_nG_n(1)}{An(1-x_nt)}\right| =& \frac{1}{1-x_nt}\frac{n+1}{n+2}x_nP_n(1)\\ \leq& \frac{n+1}{n+2}\left\{x_n\left[1 - \frac{A+2B}{A}\frac{n+2}{n(n+1)}\right] - \frac{n+2}{n+1}\right\}.\end{aligned} \tag{2.6.68}$$

Using $x_\star$, defined in Proposition 2.6.8, and the fact that $\frac{A+2B}{A}\frac{n+2}{n(n+1)} - 1 > 0$, we obtain

$$\begin{aligned}\left|\frac{4x_nG_n(1)}{An(1-x_nt)}\right| &\leq \frac{n+1}{n+2}\left\{x_\star\left[1-\frac{A+2B}{A}\frac{n+2}{n(n+1)}\right]-\frac{n+2}{n+1}\right\}\\ &\leq \frac{n+1}{n+2}\left\{\frac{2n+1}{2(n+1)}\frac{\frac{A+2B}{A}\frac{n+2}{n(n+1)}-1}{\frac{A+2B}{A(n+1)}-\frac{n+1}{n+2}}-\frac{n+2}{n+1}\right\}.\end{aligned}$$

Consequently (2.6.67) is satisfied provided that

$$\frac{\frac{A+2B}{A}\frac{n+2}{n(n+1)}-1}{\frac{A+2B}{A(n+1)}-\frac{n+1}{n+2}} \leq 4\frac{n+2}{2n+1}.$$

Since $\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2} > 0$, then the preceding inequality is true if and only if

$$\frac{A+2B}{A(n+1)} \geq \frac{2n^2+3n}{2n^2+3n-2}.$$

It is easy to check that the sequence $n \geq 2 \mapsto \frac{2n^2+3n}{2n^2+3n-2}$ is decreasing, and then the foregoing inequality is satisfied for any $n \geq 2$ if

$$\frac{A+2B}{A(n+1)} \geq \frac{7}{6}.$$

Fom the assumption $n+1 \leq \frac{B}{A} + \frac{9}{8}$, the above inequality holds if

$$\frac{B}{A} \geq \frac{5}{16},$$

which follows from the condition $B > A$.

Let us now move on the case $n \in \left[2, \frac{2B}{A} - \frac{9}{2}\right]$. Note that we cannot use $x_\star$ coming from Proposition 2.6.8 since we do not know if $x_n \in (-1, 0)$. In fact Proposition 2.6.7 gives us that $x_n < -1$, for $n \geq \frac{B}{A}+1$. Hence, we should slightly modify the arguments used to (2.6.67). From (2.6.68), we deduce that (2.6.67) is satisfied if

$$x_n \geq \frac{2\frac{n+2}{n+1}}{1-\frac{A+2B}{A}\frac{n+2}{n(n+1)}}.$$

By (2.6.31), it is enough to prove that

$$\frac{1}{1-\frac{A+2B}{A(n+1)}} \geq \frac{2\frac{n+2}{n+1}}{1-\frac{A+2B}{A}\frac{n+2}{n(n+1)}}.$$

Straightforward computations give that the last inequality is equivalent to

$$\frac{A+2B}{A(n+1)} \geq \frac{(n+3)n}{(n+2)(n-1)}.$$

Therefore, if n satisfies $2 \leq n \leq \frac{2B}{A} - \frac{9}{2}$, then we have

$$\frac{n+1+\frac{9}{2}}{n+1} \leq \frac{A+2B}{A(n+1)}.$$

Hence, one can check that

$$\frac{n+1+\frac{9}{2}}{n+1} \geq \frac{(n+3)n}{(n+2)(n-1)},$$

for any $n \geq 2$, and thus (2.6.67) is verified. □

One-fold case

The main objective is to make a complete study in the case $n = 1$. It is very particular because F_1 is explicit according to Remark 2.5.5 and, therefore, we can get a compact formula for the integral of (2.6.58). Our main result reads as follows.

Proposition 2.6.19. *Let $n = 1$ and $x = -\frac{A}{2B}$, then we have the formula*

$$\int_0^1 \frac{sF_1(xs)}{1-xs}\mathcal{H}(s)ds = \frac{x-1}{2x}.$$

In particular the transversal assumption (2.6.57) *is satisfied if and only if $x \notin \{0, 1\}$.*

Proof. Note that from (2.5.11)-(2.5.12) one has

$$P_1(\mathtt{t}) = \mathtt{t}^2 - \frac{3}{2x}\mathtt{t} - \frac{3}{2}\left[1 - \frac{1}{x}\right], \quad P_n(1) = -\frac{1}{2}, \quad G_1(1) = \frac{A}{12}.$$

Moreover, we get $F_1(\mathtt{t}) = 1 - \mathtt{t}$ using Remark 2.5.5, and thus

$$\begin{aligned}\mathcal{H}(\mathtt{t}) =& \frac{1}{3}\frac{x}{(1-x\mathtt{t})}\left[\frac{P_1(\mathtt{t})}{P_1(1)} - \frac{F_1(x\mathtt{t})}{F_1(x)} + \frac{2xF_1(x\mathtt{t})}{P_1(1)}\int_\mathtt{t}^1 \frac{1}{\tau^2F_1^2(x\tau)}\int_0^\tau \frac{sF_1(xs)}{1-xs}P_1(s)dsd\tau\right] \\ &- \mathtt{t} + \frac{P_1(\mathtt{t})}{P_1(1)} \\ =:& \frac{1}{3}\frac{x}{(1-x\mathtt{t})}\widehat{\mathcal{H}}(\mathtt{t}) - \mathtt{t} + \frac{P_1(\mathtt{t})}{P_1(1)}.\end{aligned} \tag{2.6.69}$$

From straightforward computations we deduce

$$\int_\mathtt{t}^1 \frac{1}{\tau^2F_1^2(x\tau)}\int_0^\tau \frac{sF_1(xs)}{1-xs}P_1(s)dsd\tau = \int_\mathtt{t}^1 \frac{\frac{\tau^2}{4} - \frac{1}{2x}\tau - \frac{3}{4}(1-\frac{1}{x})}{(1-\tau x)^2}d\tau$$

Denoting by $\varphi(\tau) = \frac{\tau^2}{4} - \frac{1}{2x}\tau - \frac{3}{4}(1-\frac{1}{x})$ and integrating by parts, we get

$$\int_\mathtt{t}^1 \frac{\varphi(\tau)}{(1-\tau x)^2}d\tau = \frac{1}{x}\left[\frac{\varphi(1)}{1-x} - \frac{\varphi(\mathtt{t})}{1-\mathtt{t}x}\right] + \frac{1-\mathtt{t}}{2x^2} = \frac{1}{x}\left[\frac{1-2x}{4x(1-x)} + \frac{1-\mathtt{t}}{2x} - \frac{\varphi(\mathtt{t})}{1-\mathtt{t}x}\right].$$

Therefore, after standard computations, we get the simplified formula

$$\widehat{\mathcal{H}}(\mathtt{t}) = \frac{3(1-\mathtt{t}x)(\mathtt{t}-1)}{x}.$$

Inserting this into (2.6.69), we find

$$\mathcal{H}(\mathtt{t}) = -1 + \frac{P_1(\mathtt{t})}{P_1(1)} = -2\mathtt{t}^2 + \frac{3}{x}\mathtt{t} + 2 - \frac{3}{x}.$$

Plugging it into the integral of (2.6.58), it yields

$$\int_0^1 s\mathcal{H}(s)ds = \frac{x-1}{2x}.$$

□

2.7 Existence of non–radial time–dependent rotating solutions

At this stage we are able to give the full statement of our main result, by using the analysis of the previous sections. In order to apply the Crandall–Rabinowitz Theorem, let us introduce the m–fold symmetries in our spaces.

$$\mathscr{C}^{k,\alpha}_{s,m}(\mathbb{D}) := \left\{ g \in \mathscr{C}^{k,\alpha}_{s}(\mathbb{D}) : \quad g(e^{i\frac{2\pi}{m}} z) = g(z), \quad \forall z \in \mathbb{D} \right\},$$
$$\mathscr{C}^{k,\alpha}_{a,m}(\mathbb{T}) := \left\{ \rho \in \mathscr{C}^{k,\alpha}_{a}(\mathbb{T}) : \quad \rho(e^{i\frac{2\pi}{m}} w) = \rho(w), \quad \forall w \in \mathbb{T} \right\},$$
$$\mathscr{H}\mathscr{C}^{k,\alpha}_{m}(\mathbb{D}) := \left\{ \phi \in \mathscr{H}\mathscr{C}^{k,\alpha}(\mathbb{D}) : \quad \phi(e^{i\frac{2\pi}{m}} z) = e^{i\frac{2\pi}{m}}\phi(z), \quad \forall z \in \mathbb{D} \right\}.$$

Note that the functions $g \in \mathscr{C}^{k,\alpha}_{s,m}(\mathbb{D})$, $\rho \in \mathscr{C}^{k,\alpha}_{a,m}(\mathbb{T})$ and $\phi \in \mathscr{H}\mathscr{C}^{k,\alpha}_{m}(\mathbb{D})$ admit the following representation:

$$g(re^{i\theta}) = \sum_{n\geq 0} g_{nm}(r)\cos(nm\theta), \quad \rho(e^{i\theta}) = \sum_{n\geq 0} \rho_n \sin(nm\theta), \quad \text{and} \quad \phi(z) = z\sum_{n\geq 1} z^{nm},$$

where $z \in \mathbb{D}$, $r \in [0,1]$ and $\theta \in [0,2\pi]$. With these spaces, the functional F defined in (2.3.1), concerning the boundary equation, is also well–defined:

Proposition 2.7.1. *Let $\varepsilon \in (0,1)$, then*

$$F : \mathbb{R} \times B_{\mathscr{C}^{1,\alpha}_{s,m}}(0,\varepsilon) \times B_{\mathscr{H}\mathscr{C}^{2,\alpha}_{m}}(0,\varepsilon) \mapsto \mathscr{C}^{1,\alpha}_{a,m}(\mathbb{T})$$

is well–defined and of class $\mathscr{C}^1$, where the balls were defined in (2.3.2).

Proof. Thanks to Proposition 2.3.1, it remains to prove that $F(\Omega, g, \phi)$ satisfies $F(\Omega,g,\phi)(e^{i\frac{2\pi}{m}}w) = F(w)$, with $w \in \mathbb{T}$:

$$\begin{aligned}
F(\Omega,g,\phi)(e^{i\frac{2\pi}{m}}w) &= \text{Im}\left[\left(\Omega\,\overline{\Phi(e^{i\frac{2\pi}{m}}w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)|\Phi'(y)|^2}{\Phi(e^{i\frac{2\pi}{m}}w) - \Phi(y)}dA(y)\right)\Phi'(e^{i\frac{2\pi}{m}}w)e^{i\frac{2\pi}{m}}w\right]\\
&= \text{Im}\left[\left(\Omega\, e^{-i\frac{2\pi}{m}}\overline{\Phi(w)} - \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(e^{i\frac{2\pi}{m}}y)|\Phi'(e^{i\frac{2\pi}{m}}y)|^2}{\Phi(e^{i\frac{2\pi}{m}}w) - \Phi(e^{i\frac{2\pi}{m}}y)}dA(y)\right)\Phi'(w)e^{i\frac{2\pi}{m}}w\right]\\
&= \text{Im}\left[\left(\Omega\, e^{-i\frac{2\pi}{m}}\overline{\Phi(w)} - \frac{e^{-i\frac{2\pi}{m}}}{2\pi}\int_{\mathbb{D}} \frac{f(y)|\Phi'(y)|^2}{\Phi(w) - \Phi(y)}dA(y)\right)\Phi'(w)e^{i\frac{2\pi}{m}}w\right]\\
&= F(\Omega,g,\phi)(w),
\end{aligned}$$

where we have used that $\phi(e^{i\frac{2\pi}{m}}z) = e^{i\frac{2\pi}{m}}\phi(z)$, $\phi'(e^{i\frac{2\pi}{m}}z) = \phi'(z)$ and $g(e^{i\frac{2\pi}{m}}z) = g(z)$. □

We must define the singular set (2.4.10) once we have introduced the symmetry in the spaces. Fixing f_0 as a quadratic profile (2.1.6), the singular set (2.4.10) becomes

$$\mathcal{S}^m_{\text{sing}} := \left\{ \widehat{\Omega}_{mn} := \frac{A}{4} + \frac{B}{2} - \frac{A(nm+1)}{2nm(nm+2)} - \frac{B}{2nm}, \quad n \in \mathbb{N}^\star \cup \{+\infty\} \right\}.$$

For the density equation defined in (2.4.6) and the new spaces we obtain the following result.

Proposition 2.7.2. *Let I be an open interval with $\overline{I} \subset \mathbb{R}\backslash\mathcal{S}^m_{\text{sing}}$. Then, there exists $\varepsilon > 0$ such that*

$$\widehat{G} : I \times B_{\mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D})}(0,\varepsilon) \to \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D})$$

is well–defined and of class $\mathscr{C}^1$, where $\widehat{G}$ is defined in (2.4.6) *and $B_{\mathscr{C}^{1,\alpha}_s(\mathbb{D})}(0,\varepsilon)$ in* (2.3.2).

Proof. Similarly to the previous result, from Proposition 2.4.1 we just have to check that $\widehat{G}(\Omega,g)(e^{i\frac{2\pi}{m}}z) = \widehat{G}(\Omega,g)(z)$, for $z \in \mathbb{D}$. From Proposition 2.3.3, there exists $\varepsilon > 0$ such that the conformal map ϕ is given by (Ω, g), and lies in $B_{\mathscr{H}\mathscr{C}^{2,\alpha}_m}(0,\varepsilon)$. Then, using $\widehat{G}(\Omega,g) = G(\Omega,g,\phi(\Omega,g))$, we have

$$\begin{aligned}
G(\Omega,g,\phi)(e^{i\frac{2\pi}{m}}z) =& \frac{4\Omega - B}{8A} f(e^{i\frac{2\pi}{m}}z) - \frac{f(e^{i\frac{2\pi}{m}}z)^2}{16A} \\
&+ \frac{1}{2\pi}\int_{\mathbb{D}} \log|\Phi(e^{i\frac{2\pi}{m}}z) - \Phi(y)| f(y)|\Phi'(y)|^2 dA(y) - \frac{\Omega|\Phi(e^{i\frac{2\pi}{m}}z)|^2}{2} - \lambda \\
=& \frac{4\Omega - B}{8A} f(z) - \frac{f(z)^2}{16A} \\
&+ \frac{1}{2\pi}\int_{\mathbb{D}} \log|\Phi(e^{i\frac{2\pi}{m}}z) - \Phi(e^{i\frac{2\pi}{m}}y)| f(e^{i\frac{2\pi}{m}}y)|\Phi'(e^{i\frac{2\pi}{m}}y)|^2 dA(y) \\
&- \frac{\Omega|e^{i\frac{2\pi}{m}}\Phi(z)|^2}{2} - \lambda \\
=& \; G(\Omega,g,\phi)(z),
\end{aligned}$$

where we have used the properties of functions g and ϕ. □

We have now all the tools we need to prove the first three points of Theorem 2.1.1, which can be detailed as follows:

Theorem 2.7.3. *Let $A > 0$, $B \in \mathbb{R}$ and f_0 a quadratic profile* (2.1.6)*. Then the following results hold true.*

1. *If $A + B < 0$, then there is $m_0 \in \mathbb{N}$ (depending only on A and B) such that for any $m \geq m_0$ there exists*
   - $\Omega_m = \frac{A+2B}{4} + \frac{A\kappa}{4m} + \frac{A}{4}\frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right)$,
   - *V a neighborhood of $(\Omega_m, 0, 0)$ in $\mathbb{R} \times \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D}) \times \mathscr{H}\mathscr{C}^{2,\alpha}_m(\mathbb{D})$,*
   - *a continuous curve $\xi \in (-a,a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$,*

   such that (2.7.2),

   $$\omega_0 = (f \circ \Phi^{-1})\mathbf{1}_{\Phi(\mathbb{D})}, \quad f = f_0 + f_\xi, \quad \Phi = \text{Id} + \phi_\xi,$$

   defines a curve of non radial solutions of Euler equations that rotates at constant angular velocity Ω_ξ. The constants κ and c_κ are defined in (2.6.13).

2. *If $B > A > 0$, then for any integer $m \in \left[1, \frac{2B}{A} - \frac{9}{2}\right]$ or $m \in \left[1, \frac{B}{A} + \frac{1}{8}\right]$ there exists*
   - $0 \leq \Omega_m < \frac{B}{2}$,
   - *V a neighborhood of $(\Omega_m, 0, 0)$ in $\mathbb{R} \times \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D}) \times \mathscr{H}\mathscr{C}^{2,\alpha}_m(\mathbb{D})$,*
   - *a continuous curve $\xi \in (-a,a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$,*

such that (2.7.2) *defines a curve of non radial solutions of Euler equations. However, there is no bifurcation with any symmetry* $m \geq \frac{2B}{A} + 2$.

3. *If* $B > 0$ *or* $B \leq -\frac{A}{1+\epsilon}$, *for some* $0,0581 < \epsilon < 1$, *then there exists*
   - V *a neighborhood of* $(0,0,0)$ *in* $\mathbb{R} \times \mathscr{C}_s^{1,\alpha}(\mathbb{D}) \times \mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$,
   - *a continuous curve* $\xi \in (-a,a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$,

   such that (2.7.2) *defines a curve of one-fold non radial solutions of Euler equations.*

4. *If* $-\frac{A}{2} \leq B \leq 0$, *then there is no bifurcation with any symmetry* $m \geq 1$. *However, in the case that* $0 < B < \frac{A}{4}$, *there is no bifurcation with any symmetry* $m \geq 2$.

Proof. **(1)** Let us prove the first assertion in the case $A+B < 0$. We will implement the Crandall–Rabinowitz Theorem (A.0.3) to $\widehat{G}$, defined in (2.4.6). First, we must concrete the domain of Ω. From Proposition 2.6.6, there exist $n_{0,1}$ and a unique solution $x_m \in (0,1)$ of $\zeta_m(x) = 0$ for any $m \geq n_{0,1}$. Then, the sequence defined by $\Omega_m = \frac{A}{4x_m} + \frac{B}{2} > \frac{A}{4} + \frac{B}{2}$ decreases to $\frac{A}{4} + \frac{B}{2}$ since (x_m) increases to 1, see Proposition 2.6.6. This limit point $\frac{A}{4} + \frac{B}{2}$ is different from $\Omega_0 = \frac{B}{2} + \frac{A}{4x_0}$ because x_0, the unique root to (2.4.13), belongs to $(0,1)$. As a consequence, by taking $n_{0,1}$ large enough we can guarantee that $\Omega_m \neq \Omega_0$ for any $m \geq n_{0,1}$. Moreover, Proposition 2.6.14 gives us that $\Omega_m \neq \hat{\Omega}_{mn}$, for $n \in \mathbb{N}$, with $\widehat{\Omega}_{mn} \in \mathcal{S}_{\text{sing}}^m$. Therefore, let I be an interval with $\Omega_m \in I$ and

$$\overline{I} \cap \mathcal{S}_{\text{sing}}^m = \varnothing, \quad \Omega_0 \notin \overline{I}.$$

By virtue of Proposition 2.3.3, we know that there exists $\varepsilon > 0$ and a $\mathscr{C}^1$ function $\mathcal{N} : I \times B_{\mathscr{C}_{s,m}^{1,\alpha}}(0,\varepsilon) \longrightarrow B_{\mathscr{H}\mathscr{C}_m^{2,\alpha}}(0,\varepsilon)$, such that

$$F(\Omega, g, \phi) = 0 \Longleftrightarrow \phi = \mathcal{N}(\Omega, g)$$

holds, for any $(\Omega, g, \phi) \in I \times B_{\mathscr{C}_{s,m}^{1,\alpha}}(0,\varepsilon) \times B_{\mathscr{H}\mathscr{C}_m^{2,\alpha}}(0,\varepsilon)$. Hence, the conformal map is defined through the density for that ε. We define the density equation,

$$\widehat{G} : I \times B_{\mathscr{C}_{s,m}^{1,\alpha}}(0,\varepsilon) \to \mathscr{C}_{s,m}^{1,\alpha}(\mathbb{D}),$$

with the expression given in (2.4.6). Thanks to Proposition 2.7.2, the function $\widehat{G}$ is well-defined in these spaces and is $\mathscr{C}^1$ with respect to (Ω, g). It remains to check the spectral properties of the Crandall–Rabinowitz Theorem. Using Proposition 2.5.3, we know that the dimension of the kernel of the linearized operator $D_g\widehat{G}(\Omega, 0)$ is given by the number of elements of the set $\mathcal{A}_x$ defined in (2.5.21). Note that we have introduced the symmetry m in our spaces, and therefore we should take into consideration this fact. Hence in the kernel study we should restrict the analysis of the resonance to the roots of ζ_{nm}. Thus, instead of dealing with the set $\mathcal{A}_x$ defined in (2.5.21) we should consider the set

$$\mathcal{A}_x^m := \Big\{ nm \in \mathbb{N} \quad \text{s.t.} \quad \zeta_{nm}(x) = 0, \ n \geq 1 \Big\}.$$

Recall that x_m is the unique root of $\zeta_m(x)$ and the sequence $n \in [n_{0,1}, +\infty[\mapsto x_n$ is strictly increasing. Therefore we deduce that $\mathcal{A}_{x_m}^m = \{m\}$, and since $\Omega_m \notin \mathcal{S}_{\text{sing}}^m$ we obtain that the kernel is one dimensional. Moreover we know from Proposition 2.5.1 that $D_g\widehat{G}(\Omega_m, 0)$ is a Fredholm operator with zero index . As to the transversality condition, note that using Proposition 2.6.16

and Proposition 2.6.17 we may find $n_{0,2}$ such that the transversality condition is satisfied provided that $m > n_{0,2}$. Taking $n_0 = \max\{n_{0,1}, n_{0,2}\}$, then Crandall-Rabinowitz Theorem can be applied to $\widehat{G}$ obtaining a small neighborhood V of (Ω_m, f_0) in $\mathbb{R} \times \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D})$, and a continuous curve $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi) \in V$, of solutions to $\widehat{G}(\Omega, g) = 0$ with

$$\forall \xi \in (-a, a), \quad f_\xi = \xi \hbar_m + \xi \beta(\xi), \lim_{\xi \to 0} \beta(\xi) = 0,$$

where $\hbar_m$ is the generator of the kernel defined in Proposition 2.5.3. Notice that for $\xi \neq 0$ we have that $f_\xi \neq 0$ and it is not radial because $\hbar_m$ is not radial. Hence the density $f_0 + f_\xi$ can not also be radial too. Furthermore, by Proposition 2.6.13 we know the asymtotics of x_m obtaining

$$\frac{1}{x_m} = 1 + \frac{\kappa}{m} + \frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right),$$

where κ and c_κ are defined in (2.6.13). Using the relation between x_m and Ω_m in (2.4.12), we get

$$\frac{A + 2B}{4} < \Omega_m = \frac{A + 2B}{4} + \frac{A\kappa}{4m} + \frac{A}{4}\frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right). \tag{2.7.1}$$

Therefore, we obtain that

$$\omega_0 = (f \circ \Phi^{-1})\mathbf{1}_{\Phi(\mathbb{D})}, \quad f = f_0 + f_\xi, \quad \Phi = \text{Id } + \phi_\xi, \tag{2.7.2}$$

defines a solution to Euler equations that rotates at constant angular velocity Ω_ξ. Moreover, we claim that this solution is not radial for all $\xi \in (-a, a)\backslash\{0\}$. Indeed, as Φ is conformal close in our functional setting to Id then the only case where the shape $\Phi(\mathbb{D})$ is radial corresponds to $\Phi = \text{Id}$. In this case the density given by $f_0 + f_\xi$ is not radial from the above discussion and hence we get a non radial solution. On the other hand, if $\Phi \neq \text{Id}$ then the support of ω_0, given by $\Phi(\mathbb{D})$ because the density f is not vanishing close to the boundary of $\Phi(\mathbb{D})$, is not a radial domain. In this cas we still get a non radial solution. So in all the cases the solutions that we have constructed are not radial.

(2) Now, we are concerned with the existence of m–fold non radial solutions of the type (2.7.2) in the case $B > A > 0$, for any integer $m \in \left[1, \frac{2B}{A} - \frac{9}{2}\right]$ or $m \in \left[1, \frac{B}{A} + \frac{1}{8}\right]$. In this part of the theorem we also prove that there is no bifurcation with the symmetry m, for any $m \geq \frac{2B}{A} + 2$. As in Assertion **(1)**, we check that the Crandall–Rabinowitz Theorem can be applied. From Proposition 2.6.7 and Proposition 2.6.6, there is a unique solution $x_m \in (-\infty, 1)$ of $\zeta_m(x) = 0$. In fact, $x_m < 0$. Then, we fix $\Omega_m = \frac{A}{4x_m} + \frac{B}{2}$. Note that by (2.6.31) and Proposition 2.6.8 we get the bounds for Ω_m. Moreover, Proposition 2.6.15 gives that $\Omega_m \notin \mathcal{S}^m_{\text{sing}}$. Then, let I be an interval such that $\Omega_m \in I$ and $\overline{I} \cap \mathcal{S}^m_{\text{sing}} = \varnothing$. Using again Proposition 2.3.3 and Proposition 2.7.2, we get that $\widehat{G} : I \times B_{\mathscr{C}^{1,\alpha}_{s,m}}(0, \varepsilon) \to \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D})$, is well-defined and $\mathscr{C}^1$ with respect to (Ω, g).

As to the spectral properties, we have stated in the previous proof that the dimension of the kernel of the linearized operator is given by the roots of ζ_{nm}. Taking $n = 1$, we know that x_m is the unique root of $\zeta_m(x)$. By Corollary 2.6.11 we get that $\zeta_{nm}(x_m) \neq 0$, for any $n \geq 2$. Hence $\mathcal{A}_{x_m} = \{m\}$. Due to $\Omega_m \notin \mathcal{S}^m_{\text{sing}}$, we have that the kernel is one dimensional. In addition we have seen in Proposition 2.5.1 that $D_g\widehat{G}(\Omega, 0)$ is a Fredholm operator of zero index. Concerning the transversal condition, note that using Proposition 2.6.18, we have that the transversal condition is satisfied. Similarly to the previous proof, the Crandall-Rabinowitz Theorem can be applied to $\widehat{G}$ obtaining a curve $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi)$ solutions of $\widehat{G}(\Omega, g) = 0$. Moreover, thanks to $\Omega_\xi \neq \Omega_0 \in (0, 1)$, Proposition 2.4.3 gives that $f_0 + f_\xi$ can not be radial since $f_\xi \neq 0$.

First, note that $\Omega \notin \left[\frac{B}{2}, \frac{B}{2}+\frac{A}{4}\right]$ is equivalent to $x<1$. By Proposition 2.6.6 and Proposition 2.6.7 we get that ζ_n has not solutions in $(-\infty, 1]$, for $m \geq \frac{2B}{A}+2$, and then there is no bifurcation with that symmetry by Proposition 2.5.1. In the opposite case, $x>1$, there is no bifurcation according to Theorem 2.5.6.

(3) Here, we are concerning with the case $A, B>0$ or $B \leq -\frac{A}{1+\epsilon}$ for some $\epsilon \in (0,1)$, with $-\frac{A}{2B} \neq x_0$, where x_0 is defined through (2.4.14). We work as in **(1)-(2)** checking the hypothesis of Crandall–Rabinowitz Theorem. Fixing $\Omega_1=0$ agrees with $x_1=-\frac{A}{2B}$, where we use (2.4.12). Proposition 2.6.15 allows us to have that $x_1 \notin \widehat{\mathcal{S}}_{\text{sing}}$. Then, we can take an interval I such that $0 \in I$, and

$$\overline{I} \cap \mathcal{S}_{\text{sing}} = \varnothing, \quad \Omega_0 \notin \overline{I}.$$

Again, Proposition 2.3.3 and Proposition 2.4.1 imply that

$$\widehat{G} : I \times B_{\mathscr{C}_s^{1,\alpha}}(0,\varepsilon) \to \mathscr{C}_s^{1,\alpha}(\mathbb{D}),$$

is well-defined and is $\mathscr{C}^1$ in (Ω, g).

We must check the spectral properties. Due to the assumptions on A and B, we get that $x_1 \leq 1$. By Proposition 2.6.6, Proposition 2.6.8 and Proposition 2.6.10 we have that $x_1 \neq x_n$ if there exists $x_n \in (-\infty, 1)$ solution of ζ_n. Note that such ϵ comes from the Proposition 2.6.10. Hence, by Corollary 2.6.11, we obtain that the kernel of $D_g\widehat{G}(0,0)$ is one dimensional, and is generated by (2.5.34), for $n=1$. Moreover, Proposition 2.5.1 implies that $D_g\widehat{G}(0,0)$ is a Fredholm operator of zero index. The transversal property is verified by virtue of Proposition 2.6.19. Then, Crandall–Rabinowitz Theorem can be applied obtaining the announced result. Note that the bifurcated solutions are not radial due to Proposition 2.4.3.

(4) The first assertion concerning the non bifurcation result comes from Proposition 2.6.6 and Proposition 2.6.7 due to the fact that ζ_m has not solutions in $(-\infty, 1]$, for $m \geq 2$. Moreover, by Corollary 2.6.9 and Theorem 2.5.6 we get that there is no bifurcation for $m=1$ since the only possibility agrees with $\Omega=0$, which satisfies (2.5.35). Finally, the bifurcation with $x>1$ is forbidden due again to Theorem 2.5.6.

The last assertion follows from Corollary 2.6.9 and Theorem 2.5.6. □

2.8 Dynamical system and orbital analysis

In this section we wish to investigate the particle trajectories inside the support of the rotating vortices that we have constructed in Theorem 2.1.1. We will show that in the frame of these V-states the trajectories are organized through concentric periodic orbits around the origin. This allows to provide an equivalent reformulation of the density equation (2.2.3) via the study of the associated dynamical system. It is worth pointing out that some of the material developed in this section about periodic trajectories and the regularity of the period is partially known in the literature and for the convenience of the reader we will provide the complete proofs.

Assuming that (2.1.1) is a solution of the Euler equations, the level sets of $\psi(x)-\Omega\frac{|x|^2}{2}$, where ψ is the stream function associated to (2.1.1), are given by the collection of the particle trajectories,

$$\begin{aligned}\partial_t\varphi(t,x) =& (v(\varphi(t,x)) - \Omega\varphi(t,x)^{\perp}) = \nabla^{\perp}\left(\psi - \Omega\frac{|\cdot|^2}{2}\right)(\varphi(t,x)),\\ \varphi(0,x) =& x \in \Phi(\overline{\mathbb{D}}).\end{aligned}$$

In the same way we have translated the problem to the unit disc $\mathbb{D}$ using the conformal map Φ via the vector field $W(\Omega, f, \Phi)$ in (2.2.5), we analyze the analogue in the level set context. We define the flow associated to W as

$$\partial_t \Psi(t,z) = W(\Omega, f, \Phi)(\Psi(t,z)), \quad \Psi(0,z) = z \in \overline{\mathbb{D}}. \tag{2.8.1}$$

Since $v(x) - \Omega x^{\perp}$ is divergence free, via Lemma 2.2.1, we obtain that $W(\Omega, f, \Phi)$ is incompressible, and then the last system is also Hamiltonian. In the following result, we highlight the relation between φ and Ψ.

Lemma 2.8.1. *The following identity*

$$\varphi(\eta_z(t), \Phi(z)) = \Phi(\Psi(t,z)), \quad \forall z \in \overline{\mathbb{D}},$$

holds, where

$$\eta_z'(t) = |\Phi'(\Phi^{-1}(\varphi(\eta_z(t), \Phi(z)))|^2, \quad \eta_z(0) = 0.$$

Proof. Let us check that $Y(t,z) = \Phi^{-1}(\varphi(t, \Phi(z)))$ verifies a similar equation as $\Psi(t,z)$ sets,

$$\begin{aligned}\partial_t Y(t,z) =& (\Phi^{-1})'(\Phi(Y(t,z))\partial_t \varphi(t, \Phi(z)) = \frac{\overline{\Phi'(Y(t,z))}}{|\Phi'(Y(t,z))|^2}(v(\Phi(Y(t,z))) - \Omega \Phi(Y(t,z))^{\perp})\\ =& \frac{W(\Omega, f, \Phi)(Y(t,z))}{|\Phi'(Y(t,z))|^2}.\end{aligned}$$

Now, we rescale the time through the function η_z, and $Y(\eta_z(t), z)$ satisfies,

$$\partial_t Y(\eta_z(t), z) = \eta_z'(t)(\partial_t Y)(\eta_z(t), z)) = \eta_z'(t)\frac{W(\Omega, f, \Phi)(Y(\eta(t), z))}{|\Phi'(Y(\eta(t), z))|^2} = W(\Omega, f, \Phi)(Y(\eta(t), z)).$$

Since $Y(\eta_z(0), z) = Y(0, z) = \Phi^{-1}(\varphi(0, \Phi(z)) = z$, we obtain the announced result. □

The next task is to connect the solutions constructed in Theorem 2.1.1 with the orbits of the associated dynamical system through the following result:

Theorem 2.8.2. *Let $m \geq 1$ and $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi)$ be one of the solutions constructed in Theorem* 2.1.1*. The flow Ψ associated to $W(\Omega_\xi, f_\xi, \Phi_\xi)$, defined in* (2.8.1)*, verifies the following properties:*

1. *$\Psi \in \mathscr{C}^1(\mathbb{R}, \mathscr{C}^{1,\alpha}(\mathbb{D}))$.*
2. *The trajectory $t \mapsto \Psi(t,z)$ is T_z periodic, located inside the unit disc and invariant by the dihedral group D_m. Moreover, if $m \geq 4$ then the period map $z \in \overline{\mathbb{D}} \mapsto T_z$ belongs to $\mathscr{C}^{1,\alpha}(\mathbb{D})$.*
3. *The family $(\Psi(t))_{t \in \mathbb{R}}$ generates a group of diffeomorphisms of the closed unit disc.*

The proof will be given in Subsection 2.8.5.

2.8.1 Periodic orbits

Here we explore sufficient conditions for Hamiltonian vector fields defined on the unit disc whose orbits are all periodic. More precisely, we shall establish the following result.

Proposition 2.8.3. *Let $W : \overline{\mathbb{D}} \to \mathbb{C}$ be a vector field in $\mathscr{C}^1(\overline{\mathbb{D}})$ satisfying the following conditions:*

i) It has divergence free.

ii) It is tangential to the boundary $\mathbb{T}$*, i.e.* $\text{Re}\,(W(z)\overline{z}) = 0, \quad \forall\, z \in \mathbb{T}$.

iii) It vanishes only at the origin.

Then, we have

1. *All the trajectories are periodic orbits located inside the unit disc, enclosing a simply connected domain containing the origin.*

2. *The family* $(\Psi(t))_{t\in\mathbb{R}}$ *generates a group of diffeomorphisms of the closed unit disc.*

3. *If* W *is antisymmetric with respect to the real axis, that is,*

$$\overline{W(z)} = -W(\overline{z}), \quad \forall z \in \overline{\mathbb{D}}. \tag{2.8.2}$$

then the orbits are symmetric with respect to the real axis.

4. *If* W *is invariant by a rotation centered at zero with angle* θ_0*, i.e.* $W(e^{i\theta_0}z) = e^{i\theta_0}W(z)$, $\forall z \in \overline{\mathbb{D}}$, *then all the orbits are invariant by this rotation.*

Proof. **(1)** Let Ψ be the solution associated to the flux W

$$\begin{cases} \partial_t\Psi(t,z) = W(\Psi(t,z)), \\ \Psi(0,z) = z \in \overline{\mathbb{D}}. \end{cases} \tag{2.8.3}$$

From the Cauchy–Lipschitz Theorem we know that the trajectory $t \mapsto \Psi(t,z)$ is defined in a maximal time interval $(-T_\star, T^\star)$, with $T_\star, T^\star > 0$, for each $z \in \overline{\mathbb{D}}$. Note that when z belongs to the boundary, then the second condition listed above implies necessarily that its trajectory does not leave the boundary. Since the vector field does not vanish anywhere on the boundary according to the third condition, the trajectory will cover all the unit disc. As the equation is autonomous, this ensures that the unit disc is a periodic orbit.
By condition i) we get that (2.8.3) is a Hamiltonian system. Let H be the Hamiltonian function such that $W = \nabla^\perp H$. Since H is $\mathscr{C}^1$ in $\overline{\mathbb{D}}$ and constant on the boundary $\mathbb{T}$ according to the assumption ii), then from iii) the origin corresponds to an extremum point.
Now, taking $|z| < 1$, the solution is globally well–posed in time, that is, $T_\star = T^\star = +\infty$. This follows easily from the fact that different orbits never intersect and consequently we should get

$$|\Psi(t,z)| < 1, \quad \forall t \in (-T_\star, T^\star),$$

meaning that the solution is bounded and does not touch the boundary so it is globally defined according to a classical blow–up criterion.

We will check that all the orbits are periodic inside the unit disc. This follows from some straightforward considerations on the level sets of the Hamiltonian H. Indeed, the $\omega-$limit of a point $z \neq 0$ cannot contain the origin because it is the only critical point and the level sets of H around this point are periodic orbits. Thus we deduce from Poincar–Bendixon Theorem that the $\omega-$limit of z will be a periodic orbit. As the level sets cannot be limit cycles then we find that the trajectory of z coincides with the periodic orbit.
(2) This follows from classical results on autonomous differential equation. In fact, we know that the flow $\Psi : \mathbb{R} \times \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ is well–defined and $\mathscr{C}^1$. For any $t \in \mathbb{R}$, it realizes a bijection with $\Psi^{-1}(t,\cdot) = \Psi(-t,\cdot)$, and $(\Psi(t))_{t\in\mathbb{R}}$ generates a group of diffeomorphisms on $\overline{\mathbb{D}}$.
(3) The symmetry of the orbits with respect to the real axis is a consequence of the following

elementary fact. Given $z \in \mathbb{D}$ and $t \mapsto \Psi(t, z)$ its trajectory, then it follows that $t \mapsto \overline{\Psi(-t, z)}$ is also a solution of the same Cauchy problem and by uniqueness we find the identity

$$\Psi(t, z) = \overline{\Psi(-t, z)}, \quad \forall t \in \mathbb{R}.$$

(4) Assume that W is invariant by the rotation R_{θ_0} centered at zero and with angle θ_0. Let $z \in \mathbb{D}$, then we shall first check the identity

$$e^{i\theta_0}\Psi(t, z) = \Psi(t, e^{i\theta_0}z), \quad \forall t \in \mathbb{R}. \tag{2.8.4}$$

To do that, it suffices to verify that both functions satisfy the same differential equation with the same initial data, and thus the identity follows from the uniqueness of the Cauchy problem. Note that (2.8.4) means that the rotation of a trajectory is also a trajectory. Denote by D_{z_0} and $e^{i\theta_0}D_{z_0}$ the domains delimited by the curves $t \mapsto \Psi(t, z_0)$ and $t \mapsto e^{i\theta_0}\Psi(t, z_0)$, respectively. Then, it is a classical result that those domains are necessary simply connected and they contain the origin according to 1). Since different trajectories never intersect, then we have only two possibilities: $D_{z_0} \subset e^{i\theta_0}D_{z_0}$ or the converse. Since the rotation is a Lebesgue preserving measure, then $D_{z_0} = e^{i\theta_0}D_{z_0}$, which implies that the periodic orbit $t \mapsto \Psi(t, z_0)$ is invariant by the rotation R_{θ_0}. □

2.8.2 Reformulation with the trajectory map

In this section we discuss a new representation of solutions to the equations of the type

$$W(z) \cdot \nabla f(z) = 0, \quad \forall z \in \overline{\mathbb{D}}, \tag{2.8.5}$$

with W a vector field as in Proposition 2.8.3 and $f : \overline{\mathbb{D}} \to \mathbb{R}$ a $\mathscr{C}^1$ function.

Proposition 2.8.4. *Let $W : \overline{\mathbb{D}} \to \mathbb{C}$ be a vector field satisfying the assumptions* i), ii) *and* iii) *of Proposition 2.8.3. Then,* (2.8.5) *is equivalent to the formulation*

$$f(z) - \frac{1}{T_z}\int_0^{T_z} f(\Psi(\tau, z))d\tau = 0, \quad \forall\, z \in \overline{\mathbb{D}}, \tag{2.8.6}$$

with T_z being the period of the trajectory $t \mapsto \Psi(t, z)$.

Proof. We first check that (2.8.5) is equivalent to

$$f(\Psi(t, z)) = f(z), \quad \forall t \in \mathbb{R}, \forall |z| \leq 1. \tag{2.8.7}$$

Although for simplicity we can assume that the equivalence is done pointwise, where we need $f \in \mathscr{C}^1$, the equivalence is perfectly valid in a weak sense without nothing more than assuming Hölder regularity on f. Indeed, if f is a $\mathscr{C}^1$ function satisfying (2.8.7), then by differentiating in time we get

$$(W \cdot \nabla f)(\Psi(t, z)) = 0, \quad \forall t \in \mathbb{R}, \forall |z| \leq 1.$$

According to Proposition 2.8.3, we have that (2.8.5) is satisfied everywhere in the closed unit disc, for any t, $\Psi(t, \overline{\mathbb{D}}) = \overline{\mathbb{D}}$. Conversely, if f is a $\mathscr{C}^1$ solution to (2.8.5), then differentiating with respect to t the function $t \mapsto f(\Psi(t, z))$ we get

$$\frac{d}{dt}f(\Psi(t, z)) = (W \cdot \nabla f)(\Psi(t, z)) = 0.$$

Therefore, we have (2.8.7). Now, we will verify that (2.8.6) is in fact equivalent to (2.8.7). The implication (2.8.7) $\Longrightarrow$ (2.8.6) is elementary. So it remains to check the converse. From (2.8.6) one has

$$f(\Psi(t,z)) - \frac{1}{T_{\Psi(t,z)}} \int_0^{T_{\Psi(t,z)}} f(\Psi(\tau,\Psi(t,z))d\tau = 0. \tag{2.8.8}$$

Since the vector field is autonomous, then all the points located at the same orbit generate periodic trajectories with the same period, and of course with the same orbit. Therefore, we have $T_{\Psi(t,z)} = T_z$. Using $\Psi(\tau,\Psi(t,z)) = \Psi(t+\tau,z)$, a change of variables, and the T_z-periodicity of $\tau \mapsto f(\Psi(\tau,z))$, then we deduce

$$\frac{1}{T_{\Psi(t,z)}} \int_0^{T_{\Psi(t,z)}} f(\Psi(\tau,\Psi(t,z))d\tau = \frac{1}{T_z}\int_t^{t+T_z} f(\Psi(\tau,z))d\tau = \frac{1}{T_z}\int_0^{T_z} f(\Psi(\tau,z))d\tau = f(z).$$

Combining this with (2.8.8), we get (2.8.7). This completes the proof. □

2.8.3 Persistence of the symmetry

We shall consider a vector field W satisfying the assumptions of Proposition 2.8.3 and (2.8.2) and let Ψ be its associated flow. We define the operator $f \mapsto Sf$ by

$$Sf(z) = f(z) - \frac{1}{T_z}\int_0^{T_z} f(\Psi(\tau,z))d\tau, \quad \forall z \in \overline{\mathbb{D}}.$$

We shall prove that f and Sf share the same planar group of invariance in the following sense.

Proposition 2.8.5. *Let $f : \overline{\mathbb{D}} \mapsto \mathbb{R}$ be a smooth function. The following assertions hold true:*

1. *If f is invariant by reflection with respect to the real axis, then Sf is invariant too. This means that*
$$f(\overline{z}) = f(z), \quad \forall z \in \overline{\mathbb{D}} \Longrightarrow Sf(\overline{z}) = Sf(z), \quad \forall z \in \overline{\mathbb{D}}.$$
2. *If W and f are invariant by the rotation R_{θ_0} centered at zero with angle $\theta_0 \in \mathbb{R}$, then Sf commutes with the same rotation. This means that*
$$f(e^{i\theta_0}z) = f(z), \quad \forall z \in \overline{\mathbb{D}} \Longrightarrow (Sf)(e^{i\theta_0}z) = Sf(z), \quad \forall z \in \overline{\mathbb{D}}.$$

Proof. (**1**) Let $z \in \overline{\mathbb{D}}$, it is a simple matter to check that $\Psi(t,\overline{z}) = \overline{\Psi(-t,z)}$, which implies $T_{\overline{z}} = T_z$, and then

$$Sf(\overline{z}) = f(z) - \frac{1}{T_z}\int_0^{T_z} f\left(\overline{\Psi(-\tau,z)}\right)d\tau = f(z) - \frac{1}{T_z}\int_0^{T_z} f\left(\Psi(-\tau,z)\right)d\tau = Sf(z).$$

(**2**) According to Proposition (2.8.3) and the fact that the vector-field W is invariant by the rotation R_{θ_0}, then we have that the orbits are symmetric with respect to this rotation and

$$T_{e^{i\theta_0}z} = T_z \quad \text{and} \quad \Psi(t, e^{i\theta_0}z) = e^{i\theta_0}\Psi(t,z),$$

where we have used (2.8.4), which implies that

$$Sf(e^{i\theta_0}z) = f(z) - \frac{1}{T_z}\int_0^{T_z} f\left(e^{i\theta_0}\Psi(\tau,z)\right)d\tau = f(z) - \frac{1}{T_z}\int_0^{T_z} f\left(\Psi(\tau,z)\right)d\tau = Sf(z).$$

This concludes the proof. □

2.8.4 Analysis of the regularity

Next, we are interested in studying the regularity of the the flow map (2.8.3) and the period map. The following result is classical, see for instance [75].

Proposition 2.8.6. *Let* $\alpha \in (0,1)$, $W : \overline{\mathbb{D}} \mapsto \mathbb{R}^2$ *be a vector-field in* $\mathscr{C}^{1,\alpha}(\mathbb{D})$ *satisfying the condition* ii) *of Proposition* 2.8.3 *and* $\Psi : \mathbb{R} \times \overline{\mathbb{D}} \mapsto \overline{\mathbb{D}}$ *its flow map. Then* $\Psi \in \mathscr{C}^1(\mathbb{R}, \mathscr{C}^{1,\alpha}(\mathbb{D}))$ *and there exists* $C > 0$ *such that*

$$\|\Psi^{\pm 1}(t)\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})} \leq e^{C\|W\|_{\mathscr{C}^{1,\alpha}(\mathbb{D}}|t|}\left(1 + \|W\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}|t|\right), \quad \forall t \in \mathbb{R},$$

holds.

Now we intend to study the regularity of the function $z = (x,y) \in \overline{\mathbb{D}} \mapsto T_z$. This is a classical subject in dynamical systems and several results are obtained in this direction for smooth Hamiltonians. Notice that in the most studies in the literature the regularity is measured with respect to the energy and not with respect to the positions as we propose to do here.

Let $z \in \overline{\mathbb{D}}$ be a given non equilibrium point, the orbit $t \mapsto \Psi(t,z)$ is periodic and T_z is the first strictly positive time such that

$$\Psi(T_z, z) - z = 0. \tag{2.8.9}$$

This is an implicit function equation, from which we expect to deduce some regularity properties. Our result reads as follows.

Proposition 2.8.7. *Let* W *be a vector field in* $\mathscr{C}^1(\overline{\mathbb{D}})$, *satisfying the assumptions of Proposition* 2.8.3 *and* (2.8.2) *and such that*

$$W(z) = izU(z), \quad \forall\, z \in \overline{\mathbb{D}},$$

with

$$\text{Re}\{U(z)\} \neq 0, \quad \forall\, z \in \overline{\mathbb{D}}. \tag{2.8.10}$$

Then the following assertions hold true:

1. *The map* $z \in \overline{\mathbb{D}} \mapsto T_z$ *is continuous and verifies the upper bound*

$$0 < T_z \leq \frac{2\pi}{\inf_{z\in\overline{\mathbb{D}}} |\text{Re}U(z)|}, \quad \forall z \in \overline{\mathbb{D}}. \tag{2.8.11}$$

2. *If in addition* $U \in \mathscr{C}^{1,\alpha}(\mathbb{D})$, *then* $z \mapsto T_z$ *belongs to* $\mathscr{C}^{1,\alpha}(\mathbb{D})$.

Remark 2.8.8. *Since the origin is an equilibrium point for the dynamical system, then its trajectory is periodic with any period. However, as we will see in the proof, there is a minimal strictly positive period denoted by* T_z, *for any curves passing through a non vanishing point* z. *The mapping* $z \in \overline{\mathbb{D}}\backslash\{0\} \mapsto T_z$ *is not only well–defined, but it can be extended continuously to zero. Thus we shall make the following convention*

$$T_0 \equiv \lim_{z\to 0} T_z.$$

Remark 2.8.9. *The upper bound in* (2.8.11) *is "almost optimal" for radial profiles, where* $U(z) = U_0(|z|) \in \mathbb{R}$, *and explicit computations yield*

$$T_z = \frac{2\pi}{|U_0(|z|)|}.$$

Proof. (**1**) We shall describe the trajectory parametrization using polar coordinates. Firstly, we may write for $z = re^{i\theta}$

$$W(z) = \Big[W^r(r,\theta) + iW^\theta(r,\theta)\Big]\, e^{i\theta},$$

with

$$W^\theta(r,\theta) = r\mathrm{Re}(U(re^{i\theta})) \quad \text{and} \quad W^r(r,\theta) = -r\mathrm{Im}(U(re^{i\theta})).$$

Given $0 < |z| \le 1$, we look for a polar parametrization of the trajectory passing through z,

$$\Psi(t,z) = r(t)e^{i\theta(t)},\ r(0) = |z|,\ \theta(0) = \mathrm{Arg}(z).$$

Inserting into (2.8.3) we obtain the system

$$\begin{aligned} \dot r(t) =& - r(t)\mathrm{Im}\left\{U\left(r(t)e^{i\theta(t)}\right)\right\} =: P(r(t),\theta(t)) \\ \dot\theta(t) =& \mathrm{Re}\left\{U\left(r(t)e^{i\theta(t)}\right)\right\} =: Q(r(t),\theta(t)). \end{aligned} \tag{2.8.12}$$

From the assumption (2.8.2) we find

$$\overline{U(z)} = U(\overline{z}), \quad \forall\, z \in \overline{\mathbb{D}},$$

which implies in turn that

$$P(r,-\theta) = -P(r,\theta) \quad \text{and} \quad Q(r,-\theta) = Q(r,\theta), \quad \forall r \in [0,1], \forall \theta \in \mathbb{R}.$$

Thus, we have the Fourier expansions

$$P(r,\theta) = \sum_{n\in\mathbb{N}^\star} P_n(r)\sin(n\theta), \quad \text{and} \quad Q(r,\theta) = \sum_{n\in\mathbb{N}} Q_n(r)\cos(n\theta).$$

Denoting by T_n and U_n the classical Chebyshev polynomials that satisfy the identities

$$\begin{aligned} \cos(n\theta)) =& T_n(\cos\theta) \\ \sin(n\theta) =& \sin(\theta)U_{n-1}(\cos\theta), \end{aligned}$$

we obtain

$$\begin{aligned} P(r,\theta) =& \sin\theta \sum_{n\in\mathbb{N}^\star} P_n(r)U_{n-1}(\cos\theta) \equiv \sin\theta\ F_1(r,\cos\theta). \\ Q(r,\theta) =& \sum_{n\in\mathbb{N}} Q_n(r)T_n(\cos\theta) \equiv F_2(r,\cos\theta). \end{aligned}$$

Coming back to (2.8.12), we get

$$\begin{aligned} \dot r(t) &= \sin\theta(t)\ F_1(r(t),\cos\theta(t)) \\ \dot\theta(t) &= F_2(r(t),\cos\theta(t)). \end{aligned} \tag{2.8.13}$$

We look for solutions in the form

$$r(t) = f_z(\cos(\theta(t)), \quad \text{with} \quad f_z : [-1,1] \to \mathbb{R},$$

and then f_z satisfies the differential equation

$$f_z'(s) = -\left(\frac{F_1}{F_2}\right)(f_z(s),s)\,, f_z(\cos\theta) = |z|. \tag{2.8.14}$$

Note that the preceding fraction is well–defined since the assumption (2.8.10) is equivalent to

$$F_2(r, \cos\theta) \neq 0, \quad \forall r \in [0,1], \theta \in \mathbb{R}.$$

Theorem 2.8.3-ii) agrees with

$$F_1(0,s) = F_1(1,s) = 0, \quad \forall |s| \leq 1,$$

which implies that the system (2.8.14) admits a unique solution $f_z : [-1,1] \to \mathbb{R}_+$ such that

$$0 \leq f_z(s) \leq 1, \quad \forall\, s \in [-1,1].$$

Hence, integrating the second equation of (2.8.13) we find after a change of variable

$$\int_{\theta_0}^{\theta(t)} \frac{1}{F_2\left(f_z(\cos s), \cos s\right)} ds = t,$$

and, therefore, the following formula for the period is obtained

$$T_z = \Big| \int_{\theta_0}^{\theta_0+2\pi} \frac{1}{F_2\left(f_z(\cos s), \cos s\right)} ds \Big| = \int_0^{2\pi} \frac{1}{\left|F_2\left(f_z(\cos s), \cos s\right)\right|} ds. \tag{2.8.15}$$

This gives the bound of the period stated in (2.8.11). The continuity $z \mapsto T_z$ follows from the same property of $z \mapsto f_z$, which can be derived from the continuous dependence with respect to the initial conditions.

(**2**) Now, we will study the regularity of the period in $\mathscr{C}^{1,\alpha}$. Note that (2.8.15) involves the function f_z which is not smooth enough because the initial condition $z \mapsto f_z(\cos\theta)$ is only Lipschitz. So it seems quite complicate to follow the regularity in $\mathscr{C}^{1,\alpha}$ from that formula. The alternative way is to study the regularity of the period using the implicit equation (2.8.9). Thus, differentiating this equation with respect to x and y we obtain

$$\begin{aligned}(\partial_x T_z)\partial_t \Psi(T_z, x, y) + \partial_x \Psi(T_z, x, y) - 1 =& 0,\\ (\partial_y T_z)\partial_t \Psi(T_z, x, y) + \partial_y \Psi(T_z, x, y) - i =& 0.\end{aligned}$$

From the flow equation and the periodicity condition we get

$$\partial_t \Psi(T_z, z) = W(z),$$

which implies

$$\begin{cases} (\partial_x T_z)W(z) + \partial_x \Psi(T_z, x, y) - 1 = & 0, \\ (\partial_y T_z)W(z) + \partial_y \Psi(T_z, x, y) - i = & 0. \end{cases} \tag{2.8.16}$$

Due to the assumption on W, the flow equation can be written as

$$\partial_t \Psi(t,z) = i\Psi(t,z)U(\Psi(t,z)), \quad \Psi(0,z) = z,$$

which can be integrated, obtaining

$$\Psi(t,z) = z\, e^{i\int_0^t U(\Psi(\tau,z))d\tau}.$$

By differentiating this identity with respect to x, it yields

$$\partial_x \Psi(t,z) = e^{i\int_0^t U(\Psi(\tau,z))d\tau} \left[1 + iz\int_0^t \partial_x \left\{U(\Psi(\tau,z))\right\} d\tau\right].$$

Since $\Psi(T_z, z) = z$, we have

$$e^{i\int_0^{T_z} U(\Psi(\tau,z))d\tau} = 1,$$

and thus

$$\partial_x \Psi(T_z, z) = 1 + iz \int_0^{T_z} \partial_x \{U(\Psi(\tau, z))\} d\tau.$$

Similarly, we find

$$\partial_y \Psi(T_z, z) = i + iz \int_0^{T_z} \partial_y \{U(\Psi(\tau, z))\} d\tau.$$

Combining these identities with (2.8.16), we obtain

$$\begin{cases} (\partial_x T_z)W(z) = & -iz \int_0^{T_z} \partial_x \{U(\Psi(\tau, z))\} d\tau, \\ (\partial_y T_z)W(z) = & -iz \int_0^{T_z} \partial_y \{U(\Psi(\tau, z))\} d\tau, \end{cases}$$

which, using the structure of W, reads as

$$\begin{cases} (\partial_x T_z)U(z) = & -i \int_0^{T_z} \partial_x \{U(\Psi(\tau, z))\} d\tau, \\ (\partial_y T_z)U(z) = & -i \int_0^{T_z} \partial_y \{U(\Psi(\tau, z))\} d\tau. \end{cases}$$

Now, notice that from Theorem 2.8.3-iv), the vector field W vanishes only at zero and since $U(0) \neq 0$, we find

$$U(z) \neq 0, \quad \forall z \in \overline{\mathbb{D}}.$$

This implies that $z \in \overline{\mathbb{D}} \mapsto \frac{1}{U(z)}$ is well–defined and belongs to $\mathscr{C}^{1,\alpha}(\mathbb{D})$. Therefore, we can write

$$\nabla_z T_z = -\frac{i}{U(z)} \int_0^{T_z} \nabla_z \{U(\Psi(\tau, z))\} d\tau, \tag{2.8.17}$$

where we have used the notation $\nabla_z = (\partial_x, \partial_y)$. According to Proposition 2.8.6 and classical composition laws, we obtain

$$\tau \mapsto \nabla_z \{U(\Psi(\tau, \cdot))\} \in \mathscr{C}(\mathbb{R}; \mathscr{C}^{0,\alpha}(\mathbb{D})).$$

Since $z \mapsto T_z$ is continuous, then we find by composition that

$$\varphi : z \in \overline{\mathbb{D}} \mapsto \int_0^{T_z} \nabla_z \{U(\Psi(\tau, z))\} d\tau \in \mathscr{C}(\overline{\mathbb{D}}).$$

Combining this information with (2.8.17), we deduce that $z \mapsto T_z \in \mathscr{C}^1(\overline{\mathbb{D}})$. Hence, we find in turn that $\varphi \in \mathscr{C}^{0,\alpha}(\mathbb{D})$ by composition. Using (2.8.17) again, it follows that $z \mapsto \nabla_z T_z \in \mathscr{C}^{0,\alpha}(\mathbb{D})$. Thus, $z \mapsto T_z \in \mathscr{C}^{1,\alpha}(\mathbb{D})$. This achieves the proof. □

2.8.5 Application to the nonlinear problem

We intend in this section to prove Theorem 2.8.2. Let us point out that from Proposition 2.3.3, the nonlinear vector field $W(\Omega, f, \Phi)$ is chosen in order to be tangent to the boundary everywhere. We will see that not only this assumption but all the assumptions of Proposition 2.8.3 are satisfied if f is chosen close to a suitable radial profile.

Lemma 2.8.10. *Let $g \in \mathscr{C}^{1,\alpha}_{s,m}(\mathbb{D})$ and $\phi \in \mathscr{H}\mathscr{C}^{2,\alpha}_m(\mathbb{D})$, then $W(\Omega, f, \Phi) \in \mathscr{C}^{1,\alpha}(\mathbb{D})$ and satisfies the symmetry properties* (2.8.2) *and*

$$W(\Omega, f, \Phi)(e^{i\frac{2\pi}{m}}z) = e^{i\frac{2\pi}{m}}W(\Omega, f, \Phi)(z).$$

Moreover, if $m \geq 4$ then

$$W(\Omega, f, \Phi)(z) = izU(z), \tag{2.8.18}$$

with $U \in \mathscr{C}^{1,\alpha}(\mathbb{D})$

Proof. Using Proposition 2.3.1 and Proposition 2.7.1, then the regularity and the symmetry properties of $W(\Omega, f, \Phi)$ are verified. Let us now check (2.8.18). Firstly, since $\Phi(0) = 0$ and $\phi \in \mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$, we have

$$\Phi(z) = z\Phi_1(z), \quad \Phi_1 \in \mathscr{C}^{1,\alpha}(\mathbb{D}).$$

In addition, $\overline{\Phi'} \in \mathscr{C}^{1,\alpha}(\mathbb{D})$, and thus we find

$$i\Omega\,\Phi(z)\,\overline{\Phi'}(z) = izU_1(z), \quad \text{with} \quad U_1 \in \mathscr{C}^{1,\alpha}(\mathbb{D}).$$

Now, to get (2.8.18) it is enough to check that

$$I(f, \Phi)(z) = zU_2(z), \quad \text{with} \quad U_2 \in \mathscr{C}^{1,\alpha}(\mathbb{D}),$$

where

$$I(f, \Phi)(z) := \frac{1}{2\pi}\int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}}|\Phi'(y)|^2 dA(y).$$

We look for the first order Taylor expansion around the origin of $I(f, \Phi)(z)$. Using the m-fold symmetry of f and Φ, it is clear that

$$I(f, \Phi)(0) = -\int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(y)}}|\Phi'(y)|^2 dA(y) = -\int_{\mathbb{D}} \frac{f(e^{i\frac{2\pi}{m}}y)}{\overline{\Phi(e^{i\frac{2\pi}{m}}y)}}|\Phi'(e^{i\frac{2\pi}{m}}y)|^2 dA(y) = e^{i\frac{2\pi}{m}}I(f, \Phi)(0),$$

which leads to

$$I(f, \Phi)(0) = 0, \tag{2.8.19}$$

for $m \geq 2$. This implies that one can always write $I(f, \Phi)(z) = zU_2(z)$, but U_2 is only bounded:

$$\|U_2\|_{L^\infty(\overline{\mathbb{D}})} \leq \|\nabla_z I_2\|_{L^\infty(\mathbb{D})} \leq \|I\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}.$$

We shall see how the extra symmetry helps to get more regularity for U_2. According to Taylor expansion one gets

$$I(f, \Phi)(z) = az + b\overline{z} + cz^2 + d\overline{z}^2 + e|z|^2 + \text{l.o.t.} \quad \text{with} \quad a, b, c, d, e \in \mathbb{C}.$$

Using the reflection invariance with respect to the real axis of f and Φ, we obtain

$$\overline{I(f, \Phi)(z)} = I(\overline{z}),$$

which implies that $a, b, c, d, e \in \mathbb{R}$. Now, the rotation invariance leads to

$$I(f, \Phi)(e^{i\frac{2\pi}{m}}z) = e^{i\frac{2\pi}{m}}I(z).$$

Then, we obtain

$$e = 0, \quad b(e^{i\frac{4\pi}{m}} - 1) = 0, \quad c(e^{i\frac{2\pi}{m}} - 1) = 0, \quad \text{and} \quad d(e^{i\frac{6\pi}{m}} - 1) = 0.$$

This implies that $b=c=d=0$, whenever $m\geq 4$. Thus, we have

$$I(f,\Phi)(z)=az+h(z),$$

with $h\in\mathscr{C}^{2,\alpha}(\mathbb{D})$, and

$$h(0)=0,\ \nabla_z h(0)=0,\quad \text{and}\quad \nabla_z^2 h(0)=0.$$

From this we claim that

$$h(z)=zk(z),\quad \text{with}\quad k\in\mathscr{C}^{1,\alpha}(\mathbb{D}),$$

which concludes the proof. □

Now we are in a position to prove Theorem 2.8.2.

Proof of Theorem 2.8.2. The existence of the conformal map Φ comes directly from Proposition 2.3.3, which gives the boundary equation (2.2.4). Moreover, Lemma 2.8.10 gives the decomposition (2.8.18) and provides the necessary properties to apply Proposition 2.8.3. Furthermore, we can use Proposition 2.8.4 in order to obtain the equivalence between a rotating solution (2.1.1) of the Euler equations and the solution of (2.8.6). Proposition 2.8.6 gives the regularity of the flow map, and it remains to check (2.8.10), using Proposition 2.8.7, in order to get the regularity of the period function. In the case of radial profile, we know that

$$W(\Omega,f_0,\mathrm{Id})(z)=izU_0(z),\quad U_0(z)=-\Omega+\frac{1}{r^2}\int_0^r sf_0(s)ds,$$

which implies

$$\begin{aligned}W(\Omega,f,\Phi)(z)-W(\Omega,f_0,\mathrm{Id})(z)=&-\Omega\phi(z)\left(1+\overline{g'(z)}\right)-\Omega z\overline{g'(z)}+\overline{g'(z)}\,I(f,\Phi)(z)\\&+I(f,\Phi)(z)-I(f_0,\mathrm{Id})(z).\end{aligned}$$

It is easy to see that

$$\left|\frac{\phi(z)}{z}\left(1+\overline{g'(z)}\right)\right|\leq\|\phi'\|_{L^\infty}\left(1+\|g'\|_{L^\infty(\mathbb{D})}\right).$$

From (2.8.19) and Lemma B.0.5, we have

$$\left|\overline{g'(z)}\frac{I(f,\Phi)(z)}{z}\right|\leq\|g'\|_{L^\infty(\mathbb{D})}|\nabla_z I(f,\Phi)\|_{L^\infty(\mathbb{D})}\leq C(\Phi)\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}\|g'\|_{L^\infty(\mathbb{D})}.$$

Using again (2.8.19), we deduce

$$\left|\frac{I(f,\Phi)(z)-I(f_0,\Phi_0)(z)}{z}\right|\quad\leq\quad\left\|\nabla_z\left[I(f,\Phi)-I(f_0,\Phi_0)\right]\right\|_{L^\infty(\mathbb{D})}.$$

Straightforward computations imply

$$\begin{aligned}I(f,\Phi)(z)-I(f_0,\Phi_0)(z)=&\frac{1}{2\pi}\int_{\mathbb{D}}\frac{g(y)}{\overline{z}-\overline{y}}dA(y)+\frac{1}{2\pi}\int_{\mathbb{D}}\frac{f(y)\left[\phi(\overline{y})-\phi(\overline{z})\right]}{(\overline{y}-\overline{z})\left(\Phi(\overline{y})-\Phi(\overline{z})\right)}dA(y)\\&+\frac{1}{2\pi}\int_{\mathbb{D}}\frac{2\mathrm{Re}\{\phi'(y)\}+|\phi'(y)|^2}{\Phi(\overline{z})-\Phi(\overline{y})}f(y)dA(y).\end{aligned}$$

From this and using Lemma B.0.5, we claim that

$$\left\|\nabla_z\left[I(f,\Phi)-I(f_0,\Phi_0)\right]\right\|_{L^\infty(\mathbb{D})}\leq C(\Phi)\|g\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}.$$

Combining the preceding estimates we find

$$W(\Omega, f, \Phi)(z) = z\left[iU_0(z) + \frac{W(\Omega, f, \Phi)(z) - W(\Omega, f_0, \mathrm{Id})(z)}{z}\right] \equiv z\left[iU_0(z) + \widehat{W}(\Omega, f, \Phi)(z)\right],$$

with

$$\|\widehat{W}(\Omega, f, \Phi)\|_{L^\infty(\mathbb{D})} \leq C(f, \Phi)\left(\|h\|_{\mathscr{C}^{1,,\alpha}(\mathbb{D})} + \|\phi\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\right).$$

Now, we take h, ϕ small enough such that

$$C(f, \Phi)\left(\|h\|_{\mathscr{C}^{1,,\alpha}(\mathbb{D})} + \|\phi\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}\right) \leq \varepsilon,$$

and ε verifies

$$0 < 2\varepsilon \leq \inf_{0<r\leq 1}\left|\Omega - \frac{1}{r^2}\int_0^r s f_0(s) ds\right| = \inf_{z\in\mathbb{D}} |U_0(z)|,$$

in order to have

$$W(\Omega, f, \Phi)(z) = izU(z), \quad \text{with} \quad 2\left|\mathrm{Re}\{U(z)\}\right| \geq \inf_{0<r\leq 1}\left|\Omega - \frac{1}{r^2}\int_0^r s f_0(s) ds\right|.$$

To end the proof let us check that this infimum is strictly positive for the quadratic profile $f_0(r) = Ar^2 + B$, where $A > 0$. Take $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi)$ the bifurcating curve from Theorem 2.1.1. Then for a small enough

$$\Omega_\xi = \frac{B}{2} + \frac{A}{4x_\xi}, x_\xi < 1 \quad \text{and} \quad \Omega_\xi \notin \mathcal{S}_{\text{sing}}$$

Thus

$$\Omega_\xi - \frac{1}{r^2}\int_0^r s f_0(s) ds = \frac{A}{4}\left(\frac{1}{x_\xi} - r^2\right).$$

Consequently

$$\inf_{r\in[0,1]}\left|\Omega_\xi - \frac{1}{r^2}\int_0^r s f_0(s) ds\right| = \inf_{r\in[0,1]} \frac{A}{4}\left|\frac{1}{x_\xi} - r^2\right| > 0.$$

This achieves the proof. □

Chapter 3

Kármán Vortex Street in incompressible fluid models

This chapter is the subject of the following publication:
C. García, *Kármán Vortex Street in incompressible fluid models.*
Nonlinearity **33**(4), (2020), 1625–1676.

Contents

3.1 Introduction

A distinctive pattern is observed in the wake of a two–dimensional bluff body placed in a uniform stream at certain velocities: this is called the von Kármán Vortex Street. It consists of vortices of high vorticity in an irrotational fluid. This complex phenomenon occurs in a large number of circumstances. For instance, the singing of power transmission lines or kite strings in the wind. It can also be observed in atmospheric flow about islands, in ocean flows about pipes and in aeronautical systems. See [89, 90, 127, 134, 135, 136, 139].

The phenomenon of periodic vortex shedding can be described as an oscillating flow that appears when a fluid passes across a bluff body. It was first studied experimentally in [127, 139]. The first theoretical model was later presented by VON KÁRMÁN in [89, 90] and nowadays it is called Kármán Vortex Street in the literature, see also [9, 102, 130] for further studies of such a model. Specifically, the author considered a distribution of point vortices distributed along two parallel staggered rows, where vortex strength at each row has opposite sign. Since the exact problem seems to be complex from a theoretical point of view, there has been many different approximations.

The generation of vortex shedding phenomena has been subject of study of many authors. The main heuristic idea is that bluff bodies often induce separated flow over a substantial proportion of their surfaces when placed within a fluid stream. Examples of bluff bodies are sharp edges bodies. At low Reynold number, the flow stays stable around the body. However, for high Reynold number (higher than a critical value) the body generates instabilities. After a transient of time, an organized steady motion is often created, thus giving rise to vortex shedding dynamics. Notice that both viscosity and bluff body are involved in the generation of the shedding process. However, once created, they seem not to influence anymore the evolution of the vortex street. This suggests that, if we focus on the later evolution of the street (but not on the initial shedding formation process), an inviscid incompressible fluid model may be proposed to describe this situation. See [88, 107, 136] for more details.

In the context of the Euler equations, Kármán Vortex Street structures arose in the works by SAFFMAN and SCHATZMAN [134, 135, 136]. Inspired by the ideas by VON KÁRMÁN [89, 90], the authors considered two parallel infinite arrays of vortices with finite area and uniform vorticity. After some numerical studies, they found the existence of this kind of solutions that translate at constant speed. Also, they discussed about the stability properties of these steady solutions with respect to two-dimensional disturbances by playing with two free parameters: size of vortices and distance between vortices in the street. More specifically, they found linear stability of the street for infinitesimal disturbances under an appropriate relation between such free parameters.

In this chapter, we focus on the study of these structures in different inviscid incompressible fluid models via a desingularization of the point vortex model proposed by VON KÁRMÁN [89, 90]. We obtain two infinite arrows of vortex patches, i.e. vortices with finite area and uniform vorticity, that translate. Then, we recover analytically the solutions found numerically by SAFFMAN and SCHATZMAN, not only in the Euler equations framework, but also in other incompressible models, that we recall in the following.

Let q be a scalar magnitude of the fluid satisfying the following transport equation

$$\begin{cases} \partial_t q + (v\cdot\nabla)q = 0, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ v = \nabla^\perp\psi, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ \psi = G * q, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ q(0,x) = q_0(x), & \text{with } x\in\mathbb{R}^2. \end{cases} \tag{3.1.1}$$

Note that the velocity field v is given in terms of q via the interaction kernel G, which is assumed to be a smooth off zero function. Here, $(x_1,x_2)^\perp = (-x_2,x_1)$. In the case that $G = \frac{1}{2\pi}\ln|\cdot|$, we arrive at the Euler equations. On the other hand, if $G = -K_0(|\lambda||\cdot|)$, with $\lambda\neq 0$ and K_0 the Bessel function, the quasi–geostrophic shallow water (QGSW) equations are found. Finally, the generalized surface quasi–geostrophic (gSQG) equations appears in the case that $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\beta\in[0,1]$ and $C_\beta = \frac{\Gamma(\frac{\beta}{2})}{2^{1-\beta}\Gamma(\frac{2-\beta}{2})}$. Note that all the previous models have a common property: the kernel G is radial, which will be crucial in this chapter.

The Euler equations deal with uniform incompressible ideal fluids and q represents the vorticity of the fluid, usually denoted by ω. Yudovich solutions, that are bounded and integrable solutions, are known to exist globally in time, see [105, 147]. When the initial data is the characteristic function of a simply–connected bounded domain D_0, the solution continues being a characteristic function over D_0, that propagates along the flow. These are known as the vortex patches solutions. If the initial domain is $\mathscr{C}^{1,\alpha}$, with $0 < \alpha < 1$, the regularity persists for any time, see [17, 35, 137]. The only explicit simply–connected vortex patches known up to now are the Rankine vortex (the circular patch), which are stationary, and the Kirchhoff ellipses [96], that rotate. Later, DEEM and ZABUSKY [49] gave some numerical observations of the existence of V–states, i.e. rotating vortex patches, with $m-$fold symmetry. Using the bifurcation theory, BURBEA [20] proved analytically the existence of these V–states close to the Rankine vortex. There has been several works concerning the V–states following the approach of BURBEA: doubly–conected V–states, corotating and counter–rotating vortex pairs, non uniform rotating vortices and global bifurcation, see [27, 51, 67, 77, 82, 84]. We refer also to Chapter 2 about non uniform rotating vortices in the Euler equations.

The quasi–geostrophic shallow water equations are found in the limit of fast rotation and weak variations of the free surface in the rotating shallow water equations, see [141]. In this context, the function q is called the potential vorticity. The parameter λ is known as the inverse "Rossby deformation length", and when it is small, it corresponds to a free surface that is nearly rigid. In the case $\lambda = 0$, we recover the Euler equations. Although vortex patches solutions are better known in the Euler equations, there are also some results in the quasi–geostrophic shallow water equations. For the analogue to the Kirchhoff ellipses in these equations, we refer to the works of DRISTCHHEL, FLIERL, POLVANI and ZABUSKY [121, 122, 123]. In [56], DRISTCHEL, HMIDI and RENAULT proved the existence of V–states bifurcating from the circular patch.

In the case of the generalized surface quasi–geostrophic, q describes the potential temperature. This model has been proposed by CÓRDOBA et al. in [43] as an interpolation between the Euler equations and the surface quasi–geostrophic (SQG) equations, corresponding to $\beta = 0$ and $\beta = 1$, respectively. The mathematical analogy with the classical three–dimensional incompressible Euler equations can be found in [42]. Some works concerning V–states in the gSQG equations are [25, 26, 76, 82].

The point model for a Kármán Vortex Street consists in two infinite arrays of point vortices with opposite strength. More specifically, consider a uniformly distributed arrow of points, with same strength in every point, located in the horizontal axis, i.e., $(kl, 0)$, with $l > 0$ and $k \in \mathbb{Z}$. The other arrow contains an infinite number of points, with opposite strength, which will be parallel to the other one and with arbitrary stagger: $(a + kl, -h)$ with $a \in \mathbb{R}$ and $h \neq 0$. We refer to Figure 1.5 for a better understanding of the localization of the points. Hence, we consider the following distribution:

$$q_0(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x), \tag{3.1.2}$$

where $a \in \mathbb{R}$, $l > 0$ and $h \neq 0$. If the kernel G is radial, then we will show that every point translates in time, with the same constant speed. Moreover, if $a = 0$ or $a = \frac{l}{2}$, the translation is parallel to the real axis. In the typical case of the Newtonian interaction, that is $G = \frac{1}{2\pi}\ln|\cdot|$, the evolution of (3.1.2) has been fully studied, see [9, 102, 89, 90, 114, 120, 130]. The problem consists in a first order Hamiltonian system and the evolution of every point is given by the following system:

$$\frac{d}{dt}z_m(t) = \sum_{m\neq k\in\mathbb{Z}} \frac{(z_m(t) - z_k(t))^{\perp}}{|z_m(t) - z_k(t)|^2} - \sum_{k\in\mathbb{Z}} \frac{(z_m(t) - \tilde{z}_k(t))^{\perp}}{|z_m(t) - \tilde{z}_k(t)|^2},$$

$$\frac{d}{dt}\tilde{z}_m(t) = \sum_{k\in\mathbb{Z}} \frac{(\tilde{z}_m(t)-z_k(t))^\perp}{|\tilde{z}_m(t)-z_k(t)|^2} - \sum_{m\neq k\in\mathbb{Z}} \frac{(\tilde{z}_m(t)-\tilde{z}_k(t))^\perp}{|\tilde{z}_m(t)-\tilde{z}_k(t)|^2},$$

with initial conditions

$$\begin{aligned} z_m(0) =& ml, \\ \tilde{z}_m(0) =& a+ml-ih, \end{aligned}$$

for $m \in \mathbb{Z}$. It can be checked that the velocity at every point is the same, providing us with a translating motion. Indeed, if $a = 0$ or $a = \frac{l}{2}$, the speed can be expressed by elementary functions, where we observe that the translation is horizontal:

$$\begin{aligned} V_0 =& \frac{1}{2l}\coth\left(\frac{\pi h}{l}\right), \quad \text{for } a = 0, \\ V_0 =& \frac{1}{2l}\tanh\left(\frac{\pi h}{l}\right), \quad \text{for } a = \frac{l}{2}. \end{aligned}$$

The works of SAFFMANN and SCHATZMAN [134, 135, 136] refer to the study of these structures in the Euler equations. In fact, they consider two infinite arrows of vortices with finite area, which have uniform vorticity inside. These are two infinite arrows of vortex patches distributed as in (3.1.2). They found existence of this kind of solutions numerically and they studied their linear stability, see [134, 135, 136]. The problem can be studied not only in the Euler equations for the full space, but in the Euler equations in the periodic setting. A theory for the Euler equations in an infinite strip can be found in [16]. In [71], GRYANIK, BORTH and OLBERS studied the quasi–geostrophic Kármán Vortex Street in two–layer fluids.

The aim of this chapter is to find analitically solutions to the model proposed by SAFFMAN and SCHATZMAN in [134, 135, 136]. We will do it not only for the Euler equations, but also for other inviscid incompressible models. Here, we follow the idea of HMIDI and MATEU in [83], where they desingularize a vortex pairs obtaining a pair of vortex patches that rotate or translate (depending on the strength of the points). The plan is the following. For $\varepsilon > 0$ and $l > 0$, consider

$$q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_1+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_2+kl}(x_1,x_2), \tag{3.1.3}$$

where D_1 and D_2 are simply–connected bounded domains. In the case $|D_1| = |\mathbb{D}|$ and $D_2 = -D_1 + a - ih$, with $a = 0$ or $a = \frac{l}{2}$, and $h \neq 0$, we find the Kármán Vortex Street (3.1.2) in the limit $\varepsilon \to 0$. This means

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_1+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D_1+a+kl-ih}(x_1,x_2)\right\} = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x),$$

in the distribution sense. Then, we connect the vortex patch model (3.1.3) with the point vortex model (3.1.2). In the following, we will refer to Kármán Vortex Street or Kármán Point Vortex Street when having the point vortex model (3.1.2). Otherwise, we denote by Kármán Vortex Patch Street in the case of (3.1.3). Some relation between the two domains is needed, and the suitable one is mentioned before: $D_2 = -D_1 + a - ih$. Assuming that $q(t,x) = q_0(x - Vt)$, for some $V \in \mathbb{R}$, and using a conformal map $\Phi : \mathbb{T} \to \partial D_1$, we arrive at the following equation

$$F(\varepsilon, f, V)(w) := \text{Re}\left[\left\{\overline{I(\varepsilon,f)(w)} - \overline{V}\right\} w\Phi'(w)\right] = 0, \quad w \in \mathbb{T},$$

where

$$
\begin{aligned}
I(\varepsilon, f)(w) := &-\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi \\
&-\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi.
\end{aligned}
$$

Let us explain the meaning of f. Assume that the conformal map is a perturbation of the identity in the following way

$$
\Phi(w) = i\left(w+\varepsilon f(w)\right), \quad f(w)=\sum_{n\geq 1} a_n w^{-n}, \quad a_n\in\mathbb{R}, w\in\mathbb{T}. \tag{3.1.4}
$$

for $G=\frac{1}{2\pi}\ln|\cdot|$ and $G=-K_0(|\lambda||\cdot|)$. Whereas, it will be consider as

$$
\Phi(w) = i\left(w+\frac{\varepsilon}{G(\varepsilon)}f(w)\right), \quad f(w)=\sum_{n\geq 1} a_n w^{-n}, \quad a_n\in\mathbb{R}, w\in\mathbb{T}. \tag{3.1.5}
$$

for more singular kernels, such as $G=\frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\beta\in(0,1)$. Moreover, we have that $F(0,0,V_0)(w)=0$, for any $w\in\mathbb{T}$. Here, V_0 is the speed of the point model (3.1.2). The nonlinear functional F is well-defined from $\mathbb{R}\times X_\alpha\times\mathbb{R}$ to $\tilde{Y}_\alpha$, where

$$
X_\alpha = \left\{ f\in\mathscr{C}^{1,\alpha}(\mathbb{T}), \quad f(w)=\sum_{n\geq 1} a_n w^{-n},\, a_n\in\mathbb{R} \right\},
$$

$$
\tilde{Y}_\alpha = \left\{ f\in\mathscr{C}^{0,\alpha}(\mathbb{T}), \quad f(e^{i\theta})=\sum_{n\geq 1} a_n \sin(n\theta),\, a_n\in\mathbb{R} \right\},
$$

for some $\alpha\in(0,1)$. However, $\partial_f F(0,0,V)$ is not an isomorphism in these spaces. Defining V as a function of ε and f in the following way

$$
V(\varepsilon,f) = \frac{\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw}{\int_{\mathbb{T}} w\Phi'(w)(1-\overline{w}^2)dw}, \tag{3.1.6}
$$

then, F is also well-defined from $\mathbb{R}\times X_\alpha$ to Y_α, where

$$
Y_\alpha = \left\{ f\in\mathscr{C}^{0,\alpha}(\mathbb{T}), \quad f(e^{i\theta})=\sum_{n\geq 2} a_n \sin(n\theta),\, a_n\in\mathbb{R} \right\}.
$$

Indeed, $\partial_f F(0,0,V)$ is an isomorphism in these spaces. Then, we fix the velocity V depending on ε and f as in (3.1.6). In this way, the Implicit Function theorem can be applied in order to desingularize the point model (3.1.2) obtaining our main result:

Theorem 3.1.1. *Consider $G=\frac{1}{2\pi}\ln|\cdot|$, $G=-K_0(|\lambda||\cdot|)$, or $G=\frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\lambda\neq 0$ and $\beta\in(0,1)$. Let $h,l\in\mathbb{R}$, with $h\neq 0$ and $l>0$, and $a=0$ or $a=\frac{l}{2}$. Then, there exists a $\mathscr{C}^1$ simply–connected bounded domain D^ε such that*

$$
q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}}\mathbf{1}_{\varepsilon D^\varepsilon+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}}\mathbf{1}_{-\varepsilon D^\varepsilon+a-ih+kl}(x),
$$

defines a horizontal translating solution of (3.1.1), *with constant speed, for any $\varepsilon\in(0,\varepsilon_0)$ and small enough $\varepsilon_0>0$.*

The proof of the theorem has to be adapted to each case. For the cases $G = \frac{1}{2\pi}\ln|\cdot|$ and $G = -K_0(|\lambda||\cdot|)$, the kernel has a logarithm singularity at the origin, which will play an important role in the choice of the scaling of the conformal map (3.1.4). Otherwise, if $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\beta \in (0,1)$, or for more general kernels, the scaling of the conformal map will depend on the singularity at the origin as depicted in (3.1.5).

This chapter is organized as follows. In Section 2, we introduce some preliminary results about the N–vortex problem and, in particular, the point model for the Kármán Vortex Street (3.1.2) with general interactions. Section 3.3 refers to the Euler and QGSW equations, where the singularity of the kernel is logarithmic. We will deal with the general case in Section 3.4, following the same ideas as in the Euler equations. The gSQG equations will become a particular case of this study. Finally, Appendix C and Appendix B will be devoted to provide some definitions and properties concerning special functions and complex integrals.

Let us end this part by summarizing some notation to be used along the chapter. We will denote the unit disc by $\mathbb{D}$ and its boundary by $\mathbb{T}$. The integral

$$\int_{\mathbb{T}} f(\xi)\, d\xi,$$

denotes the usual complex integral, for some complex function f. Moreover, we will use the symmetry sums defined by

$$\sum_{k\in\mathbb{Z}} a_k = \lim_{K\to+\infty}\sum_{|k|\leq K} a_k, \tag{3.1.7}$$

except being specified.

3.2 The N–vortex problem

The N–vortex problem consists in the study of the evolution of a set of points that interact according to some laws. Originally, the Newtonian interaction for the N–vortex problem is considered. But here, we start by assuming that the interaction between the points is due to a general function $G:\mathbb{R}\to\mathbb{R}$, which is smooth off zero. The problem is a first order Hamiltonian system of the form

$$\frac{d}{dt}z_m(t) = \sum_{\substack{k=1\\k\neq m}}^{N}\Gamma_k\nabla^{\perp}_{z_m}G(|z_m(t)-z_k(t)|), \tag{3.2.1}$$

with some initial conditions at $t=0$ and where $m=1,\ldots,N$. Moreover, $z_k\neq z_m$ if $k\neq m$, are N–points located in the plane $\mathbb{R}^2$. The constants Γ_k mean the strength of each point and the interaction between them is due to G. In the case that $G=\frac{1}{2\pi}\ln|\cdot|$, we arrive at the typical N–vortex problem:

$$\frac{d}{dt}z_m(t) = \sum_{\substack{k=1\\k\neq m}}^{N}\Gamma_k\frac{(z_m(t)-z_k(t))^{\perp}}{|z_m(t)-z_k(t)|^2}.$$

First, we deal with the evolution of two points. We can check that they rotate or translate depending on their strength: Γ_1 and Γ_2. Later, we will move on the configuration that we concern: an infinite number of points with a periodic pattern in space. For more details about the N–vortex problem in the case of the Newtonian interaction see [114].

3.2.1 Finite configurations

There are some stable situations yielding to steady configurations. Here, we illustrate the evolution of 2–points since the same idea is used later in the case of periodic configurations.

Proposition 3.2.1. *Let us consider two initial points $z_1(0)$ and $z_2(0)$, with $z_1(0) \neq z_2(0)$, located in the real axis, with strengths Γ_1 and Γ_2 respectively. We have:*

1. *If $\Gamma_1 + \Gamma_2 \neq 0$ and $\Gamma_1 z_1(0) + \Gamma_2 z_2(0) = 0$, then the solution behaves as $z_k(t) = e^{i\Omega t} z_k(0)$, for $k = 1, 2$, with $\Omega = \frac{(\Gamma_1+\Gamma_2)G'(|z_1(0)-z_2(0)|)}{|z_1(0)-z_2(0)|}$.*
2. *If $\Gamma_1+\Gamma_2 = 0$, then $z_k(t) = z_k(0)+Ut$, for $k = 1, 2$, with $U = i\Gamma_2 G'(|z_1(0)-z_2(0)|)\text{sign}(z_1(0)-z_2(0))$.*

Proof. According to (3.2.1), the evolution of the two points is given by the following system:

$$\begin{cases} \frac{d}{dt} z_1(t) = i\Gamma_2 G'(|z_1(t) - z_2(t)|)\frac{z_1(t)-z_2(t)}{|z_1(t)-z_2(t)|}, \\ \frac{d}{dt} z_2(t) = -i\Gamma_1 G'(|z_1(t) - z_2(t)|)\frac{z_1(t)-z_2(t)}{|z_1(t)-z_2(t)|}. \end{cases} \tag{3.2.2}$$

(1) By the above system, it is clear that $\frac{d}{dt}(\Gamma_1 z_1(t) + \Gamma_2 z_2(t)) = 0$, which implies that

$$\Gamma_1 z_1(t) + \Gamma_2 z_2(t) = \Gamma_1 z_1(0) + \Gamma_2 z_2(0) = 0.$$

Assuming that $z_1(t) = e^{i\Omega t} z_1(0)$, and using the above equation, one arrives at $z_2(t) = e^{i\Omega t} z_2(0)$. Then, the system (3.2.2) yields

$$\begin{cases} i\Omega e^{i\Omega t} z_1(0) = ie^{i\Omega t}\Gamma_2 G'(|z_1(0) - z_2(0)|)\frac{z_1(0)-z_2(0)}{|z_1(0)-z_2(0)|}, \\ i\Omega e^{i\Omega t} z_2(0) = -ie^{i\Omega t}\Gamma_1 G'(|z_1(0) - z_2(0)|)\frac{z_1(0)-z_2(0)}{|z_1(0)-z_2(0)|}. \end{cases}$$

Since $z_1(0)$ and $z_2(0)$ are located in the real axis, one has that $z_1(0) - z_2(0) \in \mathbb{R}$, and then subtracting the above two equations amounts to

$$\Omega = \frac{(\Gamma_1 + \Gamma_2)G'(|z_1(0) - z_2(0)|)}{|z_1(0) - z_2(0)|}.$$

(2) In this case, we have that $\frac{d}{dt}(z_1(t) - z_2(t)) = 0$, and thus

$$z_1(t) - z_2(t) = z_1(0) - z_2(0).$$

As a consequence, (3.2.2) agrees with

$$\begin{cases} \frac{d}{dt} z_1(t) = i\Gamma_2 G'(|z_1(0) - z_2(0)|)\text{sign}(z_1(0) - z_2(0)), \\ \frac{d}{dt} z_2(t) = i\Gamma_2 G'(|z_1(0) - z_2(0)|)\text{sign}(z_1(0) - z_2(0)), \end{cases}$$

which can be solved as

$$\begin{cases} z_1(t) = z_1(0) + i\Gamma_2 G'(|z_1(0) - z_2(0)|)\text{sign}(z_1(0) - z_2(0))t, \\ z_2(t) = z_2(0) + i\Gamma_2 G'(|z_1(0) - z_2(0)|)\text{sign}(z_1(0) - z_2(0))t. \end{cases}$$

□

The above result gives us that two vortex points with $\Gamma_1+\Gamma_2 \neq 0$, have a rotating evolution. Otherwise, they *translate*. From now on, we refer that a structure *translates* when the evolution of every point (or every patch, in the case of (3.1.1)) is a translation, with the same constant speed.

In the usual N–vortex problem, meaning $G = \frac{1}{2\pi}\ln|\cdot|$, the above result is well–known. Here, we have seen that if the interaction of the points is due to a kernel that is radial, we get the same evolution.

3.2.2 Periodic setting

This section deals with the evolution of two infinite arrays of points with opposite strength, which are periodic in space. More specifically, the points of the first arrow, which have the same strength, will be located in the horizontal axis. Let us assume that we have one point at the origin, and the next one differs of it a distance l. This means that we have the points $(kl, 0)$, for $l > 0$ and $k \in \mathbb{Z}$. The second arrow, with opposite strength to the previous arrow, will be parallel to the horizontal axis but with a height $h \neq 0$, having the following distribution of points: $(a + kl, -h)$, for $a \in \mathbb{R}$ and $k \in \mathbb{Z}$. For the moment, let us consider that a is any real number.

Then, we focus on

$$q(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x), \tag{3.2.3}$$

with $h \neq 0$, $l > 0$ and $a \in \mathbb{R}$. In the following results, we check that the above initial configuration translate when $G = \frac{1}{2\pi}\ln|\cdot|$, $G = -K_0(|\lambda||\cdot|)$, $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$ for $\beta \in (0,1)$, or for G satisfying some general conditions. Moreover, if $a = 0$ or $a = \frac{l}{2}$, the translation is horizontal.

We are going to differentiate two cases depending on the behavior of the interaction G at infinity. This is important in order to give a meaning to the infinite sum coming from (3.2.3), whose equations are given by

$$\begin{aligned}
\frac{d}{dt}z_m(t) &= \sum_{m\neq k\in\mathbb{Z}} G'(|z_m(t)-z_k(t)|)\frac{(z_m(t)-z_k(t))^\perp}{|z_m(t)-z_k(t)|} - \sum_{k\in\mathbb{Z}} G'(|z_m(t)-\tilde{z}_k(t)|)\frac{(z_m(t)-\tilde{z}_k(t))^\perp}{|z_m(t)-\tilde{z}_k(t)|},\\
\frac{d}{dt}\tilde{z}_m(t) &= \sum_{k\in\mathbb{Z}} G'(|\tilde{z}_m(t)-z_k(t)|)\frac{(\tilde{z}_m(t)-z_k(t))^\perp}{|\tilde{z}_m(t)-z_k(t)|} - \sum_{m\neq k\in\mathbb{Z}} G'(|\tilde{z}_m(t)-\tilde{z}_k(t)|)\frac{(\tilde{z}_m(t)-\tilde{z}_k(t))^\perp}{|\tilde{z}_m(t)-\tilde{z}_k(t)|},
\end{aligned}$$

with initial conditions

$$\begin{aligned}
z_m(0) =& ml,\\
\tilde{z}_m(0) =& a + ml - ih,
\end{aligned}$$

for $m \in \mathbb{Z}$. Then, we refer to the critical case in the case of the Newtonian interaction

$$G = \frac{1}{2\pi}\ln|\cdot|.$$

Here, we must use the structure of the logarithm in order to have a convergence sum. Note that here we need to use strongly the symmetry sum. Otherwise, the subcritical cases will use the faster decay of G at infinity as it is the case of the QGSW or gSQG interactions.

• *Critical case:* Let us first show the result for the Newtonian interaction, that is, $G = \frac{1}{2\pi}\ln|\cdot|$.

Here, we denote ω to q to emphasize that we are working with the vorticity.

Proposition 3.2.2. *Given the point vortex street* (3.2.3) *with* $G = \frac{1}{2\pi}\ln|\cdot|$, *for* $h \neq 0$, $l > 0$ *and* $a \in \mathbb{R}$, *then the street is moving with the following constant velocity speed*

$$V_0 = \frac{1}{2li}\overline{\cot\left(\frac{\pi(ih-a)}{l}\right)}. \tag{3.2.4}$$

In the case that $a = 0$ *or* $a = \frac{l}{2}$, *the translation is parallel to the horizontal axis with velocity*

$$V_0 = \frac{1}{2l}\coth\left(\frac{\pi h}{l}\right), \quad \text{for } a = 0,$$

$$V_0 = \frac{1}{2l} \tanh\left(\frac{\pi h}{l}\right), \quad \text{for } a = \frac{l}{2}.$$

Remark 3.2.3. *If $\omega_\kappa(x,y) = \kappa\omega(x,y)$, with $\kappa \in \mathbb{R}$ and ω given by* (3.2.3)*, then the velocity of the street is $V_{0,\kappa} = \kappa V_0$.*

Proof. Define

$$\omega_K(x) = \sum_{|k|\leq K} \delta_{(kl,0)}(x) - \sum_{|k|\leq K} \delta_{(a+kl,-h)}(x).$$

The idea is to consider $K \to +\infty$, getting $\omega_K \to \omega$ in the distribution sense. The associated stream function to ω_K is given by

$$\psi_K(x) = \frac{1}{2\pi} \sum_{|k|\leq K} \ln|x - kl| - \frac{1}{2\pi} \sum_{|k|\leq K} \ln|x - a - kl + ih|, \tag{3.2.5}$$

where we are using complex notation. In order to pass to the limit, we need to use the structure of the logarithm. Let us work with the sum in the following way

$$\begin{aligned}
\sum_{|k|\leq K} \ln|x - a - kl + ih| &= \ln\left|\prod_{|k|\leq K} (x - a - kl + ih)\right| \\
&= \ln\left|(x - a + ih) \prod_{k=1}^{K} \left((x - a + ih)^2 - k^2 l^2\right)\right| \\
&= \ln\left|\frac{\pi(x - a + ih)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x - a + ih)^2}{k^2 l^2}\right)\right| + \ln\left|\frac{l}{\pi} \prod_{k=1}^{K} k^2 l^2\right|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lim_{K\to\infty} \psi_K(x) = \lim_{K\to\infty} \Bigg\{ &\frac{1}{2\pi} \ln\left|\frac{\pi x}{l} \prod_{k=1}^{K} \left(1 - \frac{x^2}{k^2 l^2}\right)\right| \\
&- \frac{1}{2\pi} \ln\left|\frac{\pi(x - a + ih)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x - a + ih)^2}{k^2 l^2}\right)\right| \Bigg\}.
\end{aligned}$$

Using the product expression for the sine, that is

$$\sin(\pi x) = \pi x \prod_{k\geq 1} \left(1 - \frac{x^2}{k^2}\right), \tag{3.2.6}$$

we get that

$$\psi(x) := \lim_{K\to\infty} \psi_K(x) = \frac{1}{2\pi} \ln\left|\sin\left(\frac{\pi x}{l}\right)\right| - \frac{1}{2\pi} \ln\left|\sin\left(\frac{\pi(x - a + ih)}{l}\right)\right|.$$

In this way, we achieve that the sum in (3.2.5) converges. In the same way, the corresponding velocity agrees with

$$v_K(x) = \frac{i}{2\pi} \sum_{|k|\leq K} \frac{x - kl}{|x - kl|^2} - \frac{i}{2\pi} \sum_{|k|\leq K} \frac{x - a - kl + ih}{|x - a - kl + ih|^2},$$

where x is none of the points vortex. In each of the points, the velocity is given by

$$\begin{aligned} v_K(ml) =& \frac{i}{2\pi} \sum_{k\neq m, |k|\leq K} \frac{ml-kl}{|ml-kl|^2} - \frac{i}{2\pi} \sum_{|k|\leq K} \frac{ml-a-kl+ih}{ml-a-kl+ih|^2}, \\ v_K(a-ih+ml) =& \frac{i}{2\pi} \sum_{|k|\leq K} \frac{a-ih+ml-kl}{|x-kl|^2} - \frac{i}{2\pi} \sum_{k\neq m, |k|\leq K} \frac{ml-kl}{|x-a-kl+ih|^2}, \end{aligned}$$

for any $m \in \mathbb{Z}$. We define v as the limit of the above function.

First, let us show that the velocity at every point is the same. We begin with the first arrow

$$\begin{aligned} v(ml) =& \frac{i}{2\pi} \sum_{m\neq k\in\mathbb{Z}} \frac{ml-kl}{|ml-kl|^2} - \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{ml-a-kl+ih}{|ml-a-kl+ih|^2} \\ =& -\frac{i}{2\pi} \sum_{0\neq k\in\mathbb{Z}} \frac{kl}{|kl|^2} + \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a+kl-ih}{|a+kl-ih|^2} \\ =& \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a+kl-ih}{|a+kl-ih|^2} \\ =& v(0). \end{aligned}$$

Note that

$$\sum_{0\neq k\in\mathbb{Z}} \frac{kl}{|kl|^2} = 0,$$

since we are using the symmetry sum (3.1.7). For the second arrow, we obtain

$$\begin{aligned} v(a-ih+ml) =& \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a-ih+ml-kl}{|a-ih+ml-kl|^2} - \frac{i}{2\pi} \sum_{m\neq k\in\mathbb{Z}} \frac{ml-kl}{|ml-kl|^2} \\ =& \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a+kl-ih}{|a+kl-ih|^2} - \frac{i}{2\pi} \sum_{0\neq k\in\mathbb{Z}} \frac{kl}{|kl|^2} \\ =& v(0). \end{aligned}$$

Then, the velocity speed of the street is given by

$$V_0 = \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a+kl-ih}{|a+kl-ih|^2} - \frac{i}{2\pi} \sum_{0\neq k\in\mathbb{Z}} \frac{kl}{|kl|^2} = \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \frac{a+kl-ih}{|a+kl-ih|^2}.$$

In order to find a better expression for V, let us come back to the stream function. Using

$$\nabla \ln|\sin(x)| = \frac{\sin x_1 \cos x_1 + i \sinh x_2 \cosh x_2}{|\sin x|^2} = \overline{\cot x},$$

we achieve

$$V_0 = \frac{1}{2li} \overline{\cot\left(\frac{\pi(ih-a)}{l}\right)}.$$

Let us now work with $a = \frac{l}{2}$. Using the definition of the complex cotangent, we obtain

$$\cot\left(\frac{\pi(ih-\frac{l}{2})}{l}\right) = -i\frac{\sinh\left(\frac{\pi h}{l}\right)}{\cosh\left(\frac{\pi h}{l}\right)} = -i\tanh\left(\frac{\pi h}{l}\right),$$

which is the announced expression for the velocity. The same idea can be applied to get the expression when $a = 0$. □

For the cases $a = 0$ and $a = \frac{l}{2}$, we notice that the velocity increases as h goes to 0. Moreover, considering $a = \frac{l}{2}$ and $h \to 0$ in the above proposition, one obtains the following corollary.

Corollary 3.2.4. *The vortex arrow given by*

$$\omega(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(\frac{l}{2}+kl,0)}(x),$$

is stationary, for any $l > 0$.

Similar ideas can be applied to find that a horizontal arrow of points with the same strength is stationary.

Proposition 3.2.5. *The vortex arrow given by*

$$\omega(x) = \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x),$$

is stationary, for any $a \in \mathbb{R}$ and $h \in \mathbb{R}$.

• *Subcritical case:* We finish this section by showing the result for faster decays interactions. This case will cover the QGSW and gSQG interactions: $G = -K_0(|\lambda||\cdot|)$ or $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$ for $\beta \in (0,1)$. The result reads as follows.

Proposition 3.2.6. *Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a smooth off zero function satisfying*

(H1) G is radial such that $G(x) = \tilde{G}(|x|)$,

(H2) there exists $R > 0$ and $\beta_1 \in (0,1]$ such that $|\tilde{G}'(r)| \le \frac{C}{r^{1+\beta_1}}$, for $r \ge R$.

Then,

$$q(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x), \tag{3.2.7}$$

with $h \neq 0$, $l > 0$ and $a \in \mathbb{R}$, translates with constant velocity speed

$$V_0 = i \sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}. \tag{3.2.8}$$

In the case $a = 0$ or $a = \frac{l}{1}$, the translation is parallel to the horizontal axis.

Remark 3.2.7. *From now on, we will assume that G is radial via (H1), we will write G for $\tilde{G}$ when there is no confusion in order to simplify notation.*

Remark 3.2.8. *The second hypothesis is required to give a meaning to the infinite sum, which converges absolutely. This condition could be weakened by assuming*

$$\sum_{k\in\mathbb{Z}} \left|G'(a+kl-ih)\right| < +\infty.$$

Proof. As in the previous models, the velocity at the points is given by

$$-iv(ml) = \sum_{m\neq k\in\mathbb{Z}} G'(|ml-kl|)\frac{ml-kl}{|ml-kl|}$$

$$
\begin{aligned}
&- \sum_{k\in\mathbb{Z}} G'(|ml-a-kl+ih|)\frac{ml-a-kl+ih}{|ml-a-kl+ih|},\\
-iv(a+ml-ih) =& \sum_{k\in\mathbb{Z}} G'(|a+ml-ih-kl|)\frac{a+ml-ih-kl}{|a+ml-ih-kl|}\\
&- \sum_{m\neq k\in\mathbb{Z}} G'(|ml-kl|)\frac{ml-kl}{|ml-kl|},
\end{aligned}
$$

for $m \in \mathbb{Z}$. The above sums are converging due to the second assumption. We can check that the velocity is the same at every point of the street:

$$
\begin{aligned}
-iv(ml) =& \sum_{m\neq k\in\mathbb{Z}} G'(|ml-kl|)\frac{ml-kl}{|ml-kl|}\\
&- \sum_{k\in\mathbb{Z}} G'(|ml-a-kl+ih|)\frac{ml-a-kl+ih}{|ml-a-kl+ih|}\\
=& \sum_{0\neq k\in\mathbb{Z}} G'(|kl|)\frac{kl}{|kl|} + \sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}\\
=& \sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}\\
=& - iv(0),\\
-iv(a+ml-ih) =& \sum_{k\in\mathbb{Z}} G'(|a+ml-ih-kl|)\frac{a+ml-ih-kl}{|a+ml-ih-kl|}\\
&- \sum_{m\neq k\in\mathbb{Z}} G'(|ml-kl|)\frac{ml-kl}{|ml-kl|}\\
=& \sum_{k\in\mathbb{Z}} G'(|a-ih-kl|)\frac{a-ih-kl}{|a-ih-kl|} - \sum_{0\neq k\in\mathbb{Z}} G'(|kl|)\frac{kl}{|kl|}\\
=& - iv(0).
\end{aligned}
$$

Then,

$$
V_0 = v(0) = i\sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}.
$$

If $a = 0$ or $a = \frac{l}{2}$, one has that

$$
\sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl}{|a+kl-ih|} = 0,
$$

and the translation is in the horizontal direction. □

In the general case, we also have that an array is stationary.

Proposition 3.2.9. *Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a smooth off zero function satisfying (H1)-(H2), then*

$$
q(x) = \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x),
$$

is stationary for any $a \in \mathbb{R}$ and $h \in \mathbb{R}$.

It is clear that $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\beta \in (0,1)$, satisfies the hypothesis of the above results. In the case of the QGSW interaction, we obtain similar results. In this case, the stream function associated to (3.2.3) is given by

$$\psi(x) = -\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} K_0(\lambda|x-kl|) + \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} K_0(\lambda|x-a-kl+ih|). \tag{3.2.9}$$

The definition and some properties of the Bessel functions can be found in Appendix C. The above sum is convergent due to the behavior of K_0 at infinity, which is exponential:

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad |\arg(z)| < \frac{3}{2}\pi.$$

There is another representation of the stream function given in [71], where the periodicity structure is emphasized:

$$\begin{aligned}\psi(x_1,x_2) =& -\frac{1}{a}\sum_{k\in\mathbb{Z}} \frac{\exp\left(-\sqrt{\left(\frac{2\pi k}{a}\right)^2+\lambda^2}|x_2|\right)}{\sqrt{\left(\frac{2\pi k}{a}\right)^2+\lambda^2}}\cos\left(\frac{2\pi k}{a}x_1\right)\\ &+\frac{1}{a}\sum_{k\in\mathbb{Z}} \frac{\exp\left(-\sqrt{\left(\frac{2\pi k}{a}\right)^2+\lambda^2}|x_2+h|\right)}{\sqrt{\left(\frac{2\pi k}{a}\right)^2+\lambda^2}}\cos\left(\frac{2\pi k}{a}(x_1-a)\right).\end{aligned} \tag{3.2.10}$$

Then, we state the result concerning the QGSW interaction.

Proposition 3.2.10. *Given the point vortex street* (3.2.3)*, with $h \neq 0$, $l > 0$ and $a \in \mathbb{R}$, then the street translates with the following constant velocity speed*

$$V_0 = \frac{\lambda i}{2\pi}\sum_{k\in\mathbb{Z}} K_1(\lambda|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}. \tag{3.2.11}$$

In the case that $a = 0$ or $a = \frac{l}{2}$, the translation is parallel to the horizontal axis.

3.3 Periodic patterns in the Euler and QGSW equations

This section is devoted to show the full construction of the Kármán Vortex Street structures in the Euler equations. Instead of considering two arrows of points as in Section 3.2, we consider two infinite arrows of patches distributed in the same way than the arrows of points (3.2.3). We will refer to this configuration in the Euler equations as Kármán Vortex Patch Street.

In the case of arrows of points, we showed in the last section that they translate. Here, we want to find a similar evolution in the Euler equations. Since these structures are periodic is space, first we will have to look for the green function associated to the $-\Delta$ operator in $\mathbb{T}\times\mathbb{R}$, which will come as an infinite sum of functions. This infinite sum can be expressed in terms of elementary functions, which helps us in the computations. Once we have the equation that will characterize the Kármán Vortex Patch Street, we will have to deal with the Implicit Function theorem. Hence, a desingularization of the Kármán Point Vortex Street will show the existence of these structures in terms of finite area domains that translate.

At the end of this section, we will analyze the case of the QGSW equations, which will follow similarly. Let us focus now in the Euler equations:

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ v = K * \omega, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ \omega(t = 0, x) = \omega_0(x), & \text{with } x \in \mathbb{R}^2. \end{cases}$$

The second equation links the velocity to the vorticity through the Biot–Savart law, where $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$ and $x^\perp = (-x_2, x_1)$. We denote by ψ the stream function, which verifies $v = \nabla^\perp \psi$.

From now on we will use complex notation in order to simplify the computations. Then, we identify $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 + ix_2 \in \mathbb{C}$. In the same way, $x^\perp = ix$. Moreover, the gradient operator in $\mathbb{R}^2$ can be identified with the Wirtinger derivative, i.e.,

$$\nabla = 2\partial_{\overline{z}}, \quad \partial_{\overline{z}}\varphi(z) := \frac{1}{2}\left(\partial_1\varphi(z) + i\partial_2\varphi(z)\right), \tag{3.3.1}$$

for a complex function φ.

3.3.1 Velocity of the Kármán Vortex Patch Street

Consider the initial condition given by

$$\begin{aligned} \omega_0(x_1, x_2) =& \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_1}(x_1 - kl, x_2) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_2}(x_1 - kl, x_2) \\ =& \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_1+kl}(x_1, x_2) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_2+kl}(x_1, x_2), \end{aligned} \tag{3.3.2}$$

where D_1 and D_2 are simply–connected bounded domains such that $|D_1| = |D_2|$, and $l > 0$.

The velocity field can be computed through the Biot–Savart law in $\mathbb{T} \times \mathbb{R}$. For that, one must find the Green function associated to the $-\Delta$ operator in order to have an expression for the stream function ψ. Later, we just use that $v = \nabla^\perp \psi$, or with the complex notation, $v = 2i\partial_{\overline{z}}\psi$. This will be developed in the next result, obtaining different expressions for the velocity, which will be useful later.

Proposition 3.3.1. *The velocity field of the Euler equations associated to* (3.3.2) *is given by the following expressions:*

1.

$$v_0(x) = -\frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|\sin\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi + \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|\sin\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi.$$

2.

$$v_0(x) = \frac{i}{2l\pi}\overline{\int_{D_1} \cot\left[\frac{\pi(x-y)}{l}\right] dA(y)} - \frac{i}{2l\pi}\overline{\int_{D_2} \cot\left[\frac{\pi(x-y)}{l}\right] dA(y)}.$$

3.

$$\begin{aligned} v_0(x) =& \frac{1}{4\pi^2}\overline{\int_{\partial D_1} \frac{\overline{x} - \overline{\xi}}{x - \xi}\, d\xi} - \frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi \\ &- \frac{1}{4\pi^2}\overline{\int_{\partial D_2} \frac{\overline{x} - \overline{\xi}}{x - \xi}\, d\xi} - \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi, \end{aligned}$$

with

$$H(z) = 1 + \sum_{k\geq 1} \frac{(-1)^k}{(2k+1)!} z^{2k} = \frac{\sin(z)}{z}. \tag{3.3.3}$$

Proof. (1) Let us begin finding the stream function associated to

$$\omega_{0,K}(x_1,x_2) = \frac{1}{\pi} \sum_{|k|\leq K} \mathbf{1}_{D_1+kl}(x_1,x_2) - \frac{1}{\pi} \sum_{|k|\leq K} \mathbf{1}_{D_2+kl}(x_1,x_2),$$

by superposing the stream function of each one of the elements of the sum, i.e.,

$$\psi_{0,K}(x) = \frac{1}{2\pi^2} \sum_{|k|\leq K} \int_{D_1} \ln|x-y-kl|\, dA(y) - \frac{1}{2\pi^2} \sum_{|k|\leq K} \int_{D_2} \ln|x-y-kl|\, dA(y). \tag{3.3.4}$$

Using the same idea than in Proposition 3.2.2, we find that

$$\sum_{|k|\leq K} \ln|x-y-kl| = \ln\left|\frac{\pi(x-y)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x-y)^2}{k^2 l^2}\right)\right| + \ln\left|\frac{l}{\pi} \prod_{k=1}^{K} k^2 l^2\right|,$$

and hence

$$\begin{aligned}\psi_{0,K}(x) =& \frac{1}{2\pi^2} \int_{D_1} \ln\left|\frac{\pi(x-y)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x-y)^2}{k^2 l^2}\right)\right| dA(y) + \frac{1}{2\pi^2} \ln\left|\frac{l}{\pi} \prod_{k=1}^{K} k^2 l^2\right| |D_1| \\ &- \frac{1}{2\pi^2} \int_{D_2} \ln\left|\frac{\pi(x-y)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x-y)^2}{k^2 l^2}\right)\right| dA(y) - \frac{1}{2\pi^2} \ln\left|\frac{l}{\pi} \prod_{k=1}^{K} k^2 l^2\right| |D_2|.\end{aligned}$$

Using that D_1 and D_2 have same area, it follows that

$$\begin{aligned}\psi_{0,K}(x) =& \frac{1}{2\pi^2} \int_{D_1} \ln\left|\frac{\pi(x-y)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x-y)^2}{k^2 l^2}\right)\right| dA(y) \\ &- \frac{1}{2\pi^2} \int_{D_2} \ln\left|\frac{\pi(x-y)}{l} \prod_{k=1}^{K} \left(1 - \frac{(x-y)^2}{k^2 l^2}\right)\right| dA(y),\end{aligned}$$

where the sine formula (3.2.6) yields

$$\psi_0(x) = \frac{1}{2\pi^2} \int_{D_1} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y) - \frac{1}{2\pi^2} \int_{D_2} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y).$$

Then,

$$\begin{aligned}v_0(x) =& \frac{i\partial_{\overline{x}}}{\pi^2} \int_{D_1} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y) - \frac{i\partial_{\overline{x}}}{\pi^2} \int_{D_2} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y) \\ =& -\frac{1}{\pi^2} \int_{D_1} i\partial_{\overline{y}} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y) + \frac{1}{\pi^2} \int_{D_2} i\partial_{\overline{y}} \ln\left|\sin\left(\frac{\pi(x-y)}{l}\right)\right| dA(y) \\ =& -\frac{1}{2\pi^2} \int_{\partial D_1} \ln\left|\sin\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi + \frac{1}{2\pi^2} \int_{\partial D_2} \ln\left|\sin\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi.\end{aligned} \tag{3.3.5}$$

The Stokes theorem, see Appendix B, has been applied in the last line.

(2) This expression comes from (3.3.5) and

$$2\partial_{\overline{x}} \ln|\sin(x)| = \overline{\cot x},$$

used in Proposition 3.2.2.

(3) From (1), we can use the series expansion of the complex sine,

$$\sin(z) = zH(z), \quad H(z) = 1 + \sum_{k\geq 1} \frac{(-1)^k}{(2k+1)!} z^{2k},$$

in order to obtain

$$\begin{aligned}\ln\left|\sin\left(\frac{\pi(x-\xi)}{l}\right)\right| &= \ln\left|\frac{\pi(x-\xi)}{l}\right| + \ln\left|1 + \sum_{k\geq 1} \frac{(-1)^k}{(2k+1)!}\frac{\pi^{2k}}{l^{2k}}(x-\xi)^{2k}\right| \\ &= \ln\left|\frac{\pi(x-\xi)}{l}\right| + \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right|.\end{aligned}$$

Then, we have

$$\begin{aligned}v_0(x) = &-\frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|\frac{\pi(x-\xi)}{l}\right| d\xi - \frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi \\ &+ \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|\frac{\pi(x-\xi)}{l}\right| d\xi + \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi.\end{aligned}$$

Moreover, the Stokes formula (B.0.9) yields

$$\begin{aligned}v_0(x) = &\frac{i}{2\pi^2}\overline{\int_{D_1} \frac{1}{x-y}\, dA(y)} - \frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi \\ &- \frac{i}{2\pi^2}\overline{\int_{D_2} \frac{1}{x-y}\, dA(y)} + \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi.\end{aligned}$$

Finally, let us now use the Cauchy–Pompeiu's formula (B.0.10) for the first and third terms, to find

$$\begin{aligned}v_0(x) = &\frac{1}{4\pi^2}\overline{\int_{\partial D_1} \frac{\overline{x}-\overline{\xi}}{x-\xi}\, d\xi} - \frac{1}{2\pi^2}\int_{\partial D_1} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi \\ &- \frac{1}{4\pi^2}\overline{\int_{\partial D_2} \frac{\overline{x}-\overline{\xi}}{x-\xi}\, d\xi} - \frac{1}{2\pi^2}\int_{\partial D_2} \ln\left|H\left(\frac{\pi(x-\xi)}{l}\right)\right| d\xi.\end{aligned}$$

□

3.3.2 Functional setting of the problem

The first step is to scale the vorticity (3.3.2) in order to introduce the point vortices in our formulation and be able to desingularize them. Let us define

$$\omega_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_1 + kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D_2 + kl}(x), \tag{3.3.6}$$

for $l > 0$ and $\varepsilon > 0$. The domains D_1 and D_2 are simply–connected and bounded. In the case that $|D_1| = |\mathbb{D}|$ and $D_2 = -D_1 + a - ih$, with $a \in \mathbb{R}$ and $h \neq 0$, we find the point vortex street (3.2.3) passing to the limit $\varepsilon \to 0$:

$$\omega_{0,0}(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x). \tag{3.3.7}$$

Proposition 3.2.2 deals with (3.3.7) from the dynamical system point of view, showing that it translates. Moreover, if $a = 0$ or $a = \frac{l}{2}$, the translation is horizontal.

Now, we try to find the equation that characterize a translating evolution in the Euler equations. Assume that we have $\omega(t, x) = \omega_0(x - Vt)$, with $V \in \mathbb{C}$. Inserting this ansatz in the Euler equations, we arrived at

$$(v_0(x) - V) \cdot \nabla\omega_0(x) = 0, \quad x \in \mathbb{R}^2,$$

where "$\cdot$" indicates the scalar product in $\mathbb{R}^2$. As in the previous section, we want to work always in the complex sense identifying $(x_1, x_2) \in \mathbb{R}^2$ as $x_1 + ix_2 \in \mathbb{C}$. The gradient operator can be identify to the $\partial_{\overline{z}}$ derivative as in (3.3.1). Then, we above equation can be written as:

$$\text{Re}\left[\overline{(v_0(x) - V)}\partial_{\overline{x}}\omega_0(x)\right] = 0, \quad x \in \mathbb{C}.$$

When working with the scaled vorticity $\omega_{0,\varepsilon}$, this equation must be understood in the weak sense, yielding

$$(v_{0,\varepsilon}(x) - V) \cdot \vec{n}(x) = 0, \quad x \in \partial(\varepsilon D_1 + kl) \cup \partial(\varepsilon D_2 + kl), \tag{3.3.8}$$

or similarly,

$$\text{Re}\left[\overline{(v_{0,\varepsilon}(x) - V)}\vec{n}(x)\right] = 0, \quad x \in \partial(\varepsilon D_1 + kl) \cup \partial(\varepsilon D_2 + kl),$$

for any $k \in \mathbb{Z}$. Here, $\vec{n}$ is the exterior normal vector and $v_{0,\varepsilon}$ is the velocity associated to (3.3.6). The expression of $v_{0,\varepsilon}$ coming from Proposition 3.3.1–(2) gives us

$$v_{0,\varepsilon}(x) = \frac{i}{2l\pi\varepsilon^2}\overline{\int_{\varepsilon D_1} \cot\left[\frac{\pi(x-y)}{l}\right] dA(y)} - \frac{i}{2l\pi\varepsilon^2}\overline{\int_{\varepsilon D_2} \cot\left[\frac{\pi(x-y)}{l}\right] dA(y)}.$$

We can check that $v_{0,\varepsilon}(x + kl) = v_{0,\varepsilon}(x)$, for any $k \in \mathbb{Z}$. Moreover, we have that $\vec{n}_{D+kl}(x + kl) = \vec{n}_D(x)$, for any simply–connected bounded domain D. Then, the equation (3.3.8) reduces to

$$\text{Re}\left[\overline{(v_{0,\varepsilon}(x) - V)}\vec{n}(x)\right] = 0, \quad x \in \partial D_1 \cup \partial D_2.$$

Consider $D_2 = -D_1 + a - ih$, for $a = 0$ or $a = \frac{l}{2}$, and $h \neq 0$. By using the relation between D_1 and D_2, the above system reduces to just one equation:

$$\text{Re}\left[\overline{(v_{0,\varepsilon}(x) - V)}\vec{n}(x)\right] = 0, \quad x \in \partial D_1,$$

where

$$v_{0,\varepsilon}(\varepsilon x) = \frac{i}{2l\pi}\overline{\int_{D_1} \cot\left[\frac{\pi\varepsilon(x-y)}{l}\right] dA(y)} - \frac{i}{2l\pi}\overline{\int_{D_1} \cot\left[\frac{\pi(\varepsilon(x+y) - a + ih)}{l}\right] dA(y)}.$$

Other representations of the velocity field can be obtained using Proposition 3.3.1:

$$v_{0,\varepsilon}(\varepsilon x) = -\frac{1}{2\pi^2\varepsilon}\int_{\partial D_1} \ln\left|\sin\left(\frac{\pi(\varepsilon(x-\xi))}{l}\right)\right| d\xi - \frac{1}{2\pi^2\varepsilon}\int_{\partial D_1} \ln\left|\sin\left(\frac{\pi(\varepsilon(x+\xi) - a + ih)}{l}\right)\right| d\xi.$$

At this stage, we are going to introduce an exterior conformal map from $\mathbb{T}$ into ∂D_1 given by

$$\Phi(w) = i(w + \varepsilon f(w)), \quad f(w) = \sum_{n\geq 1} a_n w^{-n}, \quad a_n \in \mathbb{R}, w \in \mathbb{T}. \tag{3.3.9}$$

For others values of a, one must readjust the conformal map, but here we will consider $a = 0$ and $a = \frac{l}{2}$ having a horizontal translation in the point vortex system.

Note that $\vec{n}(\Phi(w)) = w\Phi'(w)$. Then, we can rewrite the equation with the use of the above conformal map in the following way:

$$F_E(\varepsilon, f, V)(w) := \text{Re}\left[\left\{\overline{I_E(\varepsilon, f)(w)} - \overline{V}\right\} w\Phi'(w)\right] = 0, \quad w \in \mathbb{T}, \tag{3.3.10}$$

where

$$\begin{aligned} I_E(\varepsilon, f)(w) :=& -\frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)-\Phi(\xi)))}{l}\right)\right|\Phi'(\xi)\,d\xi \\ & -\frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)+\Phi(\xi))-a+ih)}{l}\right)\right|\Phi'(\xi)\,d\xi. \end{aligned} \tag{3.3.11}$$

Note that

$$I_E(\varepsilon, f)(w) = v_{0,\varepsilon}(\varepsilon\Phi(w)).$$

As it is mentioned in the introduction, [83] deals with the desingularization of a vortex pairs in both the Euler equations and the generalized quasi-geostrophic equation. In order to relate F_E with the functional in [83], we can write I_E as

$$\begin{aligned} I_E(\varepsilon, f)(w) =& \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(w)}-\overline{\Phi(\xi)}}{\Phi(w)-\Phi(\xi)}\Phi'(\xi)\,d\xi} - \frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|H\left(\frac{\pi\varepsilon(\Phi(w)-\Phi(\xi))}{l}\right)\right|\Phi'(\xi)\,d\xi \\ & + \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\varepsilon(\overline{\Phi(w)}+\overline{\Phi(\xi)})-a+ih}{\varepsilon(\Phi(w)-\Phi(\xi))-a+ih}\Phi'(\xi)\,d\xi} \\ & - \frac{1}{2\pi^2\varepsilon}\int_{\mathbb{T}} \ln\left|H\left(\frac{\pi(\varepsilon(\Phi(w)+\Phi(\xi))-a+ih)}{l}\right)\right|\Phi'(\xi)\,d\xi \\ =& \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(w)}-\overline{\Phi(\xi)}}{\Phi(w)-\Phi(\xi)}\Phi'(\xi)\,d\xi} + \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\varepsilon(\overline{\Phi(w)}+\overline{\Phi(\xi)})-a+ih}{\varepsilon(\Phi(w)-\Phi(\xi))-a+ih}\Phi'(\xi)\,d\xi} \\ & + \tilde{I}_E(\varepsilon, f)(w), \end{aligned}$$

using the expression of the velocity written in Proposition 3.3.1-(3). The Residue Theorem amounts to

$$\begin{aligned} I_E(\varepsilon, f)(w) =& \frac{1}{4\pi^2\varepsilon}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(w)}-\overline{\Phi(\xi)}}{\Phi(w)-\Phi(\xi)}\Phi'(\xi)\,d\xi} \\ & + \frac{1}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{\Phi(\xi)}}{\varepsilon(\Phi(w)-\Phi(\xi))-a+ih}\Phi'(\xi)\,d\xi} + \tilde{I}_E(\varepsilon, f)(w) \\ =& \hat{I}_E(\varepsilon, f)(w) + \tilde{I}_E(\varepsilon, f)(w). \end{aligned} \tag{3.3.12}$$

Hence, the function $\hat{I}_E$ comes from the study of the vortex pairs in [83]. We will take advantage of the study done in that work about $\hat{I}_E$. In this model, $\hat{I}_E$ indicates the contribution of just two vortex patches.

Remark 3.3.2. *Note that for any $a \in \mathbb{R}$, we have that the point vortex street translates with constant speed V_0 given in Proposition 3.2.2. In the case of $a = 0$ or $a = \frac{l}{2}$, the translation is horizontal. In order to desingularize the point vortex street, one tries to find a "small" domain around each point. Such domain has to be symmetric with respect to the perpendicular axis to the one given by the translation and that determinate the expression of the conformal map. Then, for the case of a horizontal translation, the conformal map must have the expression 3.3.9. For other values of a, one has to readjust it in the sense*

$$\Phi(w) = \gamma(w + \varepsilon f(w)), \quad f(w) = \sum_{n\geq 1} a_n w^{-n}, \quad a_n \in \mathbb{R}, w \in \mathbb{T}, \tag{3.3.13}$$

with $\gamma \in \mathbb{C}$ and $|\gamma| = 1$. Then, one characterise the needed symmetry for the domain in terms of γ. In order to alleviate the computations, we consider here $a = 0$ and $a = \frac{l}{2}$, but we expect that similar results are obtained for other values of a.

The first step is to check that we recover the point vortex street with this model. Remind that $a = 0$ or $a = \frac{l}{2}$.

Proposition 3.3.3. *For any $h \neq 0$, $l > 0$, the following equation is verified*

$$F_E(0, 0, V_0)(w) = 0, \quad w \in \mathbb{T},$$

where V_0 is given by (3.2.4).

Proof. The equation that we must check is

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathrm{Re}\Big[\Big\{ &\frac{i}{2\pi^2\varepsilon}\int_{\mathbb{T}} \overline{\ln\left|\sin\left(\frac{\pi\varepsilon i(w-\xi)}{l}\right)\right| d\xi} \\ &+ \frac{i}{2\pi^2\varepsilon}\int_{\mathbb{T}} \overline{\ln\left|\sin\left(\frac{\pi(\varepsilon i(w+\xi) - a + ih)}{l}\right)\right| d\xi} - \overline{V_0}\Big\} wi\Big] = 0.\end{aligned}$$

Using the Stokes Theorem, it agrees with

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathrm{Re}\Big[\Big\{ &-\frac{i}{2l\pi}\int_{\mathbb{D}} \cot\left[\frac{\pi\varepsilon i(w-y)}{l}\right] dA(y) \\ &+ \frac{i}{2l\pi}\int_{\mathbb{D}} \cot\left[\frac{\pi(\varepsilon i(w+y) - a + ih)}{l}\right] dA(y) - \overline{V_0}\Big\} wi\Big] = 0.\end{aligned}$$

We study the equation in two parts. First, note that

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathrm{Re}&\left[\left\{\frac{i}{2l\pi}\int_{\mathbb{D}} \cot\left[\frac{\pi(\varepsilon i(w+y) - a + ih)}{l}\right] dA(y) - \overline{V_0}\right\} wi\right] \\ &= \mathrm{Re}\left[\left\{-\frac{1}{2l\pi i}\cot\left[\frac{\pi(ih-a)}{l}\right]|\mathbb{D}| - \overline{V_0}\right\} wi\right] \\ &= 0, \quad w \in \mathbb{T}.\end{aligned}$$

In the above limit, we may use the Dominated Convergence Theorem in order to introduce the limit inside the integral. Second, we use the expansion of the complex cotangent as

$$\cot(z) = \frac{1}{z} + zT(z), \quad T(z) = \sum_{k=1}^{\infty} \frac{2}{z^2 - \pi^2 k^2},$$

where T is a smooth function for $|z|<1$. Then, the only contribution in F_E is given by the first part:

$$\begin{aligned}\lim_{\varepsilon\to 0}\text{Re}\left[\frac{i}{2l\pi}\int_{\mathbb{D}}\cot\left[\frac{\pi\varepsilon i(w-y)}{l}\right]dA(y)wi\right] &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\text{Re}\left[\frac{i}{2\pi^2}\int_{\mathbb{D}}\frac{1}{w-y}dA(y)w\right]\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\text{Re}\left[i\right]\\ &= 0,\end{aligned}$$

for $w\in\mathbb{T}$, where we have used the Residue Theorem to compute the integral. □

We fix the Banach spaces that we will use when we apply the Implicit Function Theorem. For $\alpha\in(0,1)$, we define

$$X_\alpha=\left\{f\in\mathscr{C}^{1,\alpha}(\mathbb{T}),\quad f(w)=\sum_{n\geq 1}a_n w^{-n},\ a_n\in\mathbb{R}\right\}, \tag{3.3.14}$$

$$Y_\alpha=\left\{f\in\mathscr{C}^{0,\alpha}(\mathbb{T}),\quad f(e^{i\theta})=\sum_{n\geq 2}a_n\sin(n\theta),\ a_n\in\mathbb{R}\right\}. \tag{3.3.15}$$

Remark 3.3.4. *Let us explain why we need that the first frequency in the domain Y_α is vanishing. In the case that $a=0$ or $a=\frac{l}{2}$, it can be checked that $F_E(\varepsilon,f,V)$ is well–defined and $\mathscr{C}^1$ from $\mathbb{R}\times X_\alpha\times\mathbb{R}$ to*

$$\tilde{Y}_\alpha=\left\{f\in\mathscr{C}^{0,\alpha}(\mathbb{T}),\quad f(e^{i\theta})=\sum_{n\geq 1}a_n\sin(n\theta),\ a_n\in\mathbb{R}\right\}.$$

But, when we linerarize F_E and obtain $\partial_f F_E(0,0,V)$, this is not an isomorphism from X_α to $\tilde{Y}_\alpha$. However, it does from X_α to Y_α. We are using Y_α instead of $\tilde{Y}_\alpha$ in order to implement later the Implicit Function Theorem.

Remark 3.3.5. *Note that if $f\in B_{X_\alpha}(0,\sigma)$, with $\sigma<1$, then Φ is bilipschitz.*

Proposition 3.3.6. *The function $V:(-\varepsilon_0,\varepsilon_0)\times B_{X_\alpha}(0,\sigma)\longrightarrow\mathbb{R}$, given by*

$$V(\varepsilon,f)=\frac{\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw}{\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw}, \tag{3.3.16}$$

fulfills $V(0,f)=V_0$, where V_0 is defined in (3.2.4). *The parameters satisfy: $\varepsilon_0\in(0,\min\{1,\frac{l}{4}\})$, $\sigma<1$, $\alpha\in(0,1)$, and X is defined in* (3.3.14).

Proof. In the expression (3.3.16), let us work with the denominator. The Residue Theorem amounts to

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw=i\int_{\mathbb{T}}w(1-\overline{w}^2)dw=2\pi.$$

From (3.3.11) and the ideas in Proposition 3.3.3, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}I_E(\varepsilon,f)(w)&=\lim_{\varepsilon\to 0}\left\{-\frac{i}{2\pi^2\varepsilon}\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon i(w-\xi))}{l}\right)\right|d\xi+V_0\right\}\\ &=\lim_{\varepsilon\to 0}\left\{\frac{1}{2\pi^2\varepsilon}\overline{\int_{\mathbb{D}}\frac{1}{w-y}dA(y)}+V_0\right\}\end{aligned}$$

$$= \lim_{\varepsilon \to 0} \left\{ \frac{w}{2\pi\varepsilon} + V_0 \right\}. \tag{3.3.17}$$

Note also that

$$\int_{\mathbb{T}} \overline{w} w(1 - \overline{w}^2) dw = \int_{\mathbb{T}} (1 - \overline{w}^2) dw = 0,$$

via again the Residue Theorem. Then, the first term in (3.3.17) does not provide any contribution. It implies that

$$V(0, f) = V_0 \frac{\int_{\mathbb{T}} w(1 - \overline{w}^2) dw}{\int_{\mathbb{T}} w(1 - \overline{w}^2) dw} = V_0.$$

□

Proposition 3.3.7. *If* V *sets* (3.3.16)*, then*

$$\tilde{F}_E : (-\varepsilon_0, \varepsilon_0) \times B_{X_\alpha}(0, \sigma) \to Y_\alpha,$$

with $\tilde{F}_E(\varepsilon, f) = F_E(\varepsilon, f, V(\varepsilon, f))$*, is well–defined and* $\mathscr{C}^1$*. The parameters satisfy that* $\alpha \in (0, 1)$*,* $\varepsilon_0 \in (0, \min\{1, \frac{l}{4}\})$ *and* $\sigma < 1$*.*

Remark 3.3.8. *Let us clarify why we need the condition* $\varepsilon_0 < \frac{l}{4}$*. In some point of the proof we need to use Taylor formula in the following way*

$$G(|z_1 + z_2|) = G(|z_1|) + \int_0^1 G'(|z_1 + tz_2|) \frac{\text{Re}\left[(z_1 + tz_2)\overline{z_2}\right]}{|z_1 + tz_2|} dt, \tag{3.3.18}$$

for $z_1, z_2 \in \mathbb{C}$ *and* $|z_2| < |z_1|$*. Here, the use of this formula is not explicit since we are referring to the work [83], and this condition is needed in order to check* $|z_2| < |z_1|$*. Although we are not using it explicitly in this proof, we will use it for the general equation in the following section.*

Proof. We will divide the proof in three steps.

• *First step: Symmetry of* F_E*.* Note that Φ given by (3.3.9) verifies

$$\Phi(\overline{w}) = -\overline{\Phi(w)},$$

where we are taking $\vartheta = i$ in order to work with $a = 0$ or $a = \frac{l}{2}$. We are going to check that $F_E(\varepsilon, f, V)(e^{i\theta}) = \sum_{n \geq 1} f_n \sin(n\theta)$, with $f_n \in \mathbb{R}$. To do that, it is enough to prove that

$$F_E(\varepsilon, f, V)(\overline{w}) = -F_E(\varepsilon, f, V)(w).$$

Recall the following property of the complex integrals over $\mathbb{T}$:

$$\overline{\int_{\mathbb{T}} f(w) dw} = -\int_{\mathbb{T}} \overline{f(\overline{w})} dw, \tag{3.3.19}$$

for a complex function f.

Let us start with the expression of $I_E(\varepsilon, f)$ and note that

$$\ln \left| \sin \left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l} \right) \right| = \ln \left| \sin \left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) + ih)}{l} \right) \right|, \quad a = 0,$$

$$\ln \left| \sin \left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l} \right) \right| = \ln \left| \cos \left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) + ih)}{l} \right) \right|, \quad a = \frac{l}{2}.$$

Then,

$$\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(\overline{w})+\Phi(\overline{\xi}))-a+ih)}{l}\right)\right| = \ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)+\Phi(\xi))-a+ih)}{l}\right)\right|,$$

for $a=0$ and $a=\frac{l}{2}$. Notice that $I_E(\varepsilon,f)(\overline{w})=\overline{I_E(\varepsilon,f)(w)}$, which implies

$$\begin{aligned}
-2\pi^2\varepsilon\overline{I_E(\varepsilon,f)(w)} =&\overline{\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)-\Phi(\xi)))}{l}\right)\right|\Phi'(\xi)\,d\xi}\\
&+\overline{\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)+\Phi(\xi))-a+ih)}{l}\right)\right|\Phi'(\xi)\,d\xi}\\
=&-\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)-\Phi(\overline{\xi})))}{l}\right)\right|\overline{\Phi'(\overline{\xi})}\,d\xi\\
&-\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(w)+\Phi(\overline{\xi}))-a+ih)}{l}\right)\right|\overline{\Phi'(\overline{\xi})}\,d\xi\\
=&\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(\overline{w})-\Phi(\xi)))}{l}\right)\right|\Phi'(\xi)\,d\xi\\
&+\int_{\mathbb{T}}\ln\left|\sin\left(\frac{\pi(\varepsilon(\Phi(\overline{w})+\Phi(\xi))-a+ih)}{l}\right)\right|\Phi'(\xi)\,d\xi\\
=&-2\pi^2\varepsilon I_E(\varepsilon,f)(\overline{w}).
\end{aligned}$$

Next, if V is given by (3.3.16), then we are going to check that $V\in\mathbb{R}$. Let us analyze the denominator and the numerator of the expression of V:

$$\begin{aligned}
&2i\text{Im}\Big[\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw\Big]\\
&\qquad=\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw-\overline{\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw}\\
&\qquad=\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw+\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\overline{\Phi'(\overline{w})}(1-\overline{w}^2)dw\\
&\qquad=\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw-\int_{\mathbb{T}}\overline{I_E(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw\\
&\qquad=0,
\end{aligned}$$

and

$$\begin{aligned}
2i\text{Im}\left[\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw\right]&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw-\overline{\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw}\\
&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw+\int_{\mathbb{T}}w\overline{\Phi'(\overline{w})}(1-\overline{w}^2)dw\\
&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw-\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw\\
&=0.
\end{aligned}$$

Then, $V\in\mathbb{R}$. Hence,

$$F_E(\varepsilon,f,V)(\overline{w})=\text{Re}\left[\left\{\overline{I_E(\varepsilon,f)(\overline{w})-V}\right\}\overline{w}\Phi'(\overline{w})\right]$$

$$
\begin{aligned}
&= - \operatorname{Re}\left[\{I_E(\varepsilon, f)(w) - V\}\, \overline{w}\overline{\Phi'(w)}\right] \\
&= - F_E(\varepsilon, f, V)(w).
\end{aligned}
$$

In order to check that $\tilde{F}_E(\varepsilon, f) \in Y_\alpha$, we need $f_1 = 0$. For that, we ask the condition

$$
\int_0^{2\pi} F_E(\varepsilon, f, V)(e^{i\theta}) \sin(\theta) d\theta = -\frac{1}{2} \int_{\mathbb{T}} F_E(\varepsilon, f, V)(w)(1 - \overline{w}^2) dw = 0,
$$

which agrees with

$$
\int_{\mathbb{T}} \left\{ \overline{I_E(\varepsilon, f)(w)} - V \right\} w\Phi'(w)(1 - \overline{w}^2) dw = 0.
$$

Using that V verifies (3.3.16), the last equation is clearly set.

• *Second step: Regularity of V.* Let us begin with the denominator, noting that

$$
\int_{\mathbb{T}} w\Phi'(w)(1-\overline{w}^2) dw = i \int_{\mathbb{T}} w(1+\varepsilon f'(w))(1-\overline{w}^2) dw = 2\pi + i\varepsilon \int_{\mathbb{T}} w f'(w) dw = 2\pi - i\varepsilon \int_{\mathbb{T}} f(w) dw,
$$

by using the Residue Theorem. Then, if $|\varepsilon| < \varepsilon_0$ and $f \in B_{X_\alpha}(0, \sigma)$, the denominator is not vanishing. Moreover, the denominator is clearly $\mathscr{C}^1$ in ε and f.

We continue with the numerator denoting

$$
\begin{aligned}
J(\varepsilon, f) &= \int_{\mathbb{T}} \overline{I_E(\varepsilon, f)(w)} w\Phi'(w)(1 - \overline{w}^2) dw \\
&= \int_{\mathbb{T}} \overline{\hat{I}_E(\varepsilon, f)(w)} w\Phi'(w)(1 - \overline{w}^2) dw + \int_{\mathbb{T}} \overline{\tilde{I}_E(\varepsilon, f)(w)} w\Phi'(w)(1 - \overline{w}^2) dw \\
&=: J_1(\varepsilon, f)(w) + J_2(\varepsilon, f)(w),
\end{aligned}
$$

using the decomposition of I_E done in (3.3.12). Note that $\hat{I}_E$ is the part of I_E coming from the vortex pairs analyzed in [83]. In that work J_1 is analyzed showing that it is $\mathscr{C}^1$ in ε and f. Note that the spaces used in [83] are also (3.3.14)–(3.3.15) and the condition $\varepsilon_0 < \frac{l}{4}$ is needed in their computations, see Remark 3.3.8.

Then, it remains to study the regularity of $J_2(\varepsilon, f)$. We should analyze $\tilde{I}_E$, i.e.,

$$
\begin{aligned}
\tilde{I}_E(\varepsilon, f)(w) =& - \frac{1}{2\pi^2 \varepsilon} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi\varepsilon(\Phi(w) - \Phi(\xi))}{l} \right) \right| \Phi'(\xi)\, d\xi \\
& - \frac{1}{2\pi^2 \varepsilon} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l} \right) \right| \Phi'(\xi)\, d\xi \\
=& - \frac{i}{2\pi^2 \varepsilon} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi\varepsilon(\Phi(w) - \Phi(\xi))}{l} \right) \right| d\xi \\
& - \frac{i}{2\pi^2} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi\varepsilon(\Phi(w) - \Phi(\xi))}{l} \right) \right| f'(\xi)\, d\xi \\
& - \frac{i}{2\pi^2 \varepsilon} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l} \right) \right| d\xi \\
& - \frac{i}{2\pi^2} \int_{\mathbb{T}} \ln \left| H\left(\frac{\pi(\varepsilon(\Phi(w) + \Phi(\xi)) - a + ih)}{l} \right) \right| f'(\xi)\, d\xi \\
=:& - \left(I_1(\varepsilon, f) + I_2(\varepsilon, f) + I_3(\varepsilon, f) + I_4(\varepsilon, f) \right).
\end{aligned}
$$

Note that I_2 and I_4 are smooth in both variables, due to that $H(z) = \frac{\sin(z)}{z}$, see (3.3.3). Then they are $\mathscr{C}^1$ in ε and Φ. Let us analyze the others terms. Using (3.3.3) and the expansion of the logarithm,

$$\ln|1+f(z)| = \text{Re}\sum_1^\infty \frac{(-1)^{1+n}f(z)^n}{n},$$

one has

$$\ln|H(\varepsilon z)| = \varepsilon^2 G_1(\varepsilon, z),$$

with G_1 smooth in both variables. This implies that I_1 is $\mathscr{C}^1$ in ε and f. On the other way,

$$\ln|H(\varepsilon z + z')| = \varepsilon G_2(\varepsilon, z, z') + G_3(z'),$$

with G_2 and G_3 smooth. Then, we find

$$I_3(\varepsilon, f)(w) = \frac{i}{2\pi^2}\int_{\mathbb{T}} G_2\left(\varepsilon, \frac{\pi(\Phi(w)+\Phi(\xi))}{l}, \frac{-a+ih}{l}\right) d\xi,$$

which is smooth in f and ε. Hence, we achieve that V is $\mathscr{C}^1$ in both variables.

• *Third step: Regularity of $\tilde{F}_E$.* Decomposing I again as in (3.3.12), we get that

$$\tilde{F}(\varepsilon, f) = \text{Re}\left[\left\{\overline{\hat{I}_E(\varepsilon, f)(w)} - \overline{\tilde{I}_E(\varepsilon, f)(w)} - V(\varepsilon, f)\right\} w\Phi'(w)\right].$$

Again, the part coming from $\hat{I}$ is analyzed in [83], where it is shown that is $\mathscr{C}^1$ in both variables. From the second step, we got that $\tilde{I}$ and V are also smooth completing the proof. □

3.3.3 Desingularization of the Kármán Point Vortex Street

In this section, we provide the proof of the existence of Kármán Vortex Patch Street via a desingularization of the point vortex model given by the Kármán Point Vortex Street. The idea is to implement the Implicit Function Theorem to the functional $\tilde{F}_E$ defined in Proposition 3.3.7.

Theorem 3.3.9. *Let $h, l \in \mathbb{R}$, with $h \neq 0$ and $l > 0$, and $a = 0$ or $a = \frac{l}{2}$. Then, there exist D^ε such that*

$$\omega_0(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D^\varepsilon + kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D^\varepsilon + a - ih + kl}(x), \tag{3.3.20}$$

defines a horizontal translating solution of the Euler equations, with constant speed, for any $\varepsilon \in (0, \varepsilon_0)$ and small enough $\varepsilon_0 > 0$. Moreover, D^ε is at least $\mathscr{C}^1$.

Proof. In order to look for solutions in the form (3.3.20), we need to study the functional F_E defined in (3.3.10), where Φ is given by (3.3.9). Moreover, V is a function of (ε, f) described by (3.3.16).

In Proposition 3.3.7, we have that $\tilde{F}_E : \mathbb{R}\times B_{X_\alpha}(0,\sigma) \to Y_\alpha$, with $\tilde{F}_E(\varepsilon, f) = F_E(\varepsilon, f, V(\varepsilon, f))$, is well–defined and $\mathscr{C}^1$, for $\varepsilon_0 \in (0, \min\{1, \frac{l}{4}\})$ and $\sigma < 1$. Then, we wish to apply the Implicit Function Theorem to $\tilde{F}_E$. By Proposition 3.3.3 and Proposition 3.3.6, we have that $\tilde{F}_E(0,0)(w) = 0$, for any $w \in \mathbb{T}$.

Let us show that $\partial_f \tilde{F}_E(0,0)$ is an isomorphism:

$$\begin{aligned}\partial_f \tilde{F}_E(0,0)h(w) = \lim_{\varepsilon\to 0} \text{Re}\Big[&\left\{\partial_f \overline{I_E(0,f)(w)}h(w) - \partial_f V(0,0)h(w)\right\} iw \\ &+ \left\{\overline{I_E(\varepsilon,0)(w)} - V_0\right\} iw\varepsilon h'(w)\Big],\end{aligned}$$

By Proposition 3.3.6, we obtain $\partial_f V(0,f)h(w) \equiv 0$. Note also that

$$\lim_{\varepsilon\to 0} I_E(\varepsilon,0)(w) = \lim_{\varepsilon\to 0}\left\{\frac{w}{2\pi\varepsilon} + V_0\right\}.$$

Moreover, by expression (3.3.12), we have

$$\begin{aligned}
\partial_f I_E(0,0)(w)h(w) =& \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{h(w)}-\overline{h(\xi)}}{w-\xi}\,d\xi} - \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{(h(w)-h(\xi))(\overline{w}-\overline{\xi})}{(w-\xi)^2}\,d\xi} \\
&+ \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{w}-\overline{\xi}}{w-\xi}h'(\xi)\,d\xi} + \partial_f \tilde{I}_E(0,0)h(w) \\
=& \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{h(w)}-\overline{h(\xi)}}{w-\xi}\,d\xi} - \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{(h(w)-h(\xi))(\overline{w}-\overline{\xi})}{(w-\xi)^2}\,d\xi} \\
&+ \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{w}-\overline{\xi}}{w-\xi}h'(\xi)\,d\xi}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\partial_f \tilde{F}_E(0,0)h(w) =& \mathrm{Re}\left[\left\{\frac{1}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{h(w)}-\overline{h(\xi)}}{w-\xi}\,d\xi} - \frac{1}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{(h(w)-h(\xi))(\overline{w}-\overline{\xi})}{(w-\xi)^2}\,d\xi}\right.\right. \\
&\left.\left.+\frac{1}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{w}-\overline{\xi}}{w-\xi}h'(\xi)\,d\xi}\right\} w + \frac{i}{2\pi}h'(w)\right] \\
=& \mathrm{Im}\left[\left\{\frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{h(w)}-\overline{h(\xi)}}{w-\xi}\,d\xi} - \frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{(h(w)-h(\xi))(\overline{w}-\overline{\xi})}{(w-\xi)^2}\,d\xi}\right.\right. \\
&\left.\left.+\frac{i}{4\pi^2}\overline{\int_{\mathbb{T}} \frac{\overline{w}-\overline{\xi}}{w-\xi}h'(\xi)\,d\xi}\right\} w - \frac{1}{2\pi}h'(w)\right].
\end{aligned}$$

By the Residue Theorem, we have

$$\begin{aligned}
\int_{\mathbb{T}} \frac{\overline{w}-\overline{\xi}}{w-\xi}h'(\xi)\,d\xi =& 0, \\
\int_{\mathbb{T}} \frac{\overline{h(w)}-\overline{h(\xi)}}{w-\xi}\,d\xi - \int_{\mathbb{T}} \frac{(h(w)-h(\xi))(\overline{w}-\overline{\xi})}{(w-\xi)^2}\,d\xi =& 2i\int_{\mathbb{T}} \frac{\mathrm{Im}\left[\overline{(h(w)-h(\xi))}(w-\xi)\right]}{(w-\xi)^2}\,d\xi = 0.
\end{aligned}$$

Finally, we find

$$\partial_f \tilde{F}_E(0,0)h(w) = -\frac{1}{2\pi}\mathrm{Im}\left[h'(w)\right], \tag{3.3.21}$$

which is an isomorpshim from X_α to Y_α. □

Remark 3.3.10. *Analyzing [83], we realize that the above linearized operator* (3.3.21) *agrees with the linearized operator in [83] for the vortex pairs. This tells us that the only real contribution in the linearized operator is due to two vortex patches:* $\mathbf{1}_{\varepsilon D_1}$ *and* $\mathbf{1}_{-\varepsilon D_1+a-ih}$.

3.3.4 Quasi–geostrophic shallow water equation

In this section, we investigate the case of the quasi-geostrophic shallow water (QGSW) equations. Let q be the potential vorticity, then the QGSW equations are given by

$$\begin{cases} q_t + (v \cdot \nabla) q = 0, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ v = \nabla^{\perp} \psi, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ \psi = (\Delta - \lambda^2)^{-1} q, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ q(t=0, x) = q_0(x), & \text{with } x \in \mathbb{R}^2, \end{cases}$$

with $\lambda \neq 0$. The same results to the Euler equations are obtained in this case. That is due to the similarity of the kernel in both cases, in particular, they have the same behavior close to 0. In Section 3.2 we analyzed the case of the N-vortex problem, see Proposition 3.2.10. Here, we want to desingularize (3.2.3) in order to obtain periodic in space solutions that translate in the QGSW equation.

The stream function ψ can be recovered in terms of q in the following way

$$\psi(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} K_0(|\lambda||x - y|) q(t, y) \, dA(y).$$

The function K_0 is the Modified Bessel function of order zero, whose definition and some of their properties can be found in Appendix C. It is of great interest the expansion of K_0 given in (C.0.18) as

$$K_0(z) = -\ln\left(\frac{z}{2}\right) I_0(z) + \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2} \varphi(k+1),$$

where

$$\varphi(1) = -\gamma \quad \text{and} \quad \varphi(k+1) = \sum_{m=1}^{k} \frac{1}{m} - \gamma, \ k \in \mathbb{N}^*.$$

The constant γ is the Euler's constant and the function I_0 is defined in Appendix C, but we recall it as

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \Gamma(k+1)}.$$

Via this expansion, one notice that

$$K_0(z) = -\ln(z) + g_0(z) + g_1, \tag{3.3.22}$$

where

$$\begin{aligned} g_0(z) =& -z^2 \left(\ln(2) - \ln(z)\right) \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k-2}}{k! \Gamma(k+1)} + z^2 \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k-2}}{(k!)^2} \varphi(k+1), \\ g_1 =& -\gamma - \ln(2). \end{aligned}$$

Note that g_0 is smooth and $g_0(z) = O(z^2 \ln(z))$ close to 0.

Consider a Kármán Vortex Patch Street in the QGSW equations in the sense

$$q_0(x) = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \mathbf{1}_{D_1 + kl}(x) - \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \mathbf{1}_{D_2 + kl}(x),$$

where D_1 and D_2 are simply–connected bounded domains such that $|D_1| = |D_2|$, and $l > 0$. Motivated by Euler equations, assume $D_2 = -D_1 + a - ih$, having the following distribution

$$q_0(x) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D+kl}(x) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-D+a+kl-ih}(x), \tag{3.3.23}$$

where we are rewriting D_1 by D. The velocity field is given by

$$\begin{aligned} v_0(x) =& \frac{\lambda i}{2\pi^2}\sum_{k\in\mathbb{Z}}\int_D K_1(\lambda|x-y-kl|)\frac{x-y-kl}{|x-y-kl|}\,dA(y)\\ &-\frac{\lambda i}{2\pi^2}\sum_{k\in\mathbb{Z}}\int_D K_1(\lambda|x+y-a-kl+ih|)\frac{x+y-a-kl+ih}{|x+y-a-kl+ih|}\,dA(y)\\ =&\frac{1}{2\pi^2}\sum_{k\in\mathbb{Z}}\int_{\partial D} K_0(\lambda|x-y-kl|)dy + \frac{1}{2\pi^2}\sum_{k\in\mathbb{Z}}\int_{\partial D} K_0(\lambda|x+y-a-kl+ih|)dy. \end{aligned} \tag{3.3.24}$$

We scale the expression (3.3.23) in the following way

$$q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D+a+kl-ih}(x).$$

As in the Euler equations, if $|D| = |\mathbb{D}|$, with $a \in \mathbb{R}$ and $h \neq 0$, we obtain the point model:

$$q_{0,0}(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x),$$

which has been studied in Proposition 3.2.10. Considering now a translating motion in the form $q(t,x) = q_0(x - Vt)$, with $V \in \mathbb{C}$, then we arrive at

$$(v_0(x) - V)\cdot\nabla q_0(x) = 0, \quad x \in \mathbb{R}^2.$$

In the case of $q_{0,\varepsilon}$, we need to solve the above equation understood in the weak sense, i.e.,

$$(v_{0,\varepsilon}(x) - V)\cdot\vec{n}(x) = 0, \quad x \in \partial\,(\varepsilon D + kl) \cup \partial\,(-\varepsilon D + a + kl - ih), \tag{3.3.25}$$

which, as for the Euler equations, reduces to

$$\text{Re}\left[\overline{(v_{0,\varepsilon}(x) - V)}\vec{n}(x)\right] = 0, \quad x \in \partial\,(\varepsilon D), \tag{3.3.26}$$

written in the complex sense. With the use of the conformal map (3.3.9):

$$\Phi(w) = i(w + \varepsilon f(w)), \quad f(w) = \sum_{n\geq 1} a_n w^{-n}, \quad a_n \in \mathbb{R}, w \in \mathbb{T},$$

it agrees with

$$F_{QGSW}(\varepsilon, f, V)(w) := \text{Re}\left[\left\{\overline{I_{QGSW}(\varepsilon, f)(w)} - \overline{V}\right\} w\Phi'(w)\right] = 0, \quad w \in \mathbb{T}, \tag{3.3.27}$$

where

$$I_{QGSW}(\varepsilon, f)(w) = v_{0,\varepsilon}(\varepsilon\Phi(w)). \tag{3.3.28}$$

Then,

$$\begin{aligned}I_{QGSW}(\varepsilon,f)(w):=&\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}K_0(\lambda|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi\\&+\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}K_0(\lambda|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi.\end{aligned}$$

Via the expansion of K_0, given in (3.3.22), one has

$$\begin{aligned}I_{QGSW}(\varepsilon,f)(w)=&-\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\ln|\lambda(\varepsilon(\Phi(w)-\Phi(\xi))-kl)|\,\Phi'(\xi)\,d\xi\\&-\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\ln|(\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih)|\,\Phi'(\xi)\,d\xi\\&+\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}g_0\left(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|\right)\Phi'(\xi)\,d\xi\\&+\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}g_0\left(\lambda|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|\right)\Phi'(\xi)\,d\xi\\=&I_E(\varepsilon,f)(w)+\tilde{I}_{QGSW}(\varepsilon,f)(w),\end{aligned}\tag{3.3.29}$$

where I_E is the corresponding function associated to Euler equations, see (3.3.11).

The analogue to Propositions 3.3.3, 3.3.6 and 3.3.7, and Theorem 3.3.9 are obtained, whose proofs are very similar and so here we omit many details. Remark that $a=0$ or $a=\frac{l}{2}$.

Proposition 3.3.11. *For any $h\neq 0$ and $l>0$, the following equation is verified*

$$F_{QGSW}(0,0,V_0)(w)=0,\quad w\in\mathbb{T},$$

where V_0 is given by (3.2.11)*:*

$$V_0=\frac{\lambda i}{2\pi}\sum_{k\in\mathbb{Z}}K_1(\lambda|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}.$$

Proof. Using definition (3.3.27), we need to check that

$$\begin{aligned}\lim_{\varepsilon\to0}\mathrm{Re}\Big[\Big\{&-\frac{i}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}}K_0(\lambda|\varepsilon i(w-\xi)-kl|)\,d\xi}\\&-\frac{i}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}}K_0(\lambda|\varepsilon i(w+\xi)-a-kl+ih|)\,d\xi}-\overline{V_0}\Big\}wi\Big]=0.\end{aligned}$$

Via the Stokes Theorem, the expansion of K_0 given in (3.3.22) and noting that

$$2\partial_{\overline{x}}G(|ix+x'|)=-iG'(|ix+x'|)\frac{ix+x'}{|ix+x'|^2},\tag{3.3.30}$$

for $x,x'\in\mathbb{C}$ and some function $G:\mathbb{R}\to\mathbb{R}$, it reduces to

$$\lim_{\varepsilon\to0}\mathrm{Re}\Big[\Big\{\frac{\lambda}{2\pi^2\varepsilon}\int_{\mathbb{D}}\frac{dA(y)}{w-y}-\frac{i}{2\pi^2\varepsilon}\overline{\int_{\mathbb{T}}g_0(\varepsilon\lambda|w-\xi|)\,d\xi}-\frac{i}{2\pi^2\varepsilon}\sum_{0\neq k\in\mathbb{Z}}\overline{\int_{\mathbb{T}}K_0(\lambda|\varepsilon i(w-\xi)-kl|)\,d\xi}$$

$$-\frac{\lambda\varepsilon}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\overline{i\int_{\mathbb{D}}K_1(\lambda|\varepsilon i(w+y)-a-kl+ih|)\frac{\varepsilon i(w-y)-a-kl+ih}{|\varepsilon i(w-y)-a-kl+ih|}\,dA(y)}-\overline{V_0}\Big\}wi\Big].$$

Note that

$$\lim_{\varepsilon\to 0}\frac{g_0(\varepsilon\lambda|w-\xi|)}{\varepsilon}=0,$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{0\neq k\in\mathbb{Z}}\frac{i}{\varepsilon}\int_{\mathbb{T}}K_0(\lambda|\varepsilon i(x-\xi)-kl|)\,d\xi=&\lim_{\varepsilon\to 0}i\lambda\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{D}}K_1(\lambda|\varepsilon i(w-\xi)-kl|)\frac{\varepsilon i(w-\xi)-kl}{|\varepsilon i(w-\xi)-kl|^2}\\ =&i\lambda\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{D}}K_1(\lambda|kl|)\frac{kl}{|kl|^2}=0,\end{aligned}$$

making use of the Dominated Convergence Theorem. Secondly, via the definition of V_0 in (3.2.11), one has that

$$-\frac{\lambda}{2\pi^2}\sum_{k\in\mathbb{Z}}\overline{i\int_{\mathbb{D}}K_1(\lambda|-a-kl+ih|)\frac{-a-kl+ih}{|-a-kl+ih|}\,dA(y)}-\overline{V_0}=0.$$

The left term is also zero using the computations in Proposition 3.3.3. □

We avoid the proof of the following result, due to the similarity with Proposition 3.3.6.

Proposition 3.3.12. *The function* $V:(-\varepsilon_0,\varepsilon_0)\times B_{X_\alpha}(0,\sigma)\longrightarrow\mathbb{R}$, *given by*

$$V(\varepsilon,f)=\frac{\int_{\mathbb{T}}\overline{I_{QGSW}(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw}{\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw},\tag{3.3.31}$$

fulfills $V(0,f)=V_0$, *where* V_0 *is defined in* (3.2.11). *The parameters satisfy:* $\varepsilon_0\in(0,\min\{1,\frac{l}{4}\})$, $\sigma<1$, $\alpha\in(0,1)$, *and* X *is defined in* (3.3.14).

The next result concerns the well-definition of F_{QGSW} in the spaces defined in (3.3.14)–(3.3.15).

Proposition 3.3.13. *If* V *sets* (3.3.31), *then*

$$\tilde{F}_{QGSW}:(-\varepsilon_0,\varepsilon_0)\times B_{X_\alpha}(0,\sigma)\to Y_\alpha,$$

with $\tilde{F}_{QGSW}(\varepsilon,f)=F_{QGSW}(\varepsilon,f,V(\varepsilon,f))$, *is well–defined and* $\mathscr{C}^1$. *The parameters satisfy* $\alpha\in(0,1)$, $\varepsilon_0\in(0,\min\{1,\frac{l}{4}\})$ *and* $\sigma<1$.

Proof. Note the similarity of this proposition to Proposition 3.3.7. Following its ideas, in order to check the symmetry of F_{QGSW}, it is enough to prove that

$$I_{QGSW}(\varepsilon,f)(\overline{w})=\overline{I_{QGSW}(\varepsilon,f)(w)}.$$

We take advantage of (3.3.19). Via its definition, note that

$$\begin{aligned}\overline{I_{QGSW}(\varepsilon,f)(w)}=&-\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}K_0(\lambda|\varepsilon(\Phi(w)-\Phi(\overline{\xi}))-kl|)\overline{\Phi'(\overline{\xi})}\,d\xi\\ &-\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}K_0(\lambda|\varepsilon(\Phi(w)+\Phi(\overline{\xi}))-a-kl+ih|)\overline{\Phi'(\overline{\xi})}\,d\xi\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} K_0(\lambda|\varepsilon(\Phi(\overline{w})-\Phi(\xi))+kl|)\Phi'(\xi)\,d\xi\\
&+\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} K_0(\lambda|\varepsilon(\Phi(\overline{w})+\Phi(\xi))-a+kl+ih|)\Phi'(\xi)\,d\xi\\
=&I_{QGSW}(\varepsilon,f)(\overline{w}).
\end{aligned}
$$

Note that using the decomposition of I_{QGSW} given in (3.3.29), the regularity problem reduces to the same one for the Euler equations, done in Proposition 3.3.7. □

Finally, we state the result concerning the desingularization of the Kármán Vortex Street.

Theorem 3.3.14. *Let $h, l \in \mathbb{R}$, with $h \neq 0$ and $l > 0$, and $a = 0$ or $a = \frac{l}{2}$. Then, there exist D^ε such that*

$$
q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}}\mathbf{1}_{\varepsilon D^\varepsilon+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}}\mathbf{1}_{-\varepsilon D^\varepsilon+a-ih+kl}(x), \tag{3.3.32}
$$

defines a horizontal translating solution of the quasi–geostrophic shallow water equations, with constant velocity speed, for any $\varepsilon \in (0, \varepsilon_0)$ and small enough $\varepsilon_0 > 0$. Moreover, D^ε is at least $\mathscr{C}^1$.

Proof. By Proposition 3.3.13, we have that $\tilde{F}_{QGSW} : \mathbb{R} \times B_{X_\alpha}(0,\sigma) \to Y_\alpha$, with $\tilde{F}_{QGSW}(\varepsilon, f) = F_{QGSW}(\varepsilon, f, V(\varepsilon, f))$, is well–defined and $\mathscr{C}^1$, for $\varepsilon_0 \in (0, \min\{1, \frac{l}{4}\})$ and $\sigma < 1$. Moreover, Proposition 3.3.11 and Proposition 3.3.12 give us that $\tilde{F}_{QGSW}(0,0)(w) = 0$, for any $w \in \mathbb{T}$. In order to apply the Implicit Function Theorem, let us check that $\partial_f \tilde{F}_{QGSW}(0,0)$ is an isomorphism.

First, using (3.3.29), one achieves

$$
\begin{aligned}
I_{QGSW}(\varepsilon,f)(w) =&I_E(\varepsilon,f)(w) + \frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} g_0\left(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|\right)\Phi'(\xi)\,d\xi\\
&+\frac{1}{2\pi^2\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} g_0\left(\lambda|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|\right)\Phi'(\xi)\,d\xi,
\end{aligned}
$$

where g_0 is a smooth function such that $g_0(z) = O(z^2\ln(z))$ for small z, see (3.3.22). Note that

$$
\lim_{\varepsilon\to 0}\varepsilon I_{QGSW}(\varepsilon,f)(w) = \lim_{\varepsilon\to 0}\varepsilon I_E(\varepsilon,f)(w)
$$

and

$$
\lim_{\varepsilon\to 0}\partial_f\overline{I_{QGSW}(\varepsilon,0)} = \lim_{\varepsilon\to 0}\partial_f\overline{I_E(\varepsilon,0)}.
$$

Thus, we have

$$
\partial_f\tilde{F}_{QGSW}(\varepsilon,0)h(w) = \partial_f\tilde{F}_E(\varepsilon,0)h(w) = -\frac{1}{2\pi}\mathrm{Im}\left[h'(w)\right],
$$

which is an isomorphism from X to Y. □

3.4 Kármán Vortex Street in general models

Kármán Vortex Patch Street structures are found both in the Euler equations and in the QGSW equations. The important fact in both models is that the Green functions associated to the elliptic problem of the stream function have the same behavior close to 0, having then the same linearized operator. We can extend it to other models, where the generalized surface quasi–geostrophic equations are a particular case, see Theorem 3.4.7 for more details. Here, let us work with the general model:

$$\begin{cases} q_t + (v\cdot\nabla)q = 0, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ v = \nabla^\perp\psi, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ \psi = G*q, & \text{in } [0,+\infty)\times\mathbb{R}^2,\\ q(t=0,x) = q_0(x), & \text{with } x\in\mathbb{R}^2. \end{cases} \tag{3.4.1}$$

3.4.1 Scaling the equation

The aim of this section is to look for solutions of the type

$$q_0(x) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_1+kl}(x) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D_2+kl}(x).$$

The domains D_1 and D_2 are simply–connected bounded domains such that $|D_1| = |D_2|$, and $l>0$. Consider $D_2 = -D_1 + a - ih$, having the following distribution

$$q_0(x) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{D+kl}(x) - \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-D+a+kl-ih}(x), \tag{3.4.2}$$

where we are rewriting D_1 by D. The velocity field is given by

$$\begin{aligned}\pi v_0(x) =& 2i\partial_{\overline{x}}\sum_{k\in\mathbb{Z}}\int_D G(|x-y-kl|)\,dA(y) - 2i\partial_{\overline{x}}\sum_{k\in\mathbb{Z}}\int_D G(|x+y-a-kl+ih|)\,dA(y)\\ =& -2i\sum_{k\in\mathbb{Z}}\int_D \partial_{\overline{y}}G(|x-y-kl|)\,dA(y) - 2i\sum_{k\in\mathbb{Z}}\int_D \partial_{\overline{y}}G(|x+y-a-kl+ih|)\,dA(y)\\ =& -\sum_{k\in\mathbb{Z}}\int_{\partial D} G(|x-y-kl|)dy - \sum_{k\in\mathbb{Z}}\int_{\partial D} G(|x+y-a-kl+ih|)dy, \end{aligned} \tag{3.4.3}$$

where $\partial_{\overline{x}}$ is defined in (3.3.1) and the Stokes Theorem (B.0.9) is used. In order to introduce the point model configuration, let us scale the equation in the following way. For any $\varepsilon>0$, define

$$q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D+kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D+a+kl-ih}(x), \tag{3.4.4}$$

for $l>0$, $h\neq 0$ and $a\in\mathbb{R}$. If $|D| = |\mathbb{D}|$, then we arrive at the point vortex street (3.2.7) in the limit when $\varepsilon\to 0$, i.e.,

$$q_{0,0}(x) = \sum_{k\in\mathbb{Z}} \delta_{(kl,0)}(x) - \sum_{k\in\mathbb{Z}} \delta_{(a+kl,-h)}(x). \tag{3.4.5}$$

This configuration of points is studied in Proposition 3.2.6, from the dynamical system point of view, showing that (3.4.5) translates. From now on, take $a=0$ or $a=\frac{l}{2}$ having a horizontal translation in the point model. The associated velocity field to (3.4.4) is given by

$$v_{0,\varepsilon}(\varepsilon x) = -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\partial D} G(|\varepsilon(x-y)-kl|)dy - \frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\partial D} G(|\varepsilon(x+y)-a-kl+ih|)dy.$$

We introduce now the conformal map. Consider $\Phi:\mathbb{T}\to\partial D$ such that

$$\Phi(w) = i\left(w + \frac{\varepsilon}{G(\varepsilon)}f(w)\right), \quad f(w) = \sum_{n\geq 1} a_n w^{-n}, \quad a_n\in\mathbb{R}, w\in\mathbb{T}. \tag{3.4.6}$$

Hence

$$v_{0,\varepsilon}(\varepsilon\Phi(w)) = -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi$$
$$-\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi.$$

Remark 3.4.1. *The constant $G(\varepsilon)$ in the definition of the conformal map* (3.3.9) *comes from the singularity of the kernel in the general case. For the logarithmic singularities, we do not need to add this constant because there we use the structure of the logarithm. When having more singular kernels, as in this case, we need to introduce $G(\varepsilon)$.*

Assuming that we look for translating solutions, i.e., $q(t,x) = q_0(x-Vt)$, we arrive at the equation

$$F(\varepsilon, f, V)(w) := \text{Re}\left[\left\{\overline{I(\varepsilon,f)(w)} - \overline{V}\right\} w\Phi'(w)\right] = 0, \quad w\in\mathbb{T}, \tag{3.4.7}$$

where

$$I(\varepsilon,f)(w) := v_{0,\varepsilon}(\varepsilon\Phi(w)).$$

The next step is to check that if $\varepsilon = 0$, $D = \mathbb{D}$ and $V = V_0$ (referring to the Kármán Point Vortex Street), equation (3.4.7) is verified.

Proposition 3.4.2. *Let G satisfies*

(H1) G is radial such that $G(x) = \tilde{G}(|x|)$,

(H2) there exists $R > 0$ and $\beta_1 \in (0,1]$ such that $|\tilde{G}'(r)| \leq \frac{C}{r^{1+\beta_1}}$, for $r \geq R$,

(H3) there exists $\beta_2 \in (0,1)$ such that $G(z) = O\left(\frac{1}{z^{\beta_2}}\right)$ and $\log|z| = o\left(G(z)\right)$, as $z \to 0$.

For any $h \neq 0$ and $l > 0$, the following equation is verified

$$F(0,0,V_0)(w) = 0, \quad w\in\mathbb{T},$$

where V_0 is given by (3.2.8)*:*

$$V_0 = i\sum_{k\in\mathbb{Z}} G'(|a+kl-ih|)\frac{a+kl-ih}{|a+kl-ih|}.$$

Remark 3.4.3. *From Hypothesis (H3) we are assuming that the kernel is more singular than the logarithmic kernel analyzed in the previous sections, but less singular than the kernel of the surface quasi–geostrophic equation. A typical kernel satisfying (H3) is the one coming from the generalized surface quasi–geostrophic equation: $G(x) = \frac{C_\beta}{2\pi}\frac{1}{|x|^\beta}$, for $\beta\in(0,1)$.*

Proof. By definition,

$$F(\varepsilon,0,V_0)(w) = \text{Re}\Big[\Bigg\{\frac{i}{\pi\varepsilon}\overline{\int_{\mathbb{T}} G(\varepsilon|w-\xi|)\,d\xi} + \frac{i}{\pi\varepsilon}\sum_{0\neq k\in\mathbb{Z}}\overline{\int_{\mathbb{T}} G(|\varepsilon i(w-\xi)-kl|)\,d\xi}$$
$$+\frac{i}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}} G(|\varepsilon i(w+\xi)-a-kl+ih|)\,d\xi} - V_0\Bigg\} iw\Big].$$

Concerning the first term, we can compute it using

$$\operatorname{Re}\left[\frac{w}{\pi\varepsilon}\overline{\int_{\mathbb{T}} G(\varepsilon|w-\xi|)\,d\xi}\right]=\operatorname{Re}\left[\frac{w}{\pi\varepsilon}\overline{w}\overline{\int_{\mathbb{T}} G(\varepsilon|1-\xi|)\,d\xi}\right]=\frac{1}{\pi\varepsilon}\operatorname{Re}\left[\overline{\int_{\mathbb{T}} G(\varepsilon|1-\xi|)\,d\xi}\right]=0, \tag{3.4.8}$$

since the integral is a pure complex number. For the second term, note that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}&\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon i(w-\xi)-kl|)\,d\xi\\
&=-\lim_{\varepsilon\to 0}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{D}} G'(|\varepsilon i(w-\xi)-kl|)\frac{\varepsilon i(w-\xi)-kl}{|\varepsilon i(w-\xi)-kl|}\,d\xi\\
&=\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{D}} G'(|kl|)\frac{kl}{|kl|}\,d\xi\\
&=0,
\end{aligned} \tag{3.4.9}$$

via the Stokes Theorem (B.0.9) and taking into account (3.3.30). We have computed the above limit by using the Convergence Dominated Theorem and (H3). Moreover, the sum is vanishing because we are using the symmetry sum. Using again the Stokes Theorem for the third term, one arrives at

$$\begin{aligned}
\lim_{\varepsilon\to 0}&\left\{\frac{i}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}} G(|\varepsilon i(w+\xi)-a-kl+ih|)\,d\xi}-V_0\right\}\\
&=\lim_{\varepsilon\to 0}\left\{\frac{i}{\pi}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{D}} G'(|\varepsilon i(w+\xi)-a-kl+ih|)\frac{\varepsilon i(w+\xi)-a-kl+ih}{|\varepsilon i(w+\xi)-a-kl+ih|}\,d\xi}-V_0\right\}\\
&=\left\{\frac{i}{\pi}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{D}} G'(|-a-kl+ih|)\frac{-a-kl+ih}{|-a-kl+ih|}\,d\xi}-V_0\right\}\\
&=\left\{i\sum_{k\in\mathbb{Z}}\overline{G'(|-a-kl+ih|)\frac{-a-kl+ih}{|-a-kl+ih|}\,d\xi}-V_0\right\}\\
&=0,
\end{aligned}$$

by definition of V_0, which is given in (3.2.8). Again, the above limit is justified with the Convergence Dominated Theorem and (H3). □

In what follows, we will have to deal with the singularity in ε. But, there is a way to simplify the equation in order to control this singularity. We decompose $I(\varepsilon,f)$ as

$$\begin{aligned}
-\pi I(\varepsilon,f)(w)=&\frac{1}{\varepsilon}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)-\Phi(\xi))|)\Phi'(\xi)\,d\xi+\frac{1}{\varepsilon}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi\\
&+\frac{1}{\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi\\
=&:I_1(\varepsilon,f)(w)+I_2(\varepsilon,f)(w)+I_3(\varepsilon,f)(w).
\end{aligned} \tag{3.4.10}$$

We can use Taylor formula (3.3.18):

$$G(|z_1+z_2|)=G(|z_1|)+\int_0^1 G'(|z_1+tz_2|)\frac{\operatorname{Re}\left[(z_1+tz_2)\overline{z_2}\right]}{|z_1+tz_2|}dt,$$

for $z_1, z_2 \in \mathbb{C}$ and $|z_2| < |z_1|$. In the case of I_1, take $z_1 = i\varepsilon(w-\xi)$ and $z_2 = i\frac{\varepsilon^2}{K(\varepsilon)}(f(w)-f(\xi))$, implying

$$\begin{aligned}
I_1(\varepsilon, f) =& \frac{i}{\varepsilon}\int_{\mathbb{T}} G(\varepsilon|w-\xi|)\, d\xi + i\frac{\varepsilon}{G(\varepsilon)}\int_{\mathbb{T}}\int_0^1 G'\left(\varepsilon\left|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right)\\
&\times \frac{\text{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\overline{(f(w)-f(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|}dt\, d\xi\\
&+\frac{i}{G(\varepsilon)}\int_{\mathbb{T}} G(\varepsilon|\Phi(w)-\Phi(\xi)|)f'(\xi)\, d\xi\\
=& I_{1,1}(\varepsilon, f)(w) + I_{1,2}(\varepsilon, f)(w) + I_{1,3}(\varepsilon, f)(w).
\end{aligned}$$

Let us check that $|z_2| < |z_1|$:

$$|z_2| = \frac{\varepsilon^2}{G(\varepsilon)}|f(w)-f(\xi)| \leq \frac{\varepsilon^2}{G(\varepsilon)}||f||_{\mathscr{C}^1}|w-\xi| \leq \frac{\varepsilon^2}{G(\varepsilon)}|w-\xi| < |z_1|.$$

In virtue of (3.4.8), we get

$$\text{Re}\left[\frac{w}{\pi\varepsilon}\overline{\int_{\mathbb{T}} G(\varepsilon|w-\xi|)\, d\xi}\right] = 0,$$

and then the nonlinear function $F(\varepsilon, f, V)$ can be simplified as follows

$$\begin{aligned}
F(\varepsilon, f, V) =& \text{Re}\left[\left\{\overline{I(\varepsilon, f)(w)} - \overline{V}\right\} w\Phi'(w)\right] - \text{Re}\left[\frac{wf'(w)}{\pi G(\varepsilon)}\overline{\int_{\mathbb{T}} G(\varepsilon|w-\xi|)\, d\xi}\right]\\
=& \text{Re}\left[\left\{\overline{I(\varepsilon, f)(w)} - \overline{V}\right\} w\Phi'(w)\right] + \frac{i\int_{\mathbb{T}} G(\varepsilon|1-\xi|)\, d\xi}{\pi G(\varepsilon)}\text{Im}\left[f'(w)\right].
\end{aligned} \tag{3.4.11}$$

We use the descomposition of $I(\varepsilon, f)$ in (3.4.10), and we are rewriting $I_1(\varepsilon, f)$ as

$$I_1(\varepsilon, f) := I_{1,2}(\varepsilon, f) + I_{1,3}(\varepsilon, f), \tag{3.4.12}$$

since there is any contribution of $I_{1,1}(\varepsilon, f)$.

In the next result, we provide how V must depend on ε and f.

Proposition 3.4.4. *Let G satisfies the hypothesis (H1)–(H3) of Proposition 3.2.6 and Proposition 3.4.2, and*

(H4) $\frac{\tilde{G}(\varepsilon r)}{\tilde{G}(\varepsilon)\tilde{G}(r)} \to 1$, $\frac{\varepsilon\tilde{G}'(\varepsilon r)}{\tilde{G}(\varepsilon)\tilde{G}'(r)} \to 1$, when $\varepsilon \to 0$, *uniformly in* $r \in (0,2)$.

The function $V : (-\varepsilon_0, \varepsilon_0) \times B_{X_{1-\beta_2}}(0,\sigma) \longrightarrow \mathbb{R}$, given by

$$V(\varepsilon, f) = \frac{\int_{\mathbb{T}} \overline{I(\varepsilon, f)(w)} w\Phi'(w)(1-\overline{w}^2)dw}{\int_{\mathbb{T}} w\Phi'(w)(1-\overline{w}^2)dw}, \tag{3.4.13}$$

fulfills $V(0,f) = V_0$, where V_0 is defined in (3.2.8). *The parameters satisfy:* $\varepsilon_0 \in (0, \min\{1, \frac{l}{4}\})$, $\sigma < 1$, *and X is defined in* (3.3.14).

Proof. Note that

$$V(0,f) = \lim_{\varepsilon\to 0}\frac{\int_{\mathbb{T}} \overline{I(\varepsilon, f)(w)} w\Phi'(w)(1-\overline{w}^2)dw}{i\int_{\mathbb{T}} w(1-\overline{w}^2)dw} = -\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{T}} \overline{I(\varepsilon, f)(w)} w(1+\frac{\varepsilon}{G(\varepsilon)}f'(w))(1-\overline{w}^2)dw}{2\pi i},$$

via the Residue Theorem. We use the decomposition of $I(\varepsilon, f)$ given in (3.4.10). Let us begin with $I_1(\varepsilon, f)$:

$$\begin{aligned} I_1(\varepsilon, f) =& I_{1,2}(\varepsilon, f)(w) + I_{1,3}(\varepsilon, f)(w) \\ =& i\frac{\varepsilon}{G(\varepsilon)} \int_{\mathbb{T}} \int_0^1 G'\left(\varepsilon \left|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w) - f(\xi))\right|\right) \\ &\times \frac{\mathrm{Re}\left[\left((w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w) - f(\xi))\right)\overline{(f(w) - f(\xi))}\right]}{|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w) - f(\xi))|} dt\, d\xi \\ &+ \frac{i}{G(\varepsilon)} \int_{\mathbb{T}} G(\varepsilon|\Phi(w) - \Phi(\xi)|) f'(\xi)\, d\xi. \end{aligned}$$

Using (H4) and the Dominated Convergence Theorem, we get

$$I_1(0, f) = i\int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \mathrm{Re}\left[(w-\xi)\overline{(f(w) - f(\xi))}\right] d\xi + i\int_{\mathbb{T}} G(|w-\xi|) f'(\xi)\, d\xi. \tag{3.4.14}$$

Note that the Dominated Convergence Theorem can be applied since the limit in (H4) is uniform. We can compute the above integrals in the following way

$$\begin{aligned} \int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \mathrm{Re}\left[(w-\xi)\overline{(f(w) - f(\xi))}\right] d\xi =& \sum_{n\geq 1} a_n \int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \mathrm{Re}\left[(w-\xi)(w^n - \xi^n)\right] d\xi \\ =& \sum_{n\geq 1} \frac{a_n}{2}\Big\{\overline{w}^n \int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} \overline{[(1-\xi)(1-\xi^n)]}\, d\xi \\ &+ w^{n+2} \int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} [(1-\xi)(1-\xi^n)]\, d\xi\Big\}, \\ \int_{\mathbb{T}} G(|w-\xi|) f'(\xi)\, d\xi =& -\sum_{n\geq 1} a_n n \int_{\mathbb{T}} G(|w-\xi|) \frac{1}{\xi^{n+1}}\, d\xi \\ =& -\sum_{n\geq 1} a_n n \overline{w}^n \int_{\mathbb{T}} G(|1-\xi|) \frac{1}{\xi^{n+1}}\, d\xi, \end{aligned} \tag{3.4.15}$$

where $f(w) = \sum_{n\geq 1} a_n w^{-n}$. Note also that

$$\int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} \overline{[(1-\xi)(1-\xi^n)]}\, d\xi, \int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} [(1-\xi)(1-\xi^n)]\, d\xi, \int_{\mathbb{T}} G(|1-\xi|) \frac{1}{\xi^{n+1}}\, d\xi \in i\mathbb{R}.$$

In this way,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\mathbb{T}} \overline{I_1(\varepsilon, f)(w)} w \left(1 + \frac{\varepsilon}{G(\varepsilon)} f'(w)\right)(1 - \overline{w}^2) dw =& \int_{\mathbb{T}} \overline{I_1(0, f)(w)} w (1 - \overline{w}^2) dw \\ =& \sum_{n\geq 1} a_n \Big\{\frac{i}{2} \int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} \overline{[(1-\xi)(1-\xi^n)]}\, d\xi \int_{\mathbb{T}} w^n w (1 - \overline{w}^2) dw \\ &+ i\int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} [(1-\xi)(1-\xi^n)]\, d\xi \int_{\mathbb{T}} \overline{w}^{n+2} w (1 - \overline{w}^2) dw \\ &- in \int_{\mathbb{T}} G(|1-\xi|) \frac{1}{\xi^{n+1}}\, d\xi \int_{\mathbb{T}} w^n w (1 - \overline{w}^2) dw\Big\} \\ =& 0, \end{aligned}$$

by the Residue Theorem. Let us move on $I_3(\varepsilon, f)$. Here, we use also Taylor formula (3.3.18) for $z_1 = -a - kl + ih$ and $z_2 = \varepsilon(\Phi(w) + \Phi(\xi))$, finding

$$\begin{aligned}
I_3(\varepsilon, f)(w) =& \frac{i}{\varepsilon} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G(|-a-kl+ih|)\, d\xi \\
&+ i \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \int_0^1 G'\left(|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|\right) \\
&\times \frac{\text{Re}\left[(-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi)))\,\overline{(\Phi(w)+\Phi(\xi))}\right]}{|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|}\, dt\, d\xi \\
&+ \frac{i}{G(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi)) - a - kl + ih|) f'(\xi)\, d\xi \\
=& i \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \int_0^1 G'\left(|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|\right) \\
&\times \frac{\text{Re}\left[(-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi)))\,\overline{(\Phi(w)+\Phi(\xi))}\right]}{|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|}\, dt\, d\xi \\
&+ \frac{i}{G(\varepsilon)} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi)) - a - kl + ih|) f'(\xi)\, d\xi \\
=:& I_{3,1}(\varepsilon, f)(w) + I_{3,2}(\varepsilon, f)(w).
\end{aligned} \tag{3.4.16}$$

Let us check that $|z_2| < |z_1|$, which is necessary to use Taylor formula:

$$|z_2| = \varepsilon|\Phi(w) + \Phi(\xi)| \leq 2\varepsilon||\Phi||_{L^\infty} \leq 4\varepsilon < |a + l - ih| \leq |z_1|.$$

Note that

$$I_{3,2}(0, f) = 0.$$

For the other term, we achieve using (H4) that

$$\begin{aligned}
I_{3,1}(0, f)(w) =& i \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G'\left(|-a-kl+ih|\right) \frac{\text{Im}\left[(-a-kl+ih)\,\overline{(w+\xi)}\right]}{|-a-kl+ih|}\, d\xi \\
=& \frac{1}{2} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G'\left(|-a-kl+ih|\right) \frac{(-a-kl+ih)\,\overline{(w+\xi)}}{|-a-kl+ih|}\, d\xi \\
&- \frac{1}{2} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G'\left(|-a-kl+ih|\right) \frac{\overline{(-a-kl+ih)}(w+\xi)}{|-a-kl+ih|}\, d\xi \\
=& i\pi \sum_{k\in\mathbb{Z}} G'\left(|-a-kl+ih|\right) \frac{(-a-kl+ih)}{|-a-kl+ih|} \\
=& -\pi V_0,
\end{aligned}$$

by the Residue Theorem and the definition of V_0 given in (3.2.8). Using the same ideas for $I_2(\varepsilon, f)$, we find that

$$I_2(\varepsilon, f)(w) = i \sum_{0\neq k\in\mathbb{Z}} \int_{\mathbb{T}} \int_0^1 G'\left(|-kl+\varepsilon t(\Phi(w)+\Phi(\xi))|\right)$$

$$\times \frac{\operatorname{Re}\left[(-kl+\varepsilon t(\Phi(w)+\Phi(\xi)))\,\overline{(\Phi(w)+\Phi(\xi))}\right]}{|-kl+\varepsilon t(\Phi(w)+\Phi(\xi))|}dt\,d\xi$$
$$+\frac{i}{G(\varepsilon)}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi))-kl|)f'(\xi)\,d\xi,$$

and then,

$$I_2(0,f)=-i\sum_{0\neq k\in\mathbb{Z}} kl\frac{G'(|kl|)}{|kl|}\operatorname{Re}\left[\overline{w-\xi}\right]\,d\xi=0.$$

This implies that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\overline{I_2(\varepsilon,f)(w)}w\left(1+\frac{\varepsilon}{G(\varepsilon)}f'(w)\right)(1-\overline{w}^2)dw=0.$$

Finally, we find

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\left(1+\frac{\varepsilon}{G(\varepsilon)}f'(w)\right)(1-\overline{w}^2)dw=&-\frac{1}{\pi}\int_{\mathbb{T}}\overline{I_{3,1}(0,f)(w)}w(1-\overline{w}^2)dw\\ =&V_0\int_{\mathbb{T}}w(1-\overline{w}^2)dw\\ =&-2\pi i V_0,\end{aligned}$$

getting the announced result, i.e., $V(0,f)=V_0$. □

Proposition 3.4.5. *Let G satisfies the hypothesis (H1)–(H4) of Proposition 3.2.6, Proposition 3.4.2 and Proposition 3.4.4, and*

(H5) $\frac{d}{d\varepsilon}\frac{\tilde{G}(\varepsilon r)}{\tilde{G}(\varepsilon)\tilde{G}(r)}\to 0$, $\frac{d}{d\varepsilon}\frac{\varepsilon\tilde{G}'(\varepsilon r)}{\tilde{G}(\varepsilon)\tilde{G}'(r)}\to 0$, when $\varepsilon\to 0$, *uniformly in* $r\in(0,2)$.

If V sets (3.4.13), *then*

$$\tilde{F}:(-\varepsilon_0,\varepsilon_0)\times B_{X_{1-\beta_2}}(0,\sigma)\to Y_{1-\beta_2},$$

with $\tilde{F}(\varepsilon,f)=F(\varepsilon,f,V(\varepsilon,f))$, is well–defined and $\mathscr{C}^1$. The spaces X and Y are defined in (3.3.14)-(3.3.15) *taking $\alpha=1-\beta_2$, and the parameters satisfy $\varepsilon_0\in(0,\min\{1,\frac{l}{4}\})$ and $\sigma<1$.*

Proof. The proof has three steps: the symmetry of F, regularity of V, and regularity of $\tilde{F}$.

• *First step: Symmetry of F.* Let us prove that $F(\varepsilon,f,V)(e^{i\theta})=\sum_{n\geq 1} f_n\sin(n\theta)$ with $f_n\in\mathbb{R}$, i.e., checking that F verifies $F(\varepsilon,f,V)(\overline{w})=-F(\varepsilon,f,V)(w)$. First, we work with $I(\varepsilon,f)$ showing that $I(\varepsilon,f)(\overline{w})=\overline{I(\varepsilon,f)(w)}$:

$$\begin{aligned}\overline{I(\varepsilon,f)(w)}=&-\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi}\\ &-\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\overline{\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi}\\ =&\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)-\Phi(\overline{\xi}))-kl|)\overline{\Phi'(\overline{\xi})}\,d\xi\\ &+\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)+\Phi(\overline{\xi}))-a-kl+ih|)\overline{\Phi'(\overline{\xi})}\,d\xi\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)-\Phi(\overline{\xi}))-kl|)\Phi'(\xi)\,d\xi\\
&\quad -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\overline{\xi}))-a-kl+ih|)\Phi'(\xi)\,d\xi\\
&= -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(\overline{w})-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi\\
&\quad -\frac{1}{\pi\varepsilon}\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}} G(|\varepsilon(\Phi(\overline{w})+\Phi(\xi))-a-kl+ih|)\Phi'(\xi)\,d\xi\\
&=I(\varepsilon,f)(\overline{w}).
\end{aligned}
$$

Second, we prove that $V\in\mathbb{R}$ analyzing the denominator and the numerator of its expression:

$$
\begin{aligned}
&2i\mathrm{Im}\Big[\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw\Big]\\
&\quad=\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw-\overline{\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw}\\
&\quad=\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw+\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\overline{\Phi'(\overline{w})}(1-\overline{w}^2)dw\\
&\quad=\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw-\int_{\mathbb{T}}\overline{I(\varepsilon,f)(w)}w\Phi'(w)(1-\overline{w}^2)dw\\
&\quad=0,
\end{aligned}
$$

and

$$
\begin{aligned}
2i\mathrm{Im}\left[\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw\right]&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw-\overline{\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw}\\
&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw+\int_{\mathbb{T}}w\overline{\Phi'(\overline{w})}(1-\overline{w}^2)dw\\
&=\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw-\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw\\
&=0.
\end{aligned}
$$

Thirdly, from the above computations we arrive at

$$
\begin{aligned}
F(\varepsilon,f,V)(\overline{w})&=\mathrm{Re}\left[\left\{\overline{I(\varepsilon,f)(\overline{w})}-\overline{V}\right\}\overline{w}\Phi'(\overline{w})\right]\\
&=-\mathrm{Re}\left[\left\{I(\varepsilon,f)(w)-V\right\}\overline{w}\overline{\Phi'(w)}\right]\\
&=-F(\varepsilon,f,V)(w).
\end{aligned}
$$

Now, we have that $F(\varepsilon,f,V)(e^{i\theta})=\sum_{n\geq1}f_n\sin(n\theta)$. Moreover, the condition (3.4.13) agrees with the fact that $f_1=0$, we refer to the first step in the proof of Proposition 3.3.7 for more details.

• *Second step: Regularity of V.* In similarity with the Euler equations, we need to study first the denominator:

$$
\int_{\mathbb{T}}w\Phi'(w)(1-\overline{w}^2)dw=i\int_{\mathbb{T}}w(w+\varepsilon f'(w))(1-\overline{w}^2)dw=2\pi+i\varepsilon\int_{\mathbb{T}}wf'(w)dw=2\pi-i\varepsilon\int_{\mathbb{T}}f(w)dw,
$$

where we have used the Residue Theorem. Then, if $|\varepsilon| < \varepsilon_0$ and $f \in B_{X_{1-\beta_2}}(0,\sigma)$, the denominator is not vanishing. Moreover, it is $\mathscr{C}^1$ in f and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. By the expression of V, it remains to study the regularity of $J(\varepsilon, f)$:

$$J(\varepsilon,f)(w) = \int_{\mathbb{T}} \overline{I(\varepsilon,f)(w)} w\Phi'(w)(1-\overline{w}^2)dw.$$

We use the decomposition of $I(\varepsilon, f)$ given in (3.4.10). First, we began with the continuity is both variables. Fixing $f \in B_{X_{1-\beta_2}}(0,\sigma)$, note from Proposition 3.4.4 that

$$J(0,f) = 2\pi V_0,$$

and then J is continuous in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Now, fixing $\varepsilon \neq 0$, by Lemma B.0.1 and (H3), we get easily that $I_i(\varepsilon, f) \in \mathscr{C}^{1-\beta_2}(\mathbb{T})$, for $i = 1, 2, 3$.

Secondly, we study the differentiability properties. Fix again $f \in B_{X_{1-\beta_2}}(0,\sigma)$, and we differentiate with respect to ε:

$$\begin{aligned}\frac{d}{d\varepsilon} J(\varepsilon,f)(w) &= \int_{\mathbb{T}} \overline{d_\varepsilon I(\varepsilon,f)(w)} w\Phi'(w)(1-\overline{w}^2)dw \\ &+ i\left(\frac{1}{G(\varepsilon)} - \frac{\varepsilon G'(\varepsilon)}{G(\varepsilon)^2}\right) \int_{\mathbb{T}} \overline{I(\varepsilon,f)(w)} w f'(w)(1-\overline{w}^2)dw. \end{aligned} \tag{3.4.17}$$

The last expression is continuous when $\varepsilon \neq 0$ and now we aim to pass to the limit $\varepsilon \to 0$. By (H3) and (H4), we have that

$$\lim_{\varepsilon\to 0}\left(\frac{1}{G(\varepsilon)} - \frac{\varepsilon G'(\varepsilon)}{G(\varepsilon)^2}\right) = \lim_{\varepsilon\to 0}\left(\frac{1}{G(\varepsilon)} - \frac{G'(1)}{G(\varepsilon)}\right) = 0.$$

By using Proposition 3.4.4, we find

$$\begin{aligned}\lim_{\varepsilon\to 0} I(\varepsilon,f)(w) = &-\frac{i}{\pi}\int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \mathrm{Re}\left[(w-\xi)\overline{(f(w)-f(\xi))}\right] d\xi \\ &- \frac{i}{\pi}\int_{\mathbb{T}} G(|w-\xi|) f'(\xi)\, d\xi + V_0, \end{aligned}\tag{3.4.18}$$

which implies that

$$\lim_{\varepsilon\to 0}\left(\frac{1}{G(\varepsilon)} - \frac{\varepsilon G'(\varepsilon)}{G(\varepsilon)^2}\right) \int_{\mathbb{T}} \overline{I(\varepsilon,f)(w)} w f'(w)(1-\overline{w}^2)dw = 0.$$

Let us analyze the first term of (3.4.17). Using the decomposition (3.4.10), we begin with

$$\int_{\mathbb{T}} \overline{d_\varepsilon I_1(\varepsilon,f)(w)} w\Phi'(w)(1-\overline{w}^2)dw.$$

We use also the decomposition for I_1 given in (3.4.12). For the last term, one has that

$$\frac{d}{d\varepsilon} I_{1,3}(\varepsilon,f)(w) = i\int_{\mathbb{T}} \frac{d}{d\varepsilon}\left(\frac{G(\varepsilon|\Phi(w)-\Phi(\xi)|)}{G(\varepsilon)}\right) f'(\xi)\, d\xi,$$

which is continuous in $\varepsilon \neq 0$ and $f \in B_{X_{1-\beta_2}}(0,\sigma)$. Moreover, (H5) implies that

$$\lim_{\varepsilon\to 0}\frac{d}{d\varepsilon} I_{1,3}(\varepsilon,f)(w) = 0,$$

for any $w \in \mathbb{T}$. Note that we can use the Dominated Convergence Theorem since the limit in (H5) is uniform. We can differentiate $I_{1,2}$ for $\varepsilon \neq 0$ and using once again (H5), we have that

$$\lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} I_{1,2}(\varepsilon, f)(w) = 0.$$

Let us now show the idea of $I_3(\varepsilon, f)$, and $I_2(\varepsilon, f)$ will read similarly. Here, we use Taylor formula (3.3.18) as it was done in (3.4.16):

$$\begin{aligned} I_3(\varepsilon, f)(w) =& I_{3,1}(\varepsilon, f)(w) + I_{3,2}(\varepsilon, f)(w)\\ =& i\sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \int_0^1 G'\left(|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|\right)\\ &\times \frac{\text{Re}\left[(-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi)))\,\overline{(\Phi(w)+\Phi(\xi))}\right]}{|-a-kl+ih+\varepsilon t(\Phi(w)+\Phi(\xi))|}\,dt\,d\xi\\ &+ \frac{i}{G(\varepsilon)}\sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} G(|\varepsilon(\Phi(w)+\Phi(\xi)) - a - kl + ih|) f'(\xi)\,d\xi. \end{aligned}$$

These two expressions are smooth in ε. In fact, we can check that $\frac{d}{d\varepsilon} I_3(\varepsilon, f)$ is continuous in $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $f \in B_{X_{1-\beta_2}}(0, \sigma)$.

Now, fix $\varepsilon \neq 0$ and we focus on the regularity with respect to f. The integral I_1 is the more delicate one since the kernel is singular. Remark the expression of this term:

$$\begin{aligned} I_1(\varepsilon, f) =& i\frac{\varepsilon}{G(\varepsilon)} \int_{\mathbb{T}} \int_0^1 G'\left(\varepsilon\left|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right)\\ &\times \frac{\text{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\overline{(f(w)-f(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|}\,dt\,d\xi\\ &+ \frac{i}{G(\varepsilon)} \int_{\mathbb{T}} G(\varepsilon|\Phi(w)-\Phi(\xi)|) f'(\xi)\,d\xi. \end{aligned}$$

Then,

$$\begin{aligned} \partial_f I_1(\varepsilon, f)h(w) =& i\frac{\varepsilon^3}{G(\varepsilon)^2} \int_{\mathbb{T}} \int_0^1 tG''\left(\varepsilon\left|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right)\\ &\times \frac{\text{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\overline{(f(w)-f(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|^2}\\ &\times \left\{\left((w-\xi)+\frac{t\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\cdot(h(w)-h(\xi))\right\}dt\,d\xi\\ &- i\frac{\varepsilon^2}{G(\varepsilon)^2} \int_{\mathbb{T}} \int_0^1 tG'\left(\varepsilon\left|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right)\\ &\times \frac{\text{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\overline{(f(w)-f(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|^3}\,dt\,d\xi\\ &\times \left\{\left((w-\xi)+\frac{t\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\cdot(h(w)-h(\xi))\right\}dt\,d\xi\\ &+ i\frac{\varepsilon}{G(\varepsilon)} \int_{\mathbb{T}} \int_0^1 G'\left(\varepsilon\left|(w-\xi) + t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right) \end{aligned}$$

$$
\begin{aligned}
&\times \frac{\mathrm{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(h(w)-h(\xi))\right)\overline{(f(w)-f(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|}dt\,d\xi\\
&+i\frac{\varepsilon}{G(\varepsilon)}\int_{\mathbb{T}}\int_0^1 G'\left(\varepsilon\left|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right|\right)\\
&\times \frac{\mathrm{Re}\left[\left((w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\overline{(h(w)-h(\xi))}\right]}{|(w-\xi)+t\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))|}dt\,d\xi\\
&+\frac{i}{G(\varepsilon)}\int_{\mathbb{T}}G(\varepsilon|\Phi(w)-\Phi(\xi)|)h'(\xi)\,d\xi\\
&+\frac{i\varepsilon^2}{G(\varepsilon)^2}\int_{\mathbb{T}}\frac{G'(\varepsilon|\Phi(w)-\Phi(\xi)|)}{|\Phi(w)-\Phi(\xi)|}f'(\xi)\\
&\times\left\{\left((w-\xi)+\frac{\varepsilon}{G(\varepsilon)}(f(w)-f(\xi))\right)\cdot(h(w)-h(\xi))\right\}\,d\xi.
\end{aligned}
$$

For any $\varepsilon\neq 0$, the above expression is continuous in f by using Lemma B.0.1. Moreover, using (H4) we can obtain the limit when $\varepsilon\to 0$ as

$$
\partial_f I_1(0,f)h(w)=i\int_{\mathbb{T}}\frac{G'(|w-\xi|)}{|w-\xi|}\mathrm{Re}\left[(w-\xi)\overline{(h(w)-h(\xi))}\right]\,d\xi+i\int_{\mathbb{T}}G(|w-\xi|)h'(\xi)\,d\xi. \tag{3.4.19}
$$

Note that it agrees when differentiating with respect to f in (3.4.14).

For the other two integrals, notice that I_2 and I_3 are not singular integrals due to $|\varepsilon(\Phi(w)-\Phi(\xi))-kl|$ it not vanishing for $k\neq 0$ and neither $|\varepsilon(\Phi(w)-\Phi(\xi))-a-kl+ih|$, for any $k\in\mathbb{Z}$. This gives us that I_2 and I_3 are $\mathscr{C}^1$. Let us show the idea of I_2:

$$
\begin{aligned}
\partial_f I_2(\varepsilon,f)h(w)=&\partial_f\frac{1}{\varepsilon}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\Phi'(\xi)\,d\xi\\
=&\frac{i}{\varepsilon}\frac{\varepsilon^2}{G(\varepsilon)}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}}G'(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)\\
&\times\frac{(\varepsilon(\Phi(w)-\Phi(\xi))-kl)\cdot(h(w)-h(\xi))}{|\varepsilon(\Phi(w)-\Phi(\xi))-kl|}\Phi'(\xi)\,d\xi\\
&+\frac{i}{\varepsilon}\frac{\varepsilon}{G(\varepsilon)}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{T}}G(|\varepsilon(\Phi(w)-\Phi(\xi))-kl|)h'(\xi)\,d\xi.
\end{aligned}
$$

By Lemma B.0.1, the last expression is continuous in f. Moreover, it does when $\varepsilon\neq 0$ and it is easy to check, with the help of the Convergence Dominated Theorem, that

$$
\partial_f I_2(0,f)h(w)=0. \tag{3.4.20}
$$

In the same way, one can check that $\partial_f I_3(\varepsilon,f)$ is continuous in both variables and

$$
\partial_f I_3(0,f)h(w)=0. \tag{3.4.21}
$$

• *Third step: Regularity of $\tilde{F}$.* Since $V(\varepsilon,f)$ is $\mathscr{C}^1$ in both variables and using the computations above concerning $I(\varepsilon,f)$, one can easily check that $\tilde{F}$ is $\mathscr{C}^1$. □

3.4.2 Main result

Finally, we can announce the result concerning the desingularization of the point model (3.2.7) in the general system. We need to impose an extra condition to G in order to obtain that the linearized operator is an isomorphism.

Theorem 3.4.6. *Consider G satisfying (H1)–(H5) of Proposition* (3.2.6)*, Proposition* (3.4.2) *and Proposition* (3.4.4)*, and*

(H6) $0 \notin \left\{ n \int_{\mathbb{T}} G(|1-\xi|)(1-\overline{\xi}^{n+1})\, d\xi - i \int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} \text{Im}\left[(1-\xi)(1-\xi^n)\right] d\xi, \quad n \geq 1 \right\}.$

Let $h, l \in \mathbb{R}$, with $h \neq 0$ and $l > 0$, and $a = 0$ or $a = \frac{l}{2}$. Then, there exist D^ε such that

$$q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\varepsilon D^\varepsilon + kl}(x) - \frac{1}{\pi\varepsilon^2} \sum_{k \in \mathbb{Z}} \mathbf{1}_{-\varepsilon D^\varepsilon + a - ih + kl}(x), \tag{3.4.22}$$

defines a horizontal translating solution of (3.4.1)*, with constant speed, for any $\varepsilon \in (0, \varepsilon_0)$ and small enough $\varepsilon_0 > 0$. Moreover, D^ε is at least $\mathscr{C}^1$.*

Proof. Let us consider $\tilde{F} : (-\varepsilon_0, \varepsilon_0) \times B_{X_{1-\beta_2}}(0, \sigma) \to Y_{1-\beta_2}$, with $\varepsilon \in (0,1)$, $\varepsilon < \frac{l}{4}$ and $\sigma < 1$, defined in Proposition 3.4.5. By that proposition, it is $\mathscr{C}^1$ in both variables. Moreover, Proposition 3.4.2 and Proposition and 3.4.4, give us that $\tilde{F}(0,0) = 0$. In order to implement the Implicit Function Theorem, let us compute the linearized operator:

$$\begin{aligned} \partial_f \tilde{F}(0,0)h(w) = \lim_{\varepsilon \to 0} \text{Re} \Big[& \left\{ \partial_f \overline{I(0,0)} h(w) - \partial_f V(0,0)h(w) \right\} iw \\ & + \left\{ \overline{I(0,0)(w)} - V_0 \right\} iw \frac{\varepsilon}{G(\varepsilon)} h'(w) \Big] + \frac{i \int_{\mathbb{T}} G(\varepsilon|w-\xi|)\, d\xi}{\pi G(\varepsilon)} \text{Im}\left[h'(w)\right]. \end{aligned}$$

By Proposition 3.4.4, then $\partial_f V(0,f)h(w) \equiv 0$. In virtue of (3.4.18), we have

$$I(0,f)(w) = -\frac{i}{\pi} \int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \text{Re}\left[(w-\xi)\overline{(f(w)-f(\xi))}\right] d\xi - \frac{i}{\pi} \int_{\mathbb{T}} G(|w-\xi|) f'(\xi)\, d\xi + V_0,$$

which implies

$$I(0,0)(w) = V_0.$$

Then, we have

$$\partial_f \tilde{F}(0,0)h(w) = \text{Re}\left[iw \partial_f \overline{I(0,0)} h(w) - \frac{1}{\pi} \overline{\int_{\mathbb{T}} G(|1-\xi|)\, d\xi}\, h'(w) \right].$$

On the other hand, using (3.4.19)-(3.4.20)-(3.4.21) we obtain

$$\begin{aligned} -\pi \partial_f I(0,0)h(w) &= \partial_f I_1(0,0)h(w) \\ &= i \int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \text{Re}\left[(w-\xi)\overline{(h(w)-h(\xi))}\right] d\xi + i \int_{\mathbb{T}} G(|w-\xi|) h'(\xi)\, d\xi, \end{aligned}$$

which amounts to

$$\begin{aligned} \partial_f \tilde{F}(0,0)h(w) = \text{Re}\Big[& -\frac{w}{\pi} \overline{\int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|} \text{Re}\left[(w-\xi)\overline{(h(w)-h(\xi))}\right] d\xi} \\ & - \frac{w}{\pi} \overline{\int_{\mathbb{T}} G(|w-\xi|) h'(\xi)\, d\xi} - \frac{h'(w)}{\pi} \overline{\int_{\mathbb{T}} G(|1-\xi|)\, d\xi} \Big]. \end{aligned}$$

Let us define $\mathcal{K} : X_{1-\beta_2} \to Y_{1-\beta_2}$, as

$$\begin{aligned}\mathcal{K}(h)(w) &= \int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|}\mathrm{Re}\left[(w-\xi)\overline{(h(w)-h(\xi))}\right]\,d\xi\\&+\int_{\mathbb{T}} G(|w-\xi|)h'(\xi)\,d\xi.\end{aligned}$$

Then,

$$\partial_f \tilde{F}(0,0)h(w) = \; -\tfrac{1}{\pi}\mathrm{Re}\Big[h'(w)\overline{\textstyle\int_{\mathbb{T}} G(|1-\xi|)} + w\overline{\mathcal{K}(w)}\Big].$$

For any $h \in X_{1-\beta_2}$ we have that $h \in \mathscr{C}^{2-\beta_2}(\mathbb{T})$, and as a consequence of Lemma B.1 we achieve that $\mathcal{K}(h) \in \mathscr{C}^{2-\beta_2}(\mathbb{T})$. Since $\mathscr{C}^{2-\beta_2}(\mathbb{T})$ is compactly embedded in $\mathscr{C}^{1-\beta_2}(\mathbb{T})$, one has that the operator $\mathcal{K}$ defined above is compact. On the other hand, $h \in X_{1-\beta_2} \mapsto h' \in Y_{1-\beta_2}$ is an isomorphism, which implies that it is a Fredholm operator of zero index. Since compact perturbations of Fredholm operators are also Fredholm operators of same index, we conclude that $\partial_f \tilde{F}(0,0)$ is Fredholm with zero index. As a consequence, checking that $\partial_f \tilde{F}(0,0)$ is an isomorphism, it is enough to check that the kernel is trivial. We can compute the integrals involved in the linearized operator using (3.4.15), and find

$$\begin{aligned}\int_{\mathbb{T}} \frac{G'(|w-\xi|)}{|w-\xi|}\mathrm{Re}\left[(w-\xi)\overline{(h(w)-h(\xi))}\right]\,d\xi =& \sum_{n\geq 1}\frac{a_n}{2}\Big\{\overline{w}^n\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}\overline{[(1-\xi)(1-\xi^n)]}\,d\xi\\&+w^{n+2}\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}[(1-\xi)(1-\xi^n)]\,d\xi\Big\},\\\int_{\mathbb{T}} G(|w-\xi|)h'(\xi)\,d\xi =& -\sum_{n\geq 1}a_n n\overline{w}^n\int_{\mathbb{T}}G(|1-\xi|)\frac{1}{\xi^{n+1}}\,d\xi,\end{aligned}$$

where $h(w) = \sum_{n\geq 1} a_n w^{-n}$. Note also that

$$\begin{aligned}&\int_{\mathbb{T}} G(\varepsilon|1-\xi|)\,d\xi, \int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}\overline{[(1-\xi)(1-\xi^n)]}\,d\xi,\\&\qquad\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}[(1-\xi)(1-\xi^n)]\,d\xi, \int_{\mathbb{T}}G(|1-\xi|)\frac{1}{\xi^{n+1}}\,d\xi \in i\mathbb{R}.\end{aligned}$$

Finally, we achieve

$$\begin{aligned}\partial_f \tilde{F}(0,0)h(w) =& \sum_{n\geq 1}\frac{a_n}{\pi}\mathrm{Re}\Big[\frac{w^{n+1}}{2}\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}\overline{[(1-\xi)(1-\xi^n)]}\,d\xi\\&+\frac{\overline{w}^{n+1}}{2}\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}[(1-\xi)(1-\xi^n)]\,d\xi\\&-nw^{n+1}\int_{\mathbb{T}}G(|1-\xi|)\frac{1}{\xi^{n+1}}\,d\xi - n\overline{w}^{n+1}\int_{\mathbb{T}}G(\varepsilon|1-\xi|)\,d\xi\Big]\\=&\sum_{n\geq 1}\frac{a_n i}{\pi}\sin((n+1)\theta)\Big\{\frac{1}{2}\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}\overline{[(1-\xi)(1-\xi^n)]}\,d\xi\\&-\frac{1}{2}\int_{\mathbb{T}}\frac{G'(|1-\xi|)}{|1-\xi|}[(1-\xi)(1-\xi^n)]\,d\xi\\&-n\int_{\mathbb{T}}G(|1-\xi|)\frac{1}{\xi^{n+1}}\,d\xi + n\int_{\mathbb{T}}G(\varepsilon|1-\xi|)\,d\xi\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 1} \frac{a_n i}{\pi} \sin((n+1)\theta)\Big\{ n\int_{\mathbb{T}} G(|1-\xi|)(1-\overline{\xi}^{n+1})\, d\xi \\
&\quad - i\int_{\mathbb{T}} \frac{G'(|1-\xi|)}{|1-\xi|} \mathrm{Im}\left[(1-\xi)(1-\xi^n)\right]\, d\xi \Big\}.
\end{aligned}
$$

By (H6), we get that the kernel is trivial and then the linerized operator is an isomorphism. □

As a consequence, we get the result for the generalized surface quasi–geostrophic equation, meaning $G = \frac{C_\beta}{2\pi}\frac{1}{|\cdot|^\beta}$, for $\beta \in (0,1)$ and $C_\beta = \frac{\Gamma(\frac{\beta}{2})}{2^{1-\beta}\Gamma(\frac{2-\beta}{2})}$. We just have to check that (H6) is verified and these computations are done in [83] for the vortex pairs.

Theorem 3.4.7. *Let $h, l \in \mathbb{R}$, with $h \neq 0$ and $l > 0$, and $a = 0$ or $a = \frac{l}{2}$. Then, there exists D^ε such that*

$$
q_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{\varepsilon D^\varepsilon + kl}(x) - \frac{1}{\pi\varepsilon^2}\sum_{k\in\mathbb{Z}} \mathbf{1}_{-\varepsilon D^\varepsilon + a - ih + kl}(x), \tag{3.4.23}
$$

defines a horizontal translating solution of the generalized surface quasi-geostrophic equations for $\beta \in (0,1)$, with constant velocity speed, for any $\varepsilon \in (0,\varepsilon_0)$ and small enough $\varepsilon_0 > 0$. Moreover, D^ε is at least $\mathscr{C}^1$.

Chapter 4

Time periodic solutions for 3D quasi–geostrophic model

This chapter is the subject of the following publication:
C. García, T. Hmidi, J. Mateu, *Time periodic solutions for 3D quasi–geostrophic model*, arXiv:2004.01644.

Contents

4.1 Introduction

The large scale dynamics of an inviscid three–dimensional fluid subject to rapid background rotation and strong stratification can be described through the so–called quasi–geostrophic model. It is an asymptotic model derived from the Boussinesq system for vanishing Rossby and Froude numbers, for more details about its formal derivation we refer to [118]. Rigorous derivation can be found in [14, 33, 87].

We point out that this system is a pertinent model commonly used in the ocean and atmosphere circulations to describe the vortices and to track the emergence of long–lived structures. The quasi–geostrophic system is described by the potential vorticity q which is merely advected by the fluid,

$$\begin{cases} \partial_t q + u\partial_1 q + v\partial_2 q = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ \Delta\psi = q, \\ u = -\partial_2\psi, \; v = \partial_1\psi, \\ q(t=0,x) = q_0(x). \end{cases} \tag{4.1.1}$$

The second equation involving the standard Laplacian of $\mathbb{R}^3$ can be formally inverted using Green's function leading to the following representation of the stream function ψ,

$$\psi(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(t,y)}{|x-y|} dA(y),$$

where dA denotes the usual Lebesgue measure. The velocity field $(u,v,0)$ is solenoidal and can be recovered from q through the Biot–Savart law,

$$(u,v)(t,x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x_1-y_1, x_2-y_2)^\perp}{|x-y|^3} q(t,y)dA(y).$$

Notice that the velocity field is planar but its components depends on the all spatial variables and the potential vorticity is transported by the associated flow. The incompressibility of the velocity allows to adapt without any difficulties the classical results known for 2D Euler equations. For instance, see [105], one may get global unique strong solutions when the initial data q_0 belongs to Hölder class $\mathscr{C}^\alpha$, for $\alpha \geq 0$. Yudovich theory [147] can also be implemented and one gets global unique solution when $q_0 \in L^1 \cap L^\infty$. This latter context allows to deal with discontinuous vortices of the patch form, meaning a characteristic function of a bounded domain. This structure is preserved in time and the vortex patch problem consists in studying the regularity of the boundary and to analyze whether singularities can be formed in finite time on the boundary.

For the 2D Euler equations, the $\mathscr{C}^{1,\alpha}$ regularity of the boundary of the patch, with $\alpha \in (0,1)$, is preserved in time, see [35, 17, 137]. The contour dynamics of the patch is in general hardly to track and filamentation may occur. Therefore it is of important interest to look for ordered structure in turbulent flows like relative equilibria. It seems that, only few explicit examples are known in the literature in the patch form: the circular patches which are stationary and the elliptic ones which rotate uniformly with a constant angular velocity. This latter example is known as the Kirchhoff ellipses. However lot of implicit examples with higher symmetry have been constructed during the last decades and the first ones are discovered numerically by DEEM and ZABUSKY [49]. Having this kind of V-states solutions in mind, BURBEA [20] designed a rigorous approach to generate them close to Rankine vortices through complex analysis tools and bifurcation theory. Later this idea was fruitfully improved and extended in various directions generating lot of contributions dealing, for instance, with interesting topics like the regularity problem of the relative equilibria, their existence with different topological structure or for different active scalar equations and so forth. For more details about this active area we refer the reader to the works [25, 26, 27, 28, 50, 51, 53, 56, 65, 67, 69, 76, 77, 81, 82, 83, 84] and the references therein. See also Chapters 2 and 3.

The main concern of this chapter is to investigate the existence of non trivial relative equilibria of the 3D quasi–geostrophic system close to the stationary revolution shapes. In our context, we mean by relative equilibria periodic solutions in the patch form, rotating uniformly about the vertical axis without any deformation. Very recently, REINAUD has explored numerically in [129] the existence and the linear stability of finite volume relative equilibria distributed around circular point vortex arrays. Similar analysis has been implemented in [128] for toroidal vortices. Apart from the numerical experiments, no analytical results had been yet developed and the main inquiry of this paper is to design some technical material allowing to construct relative equilibria close to general smooth stationary revolution shapes. The basic tool is bifurcation theory but as we shall see its implementation is an involved task which requires refined and careful analysis. Let us explain more our strategy and how to proceed. First, we start with deriving the contour dynamic equation for rotating finite volume patches $\mathbf{1}_D$. To do so, we look for smooth domains D with the following parametrization,

$$D = \left\{(re^{i\theta}, \cos(\phi)) \, : \, 0 \leq r \leq r(\phi, \theta), 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\right\},$$

where the shape is sufficiently close to a revolution shape domain, meaning that

$$r(\phi, \theta) = r_0(\phi) + f(\phi, \theta),$$

with small perturbation f. Since the domain is assumed to be smooth then we should prescribe the Dirichlet boundary conditions,

$$r_0(0) = r_0(\pi) = f(0, \theta) = f(\pi, \theta) = 0.$$

Notice that without any perturbation, that is, $f \equiv 0$, the initial data $q_0 = \mathbf{1}_D$ defines a stationary solution for (4.1.1), as we will prove in Lemma 4.2.1. Now a rotating solution about the vertical axis is a time–dependent solution taking the form,

$$q(t, x) = q_0(e^{-i\Omega t} x_h, x_3), \quad q_0 = \mathbf{1}_D, \; x_h = (x_1, x_2).$$

We shall see later that this is equivalent to check that

$$F(\Omega, f)(\phi, \theta) := \psi_0(r(\phi, \theta)e^{i\theta}, \cos(\phi)) - \frac{\Omega}{2} r^2(\phi, \theta) - m(\Omega, f)(\phi) = 0,$$

for any $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$, where

$$m(\Omega, f)(\phi) := \frac{1}{2\pi} \int_0^{2\pi} \left\{\psi_0(r(\phi, \theta)e^{i\theta}, \cos(\phi)) - \frac{\Omega}{2} r^2(\phi, \theta)\right\} d\theta,$$

where ψ_0 stands for the stream function associated to q_0. With this reformulation we visualize the smooth rotating surface as a collection of interacting stratified horizontal sections rotating with the same angular velocity but their size degenerates when we approach the north and south poles corresponding to $\phi \in \{0, \pi\}$.

In order to apply a bifurcation argument, one has to deal with the linearized operator of F around $f=0$. From Proposition 4.3.3 such linearized operator has a compact expression in terms of hypergeometric functions. Indeed, for $h(\phi,\theta)=\sum_{n\geq 1}h_n(\phi)\cos(n\theta)$, one gets

$$\partial_f F(\Omega,0)h(\phi,\theta)=r_0(\phi)\nu_\Omega(\phi)\sum_{n\geq 1}\cos(n\theta)\mathcal{L}_n(h_n)(\phi),$$

where

$$\begin{aligned}\mathcal{L}_n(h_n)(\phi)=&h_n(\phi)-\mathcal{K}_n^\Omega(h_n)(\phi)\\ :=&h_n(\phi)-\int_0^\pi \nu_\Omega^{-1}(\phi)H_n(\phi,\varphi)h_n(\varphi)d\varphi,\end{aligned}$$

with

$$\nu_\Omega(\phi):=\int_0^\pi H_1(\phi,\varphi)d\varphi-\Omega,\quad R(\phi,\varphi):=(r_0(\phi)+r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2$$

and for $n\geq 1$,

$$H_n(\phi,\varphi):=\frac{2^{2n-1}\left(\frac{1}{2}\right)_n^2}{(2n)!}\frac{\sin(\varphi)r_0^{n-1}(\phi)r_0^{n+1}(\varphi)}{[R(\phi,\varphi)]^{n+\frac{1}{2}}}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right).$$

Here F_n denotes the hypergeometric function

$$F_n(x)=F\left(n+\frac{1}{2},n+\frac{1}{2},2n+1,x\right),\quad x\in[0,1).$$

An important observation is that the kernel study of $\partial_f F(\Omega,0)$ amounts to checking whether 1 is an eigenvalue for $\mathcal{K}_n^\Omega$. To do that we first start with symmetrizing this operator by working on suitable weighted Hilbert spaces. A natural candidate for that is the Hilbert space $L^2_{\mu_\Omega}(0,\pi)$ of square integrable functions with respect to the measure

$$d\mu_\Omega(\varphi):=\sin(\varphi)r_0^2(\varphi)\nu_\Omega(\varphi)d\varphi.$$

In general this defines a signed measure and to get a positive one we should restrict the values of Ω to the set $(-\infty,\kappa)$, where $\kappa:=\inf_{\phi\in(0,\pi)}\int_0^\pi H_1(\phi,\varphi)d\varphi$.

In the next step we prove that for any $n\geq 1$, the operator $\mathcal{K}_n^\Omega:L^2_{\mu_\Omega}\to L^2_{\mu_\Omega}$ acts as a compact self–adjoint integral operator. This answers to the structure of the eigenvalues which is a discret set and we establish from the positivity of the kernel that the largest eigenvalue $\lambda_n(\Omega)$ giving the spectral radius is positive and simple. For given integer $n\geq 1$, we define the set

$$\mathscr{S}_n:=\Big\{\Omega\in(-\infty,\kappa)\quad\text{s.t.}\quad\lambda_n(\Omega)=1\Big\},$$

and in Proposition 4.4.3 we shall describe some basic properties on λ_n through precise study of the kernel. Those properties show in particular that the set $\mathscr{S}_n$ is formed by a single point denoted by Ω_n, see Proposition 4.4.4 for more details. In addition, we show that the sequence $n\in N^\star\mapsto\Omega_n$ is strictly increasing which ensures that the kernel of the linearized operator is a one–dimensional vector space, see Proposition 4.4.7. Notice that the preceding weighted space $L^2_{\mu_\Omega}$ is so weak in order to get its stability by the nonlinear functional F. So we need to reinforce the regularity by selecting the standard Hölder spaces $\mathscr{C}^{1,\alpha}$ with Dirichlet boundary condition and $\alpha \in (0,1)$. However this choice generates two delicate problems. The first one is to check that the eigenfunctions constructed in $L^2_{\mu_\Omega}$ are sufficient smooth and belongs to the new spaces. To reach this regularity we need to check that the function ν_Ω is $\mathscr{C}^{1,\alpha}$ and this requires more careful analysis due to the logarithmic singularity, see Proposition 4.4.1. Notice that the eigenfunctions satisfy the boundary condition provided that $n \geq 2$ and which fails for $n = 1$. The second difficulty concerns the stability of the Hölder spaces by the nonlinear functional F, in fact not F but another modified functional $\tilde{F}$ deduced from the preceding one by removing the singularities coming from of the north and south poles, see (4.2.13). The deformation of the Euclidean kernel through the cylindrical coordinates generates singularities on the poles because the size of horizontal sections degenerates on those points. That is the central difficulty when we try to implement potential theory arguments to get the stability of the function spaces and will be discussed in Section 4.5.

Before stating our result, we need to make the following assumptions on the initial profile r_0 and denoted throughout this paper by **(H)** :

(H1) $r_0 \in \mathscr{C}^2([0,\pi])$, with $r_0(0) = r_0(\pi) = 0$ and $r_0(\phi) > 0$ for $\phi \in (0,\pi)$.

(H2) There exists $C > 0$ such that

$$\forall\, \phi \in [0,\pi], \quad C^{-1}\sin\phi \leq r_0(\phi) \leq C\sin(\phi).$$

(H3) r_0 is symmetric with respect to $\phi = \frac{\pi}{2}$, i.e., $r_0\left(\frac{\pi}{2} - \phi\right) = r_0\left(\frac{\pi}{2} + \phi\right)$, for any $\phi \in [0, \frac{\pi}{2}]$.

Now we are ready to give a short version of the main result of this paper and the precise one is detailed in Theorem 4.6.1.

Theorem 4.1.1. *Assume that r_0 satisfies the assumptions* **(H)**. *Then for any $m \geq 2$, there exists a curve of non trivial rotating solutions with m-fold symmetry to the equation* (4.1.1) *bifurcating from the trivial revolution shape associated to r_0 at the angular velocity Ω_m, the unique point of the set $\mathscr{S}_m$.*

We precise that we mean by m-fold shape symmetry of $\mathbb{R}^3$ a surface invariant by rotation with axis (oz) and angle $\frac{2\pi}{m}$.

There is the particular case of $r_0(\phi) = \sin(\phi)$ defining the unit sphere. Here, its associated stream function can be explicitly computed (see [94]) and it is quadratic inside the shape, that is,

$$\psi_0(x) = \frac{1}{6}(x_1^2 + x_2^3 + x_3^2 - 3).$$

That gives us some interesting properties on the eigenvalues Ω_m of the above Theorem 4.1.1. In particular, we achieve that the above eigenvalues Ω_m belongs to $(0, \frac{1}{3})$. Same properties occur also in the case of an ellipsoid of equal x and y axes defining a revolution shape around the z–axis. In this case, the associated stream function is also quadratic. See Section 4.6.1 for a more detailed discussion about those cases.

The paper is structured as follows. In Section 2, we provide different reformulations for the rotating patch problem and we introduce the appropriate function spaces. Section 3 is devoted to different useful expressions of the linearized operator around a stationary solution. The spectral study of the linearized operator will be developed in Section 4. In Section 5, we shall discuss the well–definition of the nonlinear functional and its regularity. In Section 6, we give the general statement of our result and provide its proof. We end this paper with three appendices concerning special functions, bifurcation theory and potential theory.

4.2 Vortex patch equations

Take an initial data uniformly distributed in a bounded domain of $\mathbb{R}^3$, that is, $q_0 = \mathbf{1}_D$. Then, this structure is preserved by the evolution and one gets for any time $t \geq 0$

$$q(t,x) = \mathbf{1}_{D(t)}(x), \tag{4.2.1}$$

for some bounded domain $D(t)$. To track the dynamics of the boundary (which is a surface here) we can implement the contour dynamics method introduced by DEEM and ZABUSKY for Euler equations [49]. Indeed, let $\gamma_t : (\phi,\theta) \in \mathbb{T}^2 \mapsto \gamma_t(\phi,\theta) \in \mathbb{R}^3$ be any parametrization of the boundary ∂D_t. Since the boundary is transported by the flow then

$$\big(\partial_t \gamma_t - U(t,\gamma_t)\big) \cdot n(\gamma_t) = 0, \tag{4.2.2}$$

where $U = (u,v,0)$ and $n(\gamma_t)$ is a unit normal vector to the boundary at the point γ_t. There is a special parametrization called Lagrangian parametrization given by

$$\partial_t \gamma_t = U(t,\gamma_t),$$

which is commonly used to follow the boundary motion. From Biot–Savart law we deduce that

$$U\big(t,\gamma_t(\phi,\theta)\big) = \frac{1}{4\pi}\int_{D_t} \frac{\big(\gamma_t(\phi,\theta)-y\big)^\perp}{|\gamma_t(\phi,\theta)-y|^3} dA(y) = \frac{1}{4\pi}\int_{\partial D_t} \frac{n^\perp(y)}{|\gamma_t(\phi,\theta)-y|} d\sigma(y), \tag{4.2.3}$$

where $d\sigma$ denotes the Lebesgue surface measure of ∂D_t. We have used the notation $x^\perp = (-x_2, x_1, 0) \in \mathbb{R}^3$ for $x = (x_1,x_2,x_3) \in \mathbb{R}^3$.

4.2.1 Stationary patches

Our next goal is to check that any initial patch with revolution shape around the vertical axis generates a stationary solution. More precisely, we have the following result.

Lemma 4.2.1. *Let $r : [-1,1] \to \mathbb{R}_+$ be a continuous function with $r(-1) = r(1) = 0$ and let D be the domain enclosed by the surface $\big\{(r(z)e^{i\theta}, z),\ \theta \in [0,2\pi], z \in [-1,1]\big\}$, then $q(t,x) = \mathbf{1}_D(x)$ defines a stationary solution for* (4.1.1).

Proof. Recall from (4.2.3) that

$$U(x) = \frac{1}{4\pi}\int_D \frac{(x-y)^\perp}{|x-y|^3} dA(y). \tag{4.2.4}$$

Define

$$G(x) := U(x)\cdot x = \frac{1}{4\pi}\int_D \frac{x\cdot y^\perp}{|x-y|^3} dA(y), \quad x \in \mathbb{R}^3,$$

and let us prove that $G \equiv 0$. Take $\theta \in \mathbb{R}$ and denote by $\mathcal{R}_\theta$ the rotation: $x = (x_h, x_3) \mapsto (e^{i\theta}x_h, x_3)$. Since D is invariant by $\mathcal{R}_\theta$, changing variables leads to

$$G(\mathcal{R}_\theta x) = G(x).$$

Therefore $G(x) = G(|x_h|, 0, x_3)$, which means that

$$G(x) = \frac{-|x_h|}{4\pi}\int_D \frac{y_2\, dA(y)}{\big((|x_h|-y_1)^2 + y_2^2 + (x_3-y_3)^2\big)^{\frac{3}{2}}}.$$

Since D is invariant by the reflexion: $y \mapsto (y_1, -y_2, y_3)$ then a change of variables implies that $G(x_1, x_2, x_3) = G(x_1, -x_2, x_3) = -G(x_1, x_2, x_3)$ and thus $G(x) = 0$. Consequently we get in particular that

$$U(x) \cdot x = 0, \quad \forall x \in \partial D.$$

On the other hand, we get from the revolution shape property of D that the horizontal component of the normal vector is $\vec{n}_h(x) = (x_1, x_2)$, which implies

$$U(x) \cdot \vec{n}(x) = (u, v)(x) \cdot \vec{n}_h(x) = 0, \quad \forall\, x \in \partial D. \tag{4.2.5}$$

This implies that $\mathbf{1}_D$ is a stationary solution in the weak sense. □

4.2.2 Reformulations for periodic patches

In this section, we shall give two ways to write down rotating patches using respectively the velocity field and the stream function. Assume that we have a rotating patch around the x_3 axis with constant angular velocity $\Omega \in \mathbb{R}$, that is $D_t = \mathcal{R}_{\Omega t} D$, with $\mathcal{R}_{\Omega t}$ being the rotation of angle Ωt around the vertical axis. Inserting this expression into the equation (4.2.2) we get

$$\big(U(x) - \Omega x^\perp\big) \cdot \vec{n}(x) = 0, \quad \forall\, x \in \partial D.$$

Since U is horizontal then this equation means also that each horizontal section $D_{x_3} := \{y \in \mathbb{R}^2,\ (y, x_3) \in D\}$ rotates with the same angular velocity Ω. Hence the horizontal sections satisfy the equation

$$\big(U(x) - \Omega x^\perp\big) \cdot \vec{n}_{D_{x_3}}(x_h) = 0, \quad x_h = (x_1, x_2) \in \partial D_{x_3},\ x_3 \in \mathbb{R},$$

where $\vec{n}_{D_{x_3}}$ denotes a normal vector to the planar curve ∂D_{x_3}. Next we shall write down this equation in the particular case of simply connected domains that can be described through polar parametrization in the following way:

$$D = \Big\{ (re^{i\theta}, \cos(\phi)) \,:\, 0 \le r \le r(\phi, \theta), 0 \le \theta \le 2\pi, 0 \le \phi \le \pi \Big\}. \tag{4.2.6}$$

Notice that we have assumed in this description, and without any loss of generality, that orthogonal projection over the vertical axis is the segment $[-1, 1]$. The horizontal sections are indexed by ϕ and parametrized by the polar coordinates as $\theta \mapsto r(\phi, \theta)$ and it is obvious that

$$\vec{n}_{\partial D_{x_3}}(r(\phi, \theta) e^{i\theta}) = \left(i\partial_\theta r(\phi, \theta) - r(\phi, \theta)\right) e^{i\theta}.$$

Then, the equation of the sections reduces to

$$\operatorname{Re}\Big[\Big\{ U_h(\phi, \theta) - i\Omega r(\phi, \theta) e^{i\theta} \Big\} \Big\{ \big[i\partial_\theta r(\phi, \theta) + r(\phi, \theta)\big] e^{-i\theta} \Big\}\Big] = 0, \quad \forall (\phi, \theta) \in [0, \pi] \times [0, 2\pi], \tag{4.2.7}$$

with, according to (4.2.3) and the change of variable $y_3 = \cos\varphi$,

$$\begin{aligned}
U_h(\phi, \theta) :=& (U_1, U_2)(r(\phi, \theta), \cos\phi) \\
=& \frac{1}{4\pi} \int_{-1}^{1} \int_{\partial D_{y_3}} \frac{dy_h dy_3}{|(r(\phi, \theta) e^{i\theta}, \cos(\phi)) - y|} \\
=& \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)(\partial_\eta r(\varphi, \eta) e^{i\eta} + i r(\varphi, \eta) e^{i\eta})}{|(r(\phi, \theta) e^{i\theta}, \cos(\phi)) - (r(\varphi, \eta) e^{i\eta}, \cos(\varphi))|} d\eta d\varphi.
\end{aligned} \tag{4.2.8}$$

We shall look for a rotating solution close to a stationary one described by a given revolution shape $(\theta,\phi)\mapsto(r_0(\phi)e^{i\theta},\cos(\phi))$. This means that we are looking for a parametrization in the form

$$r(\phi,\theta)=r_0(\phi)+f(\phi,\theta),\quad f(\phi,\theta)=\sum_{n\geq 1}f_n(\phi)\cos(n\theta). \tag{4.2.9}$$

Implicitly, we have assumed that the domain D is symmetric with respect to the plane $x_2=0$. In addition, we ask the following boundary conditions,

$$r_0(0)=r_0(\pi)=f(0,\theta)=f(\pi,\theta)=0,$$

meaning that the domain D intersects the vertical axis at the points $(0,0,-1)$ and $(0,0,1)$.

Define the functionals

$$F_{\mathbf{v}}(\Omega,f)(\phi,\theta):=\mathrm{Re}\left[\left\{I_{\mathbf{v}}(f)(\phi,\theta)-i\Omega r(\phi,\theta)e^{i\theta}\right\}\left\{i\partial_\theta r(\phi,\theta)e^{-i\theta}+r(\phi,\theta)e^{-i\theta}\right\}\right],$$

with

$$I_{\mathbf{v}}(f)(\phi,\theta):=U_h(\phi,\theta)=\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)(\partial_\eta r(\varphi,\eta)e^{i\eta}+ir(\varphi,\eta)e^{i\eta})}{|(r(\phi,\theta)e^{i\theta},\cos(\phi))-(r(\varphi,\eta)e^{i\eta},\cos(\varphi))|}d\eta d\varphi. \tag{4.2.10}$$

The subscript $\mathbf{v}$ refers to the velocity formulation and we use it to compare it later to the stream function formulation. Hence, we need to study the equation:

$$F_{\mathbf{v}}(\Omega,f)(\phi,\theta)=0,\quad (\phi,\theta)\in[0,\pi]\times[0,2\pi].$$

By Lemma 4.2.1, one has $F_{\mathbf{v}}(\Omega,0)(\phi,\theta)\equiv 0$, for any $\Omega\in\mathbb{R}$.

4.2.3 Stream function formulation

There is another way to characterize the rotating solutions described in the previous subsection by virtue of the stream function formulation.

For $\phi\in[0,\pi]$, let $\theta\in[0,2\pi]\mapsto\gamma_\phi(\theta):=r(\phi,\theta)e^{i\theta}$, be the parametrization of ∂D_z, where $z=\cos(\phi)$. Then one can check without difficulties that (4.2.7) agrees with

$$\partial_\theta\left\{\psi_0(\gamma_\phi(\theta),\cos(\phi))-\frac{\Omega}{2}|\gamma_\phi(\theta)|^2\right\}=0,\quad \forall(\phi,\theta)\in[0,\pi]\times[0,2\pi].$$

Then, the equation can be integrated obtaining

$$\psi_0(\gamma_\phi(\theta),\cos(\phi))-\frac{\Omega}{2}|\gamma_\phi(\theta)|^2=m(\Omega,f)(\phi),$$

where $m(\Omega,f)(\phi)$ is a function depending only on ϕ and given by

$$m(\Omega,f)(\phi):=\frac{1}{2\pi}\int_0^{2\pi}\left\{\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))-\frac{\Omega}{2}r^2(\phi,\theta)\right\}d\theta,\quad r=r_0+f. \tag{4.2.11}$$

Let us consider the functional

$$F_{\mathbf{s}}(\Omega,f)(\phi,\theta):=\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))-\frac{\Omega}{2}r^2(\phi,\theta)-m(\Omega,f)(\phi)$$

$$=G(\Omega,f)(\phi,\theta)-\frac{1}{2\pi}\int_0^{2\pi}G(\Omega,f)(\phi,\eta)d\eta, \tag{4.2.12}$$

where

$$G(\Omega,f)(\phi,\theta):=\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))-\frac{\Omega}{2}r^2(\phi,\theta),$$

and the stream function is given by

$$\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))=-\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)}\frac{\sin(\varphi)rdrd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta)e^{i\theta},\cos(\phi))|}.$$

Then, finding a rotating solution amounts to solving in f, for some specific angular velocity constant Ω, the equation

$$F_{\mathbf{s}}(\Omega,f)(\phi,\theta)=0,\quad \forall(\phi,\theta)\in[0,\pi]\times[0,2\pi].$$

Remark that one may check directly from this reformulation that any revolution shape is a solution for any angular velocity Ω, meaning that, $F_{\mathbf{s}}(\Omega,0)=0$, for any Ω. Motivated by the Section 3 on the structure of the linearized operator, we find better to filter the singularities of the poles and work with the modified functional

$$\tilde{F}(\Omega,f)(\phi,\theta):=\frac{F(\Omega,f)(\phi,\theta)}{r_0(\phi)}.$$

Therefore, we get

$$\tilde{F}(\Omega,f)(\phi,\theta)=\frac{1}{r_0(\phi)}\left\{I(f)(\phi,\theta)-\frac{\Omega}{2}r(\phi,\theta)^2-m(\Omega,f)(\phi)\right\}, \tag{4.2.13}$$

with

$$I(f)(\phi,\theta):=-\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)}\frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta)e^{i\theta},\cos(\phi))|}, \tag{4.2.14}$$

and

$$r(\phi,\theta)=r_0(\phi)+f(\phi,\theta).$$

4.2.4 Functions spaces

First we shall recall the Hölder spaces defined on an open non void set $\mathscr{O}\subset\mathbb{R}^d$. Let $\alpha\in(0,1)$ then

$$\mathscr{C}^{1,\alpha}(\mathscr{O})=\Big\{f:\mathscr{O}\mapsto\mathbb{R},\|f\|_{\mathscr{C}^{1,\alpha}}<\infty\Big\},$$

with

$$\|f\|_{\mathscr{C}^{1,\alpha}}=\|f\|_{L^\infty}+\|\nabla f\|_{L^\infty}+\sup_{x\neq y\in\mathscr{O}}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^\alpha}.$$

It is known that $\mathscr{C}^{1,\alpha}(\mathscr{O})$ is a Banach algebra, meaning a complete space satisfying

$$\|fg\|_{\mathscr{C}^{1,\alpha}}\leq C\|f\|_{\mathscr{C}^{1,\alpha}}\|g\|_{\mathscr{C}^{1,\alpha}}.$$

Denote by $\mathbb{T}$ the one–dimensional torus and we identify the space $\mathscr{C}^{1,\alpha}(\mathbb{T})$ to the space $\mathscr{C}^{1,\alpha}_{2\pi}(\mathbb{R})$ of 2π–periodic functions that belongs to $\mathscr{C}^{1,\alpha}(\mathbb{R})$. The space $\mathscr{C}^{1,\alpha}(\mathbb{T})$ is equipped with the same norm of $\mathscr{C}^{1,\alpha}((0,2\pi))$. Next, we shall introduce the function spaces that we use in a crucial way to study the stability of the functional $\tilde{F}$ defined in (4.2.13). For $\alpha \in (0,1)$ and $m \in \mathbb{N}^\star$, set

$$X_m^\alpha := \Big\{ f \in \mathscr{C}^{1,\alpha}\big((0,\pi)\times\mathbb{T}\big),\ f(\phi,\theta) = \sum_{n\geq 1} f_n(\phi)\cos(nm\theta) \Big\}, \tag{4.2.15}$$

supplemented with the conditions

$$\forall \theta \in [0,2\pi] \quad f(0,\theta) = f(\pi,\theta) = 0 \quad \text{and} \quad \forall (\phi,\theta) \in [0,\pi]\times[0,2\pi] \quad f(\pi-\phi,\theta) = f(\phi,\theta). \tag{4.2.16}$$

This space is equipped with the same norm as $\mathscr{C}^{1,\alpha}((0,\pi)\times(0,2\pi))$. The first assumption in (4.2.16) is a kind of partial Dirichlet condition and the second one is a symmetry property with respect to the equatorial $\phi = \frac{\pi}{2}$. Notice that any function $f \in \mathscr{C}^{1,\alpha}\big((0,\pi)\times\mathbb{T}\big)$ admits a continuous extension up to the boundary, so the foregoing conditions are meaningful. Furthermore, the Dirichlet boundary conditions allow in view of Taylor formula to get a constant $C > 0$ such that for any $f \in X_m^\alpha$

$$\begin{aligned} |f(\varphi,\eta)| &\leq C\|f\|_{\mathrm{Lip}} \sin\varphi, \\ \partial_\eta f(0,\eta) = \partial_\eta f(\pi,\eta) = 0 \quad &\text{and} \quad |\partial_\eta f(\varphi,\eta)| \leq C\|f\|_{\mathscr{C}^{1,\alpha}} \sin^\alpha(\varphi). \end{aligned} \tag{4.2.17}$$

The notation $B_{X_m^\alpha}(\varepsilon)$ means the ball of X_m^α centered in 0 with radius ε.

Next we shall discuss quickly some consequences needed for later purposes and following from the assumptions (**H**) on r_0, given in the Introduction before our main statement.

- From **(H2)** we have that $r_0'(0) > 0$ and by continuity of the derivative there exists $\delta > 0$ such that $r_0'(\phi) > 0$ for $\phi \in [0,\delta]$. Combining this with the mean value theorem, we deduce the arc-chord estimate: there exists $C > 0$ such that

$$C^{-1}(\phi-\varphi)^2 \leq (r_0(\varphi) - r_0(\phi))^2 + (\cos(\phi) - \cos(\varphi))^2 \leq C(\phi-\varphi)^2, \tag{4.2.18}$$

  for any $\phi, \varphi \in [0,\pi]$.

- We have that $\frac{\sin(\cdot)}{r_0(\cdot)} \in \mathscr{C}^\alpha([0,\pi])$, and then $\phi \in [0,\pi] \mapsto \frac{\phi}{r_0(\phi)} \in \mathscr{C}^\alpha([0,\pi])$.

4.3 Linearized operator

This section is devoted to show different expressions of the linearized operator around a revolution shape. We can find an useful one in terms of hypergeometric functions. See Appendix C for details about these special functions.

From now on, we will use the stream function formulation and then we omit the subscript s to F_{s} in order to alleviate the notation. The linearized operator of the velocity formulation is closely related to this one, see the previous section.

4.3.1 First representation

In the following, we provide the structure of the linearized operator of F around the trivial solution $(\Omega, 0)$.

Proposition 4.3.1. *Let $\tilde{F}$ be as in* (4.2.13) *and $(\phi,\theta) \in [0,\pi]\times[0,2\pi] \mapsto h(\phi,\theta) = \sum_{n\geq 1} h_n(\phi)\cos(n\theta)$ be a smooth function. Then,*

$$\partial_f \tilde{F}(\Omega,0)h(\phi,\theta) = -\Omega \sum_{n\geq 1} h_n(\phi)\cos(n\theta)$$

$$+\sum_{n\geq 1}\cos(n\theta)\Bigg\{\frac{h_n(\phi)r_0(\phi)}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(\eta)\ d\eta d\varphi}{\sqrt{r_0^2(\phi)+r_0^2(\varphi)+(\cos\phi-\cos\varphi)^2-2r_0(\phi)r_0(\varphi)\cos(\eta)}}$$
$$-\frac{1}{4\pi r_0(\phi)}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)h_n(\varphi)r_0(\varphi)\cos(n\eta)}{\sqrt{r_0^2(\phi)+r_0^2(\varphi)+(\cos\phi-\cos\varphi)^2-2r_0(\phi)r_0(\varphi)\cos(\eta)}}d\eta d\varphi\Bigg\}. \tag{4.3.1}$$

Proof. First, note that

$$|(re^{i\eta},\cos(\varphi))-(r_0(\phi)e^{i\theta},\cos(\phi))|^2=r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\theta-\eta).$$

The linearized operator at a state r_0 is defined by Gateaux derivative,

$$\begin{aligned}\partial_f\tilde{F}(\Omega,0)h(\phi,\theta):=&\frac{d}{dt}\tilde{F}(\Omega,th)\Big|_{t=0}(\phi,\theta)\\ =&\frac{1}{r_0(\phi)}\left(\frac{d}{dt}G(\Omega,th)\Big|_{t=0}(\phi,\theta)-\frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dt}G(\Omega,th)\Big|_{t=0}(\phi,\eta)d\eta\right).\end{aligned}$$

Thus straightforward computations yield

$$\frac{d}{dt}G(\Omega,th)\Big|_{t=0}(\phi,\theta)=-\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h(\varphi,\eta)d\eta d\varphi}{A(\phi,\theta,\varphi,\eta)^{\frac12}}-\Omega r_0(\phi)h(\phi,\theta)$$
$$+\frac{h(\phi,\theta)}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\sin(\varphi)r(r_0(\phi)-r\cos(\eta))\ drd\eta d\varphi}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}},$$

with

$$A(\phi,\theta,\varphi,\eta):=r_0^2(\varphi)+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\varphi)r_0(\phi)\cos(\theta-\eta).$$

By expanding h in Fourier series we get

$$\partial_fG(\Omega,0)h(\phi,\theta)=-\sum_{n\geq1}\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(n\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac12}}h_n(\varphi)d\eta d\varphi+\Omega r_0(\phi)h_n(\phi)\cos(n\theta)$$
$$+\frac{1}{4\pi}\sum_{n\geq1}h_n(\phi)\cos(n\theta)\int_0^{\pi}\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\sin(\varphi)r(r_0(\phi)-r\cos(\eta))drd\eta d\varphi}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr(\phi,\theta)\cos(\eta))^{\frac32}}.$$

Let us analyze every term. For the first one, making the change of variable $\theta-\eta\mapsto\eta$ we get using a symmetry argument,

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h_n(\varphi)\cos(n\eta)d\eta d\varphi}{A(\phi,\theta,\varphi,\eta)^{\frac12}}=&\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h_n(\varphi)\cos(n(\eta-\theta))d\eta d\varphi}{A(\phi,\theta,\varphi,\theta-\eta)^{\frac12}}\\ =\cos(n\theta)&\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h_n(\varphi)\cos(n\eta)d\eta d\varphi}{(r_0^2(\varphi)+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\varphi)r_0(\phi)\cos(\eta))^{\frac12}}.\end{aligned}$$

Concerning the last integral term, we first use the identity

$$\partial_r\frac{r}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}=$$

$$\frac{1}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}$$
$$-\frac{r(r-r_0(\phi)\cos\eta)}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}.$$

Consequently

$$\begin{aligned}\mathcal{I}(\phi,\varphi)&:=\int_0^{2\pi}\int_0^{r_0(\varphi)}\partial_r\frac{r\cos(\eta)drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}\\&=\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\cos(\eta)drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}\\&\quad-\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{r^2\cos(\eta)-rr_0(\phi)(1-\sin^2(\eta))drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{I}(\phi,\varphi)&=\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{r(r_0(\phi)-r\cos(\eta))drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}\\&\quad+\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\cos(\eta)drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}\\&\quad-\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{rr_0(\phi)\sin^2(\eta)drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}.\end{aligned}$$

Integrating by parts with respect to η gives

$$\begin{aligned}&\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\cos(\eta)drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}\\&\qquad-\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{rr_0(\phi)\sin(\eta)^2drd\eta}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}=0.\end{aligned}$$

Putting together the preceding identities allows to get

$$\begin{aligned}&\int_0^{\pi}\int_0^{2\pi}\int_0^{r_0(\varphi)}\frac{\sin(\varphi)r(r_0(\phi)-r\cos(\eta))drd\eta d\varphi}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac32}}\\&=\int_0^{\pi}\int_0^{2\pi}\int_0^{r_0(\varphi)}\partial_r\frac{\sin(\varphi)r\cos(\eta)drd\eta d\varphi}{(r^2+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2rr_0(\phi)\cos(\eta))^{\frac12}}\\&=\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(\eta)drd\eta d\varphi}{(r_0^2(\varphi)+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\varphi)r_0(\phi)\cos(\eta))^{\frac12}}.\end{aligned}$$

Therefore we obtain

$$\partial_fG(\Omega,0)h(\phi,\theta)=-\sum_{n\geq1}\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(n\eta)}{A(\phi,\theta,\varphi,\theta-\eta)^{\frac12}}h_n(\varphi)d\eta d\varphi\cos(n\theta)$$

$$+\frac{1}{4\pi}\sum_{n\geq 1}h_n(\phi)\cos(n\theta))\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(\eta)drd\eta d\varphi}{(r_0^2(\varphi)+r_0^2(\phi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\varphi)r_0(\phi)\cos(\eta))^{\frac{1}{2}}}$$
$$+\Omega r_0(\phi)\sum_{n\geq 1}h_n(\phi)\cos(n\theta).$$

Now it is clear that

$$\frac{1}{2\pi}\int_0^{2\pi}\partial_f G(\Omega,0)h(\phi,\eta)d\eta=0,$$

and so (4.3.1) is given. □

4.3.2 Second representation with hypergeometric functions

The main purpose of this subsection is to provide a suitable representation of the linearized operator. First we need to use some notations. For $n\geq 1$, set

$$F_n(x):=F\left(n+\frac{1}{2},n+\frac{1}{2},2n+1,x\right),\quad x\in[0,1),$$

where the hypergeometric functions are defined in the Appendix C. Other useful notations are listed below,

$$R(\phi,\varphi):=(r_0(\phi)+r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2,\quad 0<\phi,\varphi<\pi, \tag{4.3.2}$$

and

$$H_n(\phi,\varphi):=\frac{2^{2n-1}\left(\frac{1}{2}\right)_n^2}{(2n)!}\frac{\sin(\varphi)r_0^{n-1}(\phi)r_0^{n+1}(\varphi)}{[R(\phi,\varphi)]^{n+\frac{1}{2}}}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right). \tag{4.3.3}$$

Remark 4.3.2. *Note from the above expression that*

$$\int_0^{\pi}H_1(\phi,\varphi)d\varphi=\frac{1}{r_0(\phi)}\partial_R\psi_0(Re^{i\theta},\cos(\phi))\left|_{R=r_0(\phi)}\right., \tag{4.3.4}$$

where ψ_0 is the stream function at $t=0$ associated to the domain parametrized by $(r_0(\phi)e^{i\theta},\cos(\phi))$, for $(\phi,\theta)\in[0,\pi]\times[0,2\pi]$.

Now we are ready to state the main result of this section.

Proposition 4.3.3. *Let $\tilde{F}$ be as in* (4.2.13) *and $h(\phi,\theta)=\sum_{n\geq 1}h_n(\phi)\cos(n\theta)$, $(\phi,\theta)\in[0,\pi]\times[0,2\pi]$, be a smooth function. Then,*

$$\partial_f\tilde{F}(\Omega,0)h(\phi,\theta)=\sum_{n\geq 1}\cos(n\theta)\mathcal{L}_n^{\Omega}(h_n)(\phi), \tag{4.3.5}$$

where

$$\mathcal{L}_n^{\Omega}(h_n)(\phi)=h_n(\phi)\left[\int_0^{\pi}H_1(\phi,\varphi)d\varphi-\Omega\right]-\int_0^{\pi}H_n(\phi,\varphi)h_n(\varphi)d\varphi,\quad \phi\in(0,\pi).$$

Proof. With the help of Lemma C.0.1, we can simplify more the expression of the linearized operator given in Proposition 4.3.1. We shall first give another representation of the first integral of (4.3.1),

$$\begin{aligned}\mathscr{I}_1(\phi) :=&\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\cos(\eta)}{\sqrt{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\phi)r_0(\varphi)\cos(\eta)}}d\eta d\varphi\\ =&\frac{1}{4\pi}\int_0^{\pi}\frac{\sin(\varphi)r_0(\varphi)}{\sqrt{2r_0(\phi)r_0(\varphi)}}\int_0^{2\pi}\frac{\cos(\eta)}{\sqrt{\frac{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2}{2r_0(\phi)r_0(\varphi)}-\cos(\eta)}}d\eta d\varphi.\end{aligned}$$

From Lemma C.0.1 we infer

$$\int_0^{2\pi}\frac{\cos(\eta)\ d\eta}{\sqrt{\frac{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2}{2r_0(\phi)r_0(\varphi)}-\cos(\eta)}}=2\pi\frac{2\left(\frac{1}{2}\right)_1^2}{2!}\frac{F_1\left(\frac{2}{1+\frac{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2}{2r_0(\phi)r_0(\varphi)}}\right)}{\left(1+\frac{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2}{2r_0(\phi)r_0(\varphi)}\right)^{\frac{3}{2}}}.$$

Thus we deduce

$$\mathscr{I}_1(\phi)=r_0(\phi)\frac{1}{4}\int_0^{\pi}\frac{\sin(\varphi)r_0^2(\varphi)}{R^{\frac{3}{2}}(\phi,\varphi)}F_1\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)d\varphi=r_0(\phi)H_1(\phi,\varphi).$$

Remark that the validity of Lemma C.0.1 is guaranteed since the inequality

$$\left|\frac{2}{1+\frac{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2}{2r_0(\phi)r_0(\varphi)}}\right|=\left|\frac{4r_0(\phi)r_0(\varphi)}{(r_0(\phi)+r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2}\right|<1,$$

is satisfied provided that $\phi\neq\varphi$ which leads to a negligible set. For the last integral in (4.3.1), we apply once again Lemma C.0.1,

$$\begin{aligned}&\int_0^{2\pi}\frac{\cos(n\eta)}{\sqrt{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\phi)r_0(\varphi)\cos(\eta)}}d\eta\\ &=\frac{2^{2n+1}\pi\left(\frac{1}{2}\right)_n^2}{(2n)!}\frac{r_0^n(\phi)r_0^n(\varphi)}{R^{n+\frac{1}{2}}(\phi,\varphi)}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)h_n(\varphi)r_0(\varphi)\cos(n\eta)}{\sqrt{r_0^2(\phi)+r_0^2(\varphi)+(\cos(\phi)-\cos(\varphi))^2-2r_0(\phi)r_0(\varphi)\cos(\eta)}}d\eta d\varphi\\ &=\frac{2^{2n-1}\left(\frac{1}{2}\right)_n^2}{(2n)!}\int_0^{\pi}\frac{r_0^n(\phi)r_0^{n+1}(\varphi)\sin(\varphi)}{R^{n+\frac{1}{2}}(\phi,\varphi)}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)h_n(\varphi)d\varphi\\ &=r_0(\phi)H_n(\phi,\varphi),\end{aligned}$$

which gives the announced result. □

4.3.3 Qualitative properties of some auxiliary functions

In the following lemma, we shall study some specific properties of the sequence of functions $\{H_n\}_n$ introduced in (4.3.3). We shall study the monotonicity of the sequence $n \mapsto H_n(\phi,\varphi)$ which will be crucial later in the study of the monotonicity of the eigenvalues associated to the operators family $\{\mathcal{L}_n, n\geq 1\}$. We will also study the decay rate of this sequence for large n.

Lemma 4.3.4. *For any $\varphi\neq\phi\in(0,\pi)$, the sequence $n\in\mathbb{N}^\star\mapsto H_n(\phi,\varphi)$ is strictly decreasing.*

Moreover, if we assume that r_0 satisfies **(H2)***, then, for any $0\leq\alpha<\beta\leq 1$ there exists a constant $C>0$ such that*

$$|H_n(\phi,\varphi)|\leq Cn^{-\alpha}\frac{\sin(\varphi)r_0^{\frac{1}{2}}(\varphi)}{r_0^{\frac{3}{2}}(\phi)}|\phi-\varphi|^{-\beta},\quad \forall n\geq 1, \phi\neq\varphi\in(0,\pi). \tag{4.3.6}$$

Proof. By virtue of (4.3.3) we may write

$$H_n(\phi,\varphi)=\frac{2^{2n-1}\Gamma^2\left(n+\frac{1}{2}\right)}{(2n)!\pi}\frac{\sin(\varphi)r_0(\varphi)^{\frac{1}{2}}}{4^{n+\frac{1}{2}}r_0(\phi)^{\frac{3}{2}}}x^{n+\frac{1}{2}}F_n(x),$$

where $x:=\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}$ belongs to $[0,1)$ provided that $\varphi\neq\phi$. Now using the integral representation of hypergeometric functions (C.0.2) we obtain

$$\begin{aligned}
H_n(\phi,\varphi)=&\frac{2^{2n-1}\Gamma^2\left(n+\frac{1}{2}\right)}{(2n)!\pi}\frac{\sin(\varphi)r_0^{\frac{1}{2}}(\varphi)}{4^{n+\frac{1}{2}}r_0^{\frac{3}{2}}(\phi)}\frac{(2n)!}{\Gamma^2\left(n+\frac{1}{2}\right)}x^{n+\frac{1}{2}}\int_0^1 t^{n-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}(1-tx)^{-n-\frac{1}{2}}dt\\
=&\frac{2^{2n-1}\Gamma^2\left(n+\frac{1}{2}\right)}{(2n)!\pi}\frac{\sin(\varphi)r_0^{\frac{1}{2}}(\varphi)}{4^{n+\frac{1}{2}}r_0^{\frac{3}{2}}(\phi)}\frac{(2n)!}{\Gamma^2\left(n+\frac{1}{2}\right)}\mathcal{H}_n(x)\\
=&\frac{1}{4\pi}\frac{\sin(\varphi)r_0(\varphi)^{\frac{1}{2}}}{r_0^{\frac{3}{2}}(\phi)}\mathcal{H}_n(x),
\end{aligned} \tag{4.3.7}$$

with the notation

$$\mathcal{H}_n(x):=x^{n+\frac{1}{2}}\int_0^1 t^{n-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}(1-tx)^{-n-\frac{1}{2}}dt.$$

Therefore the announced result amounts to checking that $n\mapsto\mathcal{H}_n(x)$ is strictly decreasing for any $x\in(0,1)$. This follows from the fact that $n\mapsto x^{n+\frac{1}{2}}$ is strictly decreasing combined with the identity

$$\int_0^1 t^{n-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}(1-tx)^{-n-\frac{1}{2}}dt=\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-tx)^{-\frac{1}{2}}\left(\frac{t(1-t)}{1-tx}\right)^n dt,$$

which shows the strict decreasing of this sequence since $0<\frac{t(1-t)}{1-tx}<1$, for any $t,x\in(0,1)$.

It follows that for any $\phi\neq\varphi$, the sequence $n\mapsto H_n(\phi,\varphi)$ is strictly decreasing.

It remains to prove the decay estimate of H_n for large n. It is an immediate consequence of the following more precise estimate: for any $\alpha\in[0,1]$, we get

$$|\mathcal{H}_n(z)|\leq\sqrt{2}|z|^{n-\frac{1}{2}+\alpha}\frac{|\ln(1-|z|)|^{1-\alpha}}{n^\alpha(1-|z|)^\alpha}, \tag{4.3.8}$$

for $n \geq 1$ and $|z| < 1$. To see the connection with (4.3.6) recall first from (4.3.7) that

$$|H_n(\phi, \varphi)| \lesssim \frac{\sin(\varphi) r_0(\varphi)^{\frac{1}{2}}}{r_0(\phi)^{\frac{3}{2}}} |\mathcal{H}_n(x)|.$$

Since $0 \leq x \leq 1$ then we obtain from (4.3.8) that for any $1 \geq \beta > \alpha \geq 0$

$$|\mathcal{H}_n(x)| \lesssim \frac{|\ln(1-x)|^{1-\alpha}}{n^\alpha (1-x)^\alpha} \lesssim n^{-\alpha}(1-x)^{-\beta}.$$

According to (4.4.8) we deduce that

$$|\mathcal{H}_n(x)| \lesssim \frac{1}{n^\alpha |\phi - \varphi|^\beta}.$$

which is the announced inequality. Let us now turn to the proof of (4.3.8). We write

$$|\mathcal{H}_n(z)| \leq \sqrt{2}|z|^{n+\frac{1}{2}} \int_0^1 t^{n-\frac{1}{2}} \frac{(1-t)^{n-\frac{1}{2}}}{|1-tz|^{n-\frac{1}{2}}} \frac{dt}{|1-tz|} \leq \sqrt{2}|z|^{n+\frac{1}{2}} \int_0^1 t^{n-\frac{1}{2}} \frac{dt}{1-t|z|},$$

where we have used that

$$|1-tz| \geq 1 - t|z| \geq 1 - t,$$

for any $t \in [0,1]$ and $|z| < 1$. Observe that we get easily the identity

$$\int_0^1 t^{n-\frac{1}{2}} \frac{dt}{1-t|z|} = \sum_{k \geq 0} \frac{|z|^k}{n + \frac{1}{2} + k}, \tag{4.3.9}$$

which implies

$$\int_0^1 t^{n-\frac{1}{2}} \frac{dt}{1-t|z|} \leq \frac{1}{n(1-|z|)},$$

and

$$\int_0^1 t^{n-\frac{1}{2}} \frac{dt}{1-t|z|} \leq -\frac{\ln(1-|z|)}{|z|}.$$

By using interpolation, we obtain

$$\int_0^1 t^{n-\frac{1}{2}} \frac{dt}{1-t|z|} \leq \frac{1}{n^\alpha} \frac{|\ln(1-|z||^{1-\alpha}}{|z|^{1-\alpha}(1-|z|)^\alpha},$$

which gives us

$$|\mathcal{H}_n(z)| \leq \sqrt{2}|z|^{n+\frac{1}{2}} \frac{|\ln(1-|z||^{1-\alpha}}{|z|^{1-\alpha}(1-|z|)^\alpha} \frac{1}{n^\alpha},$$

for $n \in \mathbb{N}^\star$ and $|z| < 1$. □

4.4 Spectral study

In this section, we aim to investigate some fundamental spectral properties of the linearized operator $\partial_f F(\Omega, 0)$ in order to apply the Crandall–Rabinowitz theorem. For this goal one must check that the kernel and the co–image of the linearized operator are one dimensional vector spaces. Noting that the study of the kernel agrees with the eigenvalue problem of a Hilbert–Schmidt operator, we achieve that the dimension is one. Moreover, we will study the Fredholm structure of the linearized operator, which will imply that the codimension of the image is one. At the end of the section, we characterize also the transversal condition.

4.4.1 Symmetrization of the linearized operator

The main strategy to explore some spectral properties of the linearized operator at each frequency level n is to construct a suitable Hilbert space, basically an L^2 space with respect to a special Borel measure, on which it acts as a self–adjoint compact operator. Later we investigate the eigenspace associated with the largest eigenvalue and prove in particular that this space is one–dimensional.

Let us explain how to symmetrize the operator. Recall from (4.3.5) that for any smooth function $h(\phi,\theta)=\sum_{n\geq 1} h_n(\phi)\cos(n\theta)$, we may write the operator $\mathcal{L}_n$ under the form

$$\mathcal{L}_n^{\Omega}(h)(\phi)=\nu_\Omega(\phi)\left\{h(\phi)-\int_0^\pi K_n(\phi,\varphi)h(\varphi)d\mu_\Omega(\varphi)\right\},\quad \phi\in[0,\pi], \tag{4.4.1}$$

with

$$K_n(\phi,\varphi):=\frac{H_n(\phi,\varphi)}{\sin(\varphi)\nu_\Omega(\phi)\nu_\Omega(\varphi)r_0^2(\varphi)}, \tag{4.4.2}$$

$$\nu_\Omega(\phi):=\int_0^\pi H_1(\phi,\varphi)d\varphi-\Omega, \tag{4.4.3}$$

and the signed measure

$$d\mu_\Omega(\varphi):=\sin(\varphi)r_0^2(\varphi)\nu_\Omega(\varphi)d\varphi. \tag{4.4.4}$$

Define the quantity

$$\kappa:=\inf_{\phi\in[0,\pi]}\int_0^\pi H_1(\phi,\varphi)d\varphi. \tag{4.4.5}$$

We shall discuss in Proposition 4.4.1 below the existence of κ which allows to guarantee the positivity of the measure $d\mu_\Omega$ provided that the parameter Ω is restricted to lie in the interval $(-\infty,\kappa)$. We shall also study the regularity of the function ν_Ω which is delicate and more involved. In particular, we prove that, under reasonable assumptions on the profile r_0, this function is at least in the Hölder space $\mathscr{C}^{1,\alpha}$ for any $\alpha\in(0,1)$.

Notice that the kernel K_n is symmetric. Indeed, according to (4.3.3) we get the formula

$$K_n(\phi,\varphi)=\frac{2^{2n-1}\left(\frac{1}{2}\right)_n^2}{(2n)!}\frac{r_0^{n-1}(\phi)r_0^{n-1}(\varphi)}{\nu_\Omega(\phi)\nu_\Omega(\varphi)\left[R(\phi,\varphi)\right]^{n+\frac{1}{2}}}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right), \tag{4.4.6}$$

which gives the announced property in view of the symmetry of R, that is, $R(\phi,\varphi)=R(\varphi,\phi)$.

We shall explore in Section 4.4.3 more spectral properties of the symmetric operator associated to the kernel K_n.

4.4.2 Regularity of ν_Ω

The main goal of this paragraph is to analyze the regularity of the function ν_Ω. For lower regularities than Lipschitz class, this can be implemented in a standard way using some boundary behavior of the hypergeometric functions. However for higher regularity of type $\mathscr{C}^{1,\alpha}$, the problem turns out to be more delicate due to some logarithmic singularity induced by H_1. To get rid of this singularity we use some specific cancellation coming from the structure of the kernel. We shall also develop the local structure of ν_Ω near its minimum which appears to be crucial later especially in Proposition 4.4.3.

The main result of this section reads as follows.

Proposition 4.4.1. *Let r_0 be a profile satisfying* **(H1)** *and* **(H2)**. *Then the following properties hold true.*

1. *The function $\phi \in [0,\pi] \mapsto \nu_\Omega(\phi)$ belongs to $\mathscr{C}^\beta([0,\pi])$, for all $\beta \in [0,1)$.*
2. *We have $\kappa > 0$ and for any $\Omega \in (-\infty,\kappa)$ we get*
$$\forall \phi \in [0,\pi], \quad \nu_\Omega(\phi) \geq \kappa - \Omega > 0.$$
3. *The function ν_Ω belongs to $\mathscr{C}^{1,\alpha}([0,\pi])$, for any $\alpha \in (0,1)$, with*
$$\nu'_\Omega(0) = \nu'_\Omega(\pi) = 0.$$
4. *Let $\Omega \in (-\infty,\kappa]$ and assume that ν_Ω reaches its minimum at a point $\phi_0 \in [0,\pi]$ then there exists $C > 0$ independent of Ω such that,*
$$\forall \phi \in [0,\pi], \quad 0 \leq \nu_\Omega\big(\phi\big) - \nu_\Omega\big(\phi_0\big) \leq C|\phi - \phi_0|^{1+\alpha}.$$
Moreover, for $\Omega = \kappa$ this result becomes
$$\forall \phi \in [0,\pi], \quad 0 \leq \nu_\kappa\big(\phi\big) \leq C|\phi - \phi_0|^{1+\alpha}.$$

Proof. (**1**) To start, notice first that according to (4.3.3)

$$\begin{aligned} H_1(\phi,\varphi) =& \frac{1}{4\pi}\frac{\sin(\varphi) r_0^2(\varphi)}{[R(\phi,\varphi)]^{\frac{3}{2}}} F_1\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right), \quad \forall\, \varphi \neq \phi \in (0,\pi) \\ =& \mathscr{K}_1(\phi,\varphi)\mathscr{K}_2(\phi,\varphi), \end{aligned} \tag{4.4.7}$$

where
$$\mathscr{K}_1(\phi,\varphi) := \frac{1}{4\pi}\frac{\sin(\varphi) r_0^2(\varphi)}{R^{\frac{3}{2}}(\phi,\varphi)}.$$

Therefore we may write

$$\forall \phi \in (0,\pi), \quad \nu_\Omega(\phi) = \int_0^\pi \mathscr{K}_1(\phi,\varphi)\mathscr{K}_2(\phi,\varphi) d\varphi - \Omega.$$

Using the boundary behavior of hypergeometric functions stated in Proposition C.0.2 we deduce that

$$\begin{aligned} 1 \leq F_1\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right) \leq & C + C\ln\left(1 - \frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right) \\ \leq & C + C\ln\left(\frac{(r_0(\phi)+r_0(\varphi))^2 + (\cos\phi - \cos\varphi)^2}{(r_0(\phi)-r_0(\varphi))^2 + (\cos\phi - \cos\varphi)^2}\right). \end{aligned}$$

From the assumption **(H2)** on r_0 we can write, using the mean value theorem

$$(r_0(\phi)+r_0(\varphi))^2 + (\cos\phi - \cos\varphi)^2 \leq C(\sin\phi + \sin\varphi)^2 + (\phi - \varphi)^2.$$

In view of (4.2.18) we get for all $\phi \neq \varphi \in [0,\pi]$,

$$1 \leq \frac{(r_0(\phi)+r_0(\varphi))^2 + (\cos\phi - \cos\varphi)^2}{(r_0(\phi)-r_0(\varphi))^2 + (\cos\phi - \cos\varphi)^2} \leq C\frac{(\sin\phi + \sin\varphi)^2}{(\phi-\varphi)^2} + C. \tag{4.4.8}$$

Consequently, we get

$$1 \leq \mathscr{K}_2(\phi,\varphi) \leq C + C\ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right). \tag{4.4.9}$$

On the other hand, it is obvious using the assumption **(H2)** on r_0 that

$$0 \leq \frac{\sin(\varphi)r_0^2(\varphi)}{[R(\phi,\varphi)]^{\frac{3}{2}}} \leq \frac{\sin\varphi}{r_0(\varphi)} \leq C, \quad \forall \varphi,\phi \in (0,\pi).$$

It follows that

$$\sup_{\phi\in(0,\pi)} |\nu_\Omega(\phi)| \leq C + C\sup_{\phi\in(0,\pi)}\int_0^\pi \ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right)d\varphi \leq C,$$

which ensures that ν_Ω is bounded. Now let us check the Hölder continuity. First that

$$\partial_\phi R(\phi,\varphi) = 2r_0'(\phi)(r_0(\phi)+r_0(\varphi)) + 2\sin\phi(\cos\varphi - \cos\phi), \tag{4.4.10}$$

which implies that

$$|\partial_\phi R(\phi,\varphi)| \leq CR^{\frac{1}{2}}(\phi,\varphi).$$

It follows that

$$|\partial_\phi R^{-\frac{3}{2}}(\phi,\varphi)| \lesssim R^{-2}(\phi,\varphi) \lesssim r_0^{-4}(\varphi).$$

Differentiating $\mathscr{K}_1$ with respect to ϕ yields

$$4\pi\partial_\phi \mathscr{K}_1(\phi,\varphi) = \partial_\phi(R^{-\frac{3}{2}})(\phi,\varphi)\sin\varphi\, r_0^2(\varphi).$$

Hence using **(H2)** we deduce that

$$\sup_{\phi\in(0,\pi)} |\partial_\phi \mathscr{K}_1(\phi,\varphi)| \leq C\frac{\sin\varphi}{r_0^2(\varphi)} \lesssim \frac{1}{\sin\varphi}\cdot \tag{4.4.11}$$

From an interpolation argument using the boundedness of $\mathscr{K}_1$ we find, according to the mean value theorem,

$$\begin{aligned}|\mathscr{K}_1(\phi_1,\varphi)-\mathscr{K}_1(\phi_2,\varphi)| &= |\mathscr{K}_1(\phi_1,\varphi)-\mathscr{K}_1(\phi_2,\varphi)|^{1-\beta}|\mathscr{K}_1(\phi_1,\varphi)-\mathscr{K}_1(\phi_2,\varphi)|^{\beta}\\ &\leq \big(2\|\mathscr{K}_1\|_{L^\infty}\big)^{1-\beta}\|\mathscr{K}_1\|_{\text{Lip}}^{\beta}|\phi_1-\phi_2|^\beta\\ &\lesssim \frac{|\phi_1-\phi_2|^\beta}{\sin^\beta\varphi}\cdot\end{aligned} \tag{4.4.12}$$

Next we shall proceed in a similar way to the estimate $\mathscr{K}_2$. Using Leibniz rule implies that

$$\partial_\phi \mathscr{K}_2(\phi,\varphi) = 4r_0(\varphi)\frac{r_0'(\phi)R(\phi,\varphi) - r_0(\phi)\partial_\phi R(\phi,\varphi)}{R^2(\phi,\varphi)}F_1'\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right).$$

We know that

$$\forall x \in (-1,1), \quad F_1'(x) = \frac{3}{4}F(5/2,5/2;4;x).$$

Hence by virtue of the boundary behavior stated in Proposition C.0.2 we get

$$\forall x \in [0,1), \quad |F_1'(x)| \lesssim (1-x)^{-1}.$$

It follows that, using (4.2.18),

$$\forall \phi \neq \varphi \in (0,\pi) \quad \Big| F_1'\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)\Big| \lesssim \frac{(r_0(\phi)+r_0(\varphi))^2+(\cos\phi-\cos\varphi)^2}{(r_0(\phi)-r_0(\varphi))^2+(\cos\phi-\cos\varphi)^2} \lesssim \frac{R(\phi,\varphi)}{|\phi-\varphi|^2}. \tag{4.4.13}$$

By explicit calculation using (4.4.10) we get

$$\begin{aligned} r_0'(\phi)R(\phi,\varphi) - r_0(\phi)\partial_\phi R(\phi,\varphi) =& r_0'(\phi)\left(r_0^2(\varphi)-r_0^2(\phi)+(\cos\phi-\cos\varphi)^2\right) \\ &+ 2r_0(\phi)\sin\phi(\cos\phi-\cos\varphi). \end{aligned}$$

Then using the mean value theorem we get

$$\begin{aligned} |r_0'(\phi)R(\phi,\varphi) - r_0(\phi)\partial_\phi R(\phi,\varphi)| \lesssim& |\phi-\varphi|\left(r_0(\varphi)+r_0(\phi)+|\cos\phi-\cos\varphi|\right) \\ &+ r_0(\phi)\sin\phi|\cos\phi-\cos\varphi| \\ \lesssim& |\phi-\varphi|R^{\frac12}(\phi,\varphi). \end{aligned}$$

Putting together the preceding estimates we find

$$|\partial_\phi \mathscr{K}_2(\phi,\varphi)| \lesssim r_0(\varphi)|\phi-\varphi|R^{-\frac32}(\phi,\varphi)\Big|F_1'\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)\Big| \lesssim |\phi-\varphi|^{-1}.$$

Then using again the mean value theorem, we get $\phi\in(0,\pi)$ such that

$$|\mathscr{K}_2(\phi_1,\varphi)-\mathscr{K}_2(\phi_2,\varphi)| \lesssim |\phi_1-\phi_2||\phi-\varphi|^{-1}.$$

Combining this estimate with (4.4.9) and using an interpolation argument we get for $\varepsilon>0$,

$$\begin{aligned} |\mathscr{K}_2(\phi_1,\varphi)-\mathscr{K}_2(\phi_2,\varphi)| \lesssim& |\phi_1-\phi_2|^\beta|\phi-\varphi|^{-\beta} \\ &\times\left(C+\ln\left(\frac{\sin\phi_1+\sin\varphi}{|\phi_1-\varphi|}\right)+\ln\left(\frac{\sin\phi_2+\sin\varphi}{|\phi_2-\varphi|}\right)\right)^{1-\beta} \\ \lesssim& |\phi_1-\phi_2|^\beta\left(C+\big|\ln(\sin\varphi)^{1-\beta}\big|\right)\left(|\phi-\varphi|^{-\beta-\varepsilon}\right). \end{aligned} \tag{4.4.14}$$

Putting together (4.4.12) and (4.4.14) we deduce that

$$\begin{aligned} |(\mathscr{K}_1\mathscr{K}_2)(\phi_1,\varphi)-(\mathscr{K}_1\mathscr{K}_2)(\phi_2,\varphi)| \leq& \mathscr{K}_1(\phi_1,\varphi)|\mathscr{K}_2(\phi_1,\varphi)-\mathscr{K}_2(\phi_2,\varphi)| \\ &+ \mathscr{K}_2(\phi_2,\varphi)|\mathscr{K}_1(\phi_1,\varphi)-\mathscr{K}_1(\phi_2,\varphi)| \\ \lesssim& |\phi_1-\phi_2|^\beta\left(C+\big|\ln(\sin\varphi)\big|\right)^{1-\beta} \\ &\times\max\left(|\phi-\varphi|^{-\beta-\varepsilon},(\sin\varphi)^{-\beta}\right). \end{aligned}$$

Since $\beta\in(0,1)$, then if ε is small enough we obtain

$$\sup_{\phi\in(0,\pi)}\int_0^\pi |\ln\sin\varphi||\phi-\varphi|^{-\beta-\varepsilon}d\varphi<\infty.$$

Consequently we get

$$|\nu_\Omega(\phi_1)-\nu_\Omega(\phi_2)| \leq C|\phi_1-\phi_2|^\beta,$$

which implies that $\nu_\Omega\in\mathscr{C}^\beta([0,\pi])$.

(**2**) The function $\phi \mapsto \Omega + \nu_\Omega(\phi)$ is continuous over the compact set $[0, \pi]$ then it reaches its minimum at some point $\phi_0 \in [0, \pi]$. Thus from the definition of κ in (4.4.5) we deduce that

$$\kappa = \inf_{\phi \in (0,\pi)} \int_0^\pi H_1(\phi, \varphi) d\varphi = \int_0^\pi H_1(\phi_0, \varphi) d\varphi > 0,$$

which implies that

$$\forall\, \phi \in [0, \pi], \quad \nu_\Omega(\phi) \geq \int_0^\pi H_1(\phi_0, \varphi) d\varphi - \Omega \geq \kappa - \Omega.$$

Hence we infer that for any $\Omega \in (-\infty, \kappa)$

$$\forall\, \phi \in [0, \pi], \quad \nu_\Omega(\phi) \geq \kappa - \Omega > 0.$$

(**3**) The proof is long and technical and for the clarity of the presentation it will be divided into two steps. In the first one we prove that ν_Ω is $\mathscr{C}^1$ in the full closed interval $[0, \pi]$. This is mainly based on two principal ingredients. The first one is an important algebraic cancellation in the integrals allowing to get rid of the logarithmic singularity coming from the boundary and the second one is the boundary behavior of the hypergeometric functions allowing to deal with the diagonal singularity lying inside the domain of integration. Notice that in order to apply Lebesgue theorem and recover the continuity of the derivative up to the boundary we use a rescaling argument. This rescaling argument shows in addition a surprising effect concerning the derivative at the boundary points $\nu'_\Omega(0)$ and $\nu'_\Omega(\pi)$: they are independent of the global structure of the profile r_0 and they do depend only on the derivative $r'_0(0)$. This propriety allows to compute $\nu'_\Omega(0)$ using the special geometry of the sphere where this derivative is vanishing. As to the second step it is devoted to the proof of $\nu'_\Omega \in \mathscr{C}^\alpha(0, \pi)$ which is involved and requires more refined analysis.

• **Step 1:** $\nu_\Omega \in \mathscr{C}^1([0, \pi])$. The first step is to check that ν_Ω is $\mathscr{C}^1$ on $[0, \pi]$. Set

$$\varrho(\phi, \varphi) := \frac{4 r_0(\phi) r_0(\varphi)}{R(\phi, \varphi)},$$

then we can check that

$$\partial_\phi H_1(\phi, \varphi) = \mathscr{K}_1(\phi, \varphi) \left(-\frac{3}{2} R^{-1}(\phi, \varphi) \partial_\phi R(\phi, \varphi) F_1(\varrho(\phi, \varphi)) + F'_1(\varrho(\phi, \varphi)) \partial_\phi \varrho(\phi, \varrho) \right),$$

which implies after simple manipulations that

$$\begin{aligned} \partial_\phi H_1 = & \mathscr{K}_1 \left(-\frac{3}{2} \frac{\partial_\phi R}{R} + \frac{3}{4} \partial_\phi \rho - \frac{3}{2} \frac{\partial_\phi R}{R} \big(F_1(\rho) - 1\big) + \big(F'_1(\rho) - 3/4\big) \big[\partial_\phi \rho + \partial_\varphi \rho\big] \right) \\ & - \mathscr{K}_1 \big(F'_1(\rho) - 3/4\big) \partial_\varphi \rho. \end{aligned}$$

In addition using the identity

$$\begin{aligned} \mathscr{K}_1 \big(F'_1(\rho) - 3/4\big) \partial_\varphi \rho = & \mathscr{K}_1 \partial_\varphi \big[F(\rho) - 3/4\rho - 1\big] \\ = & \partial_\varphi \left(\mathscr{K}_1 \big[F(\rho) - 3/4\rho - 1\big] \right) - (\partial_\varphi \mathscr{K}_1) \big[F(\rho) - 3/4\rho - 1\big], \end{aligned}$$

we find

$$\partial_\phi H_1 = \varkappa_0 + \varkappa_1 \big(F_1(\varrho) - 1\big) + \varkappa_2 \big(F'_1(\varrho) - 3/4\big) - \partial_\varphi \left(\mathscr{K}_1 \big[F(\rho) - 3/4\rho - 1\big] \right),$$

with

$$\begin{aligned}\varkappa_0 :=&\mathscr{K}_1\left(-\frac{3}{2}\frac{\partial_\phi R}{R}+\frac{3}{4}\partial_\phi\rho\right)-\frac{3}{4}\rho\partial_\varphi\mathscr{K}_1,\\ \varkappa_1 :=&-\frac{3}{2}\frac{\partial_\phi R}{R}\mathscr{K}_1+\partial_\varphi\mathscr{K}_1,\\ \varkappa_2 :=&\mathscr{K}_1\left(\partial_\phi\rho+\partial_\varphi\rho\right).\end{aligned}\tag{4.4.15}$$

Notice that $F_1(0)=1,\ F_1'(0)=\frac{3}{4}$. Assuming that the following functions are well–defined and using the boundary conditions then we can write

$$\begin{aligned}\nu_\Omega'(\phi)=&\int_0^\pi\Big(\varkappa_0(\phi,\varphi)+\varkappa_1(\phi,\varphi)\big[F_1(\varrho(\phi,\varphi))-1\big]+\varkappa_2(\phi,\varphi)\big[F_1'(\varrho(\phi,\varphi))-3/4\big]\Big)d\varphi\\ :=&\zeta_1(\phi)+\zeta_2(\phi)+\zeta_3(\phi),\end{aligned}\tag{4.4.16}$$

with

$$\zeta_1(\phi):=\int_0^\pi\varkappa_0(\phi,\varphi)d\varphi,\quad \zeta_2(\phi):=\int_0^\pi\varkappa_1(\phi,\varphi)\big[F_1(\varrho(\phi,\varphi))-1\big]d\varphi$$

and

$$\zeta_3(\phi)=\int_0^\pi\varkappa_2(\phi,\varphi)\big[F_1'(\varrho(\phi,\varphi))-3/4\big]d\varphi.$$

Direct computations show that

$$\partial_\phi\rho(\phi,\varphi)=\frac{4r_0(\varphi)r_0'(\phi)}{R^2(\phi,\varphi)}\Big(R(\phi,\varphi)-2r_0(\phi)\big(r_0(\varphi)+r_0(\phi)\big)\Big)-\frac{8r_0(\phi)r_0(\varphi)}{R^2(\phi,\varphi)}\sin\phi(\cos\varphi-\cos\phi).$$

According to (4.4.10) and using some cancellation, it implies that

$$\begin{aligned}\varkappa_0(\phi,\varphi)=&-3r_0'(\phi)\frac{r_0(\phi)\mathscr{K}_1(\phi,\varphi)}{R(\phi,\varphi)}\left(1+2\frac{r_0(\varphi)(r_0(\phi)+r_0(\varphi))}{R(\phi,\varphi)}\right)\\ &+6\frac{\mathscr{K}_1(\phi,\varphi)}{R^2(\phi,\varphi)}\sin(\phi)\,r_0(\phi)\,r_0(\varphi)\big(\cos\phi-\cos\varphi\big)\\ &+3\frac{\mathscr{K}_1(\phi,\varphi)}{R(\phi,\varphi)}\sin(\phi)\big(\cos\phi-\cos\varphi\big)-3\partial_\varphi\mathscr{K}_1\frac{r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}.\end{aligned}\tag{4.4.17}$$

We point out that this simplification is crucial and allows to get rid of the logarithmic singularity.

Now we shall start with the regularity of the function

$$\zeta_1:\phi\in(0,\pi)\mapsto\int_0^\pi\varkappa_0(\phi,\varphi)d\varphi,$$

and prove first that it is continuous in $[0,\pi]$. It is obvious from (4.4.17) that $\varkappa_0$ is $\mathscr{C}^1$ over any compact set contained in $(0,\pi)\times[0,\pi]$ and therefore ζ_1 is $\mathscr{C}^1$ over any compact set contained in $(0,\pi)$. Thus it remains to check that this function is continuous at the points 0 and π. The proofs for both cases are quite similar and we shall only check the continuity at the origin. For this purpose it is enough to check that ζ_1 admits a limit at zero. Before that let us check that ζ_1 is bounded in $(0,\pi)$. From the definition of R stated in (4.3.2) and using elementary inequalities it is easy to verify the following estimates: for any $(\phi,\varphi)\in(0,\pi)^2$

$$\frac{r_0(\phi)(r_0(\phi)+r_0(\varphi))}{R(\phi,\varphi)}\le 1,$$

$$\frac{r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)} \leq \frac{1}{2},$$
$$r_0(\phi)r_0(\varphi)\big|\cos\phi-\cos\varphi\big| \leq R(\phi,\varphi).$$

In addition, the assumption **(H2)** implies that

$$\sup_{\phi,\varphi\in(0,\pi)} \mathscr{K}_1(\phi,\varphi)<\infty.$$

Thus we find according to (4.4.11) and **(H2)**

$$\forall(\phi,\varphi)\in(0,\pi)^2,\ |\varkappa_0(\phi,\varphi)| \lesssim \frac{\sin(\phi)}{R(\phi,\varphi)}+|\partial_\varphi\mathscr{K}_1(\phi,\varphi)|\frac{r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)} \lesssim \frac{\sin(\phi)}{R(\phi,\varphi)}. \tag{4.4.18}$$

Hence we deduce that

$$\forall\phi\in(0,\pi),\quad |\zeta_1(\phi)|\lesssim\int_0^\pi\frac{\sin(\phi)}{R(\phi,\varphi)}d\varphi\lesssim\int_0^{\frac{\pi}{2}}\frac{\sin\phi}{(\sin\phi+\sin\varphi)^2}d\varphi.$$

Making the change of variables $\sin\varphi=x$ we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{\sin\phi}{(\sin\phi+\sin\varphi)^2}d\varphi &= \sin\phi\int_0^1\frac{1}{(\sin\phi+x)^2}\frac{dx}{\sqrt{1-x^2}}\\ &\lesssim \sin\phi\int_0^{\frac{1}{2}}\frac{1}{(\sin\phi+x)^2}dx+\sin\phi\lesssim 1.\end{aligned}$$

Thus

$$\sup_{\phi\in(0,\pi)}|\zeta_1(\phi)|<\infty. \tag{4.4.19}$$

Let us now prove that ζ_1 admits a limit at the origin and compute its value. For this goal, take $0<\delta<<1$ small enough and write

$$\zeta_1(\phi)=\int_0^\delta\varkappa_0(\phi,\varphi)d\varphi+\int_\delta^\pi\varkappa_0(\phi,\varphi)d\varphi.$$

The assumption **(H2)** combined with standard trigonometric formula allow to get the estimate

$$R(\phi,\varphi)\gtrsim(\sin\phi+\sin\varphi)^2+(\cos\phi-\cos\varphi)^2\gtrsim 1-\cos(\phi+\varphi). \tag{4.4.20}$$

From this we infer that

$$\forall\,\phi\in[0,\pi/2],\forall\varphi\in(\delta,\pi),\quad R(\phi,\varphi)\gtrsim 1-\cos\delta. \tag{4.4.21}$$

Thus we get from (4.4.18)

$$\int_\delta^\pi\varkappa_0(\phi,\varphi)d\varphi\lesssim\int_\delta^\pi\frac{\phi}{(1-\cos\delta)^2}d\varphi\lesssim\phi\delta^{-4}.$$

This implies that for given small parameter δ one has

$$\lim_{\phi\to0}\int_\delta^\pi\varkappa_0(\phi,\varphi)d\varphi=0.$$

Therefore

$$\limsup_{\phi\to 0}\zeta_1(\phi)=\limsup_{\phi\to 0}\int_0^\delta \varkappa_0(\phi,\varphi)d\varphi.$$

Making the change of variables $\varphi=\phi\theta$ we get

$$\int_0^\delta \varkappa_0(\phi,\varphi)d\varphi=\int_0^{\frac{\delta}{\phi}}\phi\varkappa_0(\phi,\phi\theta)\,d\theta.$$

From (4.4.18) and **(H2)** one may write

$$\forall(\phi,\varphi)\in(0,\delta)^2,\quad |\varkappa_0(\phi,\varphi)|\lesssim\frac{\phi}{(\phi+\varphi)^2},$$

which yields after simplification to the uniform bound on ϕ,

$$\forall\theta\in[0,\delta/\phi],\quad \phi\varkappa_0(\phi,\phi\theta)\lesssim\frac{1}{(1+\theta)^2}.$$

This gives a domination which is integrable over $(0,+\infty)$. In order to apply classical dominated Lebesgue theorem, it remains to check the convergence almost everywhere in θ as ϕ goes to zero. This can be done through the first–order Taylor expansion around zero. First one has the expansion

$$r_0(\phi\theta)=c_0\phi\theta+\phi\theta\epsilon(\phi\theta);\quad R(\phi,\phi\theta)=c_0^2\phi^2\big(1+\theta+\theta\epsilon(\phi\theta)\big)^2,$$

with $c_0=r_0'(0)$ and $\lim\limits_{x\to 0}\epsilon(x)=0$. Thus, from the definitions (4.3.2) and (4.4.7) it is straightforward that

$$4\pi\lim_{\phi\to 0}\mathscr{K}_1(\phi,\phi\theta)=\frac{c_0^{-1}\theta^3}{(1+\theta)^3},\quad \lim_{\phi\to 0}\frac{\phi r_0(\phi)}{R(\phi,\phi\theta)}=\frac{c_0^{-1}}{(1+\theta)^2}. \tag{4.4.22}$$

Hence

$$4\pi\lim_{\phi\to 0}r_0'(\phi)\frac{\phi r_0(\phi)\mathscr{K}_1(\phi,\phi\theta)}{R(\phi,\phi\theta)}\left(1+2\frac{r_0(\phi\theta)(r_0(\phi)+r_0(\phi\theta))}{R(\phi,\phi\theta)}\right)=\frac{c_0^{-1}\theta^3(1+3\theta)}{(1+\theta)^6}. \tag{4.4.23}$$

Similarly we get

$$4\pi\lim_{\phi\to 0}\frac{\mathscr{K}_1(\phi,\phi\theta)}{R^2(\phi,\phi\theta)}\sin\phi\, r_0(\phi)r_0(\phi\theta)\big(\cos\phi-\cos(\phi\theta)\big)=0,$$

and

$$4\pi\lim_{\phi\to 0}\frac{\mathscr{K}_1(\phi,\phi\theta)}{R(\phi,\phi\theta)}\sin(\phi)\big(\cos\phi-\cos(\phi\theta)\big)=0.$$

Standard computations yield

$$\begin{aligned}4\pi\partial_\varphi\mathscr{K}_1(\phi,\varphi)=&-3R^{-\frac{5}{2}}(\phi,\varphi)\sin(\varphi)\,r_0^2(\varphi)\Big(r_0'(\varphi)(r_0(\phi)+r_0(\varphi))+\sin(\varphi)\,(\cos\phi-\cos\varphi)\Big)\\&+\frac{\cos(\varphi)\,r_0^2(\varphi)+2\sin(\varphi)r_0(\varphi)\,r_0'(\varphi)}{R^{\frac{3}{2}}(\phi,\varphi)}.\end{aligned} \tag{4.4.24}$$

Thus

$$4\pi\lim_{\phi\to 0}\phi\partial_\varphi\mathscr{K}_1(\phi,\phi\theta)=c_0^{-1}\left(-3\frac{\theta^3}{(1+\theta)^4}+\frac{3\theta^2}{(1+\theta)^3}\right)=3c_0^{-1}\frac{\theta^2}{(1+\theta)^4}. \tag{4.4.25}$$

Therefore

$$4\pi \lim_{\phi\to 0} \phi\partial_\varphi \mathscr{K}_1(\phi,\phi\theta)\frac{r_0(\phi)r_0(\phi\theta)}{R(\phi,\phi\theta)} = 3c_0^{-1}\frac{\theta^3}{(1+\theta)^6}. \tag{4.4.26}$$

Plugging (4.4.23), (4.4.24) and (4.4.26) into (4.4.17)

$$4\pi \lim_{\phi\to 0} \phi\varkappa_0(\phi,\phi\theta) = -\frac{3c_0^{-1}\theta^3}{(1+\theta)^6}\Big(2+3\theta\Big).$$

Using Lebesgue dominated theorem we deduce that

$$4\pi \lim_{\phi\to 0}\int_0^{\frac{\delta}{\phi}} \phi\varkappa_0(\phi,\phi\theta)d\theta = -3c_0^{-1}\int_0^{+\infty}\frac{2\theta^3+3\theta^4}{(1+\theta)^6}d\theta.$$

Computing the integrals we finally get

$$4\pi \lim_{\phi\to 0}\zeta_1(\phi) = -\frac{7}{10}c_0^{-1}. \tag{4.4.27}$$

Let us now move to the regularity of the function ζ_2 defined in (4.4.16) through

$$\phi\in(0,\pi),\quad \zeta_2(\phi) = \int_0^\pi \varkappa_1(\phi,\varphi)\big(F_1(\rho(\phi,\varphi))-1\big)d\varphi,$$

where $\varkappa_1$ is defined in(4.4.15). From direct computations using $|\partial_\phi R|\lesssim R^{\frac{1}{2}}$, the boundedness of $\mathscr{K}_1$, the assumption **(H2)** and (4.4.24) one can check that

$$\begin{aligned}|\varkappa_1(\phi,\varphi)| \lesssim&\frac{|\partial_\phi R(\phi,\varphi)|\mathscr{K}_1(\phi,\varphi)}{R(\phi,\varphi)} + |\partial_\varphi\mathscr{K}_1(\phi,\varphi)|\\ \lesssim& R^{-\frac{1}{2}}(\phi,\varphi).\end{aligned}$$

Using Proposition C.0.2 we get

$$\begin{aligned}|F_1(\varrho(\phi,\varphi))-1| \lesssim&\varrho(\phi,\varphi)\left(1+|\ln(1-\varrho(\phi,\varphi))|\right)\\ \lesssim&\frac{r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\left(1+\ln\left(\frac{(r_0(\phi)+r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2}{(r_0(\phi)-r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2}\right)\right).\end{aligned} \tag{4.4.28}$$

Thus

$$|\varkappa_1(\phi,\varphi)[F_1(\varrho(\phi,\varphi))-1]| \lesssim\frac{r_0(\phi)r_0(\varphi)}{R^{\frac{3}{2}}(\phi,\varphi)}\left(1+\ln\left(\frac{(r_0(\phi)+r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2}{(r_0(\phi)-r_0(\varphi))^2+(\cos(\phi)-\cos(\varphi))^2}\right)\right).$$

Hence using the arc-chord property (4.2.18) we find

$$|\varkappa_1(\phi,\varphi)[F_1(\varrho(\phi,\varphi))-1]| \lesssim \frac{r_0(\phi)r_0(\varphi)}{R^{\frac{3}{2}}(\phi,\varphi)}\left(C+\ln\left(\frac{R(\phi,\varphi)}{|\phi-\varphi|^2}\right)\right). \tag{4.4.29}$$

for some constant $C>0$. In addition, using (4.4.20) we get

$$\inf_{\substack{\phi\in[0,\frac{\pi}{2}],\\ \varphi\in[\frac{\pi}{2},\pi]}} R(\phi,\varphi)>0, \tag{4.4.30}$$

which leads to

$$\forall\, \phi \in [0, \pi/2],\, \varphi \in [\pi/2, \pi], \quad |\varkappa_1(\phi, \varphi)[F_1(\varrho(\phi, \varphi)) - 1]| \lesssim 1 + |\ln|\phi - \varphi||.$$

This implies that

$$\sup_{\phi \in [0,\pi/2]} \int_{\frac{\pi}{2}}^{\pi} |\varkappa_1(\phi, \varphi)[F_1(\varrho(\phi, \varphi)) - 1]| d\varphi \lesssim 1 + \sup_{\phi \in [0,\pi/2]} \int_0^{\pi} |\ln|\phi - \varphi|| d\varphi < \infty. \tag{4.4.31}$$

Now in the region $\phi, \varphi \in [0, \pi/2]$, we use the estimate **(H2)** leading to

$$(\phi + \varphi)^2 \lesssim R(\phi, \varphi) \lesssim (\phi + \varphi)^2.$$

Plugging this into (4.4.29) we find

$$\sup_{\phi \in [0,\pi/2]} \int_0^{\frac{\pi}{2}} |\varkappa_1(\phi, \varphi)[F_1(\varrho(\phi, \varphi)) - 1]| d\varphi \lesssim 1 + \sup_{\phi \in [0,\pi/2]} \int_0^{\frac{\pi}{2}} \frac{\phi\varphi}{(\phi + \varphi)^3} |\ln\left(\frac{\phi + \varphi}{|\phi - \varphi|}\right)| d\varphi.$$

Making the change of variables $\varphi = \phi\theta$ we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\phi\varphi}{(\phi + \varphi)^3} |\ln\left(\frac{\phi + \varphi}{|\phi - \varphi|}\right)| d\varphi &= \int_0^{\frac{\pi}{2\phi}} \frac{\theta}{(1 + \theta)^3} \ln\Big(\frac{1 + \theta}{|1 - \theta|}\Big) d\varphi \\ &\leq \int_0^{+\infty} \frac{\theta}{(1 + \theta)^3} \ln\Big(\frac{1 + \theta}{|1 - \theta|}\Big) d\varphi < \infty,\end{aligned}$$

which implies that

$$\sup_{\phi \in [0,\pi/2]} \int_0^{\frac{\pi}{2}} |\varkappa_1(\phi, \varphi)[F_1(\varrho(\phi, \varphi)) - 1]| d\varphi < \infty.$$

Therefore we obtain by virtue of (4.4.31)

$$\sup_{\phi \in [0,\pi/2]} \int_0^{\pi} |\varkappa_1(\phi, \varphi)[F_1(\varrho(\phi, \varphi)) - 1]| d\varphi < \infty.$$

By symmetry we get similar estimate for $\phi \in [\frac{\pi}{2}, \pi]$ and hence

$$\sup_{\phi \in (0,\pi)} |\zeta_2(\phi)| < \infty. \tag{4.4.32}$$

Let us now calculate the limit when ϕ goes to 0 of ζ_2 at zero. We shall proceed in a similar way to ζ_1. Let $0 < \delta \ll 1$ enough small, then using (4.4.29) combined with (4.4.21) we obtain

$$\lim_{\phi \to 0} \int_{\delta}^{\pi} \varkappa_1(\phi, \varphi) |F_1(\rho(\phi, \varphi)) - 1| d\varphi = 0.$$

Hence

$$\limsup_{\phi \to 0} \zeta_2(\phi) = \limsup_{\phi \to 0} \int_0^{\delta} \varkappa_1(\phi, \varphi) \big(F_1(\rho(\phi, \varphi)) - 1\big) d\varphi.$$

Now we make the change of variables $\varphi = \phi\theta$ and then

$$\limsup_{\phi \to 0} \zeta_2(\phi) = \limsup_{\phi \to 0} \int_0^{\frac{\delta}{\phi}} \phi\varkappa_1(\phi, \phi\theta) \big(F_1(\rho(\phi, \phi\theta)) - 1\big) d\theta.$$

According to (4.4.15) one has

$$\phi\varkappa_1(\phi,\phi\theta) = -\frac{3}{2}\frac{\phi\partial_\phi R(\phi,\phi\theta)}{R(\phi,\phi\theta)}\mathscr{K}_1(\phi,\phi\theta) + \phi\partial_\varphi\mathscr{K}_1(\phi,\phi\theta).$$

From the differentiating of the expression of R stated in (4.3.2) we get

$$\frac{\phi\partial_\phi R(\phi,\phi\theta)}{R(\phi,\phi\theta)} = 2\frac{\phi r_0'(\phi)(r_0(\phi)+r_0(\phi\theta)) + \phi\sin\phi(\cos(\phi\theta)-\cos\phi)}{(r_0(\phi)+r_0(\phi\theta))^2+(\cos\phi-\cos(\phi\theta))^2}\cdot$$

Taking Taylor expansion at the first order we deduce the pointwise convergence,

$$\lim_{\phi\to 0}\frac{\phi\partial_\phi R(\phi,\phi\theta)}{R(\phi,\phi\theta)} = \frac{2}{1+\theta}\cdot$$

Combined with (4.4.22) it implies that

$$4\pi\lim_{\phi\to 0} -\frac{3}{2}\frac{\phi\partial_\phi R(\phi,\phi\theta)}{R(\phi,\phi\theta)}\mathscr{K}_1(\phi,\phi\theta) = -\frac{3c_0^{-1}\theta^3}{(1+\theta)^4}\cdot$$

Plugging (4.4.25) and the preceding estimates into the expression of $\varkappa_1$ given by (4.4.15) we find

$$\begin{aligned}4\pi\lim_{\phi\to 0}\phi\varkappa_1(\phi,\phi\theta) &= -\frac{3c_0^{-1}\theta^3}{(1+\theta)^4} + 3c_0^{-1}\frac{\theta^2}{(1+\theta)^4}\\ &= 3c_0^{-1}\frac{\theta^2(1-\theta)}{(1+\theta)^4}\cdot\end{aligned}$$

From the result

$$\lim_{\phi\to 0}\frac{r_0(\phi)r_0(\phi\theta)}{R(\phi,\phi\theta)} = \frac{\theta}{(1+\theta)^2},$$

we deduce the point-wise convergence

$$\lim_{\phi\to 0}F_1(\rho(\phi,\phi\theta)) = F_1\left(\frac{4\theta}{(1+\theta)^2}\right).$$

Consequently,

$$4\pi\lim_{\phi\to 0}\phi\varkappa_1(\phi,\phi\theta)\Big[F_1(\rho(\phi,\phi\theta))-1\Big] = 3c_0^{-1}\frac{\theta^2(1-\theta)}{(1+\theta)^4}\left(F_1\left(\frac{4\theta}{(1+\theta)^2}\right)-1\right)\cdot$$

Therefore,

$$4\pi\lim_{\phi\to 0}\zeta_2(\phi) = 3c_0^{-1}\int_0^{+\infty}\frac{\theta^2(1-\theta)}{(1+\theta)^4}\left(F_1\left(\frac{4\theta}{(1+\theta)^2}\right)-1\right)d\theta. \tag{4.4.33}$$

Next, we shall implement similar study for ζ_3 defined in (4.4.16). Straightforward computations yield

$$\partial_\phi\varrho(\phi,\varphi) + \partial_\varphi\varrho(\phi,\varphi) = \varrho_1(\phi,\varphi) + \varrho_2(\phi,\varphi), \tag{4.4.34}$$

with

$$\varrho_1(\phi,\varphi) := 4\frac{r_0^2(\varphi)-r_0^2(\phi)}{R^2(\phi,\varphi)}r_0'(\phi)\big(r_0(\varphi)-r_0(\phi)\big), \tag{4.4.35}$$

and

$$\begin{aligned}\varrho_2(\phi,\varphi) :=& 4\frac{r_0^2(\varphi)-r_0^2(\phi)}{R^2(\phi,\varphi)}r_0(\phi)\Big(r_0'(\phi)-r_0'(\varphi)\Big)\\&+8\frac{r_0(\varphi)r_0(\phi)}{R^2(\phi,\varphi)}\big(\cos\phi-\cos\varphi\big)\big(\sin\phi-\sin\varphi\big)\\&+4\frac{(\cos\phi-\cos\varphi)^2}{R^2(\phi,\varphi)}\Big(r_0(\varphi)r_0'(\phi)+r_0(\phi)r_0'(\varphi)\Big).\end{aligned}\tag{4.4.36}$$

Since r_0' is Lipschitz then using the mean value theorem we get

$$\forall\,\phi,\varphi\in(0,\pi),\quad |\varrho_1(\phi,\varphi)|+|\varrho_2(\phi,\varphi)|\lesssim\frac{(\phi-\varphi)^2}{R^{\frac32}(\phi,\varphi)}.\tag{4.4.37}$$

From Proposition C.0.2 combined with (4.2.18) we get

$$\begin{aligned}|F_1'(\varrho(\phi,\varphi))-3/4| =& \frac34|F(5/2,5/2;4;\rho(\phi,\varphi))-1|\\ \lesssim&\frac{\varrho(\phi,\varphi)R(\phi,\varphi)}{(r_0(\phi)-r_0(\varphi)^2+(\cos\phi-\cos\varphi)^2}\\ \lesssim&\frac{r_0(\phi)r_0(\varphi)}{(\phi-\varphi)^2}.\end{aligned}\tag{4.4.38}$$

In addition

$$\begin{aligned}|\varkappa_2(\phi,\varphi))| =& |\mathscr{K}_1(\phi,\varphi)||\varrho_1(\phi,\varphi)+\varrho_2(\phi,\varphi)|\\ \lesssim&\frac{\sin\varphi\, r_0^2(\varphi)}{R^{\frac32}(\phi,\varphi)}\frac{(\phi-\varphi)^2}{R^{\frac32}(\phi,\varphi)}\\ \lesssim&\frac{\sin\varphi\,(\varphi-\phi)^2}{R^2(\phi,\varphi)}.\end{aligned}$$

Consequently, we obtain in view of **(H2)**

$$|\varkappa_2(\phi,\varphi)[F_1'(\varrho(\phi,\varphi))-3/4]|\lesssim\frac{\sin(\varphi)r_0(\varphi)r_0(\phi)}{R^2(\phi,\varphi)}\lesssim\frac{\sin\phi}{R(\phi,\varphi)}.\tag{4.4.39}$$

As before, we can assume without any loss of generality that $\phi\in[0,\,\pi/2]$, then by **(H2)**

$$\begin{aligned}\int_0^\pi|\varkappa_2(\phi,\varphi)[F_1'(\varrho(\phi,\varphi))-3/4]|d\varphi\lesssim&\int_0^{\frac{\pi}{2}}\frac{\sin\phi}{(\sin\phi+\sin\varphi)^2}d\varphi\\ \lesssim&\int_0^{\frac{\pi}{2}}\frac{\phi}{\big(\phi+\varphi\big)^2}d\varphi.\end{aligned}$$

By the change of variables $\varphi=\phi\theta$ we get

$$\int_0^{\frac{\pi}{2}}\frac{\phi}{\big(\phi+\varphi\big)^2}d\varphi=\int_0^{\frac{\pi}{2\phi}}\frac{1}{\big(1+\theta\big)^2}d\theta\le\int_0^{+\infty}\frac{1}{\big(1+\theta\big)^2}d\theta<\infty.$$

Therefore

$$\sup_{\phi\in[0,\pi/2]}\int_0^\pi|\varkappa_2(\phi,\varphi)[F_1'(\varrho(\phi,\varphi))-3/4]|d\varphi<\infty.$$

Consequently

$$\sup_{\phi\in[0,\pi]} |\zeta_3(\phi)| < \infty. \tag{4.4.40}$$

Now, we shall calculate the limit of ζ_3 at the origin. Let $0 < \delta \ll 1$ enough small, then using (4.4.39) combined with (4.4.21) and **(H2)** we obtain

$$\lim_{\phi\to 0} \int_\delta^\pi \varkappa_2(\phi,\varphi)[F_1'(\varrho(\phi,\varphi)) - 3/4]|d\varphi = 0.$$

It follows that

$$\limsup_{\phi\to 0} \zeta_3(\phi) = \limsup_{\phi\to 0} \int_0^\delta \varkappa_2(\phi,\varphi)[F_1'(\varrho(\phi,\varphi)) - 3/4]|d\varphi.$$

Making the change of variables $\varphi = \phi\theta$ yields

$$\limsup_{\phi\to 0} \zeta_3(\phi) = \limsup_{\phi\to 0} \int_0^{\frac{\delta}{\phi}} \phi\varkappa_2(\phi,\phi\theta)\big[F_1'(\rho(\phi,\phi\theta)) - 3/4\big]d\theta.$$

Using Taylor expansion at the order one in (4.4.35) and (4.4.36) we can check that

$$\lim_{\phi\to 0} \phi\varrho_1(\phi,\phi\theta) = 4\frac{(\theta-1)^2}{(1+\theta)^3}, \quad \lim_{\phi\to 0} \phi\varrho_2(\phi,\phi\theta) = 0.$$

Hence we get in view of the definition of $\varkappa_2$ and (4.4.22) the point-wise limit

$$4\pi \lim_{\phi\to 0} \phi\varkappa_2(\phi,\phi\theta) = 4\pi \lim_{\phi\to 0} \phi\mathscr{K}_1(\phi,\phi\theta)\Big(\varrho_1(\phi,\phi\theta) + \varrho_2(\phi,\phi\theta)\Big) = 4c_0^{-1}\frac{\theta^3(\theta-1)^2}{(1+\theta)^6}.$$

It follows that

$$4\pi \lim_{\phi\to 0} \phi\varkappa_2(\phi,\phi\theta)\Big(F_1'(\rho(\phi,\phi\theta) - 3/4\Big) = 4c_0^{-1}\frac{\theta^3(\theta-1)^2}{(1+\theta)^6}\left(F_1'\Big(\frac{4\theta}{(1+\theta)^2}\Big) - 3/4\right).$$

Applying Lebesgue theorem yields

$$4\pi \lim_{\phi\to 0} \zeta_3(\phi) = 4c_0^{-1}\int_0^{+\infty} \frac{\theta^3(\theta-1)^2}{(1+\theta)^6}\left(F_1'\Big(\frac{4\theta}{(1+\theta)^2}\Big) - 3/4\right)d\theta. \tag{4.4.41}$$

Putting together (4.4.16), (4.4.27), (4.4.33) and (4.4.41) we find

$$\begin{aligned}4\pi \lim_{\phi\to 0} \nu_\Omega'(\phi) =& -\frac{7}{10}c_0^{-1} + 4c_0^{-1}\int_0^{+\infty} \frac{\theta^3(\theta-1)^2}{(1+\theta)^6}\left(F_1'\Big(\frac{4\theta}{(1+\theta)^2}\Big) - 3/4\right)d\theta \\ &+ 3c_0^{-1}\int_0^{+\infty} \frac{\theta^2(1-\theta)}{(1+\theta)^4}\left(F_1\left(\frac{4\theta}{(1+\theta)^2}\right) - 1\right)d\theta := \eta c_0^{-1}.\end{aligned}$$

Notice that the real number η is well-defined since all the integrals converge. This shows the existence of the derivative of ν_Ω at the origin. It is important to emphasize that number η is independent of the profile r_0 and we claim that the number η is zero. It is slightly difficult to check this result directly from the integral representation of η, however we shall check it in a different way by commuting its value for the unit ball

$$\big\{(re^{i\theta},z),\ r^2+z^2 \le 1, \theta\in\mathbb{R}\big\}$$

whose boundary can be parametrized by $(\phi,\theta)\mapsto(r_0(\phi)e^{i\theta},\cos\phi)$ with $r_0(\phi)=\sin\phi$. Now according to the identity (4.3.4) one has

$$\int_0^\pi H_1(\phi,\varphi)d\varphi=\frac{1}{r_0(\phi)}\partial_r\psi_0(re^{i\theta},\cos(\phi))\,\big|_{r=r_0(\phi)}\,.$$

However it is known [94] that the stream function ψ_0 is radial and quadratic inside the domain taking the form

$$0\le r\le\sin\phi,\quad \psi_0(re^{i\theta},\cos(\phi)=\frac{1}{6}\big(r^2+\cos^2\phi\big).$$

Consequently, with this special geometry the function ν_Ω is constant and therefore

$$\nu_\Omega'(0)=\nu_\Omega'(\pi)=0.$$

• **Step 2:** $\nu_\Omega'\in\mathscr{C}^\alpha(0,\pi)$. We shall prove that ν_Ω' is $\mathscr{C}^\alpha(0,\pi)$ and for this purpose we start with the first term in (4.4.16), i.e., ζ_1. According to (4.4.17) it can be split into several terms and to fix the ideas let us describe how to proceed with the first term given by

$$\phi\mapsto 4\pi r_0'(\phi)r_0(\phi)\int_0^\pi\frac{\mathscr{K}_1(\phi,\varphi)}{R(\phi,\varphi)}d\varphi=r_0'(\phi)r_0(\phi)\int_0^\pi\frac{\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi,$$

and check that it belongs to $\mathscr{C}^\alpha(0,\pi)$. The remaining terms of ζ_1 can be treated in a similar way and to alleviate the discussion we leave them to the reader.

From the assumptions **(H)** on r_0 we have r_0', $\phi\mapsto\frac{r_0(\phi)}{\sin\phi}\in\mathscr{C}^\alpha(0,\pi)$, then using classical law products it suffices to verify that

$$\phi\mapsto\int_0^\pi\frac{\sin(\phi)\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi\in\mathscr{C}^\alpha(0,\pi).$$

This function is locally $\mathscr{C}^1$ in $(0,\pi)$ and so the problem reduces to check the regularity close to the boundary $\{0,\pi\}$. By symmetry it suffices to check the regularity near the origin. Decompose the integral as follows

$$\int_0^\pi\frac{\sin(\phi)\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi=\int_0^{\frac{\pi}{2}}\frac{\sin(\phi)\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi+\int_{\frac{\pi}{2}}^\pi\frac{\sin(\phi)\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi.$$

Since we are considering $\phi\in(0,\pi/2)$, it is easy to check that the last integral term defines a $\mathscr{C}^1$ function in $[0,\pi]$ and therefore the problem amounts to checking that the function

$$\zeta_{1,1}:\phi\mapsto\int_0^{\frac{\pi}{2}}\frac{\phi\sin(\varphi)r_0^2(\varphi)}{R^{\frac52}(\phi,\varphi)}d\varphi,$$

is $\mathscr{C}^\alpha$ close to zero. Making the change of variables $\varphi=\phi\theta$ we get

$$\zeta_{1,1}(\phi)=\int_0^{\frac{\pi}{2\phi}}\frac{\phi^2\sin(\phi\theta)r_0^2(\phi\theta)}{R^{\frac52}(\phi,\phi\theta)}d\theta=\int_0^{\frac{\pi}{2\phi}}\frac{\frac{\sin(\phi\theta)}{\phi}\Big(\frac{r_0(\phi\theta)}{\phi}\Big)^2}{\Big(\big(\frac{r_0(\phi)+r_0(\phi\theta)}{\phi}\big)^2+\big(\frac{\cos(\phi)-\cos(\phi\theta)}{\phi}\big)^2\Big)^{\frac52}}d\theta.$$

Let us now define the following functions

$$\forall\, s\in[\phi,\pi/2],\quad T_{1,\phi}(s):=\int_0^{\frac{\pi}{2s}}\frac{\phi^2\sin(\phi\theta)r_0^2(\phi\theta)}{R^{\frac52}(\phi,\phi\theta)}d\theta,$$

and

$$\forall\, s\in(0,\phi],\quad T_{2,\phi}(s):=\int_0^{\frac{\pi}{2\phi}}\frac{\frac{\sin(s\theta)}{s}\left(\frac{r_0(s\theta)}{s}\right)^2}{\left(\left(\frac{r_0(s)+r_0(s\theta)}{s}\right)^2+\left(\frac{\cos(s)-\cos(s\theta)}{s}\right)^2\right)^{\frac52}}d\theta. \tag{4.4.42}$$

We will show that $T_{1,\phi}\in\mathscr{C}^\alpha[\phi,\pi/2]$ and $T_{2,\phi}\in\mathscr{C}^\alpha(0,\phi]$ uniformly in $\phi\in(0,\frac{\pi}{2})$. Thus we get in particular a constant $C>0$ such that for any $0<\phi_1\le\phi_2<\frac{\pi}{2}$,

$$|T_{1,\phi_1}(\phi_1)-T_{1,\phi_1}(\phi_2)|\le C|\phi_1-\phi_2|^\alpha, \tag{4.4.43}$$

and

$$|T_{2,\phi_2}(\phi_1)-T_{2,\phi_2}(\phi_2)|\le C|\phi_1-\phi_2|^\alpha. \tag{4.4.44}$$

By combining (4.4.43) and (4.4.44), we are able to get

$$|\zeta_{1,1}(\phi_1)-\zeta_{1,1}(\phi_2)|\le|T_{1,\phi_1}(\phi_1)-T_{1,\phi_1}(\phi_2)|+|T_{2,\phi_2}(\phi_1)-T_{2,\phi_2}(\phi_2)|\le C|\phi_1-\phi_2|^\alpha.$$

This ensures that $\zeta_{1,1}\in\mathscr{C}^\alpha(0,\pi/2)$.

It remains to show that $T_{1,\phi}\in\mathscr{C}^\alpha([\phi,\pi/2])$ and $T_{2,\phi}\in\mathscr{C}^\alpha((0,\phi])$ uniformly in $\phi\in(0,\frac{\pi}{2})$. We start with the term $T_{1,\phi}$. Then straightforward computations imply

$$\begin{aligned}\forall\, s\in[\phi,\pi/2],\quad |\partial_s T_{1,\phi}(s)|=&\frac{\pi}{2s^2}\frac{\phi^2\sin(\phi\pi/2s)r_0^2(\phi\pi/2s)}{R^{\frac52}(\phi,\phi\pi/2s)}\\ \le& C\frac{1}{s^5}\frac{\phi^5}{\phi^5(1+\frac{\pi}{2s})^5}\le C,\end{aligned}$$

for any $\phi,s\in(0,\pi/2]$. Notice that we have used in the last line the following inequalities which follow from the assumptions **(H2)**,

$$\phi\lesssim r_0(\phi)\lesssim\phi,\quad\forall\phi\in[0,\pi/2]$$

and

$$\theta\lesssim\frac{r_0(\phi\theta)}{\phi}\lesssim\theta,\quad\forall\theta\in[0,\pi/2\phi]. \tag{4.4.45}$$

Hence $T_{1,\phi}\in\mathrm{Lip}([\phi,\pi/2])$, uniformly with respect to $\phi\in(0,\pi/2)$.

Let us move to the term $T_{2,\phi}$. First, we write

$$\frac{\sin(\phi\theta)}{\phi}=\theta\int_0^1\cos(\phi\theta\tau)d\tau,$$

and taking the derivative with respect to ϕ we obtain

$$\partial_\phi\left(\frac{\sin(\phi\theta)}{\phi}\right)=-\theta^2\int_0^1\sin(\phi\theta\tau)\tau\,d\tau.$$

Hence,

$$\Big|\partial_\phi\left(\frac{\sin(\phi\theta)}{\phi}\right)\Big|\le\theta^2. \tag{4.4.46}$$

By the mean value theorem we infer

$$\left|\frac{\sin(s_1\theta)}{s_1}-\frac{\sin(s_2\theta)}{s_2}\right|\le |s_1-s_2|\theta^2$$

Interpolating between (4.4.45), which is also true for $r_0=\sin$, and (4.4.46) we obtain

$$\left|\frac{\sin(s_1\theta)}{s_1}-\frac{\sin(s_2\theta)}{s_2}\right|\le C|s_1-s_2|^\alpha\theta^{1-\alpha}\theta^{2\alpha}=C|s_1-s_2|^\alpha\theta^{1+\alpha}. \tag{4.4.47}$$

Using Taylor formula

$$r_0(\phi\theta)=\phi\theta\int_0^1 r_0'(\tau\phi\theta)d\tau,$$

one finds that if $0\le\phi\theta\le\pi/2$ then

$$\left|\partial_\phi\left(\frac{r_0(\phi\theta)}{\phi}\right)\right|\le C\theta^2. \tag{4.4.48}$$

As before, one gets that if $0\le s_1\theta,\ s_2\theta\le\pi/2$ hence

$$\left|\frac{r_0(s_1\theta)}{s_1}-\frac{r_0(s_2\theta)}{s_2}\right|\le C|s_1-s_2|^\alpha\theta^{1+\alpha}. \tag{4.4.49}$$

Now, let us check that $T_{2,\phi}$ is $\mathscr{C}^\alpha(0,\phi]$ uniformly in $\phi\in(0,\pi/2)$. Let $s_1,s_2\in(0,\phi]$, then using the estimates (4.4.47) and (4.4.45), we achieve for any $s\in(0,\phi]$,

$$\begin{aligned}\left|\int_0^{\frac{\pi}{2\phi}}\frac{\left(\frac{\sin(s_1\theta)}{s_1}-\frac{\sin(s_2\theta)}{s_2}\right)\left(\frac{r_0(s\theta)}{s}\right)^2}{\left((\frac{r_0(s)+r_0(s\theta)}{s})^2+(\frac{\cos(s)-\cos(s\theta)}{s})^2\right)^{\frac52}}d\theta\right|&\le C|s_1-s_2|^\alpha\int_0^{\frac{\pi}{2\phi}}\frac{\theta^{1+\alpha}\theta^2}{(1+\theta)^5}\\&\le C|s_1-s_2|^\alpha,\end{aligned}$$

for $\alpha\in(0,1)$. In the same way

$$\begin{aligned}\left|\int_0^{\frac{\pi}{2\phi}}\frac{\frac{\sin(s\theta)}{s}\frac{r_0(s\theta)}{s}\left(\frac{r_0(s_1\theta)}{s_1}-\frac{r_0(s_2\theta)}{s_2}\right)}{\left((\frac{r_0(s)+r_0(s\theta)}{s})^2+(\frac{\cos(s)-\cos(s\theta)}{s})^2\right)^{\frac52}}d\theta\right|&\le C|s_1-s_2|^\alpha\int_0^{\frac{\pi}{2\phi}}\frac{\theta^{1+\alpha}\theta^2}{(1+\theta)^5}\\&\le C|s_1-s_2|^\alpha.\end{aligned}$$

To analyze the difference of the denominator in $T_{2,\phi}$ we first write that for any $0\le s\theta\le\frac{\pi}{2}$,

$$s^5R^{-\frac52}(s,s\theta)=\left(\left(\frac{r_0(s)+r_0(s\theta)}{s}\right)^2+\left(\frac{\cos(s)-\cos(s\theta)}{s}\right)^2\right)^{-\frac52}\lesssim(1+\theta)^{-5}$$

and by differentiation using (4.4.46) and (4.4.48) we find that if $0\le s\theta\le\frac{\pi}{2}$ hence

$$\left|\partial_s(\left(s^5R^{-\frac52}(s,s\theta)\right)\right|\lesssim(1+\theta)^{-4}.$$

This implies in view of the mean value theorem

$$\forall\,0\le s_1\theta,s_2\theta\le\frac{\pi}{2},\quad\left|s_1^5R^{-\frac52}(s_1,s_1\theta)-s_2^5R^{-\frac52}(s_2,s_2\theta)\right|\lesssim(1+\theta)^{-4}|s_1-s_2|.$$

Moreover

$$\forall\, 0 \le s_1\theta, s_2\theta \le \frac{\pi}{2}, \quad \left| s_1^5 R^{-\frac{5}{2}}(s_1, s_1\theta) - s_2^5 R^{-\frac{5}{2}}(s_2, s_2\theta) \right| \lesssim (1+\theta)^{-5}.$$

Then by interpolation we get

$$\forall\, 0 \le s_1\theta, s_2\theta \le \frac{\pi}{2}, \quad \left| s_1^5 R^{-\frac{5}{2}}(s_1, s_1\theta) - s_2^5 R^{-\frac{5}{2}}(s_2, s_2\theta) \right| \lesssim (1+\theta)^{\alpha-5}|s_1 - s_2|^{\alpha}.$$

Therefore we obtain

$$\int_0^{\frac{\pi}{2\phi}} \frac{\sin(s\theta)}{s}\left(\frac{r_0(s\theta)}{s}\right)^2 \left| s_1^5 R^{-\frac{5}{2}}(s_1, s_1\theta) - s_2^5 R^{-\frac{5}{2}}(s_2, s_2\theta) \right| d\theta \lesssim |s_1 - s_2|^{\alpha} \int_0^{\infty} (1+\theta)^{\alpha-2} d\theta,$$

which converges since $\alpha \in (0,1)$.

Combining the preceding estimates one deduces that

$$\forall\, s_1, s_0 \in (0, \phi], \quad |T_{2,\phi}(s_1) - T_{2,\phi}(s_2)| \le C|s_1 - s_2|^{\alpha},$$

uniformly in $\phi \in (0, \pi/2)$. Hence, we conclude that $\zeta_{1,1}$ is $\mathscr{C}^{\alpha}(0, \pi/2)$, for any $\alpha \in (0,1)$.

Let us now move to the regularity of ζ_2 defined in (4.4.16) which takes the form

$$\zeta_2(\phi) = -\frac{3}{8\pi}\zeta_{2,1}(\phi) + \zeta_{2,2}(\phi), \quad \zeta_{2,1}(\phi) := \int_0^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi)[F_1(\rho(\phi,\varphi)) - 1] d\varphi,$$

and

$$\zeta_{2,2}(\phi) := \int_0^{\pi} \partial_\varphi \mathscr{K}_1(\phi,\varphi)\big(F_1(\rho(\phi,\varphi)) - 1\big) d\varphi.$$

We give the details for the first function which can be split into two parts as follows

$$\begin{aligned}
\zeta_{2,1}(\phi) = & \int_0^{\frac{\pi}{2}} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi)[F_1(\rho(\phi,\varphi)) - 1] d\varphi \\
& + \int_{\frac{\pi}{2}}^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi)[F_1(\rho(\phi,\varphi)) - 1] d\varphi \\
=: & I_1(\phi) + I_2(\phi).
\end{aligned}$$

As before by evoking the symmetry property or r we can restrict the study to $\phi \in [0, \frac{\pi}{2}]$. The second term is the easiest one and we claim that $I_2 \in W^{1,\infty}$. Indeed,

$$\begin{aligned}
I_2'(\phi) = & \int_{\frac{\pi}{2}}^{\pi} \partial_\phi \left(\frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \right) \sin(\varphi) r_0^2(\varphi)[F_1(\rho(\phi,\varphi)) - 1] d\varphi \\
& + \int_{\frac{\pi}{2}}^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi) F_1'(\rho(\phi,\varphi)) \partial_\phi \rho(\phi,\varphi) d\varphi.
\end{aligned}$$

It can be transformed into

$$
\begin{aligned}
I_2'(\phi) =& \int_{\frac{\pi}{2}}^{\pi} \partial_\phi \left(\frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \right) \sin(\varphi) r_0^2(\varphi) \Big(F_1(\rho(\phi,\varphi)) - 1 \Big) d\varphi \\
&+ \int_{\frac{\pi}{2}}^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi) F_1'(\rho(\phi,\varphi))(\partial_\phi \rho(\phi,\varphi) + \partial_\varphi \rho(\phi,\varphi)) d\varphi \\
&- \int_{\frac{\pi}{2}}^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi) \partial_\varphi (F_1(\rho(\phi,\varphi) - 1) d\varphi.
\end{aligned}
$$

Integrating by parts yields

$$
\begin{aligned}
I_2'(\phi) =& \int_{\frac{\pi}{2}}^{\pi} \partial_\phi \left(\frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \right) \sin(\varphi) r_0^2(\varphi) \Big(F_1(\rho(\phi,\varphi)) - 1 \Big) d\varphi \\
&+ \int_{\frac{\pi}{2}}^{\pi} \frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi) F_1'(\rho(\phi,\varphi))(\partial_\phi \rho(\phi,\varphi) + \partial_\varphi \rho(\phi,\varphi)) d\varphi \\
&+ \int_{\frac{\pi}{2}}^{\pi} \partial_\varphi \left(\frac{\partial_\phi R(\phi,\varphi)}{R^{\frac{5}{2}}(\phi,\varphi)} \sin(\varphi) r_0^2(\varphi) \right) \Big(F_1(\rho(\phi,\varphi) - 1 \Big) d\varphi \\
&+ \frac{\partial_\phi R(\phi,\frac{\pi}{2})}{R^{\frac{5}{2}}(\phi,\frac{\pi}{2})} r_0^2\left(\pi/2\right) \Big(F_1\left(\rho\left(\phi,\pi/2\right)\right) - 1 \Big).
\end{aligned}
$$

Notice that the last term is bounded uniformly on $\phi \in [0,\pi/2]$. In fact, one has from the definition of R in (4.3.2)

$$
\forall \phi \in [0,\pi/2], \quad \frac{1}{R(\phi,\frac{\pi}{2})} \leq \frac{1}{r_0(\pi/2)}.
$$

Using (4.4.10) we get

$$
\partial_\phi R(\phi,\pi/2) = 2r_0'(\phi)(r_0(\phi) + r_0(\pi/2)) - 2\sin\phi\cos\phi.
$$

Moreover, since r_0 is symmetric with respect to $\pi/2$ then we get $r_0'\left(\frac{\pi}{2}\right) = 0$, which implies that $\partial_\phi R(\pi/2,\pi/2) = 0$ and by the mean value theorem,

$$
\forall \phi \in (0,\pi), \quad \Big| \left(\partial_\phi R\right)\left(\phi,\pi/2\right) \Big| \lesssim \Big| \phi - \frac{\pi}{2} \Big|.
$$

Hence, combining (4.4.28) and (4.4.8) we find

$$
\begin{aligned}
\forall \phi \in \left(0,\frac{\pi}{2}\right), \quad \Big| F_1\left(\rho\left(\phi,\frac{\pi}{2}\right)\right) - 1 \Big| \lesssim& \rho\left(\phi,\frac{\pi}{2}\right) \left(1 - \ln\Big[1 - \rho\left(\phi,\frac{\pi}{2}\right) \Big] \right) \\
\lesssim& \Big| \ln\left(\frac{\pi}{2} - \phi\right) \Big|.
\end{aligned}
$$

Consequently

$$
\forall \phi \in \left(0,\frac{\pi}{2}\right), \quad \frac{\Big|\partial_\phi R(\phi,\frac{\pi}{2})\Big|}{R^{\frac{5}{2}}(\phi,\frac{\pi}{2})} r_0^2\left(\frac{\pi}{2}\right) \Big| F_1\left(\rho\left(\phi,\frac{\pi}{2}\right)\right) - 1 \Big| \lesssim \left(\frac{\pi}{2} - \phi\right) \Big| \ln\left(\frac{\pi}{2} - \phi\right) \Big|, \tag{4.4.50}
$$

which ensures that this quantity is bounded in the interval $(0,\frac{\pi}{2})$.

Next, let us check the boundedness of the integral terms of I_2'. Inequality (4.4.30) allows to get

$$\sup_{\substack{\phi\in[0,\pi/2]\\ \varphi\in[\pi/2,\pi]}}\left|\partial_\phi\left(\frac{\partial_\phi R(\phi,\varphi)}{R(\phi,\varphi)^{\frac52}}\right)\right|+\left|\frac{\partial_\phi R(\phi,\varphi)}{R(\phi,\varphi)^{\frac52}}\right|+\left|\partial_\varphi\left(\frac{\partial_\phi R(\phi,\varphi)}{R(\phi,\varphi)^{\frac52}}\sin(\varphi)r_0(\varphi)^2\right)\right|<\infty,$$

which implies

$$|I_2'(\phi)|\lesssim 1+\int_{\frac{\pi}{2}}^{\pi}\Big|F_1(\rho(\phi,\varphi))-1\Big|d\varphi+\int_{\frac{\pi}{2}}^{\pi}\big|F_1'(\rho(\phi,\varphi))(\partial_\phi\rho(\phi,\varphi)+\partial_\varphi\rho(\phi,\varphi))\big|d\varphi.$$

Therefore, (4.4.8) combined with (4.4.13) and (4.4.37) yield

$$\forall\phi\in[0,\pi/2],\quad |I_2'(\phi)|\leq C+C\int_{\frac{\pi}{2}}^{\pi}\ln\left(\frac{\phi+\varphi}{|\phi-\varphi|}\right)d\varphi+C\int_{\frac{\pi}{2}}^{\pi}\frac{R(\phi,\varphi)}{(\phi-\varphi)^2}\frac{(\phi-\varphi)^2}{R^{\frac32}(\phi,\varphi)}d\varphi\leq C.$$

Let us move to I_1. First, we do the change of variables $\varphi=\phi\theta$ leading to

$$I_1(\phi)=\int_0^{\frac{\pi}{2\phi}}\frac{\phi(\partial_\phi R)(\phi,\phi\theta)}{R^{\frac52}(\phi,\phi\theta)}\sin(\phi\theta)r_0^2(\phi\theta)\Big(F_1(\rho(\phi,\phi\theta))-1\Big)d\theta.$$

We will check that I_1 is $\mathscr{C}^\alpha(0,\pi/2)$, for any $\alpha\in(0,1)$. Indeed, take $\phi_1\leq\phi_2\in(0,\frac{\pi}{2})$, then

$$\begin{aligned}
I_1(\phi_1)-I_1(\phi_2)=&\int_{\frac{\pi}{2\phi_2}}^{\frac{\pi}{2\phi_1}}\frac{\phi_1(\partial_\phi R)(\phi_1,\phi_1\theta)}{R^{\frac52}(\phi_1,\phi_1\theta)}\sin(\phi_1\theta)r_0^2(\phi_1\theta)\Big(F_1(\rho(\phi_1,\phi_1\theta))-1\Big)d\theta\\
&+\int_0^{\frac{\pi}{2\phi_2}}\frac{\phi_1(\partial_\phi R)(\phi_1,\phi_1\theta)}{R^{\frac52}(\phi_1,\phi_1\theta)}\sin(\phi_1\theta)r_0^2(\phi_1\theta)\Big(F_1(\rho(\phi_1,\phi_1\theta))-F_1(\rho(\phi_2,\phi_2\theta))\Big)d\theta\\
&+\int_0^{\frac{\pi}{2\phi_2}}\left(\frac{\phi_1(\partial_\phi R)(\phi_1,\phi_1\theta)}{R^{\frac52}(\phi_1,\phi_1\theta)}\sin(\phi_1\theta)r_0^2(\phi_1\theta)-\frac{\phi_2(\partial_\phi R)(\phi_2,\phi_2\theta)}{R^{\frac52}(\phi_2,\phi_2\theta)}\sin(\phi_2\theta)r_0^2(\phi_2\theta)\right)\\
&\qquad\times\Big(F_1(\rho(\phi_2,\phi_2\theta))-1\Big)d\theta\\
=:&I_{1,1}+I_{1,2}+I_{1,3},
\end{aligned}\tag{4.4.51}$$

where

$$I_{1,1}:=\int_{\frac{\pi}{2\phi_2}}^{\frac{\pi}{2\phi_1}}\frac{\phi_1(\partial_\phi R)(\phi_1,\phi_1\theta)}{R^{\frac52}(\phi_1,\phi_1\theta)}\sin(\phi_1\theta)r_0^2(\phi_1\theta)\Big(F_1(\rho(\phi_1,\phi_1\theta))-1\Big)d\theta.$$

We follow the ideas done for ζ_1. In order to estimate $I_{1,1}$, define

$$G_{1,\phi}(s):=\int_0^{\frac{\pi}{2s}}\frac{\phi(\partial_\phi R)(\phi,\phi\theta)}{R^{\frac52}(\phi,\phi\theta)}\sin(\phi\theta)r_0^2(\phi\theta)\Big(F_1(\rho(\phi,\phi\theta))-1\Big)d\theta.$$

Then

$$\forall\, s\in[\phi,\pi/2),\quad \partial_s G_{1,\phi}(s)=-\frac{\pi}{2s^2}\frac{\phi(\partial_\phi R)(\phi,\frac{\pi\phi}{2s})}{R^{\frac52}(\phi,\frac{\pi\phi}{2s})}\sin\left(\frac{\pi\phi}{2s}\right)r_0^2\left(\frac{\pi\phi}{2s}\right)\left[F_1\left(\rho\left(\phi,\frac{\pi\phi}{2s}\right)\right)-1\right].$$

Applying the mean value theorem with

$$(\partial_\phi R)(\phi,\phi\theta)=2r_0'(\phi)(r_0(\phi)+r_0(\phi\theta))+2\sin(\phi)(\cos(\phi\theta)-\cos(\phi)),$$

we get

$$|(\partial_\phi R)(\phi,\phi\theta)|\leq C\phi(1+\theta)+\phi|1-\theta|. \tag{4.4.52}$$

Moreover, using as before (4.4.28) combined with the assumptions **(H)** we find

$$\big|F_1(\rho(\phi,\phi\theta))-1\big|\lesssim\frac{1}{1+\theta}\Big(1+\ln\left|\frac{1+\theta}{1-\theta}\right|\Big). \tag{4.4.53}$$

Putting together the preceding estimates allows to get

$$\begin{aligned}|\partial_s G_{1,\phi}(s)|\lesssim&\frac{1}{s^2}\frac{\phi\left\{\phi(1+\frac{\pi}{2s})+\phi|1-\frac{\pi}{2s}|\right\}}{\phi^5(1+\frac{\pi}{2s})^5}\frac{\phi^3}{s^3}\frac{1}{1+\frac{\pi}{2s}}\left(1+\ln\left|\frac{1+\frac{\pi}{2s}}{1-\frac{\pi}{2s}}\right|\right)\\ \lesssim&\frac{(s+\frac{\pi}{2})+|s-\frac{\pi}{2}|}{(s+\frac{\pi}{2})^6}\left(1+\ln\left|\frac{1+\frac{\pi}{2s}}{1-\frac{\pi}{2s}}\right|\right).\end{aligned}$$

It follows that

$$\forall s\in(0,\pi/2),\quad\sup_{\phi\in(0,s]}|\partial_s G_{1,\phi}(s)|\lesssim 1+\left|\ln\left(\frac{\pi}{2}-s\right)\right|.$$

Now using this estimate combined with the mean value theorem we get for $0<\phi_1\leq\phi_2<\frac{\pi}{2}$

$$\begin{aligned}|I_{1,1}|\leq&\int_{\phi_1}^{\phi_2}|\partial_s G_{1,\phi_1}(s)|\,ds\\ \leq& C|\phi_1-\phi_2|+\int_{\phi_1}^{\phi_2}\left|\ln\left(\frac{\pi}{2}-s\right)\right|ds.\end{aligned}$$

Using Hölder inequality yields for any $\alpha\in(0,1)$,

$$\int_{\phi_1}^{\phi_2}\left|\ln\left(\frac{\pi}{2}-s\right)\right|ds\leq|\phi_1-\phi_2|^\alpha\left(\int_0^{\frac{\pi}{2}}\left|\ln\left(\frac{\pi}{2}-s\right)\right|^{\frac{1}{1-\alpha}}ds\right)^{1-\alpha}\leq C_\alpha|\phi_1-\phi_2|^\alpha.$$

Notice that the constant C_α blows up when α approaches 1. Thus

$$\forall\,\phi_1,\phi_2\in(0,\pi/2),\quad|I_{1,1}|\leq C_\alpha|\phi_1-\phi_2|^\alpha.$$

Next, let us move to the estimate of $I_{1,2}$. Using (4.4.13) we arrive at

$$|F_1'(\rho(\phi,\phi\theta))|\leq C\frac{R(\phi,\phi\theta)}{\phi^2(1-\theta)^2}\leq C\frac{(1+\theta)^2}{(1-\theta)^2}. \tag{4.4.54}$$

Set

$$\mathscr{R}(\theta,\phi):=\rho(\phi,\phi\theta)=\frac{4r_0(\phi)r_0(\phi\theta)}{R(\phi,\phi\theta)},$$

then differentiating with respect to θ we get

$$\partial_\theta\mathscr{R}(\theta,\phi)=-\frac{8r_0(\phi)r_0(\phi\theta)}{R(\phi,\phi\theta)^2}\Big((r_0(\phi)+r_0(\phi\theta))\phi r_0'(\phi\theta)+(\cos(\phi)-\cos(\phi\theta))\phi\sin(\phi\theta)\Big)$$

$$+\frac{4r_0(\phi)\phi r_0'(\phi\theta)}{R(\phi,\phi\theta)}.$$

Using the assumption **(H2)** we may check that

$$\forall\, 0\leq \phi\theta\leq \pi/2,\quad |\partial_\theta\mathscr{R}(\theta,\phi)|\leq \frac{C}{(1+\theta)^2},$$

where C depends only on $\|r_0'\|_{L^\infty}$. Now by rewriting

$$\begin{aligned}\partial_\theta\mathscr{R}(\theta,\phi)=&\frac{\frac{4r_0(\phi)}{\phi}r_0'(\phi\theta)}{\left(\frac{r_0(\phi)+r_0(\phi\theta)}{\phi}\right)^2+\left(\frac{\cos(\phi)-\cos(\phi\theta)}{\phi}\right)^2}\\&-\frac{8\frac{r_0(\phi)}{\phi}\frac{r_0(\phi\theta)}{\phi}}{\left\{\left(\frac{r_0(\phi)+r_0(\phi\theta)}{\phi}\right)^2+\left(\frac{\cos(\phi)-\cos(\phi\theta)}{\phi}\right)^2\right\}^2}\\&\times\left[\frac{(r_0(\phi)+r_0(\phi\theta))}{\phi}r_0'(\phi\theta)+(\cos(\phi)-\cos(\phi\theta))\frac{\sin(\phi\theta)}{\phi}\right],\end{aligned}$$

and differentiating on ϕ we get the estimate

$$\forall\, 0\leq \phi\theta\leq \pi/2,\quad |\partial_\phi\partial_\theta\mathscr{R}(\theta,\phi)|\leq \frac{C}{(1+\theta)},$$

where C depends only on $\|r_0\|_{\mathscr{C}^2}$. Taylor formulae

$$\mathscr{R}(\theta,\phi)=\mathscr{R}(1,\phi)+\int_1^\theta\partial_\theta\mathscr{R}(\tau,\phi)d\tau$$

combined with $\mathscr{R}(1,\phi)=1$ yields

$$\partial_\phi\mathscr{R}(\theta,\phi)=\int_1^\theta\partial_\phi\partial_\theta\mathscr{R}(\tau,\phi)d\tau.$$

This implies in turn that

$$\sup_{\phi\in(0,\frac{\pi}{2\theta})}|\partial_\phi\mathscr{R}(\theta,\phi)|\leq C\left|\ln\left(\frac{1+\theta}{2}\right)\right|. \tag{4.4.55}$$

Combining this estimate with (4.4.54) we deduce that

$$\sup_{\phi\in(0,\frac{\pi}{2\theta})}\left|\partial_\phi\big[F_1(\rho(\phi,\phi\theta))\big]\right|\leq C\frac{(1+\theta)^2}{(1-\theta)^2}\left|\ln\left(\frac{1+\theta}{2}\right)\right|.$$

Following an interpolation argument combining the preceding estimate with (4.4.53) yields for any $\alpha\in[0,1]$ and for $0<\phi_1\leq\phi_2\leq\frac{\pi}{2\theta}$

$$\left|F_1(\rho(\phi_1,\phi_1\theta))-F_1(\rho(\phi_2,\phi_2\theta))\right|\leq C|\phi_1-\phi_2|^\alpha\frac{(1+\theta)^{3\alpha-1}}{|1-\theta|^{2\alpha}}\left|\ln\left(\frac{1+\theta}{2}\right)\right|^\alpha\left(1+\ln\left|\frac{1+\theta}{1-\theta}\right|\right)^{1-\alpha}.$$

Plugging this estimate into the definition of $I_{1,2}$ given in (4.4.51) implies

$$|I_{1,2}|\leq C|\phi_1-\phi_2|^\alpha\int_0^\infty\frac{\theta^3}{(1+\theta)^5}\frac{(1+\theta)^{3\alpha-1}}{|1-\theta|^{2\alpha}}\left|\ln\left(\frac{1+\theta}{2}\right)\right|^\alpha\left(1+\ln\left|\frac{1+\theta}{1-\theta}\right|\right)^{1-\alpha}d\theta.$$

This integral converges, close to 1 and at ∞, provided that $0 \le \alpha < 1$. We mention that to get the integrability close to 1 we use the approximation

$$\ln\left(\frac{1+\theta}{2}\right) \overset{1}{\sim} \frac{\theta-1}{2}.$$

As to the estimate of the term $I_{1,3}$ described in (4.4.51) we roughly implement similar ideas. For that purpose, we introduce the function

$$\begin{aligned}\forall\, 0 \le s \le \phi \le \frac{\pi}{2}, \quad G_{2,\phi}(s) =: & \int_0^{\frac{\pi}{2\phi}} \frac{s(\partial_\phi R)(s,s\theta)}{R^{\frac52}(s,s\theta)} \sin(s\theta) r_0^2(s\theta)[F_1(\rho(\phi,\phi\theta)) - 1] d\theta \\ = & \int_0^{\frac{\pi}{2\phi}} \frac{\frac{(\partial_\phi R)(s,s\theta)}{s}}{\frac{R^{\frac52}(s,s\theta)}{s^5}} \frac{\sin(s\theta)}{s} \Big(\frac{r_0(s\theta)}{s}\Big)^2 \Big(F_1(\rho(\phi,\phi\theta)) - 1\Big) d\theta.\end{aligned}$$

Then combining (4.4.47), (4.4.52) and (4.4.53), we deduce that

$$\begin{aligned}&\int_0^{\frac{\pi}{2\phi}} \frac{\frac{|\partial_\phi R(s,s\theta)|}{s}\left|\frac{\sin(s_1\theta)}{s_1} - \frac{\sin(s_2\theta)}{s_2}\right|}{\left(\left(\frac{r_0(s)+r_0(s\theta)}{s}\right)^2 + \left(\frac{\cos(s)-\cos(s\theta)}{s}\right)^2\right)^{\frac52}} \frac{r_0(s\theta)^2}{s^2}\big|F_1(\rho(\phi,\phi\theta)) - 1\big| d\theta \\ &\le C|s_1 - s_2|^\alpha \int_0^\infty \frac{1+\theta}{(1+\theta)^5} \theta^{1+\alpha}\theta^2 \frac{1}{1+\theta}\left(1 + \ln\left|\frac{1+\theta}{1-\theta}\right|\right) d\theta \\ &\le C|s_1 - s_2|^\alpha,\end{aligned}$$

provided that $\alpha \in (0,1)$. Implementing the same analysis for the remaining terms and using in particular (4.4.49) as for ζ_1, we find

$$\forall\, 0 \le s_1, s_2 \le \phi, \quad |G_{2,\phi}(s_1) - G_{2,\phi}(s_2)| \le C|s_1 - s_2|^\alpha,$$

uniformly for $\phi \in (0,\pi/2)$. Therefore from the definition (4.4.51) we obtain for any $0 \le \phi_1 \le \phi_2 \le \frac{\pi}{2}$,

$$|I_{1,3}| = \big|G_{2,\phi_2}(\phi_1) - G_{2,\phi_2}(\phi_1)\big| \le C|\phi_1 - \phi_2|^\alpha.$$

It remains to estimate the term ζ_3 defined by (4.4.16). It can be split as follows,

$$\begin{aligned}\zeta_3(\phi) =& \frac{1}{4\pi}\int_0^\pi \frac{\sin(\varphi) r_0^2(\varphi)}{R^{\frac32}(\phi,\varphi)} (\partial_\phi \rho(\phi,\varphi) + \partial_\varphi \rho(\phi,\varphi)) \left[F_1'(\rho(\phi,\varphi)) - \frac34\right] d\varphi \\ =& \frac{1}{4\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(\varphi) r_0^2(\varphi)}{R^{\frac32}(\phi,\varphi)} (\partial_\phi \rho(\phi,\varphi) + \partial_\varphi \rho(\phi,\varphi)) \left[F_1'(\rho(\phi,\varphi)) - \frac34\right] d\varphi \\ &+ \frac{1}{4\pi}\int_{\frac{\pi}{2}}^\pi \frac{\sin(\varphi) r_0^2(\varphi)}{R^{\frac32}(\phi,\varphi)} (\partial_\phi \rho(\phi,\varphi) + \partial_\varphi \rho(\phi,\varphi)) \left[F_1'(\rho(\phi,\varphi)) - \frac34\right] d\varphi \\ =& I_3 + I_4.\end{aligned}$$

We will show that I_3 belongs to $\mathscr{C}^\alpha(0,\frac{\pi}{2})$ and the same procedure works as well to check the regularity for I_4 that we skip here. To estimate I_3 we proceed as before through the use of the change of variables $\varphi=\phi\theta$,

$$I_3(\phi)=\frac{1}{4\pi}\int_0^{\frac{\pi}{2\phi}}\frac{\phi\sin(\phi\theta)r_0(\phi\theta)^2}{R(\phi,\phi\theta)^{\frac32}}((\partial_\phi\rho)(\phi,\theta\phi)+(\partial_\varphi\rho)(\phi,\phi\theta))\left[F_1'(\rho(\phi,\phi\theta))-\frac34\right]d\theta.$$

Define the functions

$$\begin{aligned}
H_{1,\phi}(s)&:=\int_0^{\frac{\pi}{2s}}\frac{\phi\sin(\phi\theta)r_0^2(\phi\theta)}{R^{\frac32}(\phi,\phi\theta)}\Big((\partial_\phi\rho)(\phi,\theta\phi)+(\partial_\varphi\rho)(\phi,\phi\theta)\Big)\left[F_1'(\rho(\phi,\phi\theta))-\frac34\right]d\theta,\\
H_{2,\phi}(s)&:=\int_0^{\frac{\pi}{2\phi}}\frac{s\sin(s\theta)r_0^2(s\theta)}{R^2(s,s\theta)}R^{\frac12}(\phi,\phi\theta)\Big((\partial_\phi\rho)(\phi,\theta\phi)+(\partial_\varphi\rho)(\phi,\phi\theta)\Big)\left[F_1'(\rho(\phi,\phi\theta))-\frac34\right]d\theta,\\
H_{3,\phi}(s)&:=\int_0^{\frac{\pi}{2\phi}}\frac{\phi\sin(\phi\theta)r_0^2(\phi\theta)}{R^2(\phi,\phi\theta)}R^{\frac12}(s,s\theta)\Big((\partial_\phi\rho)(s,\theta s)+(\partial_\varphi\rho)(s,s\theta)\Big)\left[F_1'(\rho(\phi,\phi\theta))-\frac34\right]d\theta,\\
H_{4,\phi}(s)&:=\int_0^{\frac{\pi}{2\phi}}\frac{\phi\sin(\phi\theta)r_0^2(\phi\theta)}{R^2(\phi,\phi\theta)}R^{\frac12}(\phi,\phi\theta)\Big((\partial_\phi\rho)(\phi,\theta\phi)+(\partial_\varphi\rho)(\phi,\phi\theta)\Big)\left[F_1'(\rho(s,s\theta))-\frac34\right]d\theta.
\end{aligned}$$

In order to check that I_3 belongs to $\mathscr{C}^\alpha(0,\frac{\pi}{2})$, it suffices to prove that each function $H_{i,\phi}$ is in $\mathscr{C}^\alpha(0,\frac{\pi}{2})$ uniformly in $\phi\in(0,\pi/2)$, for any $i=1,\dots,4$. Let us start with $H_{1,\phi}$ showing that its derivative is bounded.

From straightforward calculus it is easy to check that for any $0\le\phi\le s<\frac{\pi}{2}$,

$$|H_{1,\phi}'(s)|\le\frac{\pi}{2s^2}\frac{\phi\sin\left(\frac{\phi\pi}{2s}\right)r_0^2\left(\frac{\phi\pi}{2s}\right)}{R^{\frac32}\left(\phi,\frac{\phi\pi}{2s}\right)}\Big|(\partial_\phi\rho)(\phi,\phi\pi/2s)+(\partial_\varphi\rho)(\phi,\phi\pi/2s)\Big|\left|F_1'\left(\rho\big(\phi,\phi\pi/2s\big)\right)-\frac34\right|.$$

Hence, we obtain

$$|H_{1,\phi}'(s)|\lesssim\frac{1}{s^5}\frac{\phi^4}{\phi^3(1+\frac{\pi}{2s})^3}\Big|(\partial_\phi\rho)(\phi,\phi\pi/2s)+(\partial_\varphi\rho)(\phi,\phi\pi/2s)\Big|\left|F_1'\left(\rho\big(\phi,\phi\pi/2s\big)\right)-\frac34\right|.$$

Using (4.4.34)–(4.4.37)–(4.4.38) allows to get

$$|H_{1,\phi}'(s)|\lesssim\frac{1}{s^5}\frac{\phi^4}{\phi^3(1+\frac{\pi}{2s})^3}\frac{\phi^2\left(1-\frac{\pi}{2s}\right)^2}{\phi^3\left(1+\frac{\pi}{2s}\right)^3}\frac{\phi^2\frac{\pi}{2s}}{\phi^2\left(1-\frac{\pi}{2s}\right)^2}\lesssim1,$$

which is uniformly bounded on $0\le\phi\le s<\frac{\pi}{2}$. We shall skip the details for $H_{2,\phi}$ which can be analyzed following the same lines of the term $T_{2,\phi}$ introduced in (4.4.42).

Let us now focus on the estimate of $H_{3,\phi}$. Set

$$\mathscr{T}(\theta,s):=R^{\frac12}(s,s\theta)\big((\partial_\phi\rho)(s,s\theta)+(\partial_\varphi\rho)(s,s\theta)\big),$$

then using (4.4.37), we deduce

$$|\mathscr{T}(\theta,s)|\le C\frac{(1-\theta)^2}{(1+\theta)^2}. \tag{4.4.56}$$

By ranging the expression of $\mathscr{T}$ as follows

$$\begin{aligned}\mathscr{T}(\theta,s) =& 4\frac{\frac{r_0^2(s\theta)}{s^2}-\frac{r_0^2(s)}{s^2}}{\frac{R^{\frac{3}{2}}(s,s\theta)}{s^3}}r_0'(s)\left(\frac{r_0(s\theta)}{s}-\frac{r_0(s)}{s}\right)\\ &+4\frac{\frac{r_0^2(s\theta)}{s^2}-\frac{r_0^2(s)}{s^2}}{\frac{R^{\frac{3}{2}}(s,s\theta)}{s^3}}\frac{r_0(s)}{s}\Big(r_0'(s)-r_0'(s\theta)\Big)\\ &+8\frac{\frac{r_0(s\theta)}{s}\frac{r_0(s)}{s}}{\frac{R^{\frac{3}{2}}(s,s\theta)}{s^3}}\big(\cos s-\cos s\theta\big)\left(\frac{\sin s}{s}-\frac{\sin s\theta}{s}\right)\\ &+4\frac{\frac{(\cos s-\cos s\theta)^2}{s^2}}{\frac{R^{\frac{3}{2}}(s,s\theta)}{s^3}}\left(\frac{r_0(s\theta)}{s}r_0'(s)+\frac{r_0(s)}{s}r_0'(s\theta)\right),\end{aligned}$$

and differentiating with respect to s we find

$$|\partial_s\mathscr{T}(\theta,s)|\le C\frac{(1-\theta)^2}{(1+\theta)}. \tag{4.4.57}$$

We will not give the full details for this estimate because the computations are long and tedious, but to get a more precise idea how this works we shall just explain the estimate of the first term in $\partial_s\mathscr{T}$ given by (the other terms are treated similarly)

$$\forall\ 0<\theta s<\frac{\pi}{2},\quad \mathscr{T}_1(\theta,s)=:4\frac{\partial_s\left(\frac{r_0(s\theta)-r_0(s)}{s}\right)\frac{r_0(s\theta)+r_0(s)}{s}}{\frac{R^{\frac{3}{2}}(s,s\theta)}{s^3}}r_0'(s)\frac{r_0(s\theta)-r_0(s)}{s}.$$

Define

$$\forall\ 0<\theta s<\frac{\pi}{2},\quad g(\theta,s):=\frac{r_0(s\theta)-r_0(s)}{s}.$$

Then, one has $\partial_\theta g(\theta,s)=r_0'(s\theta)$ and then

$$|\partial_s\partial_\theta g(\theta,s)|=|\theta r_0''(s\theta)|\le C\theta. \tag{4.4.58}$$

Since $g(1,s)=0$, we can write by Taylor formulae

$$g(\theta,s)=\int_1^\theta \partial_\theta g(\tau,s)d\tau,$$

and hence

$$\partial_s g(\theta,s)=\int_1^\theta \partial_s\partial_\theta g(\tau,s)d\tau.$$

Using (4.4.58), we achieve

$$\forall\ 0<\theta s<\frac{\pi}{2},\quad |\partial_s g(\theta,s)|\le C|1-\theta|\theta.$$

Plugging this into the the definition of $\mathscr{T}_1$ and using the mean value theorem yield to the estimate

$$|\mathscr{T}_1|\le C\frac{|1-\theta|\theta(1+\theta)|1-\theta|}{(1+\theta)^3}\le C\frac{(1-\theta)^2}{(1+\theta)}.$$

Now, interpolating between (4.4.56) and (4.4.57), we find that for any $\alpha \in (0,1)$

$$|\mathscr{T}(\theta, s_1) - \mathscr{T}(\theta, s_2)| \leq C|s_1 - s_2|^{\alpha} \frac{(1-\theta)^2}{(1+\theta)^{2-\alpha}}. \tag{4.4.59}$$

Using (4.4.38) we get

$$\forall\, 0 \leq \phi\theta \leq \frac{\pi}{2}, \quad \left|F_1'(\rho(\phi, \phi\theta)) - \frac{3}{4}\right| \leq C \frac{\theta}{(1-\theta)^2}. \tag{4.4.60}$$

Combining this estimate with (4.4.59) and (4.4.38), we conclude that for any $0 \leq s_1, s_2 \leq \phi \leq \frac{\pi}{2}$

$$|H_{3,\phi}(s_1) - H_{3,\phi}(s_2)| \leq C|s_1 - s_2|^{\alpha} \int_0^{+\infty} \frac{\phi^4\theta^3}{\phi^4(1+\theta)^4} \frac{(1-\theta)^2}{(1+\theta)^{2-\alpha}} \frac{\theta}{(1-\theta)^2} d\theta C|s_1 - s_2|^{\alpha}, \tag{4.4.61}$$

for any $\alpha \in (0,1)$. Let us finish working with $H_{4,\phi}$. Moreover, as a consequence of (C.0.13) and (4.4.55) one has

$$\left|\partial_s \left(F_1'(\rho(s, s\theta)) - \frac{3}{4}\right)\right| \leq C|F_1''(\rho(s, s\theta))||\partial_s(\rho(s, s\theta))| \leq C \frac{(1+\theta)^4}{(1-\theta)^4} \left|\ln\left(\frac{1+\theta}{2}\right)\right|. \tag{4.4.62}$$

Interpolating between (4.4.60) and (4.4.62) we achieve

$$\begin{aligned}\left|F_1'(\rho(s_1, s_1\theta)) - F_1'(\rho(s_2, s_2\theta))\right| \leq & C|s_1 - s_2|^{\alpha} \frac{(1+\theta)^{4\alpha}}{(1-\theta)^{4\alpha}} \left|\ln\left(\frac{1+\theta}{2}\right)\right|^{\alpha} \frac{\theta^{1-\alpha}}{(1-\theta)^{2(1-\alpha)}} \\ \leq & C|s_1 - s_2|^{\alpha} \frac{(1+\theta)^{1+3\alpha}}{(1-\theta)^{2+2\alpha}} \left|\ln\left(\frac{1+\theta}{2}\right)\right|^{\alpha}.\end{aligned} \tag{4.4.63}$$

Finally, using (4.4.35), (4.4.37), (4.4.56) and (4.4.63) we obtain for any $0 \leq s_1, s_2 \leq \phi$

$$\begin{aligned}|H_{4,\phi}(s_1) - H_{4,\phi}(s_2)| \leq & C|s_1 - s_2|^{\alpha} \int_0^{+\infty} \frac{\theta^3}{(1+\theta)^4} \frac{(1-\theta)^2}{(1+\theta)^2} \frac{(1+\theta)^{1+3\alpha}}{(1-\theta)^{2+2\alpha}} \left|\ln\left(\frac{1+\theta}{2}\right)\right|^{\alpha} d\theta \\ \leq & C|s_1 - s_2|^{\alpha} \int_0^{+\infty} (1+\theta)^{3\alpha-2}|1-\theta|^{-2\alpha} \left|\ln\left(\frac{1+\theta}{2}\right)\right|^{\alpha} d\theta \\ \leq & C|s_1 - s_2|^{\alpha},\end{aligned}$$

the convergence of the integral is guaranteed provided that $\alpha \in (0,1)$. This achieves the proof of $\nu_\Omega \in \mathscr{C}^{1,\alpha}(0,\pi)$ for any $\alpha \in (0,1)$.

(4) Since the function ν_Ω reaches its minimum at a point $\phi_0 \in [0,\pi]$, we have that if this point belongs to the open set $(0,\pi)$ then necessary $\nu_\Omega'(\phi_0) = 0$. However when $\phi_0 \in \{0,\pi\}$ then from the point **(3)** of Proposition 4.4.1 we deduce also that the derivative is vanishing at ϕ_0. Using the mean value theorem, we obtain for any $\phi \in [0,\pi]$

$$\nu_\Omega(\phi) = \nu_\Omega(\phi_0) + \nu_\Omega'(\overline{\phi})(\phi - \phi_0) = \nu_\Omega(\phi_0) + \big(\nu_\Omega'(\overline{\phi}) - \nu_\Omega'(\phi_0)\big)(\phi - \phi_0),$$

for some $\overline{\phi} \in (\phi_0, \phi)$. Since $\nu_\Omega' \in \mathscr{C}^{\alpha}$ then

$$\left|\nu_\Omega'(\overline{\phi}) - \nu_\Omega'(\phi_0)\right| \leq \|\nu_\Omega'\|_{\mathscr{C}^\alpha} |\phi - \phi_0|^{\alpha}.$$

Notice that $\|\nu'_\Omega\|_{\mathscr{C}^\alpha}$ is independent of Ω. Consequently

$$\forall \phi \in [0,\pi], \quad 0 \le \nu_\Omega(\phi) - \nu_\Omega(\phi_0) \le C|\phi - \phi_0|^{1+\alpha},$$

for some absolute constant C. In the particular case $\Omega = \kappa$ we get from the definition (4.4.5) that $\nu_\kappa(\phi_0) = 0$ and therefore the preceding result becomes

$$\forall \phi \in [0,\pi], \quad 0 \le \nu_\kappa(\phi) \le C|\phi - \phi_0|^{1+\alpha}, \quad \nu_\kappa(\phi_0) = 0.$$

□

4.4.3 Eigenvalue problem

In Section 4.4.1 we have checked that the operator $\mathcal{L}_n^\Omega$ defined in (4.4.1) is of integral type. Then studying the kernel of this operator reduces to solving the integral equation

$$\mathcal{K}_n^\Omega h_n(\phi) := \int_0^\pi K_n(\phi,\varphi) h_n(\varphi) d\mu_\Omega(\varphi) = h_n(\phi), \quad \forall \phi \in [0,\pi], \tag{4.4.64}$$

where the kernel K_n and the measure $d\mu_\Omega$ are defined successively in (4.4.2) and (4.4.4). The parameter Ω ranges over the interval $(-\infty,\kappa)$. This latter condition is imposed to guarantee the positivity of the measure $d\mu_\Omega$ through the positivity of ν_Ω according to Lemma 4.4.1. We point out that studying the kernel of $\mathcal{L}_n^\Omega$ amounts to finding the values of Ω such that 1 is an eigenvalue of $\mathcal{K}_n^\Omega$. To investigate the spectral study of $\mathcal{K}_n^\Omega$ we need to introduce the Hilbert space $L^2_{\mu_\Omega}$ of measurable functions $f : [0,\pi] \to \mathbb{R}$ such that

$$\|f\|_{\mu_\Omega} := \left(\int_0^\pi |f(\varphi)|^2 d\mu_\Omega(\varphi)\right)^{\frac{1}{2}} < \infty. \tag{4.4.65}$$

Notice that the space $L^2_{\mu_\Omega}$ is equipped with the usual inner product:

$$\langle f, g\rangle_\Omega = \int_0^\pi f(\varphi) g(\varphi) d\mu_\Omega(\varphi), \quad \forall f, g \in L^2_{\mu_\Omega}. \tag{4.4.66}$$

Remark 4.4.2. 1. *Since $d\mu_\Omega$ is a nonnegative bounded Borel measure for any $\Omega \in (-\infty,\kappa)$, then the Hilbert space $L^2_{\mu_\Omega}$ is separable.*

2. *For any $\Omega \in (-\infty,\kappa)$, the space $L^2_{\mu_\Omega}$ is isomorphic to the space L^2_μ where*

$$d\mu(\varphi) = \sin(\varphi)\, r_0^2(\varphi)\, d\varphi.$$

This follows from Proposition 4.4.1-(2)which ensures that ν_Ω is nowhere vanishing. However this property fails for the critical value $\Omega = \kappa$ because ν_κ is vanishing at some points.

The next proposition deals with some basic properties of the operator $\mathcal{K}_n^\Omega$.

Proposition 4.4.3. *Let $\Omega \in (-\infty,\kappa)$ and r_0 satisfies the assumptions* **(H1)** *and* **(H2)**. *Then, the following assertions hold true.*

1. *For any $n \ge 1$, the operator $\mathcal{K}_n^\Omega : L^2_{\mu_\Omega} \to L^2_{\mu_\Omega}$ is Hilbert–Schmidt and self-adjoint.*

2. *For any $n \geq 1$, the eigenvalues of $\mathcal{K}_n^\Omega$ form a countable family of real numbers. Let $\lambda_n(\Omega)$ be the largest eigenvalue, then it is strictly positive and satisfies*

$$\int_0^\pi \int_0^\pi \frac{H_n(\phi,\varphi)\sin^{\frac12}(\phi)r_0(\phi)\varrho(\varphi)\varrho(\phi)}{\nu_\Omega^{\frac12}(\varphi)\nu_\Omega^{\frac12}(\phi)\sin^{\frac12}(\varphi)r_0(\varphi)}d\varphi d\phi \leq \lambda_n(\Omega) \leq \int_0^\pi \int_0^\pi K_n^2(\phi,\varphi)d\mu_\Omega(\varphi)\,d\mu_\Omega(\phi),$$

for any function ϱ such that $\int_0^\pi \varrho^2(\varphi)d\varphi = 1$.

3. *We have the following decay: for any $\alpha \in [0,1)$ there exists $C > 0$ such that*

$$\forall\,\Omega \in (-\infty,\kappa),\,\forall\,n \geq 1, \qquad \int_0^\pi \int_0^\pi K_n^2(\phi,\varphi)d\mu_\Omega(\varphi)\,d\mu_\Omega(\phi) \leq C(\kappa-\Omega)^{-2}n^{-\alpha}.$$

4. *The eigenvalue $\lambda_n(\Omega)$ is simple and the associated nonzero eigenfunctions do not vanish in $(0,\pi)$.*

5. *For any $\Omega \in (-\infty,\kappa)$, the sequence $n \in \mathbb{N}^\star \mapsto \lambda_n(\Omega)$ is strictly decreasing.*

6. *For any $n \geq 1$ the map $\Omega \in (-\infty,\kappa) \mapsto \lambda_n(\Omega)$ is differentiable and strictly increasing.*

Proof. **(1)** In order to check that $\mathcal{K}_n^\Omega$ is a Hilbert–Schmidt operator, we need to verify that the kernel K_n satisfies the integrability condition

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega} := \left(\int_0^\pi \int_0^\pi |K_n(\phi,\varphi)|^2 d\mu_\Omega(\varphi)d\mu_\Omega(\phi)\right)^{\frac12} < +\infty.$$

Indeed, by (4.4.2) anf (4.3.3), one gets

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 = C_n \int_0^\pi \int_0^\pi \frac{\sin(\varphi)\sin(\phi)\,r_0^{2n}(\phi)r_0^{2n}(\varphi)}{R^{2n+1}(\phi,\varphi)\,\nu_\Omega(\varphi)\nu_\Omega(\phi)}\,F_n^2\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)d\varphi d\phi,$$

for some constant C_n and R was defined in (4.3.2). Remark that

$$\left|\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right| \leq 1.$$

Moreover, according to Lemma 4.4.1 the function $\nu_\Omega(\varphi)$ is not vanishing in the interval $[0,\pi]$ provided that $\Omega < \kappa$. Therefore we get

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \lesssim \int_0^\pi \int_0^\pi \frac{\sin(\varphi)\sin(\phi)}{R(\phi,\varphi)}F_n^2\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)d\varphi d\phi.$$

By (C.0.11) and the assumption **(H2)** we deduce that

$$\begin{aligned}\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \leq & C + C\int_0^\pi \int_0^\pi \ln^2\left(1-\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)d\varphi d\phi\\ \leq & C + C\int_0^\pi \int_0^\pi \ln^2\left(\frac{(r_0(\phi)-r_0(\varphi))^2+(\cos\phi-\cos\varphi)^2}{R(\phi,\varphi)}\right)d\varphi d\phi.\end{aligned}$$

It suffices now to use the inequality (4.4.8) to get

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \leq C + C\int_0^\pi\int_0^\pi \ln^2\left(\frac{\sin\phi+\sin\varphi}{|\phi-\varphi|}\right)d\varphi d\phi < \infty.$$

This concludes that the operator $\mathcal{K}_n^\Omega$ is bounded and is of Hilbert–Schmidt type. As a consequence from the general theory this operator is necessary compact.

On the other hand, as we have mentioned before the kernel K_n is symmetric in view of the formula (4.4.6) and the symmetry of R defined in (4.3.2). Therefore we deduce that $\mathcal{K}_n^\Omega$ is a self–adjoint operator

(2) From the spectral theorem on self-adjoint compact operators, we know that the eigenvalues of $\mathcal{K}_n^\Omega$ form a countable family of real numbers. Define the real numbers

$$m = \inf_{\|h\|_{\mu_\Omega}=1}\langle \mathcal{K}_n^\Omega h, h\rangle_\Omega \quad \text{and} \quad M = \sup_{\|h\|_{\mu_\Omega}=1}\langle \mathcal{K}_n^\Omega h, h\rangle_\Omega.$$

Since $\mathcal{K}$ is self-adjoint, we obtain $\sigma(\mathcal{K}_n^\Omega) \subset [m, M]$, with $m \in \sigma(\mathcal{K}_n^\Omega)$ and $M \in \sigma(\mathcal{K}_n^\Omega)$, where the set $\sigma(\mathcal{K}_n^\Omega)$ denotes the spectrum of $\mathcal{K}_n^\Omega$. Since $\lambda_n(\Omega)$ is the largest eigenvalue, then

$$\lambda_n(\Omega) = M = \sup_{\|h\|_{\mu_\Omega}=1}\langle \mathcal{K}_n^\Omega h, h\rangle_\Omega. \tag{4.4.67}$$

We shall prove that $M > 0$ and $|m| \leq M$. Indeed, for any $h \in L^2_{\mu_\Omega}$, the positive function $|h|$ belongs also to $L^2_{\mu_\Omega}$ with the same norm and using the positivity of the kernel K_n we obtain

$$\sup_{\|h\|_{\mu_\Omega}=1}\langle \mathcal{K}_n^\Omega h, h\rangle_\Omega = \sup_{h\geq 0, \|h\|_{\mu_\Omega}=1}\langle \mathcal{K}_n^\Omega h, h\rangle_\Omega.$$

Using once again the positivity of the kernel one deduces that

$$\forall h \geq 0, \|h\|_{\mu_\Omega} = 1 \Longrightarrow \langle \mathcal{K}_n^\Omega h, h\rangle_\Omega > 0.$$

Consequently, we obtain that $M > 0$. In order to prove that $|m| \leq M$, we shall proceed as follows. Using the positivity of the kernel, we achieve

$$|m| \leq \langle \mathcal{K}_n^\Omega |h|, |h|\rangle_\Omega \leq M, \quad \forall \|h\|_{\mu_\Omega} = 1.$$

This implies that M is nothing but the spectral radius of the operator $\mathcal{K}_n^\Omega$, that is,

$$M = \|\mathcal{K}_n^\Omega\|_{\mathcal{L}(L^2_{\mu_\Omega})}.$$

From the Cauchy–Schwarz inequality, one deduces that

$$\|\mathcal{K}_n^\Omega\|^2_{\mathcal{L}(L^2_{\mu_\Omega})} \leq \int_0^\pi\int_0^\pi |K_n(\phi,\varphi)|^2 d\mu_\Omega(\phi)d\mu_\Omega(\varphi),$$

which implies that

$$\lambda_n^2(\Omega) \leq \int_0^\pi\int_0^\pi |K_n(\phi,\varphi)|^2 d\mu_\Omega(\phi)d\mu_\Omega(\varphi).$$

For the lower bound, we shall work with the special function

$$f(\varphi) = \frac{\varrho(\varphi)}{\sin(\varphi)^{\frac{1}{2}} r_0(\varphi)\nu_\Omega(\varphi)^{\frac{1}{2}}}, \quad \varphi \in (0,\pi),$$

with the normalized condition $\|f\|_{\mu_\Omega} = 1$ which is equivalent to

$$\int_0^\pi \varrho^2(\varphi)d\varphi = 1$$

and

$$\lambda_n(\Omega) \geq \langle \mathcal{K}_n^\Omega f, f\rangle_\Omega = \int_0^\pi \int_0^\pi \frac{H_n(\phi,\varphi)}{\nu_\Omega^{\frac12}(\varphi)\nu_\Omega^{\frac12}(\phi)} \frac{\sin^{\frac12}(\phi)r_0(\phi)}{\sin^{\frac12}(\varphi)r_0(\varphi)}\varrho(\varphi)\varrho(\phi)d\varphi d\phi.$$

This gives the announced lower bound for the largest eigenvalue.

(3) From the expression of K_n given by (4.4.2) we easily get

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \leq \int_0^\pi \int_0^\pi K_n^2(\phi,\varphi)d\mu_\Omega(\varphi)\, d\mu_\Omega(\phi) = \int_0^\pi \int_0^\pi \frac{H_n^2(\phi,\varphi)}{\nu_\Omega(\phi)\nu_\Omega(\varphi)} \frac{\sin(\phi)r_0^2(\phi)}{\sin(\varphi)r_0^2(\varphi)}d\phi d\varphi.$$

Using the definition (4.4.5) of κ we infer

$$\forall\, \phi \in [0,\pi], \quad \nu_\Omega(\phi) \geq \kappa - \Omega$$

and combined with the assumption **(H2)** we obtain

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \lesssim (\kappa-\Omega)^{-2} \int_0^\pi \int_0^\pi H_n^2(\phi,\varphi)\frac{\sin(\phi)r_0^2(\phi)}{\sin(\varphi)r_0^2(\varphi)}d\phi d\phi d\varphi.$$

Applying Lemma 4.3.4 yields for any $0 \leq \alpha < \beta \leq 1$

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \lesssim (\kappa-\Omega)^{-2} n^{-2\alpha} \int_0^\pi \int_0^\pi |\phi-\varphi|^{-2\beta} d\phi d\varphi.$$

By taking $\beta < \frac12$ we get the convergence of the integral and consequently we obtain the announced result,

$$\|\mathcal{K}_n^\Omega\|_{\mu_\Omega}^2 \lesssim (\kappa-\Omega)^{-2} n^{-2\alpha}.$$

(4) First, let us check that any nonzero eigenfunction associated to the largest eigenvalue $\lambda_n(\Omega)$ should be with a constant sign. Indeed, let f be a nonzero normalized eigenfunction and assume that it changes the sign over a non negligible set. From the strict positivity of the kernel in the interval $(0,\pi)$, we deduce that

$$\mathcal{K}_n^\Omega f(\phi) < \mathcal{K}_n^\Omega |f|(\phi), \quad \forall \phi \in (0,\pi).$$

First, by the assumption on f we get

$$\int_0^\pi \mathcal{K}_n^\Omega(f)(\phi) f(\phi) d\mu_\Omega(\phi) = \lambda_n(\Omega) \int_0^\pi f^2(\phi)d\mu_\Omega(\phi) = \lambda_n(\Omega).$$

Second, from (4.4.67) we have that

$$\int_0^\pi \mathcal{K}_n^\Omega(|f|)(\phi)|f(\phi)|d\mu_\Omega(\phi) \leq \lambda_n(\Omega).$$

Consequently,

$$\lambda_n(\Omega) = \int_0^\pi \mathcal{K}_n^\Omega(f)(\phi)f(\phi)d\mu_\Omega(\phi) < \int_0^\pi \mathcal{K}_n^\Omega(|f|)(\phi)|f(\phi)|d\mu_\Omega(\phi) \leq \lambda_n(\Omega),$$

achieving a contradiction. Hence, any nonzero eigenfunction of $\lambda_n(\Omega)$ must have a constant sign. Now let us check that f is not vanishing in $(0, \pi)$. First we write

$$f(\phi) = \frac{1}{\lambda_n(\Omega)}\mathcal{K}_n^\Omega f(\phi) = \frac{1}{\lambda_n(\Omega)\nu_\Omega(\phi)}\int_0^\pi H_n(\phi, \varphi)f(\varphi)d\varphi.$$

From (4.3.3) and Lemma 4.4.1 we get

$$\forall \phi, \varphi \in (0, \pi), \quad H_n(\phi, \varphi) > 0, \quad \nu_\Omega(\phi) > 0.$$

The first assertion follows from the strict positivity of the associated hypergeometric function. Combined with the positivity of f we deduce that

$$\forall \phi \in (0, \pi), \quad f(\phi) > 0.$$

Finally, we shall check that the dimension of the subspace generated by the eigenfunctions associated to $\lambda_n(\Omega)$ is one–dimensional. Assume that we have two independent eigenfunctions f_0 and f_1, which are necessary with constant sign, then there exists $a, b \in \mathbb{R}$ such that the eigenfunction $af_0 + bf_1$ changes its sign. This is a contradiction.

(5) Using (4.4.2) combined with Lemma 4.3.4, we get that $n \in \mathbb{N}^\star \mapsto K_n(\phi, \varphi)$ is strictly decreasing for any $\varphi \neq \phi \in (0, \pi)$. Then, for any $\Omega \in (-\infty, \kappa)$ and for any nonnegative function f, we get

$$\forall\, \phi \in (0, \pi), \quad \mathcal{K}_n^\Omega f(\phi) > \mathcal{K}_{n+1}^\Omega f(\phi),$$

which implies in turn that

$$\int_0^\pi \mathcal{K}_n^\Omega(f)(\phi)f(\phi)d\mu_\Omega(\phi) > \int_0^\pi \mathcal{K}_{n+1}^\Omega(f)(\phi)f(\phi)d\mu_\Omega(\phi).$$

Since the largest eigenvalue $\lambda_{n+1}(\Omega)$ is reached at some positive normalized function $f_{n+1} \geq 0$, then

$$\begin{aligned}
\lambda_{n+1}(\Omega) &= \int_0^\pi \mathcal{K}_{n+1}^\Omega(f_{n+1})(\phi)f_{n+1}(\phi)d\mu_\Omega(\phi)\\
&< \int_0^\pi \mathcal{K}_n^\Omega(f_{n+1})(\phi)f_{n+1}(\phi)d\mu_\Omega(\phi)\\
&< \sup_{\|f\|_{\mu_\Omega}=1} \int_0^\pi \mathcal{K}_n^\Omega(f)(\phi)f(\phi)d\mu_\Omega(\phi)\\
&< \lambda_n(\Omega).
\end{aligned}$$

This provides the announced result.

(6) Fix $\Omega_0 \in (-\infty, \kappa)$ and denote by f_n^Ω the positive normalized eigenfunction associated to the eigenvalue $\lambda_n(\Omega)$. Using the definition of eigenfunction, then

$$\lambda_n(\Omega) = \frac{\langle \mathcal{K}_n^\Omega f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}, \quad \|f_n^\Omega\|_{\mu_{\Omega_0}} = 1.$$

The regularity follows from the general theory using the fact this eigenvalue is simple. However we can in our special case give a direct proof for its differentiability in the following way. From the decomposition

$$\frac{1}{\nu_\Omega(\phi)} = \frac{1}{\nu_{\Omega_0}(\phi)} + \frac{\Omega - \Omega_0}{\nu_\Omega(\phi)\nu_{\Omega_0}(\phi)},$$

we get according to the expression of $\mathcal{K}_n^\Omega$

$$\begin{aligned}\mathcal{K}_n^\Omega f(\phi) =& \frac{1}{\nu_{\Omega_0}(\phi)} \int_0^\pi H_n(\phi,\varphi) f(\varphi) d\varphi + (\Omega - \Omega_0) \int_0^\pi \frac{H_n(\phi,\varphi)}{\nu_\Omega(\phi)\nu_{\Omega_0}(\phi)} f(\varphi) d\varphi \\ =& \mathcal{K}_n^{\Omega_0} f(\phi) + (\Omega - \Omega_0) \mathscr{R}_n^{\Omega_0,\Omega} f(\phi)\end{aligned}$$

with

$$\mathscr{R}_n^{\Omega_0,\Omega} f(\phi) := \int_0^\pi \frac{H_n(\phi,\varphi)}{\nu_\Omega(\phi)\nu_{\Omega_0}(\phi)} f(\varphi) d\varphi.$$

Therefore we obtain

$$\lambda_n(\Omega) = \frac{\langle \mathcal{K}_n^{\Omega_0} f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}} + (\Omega - \Omega_0) \frac{\langle \mathscr{R}_n^{\Omega_0,\Omega} f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}.$$

As $\mathcal{K}_n^{\Omega_0}$ is self-adjoint on the Hilbert space $L^2_{\mu_{\Omega_0}}$ then

$$\frac{\langle \mathcal{K}_n^{\Omega_0} f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}} = \frac{\langle f_n^\Omega, \mathcal{K}_n^{\Omega_0} f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^\Omega, f_n^{\Omega_0}\rangle_{\Omega_0}} = \lambda_n(\Omega_0).$$

Therefore we deuce that $\Omega \mapsto \lambda_n(\Omega)$ is differentiable at Ω_0 and

$$\lambda_n'(\Omega_0) = \frac{\langle \mathscr{R}_n^{\Omega_0,\Omega_0} f_n^{\Omega_0}, f_n^{\Omega_0}\rangle_{\Omega_0}}{\langle f_n^{\Omega_0}, f_n^{\Omega_0}\rangle_{\Omega_0}} = \int_0^\pi \frac{H_n(\phi,\varphi)}{\nu_{\Omega_0}(\phi)} f_n^{\Omega_0}(\varphi) f_n^{\Omega_0}(\phi) \sin(\phi) r_0^2(\phi) d\phi d\varphi.$$

This formula is nothing but the Feynman-Hellman formula. Since

$$\forall \varphi, \phi \in (0,\pi), \quad H_n(\phi,\varphi) > 0,\ f_n^{\Omega_0}(\phi) > 0,\ \nu_{\Omega_0}(\phi) > 0,$$

we find that $\lambda_n'(\Omega_0) > 0$, which achieves the proof of the suitable result. □

Next we shall establish the following result.

Proposition 4.4.4. *Let $n \geq 1$ and r_0 satisfies the assumptions* **(H1)** *and* **(H2)**. *Set*

$$\mathscr{S}_n := \Big\{\Omega \in (-\infty, \kappa) \quad \text{s.t.} \quad \lambda_n(\Omega) = 1\Big\}. \tag{4.4.68}$$

Then the following holds true

1. *The set $\mathscr{S}_n$ is formed by a single point denoted by Ω_n.*
2. *The sequence $(\Omega_n)_{n\geq 1}$ is strictly increasing and satisfies*

$$\lim_{n\to+\infty} \Omega_n = \kappa.$$

Proof. **(1)** To check that the set $\mathscr{S}_n$ is non empty we shall use the intermediate value theorem. From the upper bound in Proposition 4.4.3–(2) and (4.4.2)we find that

$$0 \leq \lambda_n(\Omega) \leq \int_0^\pi \int_0^\pi \frac{H_n^2(\phi,\varphi)\sin\phi\, r_0^2(\phi)}{\nu_\Omega(\phi)\nu_\Omega(\varphi)\, \sin(\varphi)\, r_0^2(\varphi)} d\mu_\Omega(\varphi)\, d\mu_\Omega(\phi).$$

Thus by taking the limit as $\Omega \to -\infty$ we deduce that

$$\lim_{\Omega\to-\infty} \lambda_n(\Omega) = 0. \tag{4.4.69}$$

Next, we intend to show that

$$\lim_{\Omega\to\kappa} \lambda_n(\Omega) = +\infty. \tag{4.4.70}$$

Using the lower bound of $\lambda_n(\Omega)$ in Proposition 4.4.3–(2), we find by virtue of Fatou Lemma

$$\int_0^\pi \int_0^\pi \frac{H_n(\phi,\varphi)}{\nu_\kappa^{\frac12}(\varphi)\nu_\kappa^{\frac12}(\phi)} \frac{\sin^{\frac12}(\phi) r_0(\phi)}{\sin^{\frac12}(\varphi) r_0(\varphi)} \varrho(\phi)\varrho(\varphi) d\varphi d\phi \leq \liminf_{\Omega\to\kappa} \lambda_n(\Omega),$$

for any ϱ satisfying $\displaystyle\int_0^\pi \varrho^2(\phi)d\phi = 1$. According to Lemma 4.4.1–(4), the function ν_κ reaches its minimum at a point $\phi_0 \in [0,\pi]$ and

$$\forall\, \phi \in [0,\pi], \quad 0 \leq \nu_\kappa(\phi) \leq C|\phi-\phi_0|^{1+\alpha}.$$

There are two possibilities: $\phi_0 \in (0,\pi)$ or $\phi_0 \in \{0,\pi\}$. Let us start with the first case and we shall take ϱ as follows

$$\varrho(\phi) = \frac{c_\beta}{|\phi-\phi_0|^\beta},$$

with $\beta < \frac12$ and the constant c_β is chosen such that ϱ is normalized. Hence using the preceding estimates we get

$$C\int_0^\pi \int_0^\pi \frac{H_n(\phi,\varphi)}{|\phi-\phi_0|^{\frac{1+\alpha}{2}+\beta}|\varphi-\phi_0|^{\frac{1+\alpha}{2}+\beta}} \frac{\sin^{\frac12}(\phi) r_0(\phi)}{\sin^{\frac12}(\varphi) r_0(\varphi)} d\varphi d\phi \leq \liminf_{\Omega\to\kappa} \lambda_n(\Omega). \tag{4.4.71}$$

Let $\varepsilon > 0$ such that $[\phi_0-\varepsilon, \phi_0+\varepsilon] \subset (0,\pi)$. According to (4.3.3) the function H_n is strictly positive in the domain $(0,\pi)^2$, hence there exists $\delta > 0$ such

$$\forall\, (\phi,\varphi) \in [\phi_0-\varepsilon, \phi_0+\varepsilon]^2, \quad \frac{H_n(\phi,\varphi)\sin^{\frac12}(\phi) r_0(\phi)}{\sin^{\frac12}(\varphi) r_0(\varphi)} \geq \delta.$$

Thus we obtain

$$C\int_{\phi_0-\varepsilon}^{\phi_0+\varepsilon} \int_{\phi_0-\varepsilon}^{\phi_0+\varepsilon} \frac{d\phi\, d\varphi}{|\phi-\phi_0|^{\frac{1+\alpha}{2}+\beta}|\varphi-\phi_0|^{\frac{1+\alpha}{2}+\beta}} \leq \liminf_{\Omega\to\kappa} \lambda_n(\Omega).$$

By taking $\frac{1+\alpha}{2} + \beta > 1$, which is an admissible configuration, we find

$$\lim_{\Omega\to\kappa} \lambda_n(\Omega) = +\infty.$$

Now let us move to the second possibility where $\phi_0 \in \{0, \pi\}$ and without any loss of generality we can only deal with the case $\phi_0 = 0$. From (4.3.3) and using the inequality

$$\forall\, x \in [0, 1), \quad F_n(x) \geq 1,$$

we obtain

$$\forall \phi, \varphi \in (0, \pi), \quad H_n(\phi, \varphi) \geq c_n \frac{\sin(\varphi) r_0^{n-1}(\phi) r_0^{n+1}(\varphi)}{[R(\phi, \varphi)]^{n+\frac{1}{2}}}.$$

Combined with the assumption (**H2**), it implies

$$\forall \phi, \varphi \in (0, \pi), \quad H_n(\phi, \varphi) \geq c_n \frac{\sin^{n+2}(\varphi) \sin^{n-1}(\phi)}{[R(\phi, \varphi)]^{n+\frac{1}{2}}}.$$

Plugging this into (4.4.71) we find

$$C_n \int_0^\pi \int_0^\pi \frac{1}{\phi^{\frac{1+\alpha}{2}+\beta} \varphi^{\frac{1+\alpha}{2}+\beta}} \frac{\sin^{n+\frac{1}{2}}(\varphi) \sin^{n+\frac{1}{2}}(\phi)}{[R(\phi, \varphi)]^{n+\frac{1}{2}}} d\varphi d\phi \leq \liminf_{\Omega \to \kappa} \lambda_n(\Omega).$$

Let $\varepsilon > 0$ sufficiently small, then using Taylor expansion we get according to (4.3.2)

$$0 \leq \phi, \varphi \leq \varepsilon \Longrightarrow R(\phi, \varphi) \leq C(\phi + \varphi)^2.$$

Thus

$$C_n \int_0^\varepsilon \int_0^\varepsilon \frac{1}{\phi^{\frac{1+\alpha}{2}+\beta} \varphi^{\frac{1+\alpha}{2}+\beta}} \frac{\varphi^{n+\frac{1}{2}} \phi^{n+\frac{1}{2}}}{(\phi + \varphi)^{2n+1}} d\varphi d\phi \leq \liminf_{\Omega \to \kappa} \lambda_n(\Omega).$$

which gives after simplification

$$C_n \int_0^\varepsilon \int_0^\varepsilon \frac{\varphi^{n-\frac{\alpha}{2}-\beta} \phi^{n-\frac{\alpha}{2}-\beta}}{(\phi + \varphi)^{2n+1}} d\varphi d\phi \leq \liminf_{\Omega \to \kappa} \lambda_n(\Omega).$$

Making the change of variables $\varphi = \phi\theta$ we obtain

$$\int_0^\varepsilon \int_0^\varepsilon \frac{\varphi^{n-\frac{\alpha}{2}-\beta} \phi^{n-\frac{\alpha}{2}-\beta}}{(\phi + \varphi)^{2n+1}} d\varphi d\phi = \int_0^\varepsilon \phi^{-\alpha-2\beta} \int_0^{\frac{\varepsilon}{\phi}} \frac{\theta^{n-\frac{\alpha}{2}-\beta}}{(1 + \theta)^{2n+1}} d\theta d\phi.$$

This integral diverges provided that $\alpha + 2\beta > 1$ and thus under this assumption

$$\lim_{\Omega \to \kappa} \lambda_n(\Omega) = +\infty.$$

Hence we obtain (4.4.70). By the intermediate mean value, we achieve the existence of at least one solution for the equation

$$\lambda_n(\Omega) = 1.$$

Consequently, using Proposition 4.4.3 we deduce by the intermediate value theorem that the set $\mathscr{S}_n$ contains only one element.

(2) Since Ω_n satisfies the equation

$$\lambda_n(\Omega_n) = 1.$$

According to Proposition 4.4.3–(5) the sequence $k \mapsto \lambda_k(\Omega_n)$ is strictly decreasing. It implies in particular that

$$\lambda_{n+1}(\Omega_n) < \lambda_n(\Omega_n) = 1.$$

Hence by (4.4.70) one may apply the intermediate value theorem and find an element of the set $\mathscr{S}_{n+1}$ in the interval (Ω_n, κ). This means that $\Omega_{n+1} > \Omega_n$ and thus this sequence is strictly increasing. It remains to prove that this sequence is converging to κ. The convergence of this sequence to some element $\overline{\Omega} \leq \kappa$ is clear. To prove that $\overline{\Omega} = \kappa$ we shall argue by contradiction by assuming that $\overline{\Omega} < \kappa$. By the construction of Ω_n one has necessary

$$\forall n \geq 1, \quad \lambda_n(\overline{\Omega}) > 1.$$

Using the upper-bound estimate stated in Proposition 4.4.3–(2) combined with the point (3) we obtain for any $\alpha \in (0, 1)$

$$\forall\, n \geq 1, \quad 0 < \lambda_n(\overline{\Omega}) \lesssim (\kappa - \overline{\Omega})^{-2} n^{-\alpha}. \tag{4.4.72}$$

By taking the limit as $n \to +\infty$ we find

$$\lim_{n \to +\infty} \lambda_n(\overline{\Omega}) = 0.$$

This contradicts (4.4.72) which achieves the proof. □

4.4.4 Eigenfunctions regularity

This section is devoted to the strong regularity of the eigenfunctions associated to the operator $\mathcal{K}_n^{\Omega}$ and constructed in Proposition 4.4.3. We have already seen that these eigenfunctions belongs to a weak function space $L^2_{\mu_\Omega}$. Here we shall show first their continuity and later their Hölder regularity.

Continuity

The main result of this section reads as follows.

Proposition 4.4.5. *Let $\Omega \in (-\infty, \kappa)$, $n \geq 1$, r_0 satisfies the assumptions* **(H1)** *and* **(H2)***, and f be an eigenfunction for $\mathcal{K}_n^{\Omega}$ associated to a non-vanishing eigenvalue. Then f is continuous over $[0, \pi]$, and for $n \geq 2$ it satisfies the boundary condition $f(0) = f(\pi) = 0$. However this boundary condition fails for $n = 1$ at least with the eigenfunctions associated to the largest eigenvalue $\lambda_1(\Omega)$.*

Proof. Let $f \in L^2_{\mu_\Omega}$ be any non trivial eigenfunction of the operator $\mathcal{K}_n^{\Omega}$ defined in (4.4.64) and associated to an eigenvalue $\lambda \neq 0$, then

$$f(\phi) = \frac{1}{\lambda\, \nu_\Omega(\phi)} \int_0^{\pi} H_n(\phi, \varphi) f(\varphi) d\varphi, \quad \forall \phi \in (0, \pi)\, \text{a.e.} \tag{4.4.73}$$

Since $f \in L^2_{\mu_\Omega}$, then the function $g : \varphi \in [0, \pi] \mapsto r_0^{\frac{3}{2}}(\varphi) f(\varphi)$belongs to $L^2((0, \pi); d\varphi)$. Therefore the equation (4.4.73) can be written in terms of g as follows

$$g(\phi) = \frac{1}{\lambda\, \nu_\Omega(\phi)} \int_0^{\pi} r_0^{-\frac{3}{2}}(\varphi) r_0^{\frac{3}{2}}(\phi) H_n(\phi, \varphi) g(\varphi) d\varphi, \quad \forall \phi \in (0, \pi)\, \text{a.e.}$$

Coming back to the definition of H_n in (4.3.3) we obtain for some constant c_n the formula

$$r_0^{-\frac{3}{2}}(\varphi)r_0^{\frac{3}{2}}(\phi)H_n(\phi,\varphi) = c_n\frac{\sin(\varphi)r_0^{n-\frac{1}{2}}(\varphi)r_0^{n+\frac{1}{2}}(\phi)}{[R(\phi,\varphi)]^{n+\frac{1}{2}}}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right).$$

Using (4.4.9) and the assumption **(H2)** yields

$$\begin{aligned} r_0^{-\frac{3}{2}}(\varphi)r_0^{\frac{3}{2}}(\phi)H_n(\phi,\varphi) \lesssim & \frac{r_0^{n+\frac{1}{2}}(\varphi)r_0^{n+\frac{1}{2}}(\phi)}{R^{n+\frac{1}{2}}(\phi,\varphi)}\left(1+\ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right)\right) \\ \lesssim & 1+\ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right). \end{aligned} \tag{4.4.74}$$

This implies, using Cauchy-Schwarz inequality and the fact that ν_Ω is bounded away from zero

$$\begin{aligned} |g(\phi)| \lesssim & \int_0^\pi \left(1+\ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right)\right)g(\varphi)d\varphi \\ \lesssim & \|g\|_{L^2(d\varphi)}\left(1+\int_0^\pi \ln^2\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right)d\varphi\right)^{\frac{1}{2}} \\ \lesssim & \|f\|_{\mu_\Omega}, \quad \forall\phi\in(0,\pi)\ \text{a.e.} \end{aligned}$$

It follows that g is bounded. Now inserting this estimate into (4.4.73) allows to get

$$\begin{aligned} |f(\phi)| \lesssim & \|g\|_{L^\infty}\int_0^\pi r_0^{-\frac{3}{2}}(\varphi)H_n(\phi,\varphi)d\varphi, \\ \lesssim & \|f\|_{\mu_\Omega}\int_0^\pi \frac{\sin(\varphi)r_0^{n-\frac{1}{2}}(\varphi)r_0^{n-1}(\phi)}{[R(\phi,\varphi)]^{n+\frac{1}{2}}}F_n\left(\frac{4r_0(\phi)r_0(\varphi)}{R(\phi,\varphi)}\right)d\varphi. \end{aligned} \tag{4.4.75}$$

Using once again (4.4.9) and the assumption **(H2)** we deduce that

$$|f(\phi)| \lesssim \|f\|_{\mu_\Omega}\int_0^\pi \frac{\sin^{n-1}(\phi)\sin^{n+\frac{1}{2}}(\varphi)}{\big((\sin(\phi)+\sin(\varphi))^2+(\cos\phi-\cos\varphi)^2\big)^{n+\frac{1}{2}}}\Big(1+\ln\Big(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\Big)\Big)d\varphi.$$

By symmetry we may restrict the analysis to $\phi\in[0,\frac{\pi}{2}]$. Thus, splitting the integral given in (4.4.75) and using that

$$\inf_{\substack{\varphi\in[\pi/2,\pi]\\ \phi\in[0,\pi/2]}} R(\phi,\varphi)>0,$$

we obtain

$$\begin{aligned} |f(\phi)| \lesssim & \|f\|_{\mu_\Omega}\int_0^{\frac{\pi}{2}} \frac{\sin^{n-1}(\phi)\sin^{n+\frac{1}{2}}(\varphi)}{\big(\sin(\phi)+\sin(\varphi)\big)^{2n+1}}\Big(1+\ln\Big(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\Big)\Big)d\varphi \\ & +\|f\|_{\mu_\Omega}\int_{\frac{\pi}{2}}^{\pi}\Big(1+\ln\Big(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\Big)\Big)d\varphi. \end{aligned}$$

It follows that

$$|f(\phi)| \lesssim \|f\|_{\mu\Omega} \int_0^{\frac{\pi}{2}} \frac{\phi^{n-1}\varphi^{n+\frac{1}{2}}}{(\phi+\varphi)^{2n+1}} \Big(1+\ln\Big(\frac{\phi+\varphi}{|\phi-\varphi|}\Big)\Big) d\varphi + \|f\|_{\mu\Omega}.$$

Using the change of variables $\varphi = \phi\theta$ we get

$$\begin{aligned}|f(\phi)| \lesssim& \|f\|_{\mu\Omega}\phi^{-\frac{1}{2}} \int_0^{\frac{\pi}{2\phi}} \frac{\theta^{n-1}}{(1+\theta)^{2n+1}} \Big(1+\ln\Big(\frac{1+\theta}{|1-\theta|}\Big)\Big) d\theta + \|f\|_{\mu\Omega}\\ \lesssim& \|f\|_{\mu\Omega}\phi^{-\frac{1}{2}}.\end{aligned}$$

Consequently we find

$$\sup_{\phi\in(0,\pi)} r_0^{\frac{1}{2}}(\phi)|f(\phi)| \lesssim \|f\|_{\mu\Omega}.$$

Inserting this estimate into (4.4.73) and using (4.4.74) yields

$$\begin{aligned}|f(\phi)| \lesssim& \|f\|_{\mu\Omega} \int_0^{\pi} r_0^{-\frac{1}{2}}(\varphi) H_n(\phi,\varphi) d\varphi\\ \lesssim& \|f\|_{\mu\Omega} \int_0^{\pi} \frac{r_0^{n+\frac{3}{2}}(\varphi) r_0^{n-1}(\phi)}{R^{n+\frac{1}{2}}(\phi,\varphi)} \left(1+\ln\left(\frac{\sin(\phi)+\sin(\varphi)}{|\phi-\varphi|}\right)\right) d\varphi.\end{aligned}$$

As before we can restrict $\phi \in [0, \frac{\pi}{2}]$ and by using the fact

$$\inf_{\substack{\varphi\in[\pi/2,\pi]\\ \phi\in[0,\pi/2]}} R(\phi,\varphi) > 0,$$

we deduce after splitting the integral

$$\begin{aligned}|f(\phi)| \lesssim& \|f\|_{\mu\Omega} \int_0^{\pi} r_0^{-\frac{1}{2}}(\varphi) H_n(\phi,\varphi) d\varphi\\ \lesssim& \|f\|_{\mu\Omega} r_0^{n-1}(\phi) + \|f\|_{\mu\Omega} \int_0^{\frac{\pi}{2}} \frac{\varphi^{n+\frac{3}{2}}\phi^{n-1}}{(\phi+\varphi)^{2n+1}} \left(1+\ln\left(\frac{\phi+\varphi}{|\phi-\varphi|}\right)\right) d\varphi.\end{aligned}$$

Making the change of variables $\varphi = \phi\theta$ leads to

$$\begin{aligned}\forall\phi\in[0,\pi/2], \quad |f(\phi)| \lesssim& |f\|_{\mu\Omega} r_0^{n-1}(\phi) + \|f\|_{\mu\Omega}\phi^{\frac{1}{2}} \int_0^{\frac{\pi}{2\phi}} \frac{\theta^{n+\frac{3}{2}}}{(1+\theta)^{2n+1}} \left(1+\ln\left(\frac{\theta+1}{|\theta-1|}\right)\right) d\theta\\ \lesssim& |f\|_{\mu\Omega}(\phi^{n-1}+\phi^{\frac{1}{2}}).\end{aligned}$$

Consequently we get

$$\forall\phi\in(0,\pi), \quad |f(\phi)| \lesssim |f\|_{\mu\Omega}(r_0^{n-1}(\phi) + r_0^{\frac{1}{2}}(\phi)).$$

This shows that f is bounded over $(0,\pi)$ and by Lebesgue theorem one can show that f is in fact continuous on $[0,\pi]$ and satisfies for $n\geq 2$ the boundary condition

$$f(0)=f(\pi)=0.$$

Last, we shall check that this boundary condition fails for $n=1$ with the largest eigenvalue $\lambda_1(\Omega)$. Indeed, according to (4.4.73) we have

$$f(0)=\frac{1}{\lambda\nu_\Omega(0)}\int_0^\pi H_1(0,\varphi)f(\varphi)d\varphi.$$

However, from (4.3.3) we get

$$\forall\,\varphi\in(0,\pi),\quad H_1(0,\varphi)=c_1\frac{r_0^3(\varphi)}{R^{\frac{3}{2}}(0,\varphi)}>0.$$

Combining this with the fact that f does not change the sign allows to get that $f(0)\neq 0$. □

Hölder continuity

The main goal of this section is to prove the Hölder regularity of the eigenfunctions.

Proposition 4.4.6. *Assume that r_0 satisfies the conditions* **(H)** *and let $\Omega\in(-\infty,\kappa)$, then any solution h of the equation*

$$h(\phi)=\frac{1}{\lambda\,\nu_\Omega(\phi)}\int_0^\pi H_n(\varphi,\phi)h(\varphi)d\varphi,\quad\forall\phi\in(0,\pi),\tag{4.4.76}$$

with $\lambda\neq 0$, belongs to $\mathscr{C}^{1,\alpha}(0,\pi)$, for any $n\geq 2$. The functions involved in the above expression can be found in (4.3.3)–(4.4.1)–(4.4.2).

Proof. From the initial expression of the linearized operator (4.3.1) in Proposition 4.3.1 and combining it with Proposition 4.3.3, one has

$$\mathcal{F}_n(h)(\phi):=\int_0^\pi H_n(\varphi,\phi)h(\varphi)d\varphi=\frac{1}{4\pi}\frac{1}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\mathcal{H}_n(\phi,\varphi,\eta)h(\varphi)d\eta d\varphi,\tag{4.4.77}$$

with

$$\widehat{R}(\phi,\varphi,\eta):=(r_0(\phi)-r_0(\varphi))^2+2r_0(\phi)r_0(\varphi)(1-\cos\eta)+(\cos\phi-\cos\varphi)^2,$$
$$\mathcal{H}_n(\phi,\varphi,\eta):=\frac{\sin(\varphi)r_0(\varphi)\cos(n\eta)}{\widehat{R}^{\frac{1}{2}}(\phi,\varphi,\eta)}.$$

It is clear that any solution h of (4.4.76) is equivalent to a solution of

$$\forall\,\phi\in(0,\pi),\quad h(\phi)=\frac{\mathcal{F}_n(h)(\phi)}{\lambda\,\nu_\Omega(\phi)}.$$

From Proposition 4.4.1 we know that $\nu_\Omega\in\mathscr{C}^{1,\alpha}(0,\pi)$ and does not vanish when $\Omega\in(-\infty,\kappa)$. Therefore to check the regularity $h\in\mathscr{C}^{1,\alpha}(0,\pi)$ it is enough from the classical law products to establish that $\mathcal{F}_n(h)\in\mathscr{C}^{1,\alpha}(0,\pi)$. Since h is symmetric with respect to $\phi=\frac{\pi}{2}$, then one can verify that $\mathcal{F}_n(h)$ preserves this symmetry and hence we shall only check the regularity in the interval $[0,\frac{\pi}{2}]$. Notice that Proposition 4.4.5 tells us that h is continuous in $[0,\pi]$, for any $n\geq 1$.

In order to prove such regularity, let us first check that

$$\widehat{R}(\phi,\varphi,\eta)\geq C\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\},\quad \forall\,\phi,\varphi\in(0,\pi),\forall\eta\in(0,2\pi),\tag{4.4.78}$$

which is the key point in this proof. In order to do so, recall first from (4.2.18) that

$$\widehat{R}(\phi,\varphi,\eta)\geq C(\phi-\varphi)^2.\tag{4.4.79}$$

On the other hand, define the function

$$g_1(x)=x^2+r_0(\varphi)^2-2xr_0(\varphi)\cos(\eta)+(\cos(\varphi)-\cos(\phi))^2,$$

which obviously verifies $g_1(r_0(\phi))=\widehat{R}(\phi,\varphi,\eta)$. Such function has a minimum located at

$$x_c=r_0(\varphi)\cos(\eta).$$

Now we shall distinguish two cases: $\cos\eta\in[0,1]$ and $\cos\eta\in[-1,0]$. In the first case we get

$$\begin{aligned}g_1(x)\geq&g_1(x_c)=r_0^2(\varphi)\sin^2(\eta)+(\cos(\varphi)-\cos(\phi))^2\\ \geq&r_0^2(\varphi)\sin^2(\eta).\end{aligned}$$

From elementary trigonometric relations we deduce that

$$\sin^2\eta=2\sin^2(\eta/2)\big(1+\cos(\eta)\big)\geq 2\sin^2(\eta/2).$$

This implies in particular that, for $\cos\eta\in[0,1]$

$$\widehat{R}(\phi,\varphi,\eta)\geq 2r_0^2(\varphi)\sin^2(\eta/2).$$

As to the second case $\cos\eta\in[-1,0]$, we simply notice that the critical point x_c is negative and therefore the second degree polynomial g_1 is strictly increasing in $\mathbb{R}_+$. This implies that

$$\widehat{R}(\phi,\varphi,\eta)=g_1(r_0(\phi))\geq g_1(0)\geq r_0^2(\varphi)\geq r_0^2(\varphi)\sin^2(\eta/2).$$

Therefore we get in both cases

$$\widehat{R}(\phi,\varphi,\eta)\geq r_0^2(\varphi)\sin^2(\eta/2).\tag{4.4.80}$$

By the symmetry property $\widehat{R}(\phi,\varphi,\eta)=\widehat{R}(\varphi,\phi,\eta)$ we also get

$$\widehat{R}(\phi,\varphi,\eta)\geq r_0^2(\phi)\sin^2(\eta/2).\tag{4.4.81}$$

Adding together (4.4.79)–(4.4.80)–(4.4.81), we achieve

$$3\widehat{R}(\phi,\varphi,\eta)\geq C(\phi-\varphi)^2+(r_0^2(\phi)+r_0^2(\varphi))\sin^2(\eta/2).\tag{4.4.82}$$

It suffices now to combine this inequality with the assumption **(H2)** on r_0 in order to get the announced estimate (4.4.78).

Let us now prove that $\mathcal{F}_n(h)\in\mathscr{C}^{1,\alpha}$ and for this aim we shall proceed into four steps.

- **Step 1:** If $h\in L^\infty$ then $\mathcal{F}_n(h)\in\mathscr{C}^\alpha(0,\pi)$.

Here we check that $\mathcal{F}_n(h) \in \mathscr{C}^\alpha(0,\pi)$ for any $n \geq 1$. In order to avoid the singularity in the denominator coming from r_0, we integrate by parts in the variable η

$$\mathcal{F}_n(h)(\phi) = -\frac{1}{4\pi n}\int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi) r_0^2(\varphi)\sin(n\eta)\sin(\eta)h(\varphi)}{\widehat{R}^{\frac{3}{2}}(\phi,\varphi,\eta)} d\eta d\varphi.$$

Introduce

$$K_1(\phi,\varphi,\eta) := \frac{\sin(\varphi) r_0^2(\varphi)\sin(n\eta)\sin(\eta)h(\varphi)}{\widehat{R}^{\frac{3}{2}}(\phi,\varphi,\eta)},$$

and according to Chebyshev polynomials we know that

$$\sin(n\eta) = \sin(\eta)\, U_{n-1}(\cos\eta), \tag{4.4.83}$$

with U_n being a polynomial of degree n. Thus

$$K_1(\phi,\varphi,\eta) = \frac{\sin(\varphi) r_0^2(\varphi) U_{n-1}(\cos\eta)\sin^2(\eta)h(\varphi)}{\widehat{R}^{\frac{3}{2}}(\phi,\varphi,\eta)}.$$

Using the assumption **(H2)** combined with the estimate (4.4.78) for the denominator $\widehat{R}(\phi,\varphi,\eta)$, we achieve

$$\begin{aligned}
|K_1(\phi,\varphi,\eta)| \lesssim & \frac{\|h\|_{L^\infty}\sin^3(\varphi)\sin^2(\eta/2)}{\big((\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac{3}{2}}} \\
\lesssim & \frac{\sin(\varphi)}{\big((\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac{1}{2}}}.
\end{aligned}$$

Interpolating between the two inequalities

$$\frac{\sin(\varphi)}{\big((\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac{1}{2}}} \leq |\varphi-\phi|^{-1}$$

and

$$\frac{\sin(\varphi)}{\big((\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac{1}{2}}} \leq \sin^{-1}(\eta/2),$$

we deduce that for any $\beta \in [0,1]$

$$\frac{\sin(\varphi)}{\big((\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac{1}{2}}} \lesssim |\varphi-\phi|^{-(1-\beta)}\sin^{-\beta}(\eta/2). \tag{4.4.84}$$

Then,

$$|K_1(\phi,\varphi,\eta)| \lesssim \frac{1}{|\phi-\varphi|^{1-\beta}\sin^\beta(\eta/2)}.$$

Let us now bound the derivative $\partial_\phi K_1(\phi,\varphi,\eta)$. For this purpose, let us first show that

$$|\partial_\phi \widehat{R}(\phi,\varphi,\eta)| \lesssim \widehat{R}^{\frac{1}{2}}(\phi,\varphi,\eta). \tag{4.4.85}$$

Indeed

$$\frac{\partial_\phi \widehat{R}(\phi,\varphi,\eta)}{\widehat{R}^{\frac12}(\phi,\varphi,\eta)}=\frac{2r_0'(\phi)(r_0(\phi)-r_0(\varphi))+2r_0'(\phi)r_0(\varphi)(1-\cos(\eta))+2\sin(\phi)(\cos(\varphi)-\cos(\phi))}{\widehat{R}^{\frac12}(\phi,\varphi,\eta)}.$$

Using the identity $1-\cos(\eta)=2\sin^2(\eta/2)$ and (4.4.78), we get a constant C such that

$$\frac{\big|\partial_\phi \widehat{R}(\phi,\varphi,\eta)\big|}{\widehat{R}^{\frac12}(\phi,\varphi,\eta)}\lesssim\frac{|\phi-\varphi|+\sin(\varphi)\sin^2(\eta/2)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac12}}\leq C,$$

achieving (4.4.85). Therefore, taking the derivative in ϕ of K_1 yields

$$\begin{aligned}|\partial_\phi K_1(\phi,\varphi,\eta)|\leq& C\|h\|_{L^\infty}\frac{\sin^3(\varphi)\sin^2(\eta/2)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^2}\\ \lesssim&\frac{\sin(\varphi)}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)}\\ \lesssim&\frac{|\phi-\varphi|^{-1}\sin(\varphi)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac12}}.\end{aligned}$$

Hence (4.4.84) allows to get or any $\beta\in(0,1)$,

$$|\partial_\phi K_1(\phi,\varphi,\eta)|\lesssim\frac{1}{|\phi-\varphi|^{2-\beta}\sin^\beta(\eta/2)}.$$

By adapting Proposition B.0.2 to the case where the operator $\mathcal{K}$ depends only on one variable, we infer $\mathcal{F}_n(h)\in\mathscr{C}^\beta(0,\pi)$ for any $\beta\in(0,1)$.

• **Step 2:** For $n\geq 2$, if h is bounded then $\mathcal{F}_n(h)(0)=0$.

Notice that this property was shown in Proposition 4.4.5 and we give here an alternative proof. Since ν_Ω is not vanishing then this amounts to checking that $\mathcal{F}_n(h)(0)=0$. By continuity, it is clear by Fubini that

$$\begin{aligned}\mathcal{F}_n(h)(0)=&-\frac{1}{4\pi n}\int_0^\pi\int_0^{2\pi}\frac{\big(\sin(\varphi)r_0^2(\varphi)\sin(n\eta)\sin(\eta)h(\varphi)}{\big(r_0^2(\varphi)+(1-\cos\varphi)^2\big)^{\frac32}}d\eta d\varphi\\ =&-\frac{1}{4\pi n}\int_0^\pi\frac{\sin(\varphi)r_0^2(\varphi)h(\varphi)d\varphi}{\big(r_0^2(\varphi)+(1-\cos\varphi)^2\big)^{\frac32}}\int_0^{2\pi}\sin(n\eta)\sin(\eta)d\eta,\end{aligned}$$

which is vanishing if $n\geq 2$. Hence, $h(0)=0$, for any $n\geq 2$.

• **Step 3:** If $h\in\mathscr{C}^\alpha(0,\pi)$ and $h(0)=0$, then $\mathcal{F}_n(h)\in W^{1,\infty}(0,\pi)$.

Since we have shown before that $\mathcal{F}_n(h)\in\mathscr{C}^\beta(0,\pi)$ for any $\beta\in(0,1)$. Then it is enough to check that $\mathcal{F}_n(h)'\in L^\infty(0,\pi)$. For this aim, we write

$$\mathcal{F}_n(h)'(\phi)=\frac38\frac{1}{\pi n}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)^2\sin(n\eta)\sin(\eta)h(\varphi)\partial_\phi\widehat{R}(\phi,\varphi,\eta)}{\widehat{R}^{\frac52}(\phi,\varphi,\eta)}d\eta d\varphi.$$

Adding and subtracting some appropriated terms, we find

$$\mathcal{F}_n(h)'(\phi)=\frac{3}{8\pi n}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(n\eta)\sin(\eta)\big(h(\varphi)-h(\phi)\big)\partial_\phi\widehat{R}(\phi,\varphi,\eta)}{\widehat{R}^{\frac52}(\phi,\varphi,\eta)}d\eta d\varphi$$

$$\begin{aligned}
&+\frac{3h(\phi)}{8\pi n}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(n\eta)\sin(\eta)\big[\partial_\phi\widehat{R}(\phi,\varphi,\eta)+\partial_\varphi\widehat{R}(\phi,\varphi,\eta)\big]}{\widehat{R}^{\frac52}(\phi,\varphi,\eta)}d\eta d\varphi\\
&-\frac{3h(\phi)}{8\pi n}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(n\eta)\sin(\eta)\partial_\varphi\widehat{R}(\phi,\varphi,\eta)}{\widehat{R}^{\frac52}(\phi,\varphi,\eta)}d\eta d\varphi\\
:=&\frac{3}{8\pi n}(I_1+I_2-I_3)(\phi). \qquad (4.4.86)
\end{aligned}$$

Let us bound each term separately. Using (4.4.78), (4.4.83) and (4.4.85) we achieve

$$\begin{aligned}
|I_1(\phi)|\leq&C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{\sin^3(\varphi)\sin^2(\eta)|\varphi-\phi|^\alpha d\eta d\varphi}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^2}\\
\leq&C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)|\varphi-\phi|^\alpha d\eta d\varphi}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)}.
\end{aligned}$$

We write in view of (4.4.84)

$$\begin{aligned}
\frac{\sin(\varphi)}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)}\leq&\frac{|\phi-\varphi|^{-1}\sin(\varphi)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac12}}\\
\lesssim&\frac{1}{|\phi-\varphi|^{2-\beta}\sin^\beta(\eta/2)}.
\end{aligned}$$

Therefore by imposing $1-\alpha<\beta<1$ we get

$$|I_1(\phi)|\leq C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{d\eta d\varphi}{|\phi-\varphi|^{2-\alpha-\beta}\sin^\beta(\eta/2)}\leq C\|h\|_{\mathscr{C}^\alpha},$$

which implies immediately that $I_1\in L^\infty$. Now let us move to the boundedness of I_2. From direct computations we get

$$\begin{aligned}
\Big|\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\Big|=&\Big|2(r_0(\phi)-r_0(\varphi))(r_0'(\phi)-r_0'(\varphi))+2(\cos\phi-\cos\varphi)(\sin\varphi-\sin\phi)\\
&+2(1-\cos\eta)\big(r_0'(\phi)r_0(\varphi)+r_0'(\varphi)r_0(\phi)\big)\Big|. \qquad (4.4.87)
\end{aligned}$$

Combining the assumption **(H2)** with $r_0'\in W^{1,\infty}$ and the intermediate value theorem yields

$$\begin{aligned}
\Big|\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\Big|\leq&C\Big(|\phi-\varphi|^2+(\sin\varphi+\sin\phi)\sin^2(\eta/2)\Big)\\
\leq&C\Big(|\phi-\varphi|^2+(\sin\varphi+\sin\phi)\sin^2(\eta/2)\Big). \qquad (4.4.88)
\end{aligned}$$

Hence, using (4.4.78) and (4.4.83) we obtain

$$\begin{aligned}
|I_2(\phi)|\leq&C|h(\phi)-h(0)|\int_0^\pi\int_0^{2\pi}\frac{\sin^3(\varphi)\sin^2(\eta)\big(|\phi-\varphi|^2+(\sin\varphi+\sin\phi)\sin^2(\eta/2)\big)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac52}}d\eta d\varphi\\
\leq&C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{\phi^\alpha d\eta d\varphi}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big)^{\frac12}}.
\end{aligned}$$

Interpolation inequalities imply

$$\frac{1}{\big((\phi-\varphi)^2+(\sin^2(\varphi)+\sin^2(\phi))\sin^2(\eta/2)\big)^{\frac12}}\leq|\phi-\varphi|^{\alpha-1}\sin^{-\alpha}(\phi)\sin^{-\alpha}(\eta/2). \tag{4.4.89}$$

Therefore we get for any $\phi\in(0,\pi/2),\varphi\in(0,\pi)$ and $\eta\in(0,2\pi)$,

$$\frac{\phi^\alpha}{\big((\phi-\varphi)^2+(\sin(\phi)+\sin(\varphi))^2\sin^2(\eta/2)\big)^{\frac12}}\lesssim|\phi-\varphi|^{\alpha-1}\sin^{-\alpha}(\eta/2). \tag{4.4.90}$$

It follows that

$$|I_2(\phi)|\leq C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{d\eta d\varphi}{|\phi-\varphi|^{1-\alpha}\sin^\alpha(\eta/2)}\leq C||h||_{\mathscr{C}^\alpha},$$

which gives the boundedness of I_2. It remains to bound the last term I_3. Then integrating by parts we infer

$$I_3(\phi)=\frac23 h(\phi)\int_0^\pi\int_0^{2\pi}\frac{\partial_\varphi\big(\sin(\varphi)r_0^2(\varphi)\big)\sin(n\eta)\sin(\eta)}{\widehat{R}^{\frac32}(\phi,\varphi,\eta)}d\eta d\varphi.$$

Then, since $h(0)=0$ and $h\in\mathscr{C}^\alpha$ we find according to the assumptions **(H)**, (4.4.78) and (4.4.83)

$$\begin{aligned}|I_3(\phi)|\leq&C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{\phi^\alpha\sin^2(\varphi)\sin^2(\eta)d\eta d\varphi}{\big[(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big]^{\frac32}}\\ \leq&C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{\phi^\alpha d\eta d\varphi}{\big[(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big]^{\frac12}}.\end{aligned}$$

Applying (4.4.90) yields

$$|I_3(\phi)|\leq C\|h\|_{\mathscr{C}^\alpha}\int_0^\pi\int_0^{2\pi}\frac{d\eta d\varphi}{|\phi-\varphi|^{1-\alpha}\sin^\alpha(\eta/2)}\leq C\|h\|_{\mathscr{C}^\alpha}.$$

Finally, we get the announced result, that is, $h\in W^{1,\infty}(0,\pi)$.

• **Step 4:** If $h'\in L^\infty(0,\pi)$ and $h(0)=0$, then $\mathcal{F}_n(h)'\in\mathscr{C}^\beta(0,\pi)$ for any $\beta\in(0,1)$.
Coming back to (4.4.86) and integrating by parts in the last integral we deduce

$$\begin{aligned}\mathcal{F}_n(h)'(\phi)=&\frac{3}{8\pi n}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)h(\varphi)\sin(n\eta)\sin(\eta)\big[\partial_\phi\widehat{R}(\phi,\varphi,\eta)+\partial_\varphi\widehat{R}(\phi,\varphi,\eta)\big]}{\widehat{R}^{\frac52}(\phi,\varphi,\eta)}d\eta d\varphi\\ &-\frac{1}{4\pi n}\int_0^\pi\int_0^{2\pi}\frac{\partial_\varphi\big(\sin(\varphi)r_0^2(\varphi)h(\varphi)\big)\sin(n\eta)\sin(\eta)}{\widehat{R}^{\frac32}(\phi,\varphi,\eta)}d\eta d\varphi\\ :=&\frac{3}{8\pi n}\int_0^\pi\int_0^{2\pi}\big(T_1-\frac23T_2\big)(\phi,\varphi,\eta)d\eta d\varphi,\end{aligned} \tag{4.4.91}$$

with

$$T_2(\phi,\varphi,\eta)=\frac{\partial_\varphi\big(\sin(\varphi)r_0(\varphi)^2h(\varphi)\big)\sin(n\eta)\sin(\eta)}{\widehat{R}^{\frac{3}{2}}(\phi,\varphi,\eta)}.$$

We want to apply Proposition B.0.2 to each of those terms. First, for T_1 we use **(H)** and (4.4.88) combined with (4.4.78), we arrive at

$$\begin{aligned}|T_1(\phi,\varphi,\eta)|&\lesssim\frac{\sin^3(\varphi)|h(\varphi)|\sin^2(\eta)\big(|\phi-\varphi|^2+(\sin\phi+\sin\varphi)\sin^2(\eta)\big)}{\big\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big\}^{\frac{5}{2}}}\\&\lesssim\frac{|h(\varphi)|}{\big\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big\}^{\frac{1}{2}}}.\end{aligned}$$

Since $h'\in L^\infty$ and $h(0)=h(\pi)=0$ then we can write $h(\varphi)=\sin(\varphi)\widehat{h}(\varphi)$, with $\widehat{h}\in L^\infty(0,\pi)$.
Consequently,

$$\begin{aligned}|T_1(\phi,\varphi,\eta)|\lesssim&||\widehat{h}||_{L^\infty}\frac{\sin(\varphi)}{\big\{(\phi-\varphi)^2+\sin^2(\varphi)\sin^2(\eta/2)\big\}^{\frac{1}{2}}}\\\lesssim&\frac{1}{\big\{(\phi-\varphi)^2+\sin^2(\eta/2)\big\}^{\frac{1}{2}}}.\end{aligned}$$

Interpolating again, we find that for any $\beta\in[0,1]$

$$|T_1(\phi,\varphi,\eta)|\lesssim\frac{1}{|\phi-\varphi|^{1-\beta}\sin^\beta(\eta/2)}.$$

Let us mention that we have proven that

$$\frac{|h(\varphi)|}{\big\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\big\}^{\frac{1}{2}}}\leq C\frac{1}{|\phi-\varphi|^{1-\beta}\sin^\beta(\eta/2)},\tag{4.4.92}$$

for any $\beta\in(0,1)$, $\phi\in(0,\pi/2),\varphi\in(0,\pi)$ and $\eta\in(0,2\pi)$, which will be useful later.
Now we shall estimate the derivative of T_1 with respect to ϕ. We start with

$$\begin{aligned}\Big|\partial_\phi\Big\{\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\Big\}\Big|=&\Big|2r_0'(\phi)(r_0'(\phi)-r_0'(\varphi))+2(r_0(\phi)-r_0(\varphi))r_0''(\phi)\\&-2\sin(\phi)(\sin\varphi-\sin\phi)-2(\cos\phi-\cos\varphi)\cos(\phi)\\&+2(1-\cos\eta)\big(r_0''(\phi)r_0(\varphi)+r_0'(\varphi)r_0'(\phi)\big)\Big|.\end{aligned}\tag{4.4.93}$$

Using that $r_0''\in L^\infty$, we find

$$\Big|\partial_\phi\Big\{\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\Big\}\Big|\leq C\left(|\phi-\varphi|+\sin^2(\eta/2)\right).\tag{4.4.94}$$

Thus,

$$\begin{aligned}|\partial_\phi T_1(\phi,\varphi,\eta)|\lesssim&\frac{\sin^3(\varphi)|h(\varphi)|\sin^2(\eta)\Big|\partial_\phi\Big\{\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\Big\}\Big|}{\widehat{R}^{\frac{5}{2}}(\phi,\varphi,\eta)}\\&+\frac{\sin^3(\varphi)|h(\varphi)|\sin^2(\eta)\left|\big(\partial_\phi+\partial_\varphi\big)\widehat{R}(\phi,\varphi,\eta)\right|\left|\partial_\phi\widehat{R}(\phi,\varphi,\eta)\right|}{\widehat{R}^{\frac{7}{2}}(\phi,\varphi,\eta)}.\end{aligned}$$

Using (4.4.78)–(4.4.88)–(4.4.94), we find

$$
\begin{aligned}
|\partial_\phi T_1(\phi,\varphi,\eta)| \lesssim & \frac{\sin^3(\varphi)|h(\varphi)|\sin^2(\eta)\left\{|\phi-\varphi|+\sin^2(\eta/2)\right\}}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{\frac{5}{2}}}\\
& + \frac{\sin^3(\varphi)|h(\varphi)|\sin^2(\eta)\left\{|\phi-\varphi|^2+(\sin(\varphi)+\sin(\phi))\sin^2(\eta/2)\right\}}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{3}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|\partial_\phi T_1(\phi,\varphi,\eta)| \lesssim & \frac{\sin(\varphi)|h(\varphi)|\left\{|\phi-\varphi|+\sin^2(\eta/2)\right\}}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{\frac{3}{2}}}\\
& + \frac{\sin(\varphi)|h(\varphi)|\left\{|\phi-\varphi|^2+(\sin(\varphi)+\sin(\phi))\sin^2(\eta/2)\right\}}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{2}}\\
\lesssim & \frac{|h(\varphi)|}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}}.
\end{aligned}
$$

Putting together this estimate with (4.4.92) we infer

$$
\begin{aligned}
|\partial_\phi T_1(\phi,\varphi,\eta)| \lesssim & \frac{|\phi-\varphi|^{-1}|h(\varphi)|}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{\frac{1}{2}}}\\
\lesssim & C|\phi-\varphi|^{-(2-\beta)}\sin^{-\beta}(\eta/2),
\end{aligned}
$$

for any $\beta\in(0,1)$.

Concerning the estimate of the term T_2, we first make appeal to (4.4.78) and (4.4.83) leading to

$$
\begin{aligned}
|T_2(\phi,\varphi,\eta)| \lesssim & \frac{(\sin^3(\varphi)+\sin^2(\varphi)|h(\varphi)|)\sin^2(\eta/2)}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{\frac{3}{2}}}\\
\lesssim & \frac{(\sin(\varphi)+|h(\varphi)|)}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{\frac{1}{2}}}.
\end{aligned}
$$

Applying (4.4.84) and (4.4.92), one finds

$$
|T_2(\phi,\varphi,\eta)| \leq C|\phi-\varphi|^{1-\beta}\sin^{\beta}(\eta/2),
$$

for any $\beta\in(0,1)$. The next stage is devoted to the estimate of $\partial_\phi T_2$ and one gets from direct computations

$$
|\partial_\phi T_2(\phi,\varphi,\eta)| \lesssim \frac{(\sin^3(\varphi)+\sin^2(\varphi)|h(\varphi)|)\sin^2(\eta/2)\left|\partial_\phi\widehat{R}(\phi,\varphi,\eta)\right|}{\widehat{R}^{\frac{5}{2}}(\phi,\varphi,\eta)}.
$$

Using (4.4.85) and (4.4.78), it implies

$$
\begin{aligned}
|\partial_\phi T_2(\phi,\varphi,\eta)| \lesssim & \frac{(\sin^3(\varphi)+\sin^2(\varphi)|h(\varphi)|)\sin^2(\eta/2)}{\left\{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)\right\}^{2}}\\
\lesssim & \frac{(\sin(\varphi)+|h(\varphi)|)}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2(\eta/2)}.
\end{aligned}
$$

Therefore (4.4.84) and (4.4.92) allows to get

$$
|\partial_\phi T_2(\phi,\varphi,\eta)| \lesssim |\phi-\varphi|^{-(2-\beta)}\sin^{\beta}(\eta/2),
$$

for any $\beta\in(0,1)$. Hence, by Proposition B.0.2 we achieve that $\mathcal{F}_n(h)'\in\mathscr{C}^{\beta}$, for any $\beta\in(0,1)$, which achieves the proof of the announced result. □

4.4.5 Fredholm structure

In this section we shall be concerned with the Fredholm structure of the linearized operator $\partial_f \tilde{F}(\Omega, 0)$ defined through (4.3.5) and (4.4.1). Our main result reads as follows.

Proposition 4.4.7. *Let $m \geq 2$, $\alpha \in (0,1)$ and $\Omega \in (-\infty, \kappa)$, then $\partial_f \tilde{F}(\Omega, 0) : X_m^\alpha \to X_m^\alpha$ is a well–defined Fredholm operator with zero index. In addition, for $\Omega = \Omega_m$, the kernel of $\partial_f \tilde{F}(\Omega_m, 0)$ is one–dimensional and its range is closed and of co-dimension one.*

Recall that the spaces X_m^α have been introduced in (4.2.15) *and Ω_m in Proposition* 4.4.4.

Proof. We shall first prove the second part, assuming the first one. The structure of the linearized operator is detailed in (4.3.5) and one has for $h(\phi, \theta) = \sum_{n \geq 1} h_n(\phi) \cos(n\theta)$

$$\partial_f \tilde{F}(\Omega, 0) h(\phi, \theta) = \sum_{n \geq 1} \cos(n\theta) \mathcal{L}_n^\Omega(h_n)(\phi),$$

where

$$\mathcal{L}_n^\Omega(h_n)(\phi) = \nu_\Omega(\phi) h_n(\phi) - \int_0^\pi H_n(\phi, \varphi) h_n(\varphi) d\varphi, \quad \phi \in [0, \pi].$$

In view of (4.4.1) and (4.4.64), this integral equation can be written in the form

$$\mathcal{K}_n^\Omega h = h.$$

We define the dispersion set by

$$\mathcal{S} = \big\{\Omega \in (-\infty, \kappa), \quad \mathrm{Ker} \partial_f \tilde{F}(\Omega, 0) \neq \{0\}\big\}.$$

Hence $\Omega \in \mathcal{S}$ if and only if there exists $m \geq 1$ such that the equation

$$\forall \phi \in [0, \pi], \quad \mathcal{K}_m^\Omega(h_m)(\phi) = h_m(\phi),$$

admits a nontrivial solution satisfying the regularity $h_m \in \mathscr{C}^{1,\alpha}(0, \pi)$ and the boundary condition $h_m(0) = h_m(\pi) = 0$. By virtue of Proposition 4.4.5 and Proposition 4.4.6 the foregoing conditions are satisfied for any eigenvalue provided that $m \geq 2$. On the other hand, we have shown in Proposition 4.4.3-(4) that for $\Omega = \Omega_m$ the kernel of $\mathcal{L}_m$ is one–dimensional. Moreover, Proposition 4.4.3-(5) ensures that for any $n > m$ we have $\lambda_n(\Omega_m) < \lambda_m(\Omega_m) = 1$. Since by construction $\lambda_n(\Omega_m)$ is the largest eigenvalue of $\mathcal{K}_n^{\Omega_m}$, then 1 could not be an eigenvalue of this operator and the equation

$$\mathcal{K}_n^{\Omega_m} h = h,$$

admits only the trivial solution. Thus the kernel of the restricted operator $\partial_f \tilde{F}(\Omega_m, 0) : X_m^\alpha \to X_m^\alpha$ is one-dimensional and is generated by the eigenfunction

$$(\phi, \theta) \mapsto h_m(\phi) \cos(m\theta).$$

We emphasize that this element belongs to the space X_m^α because it belongs to the function space $\mathscr{C}^{1,\alpha}((0, \pi) \times (0, 2\pi))$ since $\phi \mapsto h_m(\phi) \in \mathscr{C}^{1,\alpha}(0, \pi)$. The range of $\partial_f \tilde{F}(\Omega_m, 0)$ is closed and of co-dimension one follows from the fact this operator is Fredholm of zero index.

Next, let us show that $\partial_f \tilde{F}(\Omega, 0)$ is Fredholm of zero index. By virtue of the computations developed in Proposition 4.3.1, we assert that

$$\partial_f \tilde{F}(\Omega, 0) h(\phi, \theta) = \nu_\Omega(\phi) h(\phi, \theta) - \frac{1}{4\pi} G(h)(\phi, \theta),$$

with

$$G(h)(\phi,\theta) = \frac{1}{r_0(\phi)}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h(\varphi,\eta)d\eta d\varphi}{A(\phi,\theta,\varphi,\eta)^{\frac{1}{2}}}, \tag{4.4.95}$$
$$A(\phi,\theta,\varphi,\eta) = (r_0(\phi)-r_0(\varphi))^2+2r_0(\phi)r_0(\varphi)(1-\cos(\theta-\eta))+(\cos(\phi)-\cos(\varphi))^2.$$

Since $\Omega\in(-\infty,\kappa)$, the function ν_Ω is not vanishing. Moreover, by Proposition 4.4.1 one has that $\nu_\Omega\in\mathscr{C}^{1,\beta}$, for any $\beta\in(0,1)$.

Define the linear operator $\nu_\Omega\text{Id}:X_m^\alpha\to X_m^\alpha$ by

$$(\nu_\Omega\text{Id})(h)(\phi,\theta)=\nu_\Omega(\phi)h(\phi,\theta)\cdot$$

We shall check that it defines an isomorphism. The continuity of this operator follows from the regularity $\nu_\Omega\in\mathscr{C}^{1,\alpha}(0,\pi)$ combined with the fact $\mathscr{C}^{1,\alpha}((0,\pi)\times(0,2\pi))$ is an algebra. The Dirichlet boundary condition, the m–fold symmetry and the absence of the frequency zero are immediate for the product $\nu_\Omega h$, which finally belongs to X_m^α. Moreover, since ν_Ω is not vanishing, one has that $\nu_\Omega\text{Id}$ is injective. In order to check that such an operator is an isomorphism, it is enough to check that it is surjective, as a consequence of the Banach theorem. Take $k\in X_m^\alpha$, and we will find $h\in X_m^\alpha$ such that $(\nu_\Omega\text{Id})(h)=k$. Indeed, h is given by

$$h(\phi,\theta)=\frac{d(\phi,\theta)}{\nu_\Omega(\phi)}\cdot$$

Using the regularity of ν_Ω and the fact that it is not vanishing, it is easy to check that its inverse $\frac{1}{\nu_\Omega}$ still belongs to $\mathscr{C}^{1,\alpha}(0,\pi)$. Similar arguments as before allow to get $h\in X_m^\alpha$. Hence $\nu_\Omega\text{Id}$ is an isomorphism, and thus it is a Fredholm operator of zero index. From classical results on index theory, it is known that to get $\partial_f\tilde{F}(\Omega,0)$ is Fredholm of zero index, it is enough to establish that the perturbation $G:X_m^\alpha\to X_m^\alpha$ is compact. To do so, we prove that for any $\beta\in(\alpha,1)$ one has the smoothing effect

$$\forall\, h\in X_m^\alpha,\quad \|G(h)\|_{\mathscr{C}^{1,\beta}}\le C\|h\|_{\mathscr{C}^{1,\alpha}},$$

that we combine with the compact embedding $\mathscr{C}^{1,\beta}\big((0,\pi)\times(0,2\pi)\big)\hookrightarrow\mathscr{C}^{1,\alpha}\big((0,\pi)\times(0,2\pi)\big)$.

Take $h\in X_m^\alpha$ and let us show that $G(h)\in\mathscr{C}^{1,\beta}\big((0,\pi)\times(0,2\pi)\big)$, for any $\beta\in(0,1)$. We shall first deal with a preliminary fact. Define the following function

$$\forall\,\varphi\in[0,\pi],\ \theta,\eta\in\mathbb{R},\quad g_\theta(\varphi,\eta)=\int_\theta^\eta h(\varphi,\tau)d\tau. \tag{4.4.96}$$

By (4.2.17) we infer

$$|g_\theta(\varphi,\eta)|\le C\|h\|_{\text{Lip}}|\theta-\eta|\sin(\varphi),$$

According to the definition of the space X_m^α, the partial function $\tau\mapsto h(\varphi,\tau)$ is 2π-periodic and with zero average, that is, $\displaystyle\int_0^{2\pi}h(\varphi,\tau)d\tau=0$. This allows to get that $\eta\mapsto g_\theta(\varphi,\eta)$ is also 2π-periodic, and from elementary arguments we find

$$|g_\theta(\varphi,\eta)|\le C\|h\|_{\text{Lip}}\,|\sin((\theta-\eta)/2)|\sin(\varphi), \tag{4.4.97}$$

for any $\varphi\in[0,\pi]$ and $\theta,\eta\in[0,2\pi]$. In addition, it is immediate that $g_\theta\in\mathscr{C}^{1,\alpha}\big((0,\pi)\times(0,2\pi)\big)$ and

$$\partial_\varphi g_\theta(\varphi,\eta)=\int_\theta^\eta\partial_\varphi h(\varphi,\tau)d\tau.$$

The same arguments as before show that the partial function $\tau \mapsto \partial_\varphi h(\varphi,\tau)$ is 2π-periodic and with zero average. Moreover, $\eta \mapsto \partial_\varphi g_\theta(\varphi,\eta)$ is also 2π-periodic and

$$|\partial_\varphi g_\theta(\varphi,\eta)| \le C\|h\|_{\text{Lip}}\,|\sin((\theta-\eta)/2)|, \tag{4.4.98}$$

for any $\varphi\in[0,\pi]$ and $\theta,\eta\in[0,2\pi]$. Using the auxiliary function g_θ, one can integrate by parts in $G(h)$ in the variable η obtaining

$$G(h)(\phi,\theta) = \int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)r_0^2(\varphi)\sin(\eta-\theta)g_\theta(\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi. \tag{4.4.99}$$

The boundary term in the above integration by parts is vanishing due to the periodicity in η of the involved functions. It follows from (4.4.78),

$$A(\phi,\theta,\varphi,\eta) \gtrsim (\phi-\varphi)^2 + (\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2), \tag{4.4.100}$$

for any $\phi,\varphi\in(0,\pi)$ and $\theta,\eta\in(0,2\pi)$, and this estimate is crucial in the proof.

The boundedness of $G(h)$ can be implemented by using (4.4.97) and (4.4.100). Indeed, we write

$$\begin{aligned}|G(h)(\phi,\theta)| \lesssim&\|h\|_{\text{Lip}}\int_0^\pi\int_0^{2\pi}\frac{\sin^2(\varphi)r_0^2(\varphi)|\sin(\theta-\eta)||\sin((\theta-\eta)/2)|d\eta d\varphi}{\big((\phi-\varphi)^2+(\sin^2(\varphi)+\sin^2(\phi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}\\ \lesssim&\|h\|_{\text{Lip}}\int_0^\pi\int_0^{2\pi}\frac{r_0^2(\varphi)\sin^2(\varphi)\sin^2((\theta-\eta)/2)|d\eta d\varphi}{\big((\phi-\varphi)^2+(\sin^2(\varphi)+\sin^2(\phi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}.\end{aligned}$$

Therefore, we obtain

$$|G(h)(\phi,\theta)| \lesssim\|h\|_{\text{Lip}}\int_0^\pi\int_0^{2\pi}\frac{r_0^2(\varphi)|d\eta d\varphi}{\big((\phi-\varphi)^2+(\sin^2(\varphi)+\sin^2(\phi))\sin^2((\theta-\eta)/2)\big)^{\frac{1}{2}}}.$$

From the assumption (**H2**) on r_0 combined with (4.4.84) we get for any $\beta\in(0,1)$, and then

$$|G(h)(\phi,\theta)| \lesssim\|h\|_{\text{Lip}}\int_0^\pi\int_0^{2\pi}|\phi-\varphi|^{\beta-1}|\sin((\theta-\eta)/2)|^{-\beta}d\eta d\varphi \lesssim \|h\|_{\text{Lip}}.$$

Therefore

$$\|G(h)\|_{L^\infty} \lesssim \|h\|_{\mathscr{C}^{1\alpha}}. \tag{4.4.101}$$

The next step is to check now that $\partial_\phi G(h)\in\mathscr{C}^\alpha$ by making appeal to Proposition B.0.2. From direct computations using (4.4.99) we infer

$$\partial_\phi G(h)(\phi,\theta) = \frac{3}{2}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)\partial_\phi A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}d\eta d\varphi.$$

Adding and subtracting in the numerator $\partial_\varphi A(\phi,\theta,\varphi,\eta)$, it can be written in the form

$$\partial_\phi G(h)(\phi,\theta) =\frac{3}{2}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)\big(\partial_\phi A(\phi,\theta,\varphi,\eta)+\partial_\varphi A(\phi,\theta,\varphi,\eta)\big)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}d\eta d\varphi$$

$$-\frac{3}{2}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)\partial_\varphi A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}d\eta d\varphi.$$

Integrating by by parts in φ in the last term yields

$$\begin{aligned}\partial_\phi G(h)(\phi,\theta)=&\frac{3}{2}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)\big(\partial_\phi A(\phi,\theta,\varphi,\eta)+\partial_\varphi A(\phi,\theta,\varphi,\eta)\big)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}d\eta d\varphi\\&-\int_0^{\pi}\int_0^{2\pi}\frac{\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)g_\theta(\varphi,\eta)\right)\sin(\theta-\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi\\=:&\frac{3}{2}\mathscr{G}_1(\phi,\theta)-\mathscr{G}_2(\phi,\theta).\end{aligned}$$

The goal is check the kernel assumptions for Proposition B.0.2 in order to prove that $\mathscr{G}_1$ and $\mathscr{G}_2$ belong to $\mathscr{C}^\beta$, for any $\beta\in(0,1)$. For this aim, we define the kernels

$$K_1(\phi,\theta,\varphi,\eta):=\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}},$$

and

$$K_2(\phi,\theta,\varphi,\eta):=\frac{\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)g_\theta(\varphi,\eta)\right)\sin(\theta-\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}.$$

Let us start with K_1 and show that it satisfies the hypothesis of Proposition B.0.2. From straightforward calculus we obtain in view of the assumptions (**H**) and the mean value theorem

$$\begin{aligned}|(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)|=&|2(r_0'(\phi)-r_0'(\varphi))(r_0(\phi)-r_0(\varphi))\\&+2(r_0(\phi)r_0'(\varphi)+r_0(\varphi)r_0'(\phi))(1-\cos(\theta-\eta))\\&+2(\sin(\phi)-\sin(\varphi))(\cos(\phi)-\cos(\varphi))|\\\lesssim&(\phi-\varphi)^2+\big(\sin\varphi+\sin\phi\big)\sin^2((\theta-\eta)/2).\end{aligned}\tag{4.4.102}$$

Using the inequality $|ab|\leq\frac{1}{2}(a^2+b^2)$ allows to get

$$\sin\varphi|(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)|\lesssim(\phi-\varphi)^2+\big(\sin^2\varphi+\sin^2\phi\big)\sin^2((\theta-\eta)/2).$$

Thus, applying (4.4.100) we deduce that

$$\sin\varphi\big|(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)\big|\lesssim|A(\phi,\theta,\varphi,\eta)|.\tag{4.4.103}$$

Then, putting together (4.4.97), (**H2**), (4.4.100) and (4.4.103) we find

$$\begin{aligned}|K_1(\phi,\theta,\varphi,\eta)|\lesssim&\|h\|_{\text{Lip}}\frac{\sin(\varphi)r_0^2(\varphi)\sin^2((\theta-\eta)/2)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}\\\lesssim&\|h\|_{\text{Lip}}\frac{\sin(\varphi)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{1}{2}}}.\end{aligned}$$

As a consequence of (4.4.84), we immediately get

$$|K_1(\phi,\theta,\varphi,\eta)|\lesssim\|h\|_{\text{Lip}}|\phi-\varphi|^{\beta-1}|\sin((\theta-\eta)/2)|^{-\beta},\tag{4.4.104}$$

for any $\beta \in (0,1)$. Let us compute the derivative with respect to ϕ of K_1,

$$\partial_\phi K_1(\phi,\theta,\varphi,\eta) = \frac{\sin(\varphi) r_0^2(\varphi)\sin(\theta-\eta) g_\theta(\varphi,\eta)\partial_\phi((\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta))}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}} - \frac{5}{2}\frac{\sin(\varphi) r_0^2(\varphi)\sin(\theta-\eta) g_\theta(\varphi,\eta)((\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta))\partial_\phi A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{7}{2}}}.$$

From direct computations, we easily get

$$|\partial_\phi((\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta))| \lesssim |\phi-\varphi| + \sin^2((\theta-\eta)/2) \tag{4.4.105}$$

and

$$|\partial_\phi A(\phi,\theta,\varphi,\eta)| \lesssim |\phi-\varphi| + \sin(\varphi)\sin^2((\theta-\eta)/2). \tag{4.4.106}$$

Then, it is clear from (4.4.100) that

$$|\partial_\phi A(\phi,\theta,\varphi,\eta)| \lesssim A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta). \tag{4.4.107}$$

In addition, one may check that

$$\begin{aligned}|(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)| &\lesssim (\phi-\varphi)^2 + \big(\sin\varphi+\sin\phi\big)\sin^2((\theta-\eta)/2)\\ &\lesssim A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)\big(|\varphi-\phi| + |\sin((\theta-\eta)/2)|\big).\end{aligned} \tag{4.4.108}$$

By using (4.4.97), (**H2**), (4.4.100) (4.4.100), (4.4.103), (4.4.105), (4.4.106) and (4.4.108), one achieves

$$\begin{aligned}|\partial_\phi K_1(\phi,\theta,\varphi,\eta)| \lesssim& \|h\|_{\textnormal{Lip}}\frac{\sin^2(\varphi)\left(|\phi-\varphi|+\sin^2((\theta-\eta)/2)\right)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}\\ &+\|h\|_{\textnormal{Lip}}\frac{\sin^2(\varphi)\big(|\varphi-\phi|+|\sin((\theta-\eta)/2)|\big)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}\\ \lesssim& \|h\|_{\textnormal{Lip}}\frac{\sin^2(\varphi)\left(|\phi-\varphi|+|\sin((\theta-\eta)/2)|\right)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}.\end{aligned}$$

Therefore, using some elementary inequalities allow to get

$$\begin{aligned}|\partial_\phi K_1(\phi,\theta,\varphi,\eta)| \lesssim& \|h\|_{\textnormal{Lip}}\frac{\sin\varphi}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)}\\ \lesssim& \|h\|_{\textnormal{Lip}}\frac{|\phi-\varphi|^{-1}\,\sin\varphi}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{1}{2}}}.\end{aligned}$$

Applying (4.4.84)implies for any $\beta \in (0,1)$

$$|\partial_\phi K_1(\phi,\theta,\varphi,\eta)| \leq C\|h\|_{\textnormal{Lip}}|\phi-\varphi|^{-(2-\beta)}|\sin((\theta-\eta)/2)|^{-\beta}.$$

Now, let us move to the estimate of the partial derivative $\partial_\theta K_1$, given by

$$\partial_\theta K_1(\phi,\theta,\varphi,\eta) = \frac{\sin(\varphi) r_0^2(\varphi)\partial_\theta(\sin(\theta-\eta)g_\theta(\varphi,\eta))(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}} + \frac{\sin(\varphi) r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)\partial_\theta\left\{(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)\right\}}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}$$

$$-\frac{5}{2}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\theta-\eta)g_\theta(\varphi,\eta)((\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta))(\partial_\theta A(\phi,\theta,\varphi,\eta))}{A(\phi,\theta,\varphi,\eta)^{\frac{7}{2}}}.$$

By definition of g_θ in (4.4.96) and (4.4.97), one concludes in view of (4.4.100) that

$$|\partial_\theta(\sin(\theta-\eta)g_\theta(\varphi,\eta))| \lesssim \|h\|_{\text{Lip}}\sin(\varphi)|\sin((\theta-\eta)/2)| \lesssim \|h\|_{\text{Lip}}A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta). \tag{4.4.109}$$

Moreover, using (4.4.102) one gets

$$|\partial_\theta\left\{(\partial_\phi+\partial_\varphi)A(\phi,\theta,\varphi,\eta)\right\}| \lesssim (\sin(\phi)+\sin(\varphi))|\sin(\theta-\eta)| \lesssim A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta). \tag{4.4.110}$$

Using also the definition of A, we obtain

$$|\partial_\theta A(\phi,\theta,\varphi,\eta)| \le C\sin(\phi)\sin(\varphi)|\sin(\theta-\eta)| \lesssim \sin(\varphi)A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta). \tag{4.4.111}$$

Then, with the help of (4.4.100), (4.4.102) (4.4.109), (4.4.110) and (4.4.111), we can estimate $\partial_\theta K_1$ as

$$\begin{aligned}|\partial_\theta K_1(\phi,\theta,\varphi,\eta)| \lesssim& \|h\|_{\text{Lip}}\frac{\sin^3(\varphi)\left((\phi-\varphi)^2+(\sin(\varphi)+\sin(\phi))\sin^2((\theta-\eta)/2)\right)}{\left((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\right)^2}\\ &+\|h\|_{\text{Lip}}\frac{\sin^2(\varphi)}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)}\\ &+\|h\|_{\text{Lip}}\frac{\sin^3(\varphi)\left((\phi-\varphi)^2+(\sin(\varphi)+\sin(\phi))\sin^2((\theta-\eta)/2)\right)}{\left((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\right)^2}.\end{aligned}$$

Consequently we get

$$\begin{aligned}|\partial_\theta K_1(\phi,\theta,\varphi,\eta)| \lesssim& \frac{\|h\|_{\text{Lip}}\ \sin^2(\varphi)}{(\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)}\\ \lesssim& \frac{\|h\|_{\text{Lip}}\,|\sin((\theta-\eta)/2)|^{-1}\ \sin(\varphi)}{\left((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\right)^{\frac{1}{2}}}.\end{aligned}$$

Therefore we obtain by virtue of (4.4.84)

$$|\partial_\theta K_1(\phi,\theta,\varphi,\eta)| \le C\|h\|_{\text{Lip}}|\phi-\varphi|^{-\beta}|\sin((\theta-\eta)/2|^{-(2-\beta)}, \tag{4.4.112}$$

for any $\beta\in(0,1)$. Hence, all the hypothesis of Proposition B.0.2 are satisfied and therefore we deduce that $\mathscr{G}_1$ belongs to $\mathscr{C}^\beta\big((0,\pi)\times(0,2\pi)\big)$, for any $\beta\in(0,1)$. The estimates of the kernel K_2 we are quite similar to those of K_1 modulo some slight adaptations. We shall not develop all the estimates which are straightforward and tedious. We will restrict this discussion to the analogous estimate to (4.4.104) and (4.4.112). First note that thanks to (4.4.97) and (4.4.98) one gets

$$\left|\partial_\varphi(\sin(\varphi)\,r_0^2(\varphi)g_\theta(\varphi,\theta))\right| \lesssim \|h\|_{\text{Lip}}\sin^3(\varphi)|\sin((\theta-\eta)/2)|.$$

This implies that

$$\begin{aligned}\left|K_2(\phi,\theta,\varphi,\eta)\right| \lesssim& \|h\|_{\text{Lip}}\frac{\sin^3(\varphi)\sin^2((\theta-\eta)/2)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}\\ \lesssim& \|h\|_{\text{Lip}}\frac{\sin(\varphi)}{A(\phi,\theta,\varphi,\eta)^{\frac{1}{2}}}.\end{aligned}$$

It follows from (4.4.100) and (4.4.84) that

$$|K_2(\phi,\theta,\varphi,\eta)| \lesssim \|h\|_{\text{Lip}}|\phi-\varphi|^{-\beta}|\sin((\theta-\eta)/2|^{-(1-\beta)}, \tag{4.4.113}$$

which is the announced estimate. As to the estimate of $\partial_\theta K_2$ we first write

$$\begin{aligned}\partial_\theta K_2(\phi,\theta,\varphi,\eta) =& -\frac{\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)h(\varphi,\eta)\right)\sin(\theta-\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac32}} + \frac{\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)g_\theta(\varphi,\eta)\right)\cos(\theta-\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\\ &-\frac{3}{2}K_2(\phi,\theta,\varphi,\eta)\frac{\partial_\theta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)}\end{aligned} \tag{4.4.114}$$

Straightforward calculations using (**H2**) and (4.2.17) show that

$$\begin{aligned}\frac{\left|\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)\partial_\theta g_\theta(\varphi,\eta)\right)\sin(\theta-\eta)\right|}{A(\phi,\theta,\varphi,\eta)^{\frac32}} \lesssim& \|h\|_{\text{Lip}}\frac{\sin^3(\varphi)|\sin((\theta-\eta)/2)|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\\ \lesssim& \|h\|_{\text{Lip}}\frac{\sin^2(\varphi)}{A(\phi,\theta,\varphi,\eta)}.\end{aligned}$$

Putting together (4.4.100) and (4.4.84) implies

$$\begin{aligned}\frac{\sin^2(\varphi)}{A(\phi,\theta,\varphi,\eta)} \lesssim& |\sin((\theta-\eta)/2)|^{-1}\frac{\sin(\varphi)}{A^{\frac12}(\phi,\theta,\varphi,\eta)}\\ \lesssim& \frac{1}{|\phi-\varphi|^\beta|\sin((\theta-\eta)/2)|^{2-\beta}}.\end{aligned} \tag{4.4.115}$$

Therefore we find

$$\frac{\left|\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)\partial_\theta g_\theta(\varphi,\eta)\right)\sin(\theta-\eta)\right|}{A(\phi,\theta,\varphi,\eta)^{\frac32}} \lesssim \frac{\|h\|_{\text{Lip}}}{|\phi-\varphi|^\beta|\sin((\theta-\eta)/2)|^{2-\beta}}.$$

As to the second term of the right-hand side of (4.4.114), we get in view of (**H2**), (4.4.97) and (4.4.98)

$$\begin{aligned}\frac{\left|\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)g_\theta(\varphi,\eta)\right)\cos(\theta-\eta)\right|}{A(\phi,\theta,\varphi,\eta)^{\frac32}} \lesssim& \|h\|_{\text{Lip}}\frac{\sin^3(\varphi)|\sin((\theta-\eta)/2)|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\\ \lesssim& \|h\|_{\text{Lip}}\frac{\sin^2(\varphi)}{A(\phi,\theta,\varphi,\eta)}.\end{aligned}$$

It follows from (4.4.115) that

$$\frac{\left|\partial_\varphi\left(\sin(\varphi)r_0^2(\varphi)g_\theta(\varphi,\eta)\right)\cos(\theta-\eta)\right|}{A(\phi,\theta,\varphi,\eta)^{\frac32}} \lesssim \|h\|_{\text{Lip}}\frac{\|h\|_{\text{Lip}}}{|\phi-\varphi|^\beta|\sin((\theta-\eta)/2)|^{2-\beta}}.$$

Concerning the last term of (4.4.114), we put together (4.4.111), (4.4.113), (4.4.100) and (4.4.84) that

$$\begin{aligned}|K_2(\phi,\theta,\varphi,\eta)|\frac{|\partial_\theta A(\phi,\theta,\varphi,\eta)|}{A(\phi,\theta,\varphi,\eta)} \lesssim& \|h\|_{\text{Lip}}|\phi-\varphi|^{-\beta}|\sin((\theta-\eta)/2|^{-(1-\beta)}\frac{\sin\varphi}{A^{\frac12}(\phi,\theta,\varphi,\eta)}\\ \lesssim& \|h\|_{\text{Lip}}|\phi-\varphi|^{-\beta}|\sin((\theta-\eta)/2|^{-(2-\beta)}.\end{aligned}$$

Therefore collecting the preceding estimates allows to get the suitable estimate for $\partial_\theta K_2$,

$$|\partial_\theta K_2(\phi,\theta,\varphi,\eta)| \lesssim \|h\|_{\text{Lip}}|\phi-\varphi|^{-\beta}|\sin((\theta-\eta)/2|^{-(2-\beta)}.$$

The estimate for $\partial_\phi K_2$ can be done similarly in a straightforward way. Consequently the assumptions of Proposition B.0.2 hold true and one deduces that $\mathscr{G}_1 \in \mathscr{C}^\beta\big((0,\pi)\times(0,2\pi)\big)$. Hence we obtain $\partial_\phi G \in \mathscr{C}^\beta\big((0,\pi)\times(0,2\pi)\big)$, with the estimate

$$\|\partial_\phi G(h)\|_{\mathscr{C}^\beta} \lesssim \|h\|_{\mathscr{C}^{1,\alpha}}. \tag{4.4.116}$$

The next stage is to show that $\partial_\theta G(h) \in \mathscr{C}^\beta\big((0,\pi)\times(0,2\pi)\big)$ following the same strategy as before. From (4.4.95), we get

$$\partial_\theta G(h)(\phi,\theta) = -\frac{1}{2}\frac{1}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h(\varphi,\eta)\partial_\theta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi.$$

Direct computations show that

$$\partial_\theta A(\phi,\theta,\varphi,\eta) = 2r_0(\phi)r_0(\varphi)\sin(\theta-\eta) = -\partial_\eta A(\phi,\theta,\varphi,\eta).$$

It follows

$$\partial_\theta G(h)(\phi,\theta) = \frac{1}{2}\frac{1}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h(\varphi,\eta)\partial_\eta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi.$$

On the other hand, integration by parts in η yields

$$\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)h(\varphi,\theta)\partial_\eta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi = 0.$$

Thus we deduce by subtraction

$$\begin{aligned}\partial_\theta G(h)(\phi,\theta) &= \frac{1}{2}\frac{1}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0(\varphi)\big(h(\varphi,\eta)-h(\varphi,\theta)\big)\partial_\eta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi\\ &= \int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)r_0^2(\varphi)\big(h(\varphi,\eta)-h(\varphi,\theta)\big)\sin(\eta-\theta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}d\eta d\varphi.\end{aligned} \tag{4.4.117}$$

Since $h \in \mathscr{C}^{1,\alpha}$, then the mean value theorem implies

$$|h(\varphi,\theta)-h(\varphi,\eta)| \lesssim \|h\|_{\text{Lip}}|\theta-\eta|.$$

Moreover, by the 2π-periodicity of h in η one obtains

$$|h(\varphi,\theta)-h(\varphi,\eta)| \lesssim \|h\|_{\mathscr{C}^{1,\alpha}}\big|\sin\big((\theta-\eta)/2\big)\big|. \tag{4.4.118}$$

Define the kernel

$$K_3(\phi,\theta,\varphi,\eta) := \frac{\sin(\varphi)r_0^2(\varphi)\sin(\eta-\theta)\big(h(\varphi,\eta)-h(\varphi,\theta)\big)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}},$$

and let us check the hypothesis of Proposition B.0.2. First using (4.4.100), (**H2**) and (4.4.118) we obtain

$$\begin{aligned}|K_3(\phi,\theta,\varphi,\eta)| \lesssim & \frac{\sin^3(\varphi)\sin^2((\theta-\eta)/2)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{3}{2}}}\\ \lesssim & \frac{\sin(\varphi)}{\big((\phi-\varphi)^2+(\sin^2(\phi)+\sin^2(\varphi))\sin^2((\theta-\eta)/2)\big)^{\frac{1}{2}}}.\end{aligned}$$

Applying (4.4.84) yields

$$|K_3(\phi,\theta,\varphi,\eta)| \lesssim \|h\|_{\mathscr{C}^{1,\alpha}}|\phi-\varphi|^{-(1-\beta)}|\sin((\theta-\eta)/2)|^{-\beta}, \tag{4.4.119}$$

for any $\beta\in(0,1)$. Let us estimate $\partial_\phi K_3$ which is explicitly given by,

$$\begin{aligned}\partial_\phi K_3(\phi,\theta,\varphi,\eta) =& -\frac{3}{2}\frac{\sin(\varphi)r_0^2(\varphi)\sin(\eta)\big(h(\varphi,\eta)-h(\varphi,\theta)\big)\partial_\phi A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)^{\frac{5}{2}}}\\ =& -\frac{3}{2}K_3(\phi,\theta,\varphi,\eta)\frac{\partial_\phi A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)}.\end{aligned}$$

By virtue of (4.4.107) and (4.4.119), we achieve

$$\begin{aligned}|\partial_\phi K_3(\phi,\theta,\varphi,\eta)| \lesssim & \|h\|_{\mathscr{C}^{1,\alpha}}|\phi-\varphi|^{-(1-\beta)}\big|\sin\big((\theta-\eta)/2\big)\big|^{-\beta}A^{-\frac{1}{2}}(\phi,\theta,\varphi,\eta),\\ \lesssim & \frac{\|h\|_{\mathscr{C}^{1,\alpha}}}{|\phi-\varphi|^{2-\beta}|\sin((\theta-\eta)/2)|^{\beta}},\end{aligned}$$

for any $\beta\in(0,1)$. It remains to establish the suitable estimates for $\partial_\theta K_3$. First we have

$$\begin{aligned}\partial_\theta K_3(\phi,\theta,\varphi,\eta) =& -\frac{3}{2}K_3(\phi,\theta,\varphi,\eta)\frac{\partial_\theta A(\phi,\theta,\varphi,\eta)}{A(\phi,\theta,\varphi,\eta)} - \frac{\sin(\varphi)r_0^2(\varphi)\sin(\eta-\theta)\partial_\theta h(\varphi,\theta)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}\\ &- \frac{\sin(\varphi)r_0^2(\varphi)\cos(\eta-\theta)\big(h(\varphi,\eta)-h(\varphi,\theta)\big)}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}.\end{aligned}$$

Using (4.4.111) and (4.4.119) (where we exchange β by $1-\beta$) we get

$$\begin{aligned}\big|K_3(\phi,\theta,\varphi,\eta)\big|\frac{\big|\partial_\theta A(\phi,\theta,\varphi,\eta)\big|}{A(\phi,\theta,\varphi,\eta)} \lesssim & \frac{\|h\|_{\mathscr{C}^{1,\alpha}}}{|\phi-\varphi|^{\beta}|\sin((\theta-\eta)/2)|^{1-\beta}}\frac{\sin\varphi}{A^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)}\\ \lesssim & \frac{\|h\|_{\mathscr{C}^{1,\alpha}}}{|\phi-\varphi|^{\beta}|\sin((\theta-\eta)/2)|^{2-\beta}}.\end{aligned}$$

For the second term of the right-hand side of $\partial_\theta K_3$ we write in view of (**H2**)

$$\begin{aligned}\frac{\sin(\varphi)r_0^2(\varphi)|\sin(\eta-\theta)||\partial_\theta h(\varphi,\theta)|}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}} \lesssim & \|h\|_{\mathrm{Lip}}\frac{\sin^3(\varphi)|\sin((\theta-\eta)/2)|}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}}\\ \lesssim & \|h\|_{\mathrm{Lip}}\frac{\sin^2(\varphi)}{A(\phi,\theta,\varphi,\eta)}.\end{aligned}$$

Applying (4.4.115) yields

$$\frac{\sin(\varphi)r_0^2(\varphi)|\sin(\eta-\theta)||\partial_\theta h(\varphi,\theta)|}{A(\phi,\theta,\varphi,\eta)^{\frac{3}{2}}} \lesssim \frac{\|h\|_{\mathrm{Lip}}}{|\phi-\varphi|^{\beta}|\sin((\theta-\eta)/2)|^{2-\beta}}.$$

Concerning the last term of the right-hand side of $\partial_\theta K_3$, it is similar to the foregoing one. Indeed, using (4.4.118) and (**H2**) we get

$$\begin{aligned}\frac{\sin(\varphi)r_0^2(\varphi)|\cos(\eta-\theta)|\big|h(\varphi,\eta)-h(\varphi,\theta)\big|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\lesssim&\|h\|_{\mathscr{C}^{1,\alpha}}\frac{\sin^3(\varphi)|\sin((\eta-\theta)/2)|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\\ \lesssim&\|h\|_{\mathscr{C}^{1,\alpha}}\frac{\sin^2(\varphi)}{A(\phi,\theta,\varphi,\eta)}.\end{aligned}$$

It suffices to use (4.4.115) to obtain

$$\begin{aligned}\frac{\sin(\varphi)r_0^2(\varphi)|\cos(\eta-\theta)|\big|h(\varphi,\eta)-h(\varphi,\theta)\big|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\lesssim&\|h\|_{\mathscr{C}^{1,\alpha}}\frac{\sin^3(\varphi)|\sin((\eta-\theta)/2)|}{A(\phi,\theta,\varphi,\eta)^{\frac32}}\\ \lesssim&\frac{\|h\|_{\mathscr{C}^{1,\alpha}}}{|\phi-\varphi|^\beta|\sin((\theta-\eta)/2)|^{2-\beta}}.\end{aligned}$$

Therefore we get from the preceding estimates

$$|\partial_\theta K_3(\phi,\theta,\varphi,\eta)|\lesssim\frac{\|h\|_{\mathscr{C}^{1,\alpha}}}{|\phi-\varphi|^\beta|\sin((\theta-\eta)/2)|^{2-\beta}}.$$

Consequently, all the assumptions of Proposition B.0.2 are verified by the kernel K_3 and thus we deduce that $\partial_\theta G(h)\in\mathscr{C}^\beta\big((0,\pi)\times(0,2\pi)\big)$ for any $\beta\in(0,1)$, with the estimate

$$\|\partial_\theta G(h)\|_{\mathscr{C}^\beta}\lesssim\|h\|_{\mathscr{C}^{1,\alpha}}.$$

Putting together this estimate with (4.4.116) and (4.4.101) yields

$$\|G(h)\|_{\mathscr{C}^{1,\beta}}\lesssim\|h\|_{\mathscr{C}^{1,\alpha}},$$

and this achieves the proof of the proposition. □

4.4.6 Transversality

We have shown in Proposition 4.4.7 that when Ω belongs to the discrete set $\{\Omega_m, m\geq 2\}$ then the linearized operator $\partial_f\tilde F(\Omega,0)$ is of Fredholm type with one–dimensional kernel. This property is not enough to bifurcate to nontrivial solutions for the nonlinear problem. A sufficient condition for that, according to Theorem A.0.3, is the the transversal assumption which amounts to checking

$$\partial^2_{\Omega,f}\tilde F(\Omega_m,0)f_m^\star\notin\mathrm{Im}(\partial_f\tilde F(\Omega_m,0)),$$

where $f_m^\star$ is a generator of the kernel of $\partial_f\tilde F(\Omega_m,0)$. Note that as a consequence of (4.4.1) and (4.4.2), for a function $h:(\phi,\theta)\mapsto\sum_{n\geq1}h_n(\phi)\cos(n\theta)\in X_m^\alpha$, we get

$$\partial_f\tilde F(\Omega,0)h(\phi,\theta)=\sum_{n\geq1}\mathcal{L}_n^\Omega h_n(\phi)\cos(n\theta),$$

with

$$\begin{aligned}\mathcal{L}_n^\Omega h_n(\phi)=&\nu_\Omega(\phi)h_n(\phi)-\int_0^\pi H_n(\phi,\varphi)h_n(\phi,\varphi)d\varphi\\ =&\nu_\Omega(\phi)\big(h_n(\phi)-\mathcal{K}_n^\Omega h(\phi)\big),\end{aligned}$$

where $\mathcal{K}_n^{\Omega}$ is defined in (4.4.64). Hence, the second mixed derivative takes the form,

$$\partial^2_{\Omega,f}\tilde{F}(\Omega,0)h(\phi,\theta)=-h(\phi,\theta).$$

Our main result of this section reads as follows.

Proposition 4.4.8. *Let $m\geq 2$, then the transversal condition holds true, that is,*

$$\partial^2_{\Omega,f}\tilde{F}(\Omega_m,0)f_m^\star\notin \mathrm{Im}(\partial_f\tilde{F}(\Omega_m,0)),$$

where $f_m^\star$ is a generator of the kernel of $\partial_f\tilde{F}(\Omega_m,0)$.

Proof. Recall from the proof of Proposition 4.4.7 that the function $f_m^\star$ has the form

$$f_m^\star(\phi,\theta)=h_m^\star(\phi)\cos(m\theta)$$

and $h_m^\star$ is a nonzero solution to the equation

$$\mathcal{K}_m^{\Omega_m}h_m^\star(\phi)=h_m^\star(\phi).$$

It follows that

$$\partial^2_{\Omega,f}\tilde{F}(\Omega_m,0)f_m^\star(\phi,\theta)=-h_m^\star(\phi)\cos(m\theta).$$

Assume that this element belongs to the range of $\partial_f\tilde{F}(\Omega_m,0)$. Then we can find h_m such that

$$h_m^\star(\phi)=\nu_{\Omega_m}(\phi)\big(h_m(\phi)-\mathcal{K}_m^{\Omega_m}h_m(\phi)\big).$$

Dividing this equality by ν_{Ω_m} and taking the inner product with $h_m^\star$, with respect to $\langle\cdot,\cdot\rangle_{\Omega_m}$ defined in (4.4.66) yields by the symmetry of $\mathcal{K}_m^{\Omega_m}$

$$\begin{aligned}\Big\langle\frac{h_m^\star}{\nu_{\Omega_m}},h_m^\star\Big\rangle_{\Omega_m}=&\Big\langle h_m,h_m^\star\Big\rangle_{\Omega_m}-\Big\langle\mathcal{K}_m^{\Omega_m}h_m,h_m^\star\Big\rangle_{\Omega_m}\\=&\Big\langle h_m,h_m^\star\Big\rangle_{\Omega_m}-\Big\langle h_m,\mathcal{K}_m^{\Omega_m}h_m^\star\Big\rangle_{\Omega_m}\\=&\Big\langle h_m,h_m^\star-\mathcal{K}_m^{\Omega_m}h_m^\star\Big\rangle_{\Omega_m}\\=&0.\end{aligned}$$

Coming back to the definition of the inner product (4.4.66) and (4.4.4), we find

$$\int_0^\pi (h_m^\star(\varphi))^2\sin(\varphi)\,r_0^2(\varphi)d\varphi=0.$$

From the assumption ($\mathbf{H}$) we know that r_0 does not vanish in $(0,\pi)$. Then we get from the continuity of $h_m^\star$ that this latter function should vanish everywhere in $(0,\pi)$, which is a contradiction. Hence, we deduce that $f_m^\star$ does not belong to the range of $\partial_f\tilde{F}(\Omega_m,0)$ and then the transversal condition is satisfied. □

4.5 Nonlinear action

This section is devoted to the regularity study of the nonlinear functional $\tilde{F}$ defined in (4.2.13) that we recall for the convenience of the reader,

$$\tilde{F}(\Omega, f)(\phi, \theta) = \frac{1}{r_0(\phi)} \left\{ I(f)(\phi, \theta) - \frac{\Omega}{2} r^2(\phi, \theta) - m(\Omega, f)(\phi) \right\},$$

for any $(\phi, \theta) \in (0, \pi) \times (0, 2\pi)$ and where

$$I(f)(\phi, \theta) = -\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \int_0^{r(\varphi, \eta)} \frac{r \sin(\varphi) dr d\eta d\varphi}{|(re^{i\eta}, \cos(\varphi)) - (r(\phi, \theta)e^{i\theta}, \cos(\phi))|},$$

the mean m is defined in (4.2.11) and

$$r(\phi, \theta) = r_0(\phi) + f(\phi, \theta).$$

We would like in particular to analyze the symmetry/regularity persistence of the function spaces X_m^α defined in (4.2.15) and (4.2.16) through the action of the nonlinear functional $\tilde{F}$.

4.5.1 Symmetry persistence

The main task here is to check the symmetry persistence of the function spaces X_m^α defined in (4.2.15) through the nonlinear action of $\tilde{F}$. Notice that at this level, we do not raise the problem of whether or not this functional is well-defined and this target is postponed later in Section 4.5. First recall that

$$X_m^\alpha = \Big\{ f : [0, \pi] \times [0, 2\pi] \to \mathbb{R} : f \in \mathscr{C}^{1,\alpha}, f(0, \theta) = f(\pi, \theta) \equiv 0, \\ f\left(\frac{\pi}{2} - \phi, \theta\right) = f\left(\frac{\pi}{2} + \phi, \theta\right), \; f(\phi, \theta) = \sum_{n \geq 1} f_n(\phi) \cos(nm\theta) \Big\}.$$

Proposition 4.5.1. *Let $\Omega \in \mathbb{R}$, $f \in X_m^\alpha$ with $m \geq 1$ and assume that r_0 satisfies the conditions* $(\mathbf{H})$. *Then the following assertions hold true.*

1. *The equatorial symmetry:*

$$\tilde{F}(\Omega, f)(\pi - \phi, \theta) = \tilde{F}(\Omega, f)(\phi, \theta), \quad \forall (\phi, \theta) \in [0, \pi] \times \mathbb{R}.$$

2. *We get the algebraic structure,*

$$\tilde{F}(\Omega, f)(\phi, \theta) = \sum_{n \geq 1} f_n(\phi) \cos(n\theta),$$

for some functions f_n and for any $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$.

3. *The m-fold symmetry: $\tilde{F}(\Omega, f)(\phi, \theta + \frac{2\pi}{m}) = \tilde{F}(\Omega, f)(\phi, \theta)$, for any $(\phi, \theta) \in [0, \pi] \times \mathbb{R}$.*

Proof. (**1**) From the expression of $\tilde{F}$ in (4.2.13), it is suffices to check the property for $I(f)$. One can easily verify using the symmetry of the functions cos and r combined with the change of variables $\varphi \mapsto \pi - \varphi$

$$\begin{aligned} I(f)\,(\pi-\phi,\theta) =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\pi-\phi,\theta)e^{i\theta},\cos(\pi-\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\pi-\varphi,\eta)} \frac{r\sin(\pi-\varphi)drd\eta d\varphi}{|(re^{i\eta},-\cos(\varphi))-(r(\phi,\theta)e^{i\theta},-\cos(\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta)e^{i\theta},\cos(\phi))|}\\ =& I(f)\,(\phi,\theta)\,. \end{aligned}$$

(**2**) In order to get the desired structure, it suffices to check the following symmetry

$$I(f)(\phi,-\theta)=I(f)(\phi,\theta),\quad \forall\,(\phi,\theta)\in[0,\pi]\times\mathbb{R}.$$

To do that, we use the symmetry of r, that is $r(\varphi,-\theta)=r(\varphi,\theta)$, combined with the change of variables $\eta\mapsto-\eta$ allowing to get

$$\begin{aligned} I(f)(\phi,-\theta) =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,-\theta)e^{-i\theta},\cos(\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,-\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{-i\eta},\cos(\varphi))-(r(\phi,\theta)e^{-i\theta},\cos(\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta)e^{i\theta},\cos(\phi))|}\\ =& I(f)(\phi,\theta). \end{aligned}$$

(**3**) First, since r belongs to X_m^α then it satisfies $r(\varphi,\theta+\frac{2\pi}{m})=r(\varphi,\theta)$. Thus we get by the change of variables $\eta\mapsto\eta+\frac{2\pi}{m}$

$$\begin{aligned} I(f)\left(\phi,\theta+\frac{2\pi}{m}\right) =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta+\frac{2\pi}{m})e^{i(\theta+\frac{2\pi}{m})},\cos(\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta+\frac{2\pi}{m})} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i(\eta+\frac{2\pi}{m})},\cos(\varphi))-(r(\phi,\theta)e^{i(\theta+\frac{2\pi}{m})},\cos(\phi))|}\\ =& -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)} \frac{r\sin(\varphi)drd\eta d\varphi}{|(re^{i\eta},\cos(\varphi))-(r(\phi,\theta)e^{i\theta},\cos(\phi))|}\\ =& I(f)(\phi,\theta). \end{aligned}$$

Notice that we have used the fact that the Euclidean distance id $\mathbb{C}$ is invariant by the rotation action $z\mapsto e^{i\frac{2\pi}{m}}z$. □

The next discussion is devoted to the symmetry effects of the surface of the vortices on the velocity structure. We shall show the following.

Lemma 4.5.2. *If* r_0 *satisfies* $(\mathbf{H})$ *and* $f \in X_m^\alpha$*, with* $m \geq 2$*, then*

$$\forall\, z \in \mathbb{R}, \qquad \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta(r(\varphi,\eta)\cos(\eta))d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}} = 0,$$

$$\forall\, z \in \mathbb{R}, \qquad \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta(r(\varphi,\eta)\sin(\eta))d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}} = 0.$$

As a consequence, the velocity field defined in (4.2.4) *is vanishing at the vertical axis, that is,*

$$U(0,0,z) = 0,$$

for any $z \in \mathbb{R}$.

Proof. Set for any $z \in \mathbb{R}$,

$$I_1(z) := \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta(r(\varphi,\eta)\cos(\eta))d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}},$$

$$I_2(z) := \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta(r(\varphi,\eta)\sin(\eta))d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}}.$$

Observe that from the periodicity in η we may write

$$\begin{aligned} I_1(z) = &\int_0^\pi \int_{-\pi}^{\pi} \frac{\sin(\varphi)(\partial_\eta r)(\varphi,\eta)\cos(\eta)d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}} \\ &- \int_0^\pi \int_{-\pi}^{\pi} \frac{\sin(\varphi)r(\varphi,\eta)\sin(\eta)d\eta d\varphi}{\Big(r^2(\varphi,\eta)+\big(z-\cos\varphi\big)^2\Big)^{\frac12}}. \end{aligned}$$

Since $f \in X_m^\alpha$, then $r(\varphi,-\eta) = r(\varphi,\eta)$ and so $(\partial_\eta r)(\varphi,-\eta) = -(\partial_\eta r)(\varphi,\eta)$. Therefore making the change of variables $\eta \mapsto -\eta$ allows to get $I_1(z) = 0$.

To check $I_2(z) = 0$ we shall use the m-fold symmetry of r. In fact by the change of variables $\eta \mapsto \eta + \frac{2\pi}{m}$ and using the 2π-periodicity in η and some elementary trigonometric identity, we find

$$\begin{aligned} I_2(z) &= \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta+\frac{2\pi}{m})\big)}{(r(\varphi,\eta)^2+(z-\cos(\varphi)^2)^{\frac12}} d\eta d\varphi \\ &= \cos(2\pi/m)I_2(z) + \sin(2\pi/m)I_1(z). \end{aligned}$$

Since $m \geq 2$ and $I_1(z) = 0$ then we get $I_2(z) = 0$.

Coming back to (4.2.4) and following the change of variables giving (4.2.8) we easily get

$$U(0,0,z) = \big(I_1(z), I_2(z), 0\big),$$

which gives the announced result. $\square$

4.5.2 Deformation of the Euclidean norm

The spherical change of coordinates used to recover both the velocity and the stream function from the surface geometry of the patch yields to a deformation of the Green function. Notice that in the usual Cartesian coordinates the Green kernel is radial and thus it is isotropic with respect to all the variables. In the new coordinates we loose such property and the Green kernel becomes anisotropic and the north and south poles are degenerating points. To deal with these defects one needs refined treatments in the behavior of the kernel or also the adaptation of the function spaces which are of Dirichlet type. The following lemma is crucial to deal with the anisotropy of the kernel.

Lemma 4.5.3. *Let $m \geq 1, \alpha \in (0,1)$, r_0 satisfies $(\mathbf{H})$, $f \in X_m^\alpha$ such that $\|f\|_{\text{Lip}} \leq \varepsilon$ with ε small enough and set $r = r_0 + f$. Define for any $\phi \in [0, \frac{\pi}{2}]$, $\varphi \in [0,\pi]$, $\theta, \eta \in [0, 2\pi]$ and $s \in [0,1]$*

$$J_s(\phi,\theta,\varphi,\eta) := (r(\varphi,\eta) - sr(\phi,\theta))^2 + 2sr(\phi,\theta)r(\varphi,\eta)(1-\cos(\theta-\eta)) + (\cos(\phi)-\cos(\varphi))^2.$$

Then

$$\begin{aligned} |J_0(\phi,\theta,\varphi,\eta)| \geq& C\sin^2(\varphi), && (4.5.1)\\ |J_s(\phi,\theta,\varphi,\eta)| \geq& C\Big(\big(\sin^2(\varphi) + s^2\phi^2\big)\sin^2((\theta-\eta)/2) + (\varphi+\phi)^2(\phi-\varphi)^2\Big), && (4.5.2) \end{aligned}$$

with C an absolute constant. Remark that we have restricted ϕ to $\in [0,\pi/2]$ instead of $[0,\pi]$ because the symmetry of r with respect to $\frac{\pi}{2}$.

Proof. Since $f \in B_{X_m^\alpha}(\varepsilon)$, for some $\varepsilon < 1$, and r_0 verifies **(H2)** then

$$r(\varphi,\eta) = r_0(\varphi) + f(\varphi,\eta) \geq 2C\sin\varphi - |f(\varphi,\eta)|.$$

In addition f satisfies (4.2.17) and in particular

$$\frac{|f(\varphi,\eta)|}{\sin(\varphi)} \leq C_1\|f\|_{\text{Lip}}.$$

It follows that,

$$r(\varphi,\eta) \geq \big(2C - C_1\|f\|_{\text{Lip}}\big)\sin(\varphi).$$

By imposing $\|f\|_{\text{Lip}} \leq \varepsilon = \frac{C}{C_1}$, we infer

$$r(\varphi,\eta) \geq C\sin(\varphi). \tag{4.5.3}$$

Consequently, we obtain

$$J_0(\phi,\theta,\varphi,\eta) = r^2(\varphi,\eta) + (\cos(\varphi)-\cos(\phi))^2 \geq C\sin^2(\varphi),$$

which gives the estimate (4.5.1). Let us now check the validity of (4.5.2). First, we remark that

$$J_s(\phi,\theta,\varphi,\eta) = r^2(\varphi,\eta) + s^2r^2(\phi,\theta) - 2sr(\varphi,\eta)r(\phi,\theta)\cos(\theta-\eta) + (\cos(\varphi)-\cos(\phi))^2.$$

Denote

$$g_1(x) := r^2(\varphi,\eta) + x^2 - 2xr(\varphi,\eta)\cos(\theta-\eta) + (\cos(\varphi)-\cos(\phi))^2,$$

and therefore we get the relation $g_1(sr(\phi,\theta)) = J_s(\phi,\theta,\varphi,\eta)$. From the variation arguments we infer that the function g_1 reaches its global minimum at the point

$$x_c = r(\varphi,\eta)\cos(\theta-\eta).$$

Let us distinguish the cases $\cos(\theta-\eta)\in[0,1]$ and $\cos(\theta-\eta)\in[-1,0]$. In the first case, one has according to (4.5.3)

$$\begin{aligned} J_s(\phi,\theta,\varphi,\eta) =& g_1(sr(\phi,\theta)) \\ \geq& g_1(x_c) = r^2(\varphi,\eta)\sin^2(\theta-\eta) + (\cos(\varphi)-\cos(\phi))^2 \\ \geq& C(\sin^2(\varphi)\sin^2(\theta-\eta) + (\cos(\varphi)-\cos(\phi))^2). \end{aligned}$$

Using that $\cos(\theta-\eta)\in[0,1]$, one gets

$$\sin^2(\theta-\eta) = 2\sin^2((\theta-\eta)/2)(1+\cos(\theta-\eta)) \geq 2\sin^2((\theta-\eta)/2).$$

Moreover, since $\phi\in[0,\frac{\pi}{2}]$ and $\varphi\in[0,\pi]$, we obtain

$$\begin{aligned} |\cos(\varphi)-\cos(\phi)| =& |(1-\cos(\phi)) - (1-\cos(\varphi))| \\ =& 2|\sin^2(\phi/2) - \sin^2(\varphi/2)| \\ =& 2|\sin(\phi/2)-\sin(\varphi/2)|(\sin(\phi/2)+\sin(\varphi/2)) \\ \geq& C|\phi-\varphi||\phi+\varphi|. \end{aligned} \tag{4.5.4}$$

Hence

$$J_s(\phi,\theta,\varphi,\eta) \geq C\Big(\sin^2(\varphi)\sin^2((\theta-\eta)/2) + (\phi+\varphi)^2(\phi-\varphi)^2\Big). \tag{4.5.5}$$

In the second case where $\cos(\theta-\eta)\in[-1,0]$, the critical point is negative, $x_c \leq 0$, and one has from the variations of g_1, the estimate (4.5.3) and (4.5.4)

$$\begin{aligned} J_s(\phi,\theta,\varphi,\eta) =& g_1(sr(\phi,\theta)) \\ \geq& g_1(0) = r(\varphi,\eta)^2 + (\cos(\varphi)-\cos(\phi))^2 \\ \geq& C(\sin^2(\varphi)\sin^2((\theta-\eta)/2) + (\phi+\varphi)^2(\phi-\varphi)^2). \end{aligned} \tag{4.5.6}$$

Putting together (4.5.5) and (4.5.6), one deduces that

$$J_s(\phi,\theta,\varphi,\eta) \geq C\Big(\sin^2(\varphi)\sin^2((\theta-\eta)/2) + (\phi+\varphi)^2(\phi-\varphi)^2\Big), \tag{4.5.7}$$

for any $\phi\in[0,\pi/2]$, $\varphi\in[0,\pi]$ and $\theta,\eta\in[0,2\pi]$.

Following the same ideas, we introduce the function

$$g_2(x) := x^2 + s^2r^2(\phi,\theta) - 2sxr(\phi,\theta)\cos(\theta-\eta) + (\cos(\varphi)-\cos(\phi))^2,$$

which satisfies $g_2(r(\varphi,\eta)) = J_s(\phi,\theta,\varphi,\eta)$. Then as before we can check easily that the function g_2 reaches its minimum at the point $\tilde{x}_c = sr(\phi,\theta)\cos(\theta-\eta)$. Similarly we distinguishing between two cases $\cos(\theta-\eta)\in[0,1]$ and $\cos(\theta-\eta)\in[-1,0]$. For the first case, using (4.5.4), we have

$$J_s(\phi,\theta,\varphi,\eta) \geq C(s^2\sin^2(\phi)\sin^2((\theta-\eta)/2) + (\phi+\varphi)^2(\phi-\varphi)^2).$$

Since $\phi\in[0,\pi/2]$, we have that $\sin(\phi)\geq\frac{2}{\pi}\phi$, and then

$$J_s(\phi,\theta,\varphi,\eta) \geq C(s^2\phi^2\sin^2((\theta-\eta)/2) + (\phi+\varphi)^2(\phi-\varphi)^2). \tag{4.5.8}$$

By summing up (4.5.7)–(4.5.8) we achieve (4.5.2). □

4.5.3 Regularity persistence

In this section we shall investigate the regularity of the function $\tilde{F}$ introduced in (4.2.13). The main result reads as follows.

Proposition 4.5.4. *Let $m \geq 2, \alpha \in (0,1)$ and r_0 satisfy $(\mathbf{H})$. There exists $\varepsilon \in (0,1)$ small enough such that the functional*

$$\tilde{F} : \mathbb{R} \times B_{X_m^\alpha}(\varepsilon) \to X_m^\alpha$$

is well-defined and of class $\mathscr{C}^1$. The function spaces X_m^α are defined in (4.2.15) *and* (4.2.16).

Proof. First we shall split the functional $\tilde{F}$ into two pieces

$$\tilde{F}(\Omega, f)(\phi,\theta) = F_1(f)(\phi,\theta) - \frac{\Omega}{2}F_2(f)(\phi,\theta) - \frac{1}{2\pi}\int_0^{2\pi}\Big[F_1(f)(\phi,\theta) - \frac{\Omega}{2}F_2(f)(\phi,\theta)\Big]\, d\theta,$$

with

$$\begin{aligned} F_1(f)(\phi,\theta) =& \frac{I(f)(\phi,\theta)}{r_0(\phi)}, \\ F_2(f)(\phi,\theta) =& 2f(\phi,\theta) + \frac{f^2(\phi,\theta)}{r_0(\phi)}\cdot \end{aligned}$$

Note that $I(f)$ is defined (4.2.14) and it is nothing but the stream function ψ_0 associated to the domain parametrized by

$$(\phi,\theta) \in [0,\pi]\times[0,2\pi] \mapsto \Big(\big(r_0(\phi) + f(\phi,\theta)\big)e^{i\theta}, \cos\phi\Big)\cdot$$

Thus

$$F_1(f)(\phi,\theta) = \frac{\psi_0\left(\big(r_0(\phi) + f(\phi,\theta)\big)e^{i\theta}, \cos\phi\right)}{r_0(\phi)}\cdot \tag{4.5.9}$$

We point out that according to the general potential theory the steam function ψ_0 belongs at least to the space $\mathscr{C}^{1,\alpha}(\mathbb{R}^3)$. The proof will be divided into three steps.

Step 1: $f \mapsto F_2(f)$ is $\mathscr{C}^1$. In this step, we check that F_2 is well-defined and of class $\mathscr{C}^1$. The first term is trivial to check. As to the second one, it is clear by Taylor formula using the boundary conditions and $(\mathbf{H2})$ that the function $\frac{f^2}{r_0}$ is bounded and vanishes at the points $\phi = 0, \pi$. For the regularity, we differentiate with respect to ϕ,

$$\partial_\phi\left(\frac{f^2(\phi,\theta)}{r_0(\phi)}\right) = -r_0'(\phi)\left(\frac{f(\phi,\theta)}{r_0(\phi)}\right)^2 + 2\frac{f(\phi,\theta)}{r_0(\phi)}\partial_\phi f(\phi,\theta).$$

Using again Taylor formula and the assumptions $(\mathbf{H})$ on r_0 we deduce that the functions $(\phi,\theta) \mapsto \frac{f(\phi,\theta)}{\sin\phi}$ and $(\phi,\theta) \mapsto \frac{\sin\phi}{r_0(\phi)}$ belongs to $\mathscr{C}^\alpha$. Thus using the algebra structure of this latter space we infer that $(\phi,\theta) \mapsto \frac{f(\phi,\theta)}{r_0(\phi)}$ belongs also to $\mathscr{C}^\alpha$. The same algebra structure allows to get $\partial_\phi\left(\frac{f^2}{r_0}\right) \in \mathscr{C}^\alpha$. Following the same argument we obtain $\partial_\theta F_2$ belongs to $\mathscr{C}^\alpha$. Concerning the symmetry; it can be derived from Proposition 4.5.1 combined with the fact that frequency $n = 0$ is eliminated in the definition of $\tilde{F}$ by subtracting the mean value in θ.

Now let us check the $\mathscr{C}^1$ dependence in f of F_1. First we can check that its Frechet derivative takes the form

$$\partial_f F_2(f)h(\phi,\theta) = 2h(\phi,\theta) + 2\frac{f(\phi,\theta)h(\phi,\theta)}{r_0(\phi)}\cdot$$

Using similar ideas as before, we can easily get that

$$\|\partial_f F_2(f_1)h - \partial_f F_2(f_2)h\|_{X_m^\alpha} \le C\|f_1 - f_2\|_{X_m^\alpha}\|h\|_{X_m^\alpha}.$$

This implies that $f \mapsto \partial_f F_2(f)$ is continuous and therefore F_2 is of class $\mathscr{C}^1$.

Step 2: $f \mapsto F_1(f)$ *is well-defined*. This is more involved than F_2. According to Proposition 4.5.1 the functional F_1 is symmetric with respect to $\phi = \frac{\pi}{2}$ and therefore it suffices to check the desired regularity in the range $\phi \in (0, \pi/2)$. Let us emphasize that we need to check the regularity not for F_1 but for its fluctuation with respect to the mean rate, that is,

$$\mathscr{F}_1 : (\phi,\theta) \mapsto F_1(f)(\phi,\theta) - \langle F_1(f)\rangle_\theta \quad \text{with} \quad \langle F_1(f)\rangle_\theta := \frac{1}{2\pi}\int_0^{2\pi} F_1(f)(\phi,\theta)d\theta.$$

First, we shall check that $\mathscr{F}_1$ is bounded and satisfies the boundary condition $\mathscr{F}_1(0,\theta) = 0$, for any $\theta \in (0, 2\pi)$. The remaining boundary condition $\mathscr{F}_1(\pi,\theta) = 0$ follows from the symmetry with respect to the equatorial. For this purpose, we write by virtue of Taylor formula

$$\forall x_h \in \mathbb{R}^2, \quad \psi_0(x_h, \cos\phi) = \psi_0(0,0,\cos\phi) + x_h \cdot \int_0^1 \nabla_h\psi_0\big(\tau x_h, \cos\phi\big)d\tau. \tag{4.5.10}$$

Making the substitution $x_h = (r_0(\phi) + f(\phi,\theta))e^{i\theta}$ and using (4.5.9) we infer

$$\begin{aligned} F_1(f)(\phi,\theta) =& \frac{\psi(0,0,\cos\phi)}{r_0(\phi)} + \left(1 + \frac{f(\phi,\theta)}{r_0(\phi)}\right)e^{i\theta} \cdot \int_0^1 \nabla_h\psi_0\left(\tau(r_0(\phi) + f(\phi,\theta))e^{i\theta}, \cos\phi\right)d\tau \\ =:& \frac{\psi(0,0,\cos\phi)}{r_0(\phi)} + \mathscr{F}_{1,1}(\phi,\theta). \end{aligned}$$

We observe that the $\cdot$ denotes the usual Euclidean inner product of $\mathbb{R}^2$. Consequently, we obtain

$$\mathscr{F}_1(\phi,\theta) = \mathscr{F}_{1,1}(\phi,\theta) - \langle\mathscr{F}_{1,1}\rangle_\theta. \tag{4.5.11}$$

Let us analyze the term $\mathscr{F}_{1,1}$ and check its continuity and the Dirichlet boundary condition. First we observe from the assumption (**H2**) that 0 is a simple zero for r_0 and we know that $f(0,\theta) = 0$, then one may easily obtain the bound

$$|\mathscr{F}_{1,1}(\phi,\theta)| \le C(1 + \|\partial_\phi f\|_{L^\infty})\|\nabla_h\psi_0\|_{L^\infty(\mathbb{R}^3)}.$$

Furthermore, according to Lebesgue dominated convergence theorem we infer

$$\lim_{\phi\to 0}\mathscr{F}_{1,1}(\phi,\theta) = \left(1 + \frac{\partial_\phi f(0,\theta)}{r_0'(0)}\right)e^{i\theta} \cdot \nabla_h\psi_0\,(0,0,1)\,,$$

and this convergence is uniform in $\theta \in (0, 2\pi)$. Notice that the same tool gives the continuity of $\mathscr{F}_{1,1}$ in $[0;\pi/2] \times [0;2\pi]$.

Now, applying Lemma 4.5.2 we get $\nabla_h\psi_0\,(0,0,1) = 0$, and therefore

$$\forall\,\theta \in (0,2\pi), \quad \lim_{\phi\to 0}\mathscr{F}_{1,1}(\phi,\theta) = \lim_{\phi\to 0}\langle\mathscr{F}_{1,1}\rangle_\theta = 0.$$

This implies that $\mathscr{F}_1$ is continuous in $[0,\pi] \times [0,2\pi]$ and it satisfies the required Dirichlet boundary condition $\mathscr{F}_1(0,\theta) = \mathscr{F}_1(\pi,\theta) = 0$.

The next step is to establish that $\partial_\theta \mathscr{F}_1$ and $\partial_\phi \mathscr{F}_1$ are $\mathscr{C}^\alpha$. We will relate such derivatives to the two-components velocity field $U = \nabla_h^\perp \psi_0$. Differentiating (4.5.9) with respect to θ leads to

$$\begin{aligned}
\partial_\theta \mathscr{F}_1(\phi,\theta) =& \partial_\theta F_1(f)(\phi,\theta)\\
=& r_0^{-1}(\phi)\, \nabla_h \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)) \cdot \Big(r(\phi,\theta) i e^{i\theta} + \partial_\theta r(\phi,\theta) e^{i\theta}\Big)\\
=& -\frac{r(\phi,\theta)}{r_0(\phi)} U(r(\phi,\theta)e^{i\theta},\cos(\phi)) \cdot e^{i\theta} + \partial_\theta r(\phi,\theta) \frac{U(r(\phi,\theta)e^{i\theta},\cos(\phi))}{r_0(\phi)} \cdot i e^{i\theta},
\end{aligned} \tag{4.5.12}$$

where $r(\phi,\theta) = r_0(\phi) + f(\phi,\theta)$ and recall that $\cdot$ is the usual Euclidean inner product of $\mathbb{R}^2$.

Concerning the regularity of the partial derivative in ϕ, we achieve

$$\begin{aligned}
\partial_\phi F_1(f)(\phi,\theta) =& -\frac{r_0'(\phi)}{r_0^2(\phi)} \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)) + \frac{\partial_\phi r(\phi,\theta)}{r_0(\phi)} \nabla_h \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)) \cdot e^{i\theta}\\
& - \frac{\sin(\phi)}{r_0(\phi)} \partial_z \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))\\
=& -\frac{r_0'(\phi)}{r_0^2(\phi)} \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)) - \partial_\phi r(\phi,\theta) \frac{U(r(\phi,\theta)e^{i\theta},\cos(\phi))}{r_0(\phi)} \cdot i e^{i\theta}\\
& - \frac{\sin(\phi)}{r_0(\phi)} \partial_z \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)).
\end{aligned} \tag{4.5.13}$$

Define

$$\mathscr{J}_1(\phi,\theta) = \frac{r_0'(\phi)}{r_0^2(\phi)} \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)),$$

and

$$\mathscr{J}_2(\phi,\theta) = \partial_\phi r(\phi,\theta) \frac{U(r(\phi,\theta)e^{i\theta},\cos(\phi))}{r_0(\phi)} \cdot i e^{i\theta}.$$

Then from (4.5.13) we may write

$$\partial_\phi F_1(f)(\phi,\theta) = -\mathscr{J}_1(\phi,\theta) - \mathscr{J}_2(\phi,\theta) - \frac{\sin(\phi)}{r_0(\phi)} \partial_z \psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)).$$

Note that the last term belongs to $\mathscr{C}^\alpha$. Indeed, as $(\phi,\theta) \mapsto \big(r(\phi,\theta)e^{i\theta},\cos(\phi)\big)$ belongs to $\mathscr{C}^{1,\alpha}$ and $\partial_z \psi_0 \in \mathscr{C}^\alpha(\mathbb{R}^3)$ then by composition we infer $(\phi,\theta) \mapsto \partial_z \psi_0\big(r(\phi,\theta)e^{i\theta},\cos(\phi)\big)$ is in $\mathscr{C}^\alpha\big((0,\pi)\times(0,2\pi)\big)$. On the other hand, the function $\frac{\sin}{r_0}$ belongs to $\mathscr{C}^\alpha$ and thus by the algebra structure of $\mathscr{C}^\alpha$ we obtain the announced result.

Concerning the term $\mathscr{J}_1$, we use Taylor formula for the stream function ψ_0 as in (4.5.10) finding that

$$\begin{aligned}
\mathscr{J}_1(\phi,\theta) =& \frac{r_0'(\phi)\psi_0(0,0,\cos\phi)}{r_0^2(\phi)} + r_0^{-1}(\phi)\left(1 + \frac{f(\phi,\theta)}{r_0(\phi)}\right) \int_0^1 \nabla_h \psi_0\big(s\, r(\phi,\theta)e^{i\theta},\cos\phi\big) ds \cdot e^{i\theta}\\
=& \frac{r_0'(\phi)\psi_0(0,0,\cos\phi)}{r_0^2(\phi)} - \left(1 + \frac{f(\phi,\theta)}{r_0(\phi)}\right) r_0^{-1}(\phi) \int_0^1 U\big(s\, r(\phi,\theta)e^{i\theta},\cos\phi\big) ds \cdot i e^{i\theta}.
\end{aligned}$$

We observe that the first term is singular and depends only in ϕ and therefore it does not contribute in $\mathscr{J}_1 - \langle \mathscr{J}_1 \rangle_\theta$. Since $(\phi,\theta) \mapsto \frac{r(\phi,\theta)}{r_0(\phi)}$ belongs to $\mathscr{C}^\alpha$ then to get $\mathscr{J}_1 - \langle \mathscr{J}_1 \rangle_\theta \in \mathscr{C}^\alpha$ it suffices to prove that

$$(\phi,\theta) \mapsto \int_0^1 \frac{U(sr(\phi,\theta)e^{i\theta},\cos(\phi))}{r_0(\phi)} \cdot i e^{i\theta} ds \in \mathscr{C}^\alpha. \tag{4.5.14}$$

On the other hand to obtain $\mathscr{J}_2 \in \mathscr{C}^\alpha$ it is enough to get

$$(\phi,\theta) \mapsto \frac{U(r(\phi,\theta)r^{i\theta},\cos(\phi))}{r_0(\phi)} \cdot i\, e^{i\theta} \in \mathscr{C}^\alpha. \tag{4.5.15}$$

From (4.5.12) we get that $\partial_\theta F_1(f) \in \mathscr{C}^\alpha$ provided that (4.5.14) and (4.5.15) are satisfied together with

$$(\phi,\theta) \mapsto U(r(\phi,\theta)r^{i\theta},\cos(\phi)) \cdot e^{i\theta} \in \mathscr{C}^\alpha. \tag{4.5.16}$$

By virtue of (4.2.10) and the fact that $U = \nabla_h^\perp \psi$, we find that

$$U(r(\phi,\theta)e^{i,\theta},\cos(\phi)) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)\big(\partial_\eta r(\varphi,\eta)e^{i\eta} + ir(\varphi,\eta)e^{i\eta}\big)\,d\eta d\varphi}{|(r(\phi,\theta)e^{i\theta},\cos(\phi)) - (r(\varphi,\eta)e^{i\eta},\cos(\varphi))|}. \tag{4.5.17}$$

Next we intend to prove (4.5.14), (4.5.15) and (4.5.16).

• *Proof of* (4.5.16). Using (4.5.17), we deduce that

$$U(r(\phi,\theta)e^{i,\theta},\cos(\phi)) \cdot e^{i\theta} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\cos(\eta-\theta)\big)\,d\eta d\varphi}{|(r(\phi,\theta)e^{i\theta},\cos(\phi)) - (r(\varphi,\eta)e^{i\eta},\cos(\varphi))|}.$$

Using the notation of Lemma (4.5.3) we find that

$$|(r(\phi,\theta)e^{i\theta},\cos(\phi)) - (r(\varphi,\eta)e^{i\eta},\cos(\varphi))| = J_1^{\frac{1}{2}}(\phi,\theta,\varphi,\eta),$$

and therefore we may write

$$U(r(\phi,\theta)e^{i,\theta},\cos(\phi)) \cdot e^{i\theta} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\cos(\eta-\theta)\big)\,d\eta d\varphi}{J_1^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)}.$$

This can be split into two integral terms

$$\begin{aligned} U(r(\phi,\theta)e^{i,\theta},\cos(\phi)) \cdot e^{i\theta} &= \int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)\partial_\eta r(\varphi,\eta)\cos(\eta-\theta)d\eta d\varphi}{J_1^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)} \\ &\quad - \int_0^\pi\int_0^{2\pi} \frac{\sin(\varphi)r(\varphi,\eta)\sin(\eta-\theta)d\eta d\varphi}{J_1^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)} \\ &:= \mathcal{I}_1(\phi,\theta) - \mathcal{I}_2(\phi,\theta). \end{aligned} \tag{4.5.18}$$

Next, we shall prove that $\mathcal{I}_1$ is $\mathscr{C}^\alpha$. Notice that the second term $\mathcal{I}_2$ is more easier to deal with than $\mathcal{I}_1$ because its kernel is more regular on the diagonal than that of $\mathcal{I}_1$. To get $\mathcal{I}_2 \in \mathscr{C}^\alpha$ it suffices to use in a standard way Proposition B.0.2. We shall skip this part and focus our attention to the proof to the delicate part $\mathcal{I}_1$. For this aim let us define the kernel

$$\mathscr{K}_1(\phi,\theta,\varphi,\eta) = \frac{\sin(\varphi)\partial_\eta r(\varphi,\eta)\cos(\eta-\theta)}{J_1^{\frac{1}{2}}(\phi,\theta,\varphi,\eta)}.$$

We shall start with checking that $\mathscr{K}_1$ is bounded. For this goal we use Lemma 4.5.3 which implies

$$|\mathscr{K}_1(\phi,\theta,\varphi,\eta)| \leq \frac{C\,\sin(\varphi)|\partial_\eta r(\varphi,\eta)|}{\{(\varphi+\phi)^2(\phi-\varphi)^2 + (\sin^2(\varphi)+\phi^2)\sin^2((\theta-\eta)/2)\}^{\frac{1}{2}}}.$$

It is easy to check the inequality

$$\frac{\sin(\varphi)}{\Big((\varphi+\phi)^2(\phi-\varphi)^2+(\sin^2(\varphi)+\phi^2)\sin^2((\theta-\eta)/2)\Big)^{\frac12}}\leq\frac{1}{\Big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\Big)^{\frac12}}. \tag{4.5.19}$$

By interpolating between

$$\Big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\Big)^{\frac12}\geq|\phi-\varphi|,$$

and

$$\Big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\Big)^{\frac12}\geq|\sin((\theta-\eta)/2)|,$$

one finds that for any $\beta\in[0,1]$,

$$\frac{\sin(\varphi)}{\Big((\varphi+\phi)^2(\phi-\varphi)^2+(\sin^2(\varphi)+\phi^2)\sin^2((\theta-\eta)/2)\Big)^{\frac12}}\leq\frac{1}{|\phi-\varphi|^{1-\beta}|\sin((\theta-\eta)/2)|^{\beta}}, \tag{4.5.20}$$

implying that

$$|\mathscr{K}_1(\phi,\theta,\varphi,\eta)|\leq\frac{C}{|\phi-\varphi|^{1-\beta}|\sin((\theta-\eta)/2)|^{\beta}}. \tag{4.5.21}$$

Therefore, we easily achieve that $\mathcal{I}_1\in L^\infty$. To establish that $\mathcal{I}_1\in\mathscr{C}^\alpha$, we proceed in a direct way using the definition. Before that we remark that to get the $\mathscr{C}^\alpha$ regularity in both variables (ϕ,θ) it is enough to check the $\mathscr{C}^\alpha$-regularity separately in the partial variables. Thus we shall check that $\phi\mapsto\mathcal{I}_1(\phi,\theta)$ is $\mathscr{C}^\alpha(0,\pi)$ uniformly in $\theta\in[0,2\pi]$. The $\mathscr{C}^\alpha$-regularity of the mapping $\theta\mapsto\mathcal{I}_1(\phi,\theta)$ uniformly in $\phi\in[0,\pi]$ can be done in a similar way to the preceding one, and to alleviate the discussion we shall skip this part. Take $\phi_1,\phi_2\in[0,\frac{\pi}{2}]$ with $0<\phi_1<\phi_2$, then it is easy to check from some algebraic considerations that

$$\mathcal{I}_1(\phi_2,\theta)-\mathcal{I}_1(\phi_1,\theta)=\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)\partial_\eta r(\varphi,\eta)\cos(\eta-\theta)(J_1(\phi_1,\theta,\varphi,\eta)-J_1(\phi_2,\theta,\varphi,\eta))d\eta d\varphi}{J_1^{\frac12}(\phi_2,\theta,\varphi,\eta)J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)(J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)+J_1^{\frac12}(\phi_2,\theta,\varphi,\eta))}.$$

Coming back to the definition of J_1 seen in Lemma 4.5.3, we can check that

$$\begin{aligned}J_1(\phi_1,\theta,\varphi,\eta)-J_1(\phi_2,\theta,\varphi,\eta)=&\big(r(\phi_1,\theta)-r(\phi_2,\theta)\big)\big(r(\phi_1,\theta)-r(\varphi,\theta)+r(\phi_2,\theta)-r(\varphi,\eta)\big)\\&+2r(\varphi,\eta)\big(r(\phi_1,\theta)-r(\phi_2,\theta)\big)\big(1-\cos(\theta-\eta)\big)\\&-\big(\cos\varphi-\cos\phi_1+\cos\varphi-\cos\phi_2\big)\big(\cos\phi_1-\cos\phi_2\big).\end{aligned}$$

Since $r\in\text{Lip}$ we infer by interpolation

$$\big|r(\phi_1,\theta)-r(\phi_2,\theta)\big|\leq C|\phi_1-\phi_2|^\alpha\Big(|r(\phi_1,\theta)-r(\varphi,\eta)|^{1-\alpha}+|r(\phi_2,\theta)-r(\varphi,\eta)|^{1-\alpha}\Big),$$

and

$$\big|r(\phi_1,\theta)-r(\phi_2,\theta)\big|\leq C|\phi_1-\phi_2|^\alpha\Big(r^{1-\alpha}(\phi_1,\theta)|+r^{1-\alpha}(\phi_2,\theta)\Big).$$

Consequently we find

$$|J_1(\phi_1,\theta,\varphi,\eta)-J_1(\phi_2,\theta,\varphi,\eta)|\leq C|\phi_1-\phi_2|^\alpha\Big(|r(\phi_1,\theta)-r(\varphi,\eta)|^{2-\alpha}+|r(\phi_2,\theta)-r(\varphi,\eta)|^{2-\alpha}$$

$$+ r(\varphi,\eta)\big(r^{1-\alpha}(\phi_1,\theta)+r^{1-\alpha}(\phi_2,\theta)\big)\big(1-\cos(\eta-\theta)\big)$$
$$+|\cos(\varphi)-\cos(\phi_1)|^{2-\alpha}+|\cos(\varphi)-\cos(\phi_2)|^{2-\alpha}\Big).$$

From straightforward calculus we observe that

$$\frac{|r(\phi_i,\theta)-r(\varphi,\eta)|^{2-\alpha}+r(\varphi,\eta)r(\phi_i,\theta)^{1-\alpha}\big(1-\cos(\eta-\theta)\big)+|(\cos(\varphi)-\cos(\phi_i)|^{2-\alpha}}{J_1(\phi_i,\theta,\varphi,\eta)^{\frac{2-\alpha}{2}}}\leq C,$$

and then we find

$$|J_1(\phi_1,\theta,\varphi,\eta)-J_1(\phi_2,\theta,\varphi,\eta)|\leq C|\phi_1-\phi_2|^{\alpha}\Big(J_1^{\frac{2-\alpha}{2}}(\phi_1,\theta,\varphi,\eta)+J_1^{\frac{2-\alpha}{2}}(\phi_2,\theta,\varphi,\eta)\Big).$$

It follows that

$$\begin{aligned}&\frac{|J_1(\phi_1,\theta,\varphi,\eta)-J_1(\phi_2,\theta,\varphi,\eta)|}{J_1^{\frac12}(\phi_2,\theta,\varphi,\eta)J_1(\phi_1,\theta,\varphi,\eta)^{\frac12}(J_1(\phi_1,\theta,\varphi,\eta)^{\frac12}+J_1^{\frac12}(\phi_2,\theta,\varphi,\eta))}\\ &\lesssim\frac{|\phi_1-\phi_2|^{\alpha}}{J_1^{\frac{\alpha}{2}}(\phi_2,\theta,\varphi,\eta)J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)}+\frac{|\phi_1-\phi_2|^{\alpha}}{J_1^{\frac12}(\phi_2,\theta,\varphi,\eta)J_1^{\frac{\alpha}{2}}(\phi_1,\theta,\varphi,\eta)}.\end{aligned}\tag{4.5.22}$$

Using (4.5.22), one finds

$$\begin{aligned}|\mathcal{I}_1(\phi_2,\theta)-\mathcal{I}_1(\phi_1,\theta)|\lesssim&|\phi_1-\phi_2|^{\alpha}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)|\partial_\eta r(\varphi,\eta)|d\eta d\varphi}{J_1^{\frac{\alpha}{2}}(\phi_2,\theta,\varphi,\eta)J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)}\\ &+|\phi_1-\phi_2|^{\alpha}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)|\partial_\eta r(\varphi,\eta)|d\eta d\varphi}{J_1^{\frac12}(\phi_2,\theta,\varphi,\eta)J_1^{\frac{\alpha}{2}}(\phi_1,\theta,\varphi,\eta)}.\end{aligned}$$

By virtue of (4.5.20), for any $\beta\in(0,1)$ we obtain

$$\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)|\partial_\eta r(\varphi,\eta)|d\eta d\varphi}{J_1^{\frac{\alpha}{2}}(\phi_2,\theta,\varphi,\eta)J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)}\leq\int_0^{\pi}\int_0^{2\pi}\frac{|\partial_\eta r(\varphi,\eta)|\,J_1^{-\frac{\alpha}{2}}(\phi_2,\theta,\varphi,\eta)d\eta d\varphi}{|\phi_1-\varphi|^{1-\beta}|\sin((\theta-\eta)/2)|^{\beta}}.$$

Hence, we get in view of Lemma 4.5.3 and (4.2.17)

$$\begin{aligned}\frac{|\partial_\eta r(\varphi,\eta)|}{J_1(\phi_i,\theta,\varphi,\eta)^{\frac{\alpha}{2}}}\lesssim&\frac{\varphi^{\alpha}}{\big((\varphi+\phi_i)^2(\phi_i-\varphi)^2+(\sin^2(\varphi)+\phi_i^2)\sin^2((\theta-\eta)/2)\big)^{\frac{\alpha}{2}}}\\ \lesssim&|\phi_i-\varphi|^{-\alpha}.\end{aligned}$$

for any $i=1,2$. This implies that

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi}\frac{\sin(\varphi)|\partial_\eta r(\varphi,\eta)|d\eta d\varphi}{J_1^{\frac{\alpha}{2}}(\phi_2,\theta,\varphi,\eta)J_1^{\frac12}(\phi_1,\theta,\varphi,\eta)}\lesssim&\int_0^{\pi}\int_0^{2\pi}\frac{|\phi_2-\varphi|^{-\alpha}d\eta d\varphi}{|\phi_1-\varphi|^{1-\beta}|\sin((\theta-\eta)/2)|^{\beta}}\\ \lesssim&\int_0^{\pi}\frac{|\phi_2-\varphi|^{-\alpha}d\varphi}{|\phi_1-\varphi|^{1-\beta}}.\end{aligned}$$

It is classical that

$$\sup_{\phi_1,\phi_2\in(0,\pi)}\int_0^\pi \frac{|\phi_2-\varphi|^{-\alpha}}{|\phi_1-\varphi|^{1-\beta}}d\varphi<\infty,$$

provided that $0\le\alpha<\beta<1$. Similarly we prove under the same condition that

$$\sup_{\substack{\phi_1,\phi_2\in(0,\pi)\\ \theta\in(0,2\pi)}}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)|\partial_\eta r(\varphi,\eta)|d\eta d\varphi}{J_1^{\frac12}(\phi_2,\theta,\varphi,\eta)J_1^{\frac{\alpha}{2}}(\phi_1,\theta,\varphi,\eta)}<\infty.$$

Putting together the preceding estimates yields for any $0<\alpha<\beta<1$

$$\forall\theta\in(0,2\pi),\phi_1,\phi_2\in(0,\pi),\quad |\mathcal{I}_1(\phi_2,\theta)-\mathcal{I}_1(\phi_1,\theta)|\le C|\phi_1-\phi_2|^\alpha,$$

where the constant C is independent of θ,ϕ_1 and ϕ_2. Using similar ideas, we can also prove that

$$|\mathcal{I}_1(\phi,\theta_2)-\mathcal{I}_1(\phi,\theta_1)|\le C|\theta_1-\theta_2|^\alpha.$$

Finally, this allows to get that $\mathcal{I}_1$ is $\mathscr{C}^\alpha(0,\pi)\times(0,2\pi)$.

• *Proof of* (4.5.15). In fact we shall establish a more refined result:

$$(\phi,\theta)\mapsto\frac{U\big(sr(\phi,\theta)e^{i\theta},\cos(\phi)\big)}{r_0(\phi)}\cdot ie^{i\theta}\in\mathscr{C}^\alpha([0,\pi]\times[0,2\pi]).$$

uniformly with respect to $s\in[0,1]$. This allows to get the results (4.5.15) and (4.5.14).

Coming back to (4.5.17) and using J_s introduced in Lemma 4.5.3 we find the expression

$$\frac{U\big(sr(\phi,\theta)e^{i\theta},\cos(\phi)\big)}{r_0(\phi)}\cdot ie^{i\theta}=\frac{1}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta-\theta)\big)}{J_s^{\frac12}(\phi,\theta,\varphi,\eta)}d\eta d\varphi.\tag{4.5.23}$$

Moreover, by Lemma 4.5.2 we have

$$\forall\,\phi,\theta)\in(0,\pi)\times(0,2\pi),\qquad\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta-\theta)\big)d\eta d\varphi}{J_0(\phi,\theta,\varphi,\eta)^{\frac12}}=0,$$

and then we can subtract this vanishing term obtaining

$$\frac{U\big(sr(\phi,\theta)e^{i\theta},\cos(\phi)\big)}{r_0(\phi)}\cdot ie^{i\theta}=$$
$$-\frac{r(\phi,\theta)}{r_0(\phi)}\int_0^\pi\int_0^{2\pi}\frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta-\theta)\big)s\big(sr(\phi,\theta)-2r(\varphi,\eta)\cos(\theta-\eta)\big)}{J_s^{\frac12}(\phi,\theta,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)}d\eta d\varphi.$$

Notice that we eliminate .the variable θ from the definition of J_0 because it is independent of this parameter. Since $(\phi,\theta)\mapsto\frac{r(\phi,\theta)}{r_0(\phi)}$ is $\mathscr{C}^\alpha$, then to get the desired regularity it is enough to check it for the the integral term. Denote

$$\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)=\frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta-\theta)\big)s\big(sr(\phi,\theta)-2r(\varphi,\eta)\cos(\theta-\eta)\big)}{J_s^{\frac12}(\phi,\theta,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)},\tag{4.5.24}$$

and let us show first that

$$(\phi,\theta)\mapsto\int_0^\pi\int_0^{2\pi}\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)d\eta d\varphi,$$

belongs to L^∞. It is plain that

$$|\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)|\lesssim\frac{\sin(\varphi)\big(r(\varphi,\eta)+|\partial_\eta r(\varphi,\eta)||\sin(\eta-\theta)|\big)s\big(sr(\phi,\theta)+r(\varphi,\eta)\big)}{J_s^{\frac12}(\phi,\theta,\varphi,\eta)J_0(\phi,\varphi,\eta)}.$$

Combined with the estimate (4.5.1), it yields

$$|\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)|\lesssim\frac{(r(\varphi,\eta)+|\partial_\eta r(\varphi,\eta)||\sin(\eta-\theta)|)s(sr(\phi,\theta)+r(\varphi,\eta))}{\sin(\varphi)J_s^{\frac12}(\phi,\theta,\varphi,\eta)}.$$

As we have mentioned before at different stages, the symmetry allows us to restrict the discussion to the interval $\phi\in(0,\pi/2)$. Then using Lemma 4.5.3 and (4.2.17) we achieve that

$$\begin{aligned}\frac{sr(\phi,\theta)+r(\varphi,\eta)}{J_s^{\frac12}(\phi,\theta,\varphi,\eta)}&\lesssim\frac{s\phi+\sin\varphi}{\big\{(\varphi+\phi)^2(\phi-\varphi)^2+(\sin^2(\varphi)+s^2\phi^2)\sin^2((\theta-\eta)/2)\big\}^{\frac12}}\\&\lesssim\frac{1}{\big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big)^{\frac12}}.\end{aligned}\tag{4.5.25}$$

Hence, the estimate of (4.2.17) allows to get

$$\begin{aligned}|\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)|&\lesssim\frac{\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta-\eta)|}{\sin(\varphi)\big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big)^{\frac12}}\\&\lesssim\frac{1}{\big((\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big)^{\frac12}}+\frac{1}{\sin^{1-\alpha}(\varphi)}.\end{aligned}$$

By interpolation we deduce for any $\beta\in(0,1)$,

$$|\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)|\lesssim\frac{1}{|\phi-\varphi|^{1-\beta}|\sin((\theta-\eta)/2)|^\beta}+\sin^{\alpha-1}(\varphi).$$

It follows that

$$(\phi,\theta)\mapsto\int_0^\pi\int_0^{2\pi}\mathscr{K}_2(s,\phi,\theta,\varphi,\eta)d\eta d\varphi\in L^\infty\big((0,\pi)\times(0,2\pi)\big).$$

Let us move to the $\mathscr{C}^\alpha$-regularity of this latter function. This amounts to checking the partial regularity separately in ϕ and θ. The strategy is the same for both of them and to alleviate the discussion, we shall establish the regularity in the variable θ, contrary to the preceding section where it wass established for $\mathcal{I}_1$ in the direction of φ. The goal is to get a convenient estimate for the difference

$$\int_0^\pi\int_0^{2\pi}\big(\mathscr{K}_2(s,\phi,\theta_1,\varphi,\eta)-\mathscr{K}_2(s,\phi,\theta_2,\varphi,\eta)\big)d\eta d\varphi,$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$. Coming back to the definition of the kernel $\mathscr{K}_2$ in (4.5.24) one deduces through straightforward algebraic computations that

$$\mathscr{K}_2(s,\phi,\theta_1,\varphi,\eta) - \mathscr{K}_2(s,\phi,\theta_2,\varphi,\eta) = \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6.$$

with

$$\mathcal{I}_3 = \frac{\sin(\varphi)\partial_\eta\big[r(\varphi,\eta)\big(\sin(\eta-\theta_1)-\sin(\eta-\theta_2)\big)\big]s\big[sr(\phi,\theta_1)-2r(\varphi,\eta)\cos(\theta_1-\eta)\big]}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)},$$

$$\begin{aligned}\mathcal{I}_4 =&\frac{\sin(\varphi)\partial_\eta\big[r(\varphi,\eta)\sin(\eta-\theta_2)\big]s\big[sr(\phi,\theta_1)-2r(\varphi,\eta)\cos(\theta_1-\eta)\big]}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)}\\ &\times\frac{J_s(\phi,\theta_2,\varphi,\eta)-J_s(\phi,\theta_1,\varphi,\eta)}{\big[J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big]\big[J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)\big]},\end{aligned}$$

$$\mathcal{I}_5 = \frac{\sin(\varphi)\partial_\eta\big[r(\varphi,\eta)\sin(\eta-\theta_2)\big]s\big[sr(\phi,\theta_1)-sr(\phi,\theta_2)-2r(\varphi,\eta)\big(\cos(\theta_1-\eta)-\cos(\theta_2-\eta)\big)\big]}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)}$$

and

$$\begin{aligned}\mathcal{I}_6 =&\frac{\sin(\varphi)\partial_\eta\big(r(\varphi,\eta)\sin(\eta-\theta_2)\big)s\big[sr(\phi,\theta_2)-2r(\varphi,\eta)\cos(\theta_2-\eta)\big]}{J_s(\phi,\theta_1,\varphi,\eta)^{\frac12}J_0(\phi,\varphi,\eta)^{\frac12}(J_s(\phi,\theta_2,\varphi,\eta)^{\frac12}+J_0(\phi,\varphi,\eta)^{\frac12})}\\ &\times\frac{J_s(\phi,\theta_2,\varphi,\eta)-J_s(\phi,\theta_1,\varphi,\eta)}{J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big)}.\end{aligned}$$

We shall estimate independently each one of those terms. Concerning the term $\mathcal{I}_3$ it can be estimated using (4.5.1)

$$\begin{aligned}|\mathcal{I}_3| \lesssim&|\theta_1-\theta_2|\frac{\sin(\varphi)\big(r(\varphi,\eta)+|\partial_\eta r(\varphi,\eta)|\big)s\big(sr(\phi,\theta_1)+r(\varphi,\eta)\big)}{\sin^2(\varphi)J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)}\\ \lesssim&|\theta_1-\theta_2|\frac{\big(r(\varphi,\eta)+|\partial_\eta r(\varphi,\eta)|\big)s\big(sr(\phi,\theta_1)+r(\varphi,\eta)\big)}{\sin(\varphi)J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)}.\end{aligned}$$

Then by virtue of (4.5.25) and (4.2.17), we find

$$\begin{aligned}|\mathcal{I}_3| \lesssim&\frac{|\theta_1-\theta_2|\big(\sin(\varphi)+\sin^\alpha(\varphi)\big)}{\sin(\varphi)\,\big((\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big)^{\frac12}}\\ \lesssim&\frac{|\theta_1-\theta_2|\sin^{\alpha-1}(\varphi)}{\big((\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big)^{\frac12}}.\end{aligned}$$

Combining this estimate with the interpolation inequality: for any $\beta\in[0,1]$

$$\frac{1}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)^2)\big\}^{\frac12}} \le \frac{1}{|\varphi-\phi|^\beta|\sin((\eta-\theta_1)/2)|^{1-\beta}},$$

we infer

$$|\mathcal{I}_3| \lesssim\frac{|\theta_1-\theta_2|}{\sin^{1-\alpha}(\varphi)|\varphi-\phi|^\beta|\sin((\eta-\theta_1)/2)|^{1-\beta}}.$$

Thus

$$\int_0^\pi \int_0^{2\pi} |\mathcal{I}_3| d\eta d\varphi \lesssim \int_0^\pi \int_0^{2\pi} \frac{|\theta_1-\theta_2| d\eta d\varphi}{\sin^{1-\alpha}(\varphi)|\varphi-\phi|^\beta |\sin((\eta-\theta_1)/2)|^{1-\beta}}$$
$$\lesssim |\theta_1-\theta_2|,$$

uniformly in $\phi \in (0,\pi)$ and $\theta_1, \theta_2 \in (0, 2\pi)$, provided that $0 < \beta < \alpha \leq 1$.

Concerning the term $\mathcal{I}_4$, we first use the definition of J_s in Lemma 4.5.3 and one may check

$$\begin{aligned} |J_s(\phi,\theta_2,\varphi,\eta) - J_s(\phi,\theta_1,\varphi,\eta)| \leq & |sr(\phi,\theta_1) - sr(\phi,\theta_2)|(|sr(\phi,\theta_1) - r(\varphi,\eta)| + |sr(\phi,\theta_2) - r(\varphi,\eta)|) \\ & + 2(sr(\phi,\theta_1) - sr(\phi,\theta_2)) r(\varphi,\eta)(1-\cos(\theta_1-\eta)) \\ & + 2sr(\phi,\theta_2) r(\varphi,\eta) \big|\cos(\theta_2-\eta) - \cos(\theta_1-\eta)\big|. \end{aligned} \tag{4.5.26}$$

Using the trigonometric identity

$$1-\cos(\theta-\eta) = 2\sin^2((\theta-\eta)/2), \tag{4.5.27}$$

we get

$$\begin{aligned} |J_s(\phi,\theta_2,\varphi,\eta) - J_s(\phi,\theta_1,\varphi,\eta)| \lesssim & |sr(\phi,\theta_1) - sr(\phi,\theta_2)| \Big(J_s^{\frac{1}{2}}(\phi,\theta_2,\varphi,\eta) + J_s^{\frac{1}{2}}(\phi,\theta_1,\varphi,\eta) \Big) \\ & + 2sr(\phi,\theta_2) r(\varphi,\eta) \big|\cos(\theta_2-\eta) - \cos(\theta_1-\eta)\big|. \end{aligned} \tag{4.5.28}$$

From (4.2.17) combined with Taylor formula we find

$$\begin{aligned} |r(\phi,\theta_2) - r(\phi,\theta_1)| &\leq \left| \int_{\theta_1}^{\theta_2} \partial_\eta r(\phi,\eta) d\eta \right| \\ &\lesssim |\theta_2-\theta_1| \sin^\alpha(\phi). \end{aligned}$$

Therefore we get by interpolation inequality

$$\begin{aligned} |sr(\phi,\theta_1) - sr(\phi,\theta_2)| \lesssim |\theta_1-\theta_2|^\alpha s^\alpha \phi^{\alpha^2} \Big[& |sr(\phi,\theta_1) - r(\varphi,\eta)|^{1-\alpha} \\ & + |sr(\phi,\theta_2) - r(\varphi,\eta)|^{1-\alpha} \Big]. \end{aligned}$$

Hence

$$|sr(\phi,\theta_1) - sr(\phi,\theta_2)| \lesssim |\theta_1-\theta_2|^\alpha s^\alpha \phi^{\alpha^2} \Big[J_s^{\frac{1-\alpha}{2}}(\phi,\theta_1,\varphi,\eta) + J_s^{\frac{1-\alpha}{2}}(\phi,\theta_2,\varphi,\eta) \Big]. \tag{4.5.29}$$

Combining (4.5.29) together with (4.5.26) implies

$$\begin{aligned} |J_s(\phi,\theta_2,\varphi,\eta) - J_s(\phi,\theta_1,\varphi,\eta)| \lesssim & |\theta_1-\theta_2|^\alpha s^\alpha \phi^{\alpha^2} \Big(J_s^{1-\frac{\alpha}{2}}(\phi,\theta_2,\varphi,\eta) + J_s^{1-\frac{\alpha}{2}}(\phi,\theta_1,\varphi,\eta) \Big) \\ & + 2sr(\phi,\theta_2) r(\varphi,\eta) \big|\cos(\theta_2-\eta) - \cos(\theta_1-\eta)\big|. \end{aligned} \tag{4.5.30}$$

Using once again (4.5.27) we get successively

$$\xi := |\cos(\theta_2-\eta) - \cos(\theta_1-\eta)| \leq 2\Big(\sin^2((\theta_2-\eta)/2) + \sin^2((\theta_1-\eta)/2) \Big)$$

and

$$\xi = \sqrt{2} \Big| \sqrt{1-\cos(\theta_2-\eta)} - \sqrt{1-\cos(\theta_1-\eta)} \Big| \Big(|\sin((\theta_1-\eta)/2)| + |\sin((\theta_2-\eta)/2)| \Big)$$

$$\leq 2|\theta_1-\theta_2|\Big(|\sin((\theta_1-\eta)/2)|+|\sin((\theta_2-\eta)/2)|\Big),$$

where we have used the inequality

$$||\sin x|-|\sin y||\leq |x-y|.$$

Hence we deduce by interpolation

$$\xi\lesssim|\theta_1-\theta_2|^\alpha\Big(|\sin((\theta_1-\eta)/2)|^{2-\alpha}+|\sin((\theta_2-\eta)/2)|^{2-\alpha}\Big).\tag{4.5.31}$$

Using the assumption (4.2.17) and (4.5.2)we find

$$\begin{aligned}sr(\phi,\theta_2)r(\varphi,\eta)\xi&\lesssim s\phi\varphi|\theta_1-\theta_2|^\alpha\Big(|\sin((\theta_1-\eta)/2)|^{2-\alpha}+|\sin((\theta_2-\eta)/2)|^{2-\alpha}\Big)\\&\lesssim s^\alpha\phi^\alpha|\theta_1-\theta_2|^\alpha\Big[J_s^{\frac{1-\alpha}{2}}(\phi,\theta_1,\varphi,\eta)+J_s^{\frac{1-\alpha}{2}}(\phi,\theta_2,\varphi,\eta)\Big].\end{aligned}$$

Inserting this inequality into (4.5.30) implies

$$|J_s(\phi,\theta_2,\varphi,\eta)-J_s(\phi,\theta_1,\varphi,\eta)|\lesssim|\theta_1-\theta_2|^\alpha s^\alpha\phi^{\alpha^2}\Big(J_s^{1-\frac{\alpha}{2}}(\phi,\theta_2,\varphi,\eta)+J_s^{1-\frac{\alpha}{2}}(\phi,\theta_1,\varphi,\eta)\Big).$$

It follows that

$$\frac{|J_s(\phi,\theta_2,\varphi,\eta)-J_s(\phi,\theta_1,\varphi,\eta)|}{J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)}\lesssim|\theta_1-\theta_2|^\alpha s^\alpha\phi^{\alpha^2}\Big(J_s^{\frac{1-\alpha}{2}}(\phi,\theta_2,\varphi,\eta)+J_s^{\frac{1-\alpha}{2}}(\phi,\theta_1,\varphi,\eta)\Big).$$

Thus we get

$$\begin{aligned}|\mathcal{I}_4|&\lesssim|\theta_1-\theta_2|^\alpha\frac{\big(r(\varphi,\eta)+\partial_\eta r(\varphi,\eta)|\sin(\theta_2-\eta)|\big)s(sr(\phi,\theta_1)+r(\varphi,\eta))s^\alpha\phi^{\alpha^2}}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big)^\alpha\big(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)\big)}\\&+|\theta_1-\theta_2|^\alpha\frac{(r(\varphi,\eta)+\partial_\eta r(\varphi,\eta)|\sin(\theta_2-\eta)|)s(sr(\phi,\theta_1)+r(\varphi,\eta))s^\alpha\phi^{\alpha^2}}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big)\big(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)\big)^\alpha}.\end{aligned}$$

Applying (4.5.25) combined with (4.2.17) we arrive at

$$\begin{aligned}|\mathcal{I}_4|\lesssim&\frac{|\theta_1-\theta_2|^\alpha\big(\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|\big)ss^\alpha\phi^{\alpha^2}}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta))^\alpha(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta))}\\&+\frac{|\theta_1-\theta_2|^\alpha\big(\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|\big)ss^\alpha\phi^{\alpha^2}}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_1,\varphi,\eta))(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta))^\alpha}\\\lesssim&\quad\mathcal{J}_1+\mathcal{J}_2.\end{aligned}\tag{4.5.32}$$

The right hand side terms $\mathcal{J}_1$ and $\mathcal{J}_2$ are treated similarly and we shall only focus on the first one. Using (4.5.2) allows to get

$$\begin{aligned}|\mathcal{J}_1|\lesssim&\frac{|\theta_1-\theta_2|^\alpha}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}\sin^\alpha(\varphi)}\\&+\frac{|\theta_1-\theta_2|^\alpha|\sin(\theta_2-\eta)|\,s^\alpha\phi^{\alpha^2}}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}(\sin(\varphi)+J_s^{\frac12}(\phi,\theta_2,\varphi,\eta))}.\end{aligned}$$

Using (4.5.2) we deduce, since $\alpha \in (0,1)$, that

$$\begin{aligned}|\sin(\theta_2-\eta)|\, s^{\alpha}\phi^{\alpha^2} &\le |\sin(\theta_2-\eta)|^{\alpha^2} s^{\alpha^2}\phi^{\alpha^2}\\ &\lesssim J_s^{\frac{\alpha^2}{2}}(\phi,\theta_2,\varphi,\eta).\end{aligned} \tag{4.5.33}$$

Thus

$$\begin{aligned}|\mathcal{J}_1| \lesssim & \frac{|\theta_1-\theta_2|^{\alpha}}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}\sin^{\alpha}(\varphi)}\\ &+\frac{|\theta_1-\theta_2|^{\alpha}}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}\sin^{1-\alpha^2}(\varphi)}.\end{aligned}$$

The same estimate holds true for $\mathcal{J}_1$. Therefore we find

$$|\mathcal{I}_4| \lesssim |\theta_1-\theta_2|^{\alpha}\frac{\sin^{-\alpha}(\varphi)+\sin^{\alpha^2-1}(\varphi)}{\big\{(\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big\}^{\frac12}}.$$

Hence by interpolation inequality we get for all $\gamma \in (0,1)$

$$|\mathcal{I}_4| \lesssim |\theta_1-\theta_2|^{\alpha}\frac{\sin^{-\alpha}(\varphi)+\sin^{\alpha^2-1}(\varphi)}{|\phi-\varphi|^{\gamma}|\sin((\theta_1-\eta)/2)|^{1-\gamma}}.$$

It follows that

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi}|\mathcal{I}_4|d\eta d\varphi \lesssim & |\theta_1-\theta_2|\int_0^{\pi}\int_0^{2\pi}\frac{\sin^{-\alpha}(\varphi)+\sin^{\alpha^2-1}(\varphi)}{|\phi-\varphi|^{\gamma}|\sin((\theta_1-\eta)/2)|^{1-\gamma}}d\varphi d\eta\\ \lesssim & |\theta_1-\theta_2|,\end{aligned}$$

uniformly in $\phi \in (0,\pi)$ and $\theta_1,\theta_2 \in (0,2\pi)$, provided that $0<\gamma<\min(1-\alpha,\alpha^2)$.

As to the treatment of $\mathcal{I}_6$, it is quite similar to $\mathcal{I}_4$. Thus we shall omit the details for this term and focus on the estimate of the term $\mathcal{I}_5$. First we make the decomposition

$$\mathcal{I}_5 = \mathcal{J}_3 + \mathcal{J}_4,$$

with

$$\mathcal{J}_3 = \frac{\sin(\varphi)\partial_\eta\big[r(\varphi,\eta)\sin(\eta-\theta_2)\big]s\big[sr(\phi,\theta_1)-sr(\phi,\theta_2))\big]}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)},$$

and

$$\mathcal{J}_4 = -2\frac{\sin(\varphi)\partial_\eta\big[r(\varphi,\eta)\sin(\eta-\theta_2)\big]sr(\varphi,\eta)\big[\cos(\theta_1-\eta)-\cos(\theta_2-\eta)\big]}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)J_0^{\frac12}(\phi,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+J_0^{\frac12}(\phi,\varphi,\eta)\big)}.$$

Let us start with the last term $\mathcal{J}_4$. Using Lemma 4.5.3 combined with (4.5.31) and (4.2.17) yields

$$|\mathcal{J}_4| \lesssim |\theta_1-\theta_2|^{\alpha}\frac{\big[\sin\varphi+\sin^{\alpha}(\varphi)|\sin\big((\eta-\theta_2)/2\big)|\big]\sin\varphi}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)}.$$

From (4.5.2) we infer

$$|\mathcal{J}_4| \lesssim |\theta_1-\theta_2|^{\alpha}\frac{\sin^2(\varphi)+\sin^{\alpha}(\varphi)J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)|}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)}$$

$$\lesssim |\theta_1-\theta_2|^\alpha \frac{\sin^\alpha(\varphi)}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)}.$$

Applying (4.5.25) leads to

$$\begin{aligned}|\mathcal{J}_4| \lesssim& |\theta_1-\theta_2|^\alpha \frac{\varphi^{\alpha-1}}{\big((\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big)^{\frac12}}\\ \lesssim& |\theta_1-\theta_2|^\alpha \frac{\varphi^{\alpha-1}}{|\phi-\varphi|^\beta|\sin((\theta_1-\eta)/2)|^{1-\beta}}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\pi\int_0^{2\pi}|\mathcal{J}_4|d\eta d\varphi \lesssim& |\theta_1-\theta_2|^\alpha\int_0^\pi\int_0^{2\pi}\frac{\varphi^{\alpha-1}d\eta d\varphi}{|\phi-\varphi|^\beta|\sin((\theta_1-\eta)/2)|^{1-\beta}}\\ \lesssim& |\theta_1-\theta_2|^\alpha,\end{aligned}$$

uniformly in $\phi\in(0,2\pi)$ and $\theta_1,\theta_2\in(0,2\pi)$ provided that $0<\beta<\alpha<1$. Next we shall deal with the term $\mathcal{J}_5$. Then by virtue of (4.5.29) combined with (4.2.17) we may write

$$\begin{aligned}|\mathcal{J}_3| \lesssim& |\theta_1-\theta_2|^\alpha \frac{\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|}{J_s(\phi,\theta_1,\varphi,\eta)^{\frac12}(J_s(\phi,\theta_2,\varphi,\eta)^{\frac12}+\sin\varphi)}\\ &\times(s^{1+\alpha}\phi^{\alpha^2}\Big[J_s^{\frac{1-\alpha}{2}}(\phi,\theta_1,\varphi,\eta)+J_s^{\frac{1-\alpha}{2}}(\phi,\theta_2,\varphi,\eta)\Big].\end{aligned}$$

It follows that

$$\begin{aligned}|\mathcal{J}_3| \lesssim& |\theta_1-\theta_2|^\alpha \frac{\big(\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|\big)s^{1+\alpha}\phi^{\alpha^2}}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)^\alpha}\\ &+|\theta_1-\theta_2|^\alpha \frac{\big(\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|\big)s^{1+\alpha}\phi^{\alpha^2}}{J_s^{\frac{\alpha}{2}}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)}\\ \lesssim& |\theta_1-\theta_2|^\alpha\mathcal{J}_{3,1}+|\theta_1-\theta_2|^\alpha\mathcal{J}_{3,2}.\end{aligned}$$

According to (4.5.33) and since $\alpha\in(0,1)$ we find that

$$\begin{aligned}|\mathcal{J}_{3,1}| \lesssim& \frac{\sin(\varphi)+\sin^\alpha(\varphi)|\sin(\theta_2-\eta)|^{\alpha^2}s^{\alpha^2}\phi^{\alpha^2}}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)^\alpha}\\ \lesssim& \frac{\sin(\varphi)+\sin^\alpha(\varphi)J_s^{\frac{\alpha^2}{2}}(\phi,\theta_2,\varphi,\eta)}{J_s^{\frac12}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)^\alpha}\end{aligned}$$

and similarly we obtain

$$|\mathcal{J}_{3,2}| \lesssim \frac{\sin(\varphi)+\sin^\alpha(\varphi)J_s^{\frac{\alpha^2}{2}}(\phi,\theta_2,\varphi,\eta)}{J_s^{\frac{\alpha}{2}}(\phi,\theta_1,\varphi,\eta)\big(J_s^{\frac12}(\phi,\theta_2,\varphi,\eta)+\sin\varphi\big)}.$$

Consequently, we get from (4.5.25)

$$|\mathcal{J}_{3,1}| \lesssim \frac{\sin^{-\alpha}(\varphi)+\sin^{\alpha^2-1}(\varphi)}{\big((\phi-\varphi)^2+\sin^2((\theta_1-\eta)/2)\big)^{\frac12}}$$

$$\lesssim \frac{\sin^{-\alpha}(\varphi) + \sin^{\alpha^2-1}(\varphi)}{|\phi - \varphi|^{\gamma} |\sin((\theta_1 - \eta)/2)|^{1-\gamma}}.$$

By integration, we obtain

$$\int_0^{\pi}\int_0^{2\pi} |\mathcal{J}_{3,1}| d\eta d\varphi \lesssim \int_0^{\pi}\int_0^{2\pi} \frac{\sin^{-\alpha}(\varphi) + \sin^{\alpha^2-1}(\varphi)}{|\phi - \varphi|^{\gamma} |\sin((\theta_1 - \eta)/2)|^{1-\gamma}} d\eta d\varphi$$
$$\lesssim 1,$$

uniformly in $\phi \in (0, 2\pi)$ and $\theta_1 \in (0, 2\pi)$ provided that $0 < \gamma < \min(\alpha^2, 1 - \alpha)$.

Following the same ideas as before and using (4.5.25) we get

$$|\mathcal{J}_{3,2}| \lesssim \left(\frac{\sin\varphi}{J_s^{\frac{1}{2}}(\phi, \theta_1, \varphi, \eta)} \right)^{\alpha} \left(\sin^{-\alpha}(\varphi) + \big(J_s^{\frac{1}{2}}(\phi, \theta_2, \varphi, \eta) + \sin\varphi\big)^{\alpha^2-1} \right)$$
$$\lesssim \frac{\sin^{-\alpha}(\varphi) + \sin^{\alpha^2-1}(\varphi)}{\big((\phi - \varphi)^2 + \sin^2((\theta_1 - \eta)/2)\big)^{\frac{\alpha}{2}}}.$$

It follows that

$$|\mathcal{J}_{3,2}| \lesssim \frac{\sin^{-\alpha}(\varphi) + \sin^{\alpha^2-1}(\varphi)}{|\sin((\theta_1 - \eta)/2)|^{\alpha}}.$$

By integration, we deduce that

$$\int_0^{\pi}\int_0^{2\pi} |\mathcal{J}_{3,2}| d\eta d\varphi \lesssim \int_0^{\pi}\int_0^{2\pi} \frac{\sin^{-\alpha}(\varphi) + \sin^{\alpha^2-1}(\varphi)}{|\sin((\theta_1 - \eta)/2)|^{\alpha}} d\eta d\varphi.$$
$$\lesssim 1,$$

uniformly in $\phi \in (0, 2\pi)$ and $\theta_1 \in (0, 2\pi)$ provided that $0 < \alpha < 1$. Putting together the preceding estimates allows to get

$$\int_0^{\pi}\int_0^{2\pi} |\mathcal{I}_5| d\eta d\varphi \lesssim |\theta_1 - \theta_2|^{\alpha}.$$

Therefore, we obtain

$$\left| \int_0^{\pi}\int_0^{2\pi} (\mathscr{K}_2(s, \phi, \theta_1, \varphi, \eta) - \mathscr{K}_2(s, \phi, \theta_2, \varphi, \eta)) d\eta d\varphi \right| \le C|\theta_1 - \theta_2|^{\alpha},$$

uniformly in $\phi \in (0, \pi)$. This concludes the proof of the stability of the function spaces by $\tilde{F}$.

Step 3: F_1 *is* $\mathscr{C}^1$. In this last step, we check that F_1 is $\mathscr{C}^1$.

More precisely, we intend to prove the following

$$\|\partial_f F_1(f_1)h - \partial_f F_1(f_2)h\|_{L^\infty} \lesssim \|h\|_{\mathscr{C}^{1,\alpha}} \|f_1 - f_2\|_{\mathscr{C}^{1,\alpha}}^{\gamma}, \tag{4.5.34}$$
$$\|\partial_\theta\big(\partial_f F_1(f_1)h - \partial_f F_1(f_2)h\big)\|_{\mathscr{C}^{\alpha}} \lesssim \|h\|_{\mathscr{C}^{1,\alpha}} \|f_1 - f_2\|_{\mathscr{C}^{1,\alpha}}^{\gamma}, \tag{4.5.35}$$

and

$$\|\partial_\phi\big(\partial_f F_1(f_1)h - \partial_f F_1(f_2)h\big)\|_{\mathscr{C}^{\alpha}} \lesssim \|h\|_{\mathscr{C}^{1,\alpha}} \|f_1 - f_2\|_{\mathscr{C}^{1,\alpha}}^{\gamma}, \tag{4.5.36}$$

for some $\gamma > 0$, and where $f_1, f_1 \in X_m^\alpha$ and are small enough.

Notice that this is more stronger than the $\mathscr{C}^1$-regularity. Denote $r_i(\phi,\theta) = r_0(\phi)+f_i(\phi,\theta)$, for $i = 1, 2$. We will check directly the estimates for the derivatives, i.e., (4.5.35)–(4.5.36) and leave the first estimate which is less delicate. From the expressions (4.5.12) and (4.5.13), it is enough to check the estimates for the terms: $U \cdot e^{i\theta}$ and $\frac{U}{r_0} \cdot ie^{i\theta}$. As we can guess the computations are very long, tedious and share lot of similarities. For this reason we shall focus only on one significant term given by (4.5.18) to illustrate how the estimates work, and restrict the discussion to the part $\mathcal{I}_2$. One has

$$\begin{aligned}\partial_f \mathcal{I}_2(f)h(\phi,\theta) =& \int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)h(\varphi,\eta)\sin(\eta-\theta)d\eta d\varphi}{J_1^{\frac{1}{2}}(f)(\phi,\theta,\varphi,\eta)}\\ &- \frac{1}{2}\int_0^\pi \int_0^{2\pi} \frac{\sin(\varphi)r(\varphi,\eta)\sin(\eta-\theta)}{J_1^{\frac{3}{2}}(f)(\phi,\theta,\varphi,\eta)}\partial_f J_1(f)h(\phi,\theta,\varphi,\eta)d\eta d\varphi\\ =:& \mathcal{T}_1(f)h(\phi,\theta) - \mathcal{T}_2(f)h(\phi,\theta),\end{aligned}$$

where

$$|(r(\phi,\theta)e^{i\theta}, \cos(\phi)) - (r(\varphi,\eta)e^{i\eta}, \cos(\varphi))|^2 = J_1(f)(\phi,\theta,\varphi,\eta), \quad r = r_0 + f,$$

and

$$\begin{aligned}\frac{1}{2}\partial_f J_1(f)h(\phi,\theta,\varphi,\eta) =& (r(\varphi,\eta) - r(\phi,\theta))(h(\varphi,\eta) - h(\phi,\theta))\\ &+ (r(\varphi,\eta)h(\phi,\theta) + h(\varphi,\eta)r(\phi,\theta))(1-\cos(\eta-\theta)).\end{aligned}$$

We will analyze only the first term $\mathcal{T}_1$ to exhibit the main ideas. The goal is to check

$$\|\mathcal{T}_1(f_1)h - \mathcal{T}_1(f_2)h\|_{\mathscr{C}^\alpha} \lesssim \|h\|_{\mathscr{C}^{1,\alpha}} \|f_1 - f_2\|_{\mathscr{C}^{1,\alpha}}^\gamma,$$

for some $\gamma > 0$. We observe that

$$\begin{aligned}(J_1(f_1) - J_1(f_2))(\phi,\theta,\varphi,\eta) =& (r_1(\varphi,\eta) - r_1(\phi,\theta))^2 - (r_2(\varphi,\eta) - r_2(\phi,\theta))^2\\ &+ 2(r_1 - r_2)(\phi,\theta)r_1(\varphi,\eta)(1-\cos(\theta-\eta))\\ &+ 2r_2(\phi,\theta)(r_1 - r_2)(\varphi,\eta)(1-\cos(\theta-\eta)).\end{aligned} \tag{4.5.37}$$

Now we write for any $\phi, \varphi \in (0,\pi)$ and $\theta, \eta \in (0, 2\pi)$

$$|r(\phi,\theta) - r(\varphi,\eta)| \leq |r(\phi,\theta) - r(\varphi,\theta)| + |r(\varphi,\theta) - r(\varphi,\eta)|,$$

By the $\mathscr{C}^{1,\alpha}$ regularity of r one has

$$|r(\phi,\theta) - r(\varphi,\theta)| \lesssim |\phi - \varphi|\|r\|_{\text{Lip}}.$$

In addition, we claim that

$$|r(\varphi,\theta) - r(\varphi,\eta)| \lesssim |\sin((\theta-\eta)/2)|\|r\|_{\text{Lip}}.$$

Indeed, and without any restriction to the generality we can impose that $0 \leq \eta \leq \theta \leq 2\pi$. We shall discuss two cases: $0 \leq \theta - \eta \leq \pi$ and $\pi \leq \theta - \eta \leq 2\pi$. In the first case, we simply write

$$\frac{|r(\varphi,\theta) - r(\varphi,\eta)|}{|\sin((\theta-\eta)/2)|} = \frac{|r(\varphi,\theta) - r(\varphi,\eta)|}{|\theta-\eta|}\frac{|\theta-\eta|}{|\sin((\theta-\eta)/2)|} \leq C\|r\|_{\text{Lip}},$$

with C a constant. As to the second case $\pi \le \theta - \eta \le 2\pi$, by setting $\widehat{\eta} = \eta + 2\pi$ we get

$$\widehat{\eta} - \theta \in [0, \pi], \quad \sin((\theta - \eta)/2) = -\sin((\theta - \widehat{\eta})/2).$$

Since $\eta \mapsto r(\varphi, \eta)$ is 2π-periodic then using the result of the first case yields

$$\begin{aligned}\frac{|r(\varphi,\theta) - r(\varphi,\eta)|}{|\sin((\theta-\eta)/2)|} &= \frac{|r(\varphi,\theta) - r(\varphi,\widehat{\eta})|}{|\sin((\theta-\widehat{\eta})/2)|} \\ &\le C\|r\|_{\text{Lip}}.\end{aligned}$$

This achieves the proof of the claim. Consequently we find

$$|r(\phi,\theta) - r(\varphi,\eta)| \lesssim \|r\|_{\text{Lip}}\big(|\phi - \varphi| + |\sin((\theta-\eta)/2)|\big). \tag{4.5.38}$$

From algebraic calculus we easily get

$$\begin{aligned}|(r_1(\varphi,\eta) - r_1(\phi,\theta))^2 - (r_2(\varphi,\eta) - r_2(\phi,\theta))^2| =& \big|((r_1 - r_2)(\varphi,\eta) - (r_1 - r_2)(\phi,\theta)\big| \\ &\times \big|((r_1 + r_2)(\varphi,\eta) - (r_1 + r_2)(\phi,\theta))\big|.\end{aligned}$$

Therefore we deduce successively from (4.5.38)

$$\begin{aligned}|(r_1(\varphi,\eta) - r_1(\phi,\theta))^2 - (r_2(\varphi,\eta) - r_2(\phi,\theta))^2| \lesssim& \|r_1 - r_2\|_{\text{Lip}}\big(|\phi - \varphi| + |\sin((\theta-\eta)/2)|\big) \\ &\times \Big(|r_1(\varphi,\eta) - r_1(\phi,\theta)| + |r_2(\varphi,\eta) - r_2(\phi,\theta)|\Big),\end{aligned}$$

and

$$|(r_1(\varphi,\eta) - r_1(\phi,\theta))^2 - (r_2(\varphi,\eta) - r_2(\phi,\theta))^2| \lesssim |r_1(\varphi,\eta) - r_1(\phi,\theta)|^2 + |r_2(\varphi,\eta) - r_2(\phi,\theta)|^2.$$

By interpolation, we infer for any $\gamma \in [0,1]$,

$$\begin{aligned}\Big|(r_1(\varphi,\eta) - r_1(\phi,\theta))^2 - (r_2(\varphi,\eta) - r_2(\phi,\theta))^2\Big| \lesssim& \|r_1 - r_2\|_{\text{Lip}}^{\gamma}\Big(|\varphi - \phi|^{\gamma} + |\sin((\theta-\eta)/2)|^{\gamma}|\Big) \\ &\times\Big(|r_1(\varphi,\eta) - r_1(\phi,\theta)|^{2-\gamma} + |r_2(\varphi,\eta) - r_2(\phi,\theta)|^{2-\gamma}\Big),\end{aligned} \tag{4.5.39}$$

On the other hand, coming back to the definition of J_1 we get

$$J_1(f_1)(\phi,\theta,\varphi,\eta) \ge |r_1(\phi,\theta) - r_1(\varphi,\eta)|^2.$$

Thus, putting together this inequality with (4.5.39) and (4.5.37) yield

$$\begin{aligned}\frac{|(J_1(f_1) - J_1(f_2))(\phi,\theta,\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi,\theta,\varphi,\eta) + J_1^{\frac12}(f_2)(\phi,\theta,\varphi,\eta)} \lesssim& \|r_1 - r_2\|_{\mathscr{C}^1}^{\gamma}\Big\{\big(|\varphi - \phi|^{\gamma} + |\sin((\theta-\eta)/2)|^{\gamma}\big) \\ &\times\big[J_1^{\frac{1-\gamma}{2}}(f_1)(\phi,\theta,\varphi,\eta) + J_1^{\frac{1-\gamma}{2}}(f_2)(\phi,\theta,\varphi,\eta)\big] + \phi|\sin((\theta-\eta)/2)|\Big\}.\end{aligned} \tag{4.5.40}$$

Now, we shall give an estimate of $\mathcal{T}_1(f_1) - \mathcal{T}_1(f_2)$ in L^∞. For this purpose, define the quantity

$$\mathscr{K}_3(f)(\phi,\theta,\varphi,\eta) = \frac{\sin(\varphi)h(\varphi,\eta)\sin(\eta-\theta)}{J_1^{\frac12}(f)(\phi,\theta,\varphi,\eta)},$$

then one can easily check that

$$\mathcal{I}_7(\phi,\theta,\varphi,\eta) := \mathscr{K}_3(f_1)(\phi,\theta,\varphi,\eta) - \mathscr{K}_3(f_2)(\phi,\theta,\varphi,\eta)$$

$$=\frac{\sin(\varphi)h(\varphi,\eta)\sin(\eta-\theta)}{J_1^{\frac12}(f_1)(\phi,\theta,\varphi,\eta)J_1^{\frac12}(f_2)(\phi,\theta,\varphi,\eta)}\frac{J_1(f_2)(\phi,\theta,\varphi,\eta)-J_1(f_1)(\phi,\theta,\varphi,\eta)}{J_1^{\frac12}(f_1)(\phi,\theta,\varphi,\eta)+J_1^{\frac12}(f_2)(\phi,\theta,\varphi,\eta)}. \tag{4.5.41}$$

From this definition, it follows that

$$\big(\mathcal{T}_1(f_1)-\mathcal{T}_1(f_2)\big)(\phi,\theta)=\int_0^\pi\int_0^{2\pi}\mathcal{I}_7(\phi,\theta,\varphi,\eta)d\varphi d\eta.$$

According to (4.5.2)

$$\begin{aligned}\phi|\sin((\theta-\eta)/2)|\lesssim&|\sin((\theta-\eta)/2)|^\gamma\phi^{1-\gamma}|\sin((\theta-\eta)/2)|^{1-\gamma}\\ \lesssim&|\sin((\theta-\eta)/2)|^\gamma J_1^{\frac{1-\gamma}{2}}(f_1)(\phi,\theta,\varphi,\eta).\end{aligned}$$

Combining this inequality with (4.5.40) and (4.5.41) leads to

$$\begin{aligned}|\mathcal{I}_7(\phi,\theta,\varphi,\eta)|\lesssim\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma&\frac{\sin(\varphi)|h(\varphi,\eta)|\big(|\varphi-\phi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma\big)}{J_1^{\frac12}(f_1)(\phi,\theta,\varphi,\eta)J_1^{\frac12}(f_2)(\phi,\theta,\varphi,\eta)}\\ &\times\Big(J_1^{\frac{1-\gamma}{2}}(f_1)(\phi,\theta,\varphi,\eta)+J_1^{\frac{1-\gamma}{2}}(f_2)(\phi,\theta,\varphi,\eta)\Big).\end{aligned} \tag{4.5.42}$$

Applying Lemma 4.5.3, we infer

$$\begin{aligned}|\mathcal{I}_7(\phi,\theta,\varphi,\eta)|\lesssim&\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma\frac{\sin(\varphi)|h(\varphi,\eta)|(|\varphi-\phi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma)}{J_1^{\frac12}(f_1)(\phi,\theta,\varphi,\eta)J_1^{\frac{\gamma}{2}}(f_2)(\phi,\theta,\varphi,\eta)}\\ &+\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma\frac{\sin(\varphi)|h(\varphi,\eta)|(|\varphi-\phi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma)}{J_1^{\frac{\gamma}{2}}(f_1)(\phi,\theta,\varphi,\eta)J_1^{\frac12}(f_2)(\phi,\theta,\varphi,\eta)}\\ \lesssim&\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma\frac{\sin(\varphi)|h(\varphi,\eta)|(|\varphi-\phi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma)}{\big\{(\varphi+\phi)^2(\phi-\varphi)^2+(\sin^2(\varphi)+\phi^2)\sin^2((\theta-\eta)/2)\big\}^{\frac{1+\gamma}{2}}}.\end{aligned}$$

Using the inequality $\varphi^2\geq\sin^2(\varphi)$ for any $\varphi\in\mathbb{R}$, one achieves

$$\begin{aligned}|\mathcal{I}_7(\phi,\theta,\varphi,\eta)|\lesssim&\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma\frac{\sin(\varphi)|h(\varphi,\eta)|(|\varphi-\phi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma)}{(\sin(\varphi)+\phi)^{1+\gamma}\big\{(\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big\}^{\frac{1+\gamma}{2}}}\\ \lesssim&\frac{|h(\varphi,\eta)|\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma}{\sin^\gamma(\varphi)\big\{(\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big\}^{\frac12}}.\end{aligned}$$

The boundary conditions $h(0,\eta)=h(\pi,\eta)=0$ allow to cancel the singularity and one gets

$$|\mathcal{I}_7(\phi,\theta,\varphi,\eta)|\lesssim\frac{\|h\|_{\mathscr{C}^{1,\alpha}}\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma}{\big\{(\phi-\varphi)^2+\sin^2((\theta-\eta)/2)\big\}^{\frac12}}.$$

Interpolating we find that for any $\beta\in(0,1)$,

$$|\mathcal{I}_7(\phi,\theta,\varphi,\eta)|\lesssim\frac{\|h\|_{\mathscr{C}^{1,\alpha}}\|r_1-r_2\|_{\mathscr{C}^{1,\alpha}}^\gamma}{|\phi-\varphi|^{1-\beta}|\sin((\eta-\theta)/2)|^\beta}.$$

Thus, we have that $\mathcal{I}_7$ is integrable in the variable (φ,η) uniformly in (ϕ,θ), and then

$$\|\mathcal{T}_1(f_1)h-\mathcal{T}_1(f_2)h\|_{L^\infty}\lesssim\|h\|_{\mathscr{C}^{1,\alpha}}\|f_1-f_2\|_{\mathscr{C}^{1,\alpha}}^\gamma.$$

The next purpose is establish the partial $\mathscr{C}^\alpha$-regularity in ϕ and the partial regularity in θ can be done similarly. We want to prove the following

$$|(\mathcal{T}_1(f_1)-\mathcal{T}_1(f_2))h(\phi_1,\theta)-(T_1(f_1)-T_1(f_2))h(\phi_2,\theta)|\lesssim\|h\|_{\mathscr{C}^{1,\alpha}}\|f_1-f_2\|^\gamma_{\mathscr{C}^{1,\alpha}}|\phi_1-\phi_2|^\alpha. \tag{4.5.43}$$

For this goal we need to study the kernel $|\mathcal{I}_7(\phi_1)-\mathcal{I}_7(\phi_2)|$. To alleviate the notation we simply denote $\mathcal{I}_7(\phi,\theta,\varphi,\eta)$ by $\mathcal{I}_7(\phi)$ and $J_1(f_i)(\phi_i,\theta,\varphi,\eta)$ by $J_1(f_i)(\phi_i)$. Adding and subtracting some appropriate terms, one finds

$$|\mathcal{I}_7(\phi_1)-\mathcal{I}_7(\phi_2)|\lesssim\mathcal{I}_8+\mathcal{I}_9+\mathcal{I}_{10}+\mathcal{I}_{11}+\mathcal{I}_{12}$$

with

$$\begin{aligned}\mathcal{I}_8=&\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi_1)J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\frac{|J_1(f_2)(\phi_1)-J_1(f_1)(\phi_1)|}{J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_1)}\\&\times\frac{|J_1(f_1)(\phi_1)-J_1(f_1)(\phi_2)|}{J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_1)(\phi_2)},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_9=&\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1(f_1)(\phi_2)^{\frac12}J_1(f_2)(\phi_1)^{\frac12}}\frac{|J_1(f_2)(\phi_1)-J_1(f_1)(\phi_1)|}{(J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_1))(J_1(f_1)(\phi_1)^{\frac12}+J_1(f_2)(\phi_2)^{\frac12})}\\&\times\frac{|J_1(f_2)(\phi_1)-J_1(f_2)(\phi_2)|}{J_1^{\frac12}(f_2)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2)},\end{aligned}$$

$$\mathcal{I}_{10}=\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\frac{|(J_1(f_2)-J_1(f_1))(\phi_1)-(J_1(f_2)-J_1(f_1))(\phi_2)|}{J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2)},$$

$$\begin{aligned}\mathcal{I}_{11}=&\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\frac{|J_1(f_2)(\phi_2)-J_1(f_1)(\phi_2)|}{(J_1^{\frac12}(f_1)(\phi_2)+J_1^{\frac12}(f_2)(\phi_2))(J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2))}\\&\times\frac{|J_1(f_1)(\phi_1)-J_1(f_1)(\phi_2)|}{J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_1)(\phi_2)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{12}=&\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)J_1^{\frac12}(f_2)(\phi_2)}\frac{|J_1(f_2)(\phi_2)-J_1(f_1)(\phi_2)|}{J_1^{\frac12}(f_1)(\phi_2)+J_1^{\frac12}(f_2)(\phi_2)}\\&\times\frac{|J_1(f_2)(\phi_1)-J_1(f_2)(\phi_2)|}{J_1^{\frac12}(f_2)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2)}.\end{aligned}$$

The estimate of those terms are quite similar and we shall restrict the discussion to the term $\mathcal{I}_{10}$ which involves more computations. The analysis is straightforward and we will just give the basic ideas. First one should give a suitable estimate for the quantity

$$|(J_1(f_2)-J_1(f_1))(\phi_1)-(J_1(f_2)-J_1(f_1))(\phi_2)|.$$

By using (4.5.37)–(4.5.39), one finds

$$\begin{aligned}
&|(J_1(f_2)-J_1(f_1))(\phi_1)-(J_1(f_2)-J_1(f_1))(\phi_2)|\\
\lesssim&|(r_1-r_2)(\phi_1,\theta)-(r_1-r_2)(\phi_2,\theta)||(r_1+r_2)(\phi_1,\theta)-(r_1+r_2)(\varphi,\eta)|\\
&+|(r_1-r_2)(\phi_2,\theta)-(r_1-r_2)(\varphi,\eta)||(r_1+r_2)(\phi_1,\theta)-(r_1+r_2)(\phi_2,\theta)|\\
&+|(r_1-r_2)(\phi_1,\theta)-(r_1-r_2)(\phi_2,\theta)|r_1(\varphi,\eta)\sin^2((\theta-\eta)/2)\\
&+|r_2(\phi_1,\theta)-r_2(\phi_2,\theta)||(r_1-r_2)(\varphi,\eta)|\sin^2((\theta-\eta)/2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|(r_1-r_2)(\phi_1,\theta)-(r_1-r_2)(\phi_2,\theta)|\lesssim\|r_1-r_2\|^\alpha|\phi_1-\phi_2|^\alpha\left(|r_1(\phi_1,\theta)-r_1(\varphi,\eta)|^{1-\alpha}\right.\\
\left.+|r_2(\phi_1,\theta)-r_2(\varphi,\eta)|^{1-\alpha}+|r_1(\phi_2,\theta)-r_1(\varphi,\eta)|^{1-\alpha}+|r_2(\phi_2,\theta)-r_2(\varphi,\eta)|^{1-\alpha}\right),
\end{aligned}$$

and

$$\begin{aligned}
|(r_1+r_2)(\phi_1,\theta)-(r_1+r_2)(\phi_2,\theta)|\lesssim|\phi_1-\phi_2|^\alpha(&|r_1(\phi_1,\theta)-r_1(\varphi,\eta)|^{1-\alpha}\\
&+|r_2(\phi_1,\theta)-r_2(\varphi,\eta)|^{1-\alpha}+|r_1(\phi_2,\theta)-r_1(\varphi,\eta)|^{1-\alpha}\\
&+|r_2(\phi_2,\theta)-r_2(\varphi,\eta)|^{1-\alpha}).
\end{aligned}$$

In a similar way, we deduce first by triangular inequality

$$|(r_1+r_2)(\phi_1,\theta)-(r_1+r_2)(\varphi,\eta)|\leq|r_1(\phi_1,\theta)-r_1(\varphi,\eta)|+|r_2(\phi_1,\theta)-r_1(\varphi,\eta)|$$

and second from (4.5.39)

$$\begin{aligned}
|(r_1-r_2)(\phi_2,\theta)-(r_1-r_2)(\varphi,\eta)|\lesssim&\|r_1-r_2\|^\gamma(|\phi_2-\varphi|^\gamma+|\sin((\theta-\eta)/2)|^\gamma)\\
&\times\left(|r_1(\phi_2,\theta)-r_1(\varphi,\eta)|^{1-\gamma}+|r_2(\phi_2,\theta)-r_2(\varphi,\eta)|^{1-\gamma}\right).
\end{aligned}$$

Combining the preceding estimate we achieve

$$\begin{aligned}
&|(J_1(f_2)-J_1(f_1))(\phi_1)-(J_1(f_2)-J_1(f_1))(\phi_2)|\\
\lesssim&|\phi_1-\phi_2|^\alpha\|f_1-f_2\|^\gamma\left(|r_1(\phi_1,\theta)-r_1(\varphi,\eta)|^{2-\alpha}+|r_2(\phi_1,\theta)-r_2(\varphi,\eta)|^{2-\alpha}\right.\\
&\left.+|r_1(\phi_2,\theta)-r_1(\varphi,\eta)|^{2-\alpha}+|r_2(\phi_2,\theta)-r_2(\varphi,\eta)|^{2-\alpha}\right)\\
\lesssim&|\phi_1-\phi_2|^\alpha\|f_1-f_2\|^\gamma(\mathscr{E}_{1,1}^{2-\alpha}+\mathscr{E}_{2,1}^{2-\alpha}+\mathscr{E}_{1,2}^{2-\alpha}+\mathscr{E}_{2,2}^{2-\alpha}).
\end{aligned}$$

where we use the notation

$$\mathscr{E}_{i,j}=|r_i(\phi_j,\theta)-r_i(\varphi,\eta)|;\ i,j\in\{1,2\}$$

Hence,

$$|\mathcal{I}_{10}|\lesssim|\phi_1-\phi_2|^\alpha\|f_1-f_2\|^\gamma\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\frac{\sum_{i,j=1}^2\mathscr{E}_{i,j}^{2-\alpha}}{J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2)}.$$

By using the definition of J_1 in Lemma 4.5.3, we immediately get

$$\mathscr{E}_{i,j}\leq J_1^{\frac12}(f_i)(\phi_j),$$

that we combine with (4.5.38) in order to get

$$\mathscr{E}_{i,j}\lesssim|\phi_j-\varphi|+|\sin((\theta-\eta)/2)|.$$

We shall analyze the term associated to $\mathscr{E}_{1,1}$ and the treatment of the other ones are quite similar. First we note

$$\frac{\mathscr{E}_{1,1}^{2-\alpha}}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)\big(J_1^{\frac12}(f_1)(\phi_1)+J_1^{\frac12}(f_2)(\phi_2)\big)}\lesssim\frac{|\phi_1-\varphi|^{1-\alpha}+|\sin((\theta-\eta)/2)|^{1-\alpha}}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}.$$

Making appeal to (4.5.25) and (4.2.17), we infer

$$\frac{\sin(\varphi)|h(\varphi)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\lesssim\frac{\|h\|_{\text{Lip}}}{\big\{(\phi_1-\varphi)^2+\sin^2((\theta-\eta)/2)\big\}^{\frac12}\big\{(\phi_2-\varphi)^2+\sin^2((\theta-\eta)/2)\big\}^{\frac12}}.$$

By interpolation we obtain for any $\gamma,\beta\in[0,1]$,

$$\frac{\sin(\varphi)|h(\varphi)|}{J_1^{\frac12}(f_1)(\phi_2)J_1^{\frac12}(f_2)(\phi_1)}\lesssim\|h\|_{\text{Lip}}\frac{|\phi_1-\varphi|^{-\gamma}|\phi_2-\varphi|^{-\beta}}{|\sin((\theta-\eta)/2)|^{2-\gamma-\beta}}.$$

Combining the preceding inequalities gives for any $\gamma_1,\gamma_2,\beta_1,\beta_2\in[0,1]$

$$\begin{aligned}\frac{\sin(\varphi)|h(\varphi,\eta)|}{J_1(f_1)(\phi_2)^{\frac12}J_1(f_2)(\phi_1)^{\frac12}}\frac{\mathscr{E}_{1,1}^{2-\alpha}}{J_1(f_1)(\phi_1)^{\frac12}+J_1(f_2)(\phi_2)^{\frac12}}\lesssim&\|h\|_{\text{Lip}}\frac{|\phi_1-\varphi|^{1-\alpha-\gamma_1}|\phi_2-\varphi|^{-\beta_1}}{|\sin((\theta-\eta)/2)|^{2-\gamma_1-\beta_1}}\\&+\|h\|_{\text{Lip}}\frac{|\phi_1-\varphi|^{-\gamma_2}|\phi_2-\varphi|^{-\beta_2}}{|\sin((\theta-\eta)/2)|^{1+\alpha-\gamma_2-\beta_2}}.\end{aligned}$$

The majorant functions are integrable in the variable (φ,η) uniformly in ϕ_1,ϕ_2,θ provided that

$$1<\gamma_1+\beta_1<2-\alpha\quad\text{and}\quad\alpha<\gamma_2+\beta_2<1,$$

and under these constraints one can find admissible parameters. Consequently,

$$\int_0^\pi\int_0^{2\pi}\mathcal{I}_{10}d\varphi d\eta\lesssim\|h\|_{\mathscr{C}^{1,\alpha}}\|f_1-f_2\|_{\mathscr{C}^{1,\alpha}}^{\gamma}|\phi_1-\phi_2|^{\alpha}.$$

This achieves the proof. ☐

4.6 Main result

In this section we shall provide a general statement that precise Theorem 4.1.1 and give its proof using all the previous results. Recall that the search of rotating solutions in the patch form to the equation (4.1.1), that is, solutions in the form

$$q(t,x)=q_0(e^{-i\Omega t}(x_1,x_2),x_3),\quad q_0=\mathbf{1}_D,$$

where D is a bounded simply–connected domain surrounded by a surface parametrized by

$$(\phi,\theta)\in[0,\pi]\times[0,2\pi]\mapsto((r_0(\phi)+f(\phi,\theta))e^{i\theta},\cos(\phi)),$$

reduces to solving the following infinite-dimensional equation

$$\tilde{F}(\Omega,f)=0$$

with f in a small neighborhood of the origin in the Banach space X_m^α and $\tilde{F}$ is introduced in (4.2.13). Notice that a solution is nontrivial means that the associated shape is not invariant by rotation along the vertical axis. Looking to the structure of the elements of space X_m^α one can easily see that a nonzero element guarantees a nontrivial shape. Our result stated below asserts that solutions to this functional equation do exist and are organized in a countable family of one-dimensional curves bifurcating from the trivial solution at the largest eigenvalues of the linearized operator at the origin. More precisely, we have the following.

Theorem 4.6.1. *Let $m \geq 2$ be a fixed integer and $r_0 : [0,\pi] \to \mathbb{R}$ satisfies the conditions:*

(H1) $r_0 \in \mathscr{C}^2([0,\pi])$, with $r_0(0) = r_0(\pi) = 0$ and $r_0(\phi) > 0$ for $\phi \in (0,\pi)$.

(H2) There exists $C > 0$ such that

$$\forall\, \phi \in [0,\pi], \quad C^{-1} \sin\phi \leq r_0(\phi) \leq C \sin(\phi).$$

(H3) r_0 is symmetric with respect to $\phi = \frac{\pi}{2}$, i.e., $r_0\left(\frac{\pi}{2} - \phi\right) = r_0\left(\frac{\pi}{2} + \phi\right)$, for any $\phi \in [0, \frac{\pi}{2}]$.

Then there exist $\delta > 0$ and two one–dimensonal $\mathscr{C}^1$-curves $s \in (-\delta,\delta) \mapsto f_m(s) \in X_m^\alpha$ and $s \in (-\delta,\delta) \mapsto \Omega_m(s) \in \mathbb{R}$, with

$$f_m(0) = 0, \quad f_m(s) \neq 0, \forall\, s \neq 0 \quad \textit{and} \quad \Omega_m(0) = \Omega_m,$$

where Ω_m is defined in Proposition 4.4.4, such that

$$\forall\, s \in (-\delta,\delta), \quad \tilde{F}\big(\Omega_m(s), f_m(s)\big) = 0.$$

Proof. The main material to prove this result is Crandall–Rabinowitz theorem, recalled in Theorem A.0.3. First the well–possednes and the regularity of $\tilde{F} : X_m^\alpha \to X_m^\alpha$ were discussed in Proposition 4.5.4. Thus it remains to check the suitable spectral properties of the linearized operator at the origin. The expression of this operator is detailed in Proposition 4.3.3 and it is a of Fredholm type of zero index according to Proposition 4.4.7. In addition for $\Omega = \Omega_m$ the kernel is a one-dimensional vector space. Finally, the transversal condition is satisfied by virtue of Proposition 4.4.8. ☐

4.6.1 Special case: sphere and ellipsoid

In this section we aim to show the particular case of bifurcating from spherical or ellipsoidal shapes. The main particularity of these shapes is that their associated stream function is well–known in the literature, see [94]. More specifically, let $\mathscr{E}$ be an ellipsoid inside the region

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

The associated stream function given by

$$\psi_0(x) = -\frac{1}{4\pi}\int_{\mathscr{E}} \frac{dA(y)}{|x-y|},$$

can be computed inside the ellipsoid as

$$\psi_0(x) = \frac{abc}{4}\int_0^\infty \left\{\frac{x_1^2}{a^2+s} + \frac{x_2^2}{b^2+s} + \frac{x_3^2}{c^2+s} - 1\right\} \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}.$$

In the case that $a = b$ we have that the ellipsoid is invariant under rotations about the z–axis and then it defines a stationary patch, see Lemma 4.2.1. Moreover and without loss of generality we can take $c = 1$. Note that in this case

$$\psi_0(x) = \alpha_1(a)(x_1^2 + x_2^3) + \alpha_2(a)x_3^2 + \alpha_3(a),$$

where

$$\alpha_1(a) := \frac{a^2}{4}\int_0^\infty \frac{ds}{(a^2+s)^2\sqrt{1+s}},$$

$$\alpha_2(a) := \frac{a^2}{4}\int_0^\infty \frac{ds}{(a^2+s)\sqrt{(1+s)^3}},$$

and

$$\alpha_3(a) := -\frac{a^2}{4}\int_0^\infty \frac{ds}{(a^2+s)^2\sqrt{1+s}}.$$

The sphere coincides with the case $a = 1$ having $\alpha_1(1) = \alpha_2(1) = \frac{1}{6}$ and $\alpha_3(1) = \frac{1}{2}$.
The above expression of the stream function together with Remark 4.3.2 gives us that

$$\int_0^\pi H_1(\phi,\varphi)d\varphi = 2\alpha_1(a),$$

for any $\phi \in [0,\pi]$. Recall that H_n is defined in (4.3.3). Now, by virtue of Proposition 4.3.3 one has

$$\partial_f \tilde{F}(\Omega,0)h(\phi,\theta) = \sum_{n\geq 1}\cos(n\theta)\mathcal{L}_n^\Omega(h_n)(\phi),$$

where

$$\mathcal{L}_n^\Omega(h_n)(\phi) = h_n(\phi)\left[2\alpha_1(a) - \Omega\right] - \int_0^\pi H_n(\phi,\varphi)h_n(\varphi)d\varphi, \quad \phi \in (0,\pi).$$

Moreover, the function ν_Ω used in the spectral study and defined in (4.4.3) agrees with

$$\nu_\Omega(\phi) = 2\alpha_1(a) - \Omega,$$

which now is constant on ϕ. Also the constant κ in (4.4.5) equals now to $2\alpha_1(a)$. Hence, the key point in Section 4.4.1 is the symmetrization of the above operator. For that reason, we have defined the signed measure $d\mu_\Omega$ as

$$d\mu_\Omega(\varphi) = \sin(\varphi)r_0^2(\varphi)\nu_\Omega(\varphi)d\varphi,$$

in (4.4.4) and the operator $\mathcal{K}_n^\Omega$ in (4.4.64). However, since in this case $\nu_\Omega(\varphi)$ is constant on φ, there is no need to introduce it in the measure with the goal of symmetryzing the operator. Following the ideas developed above, we deduce that the kernel study of the linearized operator agrees in this case with the following eigenvalue problem

$$\tilde{\mathcal{K}}_n(\phi) = (2\alpha_1(a) - \Omega)h(\phi).$$

Here, we define

$$\tilde{\mathcal{K}}_n(\phi) := (2\alpha_1(a) - \Omega)\mathcal{K}_n^\Omega(\phi),$$

which does not depend now on Ω by definition of $\mathcal{K}_n^{\Omega}$. Note that both operators have similar properties. Hence $\tilde{\mathcal{K}}_n$ sets the properties given in Proposition 4.4.3 taking the Lebesgue space $L^2_{\tilde{\mu}_\Omega}$ with

$$d\tilde{\mu}_\Omega(\varphi) = \sin(\varphi) r_0^2(\varphi) d\varphi.$$

Denote by $\beta_{n,i}$ the eigenvalues of $\tilde{\mathcal{K}}_n$ (for each n we have a family of eigenvalues). Then, we have necessary that

$$\Omega_n = 2\alpha_1(a) - \beta_{n,i}.$$

In Theorem 4.6.1, bifurcation occurs from $\Omega_n^\star$ given by

$$\Omega_n^\star = 2\alpha_1(a) - \beta_n^\star,$$

with

$$\beta_n^\star = \max_i \beta_{n,i}.$$

Moreover, we know that $\beta_n^\star$ is positive and then $\Omega_n^\star < 2\alpha_1(a)$. In particular, by Proposition 4.4.4, we have that $\Omega_n^\star$ tends to $\kappa = 2\alpha_1(a)$. Furthermore, $\Omega_n^\star$ increases in n and then we can bound it below by $\Omega_1^\star$. Using the equation for $\beta_1^\star$, that is

$$\int_0^\pi H_1(\varphi, \phi) h(\varphi) d\varphi = \beta_1^\star h(\phi),$$

one finds that $\beta_1^\star \leq 2\alpha_1(a)$ and then $\Omega_1^\star$ is positive. This implies that $\Omega_n^\star$ is positive for any n. Then, in Theorem 4.6.1 bifurcation holds at some $\Omega_n^\star \in (0, 2\alpha_1(a))$. Let us remark that in the case of the sphere, meaning $a = 1$, one has $2\alpha_1(a) = \frac{1}{3}$.

There is an interesting open problem concerning, first the spectral distribution of the eigenvalues $\beta_{n,i}$ (whether or not they are finite, simple or multiple), and second if bifurcation occurs at the eigenvalues $\Omega_n = 2\alpha_1(a) - \beta_{n,i}$ (which is shown to happen only for the largest eigenvalue $\beta_n^\star$). Notice that the simplicity and the monotonicity of the eigenvalues is a delicate problem and could be related to the geometry of the revolution shape. Finally we observe that since $\beta_{n,i} < \beta_n^\star$ then $\Omega_n = 2\alpha_1(a) - \beta_{n,i} > \frac{1}{3} - \beta_n^\star > 0$.

Chapter 5

Other works of the thesis and conclusions

5.1 Remarks on stationary solutions for the Euler equations

This section aims to show a work in progress about some remarks on stationary solutions for the 2D Euler equations that is motivated by the works described in Section 1.1.1.

As it is already presented in the previous chapters, the motion of an ideal inviscid incompressible fluid in two dimensions is described through the Euler equations

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \mathrm{div}(v) = 0, \\ v(0,\cdot) = v_0, \end{cases}$$

where v is the velocity field and p is the scalar pressure. It is sometimes worthy to work with the scalar vorticity ω defined as

$$\omega = \nabla^\perp \cdot v = \partial_1 v_2 - \partial_2 v_1.$$

Hence, applying the last operator to the above system, the vorticity–velocity formulation is obtained

$$\begin{cases} \partial_t \omega + v\cdot\nabla\omega = 0, \\ \omega = \nabla^\perp\cdot v, \\ \mathrm{div}(v) = 0, \\ \omega(0,\cdot) = \omega_0. \end{cases}$$

Since $\mathrm{div}(v) = 0$, then $v = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$, for some function ψ called the stream function. Thus, by the definition of ω, one arrives at

$$\Delta\psi = \omega. \tag{5.1.1}$$

In order to solve the Poisson equation, one has to assume some spacial decay for v, for instance, $|v|$ goes to 0 at infinity. Then, it yields the so called Biot-Savart law, which establishes that

$$v = K*\omega, \quad K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}.$$

In order to look for stationary solutions, we need to study the equations

$$v \cdot \nabla\omega = 0 \quad \text{and} \quad v = K * \omega,$$

or equivalently

$$\nabla^{\perp}\psi \cdot \nabla\Delta\psi = 0. \tag{5.1.2}$$

Note that if $\Delta\psi = F(\psi)$, for some scalar function F, the equation (5.1.2) is automatically satisfied, see for instance [105].

Here, we would like to provide an alternative way of obtaining stationary solutions. Our motivation is founded on the following observation. If the vorticity is radial $\omega(x) = f(|x|)$, then it is a solution of the Euler equations and it is stationary. Hence, we look for solutions taking the form $\omega(t,x) = f_1(t, R(x))$ and $\psi(t,x) = f_2(t, R(x))$, for some scalar function R. We find that if (5.1.1) is verified, then they define a stationary solution. That is described in the following lemma:

Lemma 5.1.1. *Let (ω, ψ) be a solution of Euler equation. If there exists $R : \mathbb{R}^2 \to \mathbb{R}$ smooth enough such that $\omega = \omega(t, R)$ and $\psi = \psi(t, R)$, then (ω, ψ) is stationary.*

Once we have that the solutions depending on a scalar function are always stationary, we work with the initial data checking that (5.1.1) is verified.

Theorem 5.1.2. *Let $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ smooth off the origin, and $R : \mathbb{R}^2 \to \mathbb{R}$ satisfying*

$$\Delta R + F_1(R)|\nabla R|^2 = F_2(R). \tag{5.1.3}$$

Then, $\omega_0 = e^{\int_1^R F_1(s)ds} F_2(R)$ and $\psi_0 = \int_{R_0}^R e^{\int_1^\tau F_1(s)ds} d\tau$ defines a stationary solution of the Euler equations.

The idea of the previous theorem in the following. By Lemma 5.1.1, we have that $v \cdot \nabla\omega = \nabla^{\perp}\psi \cdot \nabla\omega = 0$. For this reason, it remains to check the relation between ω and ψ, i.e., that $\Delta\psi = \omega$. Since $\psi = \psi(R)$ and $\omega = \omega(R)$, then we have that $\Delta\psi = \omega$ agrees with (5.1.3).

The well-posedness of (5.1.3) has been studied in bounded domains, see [7, 8, 125], where they use a change of variable in order to simplify the equation. Here, we take advantage of the same idea deriving some known equations and, when it is possible, come back to reinterpret the results in the framework of the Euler equations. More precisely, let us consider a general elliptic equation of the type

$$\Delta u + f_1(u)|\nabla u|^2 = f_2(u),$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. Let

$$G(t) = \int_1^t f_1(\tau)d\tau, \tag{5.1.4}$$

and $h(s)$ be defined as $\int_0^{h(s)} e^{G(t)}dt = s$, for all $s \in \mathbb{R}$. If $u = h \circ z$, then z verifies

$$\Delta z - \tilde{f}(z) = 0,$$

where $\tilde{f}(z) = f_2(h(z))e^{G(h(z))}$.

By using the previous change of variable, we can make explicit connections between the solutions of the 2D Euler equations with those of other PDEs systems and though this procedure obtaining particular solutions of the 2D Euler system. Indeed, we can relate it with Allen–Cahn, the stationary Schrödinger or Helmholtz equation, obtaining interesting solutions.

Let us explain here the relation with Allen–Cahn equation. Choosing the functions F_1 and F_2 as follow

$$\begin{aligned} F_1(R) =& \frac{2}{R}, \\ F_2(R) =& -\frac{R}{3}\left(1 - \frac{1}{9}R^6\right), \end{aligned}$$

we obtain the vorticity and the velocity in terms of R as

$$\begin{aligned} \omega(R(x)) =& \frac{R^3(x)}{3}\left(1 - \frac{1}{9}R^6(x)\right), \\ v(x_1, x_2) =& -R^2(x)\nabla^{\perp}R(x). \end{aligned}$$

Using the change of variables $R = \sqrt[3]{3S}$, we get that S verifies

$$\Delta S + S - S^3 = 0.$$

This is the Allen-Cahn equation that has been a main subject of study due to its relation to the De Giorgi conjecture.

Recently, in [61] the authors have proved the existence of solution of Allen-Cahn in dimension four with a finite number of compact connected component by means of the link with Helmholtz equation. The case of our interest, $n = 2$, is studied in [117]. In this case, it is important that there exists a family of explicit solutions

$$\tanh\left(\frac{a \cdot x + b}{\sqrt{2}}\right),$$

with a a unit vector in $\mathbb{R}^2$ and b in $\mathbb{R}$. Let us consider the particular case

$$S(x_1, x_2) = \tanh\left(\frac{x_1}{\sqrt{2}}\right).$$

This is an odd function, which has the value 0 when $x_1 = 0$ and it is invariant under translations in the x_2-axis. Hence, we can recover the vorticity and the velocity in terms of this explicit solution

$$\begin{aligned} \omega(x_1, x_2) =& \tanh\left(\frac{x_1}{\sqrt{2}}\right) - \tanh^3\left(\frac{x_1}{\sqrt{2}}\right), \\ v(x_1, x_2) =& \left(0, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tanh^2\left(\frac{x_1}{\sqrt{2}}\right)\right), \\ \psi(x_1, x_2) =& \tanh\left(\frac{x_1}{\sqrt{2}}\right). \end{aligned}$$

Notice that the above solution is in fact a shear flow. Moreover, it is clear that $|v| \to 0$, when $|x_1| \to +\infty$, achieving a stationary solution of Euler equations.

5.2 Vortex patches choreography

In this section, we explore another work that has been motivated by the desingularization of the Kármán Vortex Street done in Chapter 3. That is the desingularization of another configuration of point vortices that now are located at the vertices of a regular polygon.

Recall the Euler equations in the vorticity form that has been introduced in the previous chapters:

$$\begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ v = K * \omega, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ \omega(0,\cdot) = \omega_0, & \text{with } x \in \mathbb{R}^2, \end{cases} \tag{5.2.1}$$

where $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$.

Consider initially a regular polygon with N sides and N point vortices located at the vertices of the polygon. Considering that the polygon has its center at the origin and that there is a vertex in the horizontal axis, meaning $z_0(0) = l \in \mathbb{R}$, then the others vertices are described by

$$z_m(0) = e^{\frac{i2\pi m}{N}} z_0(0),$$

for any $m = 0, \ldots, N-1$. The evolution of such points is given by the classical N–vortex problem, see Chapter 3, that is,

$$\begin{aligned} z_m'(t) =& \frac{1}{2\pi} \sum_{m \neq k = 0}^{N-1} \frac{(z_m(t) - z_k(t))^\perp}{|z_m(t) - z_k(t)|^2}, \\ z_m(0) =& e^{\frac{i2\pi m}{N}} z_0(0), \end{aligned} \tag{5.2.2}$$

for any $m = 0, \ldots, N-1$. Hence, we can show that the evolution of such points is given through a rotation of constant angular velocity, and that is described in the following proposition.

Proposition 5.2.1. *Let $z_m(0) = e^{\frac{i2\pi m}{N}} z_0(0)$ and $z_0(0) = l \in \mathbb{R}$, for any $m = 0, \ldots, N-1$. Then, $z_m(t) = e^{i\Omega t} z_m(0)$, where*

$$\Omega = \frac{1}{2\pi l^2} \sum_{k=1}^{N-1} \frac{1}{1 - e^{\frac{i2\pi k}{N}}}, \tag{5.2.3}$$

for any $m = 0, \ldots, N-1$.

It is well–known in the literature that in the case of a vortex pair with strength 1 and separated by a distance d, one has that it rotates at angular velocity $\Omega = \frac{1}{\pi d^2}$, see [83] or Chapter 3. Note that it agrees with (5.2.3) by taking $N = 2$ and $d = 2l$. See also [100, 22] for the stability and numerical simulations of the polygon.

Motivated by Proposition 5.2.1 and also Chapter 3, our main task in this work is to find domains D_m^ε, for $m = 0, \ldots, N-1$, $N \geq 2$ and $\varepsilon > 0$, such that the initial data

$$\omega_{0,\varepsilon}(x) = \frac{1}{\pi\varepsilon^2} \sum_{m=0}^{N-1} \mathbf{1}_{D_m^\varepsilon}(x), \tag{5.2.4}$$

evolves as a rotation in the Euler equations. That is, there exists $\Omega \in \mathbb{R}$ and $\varepsilon > 0$ such that the evolution of (5.2.4) is given by

$$\omega(t,x) = \omega_{0,\varepsilon}(e^{-i\Omega t}x). \tag{5.2.5}$$

Indeed, we will consider that D_m^ε are located in the plane as the point vortices in Proposition 5.2.1, that is,

$$D_m^\varepsilon = e^{i\frac{2\pi m}{N}} D_0^\varepsilon, \tag{5.2.6}$$

for some bounded simply–connected D_0^ε. Note that assuming that $D_0^\varepsilon = \varepsilon\mathbb{D} + l$, for $l \in \mathbb{R}^\star$, one finds for $\varepsilon \to 0$ in (5.2.4) the point vortex distribution of Proposition 5.2.1:

$$\omega_{0,0}(x) = \sum_{m=0}^{N-1} \delta_{e^{i\frac{2\pi m}{N}} l}(x). \tag{5.2.7}$$

Now assume that the evolution of (5.2.4) is given by (5.2.5). In that case, the Euler equations agree with

$$\left(v_0(x) - \Omega x^\perp\right) \cdot n_{\partial D_m^\varepsilon}(x) = 0, \quad x \in \partial D_m^\varepsilon, \tag{5.2.8}$$

for any $m = 0, \dots N-1$. Here $n_{\partial D_m^\varepsilon}$ stands for a unit normal vector to ∂D_m^ε.

As it happens in Chapter 3, the problem reduces to find the roots of (5.2.8) and we need to overcome some difficulties in order to apply the infinite dimensional Implicit Function theorem. The main one is the persistence of the nonlinear function (5.2.8) in the function spaces where the linearized operator is an isomorphism. To tackle this problem, we need to fix Ω depending on ε and the domain D_0^ε in a suitable way. Then, after choosing the appropriate function spaces we get the following result.

Theorem 5.2.2. *Consider $l \in \mathbb{R}^\star$ and $N \geq 2$. Then, there exists $\varepsilon_0 > 0$ with the following property. For all $\varepsilon \in (0, \varepsilon_0)$, there is a simply–connected bounded domain D^ε, with center of masses l, such that*

$$\omega_0(x) = \frac{1}{\pi\varepsilon^2} \sum_{m=0}^{N-1} \mathbf{1}_{e^{\frac{i2\pi m}{N}} D^\varepsilon}(x),$$

defines a rotating solution of (5.2.1)*, with some constant angular velocity $\Omega(\varepsilon)$. Moreover, D^ε is at least $\mathscr{C}^1$.*

As for the desingularization of the Kármán Vortex Street in Chapter 3, here we are able to find this kind of structures also in other incompressible fluid models such as the generalized quasi–geostrophic equation.

5.3 Traveling waves in aggregation models

In this section we present another work of this dissertation that is under preparation. More specifically, it consists in the study of relative equilibria in biological models. This is a joint work with J. CAMPOS and my thesis advisor J. SOLER.

Chemotaxis refers to the motion of the species up or down a chemical concentration gradient. Examples of this biological process are the propagation of traveling bands of bacterial toward the oxygen [2, 3] or the outward propagation of concentric ring waves by the E. Coli [18, 19]. The prototypical chemotaxis model was proposed by KELLER and SEGEL [93] and in its general form reads as

$$\begin{cases} \partial_t u(t,x) = \partial_x \left\{ u(t,x)\Phi\left(\frac{\partial_x u(t,x)}{u(t,x)}\right) - au(t,x)\partial_x f(S)\right\}, & x \in \mathbb{R}, t>0, \\ \delta\partial_t S(t,x) = \gamma\partial^2_{xx} S(t,x) + k(u,S), & x \in \mathbb{R}, t>0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.3.1}$$

The function $u = u(t,x)$ refers to the cell density at position x and time t, whereas $S(t,x)$ means the density of the chemoattractant. Then, the above system consists in two coupled equations in terms of u and S. The parameter $a > 0$ measures the strength of the chemical signal and is called as the chemotactic coefficient. We take also δ, γ positive numbers where γ is the chemical diffusion coefficient. In the classical Keller–Segel model, the function Φ is taken to be the identity map in order to have a classical diffusion in the first term of (5.3.1). Moreover, f refers the chemosentivity function describing the signal mechanism and $k(u,S)$ characterizes the chemical growth and degradation.

The chemosensitivity function f can be chosen in different ways. The linear law agrees with $f(S) = S$, the logarithmic law is $f(S) = \log(S)$ or the receptor law refers to $f(S) = S^m/(1+S^m)$ for $m \in \mathbb{N}$. The system with linear law and $k(u,S) = S-u$ is called as the minimum chemotaxis model (see [38, 86]). The second one referring to the logarithmic law follows from the Weber–Frechner law, see [5, 12, 48, 93] for some applications. We refer to [142] for a survey concerning the logarithmic law.

Our work will focus on the case of logarithmic sensitivity, meaning $f(S) = \log(S)$ and where $k(u,S) = u - \lambda S$ with $\lambda \geq 0$. In this way, (5.3.1) agrees with

$$\begin{cases} \partial_t u(t,x) = \partial_x \left\{ u(t,x)\Phi\left(\frac{\partial_x u(t,x)}{u(t,x)}\right) - a\frac{\partial_x S(t,x)}{S(t,x)}u(t,x)\right\}, & x \in \mathbb{R}, t>0, \\ \delta\partial_t S(t,x) = \gamma\partial^2_{xx} S(t,x) - \lambda S(t,x) + u(t,x), & x \in \mathbb{R}, t>0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.3.2}$$

The work is devoted to give a comparative between the Keller–Segel with classical diffusion (where Φ is taken to be the identity map) and the flux–limited diffusion. In this latter case, we will assume that Φ verifies conditions (**H**):

(H1) $\Phi \in \mathscr{C}^2(\mathbb{R}), \Phi(-y) = -\Phi(y)$, and $\Phi'(y) > 0$.

(H2) $\lim_{y\to+\infty} \Phi(y) = c > 0$.

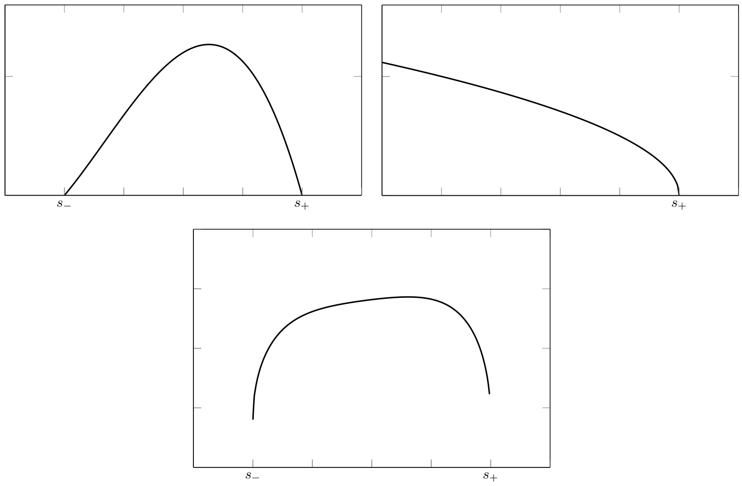

Figure 5.1: Top: solutions for Keller–Segel model with classical diffusion. Bottom: solutions for Keller–Segel model with flux–saturated diffusion.

In particular we take the relativistic model with

$$\Phi(y) = \mu \frac{y}{\sqrt{1 + \frac{\mu^2}{c^2} y^2}}, \tag{5.3.3}$$

but we can extend our results to functions Φ satisfying **(H)** with some integrability conditions. Note that the classical diffusion satisfies **(H1)** and **(H2)** with $c = +\infty$.

In this work, we will be interested in the investigation of traveling waves solutions, that is

$$u(t,x) = \tilde{u}(x - \sigma t), \quad \text{and} \quad S(t,x) = \tilde{S}(x - \sigma t), \tag{5.3.4}$$

with $\sigma > 0$, for some profiles $\tilde{u}$ and $\tilde{S}$. The search for traveling waves solutions is crucial to understand the mechanisms behind various propagating wave patters. We will show that the Keller–Segel system together with a flux–saturated mechanism exhibits new properties with respect to the classical one. Here, we will study both cases and show their differences. We refer to the shapes for traveling waves solutions in the case of classical diffusion to the top Figure 5.1 and for a flux–saturated diffusion to the bottom of Figure 5.1.

Let us briefly explain the idea of the work. Assume that we have a solution of type (5.3.4), hence (5.3.2) agrees with

$$\begin{aligned} -\sigma \tilde{u}' &= \left(\tilde{u} \Phi \left(\frac{\tilde{u}'}{\tilde{u}} \right) - a \frac{\tilde{S}'}{\tilde{S}} \tilde{u} \right)', \\ -\sigma \delta \tilde{S}' &= \gamma \tilde{S}'' - \lambda \tilde{S} + \tilde{u}, \end{aligned}$$

where $\tilde{u}'$, $\tilde{S}'$ and $\tilde{S}''$ represent the derivative with respect to the new variable $s = x - \sigma t$. Under the change of variables

$$w = \frac{\tilde{u}}{\tilde{S}}, \quad \text{and} \quad v = \frac{\tilde{S}'}{\tilde{S}}, \tag{5.3.5}$$

the above system of ordinary differential equations is related to

$$w' = w \Phi^{-1} (av - \sigma) - wv, \tag{5.3.6}$$

$$v' = -v^2 - \frac{\sigma\delta}{\gamma}v - \frac{w}{\gamma} + \frac{\lambda}{\gamma},, \tag{5.3.7}$$

with some initial conditions. The purpose of this work is to analyze the coupled system (5.3.6)–(5.3.7) and later come back to the original variables $\tilde{u}$ and $\tilde{S}$ via (5.3.5). We will separate in two cases: first we will assume that $\Phi = \text{Id}$ having a classical diffusion and later we will take Φ defined as (5.3.3). The shape of the profiles $\tilde{u}$ and $\tilde{S}$ strongly depends on the previous cases. In the following, we will briefly explain the main difference between both cases.

5.3.1 Linear diffusion

In the case of a linear diffusion, i.e., $\Phi = \text{Id}$, the system (5.3.6)–(5.3.7) is not singular and classical theory for ODEs gives us existence and uniqueness of solution. Assume that $\delta = 0$ for the sake of simplicity, and that the other parameters involved in (5.3.6)–(5.3.7) are positive real numbers. Hence, the system reduces to

$$w' = w\{(a-1)v - \sigma\}, \tag{5.3.8}$$

$$v' = \frac{\lambda}{\gamma} - v^2 - \frac{1}{\gamma}w. \tag{5.3.9}$$

Different values of a and σ will exhibit different scenarios. That can be seen from the analysis of the fixed points associated to (5.3.8)–(5.3.9). Set

$$v_\star := \sqrt{\frac{\lambda}{\gamma}}. \tag{5.3.10}$$

Hence, we get the following result.

Proposition 5.3.1. *Define*

$$(w_1, v_1) = (0, v_\star), \quad (w_2, v_2) = (0, -v_\star), \quad \text{and} \quad (w_3, v_3) = \left(\lambda - \frac{\gamma\sigma^2}{(a-1)^2}, \frac{\sigma}{a-1}\right),$$

where $v_\star$ is defined in (5.3.10). *Hence,*

1. *If $a > 0$ and $|1-a|v_\star < \sigma$, hence* (5.3.8)–(5.3.9) *has two fixed points given by (w_i, v_i), with $i = 1, 2$. The point (w_1, v_1) is a stable point, whereas (w_2, v_2) is a saddle point.*

2. *If $a \in (0,1)$ and $\sigma < (1-a)v_\star$, hence* (5.3.8)–(5.3.9) *has three fixed points given by (w_i, v_i), with $i = 1, 2, 3$. The point (w_1, v_1) is a stable point, (w_2, v_2) is an unstable point and (w_3, v_3) is a saddle point.*

3. *If $a > 1$ and $\sigma < (a-1)v_\star$, hence* (5.3.8)–(5.3.9) *has three fixed points given by (w_i, v_i), with $i = 1, 2, 3$. The points (w_1, v_1) and (w_2, v_2) are saddle points, whereas (w_3, v_3) is a stable point or a stable focus.*

The isocline map depends on the position (in case it exists) of the vertical line $v = \frac{\sigma}{a-1}$ and the parabola $w = \lambda - \gamma v^2$, with $w \geq 0$. First, consider $a < 1$. We find that the vertical line $v = \frac{\sigma}{a-1}$ can cross the parabola depending on $\sigma > (1-a)v_\star$ or $\sigma < (1-a)v_\star$. In the first case, which correspond to Proposition 5.3.1–1 we find that the area under the parabola is positively invariant. Therefore, either the solution enters the area under the parabola and the component of v of the system changes from decreasing to increasing, or the solution never touches the parabola and v is always decreasing. As a consequence, there can not be limit cycles. On the other hand, we are in the case of Proposition 5.3.1–2. Here, we have an extra fixed point (which correspond to (w_3, v_3) in the above proposition) that is the intersection between the vertical line $v = \frac{\sigma}{a-1}$ and the parabola. The phase diagram for both cases in $a < 1$ can be seen in the upper part of Figure 5.2. The case $a = 1$ is very special since we do not have any vertical line: it correspond to Proposition 5.3.1–1. As in the previous case, the area under the parabola remains positive invariant. We refer the reader to Figure 5.2–C for its phase diagram.

Finally, the case $a > 1$ is more complicate. We have again two possibilities: either the vertical line intersects the parabola. This is described in the bottom of Figure 5.2. In the case that the line does not intersect the parabola, we get a similar scenario as for $a < 1$. In the other case we can not ensure the non existence of limit cycles around the fixed point (w_3, v_3).

Moreover, by using the uniqueness of the stable manifold associated to a saddle point, we are able to prove the existence of the solution (line in blue) described in Figure 5.2 with initial datas $w_0 = w_\star$ and $v_0 > v_\star$. Hence, by doing an asymptotic analysis of (5.3.8)–(5.3.9) we are able to prove the behavior of the solutions (every line in blue) described in Figure 5.2. There the solutions are considered to have initial data $w_0 > 0$ and $v_0 > v_\star$. Note that in the case of Figure 5.2–D, since we do not know the existence of limit cycles around (w_3, v_3) we do not know if the solution with $w_0 < w_0^\star$ either converges to the limit cycle or to (w_3, v_3).

In the case $a < 1$ corresponding to the top of Figure 5.2, considering initial data $w_0 > 0$ and $v_0 < -v_\star$ we find interesting solutions. This is described in Figure 5.3 where the two blue solutions in the bottom part tends to a fixed point as $s \to -\infty$. That will provide us later traveling–waves solutions of the type in the upper right part of Figure 5.1. Finally, let us mention that there are also heteroclinic solutions joining fixed points, for instance, the one from the fixed point $(0, -v_\star)$ into $(0, v_\star)$.

Once system (5.3.8)–(5.3.9) is analyzed, we can come back to the original variables $\tilde{u}$ and $\tilde{S}$ via (5.3.5). Among other types of solutions, we are able to find profiles for travelling–waves solutions with the following shapes:

- **Type 1:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type 1 if $s_-, s_+ \in \mathbb{R}$ and $f(s_-) = f(s_+) = 0$.
- **Type 2:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type 2 if $s_- = -\infty$ and $s_+ \in \mathbb{R}$. Moreover, f satisfies $f(s_+) = 0$, and
$$\lim_{s \to -\infty} f(s) = +\infty.$$

Here, we have listed only the most interesting shapes found from the solutions in blue described in Figures 5.2 and 5.3. We refer to the top of Figure 5.1 which illustrates the shapes of the profiles. The existence of the above types of solutions strongly depends on the parameters a and σ. Note that Type 1 gives us solitons type of solutions.

5.3.2 Nonlinear flux–saturated diffusion

This section deals with the Keller–Segel equation with nonlinear flux saturated diffusion. Take that the function Φ is defined as (5.3.3), that is,

$$\Phi(y) = \mu \frac{y}{\sqrt{1 + \frac{\mu^2}{c^2} y^2}}.$$

Note that (5.3.3) satisfies **(H)** and then we can improve our results taking Φ verifying **(H)** with some integrability conditions.

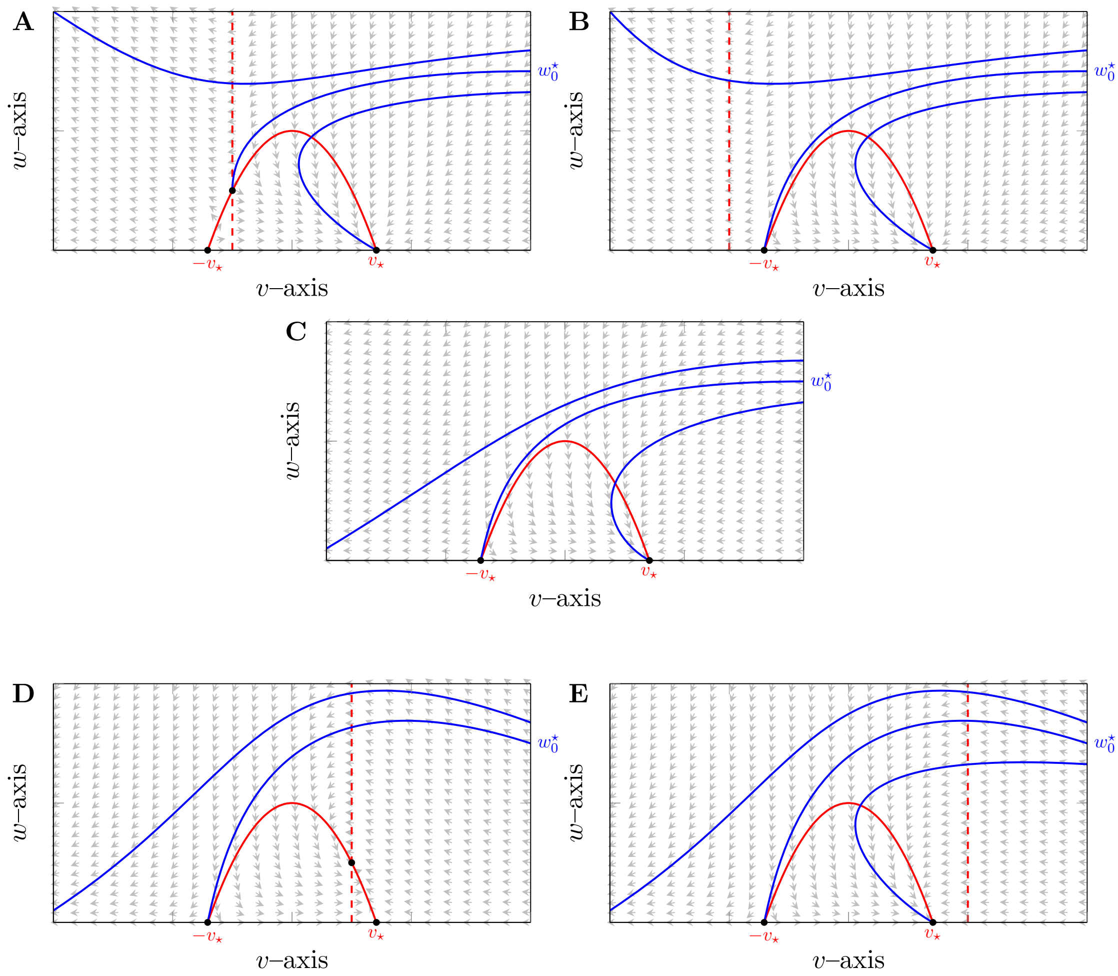

Figure 5.2: Top (A and B): $a < 1$. Center (C): $a = 1$. Bottom (D and E): $a > 1$. Left (A and D): $\sigma < |1-a|v_\star$. Right (B and E): $\sigma > |1-a|v_\star$.

Hence, the associated system for traveling–wave solutions is given by

$$w' = w\left\{\Phi^{-1}(av-\sigma) - v\right\}, \tag{5.3.11}$$

$$v' = -v^2 - \frac{w}{\gamma} + \frac{\lambda}{\gamma}. \tag{5.3.12}$$

The first consequence of **(H)** is that the domain of definition of v is restricted to

$$\frac{\sigma - c}{a} \le v \le \frac{\sigma + c}{a}.$$

Then, the above condition together with $w \ge 0$ defines the domain of definition of the solutions. Note that at the boundary, meaning $v \in \{\frac{\sigma-c}{a}, \frac{\sigma+c}{a}\}$, the value of w' blows up. Consider then that the initial condition satisfies $w(s_0) = w_0 > 0$ and $v(s_0) \in \left(\frac{\sigma-c}{a}, \frac{\sigma+c}{a}\right)$.

The study of the the fixed points and phase diagram of (5.3.11)–(5.3.12) depends on the parameters and on the number of solutions of the equation

$$\Phi^{-1}(av-\sigma) = v. \tag{5.3.13}$$

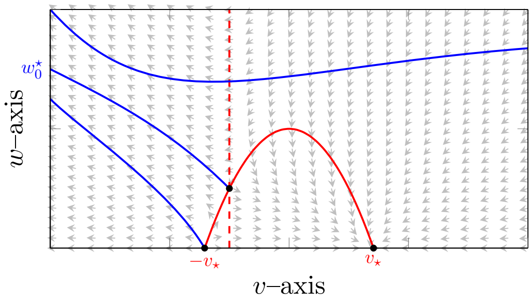

Figure 5.3: Case $a \in (0,1)$ and $\sigma < (1-a)v_\star$.

For the sake of simplicity we will assume that there is only one solution to (5.3.13), and thus we consider that $||\Phi'||_{L^\infty} < a$. Moreover, we define $\overline{v}$ as the unique solution to (5.3.13). In the case of $||\Phi'||_{L^\infty} \geq a$ we can have more than one solution and the analysis can be done in a similar way.

In the following result, we analyze the fixed points of (5.3.11)–(5.3.12).

Proposition 5.3.2. *Consider Φ satisfying* **(H)** *and $||\Phi'||_{L^\infty} < a$. Define $\overline{v} \in [0, \frac{\sigma+c}{a})$ the unique solution to*

$$\Phi^{-1}(a\overline{v} - \sigma) = \overline{v},$$

and $v_\star = \sqrt{\frac{\lambda}{\gamma}}$. We have

(A) If $\sigma < c - av_\star$ and $g(av_\star - \sigma) - v_\star > 0$, then there exist three fixed points given by

$$(w_1, v_1) = (0, v_\star), \quad (w_2, v_2) = (0, -v_\star), \quad \text{and} \quad (w_3, v_3) = \left(\lambda - \gamma\overline{v}^2, \overline{v}\right).$$

We have that (w_1, v_2) and (w_2, v_2) are saddle points, whereas (w_3, v_3) is a stable node or a stable focus.

(B) If $\sigma < c - av_\star$ and $g(av_\star - \sigma) - v_\star < 0$, then there exists two fixed points given by (w_1, v_1) and (w_2, v_2). We have that (w_1, v_2) is a stable point and (w_2, v_2) is a saddle point.

(C) If $|av_\star - c| < \sigma < c + av_\star$ and $g(av_\star - \sigma) - v_\star > 0$, then there exists two fixed points given by (w_1, v_1) and (w_3, v_3). We have that (w_1, v_1) is a saddle point, and (w_3, v_3) is a stable node or a stable focus.

(D) If $|av_\star - c| < \sigma < c + av_\star$ and $g(av_\star - \sigma) - v_\star < 0$, then there exists one fixed point given by (w_1, v_1).

(E) If $\sigma < av_\star - c$, then there exists one fixed point given by (w_3, v_3). We have that (w_3, v_3) is a stable node or a stable focus.

By analyzing the sign of w' and v' in the different cases of the previous proposition, we arrive at the phase diagram described in Figure 5.4.

Moreover, using a delicate analysis about the solutions of (5.3.11)–(5.3.12) we are able to prove the existence of singular solutions touching the boundary. That is described in blue in Figure 5.4 and more precisely in the following theorem.

Theorem 5.3.3. *For any* $v_0 \in (\frac{\sigma-c}{a}, \frac{\sigma+c}{a})$ *there exist* w_0^* *such that for any* $w_0 > w_0^*$*, any maximal solution of* (5.3.11)–(5.3.12) *with initial data* (v_0, w_0) *are defined on* (s_-, s_+) *with* $-\infty < s_- < s_+ < +\infty$*, which verifies*

1. $v(s_-) = \frac{\sigma+c}{a},\ v(s_+) = \frac{\sigma-c}{a}$,

2. $v \in \mathscr{C}^1([s_-, s_+])$ *and* $v'(s) < 0$,

3. $w(s_-), w(s_+) \in (0, +\infty)$,

4. $w'(s_-) = +\infty,\ w'(s_+) = -\infty$.

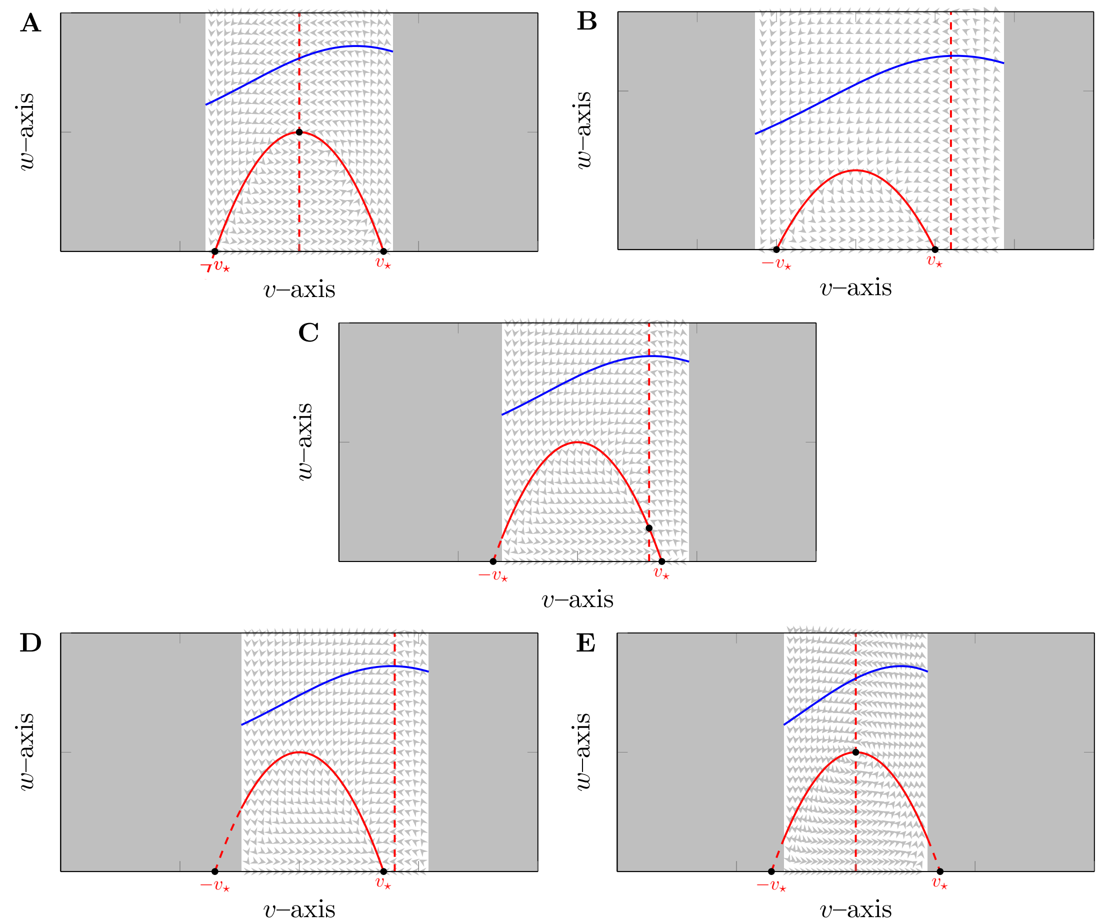

Figure 5.4: Cases given by Proposition 5.3.2

As in the classical regime, we can now come back to the original variables $\tilde{u}$ and $\tilde{S}$ via (5.3.5). Note that since the solutions (curves in blue) in Figure 5.4 touch the boundary, we have that w' blows up at (s_-, s_+), where this is the interval of maximal definition. That singularity is also translated to the original variables having that $\tilde{u}'$ blows up. Hence, we are able to find profiles of the following type:

- **Type 3:** A function $f : (s_-, s_+) \to \mathbb{R}$ is of Type 3 if $s_-, s_+ \in \mathbb{R}$ with $f(s_-), f(s_+) > 0$.

Moreover, it satisfies that

$$\lim_{s \to s_-} f'(s) = +\infty, \quad \text{and} \quad \lim_{s \to s_+} f'(s) = -\infty.$$

We refer to the top of the bottom of Figure 5.1 which illustrates the shape of the profiles.

5.4 Conclusions and perspectives

Some ongoing projects and future works that have arisen as a consequence of the results developed during this dissertation are presented here. We will focus on a specific short list containing some of the most challenging problems from our point view.

This dissertation is mainly split in three different parts. First, we have studied inhomogeneous rotating vortices for the 2D Euler equations in Chapter 2, which appear as bifurcation of some initial radial solutions. This is a joint work with HMIDI and SOLER. Second, in Chapter 3 we have worked on the existence of Kármán Vortex Street structures in different incompressible 2D models such as Euler, generalized quasi–geostrophic or the shallow water quasi–geostrophic equations. Thirdly, Chapter 4 have focused on the 3D quasi–geostrophic system where we could check the existence of homogeneous 3D patches that rotate around the z–axis. That problem is a collaboration with HMIDI and MATEU.

In Chapter 2, we provided a systematic scheme which turns to be relevant to detect non trivial rotating vortices with non uniform vorticity, far from the uniform patches but close to some known radial profiles. In particular, we fully analyzed the bifurcation from quadratic ones, thus obtaining new behaviors in the bifurcated diagram compared to the uniform case. There are two natural questions after that work. First, it is the generalization of the above result with more general profiles. The second one concerns the third statement in Theorem 2.1.1 about stationary solutions.

- **General profiles:** To the best of our knowledge, the only two works concerning inhomogeneous rotating vortices for the Euler equations are the work of CASTRO, CÓRDOBA and GÓMEZ SERRANO [27], and the content of Chapter 2 (see also [67]). While in Chapter 2 we are able to find rotating solutions around a quadratic profile that are far away the patches, in [27] they found rotating solutions that are close to the Burbea patches.

  The main difficulty in both works is the kernel study, and that forces us to restrict ourselves to some particular initial profiles. As we explained in Chapter 2, the kernel problem in such work is related to the resolution of a Volterra equation that we are able to solve in the case of quadratic profiles.

  Analyzing the expression of $\partial_f \tilde{F}_2$, there are two different ways to solve this problem. The first one is to study in more details such Volterra integro–differential equation and find some special structures that enable us to solve it for more general profiles. The second one is to try to analyze directly the kernel equation via some general results in spectral theory. In fact, the kernel equation has the following (simplified) expression

$$\begin{aligned}\mathcal{L}_n(\Omega, h_n)(r) :=& \frac{1}{T_n(\Omega,r)}\left(G_n(r)\int_0^1 s^{n+1}h_n(s)ds + \frac{1}{r^n}\int_r^1 \frac{h_n(s)}{s^{n-1}}ds + \frac{1}{r^n}\int_0^r s^{n+1}h_n(s)ds\right)\\ =& h_n(r),\end{aligned}$$

  where G_n and T_n are some functions depending on the initial density f_0. In order to achieve that the kernel is one dimensional, one must find N and Ω_N such that there exists a non trivial solution h_N of the above equation. Then, instead of differentiating the above equation and trying to solve explicitly the associated differential equation, one can study the eigenvalues of the linear operator $\mathcal{L}_n$, that strongly depends on f_0. That is strongly motivated by the work done in Chapter 4 (or [66]) where we studied rotating patches in the 3D quasi–geostrophic system.

- **Stationary solutions:** There are many studies about stationary solutions for the 2D Euler equations. In the case of stationary simply–connected vortex patches, FRAENKEL checked that the circular patch is the only stationary one, see [64]. In [69], GÓMEZ SERRANO, PARK, SHI and YAU proved that any smooth stationary solution with compactly supported and nonnegative vorticity must be radial.

An interesting future perspective is to find stationary solutions in the bifurcated branches given in Theorem 2.1.1. In the case of a simply connected vortex patch, it is possible to bifurcate the circular patch from the angular velocity $\Omega = 0$ but one just find pure translations of the circle patch. In Chapter 2, we found a branch of 1–fold symmetric rotating solutions bifurcating from $\Omega = 0$ in the third point in Theorem 2.1.1. However, as a consequence of our function spaces, we know that the bifurcated solutions are not given by a pure translation of the initial radial profile and do not contain radial profiles. Then, it is not clear from our result whether or not the branch contains stationary solutions. However, in the case that they exist then they are non trivial stationary solutions.

In order to observe these nonradial stationary solutions close to the quadratic profile $f_0(r) = Ar^2 + B$, we could try to bifurcate the initial quadratic profile only in the family of stationary solutions using the parameters A and B as bifurcation parameters (they will play the role of the angular velocity Ω). Indeed, by using the boundary and density equation (1.1.16)–(1.1.18) we can define

$$\begin{aligned} G_1(A, B, f, \phi) =& F(0, f, \phi), \\ G_2(A, B, f, \phi) =& G(0, f, \phi). \end{aligned}$$

Here, we have some trivial solutions given by $G_1(A, B, f_0, \mathrm{Id}) = G_2(A, B, f_0, \mathrm{Id}) \equiv 0$, for any $A, B \in \mathbb{R}$. Following the ideas in Chapter 2, the first equation G_1 can be solved via the Implicit Function theorem and one can try to use Crandall–Rabinowitz theorem to G_2. The kernel of the linearized operator of G_2 around f_0 is one dimensional, which follows from the work done in Chapter 2. The problem here is that the classical transversal condition in bifurcation theory is not verified and one has to work more with $\partial_f^2 \tilde{G}$.

Chapter 3 aims to provide a model for the phenomenon known as Kármán Vortex Street. The key idea is the desingularizing point vortices which turns to be very robust and can be applied to other special solutions to the N-vortex system. The motivation of Chapter 3 was the desingularization of a vortex pair in [83]. Two point vortices with same circulation rotate at a constant angular velocity. However, if the circulation are opposite they translate at a constant speed. Another relative equilibria in the N–vortex system is the case of point vortices at any vertex of a regular polygon with same circulation. All this configuration rotates around the center of the polygon at a constant angular velocity. This is explained in the previous Section 5.2. The same idea of the desingularization of the vortex pairs can be applied to other configurations of the discrete system.

To the best of our knowledge, the content of Chapter 4 is the first work concerning the bifurcation of trivial solutions of a 3D fluid model. There, we prove the existence of non uniformly rotating patches that are bifurcated from a stationary one. Such stationary patch is a revolution shape with some needed regularity. In the line of the study vortex patch type of solutions, many questions remain open.

- **Bifurcation of singular shapes:** In Chapter 4 we have bifurcated stationary patches which are revolution shapes around the vertical axis. We asked to these patches some conditions and one of them is that the shape must be located around two stationary ellipsoids (meaning that the x and y axes are equal). That ensures us that the stationary revolution shape has some regularity (we can think in an ellipsoid patch). However, such regularity is only needed in the persistence of the nonlinear function in the function spaces, and it is not needed in the spectral study. Hence, we may try to bifurcate from singular shapes where the spectral analysis is similar by using other weaker bifurcation methods.

- **Bifurcation of the rotating ellipsoidal patches:** In the case of the 2D Euler equations, the Kirchhoff ellipses define rotating patches. As for the circular patch, HMIDI and MATEU are able to bifurcate the ellipse patches in order to find a new family of rotating vortex patches, see [82]. In Chapter 4, we are able to find rotating solutions close to some stationary patches. However, there is the analogue to the Kirchhoff ellipses here and they are the ellipsoid patches with different x and y axis. Hence, we expect that the same techniques (with more involved computations) could work in order to bifurcate the ellipsoid patches and find some new rotating solutions.

- **Persistence of the regularity of the patch:** As for the 2D Euler equations, the solutions are transported along the trajectories in the 3D quasi–geostrophic system. Hence, the evolution of a patch along the trajectories is again in this type. In the 2D case, the persistence of the $\mathscr{C}^{1,\alpha}$ regularity for the patch boundary is known by CHEMIN in [35]. We believe that the same idea gives us the persistence of the regularity for the patch boundary in the quasi–geostrophic system.

- **Axisymmetric 3D Euler equation:** A more physical relevant system is the 3D Euler equations. The main different between the 2D and 3D Euler equations is the stretching term $\omega \cdot \nabla v$ that does not appear in the 2D case. As a consequence, the study of some special solutions for the 3D equations is a difficult problem.

  There is a simplification of the 3D system in terms of axisymmetric solutions. Using cylindrical coordinates, we refer to a axisymmetric solution when the velocity field does not depend on the angle θ direction, i.e.,

  $$v = v^r(t, r, x_3)e_r + v^\theta(t, r, x_3)e_\theta + v^3(t, r, x_3)e_3,$$

  where e_3, e_θ and e_3 is the standard orthonormal unit vectors defining the cylindrical coordinate system. In the case that $v^\theta = 0$ we arrive at the axisymmetric flows without swirl. In that case, the vorticity only depend on the angle component: $\omega = \omega^\theta e_\theta$. Moreover, there exists a transport equation for $\frac{\omega^\theta}{r}$, see [105].

  For the axisymmetric system without swirl, we find some special relative equilibria (in terms of patches) that behave as translations or rotations of the initial patch. Here, the spherical patch will evolve as a translation in the z axis with constant speed. This is known in the literature as the Hill spherical vortex, see [64, 79]. One can try to find special solutions around the Hill vortex by using bifurcation arguments.

Appendix A

Bifurcation theory and Fredholm operators

The main aim of bifurcation theory is to explore the topological transitions of the phase portrait through the variation of some parameters. A particular case is to understand this transition in the equilibria set for the stationary problem $F(\lambda, x) = 0$, where $F : \mathbb{R} \times X \to Y$ is a a smooth function and the spaces X and Y are Banach spaces. Assuming that one has a particular solution, $F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$, we would like to explore the bifurcation diagram close to this trivial solution, and see whether we can find multiple branches of solutions bifurcating from a given point $(\lambda_0, 0)$. When this occurs we say that the pair $(\lambda_0, 0)$ is a bifurcation point. When the linearized operator around this point generates a Fredholm operator, then one may use Lyapunov-Schmidt reduction in order to reduce the infinite-dimensional problem to a finite-dimensional one. For the latter problem we just formulate suitable assumptions so that the Implicit Function Theorem can be applied. For more discussion in this subject, we refer to see [91, 95].

In what follows we shall recall some basic results on Fredholm operators.

Definition A.0.1. *Let X and Y be two Banach spaces. A continuous linear mapping $T : X \to Y$, is a Fredholm operator if it fulfills the following properties,*

1. $\dim \operatorname{Ker} T < \infty$,
2. $\operatorname{Im} T$ *is closed in* Y,
3. $\operatorname{codim} \operatorname{Im} T < \infty$.

The integer $\dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Im} T$ *is called the Fredholm index of* T.

Next, we shall discuss the index persistence through compact perturbations, see [91, 95].

Proposition A.0.2. *The index of a Fredholm operator remains unchanged under compact perturbations.*

Now, we recall the classical Crandall–Rabinowitz Theorem whose proof can be found in [44].

Theorem A.0.3 (Crandall-Rabinowitz Theorem)**.** *Let X, Y be two Banach spaces, V be a neighborhood of 0 in X and $F : \mathbb{R} \times V \to Y$ be a function with the properties,*

1. $F(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$.

2. *The partial derivatives* $\partial_\lambda F_\lambda$*,* $\partial_f F$ *and* $\partial_\lambda \partial_f F$ *exist and are continuous.*

3. *The operator* $\partial_f F(0,0)$ *is Fredholm of zero index and* $\mathrm{Ker}(F_f(0,0)) = \langle f_0 \rangle$ *is one-dimensional.*

4. *Transversality assumption:* $\partial_\lambda \partial_f F(0,0) f_0 \notin \mathrm{Im}(\partial_f F(0,0))$*.*

If Z *is any complement of* $\mathrm{Ker}(F_f(0,0))$ *in* X*, then there is a neighborhood* U *of* $(0,0)$ *in* $\mathbb{R} \times X$*, an interval* $(-a,a)$*, and two continuous functions* $\Phi : (-a,a) \to \mathbb{R}$*,* $\beta : (-a,a) \to Z$ *such that* $\Phi(0) = \beta(0) = 0$ *and*

$$F^{-1}(0) \cap U = \{(\Phi(\xi), \xi f_0 + \xi\beta(\xi)) : |\xi| < a\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}.$$

Appendix **B**

Potential theory

This appendix is devoted to some classical estimates on potential theory that we have used through all the previous chapters. We shall deal in particular with truncated operators whose kernels are singular along the diagonal.

The action of such operators over various function spaces and its connection to the singularity of the kernel is widely studied in the literature, see [62, 78, 99, 110, 111, 144]. In the case of Calderon–Zygmund operators we refer to the recent papers [45, 46, 106] and the references therein. In what follows we shall establish some useful estimates whose proofs are classical and for the convenience of the reader we decide to provide most the details.

Let us first deal with singular integrals of the type

$$\mathcal{T}(f)(w) = \int_{\mathbb{T}} K(w,\xi) f(\xi)\, d\xi, \quad w \in \mathbb{T}, \tag{B.0.1}$$

where $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ being smooth off the diagonal. The next result focuses on the smoothness of the last operator, whose proof can be found in [76]. See also [78, 99, 103].

Lemma B.0.1. *Let $0 \le \alpha < 1$ and consider $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ with the following properties. There exists $C_0 > 0$ such that*

(i) K is measurable on $\mathbb{T} \times \mathbb{T} \setminus \{(w,w), w \in \mathbb{T}\}$ and

$$|K(w,\xi)| \le \frac{C_0}{|w-\xi|^\alpha}, \quad \forall w \neq \xi \in \mathbb{T}.$$

(ii) For each $\xi \in \mathbb{T}$, $w \mapsto K(w,\xi)$ is differentiable in $\mathbb{T} \setminus \{\xi\}$ and

$$|\partial_w K(w,\xi)| \le \frac{C_0}{|w-\xi|^{1+\alpha}}, \quad \forall w \neq \xi \in \mathbb{T}.$$

Then,

1. *The operator $\mathcal{T}$ defined by* (B.0.1) *is continuous from $L^\infty(\mathbb{T})$ to $\mathscr{C}^{1-\alpha}(\mathbb{T})$. More precisely, there exists a constant C_α depending only on α such that*

$$\|\mathcal{T}(f)\|_{1-\alpha} \le C_\alpha C_0 \|f\|_{L^\infty}.$$

2. *For $\alpha = 0$, the operator $\mathcal{T}$ is continuous from $L^\infty(\mathbb{T})$ to $\mathscr{C}^\beta(\mathbb{T})$, for any $0 < \beta < 1$. That is, there exists a constant C_β depending only on β such that*

$$\|\mathcal{T}(f)\|_\beta \leq C_\beta C_0 \|f\|_{L^\infty}.$$

Next, we study singular integrals of the type

$$\mathcal{K}(f)(x_1, x_2) = \int_0^1 \int_0^1 K(x_1, x_2, y_1, y_2) f(y_1, y_2) dy_1 dy_2, \tag{B.0.2}$$

with $(x_1, x_2) \in [0,1]^2$ and where $K : [0,1]^2 \times [0,1]^2 \to \mathbb{R}$ is smooth out the diagonal.

Proposition B.0.2. *Let $K : [0,1]^2 \times [0,1]^2 \to \mathbb{R}$ be smooth out the diagonal, satisfying*

$$|K(x_1, x_2, y_1, y_2)| \leq \frac{C_0}{|x_1 - y_1|^{1-\alpha} |x_2 - y_2|^\gamma}, \tag{B.0.3}$$

$$|K(x_1, x_2, y_1, y_2)| \leq \frac{C_0}{|x_1 - y_1|^{\gamma} |x_2 - y_2|^{1-\alpha}}, \tag{B.0.4}$$

$$|\partial_{x_1} K(x, y)| \leq \frac{C_0}{|x_1 - y_1|^{2-\alpha} |x_2 - y_2|^\gamma}, \tag{B.0.5}$$

$$|\partial_{x_2} K(x, y)| \leq \frac{C_0}{|x_1 - y_1|^{\gamma} |x_2 - y_2|^{2-\alpha}}, \tag{B.0.6}$$

with $\alpha, \gamma \in (0,1)$. Then $\mathcal{K} : L^\infty([0,1] \times [0,1]) \to \mathscr{C}^\alpha([0,1] \times [0,1])$ is well-defined and

$$||\mathcal{K}(f)||_{\mathscr{C}^\alpha} \leq C C_0 ||f||_{L^\infty}.$$

Remark B.0.3. *Note that condition* (B.0.5) *(and also* (B.0.6)*) can be replaced by*

$$|K(x_1, x_2, y_1, y_2) - K(\tilde{x_1}, x_2, y_1, y_2)| \leq C|x_1 - \tilde{x_1}|^\alpha g(x_1, \tilde{x_1}, x_2, y_1, y_2),$$

for $x_1 < \tilde{x_1}$ and $3|x_1 - \tilde{x_1}| \leq |y_1 - x_1|$. The function g must satisfy

$$\left| \int_0^1 \int_0^1 g(x_1, \tilde{x_1}, y_1, x_2, y_2) dy_1 dy_2 \right| \leq C,$$

uniformly in $x_1, \tilde{x_1}, x_2$.

Proof. The L^∞ norm of $\mathcal{K}(f)$ can be estimated as

$$\begin{aligned}
|\mathcal{K}(f)(x)| \leq & C||f||_{L^\infty} \int_0^1 \int_0^1 |K(x_1, x_2, y_1, y_2)| dy_1 dy_2 \\
\leq & C C_0 ||f||_{L^\infty} \int_0^1 \frac{dy_1}{|x_1 - y_1|^{1-\alpha}} \int_0^1 \frac{dy_2}{|x_2 - y_2|^\gamma} \\
\leq & C C_0 ||f||_{L^\infty}.
\end{aligned}$$

Remind that $\alpha, \gamma \in (0,1)$. Hence,

$$||\mathcal{K}(f)||_{L^\infty} \leq C C_0 ||f||_{L^\infty}.$$

For the Hölder regularity, take $x_1, \tilde{x_1} \in [0,1]$ with $x_1 < \tilde{x_1}$. Define $d = |x_1 - \tilde{x_1}|$, $B_{x_1}(r) = \{y \in [0,1] : |y - x_1| < r\}$ and $B^c_{x_1}(r)$ its complement set. Hence

$\mathcal{K}(f)(x_1, x_2) - \mathcal{K}(f)(\tilde{x_1}, x_2)$

$$\begin{aligned}
&= \int_0^1 \int_0^1 K(x_1,x_2,y_1,y_2) f(y_1,y_2) dy_1 dy_2 - \int_0^1 \int_0^1 K(\tilde{x_1},x_2,y_1,y_2) f(y_1,y_2) dy_1 dy_2 \\
&= \int_0^1 \int_{[0,1]\cap B_{x_1}(3d)} K(x_1,x_2,y_1,y_2) f(y_1,y_2) dy_1 dy_2 \\
&\quad - \int_0^1 \int_{[0,1]\cap B_{x_1}(3d)} K(\tilde{x_1},x_2,y_1,y_2) f(y_1,y_2) dy_1 dy_2 \\
&\quad + \int_0^1 \int_{[0,1]\cap B^c_{x_1}(3d)} (K(x_1,x_2,y_1,y_2) - K(\tilde{x_1},x_2,y_1,y_2)) f(y_1,y_2) dy_1 dy_2 \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Using (B.0.3), we arrive at

$$\begin{aligned}
|I_1| &\le CC_0 ||f||_{L^\infty} \int_{[0,1]\cap B_{x_1}(3d)} \frac{1}{|x_1-y_1|^{1-\alpha}} dy_1 \int_0^1 \frac{1}{|x_2-y_2|^\gamma} dy_2 \\
&\le CC_0 ||f||_{L^\infty} \int_{B_{x_1}(3d)} \frac{1}{|x_1-y_1|^{1-\alpha}} dy_1 \\
&\le CC_0 ||f||_{L^\infty} d^\alpha \\
&= CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}|^\alpha.
\end{aligned}$$

In order to work with I_2, note that $B_{x_1}(3d) \subset B_{\tilde{x_1}}(4d)$. Thus,

$$\begin{aligned}
|I_2| &\le CC_0 ||f||_{L^\infty} \int_{[0,1]\cap B_{x_1}(3d)} \frac{1}{|\tilde{x_1}-y_1|^{1-\alpha}} dy_1 \int_0^1 \frac{1}{|x_2-y_2|^\gamma} dy_2 \\
&\le CC_0 ||f||_{L^\infty} \int_{B_{x_1}(4d)} \frac{1}{|\tilde{x_1}-y_1|^{1-\alpha}} dy_1 \\
&\le CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}|^\alpha.
\end{aligned}$$

For the last term I_3 we use the mean value theorem and (B.0.5) achieving

$$\begin{aligned}
|I_3| &\le C \left| (x_1 - \tilde{x_1}) \int_0^1 \int_0^1 \int_{[0,1]\cap B^c_{x_1}(3d)} (\partial_{x_1} K)(x_1 + (1-s)(\tilde{x_1} - x_1), x_2, y_1, y_2) f(y_1,y_2) dy_1 dy_2 ds \right| \\
&\le CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}| \int_0^1 \int_{[0,1]\cap B^c_{x_1}(3d)} \frac{dy_1 ds}{|x_1 + (1-s)(\tilde{x_1} - x_1) - y_1|^{2-\alpha}} \int_0^1 \frac{dy_2}{|x_2-y_2|^\gamma}.
\end{aligned}$$

Note that if $y_1 \in B^c_{x_1}(3d)$, then

$$|x_1 + (1-s)(\tilde{x_1} - x_1) - y_1| \ge |x_1 - y_1| - (1-s)d \ge |x_1 - y_1| - \frac{(1-s)}{3}|x_1 - y_1| \ge \delta |x_1 - y - 1|,$$

with $\delta > 0$, which implies

$$\begin{aligned}
|I_3| &\le CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}| \int_0^1 \int_{[0,1]\cap B^c_{x_1}(3d)} \frac{dy_1 ds}{|x_1 - y_1|^{2-\alpha}} \\
&\le CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}| \frac{1}{|x_1 - \tilde{x_1}|^{1-\alpha}} \\
&\le CC_0 ||f||_{L^\infty} |x_1 - \tilde{x_1}|^\alpha.
\end{aligned}$$

The ideas for estimating

$$\mathcal{K}(f)(x_1, x_2) - \mathcal{K}(f)(x_1, \tilde{x_2}),$$

for $x_2, \tilde{x_2} \in [0,1]$ with $x_2 \neq \tilde{x_2}$, are similar. Then, we arrive at

$$||\mathcal{K}(f)||_{\mathscr{C}^\alpha} \leq CC_0||f||_{L^\infty}.$$

□

The third operator studied here is the following one:

$$\mathscr{L}(f)(z) := \int_{\mathbb{D}} K(z,y)f(y)dA(y), \quad z \in \mathbb{D} \tag{B.0.7}$$

Lemma B.0.4. *Let $\alpha \in (0,1)$ and $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is smooth off the diagonal and satisfies*

$$|K(z_1,y)| \leq \frac{C_0}{|z_1 - y|} \quad \textit{and} \quad |K(z_1,y) - K(z_2,y)| \leq C_0 \frac{|z_1 - z_2|}{|z_1 - y||z_2 - y|}, \tag{B.0.8}$$

for any $z_1, z_2 \neq y \in \mathbb{D}$, with C_0 a real positive constant. The operator defined in (B.0.7)

$$\mathscr{L} : L^\infty(\mathbb{D}) \to \mathscr{C}^{0,\alpha}(\mathbb{D})$$

is continuous, with the estimate

$$\|\mathscr{L}f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \leq CC_0\|f\|_{L^\infty(\mathbb{D})}, \quad \forall f \in L^\infty(\mathbb{D}),$$

where C is a constant depending on α.

Proof. It is easy to see that

$$\|\mathscr{L}f\|_{L^\infty(\mathbb{D})} \leq CC_0\|f\|_{L^\infty(\mathbb{D})}.$$

Using (B.0.8) combined with an interpolation argument we may write

$$\begin{aligned}|K(z_1,y) - K(z_2,y)| \leq& |K(z_1,y) - K(z_2,y)|^\alpha \big(|K(z_1,y)|^{1-\alpha} + |K(z_2,y)|^{1-\alpha}\big) \\ \leq& CC_0|z_1 - z_2|^\alpha \left[\frac{1}{|z_1 - y|^{1+\alpha}} + \frac{1}{|z_2 - y|^{1+\alpha}}\right].\end{aligned}$$

Thus from the inequality

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{|f(y)|}{|z-y|^{1+\alpha}} dA(y) \leq CC_0\|f\|_{L^\infty(\mathbb{D})}$$

we deduce the announced result. □

Before continuing to the next result, let us give the complex version of the Stokes theorem and the Cauchy–Pompeiu's formula.

The complex version of the Stokes theorem reads as follows. Let D be a simply connected domain and f a $\mathscr{C}^1$ scalar function, then

$$\int_{\partial D} f(\xi)\ d\xi = 2i \int_D \partial_{\overline{z}} f(z)\, dA(z), \tag{B.0.9}$$

where $\partial_{\overline{z}}$ can be identify to the gradient operator in the same way

$$\nabla = 2\partial_{\overline{z}}, \quad \partial_{\overline{z}}\varphi(z) := \frac{1}{2}\left(\partial_1\varphi(z) + i\partial_2\varphi(z)\right).$$

Let us introduce now the Cauchy–Pompeiu's formula. Consider $\varphi : \overline{D} \to \mathbb{C}$ be a $\mathscr{C}^1$ complex function, then

$$-\frac{1}{\pi}\int_D \frac{\partial_{\overline{y}}\varphi(y)}{w-y}dA(y) = \frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(w)-\varphi(\xi)}{w-\xi}d\xi. \tag{B.0.10}$$

The following result deals with a specific type of integrals that we have already encountered in Proposition 2.3.1. More precisely we shall be concerned with the integral

$$\mathscr{F}[\Phi](f)(z) := \int_{\mathbb{D}} \frac{f(y)}{\Phi(z)-\Phi(y)}|\Phi'(y)|^2 dA(y),\ z\in\mathbb{D}. \tag{B.0.11}$$

Lemma B.0.5. *Let $\alpha \in (0,1)$ and $\Phi : \mathbb{D} \to \Phi(\mathbb{D}) \subset \mathbb{C}$ be a conformal bi-Lipschitz function of class $\mathscr{C}^{2,\alpha}(\mathbb{D})$. Then*

$$\mathscr{F}[\Phi] : \mathscr{C}^{1,\alpha}(\mathbb{D}) \to \mathscr{C}^{1,\alpha}(\mathbb{D}),$$

is continuous. Moreover, the functional $\mathscr{F} : \Phi \in \mathscr{U} \mapsto \mathscr{F}[\Phi]$ is continuous, with

$$\mathscr{U} := \Big\{\Phi \in \mathscr{C}^{2,\alpha}(\mathbb{D}) : \Phi \text{ is bi-Lipschitz and conformal}\Big\}.$$

Proof. We start with splitting $\mathscr{F}[\Phi]f$ as follows

$$\begin{aligned}\mathscr{F}[\Phi]f(z) =& \int_{\mathbb{D}} \frac{f(y)-f(z)}{\Phi(z)-\Phi(y)}|\Phi'(y)|^2 dA(y) + f(z)\int_{\mathbb{D}}\frac{|\Phi'(y)|^2}{\Phi(z)-\Phi(y)}dA(y)\\ =:& \mathscr{F}_1[\Phi]f(z) + f(z)\mathscr{F}_2[\Phi]f(z).\end{aligned}$$

Let us estimate the first term $\mathscr{F}_1[\Phi]f$. The $L^\infty(\mathbb{D})$ bound is straightforward and comes from

$$\sup_{z,y\in\mathbb{D}} \frac{|f(y)-f(z)|}{|\Phi(z)-\Phi(y)|} \le \frac{\|f\|_{\text{Lip}}}{\|\Phi^{-1}\|_{\text{Lip}}}.$$

Setting

$$K[\Phi](z,y) = \nabla_z\left(\frac{f(y)-f(z)}{\Phi(z)-\Phi(y)}\right),$$

then one can easily check that K satisfies the assumptions

$$\begin{aligned}|K[\Phi](z,y)| \le& C\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}\frac{1}{|z-y|},\\ |K[\Phi](z_1,y)-K[\Phi](z_2,y)| \le& C\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}\max_{i,j\in\{1,2\}}\left[\frac{|z_i-z_j|^\alpha}{|z_i-y|}+\frac{|z_i-z_j|}{|z_i-y||z_j-y|}\right],\end{aligned}$$

where the constant C depends on Φ. Thus, Lemma B.0.4 yields

$$\|\nabla\mathscr{F}_1[\Phi]f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \le C\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}, \tag{B.0.12}$$

and hence we find

$$\|\mathscr{F}_1[\Phi]f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})} \le C\|f\|_{\mathscr{C}^{1,\alpha}(\mathbb{D})}.$$

Let us now check the continuity of the operator $\mathscr{F}_1 : \Phi \in \mathscr{U} \mapsto \mathscr{F}_1[\Phi]$. Taking $\Phi_1, \Phi_2 \in \mathscr{U}$ we may write,

$$|\mathscr{F}_1[\Phi_1]f(z) - \mathscr{F}_1[\Phi_2]f(z)| \le \int_{\mathbb{D}}\frac{|f(y)-f(z)|}{|\Phi_1(y)-\Phi_1(z)|}\Big||\Phi_1'(y)|^2-|\Phi_2'(y)|^2\Big|dA(y)$$

$$
\begin{aligned}
&+\int_{\mathbb{D}}|f(y)-f(z)||\Phi_2'(y)|\left|\frac{1}{\Phi_1(z)+\Phi_1(y)}-\frac{1}{\Phi_2(z)-\Phi_2(y)}\right|dA(y)\\
\leq &C\|f\|_{\text{Lip}}\left(\|\Phi_1'-\Phi_2'\|_{L^\infty(\mathbb{D})}+\|\Phi_1-\Phi_2\|_{L^\infty(\mathbb{D})}\right).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\nabla(\mathscr{F}_1[\Phi_1]f(z)-\mathscr{F}_1[\Phi_2]f(z))=&\int_D K[\Phi_1](z,y)(|\Phi_1'(y)|^2-|\Phi_2'(y)|^2)dA(y)\\
&+\int_{\mathbb{D}}\left(K[\Phi_1](z,y)-K[\Phi_2](z,y)\right)|\Phi_2'(y)|^2dA(y).
\end{aligned}
$$

Now, performing the same arguments as for (B.0.12) allows to get,

$$
\left\|\int_D K[\Phi_1](\cdot,y)(|\Phi_1'(y)|^2-|\Phi_2'(y)|^2)dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\leq C\|\Phi_1'-\Phi_2'\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}.
$$

For the second integral term, we proceed first with splitting $K[\Phi]$ in the following way

$$
\begin{aligned}
K[\Phi](z,y)=&-\frac{\nabla_z f(z)}{\Phi(z)-\Phi(y)}+(f(y)-f(z))\nabla_z\left(\frac{1}{\Phi(z)-\Phi(y)}\right)\\
=:&-\nabla_z f(z)K_1[\Phi](z,y)+K_2[\Phi](z,y).
\end{aligned}
$$

Let us check that $K_1[\Phi_1]-K_1[\Phi_2]$ obeys to the assumptions of Lemma B.0.4. For the first one, it is clear from elementary computations that

$$
|K_1[\Phi_1](z,y)-K_1[\Phi_2](z,y)|\leq C\|\Phi_1-\Phi_2\|_{\text{Lip}(\mathbb{D})}|z-y|^{-1}.
$$

Adding and subtracting adequately we obtain

$$
\begin{aligned}
&\Big|(K_1[\Phi_1]-K_1[\Phi_2])(z_1,y)-(K_1[\Phi_1]-K_1[\Phi_2])(z_2,y)\Big|\\
&\qquad\leq\left|\frac{((\Phi_1-\Phi_2)(z_1)-(\Phi_1-\Phi_2)(y))(\Phi_1(z_2)-\Phi_1(y))(\Phi_2(z_2)-\Phi_2(z_1))}{(\Phi_1(z_1)-\Phi_1(y))(\Phi_1(z_2)-\Phi_1(y))(\Phi_2(z_1)-\Phi_2(y))(\Phi_2(z_2)-\Phi_2(y))}\right|\\
&\qquad\quad+\left|\frac{(\Phi_2(z_1)-\Phi_2(y))(\Phi_1(z_2)-\Phi_1(y))((\Phi_1-\Phi_2)(z_1)-(\Phi_1-\Phi_2)(z_2))}{(\Phi_1(z_1)-\Phi_1(y))(\Phi_1(z_2)-\Phi_1(y))(\Phi_2(z_1)-\Phi_2(y))(\Phi_2(z_2)-\Phi_2(y))}\right|\\
&\qquad\quad+\left|\frac{((\Phi_1-\Phi_2)(z_2)-(\Phi_1-\Phi_2)(y))(\Phi_2(z_1)-\Phi_2(y))(\Phi_1(z_1)-\Phi_2(z_2))}{(\Phi_1(z_1)-\Phi_1(y))(\Phi_1(z_2)-\Phi_1(y))(\Phi_2(z_1)-\Phi_2(y))(\Phi_2(z_2)-\Phi_2(y))}\right|\\
&\qquad\leq C\|\Phi_1-\Phi_2\|_{\text{Lip}}\frac{|z_1-z_2|}{|z_1-y||z_2-y|}.
\end{aligned}
$$

Therefore, by applying Lemma B.0.4 we deduce that

$$
\left\|\int_{\mathbb{D}}\left(K_1[\Phi_1](\cdot,y)-K_1[\Phi_2](\cdot,y)\right)|\Phi_2'(y)|^2dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\leq C\|\Phi_1-\Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}.
$$

Let us deal with K_2. Note that $\frac{1}{\Phi(z)-\Phi(y)}$ is holomorphic, and then we can work with its complex derivative. We write it as

$$
K_2[\Phi](z,y)=\frac{f(z)-f(y)}{(\Phi(z)-\Phi(y))^2}\Phi'(z). \tag{B.0.13}
$$

We wish to apply once again Lemma B.0.4 and for this purpose we should check the suitable estimate for the kernel. From straightforward computations we find that

$$\begin{aligned}|K_2[\Phi_1](z,y)-K_2[\Phi_2](z,y)| &\le \left|\frac{f(y)-f(z)}{(\Phi_1(z)-\Phi_1(y))^2}(\Phi_1'(z)-\Phi_2'(z))\right| \\ &\quad+\left|(f(y)-f(z))\Phi_2'(z)\left(\frac{1}{(\Phi_1(z)-\Phi_1(y))^2}-\frac{1}{(\Phi_2(z)-\Phi_2(y))^2}\right)\right| \\ &\le C\|f\|_{\mathrm{Lip}}\|\Phi_1-\Phi_2\|_{\mathrm{Lip}}|z-y|^{-1}\end{aligned}$$

while for the second hypothesis we write

$$\begin{aligned}&|(K_2[\Phi_1]-K_2[\Phi_2])(z_1,y)-(K_2[\Phi_1]-K_2[\Phi_2])(z_2,y)| \\ &\le \left|\frac{f(y)-f(z_1)}{(\Phi_1(z_1)-\Phi_1(y))^2}(\Phi_1'(z_1)-\Phi_2'(z_1))-\frac{f(y)-f(z_2)}{(\Phi_1(z_2)-\Phi_1(y))^2}(\Phi_1'(z_2)-\Phi_2'(z_2))\right| \\ &\quad+\left|(f(y)-f(z_1))\Phi_2'(z_1)\frac{((\Phi_1-\Phi_2)(z_1)-(\Phi_1-\Phi_2)(y))((\Phi_1+\Phi_2)(z_1)-(\Phi_1+\Phi_2)(y))}{(\Phi_1(z_1)-\Phi_1(y))^2(\Phi_2(z_1)-\Phi_2(y))^2}\right. \\ &\quad\left.-(f(y)-f(z_2))\Phi_2'(z_2)\frac{((\Phi_1-\Phi_2)(z_2)-(\Phi_1-\Phi_2)(y))((\Phi_1+\Phi_2)(z_2)-(\Phi_1+\Phi_2)(y))}{(\Phi_1(z_2)-\Phi_1(y))^2(\Phi_2(z_2)-\Phi_2(y))^2}\right| \\ &\le C\|f\|_{\mathrm{Lip}}(\|\Phi_1-\Phi_2\|_{\mathrm{Lip}}+\|\Phi_1'-\Phi_2'\|_{\mathrm{Lip}})\frac{|z_1-z_2|}{|z_1-y||z_2-y|}.\end{aligned}$$

It follows from Lemma B.0.4 that

$$\left\|\int_{\mathbb{D}}(K_2[\Phi_1](\cdot,y)-K_2[\Phi_2](\cdot,y))\,|\Phi_2'(y)|^2dA(y)\right\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}\le C\|\Phi_1-\Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})},$$

which concludes the proof of the continuity of $\mathscr{F}_1$ with respect to Φ.

Let us now move to the second term $\mathscr{F}_2$. By using a change of variables one may write

$$\mathscr{F}_2[\Phi]f(z)=\int_{\Phi(\mathbb{D})}\frac{1}{\Phi(z)-y}dA(y).$$

First, note that

$$\|\mathscr{F}_2[\Phi]f\|_{L^\infty(\mathbb{D})}\le C,$$

and

$$\|\mathscr{F}_2[\Phi_1]f-\mathscr{F}_2[\Phi_2]f\|_{L^\infty(\mathbb{D})}\le C\|\Phi_1-\Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}.$$

By Cauchy–Pompeiu's formula (B.0.10) we get

$$\begin{aligned}\mathscr{F}_2[\Phi]f(z)=&\pi\overline{\Phi(z)}-\frac{1}{2i}\int_{\Phi(\mathbb{T})}\frac{\overline{\xi}}{\xi-\Phi(z)}d\xi \\ =&\pi\overline{\Phi(z)}-\frac{1}{2i}\int_{\mathbb{T}}\frac{\overline{\Phi(\xi)}}{\Phi(\xi)-\Phi(z)}\Phi'(\xi)d\xi \\ =&\pi\overline{\Phi(z)}-\frac{1}{2i}\mathcal{C}[\Phi](z).\end{aligned}$$

We observe that the mapping $\Phi\in\mathscr{C}^{2,\alpha}(\mathbb{D})\mapsto\overline{\Phi}\in\mathscr{C}^{2,\alpha}(\mathbb{D})$ is well-defined and continuous. So it remains to check that $\mathcal{C}[\Phi]\in\mathscr{C}^{1,\alpha}(\mathbb{D})$ and prove its continuity with respect to Φ. Note first that $\mathcal{C}[\Phi]$ is holomorphic inside the unit disc and its complex derivative is given by

$$[\mathcal{C}[\Phi]]'(z)=\Phi'(z)\int_{\mathbb{T}}\frac{\overline{\Phi(\xi)}}{(\Phi(\xi)-\Phi(z))^2}\Phi'(\xi)d\xi,\quad\forall\, z\in\mathbb{D}.$$

Using a change of variables, we deduce that

$$[\mathcal{C}[\Phi]]'(z) = -\Phi'(z)\int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)}\;\overline{\xi}^2}{\Phi(\xi)-\Phi(z)}d\xi,$$

where we have used the formula

$$\frac{d}{d\xi}\overline{\Phi(\xi)} = -\overline{\xi}^2\Phi'(\overline{\xi}).$$

For this last integral we can use the upcoming Lemma B.0.6 to obtain that $[\mathcal{C}[\Phi]]' \in \mathscr{C}^{0,\alpha}(\mathbb{D})$. Although the last is clear, we show here an alternative procedure useful to check the continuity with respect to Φ. According to [131, Lemma 6.4.8], to show that $[\mathcal{C}[\Phi]]' \in \mathscr{C}^{0,\alpha}(\mathbb{D})$ it suffices to prove that

$$\left|[\mathcal{C}[\Phi]]''(z)\right| \leq C(1-|z|)^{\alpha-1}, \quad \forall z \in \mathbb{D}. \tag{B.0.14}$$

Then, by differentiating we get

$$\begin{aligned}
[\mathcal{C}[\Phi]]''(z) =& -\Phi''(z)\int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)}\;\overline{\xi}^2}{\Phi(\xi)-\Phi(z)}d\xi - \left(\Phi'(z)\right)^2\int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)}\;\overline{\xi}^2}{\left(\Phi(\xi)-\Phi(z)\right)^2}d\xi\\
=:& -\Phi''(z)\mathcal{C}_1[\Phi](z) - \left(\Phi'(z)\right)^2\mathcal{C}_2[\Phi](z).
\end{aligned}$$

For $\mathcal{C}_1[\Phi]$ we simply write

$$\mathcal{C}_1[\Phi](z) = \int_{\mathbb{T}} \frac{\overline{\xi}^2\frac{\overline{\Phi'(\xi)}}{\Phi'(\xi)} - \overline{z}^2\frac{\overline{\Phi'(z)}}{\Phi'(z)}}{\Phi(\xi)-\Phi(z)}\Phi'(\xi)d\xi + 2i\pi\frac{\overline{\Phi'(z)}}{\Phi'(z)}\overline{z}^2.$$

Since $\xi \in \overline{\mathbb{D}} \mapsto \overline{\xi}^2\frac{\overline{\Phi'(\xi)}}{\Phi'(\xi)} \in \mathscr{C}^{0,\alpha}(\overline{\mathbb{D}})$, then we have

$$|\mathcal{C}_1[\Phi](z)| \leq C\left(\int_{\mathbb{T}}\frac{|z-\xi|^\alpha}{|z-\xi|}|d\xi| + 1\right) \leq C(1-|z|)^{\alpha-1}.$$

It remains to estimate $\mathcal{C}_2[\Phi]$. Integration by parts implies

$$\mathcal{C}_2[\Phi](z) = \int_{\mathbb{T}} \frac{\Psi(\xi)}{\Phi(\xi)-\Phi(z)}\Phi'(\xi)d\xi,$$

with

$$\Psi(\xi) = \frac{\left(\frac{\overline{\Phi'(\xi)}\overline{\xi}^2}{\Phi'(\xi)}\right)'}{\Phi'(\xi)}.$$

Since $\Phi \in \mathscr{C}^{2,\alpha}(\mathbb{D})$ and is bi-Lipschitz, then $\Psi \in \mathscr{C}^{0,\alpha}(\mathbb{D})$. Writing

$$\mathcal{C}_2[\Phi](z) = \int_{\mathbb{T}} \frac{\Psi(\xi)-\Psi(z)}{\Phi(\xi)-\Phi(z)}\Phi'(\xi)d\xi + 2i\pi\Psi(z),$$

implies that

$$|\mathcal{C}_2[\Phi]| \leq C\int_{\mathbb{T}}\frac{|\Psi(\xi)-\Psi(z)|}{|\xi-z|}|d\xi| + 2\pi\|\Psi\|_{L^\infty} \leq C\int_{\mathbb{T}}\frac{|\xi-z)|^\alpha}{|\xi-z|}|d\xi| + 2\pi\|\Psi\|_{L^\infty}.$$

Thus, we have

$$|\mathcal{C}_2[\Phi](z)| \leq C(1-|z|)^{\alpha-1}.$$

Putting together the previous estimates, we deduce (B.0.14) which implies $\mathscr{F}_2[\Phi] \in \mathscr{C}^{1,\alpha}(\mathbb{D})$.

It remains to check the continuity of $\mathscr{F}_2$ with respect to Φ. Splitting $\mathscr{F}_2[\Phi_1] - \mathscr{F}_2[\Phi_2]$ as

$$\begin{aligned}\mathscr{F}_2[\Phi_1]f(z) - \mathscr{F}_2[\Phi_2]f(z) = &\int_{\mathbb{D}} \frac{|\Phi_1'(y)|^2 - |\Phi_2'(y)|^2}{\Phi_1(z)-\Phi_1(y)} dA(y)\\ &+ \int_{\mathbb{D}} \frac{(\Phi_2-\Phi_1)(z) - (\Phi_2-\Phi_1)(y)}{(\Phi_1(z)-\Phi_1(y))(\Phi_2(z)-\Phi_2(y))} |\Phi_2'(y)|^2 dA(y),\end{aligned}$$

combined with Lemma B.0.4 yield

$$\|\mathscr{F}_2[\Phi_1] - \mathscr{F}_2[\Phi_2]\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \leq C\|\Phi_1 - \Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}.$$

Now, we need to prove a similar inequality for its derivative,

$$\|\mathscr{F}_2'[\Phi_1] - \mathscr{F}_2'[\Phi_2]\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \leq C\|\Phi_1 - \Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}.$$

Since $\Phi \in \mathscr{C}^{2,\alpha}(\mathbb{D}) \mapsto \Phi' \in \mathscr{C}^{1,\alpha}(\mathbb{D})$ is clearly continuous, we just have to prove that

$$\left|[\mathcal{C}[\Phi_1]]''(z) - [\mathcal{C}[\Phi_2]]''(z)\right| \leq C\|\Phi_1-\Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}(1-|z|)^{\alpha-1}, \quad \forall z \in \mathbb{D}.$$

It is enough to check the above estimate for $\mathcal{C}_1$ and $\mathcal{C}_2$. Let us show how dealing with the first one, and the same arguments can be applied for $\mathcal{C}_2$. We have

$$\begin{aligned}|\mathcal{C}_1[\Phi_1](z) - \mathcal{C}_1[\Phi_2](z)| \leq &\left| \int_{\mathbb{T}} \frac{\overline{\xi}^2\left(\frac{\overline{\Phi_1'(\xi)}}{\Phi_1'(\xi)} - \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)}\right) - \overline{z}^2\left(\frac{\overline{\Phi_1'(z)}}{\Phi_1'(z)} - \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)}\right)}{\Phi_1(\xi)-\Phi_1(z)} \Phi_1'(\xi) d\xi \right|\\
&+ \left| \int_{\mathbb{T}} \frac{\overline{z}^2\frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} - \overline{\xi}^2\frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)}}{\Phi_1(\xi)-\Phi_1(z)} (\Phi_1'(\xi) - \Phi_2'(\xi)) d\xi \right|\\
&+ \left| \int_{\mathbb{T}} \frac{(\Phi_1-\Phi_2)(z) - (\Phi_1-\Phi_2)(\xi)}{(\Phi_1(\xi)-\Phi_1(z))(\Phi_2(\xi)-\Phi_2(z))} \left(\overline{z}^2\frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} - \overline{\xi}^2\frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)}\right) \Phi_2'(\xi) d\xi \right|\\
&+ 2\pi\, \overline{z}^2 \left| \frac{\overline{\Phi_1'(z) - \Phi_2'(z)}}{\Phi_1'(z)} + \overline{\Phi_2'(z)} \frac{\Phi_2'(z) - \Phi_1'(z)}{\Phi_1'(z)\Phi_2'(z)} \right|\\
\leq &C\|\Phi_1 - \Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D})}|(1-|z|)^{1-\alpha},\end{aligned}$$

where we have used that

$$\left|\overline{\xi}^2\left(\frac{\overline{\Phi_1'(\xi)}}{\Phi_1'(\xi)} - \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)}\right) - \overline{z}^2\left(\frac{\overline{\Phi_1'(z)}}{\Phi_1'(z)} - \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)}\right)\right| \leq C\|\Phi_1-\Phi_2\|_{\mathscr{C}^{2,\alpha}(\mathbb{D}}|z-\xi|.$$

Then, we obtain the inequality for $\mathcal{C}_1[\Phi]$, which concludes the continuity with respect to Φ. □

The following result deals with a Calderon-Zygmund type estimate, which will be necessary in the later development. The techniques used are related to the well-known T(1)-Theorem of Wittmann, see [144]. Let us define

$$\mathscr{K}f(z) := \int_{\mathbb{T}} K(z,\xi)f(\xi)d\xi, \quad \forall z \in \mathbb{D}. \tag{B.0.15}$$

Lemma B.0.6. *Let $\alpha \in (0,1)$ and $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ smooth outside the diagonal that satisfies*

$$|K(z_1, y)| \leq C_0 |z_1 - y|^{-1}, \tag{B.0.16}$$

$$|K(z_1, y) - K(z_2, y)| \leq C_0 \frac{|z_1 - z_2|}{|z_1 - y|^2}, \quad \text{if } 2|z_1 - z_2| \leq |z_1 - y|, \tag{B.0.17}$$

$$\mathscr{K}(\mathrm{Id}) \in \mathscr{C}^{0,\alpha}(\mathbb{D}), \tag{B.0.18}$$

$$\left| \int_{\partial(\mathbb{D} \cap B_{z_1}(\rho))} K(z_1, \xi) d\xi \right| < C_0, \tag{B.0.19}$$

for any $z_1, z_2 \neq y \in \mathbb{D}$ and $\rho > 0$, where C_0 is a positive constant that does not depend on z_1, z_2, y and ρ. Then,

$$\mathscr{K} : \mathscr{C}^{0,\alpha}(\mathbb{D}) \to \mathscr{C}^{0,\alpha}(\mathbb{D})$$

is continuous, with the estimate

$$\|\mathscr{K} f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \leq C C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})},$$

where C is a constant depending only on α.

Proof. From (B.0.16) and (B.0.18), we get easily that

$$\begin{aligned} |\mathscr{K} f(z)| &\leq \left| \int_{\mathbb{T}} K(z,\xi)(f(\xi) - f(z)) d\xi \right| + \left| f(z) \int_{\mathbb{T}} K(z,\xi) d\xi \right| \\ &\leq C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \int_{\mathbb{T}} \frac{|d\xi|}{|z-\xi|^{1-\alpha}} + C\|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \\ &\leq C C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})}. \end{aligned}$$

Taking $z_1, z_2 \in \mathbb{D}$, we define $d = |z_1 - z_2|$. We write

$$\begin{aligned} &\int_{\mathbb{T}} K(z_1,\xi) f(\xi) d\xi - \int_{\mathbb{T}} K(z_2,\xi) f(\xi) d\xi \\ &\qquad = \int_{\mathbb{T}} K(z_1,\xi)(f(\xi) - f(z_1)) d\xi - \int_{\mathbb{T}} K(z_2,\xi)(f(\xi) - f(z_1)) d\xi \\ &\qquad\quad + f(z_1) \int_{\mathbb{T}} K(z_1,\xi) d\xi - f(z_1) \int_{\mathbb{T}} K(z_2,\xi) d\xi \\ &\qquad = \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_1,\xi)(f(\xi) - f(z_1)) d\xi - \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_2,\xi)(f(\xi) - f(z_1)) d\xi \\ &\qquad\quad + \int_{\mathbb{T} \cap B^c_{z_1}(3d)} (K(z_1,\xi) - K(z_2,\xi))(f(\xi) - f(z_1)) d\xi \\ &\qquad\quad + f(z_1) \int_{\mathbb{T}} K(z_1,\xi) d\xi - f(z_1) \int_{\mathbb{T}} K(z_2,\xi) d\xi \\ &\qquad =: I_1 - I_2 + I_3 + f(z_1)(I_4 - I_5). \end{aligned}$$

Using (B.0.18), we achieve

$$|I_4 - I_5| \leq C|z_1 - z_2|^\alpha.$$

Let us work with I_1 using the Layer Cake Lemma, see [103]. We use that $|\mathbb{T} \cap B_x(\rho)| \leq C\rho$, for any $\rho > 0$ and $x \in \mathbb{R}^2$, which means that it is 1-Ahlfors regular curve. In fact, taking any $z \in \mathbb{D}$ and $\rho > 0$ ones has that

$$\int_{\mathbb{T} \cap B_z(\rho)} \frac{|d\xi|}{|z-\xi|^{1-\alpha}} \quad = \quad \int_0^\infty \left| \left\{ \xi \in \mathbb{T} \cap B_z(\rho) : \frac{1}{|z-\xi|^{1-\alpha}} \geq \lambda \right\} \right| d\lambda,$$

$$
\begin{aligned}
&= \int_0^{\infty} \left|\left\{\xi \in \mathbb{T} \cap B_z(\rho) : |z-\xi| \le \lambda^{\frac{-1}{1-\alpha}}\right\}\right| d\lambda \\
&= \int_0^{\rho^{\alpha-1}} |\{\xi \in \mathbb{T} : |z-\xi| \le \rho\}|\, d\lambda + \int_{\rho^{\alpha-1}}^{+\infty} \left|\left\{\xi \in \mathbb{T} : |z-\xi| \le \lambda^{\frac{-1}{1-\alpha}}\right\}\right| d\lambda \\
&\le C\left(\rho\rho^{\alpha-1} + \int_{\rho^{\alpha-1}}^{+\infty} \lambda^{\frac{-1}{1-\alpha}} d\lambda\right) \le C\rho^{\alpha},
\end{aligned}
$$

where $|\cdot|$ inside the integral denotes the arch length measure. Applying the last estimate to I_1, we find

$$
|I_1| \le C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} \int_{\mathbb{T}\cap B_{z_1}(3d)} \frac{|d\xi|}{|z_1-\xi|^{1-\alpha}} \le CC_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2|^{\alpha}.
$$

For the term I_3, we get

$$
|I_3| \le C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2| \int_{\mathbb{T}\cap B_{z_1}(3d)^c} \frac{|d\xi|}{|z_1-\xi|^{2-\alpha}},
$$

by (B.0.17). Now, we use again the Layer Cake Lemma, obtaining

$$
\begin{aligned}
\int_{\mathbb{T}\cap B_z(\rho)^c} \frac{|d\xi|}{|z-\xi|^{2-\alpha}} &= \int_0^{\infty} \left|\left\{\xi \in \mathbb{T} : \frac{1}{|z-\xi|^{2-\alpha}} \ge \lambda, |z-\xi| \ge \rho\right\}\right| d\lambda, \\
&= \int_0^{\infty} \left|\left\{\xi \in \mathbb{T} : |z-\xi| \le \lambda^{\frac{-1}{2-\alpha}}, |z-\xi| \ge \rho\right\}\right| d\lambda \\
&= \int_0^{\rho^{\alpha-2}} \left|\left\{\xi \in \mathbb{T} : \rho \le |z-\xi| \le \lambda^{\frac{-1}{2-\alpha}}\right\}\right| d\lambda \\
&\le C \int_0^{\rho^{\alpha-2}} \lambda^{\frac{-1}{2-\alpha}} d\lambda \le C\rho^{\alpha-1}.
\end{aligned}
$$

Therefore,

$$
|I_3| \le C_0 \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2|^{\alpha}.
$$

It remains to estimate I_2. First, let us write it as

$$
\begin{aligned}
I_2 &= \int_{\mathbb{T}\cap B_{z_1}(3d)} K(z_2,\xi)(f(\xi) - f(z_2))d\xi + (f(z_2) - f(z_1)) \int_{\mathbb{T}\cap B_{z_1}(3d)} K(z_2,\xi)d\xi \\
&=: H_1 + (f(z_2) - f(z_1))H_2.
\end{aligned}
$$

H_1 can be estimated as I_1 noting that $B_{z_1}(3d) \subset B_{z_2}(4d)$. To finish, we just need to check that H_2 is bounded. Decompose it as

$$
H_2 = \int_{\mathbb{T}\cap B_{z_2}(2d)} K(z_2,\xi)d\xi + \int_{\mathbb{T}\cap B_{z_1}(3d)\cap B_{z_2}(2d)^c} K(z_2,\xi)d\xi =: J_1 + J_2,
$$

since $B_{z_2}(2d) \subset B_{z_1}(3d)$. Note that

$$
|J_2| \le C_0 \int_{\mathbb{T}\cap B_{z_1}(3d)\cap B_{z_2}(2d)^c} \frac{|d\xi|}{|z_2-\xi|} \le CC_0 d^{-1} |\mathbb{T}\cap B_{z_1}(3d)| \le CC_0.
$$

For the last term, we write

$$
J_1 = \int_{\partial(\mathbb{D}\cap B_{z_2}(2d))} K(z_2,\xi)d\xi - \int_{\mathbb{D}\cap\partial B_{z_2}(2d)} K(z_2,\xi)d\xi
$$

By using condition (B.0.19), we get that the first integral is bounded. For the second one, we obtain

$$\left|\int_{\mathbb{D}\cap\partial B_{z_2}(2d)} K(z_2,\xi)d\xi\right| \leq C_0 \int_{\mathbb{D}\cap\partial B_{z_2}(2d)} \frac{|d\xi|}{|z_2-\xi|} = \frac{1}{2d}|\mathbb{D}\cap\partial B_{z_2}(2d)| \leq C.$$

Combining all the estimates, we achieved the announced result. □

In the following result, we deal with the Cauchy integral defined as

$$\mathscr{I}[\Phi](f)(z) := \int_{\mathbb{T}} \frac{f(\xi)\Phi'(\xi)}{\Phi(z)-\Phi(\xi)}d\xi. \tag{B.0.20}$$

Note that this classical operator is fully studied in [99] in the case that $\Phi = Id$, then there we will adapt that proof.

Lemma B.0.7. *Let $\alpha \in (0,1)$ and $\Phi : \mathbb{D} \to \Phi(\mathbb{D}) \subset \mathbb{C}$ be a conformal bi-Lipschitz function of class $\mathscr{C}^{2,\alpha}(\mathbb{D})$. Therefore, we have that*

$$\mathscr{I}[\Phi] : \mathscr{C}^{0,\alpha}(\mathbb{D}) \to \mathscr{C}^{0,\alpha}(\mathbb{D}),$$

is continuous. Moreover, $\mathscr{I} : \Phi \in \mathscr{U} \mapsto \mathscr{I}[\Phi]$ is continuous, where $\mathscr{U}$ is defined in Lemma B.0.5.

Proof. Note that

$$\mathscr{I}[\Phi](f)(z) = \int_{\Phi(\mathbb{T})} \frac{(f\circ\Phi^{-1})(\xi)}{\Phi(z)-\xi}d\xi = \mathcal{C}[f\circ\Phi^{-1}](\Phi(z)),$$

where $\mathcal{C}$ is the Cauchy Integral. Then, it is classical, see [99], that

$$\begin{aligned}||\mathscr{I}[\Phi](f)||_{\mathscr{C}^{0,\alpha}(\mathbb{D})} = ||\mathcal{C}[f\circ\Phi^{-1}]\circ\Phi||_{\mathscr{C}^{0,\alpha}(\mathbb{D})} &\leq C||f\circ\Phi^{-1}||_{\mathscr{C}^{0,\alpha}(\mathbb{D})}||\Phi||_{\mathrm{Lip}(\mathbb{D})}\\ &\leq C||f||_{\mathscr{C}^{0,\alpha}(\mathbb{D})}||\Phi^{-1}||_{\mathrm{Lip}(\mathbb{D})}||\Phi||_{\mathrm{Lip}(\mathbb{D})}.\end{aligned}$$

To deal with the continuity with respect to the conformal map, we write

$$\mathscr{I}[\Phi_1]f(z) - \mathscr{I}[\Phi_2]f(z) = \int_{\mathbb{T}} f(\xi)\left(\frac{\Phi_1'(\xi)}{\Phi_1(z)-\Phi_1(\xi)} - \frac{\Phi_2'(\xi)}{\Phi_2(z)-\Phi_2(\xi)}\right)d\xi =: \int_{\mathbb{T}} f(\xi)K(z,\xi)d\xi.$$

We will check that K verifies (B.0.16)-(B.0.19) in order to use Lemma B.0.6. Straightforward computations yield

$$\begin{aligned}&|K(z_1,y)| \leq C||\Phi_1-\Phi_2||_{\mathscr{C}^{1,\alpha}(\mathbb{D})}|z_1-y|^{-1},\\ &|K(z_1,y)-K(z_2,y)| \leq C||\Phi_1-\Phi_2||_{\mathscr{C}^{1,\alpha}(\mathbb{D})}\frac{|z_1-z_2|}{|z_1-y|^2}, \quad \text{if } 2|z_1-z_2| \leq |z_1-y|,\end{aligned}$$

using that $|z_2-y| \geq |z_1-y|-|z_1-z_2| \geq \frac{1}{2}|z_1-y|$ in the second property, which concerns (B.0.16)-(B.0.17). Moreover,

$$\int_{\mathbb{T}} K(z,\xi)d\xi = \int_{\Phi_1(\mathbb{T})} \frac{d\xi}{\Phi_1(z)-\xi} - \int_{\Phi_2(\mathbb{T})} \frac{d\xi}{\Phi_2(z)-\xi} = 0,$$

which implies (B.0.18). In fact,

$$\int_{\partial(\mathbb{D}\cap B_z(\rho))} K(z,\xi)d\xi = \int_{\Phi_1(\partial(\mathbb{D}\cap B_z(\rho)))} \frac{d\xi}{\Phi_1(z)-\xi} - \int_{\Phi_2(\partial(\mathbb{D}\cap B_z(\rho)))} \frac{d\xi}{\Phi_2(z)-\xi} = C_0,$$

by applying the Residue Theorem, and where C_0 that does not depend on ρ neither z, which agrees with (B.0.19). Then, we achieve the proof using Lemma B.0.6. □

We give the explicit expressions of some integrals which appear in the analysis of the linearized operator.

Proposition B.0.8. *Let $\alpha \in (0,1)$. Given $h \in \mathscr{C}_s^{1,\alpha}(\mathbb{D})$, $k \in \mathscr{H}\mathscr{C}^{2,\alpha}(\mathbb{D})$, where the spaces are defined in Section 2.2.2, and a radial function $f_0 \in \mathscr{C}(\mathbb{D})$, the following identities*

$$\int_{\mathbb{D}} \frac{k(z)-k(y)}{(z-y)^2} f_0(y)dA(y) = 2\pi \sum_{n\geq 1} A_n z^{n-1} \left[\int_0^{|z|} r f_0(r)dr - n\int_{|z|}^1 r f_0(r)dr\right],$$

$$\int_{\mathbb{D}} \frac{f_0(y)}{z-y} \mathrm{Re}[k'(y)]dA(y) = \pi \sum_{n\geq 1} A_n(n+1)\left[-z^{n-1}\int_{|z|}^1 r f_0(r)dr + \frac{\overline{z}^{n+1}}{|z|^{2(n+1)}}\int_0^{|z|} r^{2n+1} f_0(r)dr\right],$$

$$\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) = 2\pi \frac{\overline{z}}{|z|^2}\int_0^{|z|} r f_0(r)dr,$$

$$\int_{\mathbb{D}} \frac{h(y)}{z-y} dA(y) = \pi \sum_{n\geq 1}\left[-z^{n-1}\int_{|z|}^1 \frac{1}{r^{n-1}} h_n(r)dr + \frac{\overline{z}^{n+1}}{|z|^{2n+2}}\int_0^{|z|} r^{n+1} h_n(r)dr\right],$$

$$\int_{\mathbb{D}} \log|z-y| h(y)dA(y) = -\pi \sum_{n\geq 1}\cos(n\theta)\frac{1}{n}\left[|z|^n \int_{|z|}^1 \frac{1}{r^{n-1}} h_n(r)dr + \frac{1}{|z|^n}\int_0^{|z|} |z|^{n+1} h_n(r)dr\right],$$

$$\int_{\mathbb{D}} \frac{k(z)-k(y)}{z-y} f_0(y)dA(y) = 2\pi \sum_{n\geq 1} A_n z^n \int_0^1 s f_0(s)ds,$$

$$\begin{aligned}\int_{\mathbb{D}} \log|z-y| f_0(y)\mathrm{Re}[k'(y)]dA(y) = & -\pi\sum_{n\geq 1} A_n \frac{n+1}{n}\cos(n\theta)\\ & \times\left[|z|^n\int_{|z|}^1 r f_0(r)dr + \frac{1}{|z|^n}\int_0^{|z|} r^{2n+1} f_0(r)dr\right],\end{aligned}$$

$$\int_{\mathbb{D}} \log|z-y| f_0(y)dA(y) = 2\pi\left[\int_0^{|z|}\frac{1}{\tau}\int_0^{\tau} r f_0(r)dr - \int_0^1 \frac{1}{\tau}\int_0^{\tau} r f_0(r)dr\right]$$

holds for $z \in \overline{\mathbb{D}}$.

Proof. Note that h and k can be given by

$$h(z) = \sum_{n\geq 1} h_n(r)\cos(n\theta), \quad k(z) = z\sum_{n\geq 1} A_n z^n,$$

where $z = re^{i\theta} \in \mathbb{D}$.

(1) Using the expression for the function k, the integral to be computed takes the form

$$\int_{\mathbb{D}} \frac{z^{n+1}-y^{n+1}}{z-y}\frac{f_0(y)}{z-y}dA(y).$$

An expansion of the function inside the integral provides

$$\int_{\mathbb{D}} \frac{z^{n+1}-y^{n+1}}{z-y}\frac{f_0(y)}{z-y}dA(y) = \sum_{k=0}^{n} z^{n-k}\int_{\mathbb{D}} y^k \frac{f_0(y)}{z-y}dA(y).$$

The use of polar coordinates yields

$$\int_{\mathbb{D}} y^k \frac{f_0(y)}{z-y}dA(y) = i\int_0^1 r^k f_0(r)\int_{\mathbb{T}} \frac{\xi^{k-1}}{\xi-\frac{z}{r}}d\xi. \tag{B.0.21}$$

We split our study in the cases $k=0$ and $k\geq 1$ by making use of the Residue Theorem. For $k=0$, we obtain

$$\int_{\mathbb{T}}\frac{1}{\xi}\frac{1}{\xi-\frac{z}{r}}d\xi=\begin{cases}0, & |z|\leq r,\\ -2\pi i\frac{r}{z}, & |z|\geq r,\end{cases}$$

whereas, we find

$$\int_{\mathbb{T}}\frac{\xi^{k-1}}{\xi-\frac{z}{r}}d\xi=\begin{cases}2\pi i\frac{z^{k-1}}{r^{k-1}}, & |z|\leq r,\\ 0, & |z|\geq r,\end{cases}$$

for any $k\geq 1$. This allows us to have the following expression

$$\begin{aligned}\int_{\mathbb{D}}\frac{z^{n+1}-y^{n+1}}{z-y}\frac{f_0(y)}{z-y}dA(y) &= 2\pi z^{n-1}\int_0^{|z|}rf_0(r)dr-2\pi\sum_{k=1}^{n}z^{n-k}z^{k-1}\int_{|z|}^1 rf_0(r)dr\\ &= 2\pi z^{n-1}\left[\int_0^{|z|}rf_0(r)dr-n\int_{|z|}^1 rf_0(r)dr\right].\end{aligned}$$

(2) Note that $k'(z)=\sum_{n\geq 1}A_n(n+1)z^n$. Then, the integral to be analyzed is

$$\int_{\mathbb{D}}\frac{f_0(y)}{z-y}\operatorname{Re}[k'(y)]dA(y)=\sum_{n\geq 1}\frac{A_n(n+1)}{2}\int_{\mathbb{D}}\frac{f_0(y)}{z-y}\left(y^n+\overline{y}^n\right)dA(y).$$

We study the two terms in the integral by using polar coordinates and the Residue Theorem. For the first one we have

$$\int_{\mathbb{D}}\frac{f_0(y)}{z-y}y^n dA(y)=i\int_0^1 r^n f_0(r)\int_{\mathbb{T}}\frac{\xi^{n-1}}{\xi-\frac{z}{r}}d\xi dr=-2\pi z^{n-1}\int_{|z|}^1 rf_0(r)dr.$$

In the same way, the second one can be written as

$$\begin{aligned}\int_{\mathbb{D}}\frac{f_0(y)}{z-y}\overline{y}^n dA(y) &= i\int_0^1 f_0(r)r^n\int_{\mathbb{T}}\frac{1}{\xi^{n+1}}\frac{1}{\xi-\frac{z}{r}}d\xi dr\\ &= 2\pi\frac{1}{z^{n+1}}\int_0^{|z|}f_0(r)r^{2n+1}dr=\frac{2\pi\overline{z}^{n+1}}{|z|^{2n+2}}\int_0^{|z|}f_0(r)r^{2n+1}dr,\end{aligned}$$

that concludes the proof.

(3) This integral reads as

$$\int_{\mathbb{D}}\frac{f_0(y)}{z-y}dA(y)=i\int_0^1 f_0(r)\int_{\mathbb{T}}\frac{1}{\xi}\frac{1}{\xi-\frac{z}{r}}d\xi dr=\frac{2\pi}{z}\int_0^{|z|}rf_0(r)dr=\frac{2\pi\overline{z}}{|z|^2}\int_0^{|z|}rf_0(r)dr.$$

(4) We use the expression of $h(z)$ to deduce that

$$\int_{\mathbb{D}}\frac{h(y)}{z-y}dA(y)=\frac{1}{2}\sum_{n\geq 1}\int_{\mathbb{D}}\frac{h_n(r)(e^{in\theta}+e^{-in\theta})}{z-y}dA(y).$$

The two terms involved in the integral can be computed as follows

$$\int_{\mathbb{D}}\frac{h_n(r)e^{in\theta}}{z-y}dA(y)\;=\;i\int_0^1 h_n(r)\int_{\mathbb{T}}\frac{\xi^{n-1}}{\xi-\frac{z}{r}}d\xi dr=-2\pi z^{n-1}\int_{|z|}^1\frac{1}{r^{n-1}}h_n(r)dr,$$

$$\int_{\mathbb{D}} \frac{h_n(r)e^{-in\theta}}{z-y} dA(y) = i\int_0^1 h_n(r) \int_{\mathbb{T}} \frac{1}{\xi^{n+1}} \frac{1}{\xi - \frac{z}{r}} d\xi dr$$
$$= \frac{2\pi}{z^{n+1}} \int_0^{|z|} r^{n+1} h_n(r) dr = \frac{2\pi \overline{z}^{n+1}}{|z|^{2(n+1)}} \int_0^{|z|} r^{n+1} h_n(r) dr.$$

(5) Let us differentiate with respect to r having that

$$\partial_r \int_{\mathbb{D}} \log|re^{i\theta} - y| h(y) dA(y) = \mathrm{Re}\left[\frac{z}{r}\int_{\mathbb{D}} \frac{h(y)}{z-y} dA(y)\right]$$
$$= \mathrm{Re}\left[\frac{z}{r}\sum_{n\geq 1} \pi \left[-z^{n-1}\int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{z^{n+1}}\int_0^r s^{n+1} h_n(s) ds\right]\right]$$
$$= \pi \sum_{n\geq 1} \cos(n\theta) \left[-r^{n-1}\int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{r^{n+1}}\int_0^r s^{n+1} h_n(s) ds\right].$$

This last integral was computed before by the Residue Theorem. Now, we realize that

$$-\partial_r \frac{1}{n}\left[r^n \int_r^1 \frac{1}{s^{n-1}} h_n(s) ds \; + \; \frac{1}{r^n}\int_0^r s^{n+1} h_n(s) ds\right]$$
$$= -r^{n-1}\int_r^1 \frac{1}{s^{n-1}} h_n(s) ds + \frac{1}{r^{n+1}}\int_0^r s^{n+1} h_n(s) ds.$$

Then, we obtain

$$\int_{\mathbb{D}} \log|z-y| h(y) dA(y) = -\pi \sum_{n\geq 1} \frac{1}{n}\left[r^n \int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{r^n}\int_0^r s^{n+1} h_n(s) ds\right] \cos(n\theta) + H(\theta),$$

where H is a function that only depends on θ. Taking $r = 0$ we have that

$$H(\theta) = \int_{\mathbb{D}} \log(|y|) h(y) dA(y) = 0.$$

The last is equal to zero due to the form of the function h: $h(re^{i\theta}) = \sum_{n\geq 1} h_n(r)\cos(n\theta)$.

(6) This integral can be done by spliting it as follows

$$\int_{\mathbb{D}} \frac{k(z)-k(y)}{z-y} f_0(y) dA(y) = k(z)\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) - \int_{\mathbb{D}} \frac{k(y) f_0(y)}{z-y} dA(y)$$
$$= \sum_{n\geq 1} A_n \left[z^{n+1}\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) - \int_{\mathbb{D}} \frac{y^{n+1} f_0(y)}{z-y} dA(y)\right].$$

Note that these integrals have be done before. Hence, we conclude using Integral (3) for the first one and (B.0.21) for the second one.

(7) Similarly to Integral (5), we differentiate with respect to r

$$\partial_r \int_{\mathbb{D}} \log|re^{i\theta} - y| f_0(y) \mathrm{Re}\left[k'(y)\right] dA(y) = \mathrm{Re}\left[\frac{z}{r}\int_{\mathbb{D}} \frac{f_0(y)\mathrm{Re}\left[k'(y)\right]}{z-y} dA(y)\right]$$
$$= \pi \sum_{n\geq 1} A_n (n+1) \mathrm{Re}\left[\frac{z}{r}\left[-z^{n-1}\int_r^1 s f_0(s) ds + \frac{\overline{z}^{n+1}}{r^{2(n+1)}}\int_0^r s^{2n+1} f_0(s) ds\right]\right],$$

$$= \pi \sum_{n\geq 1} A_n(n+1)\cos(n\theta)\left[-r^{n-1}\int_r^1 sf_0(s)ds + \frac{1}{r^{n+1}}\int_0^r s^{2n+1}f_0(s)ds\right],$$

where we use Integral (2). With the same argument than in Integral (5) we realize that

$$-\partial_r \frac{1}{n}\left[r^n \int_r^1 sf_0(s)ds \ + \ \frac{1}{r^n}\int_0^r s^{2n+1}f_0(s)ds\right]$$
$$= -r^{n-1}\int_r^1 sf_0(s)ds + \frac{1}{r^{n+1}}\int_0^r s^{2n+1}f_0(s)ds,$$

and hence

$$\int_{\mathbb{D}} \log|re^{i\theta}-y|f_0(y)\text{Re}\left[k'(y)\right]dA(y)$$
$$= -\pi\sum_{n\geq 1} A_n \frac{n+1}{n}\cos(n\theta)\left[r^n\int_r^1 sf_0(s)ds + \frac{1}{r^n}\int_0^r s^{2n+1}f_0(s)ds\right] + H(\theta),$$

where H is a function that only depends on θ. Evaluating in $r=0$ as in Integral (5), we get that $H\equiv 0$, obtaining the announced identity.

(8) As in Integral (5) and (7) we differentiate with respect to r having

$$\partial_r \int_{\mathbb{D}} \log|re^{i\theta}-y|f_0(y)dA(y) \quad = \quad \text{Re}\left[\frac{z}{r}\int_{\mathbb{D}}\frac{f_0(y)}{z-y}dA(y)\right] = 2\pi\frac{1}{r}\int_0^r sf_0(s)ds,$$

where the last integral is done in Integral (3). Hence,

$$\int_{\mathbb{D}} \log|re^{i\theta}-y|f_0(y)dA(y) = 2\pi\int_0^r \frac{1}{\tau}\int_0^\tau sf_0(s)dsd\tau + H(\theta),$$

where H is a function that only depends on θ. Evaluating in $r=0$ we get that

$$H(\theta) = \int_0^1\int_0^{2\pi} s\log s f_0(s)dsd\theta = -2\pi\int_0^1\frac{1}{\tau}\int_0^\tau sf_0(s)dsd\tau,$$

concluding the proof. □

Appendix C

Special functions

We give a short introduction to the Gauss hypergeometric functions and discuss some of their basic properties. Recall that for any real numbers $a, b \in \mathbb{R}$, $c \in \mathbb{R}\backslash(-\mathbb{N})$ the hypergeometric function $z \mapsto F(a,b;c;z)$ is defined on the open unit disc $\mathbb{D}$ by the power series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad \forall z \in \mathbb{D}. \tag{C.0.1}$$

The Pochhammer symbol $(x)_n$ is defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n \geq 1, \end{cases}$$

and verifies

$$(x)_n = x\,(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)\,(x)_n.$$

The series converges absolutely for all values of $|z| < 1$. For $|z| = 1$ we have that it converges absolutely if $\text{Re}(a+b-c) < 0$ and it diverges if $1 \leq \text{Re}(a+b-c)$. See [13] for more details.

We recall the integral representation of the hypergeometric function, see for instance [126, p. 47]. Assume that $\text{Re}(c) > \text{Re}(b) > 0$, then we have

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,dx, \quad \forall z \in \mathbb{C}\backslash[1,+\infty). \tag{C.0.2}$$

Notice that this representation shows that the hypergeometric function initially defined in the unit disc admits an analytic continuation to the complex plane cut along $[1,+\infty)$. Another useful identity is the following:

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right), \quad \forall\,|\text{arg}(1-z)| < \pi, \tag{C.0.3}$$

for $\text{Re}\,c > \text{Re}\,b > 0$.

The function $\Gamma : \mathbb{C}\backslash\{-\mathbb{N}\} \to \mathbb{C}$ refers to the gamma function, which is the analytic continuation to the negative half plane of the usual gamma function defined on the positive half-plane $\{\text{Re}\ z > 0\}$. It is defined by the integral representation

$$\Gamma(z) = \int_0^{+\infty} \tau^{z-1} e^{-\tau} d\tau,$$

and satisfies the relation $\Gamma(z+1) = z\,\Gamma(z),\ \forall z \in \mathbb{C}\backslash(-\mathbb{N})$. From this we deduce the identities

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_n = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)},$$

provided that all the quantities in the right terms are well-defined.
We can differentiate the hypergeometric function obtaining

$$\frac{d^k F(a,b;c;z)}{dz^k} = \frac{(a)_k (b)_k}{(c)_k} F(a+k, b+k; c+k; z), \tag{C.0.4}$$

for $k \in \mathbb{N}$. Depending on the parameters, the hypergeometric function behaves differently at 1. When $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$ and $\mathrm{Re}(c-a-b) > 0$, it can be shown that it is absolutely convergent on the closed unit disc and one finds the expression

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{C.0.5}$$

whose proof can be found in [126, Pag. 49]. However, in the case $a+b=c$, the hypergeometric function exhibits a logarithmic singularity as follows

$$\lim_{z\to 1-} \frac{F(a,b;c;z)}{-\ln(1-z)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, \tag{C.0.6}$$

see for instance [6] for more details. Next we recall some Kummer's quadratic transformations of the hypergeometric series, see [126],

$$c\,F(a,b;c;z) - (c-a)\,F(a,b;c+1;z) - a\,F(a+1,b;c+1;z) = 0, \tag{C.0.7}$$

$$(b-c)F(a,b-1;c;z) + (c-a-b)F(a,b;c;z) - a(z-1)F(a+1,b;c;z) = 0, \tag{C.0.8}$$

$$\frac{(2c-a-b+1)z-c}{c} F(a,b;c+1;z) + \frac{(a-c-1)(c-b+1)z}{c(c+1)} F(a,b;c+2;z) = F(a,b;c;z)(z-1). \tag{C.0.9}$$

Other formulas which have been used in the preceding sections are

$$\begin{cases} \displaystyle\int_0^1 F(a,b;c;\tau z)\tau^{c-1} d\tau = \frac{1}{c} F(a,b;c+1;z), \\ \displaystyle\int_0^1 F(a,b;c;\tau z)\tau^{c-1}(1-\tau) d\tau = \frac{1}{c(c+1)} F(a,b;c+2;z). \end{cases} \tag{C.0.10}$$

Let us also introduce the following lemma:

Lemma C.0.1. *Let $n \geq 0$, $\beta \in \mathbb{N}$ and $A \in \mathbb{R}$, then*

$$\int_0^{2\pi} \frac{\cos(n\theta)}{(A-\cos(\theta))^{\frac{\beta}{2}}} d\theta = \frac{2\pi}{(1+A)^{\frac{\beta}{2}+n}} \frac{\left(\frac{\beta}{2}\right)_n 2^n \left(\frac{1}{2}\right)_n}{(2n)!} F\left(n+\frac{\beta}{2}, n+\frac{1}{2}; 2n+1; \frac{2}{1+A}\right),$$

for any $A \in \mathbb{R}$ such that $\left|\frac{2}{1+A}\right| < 1$.

Proof. By a change of variables and using $\cos(2\theta) = 2\cos^2(\theta) - 1$, we arrive at

$$\int_0^{2\pi} \frac{\cos(n\theta)}{(A-\cos(\theta))^{\frac{\beta}{2}}} d\theta = \frac{2}{(1+A)^{\frac{\beta}{2}}} \int_0^{\pi} \frac{\cos(2n\theta)}{(1-\frac{2}{1+A}\cos^2(\theta))^{\frac{\beta}{2}}} d\theta.$$

Since $\left|\frac{2}{1+A}\right| < 1$, we can use Taylor series in the following way,

$$\left(1-\frac{2}{1+A}\cos^2(\theta)\right)^{-\frac{\beta}{2}} = \sum_{m\geq 0} \frac{\left(\frac{\beta}{2}\right)_m}{m!} \frac{2^m}{(1+A)^m} \cos^{2m}(\theta).$$

Then,

$$\int_0^{2\pi} \frac{\cos(n\theta)}{(A-\cos(\theta))^{\frac{\beta}{2}}} d\theta = \frac{2}{(1+A)^{\frac{\beta}{2}}} \sum_{m\geq 0} \frac{\left(\frac{\beta}{2}\right)_m}{m!} \frac{2^m}{(1+A)^m} \int_0^{\pi} \cos(2n\theta)\cos^{2m}(\theta) d\theta.$$

At this stage we use the identity

$$\int_0^{\pi} \cos^x(\theta)\cos(y\theta) d\theta = \frac{\pi\Gamma(x+1)}{2^x\Gamma(1+\frac{x+y}{2})\Gamma(1+\frac{x-y}{2})},$$

for $x > -1$ and $y \in \mathbb{R}$. That identity can be found in [50]. For $x = 2m$ and $y = 2n$, we obtain

$$\begin{aligned}\int_0^{2\pi} \frac{\cos(n\theta)}{(A-\cos(\theta))^{\frac{\beta}{2}}} d\theta =& \frac{2\pi}{(1+A)^{\frac{\beta}{2}}} \sum_{m\geq n} \frac{\left(\frac{\beta}{2}\right)_m}{m!} \frac{2^m}{(1+A)^m} \frac{\Gamma(2m+1)}{2^{2m}\Gamma(1+m+n)\Gamma(1+m-n)} \\ =& \frac{2\pi}{(1+A)^{\frac{\beta}{2}}} \sum_{m\geq 0} \frac{\left(\frac{\beta}{2}\right)_{m+n}}{(m+n)!} \frac{1}{(1+A)^{m+n}} \frac{\Gamma(2m+2n+1)}{2^{m+n}\Gamma(1+m+2n)\Gamma(1+m)}.\end{aligned}$$

We can use some properties of Gamma functions in order to find

$$\begin{aligned}\Gamma(m+1+2n)\Gamma(m+1) =& (2n)!m!(2n+1)_m, \\ \frac{\Gamma(2m+2n+1)}{(m+n)!} =& 2^{2m+2n}\left(\frac{1}{2}\right) m+n, \\ \left(\frac{\beta}{2}\right)_{m+n} =& \left(\frac{\beta}{2}\right)_n \left(n+\frac{\beta}{2}\right)_m,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{2\pi}{(1+A)^{\frac{\beta}{2}}} \sum_{m\geq 0} \frac{\left(\frac{\beta}{2}\right)_{m+n}}{(m+n)!} \frac{1}{(1+A)^{m+n}} \frac{\Gamma(2m+2n+1)}{2^{m+n}\Gamma(1+m+2n)\Gamma(1+m)} \\ &= \frac{2\pi}{(1+A)^{\frac{\beta}{2}+n}} \frac{\left(\frac{\beta}{2}\right)_n 2^n \left(\frac{1}{2}\right)_n}{(2n)!} \sum_{m\geq 0} \frac{\left(n+\frac{\beta}{2}\right)_m \left(n+\frac{1}{2}\right)_m}{m!(2n+1)_m} \left(\frac{2}{1+A}\right)^m \\ &= \frac{2\pi}{(1+A)^{\frac{\beta}{2}+n}} \frac{\left(\frac{\beta}{2}\right)_n 2^n \left(\frac{1}{2}\right)_n}{(2n)!} F\left(n+\frac{\beta}{2}, n+\frac{1}{2}, 2n+1, \frac{2}{1+A}\right).\end{aligned}$$

□

Some useful properties of Hypergeometric functions are the following.

Proposition C.0.2. *The following assertions hold true.*

1. *Bound for* $F(a,a;2a;x)$ *: for* $a>1$*, there exists* $C>0$ *such that*

$$\forall x \in [0,1), \quad F\left(a,a;2a;x\right) \leq C\frac{|\ln(1-x)|}{x} \leq C + C|\ln(1-x)|. \tag{C.0.11}$$

2. *Bound for* $F(a,a;2a-1,x)$ *: for* $a>2$*, there exists* $C>0$ *such that*

$$\forall x \in [0,1), \quad F\left(a,a;2a-1;x\right) \leq C\frac{1}{|1-x|}. \tag{C.0.12}$$

3. *Bound for* $F(a,a;2a-2,x)$ *: for* $a>3$*, there exists* $C>0$ *such that*

$$\forall x \in [0,1), \quad F\left(a,a;2a-2;x\right) \leq C\frac{1}{|1-x|^2}. \tag{C.0.13}$$

4. *For* $a>1$*, there exists* $C>0$ *such that*

$$\forall x \in [0,1), \quad 0 \leq F(a,a;2a;x) - 1 \leq Cx\big(1+|\ln(1-x)|\big).$$

5. *For* $a>2$*, there exists* $C>0$ *such that*

$$\forall x \in [0,1), \quad |F(a,a;2a-1;x) - 1| \leq C\frac{x}{1-x}.$$

6. *For* $a>1$*, there exists* $C>0$ *such that any* $\alpha \in [0,1]$

$$\forall\, x_2 \leq x_1 \in [0,1), \quad |F(a,a;2a;x_1) - F(a,a;2a;x_2)| \leq C\frac{|x_1-x_2|^\alpha}{|1-x_1|^\alpha}. \tag{C.0.14}$$

7. *For* $a>2$*, there exists* $C>0$ *such that any* $\alpha \in [0,1]$

$$\forall\, x_2 \leq x_1 \in [0,1), \quad |F(a,a;2a-1;x_1) - F(a,a;2a-1;x_2)| \leq C\frac{|x_1-x_2|^\alpha}{|1-x_1|^{1+\alpha}}. \tag{C.0.15}$$

Proof. The main tool is the integral representation of the Hypergeometric functions (C.0.2).

(1) From the integral representation (C.0.2), it is easy to get

$$\begin{aligned}|F(a,a,2a,x)| \leq& C\int_0^1 \frac{t^{a-1}(1-t)^{a-1}}{(1-xt)^a}dt\\ \leq& C\int_0^1 \left(\frac{t(1-t)}{1-xt}\right)^{a-1}\frac{1}{1-xt}dt.\end{aligned}$$

Because $t(1-t) \leq 1-tx$ for any $t,x \in [0,1]$, then we deduce

$$\begin{aligned}|F(a,a,2a,x)| \leq& C\int_0^1 \frac{dt}{1-xt}\\ \leq& C\frac{|\ln(1-x)|}{x}.\end{aligned}$$

(2) As for (1), we find

$$\begin{aligned}|F(a,a,2a-1,x)| \leq & C\int_0^1 \frac{t^{a-1}(1-t)^{a-2}}{(1-xt)^a}dt\\ \leq & C\int_0^1 \left(\frac{t(1-t)}{1-xt}\right)^{a-2}\frac{dt}{(1-xt)^2}.\end{aligned}$$

Consequently, we infer from direction calculation

$$\begin{aligned}|F(a,a,2a-1,x)| \leq & C\int_0^1 \frac{1}{(1-xt)^2}dt\\ \leq & \frac{C}{|1-x|}.\end{aligned}$$

(3) We omit here the details of the proof by similarity with (1) and (2).

(4) First note from the integral representation that $F(a,a;2a;x) > 0$ provided that $a > 0$ and $x \in [0,1)$. Moreover, it is strictly increasing function since from (C.0.4)

$$F'(a,a;2a;x) = \frac{a}{2}F(a+1,a+1;2a+1;x) > 0,\ \forall x \in [0,1).$$

According to (C.0.1) one may check by construction that $F(a,a;2a;0) = 1$ and therefore

$$F(a,a;2a;x) - 1 \geq 0.$$

By the mean value theorem, we achieve

$$F(a,a;2a;x) - 1 = \frac{a}{2}x\int_0^1 F(a+1,a+1,2a+1,\tau x)d\tau.$$

Combining this representation with (C.0.12), where we replace a by $a+1$, we achieve

$$F(a,a;2a;x) - 1 \leq Cx\int_0^1 \frac{d\tau}{1-\tau x} \leq Cx(1+|\ln(1-x)|).$$

(5) By using similar arguments as the previous point, we obtain

$$0 \leq F(a,a;2a-1;x) - 1 \leq Cx\int_0^1 F(a+1,a+1;2a;\tau x)d\tau.$$

Applying (C.0.13) by changing a with $a+1$ allows to get

$$|F(a+1,a+1;2a;x)| \leq \frac{C}{(1-x)^2}, \forall\, x \in [0,1).$$

Then,

$$F(a,a;2a-1;x) - 1 \leq Cx\int_0^1 \frac{d\tau}{(1-\tau x)^2} \leq C\frac{x}{1-x}.$$

(6) Let $t \in [0,1)$ and set $g_t(x) = (1-tx)^{-a}$. Take $0 \leq x_2 < x_1 < 1$, then direct computations, using in particular the mean value theorem, show that

$$\begin{aligned}|g_t(x_1)-g_t(x_2)| \leq& 2(1-tx_1)^{-a}\\ |g_t(x_1)-g_t(x_2)| \leq& C(1-tx_1)^{-a-1}|x_1-x_2|.\end{aligned}$$

Let $\alpha\in[0,1]$ then by interpolation between the preceding inequalities we deduce that

$$\begin{aligned}|g_t(x_1)-g_t(x_2)| =& |g_t(x_1)-g_t(x_2)|^{1-\alpha}|g_t(x_1)-g_t(x_2)|^{\alpha}\\ \leq& C(1-tx_1)^{-a-\alpha}|x_1-x_2|^{\alpha}.\end{aligned}$$

It follows that

$$\begin{aligned}|F(a,a,2a,x_1)-F(a,a,2a,x_2)| \leq& C\left|\int_0^1 t^{a-1}(1-t)^{a-1}\left(g_t(x_1)-g_t(x_2)\right)dt\right|\\ \leq& C|x_1-x_2|^{\alpha}\int_0^1 \frac{t^{a-1}(1-t)^{a-1}}{(1-x_1t)^{a+\alpha}}dt.\end{aligned}$$

Since $a>1$ and for any $t,x_1\in[0,1)$,

$$0\leq \frac{t^{a-1}(1-t)^{a-1}}{(1-x_1t)^{a+\alpha}}\leq (1-x_1t)^{-1-\alpha},$$

then

$$\begin{aligned}|F(a,a,2a,x_1)-F(a,a,2a,x_2)| \leq& C|x_1-x_2|^{\alpha}\left|\int_0^1 (1-x_1t)^{-1-\alpha}dt\right|\\ \leq& C\frac{|x_1-x_2|^{\alpha}}{|1-x_1|^{\alpha}}.\end{aligned}$$

(7) This is quite similar to the proof of the preceding one. Indeed,

$$\begin{aligned}|F(a,a,2a-1,x_1)-F(a,a,2a-1,x_2)| \leq& C\left|\int_0^1 t^{a-1}(1-t)^{a-2}\left(g_t(x_1)-g_t(x_2)\right)dt\right|\\ \leq& C|x_1-x_2|^{\alpha}\left|\int_0^1 (1-x_1t)^{-2-\alpha}dt\right|\\ \leq& C\frac{|x_1-x_2|^{\alpha}}{|1-x_1|^{1+\alpha}}.\end{aligned}$$

□

The last point that we wish to recall concerns the differential equation governing the hypergeometric equation, which is given by,

$$z(1-z)F(z)''+(c-(a+b+1)z)F(z)'-abF(z)=0, \tag{C.0.16}$$

with $a,b,\in\mathbb{R}$ and $c\in\mathbb{R}\backslash(-\mathbb{N})$ given. One of the two independent solutions of the last differential equation around $z=0$ is the hypergeometric function: $F(a,b;c;z)$. It remains to identify the second independent solution. If none of $c, c-a-b, a-b$ is an integer, then the other independent solution around the singularity $z=0$ is

$$z^{1-c}F(a-c+1,b-c+1;2-c;z). \tag{C.0.17}$$

We will be interested in the critical case when c is a negative integer. In this case, the hypergeometric differential equation has as a smooth solution given by (C.0.17). However, the second independent solution is singular and contains a logarithmic singularity, see [126, p. 55] for more details. Real solutions around $+\infty$ may be also obtained as it is done in [6]. In fact, the two independent solutions are given by

$$z^{-a}F\left(a, a+1-c; a+1-b; \frac{1}{z}\right) \quad \text{and} \quad z^{-b}F\left(b, b+1-c; b+1-a; \frac{1}{z}\right).$$

Now, let us define the Bessel function of the first kind and order ν by the expansion

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad |\arg(z)| < \pi.$$

In addition, it is known that Bessel functions admit the following integral representation

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)d\theta, \quad z \in \mathbb{C},$$

for $\nu = n \in \mathbb{Z}$. On the other hand, the Bessel functions of imaginary argument, denoted by I_ν and K_ν are defined by

$$I_\nu(z) = \sum_{k=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \quad |\arg(z)| < \pi,$$

and

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu} - I_\nu(z)}{\sin(\nu\pi)} \quad \nu \in \mathbb{C}\setminus\mathbb{Z}, \ |\arg(z)| < \pi.$$

An useful expansion for K_n can be found in [143]:

$$\begin{aligned} K_n(z) =& (-1)^{n+1}\sum_{k=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}\left(\ln\left(\frac{z}{2}\right) - \frac{1}{2}\varphi(k+1) - \frac{1}{2}\varphi(n+k+1)\right) \\ &+ \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k(n-k-1)!}{k!\left(\frac{z}{2}\right)^{n-2k}}, \end{aligned}$$

for $n \in \mathbb{N}^\star$. In the case that we concern, K_0, it reads as

$$K_0(z) = -\ln\left(\frac{z}{2}\right) I_0(z) + \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}\varphi(k+1), \tag{C.0.18}$$

where

$$\varphi(1) = -\gamma \quad \text{and} \quad \varphi(k+1) = \sum_{m=1}^{k} \frac{1}{m} - \gamma, \ k \in \mathbb{N}^*.$$

The constant γ is the Euler's constant. Moreover, the following asymtotic behavior at infinity for K_0 is given in [1]:

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad |\arg(z)| < \frac{3}{2}\pi. \tag{C.0.19}$$

Finally, the derivative of K_n can be expressed by terms of Bessel functions. In particular, $K_0' = -K_1$.

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