VARIATIONS FOR SUBMANIFOLDS OF FIXED DEGREE

Tesis Doctoral

por

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Abril de 2020

Editor: Universidad de Granada. Tesis Doctorales
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ISBN: 978-84-1306-587-8
URI: http://hdl.handle.net/10481/63492

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Riassunto

Lo scopo di questa tesi di dottorato è studiare il funzionale dell'area per sottovarietà immerse in varietà graduate equiregolari. Queste strutture, che generalizzano le varietà subrimanniane non richiedendo nessuna ipotesi di Hörmander, sono definite su una varietà liscia N ed ammettono una filtrazione $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ crescente di sottofibrati del fibrato tangente compatibile con l'operazione di commutazione. Quando fissiamo un punto p in N questa filtrazione diventa una bandiera di sottospazi ed il grado di un vettore nello spazio tangente è uguale a ℓ se tale vettore appartiene allo sottospazio $\mathcal{H}_p^\ell$ ma non appartiene a $\mathcal{H}_p^{\ell-1}$. Un sottovarietà immersa M in una struttura graduata N è ancora un varietà graduata: la sua filtrazione si ottiene intersecando ogni sottofibrato di quella originale $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ con il fibrato tangente TM. Il grado puntuale è dato dalla dimensione omogenea di questa nuova bandiera $\mathcal{H}^1 \cap T_pM \subset \ldots \subset \mathcal{H}^s \cap T_pM$. Il grado di M è il massimo tra i gradi puntuali di tutti i punti $p \in M$. Risulta che il funzionale dell'area dipende dal grado della sottovarietà. Quindi per calcolare la variazione prima dobbiamo prendere in considerazione solo variazioni ammissibili, che non aumentano il grado durante la variazione. Si verifica che ad ogni variazione ammissibile si può associare un campo vettoriale variazionale che verifica un sistema lineare di equazioni alle derivate parziali del primo ordine. Un campo vettore a supporto compatto che verifica questo sistema si dice ammissibile, la domanda naturale che ci poniamo è se è integrabile da una variazione ammissibile.

Il caso più semplice di immersione è dato da una curva $\gamma : I \to \mathbb{R}$ immersa in una varietà graduata. In questo caso L. Hsu in [56] introdusse la mappa di olonomia e scoprì che quando la sua restrizione all'intervallo $[a, b] \subset I$ è suriettiva, allora i campi vettoriali ammissibili supportati in (a, b) sono integrabili. Questa è una condizione differenziale molto difficile da verificare, mentre noi introduciamo un'ipotesi di forte regolarità, che è una condizione puntuale sul rango di una matrice, più facile da verificare ed che implica ovviamente il teorema di deformabilità precedentemente enunciato. Le curve non regolari vengono chiamate singolari e sono le geodetiche anormali introdotte da Montgomery nell'articolo [73, 74]. Tra queste curve singolari ve ne sono alcune

particolarmente interessanti che sono le curve isolate nella topologia C^1, perché non ammettono variazioni ammissibili a supporto compatto.

La condizione di forte regolarità si generalizza facilmente al caso delle sottovarietà di dimensione generica e ci permette di dedurre un teorema di deformabilità locale. Sotto questa ipotesi siamo inoltre in grado di calcolare la variazione prima e dedurre l'equazione di curvatura media che in certi casi può essere anche del terzo ordine. Ancora più interessante è il fatto che riusciamo ad esibire un esempio di sottovarietà isolata nella topologia C^1. La sottovarietà in questione è un piano di grado tre immerso nel gruppo di Engel, la cui unica variazione ammissibile trasversale coincide con l'immersione stessa.

Solo quando la sottovarietà è rigata da curve di grado ι_0, il sistema di equazioni alle derivate parziali si riduce ad un sistema di equazioni differenziali ordinarie lungo le curve caratteristiche di grado ι_0. Di conseguenza in questo caso siamo in grado di generalizzare la nozione di mappa di olonomia a dimensione più alta.

Resumen

El objetivo de esta tesis doctoral es estudiar el funcional área de subvariedades inmersas en variedades graduadas equiregulares. Estas estructuras, que generalizan las variedades subriemannianas sin asumir *a priori* la hipótesis de Hörmander, están definidas sobre una variedad diferenciable N y admiten una filtración $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ ascendente de sub-fibrados del espacio tangente compatible con el corchete de Lie. Cuando fijamos un punto p en N, esta filtración es una cadena de subespacios y el grado de un vector en el espacio tangente es igual a ℓ si dicho vector pertenece al subespacio $\mathcal{H}_p^\ell$ pero no pertenece al subespacio $\mathcal{H}_p^{\ell-1}$. Una subvariedad M inmersa en una estructura graduada es también una variedad graduada porque hereda su filtración cortando cada sub-fibrado de la primera con el fibrado tangente TM. El grado puntual se define como la dimensión homogénea de esta nueva cadena $\mathcal{H}^1 \cap T_pM \subset \ldots \subset \mathcal{H}^s \cap T_pM$. El grado de M es el máximo del grado puntual sobre todos los puntos en M. La noción de área que consideraremos, que se obtiene como límite de áreas riemannianas, depende del grado de la subvariedad. Para calcular la primera variación, tenemos que considerar sólo las variaciones admisibles, que no aumentan el grado durante la variación. Resulta que el campo variacional asociado a una variación admisible cumple un sistema lineal de ecuaciones en derivadas parciales de primer orden. Diremos que un campo vectorial con soporte compacto es admisible cuando cumpla dicho sistema de ecuaciones en derivadas parciales de primer orden. Entonces, la pregunta natural que surge es si un campo vectorial admisible es integrable por medio de una variación admisible.

El caso mas simple de inmersión viene dado por una curva $\gamma : I \to \mathbb{R}$ en un variedad graduada. L. Hsu en [56] descubrió que, cuando la aplicación de holonomía es sobreyectiva restringida al intervalo $[a, b] \subset I$, se pueden integrar los campos vectoriales admisibles con soporte en (a, b). Esta hipótesis de regularidad es muy difícil de verificar. Sin embargo, la hipótesis de regularidad fuerte, que es una condición puntual sobre el rango de la matriz de control, es mas fácil de verificar e implica claramente el teorema de deformación anteriormente enunciado. Las curvas no regulares son llamadas singulares y son las geodésicas anormales introducidas por Montgomery en los artículos [73, 74].

Entre estas curvas singulares hay algunas particularmente más interesantes, que son las curvas aisladas en la topologia C^1, porque no admiten variaciones admisibles con soporte compacto.

La condición de regularidad fuerte se puede generalizar al caso de las subvariedades de dimensión arbitraria y nos permite deducir un teorema de deformación local. Bajo esta hipótesis podemos calcular la primera variación y deducir la ecuación de curvatura media, que en algunos casos puede ser un operador de tercer orden. Aún más interesante es que exhibimos por primera vez un ejemplo de subvariedad aislada en la topología C^1. La subvariedad en cuestión es un plano de grado tres inmerso en el grupo de Engel, cuya única variación admisible transversal coincide con la misma inmersión.

Solamente cuando la subvariedad es reglada por curvas de grado ι_0, el sistema de ecuaciones en derivadas parciales se reduce a un sistema de ecuaciones diferenciales ordinarias a lo largo de las curvas características de grado ι_0. Por tanto, en este caso, podemos generalizar la noción de aplicación de holonomía a dimensiones superiores.

Introduction

The aim of this PhD thesis is to study the area functional for submanifolds immersed in an equiregular graded manifold. This setting extends the sub-Riemannian one, removing the bracket generating condition. However, even in the sub-Riemannian setting only submanifolds of dimension or codimension one have been extensively studied. We will study the general case and observe that in higher codimension new phenomena arise, which do not show up in the Riemannian case. In particular, we will prove the existence of isolated surfaces, which do not admit degree preserving variations: a phenomena observed up to now only for curves, related to the notion of abnormal geodesics.

Graded manifolds are smooth manifolds N endowed with a increasing filtration of sub-bundles $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ of the tangent bundle TN. This filtration is compatible with the Lie bracket since given $X \in \mathcal{H}^i$ and $Y \in \mathcal{H}^j$ the commutator $[X, Y]$ belongs to $\mathcal{H}^{i+j}$. Given a point $p \in N$ we have a flag of subspaces $\mathcal{H}_p^1 \subset \ldots \subset \mathcal{H}_p^s = T_pN$. The degree of a vector $v \in T_pN$ is equal to ℓ if $v \in \mathcal{H}_p^\ell$ and $v \notin \mathcal{H}_p^{\ell-1}$.

The concept of pointwise degree for a submanifold M immersed in a graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ was first introduced by Gromov in [50] as the homogeneous dimension of the tangent flag given by

$$T_pM \cap \mathcal{H}_p^1 \subset \cdots \subset T_pM \cap \mathcal{H}_p^s. \tag{0.0.1}$$

The *degree of a submanifold* $\deg(M)$ is the maximum of the pointwise degree among all points in M. An alternative way of defining the degree is the following: on an open neighborhood of a point $p \in N$ we can always consider a local basis $(X_1, \ldots, X_n)$ adapted to the filtration $(\mathcal{H}^i)_{i=1,\ldots,s}$, so that each X_j has a well defined degree. Following [69] the degree of a simple m-vector $X_{j_1} \wedge \ldots \wedge X_{j_m}$ is the sum of the degree of the vector fields of the adapted basis appearing in the wedge product. Since we can write a m-vector tangent to M with respect to the simple m-vectors of the adapted basis, the *pointwise degree* is given by the maximum of the degrees of these simple m-vectors.

Examples of graded manifolds are Carnot manifolds $(N, \mathcal{H})$, where $\mathcal{H}$ is a constant rank distribution satisfying Hörmander's rank condition: in this case $\mathcal{H}^1$ coincides with $\mathcal{H}$ and for every i, $\mathcal{H}^i$ is obtained from $\mathcal{H}$ via i commutations. Another example are Hörmander structures of type II introduced by Rothschild and Stein [93]. These structures are naturally associated to a heat subelliptic equation: $\mathcal{H}^1$ is a purely spatial Hörmander distribution, $\partial_t \in \mathcal{H}^2$ and all the elements of the flag of higher degree are obtained via commutation. In this case it is clear that regularity properties of the solution depend not only on integral curves of vector fields of degree one, but also on integral curves of the vector field ∂_t, of degree 2. Finally we can consider a sub-Riemannian manifold $(N, \mathcal{H}, h)$, which is a Carnot manifold $(N, \mathcal{H})$ endowed with a metric h on the distribution $\mathcal{H}$ and its submanifolds M. In [50, page 151] Gromov points out that, while the distance of a sub-Riemannian manifold $(N, \mathcal{H}, h)$ can be expressed in term of integral curves of vector fields of degree 1 by Chow's Theorem, the same thing is no more true for a submanifold M immersed in N, with the induced distance. Only if the new distribution $\mathcal{H} \cap TN$ verifies a Hörmander type condition on M, there exists a horizontal path tangent to M connecting any two points in M. This condition for the distribution $\mathcal{H} \cap TM$ is not satisfied even in simple cases. Nevertheless the submanifold M inherits a filtration $\mathcal{H}^1 \cap TM \subset \ldots \subset \mathcal{H}^s \cap TM$ of its tangent bundle TM by means of the flag of sub-bundles $\mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s$ in the ambient space N induced by the distribution $\mathcal{H}$. Therefore M endowed with this induced flag is a graded manifold and the induced anisotropic distance on the submanifold M can be defined as in Definition 1.1 in the paper by Nagel, Stein and Wainger [80]. As all these examples show, the category of graded manifolds seems to be the natural one to study the immersed submanifolds, since they certainly inherit the graded structure from the ambient space.

We consider a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on N. For any $p \in N$, we get an orthogonal decomposition $T_pN = \mathcal{K}_p^1 \oplus \ldots \oplus \mathcal{K}_p^s$. Then we apply to g a dilation induced by the grading, which means that, for any $r > 0$, we take the Riemannian metric g_r making the subspaces $\mathcal{K}_p^i$ orthogonal and such that

$$g_r|_{\mathcal{K}^i} = \frac{1}{r^{i-1}}\, g|_{\mathcal{K}^i} \,.$$

Whenever $\mathcal{H}^1$ is a bracket generating distribution the structure (N, g_r) converges uniformly to the sub-Riemannian structure $(N, \mathcal{H}^1, g_{|\mathcal{H}^1})$ as $r \to 0$. Therefore an immersed submanifold $M \subset N$ of degree d has Riemannian area measure $A(M, g_r)$

with respect to the metric g_r. We define area measure A_d of degree d by

$$A_d(M) := \lim_{r\downarrow 0} r^{(\deg(M)-\dim(M))/2} A(M, g_r) \tag{0.0.2}$$

when this limit exists and it is finite. In (3.1.5) we stress that the area measure A_d of degree d is given by integral of the norm the g-orthogonal projection onto the subspace of m-forms of degree equal to d of the orthonormal m-vector tangent to M. This area formula was provided in [69, 68] for C^1 submanifolds immersed in Carnot groups and in [38] for intrinsic regular submanifolds in the Heisenberg groups.

Given an immersion $\Phi : \bar{M} \to N$ of degree d into a graded manifold $(N, (\mathcal{H}^i)_i)$, we wish to compute the Euler-Lagrange equations for the area functional A_d. The problem has been intensively studied for hypersurfaces, and results appeared in [43, 28, 17, 18, 29, 6, 53, 54, 57, 89, 90, 72, 22]. For submanifolds of codimension greater than one in a sub-Riemannian structure only in the case of curves has been studied. In sub-Riemannian geometry, the existence of minimizing curves for the length functional that are not solutions of the geodesic equation was discovered by Montgomery in [73, 74]. These curves are known as abnormal extremals. The problem of their regularity has been widely considered in the literature, see for instance [75, 3, 2, 63, 61, 77, 1, 88]. The usual approach to face this problem is by means of the study of the endpoint map. However, in this work we follow an alternative approach based on the Griffiths formalism as suggested by Bryant and Hsu [11, 56], since it can be generalized to higher dimensional submanifolds showing the existence of isolated submanifolds.

In Chapter 2 we consider a curve $\gamma : I \to N$. Its degree $d = \deg(\gamma)$ is the maximum of the pointwise degree of its points, that is exactly the degree of the tangent vector $\gamma'(t)$ at each $t \in I$. Then the area functional in this case coincides with the length functional

$$L_d(\gamma, J) = \int_J \theta_d(t) dt$$

for each $J \subset I$, where $\theta_d(t)$ is the density given by the projection of $\gamma'(t)$ onto the space generated by adapted vector fields $X_{n_{d-1}+1}, \ldots, X_{n_d}$ of degree d along the curve. If we wish to compute the Euler-Lagrange equations we need to consider *admissible* variations (see 2.2.1 for the definition): the ones that preserve the degree of the initial curve γ. Then it turns out that the associated variational vector field $V(t) = \frac{\partial \Gamma_s(t)}{\partial s}\Big|_{s=0}$ associated to an admissible variation has to verify the first order condition (2.2.3) along γ. We say that a vector field along γ is admissible when it verifies the system of ODEs

(2.2.3). In [56, Theorem 3] Hsu pointed out that under a surjectivity condition of a map associated to γ each admissible vector field along γ is integrable by an admissible variation.

Roughly speaking a curve γ is regular if it admits enough compactly supported variations preserving its degree. Indeed, to integrate the vector field $V(t)$ we follow the exponential map generating the non-admissible compactly supported variation $\Gamma_s(t) = \exp_{\gamma(t)}(sV(t))$ of the initial curve γ. Let $\mathrm{supp}(V) \subset [a,b]$. By the Implicit Function Theorem there exists a vector field $Y(s,t)$ along γ vanishing at a such that the perturbations $\tilde{\Gamma}_s(t) = \exp_{\gamma(t)}(sV(t) + Y(s,t))$ of Γ are curves of the same degree of γ for each s small enough. In general $\tilde{\Gamma}$ fixes the endpoint at $\gamma(a)$ but moves the endpoint at $\gamma(b)$. Finally the surjectivity condition allows us to produce the admissible variation that moves the endpoint $\gamma(b)$ to 0, and produces the compactly supported vector field V.

This concept of surjectivity of a map associated to γ will be called regularity of γ and it deals with the controllability (see [10, Chapter 13]) of the system of ODEs (2.2.3). Indeed after splitting the admissible vector V along γ in its horizontal $V_h = \sum_{i=1}^k g_i X_i$ and vertical $V_v = \sum_{j=k+1}^n f_j X_j$ part the admissibility system of ODEs (2.2.3) is equivalent to

$$F'(t) + B(t)F(t) + A(t)G(t) = 0, \tag{0.0.3}$$

where $A(t), B(t)$ are defined in (2.3.3) and (X_i) is a global orthonormal adapted basis along γ. We control this linear system with initial condition $F(a) = 0$ on a compact interval $[a,b] \subset I$ when for each value $y_0 \in \mathbb{R}^{n-k}$ there exists a control horizontal vector field $G(t) \in C_0^{r-1}((a,b), \mathbb{R}^k)$ such that $F(t)$ solves (0.0.3) and $F(b) = y_0$. In other words if the *holonomy* map

$$H_\gamma^{a,b} : \mathcal{H}_0^{r-1}((a,b)) \to \mathcal{V}_{\gamma(b)}, \qquad H_\gamma^{a,b}(G) := F(b)$$

is surjective the system (0.0.3) is controllable. Therefore a curve γ is said to be regular restricted to $[a,b]$ when the holonomy map is surjective. It turns out that there exists a regular matrix $D(t)$ along γ solving the differential equation $D' = DB$ such that the image of the holonomy map is given by

$$H_\gamma^{a,b}(G) = -D(b)^{-1} \int_a^b D(t)A(t)G(t)dt.$$

In Corollary 5 in [56] (Proposition 2.4.6) Hsu proved that the regularity condition on γ is equivalent to maximal rank condition on the matrix $\tilde{A}(t) = D(t)A(t)$ along γ. Furthermore all singular curves are characterized by the existence of a non-vanishing row vector $\Lambda(t)$ along γ solving

$$\begin{cases} \Lambda'(t) = \Lambda(t)B(t) \\ \Lambda(t)A(t) = 0. \end{cases} \tag{0.0.4}$$

Moreover, we check that the surjectivity of the holonomy map is independent of the choice of Riemannian metric $g = \langle \cdot, \cdot \rangle$ on the tangent bundle TN, thus the regularity of a curve γ is an invariant of the graded structure $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$. Furthermore in Section 2.4.3 we proved that when we replace the space of continuous horizontal vector fields with the space of square integrable horizontal vector fields, the surjectivity of the holonomy map coincides with the surjectivity of the differential of the endpoint map, that defines the regularity in the classical setting, see [75, 3, 2].

An analysis of Hsu's regularity condition led us to introduce, in the article [23], a weaker pointwise sufficient condition named *strong regularity* to ensure the integrability of all admissible vector fields along γ (see Theorem 2.6.4) . This pointwise full rank condition does not require solving a differential equation but still ensures the regularity of the curve. Consequently the nature of Theorem 2.6.4 is purely local in the sense that guarantees variations only in neighborhoods of the point $\bar{t} \in I$ where the matrix $A(\bar{t})$ has full rank.

Even though the regularity of a curve is an invariant of the graded structure, the minimizing paths for the length functional L_d of fixed degree clearly strongly depends on the Riemannian metric g. Regarding only regular curves of fixed degree we deduce the Euler-Lagrange equation for the critical points of L_d in Theroem 2.7.2, providing some interesting application for curves of degree 2 belonging to a surfaces immersed the Heisenberg group. However there are singular curves that are not solution of the geodesic equation but they are minima for L_d. When this singular horizontal curves are minima of the sub-Riemannian length L, they are known as abnormal extremals.

In Chapter 3, we consider a C^1 m-dimensional immersion $\Phi : \bar{M} \to N$, with $m > 1$, into a graded structure $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$, $\Phi(\bar{M}) = M$. The problem of producing admissible variations that preserves the degree is analogous to the one dimensional problem previously treated. Consequently we focus on *admissible* variations, which preserve it. The associated *admissible vector fields*, $V = \frac{\partial \Gamma_t}{\partial t}\Big|_{t=0}$ satisfies the system of partial differential equations of first order (3.3.3) on M. So we are led to the central question of characterizing the admissible vector fields which are associated to an admissible variation. In this setting, in general, there does not seem to be an acceptable generalization of such an holonomy map. Indeed, the system of ODEs along the curve becomes a complicated first order system of PDEs on the submanifold. Since an existence result for compactly supported solutions of (3.3.3) is not available in general, the theory previously exhibited for curves is not simple to develop. However, in [24] we realized that the notion of *strong regularity*, introduced in [23] for curves, can be easily generalized to submanifolds of given degree. In this setting the admissibility system (3.3.3) in coordinates is given by

$$\sum_{j=1}^{m} C_j(\bar{p}) E_j(F)(\bar{p}) + B(\bar{p}) F(\bar{p}) + A(\bar{p}) G(\bar{p}) = 0, \tag{0.0.5}$$

where C_j, B, A are matrices, F are the vertical components of the admissible vector field and G are the horizontal control components. Since the strong regularity tells us that the matrix $A(\bar{p})$ has full rank we can locally write explicitly a part of the controls in terms of the vertical components and the other part of the controls, then applying the Implicit Function Theorem we produce admissible variations. This way in Theorem 3.5.2 we obtain that the strong regularity is a sufficient condition for the local integrability of admissible vector fields on M. In Remark 3.6.4 we recognize that our definition of strongly regular immersion generalizes the notion introduced by [50] of regular horizontal immersions, that are submanifolds immersed in the horizontal distribution such that the degree coincides with the topological dimension m. In [49], see also [83], the author shows a deformability theorem for regular horizontal immersions by means of the Nash's Implicit Function Theorem [81]. Our result is in the same spirit but for immersions of general degree.

We establish that this strongly regular condition holds in the case of surfaces of degree 4 and codimension 2 immersed in a four dimensional Engel structure. On the other hand we are able to show that there are isolated surfaces which does not admit degree preserving variations. Indeed, in Example 3.5.7 we exhibit an isolated plane, immersed in the Engel group, whose only admissible normal vector field is the trivial one. Moreover, Proposition 3.5.8 shows that this isolated plane is rigid in the C^1 topology, thus this plane is a local minimum for the area functional. Therefore we recognized that a similar phenomenon to the one of existence of abnormal curves can arise in higher dimension. Finally we conjecture that a bounded open set Ω contained in this isolated plane is a global minimum among all possible immersed surfaces sharing the same boundary $\partial\Omega$.

Moreover, in [47] we notice that there exist special coordinates adjusted to the admissibility system. Indeed Proposition 3.4.4 guarantees that this admissibility system is independent on the choice of metric g on TN and Proposition 3.3.5 shows that only the transversal part $V^\perp$ of the vector field $V = V^\top + V^\perp$ affects the admissibility system. Therefore, we consider an adapted tangent basis $E_1, \ldots, E_m$ for the flag (0.0.1) and then we add transversal vector fields $V_{m+1}, \ldots, V_n$ of increasing degrees so that a sorting of $\{E_1, \ldots, E_m, V_{m+1}, \ldots, V_n\}$ is a local adapted basis for N. Then we consider the metric g that makes $E_1, \ldots, E_m, V_{m+1}, \ldots, V_n$ an orthonormal basis. Hence we obtain that the admissibility system is equivalent to

$$E_j(f_i) = -\sum_{r=m+k+1}^{n} b_{ijr}\, f_r - \sum_{h=m+1}^{m+k} a_{ijh}\, g_h, \tag{0.0.6}$$

for $i = m+k+1, \ldots, n$ and $\deg(V_i) > \deg(E_j)$. In equation (0.0.6) the integer k, defined after (3.4.5), separates the horizontal control of the systems $V_h = \sum_{l=m+1}^{m+k} g_l V_l$ from the vertical component $V_v = \sum_{r=m+k+1}^{n} f_r V_r$.

As we stressed before a generalization of the *holonomy map* for general submanifolds of dimension grater than one is not easy to find but we realized that it is possible when we consider ruled m-dimensional submanifolds whose $(m-1)$ tangent vector fields $E_2, \ldots, E_m$ have degree s and fill up the last layer of the graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ and the first vector field E_1 has degree equal to ι_0, where $1 \leqslant \iota_0 \leqslant s-1$. The resulting degree is $\deg(M) = (m-1)s + \iota_0$. Therefore the ruled submanifold is foliated by curves of degree ι_0 out of the characteristic set M_0, whose points have degree strictly less than $\deg(M)$. Then, under a logarithmic change of coordinates $x = (x_1, \hat{x})$, the admissibility system (0.0.6) becomes

$$\frac{\partial F(x)}{\partial x_1} = -B(x)F(x) - A(x)G(x), \tag{0.0.7}$$

where ∂_{x_1} is the partial derivative in the direction E_1, G are the horizontal coordinates $V_h = \sum_{l=m+1}^{m+k} g_l V_l$, F are the vertical components given by $V_v = \sum_{r=m+k+1}^{n} f_r V_r$ and A, B are matrices defined at the end of Section 3.7. Therefore, this system of ODEs is easy to solve in the direction ∂_{x_1} perpendicular to the $(m-1)$ foliation generated by $E_2, \ldots, E_m$. We consider a bounded open set $\Sigma_0 \subset \{x_1 = 0\}$ in the foliation, then we build the ε-cylinder $\Omega_\varepsilon = \{(x_1, \hat{x}) \;:\; \hat{x} \in \Sigma_0, 0 < x_1 < \varepsilon\}$ over Σ_0. We consider the horizontal controls G in the space of continuous functions compactly supported in Ω_ε.

For each fixed G , F is the solution of (0.0.7) vanishing on Σ_0. Then the image of the higher dimensional *holonomy map* H_M^ε is the solution F, evaluated on the top of the cylinder Ω_ε. We say that a ruled submanifold is *regular* when the holonomy map is surjective, namely we are able to generate all possible compactly supported continuous vertical functions on $\Sigma_\varepsilon \subset \{x_1 = \varepsilon\}$ by letting vary the control G in the space of compactly supported continuous horizontal functions inside the cylinder Ω_ε. The main difference with the one dimensional case is that the target space of the holonomy map is now the Banach space of compactly supported continuous vertical vector fields on the foliation instead of the finite dimensional vertical space of vectors at the final point $\gamma(b)$ of the curve. In Theorem 3.8.7 we provide a nice characterization of singular ruled submanifolds in analogy with (0.0.4), first established in the case of curves by Hsu [56, Theorem 6]. Moreover, if $s-3 \leqslant \iota_0 \leqslant s-1$ the space of m-vector fields of degree grater than $\deg(M)$ is reasonably simple, thus in Theorem 3.9.5 we show that each admissible vector field on a regular immersed ruled submanifold is integrable in the spirit of [56, Theorem 3] (Theorem 2.5.4).

For strong regular submanifolds it is possible to compute the Euler-Lagrange equations to obtain a sufficient condition for stationary points of the area A_d of degree d. This naturally leads to a notion of mean curvature, which is not in general a second order differential operator, but can be of order three.

In the present thesis we want to show the results investigated in [23, 24, 47], develop them further and give also a general presentation of the problem by trying to make this manuscript as much self-contained as we can.

Chapter 1

Background and Preliminaries

1.1 Background

1.1.1 Carnot manifolds

Let N be an n-dimensional smooth manifold. Given two smooth vector fields X, Y on N, their *commutator* or *Lie bracket* is defined by

$$[X, Y] := XY - YX. \tag{1.1.1}$$

An *l-dimensional distribution* $\mathcal{H}$ on N assigns smoothly to every $p \in N$ an l-dimensional vector subspace $\mathcal{H}_p$ of T_pN. We say that a distribution $\mathcal{H}$ complies *Hörmander's condition* if any local frame $\{X_1, \ldots, X_l\}$ spanning $\mathcal{H}$ satisfies

$$\dim(\mathcal{L}(X_1, \ldots, X_l))(p) = n, \quad \text{for all } p \in N, \tag{1.1.2}$$

where $\mathcal{L}(X_1, \ldots, X_l)$ is the linear span of the vector fields $X_1, \ldots, X_l$ and their commutators of any order.

A *Carnot manifold* $(N, \mathcal{H})$ is a smooth manifold N endowed with an l-dimensional distribution $\mathcal{H}$ satisfying Hörmander's condition. We refer to $\mathcal{H}$ as the *horizontal distribution*. We say that a vector field on N is *horizontal* if it is tangent to the horizontal distribution at every point. A C^1 path is horizontal if the tangent vector is everywhere tangent to the horizontal distribution. A *sub-Riemannian manifold* $(N, \mathcal{H}, h)$ is a Carnot manifold $(N, \mathcal{H})$ endowed with a positive-definite inner product h on $\mathcal{H}$. Such an inner product can always be extended to a Riemannian metric on N. Alternatively, any Riemannian metric on N restricted to $\mathcal{H}$ provides a structure of sub-Riemannian manifold. The Chow–Rashevskii Theorem, proved by L.W. Chow [21] in 1939 and independently by P.K. Rashevskii [87] in 1938, assures that in a Carnot manifold $(N, \mathcal{H})$ the set of points that can be connected to a given point $p \in N$ by a horizontal path is the connected component of N containing p, see [75]. Given a Carnot manifold $(N, \mathcal{H})$, we have a flag of subbundles

$$\mathcal{H}^1 := \mathcal{H} \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^i \subset \cdots \subset TN, \tag{1.1.3}$$

defined by

$$\mathcal{H}^{i+1} := \mathcal{H}^i + [\mathcal{H}, \mathcal{H}^i], \qquad i \geqslant 1,$$

where

$$[\mathcal{H}, \mathcal{H}^i] := \{[X, Y] : X \in \mathcal{H}, Y \in \mathcal{H}^i\}.$$

The smallest integer s satisfying $\mathcal{H}_p^s = T_pN$ is called the *step* of the distribution $\mathcal{H}$ at the point p. Therefore, we have

$$\mathcal{H}_p \subset \mathcal{H}_p^2 \subset \cdots \subset \mathcal{H}_p^s = T_pN.$$

The integer list $(n_1(p), \cdots, n_s(p))$ is called the *growth vector* of $\mathcal{H}$ at p. The *homogeneous dimension* $Q(p)$ of the Carnot manifold $(N, \mathcal{H})$ at $p \in N$ is given by

$$Q(p) = \sum_{j=1}^{s} j \; n_j(p). \tag{1.1.4}$$

When the growth vector is constant in a neighborhood of a point $p \in N$ we say that p is a *regular point* for the distribution. We say that a distribution $\mathcal{H}$ on a manifold N is *equiregular* if the growth vector is constant in N.

Given a connected sub-Riemannian manifold $(N, \mathcal{H}, h)$, and a C^1 horizontal path $\gamma : [a, b] \to N$, we define the length of γ by

$$L(\gamma) = \int_a^b \sqrt{h(\dot{\gamma}(t), \dot{\gamma}(t))} \; dt. \tag{1.1.5}$$

By means of the equality

$$d_c(p, q) := \inf\{L(\gamma) : \gamma \text{ is a } C^1 \text{ horizontal path joining } p, q \in N\}, \tag{1.1.6}$$

this length defines a distance function (see [12, § 2.1.1,§ 2.1.2]) usually called the *Carnot-Carathéodory distance*, or CC-*distance* for short. See [75, Chapter 1.4] for further details.

1.1.2 Lie groups and Carnot groups

A *Lie group* $(\mathbb{G}, \cdot)$ is a differentiable manifold which is also endowed with a group structure such that the map $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ defined by $(g, h) \to g \cdot h^{-1}$ is C^∞ (see [99, Definition 3.1]) and let $\mathfrak{g}$ be its Lie algebra.

Definition 1.1.1. A *Lie algebra* $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) such that for all $X, Y, Z \in \mathfrak{g}$,

1. $[X, Y] = -[Y, X]$. (anti-commutativity)

2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. (Jacobi identity)

Definition 1.1.2. Let $\varphi : N_1 \to N_2$ be a smooth function. The vector fields X on N_1 and Y on N_2 are called φ-related if $d\varphi(X) = Y \circ \varphi$.

There is a finite dimensional Lie algebra intimately associated with each finite dimensional Lie group. Moreover all the connected, simply connected Lie groups are completely determined (up to isomorphism) by their Lie algebras, for further details see [99]. Each Lie algebra can be seen as the space of all left invariant vector fields, i.e. l_g-related to themselves where the left translation by g in $\mathbb{G}$ is given by $l_g(h) = g \cdot h$.

Let $V \subset \mathfrak{g}$ be a linear subspace of the Lie algebra. This way, V is a left invariant distribution and Hörmander's rank condition corresponds to the fact that V Lie-generates $\mathfrak{g}$. In particular $(\mathbb{G}, V)$ is a Carnot manifold. If we set an inner product h on V we obtain a sub-Riemannian metric and $(\mathbb{G}, V, h)$ is a sub-Riemannian structure.

Definition 1.1.3. We say that $\mathbb{G}$ is a *graded nilpotent Lie group* if the Lie algebra $\mathfrak{g}$ has the form

$$\mathfrak{g} = V^1 \oplus V^2 \oplus \cdots \oplus V^s$$

where $[V^i, V^j] = V^{i+j}$ and $V^r = 0$ if $r > s$. Therefore, all iterated brackets of length $r > s$ are zero. Since $V^1 = V$ Lie-generates $\mathfrak{g}$, we obtain a *Carnot group*.

Carnot groups enjoy the property of admitting *dilations* δ_t, for $t > 0$ such that $d_c(\delta_t(x), \delta_t(y)) = t\ d_c(x, y)$ for each $x, y \in \mathbb{G}$. These are first defined on the Lie algebra by the map $\delta_t : \mathfrak{g} \to \mathfrak{g}$, where $\delta_t(X) := t^i X$ when $X \in V^i$, for $i = 1, \dots, s$. Since $\mathbb{G}$ is a simply connected Lie group we define the map $\delta_t : \mathbb{G} \to \mathbb{G}$ extending the map previously defined on the Lie algebra $\mathfrak{g}$ by the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$, defined in Section 1.1.3 and [99, Definition 3.30] .

1.1.3 Exponential map

Let $\Omega \subset N$ be an open set of N and X be a smooth vector field on Ω. Fixed $p \in \Omega$, the vector field X induces a local one parameter group of transformations on Ω, $\{\sigma_X(t,p) = \sigma(t,p)\}_t$ which is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial\ \sigma(t,p)}{\partial t} = X_{|\sigma(t,p)} \\ \sigma(0,p) = p. \end{cases} \tag{1.1.7}$$

This unique solution always exists for $|t|$ sufficiently small. Moreover, if $X = X(u_1, \cdots, u_r)$ depends in smooth way on parameters $(u_1, \cdots, u_r)$ in an open set $U \subset \mathbb{R}^r$ and we consider compact sets $L \subset U$ and $K \subset \Omega$, there exists a constant ϵ_0 such that

$$\sigma : L \times]-\epsilon_0, \epsilon_0[\times K \to \Omega \tag{1.1.8}$$

is a smooth function. When it is clear that the parameters $u = (u_1, \ldots, u_r)$ and also the vector field X are fixed, we denote $\sigma_X(u,t,p)$ by $\sigma(t,p)$. For all t sufficiently small, $\sigma_X(t,x) = \sigma_{tX}(1,x) = \exp(tX)(x)$ is always well-defined. By the uniqueness of (1.1.7), there holds

$$\sigma(s, \sigma(t,p)) = \sigma(s+t,p) \qquad \text{if} \quad p \in K,\ |s+t| < \epsilon_0, \tag{1.1.9}$$

$$\sigma_{\lambda X}(t,p) = \sigma_X(\lambda t, p) \qquad \text{when} \quad p \in K,\ |\lambda t| < \epsilon_0. \tag{1.1.10}$$

Now, by equation (1.1.9) the function $x \to \sigma(-t,x)$ is a C^∞ inverse of $x \to \sigma(t,x)$. Therefore, $x \to \sigma(t,x)$ is a diffeomorphism on a compact set of Ω, for $|t|$ sufficiently small. In this sense we construct a parameter group of diffeomorphisms.

Definition 1.1.4. We define the *exponential map* by

$$\exp(X)(p) = \sigma_X(1,p)$$

whenever the right hand side is defined.

Let $X_1, \cdots, X_l$ be an orthonormal frame w.r.t. h on Ω and $(u_1, \cdots, u_l)$ be parameters in $\mathbb{R}^l$. Then, if

$$|u| = \sqrt{\sum_{i=1}^{l} u_i^2} \tag{1.1.11}$$

is sufficiently small, $|u| < \epsilon_0$, we have that the function

$$(u_1, \cdots, u_l, p) \to \exp\left(\sum_{i=1}^{l} u_i\ X_i\right)(p) \tag{1.1.12}$$

is well-defined and smooth, see [80, Appendix].

Remark 1.1.5. Notice that

1. the curve $\gamma(t) = \exp(tX)(p) = \sigma_X(t, p)$ is horizontal, when X a vector field in $\mathcal{H}$ on N.

2. the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism to a simply connected Lie group $\mathbb{G}$ from its Lie algebra $\mathfrak{g}$.

1.1.4 Sub-Riemannian differential operators

A sub-Riemannian manifold $(N, \mathcal{H}, h)$ is the underlying structure to study the degenerate subellitic operator. Let $(X_1, \ldots, X_l)$ be a local frame for the distribution $\mathcal{H}$ orthonormal with respect to the horizontal metric h and $\Omega \subset N$ be an open set, the *horizontal gradient* of a function $u : \Omega \to \mathbb{R}$ is given by

$$\nabla_{\mathcal{H}} u = \sum_{i=1}^{l} X_i(u) X_i, \tag{1.1.13}$$

where we denote by $X_i(u)(p)$ at $p \in \Omega$ the Lie derivative

$$X_i(u)(p) = \lim_{t \to 0} \frac{u(\exp(tX_i)(p)) - u(p)}{t}.$$

We say that f belongs to $C^1_{\mathcal{H}}(\Omega, \mathbb{R})$ if $X_i(f)$ are continuous functions with respect to the Carnot-Carathéodory distance d_c defined in (1.1.6) for every $i = 1, \cdots, l$. Then we define the class $C^r_{\mathcal{H}}(\Omega, \mathbb{R})$ by iteration. We will call these spaces the spaces of intrinsic function of class k w.r.t. the distribution $\mathcal{H}$.

For each $i = 1, \ldots, l$ we denote by X_i^* the *adjoint* of X_i with respect to a volume form *vol* given by

$$\int_N u\, X_i(f)\ \mathrm{dvol} = \int_N X_i^*(u)\, f\ \mathrm{dvol}.$$

Since in local coordinates dvol $=\omega \mathrm{d}\mathcal{L}$, where ω is a smooth density and $\mathcal{L}$ is the Lebesgue measure, and $X_i=\sum_{j=1}^n a_{ij}\partial_j$ we have

$$X_i^*(u)=-\omega^{-1}X_i(\omega\, u)-\sum_{j=1}^n \partial_j(a_{ij})u.$$

Therefore the sub-Laplacian defined by

$$\Delta_{\mathcal{H}}(u):=\mathrm{div}_{\mathcal{H}}(\nabla_{\mathcal{H}}(u))=\sum_{i=1}^l X_i^*X_i(u) \tag{1.1.14}$$

is a degenerate elliptic second order operator since the matrix $(a_{ij}(a_{ij})^t)$ is positive semi-definite.

Definition 1.1.6. Let P be a linear differential operator with C^∞ coefficients in an open set Ω of N. Given a relative compact open set $\Omega' \subset\subset \Omega$, we say that P is hypoellitic if for each $f\in C^\infty(\Omega')$ the solution u of the equation $Pu=f$ belongs to $C^\infty(\Omega')$.

L. Hörmander in his celebrated paper [55] showed that whenever the distribution $\mathcal{H}=\mathrm{span}\{X_1,\ldots,X_l\}$ satisfies Hörmander's rank condition (1.1.2) the degenerate second order $\Delta_{\mathcal{H}}$ is hypoelliptic. This result opened up the regularity theory and the study of the fundamental solution for degenerate second order equations mostly developed by [33, 48, 79, 59, 60]. In 1975 L. Rothschild and E. Stein [93] proved their *lifting approximation theorem*. After a lifting at not regular point to a free up algebra to a fixed level s (where all points are regular), they provided an homogeneous nilpotent approximation $(\hat{X}_1,\ldots,\hat{X}_l)$ of the system of vector fields $(X_1,\ldots,X_l)$. Then they obtain the fundamental solution of the sub-Laplacian $\Delta_{\mathcal{H}}$ using the parametrix method where the approximate operator is $\sum_{i=1}^l \hat{X}_i^*\hat{X}_i$. Similarly the Carnot-Carathéodory distance d_c can be approximated by a nilpotent distance $\hat{d}$, then, using the Hausdorff-Gromov convergence, J. Mitchell [71] proved that Carnot groups are the tangent cones for the sub-Riemannian manifolds at regular points. This result was later revisited by A. Bellaiche in [7]. Thus, we understand the importance of Carnot groups, that play the role of tangent spaces for the sub-Riemannian geometry as the Euclidean spaces $\mathbb{R}^n$ are the tangent spaces for the Riemannian manifolds.

1.1.5 Hypersurfaces immersed in sub-Riemannian manifolds

Let $(N,\,\mathcal{H},h)$ be a sub-Riemannian manifold. Consider an open set $\Omega\subset N$ and an orthonormal frame $(X_1,\ldots,X_l)$ of the distribution $\mathcal{H}$. Given a function $u\in L^1_{\mathrm{loc}}(\Omega)$

the *horizontal variation* of u with respect to Ω is defined as

$$\mathrm{Var}_{\mathcal{H}}(u,\Omega) = \sup\left\{\int_{\Omega} u\,\mathrm{div}_{\mathcal{H}}(\phi)\,\mathrm{dvol} \;:\; \phi\in C_0^1(\Omega,\mathcal{H})\right\}, \tag{1.1.15}$$

where in local coordinates $\phi=\sum_{i=1}^{l}\phi_i X_i$ for $\phi\in C_0^1(\Omega)$, $\mathrm{div}_{\mathcal{H}}(\phi)=\sum_{i=1}^{l}X_i^*(\phi_i)$ and *dvol* is the Riemannian volume form. A function u is said to have bounded $\mathcal{H}$-variation in Ω if $\mathrm{Var}_{\mathcal{H}}(u,\Omega)<\infty$, and the collection of all such functions is denoted by $BV_{\mathcal{H}}(\Omega)$. Following the celebrated paper by E. De Giorgi [30], N. Garofalo and D.M. Nhieu [43] defined the notion of perimeter in the sub-Riemannian setting.

Definition 1.1.7. Given $E\subset\Omega$ a measurable set, the $\mathcal{H}$-perimeter of E relative to Ω is defined by

$$P_{\mathcal{H}}(E,\Omega)=\mathrm{Var}_{\mathcal{H}}(\chi_E,\Omega),$$

where χ_E is the characteristic function of E. When $P_{\mathcal{H}}(E,\Omega)<\infty$ for each open set $\Omega\subset N$ we say that E is a $\mathcal{H}$-Caccioppoli set.

Inspired by the classical Plateau problem, that consists on finding hypersurfaces of least area among those that share a fix boundary, and the isoperimetric problem, that searches for the least area enclosing a fixed volume, several authors in the last twenty years have developed a rich theory in these sub-Riemannian setting. In [43, Theorem 1.24] the authors established an existence theorem for minimal surfaces under suitable assumptions on the ambient manifold N as the Poincaré inequality and the doubling property.

Fix a smooth Riemannian metric g on N such that $g_{|\mathcal{H}}=h$ and assume that $E\subset N$ has C^1 boundary Σ, it follows from the Divergence Theorem in the Riemannian manifold (N,g) that the perimeter $P_{\mathcal{H}}(E)$ coincides with the sub-Riemannian area of Σ defined by

$$A(\Sigma)=\int_{\Sigma}|\nu_h|d\Sigma, \tag{1.1.16}$$

where ν is a unit vector field normal to Σ with respect to g, ν_h the orthogonal projection of ν to the horizontal distribution, and $d\Sigma$ is the Riemannian measure of Σ.

Keeping in mind the definition of intrinsic function in section 1.1.4, the natural definition of intrinsic hypersurface in Carnot groups was provided by B. Franchi, R. Serapioni and F. Serra Cassano in [36].

Definition 1.1.8. Let $(\mathbb{G},\cdot)$ be a Carnot group. We say that $\Sigma\subset\mathbb{G}$ is a $\mathbb{G}$-intrinsic hypersurface if for any $p\in\Sigma$ there is an open set U of p and $f\in C^1_{\mathcal{H}}(U)$ such that

$$\Sigma\cap U=\{q\in U \;:\; f(q)=0\,,\;\nabla_{\mathcal{H}}f(q)\neq 0\}.$$

This definition allows to avoid the presence of singular (or characteristic) points, where the projection of the normal ν_h is equal to zero. However these intrinsic hypersurfaces can even be fractals. This notion of $\mathbb{G}$-intrinsic hypersurface is the right one to study the retificability of the $\mathcal{H}$-perimeter in Carnot groups, that was first proved by B. Franchi, R. Serapioni and F. Serra Cassano in [35] and then and then widely studied by [65, 29, 4, 25].

On the other hand, assuming that the hypersuface is a C^1 immersion, we allow the possibility of singular points.

Sub-Riemannian mean curvature equation for hypersurfaces

Let Σ be a C^2 immersed hypersurface in a sub-Riemannian manifold $(N, \mathcal{H}, h)$, $\Sigma_0 = \{p \in \Sigma \ : \ \nu_h = 0\}$ be the set of characteristic points and $\hat{\nu}_h = \frac{\nu_h}{|\nu_h|}$ outside from Σ_0. Let g be a Riemannian metric on TN such that g restricted to $\mathcal{H}$ concides with h. First of all we provide a variation Γ_t of Σ compactly supported in $\Sigma \smallsetminus \Sigma_0$. Therefore, computing the first variational formula

$$\left.\frac{d}{dt}\right|_{t=0} A(\Gamma_t(\Sigma))$$

for a C^2 immersed hypersurface, we obtain that the sub-Riemannian mean curvature is given by

$$\mathrm{div}^h_\Sigma(\hat{\nu}_h) - \sum_{j=l+1}^{n} \langle [\hat{\nu}_h, X_j], X_j \rangle \tag{1.1.17}$$

out of the characteristic set. In the previous formula the horizontal divergence

$$\mathrm{div}^h_\Sigma(\nu_h) = \sum_{i=1}^{l-1} \langle \nabla_{e_i} \hat{\nu}_h, e_i \rangle$$

is the trace of the horizontal second fundamental form $\mathit{II}_{\mathcal{H}}(e_i, e_j) = \langle \nabla_{e_i} \hat{\nu}_h, e_j \rangle$ for each $i, j = 1, \ldots, l-1$, where $e_1, \ldots, e_{l-1}$ is an orthonormal basis of $T_p\Sigma \cap \mathcal{H}_p$ such that $e_1, \ldots, e_{l-1}, \hat{\nu}_h = \frac{\nu_h}{|\nu_h|}$ is an orthonormal basis of $T_p\mathcal{H}$ and ∇ is a connection adapted to the sub-Riemannian structure, see [53, Definition 3.3]. The operator defined in [53] is not self-adjoint, but the one defined in [91] in the Heisenberg group is self-adjoint and provides a notion of principal curvatures and of mean curvature.

In the last years several papers from different fields have studied this horizontal mean curvature equation to solve Plateau's problem, Bernstein's problem and the isoperimetric problem in this new setting, only to mention a few [43, 28, 17, 18, 29, 6, 53, 54, 57, 89, 90, 72, 22].

1.1.6 Submanifolds immersed in Carnot groups

The natural question that arises is what is the natural replacement of the $\mathcal{H}$-perimeter for submanifolds of higher codimension immersed in a sub-Riemannian geometry. Since the spherical Hausdorff measure is not manageable, because it is not lower semicontinuous with respect to the Hausdorff convergence of sets, V. Magnani and D. Vittone in [69] introduced an new measure for C^1 submanifolds immersed in Carnot group $\mathbb{G}$, with Lie algebra $\mathfrak{g} = V^1 \oplus V^2 \oplus \cdots \oplus V^s$. Fix a left invariant metric g that makes the layers V^i orthogonal and an open set $\Omega \subset \mathbb{R}^m$. Let $\Phi : \Omega \to \mathbb{G}$ be an C^1 immersion, then the area measure for each Borel subset $\Omega' \subset \Omega$ is given by

$$A(\Omega') = \int_{\Omega'} |(\Phi_{x_1}(x) \wedge \ldots \wedge \Phi_{x_m}(x))_d|_g \, d\mathcal{L}(x), \tag{1.1.18}$$

where $\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^m$, $|\cdot|_g$ denotes the norm induced by g on the m-vectors and $(\cdot)_d$ is the projection of the m-tangent vector onto the degree $d = \deg(\Phi(\Omega))$. The degree of a submanifold, first introduced by Gromov [50], is the maximum over all points of $p \in M$ of the homogeneous dimension

$$\deg_M(p) = \sum_{j=1}^{s} j \dim(T_pM \cap V^j).$$

For a formal definition of the degree the reader can refer to Section 1.2.3. For $C^{1,1}$ immersions the authors in [69, Theorem 1.1] proved the equivalence between the area measure (1.1.18) and the Hausdorff measure under the key assumption that points of degree less than d are negligible with respect to the Hausdorff measure. Moreover it is worth mentioning the intrinsic approach for submanifolds in the Heisenberg group developed by [37, 38] and their replaced in [39] of the spherical Hausdorff measure by the centered m-dimensional Hausdorff measure in more general setting of metric spaces.

1.1.7 The endpoint map and Pontryagin's principle

Let $(N, \mathcal{H})$ be a Carnot manifold where $\mathcal{H}$ is a l-dimensional distribution and $I = [a, b] \subset \mathbb{R}$. Given a point $p_0 \in N$ we consider the space $\Omega_{\mathcal{H}}(I, p_0)$ of all possible absolutely continuous curves starting at p_0 whose derivatives are square integrable on I, for any metric h on the distribution. The endpoint map

$$\mathcal{E} : \Omega_{\mathcal{H}}(I, p_0) \to N$$

is defined by $\mathcal{E}(\gamma) = \gamma(b)$. Given $X_1, \dots, X_l$ a local frame for $\mathcal{H}$, the $\Omega_{\mathcal{H}}(I, p_0)$ is a Banach manifold based on $L^2(I, \mathbb{R}^l)$ with local coordinates $(u_1, \dots, u_l) \in L^2(I, \mathbb{R}^l)$ given by

$$\gamma'(t) = \sum_{i=1}^{l} u_i(t) X_i.$$

In [75, Appendix E] it is shown that that for the Cauchy problem

$$\begin{cases} \gamma'(t) = \sum_{i=1}^{l} u_i(t) X_i \\ \gamma(a) = p_0 \end{cases} \tag{1.1.19}$$

has a unique solution $\gamma = \gamma(u, p_0)$ so that in local coordinates $\mathcal{E}(u_1, \dots, u_l) = \gamma(b)$.

Definition 1.1.9. A *regular* curve is a regular point of the endpoint map $\mathcal{E}$ and a *singular* curve is a critical point of the endpoint map $\mathcal{E}$.

Remark 1.1.10. We notice that a curve $\gamma : [a, b] \to N$ given by (1.1.19) is singular if and only if there exists a co-vector $\bar{\lambda} \in T^*_{\gamma(b)} N$ such that

$$\bar{\lambda}(d\mathcal{E}(u)\, v) = 0,$$

for each $v \in L^2(I, \mathbb{R}^l)$, where $d\mathcal{E}(u)$ is the differential of $\mathcal{E}$ at $u \in L^2(I, \mathbb{R}^l)$.

Let $\mathcal{H}^{\perp *} \subset T^*N$ be the space of one-forms of rank $n - l$ that annihilate the distribution $\mathcal{H}$. Fix a local frame of $\mathcal{H}^{\perp *}$ given by $(\theta^{l+1}, \dots, \theta^n)$ in a local neighborhood U of N. Let $X_{l+1}, \dots, X_n$ be the dual frame of $(\theta^{l+1}, \dots, \theta^n)$, $X_1, \dots, X_l$ be the frame of $\mathcal{H}$ and $(\eta_1, \dots, \eta_l)$ be the dual co-frame with respect to $X_1, \dots, X_l$. The structure functions c^k_{ij} for each $i, j = 1, \dots, n$ are given by

$$[X_i, X_j] = \sum_{k=1}^{n} c^k_{ij} X_k. \tag{1.1.20}$$

Let ω be the restriction of symplectic form to $\mathcal{H}^{\perp *}$. Following the method of characteristics developed by L. Hsu in [56], R. Montgomery [75, Theorem 5.2.2] recognized that a horizontal curve γ is singular if and only if γ is the projection of a characteristic for $\mathcal{H}^{\perp *}$, that is a never vanishing absolutely continuous curve $\lambda(t) \in \mathcal{H}^{\perp *}$ such that the interior product $\iota_{\lambda'(t)} \omega = 0$ on $T_\lambda \mathcal{H}^{\perp *}$ whenever $\lambda'(t)$ exists. An absolutely continuous curve $\lambda : [a, b] \to T^*N$ that belongs to $\mathcal{H}^{\perp *}$ is given by

$$\lambda'(t) = \sum_{r=l+1}^{n} \lambda'_r(t) \frac{\partial}{\partial y_r} + \sum_{i=1}^{l} u_i(t) X_i + \sum_{j=l+1}^{n} u_j(t) X_j. \tag{1.1.21}$$

The restriction to $\mathcal{H}^{\perp *}$ of the tautological one-form is $\sum_{r=l+1}^{n} y_r\theta^r$. Since ω is the differential of this restricted form we have

$$\omega = \sum_{r=l+1}^{n} dy^r \wedge \theta^r + \sum_{j=l+1}^{n} y_j d\theta^j.$$

Then the equation $\iota_{\lambda'(t)}\omega = 0$ is equivalent to

$$0 = \frac{1}{2}\sum_{r=l+1}^{n} \lambda_r'(t)\theta^r - u_r dy^r + \sum_{j=l+1}^{n} \lambda_j d\theta^j(\lambda'(t),\cdot). \tag{1.1.22}$$

Putting the Maurer-Cartan equation (see [58]) given by

$$d\theta^j = -\frac{1}{2}\sum_{i,h=1}^{l} c_{ih}^j\, \eta^i \wedge \eta^h - \sum_{i=1}^{l}\sum_{r=l+1}^{n} c_{ir}^j\, \eta^i \wedge \theta^r - \frac{1}{2}\sum_{t,r=l+1}^{n} c_{tr}^j\, \theta^t \wedge \theta^r,$$

for $r = l+1,\dots,n$, into (1.1.22) we gain $u_r = 0$ for $r = l+1,\dots,n$ and

$$0 = \frac{1}{2}\sum_{r=l+1}^{n} \lambda_r'(t)\theta^r - \frac{1}{2}\sum_{j=l+1}^{n}\sum_{r=l+1}^{n}\sum_{i=1}^{l} \lambda_j (c_{ih}^j u_i \eta^h + c_{ir}^j u_i \theta^r).$$

Therefore we obtain that absolutely continuous curve $\lambda(t)$ with square-integrable derivative (1.1.21) is a characteristic if and only if $\lambda(t)$ satisfies the following equations

$$\begin{cases} u_j(t) = 0 \quad j = l+1,\dots,n, \\ \lambda_r' - \displaystyle\sum_{j=l+1}^{n}\sum_{i=1}^{l} \lambda_j\, c_{ir}^j\, u_i = 0, \quad r = l+1,\dots,n, \\ \displaystyle\sum_{i=1}^{l}\sum_{j=l+1}^{n} \lambda_j\, c_{ih}^j\, u_i = 0 \quad h = 1,\dots,l. \end{cases} \tag{1.1.23}$$

In control theory singular curves are the projection of *abnormal extremals* that are defined by means of the Pontryagin Maximum Principle [86] that provides necessary conditions for a horizontal curve to be a minimizer. When γ with controls u is length minimizing we call the pair (γ, u) an optimal pair.

Theorem 1.1.11 (PMP). *Let (γ, u) be an optimal pair. Then there exist $\lambda_0 \in \{0,1\}$ and Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}N$ such that*

1. $(\lambda_0, \lambda(t)) \neq (0,0)$;
2. $\lambda_0\, u_i + \langle \lambda(t), (X_i)_{\gamma(t)} \rangle = 0 \quad i = 1,\dots,l;$

3. the coordinates λ_r *for* $r = 1\dots,n$ *satisfy*

$$\lambda_r'(t) = \sum_{j=1}^{n}\sum_{i=1}^{l} \lambda_j\, c_{ir}^j u_i. \tag{1.1.24}$$

As R. Monti described in [77] the proof of Theorem 1.1.11 (see [3]) is based on the extended endpoint map

$$\mathscr{F} : \Omega_{\mathcal{H}}(I, p_0) \to \mathbb{R} \times N$$

given by

$$\mathscr{F}(\gamma) = \left(\int_a^b \|\dot\gamma(t)\| dt\, ,\, \mathcal{E}(\gamma) \right),$$

that in local coordinates is

$$\mathscr{F}(u) = \left(\int_a^b \|u(t)\| dt\, ,\, \mathcal{E}(u) \right).$$

If (γ, u) is an optimal pair for $\mathscr{F}$, then its differential $d\mathscr{F}(u)$ is not surjective. Therefore there exists a co-vector $(0,0) \neq (\lambda_0, \bar\lambda) \in \mathbb{R} \times T^*_{\gamma(b)}N$ at $\gamma(b)$ such that

$$\langle (\lambda_0, \bar\lambda), d\mathscr{F}(u)v \rangle = 0,$$

for each $v \in L^2(I, \mathbb{R}^l)$. When $\lambda_0 = 0$ we obtain the case of the abnormal extremals that are exactly the critical point of $d\mathcal{E}(u)$, thus by Definition 1.1.9 they are singular curves. The Lipschitz curve $\lambda(t)$ of Theorem 1.1.11 is obtain by

$$\lambda(t) = (P_{-t})^* \bar\lambda,$$

where $(P_{-t})^*$ is the pull-back of the optimal flow $P_t(p) = \gamma(t)$, that at each point $p \in N$ associates the solution $\gamma(t)$ at the time $t \in [a,b]$ of the Cauchy problem

$$\begin{cases} \dot\gamma(t) = \sum_{i=1}^{l} u_i X_i \\ \gamma(a) = p. \end{cases}$$

In control theory the necessary conditions 1,2,3 for the optimality of Theorem 1.1.11 are used to define the notion of extremal. We say that an horizontal curve $\gamma : [a,b] \to N$ is an *extremal* if there exist $\lambda_0 \in \{0,1\}$ and a Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}N$ such that 1,2,3 hold. When $\lambda_0 \neq 0$ we say that γ is a normal extremal and when $\lambda_0 = 0$ we say that γ is an abnormal extremal. A strictly abnormal extremal is an abnormal extremal but not a normal one.

Notice that condition 2 for an abnormal extremal ($\lambda_0 = 0$) is equivalent to

$$\langle \lambda(t), (X_i)_{\gamma(t)} \rangle = 0, \tag{1.1.25}$$

for $i = 1, \dots, l$. Therefore we have $\lambda(t) \in \mathcal{H}^{\perp *}$, that implies that the index j in (1.1.24) goes from $l + 1$ to n. Differentiating (1.1.25) we obtain

$$\sum_{h=1}^{l} \sum_{j=l+1}^{n} \lambda_j \, c^j_{hi} \, u_h = 0, \tag{1.1.26}$$

for $i = 1, \dots, l$. Putting together (1.1.25), (1.1.26) and (1.1.24) we gain, as we expected, that an abnormal extremal satisfies the characteristic equation (1.1.23).

1.2 Preliminaries

1.2.1 Graded Structure

Let N be an n-dimensional smooth manifold. Given two smooth vector fields X, Y on N, their *commutator* or *Lie bracket* [X,Y] is defined by (1.1.1). An *increasing filtration* $(\mathcal{H}^i)_{i\in\mathbb{N}}$ of the tangent bundle TN is a flag of sub-bundles

$$\mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^i \subset \cdots \subseteq TN, \tag{1.2.1}$$

such that

(i) $\cup_{i\in\mathbb{N}}\mathcal{H}^i = TN$

(ii) $[\mathcal{H}^i, \mathcal{H}^j] \subseteq \mathcal{H}^{i+j}$, for $i, j \geqslant 1$,

where $[\mathcal{H}^i, \mathcal{H}^j] := \{[X, Y] : X \in \mathcal{H}^i, Y \in \mathcal{H}^j\}$. Moreover, we say that an increasing filtration is *locally finite* when

(iii) for each $p \in N$ there exists an integer $s = s(p)$, the *step* at p, satisfying $\mathcal{H}^s_p = T_pN$. Then we have the following flag of subspaces

$$\mathcal{H}^1_p \subset \mathcal{H}^2_p \subset \cdots \subset \mathcal{H}^s_p = T_pN. \tag{1.2.2}$$

A *graded manifold* $(N, (\mathcal{H}^i))$ is a smooth manifold N endowed with a locally finite increasing filtration, namely a flag of sub-bundles (1.2.1) satisfying (i),(ii) and (iii). For the sake of brevity a locally finite increasing filtration will be simply called a filtration. Despite it may seem repetitive, for completeness reasons we will recall some concepts we previously introduced in Section 1.1.1 for Carnot manifolds. Setting $n_i(p) := \dim \mathcal{H}^i_p$, the integer list $(n_1(p), \cdots, n_s(p))$ is called the *growth vector* of the filtration (1.2.1) at p. When the growth vector is constant in a neighborhood of a point $p \in N$ we say that p is a *regular point* for the filtration. We say that a filtration $(\mathcal{H}^i)$ on a manifold N is *equiregular* if the growth vector is constant in N. From now on we suppose that N is an equiregular graded manifold.

Given a vector v in T_pN we say that the *degree* of v is equal to ℓ if $v \in \mathcal{H}^\ell_p$ and $v \notin \mathcal{H}^{\ell-1}_p$. In this case we write $\deg(v) = \ell$. The degree of a vector field is defined pointwise and can take different values at different points.

Let $(N, (\mathcal{H}^1, \ldots, \mathcal{H}^s))$ be an equiregular graded manifold. Take $p \in N$ and consider an open neighborhood U of p where a local frame $\{X_1, \cdots, X_{n_1}\}$ generating $\mathcal{H}^1$ is defined. Clearly the degree of X_j, for $j = 1, \dots, n_1$, is equal to one since the vector fields $X_1, \dots, X_{n_1}$ belong to $\mathcal{H}^1$. Moreover the vector fields $X_1, \dots, X_{n_1}$ also lie in $\mathcal{H}^2$, we add some vector fields $X_{n_1+1}, \cdots, X_{n_2} \in \mathcal{H}^2 \setminus \mathcal{H}^1$ so that $(X_1)_p, \dots, (X_{n_2})_p$ generate $\mathcal{H}^2_p$. Reducing U if necessary we have that $X_1, \dots, X_{n_2}$ generate $\mathcal{H}^2$ in U. Iterating this procedure we obtain a basis of TM in a neighborhood of p

$$(X_1, \dots, X_{n_1}, X_{n_1+1}, \dots, X_{n_2}, \dots, X_{n_{s-1}+1}, \dots, X_n), \tag{1.2.3}$$

such that the vector fields $X_{n_{i-1}+1}, \dots, X_{n_i}$ have degree equal to i, where $n_0 := 0$. The basis obtained in (1.2.3) is called an *adapted basis* to the filtration $(\mathcal{H}^1, \dots, \mathcal{H}^s)$.

Remark 1.2.1 (Carnot manifolds are graded structure)**.** The flag of sub-bundles (1.1.3) associated to a Carnot manifold $(N, \mathcal{H})$ gives rise to the graded structure $(N, (\mathcal{H}^i))$. Clearly an equiregular Carnot manifold $(N, \mathcal{H})$ of step s is a equiregular graded manifold $(N, \mathcal{H}^1, \dots, \mathcal{H}^s)$.

Submanifolds immersed in Carnot manifolds

Let M be a submanifold immersed in an equiregular Carnot manifold $(N, \mathcal{H})$ of step s. The intersection subspace $\tilde{\mathcal{H}}_p := \mathcal{H}_p \cap T_pM$ at each point $p \in M$ generates a distribution $\tilde{\mathcal{H}}$ on M. Since a priori the distribution $\tilde{\mathcal{H}}$ does not satisfy Hörmander's condition, the structure $(M, \tilde{\mathcal{H}})$ is not a Carnot manifold. Nevertheless, setting $\tilde{\mathcal{H}}^i := TM \cap \mathcal{H}^i$, the submanifold M inherits a locally finite increasing filtration $\tilde{\mathcal{H}}^1 \subset \ldots \subset \tilde{\mathcal{H}}^s = TM$, that at each point in M is given by

$$\tilde{\mathcal{H}}^1_p \subset \tilde{\mathcal{H}}^2_p \subset \cdots \subset \tilde{\mathcal{H}}^s_p = T_pM, \tag{1.2.4}$$

where $\tilde{\mathcal{H}}^j_p = T_pM \cap \mathcal{H}^j_p$ and $\tilde{n}_j(p) = \dim(\tilde{\mathcal{H}}^j_p)$. Evidently, (i) in Definition 1.2.1 is satisfied. On the other hand, if $X \in \tilde{\mathcal{H}}^i$ and $Y \in \tilde{\mathcal{H}}^j$, we can extend both vector fields in a neighborhood of M so that the extensions X_1, Y_1 lie in $\mathcal{H}^i$ and $\mathcal{H}^j$, respectively. Then $[X, Y]$ is a tangent vector to M that coincides on M with $[X_1, Y_1] \in \mathcal{H}^{i+j}$. Hence $[X, Y] \in \tilde{\mathcal{H}}^{i+j}$. This implies condition (ii) in Definition 1.2.1. Therefore $(M, \tilde{\mathcal{H}}^1, \dots, \tilde{\mathcal{H}}^s)$ is a graded manifold.

1.2.2 Degree of m-vectors

Given an adapted basis $(X_i)_{1\leqslant i\leqslant n}$, the *degree* of the *simple* m-vector field $X_{j_1}\wedge\ldots\wedge X_{j_m}$ is defined by

$$\deg(X_{j_1}\wedge\ldots\wedge X_{j_m}) := \sum_{i=1}^{m}\deg(X_{j_i}).$$

Any m-vector X can be expressed as a sum

$$X_p = \sum_J \lambda_J(p)(X_J)_p,$$

where $J=(j_1,\ldots,j_m)$, $1\leqslant j_1<\cdots<j_m\leqslant n$, is an ordered multi-index, and $X_J := X_{j_1}\wedge\ldots\wedge X_{j_m}$. The degree of X at p with respect to the adapted basis $(X_i)_{1\leqslant i\leqslant n}$ is defined by

$$\max\{\deg((X_J)_p) : \lambda_J(p)\neq 0\}.$$

It can be easily checked that the degree of X is independent of the choice of the adapted basis and it is denoted by $\deg(X)$.

If $X=\sum_J \lambda_J X_J$ is an m-vector expressed as a linear combination of simple m-vectors X_J, its projection onto the subset of m-vectors of degree d is given by

$$(X)_d = \sum_{\deg(X_J)=d}\lambda_J X_J, \tag{1.2.5}$$

and its projection over the subset of m-vectors of degree larger than d by

$$\pi_d(X) = \sum_{\deg(X_J)\geqslant d+1}\lambda_J X_J.$$

In an equiregular graded manifold with a local adapted basis $(X_1,\ldots,X_n)$, defined as in (1.2.3), the maximal degree that can be achieved by an m-vector, $m\leqslant n$, is the integer $d^m_{\max}$ defined by

$$d^m_{\max} := \deg(X_{n-m+1})+\cdots+\deg(X_n). \tag{1.2.6}$$

1.2.3 Degree of a submanifold

Let M be a submanifold of class C^1 immersed in an equiregular graded manifold $(N,(\mathcal{H}^1,\ldots,\mathcal{H}^s))$ such that $\dim(M)=m<n=\dim(N)$. Then, following [62, 69], we

define the degree of M at a point $p \in M$ by

$$\deg_M(p) := \deg(v_1 \wedge \ldots \wedge v_m), \tag{1.2.7}$$

where $v_1, \ldots, v_m$ is a basis of T_pM. Obviously, the degree is independent of the choice of the basis of T_pM. Indeed, if we consider another basis $\mathcal{B}' = (v_1', \cdots, v_m')$ of T_pM, we get

$$v_1 \wedge \cdots \wedge v_m = \det(M_{\mathcal{B},\mathcal{B}'})\ v_1' \wedge \cdots \wedge v_m'.$$

Since $\det(M_{\mathcal{B},\mathcal{B}'}) \neq 0$, we conclude that $\deg_M(p)$ is well-defined. The *degree* $\deg(M)$ of a submanifold M is the integer

$$\deg(M) := \max_{p \in M} \deg_M(p). \tag{1.2.8}$$

We define the *singular set* of a submanifold M by

$$M_0 = \{p \in M : \deg_M(p) < \deg(M)\}. \tag{1.2.9}$$

Singular points can have different degrees between m and $\deg(M) - 1$.

In [50, 0.6.B] Gromov considers the flag

$$\tilde{\mathcal{H}}_p^1 \subset \tilde{\mathcal{H}}_p^2 \subset \cdots \subset \tilde{\mathcal{H}}_p^s = T_pM, \tag{1.2.10}$$

where $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$ and $\tilde{m}_j = \dim(\tilde{\mathcal{H}}_p^j)$. Then he defines the degree at p by

$$\tilde{D}_H(p) = \sum_{j=1}^{s} j(\tilde{m}_j - \tilde{m}_{j-1}),$$

setting $\tilde{m}_0 = 0$. It is easy to check that our definition of degree is equivalent to Gromov's one, see [46, Chapter 2.2]. As we already pointed out, $(M, (\tilde{\mathcal{H}}^j)_{j \in \mathbb{N}})$ is a graded manifold.

Let us check now that the degree of a vector field and the degree of points in a submanifold are lower semicontinuous functions.

Lemma 1.2.2. *Let $(N, (\mathcal{H}^1, \ldots, \mathcal{H}^s))$ be a graded manifold regular at $p \in N$. Let V be a vector field defined on a open neighborhood U_1 of p. Then we have*

$$\liminf_{q \to p} \deg(V_q) \geqslant \deg(V_p).$$

Proof. As $p \in N$ is regular, there exists a local adapted basis $(X_1, \ldots, X_n)$ in an open neighborhood $U_2 \subset U_1$ of p. We express the smooth vector field V in U_2 as

$$V_q = \sum_{i=1}^{s} \sum_{j=n_{i-1}+1}^{n_i} c_{ij}(q)(X_j)_q \tag{1.2.11}$$

on U_2 with respect to an adapted basis $(X_1, \cdots, X_n)$, where $c_{ij} \in C^\infty(U_2)$. Suppose that the degree $\deg(V_p)$ of V at p is equal to $d \in \mathbb{N}$. Then, there exists an integer $k \in \{n_{d-1}+1, \cdots, n_d\}$ such that $c_{dk}(p) \neq 0$ and $c_{ij}(p) = 0$ for all $i = d+1, \cdots, s$ and $j = n_{i-1}+1, \cdots, n_i$. By continuity, there exists an open neighborhood $U' \subset U_2$ such that $c_{dk}(q) \neq 0$ for each q in U'. Therefore for each q in U' the degree of V_q is greater than or equal to the degree of $V(p)$,

$$\deg(V_q) \geqslant \deg(V_p) = d.$$

Taking limits we get

$$\liminf_{q \to p} \deg(V_q) \geqslant \deg(V_p). \qquad \square$$

Remark 1.2.3. In the proof of Lemma 1.2.2, $\deg(V_q)$ could be strictly greater than d in case there were a coefficient c_{ij} with $i \geqslant d+1$ satisfying $c_{ij}(q) \neq 0$.

Proposition 1.2.4. *Let M be a C^1 immersed submanifold in a graded manifold $(N, (\mathcal{H}^1, \ldots, \mathcal{H}^s))$. Assume that N is regular at $p \in M$. Then we have*

$$\liminf_{q \to p, q \in M} \deg_M(q) \geqslant \deg_M(p).$$

Proof. The proof imitates the one of Lemma 1.2.2 and it is based on the fact that the degree is defined by an open condition. Let $\tau_M = \sum_J \tau_J X_J$ be a tangent m-vector in an open neighborhood U of p, where a local adapted basis is defined. The functions τ_J are continuous on U. Suppose that the degree $\deg_M(p)$ at p in M is equal to d. This means that there exists a multi-index $\bar{J}$ such that $\tau_{\bar{J}}(p) \neq 0$ and $\deg((X_{\bar{J}})_p) = d$. Since the function $\tau_{\bar{J}}$ is continuous there exists a neighborhood $U' \subset U$ such that $\tau_{\bar{J}}(q) \neq 0$ in U'. Therefore, $\deg(\tau_M(q)) \geqslant d$ and taking limits we have

$$\liminf_{q \to p} \deg_M(q) \geqslant \deg_M(p). \qquad \square$$

Corollary 1.2.5. *Let M be a C^1 submanifold immersed in an equiregular graded manifold. Then*

1. *$\deg_M$ is a lower semicontinuous function on M.*

2. *The singular set M_0 defined in (1.2.9) is closed in M.*

Proof. The first assertion follows from Proposition 1.2.4 since every point in an equiregular graded manifold is regular. To prove 2, we take $p \in M \setminus M_0$. By 1, there exists a open neighborhood U of p in M such that each point q in U has degree $\deg_M(q)$ equal to $\deg(M)$. Therefore we have $U \subset M \setminus M_0$ and hence $M \setminus M_0$ is an open set. □

Remark 1.2.6. By Corollary 1.2.5 the set $M \setminus M_0$ is open and the growth vector $(\tilde{m}_1, \ldots, \tilde{m}_s)$ is constant in $M \setminus M_0$. Therefore we obtain that $(M \setminus M_0, \tilde{\mathcal{H}}^1, \ldots, \tilde{\mathcal{H}}^s)$ is an equiregular graded manifold.

Chapter 2

Curves of fixed degree immersed in graded manifolds

In this chapter we study the deformability properties of the simplest case of immersion, that are curves immersed in a graded manifold. First of all we recall the degree of an immersed curve and the length functional L_d for curves of degree less than or equal to d. In Section 2.2 we deal with admissible variations for curves of degree d and we deduce the system of ODEs for admissible vector fields. In Section 2.3 the invariances of this system are studied. Section 2.4 is completely devoted to description of the holonomy map and characterization of regular and singular curves. Here explicit examples of singular curve of degree greater that one are showed and we exhibit the equivalence between these singular curves and the ones defined by means of the endpoint map in Subsection 1.1.7. In Section 2.5 we provide a different proof of Theorem 3 by [56] using the Implicit Function Theorem in Banach spaces. In Section 2.6 we give in Definition 2.6.1 a weaker pointwise sufficient condition to ensure the regularity of a curve of degree d. This condition does not require solving a differential equation but still ensures the regularity of the curve, see Theorem 2.6.4. This condition will be easily generalized to submanifolds of given degree in Chapter 3. Section 2.7 is dedicated to the first variation formula for the length functional L_d. Some applications to the computation of geodesics in graded manifolds are provided. A substantial part of the content of this chapter comes from the article [23], that has already been submitted.

2.1 Length of a generic curve

In this section we shall consider an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric g. We recall the following construction from [50, 1.4.D]:

given $p \in N$, we recursively define the subspaces $\mathcal{K}_p^1 := \mathcal{H}_p^1$, $\mathcal{K}_p^{i+1} := (\mathcal{H}_p^i)^\perp \cap \mathcal{H}_p^{i+1}$, for $1 \leqslant i \leqslant (s-1)$. Here $\perp$ means perpendicular with respect to the Riemannian metric g. Therefore we have the decomposition of T_pN into orthogonal subspaces

$$T_pN = \mathcal{K}_p^1 \oplus \mathcal{K}_p^2 \oplus \cdots \oplus \mathcal{K}_p^s. \tag{2.1.1}$$

Given $r > 0$, a unique Riemannian metric g_r is defined under the conditions: (i) the subspaces $\mathcal{K}_i$ are orthogonal, and (ii)

$$g_r|_{\mathcal{K}_i} = \frac{1}{r^{i-1}} g|_{\mathcal{K}_i}, \qquad i = 1, \ldots, s. \tag{2.1.2}$$

It is well-known that when $(N, \mathcal{H})$ is a Carnot manifold the Riemannian distances of (N, g_r) uniformly converge to the Carnot-Carathéodory distance of $(N, \mathcal{H}, h)$, where $h := g_{|\mathcal{H}}$ (see [50, p. 144]).

Working on a neighborhood U of p we construct an *orthonormal* adapted basis $(X_1, \ldots, X_n)$ for the Riemannian metric g by choosing orthonormal bases in the orthogonal subspaces $\mathcal{K}_p^i$, $1 \leqslant i \leqslant s$ for each $p \in U$. Let I be a non-trivial interval, and $\gamma : I \to N$ a curve of class C^1 immersed in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$. By definition (1.2.7) the degree of γ at a point $t \in I$ is given by

$$\deg_\gamma(t) := \deg(\gamma'(t)).$$

The *degree* $\deg(\gamma)$ of a curve γ is the positive integer

$$\deg(\gamma) := \max_{t \in I} \deg_\gamma(t)$$

and the *singular set* $\gamma_0 = \gamma(I_0)$ of γ is given by

$$I_0 = \{t \in I : \deg_\gamma(t) < \deg(\gamma)\}. \tag{2.1.3}$$

By the length formula we get

$$L(\gamma, J, g_r) = \int_J |\gamma'(t)|_{g_r} \, dt, \tag{2.1.4}$$

where $J \subset I$ is a bounded measurable set on I and $L(\gamma, J, g_r)$ is the length of $\gamma(J)$ with respect to the Riemannian metric g_r. If we set $d = \deg(\gamma)$ then we have

$$\gamma'(t) = \sum_{j=1}^{n} u_j(t)(X_j)_{\gamma(t)},$$

where $u_j(t) = \langle \gamma'(t), (X_j)_{\gamma(t)} \rangle$, setting $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$. Then it follows

$$|\gamma'(t)|_{g_r} = \left(\sum_{j=1}^{n} r^{-(\deg(X_j)-1)} u_j(t)^2 \right)^{\frac{1}{2}}.$$

By Lebesgue's dominated convergence theorem we obtain

$$\lim_{r \downarrow 0} \left(r^{\frac{1}{2}(d-1)} L(\gamma, J, g_r) \right) = \int_J \left(\sum_{j=n_{d-1}+1}^{n_d} u_j(t)^2 \right)^{\frac{1}{2}} dt. \tag{2.1.5}$$

Definition 2.1.1. If $\gamma : I \to N$ is an immersed curve of degree d in a graded manifold $(N, \mathcal{H})$ endowed with a Riemannian metric g, the length L_d of degree d is defined by

$$L_d(\gamma, J) := \lim_{r \downarrow 0} \left(r^{\frac{1}{2}(d-1)} L(\gamma, J, g_r) \right),$$

for any bounded measurable set $J \subset I$.

Equation (2.1.5) provides the integral formula $L_d(\gamma, J) = \int_J \theta_d(t) dt$, where

$$\theta_d(t) = \left(\sum_{j=n_{d-1}+1}^{n_d} \langle \gamma'(t), (X_j)_{\gamma(t)} \rangle^2 \right)^{\frac{1}{2}}. \tag{2.1.6}$$

Remark 2.1.2. Clearly if $\deg(\gamma) = 1$ and $(N, \mathcal{H})$ is a Carnot manifold, the sub-Riemannian metric is given by $h = g|_{\mathcal{H}}$ then the length functional L_d coincides with the length L defined in (1.1.5).

2.2 Admissible variations for the length functional

Since the degree is defined by an open condition, the degree can not decrease along a variation $\Gamma(t, s)$ of $\gamma(t)$ in a tubular neighborhood of a curve. If it increases strictly, the length functional L_d, where d is the degree of the original curve, takes value $+\infty$ and we cannot compute the first variation of the functional.

Let us consider a curve $\gamma : I \to N$ into an equiregular graded manifold endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. In this setting we have the following definition

Definition 2.2.1. A smooth map $\Gamma : I \times (-\varepsilon, \varepsilon) \to N$ is said to be *an admissible variation* of γ if $\Gamma_s : I \to N$, defined by $\Gamma_s(t) := \Gamma(t, s)$, satisfies the following properties

(i) $\Gamma_0 = \gamma$,

(ii) $\Gamma_s(I)$ is an curve of the same degree as γ for small enough s,

(iii) $\Gamma_s(t) = \gamma(t)$ for t outside a given compact subset of I.

Definition 2.2.2. Given an admissible variation Γ, the *associated variational vector field* is defined by

$$V(t) := \frac{\partial \Gamma}{\partial s}(t, 0). \tag{2.2.1}$$

The vector field V is compactly supported in I. We shall denote by $\mathfrak{X}_0(I, N)$ the set of smooth vector fields along I. Hence $V \in \mathfrak{X}_0(I, N)$ if and only if V is a smooth map $V : I \to TN$ such that $V(t) \in T_{\gamma(t)}N$ for all $t \in I$, and is equal to 0 outside a compact subset of I.

Let us see now that the variational vector field V associated to an admissible variation Γ satisfies a differential equation of first order. Let $(X_1, \cdots, X_n)$ be an adapted frame in a neighbourhood for consistence U of $\gamma(t)$ for some $t \in I$. We denote by $d = \deg(\gamma)$ the degree of γ. As $\Gamma_s(I)$ is a curve of the same degree as $\gamma(I)$ for small s, there follows

$$\left\langle \frac{\partial \Gamma(s, t)}{\partial t}, (X_r)_{\Gamma_s(t)} \right\rangle = 0, \tag{2.2.2}$$

for all $r = n_d + 1, \ldots, n$. Taking derivative with respect to s in equality (2.2.2) and evaluating at $s = 0$ we obtain the condition

$$\langle \nabla_{\gamma'(t)} V(t), (X_r)_{\gamma(t)} \rangle + \langle \gamma'(t), \nabla_{V(t)} X_r \rangle = 0$$

for all $r = n_d + 1, \ldots, n$. In the above formula, $\langle \cdot, \cdot \rangle$ indicates the scalar product in N. The symbol ∇ denotes, in the left summand, the covariant derivative of vectors in $\mathfrak{X}(I, N)$ induced by g and, in the right summand, the Levi-Civita connection associated to g . Thus, if a variation preserves the degree then the associated variational vector field satisfies the above condition and we are led to the following definition.

Definition 2.2.3. Given an curve $\gamma : I \to N$, a vector field $V \in \mathfrak{X}_0(I, N)$ along γ is said to be *admissible* if it satisfies the system of first order ODEs

$$\langle \nabla_{\gamma'(t)} V(t), (X_r)_{\gamma(t)} \rangle + \langle \gamma'(t), \nabla_{V(t)} X_r \rangle = 0 \tag{2.2.3}$$

where $r = n_d + 1, \dots, n$ and $t \in I$. We denote by $\mathcal{A}_\gamma(I, N)$ the set of admissible vector fields.

Thus we are led naturally to a problem of integrability: given a vector field V along γ such that the first order condition (2.2.3) holds, we wish to find an admissible variation whose associated variational vector field is V.

Definition 2.2.4. We say that an admissible vector field $V \in \mathfrak{X}_0(I, N)$ is *integrable* if there exists an admissible variation such that the associated variational vector field is V.

2.3 The structure of the admissibility system of ODEs

Let $(N, (\mathcal{H}^i))$ be an equiregular graded manifold endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. We set $\mathcal{H} := \mathcal{H}^d$, where $1 \leqslant d \leqslant s$. For sake of simplicity the distribution $\mathcal{H}$ will be called horizontal as well as a curve of degree d and we set $k := n_d$. Let $\gamma : I \to N$ be a horizontal curve defined in an open interval $I \subset \mathbb{R}$. Take $a < b$ so that $[a, b] \subset \mathbb{R}$.

Given an open set U where an orthonormal adapted basis (X_i) is defined, the admissibility condition (2.2.3) for a vector field V is

$$\langle \nabla_{\gamma'} V, X_r \rangle + \langle \gamma', \nabla_V X_r \rangle = 0, \quad r = k+1, \dots, n. \tag{2.3.1}$$

Expressing V in terms of (X_i)

$$V = \sum_{i=1}^{k} g_i X_i + \sum_{j=k+1}^{n} f_j X_j,$$

we get that (2.3.1) is equivalent to the system of $(n-k)$ first order ordinary differential equations

$$f_r' + \sum_{j=k+1}^{n} b_{rj} f_j + \sum_{i=1}^{k} a_{ri} g_i = 0, \quad r = k+1, \dots, n, \tag{2.3.2}$$

where

$$\begin{aligned} a_{ri}(t) &= \langle \nabla_{\gamma'} X_i, (X_r)_\gamma \rangle + \langle \nabla_{X_i} X_r, \gamma' \rangle, \\ b_{rj}(t) &= \langle \nabla_{\gamma'} X_j, (X_r)_\gamma \rangle + \langle \nabla_{X_j} X_r, \gamma' \rangle. \end{aligned} \tag{2.3.3}$$

Remark 2.3.1. Assume that we can extend the tangent vector along γ

$$\gamma'(t) = \sum_{\ell=1}^{k} u_\ell(t)(X_\ell)_{\gamma(t)},$$

to a vector field on a tubular neighborhood of γ, then we have

$$\begin{aligned} a_{ri} &= \langle \nabla_{\gamma'} X_i, X_r \rangle + \langle \nabla_{X_i} X_r, \gamma' \rangle \\ &= \langle \nabla_{\gamma'} X_i, X_r \rangle - \langle \nabla_{X_i} \gamma', X_r \rangle \\ &= \langle [\gamma', X_i](\gamma), (X_r)_\gamma \rangle \\ &= \sum_{\ell=1}^{k} \langle [u_\ell X_\ell, X_i](\gamma), X_r \rangle = \sum_{\ell=1}^{k} u_\ell \langle [X_\ell, X_i](\gamma), (X_r)_\gamma \rangle = \sum_{\ell=1}^{k} u_\ell \, c_{\ell i}^r(\gamma) \end{aligned}$$

and

$$b_{rj} = \sum_{\ell=1}^{k} u_\ell \, c_{\ell j}^r$$

where $c_{\ell i}^r$ and $c_{\ell j}^r$ for $i, \ell = 1, \ldots, k$ and $j, r = k+1, \ldots, n$ are the structure functions defined in (1.1.20), for further details see for instance [75].

In the special case when $\mathcal{H}$ is a distribution of a Carnot manifold $(N, \mathcal{H})$ the matrix $A(t) = (a_{ir})$ represents the $\mathcal{H}_{\gamma(t)}$-curvature and $B(t) = (b_{jr})$ the H^i-curvature restricted to γ' in the first term with respect to metric g, where $H^i = \mathcal{H}^i_{\gamma(t)}/\mathcal{H}^{i-1}_{\gamma(t)}$ for $i = 2, \ldots, s$, see for instance [75, 51, 72].

The system (2.3.2) can be written in matrix form as

$$F' = -BF - AG, \tag{2.3.4}$$

where $B(t) = (b_{rj}(t))_{r=k+1,\ldots,n}^{j=k+1,\ldots,n}$ is a square matrix of order $(n-k)$ and $A(t) = (a_{ri}(t))_{r=k+1,\ldots,n}^{i=1,\ldots,k}$ is a matrix of order $(n-k) \times k$, and

$$F = \begin{pmatrix} f_{k+1} \\ \vdots \\ f_n \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}. \tag{2.3.5}$$

The system (2.3.4) has sense for any adapted orthonormal basis (Y_i) defined on the curve γ, locally extended in a tubular neighborhood of the curve. Indeed, if (X_i) and

(Y_i) are two of such adapted bases, we may write

$$Y_i = \sum_{j=1}^{n} m_{ij} X_j,$$

for some square matrix $M = (m_{ij})$ of order n. Since (X_i) and (Y_i) are adapted basis, M is a block diagonal matrix

$$M = \begin{pmatrix} M_h & 0 \\ 0 & M_v \end{pmatrix},$$

where M_h and M_v are square matrices of orders k and $(n-k)$, respectively. Let us express V as a linear combination of Y_i

$$V = \sum_{i=1}^{k} \tilde{g}_i Y_i + \sum_{j=k+1}^{n} \tilde{f}_j Y_j,$$

and let $\tilde{A}$, $\tilde{B}$ the associated matrices

$$\begin{aligned} \tilde{A} &= \Big(\langle \nabla_{\gamma'} Y_i, Y_r \rangle + \langle \nabla_{Y_i} Y_r, \gamma' \rangle \Big)_{i=1,\dots,k}^{r=k+1,\dots,n}, \\ \tilde{B} &= \Big(\langle \nabla_{\gamma'} Y_j, Y_r \rangle + \langle \nabla_{Y_j} Y_r, \gamma' \rangle \Big)_{j=k+1,\dots,n}^{r=k+1,\dots,n}. \end{aligned}$$

Letting

$$\tilde{F} = \begin{pmatrix} \tilde{f}_{k+1} \\ \vdots \\ \tilde{f}_n \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_k \end{pmatrix},$$

it is immediate to obtain the following equalities

$$\begin{aligned} \tilde{F} &= M_v F, \\ \tilde{G} &= M_h G, \\ \tilde{A} &= M_v A M_h^t, \\ \tilde{B} &= M_v (M_v')^t + M_v B M_v^t. \end{aligned} \tag{2.3.6}$$

Remark 2.3.2. We observe that the equations in (2.3.6) imply that $\tilde{F}' + \tilde{B}\tilde{F} + \tilde{A}\tilde{G} = 0$. To prove this formula it is necessary to take into account that M_h and M_v are orthogonal matrices. We also observe that the ranks of $A(t)$ and $\tilde{A}(t)$ coincide for any $t \in I$.

Remark 2.3.3. Given a smooth vector field X and a horizontal vector field Y on N, we define a covariant derivative on the bundle of horizontal vector fields by

$$\nabla^h_X Y = (\nabla_X Y)_h,$$

where $(\cdot)_h$ denotes the orthogonal projection over the horizontal distribution. This covariant derivative defines a parallel transport on any curve in N that preserves the Riemannian product of horizontal vector fields. This way we can extend any horizontal orthonormal basis at a given point of $\gamma(I)$ to a horizontal orthonormal basis in $\gamma(I)$. A similar connection can be defined on the vertical bundle using the projection over the vertical bundle that allows us to build an orthonormal basis on $\gamma(I)$ of the vertical bundle. This way we are able to produce an adapted global basis on $\gamma(I)$ (see for instance [98, 15]).

2.4 The holonomy map

In this section we first recall Hsu's construction of the holonomy map [56] and we adapt it to curves in graded manifolds. Let $(N, (\mathcal{H}^i))$ be an equiregular graded manifold endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. We set $\mathcal{H} := \mathcal{H}^d$, where $1 \leqslant d \leqslant s$. For sake of simplicity the distribution $\mathcal{H}$ will be called horizontal as well as a curve of degree d and we set $k := n_d$. Given a horizontal curve $\gamma : I \to N$, with $a \in I$, we consider the following spaces

1. $\mathfrak{X}^r_\gamma(a)$, $r \geqslant 0$, is the set of C^r vector fields along γ that vanish at a.

2. $\mathcal{H}^r_\gamma(a)$, $r \geqslant 0$, is the set of horizontal C^r vector fields along γ vanishing at a.

3. $\mathcal{V}^r_\gamma(a)$, $r \geqslant 0$, is the set of vertical vector fields of class C^r along γ vanishing at a. By a vertical vector we mean a vector in $\mathcal{H}^\perp$.

We fix an adapted orthonormal basis (X_i) along γ extended in a neighborhood of γ. The admissibility condition (2.3.1) can be expressed globally on γ using these global vector fields. We define the *admissibility* operator $\mathrm{Ad} : \mathfrak{X}(\gamma) \to \mathcal{V}(\gamma)$ by

$$\mathrm{Ad}(Y) = \sum_{i=k+1}^{n} \Big(\langle \nabla_{\gamma'} Y, X_i \rangle + \langle \gamma', \nabla_Y X_i \rangle \Big) X_i. \tag{2.4.1}$$

Observe that $\mathrm{Ad}(Y) = 0$ implies that Y is an admissible vector field on γ.

The following result is essential for the construction

Lemma 2.4.1. *Let $\gamma : I \to N$ be a horizontal curve in a graded manifold $(N, \mathcal{H})$ endowed with a Riemannian metric. Given $Z \in \mathfrak{X}_\gamma^{r-1}(a)$, there exist a unique vertical vector field $Y_v \in \mathcal{V}_\gamma^r(a)$ such that $\mathrm{Ad}(Z_h + Y_v) = Z_v$.*

Proof. We choose a global orthonormal adapted basis (X_i) on γ and write

$$Z = \sum_{i=1}^{k} g_i X_i + \sum_{r=k+1}^{n} z_r X_r,$$

The vertical vector field Y_v would be determined by their coordinates (f_r) in the vertical basis (X_r), where $r = k+1, \ldots, n$ [1].

Condition $\mathrm{Ad}(Z_h + Y_v) = Z_v$ is then equivalent to the system of $(n-k)$ ordinary differential equations

$$f_r' + \sum_{i=1}^{k} a_{ri} g_i + \sum_{j=k+1}^{n} b_{rj} f_j = z_r, \quad r = k+1, \ldots, n, \tag{2.4.2}$$

where

$$a_{ri} = \langle \nabla_{\gamma'} X_i, X_r \rangle + \langle \nabla_{X_i} X_r, \gamma' \rangle, \quad b_{rj} = \langle \nabla_{\gamma'} X_j, X_r \rangle + \langle \nabla_{X_j} X_r, \gamma' \rangle.$$

Given (g_i), the system (2.4.2) admits a unique solution defined in the whole interval I with prescribed initial conditions $f_r(a) = 0$, for $r = k+1, \ldots, n$. This concludes the proof. □

Given a horizontal curve $\gamma : I \to N$ and $[a, b] \subset I$, Lemma 2.4.1 allows us to define a holonomy type map

$$H_\gamma^{a,b} : \mathcal{H}_0^{r-1}((a,b)) \to \mathcal{V}_{\gamma(b)}$$

like in Hsu's paper [56]. Here $\mathcal{H}_0^{r-1}((a,b))$ is the space of horizontal vector fields of class $(r-1)$ with compact support in (a,b). In order to define $H_\gamma^{a,b}$ we consider a horizontal vector $V_h \in \mathcal{H}_0^{r-1}(a,b)$ with compact support in (a,b) and we take the only vector field $V_v \in \mathcal{V}_\gamma^r(a)$ such that $\mathrm{Ad}(V_h + V_v) = 0$ provided by Lemma 2.4.1. Then we define

$$H_\gamma^{a,b}(V_h) = V_v(b). \tag{2.4.3}$$

Definition 2.4.2 ([56])**.** In the above conditions, we say that γ restricted to $[a,b]$ is *regular* if the holonomy map $H_\gamma^{a,b}$ is surjective.

[1]Notice that the integer number r is an index and it is not the same r we use for the space C^r functions

Given a horizontal curve $\gamma : I \to N$, we choose an orthonormal adapted basis (X_i) along γ. A horizontal vector field V_h can be expressed in terms of this basis as $V_h = \sum_{i=1}^{k} g_i X_i$. The unique vertical vector field V_v such that $\mathrm{Ad}(V_h + V_v) = 0$ can be expressed as $V_v = \sum_{i=k+1}^{n} f_i X_i$. Defining F and G as in (2.3.5), condition $\mathrm{Ad}(V_h + V_v) = 0$ is equivalent to $F' = -BF - AG$, where A, B are the matrices defined in (2.3.3). In these conditions, the coordinates of $H_\gamma^{a,b}(V_h) = V_v(b)$ in the basis (X_i) are given by $F(b)$.

The following result allows the integration of the differential equation (2.3.4) to explicitly compute the holonomy map. As usual we first solve the homogeneous linear system (Lemma 2.4.4), then we find a solution to the associated non-homogeneous one (Proposition 2.4.3).

Proposition 2.4.3. *In the above conditions, there exists a square regular matrix $D(t)$ of order $(n-k)$ such that*

$$F(b) = -D(b)^{-1} \int_a^b (DA)(t) G(t)\, dt. \tag{2.4.4}$$

Proof. Lemma 2.4.4 below allows us to find a regular matrix $D(t)$ such that $D' = DB$. Then equation $F' = -BF - AG$ is equivalent to $(DF)' = -DAG$. Integrating between a and b, taking into account that $F(a) = 0$, and multiplying by $D(b)^{-1}$, we obtain (2.4.4). □

Lemma 2.4.4. *Let $B(t)$ be a continuous family of square matrices on the interval $[a, b]$. Let $D(t)$ be the solution of the Cauchy problem*

$$D'(t) = D(t)B(t) \text{ on } [a, b], \quad D(a) = I_d.$$

Then $\det D(t) \neq 0$ for each $t \in [a, b]$.

Proof. By the Jacobi formula we have

$$\frac{d(\det D(t))}{dt} = \mathrm{Tr}\left(\mathrm{adj}\, D(t)\, \frac{dD(t)}{dt}\right),$$

where $\mathrm{adj}D$ is the classical adjoint (the transpose of the cofactor matrix) of D and Tr is the trace operator. Therefore

$$\frac{d \det(D(t))}{dt} = \mathrm{Tr}\left((\mathrm{adj}\, D(t)) D(t) B(t)\right) = \det D(t)\, \mathrm{Tr}(B(t)). \tag{2.4.5}$$

Since $\det D(a) = 1$, the solution for (2.4.19) is given by

$$\det D(t) = e^{\int_a^t \mathrm{Tr}(B(\tau))\,d\tau} > 0,$$

for all $t \in [a, b]$. Thus, the matrix $D(t)$ is invertible for each $t \in [a, b]$. □

Definition 2.4.5. We say that the matrix $\tilde{A}(t) := (DA)(t)$ on γ defined in Proposition 2.4.3 is linearly full in $\mathbb{R}^{n-k}$ if and only if

$$\dim\left(\operatorname{span}\left\{\tilde{A}^1(t), \dots, \tilde{A}^k(t) \quad \forall t \in [a, b]\right\}\right) = n - k,$$

where $\tilde{A}^i$ for $i = 1, \dots, k$ are the columns of $\tilde{A}(t)$.

Proposition 2.4.6. *The horizontal curve γ restricted to $[a, b]$ is regular if and only if $\tilde{A}(t)$ is linearly full in $\mathbb{R}^{n-k}$.*

Proof. Assume that the holonomy map is not surjective. Then the image of $H_\gamma^{a,b}$ is contained in a hyperplane of $\mathcal{V}_{\gamma(b)}$ expressed in the coordinates associated to the basis $((X_i)_{\gamma(b)})_{i=k+1,\dots,n}$ as a row vector $\Lambda \neq 0$ with $(n-k)$ coordinates. With the notation of Proposition 2.4.3 we have

$$0 = \Lambda F(b) = -\Lambda D(b)^{-1} \int_a^b \tilde{A}(t) G(t) dt = -\int_a^b \Gamma \tilde{A}(t) G(t) dt,$$

where $\Gamma = \Lambda D(b)^{-1} \neq 0$. As this formula holds for any $G(t)$, we have $\Gamma \tilde{A}(t) = 0$ for all $t \in [a, b]$. Hence $\tilde{A}$ is not linearly full as its columns are contained in the hyperplane of $\mathbb{R}^{n-k}$ determined by Γ.

Conversely, assume that $\tilde{A}$ is not linearly full. Then there exists a row vector with $(n-k)$ coordinates $\Gamma \neq 0$ such that $\Gamma \tilde{A}(t) = 0$ for all $t \in [a, b]$. Then

$$(\Gamma D(b)) F(b) = \Gamma \int_a^b \tilde{A}(t) G(t) dt = 0.$$

Hence the image of the holonomy map is contained in a hyperplane if $\mathcal{V}_{\gamma(b)}$ and γ is not regular. □

The following result provides a technical criterion of non-regularity

Theorem 2.4.7. *The horizontal curve γ is non-regular restricted to $[a, b]$ if and only if there exists a row vector field $\Lambda(t) \neq 0$ for all $t \in [a, b]$ that solves the following system*

$$\begin{cases} \Lambda'(t) = \Lambda(t) B(t) \\ \Lambda(t) A(t) = 0. \end{cases} \tag{2.4.6}$$

Proof. Assume that γ is nonregular in $[a,b]$, then by Proposition 2.4.6 there exists a row vector $\Gamma \neq 0$ such that

$$\Gamma D(t)A(t) = 0$$

for all $t \in [a,b]$, where $D(t)$ solves

$$\begin{cases} D(t)' = D(t)B(t) \\ D(a) = I_{n-k}. \end{cases} \tag{2.4.7}$$

Since Γ is a constant vector and $D(t)$ is a regular matrix by Lemma 2.4.4 , $\Lambda(t) := \Gamma D(t)$ solves the system (2.4.6) and $\Lambda(t) \neq 0$ for all $t \in [a,b]$.

Conversely, any solution of the system (2.4.6) is given by

$$\Lambda(t) = \Gamma D(t),$$

where $\Gamma = \Lambda(0) \neq 0$ and $D(t)$ solves the equation (2.4.7). Indeed, let us consider a general solution $\Lambda(t)$ of (2.4.6). If we set

$$\Phi(t) = \Lambda(t) - \Gamma D(t),$$

where $\Gamma = \Lambda(0) \neq 0$ and $D(t)$ solves the equation (2.4.7), then we deduce

$$\begin{cases} \Phi(t)' = \Phi(t)B(t) \\ \Phi(0) = 0. \end{cases}$$

Clearly the unique solution of this system is $\Phi(t) \equiv 0$. Hence we conclude that $\Gamma\tilde{A}(t) = 0$. Thus $\tilde{A}(t)$ is not fully linear and by Proposition 2.4.6 we are done. □

Remark 2.4.8. Notice that if we write $\Lambda(t) = (\lambda_{k+1}(t), \ldots, \lambda_n(t))$ by Remark 2.3.1 the equation (2.4.6) is equivalent to

$$\begin{cases} \lambda_r'(t) = \sum_{j=k+1}^{n} \lambda_j b_{jr} = \sum_{j=k+1}^{n} \sum_{i=1}^{k} \lambda_j c_{ir}^j u_i \\ \sum_{j=k+1}^{n} \lambda_j a_{jh} = \sum_{j=k+1}^{n} \lambda_j c_{ih}^j u_i. \end{cases} \tag{2.4.8}$$

When $(N, \mathcal{H})$ is a Carnot manifold where $\mathcal{H}$ is a distribution of rank l, then we have that $k = l$ and (2.4.8) is equivalent to the characteristic system (1.1.23) where the first equations $u_j = 0$ for $j = l+1, \ldots, n$ are taken for granted since in Theorem 2.4.7

we assume γ horizontal. In the characteristic system (1.1.23) we assume γ absolutely continuous curves with square integrable derivative while in our construction γ is at least C^1, so that the coefficients of A and B are at least continuous.

2.4.1 Independence on the metric

Let g and $\tilde{g}$ be two Riemannian metrics on N and (X_i) be orthonormal adapted basis with respect to g and (Y_i) with respect to $\tilde{g}$. Clearly we have

$$Y_i = \sum_{j=1}^{n} m_{ji} X_j,$$

for some square invertible matrix $M = (m_{ji})_{j=1,\dots,n}^{i=1,\dots,n}$ of order n. Since (X_i) and (Y_i) are adapted basis, M is a block matrix

$$M = \begin{pmatrix} M_h & M_{hv} \\ 0 & M_v \end{pmatrix},$$

where M_h and M_v are square matrices of orders k and $(n-k)$, respectively, and M_{hv} is a $k \times (n-k)$ matrix.

Remark 2.4.9. One can easily check that the inverse of M is given by the block matrix

$$M^{-1} = \begin{pmatrix} M_h^{-1} & -M_h^{-1} M_{hv} M_v^{-1} \\ 0 & M_v^{-1} \end{pmatrix}.$$

Setting $\tilde{\mathbf{G}} = (\tilde{g}(X_i, X_j))_{i,j=1,\dots,n}$ we have

$$\tilde{\mathbf{G}} = \begin{pmatrix} \tilde{\mathbf{G}}_h & \tilde{\mathbf{G}}_{hv} \\ (\tilde{\mathbf{G}}_{hv})^t & \tilde{\mathbf{G}}_v \end{pmatrix} = (M^{-1})^t (M^{-1}).$$

Thus it follows

$$\begin{aligned} \tilde{\mathbf{G}}_v &= (M_v^{-1})^t M_v^{-1} + (M_v^{-1})^t M_{hv}^t (M_h^{-1})^t \, M_h^{-1} M_{hv} M_v^{-1}, \\ \tilde{\mathbf{G}}_{hv} &= -(M_h^{-1})^t M_h^{-1} M_{hv} M_v^{-1}, \\ \tilde{\mathbf{G}}_h &= (M_h^{-1})^t M_h^{-1}. \end{aligned}$$

Let $\tilde{A}$ be the associated matrix

$$\tilde{A} = \left(\tilde{g}(Y_r, [\gamma', Y_i])\right)_{r=k+1,\dots,n}^{i=1,\dots,k}.$$

Then it follows

$$\tilde{a}_{ri} = \tilde{g}(Y_r, [\gamma', Y_i]) = \tilde{g}\left(\sum_{s=1}^{n} m_{sr} X_s, [\gamma', \sum_{j=1}^{k} m_{ji} X_j]\right),$$

that it is equivalent to

$$\tilde{a}_{ri} = \sum_{s=1}^{n}\sum_{j=1}^{k} m_{sr}\, \tilde{g}(X_s, [\gamma', X_j])\, m_{ji} + m_{sr}\, \tilde{g}(X_s, X_j)\, \gamma'(m_{ji})$$

For each $j = 1, \dots, k$ we have

$$[\gamma', X_j] = \sum_{t=1}^{k} \omega_{tj} X_t + \sum_{\ell=k+1}^{n} a_{\ell j} X_\ell,$$

where $\omega_{tj} = g(X_t, [\gamma', X_j])$ and $a_{\ell j}$ defined in (2.3.3). Then we obtain

$$\begin{aligned}
\tilde{a}_{ri} =& \sum_{s=1}^{k}\sum_{j=1}^{k}\left(\sum_{t=1}^{k} m_{sr}\, \tilde{g}(X_s, X_t)\omega_{tj}\, m_{ji} + \sum_{\ell=k+1}^{n} m_{sr}\, \tilde{g}(X_s, X_\ell) a_{\ell j}\, m_{ji}\right) \\
&+ \sum_{s=1}^{k}\sum_{j=1}^{k} m_{sr}\, \tilde{g}(X_s, X_j)\, \gamma'(m_{ji}) \\
&+ \sum_{s=k+1}^{n}\sum_{j=1}^{k}\left(\sum_{t=1}^{k} m_{sr}\, \tilde{g}(X_s, X_t)\omega_{tj}\, m_{ji} + \sum_{\ell=k+1}^{n} m_{sr}\, \tilde{g}(X_s, X_\ell) a_{\ell j}\, m_{ji}\right) \\
&+ \sum_{s=k+1}^{n}\sum_{j=1}^{k} m_{sr}\, \tilde{g}(X_s, X_j)\, \gamma'(m_{ji}).
\end{aligned}$$

Setting $\Omega_h = (\omega_{tj})_{t=1,\dots,k}^{j=1,\dots,k}$ we gain

$$\begin{aligned}
\tilde{A} =& (M_{hv})^t \left(\tilde{\mathbf{G}}_h\, \Omega_h\, M_h + \tilde{\mathbf{G}}_{hv}\, A\, M_h + \tilde{\mathbf{G}}_h\, M_h'\right) \\
&+ (M_v)^t \left((\tilde{\mathbf{G}}_{hv})^t\, \Omega_h M_h + \tilde{\mathbf{G}}_v\, A\, M_h + (\tilde{\mathbf{G}}_{hv})^t M_v'\right),
\end{aligned}$$

and, by Remark 2.4.9, we obtain

$$\begin{aligned}\tilde{A} &= (M_{hv})^t \Big((M_h^{-1})^t M_h^{-1} (\Omega_h M_h + M_h') - (M_h^{-1})^t M_h^{-1} M_{hv} M_v^{-1} A M_h \Big) \\ &\quad - \Big(M_{hv}^t (M_h^{-1})^t M_h^{-1} (\Omega_h M_h + M_v') \Big) \\ &\quad + \Big(M_v^{-1} + M_{hv}^t (M_h^{-1})^t M_h^{-1} M_{hv} M_v^{-1} \Big) A M_h \\ &= M_v^{-1} A M_h.\end{aligned} \tag{2.4.9}$$

Now let $\tilde{B}$ be the associated matrix

$$\tilde{B} = \Big(\tilde{g}(Y_r, [\gamma', Y_j]) \Big)_{r=k+1,\dots,n}^{j=k+1,\dots,n}.$$

Then it follows

$$\tilde{b}_{rj} = \tilde{g}(Y_r, [\gamma', Y_j]) = \tilde{g}\left(\sum_{s=1}^{n} m_{sr} X_s, [\gamma', \sum_{t=1}^{n} m_{tj} X_t] \right),$$

that it is equivalent to

$$\tilde{b}_{rj} = \sum_{s=1}^{n} \sum_{t=1}^{n} m_{sr}\, \tilde{g}(X_s, [\gamma', X_t])\, m_{tj} + m_{sr}\, \tilde{g}(X_s, X_j)\, \gamma'(m_{tj})$$

Setting $\omega_{it} = g(X_i, [\gamma', X_t])$ for $i = 1, \dots, k$ and $t = 1, \dots, n$ we have

$$[\gamma', X_t] = \sum_{i=1}^{k} \omega_{it} X_i + \sum_{\ell=k+1}^{n} a_{\ell t} X_\ell,$$

when $t = 1, \dots, k$ and $a_{\ell t}$ as in definition (2.3.3) and

$$[\gamma', X_t] = \sum_{i=1}^{k} \omega_{it} X_i + \sum_{\ell=k+1}^{n} b_{\ell t} X_\ell,$$

when $t = k+1,\dots,n$ and $b_{\ell t}$ as in definition (2.3.3).

$$
\begin{aligned}
\tilde b_{rj} = &\sum_{s=1}^{k}\sum_{t=1}^{k}\left(\sum_{i=1}^{k} m_{sr}\,\tilde g(X_s,X_i)\omega_{it}\,m_{tj} + \sum_{\ell=k+1}^{n} m_{sr}\,\tilde g(X_s,X_\ell)a_{\ell t}\,m_{tj}\right)\\
&+\sum_{s=1}^{k}\sum_{t=k+1}^{n}\left(\sum_{i=1}^{k} m_{sr}\,\tilde g(X_s,X_i)\omega_{it}\,m_{tj} + \sum_{\ell=k+1}^{n} m_{sr}\,\tilde g(X_s,X_\ell)b_{\ell t}\,m_{tj}\right)\\
&+\sum_{s=1}^{k}\sum_{t=1}^{n} m_{sr}\,\tilde g(X_s,X_t)\,\gamma'(m_{tj})\\
&+\sum_{s=k+1}^{n}\sum_{t=1}^{k}\left(\sum_{i=1}^{k} m_{sr}\,\tilde g(X_s,X_i)\omega_{it}\,m_{tj} + \sum_{\ell=k+1}^{n} m_{sr}\,\tilde g(X_s,X_\ell)a_{\ell t}\,m_{tj}\right)\\
&+\sum_{s=k+1}^{n}\sum_{t=k+1}^{k}\left(\sum_{i=1}^{k} m_{sr}\,\tilde g(X_s,X_i)\omega_{it}\,m_{tj} + \sum_{\ell=k+1}^{n} m_{sr}\,\tilde g(X_s,X_\ell)b_{\ell t}\,m_{tj}\right)\\
&+\sum_{s=k+1}^{n}\sum_{t=1}^{n} m_{sr}\,\tilde g(X_s,X_t)\,\gamma'(m_{tj})
\end{aligned}
$$

Setting $\Omega_{hv} = (\omega_{it})_{i=1,\dots,k}^{t=k+1,\dots,n}$, it is immediate to obtain the following equality

$$
\begin{aligned}
\tilde B = M_{vh}^t\Big(&\tilde{\mathbf{G}}_h\Omega_h M_{hv} + \tilde{\mathbf{G}}_{hv}AM_{hv} + \tilde{\mathbf{G}}_h\Omega_{hv}M_v\\
&+\tilde{\mathbf{G}}_{hv}BM_v + \tilde{\mathbf{G}}_h M'_{hv} + \tilde{\mathbf{G}}_{hv}M'_v\Big)\\
&+M_v^t\Big((\tilde{\mathbf{G}}_{hv})^t\Omega_h M_{hv} + \tilde{\mathbf{G}}_v AM_{hv} + (\tilde{\mathbf{G}}_{hv})^t\Omega_{hv}M_v\\
&+\tilde{\mathbf{G}}_v BM_v + (\tilde{\mathbf{G}}_{hv})^t M'_{hv} + \tilde{\mathbf{G}}_v M'_v\Big).
\end{aligned}
$$

By Remark 2.4.9 we obtain

$$
\tilde B = M_v^{-1}\,A\,M_{hv} + M_v^{-1}BM_v + M_v^{-1}M'_v. \tag{2.4.10}
$$

Proposition 2.4.10. *Let $\gamma : I \to N$ be a curve immersed in a graded manifold $(N,\mathcal{H}^1,\dots,\mathcal{H}^s)$. Let g and $\tilde g$ be two Riemannian metrics on TN. Then γ is regular in $[a,b]$ with respect to g if and only if γ is regular in $[a,b]$ with respect to $\tilde g$.*

Proof. Let (Y_i) be an orthonormal adapted basis along γ with respect to $\tilde g$. Without loss of generality we assume that the system (2.3.4) with respect to metric $\tilde g$ is given by

$$
\tilde F'(t) = -\tilde A(t)\tilde G(t)
$$

and $\tilde{B} = 0$. This is not restrictive since starting by a generic metric $\tilde{g}_0$, by Proposition 2.4.3, there exists a matrix $D(t)$ that transforms the metric $\tilde{g}_0$ in the metric $\tilde{g}$. Let (X_i) be an orthonormal adapted basis along γ with respect to g. Then there exists an invertible block matrix

$$M = \begin{pmatrix} M_h & M_{hv} \\ 0 & M_v \end{pmatrix},$$

such that

$$Y_i = \sum_{j=1}^{n} m_{ji} X_j.$$

Then, by Theorem 2.4.7 a curve γ is non-regular with respect to $\tilde{g}$ if and only if $\tilde{\Lambda}(t) = \tilde{\Lambda} \neq 0$ is a constant row vector such that $\tilde{\Lambda}\tilde{A}(t) = 0$ for each $t \in [a, b]$. By equation (2.4.9) we obtain

$$0 = \tilde{\Lambda}\tilde{A}(t) = \tilde{\Lambda} M_v^{-1}(t)\, A(t)\, M_h(t).$$

Since M_h is invertible, setting $\Lambda(t) := \tilde{\Lambda} M_v^{-1}(t) \neq 0$ we deduce $\Lambda(t)A(t) = 0$. Moreover by (2.4.10) we have

$$0 = \tilde{B} = M_v^{-1}\, A\, M_{hv} + M_v^{-1}\, B\, M_v + M_v^{-1} M_v'.$$

Multiplying both sides by $\tilde{\Lambda}$ we deduce

$$0 = \tilde{\Lambda} M_v^{-1}\, B\, M_v + \tilde{\Lambda} M_v^{-1} M_v'. \tag{2.4.11}$$

Differentiating the identity $M_v^{-1} M_v$ it follows

$$M_v^{-1} M_v' = -(M_v^{-1})' M_v$$

Putting this identity in (2.4.11) we have

$$\tilde{\Lambda} M_v^{-1}\, B\, M_v = \tilde{\Lambda}(M_v^{-1})' M_v.$$

Since $\tilde{\Lambda}$ is constant and M_v is invertible we conclude $\Lambda'(t) = \Lambda(t)B(t)$. By Theorem 2.4.7, this means that γ is non-regular with respect to g. □

Proposition 2.4.10 shows that the definition of regularity for a curve γ is independent of the choice of Riemannian metric $g = \langle \cdot, \cdot \rangle$ on the tangent bundle TN.

2.4.2 Some low-dimensional examples and isolated curves

Remark 2.4.11. Let A be the matrix defined in (2.3.3) with respect to an adapted basis (X_i) along γ. Notice that if there exists a point $\bar{t} \in (a,b)$ such that

$$\operatorname{rank} A(\bar{t}) = n - k, \tag{2.4.12}$$

for some adapted basis (X_i), then the curve γ is regular in $[a,b]$. In particular, if we assume that (2.4.12) holds for any $\bar{t} \in (a,b)$, then the curve γ is regular in $[a,b]$. Notice that this condition implies $n - k \leqslant k$, that is $\frac{n}{2} \leqslant k$.

Example 2.4.12. Any horizontal curve $\gamma : I \to M^{2n+1}$ in a contact sub-Riemannian manifold $(M^{2n+1}, \mathcal{H} = \ker(\omega))$, is regular. Let T be the Reeb vector field. We extend the vector field γ' along γ to a vector field on M. Given a contact manifold M, one can assure the existence of a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an $(1,1)$-tensor field J so that

$$\langle T, X \rangle = \omega(X), \quad 2\langle X, J(Y) \rangle = d\omega(X,Y), \quad J^2(X) = -X + \omega(X)T. \tag{2.4.13}$$

The structure given by (M, ω, g, J) is called a contact Riemannian manifold, see [8] and [42]. In particular the structure $(M, \mathcal{H}, g)$ is a Carnot manifold. Then, we fix an orthonormal adapted basis $(X_1, \ldots, X_{2n}, T)$ along γ, where $X_i \in \mathcal{H}$ for $i = 1, \ldots, 2n$. Then the admissibility equation for $V = \sum_{i=1}^{2n} f_i X_i + f_{2n+1} T$ is given by

$$f'_{2n+1}(t) = -b\, f_{2n+1}(t) - A \begin{pmatrix} f_1(t) \\ \vdots \\ f_{2n}(t) \end{pmatrix},$$

where $b = \langle [\gamma', T], T \rangle$ and $A = (a_1, \ldots, a_{2n})$ with

$$\begin{aligned} a_i &= \langle \nabla_{\gamma'} X_i, T \rangle + \langle \nabla_{X_i} T, \gamma' \rangle \\ &= \langle [\gamma', X_i], T \rangle = \omega([\gamma', X_i]) \\ &= -d\omega(\gamma', X_i) = -2\langle \gamma', J(X_i) \rangle = 2\langle J(\gamma'), X_i \rangle. \end{aligned} \tag{2.4.14}$$

Since $J(\gamma'(t)) \in \mathcal{H}_{\gamma(t)}$ and $J(\gamma'(t)) \neq 0$ for all $t \in I$ we have $\operatorname{rank} A(t) = 1$ for all $t \in I$. Hence γ is regular in every subinterval of I.

Definition 2.4.13. We say that a curve $\gamma : I \to N$ is *isolated* when its only non-tangential admissible variation is the the trivial one.

Here we recall some basic definition given in [11]. Given two point $p, q \in (N, \mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s)$ for each $1 \leqslant d \leqslant s$ we set that $\Omega_{\mathcal{H}^d}(p,q)$ is the space of curves of degree less or equal to d that connect p, q.

Definition 2.4.14 ([11]). A curve $\gamma : [a,b] \to N$ of degree $d = \deg(\gamma)$ is $\mathcal{H}^d$*-rigid* if there is a C^1-neighborhood $\mathfrak{U}$ of γ in $\Omega_{\mathcal{H}^d}(\gamma(a), \gamma(b))$ so that every $\alpha \in \mathfrak{U}$ is a reparametrization of γ. We say that γ is locally rigid if every point of $[a,b]$ lies in a subinterval $J \subset I$ so that γ restricted to J is rigid.

Remark 2.4.15. We notice that

1. whenever $(N, \mathcal{H}^d)$ is a Carnot manifold 1.1.1, namely when $\mathcal{H}^d$ verifies Hörmander's rank condition, $\Omega_{\mathcal{H}^d}(p,q)$ is not empty by Chow's Theorem,

2. if we consider C^1 variations then an isolated curve on $I = [a,b]$ is rigid.

Here we show a well-known example of horizontal singular curve that is also rigid, first discovered by Engel.

Example 2.4.16. An Engel structure $(E, \mathcal{H})$ is 4-dimensional Carnot manifold where $\mathcal{H}$ is a two dimensional distribution of step 3. A representation of the Engel group $\mathbb{E}$, which is the tangent cone to each Engel structure, is given by $\mathbb{R}^4$ endowed with the distribution $\mathcal{H}$ generated by

$$X_1 = \partial_{x_1} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4}.$$

The second layer is generated by

$$X_3 = [X_1, X_2] = \partial_{x_3} + x_1 \partial_{x_4}$$

and the third layer by $X_4 = [X_1, X_3] = \partial_{x_4}$. Let $\gamma : \mathbb{R} \to \mathbb{R}^4$ be the horizontal curve parametrized by $\gamma(t) = (0, t, 0, 0)$ whose tangent vector $\gamma'(t)$ is given by ∂_{x_2} along the curve. We consider the Riemannian metric $g = \langle \cdot, \cdot \rangle$ that makes $(X_1, \ldots, X_4)$ an orthonormal basis. An extension of $\gamma'(t)$ to the all space is clearly ∂_{x_2} . By Remark 2.3.1 we have $b_{rj} = \langle [\partial_{x_2}, X_j], X_r \rangle = 0$ for $r, j = 3, 4$ and $a_{ri} = \langle [\partial_{x_2}, X_i], X_r \rangle = 0$ for $r = 3, 4$, $i = 1, 2$, since ∂_{x_2} does not commute with any vector. Let $[a,b] \subset \mathbb{R}$, $a < b$. Then, by Proposition 2.4.3 the holonomy map is given by

$$H_\gamma^{a,b}(G) = F(b) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for all $g_1, g_2 \in C_0^\infty([a,b])$. Therefore we deduce the holonomy map is not surjective, thus γ is non-regular restricted at each interval $[a,b]$. Clearly we observe that the constant matrix

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is not linearly full.

Bryant and Hsu [11, Proposition 3.2] proved that $\gamma(t) = (0,t,0,0)$ is rigid in the C^1 topology, here we propose an easy readjustment of their proof. In [97] Sussmann proved that this curve is a minimizer for the C-C distance between two point on this straight line.

Proposition 2.4.17. *Let $(\mathbb{E}^4, \mathcal{H})$ be the Engel group with coordinates (x_1, x_2, x_3, x_4) where the distribution $\mathcal{H}$ is generated by*

$$X_1 = \partial_{x_1} \quad \textit{and} \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4},$$

then the horizontal curve $\gamma : \mathbb{R} \to \mathbb{E}^4$, $\gamma(t) = (0,t,0,0)$ is isolated.

Proof. Let $\Gamma_s(t)$ be a compactly supported variation on $\mathbb{R}$ and K its support. Then, there exist $a > 0$ such that $K \subset [-a,a]$. Fix s small enough, we set $\Gamma_s(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$. Since $\Gamma_s(-a) = (0,-a,0,0)$, $\Gamma_s(a) = (0,a,0,0)$ we have $x_1(-a) = x_3(-a) = x_4(-a) = 0$ and $x_1(a) = x_3(a) = x_4(a) = 0$. Moreover the x_2-component of γ is an increasing smooth function of t with non-vanishing derivative which maps $[-a,a]$ diffeomorphically onto itself, therefore the curve $\Gamma_s(t)$ can be reparametrized by $(x_1(t), t, x_3(t), x_4(t))$. Since the distribution $\mathcal{H}$ is given by the kernel of the one forms

$$\omega = x_1 dx_2 - dx_3 \quad \text{and} \quad \lambda = \frac{x_1^2}{2} dx_2 - dx_4,$$

we have

$$\begin{cases} \dot{x}_3(t) = x_1(t)\dot{x}_2(t) = x_1(t) \\ \dot{x}_4(t) = \dfrac{x_1^2(t)}{2}\dot{x}_2(t) = \dfrac{x_1^2(t)}{2}. \end{cases}$$

Therefore

$$x_3(t) = \int_{-a}^t x_1(t) dt \quad \text{and} \quad x_4(t) = \int_{-a}^t \frac{x_1^2(t)}{2} dt.$$

Since $x_4(a) = 0$ we obtain that

$$\int_{-a}^t \frac{x_1^2(t)}{2} dt = 0.$$

Hence we deduce $x_1(t) \equiv 0$ and consequently $x_4(t) = x_3(t) \equiv 0$. $\square$

Slightly changing the distribution of Example 2.4.16 we show an example singular curve of degree 2.

Example 2.4.18. Let us consider $\mathbb{R}^5$ endowed with the distribution $\mathcal{H}$ generated by

$$X_1 = \partial_{x_1} \quad \text{and} \quad X_2 = \partial_{x_5} + x_1 \left(\partial_{x_2} + \frac{x_1}{2}\partial_{x_3} + \frac{x_1^3}{6}\partial_{x_4} \right).$$

The second layer is generated by

$$X_3 = [X_1, X_2] = \partial_{x_2} + x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4},$$

the third layer by $X_4 = [X_1, X_3] = \partial_{x_3} + x_1\partial_{x_4}$ and the fourth layer by $X_5 = [X_1, X_4] = \partial_{x_4}$. Now the curve $\gamma : \mathbb{R} \to \mathbb{R}^5$ parametrized by $\gamma(t) = (0, t, 0, 0, 0)$ is a curve of degree 2, its tangent vector $\gamma'(t)$ is given by ∂_{x_2}. Therefore we have $n_2 = 3$. We consider the Riemannian metric $g = \langle \cdot, \cdot \rangle$ that makes $(X_1, \ldots, X_5)$ an orthonormal basis. Since the extension ∂_{x_2} to the all space of $\gamma'(t)$ does not commute with an vector field of the basis $(X_1, \ldots, X_5)$, by Remark 2.3.1 we deduce that

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A is not linearly full we conclude that γ is a non-regular curve of degree 2. However γ is not isolated since it is possible to deform the initial curve in the direction ∂_{x_5}.

Example 2.4.19. Let $\mathcal{H}$ be 3-dimensional distribution on $\mathbb{R}^5$ generated by

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4} \quad \text{and} \quad X_3 = \partial_5.$$

The second layer is generated by $X_4 = [X_1, X_2]$ and the third layer by $X_5 = [X_1, X_4]$. Let $\gamma : \mathbb{R} \to \mathbb{R}^5$ be the horizontal curve parametrized by $\gamma(t) = (0, 0, 0, 0, t)$ whose tangent vector $\gamma'(t)$ is given by $X_3 = \partial_{x_5}$. We consider the Riemannian metric $g = \langle \cdot, \cdot \rangle$ that makes $(X_1, \ldots, X_5)$ an orthonormal basis. Since X_3 does not commute with any vector fields of the basis we deduce that $A = 0$ and $B = 0$. Then the admissibility equation (2.3.4) is given $F' = 0$ with initial condition $F(a) = 0$. Clearly the holonomy map is not surjective since $F(b) = 0$. Therefore the curve γ is non-regular on each subinterval $[a,b] \subset \mathbb{R}$. However the curve γ can be deformed in the horizontal non-tangential directions X_1 and X_2 unlike the rigid curve in Engel showed in Example 2.4.16.

Example 2.4.20 (Kolmogorov)**.** Let us consider in $\mathbb{R}^4$ the Kolmogorov operator

$$L = \partial_t + x\,\partial_y + \frac{x^2}{2}\,\partial_z - \partial^2_{xx},$$

where (x,y,z,t) is a point in $\mathbb{R}^4$. Notice that L is homogeneous of degree two under the dilation $\delta_\lambda((x,y,z,t)) = (\lambda x, \lambda^3 y, \lambda^4 z, \lambda^2 t)$. Therefore the graded structure adapted to L is given by $\mathbb{R}^4$ endowed with the filtration

$$\begin{aligned}
\mathcal{H}^1 &= \operatorname{span}\{X_1 = \partial_x\}\\
\mathcal{H}^2 &= \operatorname{span}\{X_1, X_2 = \partial_t + x\,\partial_y + \tfrac{x^2}{2}\,\partial_z\}\\
\mathcal{H}^3 &= \operatorname{span}\{X_1, X_2, X_3 = [X_1, X_2] = \partial_y + x\partial_z\}\\
\mathcal{H}^4 &= \operatorname{span}\{X_1, X_2, X_3, X_4 = [X_1, X_3] = \partial_z\}.
\end{aligned}$$

Setting $\mathcal{H} := \mathcal{H}^2$, we allow only curve of degree less than or equal to two. Due to the computations developed in Example 2.4.16, we obtain that $\gamma(s) = (0,0,0,s)$ is singular of degree two. Moreover the same argument exhibited in Proposition 2.4.17 prove that this curve is isolated.

Example 2.4.21. The 3-dimensional Heisenberg group $\mathbb{H}^1$ is a Lie group defined by a Lie algebra $\mathfrak{h}$ generated $\{X, Y, T\}$, where the only non-trivial relation is $T = [X, Y]$. Setting $\mathcal{H} = \operatorname{span}\{X, Y\}$, $(\mathbb{H}^1, \mathcal{H})$ is the simplest example of Carnot group, that clearly is a Carnot manifold. A possible presentation of the Heisenberg group is provided by $\mathbb{R}^3$ where the vector field $\{X, Y, T\}$ are given by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial_t}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial_t} \quad T = \frac{\partial}{\partial_t}.$$

As we point out in Section 1.2.1 each C^1 surface Σ immersed in the $\mathbb{H}^1$ inherits a structure of graded manifold $(\Sigma, \tilde{\mathcal{H}}^1, \tilde{\mathcal{H}}^2)$, where $\tilde{\mathcal{H}}^1 = \mathcal{H} \cap T\Sigma$ and $\tilde{\mathcal{H}}^2 = T\Sigma$. Moreover $\Sigma \smallsetminus \Sigma_0$, where Σ_0 denotes the characteristic set, is an equiregular graded manifold. The foliation properties by horizontal integral curves of $\tilde{\mathcal{H}}^1$ have been deeply studied by [84, 43, 18]. We notice that each integral curve $\gamma : I \to \Sigma$ of $\tilde{\mathcal{H}}^1$ is singular restricted to each $[a,b] \subset I$. Since the $\dim(\tilde{\mathcal{H}}^1) = 1$ the one dimensional matrix $A(t) = 0$ and the admissibility system (2.3.4) is given by

$$F' = BF.$$

Fixing the initial condition $F(a) = 0$ the unique solution of the homogeneous system is $F(t) \equiv 0$, thus the holonomy map is not surjective.

Remark 2.4.22. Let $\partial_{x_1}, \dots, \partial_{x_n}$ be the Euclidean basis in $\mathbb{R}^n$, $n \geqslant 2$. Let $1 \leqslant k \leqslant n$. Assume that we set $\mathcal{H}^1 = \operatorname{span}\{\partial_{x_1}, \dots, \partial_{x_k}\}$ and $\mathcal{H}^2 = T\mathbb{R}^n$. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a curve in $\mathcal{H}^1$ such that $\gamma'(t) = \sum_{\ell=1}^{k} h_\ell \, \partial_{x_\ell}$. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean metric in $\mathbb{R}^n$. Setting $\mathcal{H} = \mathcal{H}^1$ we obtain that matrices defined in (2.3.3) is given by

$$a_{ri} = \sum_{\ell=1}^{k} h_\ell \langle [\partial_{x_\ell}, \partial_{x_i}], \partial_{x_r} \rangle = 0$$

and

$$b_{rj} = \sum_{\ell=1}^{k} h_\ell \langle [\partial_{x_\ell}, \partial_{x_j}], \partial_{x_r} \rangle = 0,$$

for all $i = 1, \dots, k$ and $r, j = k+1, \dots, n$. Since $A = 0$ and $B = 0$ we deduce that $F'(t) = 0$ and $F(a) = 0$. Therefore $F(b) = 0$ and the holonomy map is not surjective. Hence in this setting each horizontal curve is singular but clearly is not rigid since we can deform the curve in horizontal directions.

2.4.3 The holonomy map on the space of square integrable functions

In [75, Section 3.8] R. Montgomery stressed the fact the C^1 topology is not the correct one for calculus of variations. Indeed, a rigid curve in the C^1 topology is always a local minimum (see [97]) for each functional, in particular for the length functional, since its minimality does not depend on the functional but only on its domain. Therefore R. Montgomery suggested to consider the $W^{1,2}$ topology for curves instead of the C^1 topology, introducing the endpoint map described in Section 1.1.7. Here we show how we can take into account this weakening of the regularity for the holonomy map.

Let $I \subset \mathbb{R}$ be an open interval. Let $\gamma : I \to N$ be an absolutely continuous curve of degree $d = \deg(\gamma)$ with square integrable derivative

$$\gamma'(t) = \sum_{\ell=1}^{k} u_\ell(t) (X_\ell)_{\gamma(t)},$$

where $u_\ell \in L^2_{loc}(I,\mathbb{R})$ for each $\ell = 1,\dots,k$ letting $\mathcal{H} := \mathcal{H}^d$, $k = n_d$, $((X_1)_{\gamma(t)}, \dots, (X_k)_{\gamma(t)})$ is an horizontal frame along γ for $\mathcal{H}$ and $((X_{k+1})_{\gamma(t)},\dots,(X_n)_{\gamma(t)})$ is a vertical frame for $\mathcal{V} = (\mathcal{H})^\perp$ along γ, both of them provided by Remark 2.3.3. Let $[a,b] \subset I$. A square integrable vector field $V \in L^2([a,b],TN)$ can be projected into its horizontal part $V_h \in L^2([a,b],\mathcal{H})$ given by

$$V_h = \sum_{i=1}^{k} g_i(t)(X_i)_{\gamma(t)} \quad \text{where} \quad g_i \in L^2([a,b])$$

and its vertical part $V_v \in L^2([a,b],\mathcal{V})$ given by

$$V_v = \sum_{r=k+1}^{n} f_r(t)(X_r)_{\gamma(t)} \quad \text{where} \quad f_r \in L^2([a,b]).$$

Thus, the admissibility system (2.4.2) is now equivalent to

$$f_r' + \sum_{i=1}^{k} a_{ri} g_i + \sum_{j=k+1}^{n} b_{rj} f_j = 0, \quad r = k+1,\dots,n, \tag{2.4.15}$$

where we consider distributional derivatives and the coefficients

$$a_{ri} = \sum_{\ell=1}^{k} u_\ell(t) c^r_{\ell i}(\gamma), \qquad b_{rj} = \sum_{\ell=1}^{k} u_\ell(t) c^r_{\ell j}(\gamma) \tag{2.4.16}$$

belong to $L^2([a,b])$, since by assumption $u_\ell \in L^2([a,b])$. Hence [70, Theorem 1.1] allows us to define a holonomy type map

$$\tilde{H}^{a,b}_\gamma : L^2([a,b],\mathcal{H}) \to \mathcal{V}_{\gamma(b)}$$

where $\mathcal{V}_{\gamma(b)}$ is the vector space of vertical vectors at the point $\gamma(b)$. In order to define $\tilde{H}^{a,b}_\gamma$ we consider a horizontal vector $V_h \in L^2([a,b],\mathcal{H})$ and we take the only vector field $V_v \in W^{1,2}([a,b],\mathcal{V})$ solution of (2.4.15) with initial condition $V_v(a) = 0$, thanks to [70, Theorem 1.1]. By the Sobolev Embedding Theorem [45, Corollary 7.11] the space $W^{1,2}([a,b],\mathcal{V})$ is continuously embedded in $C^{\frac{1}{2}}([a,b],\mathcal{V})$. Thus, we consider the $\frac{1}{2}$-Hölder function, denoting it always as V_v, in the class of functions of $V_v \in W^{1,2}([a,b],\mathcal{V})$ so that we define

$$\tilde{H}^{a,b}_\gamma(V_h) = V_v(b).$$

Definition 2.4.23. In the above conditions, we say that γ restricted to $[a,b]$ is *regular* if the holonomy map $\tilde{H}^{a,b}_\gamma$ is surjective.

Defining F and G as in (2.3.5), the system (2.4.15) is equivalent to $F' = -BF - AG$, where A, B are the L^2 matrices defined in (2.4.16) and the time derivative shall be understood in the distributional sense. In these conditions, the coordinates of $\tilde{H}_{\gamma}^{a,b}(V_h) = V_v(b)$ in the basis (X_i) are given by $F(b)$.

The following result allows the integration of the differential equation (2.4.15) to explicitly compute the holonomy map.

Proposition 2.4.24. *In the above conditions, there exists a square regular matrix $D(t)$ of order $(n-k)$ with coefficient in $C^{\frac{1}{2}}([a,b])$ such that*

$$F(b) = -D(b)^{-1} \int_a^b (DA)(t) G(t)\, dt. \tag{2.4.17}$$

Proof. Lemma 2.4.25 below allows us to find a regular matrix $D(t)$ with coefficient in $C^{\frac{1}{2}}([a,b])$ such that $D' = DB$. Then equation $F' = -BF - AG$ is equivalent to $(DF)' = -DAG$. Since DF belongs to $W^{1,2}([a,b], \mathcal{V})$ and the fundamental theorem of calculus still holds in $W^{1,2}$ we have

$$D(b)F(b) - D(a)F(a) = -\int_a^b (DA)(t)G(t)\, dt$$

Taking into account that $F(a) = 0$, and multiplying by $D(b)^{-1}$, we obtain (2.4.17). □

Lemma 2.4.25. *Let $B(t)$ be a L^2 family of square matrices on the interval $[a,b]$. Let $D(t)$ be the $C^{\frac{1}{2}}$ solution of the Cauchy problem*

$$D'(t) = D(t)B(t) \text{ on } [a,b], \quad D(a) = I_d. \tag{2.4.18}$$

Then $\det D(t) \neq 0$ *for each* $t \in [a,b]$.

Proof. Thanks to [70, Theorem 6.4] the Cauchy problem (2.4.18) has a unique solution $D \in W^{1,2}([a,b], \mathbb{M}^{d\times d})$, that belongs to $C^{\frac{1}{2}}([a,b], \mathbb{M}^{d\times d})$ by the Sobolev Embedding Theorem. Since the determinant is a polynomial function, therefore C^1, we apply the chain rule in Sobolev spaces to gain the Jacobi formula

$$\frac{d(\det D(t))}{dt} = \operatorname{Tr}\left(\operatorname{adj} D(t) \frac{dD(t)}{dt}\right)$$

in the distributional sense, where adj$D$ is the classical adjoint (the transpose of the cofactor matrix) of D and Tr is the trace operator. Therefore

$$\frac{d \det(D(t))}{dt} = \operatorname{Tr}\left((\operatorname{adj} D(t))D(t)B(t)\right) = \det D(t) \operatorname{Tr}(B(t)). \tag{2.4.19}$$

Since $\det D(a) = 1$, the solution for (2.4.19) is given by

$$\det D(t) = e^{\int_a^t \operatorname{Tr}(B(\tau))\,d\tau} > 0,$$

for all $t \in [a, b]$. Thus, the matrix $D(t)$ is invertible for each $t \in [a, b]$. □

Theorem 2.4.26. *The absolutely continuous curve γ of degree $d = \deg(\gamma)$, with square integrable derivative, is non-regular restricted to $[a, b]$ if and only if there exists a $C^{\frac{1}{2}}$ row vector field $\Lambda(t) \neq 0$ for all $t \in [a, b]$ that solves the following system*

$$\begin{cases} \Lambda'(t) = \Lambda(t)B(t) \\ \Lambda(t)A(t) = 0, \end{cases} \tag{2.4.20}$$

for a.e. $t \in [a, b]$.

Proof. Assume that γ is nonregular in $[a, b]$, then the image of the holonomy map is contained in a proper subspace of $\mathcal{V}_{\gamma(b)}$. Therefore there exists a row vector $\Gamma \neq 0$ such that

$$\Gamma F(b) = -\int_a^b \Gamma D(b)^{-1} D(t)A(t)G(t) = 0 \tag{2.4.21}$$

for all $G \in L^2([a, b], \mathcal{H})$, where $D(t)$ solves

$$\begin{cases} D(t)' = D(t)B(t) \\ D(a) = I_{n-k}. \end{cases} \tag{2.4.22}$$

In the previous computation we used the integral formula provided by Proposition 2.4.24. Setting $\Lambda(t) := \Gamma D(b)^{-1}D(t)$ by equation (2.4.21) we obtain $\Lambda(t)A(t) = 0$ for a.e. $t \in [a, b]$. Since Γ is a constant vector and $D(t)$ is a regular $C^{\frac{1}{2}}([a, b]) \cap W^{1,2}([a, b])$ matrix by Lemma 2.4.25, we obtain $\Lambda'(t) = \Lambda(t)B(t)$ a.e. in $[a, b]$ and $\Lambda(t) \neq 0$ for all $t \in [a, b]$.

Conversely, any solution of the system (2.4.20) is given by

$$\Lambda(t) = \Gamma D(t),$$

where $\Gamma = \Lambda(0) \neq 0$ and $D(t)$ solves the equation (2.4.22). Indeed, let us consider a general solution $\Lambda(t)$ of (2.4.20). If we set

$$\Phi(t) = \Lambda(t) - \Gamma D(t),$$

where $\Gamma = \Lambda(0) \neq 0$ and $D(t)$ solves the equation (2.4.22), then we deduce

$$\begin{cases} \Phi(t)' = \Phi(t)B(t) \\ \Phi(0) = 0. \end{cases}$$

Clearly the unique solution of this system is $\Phi(t) \equiv 0$. Hence we conclude that $\Gamma D(t)A(t) = 0$ for a.e. t in $[a,b]$. Furthermore by Proposition 2.4.24 we have that the image of the holonomy map is given by

$$F(b) = -D(b)^{-1} \int_a^b (DA)(t)G(t)\, dt$$

for each $G \in L^2([a,b], \mathcal{H})$. Setting $\tilde{\Gamma} := \Gamma D(b)$ we obtain

$$\tilde{\Gamma} F(b) = -\int_a^b \Gamma D(t)A(t)G(t)dt = 0.$$

Therefore the image of the holonomy map is contained in a proper subspace of $\mathcal{V}_{\gamma(b)}$, thus the curve γ is non-regular restricted to $[a,b]$. □

Remark 2.4.27. We notice that

- as we pointed out in Remark 2.4.8 when $(N, \mathcal{H})$ is a Carnot manifold the system (2.4.20) coincides with the characteristic system (1.1.23), but now for absolutely continuous curves with square integrable derivatives and not only for C^1 curves;

- we have the following inclusions for the holonomy map

$$\begin{array}{cccc}
\tilde{H}^{a,b}_\gamma : & L^2([a,b],\mathcal{H}) & \to & \mathcal{V}_{\gamma(b)} \\
 & \cup & & \\
H^{a,b}_\gamma : & C_0((a,b),\mathcal{H}) & \to & \mathcal{V}_{\gamma(b)} \\
 & \cup & & \\
H^{a,b}_\gamma : & C^1_0((a,b),\mathcal{H}) & \to & \mathcal{V}_{\gamma(b)} \\
 & \cup & & \\
 & \vdots & & \vdots \\
H^{a,b}_\gamma : & C^r_0((a,b),\mathcal{H}) & \to & \mathcal{V}_{\gamma(b)} \\
 & \cup & & \\
 & \vdots & & \vdots \\
H^{a,b}_\gamma : & C^\infty_0((a,b),\mathcal{H}) & \to & \mathcal{V}_{\gamma(b)},
\end{array}$$

where the suitable control space depends on the regularity $(L^2, C^1, \ldots, C^\infty)$ of the immersed curve we consider. When the curve is $W^{1,2}$ the control space is $L^2([a,b], \mathcal{H})$ and when the curve is C^r the control space for the holonomy map is $C_0^{r-1}((a,b), \mathcal{H})$ for $r \geqslant 1$.

2.5 Integrability of admissible vector fields on a regular curve

In this Section, we provide an alternative proof of the fundamental Theorem 3 in Hsu's paper [56], that implies that, when γ is a regular curve in (a,b), then any admissible vector field along γ with compact support in (a,b) is integrable. We recall that $\mathcal{H} := \mathcal{H}^d$, where $1 \leqslant d \leqslant s$ is the degree of γ. For sake of simplicity the distribution $\mathcal{H}$ will be called horizontal as well as a curve of degree d and we set $k := n_d$. We need first some preliminary results.

We consider the following spaces

1. $\mathfrak{X}_\gamma^r(a)$, $r \geqslant 0$, is the set of C^r vector fields along γ that vanish at a.
2. $\mathcal{H}_\gamma^r(a)$, $r \geqslant 0$, is the set of horizontal C^r vector fields along γ vanishing at a.
3. $\mathcal{V}_\gamma^r(a)$, $r \geqslant 0$, is the set of vertical vector fields of class C^r along γ vanishing at a. By a vertical vector we mean a vector in $\mathcal{H}^\perp$.

We shall denote by Π_v the orthogonal projection over the vertical subspace.

For $r \geqslant 1$, we consider the map

$$\mathcal{G} : \mathcal{H}_\gamma^{r-1}(a) \times \mathcal{V}_\gamma^r(a) \to \mathcal{H}_\gamma^{r-1}(a) \times \mathcal{V}_\gamma^{r-1}(a), \tag{2.5.1}$$

defined by

$$\mathcal{G}(Y_1, Y_2) = (Y_1, \mathcal{F}(Y_1 + Y_2)),$$

where $\mathcal{F}(Y) = \Pi_v(\Gamma(Y)')$, and $\Gamma(Y)(t) = \exp_{\gamma(t)}(Y(t))$. Observe that $\mathcal{F}(Y) = 0$ if and only if the curve $\Gamma(Y)$ is horizontal.

We consider on each space the corresponding $|| \cdot ||_r$ or $|| \cdot ||_{r-1}$ norm, and the corresponding product norm (it does not matter whether it is Euclidean, the sup or the 1 norm).

Then

$$D\mathcal{G}(0,0)(Y_1, Y_2) = (Y_1, D\mathcal{F}(0)(Y_1 + Y_2)),$$

where $D\mathcal{F}(0)Y$ is given by

$$D\mathcal{F}(0)Y = \sum_{i=k+1}^{n} \Big(\langle \nabla_{\gamma'} Y, X_i\rangle + \langle \gamma', \nabla_Y X_i\rangle\Big) X_i.$$

Observe that $D\mathcal{F}(0)Y = 0$ if and only if Y is an admissible vector field.

Our objective now is to prove that the map $D\mathcal{G}(0,0)$ is an isomorphism of Banach spaces. To show this, we shall need the following result.

Proposition 2.5.1. *The differential $D\mathcal{G}(0,0)$ is an isomorphism of Banach spaces.*

Proof. We first observe that $D\mathcal{G}(0,0)$ is injective, since $D\mathcal{G}(0,0)(Y_1,Y_2) = (0,0)$ implies that $Y_1 = 0$ and that the vertical vector field Y_2 satisfies the compatibility equations with initial condition $Y_2(a) = 0$. Hence $Y_2 = 0$. The map $D\mathcal{G}(0,0)$ is continuous. Indeed, if for instance we consider the 1-norm on the product space we have

$$\begin{aligned} \|D\mathcal{G}(0,0)(Y_1,Y_2)\| &= \|(Y_1, D\mathcal{F}(0)(Y_1+Y_2))\| \\ &\leqslant \|Y_1\|_{r-1} + \|D\mathcal{F}(0)(Y_1+Y_2))\|_{r-1} \\ &\leqslant (1 + \|(a_{ij})\|_{r-1})\|Y_1\|_{r-1} + (1+\|(b_{ij})\|_{r-1})\|Y_2\|_r. \end{aligned}$$

To show that $D\mathcal{G}(0,0)$ is surjective, we take (Y_1,Y_2) in the image, and we find a vector field Y along γ such that $Y(a) = 0$, $Y_h = Y_1$ and $D\mathcal{F}(0)(Y) = Y_2$ by Lemma 2.4.1. The map $D\mathcal{G}(0,0)$ is open because of the estimate (2.5.2) given in Lemma 2.5.2 below. □

Lemma 2.5.2. *In the above conditions, assume that $D\mathcal{F}(0)(Y) = Y_2$ and $Y_h = Y_1$ and $Y(a) = 0$. Then there exists a constant K such that*

$$\|Y_v\|_r \leqslant K(\|Y_2\|_{r-1} + \|Y_1\|_{r-1}). \tag{2.5.2}$$

Proof. Reasoning as in Lemma 2.4.1 we choose a global orthonormal adapted basis (X_i) on γ and write

$$Y_1 = \sum_{i=1}^{k} g_i X_i, \quad Y_2 = \sum_{r=k+1}^{n} z_r X_r \quad \text{and} \quad Y_v = \sum_{r=k+1}^{n} f_r X_r.$$

Then Y_v is a solution of the ODE (2.4.2) given by

$$F' = -B(t)F + Z(t) - \ A(t)G(t) \tag{2.5.3}$$

where $B(t), A(t)$ are defined in (2.3.3), F, G are defined in (2.3.5) and we set

$$Z = \begin{pmatrix} z_{k+1} \\ \vdots \\ z_n \end{pmatrix}.$$

Since $Y_v(a) = 0$ an Y_v solves (2.5.3) in (a, b), by Lemma 2.5.3 there exists a constant K such that

$$\begin{aligned} \|Y_v\|_{C^r([a,b])} = \|F\|_{C^r([a,b])} &\leqslant K\|Z(t) - A(t)\ G(t)\|_{C^{r-1}([a,b])} \\ &\leqslant K'(\|Y_2\|_{C^{r-1}([a,b])} + \|Y_1\|_{C^{r-1}([a,b])}). \end{aligned} \tag{2.5.4}$$

where $K' = K \max\{1, \sup_{[a,b]} \|A(t)\|_{r-1}\}$. □

Lemma 2.5.3. *Let $r \geqslant 1$ be a natural number. Let $u : [a, b] \to \mathbb{R}^d$ be the solution of the inhomogeneous problem*

$$\begin{cases} u' = A(t)u + c(t), \\ u(a) = u_0 \end{cases} \tag{2.5.5}$$

where $A(t)$ is a $d \times d$ matrix in C^{r-1} and $c(t)$ a C^{r-1} vector field. Then, there exists a constant K such that

$$\|u\|_r \leqslant K(\|c\|_{r-1} + |u_0|). \tag{2.5.6}$$

Proof. The proof is by induction. We start from the case $r = 1$. By [52, Lemma 4.1] it follows

$$u(t) \leqslant \left(|u_0| + \int_a^t |c(s)|ds\right) e^{|\int_a^t \|A(s)\| ds|},$$

where the norm of A is given by $\sup_{|x|=1} |A\ x|$. Therefore we have

$$\sup_{t\in[a,b]} |u(t)| \leqslant C_1(\sup_{t\in[a,b]} |c(t)| + |u_0|), \tag{2.5.7}$$

where we set

$$C_1 = (b - a)e^{(b-a)\sup_{t\in[a,b]} \|A(t)\|}.$$

Since u is a solution of (2.5.5) it follows

$$\begin{aligned} \sup_{t\in[a,b]} |u'(t)| &\leqslant \sup_{t\in[a,b]} \|A(t)\| \sup_{t\in[a,b]} |u(t)| + \sup_{t\in[a,b]} |c(t)| \\ &\leqslant (C_2 + 1) \sup_{t\in[a,b]} |c(t)|. \end{aligned} \tag{2.5.8}$$

Hence by (2.5.7) and (2.5.8) we obtain

$$\|u\|_1 \leqslant K(\|c\|_0 + |u_0|).$$

Assume that (2.5.6) holds for $1 \leqslant k \leqslant r$, then by (2.5.5) we have

$$u^{(r+1)}(t) = \sum_{k=0}^{r} A^{(k)}(t)\ u^{(r-k)}(t) + c^{(r)}(t).$$

By the inductive assumption we deduce

$$\begin{aligned} \sup_{t\in[a,b]} |u^{(r+1)}(t)| &\leqslant \sum_{k=0}^{r} \sup_{t\in[a,b]} \|A^{(k)}(t)\| \ \sup_{t\in[a,b]} |u^{(r-k)}(t)| + \sup_{t\in[a,b]} |c^{(r)}(t)| \\ &\leqslant (C_3+1)(\sup_{t\in[a,b]} |c^{(r)}(t)| + |u_0|). \end{aligned} \tag{2.5.9}$$

Hence the inequality (2.5.6) for $r+1$ simply follows by (2.5.9). □

Finally, we use the previous constructions to give a criterion for the integrability of admissible vector fields along a horizontal curve.

Theorem 2.5.4. *Given $r \geqslant 1$, let $\gamma : I \to N$ be a C^r curve of degree $d = \deg(\gamma)$ in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric. Assume that γ is regular in the interval $[a,b] \subset I$. Then every admissible C^{r-1} vector field with compact support in (a,b) is integrable.*

Proof. Let us take $V, V^1, \ldots, V^{n-k}$ vector fields in $\mathfrak{X}_\gamma^{r-1}(a)$ along γ vanishing at a. We consider the map

$$\tilde{\mathcal{G}} : \Big[(-\varepsilon,\varepsilon) \times (-\varepsilon,\varepsilon)^{n-k}\Big] \times \Big[\mathcal{H}_\gamma^{r-1}(a) \times \mathcal{V}_\gamma^{r}(a)\Big] \to \mathcal{H}_\gamma^{r-1}(a) \times \mathcal{V}_\gamma^{r-1}(a),$$

given by

$$\tilde{\mathcal{G}}((s,(s_i),Y_1,Y_2)) = (Y_1, F(sV + \sum_{i=1}^{n-k} s_i V^i + Y_1 + Y_2)).$$

The map $\tilde{\mathcal{G}}$ is continuous with respect to the product norms (on each factor we put the natural norm, the Euclidean one on the intervals and $||\cdot||_r$ and $||\cdot||_{r-1}$ in the spaces of vectors along γ). Moreover

$$\tilde{\mathcal{G}}(0,0,0,0) = (0,0),$$

since the curve γ has degree less or equal to $d = \deg(\gamma)$. By Proposition 2.5.1 we have that

$$D_2\tilde{\mathcal{G}}(0,0,0,0)(Y_1,Y_2) = D\mathcal{G}(0,0)(Y_1,Y_2)$$

is a linear isomorphism. We can apply the Implicit Function Theorem to obtain maps

$$Y_1 : (-\varepsilon,\varepsilon)^{n-k+1} \to \mathcal{H}_\gamma^{r-1}(a), \quad Y_2 : (-\varepsilon,\varepsilon)^{n-k+1} \to \mathcal{V}_\gamma^{r}(a),$$

such that $\tilde{\mathcal{G}}(s,(s_i),(Y_1)(s,s_i),(Y_2)(s,s_i)) = (0,0)$. This implies that $(Y_1)(s,(s_i)) = 0$ and that

$$F(sV + \sum_i s_i V^i + Y_2(s,s_i)) = 0.$$

Hence the curves

$$\Gamma(sV + \sum_i s_i V^i + Y_2(s,s_i))$$

are horizontal.

Now we assume that V is an admissible vector field with $V(a) = V(b) = 0$, and that $V^1,\ldots,V^{n-k}$ are admissible vector fields vanishing at a. Then the vector field

$$\frac{\partial Y_2}{\partial s}(0,0), \frac{\partial Y_2}{\partial s_i}(0,0)$$

along γ are vertical and admissible. Since they vanish at a, they are identically 0.

If, in addition, $V_v^1(b),\ldots,V_v^{n-k}(b)$ generate the space $\mathcal{H}^\perp(b)$. We consider the map

$$\Pi : (-\varepsilon,\varepsilon)^{n-k+1} \to N$$

given by

$$(s,(s_i)) \mapsto \Gamma(sV + \sum_i s_i V^i + Y_2(s,s_i))(b).$$

For s, (s_i) small, the image of this map is an $(n-k)$-dimensional submanifold S of N with tangent space at $\gamma(b)$ given by $\mathcal{H}^\perp_{\gamma(b)}$ (as $V(b) = 0$ and $V^i(b) = V_v^i(b)$ generate $\mathcal{H}^\perp_{\gamma(b)}$). Notice that

$$\frac{\partial \Pi(0,0)}{\partial s_i} = V^i(b) = V_v^i(b),$$

which is invertible and

$$\frac{\partial \Pi(0,0)}{\partial s} = V(b) = 0.$$

Hence we can apply the Implicit Function Theorem to conclude that there exists a family of smooth functions $s_i(s)$ so that

$$\Gamma(sV + \sum_i s_i(s)V^i + Y_2(s, s_i(s))) \tag{2.5.10}$$

are horizontal and take the value $\gamma(b)$ at b. Clearly, we have

$$\Pi(s, (s_i(s))) = \gamma(b).$$

Differentiating with respect to s at $s = 0$ we obtain

$$\frac{\partial \Pi(0,0)}{\partial s} + \sum_i \frac{\partial \Pi(0,0)}{\partial s_i} s_i'(0) = 0.$$

Therefore $s_i'(0) = 0$ for each $i = 1, \ldots, n-k$. Thus, the variational vector field to Γ is

$$\left.\frac{\Gamma(s)}{\partial s}\right|_{s=0} = V + \sum_i s_i'(0)V^i + \frac{\partial Y_2}{\partial s}(0,0) + \sum_i \frac{\partial Y_2}{\partial s_i}(0,0) = V. \tag{2.5.11}$$

Finally the variation $\Gamma(t,s) = \exp_{\gamma(t)}(sV + \sum_i s_i(s)V^i + Y_2(s, s_i(s)))$, given by (2.5.10), is C^{r-1}, since all the vector fields V, V^i, Y_2 are at least C^{r-1} (the horizontal components of V and V^i are C^{r-1} and the vertical components are C^r) and the Riemannian exponential map follows the geodesics that are smooth. □

2.6 A new integrability criterion for admissible vector fields

In this subsection we give a sufficient condition for a curve $\gamma : I \to N$ of degree d to be regular in $[a,b] \subset I$. The condition is that the matrix $A(t)$ associated to the admissibility system of differential equations (2.3.4), defined in (2.3.3), has rank $(n-k)$ for any $t \in I$. By Proposition 2.4.6, this condition implies that the curve γ is regular in $[a,b]$. The aim of this section is to give a direct proof of this fact, *that generalizes to higher dimensions*.

We recall that $\mathcal{H} := \mathcal{H}^d$, where $1 \leqslant d \leqslant s$ is the degree of γ. For sake of simplicity the distribution $\mathcal{H}$ will be called horizontal as well as a curve of degree d and we set $k := n_d$. We consider the following spaces:

1. $\mathfrak{X}^r(I, N)$, $r \geqslant 0$: the set of C^r vector fields along γ.

2. $\mathcal{H}^r(I,N)$, $r \geqslant 0$: the set of C^r horizontal vector fields along γ.

3. $\mathcal{V}^r(I,N) := \{Y \in \mathfrak{X}^r(I,N) : \langle Y, X\rangle = 0 \ \forall X \in \mathcal{H}^r(I,N)\} = \mathcal{H}^r(I,N)^\perp$.

We shall denote by Π_v the orthogonal projection over the vertical subspace.

As in the previous subsections we consider an orthonormal adapted basis (X_i) along γ and the associated admissibility system in matrix form $F' = BF + AG$, where the matrices A, B, F and G are defined in (2.3.3) and (2.3.5).

Definition 2.6.1. We say that a horizontal curve $\gamma : I \to N$ is *strongly regular* at $t \in I$ if $\operatorname{rank} A(t) = n-k$. We say that γ is *strongly regular* in $J \subset I$ if it is strongly regular at every $t \in J$.

Remark 2.6.2.

1. Given $t_0 \in I$ such that γ is strongly regular at t_0, there exists a small neighborhood J of t_0 in I where the rank of $A(t)$ is given by a fixed subset of columns of $A(t)$ for all $t \in J$. This neighborhood J can be extended to a maximal one where this property holds.

2. Notice that a curve γ can be strongly regular at $t \in I$ only when $k \geqslant \frac{n}{2}$.

The third equation in (3.4.18) implies that this definition is independent of the chosen adapted orthonormal basis along γ. With this definition, we are able to prove

Lemma 2.6.3. *Let $k \geqslant \frac{n}{2}$. Let $A(t)$ be the C^r $(n-k) \times k$ matrix defined in* (2.3.3) *with respect to $(X_1, \ldots, X_n)$. Assume that* $\operatorname{rank} A(t) = n-k$ *for $t \in [a,b]$. Then there exists a horizontal orthonormal global basis $\tilde{X}_1, \ldots, \tilde{X}_k$ on $[a,b]$ such that the matrix $\tilde{A}(t)$ with respect to the orthonormal global basis $(\tilde{X}_1, \ldots, \tilde{X}_k, X_{k+1}, \ldots, X_n)$ is given by*

$$\tilde{A}(t) = \begin{pmatrix} \tilde{A}_1(t) & 0 \end{pmatrix},$$

where $\tilde{A}_1(t)$ is an invertible square $(n-k)$ matrix.

Proof. The proof is by induction on the dimension of the kernel of A that is equal to $2k-n$.

When $\dim(\ker A(t)) = 1$ locally there exist two unitary vector fields $X(t)$ and $-X(t)$ in the kernel of $A(t)$. We define a global vector field $\tilde{X}_k(t)$ locally choosing one of the two unit vectors $X(t)$ or $-X(t)$ in the kernel and adjusting them in the overlapping intervals. Then we extend the unitary vector field $\tilde{X}_k$ to an orthonormal horizontal basis $(\tilde{X}_1, \ldots, \tilde{X}_k)$. Therefore with respect to $(\tilde{X}_1, \ldots, \tilde{X}_k, X_{k+1}, \ldots, X_n)$ the last column of the matrix $\tilde{A}$ is equal to zero.

If $\dim(\ker A(t)) > 1$, fix $\bar{t} \in]a,b[$, then by a continuation argument for the determinant there exists an open neighborhood $U_{\bar{t}} =]\bar{t} - \delta, \bar{t} + \delta[$ and a non vanishing C^r vector field $V(t)$ on U such that $A(t)V(t) = 0$. Then $\{U_t\}_{t \in [a,b]}$ is an open cover of the compact set $[a,b]$ then there exists a finite sub-cover $U_1, \dots, U_Q$ such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha, \beta \in \{1, \dots, Q\}$, $\alpha < \beta$ if and only if $\beta = \alpha + 1$. Let $\{\psi_\alpha : \alpha \in \{1, \dots, Q\}\}$ be a partition of unity subordinate to the cover $\{U_\alpha : \alpha \in \{1, \dots, Q\}\}$ for further details see [99, Definition 1.8]. For each α there exists a non vanishing C^r vector fields $V^\alpha(t)$ on U_α in $\ker A(t)$. When $U_\alpha \cap U_\beta \neq \emptyset$ we consider $V^\alpha(t)$ on U_α and $V^\beta(t)$ on U_β in $\ker A(t)$ such that they are linear independent on $U_\alpha \cap U_\beta$, since the $\dim(\ker A(t)) > 1$. Then, we set

$$X(t) = \sum_{\alpha=1}^{Q} \psi_\alpha(t) V^\alpha(t).$$

Therefore $X(t)$ is a global non vanishing vector field that belongs to $\ker(A(t))$ for all $t \in [a,b]$. Thus we can extend the global unitary vector

$$\tilde{X}_k(t) = \frac{X(t)}{|X(t)|}$$

to an orthonormal basis of the horizontal distribution. The matrix associated to this basis has a vanishing last column. We remove this column and start again until we have dimension 1. Hence in this new global horizontal basis $(\tilde{X}_1, \dots, \tilde{X}_k)$ the last $2k - n$ columns of the matrix $\tilde{A}(t)$ vanish and the rank is concentrated in the square matrix $A_1(t)$ given by the first $n - k$ columns. □

Theorem 2.6.4. *Let $\gamma : I \to N$ be a curve of degree $d = \deg(\gamma)$ in a graded manifold $(N, \mathcal{H}^1, \dots, \mathcal{H}^s)$ endowed with a Riemannian metric. Assume that γ is strongly regular in $[a,b] \subset I$. Then every admissible vector field with compact support in (a,b) is integrable.*

Proof. Let $J = [a,b]$. The admissibility system is given by

$$F'(t) + B(t)F(t) + A(t)G(t) = 0, \tag{2.6.1}$$

with respect to (X_i). By hypothesis, the rank of $A(t) \in C^r$ is maximal for all $t \in J$. By Lemma 2.6.3 there exists a global basis $(\tilde{X}_i)$ such that

$$\tilde{A}(t) = \begin{pmatrix} \tilde{A}_1(t) & 0 \end{pmatrix},$$

where $\tilde{A}_1(t)$ is an invertible square $(n-k)$ matrix. Then setting $\tilde{F}, \tilde{G}$ the new coordinates with respect to $(\tilde{X}_i)$ and $\tilde{B}$ as in (3.4.18), by Remark 2.3.2 we have

$$\tilde{F}' + \tilde{B}\tilde{F} + \tilde{A}\tilde{G} = 0 \tag{2.6.2}$$

Calling

$$\tilde{G}_1 = \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{n-k} \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} \tilde{g}_{n-k+1} \\ \vdots \\ \tilde{g}_k \end{pmatrix},$$

the admissibility system (2.6.2) can be written as

$$\tilde{F}' + \tilde{B}\tilde{F} + \tilde{A}_1\tilde{G}_1 = 0,$$

and so

$$\tilde{G}_1 = -\tilde{A}_1^{-1}\left(\tilde{F}' + \tilde{B}\tilde{F}\right). \tag{2.6.3}$$

Now let $\mathcal{H}_1^r(J)$ be the set of horizontal vector fields of class C^r in J that are linear combination of the first $(n-k)$ vectors $\tilde{X}_1, \dots, \tilde{X}_{n-k}$, and $\mathcal{H}_2^r(J)$ the set of horizontal vector fields of class C^r in J that are linear combination of the vector fields $\tilde{X}_{n-k+1}, \dots, \tilde{X}_k$.

We consider the map

$$\mathcal{G} : \mathcal{V}^r(J) \times \mathcal{H}_1^{r-1}(J) \to \mathcal{V}^r(J) \times \mathcal{V}^{r-1}(J) \tag{2.6.4}$$

defined by

$$\mathcal{G}(Y_1, Y_2) = (Y_1, \mathcal{F}(Y_1 + Y_2)).$$

We recall that, given a vector field Y along a portion of γ, we define the curve $\Gamma(Y)(t)$ by $\exp_{\gamma(t)}(Y(t))$ and we define $\mathcal{F}(Y)$ as the vertical projection of the tangent vector $\Gamma(Y)'$. We consider on each of the spaces appearing in (2.6.4) the corresponding $||\cdot||_r$ or $||\cdot||_{r-1}$ norm, and in the product one of the classical product norms.

Then

$$D\mathcal{G}(0,0)(Y_1, Y_2) = (Y_1, D\mathcal{F}(0)(Y_1 + Y_2)),$$

where $D\mathcal{F}(0)Y$ is given by

$$D\mathcal{F}(0)Y = \sum_{r=k+1}^{n} \left(\tilde{f}_r'(t) + \sum_{i=1}^{n-k} \tilde{a}_{ri}(t)\tilde{g}_i(t) + \sum_{j=k+1}^{n} \tilde{b}_{rj}(t) f_j(t) \right) X_r.$$

Observe that $D\mathcal{F}(0)Y = 0$ if and only if Y is an admissible vector field, namely Y solves (2.6.1).

Our objective now is to prove that the map $D\mathcal{G}(0,0)$ is an isomorphism of Banach spaces. Indeed suppose that $D\mathcal{G}(0,0)(Y_1, Y_2) = (0,0)$. This implies that Y_1 is equal to zero. In the coordinates previously described, Y_1 to $\tilde{F}$, and Y_2 to $\tilde{G}_1$. By the admissibility equation (2.6.3) we have that also Y_2 is equal to zero. This proves that $D\mathcal{G}(0,0)$ is injective. Let us prove now that $D\mathcal{G}(0,0)$ is surjective. Take (Z_1, Z_2), where $Z_1 \in \mathcal{V}^r(J)$, and $Z_2 \in \mathcal{V}^{r-1}(J)$ we seek Y_1, Y_2 such that $D\mathcal{G}(0,0)(Y_1, Y_2) = (Z_1, Z_2)$. Then $Y_1 = Z_1$ and Y_2 is obtained by solving the system

$$\tilde{G}_1 = -\tilde{A}_1^{-1}\Big(\tilde{F}' + \tilde{B}\tilde{F} + \tilde{Z}\Big),$$

since Y_1 and $Z_2 = \sum_{r=k+1} \tilde{z}_r \tilde{X}_r$ is already given. This proves that $D\mathcal{G}(0,0)$ is surjective.

Keeping the above notation for Y_i, Z_i, $i = 1, 2$, we notice that $D\mathcal{G}(0,0)$ is a continuous map since the identity map is continuous and there exists a constant K such that

$$\begin{aligned} \|Z_2\|_{r-1} &\leqslant K\left(\left\|\frac{d}{dt}(Y_1)\right\|_{r-1} + \|Y_1\|_{r-1} + \|Y_2\|_{r-1}\right) \\ &\leqslant K(\|Y_1\|_r + \|Y_2\|_{r-1}). \end{aligned}$$

Moreover, $D\mathcal{G}(0,0)$ is an open map since we have

$$\begin{aligned} \|Y_2\|_{r-1} &\leqslant K\left(\left\|\frac{d}{dt}(Z_1)\right\|_{r-1} + \|Z_1\|_{r-1} + \|Z_2\|_{r-1}\right) \\ &\leqslant K(\|Z_1\|_r + \|Z_2\|_{r-1}). \end{aligned}$$

This concludes the proof that $D\mathcal{G}(0,0)$ is an isomorphism of Banach spaces.

Let us finally consider an admissible vector field V compactly supported on (a, b). We consider the map

$$\tilde{\mathcal{G}} : (-\varepsilon, \varepsilon) \times \mathcal{V}^r(J) \times \mathcal{H}_1^{r-1}(J) \to \mathcal{V}^r(J) \times \mathcal{V}^{r-1}(J),$$

defined by

$$\tilde{\mathcal{G}}(s, Y_1, Y_2) = (Y_1, F(sV + Y_1 + Y_2)).$$

The map $\tilde{\mathcal{G}}$ is continuous with respect to the product norms (on each factor we put the natural norm, the Euclidean one on the intervals and $||\cdot||_r$ and $||\cdot||_{r-1}$ in the spaces of vectors along γ). Moreover

$$\tilde{\mathcal{G}}(0,0,0) = (0,0),$$

since γ is horizontal. Now we have that

$$D_2\tilde{\mathcal{G}}(0,0,0)(Y_1,Y_2) = D\mathcal{G}(0,0)(Y_1,Y_2)$$

is a linear isomorphism. We can apply the Implicit Function Theorem to obtain $\varepsilon > 0$ and unique maps

$$Y_1 : (-\varepsilon,\varepsilon) \to \mathcal{V}^r(J), \quad Y_2 : (-\varepsilon,\varepsilon) \to \mathcal{H}_1^{r-1}(J)$$

such that $\tilde{\mathcal{G}}(s, Y_1(s), Y_2(s)) = (0,0,0)$. This implies that $Y_1(s) \equiv 0$, $Y_2(0) = 0$ and that

$$\mathcal{F}(sV + Y_2(s)) = 0.$$

Hence the curve $\Gamma_s : J \to N$ defined by $\Gamma_s(t) = \exp_{\gamma(t)}(sV(t) + Y_2(s)(t))$ is horizontal for $s \in (-\varepsilon,\varepsilon)$. Differentiating the above formula at $s = 0$ we obtain

$$D\mathcal{F}(0)\left(V + \frac{\partial Y_2}{\partial s}(0)\right) = 0.$$

As V is admissible we deduce that

$$D\mathcal{F}(0)\left(\frac{\partial Y_2}{\partial s}(0)\right) = 0.$$

Since

$$\frac{\partial Y_2}{\partial s}(0) = \sum_{h=1}^{n-k} \tilde{f}_i \tilde{X}_i, \quad \tilde{f}_i \in C^{r-1}(J),$$

we get from by equation (2.6.3) that $\tilde{f}_i \equiv 0$ for each $h = 1,\ldots,n-k$. Therefore it follows that $\frac{\partial Y_3}{\partial s}(0) = 0$. This way we obtain that the variational vector field of the variation Γ_s is

$$\left.\frac{\partial \Gamma_s}{\partial s}\right|_{s=0} = V + \frac{\partial Y_2}{\partial s}(0) = V.$$

The uniqueness property of the Implicit Function Theorem provides $Y_i(s) = 0$ for $s \in (-\varepsilon,\varepsilon)$ when $V = 0$. □

2.7 The first variation formula

Let $\gamma : I \to N$ be a curve of degree d in a graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric. We fix an orthonormal adapted basis $(X_1, \ldots, X_n)$ along the curve. Recall that the length of degree d was computed in (2.1.5) as $L_d(\gamma, J) = \int_J \theta_d(t)dt$, where the *length density* θ_d of degree d is given by

$$\theta_d(t) = \left(\sum_{j=n_{d-1}+1}^{n_d} \langle \gamma'(t), (X_j)_{\gamma(t)} \rangle^2 \right)^{\frac{1}{2}}.$$

Assume that $\theta_d(t) \neq 0$ for all $t \in I$. Fixing $\bar{t} \in I$ it turns out that

$$\varphi(t) = \int_{\bar{t}}^{t} \theta_d(\tau) d\tau$$

is a diffeomorphism, then $\tilde{\gamma}(s) = \gamma(\varphi^{-1}(s))$ is a reparametrization of γ so that the new length density is identically 1. If not, by Corollary 1.2.5 we know that $I \smallsetminus I_0$ is open and we can reparametrize γ such the new length density is identically 1 on every open connected component (interval) of $I \smallsetminus I_0$.

Let $\Gamma(t, s)$ be an admissible variation of γ whose variational vector field is given by $V(t) = \frac{\partial \Gamma}{\partial s}(t, 0)$. Calling $\theta = \theta_d$, the derivative of the length functional L_d is given by

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} L_d(\Gamma_s, I) &= \frac{d}{ds}\Big|_{s=0} \int_I \left(\sum_{j=n_{d-1}+1}^{n_d} \left\langle \frac{\Gamma(t,s)}{\partial t}, (X_j)_{\Gamma(t,s)} \right\rangle^2 \right)^{\frac{1}{2}} dt \\ &= \sum_{j=n_{d-1}+1}^{n_d} \int_I \frac{\langle \gamma'(t), X_j \rangle}{\theta(t)} \frac{d}{ds}\Big|_{s=0} \left\langle \frac{\Gamma(t,s)}{\partial t}, (X_j)_{\Gamma(t,s)} \right\rangle dt \\ &= \sum_{j=n_{d-1}+1}^{n_d} \int_I \frac{\langle \gamma'(t), X_j \rangle}{\theta(t)} \left(\langle \nabla_{\gamma'} V(t), X_j \rangle + \langle \gamma'(t), \nabla_{V(t)} X_j \rangle \right) dt. \end{aligned}$$

Integrating by parts we obtain that

$$\int_I \frac{\langle \gamma'(t), X_j \rangle}{\theta(t)} \langle \nabla_{\gamma'} V(t), X_j \rangle \, dt = \int_I -\langle V(t), \nabla_{\gamma'} \left(\frac{\langle \gamma'(t), X_j \rangle}{\theta(t)} X_j \right) \rangle \, dt.$$

Since $V = \sum_{i=1}^n \langle V, X_i \rangle X_i$ we get

$$\langle \gamma'(t), \nabla_V X_j \rangle = \langle V, \sum_{i=1}^{n} \langle \gamma', \nabla_{X_i} X_j \rangle X_i \rangle,$$

and so we can write

$$\frac{d}{ds}\bigg|_{s=0} L_d(\Gamma_s, I) = \int_I \langle V, \mathbf{H}\rangle\, dt, \tag{2.7.1}$$

where

$$\mathbf{H} = \sum_{j=n_{d-1}+1}^{n_d} \left(-\nabla_{\gamma'}\left(\frac{\langle \gamma'(t), X_j\rangle}{\theta(t)} X_j\right) + \sum_{i=1}^{n} \frac{\langle \gamma'(t), X_j\rangle}{\theta(t)} \langle \gamma', \nabla_{X_i} X_j\rangle X_i \right). \tag{2.7.2}$$

Expressing $\gamma' = \sum_{\ell=1}^{n_d} \langle \gamma', X_\ell\rangle X_\ell$, we can rewrite $\mathbf{H}$ as

$$\mathbf{H} = \sum_{j=n_{d-1}+1}^{n_d} \left(-\frac{d}{dt}\left(\frac{\langle \gamma'(t), X_j\rangle}{\theta(t)}\right) X_j + \sum_{i=1}^{n}\sum_{\ell=1}^{n_d} \frac{\langle \gamma'(t), X_j\rangle \langle \gamma'(t), X_\ell\rangle}{\theta(t)}\, c_{\ell i}^{j} X_i \right), \tag{2.7.3}$$

where

$$c_{\ell i}^{j} = -\langle \nabla_{X_\ell} X_j, X_i\rangle + \langle X_\ell, \nabla_{X_i} X_j\rangle = \langle X_j, [X_\ell, X_i]\rangle. \tag{2.7.4}$$

With this preparation, we can compute the first variation formula for the length L_d of degree d for regular curves.

Remark 2.7.1. Let m be a positive integer. We remind that when x is a vector field in $\mathbb{R}^m$ with coordinates x_i for $i = 1, \ldots, m$, its transpose is the row vector $x^T = (x_1, \ldots, x_m)$.

Theorem 2.7.2. *Let $\gamma : I \to N$ be a curve of degree $d = \deg(\gamma)$ such that $\theta_d(t) = 1$ for each $t \in I \smallsetminus I_0$. Assume that the curve γ is regular restricted to $[a, b] \subset I \smallsetminus I_0$. Then γ is a critical point of the length of degree d for any admissible variation if and only if there is a constant vector $k^T \in \mathbb{R}^{n-n_d}$ such that γ satisfies along γ the following differential equation*

$$-\dot{\alpha}^T(t) + \beta_h^T(t) = \left(k - \left(\int_a^t \beta_v^T(\tau) D^{-1}(\tau)\, d\tau\right)\right) D(t) A(t), \tag{2.7.5}$$

where $u_i = \langle \gamma', X_i\rangle$ for $i = 1, \ldots, n_d$, $A(t)$ defined in (2.3.3), $D(t)$ solving (2.4.7) and we set

$$\alpha = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_{n_{d-1}+1} \\ \vdots \\ u_{n_d} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_h \\ \beta_v \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n_d} \\ \beta_{n_{d+1}} \\ \vdots \\ \beta_n \end{pmatrix}, \tag{2.7.6}$$

with

$$\beta_i = \sum_{\ell=1}^{n_d} \sum_{j=n_{d-1}+1}^{n_d} u_j\, u_\ell\, c_{\ell i}^j.$$

Proof. Fix an adapted basis (X_i) along γ, and consider the admissible vector field

$$V = \sum_{i=1}^{n_d} g_i X_i + \sum_{j=n_d+1}^{n} f_i X_i$$

solving the admissibility equation (2.3.4). Then we have

$$(DF)' = -DAG. \tag{2.7.7}$$

Since γ is regular by hypothesis, Theorem 2.5.4 implies that V is integrable, and so there exists an admissible variation $\Gamma(t,s)$ such that $V(t) = \frac{\partial \Gamma}{\partial s}(t,0)$. With the notation introduced in (2.7.6) the first variational formula with respect to (X_i) is given by

$$\frac{d}{ds}\Big|_{s=0} L_d(\Gamma_s, I) = \int_I -\dot\alpha^T(t)G(t) + \beta_h^T(t)G(t) + \beta_v^T(t)F(t)\, dt. \tag{2.7.8}$$

Since (2.7.7) holds and F is compactly supported in $[a,b]$ we have

$$\begin{aligned}\int_I \beta_v^T(t)F(t)\,dt &= \int_I \beta_v^T(t)D^{-1}(t)D(t)F(t)\,dt\\ &= -\int_I \left(\int_a^t \beta_v^T(\tau)D^{-1}(\tau)\,d\tau\right)(DF)'(t)\,dt\\ &= \int_I \left(\int_a^t \beta_v^T(\tau)D^{-1}(\tau)\,d\tau\right) D(t)A(t)G(t)\,dt.\end{aligned}$$

Therefore (2.7.8) is equivalent to

$$\int_I \left(-\dot\alpha^T(t) + \left(\int_a^t \beta_v^T(\tau)D^{-1}(\tau)\,d\tau\right) D(t)A(t) + \beta_h^T(t)\right) G(t)dt, \tag{2.7.9}$$

for each $G(t)$ that verifies

$$F(b) = -D(b)^{-1}\int_a^b D(t)A(t)G(t)dt = 0.$$

Hence a critical point of the functional L_d is given by

$$\int_I \left(-\dot\alpha^T(t) + \left(\int_a^t \beta_v^T(\tau)D^{-1}(\tau)\,d\tau\right) D(t)A(t) + \beta_h^T(t)\right) G(t)dt = 0,$$

for each G satisfying

$$H_\gamma^{a,b}(G) = D(b)^{-1} \int_a^b D(t)A(t)G(t)dt = 0.$$

Since the holonomy map is surjective by the du Bois-Reymond Lemma [56, Lemma C.1] there exists a constant vector $\tilde{k}^T \in \mathbb{R}^{n-n_d}$ such that the Euler-Lagrange equation is given by

$$-\dot{\alpha}^T(t) + \left(\int_a^t \beta_v^T(\tau)D^{-1}(\tau)\, d\tau\right) D(t)A(t) + \beta_h^T(t) = k\ D(b)^{-1}D(t)A(t). \quad (2.7.10)$$

Since $D(b)^{-1}$ is a constant matrix we have that $k = \tilde{k}D(b)^{-1}$ is a constant row vector. Hence γ satisfies equation (2.7.5).

Conversely, assume that equation (2.7.5) holds so than also (2.7.17) holds. Putting equation (2.7.17) into (2.7.9) we obtain

$$\frac{d}{ds}\Big|_{s=0} L_d(\Gamma_s, I) = \int_I k\ D(t)A(t)G(t)dt = k \int_I D(t)A(t)G(t)dt.$$

Since (2.7.7) holds and F is compactly supported in $[a, b]$ we conclude

$$\frac{d}{ds}\Big|_{s=0} L_d(\Gamma_s, I) = k\ \ (D(t)F(t))\Big|_{t=a}^{t=b} = 0.$$

Hence we proved that γ is a critical point of the length functional of degree d for any admissible variation. □

Example 2.7.3 (The Heisenberg group $\mathbb{H}^n$)**.** A well-known example of contact sub-Riemannian manifold (see Example 2.4.12) is the Heisenberg group $\mathbb{H}^n$, defined as $\mathbb{R}^{2n+1}$ endowed with the contact form

$$\omega_0 = dt + \sum_{i=1}^n (x_i dy_i - y_i dx_i).$$

Moreover $\mathbb{H}^n$ is a Lie group $(\mathbb{R}^{2n+1}, *)$ where the product is defined, for any pair of points $(z, t) = (z_1, \ldots, z_n, t)$, $(z', t') = (z'_1, \ldots, z'_n, t')$ in $\mathbb{R}^{2n+1} = \mathbb{C}^{2n} \times \mathbb{R}$, by

$$(z, t) * (z', t') = \left(z + z', t + t' + \sum_{i=1}^n \mathrm{Im}(z_i \bar{z}_i')\right).$$

A basis of left invariant vector fields is given by $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, T\}$, where

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial_t}, \quad Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial_t} \quad i = 1, \dots, n, \quad T = \frac{\partial}{\partial_t}.$$

The only non-trivial relation is $[X_i, Y_i] = T$. Here the horizontal metric h is the one that makes $\{X_i, Y_i : i = 1, \cdots, n\}$ an orthonormal basis of $\mathcal{H} = \ker(\omega_0)$. On the tangent bundle we consider the metric $g = \langle\cdot,\cdot\rangle$ so that (2.4.13) holds. Clearly, we have $\langle X_i, T\rangle = \langle Y_i, T\rangle = 0$ for all $i = 1, \dots, n$. Let ∇ be the Levi-Civita connection associated to g. From Koszul formula and the Lie bracket relations we get

$$\nabla_{X_i} X_j = \nabla_{Y_i} Y_j = \nabla_T T = 0, \quad \nabla_{X_i} Y_j = -\delta_{ij} T, \quad \nabla_{Y_i} X_j = \delta_{ij} T \tag{2.7.11}$$

For any vector field X on $\mathbb{H}^n$ we have $J(X) = \nabla_X T$. Following the previous notation we set $X_{n+i} := Y_i$ for all $i = 1, \dots, n$ and $X_{2n+1} := T$, then the only non-trivial structure constants are

$$c_{i\,n+i}^{2n+1} = -c_{n+i\,i}^{2n+1} = \langle [X_i, X_{i+n}], X_{2n+1}\rangle = 1, \tag{2.7.12}$$

for all $i = 1, \dots, n$.

Let $\gamma : I \to \mathbb{H}^n$ be an horizontal curve parameterized by arc length, i.e. $\theta(t) = 1$. By equation (2.7.11) and the linearity of ∇ on the first term we have $\langle\gamma', \nabla_{\gamma'} X_j\rangle = 0$ thus it holds

$$\dot{\alpha}_j = \langle\nabla_{\gamma'}\gamma', X_j\rangle + \langle\gamma', \nabla_{\gamma'} X_j\rangle = \langle\nabla_{\gamma'}\gamma', X_j\rangle,$$

where α is defined in Theorem 2.7.2. Since $\langle\nabla_{\gamma'}\gamma', T\rangle = -\langle\gamma', J(\gamma')\rangle = 0$, then $\nabla_{\gamma'}\gamma'$ is horizontal. By equation (2.7.12) we have $c_{\ell i}^j = 0$ for all $j = 1, \dots, 2n$, thus we deduce that

$$\beta_i = \sum_{\ell=1}^{2n}\sum_{j=1}^{2n} \langle\gamma', X_j\rangle\langle\gamma', X_\ell\rangle\, c_{\ell i}^j = 0$$

for all $i = 1, \dots, 2n+1$. In this setting we have

$$B = \langle[\gamma', T], T\rangle = 0, \quad A = (a_1, \dots, a_{2n}),$$

where $a_i = 2\langle J(\gamma'), X_i\rangle$, as we computed in (2.4.14). Since the solution of following Cauchy problem

$$\begin{cases} D'(t) = D(t)B(t) = 0 \\ D(a) = 1 \end{cases}$$

is given by $D(t) = 1$ for all $t \in [a, b]$, the right side term of (2.7.5) is given by $2kJ(\gamma')$. Then we conclude that γ is a critical point of the horizontal length functional for any admissible variation if and only if there exists a constant $k \in \mathbb{R}$ such that

$$-\nabla_{\gamma'}\gamma' = 2kJ(\gamma'). \tag{2.7.13}$$

Explicit solutions to this geodesic equation can be find in [89, p. 10], [76, p. 160] and in [7, p. 28].

Let now $\gamma : I \to \mathbb{H}^n$ be a curve such that $\deg(\gamma) = 2$. We parametrize the curve γ so that the length density $\theta_2(t) = \langle \gamma', T\rangle = 1$ for all $t \in I \smallsetminus I_0$. Since $\deg(\gamma) = 2$ is the maximal degree for a curve in Heisenberg the vertical set $\mathcal{V}_{\gamma(t)} = \{0\}$ for all $t \in I \smallsetminus I_0$. Then γ is regular restricted to each interval $[a, b] \subset I \smallsetminus I_0$. Therefore we have that $k = D = A = 0$ and

$$\alpha = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \beta_i = \sum_{\ell=1}^{2n+1} u_{2n+1}\, u_\ell\, c_{\ell i}^{2n+1}.$$

Thus we have $\beta_i = -\langle \gamma', Y_i\rangle$ for $i = 1, \ldots, n$, $\beta_i = \langle \gamma', X_i\rangle$ for $i = n + 1, \ldots, 2n$ and $\beta_{2n+1} = 0$. We deduce $\beta = J(\gamma')$. Hence the geodesic equation (2.7.5) is given by $J(\gamma') = 0$, then $\gamma' = T$. We conclude that the geodesics of degree 2 are straight lines in direction ∂_t.

2.7.1 Some properties of the length functional of degree two for surfaces immersed in the Heisenberg group

Let Σ be a surface immersed in the Heisenberg group $\mathbb{H}^1$, where a basis of left-invariant vector fields is given by

$$X = \partial_x + \frac{y}{2}\partial_t, \quad Y = \partial_y - \frac{x}{2}\partial_t, \quad T = \partial_t.$$

We consider the ambient metric $g = \langle \cdot, \cdot \rangle$ that makes (X, Y, T) an orthonormal basis, see Example 2.4.21, and $\mathcal{H} = \text{span}\{X, Y\}$. Σ inherits the Riemannian metric $\bar{g}$ induced by g. Hence $(\Sigma, \tilde{\mathcal{H}}^1, \tilde{\mathcal{H}}^2)$ is a graded manifold endowed with the Riemannian metric $\bar{g}$, where $\tilde{\mathcal{H}}^1_p = T_p\Sigma \cap \mathcal{H}_p$, $\tilde{\mathcal{H}}^2_p = T_p\Sigma$ if p belongs to $\Sigma \smallsetminus \Sigma_0$ and $\tilde{\mathcal{H}}^1_p = \tilde{\mathcal{H}}^2_p = \mathcal{H}_p = T_p\Sigma$ for $p \in \Sigma_0$. Let N be a unit vector normal to Σ w.r.t. g and $N_h = N - \langle N, T\rangle T$ its orthogonal projection onto $\mathcal{H}$. In the regular part $\Sigma \smallsetminus \Sigma_0$, the horizontal Gauss map

ν_h and the characteristic vector field Z are defined by

$$\nu_h = \frac{N_h}{|N_h|}, \quad Z = J(\nu_h), \tag{2.7.14}$$

where $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Clearly Z is horizontal and orthogonal to ν_h then it is tangent to Σ. If we define

$$S = \langle N, T\rangle \nu_h - |N_h| T, \tag{2.7.15}$$

then $(Z_p, S_p) = (e_1, e_2)$ is an orthogonal basis of $T_p\Sigma$ and it is adapted to the filtration $\mathcal{H}_p^1 \cap T_p\Sigma \subset \mathcal{H}_p^2 \cap T_p\Sigma$ for each p in $\Sigma \smallsetminus \Sigma_0$.

In the regular part $\Sigma \smallsetminus \Sigma_0$ the length functional L_2 is well-defined. Since all variation Γ_s are admissible and the first variation formula is given by

$$\left.\frac{d}{ds}\right|_{s=0} L_2(\Gamma_s, I) = \int_I \langle V, \mathbf{H}\rangle \, dt, \tag{2.7.16}$$

where V is a vector field in $T\Sigma$ and $\mathbf{H}$ is given by equation (2.7.3). Then the Euler-Lagrange equation for L_2 is given by $\mathbf{H} = 0$. Following equation (2.7.3), $\mathbf{H} = 0$ is equivalent to

$$-\frac{d}{dt}\left(\frac{\langle \gamma'(t), e_2\rangle}{|\langle \gamma'(t), e_2\rangle|}\right) e_2 + \sum_{i=1}^{2}\sum_{\ell=1}^{2}\left(\frac{\langle \gamma'(t), e_2\rangle \langle \gamma'(t), e_\ell\rangle}{\langle \gamma'(t), e_2\rangle} c_{\ell i}^{j}\right) e_i = 0. \tag{2.7.17}$$

Then a straightforward computation shows that (2.7.17) is equivalent to

$$\left(\langle \gamma'(t), e_1\rangle c_{21}^2\right) e_1 + \left(\langle \gamma'(t), e_1\rangle c_{12}^2\right) e_2 = 0$$

This means that the geodesic equation for L_2 is given by

$$\langle \gamma'(t), Z\rangle \langle [S, Z], S\rangle = 0. \tag{2.7.18}$$

Whenever $\langle [S, Z], S\rangle \neq 0$ the unique geodesic for L_2 starting from p is the integral curve of the vector field S passing through p, namely the unique solution of the following Cauchy problem

$$\begin{cases} \gamma'(t) = S_{\gamma(t)} \\ \gamma(0) = p. \end{cases}$$

The projection of the integral curve of S onto the xy-plane are called *seed curves* in the literature, see for instance [14, page 159].

Example 2.7.4. A vertical plane P_v in $\mathbb{H}^1$ is given by

$$P_v = \{(x, y, t) \in \mathbb{H}^1 \ : \ ax + by = c, \ a^2 + b^2 = 1, \ c \in \mathbb{R}\}.$$

It is easy to see that the $Z = bX - aY$ and $S = T$. Thus on a vertical plane we always have $\langle [T, Z], T\rangle = 0$, since the Lie algebra of the Heisenberg group is nilpotent. More generally each surface obtained by the product of a planar curve in the xy-plane with $\mathbb{R}$ in the t direction (see [16, Example 3.4]) verifies $\langle [S, Z], S\rangle = 0$. Therefore all curves in P_v satisfy the geodesic equation (2.7.18). Now for seek of simplicity in the computation we consider the vertical plane $\{y = 0\}$. This is not restrictive since a generic vertical plane P_v can be obtained by a rotation and a left-translation of $\{y = 0\}$. The length functional of degree 2 is given by

$$L_2(\gamma) = \int_a^b |\langle \gamma'(s), T\rangle| ds,$$

where $\gamma(s) = (x(s), t(s))$ is a piecewise C^1 curve in $P_v = \{y = 0\}$. Let $p = (x_0, t_0)$ and $q = (x_1, t_1)$ be two point in P_v. We consider the piecewise curve $\alpha(s) : [0, 2] \to P_v$ defined by

$$\alpha(s) = \begin{cases} \alpha_1(s) = (x_0, s\ t_1 + (1 - s)t_0) & \text{if} \quad s \in [0, 1] \\ \alpha_0(s) = ((s - 1)x_1 + (2 - s)x_0, t_1) & \text{if} \quad s \in [1, 2]. \end{cases}$$

We claim that $\alpha(t)$ is a minimizing curve for the length functional of degree 2, that means $L_2(\alpha) \leqslant L_2(\gamma)$ for each curve $\gamma : [a, b] \to P_v$, $\gamma(s) = (x(s), t(s))$ such that $\gamma(a) = (x_0, t_0)$ and $\gamma(b) = (x_1, t_1)$. Indeed, defining the following function

$$f : [a, b] \to \mathbb{R}, \qquad f(s) = \langle t(s) - t_0, t_1 - t_0\rangle,$$

we have $f(a) = 0$ and $f(b) = |t_1 - t_0|^2$. Then, it holds

$$f(b) = f(b) - f(a) = \int_a^b f'(s) ds. \tag{2.7.19}$$

By Cauchy-Schwarz inequality and (2.7.19) we obtain

$$|t_1 - t_0|^2 = \left| \int_a^b f'(s) ds \right| \leqslant \int_a^b |f'(s)| ds \leqslant \int_a^b |\langle t'(s), t_1 - t_0 \rangle| ds$$
$$\leqslant |t_1 - t_0| \int_a^b |t'(s)| ds.$$

Then, it follows

$$|t_1 - t_0| \leqslant \int_a^b |t'(s)| ds = \int_a^b \sqrt{\langle \gamma'(s), T \rangle^2} ds = L_2(\gamma). \tag{2.7.20}$$

Since $L_2(\alpha_0) = 0$ we have $L_2(\alpha) = L_2(\alpha_1) = |t_1 - t_0|$. By equation (2.7.20) we conclude $L_2(\alpha) \leqslant L_2(\gamma)$. However L_2 has several minimum among all curves that fix that end-points p and q, because each curve of degree 2 that has increasing t coordinate is a minimum for the length functional L_2. Indeed, when we reach the horizontal leaf of coordinates t_1 we can connect each point on the leaf leaving unchanged the value of L_2.

Example 2.7.5 (Characteristic plane)**.** Let P_c be the characteristic (or horizontal) plane in $\mathbb{H}^1$ defined by

$$P_c = \{(x, y, t) \in \mathbb{H}^1 \ : \ t = 0\}.$$

In cylindrical coordinates $x = \rho \cos(\theta)$, $y = \rho \sin(\theta)$ and $t = t$, where $\rho > 0$ and $\theta \in [0, 2\pi]$, we consider the orthonormal adapted basis (X', Y', T) in $\mathbb{H}^1$, where

$$\begin{aligned} X' &= \cos(\theta) X + \sin(\theta) Y = \frac{\partial}{\partial \rho}, \\ Y' &= -\sin(\theta) X + \cos(\theta) Y = \frac{1}{\rho} \frac{\partial}{\partial \theta} + \frac{\rho}{2} \frac{\partial}{\partial t}. \end{aligned} \tag{2.7.21}$$

Furthermore, since the tangent vector to P_c are $\frac{\partial}{\partial \rho}$ and

$$\frac{\partial}{\partial \theta} = \rho Y' - \frac{\rho^2}{2} T,$$

we deduce

$$Z = \frac{\partial}{\partial \rho} \quad \text{and} \quad S = \frac{1}{\rho \sqrt{1 + \frac{\rho^2}{4}}} \frac{\partial}{\partial \theta}.$$

Since $\langle [Z, S], S \rangle = \frac{\partial}{\partial \rho} \left((1 + \frac{\rho^2}{4})^{-\frac{1}{2}} \right) \neq 0$ for all $\rho > 0$ the geodesics for L_2 are integral curves of S. Let $\gamma(t) : [0, \bar{t}] \to \mathbb{R}^2 \setminus \{0\}$ be the integral closed curve of S such that $\gamma(0) = (\rho_0, \theta_0)$ and $\gamma(\bar{t}) = (\rho_0, \theta_0)$, $\rho_0 > 0$, $\theta_0 \in [0, 2\pi]$ and $| \cdot |$ the Euclidean metric. When ρ_0 tends to 0 the circle described by γ collapses to the characteristic point 0 and we have

$$\lim_{\rho_0\to 0}\int_I |\gamma'(t)|dt = \lim_{\rho_0\to 0}\frac{2\pi\rho_0}{\rho_0\sqrt{1+\frac{\rho_0^2}{4}}} = 2\pi.$$

Example 2.7.6 (Pansu's spheres)**.** In cylindrical coordinates the Pansu sphere $\mathbb{S}_1$ is the union of the graphs of the functions f and $-f$ defined on the plane $t=0$, where for $0<\rho\leqslant 1$

$$f(\rho,\theta)=\frac{1}{2}\left(\rho\sqrt{1-\rho^2}+\cos^{-1}(\rho)\right).$$

Then, for $0<\rho<1$ we have

$$\frac{\partial f}{\partial \rho} = -\frac{\rho^2}{\sqrt{1-\rho^2}} \quad \text{and} \quad \frac{\partial f}{\partial \theta}=0.$$

Therefore the unit normal N to the upper (lower) hemisphere, described by the graph f (respectively $-f$), is

$$N=\frac{1}{\sqrt{1-\rho^2}}\left(T\pm\frac{\rho^2}{\sqrt{1-\rho^2}}X'\right).$$

Thus, we have $\nu_h=\pm X'$, $Z=\pm Y'$, $\langle N,T\rangle=(1-\rho^2)^{-\frac{1}{2}}$ and

$$N_h=\pm\frac{\rho^2}{1-\rho^2}X' \quad \text{and} \quad S=\frac{1}{\sqrt{1-\rho^2}}X'\pm\frac{\rho^2}{1-\rho^2}T.$$

A straightforward computation shows that for $0<\rho<1$ there holds

$$\begin{aligned}\langle[Z,S],S\rangle &= \frac{1}{\sqrt{1-\rho^2}}\langle[Y',X'],S\rangle \\ &= \frac{1}{1-\rho^2}\langle\frac{1}{\rho}Y'+T, X'+\frac{\rho^2}{\sqrt{1-\rho^2}}T\rangle \\ &= \rho^2(1-\rho^2)^{-\frac{3}{2}}\neq 0.\end{aligned} \tag{2.7.22}$$

On the equator of $\mathbb{S}_1$, parametrized by $\theta\to(1,\theta,0)$, we have $\langle[Z,S],S\rangle=0$ since T is tangent to $\mathbb{S}_1$. Fix a point $p_0=(\rho_0,\theta_0,\pm f(\rho_0,\theta_0))\in\mathbb{S}_1$ with $0<\rho_0<1$. By (2.7.22) out from the equator $\langle[Z,S],S\rangle\neq 0$, then the geodesic $\gamma(s)=(\rho(s),\theta(s),t(s)=$

$f(\rho(s), \theta(s)))$ at p_0 for the L_2 functional is the solution of the following Cauchy problem

$$\begin{cases} \dot\rho(s) = \frac{1}{\sqrt{1-\rho(s)^2}} \\ \dot\theta(s) = 0 \\ \dot t(s) = \mp\frac{\rho(s)^2}{1-\rho(s)^2} \\ \gamma(0) = (\rho_0, \theta_0, \pm f(\rho_0, \theta_0)). \end{cases} \tag{2.7.23}$$

Then we have $\theta(s) = \theta_0$ and $\rho(s)$ verifies

$$\frac{1}{2}\left[y\sqrt{1-y^2} + \sin^{-1}(y)\right]_{\rho_0}^{\rho(s)} = s. \tag{2.7.24}$$

Moreover, we notice that the sign of $\dot t(s)$ is opposite to the sign of f. This means that the t coordinate of a geodesic $\gamma(s)$ decreases in the upper hemisphere and increases in lower hemisphere until γ reaches the equator. Then for each $\rho_0 > 0$ and $\theta_0 \in [0, 2\pi]$ we consider $\gamma_u : [0, \bar s] \to \mathbb{S}_1$ the solution of (2.7.23) such that $\gamma_u(0) = (\rho, \theta, f(\rho))$, $\gamma_u(\bar s) = (1, \theta_0, 0)$ and $\gamma_\ell : [0, \bar s] \to \mathbb{S}_1$ the solution of (2.7.23) such that $\gamma_u(0) = (\rho, \theta, -f(\rho))$, $\gamma_\ell(\bar s) = (1, \theta_0, 0)$. Letting $I = [0, 2\bar s]$ we set

$$\alpha(s) = \begin{cases} \gamma_u(s) & s \in [0, \bar s] \\ \gamma_\ell(2\bar s - s) & s \in [\bar s, 2\bar s] \end{cases}$$

When ρ_0 tends to 0 the end-points of α go to the poles that are characteristic points and we have

$$\lim_{\rho_0 \to 0} \int_I |\alpha'(t)| dt = \lim_{\rho_0 \to 0} 2\bar s = \frac{\pi}{2},$$

since by (2.7.24) it follows

$$\bar s = \lim_{\rho_0 \to 0} \lim_{\rho \to 1} \frac{1}{2}\left[y\sqrt{1-y^2} + \sin^{-1}(y)\right]_{\rho_0}^{\rho(s)} = \frac{\pi}{4}.$$

Chapter 3

Submanifolds of fixed degree immersed in graded manifolds

This chapter is devoted to the study of m-dimensional submanifolds of fixed degree immersed in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric, in particular we will show how to extend the notions of deformability and regularity introduced for curves in Chapter 2 to immersed submanifolds. First of all in Section 3.1 we define the area of degree d for these submanifolds. This is done as a limit of Riemannian areas. In addition, an integral formula for this area in terms of a density is given in formula (3.1.4). Section 3.2 is devoted to provide examples of submanifolds of certain degrees and the associated area functionals. In Sections 3.3 and 3.4 we introduce the notions of admissible variations, admissible vector fields and integrable vector fields and we study the system of first order partial differential equations defining the admissibility of a vector field. In particular, we show the independence of the admissibility condition for vector fields of the Riemannian metric in § 3.4.2. In Section 3.5 we give the notion of a strongly regular submanifold of degree d, see Definition 3.5.1. Then we prove in Theorem 3.5.2 that the strong regularity condition implies that any admissible vector vector is integrable. In addition, we exhibit in Example 3.5.7 an isolated plane whose only admissible normal vector field is the trivial one. In Section 3.6 we introduce suitable intrinsic coordinates to rewrite in convincing way the admissibility system. In Section 3.7 we provide the definition of ruled submanifolds. Section 3.8 is completely devoted to description of the higher-dimensional holonomy map and characterization of regular and singular ruled submanifolds. Finally, in Section 3.9 we give the proof of Theorem 3.9.5, that is a generalization of Theorem 2.5.4. Finally in Section 3.10 we compute the Euler-Lagrange equations of a strongly regular submanifold and give some examples. A substantial part of the content of this chapter comes form the articles [24] and [47], that have already been submitted.

3.1 Area for submanifolds of given degree

In this section we shall consider a graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric g, and an immersed submanifold M of dimension m.

We recall the construction introduced in Section 2.1 : given $p \in N$, we recursively define the subspaces $\mathcal{K}_p^1 := \mathcal{H}_p$, $\mathcal{K}_p^{i+1} := (\mathcal{H}_p^i)^\perp \cap \mathcal{H}_p^{i+1}$, for $1 \leqslant i \leqslant (s-1)$. Here $\perp$ means perpendicular with respect to the Riemannian metric g. Therefore we have the decomposition (2.1.1) of T_pN into orthogonal subspaces. Given $r > 0$, the unique Riemannian metric g_r is the one that makes the subspaces $\mathcal{K}_i$ orthogonal and verifies (2.1.2).

Working on a neighborhood U of p where a local frame $(X_1, \ldots, X_k)$ generating the distribution $\mathcal{H}$ is defined, we construct an *orthonormal* adapted basis $(X_1, \ldots, X_n)$ for the Riemannian metric g by choosing orthonormal bases in the orthogonal subspaces $\mathcal{K}^i$, $1 \leqslant i \leqslant s$. Thus, the m-vector fields

$$\tilde{X}_J^r = \left(r^{\frac{1}{2}(\deg(X_{j_1})-1)} X_{j_1}\right) \wedge \ldots \wedge \left(r^{\frac{1}{2}(\deg(X_{j_m})-1)} X_{j_m}\right), \tag{3.1.1}$$

where $J = (j_1, j_2, \ldots, j_m)$ for $1 \leqslant j_1 < \cdots < j_m \leqslant n$, are orthonormal with respect to the extension of the metric g_r to the space of m-vectors. We recall that the metric g_r is extended to the space of m-vectors simply defining

$$g_r(v_1 \wedge \ldots \wedge v_m, v_1' \wedge \ldots \wedge v_m') = \det\left(g_r(v_i, v_j')\right)_{1 \leqslant i,j \leqslant m}, \tag{3.1.2}$$

for $v_1, \ldots, v_m$ and $v_1', \ldots, v_m'$ in T_pN. Observe that the extension is denoted the same way.

3.1.1 Area for submanifolds of given degree

Assume now that M is an immersed C^1 submanifold of dimension m in a equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ equipped with the Riemannian metric g. We take a Riemannian metric μ on M. For any $p \in M$ we pick a μ-orthonormal basis $e_1, \ldots, e_m$ in T_pM. By the area formula we get

$$A(M', g_r) = \int_{M'} |e_1 \wedge \ldots \wedge e_m|_{g_r} \, d\mu(p), \tag{3.1.3}$$

where $M' \subset M$ is a bounded measurable set on M and $A(M', g_r)$ is the m-dimensional area of M' with respect to the Riemannian metric g_r.

Now we express

$$e_1 \wedge \ldots \wedge e_m = \sum_J \tau_J(p)(X_J)_p = \sum_J \tilde{\tau}_J^r(p)(\tilde{X}_J^r)_p, \quad r > 0.$$

From (3.1.1) we get $\tilde{X}_J^r = r^{\frac{1}{2}(\deg(X_J)-m)} X_J$, and so $\tilde{\tau}_J = r^{-\frac{1}{2}(\deg(X_J)-m)}\tau_J$. Moreover, as $\{\tilde{X}_J^r\}$ is an orthonormal basis for g_r, we have

$$|e_1 \wedge \ldots \wedge e_m|_{g_r}^2 = \sum_J (\tilde{\tau}_J^r(p))^2 = \sum_J r^{-(\deg(X_J)-m)} \tau_J^2(p).$$

Therefore, we have

$$\begin{aligned} \lim_{r \downarrow 0} r^{\frac{1}{2}(\deg(M)-m)} |e_1 \wedge \ldots \wedge e_m|_{g_r} &= \lim_{r \downarrow 0} \Big(\sum_J r^{(\deg(M)-\deg(X_J))} \tau_J^2(p) \Big)^{1/2} \\ &= \Big(\sum_{\deg(X_J)=\deg(M)} \tau_J^2(p) \Big)^{1/2}. \end{aligned}$$

By Lebesgue's dominated convergence theorem we obtain

$$\lim_{r \downarrow 0} \Big(r^{\frac{1}{2}(\deg(M)-m)} A(M', g_r) \Big) = \int_{M'} \Big(\sum_{\deg(X_J)=\deg(M)} \tau_J^2(p) \Big)^{\frac{1}{2}} d\mu(p). \tag{3.1.4}$$

Definition 3.1.1. If M is an immersed submanifold of degree d in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric g, the degree d area A_d is defined by

$$A_d(M') := \lim_{r \downarrow 0} \Big(r^{\frac{1}{2}(d-m)} A(M', g_r) \Big),$$

for any bounded measurable set $M' \subset M$.

Equation (3.1.4) provides an integral formula for the area A_d. An immediate consequence of the definition is the following

Remark 3.1.2. Setting $d := \deg(M)$ we have by equation (3.1.4) and the notation introduced in (1.2.5) that the degree d area A_d is given by

$$A_d(M') = \int_{M'} |(e_1 \wedge \ldots \wedge e_m)_d|_g \, d\mu(p). \tag{3.1.5}$$

for any bounded measurable set $M' \subset M$. When the ambient manifold is a Carnot group this area formula was obtained by [69]. Notice that the d area A_d is given by the integral of the m-form

$$\omega_d(v_1, \ldots, v_m)(p) = \langle v_1 \wedge \ldots \wedge v_m, \frac{(e_1 \wedge \ldots \wedge e_m)_d}{|(e_1 \wedge \ldots \wedge e_m)_d|_g} \rangle, \tag{3.1.6}$$

where $v_1, \ldots, v_m$ is a basis of T_pM.

In a more general setting, an m-dimensional submanifold in a Riemannian manifold is an m-current (i.e., an element of the dual of the space of m-forms), and the area is the mass of this current (for more details see [31]). Similarly, a natural generalization of an m-dimensional submanifold of degree d immersed in a graded manifold is an m-current of degree d whose mass should be given by A_d. In [38] the authors studied the theory of $\mathbb{H}$-currents in the Heisenberg group. Their mass coincides with our area (3.1.5) on intrinsic C^1 submanifolds. However in (3.1.6) we consider all possible m-forms and not only the intrinsic m-forms in the Rumin's complex [94, 82, 5].

Corollary 3.1.3. *Let M be an m-dimensional immersed submanifold of degree d in a graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ endowed with a Riemannian metric g. Let $M_0 \subset M$ be the closed set of singular points of M. Then $A_d(M_0) = 0$.*

Proof. Since M_0 is measurable, from (3.1.4) we obtain

$$A_d(M_0) = \int_{M_0} \Big(\sum_{\deg(X_J)=d} \tau_J^2(p) \Big)^{\frac{1}{2}} \, d\mu(p),$$

but $\tau_J(p) = 0$ when $\deg(X_J) = d$ and $p \in M_0$ since $\deg_M(p) < d$. □

Remark 3.1.4. Another easy consequence of the definition is the following: if M is an immersed submanifold of degree d in graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ with a Riemannian metric, then $A_{d'}(M') = \infty$ for any open set $M' \subset M$ when $d' < d$. This follows easily since in the expression

$$r^{\frac{1}{2}(d'-m)} \left| e_1 \wedge \ldots \wedge e_m \right|_{g_r}$$

we would have summands with negative exponent for r.

In the following example, we exhibit a Carnot manifold with two different Riemannian metrics that coincide when restricted to the horizontal distribution, but yield different area functionals of a given degree

Example 3.1.5. We consider the Carnot group $\mathbb{H}^1 \otimes \mathbb{H}^1$, which is the direct product of two Heisenberg groups. Namely, let $\mathbb{R}^3 \times \mathbb{R}^3$ be the 6-dimensional Euclidean space with coordinates (x, y, z, x', y', z'). We consider the 4-dimensional distribution $\mathcal{H}$ generated by

$$\begin{aligned} X &= \partial_x - \frac{y}{2}\partial_z, & Y &= \partial_y + \frac{x}{2}\partial_z, \\ X' &= \partial_{x'} - \frac{y'}{2}\partial_{z'} & Y' &= \partial_{y'} + \frac{x'}{2}\partial_{z'}. \end{aligned}$$

The vector fields $Z = [X, Y] = \partial_z$ and $Z' = [X', Y'] = \partial_{z'}$ are the only non trivial commutators that generate, together with X, Y, X', Y', the subspace $\mathcal{H}^2 = T(\mathbb{H}^1 \otimes \mathbb{H}^1)$. Let Ω be a bounded open set of $\mathbb{R}^2$ and u a smooth function on Ω such that $u_t(s, t) \equiv 0$. We consider the immersed surface

$$\begin{aligned} \Phi : \Omega &\longrightarrow \mathbb{H}^1 \otimes \mathbb{H}^1, \\ (s, t) &\longmapsto (s, 0, u(s, t), 0, t, u(s, t)), \end{aligned}$$

whose tangent vectors are

$$\begin{aligned} \Phi_s =& (1, 0, u_s, 0, 0, u_s) = X + u_s\ Z + u_s\ Z', \\ \Phi_t =& (0, 0, 0, 0, 1, 0) = Y'. \end{aligned}$$

Thus, the 2-vector tangent to M is given by

$$\Phi_s \wedge \Phi_t = X \wedge Y' + u_s(Z \wedge Y' + Z' \wedge Y').$$

When $u_s(s, t)$ is different from zero the degree is equal to 3, since both $Z \wedge Y'$ and $Z' \wedge Y'$ have degree equal to 3. Points of degree 2 corresponds to the zeroes of u_s. We define a 2-parameter family $g_{\lambda,\nu}$ of Riemannian metrics on $\mathbb{H}^1 \otimes \mathbb{H}^1$, for $(\lambda, \mu) \in \mathbb{R}^2$, by the conditions (i) (X, Y, X', Y') is an orthonormal basis of $\mathcal{H}$, (ii) Z, Z' are orthogonal to $\mathcal{H}$, and (iii) $g(Z, Z) = \lambda$, $g(Z', Z') = \mu$ and $g(Z', Z) = 0$. Therefore, the degree 3 area of Ω with respect to the metric $g_{\mu,\nu}$ is given by

$$A_3(\Omega) = \int_\Omega u_s(\lambda + \nu)\ dsdt.$$

As we shall see later, these different functionals will not have the same critical points, that would depend on the election of Riemannian metric.

However, in some cases, we can give conditions ensuring that two different extensions of a sub-Riemannian metric on a Carnot manifold provide the same area functional of a given degree up to scaling. These functionals would have the same critical points.

Remark 3.1.6. Let M be an m-dimensional submanifold immersed in an equiregular Carnot manifold $(N, \mathcal{H})$ and we set $d := \deg(M)$. Assume that for all p in M

(i) there exist $e_1, \ldots, e_{m-1}$ vectors tangent to T_pM that belong to $\mathcal{H}_p$,

(ii) there exists $e_m \in T_pM$ such that $\deg(e_m) = d - m + 1$,

(iii) $n_{d-m+1} - n_{d-m} = 1$.

Let h be a sub-Riemannian metric defined on $\mathcal{H}$ and g, $\bar{g}$ be two different metrics extending h to the whole tangent space TN and such that $\mathcal{H}_p^{d-m+1} \setminus \mathcal{H}_p^{d-m}$ is orthogonal to $\mathcal{H}_p$ with respect to the both metric $g, \bar{g}$ at each p in M. Then there exists a real number $\lambda > 0$ such that

$$A_d(M, g) = \lambda \, A_d(M, \bar{g}). \tag{3.1.7}$$

Indeed, fix an adapted basis $(X_1, \ldots, X_n)$ orthonormal in $\mathcal{H}$. Then, the degree d component of the tangent m-vector is given by

$$(e_1 \wedge \ldots \wedge e_m)_d = \sum_{1<j_1<\cdots<j_{m-1}<k} a_J X_{j_1} \wedge \ldots \wedge X_{j_{m-1}} \wedge X_{n_{d-m+1}},$$

where $J = (j_1, \ldots, j_{m-1}, n_{d-m+1})$ and $a_J \in \mathbb{R}$. Therefore there exists $\lambda > 0$ such that

$$\begin{aligned} |(e_1 \wedge \ldots \wedge e_m)_d|_g &= \sum_{1<j_1<\cdots<j_{m-1}<k} a_J^2 \, g(X_{n_{d-m+1}}, X_{n_{d-m+1}}) \\ &= \sum_{1<j_1<\cdots<j_{m-1}<k} a_J^2 \, \lambda \, \bar{g}(X_{n_{d-m+1}}, X_{n_{d-m+1}}) \\ &= \lambda \, |(e_1 \wedge \ldots \wedge e_m)_d|_{\bar{g}}. \end{aligned}$$

Hence, by the integral formula (3.1.5) we obtain (3.1.7).

3.1.2 Strongly regular submanifolds for the growth vector

Let M be a submanifold in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$. Then we consider the flag (1.2.10)

$$\tilde{\mathcal{H}}_p^1 \subset \tilde{\mathcal{H}}_p^2 \subset \cdots \subset \tilde{\mathcal{H}}_p^s = T_pM, \tag{3.1.8}$$

where $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$ and $\tilde{m}_j(p) = \dim(\tilde{\mathcal{H}}_p^j) - \dim(\tilde{\mathcal{H}}_p^{j-1})$. In [44] Ghezzi and Jean say that M is *strongly equiregular* if $\tilde{m}_j(p)$ is constant for each p in M and each $j = 1, \ldots, s$. In analogy with the definition of a regular point, we say that M is *strongly regular for the growth vector* at p in M if there exists a neighborhood $U_p \subset M$ such that $\tilde{m}_j(q)$ is constant for each q in U_p and each $j = 1, \cdots, s$.

Proposition 3.1.7. *Let M be a smooth submanifold of degree $\deg(M)$ in an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$. Suppose that p is a point of maximal degree $\deg_M$. Then M is strongly regular for the growth vector at p.*

Proof. Let p be a point of maximal degree. By Proposition 1.2.4 there exists an open neighborhood $U_p \subset N$ such that $\deg_M(q) \geqslant \deg_M(p)$ for all q in $U_p \cap M$. Since $\deg_M(p)$ achieves the maximum value $d(M)$ at p we have $\deg_M(q) = \deg_M(p)$ for all q in $U_p \cap M$. Let us consider a basis of the tangent space

$$\mathcal{B}_p = (e_1, \ldots, e_{\tilde{m}_1}, e_{\tilde{m}_1+1}, \ldots, e_{\tilde{m}_2}, \ldots, e_{\tilde{m}_{s-1}+1}, \ldots, e_{\tilde{m}_s}) \tag{3.1.9}$$

adapted to the flag (3.1.8) at p in M such that

$$\begin{aligned}
\tilde{\mathcal{H}}_p^1 &= \text{span}\{e_1, \ldots, e_{\tilde{m}_1}\}, \\
\tilde{\mathcal{H}}_p^2 &= \text{span}\{e_1, \ldots, e_{\tilde{m}_1}, e_{\tilde{m}_1+1}, \ldots, e_{\tilde{m}_2}\}, \\
&\vdots \\
\tilde{\mathcal{H}}_p^s &= \text{span}\{e_1, \ldots, e_{\tilde{m}_{s-1}}, e_{\tilde{m}_{s-1}+1}, \ldots, e_{\tilde{m}_s}\}.
\end{aligned}$$

We can consider smooth vector fields $\{E_j\}_{j=1,\ldots,\tilde{m}_s}$ defined on an open neighborhood $U'_p \subset U_p$ such that

$$\mathcal{D}_q = (E_1|_q, \ldots, E_{\tilde{m}_1}|_q, E_{\tilde{m}_1+1}|_q, \ldots, E_{\tilde{m}_2}|_q, \ldots, E_{\tilde{m}_{s-1}+1}|_q, \ldots, E_{\tilde{m}_s}|_q)$$

span all the tangent space T_qM for each q in $U'_p \cap M$ and $\mathcal{D}_p$ is equal to the basis $\mathcal{B}_p$ at p. By Lemma 1.2.2 we have that there exists an open set $U''_p \subset U'_p$ such that $\deg(E_j|_q) \geqslant \deg(e_j)$ for all $j = 1, \ldots, \tilde{m}_s$. Moreover, we claim that $\deg(E_i|_q) = \deg(e_i)$ for all $i = 1, \ldots, m = \tilde{m}_s$. Otherwise there exists an index k such that $\deg(E_k|_q) > \deg(e_k)$ and we have

$$\deg_M(q) = \sum_{i=1}^m \deg(E_i|_q) > \sum_{i=1}^m \deg(e_i) = \deg_M(p).$$

which is impossible since we know that $\deg_M(p) = \deg_M(q)$ for all q in U_p''. Hence we have $\tilde{m}_i(q) = \tilde{m}_i(p)$ for all $i = 1, \ldots, s$, what implies that the submanifold M is strongly regular for the growth vector at p. $\square$

Remark 3.1.8. In [44, Theorem 1] it is proved that the degree $\deg(M)$ of a strongly equiregular submanifold M immersed in an equiregular Carnot manifold is equal to the spherical Hausdorff dimension, induced by the C-C distance d_c defined in (1.1.6). In addition, the Radon-Nikodym derivative of the d-spherical Hausdorff measure with respect to a generic measure on M was computed.

Since the spherical Hausdorff dimension is a local property, a straightforward consequence of Proposition 3.1.7 is that the spherical Hausdorff dimension of $M \smallsetminus M_0$ is equal to $d = \deg(M)$. Moreover in [69, Theorem 1.2] the authors computed the Radon-Nikodym derivative of the Riemannian area of M induced by a graded metric in a Carnot group with respect to the d-spherical Hausdorff measure at points of maximum degree d. This quantity is equal to the ratio between a metric factor and the norm of the projection of the unit m-vector $\tau_M(p)$ tangent to M onto the degree d. The behavior of this metric factor has been deeply investigated by V. Magnani in [67, 66].

Hence, we speculate that at points of maximum degree $d = \deg(M)$ the Radon-Nikodym derivative of m-dimensional area measure on (M, μ), induced by the ambient metric g on the Carnot manifold, with respect to the d-dimensional spherical Hausdorff measure should be equal to the ratio between a metric factor and the norm of the projection of the unit m-vector $\tau_M(p)$ tangent to M onto the degree d. This should imply that our d-area measure given in (3.1.5) is absolutely continuous with respect to the spherical Hausdorff measure whenever $\mathcal{S}^d(M_0) = 0$.

3.2 Examples

3.2.1 Degree of a hypersurface in a Carnot manifold

Let M be a C^1 hypersurface immersed in an equiregular Carnot manifold $(N, \mathcal{H})$, where $\mathcal{H}$ is a bracket generating l-dimensional distribution. Let Q be the homogeneous dimension of N and $p \in M$.

Let us check that $\deg(M) = Q - 1$. The pointwise degree of M is given by

$$\deg_M(p) = \sum_{j=1}^{s} j(\tilde{m}_j - \tilde{m}_{j-1}),$$

where $\tilde{m}_j = \dim(\tilde{\mathcal{H}}_p^j)$ with $\tilde{\mathcal{H}}_p^j = T_pM \cap \mathcal{H}_p^j$. Recall that $n_i = \dim(\mathcal{H}_p^i)$. As T_pM is a hyperplane of T_pN we have that either $\tilde{\mathcal{H}}_p^i = \mathcal{H}_p^i$ and $\tilde{m}_i = n_i$, or $\tilde{\mathcal{H}}_p^i$ is a hyperplane of $\mathcal{H}_p^i$ and $\tilde{m}_i = n_i - 1$. On the other hand,

$$\tilde{m}_i - \tilde{m}_{i-1} \leqslant n_i - n_{i-1}.$$

Writing

$$n_i - n_{i-1} = \tilde{m}_i - \tilde{m}_{i-1} + z_i,$$

for non-negative integers z_i and adding up on i from 1 to s we get

$$\sum_{i=1}^{s} z_i = 1,$$

since $\tilde{m}_s = n - 1$ and $n_s = n$. We conclude that there exists $i_0 \in \{1, \ldots, s\}$ such that $z_{i_0} = 1$ and $z_j = 0$ for all $j \neq i_0$. This implies

$$\begin{aligned} \tilde{m}_i &= n_i, & i &< i_0, \\ \tilde{m}_i &= n_i - 1, & i &\geqslant i_0. \end{aligned}$$

If $i_0 > 1$ for all $p \in M$, then $\mathcal{H} \subset TM$, a contradiction since $\mathcal{H}$ is a bracket-generating distribution. We conclude that $i_0 = 1$ and so

$$\begin{aligned} \deg(M) &= \sum_{i=1}^{s} i\,(\tilde{m}_i - \tilde{m}_{i-1}) = 1 \cdot \tilde{m}_1 + \sum_{i=2}^{s} i\,(\tilde{m}_i - \tilde{m}_{i-1}) \\ &= 1 \cdot (n_1 - 1) + \sum_{i=2}^{s} i\,(n_i - n_{i-1}) = Q - 1. \end{aligned}$$

3.2.2 A_{2n+1}-area of a hypersurface in a $(2n+1)$-dimensional contact manifold

A *contact manifold* is a smooth manifold M^{2n+1} of odd dimension endowed with a one form ω such that $d\omega$ is non-degenerate when restricted to $\mathcal{H} = \ker(\omega)$. Since it holds

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]),$$

for $X, Y \in \mathcal{H}$, the distribution $\mathcal{H}$ is non-integrable and satisfies Hörmander rank condition by Frobenius theorem. When we define a horizontal metric h on the distribution $\mathcal{H}$ then $(M, \mathcal{H}, h)$ is a sub-Riemannian structure. It is easy to prove that there exists an unique vector field T on M so that

$$\omega(T) = 1, \quad \mathcal{L}_T(X) = 0,$$

where $\mathcal{L}$ is the Lie derivative and X is any vector field on M. This vector field T is called the *Reeb vector field*. We can always extend the horizontal metric h to the Riemannian metric g making T a unit vector orthogonal to $\mathcal{H}$.

Let Σ be a C^1 hypersurface immersed in M. In this setting the singular set of Σ is given by

$$\Sigma_0 = \{p \in \Sigma : T_p\Sigma = \mathcal{H}_p\},$$

and corresponds to the points in Σ of degree $2n$. Observe that the non-integrability of $\mathcal{H}$ implies that the set $\Sigma \smallsetminus \Sigma_0$ is not empty in any hypersurface Σ.

Let N be the unit vector field normal to Σ at each point, then on the regular set $\Sigma \smallsetminus \Sigma_0$ the g-orthogonal projection N_h of N onto the distribution $\mathcal{H}$ is different from zero. Therefore out of the singular set Σ_0 we define the *horizontal unit normal* by

$$\nu_h = \frac{N_h}{|N_h|},$$

and the vector field

$$S = \langle N, T \rangle \nu_h - |N_h| T,$$

which is tangent to Σ and belongs to $\mathcal{H}^2$. Moreover, $T_p\Sigma \cap (\mathcal{H}^2_p \smallsetminus \mathcal{H}^1_p)$ has dimension equal to one and $T_p\Sigma \cap \mathcal{H}^1_p$ equal to $2n-1$, thus the degree of the hypersurface Σ out of the singular set is equal to $2n+1$. Let $e_1, \ldots, e_{2n-1}$ be an orthonormal basis in $T_p\Sigma \cap \mathcal{H}^1_p$. Then $e_1, \ldots, e_{2n-1}, S_p$ is an orthonormal basis of $T_p\Sigma$ and we have

$$e_1 \wedge \ldots \wedge e_{2n-1} \wedge S = \langle N, T \rangle e_1 \wedge \ldots \wedge e_{2n-1} \wedge \nu_h - |N_h| e_1 \wedge \ldots \wedge e_{2n-1} \wedge T.$$

Hence we obtain

$$A_{2n+1}(\Sigma) = \int_\Sigma |N_h| d\Sigma. \tag{3.2.1}$$

This formula was first obtained by [43] in the general setting of hypersurfaces immersed in sub-Riemannian manifold, as we mention in Section 1.1.5 equation (1.1.16). In the case of 3-dimensional pseudo-hermitian manifolds it was deduced by [18, 84] and generalized to any dimension by [40, 95].

Example 3.2.1 (The Heisenberg group $\mathbb{H}^n$)**.** A well-known example of contact manifold is the Heisenberg group $\mathbb{H}^n$, described in Example 2.7.3, defined as $\mathbb{R}^{2n+1}$ endowed with the contact form

$$\omega_0 = dt + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i).$$

Moreover $\mathbb{H}^n$ is a Lie group $(\mathbb{R}^{2n+1}, *)$ where the product is defined, for any pair of points $(z,t) = (z_1, \ldots, z_n, t)$, $(z', t') = (z'_1, \ldots, z'_n, t')$ in $\mathbb{R}^{2n+1} = \mathbb{C}^{2n} \times \mathbb{R}$, by

$$(z,t) * (z',t') = \left(z + z', t + t' + \sum_{i=1}^{n} \mathrm{Im}(z_i \bar{z}_i') \right).$$

A basis of left invariant vector fields is given by $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, T\}$, where

$$X_i = \frac{\partial}{\partial_{x_i}} + y_i \frac{\partial}{\partial_t}, \quad Y_i = \frac{\partial}{\partial_{y_i}} - x_i \frac{\partial}{\partial_t} \quad i = 1, \ldots, n, \quad T = \frac{\partial}{\partial_t}.$$

The only non-trivial relation is $[X_i, Y_i] = -2T$. Here the horizontal metric h is the one that makes $\{X_i, Y_i : i = 1, \cdots, n\}$ an orthonormal basis of $\mathcal{H} = \ker(\omega_0)$.

Let Ω be an open set of $\mathbb{R}^{2n}$ and $u : \Omega \to \mathbb{R}$ be a function of class C^1. When we consider a graph $\Sigma = \mathrm{Graph}(u)$ given by the zero set level of the C^1 function

$$f(x_1, y_1, \ldots, x_n, y_n, t) = u(x_1, y_1, \ldots, x_n, y_n) - t = 0,$$

the unit tangent N normal to Σ is

$$N = \frac{\sum_{i=1}^{n} (u_{x_i} - y_i) X_i + (u_{y_i} + x_i) Y_i - T}{\sqrt{1 + \sum_{i=1}^{n} (u_{x_i} - y_i)^2 + (u_{y_i} + x_i)^2}}.$$

Therefore the projection of N onto the horizontal distribution is given by

$$N_h = \frac{\sum_{i=1}^{n} (u_{x_i} - y_i) X_i + (u_{y_i} + x_i) Y_i}{\sqrt{1 + \sum_{i=1}^{n} (u_{x_i} - y_i)^2 + (u_{y_i} + x_i)^2}}.$$

Then, setting the horizontal metric so that X_i, Y_i are orthonormal we have the expression

$$A_{2n+1}(\Sigma \setminus \Sigma_0, \lambda) = \int_\Omega \left(\sum_{i=1}^{n} (u_{x_i} - y_i)^2 \ + (u_{y_i} + x_i)^2 \right)^{\frac{1}{2}} d\mathcal{L}, \tag{3.2.2}$$

where $\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^{2n}$. This is exactly the area formula independently established in recent years, see for instance [28, 18, 19, 92, 53]. This formula is valid for any set $\Omega \subset \Sigma$ since $A_{2n+1}(\Sigma_0) = 0$.

Remark 3.2.2. In general when we fix a metric g we can always consider an orthonormal basis $(e_1, \dots, e_m)$ of T_pM, which is also adapted to the flag (1.2.10). However, it is not always possible to extend this basis $(e_1, \cdots, e_m)$ to an orthonormal adapted basis of T_pN, unless T_pM is contained in $\mathcal{H}_p^i$ with $i < s$. For instance, if we consider a surface Σ immersed in the Heisenberg group $(\mathbb{H}^1, g)$ where a basis of orthonormal left invariant vector fields is given by

$$X = \partial_x - \frac{y}{2}\partial_t, \quad Y = \partial_y + \frac{x}{2}\partial_t, \quad T = \partial_t.$$

Let N be unit normal vector to Σ for g and $N_h = N - \langle N, T \rangle T$ its projection onto $\mathcal{H}$. In the regular part $\Sigma \smallsetminus \Sigma_0$ the horizontal Gauss map ν_h and the characteristic vector field Z are defined by

$$\nu_h = \frac{N_h}{|N_h|}, \quad Z = J(\nu_h), \tag{3.2.3}$$

where $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Clearly Z is horizontal and orthogonal to ν_h then it is tangent to Σ.

If we define

$$S = \langle N, T \rangle \nu_h - |N_h| T, \tag{3.2.4}$$

then (Z_p, S_p) is an orthogonal basis of $T_p\Sigma$ and it is adapted to the flag $\mathcal{H}_p^1 \cap T_p\Sigma \subset \mathcal{H}_p^2 \cap T_p\Sigma$ for each p in $\Sigma \smallsetminus \Sigma_0$. The only way to extend (Z_p, S_p) to an orthogonal basis of $T_p\mathbb{H}^1$ is to add N_p. Obviously (Z_p, S_p, N_p) is orthonormal basis of $\mathbb{H}^1$ but it is not adapted to the flag $\mathcal{H}_p^1 \subset \mathcal{H}_p^2$. Notice that an adapted basis that is also an extension of (Z, S) is for instance given by (ν_p, Z, S), which is not an orthogonal basis.

Example 3.2.3 (The roto-translational group)**.** Take coordinates (x, y, θ) in the 3-dimensional manifold $\mathbb{R}^2 \times \mathbb{S}^1$. We consider the contact form

$$\omega = \sin(\theta)dx - \cos(\theta)dy,$$

the horizontal distribution $\mathcal{H} = \ker(\omega)$, is spanned by the vector fields

$$X = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad Y = \partial_\theta,$$

and the horizontal metric h that makes X and Y orthonormal.

Therefore $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with this one form ω is a contact manifold. Moreover $(\mathbb{R}^2 \times \mathbb{S}^1, \mathcal{H}, h)$ has a sub-Riemannian structure which is also a Lie group known as the roto-translational group. A mathematical model of simple cells of the visual cortex V1 using the sub-Riemannian geometry of the roto-translational Lie group was proposed by Citti and Sarti (see [26], [27]).

Here the Reeb vector field is given by

$$T = [X, Y] = \sin(\theta)\partial_x - \cos(\theta)\partial_y.$$

Let Ω be an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a function of class C^1. When we consider a graph $\Sigma = \mathrm{Graph}(u)$ given by the zero set level of the C^1 function

$$f(x, y, \theta) = u(x, y) - \theta = 0,$$

the unit normal N to Σ is given by

$$N = \frac{X(u)X - Y + T(u)T}{\sqrt{1 + X(u)^2 + T(u)^2}}.$$

Therefore the projection of N onto the horizontal distribution is given by

$$N_h = \frac{X(u)X - Y}{\sqrt{1 + X(u)^2 + T(u)^2}}.$$

Hence the 3-area functional is given by

$$A_3(\Sigma \setminus \Sigma_0, \lambda) = \int_\Omega \left(1 + X(u)^2\right)^{\frac{1}{2}} \, dxdy.$$

3.2.3 A_4-area of a ruled surface immersed in an Engel structure

Let $E = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ be a smooth manifold with coordinates $p = (x, y, \theta, k)$. We set $\mathcal{H} = \mathrm{span}\{X_1, X_2\}$, where

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y + k\partial_\theta \quad \text{and} \quad X_2 = \partial_k. \tag{3.2.5}$$

Therefore $(E, \mathcal{H})$ is a Carnot manifold, indeed $\mathcal{H}$ satisfy Hörmander's rank condition since

$$\begin{aligned} X_3 &= [X_1, X_2] = -\partial_\theta \\ X_4 &= [X_1, [X_1, X_2]] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y. \end{aligned} \tag{3.2.6}$$

and, setting $X_i = \sum_{j=1}^4 A_i^j(p)\partial_j$, we have $\det(A(p)) \neq 0$ where

$$A(p) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & k & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \end{pmatrix},$$

and

$$(\partial_x, \partial_y, \partial_\theta, \partial_k) = (\partial_1, \partial_2, \partial_3, \partial_4).$$

The sub-bundle $\mathcal{H}^2$ is generated by $\mathcal{H}$ and the vector field $X_3 = [X_1, X_2] = -\partial_\theta$. Finally, the sub-bundle $\mathcal{H}^3 = TN$ is generated by $\mathcal{H}^2$ and the vector field $X_4 = [X_1, X_3] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y$. Therefore, $(X_1, \ldots, X_4)$ is an adapted basis to the flag $\mathcal{H} \subset \mathcal{H}^2 \subset \mathcal{H}^3 = TE$.

To define a sub-Riemannian structure we need an inner product on the distribution $\mathcal{H}$. In the present work we will use two different metrics on the distribution $\mathcal{H}$:

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.7}$$

the one which makes X_1 and X_2 orthonormal and

$$h_2 = \begin{pmatrix} 1+k^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.8}$$

the one induced by the Euclidean metric. Therefore, $(E, \mathcal{H}, h_1)$ and $(E, \mathcal{H}, h_2)$ are sub-Riemannian manifolds. We can write the canonical basis respect to $X_1, \cdots, X_4$

$$\partial_i = \sum_{j=1}^4 (A(p)^{-1})_i^j X_j, \tag{3.2.9}$$

where

$$A(p)^{-1} = \begin{pmatrix} \cos(\theta) & 0 & k\cos(\theta) & -\sin(\theta) \\ \sin(\theta) & 0 & k\sin(\theta) & \cos(\theta) \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Here we provide a similar computation to the one developed by Le Donne and Magnani in [62] in the Engel group. Since the Engel group is the tangent space to E these computations are morally equivalent. Let Ω be an open set of $\mathbb{R}^2$ endowed with the

Lebesgue measure. We denote by (u_1, u_2) a point in Ω. Then we consider an embedding $\Phi : \Omega \to E$, where we set $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4)$ and $\Sigma = \Phi(\Omega)$. The tangent vector to Σ are $\Phi_{u_i} = \sum_{j=1}^4 \Phi^j_{u_i} e_j$, $i = 1, 2$. By equation (3.2.9) we obtain

$$\begin{aligned}
\Phi_{u_i} &= \sum_{j=1}^{4} \Phi^j_{u_i} \sum_{k=1}^{4} (A(p)^{-1})^k_j X_k \\
&= \Phi^1_{u_i}(\cos(\theta)X_1 + k\cos(\theta)X_3 - \sin(\theta)X_4) \\
&\quad + \Phi^2_{u_i}(\sin(\theta)X_1 + k\sin(\theta)X_3 + \cos(\theta)X_4) \\
&\quad - \Phi^3_{u_i}X_3 + \Phi^4_{u_i}X_2 \\
&= (\cos(\Phi^3)\Phi^1_{u_i} + \sin(\Phi^3)\Phi^2_{u_i})X_1 + \Phi^4_{u_i}X_2 + (\Phi^4(\cos(\Phi^3)\Phi^1_{u_i} + \sin(\Phi^3)\Phi^2_{u_i}) \\
&\quad - \Phi^3_{u_i})X_3 + (-\sin(\Phi^3)\Phi^1_{u_i} + \cos(\Phi^3)\Phi^2_{u_i})X_4.
\end{aligned}$$

Computing the wedge product, it follows

$$\begin{aligned}
\Phi_x \wedge \Phi_y = (\cos(\Phi^3)\Phi^{14}_u &+ \sin(\Phi^3)\Phi^{24}_u)X_1 \wedge X_2 \\
&- (\cos(\Phi^3)\Phi^{13}_u + \sin(\Phi^3)\Phi^{23}_u)X_1 \wedge X_3 \\
&+ \Phi^{12}_u X_1 \wedge X_4 \\
&+ (\Phi^{34}_u - \Phi^4(\cos(\Phi^3)\Phi^{14}_u + \sin(\Phi^3)\Phi^{24}_u))X_2 \wedge X_3 \\
&+ (\sin(\Phi^3)\Phi^{14}_u - \cos(\Phi^3)\Phi^{24}_u)X_2 \wedge X_4 \\
&+ (\Phi^4\Phi^{12}_u - \sin(\Phi^3)\Phi^{13}_u + \cos(\Phi^3)\Phi^{23}_u)X_3 \wedge X_4,
\end{aligned} \tag{3.2.10}$$

where we set

$$\Phi^{ij}_u = \det \begin{pmatrix} \Phi^i_x & \Phi^i_y \\ \Phi^j_x & \Phi^j_y \end{pmatrix}.$$

According to the notion of pointwise degree, we have that

$$\deg_\Sigma(\Phi(u)) = \begin{cases} 5 & \text{if } c_{34}(u) \neq 0 \\ 4 & \text{if } |c_{14}(u)| + |c_{24}(u)| > 0 \quad \text{and} \quad c_{34}(u) = 0 \\ 3 & \text{if } |c_{13}(u)| + |c_{23}(u)| > 0 \quad \text{and} \quad c_{34}(u) = c_{14}(u) = c_{24}(u) = 0 \\ 2 & \text{if } c_{34}(u) = c_{14}(u) = c_{24}(u) = c_{13}(u) = c_{23}(u) = 0 \end{cases} \tag{3.2.11}$$

where we set

$$\Phi_{u_1} \wedge \Phi_{u_2} = \sum_{1 \leqslant i < j \leqslant 4} c_{ij}(u) X_i \wedge X_j.$$

Notice that the degree of Σ can never be equal to 2. Indeed, if $\deg(\Sigma)$ was equal to 2 the submanifold Σ would be a integrable manifold for the distribution $\mathcal{H}$, then $\mathcal{H}$ would be involutive by Frobenius Theorem. However, the distribution $\mathcal{H}$ is bracket-generating and not involutive. Since we are particularly interested in applications to the visual cortex (see [46],[85, 1.5.1.4] to understand the reasons) we consider the surface $\Sigma = \Phi(\Omega)$ given by $\Phi = (x, y, \theta(x, y), \kappa(x, y))$. The tangent vectors to Σ are

$$\Phi_x = (1, 0, \theta_x, k_x), \qquad \Phi_y = (0, 1, \theta_y, \kappa_y). \tag{3.2.12}$$

In order to know the dimension of $T_p\Sigma \cap \mathcal{H}_p$ it is necessary to take in account the rank of the matrix

$$B = \begin{pmatrix} 1 & 0 & \theta_x & \kappa_x \\ 0 & 1 & \theta_y & \kappa_y \\ \cos(\theta) & \sin(\theta) & \kappa & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2.13}$$

Obviously $\operatorname{rank}(B) \geqslant 3$, indeed we have

$$\det \begin{pmatrix} 1 & 0 & \kappa_x \\ 0 & 1 & \kappa_y \\ 0 & 0 & 1 \end{pmatrix} \neq 0.$$

Moreover, it holds

$$\begin{aligned} \operatorname{rank}(B) = 3 \quad &\Leftrightarrow \quad \det \begin{pmatrix} \cos(\theta) & \sin(\theta) & \kappa \\ 1 & 0 & \theta_x \\ 0 & 1 & \theta_y \end{pmatrix} = 0 \\ &\Leftrightarrow \quad \kappa - \theta_x \cos(\theta) - \theta_y \sin(\theta) = 0 \\ &\Leftrightarrow \quad \kappa = X_1(\theta(x, y)). \end{aligned} \tag{3.2.14}$$

Since we are inspired by the foliation property of hypersurface in roto-translational group and the lifting of a retinal $2D$ image to the cortex Engel space E enjoys (3.2.14), in the present work we consider only surface $\Sigma = \{(x, y, \theta(x, y), \kappa(x, y))\}$ verifying the foliation condition $\kappa = X_1(\theta(x,y))$. Thus, thanks to (3.2.10), we have

$$\begin{aligned}\Phi_x \wedge \Phi_y =&(\cos(\theta)\kappa_y - \sin(\theta)\kappa_x)X_1 \wedge X_2 - (\cos(\theta)\theta_y - \sin(\theta)\theta_x)X_1 \wedge X_3 \\ &+ X_1 \wedge X_4 + (\theta_x\kappa_y - \theta_y\kappa_x - \kappa(\cos(\theta)\kappa_y - \sin(\theta)\kappa_x))X_2 \wedge X_3 \\ &+ (\sin(\theta)\kappa_y + \cos(\theta)\kappa_x)X_2 \wedge X_4 \\ &+ (\kappa - \sin(\theta)\theta_y - \cos(\theta)\theta_x)X_3 \wedge X_4.\end{aligned} \tag{3.2.15}$$

By the foliation condition (3.2.14) we have that the coefficient of $X_3 \wedge X_4$ is always equal to zero, then we deduce that $\deg(\Sigma) \leqslant 4$. Moreover, the coefficient of $X_1 \wedge X_4$ never vanishes, therefore $\deg(\Sigma) = 4$ and there are not singular points in Σ. When $\kappa = X_1(\theta)$ a tangent basis of $T_p\Sigma$ adapted to 1.2.10 is given by

$$\begin{aligned}e_1 &= \cos(\theta)\Phi_x + \sin(\theta)\Phi_y = X_1 + X_1(\kappa)X_2, \\ e_2 &= -\sin(\theta)\Phi_x + \cos(\theta)\Phi_y = X_4 - X_4(\theta)X_3 + X_4(\kappa)X_2.\end{aligned} \tag{3.2.16}$$

When we fix the Riemannian metric g_1 that makes $(X_1, \ldots, X_4)$ we have that the A_4-area of Σ is given by

$$A_4(\Sigma, g) = \int_\Omega \left(1 + X_1(\kappa)^2\right)^{\frac{1}{2}} \, dxdy = \int_\Omega \left(1 + X_1^2(\theta)^2\right)^{\frac{1}{2}} \, dxdy. \tag{3.2.17}$$

When we fix the Euclidean metric g_0 that makes $(\partial_1, \partial_2, \partial_\theta, \partial_k)$ we have that the A_4-area of Σ is given by

$$A_4(\Sigma, g_0) = \int_\Omega \left(1 + \kappa^2 + X_1(\kappa)^2\right)^{\frac{1}{2}} \, dxdy. \tag{3.2.18}$$

Notice that g_1 restricted to the distribution is equal to h_1 and $g_0|_{\mathcal{H}}$ is equal to h_2.

3.3 Admissible variations for submanifolds

Let us consider an m-dimensional manifold $\bar{M}$ and an immersion $\Phi : \bar{M} \to N$ into an equiregular graded manifold endowed with a Riemannian metric $g = \langle \cdot, \cdot \rangle$. We shall denote the image $\Phi(\bar{M})$ by M and $d := \deg(M)$. In this setting we have the following definition

Definition 3.3.1. A smooth map $\Gamma : \bar{M} \times (-\varepsilon, \varepsilon) \to N$ is said to be *an admissible variation* of Φ if $\Gamma_t : \bar{M} \to N$, defined by $\Gamma_t(\bar{p}) := \Gamma(\bar{p}, t)$, satisfies the following properties

(i) $\Gamma_0 = \Phi$,

(ii) $\Gamma_t(\bar{M})$ is an immersion of the same degree as $\Phi(\bar{M})$ for small enough t, and

(iii) $\Gamma_t(\bar{p}) = \Phi(\bar{p})$ for $\bar{p}$ outside a given compact subset of $\bar{M}$.

Definition 3.3.2. Given an admissible variation Γ, the *associated variational vector field* is defined by

$$V(\bar{p}) := \frac{\partial \Gamma}{\partial t}(\bar{p}, 0). \tag{3.3.1}$$

The vector field V is an element of $\mathfrak{X}_0(\bar{M}, N)$: i.e., a smooth map $V : \bar{M} \to TN$ such that $V(\bar{p}) \in T_{\Phi(\bar{p})}N$ for all $\bar{p} \in \bar{M}$. It is equal to 0 outside a compact subset of $\bar{M}$.

Let us see now that the variational vector field V associated to an admissible variation Γ satisfies a differential equation of first order. Let $p = \Phi(\bar{p})$ for some $\bar{p} \in \bar{M}$, and $(X_1, \cdots, X_n)$ an adapted frame in a neighborhood U of p. Take a basis $(\bar{e}_1, \ldots, \bar{e}_m)$ of $T_{\bar{p}}\bar{M}$ and let $e_j = d\Phi_{\bar{p}}(\bar{e}_j)$ for $1 \leqslant j \leqslant m$. As $\Gamma_t(\bar{M})$ is a submanifold of the same degree as $\Phi(\bar{M})$ for small t, there follows

$$\Big\langle (d\Gamma_t)_{\bar{p}}(e_1) \wedge \ldots \wedge (d\Gamma_t)_{\bar{p}}(e_m), (X_J)_{\Gamma_t(\bar{p})} \Big\rangle = 0, \tag{3.3.2}$$

for all $X_J = X_{j_1} \wedge \ldots \wedge X_{j_m}$, with $1 \leqslant j_1 < \cdots < j_m \leqslant n$, such that $\deg(X_J) > \deg(M)$. Taking derivative with respect to t in equality (3.3.2) and evaluating at $t = 0$ we obtain the condition

$$0 = \langle e_1 \wedge \ldots \wedge e_m, \nabla_{V(p)} X_J \rangle + \sum_{k=1}^{m} \langle e_1 \wedge \ldots \wedge \nabla_{e_k} V \wedge \ldots \wedge e_m, X_J \rangle$$

for all X_J such that $\deg(X_J) > \deg(M)$. In the above formula, $\langle \cdot, \cdot \rangle$ indicates the scalar product in the space of m-vectors induced by the Riemannian metric g. The symbol ∇ denotes, in the left summand, the Levi-Civita connection associated to g and, in the right summand, the covariant derivative of vectors in $\mathfrak{X}(\bar{M}, N)$ induced by g. Thus, if a variation preserves the degree then the associated variational vector field satisfies the above condition and we are led to the following definition.

Definition 3.3.3. Given an immersion $\Phi : \bar{M} \to N$, a vector field $V \in \mathfrak{X}_0(\bar{M}, N)$ is said to be *admissible* if it satisfies the system of first order PDEs

$$0 = \langle e_1 \wedge \ldots \wedge e_m, \nabla_{V(p)} X_J \rangle + \sum_{k=1}^{m} \langle e_1 \wedge \ldots \wedge \nabla_{e_k} V \wedge \ldots \wedge e_m, X_J \rangle \tag{3.3.3}$$

where $X_J = X_{j_1} \wedge \ldots \wedge X_{j_m}$, $\deg(X_J) > d$ and $p \in M$. We denote by $\mathcal{A}_\Phi(\bar{M}, N)$ the set of admissible vector fields.

It is not difficult to check that the conditions given by (3.3.3) are independent of the choice of the adapted basis.

Thus we are led naturally to a problem of integrability: given $V \in \mathfrak{X}_0(\bar{M}, N)$ such that the first order condition (3.3.3) holds, we ask whether an admissible variation whose associated variational vector field is V exists.

Definition 3.3.4. We say that an admissible vector field $V \in \mathfrak{X}_0(\bar{M}, N)$ is *integrable* if there exists an admissible variation such that the associated variational vector field is V.

Proposition 3.3.5. *Let $\Phi : \bar{M} \to N$ be an immersion into a graded manifold. Then a vector field $V \in \mathfrak{X}_0(\bar{M}, N)$ is admissible if and only if its normal component $V^\perp$ is admissible.*

Proof. Since the Levi-Civita connection and the covariant derivative are additive we deduce that the admissibility condition (3.3.3) is additive in V. We decompose $V = V^\top + V^\perp$ in its tangent $V^\top$ and normal $V^\perp$ components and observe that $V^\top$ is always admissible since the flow of $V^\top$ is an admissible variation leaving $\Phi(\bar{M})$ invariant with variational vector field $V^\top$. Hence, $V^\perp$ satisfies (3.3.3) if and only if V verifies (3.3.3). □

3.4 The structure of the admissibility system of first order PDEs

Let us consider an open set $U \subset N$ where a local adapted basis $(X_1, \ldots, X_n)$ is defined. We know that the simple m-vectors $X_J := X_{j_1} \wedge \ldots \wedge X_{j_m}$ generate the space $\Lambda_m(U)$ of m-vectors. At a given point $p \in U$, its dimension is given by the formula

$$\dim(\Lambda_m(U)_p) = \binom{n}{m}.$$

Given two m-vectors $v, w \in \Lambda_m(U)_p$, it is easy to check that $\deg(v + w) \leqslant \max\{\deg v, \deg w\}$, and that $\deg \lambda v = \deg v$ when $\lambda \neq 0$ and 0 otherwise. This implies that the set

$$\Lambda_m^d(U)_p := \{v \in \Lambda_m(U)_p : \deg v \leqslant d\}$$

is a vector subspace of $\Lambda_m(U)_p$. To compute its dimension we let $v_i := (X_i)_p$ and we check that a basis of $\Lambda_m^d(U)_p$ is composed of the vectors

$$v_{i_1} \wedge \ldots \wedge v_{i_m} \text{ such that } \sum_{j=i_1}^{i_m} \deg(v_j) \leqslant d.$$

To get an m-vector in such a basis we pick any of the k_1 vectors in $\mathcal{H}_p^1 \cap \{v_1, \ldots, v_n\}$ and, for $j = 2, \ldots, s$, we pick any of the k_j vectors on $(\mathcal{H}_p^j \smallsetminus \mathcal{H}_p^{j-1}) \cap \{v_1, \ldots, v_n\}$, so that

- $k_1 + \cdots + k_s = m$, and
- $1 \cdot k_1 + \cdots + s \cdot k_s \leqslant d$.

So we conclude, taking $n_0 = 0$, that

$$\dim(\Lambda_m^d(U)_p) = \sum_{\substack{k_1+\cdots+k_s=m,\\ 1\cdot k_1+\cdots+s\cdot k_s\leqslant d}} \left(\prod_{i=1}^{s} \binom{n_i - n_{i-1}}{k_i} \right).$$

When we consider two simple m-vectors $v_{i_1} \wedge \ldots \wedge v_{i_m}$ and $v_{j_1} \wedge \ldots \wedge v_{j_m}$, their scalar product is 0 or ± 1, the latter case when, after reordering if necessary, we have $v_{i_k} = v_{j_k}$ for $k = 1, \ldots, m$. This implies that the orthogonal subspace $\Lambda_m^d(U)_p^\perp$ of $\Lambda_m^d(U)_p$ in $\Lambda_m(U)_p$ is generated by the m-vectors

$$v_{i_1} \wedge \ldots \wedge v_{i_m} \text{ such that } \sum_{j=i_1}^{i_m} \deg(v_j) > d.$$

Hence we have

$$\dim(\Lambda_m^d(U)_p^\perp) = \sum_{\substack{k_1+\cdots+k_s=m,\\ 1\cdot k_1+\cdots+s\cdot k_s> d}} \left(\prod_{i=1}^{s} \binom{n_i - n_{i-1}}{k_i} \right), \tag{3.4.1}$$

with $n_0 = 0$. Since $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ is equiregular, $\ell = \dim(\Lambda_m^d(U)_p^\perp)$ is constant on N. Then we can choose an orthonormal basis $(X_{J_1}, \ldots, X_{J_\ell})$ in $\Lambda_m^d(U)_p^\perp$ at each point $p \in U$.

3.4.1 The admissibility system with respect to an adapted local basis

In the same conditions as in the previous subsection, let $\ell = \dim(\Lambda^d_m(U)^\perp_p)$ and $(X_{J_1}, \ldots, X_{J_\ell})$ an orthonormal basis of $\Lambda^d_m(U)^\perp_p$. Any vector field $V \in \mathfrak{X}(\bar{M}, N)$ can be expressed in the form

$$V = \sum_{h=1}^{n} f_h X_h,$$

where $f_1, \ldots, f_n \in C^\infty(\Phi^{-1}(U), \mathbb{R})$. We take $\bar{p}_0 \in \Phi^{-1}(U)$ and, reducing U if necessary, a local adapted basis $(E_i)_i$ of $T\bar{M}$ in $\Phi^{-1}(U)$. Hence the admissibility system (3.3.3) is equivalent to

$$\sum_{j=1}^{m}\sum_{h=1}^{n} c_{ijh}\, E_j(f_h) + \sum_{h=1}^{n} \beta_{ih}\, f_h = 0, \quad i = 1, \ldots, \ell, \tag{3.4.2}$$

where

$$c_{ijh}(\bar{p}) = \langle e_1 \wedge \ldots \wedge \overset{(j)}{(X_h)_p} \wedge \ldots \wedge e_m, (X_{J_i})_p\rangle, \tag{3.4.3}$$

and

$$\begin{aligned} \beta_{ih}(\bar{p}) = \langle e_1 \wedge \ldots \wedge e_m, \nabla_{(X_h)_p} X_{J_i}\rangle + \\ + \sum_{j=1}^{m} \langle e_1 \wedge \ldots \wedge \nabla_{e_j} X_h \wedge \ldots \wedge e_m, (X_{J_i})_p\rangle \\ = \sum_{j=1}^{m} \langle e_1 \wedge \ldots \wedge [E_j, X_h](p) \wedge \ldots \wedge e_m, (X_{J_i})_p\rangle. \end{aligned} \tag{3.4.4}$$

In the above equation we have extended the vector fields E_i in a neighborhood of $p_0 = \Phi(\bar{p}_0)$ in N, denoting them in the same way.

Definition 3.4.1. Let $\tilde{m}_\alpha(p)$ be the dimension of $\tilde{\mathcal{H}}^\alpha_p = T_p M \cap \mathcal{H}^\alpha_p$, $\alpha \in \{1, \ldots, s\}$, where we consider the flag defined in (1.2.10). Then we set

$$\iota_0(U) = \max_{p \in U} \min_{1 \leqslant \alpha \leqslant s} \{\alpha : \tilde{m}_\alpha(p) \neq 0\} \tag{3.4.5}$$

and

$$\rho := n_{\iota_0} = \dim(\mathcal{H}^{\iota_0}) \geqslant \dim(\mathcal{H}^1) = n_1. \tag{3.4.6}$$

The integer number $\iota_0(U)$ is the ambient degree of the first sub-bundle the induced filtration $(\tilde{\mathcal{H}}^\alpha)_{\alpha=1,\ldots,s}$ and ρ is the dimension of the corresponding ambient sub-bundle $\mathcal{H}^{\iota_0}$.

Remark 3.4.2. In the differential system (3.4.2), derivatives of the function f_h appear only when some coefficient $c_{ijh}(\bar{p})$ is different from 0. For fixed h, notice that $c_{ijh}(\bar{p}) = 0$, for all $i = 1, \ldots, \ell$, $j = 1, \ldots, m$ and $\bar{p}$ in $\Phi^{-1}(U)$ if and only if

$$\deg(e_1 \wedge \cdots \wedge \overset{(j)}{(X_h)_p} \wedge \cdots \wedge e_m) \leqslant d, \quad \text{for all } 1 \leqslant j \leqslant m,\, p \in \Phi^{-1}(U).$$

This property is equivalent to

$$\deg((X_h)_p) \leqslant \deg(e_j), \text{for all } 1 \leqslant j \leqslant m, p \in \Phi^{-1}(U).$$

So we have $c_{ijh} = 0$ in $\Phi^{-1}(U)$ for all i, j if and only if $\deg(X_h) \leqslant \iota_0(U)$.

We write

$$V = \sum_{h=1}^{\rho} g_h X_h + \sum_{r=\rho+1}^{n} f_r X_r,$$

so that the local system (3.4.2) can be written as

$$\sum_{j=1}^{m} \sum_{r=\rho+1}^{n} c_{ijr} E_j(f_r) + \sum_{r=\rho+1}^{n} b_{ir} f_r + \sum_{h=1}^{\rho} a_{ih} g_h = 0, \tag{3.4.7}$$

where c_{ijr} is defined in (3.4.3) and, for $1 \leqslant i \leqslant \ell$,

$$a_{ih} = \beta_{ih}, \quad b_{ir} = \beta_{ir},\ 1 \leqslant h \leqslant \rho,\ \rho + 1 \leqslant r \leqslant n, \tag{3.4.8}$$

where β_{ij} is defined in (3.4.4). We denote by B the $\ell \times (n - \rho)$ matrix whose entries are b_{ir}, by A the $\ell \times \rho$ whose entries are a_{ih} and for $j = 1, \ldots, m$ we denote by C_j the $\ell \times (n - \rho)$ matrix $C_j = (c_{ijh})^{i=1,\ldots,\ell}_{h=\rho+1,\ldots,n}$. Setting

$$F = \begin{pmatrix} f_{\rho+1} \\ \vdots \\ f_n \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ \vdots \\ g_\rho \end{pmatrix} \tag{3.4.9}$$

the admissibility system (3.4.7) is given by

$$\sum_{j=1}^{m} C_j E_j(F) + BF + AG = 0. \tag{3.4.10}$$

3.4.2 Independence on the metric

Let g and $\tilde{g}$ be two Riemannian metrics on N and (X_i) be orthonormal adapted basis with respect to g and (Y_i) with respect to $\tilde{g}$. Clearly we have

$$Y_i = \sum_{j=1}^{n} d_{ji} X_j,$$

for some square invertible matrix $D = (d_{ji})_{j=1,\ldots,n}^{i=1,\ldots,n}$ of order n. Since (X_i) and (Y_i) are adapted basis, D is a block matrix

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \ldots & D_{1s} \\ 0 & D_{22} & D_{23} & \ldots & D_{2s} \\ 0 & 0 & D_{33} & \ldots & D_{3s} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & D_{ss} \end{pmatrix},$$

where D_{ii} for $i = 1, \ldots, s$ are square matrices of orders n_i. Let ρ be the integer defined in (3.4.1), then we define $D_h = (d_{ji})_{i,j=1,\ldots,\rho}$, $D_v = (d_{ji})_{i,j=\rho+1,\ldots,n}$ and $D_{hv} = (d_{ji})_{j=1,\ldots,\rho}^{i=\rho+1,\ldots,n}$. Let us express V as a linear combination of (Y_i)

$$V = \sum_{h=1}^{\rho} \tilde{g}_h Y_h + \sum_{r=\rho+1}^{n} \tilde{f}_r Y_r,$$

then we set

$$\tilde{F} = \begin{pmatrix} \tilde{f}_{\rho+1} \\ \vdots \\ \tilde{f}_n \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_\rho \end{pmatrix}$$

and F and G as in (3.7.5).

Given $I = (i_1, \ldots, i_m)$ with $i_1 < \ldots < i_m$, we have

$$\begin{aligned} Y_I = Y_{i_1} \wedge \cdots \wedge Y_{i_m} &= \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} d_{j_1 i_1} \cdots d_{j_m i_m} X_{j_1} \wedge \cdots \wedge X_{j_m} \\ &= \sum_{j_1<\ldots<j_m} \lambda_{i_1\ldots i_m}^{j_1\ldots j_m} X_{j_1} \wedge \cdots \wedge X_{j_m} = \sum_J \lambda_{JI} X_J. \end{aligned}$$

Since the adapted change of basis preserves the degree of the m-vectors, the square matrix $\mathbf{\Lambda} = (\lambda_{JI})$ of order $\binom{n}{m}$ acting on the m-vector is given by

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_h & \mathbf{\Lambda}_{hv} \\ 0 & \mathbf{\Lambda}_v, \end{pmatrix} \tag{3.4.11}$$

where $\mathbf{\Lambda}_h$ and $\mathbf{\Lambda}_v$ are square matrices of order $\binom{n}{m} - \ell$ and ℓ respectively and $\mathbf{\Lambda}_{hv}$ is a matrix of order $\left(\binom{n}{m} - \ell\right) \times \ell$. Moreover the matrix $\mathbf{\Lambda}$ is invertible since both $\{X_J\}$ and $\{Y_I\}$ are basis of the vector space of m-vectors.

Remark 3.4.3. One can easily check that the inverse of $\mathbf{\Lambda}$ is given by the block matrix

$$\mathbf{\Lambda}^{-1} = \begin{pmatrix} \mathbf{\Lambda}_h^{-1} & -\mathbf{\Lambda}_h^{-1}\mathbf{\Lambda}_{hv}\mathbf{\Lambda}_v^{-1} \\ 0 & \mathbf{\Lambda}_v^{-1} \end{pmatrix}.$$

Setting $\tilde{\mathbf{G}} = (\tilde{g}(X_I, X_J))$ we have

$$\tilde{\mathbf{G}} = \begin{pmatrix} \tilde{\mathbf{G}}_h & \tilde{\mathbf{G}}_{hv} \\ (\tilde{\mathbf{G}}_{hv})^t & \tilde{\mathbf{G}}_v \end{pmatrix} = (\mathbf{\Lambda}^{-1})^t(\mathbf{\Lambda}^{-1}).$$

Thus it follows

$$\begin{aligned} \tilde{\mathbf{G}}_v &= (\mathbf{\Lambda}_v^{-1})^t\mathbf{\Lambda}_v^{-1} + (\mathbf{\Lambda}_v^{-1})^t\mathbf{\Lambda}_{hv}^t(\mathbf{\Lambda}_h^{-1})^t\,\mathbf{\Lambda}_h^{-1}\mathbf{\Lambda}_{hv}\mathbf{\Lambda}_v^{-1}, \\ \tilde{\mathbf{G}}_{hv} &= -(\mathbf{\Lambda}_h^{-1})^t\mathbf{\Lambda}_h^{-1}\mathbf{\Lambda}_{hv}\mathbf{\Lambda}_v^{-1}, \\ \tilde{\mathbf{G}}_h &= (\mathbf{\Lambda}_h^{-1})^t\mathbf{\Lambda}_h^{-1}. \end{aligned}$$

Let $\tilde{A}$ be the associated matrix

$$\tilde{A} = \left(\tilde{g}\Big(Y_{J_i}, \sum_{j=1}^{m} E_1 \wedge \ldots \wedge [E_j, Y_h](p) \wedge \ldots \wedge E_m\Big)\right)_{i=1,\ldots,\ell}^{h=1,\ldots,\rho}.$$

Setting

$$\omega_{Jr} = \sum_{j=1}^{m} g(X_J, E_1 \wedge \cdots \wedge [E_j, X_r] \wedge \cdots \wedge E_m),$$

and $\Omega = \begin{pmatrix}\Omega_h & \Omega_v\end{pmatrix} = (\omega_{Jr})^{r=1,\dots,n}_{\deg(J)\leqslant d}$, a straightforward computation shows

$$\begin{aligned}\tilde{A} =&(\mathbf{\Lambda}_{hv})^t\left(\tilde{\mathbf{G}}_h\,\Omega_h\,D_h + \tilde{\mathbf{G}}_{hv}\,A\,D_h + \tilde{\mathbf{G}}_h\sum_{j=1}^m C_jE_j(D_h)\right)\\ &+(\mathbf{\Lambda}_v)^t\left((\tilde{\mathbf{G}}_{hv})^t\,\Omega_hD_h + \tilde{\mathbf{G}}_v\,A\,D_h + (\tilde{\mathbf{G}}_{hv})^t\sum_{j=1}^m C_jE_j(D_h)\right)\end{aligned}$$

By Remark 3.4.3 we obtain

$$\begin{aligned}\tilde{A} =&(\mathbf{\Lambda}_{hv})^t\Big((\mathbf{\Lambda}_h^{-1})^t\mathbf{\Lambda}_h^{-1}\,(\Omega_h\,D_h + \sum_{j=1}^m C_jE_j(D_h))\\ &\quad-(\mathbf{\Lambda}_h^{-1})^t\mathbf{\Lambda}_h^{-1}\mathbf{\Lambda}_{hv}\mathbf{\Lambda}_v^{-1}\,A\,D_h\Big)\\ &\quad-\left(\mathbf{\Lambda}_{hv}^t(\mathbf{\Lambda}_h^{-1})^t\mathbf{\Lambda}_h^{-1}\,(\Omega_hD_h + \sum_{j=1}^m C_jE_j(D_h))\right)\\ &\quad+\left(\mathbf{\Lambda}_v^{-1} + \mathbf{\Lambda}_{hv}^t(\mathbf{\Lambda}_h^{-1})^t\,\mathbf{\Lambda}_h^{-1}\mathbf{\Lambda}_{hv}\mathbf{\Lambda}_v^{-1}\right)\,A\,D_h\\ =&\mathbf{\Lambda}_v^{-1}\,A\,D_h.\end{aligned} \tag{3.4.12}$$

First we notice that if $h = 1,\dots,\rho$ we have

$$\begin{aligned}\tilde{c}_{ijh} &= \tilde{g}(Y_{J_i}, E_1\wedge\dots\wedge \overset{(j)}{Y_h}\wedge\dots\wedge E_m)\\ &=\sum_I\sum_{deg(J)\leqslant d}\sum_{k=1}^{\rho}\lambda_{IJ_i}\,\tilde{g}(X_I,X_J)c_{Jjk}\,d_{kh}\\ &=\sum_{\deg(I)\leqslant d}\sum_{\deg(J)\leqslant d}\sum_{k=1}^{\rho}\lambda_{IJ_i}\,\tilde{g}(X_I,X_J)c_{Jjk}\,d_{kh}+\\ &\quad+\sum_{\deg(I)>d}\sum_{\deg(J)\leqslant d}\sum_{k=1}^{\rho}\lambda_{IJ_i}\,\tilde{g}(X_I,X_J)c_{Jjk}\,d_{kh}.\end{aligned} \tag{3.4.13}$$

Therefore, setting

$$\tilde{C}_j^H = \left(\tilde{g}(Y_J, E_1\wedge\dots\wedge\overset{(j)}{Y_h}\wedge\dots\wedge E_m)\right)^{h=1,\dots,\rho}_{\deg(J)\leqslant d}$$

and

$$\tilde{C}_j^0 = \left(\tilde{g}(Y_{J_i}, E_1\wedge\dots\wedge\overset{(j)}{Y_h}\wedge\dots\wedge E_m)\right)^{h=1,\dots,\rho}_{i=1,\dots,\ell},$$

by (3.4.13) we gain

$$\tilde{C}_j^0 = (\mathbf{\Lambda}_{hv}^t \tilde{\mathbf{G}}_h + \mathbf{\Lambda}_v^t (\tilde{\mathbf{G}}_{hv})^t)(C_j^H D_h) = 0.$$

Let $\tilde{C}_j$ be the associated matrix

$$\tilde{C}_j = \Big(\tilde{g}(Y_{J_i}, E_1 \wedge \ldots \wedge \overset{(j)}{Y_h} \wedge \ldots \wedge E_m) \Big)_{i=1,\ldots,\ell}^{h=\rho+1,\ldots,n}.$$

Setting

$$\tilde{C}_j^{HV} = \Big(\tilde{g}(Y_J, E_1 \wedge \ldots \wedge \overset{(j)}{Y_h} \wedge \ldots \wedge E_m) \Big)_{\deg(J) \leqslant d}^{h=\rho+1,\ldots,n},$$

it is immediate to obtain the following equality

$$\begin{aligned} \tilde{C}_j =& (\mathbf{\Lambda}_{hv})^t \left(\tilde{\mathbf{G}}_h (C_j^H D_{hv} + C_j^{HV} D_v) + \tilde{\mathbf{G}}_{hv} C_j D_v \right) \\ &+ (\mathbf{\Lambda}_v)^t \left((\tilde{\mathbf{G}}_{hv})^t (C_j^H D_{hv} + C_j^{HV} D_v) + \tilde{\mathbf{G}}_v C_j D_v \right) \\ =& \mathbf{\Lambda}_v^{-1} C_j D_v. \end{aligned} \tag{3.4.14}$$

Let $\tilde{B}$ be the associated matrix

$$\tilde{B} = \left(\tilde{g}\Big(Y_{J_i}, \sum_{j=1}^m E_1 \wedge \ldots \wedge [E_j, Y_h] \wedge \ldots \wedge E_m \Big) \right)_{i=1,\ldots,\ell}^{h=\rho+1,\ldots,n}.$$

A straightforward computation shows

$$\begin{aligned} \tilde{B} =& (\mathbf{\Lambda}_{hv})^t \Big(\tilde{\mathbf{G}}_h (\Omega_h \, D_{hv} + \Omega_v D_v + \sum_{j=1}^m C_j^H E_j(D_{hv}) + C_j^{HV} E_j(D_h)) \\ &+ \tilde{\mathbf{G}}_{hv} (A D_{hv} + B D_v + \sum_{j=1}^m C_j E_j(D_v)) \Big) \\ &+ (\mathbf{\Lambda}_v)^t \Big(\tilde{\mathbf{G}}_{hv}^t (\Omega_h \, D_{hv} + \Omega_v D_v + \sum_{j=1}^m C_j^H E_j(D_{hv}) + C_j^{HV} E_j(D_h)) \\ &+ \tilde{\mathbf{G}}_v (A D_{hv} + B D_v + \sum_{j=1}^m C_j E_j(D_v)) \Big) \end{aligned}$$

By Remark 3.4.3 we obtain

$$\tilde{B} = \mathbf{\Lambda}_v^{-1} \, A \, D_{hv} + \mathbf{\Lambda}_v^{-1} B D_v + \sum_{j=1}^m \mathbf{\Lambda}_v^{-1} C_j E_j(D_v). \tag{3.4.15}$$

Finally, we have $G = D_h \tilde{G} + D_{hv} \tilde{F}$ and $F = D_v \tilde{F}$.

Proposition 3.4.4. *Let g and $\tilde{g}$ be two different metrics, then a vector fields V is admissible w.r.t. g if and only if V is admissible w.r.t. $\tilde{g}$.*

Proof. We remind that an admissible vector field

$$V = \sum_{i=1}^{\rho} g_i X_i + \sum_{i=\rho+1}^{n} f_i X_i$$

w.r.t. g satisfies

$$\sum_{j=1}^{m} C_j E_j(F) + BF + AG = 0. \tag{3.4.16}$$

By (3.4.12), (3.4.15) and (3.4.14) we have

$$\begin{aligned}\sum_{j=1}^{m} \tilde{C}_j E_j(\tilde{F}) + \tilde{B}\tilde{F} + \tilde{A}\tilde{G} &= \mathbf{\Lambda}_v^{-1}\Big(\sum_{j=1}^{m} C_j (D_v E_j(\tilde{F}) + E_j(D_v)\tilde{F}) \\ &+ A\,D_{hv}\tilde{F} + A\,D_h\tilde{G} + BD_v\tilde{F}\Big) = \mathbf{\Lambda}_v^{-1}\Big(\sum_{j=1}^{m} C_j E_j(F) + BF + AG\Big)\end{aligned} \tag{3.4.17}$$

In the previous equation we used that $G = D_h\tilde{G} + D_{hv}\tilde{F}$, $F = D_v\tilde{F}$ and

$$E_j(D_v)D_v^{-1} + D_v E_j(D_v^{-1}) = 0,$$

for all $j = 1, \dots, m$, that follows by $D_v D_v^{-1} = I_{n-\rho}$. Then the admissibility system (3.4.16) w.r.t. g is equal to zero if and only if the admissibility system (3.4.17) w.r.t. $\tilde{g}$. □

Remark 3.4.5. When the metric g is fixed and (X_i) and (Y_i) are orthonormal adapted basis w.r.t g, the matrix D is a block diagonal matrix given by

$$D = \begin{pmatrix} D_h & 0 \\ 0 & D_v \end{pmatrix},$$

where D_h and D_v are square orthogonal matrices of orders ρ and $(n-\rho)$, respectively. From equations (3.4.12), (3.4.15), (3.4.14) it is immediate to obtain the following equalities

$$\begin{aligned}\tilde{F} &= D_v^{-1} F,\\ \tilde{G} &= D_h^{-1} G,\\ \tilde{A} &= \mathbf{\Lambda}_v^{-1}\ A\ D_h,\\ \tilde{B} &= \mathbf{\Lambda}_v^{-1} B D_v + \sum_{j=1}^{m} \mathbf{\Lambda}_v^{-1} C_j E_j(D_v),\\ \tilde{C}_j &= \mathbf{\Lambda}_v^{-1} C_j D_v.\end{aligned} \tag{3.4.18}$$

3.4.3 The admissibility system with respect to the intrinsic basis of the normal space

Let ℓ be the dimension of $\Lambda_m^d(U)_p^{\perp}$ and $(X_{J_1}, \ldots, X_{J_\ell})$ an orthonormal basis of simple m-vector fields of degree grater that d. Let $\bar{p}_0$ be a point in $\bar{M}$ and $\Phi(\bar{p}_0) = p_0$. Let $e_1, \ldots, e_m$ be an adapted basis of $T_{p_0}M$ that we extend to adapted vector fields $E_1, \ldots, E_m$ tangent to M on U. Let $v_{m+1}, \ldots, v_n$ be a basis of $(T_{p_0}M)^{\perp}$ that we extend to vector fields $V_{m+1}, \ldots, V_n$ normal to M on U, where we possibly reduced the neighborhood U of p_0 in N. Then any vector field in $\mathfrak{X}(\Phi^{-1}(U), N)$ is given by

$$V = \sum_{j=1}^{m} \psi_j E_j + \sum_{h=m+1}^{n} \psi_h V_h,$$

where $\psi_1, \ldots, \psi_n \in C^r(\Phi^{-1}(U), \mathbb{R})$. By Proposition 3.3.5 we deduce that V is admissible if and only if $V^{\perp} = \sum_{h=m+1}^{n} \psi_h V_h$ is admissible. Hence we obtain that the system (3.3.3) is equivalent to

$$\sum_{j=1}^{m} \sum_{h=m+1}^{n} \xi_{ijh} E_j(\psi_h) + \sum_{h=m+1}^{n} \hat{\beta}_{ih} \psi_h = 0, \quad i = 1, \ldots, \ell, \tag{3.4.19}$$

where

$$\xi_{ijh}(\bar{p}) = \langle e_1 \wedge \ldots \wedge \overset{(j)}{v_h} \wedge \ldots \wedge e_m, (X_{J_i})_p \rangle \tag{3.4.20}$$

and

$$\begin{aligned}\hat{\beta}_{ih}(\bar{p}) &= \langle e_1 \wedge \ldots \wedge e_m, \nabla_{v_h} X_{J_i} \rangle + \\ &\quad + \sum_{j=1}^{m} \langle e_1 \wedge \ldots \wedge \nabla_{e_j} V_h \wedge \ldots \wedge e_m, (X_{J_i})_p \rangle \\ &= \sum_{j=1}^{m} \langle e_1 \wedge \ldots \wedge [E_j, V_h](p) \wedge \ldots \wedge e_m, (X_{J_i})_p \rangle.\end{aligned} \tag{3.4.21}$$

Definition 3.4.6. Let $\iota_0(U)$ be the integer defined in 3.4.1. Then we set $k := n_{\iota_0} - \tilde{m}_{\iota_0}$.

Assume that $k \geqslant 1$, and write

$$V^{\perp} = \sum_{h=m+1}^{m+k} \phi_h V_h + \sum_{r=m+k+1}^{n} \psi_r V_r,$$

and the local system (3.4.19) is equivalent to

$$\sum_{j=1}^{m} \sum_{r=\rho+1}^{n} \xi_{ijr} E_j(\psi_r) + \sum_{r=\rho+1}^{n} \beta_{ir} \psi_r + \sum_{h=m+1}^{m+k} \alpha_{ih} \phi_h = 0, \tag{3.4.22}$$

where ξ_{ijr} is defined in (3.4.20) and, for $1 \leqslant i \leqslant \ell$,

$$\alpha_{ih} = \hat{\beta}_{ih}, \quad \beta_{ir} = \hat{\beta}_{ir}, \; m+1 \leqslant h \leqslant m+k, \; m+k+1 \leqslant r \leqslant n. \tag{3.4.23}$$

We denote by $B^{\perp}$ the $\ell \times (n-m-k)$ matrix whose entries are β_{ir}, by $A^{\perp}$ the $\ell \times k$ whose entries are α_{ih} and for every $j = 1, \cdots m$ by $C_j^{\perp}$ the $\ell \times (n-m-k)$ matrix with entries $(\xi_{ijh})_{h=m+k+1,\ldots,n}^{i=1,\ldots,\ell}$ Setting

$$F^{\perp} = \begin{pmatrix} \psi_{m+k+1} \\ \vdots \\ \psi_n \end{pmatrix}, \quad G^{\perp} = \begin{pmatrix} \phi_{m+1} \\ \vdots \\ \phi_{m+k} \end{pmatrix} \tag{3.4.24}$$

the admissibility system (3.4.2) is given

$$\sum_{j=1}^{m} C_j^{\perp} E_j(F^{\perp}) + B^{\perp} F^{\perp} + A^{\perp} G^{\perp} = 0. \tag{3.4.25}$$

Remark 3.4.7. We can define the matrices $A^{\top}$, $B^{\top}$, $C^{\top}$ with respect to the tangent projection $V^{\top}$ in a similar way to the matrices $A^{\perp}$, $B^{\perp}$, $C^{\perp}$. First of all we notice that the entries

$$\xi_{ij\nu}^{\top}(\bar{p}) = \langle e_1 \wedge \ldots \wedge \overset{(j)}{e_\nu} \wedge \ldots \wedge e_m, (X_{J_i})_p \rangle$$

for $i = 1, \ldots, \ell$ and $j, \nu = 1, \ldots, m$ are all equal to zero. Therefore the matrices $C^{\top}$ and $B^{\top}$ are equal to zero. On the other hand, $A^{\top}$ is the $(\ell \times m)$-matrix whose entries are given by

$$\alpha_{i\nu}^{\top}(\bar{p}) = \sum_{j=1}^{m} \langle e_1 \wedge \ldots \wedge [E_j, E_\nu](p) \wedge \ldots \wedge e_m, (X_{J_i})_p \rangle$$

for $i = 1, \ldots, \ell$ and $\nu = 1, \ldots, m$. Frobenius Theorem implies that the Lie brackets $[E_j, E_\nu]$ are all tangent to M for $j, \nu = 1, \ldots, m$, and so all the entries of $A^\top$ are equal to zero.

3.5 Integrability of admissible vector fields

In general, given an admissible vector field V, the existence of an admissible variation with associated variational vector field V is not guaranteed. The next definition is a sufficient condition to ensure the integrability of admissible vector fields.

Definition 3.5.1. Let $\Phi : \bar{M} \to N$ be an immersion of degree d of an m-dimensional manifold into a graded manifold endowed with a Riemannian metric g. Let $\ell = \dim(\Lambda_m^d(U)_q^\perp)$ for all $q \in N$ and $\rho = n_{\iota_0}$ set in (3.4.1). When $\rho \geqslant \ell$ we say that Φ is *strongly regular* at $\bar{p} \in \bar{M}$ if

$$\operatorname{rank}(A(\bar{p})) = \ell,$$

where A is the matrix appearing in the admissibility system (3.4.10).

The rank of A is independent of the local adapted basis chosen to compute the admissibility system (3.4.10) because of equations (3.4.18). Next we prove that strong regularity is a sufficient condition to ensure local integrability of admissible vector fields.

Theorem 3.5.2. *Let $\Phi : \bar{M} \to N$ be a smooth immersion of an m-dimensional manifold into an equiregular graded manifold N endowed with a Riemannian metric g. Assume that the immersion Φ of degree d is strongly regular at $\bar{p}$. Then there exists an open neighborhood $W_{\bar{p}}$ of $\bar{p}$ such every admissible vector field V with compact support on $W_{\bar{p}}$ is integrable.*

Proof. Let $p = \Phi(\bar{p})$. First of all we consider an open neighborhood $U_p \subset N$ of p such that an adapted orthonormal frame $(X_1, \ldots, X_n)$ is well defined. Since Φ is strongly regular at $\bar{p}$ there exist indexes $h_1, \ldots, h_\ell$ in $\{1, \ldots, \rho\}$ such that the submatrix

$$\hat{A}(\bar{p}) = \begin{pmatrix} a_{1h_1}(\bar{p}) & \cdots & a_{1h_\ell}(\bar{p}) \\ \vdots & \ddots & \vdots \\ a_{\ell h_1}(\bar{p}) & \cdots & a_{\ell h_\ell}(\bar{p}) \end{pmatrix}$$

is invertible. By a continuity argument there exists an open neighborhood $W_{\bar{p}} \subset \Phi^{-1}(U_p)$ such that $\det(\hat{A}(\bar{q})) \neq 0$ for each $\bar{q} \in W_{\bar{p}}$.

We can rewrite the system (3.4.10) in the form

$$\begin{pmatrix} g_{h_1} \\ \vdots \\ g_{h_\ell} \end{pmatrix} = -\hat{A}^{-1} \left(\sum_{j=1}^{m} C_j E_j(F) + BF + \tilde{A} \begin{pmatrix} g_{i_1} \\ \vdots \\ g_{i_{\rho-\ell}} \end{pmatrix} \right), \tag{3.5.1}$$

where $i_1, \ldots, i_{\rho-\ell}$ are the indexes of the columns of A that do not appear in $\hat{A}$ and $\tilde{A}$ is the $\ell \times (\rho - \ell)$ matrix given by the columns $i_1, \ldots, i_{\rho-\ell}$ of A. The vectors $(E_i)_i$ form an orthonormal basis of $T\bar{M}$ near $\bar{p}$.

On the neighborhood $W_{\bar{p}}$ we define the following spaces

1. $\mathfrak{X}_0^r(W_{\bar{p}}, N)$, $r \geqslant 0$ is the set of C^r vector fields compactly supported on $W_{\bar{p}}$ taking values in TN.
2. $\mathcal{A}_0^r(W_{\bar{p}}, N) = \{Y \in \mathfrak{X}_0^r(W_{\bar{p}}, N) : Y = \sum_{s=1}^{\rho} g_s X_s\}$.
3. $\mathcal{A}_{1,0}^r(W_{\bar{p}}, N) = \{Y \in \mathcal{A}_0^r(W_{\bar{p}}, N) : Y = \sum_{i=1}^{\ell} g_{h_i} X_{h_i}\}$.
4. $\mathcal{A}_{2,0}^r(W_{\bar{p}}, N) = \{Y \in \mathcal{A}_0^r(W_{\bar{p}}, N) : \langle Y, X\rangle = 0 \ \forall\ X \in \mathcal{A}_{1,0}^r(W_{\bar{p}}, N)\}$.
5. $\mathcal{V}_0^r(W_{\bar{p}}, N) = \{Y \in \mathfrak{X}^r(W_{\bar{p}}, N) : \langle Y, X\rangle = 0\ \forall X \in \mathcal{A}_0^r(W_{\bar{p}}, N)\}$ $= \mathcal{A}_0^r(W_{\bar{p}}, N)^\perp$.
6. $\Lambda_0^r(W_{\bar{p}}, N) = \{\sum_{i=1}^{\ell} f_i X_{J_i} : f_i \in C_0^r(W_{\bar{p}})\}$.

Given $r \geqslant 1$, we set

$$E := \mathcal{A}_{2,0}^{r-1}(W_{\bar{p}}, N) \times \mathcal{V}_0^r(W_{\bar{p}}, N),$$

and consider the map

$$\mathcal{G} : E \times \mathcal{A}_{1,0}^{r-1}(W_{\bar{p}}, N) \to E \times \Lambda_0^{r-1}(W_{\bar{p}}, N), \tag{3.5.2}$$

defined by

$$\mathcal{G}(Y_1, Y_2, Y_3) = (Y_1, Y_2, \mathcal{F}(Y_1 + Y_2 + Y_3)),$$

where Π_v is the projection in the space of m-forms with compact support in $W_{\bar{p}}$ onto $\Lambda^r(W_{\bar{p}}, N)$, and

$$\mathcal{F}(Y) = \Pi_v \left(d\Gamma(Y)(e_1) \wedge \ldots \wedge d\Gamma(Y)(e_m) \right),$$

where $\Gamma(Y)(p) = \exp_{\Phi(p)}(Y_p)$. Observe that $\mathcal{F}(Y) = 0$ if and only if the submanifold $\Gamma(Y)$ has degree less or equal to d. We consider on each space the corresponding $||\cdot||_r$ or $||\cdot||_{r-1}$ norm, and a product norm.

Then

$$D\mathcal{G}(0,0,0)(Y_1, Y_2, Y_3) = (Y_1, Y_2, D\mathcal{F}(0)(Y_1 + Y_2 + Y_3)),$$

where we write in coordinates

$$Y_1 = \sum_{t=1}^{\rho-\ell} g_{i_t}\, X_{i_t}, \quad Y_2 = \sum_{i=1}^{\ell} g_{h_i}\, X_{h_i}, \quad \text{and} \quad Y_3 = \sum_{r=\rho+1}^{n} f_r\, X_r.$$

Following the same argument we used in Section 3.3, taking the derivative at $t = 0$ of (3.3.2), we deduce that the differential $D\mathcal{F}(0)Y$ is given by

$$D\mathcal{F}(0)Y = \sum_{i=1}^{\ell} \left(\sum_{j=1}^{m} \sum_{r=\rho+1}^{n} c_{ijr} E_j(f_r) + \sum_{r=\rho+1}^{n} b_{ir} f_r + \sum_{h=1}^{\rho} a_{ih} g_h \right) X_{J_i}.$$

Observe that $D\mathcal{F}(0)Y = 0$ if and only if Y is an admissible vector field, namely Y solves (3.5.1).

Our objective now is to prove that the map $D\mathcal{G}(0,0,0)$ is an isomorphism of Banach spaces.

Indeed suppose that $D\mathcal{G}(0,0,0)(Y_1, Y_2, Y_3) = (0,0,0)$. This implies that Y_1 and Y_2 are equal zero. By the admissible equation (3.5.1) we have that also Y_3 is equal to zero, then $D\mathcal{G}(0,0,0)$ is injective. Then fix (Z_1, Z_2, Z_3), where $Z_1 \in \mathcal{A}_{2,0}^{r-1}(W_{\bar{p}}, N)$, $Z_2 \in \mathcal{V}_0^r(W_{\bar{p}}, N)$, $Z_3 \in \Lambda_0^{r-1}(W_{\bar{p}}, N)$ we seek Y_1, Y_2, Y_3 such that $D\mathcal{G}(0,0,0)(Y_1, Y_2, Y_3) = (Z_1, Z_2, Z_3)$. We notice that $D\mathcal{F}(0)(Y_1 + Y_2 + Y_3) = Z_3$ is equivalent to

$$\begin{pmatrix} z_1 \\ \vdots \\ z_\ell \end{pmatrix} = \left(\sum_{j=1}^{m} C_j\, E_j(F) + BF + \tilde{A} \begin{pmatrix} g_{i_1} \\ \vdots \\ g_{i_{\rho-\ell}} \end{pmatrix} + \hat{A} \begin{pmatrix} g_{h_1} \\ \vdots \\ g_{h_\ell} \end{pmatrix} \right),$$

where with an abuse of notation we identify $Z_3 = \sum_{i=1}^{\ell} z_i\, X_{J_i}$ and $\sum_{i=1}^{\ell} z_i\, X_{h_i}$. Since $\hat{A}$ is invertible we have the following system

$$\begin{pmatrix} g_{h_1} \\ \vdots \\ g_{h_\ell} \end{pmatrix} = -\hat{A}^{-1} \left(\sum_{j=1}^{m} C_j\, E_j(F) + BF + \tilde{A} \begin{pmatrix} g_{i_1} \\ \vdots \\ g_{i_{\rho-\ell}} \end{pmatrix} + \begin{pmatrix} z_1 \\ \vdots \\ z_\ell \end{pmatrix} \right). \tag{3.5.3}$$

Clearly $Y_1 = Z_1$ fixes $g_{i_1}, \dots, g_{i_{\rho-\ell}}$ in (3.5.3), and $Y_2 = Z_2$ fixes the first and second term of the right hand side in (3.5.3). Since the right side terms are given we have determined Y_3, i.e. $g_{h_1}, \dots, g_{h_\ell}$, such that Y_3 solves (3.5.3). Therefore $D\mathcal{G}(0,0,0)$ is surjective. Thus we have proved that $D\mathcal{G}(0,0,0)$ is a bijection.

Let us prove now that $D\mathcal{G}(0,0,0)$ is a continuous and open map. Letting $D\mathcal{G}(0,0,0)(Y_1,Y_2,Y_3) = (Z_1,Z_2,Z_3)$, we first notice $D\mathcal{G}(0,0,0)$ is a continuous map since identity maps are continuous and, by (3.5.3), there exists a constant K such that

$$\begin{aligned}\|Z_3\|_{r-1} &\leqslant K\left(\sum_{j=1}^{m} \|\nabla_j Y_2\|_{r-1} + \|Y_2\|_{r-1} + \|Y_1\|_{r-1} + \|Y_3\|_{r-1}\right)\\ &\leqslant K(\|Y_2\|_r + \|Y_1\|_{r-1} + \|Y_3\|_{r-1}).\end{aligned}$$

Moreover, $D\mathcal{G}(0,0,0)$ is an open map since we have

$$\begin{aligned}\|Y_3\|_{r-1} &\leqslant K\left(\sum_{j=1}^{m} \|\nabla_j Z_2\|_{r-1} + \|Z_2\|_{r-1} + \|Z_1\|_{r-1} + \|Z_3\|_{r-1}\right)\\ &\leqslant K(\|Z_2\|_r + \|Z_1\|_{r-1} + \|Z_3\|_{r-1}).\end{aligned}$$

This implies that $D\mathcal{G}(0,0,0)$ is an isomorphism of Banach spaces.

Let now us consider an admissible vector field V with compact support on W_p. We consider the map

$$\tilde{\mathcal{G}} : (-\varepsilon,\varepsilon) \times E \times \mathcal{A}_{0,1}^{r-1}(W_{\bar{p}}, N) \to E \times \Lambda_0^{r-1}(W_{\bar{p}}, N),$$

defined by

$$\tilde{\mathcal{G}}(s, Y_1, Y_3, Y_2) = (Y_1, \mathcal{F}(sV + Y_1 + Y_3 + Y_2)).$$

The map $\tilde{\mathcal{G}}$ is continuous with respect to the product norms (on each factor we put the natural norm, the Euclidean one on the intervals and $||\cdot||_r$ and $||\cdot||_{r-1}$ in the spaces of vectors on $\Phi(\bar{M})$). Moreover

$$\tilde{\mathcal{G}}(0,0,0,0) = (0,0),$$

since Φ has degree d. Denoting by D_Y the differential with respect to the last three variables of $\tilde{G}$ we have that

$$D_Y\tilde{\mathcal{G}}(0,0,0,0)(Y_1,Y_2,Y_3) = D\mathcal{G}(0,0,0)(Y_1,Y_2,Y_3)$$

is a linear isomorphism. We can apply the Implicit Function Theorem to obtain unique maps

$$\begin{aligned} &Y_1 : (-\varepsilon,\varepsilon) \to \mathcal{A}_{0,2}^{r-1}(W_{\bar{p}},N),\\ &Y_2 : (-\varepsilon,\varepsilon) \to \mathcal{V}_0^{r}(W_{\bar{p}},N),\\ &Y_3 : (-\varepsilon,\varepsilon) \to \mathcal{A}_{0,1}^{r-1}(W_{\bar{p}},N), \end{aligned} \tag{3.5.4}$$

such that $\tilde{\mathcal{G}}(s,Y_1(s),Y_2(s),Y_3(s)) = (0,0)$. This implies that $Y_1(s) = 0$, $Y_2(s) = 0$, $Y_3(0) = 0$ and that

$$\mathcal{F}(sV + Y_3(s)) = 0.$$

Differentiating this formula at $s = 0$ we obtain

$$D\mathcal{F}(0)\left(V + \frac{\partial Y_3}{\partial s}(0)\right) = 0.$$

Since V is admissible we deduce

$$D\mathcal{F}(0)\frac{\partial Y_3}{\partial s}(0) = 0.$$

Since $\frac{\partial Y_3}{\partial s}(0) = \sum_{i=1}^{\ell} g_{h_i} X_{h_i}$, where $g_{h_i} \in C_0^{r-1}(W_{\bar{p}})$, equation (3.5.1) implies $g_{h_i} \equiv 0$ for each $i = 1,\ldots,\ell$. Therefore it follows $\frac{\partial Y_3}{\partial s}(0) = 0$.

Hence the variation $\Gamma_s(\bar{p}) = \Gamma(sV + Y_3(s))(\bar{p})$ coincides with $\Phi(\bar{q})$ for $s = 0$ and $\bar{q} \in W_{\bar{p}}$, it has degree d and its variational vector fields is given by

$$\left.\frac{\partial \Gamma_s}{\partial s}\right|_{s=0} = V + \frac{\partial Y_3}{\partial s}(0) = V.$$

Moreover, $\text{supp}(Y_3) \subseteq \text{supp}(V)$. Indeed, if $\bar{q} \notin \text{supp}(V)$, the unique vector field $Y_3(s)$, such $\mathcal{F}(Y_3(s)) = 0$, is equal to 0 at $\bar{q}$. □

Remark 3.5.3. In Proposition 3.3.5 we stressed the fact that a vector field $V = V^{\top} + V^{\perp}$ is admissible if and only if $V^{\perp}$ is admissible. This follows from the additivity in V of the admissibility system (3.3.3) and the admissibility of $V^{\top}$. Instead of writing V with respect to the adapted basis $(X_i)_i$ we consider the basis $E_1,\ldots,E_m,V_{m+1},\ldots,V_n$ described in Section 3.4.3.

Let $A^{\perp}, B^{\perp}, C^{\perp}$ be the matrices defined in (3.4.23), $A^{\top}$ be the one described in Remark 3.4.7 and A be the matrix with respect to the basis $(X_i)_i$ defined in (3.4.8). When we change only the basis for the vector field V by (3.4.12) we obtain $\tilde{A} = AD_h$. Since $A^{\top}$ is the null matrix and $\tilde{A} = (A^{\top}|\,A^{\perp})$ we conclude that $\text{rank}(A(\bar{p})) =$

$\text{rank}(A^{\perp}(\bar{p}))$. Furthermore Φ is strongly regular at $\bar{p}$ if and only if $\text{rank}(A^{\perp}(\bar{p})) = \ell \leqslant k$, where k is the integer defined in 3.4.6.

3.5.1 Some examples of strongly regular submanifolds

Example 3.5.4. Consider a hypersurface Σ immersed in an equiregular Carnot manifold $(N, \mathcal{H})$, then we have that Σ always has degree $d := \deg(\Sigma)$ equal to $d^{n-1}_{\max} = Q - 1$, see 3.2.1. Therefore the dimension ℓ, defined in Section 3.4, of $\Lambda^d_m(U)_p$ is equal to zero. Thus any compactly supported vector field V is admissible and integrable. When the Carnot manifold N is a contact structure $(M^{2n+1}, \mathcal{H} = \ker(\omega))$, see 3.2.2, the hypersurface Σ has always degree equal to $d^{2n}_{\max} = 2n + 1$.

Example 3.5.5. Let $(E, \mathcal{H})$ be the Carnot manifold described in Section 3.2.3 where $(x, y, \theta, k) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R} = E$ and the distribution $\mathcal{H}$ is generated by

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y + k\partial_\theta, \quad X_2 = \partial_k.$$

Clearly $(X_1, \ldots, X_4)$ is an adapted basis for $\mathcal{H}$. Moreover the others no-trivial commutators are given by

$$\begin{aligned}
[X_1, X_4] &= -kX_1 - k^2X_3 \\
[X_3, X_4] &= X_1 + kX_3.
\end{aligned}$$

Let $\Omega \subset \mathbb{R}^2$ be an open set. We consider the surface $\Sigma = \Phi(\Omega)$ where

$$\Phi(x, y) = (x, y, \theta(x, y), \kappa(x, y))$$

and such that $X_1(\theta(x, y)) = \kappa(x, y)$. Therefore the $\deg(\Sigma) = 4$ and its tangent vectors are given by

$$\begin{aligned}
\tilde{E}_1 =& X_1 + X_1(\kappa)X_2, \\
\tilde{E}_2 =& X_4 - X_4(\theta)X_3 + X_4(\kappa)X_2.
\end{aligned}$$

Let $g = \langle \cdot, \cdot \rangle$ be the metric that makes orthonormal the adapted basis $(X_1, \ldots, X_4)$. Since $(\Lambda^4_2(N))^{\perp} = \text{span}\{X_3 \wedge X_4\}$ the only no-trivial coefficient c_{11r}, for $r = 3, 4$ are given by

$$\langle X_3 \wedge \tilde{E}_2, X_3 \wedge X_4 \rangle = 1, \quad \text{and} \quad \langle X_4 \wedge \tilde{E}_2, X_3 \wedge X_4 \rangle = X_4(\theta).$$

On the other hand $c_{12h} = \langle \tilde{E}_1 \wedge X_k, X_3 \wedge X_4 \rangle = 0$ for each $h = 1, \ldots, 4$, since we can not reach the degree 5 if one of the two vector fields in the wedge has degree one. Therefore the only equation in (3.4.2) is given by

$$\tilde{E}_1(f_3) + X_4(\theta)\tilde{E}_1(f_4) + \sum_{h=1}^{4} \Big(\langle X_3 \wedge X_4, \tilde{E}_1 \wedge [\tilde{E}_2, X_h] + [\tilde{E}_1, X_h] \wedge \tilde{E}_2 \rangle \Big) f_h = 0. \quad (3.5.5)$$

Since $\deg(\tilde{E}_1 \wedge [\tilde{E}_2, X_h]) \leqslant 4$ we have $\langle X_3 \wedge X_4, \tilde{E}_1 \wedge [\tilde{E}_2, X_h] \rangle = 0$ for each $h = 1, \ldots, 4$. Since $[uX, Y] = u[X, Y] - Y(u)X$ for each $X, Y \in \mathfrak{X}(N)$ and $u \in C^\infty(N)$ we have

$$\begin{aligned} [\tilde{E}_1, X_h] &= [X_1, X_h] + X_1(\kappa)[X_2, X_h] - X_h(X_1(\kappa))X_2 \\ &= \begin{cases} -X_1(\kappa)X_3 - X_1(X_1(\kappa))X_2 & h = 1 \\ X_3 - X_2(X_1(\kappa))X_2 & h = 2 \\ X_4 - X_3(X_1(\kappa))X_2 & h = 3 \\ -\kappa X_1 - \kappa^2 X_3 - X_4(X_1(\kappa))X_2 & h = 4. \end{cases} \end{aligned}$$

Thus, we deduce

$$\langle X_3 \wedge X_4, [\tilde{E}_1, X_h] \wedge \tilde{E}_2 \rangle = \begin{cases} -X_1(\kappa) & h = 1 \\ 1 & h = 2 \\ X_4(\theta) & h = 3 \\ -\kappa^2 & h = 4. \end{cases}$$

Hence the equation (3.5.5) is equivalent to

$$\tilde{E}_1(f_3) + X_4(\theta)\tilde{E}_1(f_4) - X_1(\kappa)f_1 + f_2 - X_4(\theta)f_3 - \kappa^2 f_4 = 0 \quad (3.5.6)$$

Since $\iota_0(\Omega) = 1$, we have $\rho = n_1 = 2$, where ρ is the natural number defined in (3.4.1). In this setting the matrix C is given by

$$C = \begin{pmatrix} 1 & 0 & X_4(\theta) & 0 \end{pmatrix},$$

Then the matrices A and B are given by

$$A = \begin{pmatrix} -X_1(\kappa) & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -X_4(\theta) & -\kappa^2 \end{pmatrix}.$$

Since $\mathrm{rank}(A(x,y)) = 1$ and the matrix $\hat{A}(x,y)$, defined in the proof of Theorem 3.5.2, is equal to 1 for each $(x,y) \in \Omega$ we have that Φ is strongly regular at each point (x,y) in Ω and the open set $W_{(x,y)} = \Omega$. Hence by Theorem 3.5.2 each admissible vector field on Ω is integrable.

On the other hand we notice that $k = n_1 - \tilde{m}_1 = 1$. By the Gram-Schmidt process an orthonormal basis with respect to the metric g is given by

$$\begin{aligned}
E_1 &= \frac{1}{\alpha_1}(X_1 + X_1(\kappa)X_2),\\
E_2 &= \frac{1}{\alpha_2}\left(X_4 - X_4(\theta)X_3 + \frac{X_4(\kappa)}{\alpha_1^2}(X_2 - X_1(\kappa)X_1)\right),\\
V_3 &= \frac{1}{\alpha_3}(X_3 + X_4(\theta)X_4),\\
V_4 &= \frac{\alpha_3}{\alpha_2\alpha_1}\left((-X_1(\kappa)X_1 + X_2) + \frac{X_4(\kappa)}{\alpha_3^2}(X_4(\theta)X_3 - X_4)\right),
\end{aligned}$$

where we set

$$\begin{aligned}
\alpha_1 &= \sqrt{1 + X_1(\kappa)^2}, \quad \alpha_3 = \sqrt{1 + X_4(\theta)^2}\\
\alpha_2 &= \sqrt{1 + X_4(\theta)^2 + \frac{X_4(\kappa)^2}{(1 + X_1(\kappa)^2)}} = \frac{\sqrt{\alpha_1^2\alpha_3^2 + X_4(\kappa)^2}}{\alpha_1}.
\end{aligned}$$

Since it holds

$$\begin{aligned}
\langle V_3 \wedge E_2, X_3 \wedge X_4\rangle &= \frac{\alpha_3}{\alpha_2},\\
\langle V_4 \wedge E_2, X_3 \wedge X_4\rangle &= 0,\\
\langle [E_1, V_3] \wedge E_2, X_3 \wedge X_4\rangle &= \frac{X_4(\theta)(1 - \kappa^2)}{\alpha_1\alpha_2\alpha_3},\\
\langle [E_1, V_4] \wedge E_2, X_3 \wedge X_4\rangle &= \frac{\alpha_3}{\alpha_2}\left(1 + \frac{X_4(\kappa)^2}{\alpha_1^2\alpha_3^2}\right) = \frac{\alpha_2}{\alpha_3},
\end{aligned}$$

then a vector field $V^{\perp} = \psi_3(x,y)\, V_3 + \psi_4(x,y)\, V_4$ normal to Σ is admissible if and only if $\psi_3, \psi_4 \in C_0^r(\Omega)$ verify

$$\frac{\alpha_3}{\alpha_2} E_1(\psi_3) + \frac{X_4(\theta)(1-\kappa^2)}{\alpha_1\alpha_2\alpha_3}\, \psi_3 + \frac{\alpha_2}{\alpha_3}\, \psi_4 = 0.$$

That is equivalent to

$$\bar{X}_1(\psi_3) + b^{\perp}\, \psi_3 + a^{\perp}\, \psi_4 = 0, \tag{3.5.7}$$

where $\bar{X}_1 = \cos(\theta(x,y))\partial_x + \sin(\theta(x,y))\partial_y$ and

$$\begin{aligned} b^{\perp} &= \frac{X_4(\theta)(1 - X_1(\theta)^2)}{1 + X_4(\theta)^2}, \\ a^{\perp} &= \alpha_1 \left(1 + \frac{X_4(\kappa)^2}{\alpha_1^2 \alpha_3^2} \right). \end{aligned}$$

In particular, since $a^{\perp}(x,y) > 0$ we have that $\operatorname{rank}(a^{\perp}(x,y)) = 1$ for all $(x,y) \in \Omega$. Along the integral curve $\gamma'(t) = \bar{X}_1$ on Ω the equation (3.5.7) reads

$$\psi_3'(t) + b^{\perp}(t)\psi_3(t) + a^{\perp}(t)\psi_4(t) = 0,$$

where we set $f(t) = f(\gamma(t))$ for each function $f : \Omega \to \mathbb{R}$.

3.5.2 An isolated plane in the Engel group

Definition 3.5.6. We say that an immersion $\Phi : \bar{M} \to N$ in an equiregular graded manifold $(N, \mathcal{H}^1 \subset \ldots \subset \mathcal{H}^s)$ is *isolated* if the only admissible variation normal to $M = \Phi(\bar{M})$ is the trivial one.

Here we provide an example of isolated surface immersed in the Engel group.

Example 3.5.7. Let $N = \mathbb{R}^4$ and $\mathcal{H} = \operatorname{span}\{X_1, X_2\}$, where

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4}$$

and $X_3 = \partial_{x_3}$ and $X_4 = \partial_{x_4}$. We denote by $\mathbb{E}^4$ the Engel group given by $(\mathbb{R}^4, \mathcal{H})$. Let $\Upsilon : \Omega \subset \mathbb{R}^2 \to \mathbb{E}^4$ be the immersion given by

$$\Upsilon(v,\omega) = (v, 0, \omega, 0).$$

Since $\Upsilon_v \wedge \Upsilon_w = X_1 \wedge X_3$ the degree $\deg(\Sigma) = 3$, where $\Sigma = \Upsilon(\Omega)$ is a plane. An admissible vector field $V = \sum_{k=1}^4 f_k X_k$ verifies the system (3.4.2) that is given by

$$\begin{aligned} \sum_{h=1}^{4} \frac{\partial f_h}{\partial x_1} \langle X_h \wedge X_3, X_{J_i} \rangle + \frac{\partial f_h}{\partial x_3} \langle X_1 \wedge X_h, X_{J_i} \rangle + \\ + f_h \left(\langle [X_1, X_h] \wedge X_3, X_{J_i} \rangle + \langle X_1 \wedge [X_3, X_h], X_{J_i} \rangle \right) = 0, \end{aligned} \tag{3.5.8}$$

for $X_{J_1} = X_1 \wedge X_4$, $X_{J_2} = X_2 \wedge X_4$ and $X_{J_3} = X_3 \wedge X_4$. Therefore (3.5.8) is equivalent to

$$\begin{cases} \dfrac{\partial f_4}{\partial x_3} + f_2 = 0 \\ 0 = 0 \\ -\dfrac{\partial f_4}{\partial x_1} = 0. \end{cases}$$

Let $K = \mathrm{supp}(V)$. First of all we have $\frac{\partial f_4}{\partial x_1} = 0$. Since $f_4 \in C^\infty(\Omega)$ there follows

$$\frac{\partial f_2}{\partial x_1} = -\frac{\partial^2 f_4}{\partial x_3 \partial x_1} = 0.$$

Then let $(x_1, x_2) \in K$ we consider the curve

$$\gamma : s \mapsto (x_1 + s, x_3)$$

along which f_4 and f_2 are constant. Since f_4 and f_2 are compactly supported at the end point, $(x_1 + s_0, x_3) \in \partial K$ we have $f_4(x_1 + s_0, x_3) = f_2(x_1 + s_0, x_3) = 0$. Therefore we gain $f_4 = f_2 \equiv 0$. Therefore the admissible only vector fields $f_1 X_1 + f_3 X_3$ are tangent to Σ. Assume that there exists an admissible variation Γ_s for Υ, then its associated variational vector field is admissible. However we proved that the only admissible vector fields are tangent to Σ, therefore the admissible variation Γ_s has to be tangent to Σ and the only normal one a trivial variation, hence we conclude that the plane Σ is isolated.

Moreover, we have that $k = 1$ and the matrix $A^\perp$ defined in 3.5.1 is given by

$$A(u, w) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\mathrm{rank}(A) = 1 < 3$ we deduce that Υ is not strongly regular at any point in Ω.

Here we prove that Σ is isolated without using the admissibility system.

Proposition 3.5.8. *Let $\mathbb{E}^4$ be the Engel group given by $(\mathbb{R}^4, \mathcal{H})$, where the distribution $\mathcal{H}$ is generated by*

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4}.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Then the immersion $\Upsilon : \Omega \to \mathbb{E}^4$ of degree 3 given by

$$\Upsilon(v, w) = (v, 0, w, 0)$$

is isolated.

Proof. An admissible normal variation Γ_s of Υ has to have the same degree of Υ and has to share the same boundary $\Upsilon(\partial\Omega) = \partial\Sigma$, where clearly $\Sigma = \Upsilon(\Omega)$. For a fix s, we can parametrize Γ_s by

$$\Phi : \Omega \to \mathbb{E}^4, \quad \Phi(v, w) = (v, \phi(v, w), w, \psi(v, w)),$$

where $\phi, \psi \in C_0^1(\Omega, \mathbb{R})$. Since $\deg(\Phi(\Omega)) = 3$ we gain

$$\begin{cases} \langle \Phi_v \wedge \Phi_w, X_1 \wedge X_4 \rangle = 0 \\ \langle \Phi_v \wedge \Phi_w, X_2 \wedge X_4 \rangle = 0 \\ \langle \Phi_v \wedge \Phi_w, X_1 \wedge X_4 \rangle = 0, \end{cases} \tag{3.5.9}$$

where

$$\Phi_v = \partial_1 + \phi_v\partial_2 + \psi_v\partial_4 = X_1 + \phi_v(X_2 - vX_3 + wX_4) + \psi_v$$

and

$$\Phi_v = \phi_w\partial_2 + \partial_3\psi_w\partial_4 = \phi_w(X_2 - vX_3 + wX_4) + X_3 + \psi_w.$$

Denoting by π_4 the projection over the 2-vectors of degree larger than 3, we have

$$\begin{aligned} \pi_4(\Phi_v \wedge \Phi_w) =& (\psi_w + w\phi_w)X_1 \wedge X_4 + \phi_v(\psi_w + w\phi_w)X_2 \wedge X_4 \\ &- v\phi_v(\psi_w + w\phi_w)X_3 \wedge X_4 + (\psi_v + w\phi_v)X_4 \wedge X_2 \\ &+ (1 - v\phi_w)(\psi_v + w\phi_v)X_4 \wedge X_3. \end{aligned}$$

Therefore (3.5.9) is equivalent to

$$\begin{cases} \psi_w + w\phi_w = 0 \\ \phi_v\psi_w - \psi_v\phi_w = 0 \\ v(\phi_v\psi_w - \psi_v\phi_w) - (\psi_v + w\phi_v) = 0. \end{cases} \tag{3.5.10}$$

The second equation implies that (3.5.10) is equivalent to

$$\begin{cases} \psi_w + w\phi_w = 0 \\ \phi_v\psi_w - \psi_v\phi_w = 0 \\ \psi_v + w\phi_v = 0. \end{cases} \tag{3.5.11}$$

Then we notice that the first and the third equations implies the second one as it follows

$$\phi_v\psi_w - \psi_v\phi_w = -\phi_v w\phi_w + w\phi_v\phi_w = 0.$$

Therefore the immersion Φ has degree three if and only if

$$\begin{cases} \psi_w = -w\phi_w \\ \psi_v = -w\phi_v. \end{cases} \tag{3.5.12}$$

Only when the compatibility condition for linear system of first order is given we have a solution of this system. However the compatibility condition is given by

$$0 = \psi_{wv} - \psi_{vw} = \phi_v$$

Since $\phi \in C^1_0(\Omega)$ we obtain $\phi \equiv 0$. Therefore also $\psi_v = 0$, then $\psi \equiv 0$. Hence $\Phi = \Upsilon$. □

Here we provide the generic equations for a immersed surface of degree three in the Engel group.

Proposition 3.5.9. *Let $\mathbb{E}^4$ be the Engel group given by $(\mathbb{R}^4, \mathcal{H})$, where the distribution $\mathcal{H}$ is generated by*

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} + x_3\partial_{x_4}.$$

Given $\Omega \subset \mathbb{R}^2$ an open set, a general C^1 immersion $\Phi : \Omega \to \mathbb{E}^4$ has degree three if and only if

$$\begin{cases} \phi^1_u(\phi^4_v - \phi^2_v\phi^3) = \phi^1_v(\phi^4_u - \phi^2_u\phi^3) \\ \phi^2_u(\phi^4_v - \phi^2_v\phi^3) = \phi^2_v(\phi^4_u - \phi^2_u\phi^3) \\ \phi^3_u - \phi^2_u\phi^1)(\phi^4_v - \phi^2_v\phi^3) = (\phi^3_v - \phi^2_v\phi^1)(\phi^4_u - \phi^2_u\phi^3), \end{cases} \tag{3.5.13}$$

where $\Phi(u,v) = (\phi^1(u,v), \phi^2(u,v), \phi^3(u,v), \phi^4(u,v))$, $\phi^i_u = \frac{\partial \phi^i}{\partial u}$ and $\phi^i_v = \frac{\partial \phi^i}{\partial v}$ for each $i = 1, \dots, 4$.

Proof. We recall that $X_3 = [X_1, X_2] = \partial_{x_3}$ has degree two and $X_4 = [X_3, X_2] = \partial_{x_4}$ has degree there. Therefore we have

$$\begin{aligned} \Phi_u &= \phi_u^1 \partial_{x_1} + \phi_u^2 \partial_{x_2} + \phi_u^3 \partial_{x_3} + \phi_u^4 \partial_{x_4} \\ &= \phi_u^1 X_1 + \phi_u^2 X_2 + (\phi_u^3 - \phi_u^2 \phi^1) X_3 + (\phi_u^4 - \phi_u^2 \phi^3) X_4 \end{aligned}$$

and

$$\begin{aligned} \Phi_v &= \phi_v^1 \partial_{x_1} + \phi_v^2 \partial_{x_2} + \phi_v^3 \partial_{x_3} + \phi_v^4 \partial_{x_4} \\ &= \phi_v^1 X_1 + \phi_v^2 X_2 + (\phi_v^3 - \phi_v^2 \phi^1) X_3 + (\phi_v^4 - \phi_v^2 \phi^3) X_4. \end{aligned}$$

Then the 2 tangent vector to $\Phi(\Omega) = \Sigma$ is given by

$$\begin{aligned} \Phi_v \wedge \Phi_u =& (\phi_u^1 \phi_v^2 - \phi_u^2 \phi_v^1)\, X_1 \wedge X_2 + (\phi_u^1 (\phi_v^3 - \phi_v^2 \phi^1) - \phi_v^1 (\phi_u^3 - \phi_u^2 \phi^1))\, X_1 \wedge X_3 \\ &+ (\phi_u^2 (\phi_v^3 - \phi_v^2 \phi^1) - \phi_v^2 (\phi_u^3 - \phi_u^2 \phi^1))\, X_2 \wedge X_3 \\ &+ (\phi_u^1 (\phi_v^4 - \phi_v^2 \phi^3) - \phi_v^1 (\phi_u^4 - \phi_u^2 \phi^3))\, X_1 \wedge X_4 \\ &+ (\phi_u^2 (\phi_v^4 - \phi_v^2 \phi^3) - \phi_v^2 (\phi_u^4 - \phi_u^2 \phi^3))\, X_2 \wedge X_4 \\ &+ ((\phi_u^3 - \phi_u^2 \phi^1)(\phi_v^4 - \phi_v^2 \phi^3) - (\phi_v^3 - \phi_v^2 \phi^1)(\phi_u^4 - \phi_u^2 \phi^3)) X_3 \wedge X_4. \end{aligned}$$

When the degree of Σ is less or equal the coefficients in front of the 2 simple vectors of degree 4 and 5 have to be equal to zero, that is verified if and only if the system of PDEs (3.5.13) holds. □

Here we provide some examples of surfaces of degree three that are not rigid in the C^1 topology.

Example 3.5.10. Given $\Omega \subset \mathbb{R}^2$ an open set, let $\Phi : \Omega \to \mathbb{R}^4$ be the immersion parametrized by

$$\Phi(u, v) = \left(\phi^1(u, v), \phi^2(u, v), \phi^3(u, v), \phi^4(u, v)\right), \tag{3.5.14}$$

where $\phi^2(u, v) = \psi(u+v)$, $\phi^3(u, v) = \dfrac{\varphi'(u+v)}{\psi'(u+v)}$ and $\phi^4(u, v) = \varphi(u+v)$ for $\psi, \varphi \in C^2(\Omega)$ with $\psi'(u, v) \neq 0$ for each $(u, v) \in \Omega$. Φ is an immersion whenever $\phi_u^1 \neq \phi_v^1$. We notice that

$$\phi_u^2(u, v) = \frac{\partial \psi(u+v)}{\partial u} = \psi'(u+v) = \frac{\partial \psi(u+v)}{\partial v} = \phi_v^2(u, v).$$

Then it follows

$$\frac{\partial \phi^4(u,v)}{\partial u} = \varphi'(u+v) = \psi'(u+v)\frac{\varphi'(u+v)}{\psi'(u+v)} = \phi_u^2(u,v)\phi^3(u,v)$$

and

$$\frac{\partial \phi^4(u,v)}{\partial v} = \varphi'(u+v) = \psi'(u+v)\frac{\varphi'(u+v)}{\psi'(u+v)} = \phi_v^2(u,v)\phi^3(u,v).$$

Therefore the immersion solves the system (3.5.13), thus by Proposition 3.5.9. Since the compactly supported variations

$$\Gamma_t(u,v) = \Phi(u,v) + t(g(u,v),0,0,0)$$

for every $g \in C_0^\infty(\Omega)$ are all admissible, because the first component $\phi^1 + tg$ is not involved in the system (3.5.13), we have that all the immersions Φ of the type (3.5.14) are not isolated.

3.6 Intrinsic coordinates for the admissibility system of PDEs

Let $\Phi : \bar{M} \to N$ be a $C^{1,1}$ immersion in a graded manifold, $M = \Phi(\bar{M})$ and $d = \deg(M)$. By Proposition 3.4.4 we realize that the admissibility of a vector field V is independent of the metric. Therefore we can use any metric in order to study the system. Let p be a point in $M \setminus M_0$, that is an open set thanks to Corollary 1.2.5. We consider $e_1, \ldots, e_m$ a basis of T_pM adapted to the flag (1.2.10). Then we complete this basis to a basis of the ambient space T_pN adding $v_{m+1}, \ldots, v_n$ of increasing degree such that a sorting of $\{e_1, \ldots, e_m, v_{m+1}, \ldots, v_n\}$ is an adapted basis of T_pN. Thus we extend $e_1, \ldots, e_m, v_{m+1}, \ldots, v_n$ to vector fields $E_1, \ldots, E_m, V_{m+1}, \ldots, V_n$ so that their sorting is still adapted in a neighborhood of p. Since the immersion is $C^{1,1}$, the vector fields $E_1, \ldots, E_m$ are Lipschitz, thus the vector fields $V_{m+1}, \ldots, V_n$ are also Lipschitz. Then we consider the metric $g = \langle \cdot, \cdot \rangle$ that makes $E_1, \ldots, E_m, V_{m+1}, \ldots, V_n$ an orthonormal basis in a neighborhood U of p.

Letting ι_0 be the integer defined in (3.4.5) given by

$$\iota_0(U) = \max_{p \in U} \min_{1 \leqslant \alpha \leqslant s} \{\alpha : \tilde{m}_\alpha(p) \neq 0\},$$

and $k := n_{\iota_0} - \tilde{m}_{\iota_0}$ the integer defined in 3.4.6. Given a generic vector field W transversal to TM, the only simple m-vectors of degree strictly greater than d whose scalar product with

$$E_1 \wedge \cdots \wedge \overset{(j)}{W} \wedge \cdots \wedge E_m \tag{3.6.1}$$

are candidate to be different from zero are

$$E_1 \wedge \cdots \wedge \overset{(j)}{V_i} \wedge \cdots \wedge E_m$$

for $i = m+1, \ldots, n$ and $\deg(V_i) > \deg(E_j)$. Since $(V_i)_i$ and $(E_j)_j$ have increasing degree, we obtain $\deg(V_i) > \deg(E_1) = \iota_0$ if and only if $i = m+k+1, \ldots, n$, where k is defined in (3.4.6). Therefore we deduce that the candidates simple m-vectors of degree strictly greater than d whose scalar product against (3.6.1) is different from zero are

$$E_1 \wedge \cdots \wedge \overset{(j)}{V_i} \wedge \cdots \wedge E_m$$

for $i = m+k+1, \ldots, n$ and $\deg(V_i) > \deg(E_j)$. Furthermore by Proposition 3.3.5 we know that V is admissible if and only if

$$V^\perp = \sum_{h=m+1}^{m+k} g_h V_h + \sum_{r=m+k+1}^{n} f_r V_r \tag{3.6.2}$$

is admissible. Therefore putting $V^\perp$ in (3.3.3) we obtain

$$\begin{aligned} \sum_{j=1}^{m} \Big(\sum_{r=m+k+1}^{n} \tilde{c}_{ijr\alpha} E_j(f_r) + \sum_{h=m+1}^{m+k} \tilde{c}_{ijh\alpha} E_j(g_h) \\ + \sum_{r=m+k+1}^{n} \tilde{b}_{ijr\alpha} f_r + \sum_{h=m+1}^{m+k} \tilde{a}_{ijh\alpha} g_h \Big) = 0, \end{aligned} \tag{3.6.3}$$

where

$$\begin{aligned} \tilde{c}_{ijt\alpha} &= \langle E_1 \wedge \cdots \wedge \overset{(j)}{V_t} \wedge \cdots \wedge E_m, E_1 \wedge \cdots \wedge \overset{(\alpha)}{V_i} \wedge \cdots \wedge E_m \rangle \\ \tilde{a}_{ijh\alpha} &= \langle E_1 \wedge \ldots \wedge \overset{(j)}{[E_j, V_h]} \wedge \ldots \wedge E_m, E_1 \wedge \cdots \wedge \overset{(\alpha)}{V_i} \wedge \cdots \wedge E_m \rangle \\ \tilde{b}_{ijr\alpha} &= \langle E_1 \wedge \ldots \wedge \overset{(j)}{[E_j, V_r]} \wedge \ldots \wedge E_m, E_1 \wedge \cdots \wedge \overset{(\alpha)}{V_i} \wedge \cdots \wedge E_m \rangle, \end{aligned}$$

for $t = m+1,\dots,n$, $r = m+k+1,\dots,n$, $h = m+1,\dots,m+k$, $\alpha = 1,\dots,m$, $i = m+k+1,\dots,n$ and $\deg(V_i) > \deg(E_\alpha)$. Then we have that $\tilde{c}_{ijt\alpha}$ is equal to 1 for $i = t > m+k$, $\alpha = j$ and $\deg(V_i) > \deg(E_j)$ or equal to zero otherwise. Moreover, we notice that $\tilde{a}_{ijh\alpha}$ and $\tilde{b}_{ijr\alpha}$ are different from zero only when $\alpha = j$ and in particular we have

$$a_{ijh} := \tilde{a}_{ijhj} = \langle V_i, [E_j, V_h]\rangle, \tag{3.6.4}$$

for $h = m+1,\dots,m+k$, $i = m+k+1,\dots,n$, $\deg(V_i) > \deg(E_j)$ and

$$b_{ijr} := \tilde{b}_{ijrj} = \langle V_i, [E_j, V_r]\rangle, \tag{3.6.5}$$

for $i, r = m+k+1,\dots,n$ and $\deg(V_i) > \deg(E_j)$. Therefore V is admissible if and only if

$$E_j(f_i) = -\sum_{r=m+k+1}^{n} b_{ijr}\, f_r - \sum_{h=m+1}^{m+k} a_{ijh}\, g_h, \tag{3.6.6}$$

for $i = m+k+1,\dots,n$ and $\deg(V_i) > \deg(E_j)$.

Remark 3.6.1. Notice that the coefficients a_{ijh} and b_{ijr} defined in (3.6.4) and (3.6.5) are defined almost everywhere. Indeed the vector fields $E_1,\dots,E_m, V_{m+1},\dots,V_n$ are Lipschitz, then thanks to [32] the Lie brackets $[E_j, V_h]$ and $[E_j, V_r]$ for $j = 1,\dots,m$, $h = m+1,\dots,m+k$ and $r = m+k+1,\dots,n$ are defined almost everywhere.

Example 3.6.2 (Lagrangian submanifolds)**.** Given $n > 1$ we consider the Heisenberg group $\mathbb{H}^n$, previously described in Example 2.7.3, with its distribution $\mathcal{H}$ generated by

$$X_i = \frac{\partial}{\partial_{x_i}} + \frac{y_i}{2}\frac{\partial}{\partial_t}, \quad Y_i = \frac{\partial}{\partial_{y_i}} - \frac{x_i}{2}\frac{\partial}{\partial_t} \quad i = 1,\dots,n.$$

The Reeb vector fields is provided by $T = \partial_t = [X_i, Y_i]$ for $i = 1,\dots,n$. Let $g = \langle\cdot,\cdot\rangle$ be the Riemannian metric that make $(X_1,\dots,X_n,Y_1,\dots,Y_n,T)$ an orthonormal basis. We rename the horizontal vector fields $X_1,\dots,X_n,Y_1,\dots,Y_n$ with $Z_1,\dots,Z_n,Z_{n+1},\dots,Z_{2n}$. Let Ω be an open set of $\mathbb{R}^m$, with $m \leqslant n$. Here we consider a C^1 immersion $\Phi : \Omega \to \mathbb{H}^n$ such $M = \Phi(\Omega)$ is a Lagrangian submanifold. Let $E_1,\dots,E_m$ be an orthonormal local frame of TM, then there exist $Z_{i_1},\dots,Z_{i_n}$ such that $[Z_{i_\alpha}, Z_{i_\beta}] = 0$ for each $\alpha,\beta = 1,\dots,n$ and

$$E_j = \sum_{\alpha=1}^{n} a_j^\alpha(\bar{p}) Z_{i_\alpha} \quad \text{for} \quad j = 1,\dots,m, \tag{3.6.7}$$

where the matrix $A = (a_j^\alpha(\bar{p}))_{j=1,\dots,m}^{\alpha=1,\dots,n}$ has full rank equal to m, for each $\bar{p} \in \Omega$. Therefore a vector field $V = \sum_{i=1}^n g_i X_i + g_{i+n} Y_i + f T$ is admissible if and only if it satisfies the system

$$E_j(f) = -\langle [E_j, T], T\rangle f - \sum_{i=1}^n (\langle [E_j, X_i], T\rangle g_i + \langle [E_j, Y_i], T\rangle g_{i+n}),$$

that is equivalent to

$$E_j(f) = -\sum_{\alpha=1}^n a_j^\alpha g_{i_\alpha}.$$

Since $\text{rank}(A(\bar{p})) = m$ for each $\bar{p} \in \Omega$ we have that M is strongly regular at each point $p \in M$. Thus, by Theorem 3.5.2 for each point $\bar{p} \in \Omega$ there exists a neighborhood $W_{\bar{p}} \subset \Omega$ such that each admissible vector field compactly support in $W_{\bar{p}}$ is integrable. Notice that in this case in order to write the system we only need that the coefficients a_j^α are continuous and not Lipschitz.

Here we exhibit an example of 2D plane immersed in 6D space that admits compactly supported admissible vector fields only in one direction.

Example 3.6.3. Let $N = \mathbb{R}^6$ and $\mathcal{H}^1 = \text{span}\{X_1, X_2, X_3\}$, where

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2}, \quad X_3 = \partial_{x_3} + x_1\partial_{x_4} + x_2\partial_{x_5} + \frac{x_2^2}{2}\partial_6$$

and $X_4 = [X_1, X_3] = \partial_{x_4}$, $X_5 = [X_2, X_3] = \partial_{x_4} + x_2\partial_{x_6}$ that with X_1, X_2, X_3 generate $\mathcal{H}^2$. Finally the last sub-bundle $\mathcal{H}^3$ is obtained adding $X_6 = [X_2, X_5] = \partial_{x_6}$. Notice that $(\mathbb{R}^6, \mathcal{H}^1)$ is a Carnot group whose growth vector is $(3, 5, 6)$. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Let $\Phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^6$ be the immersion given by

$$\Phi(v, \omega) = (0, u, 0, v, 0, 0).$$

Since $\Phi_v \wedge \Phi_u = X_2 \wedge X_4$ the degree $\deg(\Sigma) = 3$. Let $g = \langle \cdot, \cdot \rangle$ be the metric that makes $X_1, \dots, X_6$ an orthonormal basis. An admissible vector field $V^\perp = g_1 X_1 + g_3 X_3 + f_5 X_5 + f_6 X_6$ normal to $T\Sigma$ has to satisfies (3.6.6). In this case a straightforward computation, based on the commutator relations, shows that (3.6.6) is equivalent to

$$\begin{cases} X_2(f_5) &= -g_3 \\ X_2(f_6) &= -f_5 \\ X_4(f_6) &= 0. \end{cases} \tag{3.6.8}$$

Notice that we have $d\Phi(\partial_u) = X_2$ and $d\Phi(\partial_v) = X_4$, therefore on surface we have

$$\begin{cases} \dfrac{\partial f_5}{\partial u}(u,v) = -g_3(u,v) \\ \dfrac{\partial f_6}{\partial u}(u,v) = -f_5(u,v) \\ \dfrac{\partial f_6}{\partial v}(u,v) = 0. \end{cases} \tag{3.6.9}$$

Following [52] a sufficient and necessary condition for the existence and unicity of a solution for (3.6.9) si given by

$$\frac{\partial f_5}{\partial v}(u,v) = 0.$$

If we seek for a compactly supported admissible vector fields $V^\perp$, following the same argument of Example 3.5.7, we deduce that $f_6 = f_5 = g_3 \equiv 0$ on Ω. Hence the only compactly supported vector field is given by $V^\perp = g_1 X_1$ for each $g_1 \in C_0^1(\Omega)$.

Remark 3.6.4. Let $(N, \mathcal{H})$ be a Carnot manifold such that $\mathcal{H} = \ker(\theta)$ where θ is a $\mathbb{R}^{n-\ell}$ one form. Following [50, 83] we say that an immersion $\Phi : \bar{M} \to N$ is horizontal when the pull-back $\Phi^*\theta = 0$ and, given a point $p \in \Phi(\bar{M})$, the subspace $T_pM \subset \mathcal{H}_p$ is regular if the map

$$V \to (\iota_V d\theta)_{|T_pM} \tag{3.6.10}$$

is onto for each horizontal vector V on $\bar{M}$. Let X be an horizontal extension of V on N and Y be another horizontal vector field on N, then

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) = -\theta([X,Y])$$

Assume that the local frame $E_1, \ldots, E_m$ generate T_pM at p then the map (3.6.10) is given by $\theta([X, E_j](p))$, for each $j = 1, \ldots, m$. In our notation the surjectivity of this map coincides with the pointwise condition of maximal rank of the matrix (a_{ijh}) introduced in equation (3.6.4). Since by equation (3.4.18) the rank of A is independent of the metric g we deduce that this regularity notion introduced by [50, 49] is equivalent to strongly regularity at $\bar{p}$ (Definition 3.5.1) for the class of horizontal immersions.

3.7 Ruled submanifolds in graded manifolds

In this section we consider a particular type of submanifolds for which the admissibility system reduces to a system of ODEs along the characteristic curves, that rule these submanifolds by determining their degree since the other adapted tangent vectors tangent to M have highest degree equal to s.

Let $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ be an equiregular graded manifold. We consider an immersion $\Phi : \bar{M} \to N$ such that

$$\deg(M) = (m-1)s + \iota_0, \tag{3.7.1}$$

where $M := \Phi(\bar{M})$ and $m = \dim(\bar{M})$. Furthermore we assume $n_s - n_{s-1} = m-1$, where $n_s = \dim(\mathcal{H}^s)$ and $n_{s-1} = \dim(\mathcal{H}^{s-1})$. Let p be a point of maximum degree in M. Let $e_1, \ldots, e_m$ be a basis of T_pM adapted to the flag (1.2.10). Therefore $\deg(e_1) = \iota_0$ and $\deg(e_j) = s$ for $j = 2, \ldots, m$ and $k = n_{\iota_0} - 1$. Then we follow the construction described in Section 3.6 to provide the metric g and the orthonormal basis $E_1, \ldots, E_m, V_{m+1}, \ldots, V_n$ whose sorting is an adapted basis. Since $\deg(E_j) \geqslant \deg(V_i)$ for each $j = 2, \ldots, m$ and $i = m+k+1, \ldots, n$, the only derivative that appears in (3.6.6) is E_1. Therefore we deduce that a vector field $V^\perp$, given by equation (3.6.2), is admissible if and only if it satisfies

$$E_1(f_i) + \sum_{r=m+k+1}^{n} b_{i1r} f_r + \sum_{h=m+1}^{m+k} a_{i1h} g_h = 0, \tag{3.7.2}$$

for $i = m+k+1, \ldots, n$ and

$$a_{i1h}(p) = \langle v_i, [E_1, V_h](p) \rangle,$$

and

$$b_{i1r}(p) = \langle v_i, [E_1, V_r](p) \rangle.$$

Given p in M each point q in a local neighborhood U of p in M can be reached using the exponential map as follows

$$q = \exp(x_1 E_1) \exp\left(\sum_{j=2}^{m} x_j E_j \right)(p).$$

On this open neighborhood U we consider the local coordinates $x = (x_1, x_2, \ldots, x_m)$ given by logarithmic map Ξ. We set $\hat{x} := (x_2, \ldots, x_m)$. Given a relative compact open subset $\Omega \subset\subset \Xi(U)$ we consider

$$\Sigma_0 = \{x_1 = 0\} \cap \Omega$$

be the $(m-1)$-dimensional leaf normal to E_1. Then there exists $\varepsilon > 0$ so that the closure of the cylinder

$$\Omega_\varepsilon = \{(x_1, \hat{x}) \ : \ 0 < x_1 < \varepsilon, \ \hat{x} \in \Sigma_0\} \tag{3.7.3}$$

is contained in $\Xi(U)$. Then $\Sigma_\varepsilon = \{(\varepsilon, \hat{x}) \ : \ \hat{x} \in \Sigma_0\}$ is the top of the cylinder. Since $d\Xi(E_1) = \partial_{x_1}$ in this logarithmic coordinates the *admissibility system* (3.7.2) is given by

$$\frac{\partial F(x)}{\partial x_1} = -B(x)F(x) - A(x)G(x), \tag{3.7.4}$$

where we set

$$F = \begin{pmatrix} f_{m+k+1} \\ \vdots \\ f_n \end{pmatrix}, \quad G = \begin{pmatrix} g_{m+1} \\ \vdots \\ g_{m+k} \end{pmatrix} \tag{3.7.5}$$

and we denote by B the $(n-m-k)$ square matrix whose entries are b_{i1r}, by A the $(n-m-k) \times k$ matrix whose entries are a_{i1h}.

3.8 The high dimensional holonomy map for ruled submanifolds

For general submanifolds we are not able to provide a satisfactory generalization of the holonomy map described for curves in Section 2.4. The main difficulty is that we do not know how to verify a priori the compatibility conditions [52, Eq. (1.4), Chapter VI], that are necessary and sufficient conditions for the uniqueness and the existence of a solution of the admissibility system (3.6.6) (see [52, Theorem 3.2, Chapter VI]). However, for ruled submanifolds the system (3.6.6) reduces to the system of ODEs (3.7.2) along the characteristic curves. Therefore, a uniqueness and existence result for the solution is given by the classical Cauchy-Peano Theorem, as in the case of curves in Section 2.4. We thought it reasonable to organize this section following the same structure of Section 2.4. The main difference is that the target space of the high dimensional holonomy map is the Banach space of continuous functions on the foliation perpendicular to the characteristic curves.

Let Ω_ε be the open cylinder defined in (3.7.3) and $T_{\Sigma_0}(f) = f(0, \cdot)$ and $T_{\Sigma_\varepsilon}(f) = f(\varepsilon, \cdot)$ be the operators that evaluate functions at $x_1 = 0$ and at $x_1 = \varepsilon$, respectively. Then we consider the following spaces:

1. $\mathcal{H}_0(\Omega_\varepsilon) = \left\{ \sum_{i=m+1}^{m+k} g_i V_i \ : \ g_i \in C_0(\Omega_\varepsilon) \right\}$.
2. $\mathcal{V}^1(\Omega_\varepsilon) = \left\{ \sum_{i=m+k}^{n} f_i V_i \ : \ \partial_{x_1} f_i \in C(\bar{\Omega}_\varepsilon), f_i \in C(\bar{\Omega}_\varepsilon), T_{\Sigma_0}(f_i) = 0 \right\}$.
3. $\mathcal{V}(\Sigma_\varepsilon)$ is the set of compactly supported vertical vector fields in $C_0(\Sigma_\varepsilon, \mathbb{R}^{n-m-k})$ normal to M.

Therefore the Cauchy problem allows us to define the holonomy type map

$$H_M^\varepsilon : \mathcal{H}_0(\Omega_\varepsilon) \to \mathcal{V}(\Sigma_\varepsilon), \tag{3.8.1}$$

in the following way: we consider a horizontal compactly supported continuous vector

$$Y_h = \sum_{l=m+1}^{m+k} g_l V_l \in \mathcal{H}_0(\Omega_\varepsilon)$$

we fix the initial condition $Y_v(0, \hat{x}) = 0$. Then there exists a unique solution

$$Y_v = \sum_{r=m+k+1}^{n} f_r V_r \in \mathcal{V}^1(\Omega_\varepsilon)$$

of the admissibility system (3.7.4) with initial condition $Y_v(0, \hat{x}) = 0$. Letting

$$\mathbf{T}_{\Sigma_\varepsilon} : \mathcal{V}^1(\Omega_\varepsilon) \to \mathcal{V}(\Sigma_\varepsilon)$$

be the evaluating operator for vertical vectors fields at $x_1 = \varepsilon$ defined by $\mathbf{T}_{\Sigma_\varepsilon}(V) = V(\varepsilon, \cdot)$, we define $H_M^\varepsilon(Y_h) = \mathbf{T}_{\Sigma_\varepsilon}(Y_v)$.

Definition 3.8.1. We say that Φ restricted to $\bar{\Omega}_\varepsilon$ is *regular* if the holonomy map H_M^ε is surjective.

The following result allows the integration of the differential system (3.7.4) to explicitly compute the holonomy map.

Proposition 3.8.2. *In the above conditions, there exists a square regular matrix $D(x_1, \hat{x})$ of order $(n - k - m)$ such that*

$$F(\varepsilon, \hat{x}) = -D(\varepsilon, \hat{x})^{-1} \int_0^\varepsilon (DA)(\tau, \hat{x}) G(\tau, \hat{x}) \, d\tau, \tag{3.8.2}$$

for each $\hat{x} \in \Sigma_0$.

Proof. Lemma 3.8.3 below allows us to find a regular matrix $D(x_1, \hat{x})$ such that $\partial_{x_1} D = DB$. Then equation $\partial_{x_1} F = -BF - AG$ is equivalent to $\partial_{x_1}(DF) = -DAG$. Integrating between 0 and ε, taking into account that $F(0, \hat{x}) = 0$ for each $\hat{x} \in \Sigma_0$, and multiplying by $D(\varepsilon, \hat{x})^{-1}$, we obtain (3.8.2). □

Lemma 3.8.3. *Let E be an open set of $\mathbb{R}^{m-1}$. Let $B(t, \lambda)$ be a continuous family of square matrices on $[0, \varepsilon] \times E$. Let $D(t, \lambda)$ be the solution of the Cauchy problem*

$$\partial_t D(t, \lambda) = D(t, \lambda) B(t, \lambda) \text{ on } [0, \varepsilon] \times E, \quad D(0, \lambda) = I_d,$$

for each $\lambda \in E$. Then $\det D(t, \lambda) \neq 0$ for each $(t, \lambda) \in [0, \varepsilon] \times E$.

Definition 3.8.4. We say that the matrix $\tilde{A}(x_1, \hat{x}) := (DA)(x_1, \hat{x})$ on Ω_ε defined in Proposition 3.8.2 is linearly full $\mathbb{R}^{n-m-k}$ if and only if for each $\hat{x} \in \Sigma_0$

$$\dim \left(\operatorname{span} \left\{ \tilde{A}^1(x_1, \hat{x}), \dots, \tilde{A}^k(x_1, \hat{x}) \quad \forall\, x_1 \in [0, \varepsilon] \right\} \right) = n - m - k,$$

where $\tilde{A}^i$ for $i = m+1, \dots, m+k$ are the columns of $\tilde{A}(x_1, \hat{x})$.

Lemma 3.8.5. *Let $\mathcal{L} : X \to Y$ be a linear closed operator of Banach spaces. Then $\mathcal{L}$ is not surjective if and only if there exists $\mu \in Y^*$, $\mu \not\equiv 0$ such that $\mu(y) = 0$ for each $y \in Range(\mathcal{L})$.*

Proof. Assume that $\mathcal{L}$ is not surjective, namely the subspace $\mathrm{Range}(\mathcal{L}) = \overline{\mathrm{Range}(\mathcal{L})} \subsetneq Y$, then by [9, Corollary 1.8] we obtain the result. Conversely by contradiction assume that $\mathrm{Range}(\mathcal{L}) = Y$, but by assumption there exists a a dual function $\mu \not\equiv 0$ such that $\mu(y) = 0$ for each $y \in Y$, which is absurd. □

Proposition 3.8.6. *The immersion Φ restricted to $\bar{\Omega}_\varepsilon$ is regular if and only if $\tilde{A}(x_1, \hat{x})$ is linearly full in $\mathbb{R}^{n-m-k}$.*

Proof. Assume that the holonomy map is not surjective. The representation formula (3.8.2) allows us to deduce that the linear map H^ε_Σ is closed, since the limit of integrals of an uniform sequence of continuos functions converges to the integral of the uniform limit of the sequence. Since the dual of the space of compactly supported continuous functions is the space of Radon measures (see [34, Chapter 7]), by Lemma 3.8.5 there exists a Radon measure $\mu \neq 0$ and a continuous row vector $\Gamma(\hat{x})$ such that

$$\begin{aligned} 0 = \Gamma\mu(F(\varepsilon, \cdot)) &= -\mu \left(\Gamma(\hat{x}) D(\varepsilon, \hat{x})^{-1} \int_0^\varepsilon (DA)(\tau, \hat{x}) G(\tau, \hat{x})\, d\tau \right) \\ &= - \int_0^\varepsilon \int_{\Sigma_0} \tilde{\Gamma}(\hat{x}) (DA)(\tau, \hat{x}) G(\tau, \hat{x})\, d\mu(\hat{x}) d\tau \end{aligned}$$

where $\tilde{\Gamma} = \Gamma(\hat{x})D(\varepsilon, \hat{x})^{-1} \neq 0$. As this formula holds for any $G(t, \hat{x})$, we have $\Gamma(\hat{x})\tilde{A}(t, \hat{x}) = 0$ for all $t \in [a, b]$ and μ-a.e. in $\hat{x}$. Since the $\text{supp}(\mu) \neq \emptyset$ there exists $\hat{x}_0 \in \text{supp}(\mu)$ such that $\Gamma(\hat{x}_0)\tilde{A}(t, \hat{x}_0) = 0$, then their columns are contained in the hyperplane of $\mathbb{R}^{n-m-k}$ determined by $\Gamma(\hat{x}_0)$. Hence we deduce that $\tilde{A}$ is not linearly full.

Conversely, assume that $\tilde{A}$ is not linearly full. Then there exist a point $\hat{x}_0 \in \Sigma_0$ and a row vector with $(n - m - k)$ coordinates $\Gamma \neq 0$ such that $\Gamma\tilde{A}(x_1, \hat{x}_0) = 0$ for all $x_1 \in [0, \varepsilon]$. Then, denoting by $\delta_{\hat{x}_0}(\varphi) = \varphi(\hat{x}_0)$ the delta distribution, we have

$$\Gamma\delta_{\hat{x}_0}(D(\varepsilon, \cdot)F(\varepsilon, \cdot)) = -\int_0^{\varepsilon} \Gamma(DA)(\tau, \hat{x}_0)G(\tau, \hat{x}_0)\, d\tau = 0$$

Since the Radon measure $\delta_{\hat{x}_0}$ annihilates the image of the holonomy map by Lemma 3.8.5 we conclude that the holonomy map is not surjective. □

The following result provides a useful characterization of non-regularity

Theorem 3.8.7. *The immersion Φ restricted to $\bar{\Omega}_\varepsilon$ is non-regular if and only if there exist a point $\hat{x}_0 \in \Sigma_0$ and a row vector field $\Lambda(x_1, \hat{x}_0) \neq 0$ for all $x_1 \in [0, \varepsilon]$ that solves the following system*

$$\begin{cases} \partial_{x_1}\Lambda(x_1, \hat{x}_0) = \Lambda(x_1, \hat{x}_0)B(x_1, \hat{x}_0) \\ \Lambda(x_1, \hat{x}_0)A(x_1, \hat{x}_0) = 0. \end{cases} \tag{3.8.3}$$

Proof. Assume that Φ restricted to $\bar{\Omega}_\varepsilon$ is non-regular, then by Proposition 3.8.6 there exist a point $\hat{x}_0 \in \Sigma_0$ and a row vector $\Gamma \neq 0$ such that

$$\Gamma D(x_1, \hat{x}_0)A(x_1, \hat{x}_0) = 0$$

for all $x_1 \in [0, \varepsilon]$, where $D(x_1, \hat{x}_0)$ solves

$$\begin{cases} \partial_{x_1} D = DB \\ D(0, \hat{x}_0) = I_{n-m-k}. \end{cases} \tag{3.8.4}$$

Since Γ is a constant vector and $D(x_1, \hat{x}_0)$ is a regular matrix by Lemma 3.8.3 , $\Lambda(x_1, \hat{x}_0) := \Gamma D(x_1, \hat{x}_0)$ solves the system (3.8.3) and $\Lambda(x_1, \hat{x}_0) \neq 0$ for all $x_1 \in [0, \varepsilon]$.

Conversely, any solution of the system (3.8.3) is given by

$$\Lambda(x_1, \hat{x}_0) = \Gamma D(x_1, \hat{x}_0),$$

where $\Gamma = \Lambda(0, \hat{x}_0) \neq 0$ and $D(x_1, \hat{x}_0)$ solves the equation (3.8.4). Indeed, let us consider a general solution $\Lambda(t, \hat{x}_0)$ of (3.8.3). If we set

$$\Psi_{\hat{x}_0}(t) = \Lambda(t, \hat{x}_0) - \Gamma D(t, \hat{x}_0),$$

where $\Gamma = \Lambda(0, \hat{x}_0) \neq 0$ and $D(t, \hat{x}_0)$ solves the equation (3.8.4), then we deduce

$$\begin{cases} \partial_t \Psi_{\hat{x}_0}(t) = \Psi_{\hat{x}'}(t) B(t, \hat{x}_0) \\ \Psi_{\hat{x}_0}(0) = 0. \end{cases}$$

Clearly the unique solution of this system is $\Psi_{\hat{x}_0}(t) \equiv 0$. Hence we conclude that $\Gamma \tilde{A}(x_1, \hat{x}_0) = 0$. Thus $\tilde{A}(x_1, \hat{x}_0)$ is not fully linear and by Proposition 3.8.6 we are done. □

3.9 Integrability of admissible vector fields for a ruled regular submanifold

Since for ruled submanifolds we have the notion of regularity given by high dimensional holonomy map, in this section we deduce a deformability global result (Theorem 3.9.5) for immersed submanifolds in analogy with Theorem 2.5.4. Indeed the regularity assumption, that comes from a solution of a system of ODEs, allows us to produce admissible variations of the original immersion compactly supported in an arbitrary set. This result is sharper than the one obtained for general submanifolds (Theorem 3.5.2), where we provide only variations of the original immersion compactly supported in an open neighborhood of the strongly regular point.

Let $(N, \mathcal{H}^1, \dots, \mathcal{H}^s)$ be an equiregular graded manifold. We consider a m-dimensional immersion $\Phi : \bar{M} \to N$ such that $n_s - n_{s-1} = m - 1$. Furthermore we assume that $\deg(M) = (m-1)s + \iota_0$ and

$$s - 3 \leqslant \iota_0 \leqslant s - 1. \tag{3.9.1}$$

Remark 3.9.1. Since $n_s - n_{s-1} = m - 1$ and the condition (3.9.1) holds we have that the only simple m-vectors of degree strictly grater than $\deg(M)$ are

$$V_i \wedge E_2 \wedge \cdots \wedge E_m$$

for $i = m + k + 1, \dots, n$. When $\iota_0 = s - 1$ the submanifold has maximum degree therefore all vector fields are admissible, thus there are no singular submanifold.

Keeping the previous notation we now consider the following spaces

1. $\mathcal{H}(\Sigma_0) = \left\{ Y_H = \sum_{i=m+1}^{m+k} g_i V_i \ : \ g_i \in C(\bar{\Omega}_\varepsilon), T_{\Sigma_0}(g_i) = 0 \right\}$ where the norm is given by
$$\|Y_h\|_\infty := \max_{i=m+1,\ldots,m+k} \sup_{x\in\bar{\Omega}_\varepsilon} |g_i|$$

2. $\mathcal{V}^1(\Sigma_0) = \left\{ Y_v = \sum_{i=m+k}^{n} f_i V_i \ : \ \partial_{x_1} f_i \in C(\bar{\Omega}_\varepsilon), f_i \in C(\bar{\Omega}_\varepsilon), T_{\Sigma_0}(f_i) = 0 \right\}$, where the norm is given by
$$\|Y_v\|_1 := \max_{i=m+k,\ldots,n} (\sup_{x\in\bar{\Omega}_\varepsilon} |f_i| + \sup_{x\in\bar{\Omega}_\varepsilon} |\partial_{x_1} f_i|)$$

3. $\Lambda(\Sigma_0)$ is the set of elements given by
$$\sum_{i=m+k+1}^{n} z_i(x_1,\ldots,x_m)\ V_i \wedge E_2 \wedge \cdots \wedge E_m$$
where $z_i \in C(\bar{\Omega}_\varepsilon)$ vanishing on Σ_0.

We denote by Π_d the orthogonal projection over the space $\Lambda(\Sigma_0)$, that is the bundle over the vector space of simple m-vectors of degree strictly grater than d, thanks to Remark 3.9.1. Then we set

$$\mathcal{G} : \mathcal{H}(\Sigma_0) \times \mathcal{V}^1(\Sigma_0) \to \mathcal{H}(\Sigma_0) \times \Lambda(\Sigma_0), \tag{3.9.2}$$

defined by
$$\mathcal{G}(Y_1, Y_2) = (Y_1, \mathcal{F}(Y_1 + Y_2)),$$
where
$$\mathcal{F}(Y) = \Pi_d \left(d\Gamma(Y)(e_1) \wedge \ldots \wedge d\Gamma(Y)(e_m) \right),$$
and $\Gamma(Y)(p) = \exp_{\Phi(p)}(Y_p)$. Observe that now $\mathcal{F}(Y) = 0$ implies that the degree of the variation $\Gamma(Y)$ is less than or equal to d. Then

$$DG(0,0)(Y_1, Y_2) = (Y_1, DF(0)(Y_1 + Y_2)),$$

where $DF(0)Y$ is given by

$$DF(0)Y = \sum_{i=m+k+1}^{n} \Big(\frac{\partial f_i(x)}{\partial x_1} + \sum_{r=m+k}^{n} b_{i1r} f_r + \sum_{h=m+1}^{m+k} a_{i1h} g_h \Big) V_i \wedge E_2 \wedge \cdots \wedge E_m.$$

Observe that $DF(0)Y = 0$ if and only if Y is an admissible vector field, namely Y solves (3.7.4).
Our objective now is to prove that the map $DG(0,0)$ is an isomorphism of Banach spaces. To show this, we shall need the following result.

Proposition 3.9.2. *The differential $DG(0,0)$ is an isomorphism of Banach spaces.*

Proof. We first observe that $DG(0,0)$ is injective, since $DG(0,0)(Y_1,Y_2) = (0,0)$ implies that $Y_1 = 0$ and that the vertical vector field Y_2 satisfies the compatibility equations with initial condition $Y_2(0,\hat{x}) = 0$ for each $\hat{x} \in \Sigma_0$. Hence $Y_2 = 0$. The map $DG(0,0)$ is continuous. Indeed, if for instance we consider the 1-norm on the product space we have

$$\begin{aligned} \|DG(0,0)(Y_1,Y_2)\| &= \|(Y_1, DF(0)(Y_1+Y_2))\| \\ &\leqslant \|Y_1\|_\infty + \|DF(0)(Y_1+Y_2))\|_\infty \\ &\leqslant (1 + \|(a_{hij})\|_\infty)\|Y_1\|_\infty + (1 + \|(b_{rij})\|_\infty)\|Y_2\|_1. \end{aligned}$$

To show that $DG(0,0)$ is surjective, we take (Y_1,Y_2) in the image, and we find a vector field Y on Ω_ε such that $Y_H = Y_1$, $D\mathcal{F}(0)(Y) = Y_2$ and $Y_v(0,\hat{x}) = 0$. The map $DG(0,0)$ is open because of the estimate (3.9.3) given in Lemma 3.9.3 below. □

Lemma 3.9.3. *In the above conditions, assume that $D\mathcal{F}(0)(Y) = Y_2$ and $Y_h = Y_1$ and $Y(a) = 0$. Then there exists a constant K such that*

$$\|Y_v\|_1 \leqslant K(\|Y_2\|_\infty + \|Y_1\|_\infty) \tag{3.9.3}$$

Proof. We write

$$Y_1 = \sum_{h=m+1}^{m+k} g_h V_h, \quad Y_2 = \sum_{i=k+1}^{n} z_i V_i \wedge E_2 \wedge \cdots \wedge E_m \quad \text{and} \quad Y_v = \sum_{r=k+1}^{n} f_r V_r.$$

Then Y_v is a solution of the ODE (3.7.4) given by

$$\partial_{x_1} F(x_1,\hat{x}) = -B(x)F(x_1,\hat{x}) + Z(x_1,\hat{x}) - A(x)G(x_1,\hat{x}) \tag{3.9.4}$$

where $B(x), A(x)$ are defined after (3.7.5), F, G are defined in (3.7.5) and we set

$$Z = \begin{pmatrix} z_{m+k+1} \\ \vdots \\ z_n \end{pmatrix}.$$

Since $Y_v(0,\hat{x}) = 0$ an Y_v solves (3.9.4) in $(0,\varepsilon)$, by Lemma 3.9.4 there exists a constant K such that

$$\begin{aligned} \|Y_v\|_1 = \|F\|_1 &\leqslant K\|Z(x) - A(x)\ G(x)\|_\infty \\ &\leqslant \tilde{K}(\|Y_2\|_\infty + \|Y_1\|_\infty). \end{aligned} \tag{3.9.5}$$

where $\tilde{K} = K \max\{1, \|A(x)\|_\infty\}$. □

Lemma 3.9.4. *Let E be an open set of $\mathbb{R}^{m-1}$. Let $u : [0,\varepsilon] \times E \to \mathbb{R}^d$ be the solution of the inhomogeneous problem*

$$\begin{cases} u'(t,\lambda) = A(t,\lambda)u(t,\lambda) + c(t,\lambda), \\ u(0,\lambda) = u_0(\lambda) \end{cases} \tag{3.9.6}$$

where $A(t,\lambda)$ is a $d \times d$ continuous matrix on $[0,\varepsilon] \times E$ and $c(t,\lambda)$ a continuous vector field on $[0,\varepsilon] \times E$. We denote by u' the partial derivative $\partial_t u$. Then, there exists a constant K such that

$$\|u\|_1 := \|u\|_\infty + \|u'\|_\infty \leqslant K(\|c\|_\infty + |u_0|_\infty). \tag{3.9.7}$$

Proof. We start from the case $r = 1$. By [52, Lemma 4.1] it follows

$$u(t,\lambda) \leqslant \left(|u_0(\lambda)| + \int_0^t |c(s,\lambda)| ds \right) e^{\left| \int_0^t \|A(s,\lambda)\| ds \right|},$$

for each $\lambda \in E$ and where the norm of A is given by $\sup_{|x|=1} |A\ x|$. Therefore we have

$$\sup_{t\in[0,\varepsilon]} \sup_{\lambda\in E} |u(t,\lambda)| \leqslant C_1 (\sup_{t\in[0,\varepsilon]} \sup_{\lambda\in E} |c(t,\lambda)| + \sup_{\lambda\in E} |u_0(\lambda)|), \tag{3.9.8}$$

where we set

$$C_1 = \varepsilon e^{\varepsilon \sup_{t\in[0,\varepsilon]} \sup_{\lambda\in E} \|A(t,\lambda)\|}.$$

Since u is a solution of (3.9.6) it follows

$$\begin{aligned}\sup_{t\in[0,\varepsilon]}\sup_{\lambda\in E}|u'(t,\lambda)| &\leqslant \sup_{t\in[0,\varepsilon]}\sup_{\lambda\in E}\|A(t,\lambda)\| \sup_{t\in[0,\varepsilon]}\sup_{\lambda\in E}|u(t,\lambda)| + \sup_{t\in[0,\varepsilon]}\sup_{\lambda\in E}|c(t,\lambda)| \\ &\leqslant (C_2+1)\sup_{t\in[0,\varepsilon]}\sup_{\lambda\in E}|c(t,\lambda)|.\end{aligned} \tag{3.9.9}$$

Hence by (3.9.8) and (3.9.9) we obtain

$$\|u\|_1 \leqslant K(\|c\|_\infty + \|u_0\|_\infty).$$ □

Finally, we use the previous constructions to give a criterion for the integrability of admissible vector fields along a horizontal curve.

Theorem 3.9.5. *Let $\Phi : \bar{M} \to N$ be an immersion into an equiregular graded manifold $(N, \mathcal{H}^1, \ldots, \mathcal{H}^s)$ such as condition such that $\deg(M) = (m-1)s + \iota_0$, where $m = \dim(\bar{M})$ and $s-3 \leqslant \iota_0 \leqslant s-1$. Let $\Omega_\varepsilon = \{(x_1, \hat{x}) \ : \ 0 < x_1 < \varepsilon, \ \hat{x} \in \Sigma_0\}$. Assume that Φ is regular on the compact $\bar{\Omega}_\varepsilon$. Then every admissible vector field with compact support in Ω_ε is integrable.*

Proof. If $\iota_0 = s-1$ all vector fields are admissible, then all immersions are automatically regular. Each vector field is integrable for instance by the exponential map.

Let now $s-3 \leqslant \iota_0 \leqslant s-2$. Let us take V vector field on Ω_ε and $\{V^i\}_{i=1}^\infty$ vector fields equi-bounded in the supremum norm on $\bar{\Omega}_\varepsilon$. Let $l^1(\mathbb{R})$ the space of summable sequences. We consider the map

$$\tilde{\mathcal{G}} : \Big[(-\varepsilon, \varepsilon) \times l^1(\mathbb{R})\Big] \times \mathcal{H}(\Sigma_0) \times \mathcal{V}^1(\Sigma_0) \to \mathcal{H}(\Sigma_0) \times \Lambda(\Sigma_0),$$

given by

$$\tilde{\mathcal{G}}((s, (s_i), Y_1, Y_2)) = (Y_1, F(sV + \sum_{i=1}^{\infty} s_i V^i + Y_1 + Y_2)).$$

The map $\tilde{\mathcal{G}}$ is continuous with respect to the product norms (on each factor we put the natural norm, the Euclidean one on the interval, the l^1 norm and $||\cdot||_\infty$ and $||\cdot||_1$ in the spaces of vectors on Ω). Moreover

$$\tilde{\mathcal{G}}(0,0,0,0) = (0,0),$$

since the curve γ is horizontal. Denoting by D_Y the differential with respect to the last two variables of $\tilde{G}$ we have that

$$D_Y\tilde{\mathcal{G}}(0,0,0,0)(Y_1,Y_2) = D\mathcal{G}(0,0)(Y_1,Y_2)$$

is a linear isomorphism. We can apply the Implicit Function Theorem to obtain maps

$$Y_1 : (-\varepsilon,\varepsilon)\times l^1(\varepsilon) \to \mathcal{H}(\Sigma_0), \quad Y_2 : (-\varepsilon,\varepsilon)\times l^1(\varepsilon) \to \mathcal{V}^1(\Sigma_0),$$

such that $\tilde{\mathcal{G}}(s,(s_i),(Y_1)(s,s_i),(Y_2)(s,s_i)) = (0,0)$. We denote by $l^1(\varepsilon)$ the ball of radio ε in Banach space $l^1(\mathbb{R})$. This implies that $(Y_1)(s,(s_i)) = 0$ and that

$$F(sV + \sum_{i=1}^{\infty} s_i V^i + Y_2(s,s_i)) = 0.$$

Hence the submanifolds

$$\Gamma(sV + \sum_i s_i V^i + Y_2(s,s_i))$$

have degree equal to or less than d.

Now we assume that V is an admissible vector field compactly supported on Ω_ε, and that V^i are admissible vector fields such that V^i_v vanishing on Σ_0. Then the vector field

$$\frac{\partial Y_2}{\partial s}(0,0), \frac{\partial Y_2}{\partial s_i}(0,0)$$

on Ω_ε are vertical and admissible. Since they vanish at $(0,\hat{x})$, they are identically 0.

Since the holonomy map is surjective we choose $\{V^i\}_{i=1}^{\infty}$ on Ω_ε such that $\{\mathbf{T}_{\Sigma_\varepsilon}(V^i_v)\}_{i\in\mathbb{N}}$ is a normalized Schauder basis for $\mathcal{V}(\Sigma_\varepsilon)$. Then we consider the map

$$\mathcal{P} : (-\varepsilon,\varepsilon)\times l^1(\varepsilon) \to \mathcal{C}_0(\Sigma_\varepsilon, N)$$

given by

$$(s,(s_i)) \mapsto \Gamma(sV + \sum_{i=1}^{\infty} s_i V^i + Y_2(s,s_i))_{|\Sigma_\varepsilon},$$

where $\mathcal{C}_0(\Sigma_\varepsilon, N)$ is the Banach manifold based on $C_0(\Sigma_\varepsilon,\mathbb{R}^n)$. Notice that

$$\frac{\partial \mathcal{P}(0,0)}{\partial s_i} = \mathbf{T}_{\Sigma_\varepsilon}(V^i) = \mathbf{T}_{\Sigma_\varepsilon}(V^i_v),$$

which is invertible since the holonomy map is surjective and

$$\frac{\partial \mathcal{P}(0,0)}{\partial s} = \mathbf{T}_{\Sigma_\varepsilon}(V) = 0,$$

since V is compactly supported in Ω_ε. Hence we can apply the Implicit Function Theorem to conclude that there exist $\varepsilon' < \varepsilon$ and a family of smooth functions $s_i(s)$, with $\sum_i |s_i(s)| < \varepsilon$ for all $s \in (-\varepsilon', \varepsilon')$, so that

$$\Gamma(sV + \sum_i s_i(s)V^i + Y_2(s, s_i(s)))$$

take the value $\Phi(\bar{p})$ for almost each $\bar{p} \in \Sigma_\varepsilon$. Since the vector fields $\{V^i\}_{i=1}^\infty$ are equibounded in the supremum norm on $\bar{\Omega}_\varepsilon$, the series $\sum_i s_i(s)V^i$ is absolutely convergent on $\bar{\Omega}_\varepsilon$.

Clearly, we have

$$\mathcal{P}(s,(s_i(s)))(\bar{p}) = \Phi(\bar{p}),$$

for almost each $\bar{p} \in \Sigma_\varepsilon$. Differentiating with respect to s at $s = 0$ we obtain

$$\frac{\partial \mathcal{P}(0,0)}{\partial s} + \sum_i \frac{\partial \mathcal{P}(0,0)}{\partial s_i} s_i'(0) = 0.$$

Therefore $s_i'(0) = 0$ for each $i \in \mathbb{N}$. Thus, the variational vector field to Γ is

$$\left.\frac{\Gamma(s)}{\partial s}\right|_{s=0} = V + \sum_i s_i'(0)V^i + \frac{\partial Y_2}{\partial s}(0,0) + \sum_i \frac{\partial Y_2}{\partial s_i}(0,0) = V. \tag{3.9.10}$$

□

3.10 First variation formula for submanifolds

In this section we shall compute a first variation formula for the area A_d of a submanifold of degree d. We shall give some definitions first. Assume that $\Phi : \bar{M} \to N$ is an immersion of a smooth m-dimensional manifold into an n-dimensional equiregular graded manifold endowed with a Riemannian metric g. Let $\mu = \Phi^* g$. Fix $\bar{p} \in \bar{M}$ and let $p = \Phi(\bar{p})$. Take a μ-orthonormal basis $(\bar{e}_1, \ldots, \bar{e}_m)$ in $T_{\bar{p}}\bar{M}$ and define $e_i := d\Phi_{\bar{p}}(\bar{e}_i)$ for $i = 1, \ldots, m$. Then the degree d area density Θ is defined by

$$\Theta(\bar{p}) := |(e_1 \wedge \ldots \wedge e_m)_d| = \left(\sum_{\deg(X_J)=d} \langle e_1 \wedge \ldots \wedge e_m, (X_J)_p \rangle^2 \right)^{1/2}, \tag{3.10.1}$$

where $(X_1,\ldots,X_n)$ is an orthonormal adapted basis of TN. Then we have

$$A_d(M) = \int_{\bar M} \Theta(\bar p) d\mu(\bar p).$$

Letting M_0 be the singular set defined in (1.2.9) we set

$$\bar M_0 := \Phi^{-1}(M_0). \tag{3.10.2}$$

Assume now that $V \in \mathfrak{X}(\bar M, N)$, then we set

$$(\operatorname{div}^d_{\bar M} V)(\bar p) := \sum_{i=1}^m \langle e_1 \wedge \ldots \wedge \nabla_{e_i} V \wedge \ldots \wedge e_m, (e_1 \wedge \ldots \wedge e_m)_d \rangle. \tag{3.10.3}$$

Finally, define the linear function f by

$$f(V_{\bar p}) := \sum_{\deg(X_J)=d} \langle e_1 \wedge \ldots \wedge e_m, \nabla_{V_{\bar p}} X_J \rangle \langle e_1 \wedge \ldots \wedge e_m, (X_J)_{\bar p} \rangle. \tag{3.10.4}$$

Then we have the following result

Theorem 3.10.1. *Let $\Phi : \bar M \to N$ be an immersion of degree d of a smooth m-dimensional manifold into an equiregular graded manifold equipped with a Riemannian metric g. Assume that there exists an admissible variation $\Gamma : \bar M \times (-\varepsilon, \varepsilon) \to N$ with associated variational field V compactly supported on $\bar M \smallsetminus \bar M_0$. Then*

$$\frac{d}{dt}\Big|_{t=0} A_d(\Gamma_t(\bar M)) = \int_{\bar M \smallsetminus \bar M_0} \frac{1}{\Theta(\bar p)} \Big((\operatorname{div}^d_{\bar M} V)(\bar p) + f(V_{\bar p}) \Big) d\mu(\bar p). \tag{3.10.5}$$

Proof. Fix a point $\bar p \in \bar M$. Clearly, $\mathcal{E}_i(t,\bar p) = d\Gamma_{(\bar p,t)}(\bar e_i)$, $i = 1,\ldots,m$, are vector fields along the curve $t \mapsto \Gamma(\bar p, t)$. Therefore, the first variation is given by

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} A(\Gamma_t(\bar M)) &= \int_{\bar M \smallsetminus \bar M_0} \frac{d}{dt}\Big|_{t=0} |\left(\mathcal{E}_1(t) \wedge \ldots \wedge \mathcal{E}_m(t)\right)_d| d\mu(\bar p) \\ &= \int_{\bar M \smallsetminus \bar M_0} \frac{d}{dt}\Big|_{t=0} \left(\sum_{\deg(X_J)=d} \langle \mathcal{E}_1(t) \wedge \ldots \wedge \mathcal{E}_m(t), X_J \rangle^2 \right)^{\frac{1}{2}} d\mu(\bar p). \end{aligned}$$

The derivative of the last integrand is given by

$$\frac{1}{|(e_1 \wedge \ldots \wedge e_m)_d|} \sum_{\deg(X_J)=d} \langle e_1 \wedge \ldots \wedge e_m, (X_J)_p \rangle \times$$
$$\times \left(\langle e_1 \wedge \ldots \wedge e_m, \nabla_{V_{\bar{p}}} X_J \rangle + \sum_{i=1}^{m} \langle e_1 \wedge \ldots \wedge \nabla_{e_i} V \wedge \ldots \wedge e_m, (X_J)_p \rangle \right).$$

Using (3.10.3) and (3.10.4) we obtain (3.10.5). □

Remark 3.10.2. Let us denote by $(E_1, \ldots, E_m)$ a local frame for the tangent space $d\Phi(T\bar{M})$ such that $E_i(p) = e_i$ for $i = 1, \ldots, m$. Notice that

$$\frac{f(V_{\bar{p}})}{\Theta(\bar{p})} = \langle e_1 \wedge \ldots \wedge e_m, \nabla_{V_p} \left(\frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right) \rangle. \tag{3.10.6}$$

Indeed, since we have

$$(E_1 \wedge \ldots \wedge E_m)_d = \sum_{\deg(X_J)=d} \langle E_1 \wedge \ldots \wedge E_m, X_J \rangle X_J,$$

we get

$$V_p \left(\frac{1}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right) = - \sum_{\deg(X_J)=d} \frac{\langle e_1 \wedge \ldots \wedge e_m, X_J \rangle V_p \big(\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle \big)}{|(e_1 \wedge \ldots \wedge e_m)_d|^3},$$

and so we have that $\nabla_{V_p} \left(\frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right)$ is equal to

$$\frac{1}{\Theta(\bar{p})} \left(\sum_{\deg(X_J)=d} V_p \big(\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle \big) (X_J)_p + \langle e_1 \wedge \ldots \wedge e_m, (X_J)_p \rangle \nabla_{V_p} X_J \right)$$
$$+ V_p \left(\frac{1}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right) (e_1 \wedge \ldots \wedge e_m)_d.$$

Multiplying by $e_1 \wedge \ldots \wedge e_m$ and taking into account the above computations we get (3.10.6).

Definition 3.10.3. Let $\Phi : \bar{M} \to N$ be an immersion of degree d of a smooth m-dimensional manifold into an equiregular graded manifold equipped with a Riemannian metric g. We say that Φ is A_d-stationary, or simply stationary, if it is a critical point of the area A_d for any admissible variation.

Proposition 3.10.4. *Let $\Phi : \bar{M} \to N$ be an immersion of degree d of a smooth m-dimensional manifold into an equiregular graded manifold equipped with a Riemannian metric g. Let Γ_t be admissible variation whose variational field $V = V^{\top}$ is compactly supported and tangent to $M = \Phi(\bar{M})$. Then we have*

$$\frac{d}{dt}\Big|_{t=0} A_d(\Gamma_t(\bar{M})) = 0.$$

Proof. We consider the d area m-form ω_d defined in (3.1.6). Therefore, we have

$$\mathcal{L}_V \omega_d = \imath_V d\omega_d + d\imath_V \omega_d,$$

where $\imath_V$ is the interior product in M, $\mathcal{L}_V$ is the Lie derivative in M and d denotes the exterior derivative in M. Since ω_d is a top-dimensional form in M we have $d\omega_d = 0$. Thus, it follows

$$\begin{aligned}
\mathcal{L}_V\ \omega_d &= d\ (\imath_V \omega_d) \\
&= \sum_{i=1}^{m} (-1)^i W_i \Big(\imath_V \omega_d(W_1, \ldots, \hat{W}_i, \ldots, W_m)\Big) + \\
&\qquad + \sum_{i<j} (-1)^{i+j} \imath_V \omega_d([W_i, W_j], W_1, \ldots, \hat{W}_i, \ldots, \hat{W}_j, \ldots, W_m) \\
&= \sum_{i=1}^{m} W_i \left(\omega_d(W_1, \ldots, \overset{(i)}{V}, \ldots, W_m) \right) \\
&= \sum_{i=1}^{m} \langle W_1 \wedge \ldots \wedge \nabla_{W_i} V \wedge \ldots \wedge W_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle + \\
&\qquad + \langle W_1 \wedge \ldots \wedge W_m, \nabla_V \left(\frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right) \rangle,
\end{aligned} \tag{3.10.7}$$

where $W_1, \ldots, W_m$ are vector fields that at each point $q \in M$ provide a basis of $T_p M$ and $(E_1, \ldots, E_m)$ is an orthonormal basis of vector field such that $E_i(p) = e_i$ for $i = 1, \ldots, m$. Choosing V_i equal to E_i for $i = 1, \ldots, m$ and evaluating the pullback of (3.10.7) at $\bar{p}$ and by Remark 3.10.2 we obtain

$$d\ (\Phi^*(\imath_V \omega_d))(\bar{p}) = \frac{1}{\Theta(\bar{p})} \Big((\operatorname{div}^d_{\bar{M}} V)(\bar{p}) + f(V_{\bar{p}}) \Big).$$

By the Stokes Theorem we have

$$\int_{\bar{M}} \frac{1}{\Theta(\bar{p})} \Big((\operatorname{div}^d_{\bar{M}} V)(\bar{p}) + f(V_{\bar{p}}) \Big) d\mu(\bar{p}) = \int_{\partial \bar{M}} \Phi^*(\imath_V \omega_d) = 0,$$

since V is compactly supported in $\bar{M}$. $\square$

Remark 3.10.5. An alternative proof of Proposition 3.10.4 is the following: since $\Gamma_t(\bar{M}) \subset \Phi(M)$ for all t, the vector field $\bar{V}_p = d\Phi_{\bar{p}}^{-1}(V_{\bar{p}})$ is tangent to $\bar{M}$ and we have

$$\frac{d}{dt}\bigg|_{t=0} A_d(M) = \int_{\bar{M}} (\bar{V}(\Theta) + \Theta \operatorname{div}_{\bar{M}} \bar{V})\, d\mu = \int_{\bar{M}} \operatorname{div}_{\bar{M}}(\Theta \bar{V})\, d\mu = 0.$$

Lemma 3.10.6. *Let $f, g \in C^\infty(M)$ and X be a tangential vector field in $C^\infty(M, TM)$. Then there holds,*

(i) $f \operatorname{div}_M(X) + X(f) = \operatorname{div}_M(fX)$,

(ii) $gX(f) = \operatorname{div}_M(fgX) - gf \operatorname{div}_M(X) - fX(g)$.

Proof. By the definition of divergence we obtain (i) as follows

$$\operatorname{div}_M(fX) = \sum_{i=1}^{m} \langle \nabla_{e_i}(fX), e_i \rangle = \sum_{i=1}^{m} e_i(f)\langle X, e_i \rangle + f\langle \nabla_{e_i}(X), e_i \rangle.$$

To deduce (ii) we apply twice (i) as follows

$$\operatorname{div}_M(gfX) - fX(g) = g \operatorname{div}_M(fX) = gX(f) + gf \operatorname{div}_M(X).$$ $\square$

Theorem 3.10.7. *Let $\Phi : \bar{M} \to N$ be an immersion of degree d of a smooth m-dimensional manifold into an equiregular graded manifold equipped with a Riemannian metric g. Assume that there exists an admissible variation $\Gamma : \bar{M} \times (-\varepsilon, \varepsilon) \to N$ with associated variational field V compactly supported on $\bar{M} \smallsetminus \bar{M}_0$. Then*

$$\frac{d}{dt}\bigg|_{t=0} A_d(\Gamma_t(\bar{M})) = \int_{\bar{M} \smallsetminus \bar{M}_0} \langle V, \mathbf{H}_d \rangle d\mu, \tag{3.10.8}$$

where $\mathbf{H}_d$ is the vector field

$$\begin{aligned} &- \sum_{j=m+1}^{n} \sum_{i=1}^{m} \operatorname{div}_M \Big(\xi_{ij} E_i\Big) N_j. \\ &+ \sum_{j=m+1}^{n} \sum_{i=1}^{m} \langle E_1 \wedge \ldots \wedge \nabla_{E_i} N_j \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle\, N_j \\ &+ \sum_{j=m+1}^{n} \frac{f(N_j)}{\Theta} N_j. \end{aligned} \tag{3.10.9}$$

In this formula, $(E_i)_i$ is a local orthonormal basis of TM and $(N_j)_j$ a local orthonormal basis of $TM^\perp$. The functions ξ_{ij} are given by

$$\xi_{ij} = \langle E_1 \wedge \ldots \wedge \overset{(i)}{N_j} \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle. \tag{3.10.10}$$

Proof. Since our computations are local and immersions are local embeddings, we shall identify locally $\bar{M}$ and M to simplify the notation.

We decompose $V = V^\top + V^\perp$ in its tangential $V^\top$ and perpendicular $V^\perp$ parts. Since $\operatorname{div}^d_{\bar{M}}$ and the functional f defined in (3.10.4) are additive, we use the first variation formula (3.10.5) and Proposition 3.10.4 to obtain

$$\frac{d}{dt}\bigg|_{t=0} A_d(\Gamma_t(\bar{M})) = \int_{\bar{M}\smallsetminus \bar{M}_0} \frac{1}{\Theta(\bar{p})} \Big((\operatorname{div}^d_{\bar{M}} V^\perp)(\bar{p}) + f(V^\perp_{\bar{p}}) \Big) d\mu(\bar{p}).$$

To compute this integrand we consider a local orthonormal basis $(E_i)_i$ in TM around p and a local orthonormal basis $(N_j)_j$ of $TM^\perp$ with $(N_j)_j$. We have

$$V^\perp = \sum_{j=m+1}^{n} \langle V, N_j \rangle N_j.$$

We compute first

$$\frac{\operatorname{div}^d_{\bar{M}} V^\perp}{\Theta} = \sum_{i=1}^{m} \langle E_1 \wedge \ldots \wedge \nabla_{E_i} V^\perp \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle$$

as

$$\sum_{i=1}^{m} \sum_{j=m+1}^{n} \langle E_1 \wedge \ldots \wedge \Big(\nabla_{E_i} \langle V, N_j \rangle N_j \Big) \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle,$$

that it is equal to

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=m+1}^{n} \Bigg(& E_i\Big(\langle V, N_j \rangle\Big) \langle E_1 \wedge \ldots \wedge \overset{(i)}{N_j} \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle \\ & + \langle V, N_j \rangle \langle E_1 \wedge \ldots \wedge \overset{(i)}{\nabla_{E_i} N_j} \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle \Bigg). \end{aligned} \tag{3.10.11}$$

The group of summands in the second line of (3.10.11) is equal to $\langle V, \mathbf{H}_2 \rangle$, where

$$\mathbf{H}_2 = \sum_{i=1}^{m} \sum_{j=m+1}^{n} \langle E_1 \wedge \ldots \wedge \overset{(i)}{\nabla_{E_i} N_j} \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle \, N_j.$$

To treat the group of summands in the first line of (3.10.11) we use (ii) in Lemma 3.10.6. recalling (3.10.10) we have

$$E_i\Big(\langle V, N_j\rangle\Big)\xi_{ij} = \operatorname{div}_M\Big(\langle V, N_j\rangle \xi_{ij} E_i\Big) - \langle V, \operatorname{div}_M\Big(\xi_{ij}E_i\Big)N_j\rangle,$$

so that applying the Divergence Theorem we have that the integral in M of the first group of summands in (3.10.11) is equal to

$$\int_{\bar{M}\smallsetminus\bar{M}_0} \langle V, \mathbf{H}_1\rangle d\mu,$$

where

$$\mathbf{H}_1 = -\sum_{i=1}^{m}\sum_{j=m+1}^{n} \operatorname{div}_M\Big(\xi_{ij}E_i\Big)N_j.$$

We treat finally the summand

$$\frac{f(V^\perp)}{\Theta} = \sum_{i=m+1}^{n}\langle V, N_j\rangle\frac{f(N_j)}{\Theta} = \langle V, \mathbf{H}_3\rangle,$$

where

$$\mathbf{H}_3 = \sum_{j=m+1}^{n}\frac{f(N_j)}{\Theta}N_j.$$

This implies the result since $\mathbf{H}_d = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3$. □

In the following result we obtain a slightly different expression for the mean curvature $\mathbf{H}_d$ in terms of Lie brackets. This expression is sometimes more suitable for computations.

Corollary 3.10.8. *Let $\Phi : \bar{M} \to N$ be an immersion of degree d of a smooth m-dimensional manifold into an equiregular graded manifold equipped with a Riemannian metric g, $M = \Phi(\bar{M})$. We consider an extension $(E_i)_i$ of a local orthonormal basis of TM and respectively an extension $(N_j)_j$ of a local orthonormal basis of $TM^\perp$ to an open neighborhood of N. Then the vector field $\mathbf{H}_d$ defined in (3.10.9) is equal to*

$$\mathbf{H}_d = \sum_{j=m+1}^{n}\Bigg(\operatorname{div}_M\Big(\Theta N_j - \sum_{i=1}^{m}\xi_{ij}E_i\Big) + N_j(\Theta) + \sum_{i=1}^{m}\sum_{k=m+1}^{n}\xi_{ik}\langle[E_i, N_j], N_k\rangle\Bigg)N_j, \tag{3.10.12}$$

where ξ_{ij} is defined in (3.10.10).

Proof. Keeping the notation used in the proof of Theorem 3.10.7 we consider

$$\mathbf{H}_2 = \sum_{i=1}^{m} \sum_{j=m+1}^{n} \langle E_1 \wedge \ldots \wedge \overset{(i)}{\nabla_{E_i} N_j} \wedge \ldots \wedge E_m, \frac{(E_1 \wedge \ldots \wedge E_m)_d}{|(E_1 \wedge \ldots \wedge E_m)_d|} \rangle \, N_j.$$

Writing

$$\nabla_{E_i} N_j = \sum_{\nu=1}^{m} \langle \nabla_{E_i} N_j, E_\nu \rangle E_\nu + \sum_{k=m+1}^{m} \langle \nabla_{E_i} N_j, N_k \rangle N_k, \tag{3.10.13}$$

we gain

$$\mathbf{H}_2 = \sum_{j=m+1}^{n} \Big(\operatorname{div}_M(N_j) \, |(E_1 \wedge \ldots \wedge E_m)_d| + \sum_{i=1}^{m} \sum_{k=m+1}^{n} \xi_{ik} \langle \nabla_{E_i} N_j, N_k \rangle \Big) N_j.$$

Let us consider

$$\mathbf{H}_3 = \sum_{j=m+1}^{n} \sum_{\deg(X_J)=d} \left(\langle E_1 \wedge \ldots \wedge E_m, \nabla_{N_j} X_J \rangle \frac{\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle}{|(E_1 \wedge \ldots \wedge E_m)_d|} \right) N_j. \tag{3.10.14}$$

Since the Levi-Civita connection preserves the metric, we have

$$\langle E_1 \wedge \ldots \wedge E_m, \nabla_{N_j} X_J \rangle = N_j(\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle) - \langle \nabla_{N_j}(E_1 \wedge \cdots \wedge E_m), X_J \rangle. \tag{3.10.15}$$

Putting the first term of the right hand side of (3.10.15) in (3.10.14) we obtain

$$\sum_{\deg(X_J)=d} N_j(\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle) \frac{\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle}{|(E_1 \wedge \ldots \wedge E_m)_d|} = N_j(\Theta).$$

On the other hand writing

$$\nabla_{N_j} E_i = \sum_{\nu=1}^{m} \langle \nabla_{N_j} E_i, E_\nu \rangle E_\nu + \sum_{k=m+1}^{m} \langle \nabla_{N_j} E_i, N_k \rangle N_k$$

we deduce

$$\sum_{i=1}^{m} \sum_{\deg(X_J)=d} \langle E_1 \wedge \ldots \wedge \overset{(i)}{\nabla_{N_j} E_i} \wedge \ldots \wedge E_m, X_J \rangle \frac{\langle E_1 \wedge \ldots \wedge E_m, X_J \rangle}{|(E_1 \wedge \ldots \wedge E_m)_d|} =$$
$$= \sum_{i=1}^{m} \sum_{k=m+1}^{n} \langle \nabla_{N_j} E_i, N_k \rangle \xi_{ik}.$$

Therefore we obtain

$$\mathbf{H}_3 = \sum_{j=m+1}^{n} \Big(N_j(\Theta) - \sum_{i=1}^{m} \sum_{k=m+1}^{n} \langle \nabla_{N_j} E_i, N_k \rangle \xi_{ik} \Big) N_j.$$

Since the Levi-Civita connection is torsion-free we have

$$\mathbf{H}_2 + \mathbf{H}_3 = \sum_{j=m+1}^{n} \Big(\mathrm{div}_M(N_j)\,\Theta + N_j(\Theta) + \sum_{i=1}^{m} \sum_{k=m+1}^{n} \xi_{ik} \langle [E_i, N_j], N_k \rangle \Big).$$

Since $\mathrm{div}_M(N_j)\,\Theta = \mathrm{div}_M(\Theta\, N_j)$ we conclude that $\mathbf{H}_d = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3$ is equal to (3.10.12). □

3.10.1 First variation formula for strongly regular submanifolds

Definition 3.10.9. Let $\Phi : \bar{M} \to N$ be a strongly regular immersion (see § 3.5) at $\bar{p}$, $v_{m+1}, \ldots, v_n$ be an orthonormal adapted basis of the normal bundle and k be the integer defined in 3.4.6. Let $N_{m+1}, \ldots, N_n$ be a local adapted frame of the normal bundle so that $(N_j)_p = v_j$. By Remark 3.5.3 the immersion Φ is strongly regular at $\bar{p}$ if and only if $\mathrm{rank}(A^\perp) = \ell$. Then there exists a partition of $\{m+1, \ldots, m+k\}$ into sub-indices $h_1 < \ldots < h_\ell$ and $i_1 < \ldots < i_{m+k-\ell}$ such that the matrix

$$\hat{A}^\perp(\bar{p}) = \begin{pmatrix} \alpha_{1h_1}(\bar{p}) & \cdots & \alpha_{1h_\ell}(\bar{p}) \\ \vdots & \ddots & \vdots \\ \alpha_{\ell h_1}(\bar{p}) & \cdots & \alpha_{\ell h_\ell}(\bar{p}) \end{pmatrix} \tag{3.10.16}$$

is invertible. The mean curvature vector of degree d out of the singular set M_0 defined in Theorem 3.10.7 is given by

$$\mathbf{H}_d = \sum_{j=m+1}^{n} H_d^j N_j.$$

Then we decompose $\mathbf{H}_d$ into the following three components

$$\mathbf{H}_d^v = \begin{pmatrix} H_d^{m+k+1} \\ \vdots \\ H_d^n \end{pmatrix}^t, \quad \mathbf{H}_d^h = \begin{pmatrix} H_d^{h_1} \\ \vdots \\ H_d^{h_\ell} \end{pmatrix}^t, \quad \text{and} \quad \mathbf{H}_d^\iota = \begin{pmatrix} H_d^{i_1} \\ \vdots \\ H_d^{i_{m+k-\ell}} \end{pmatrix}^t \tag{3.10.17}$$

with respect to $N_{m+1}, \ldots, N_n$.

Theorem 3.10.10. *Let $\Phi : \bar{M} \to N$ be a strongly regular immersion at $\bar{p}$ in a graded manifold. Then Φ is a critical point of the A_d area functional if and only if the immersion Φ verifies*

$$\mathbf{H}_d^{\iota} - \mathbf{H}_d^{h}(\hat{A}^{\perp})^{-1}\tilde{A}^{\perp} = 0, \tag{3.10.18}$$

and

$$\mathbf{H}_d^{v} - \mathbf{H}_d^{h}(\hat{A}^{\perp})^{-1}B^{\perp} - \sum_{j=1}^{m} E_j^{*}\left(\mathbf{H}_d^{h}\,(\hat{A}^{\perp})^{-1}C_j^{\perp}\right) = 0, \tag{3.10.19}$$

on the open set $W_{\bar{p}} \subset \bar{M} \setminus \bar{M}_0$ introduced in Theorem 3.5.2 and where E_j^* is the adjoint operator of E_j for $j = 1, \ldots, m$ and $\mathbf{H}_d^v$, $\mathbf{H}_d^h$ and $\mathbf{H}_d^{\iota}$ are defined in (3.10.17), $B^{\perp}$, $C_j^{\perp}$ in 3.4.3, $\hat{A}^{\perp}$ in (3.10.16) and $\tilde{A}^{\perp}$ is the $\ell \times (m+k-\ell)$ matrix given by the columns $i_1, \ldots, i_{m+k-\ell}$ of $A^{\perp}$.

Proof. Since $\Phi : \bar{M} \to N$ is a normal strongly regular immersion then by Theorem 3.5.2 each normal admissible vector field

$$V^{\perp} = \sum_{i=m+1}^{m+k} \phi_i\, N_i + \sum_{r=m+k+1}^{n} \psi_r\, N_r$$

is integrable. Keeping in mind the sub-indices in Definition 3.10.9, we set

$$\Psi = \begin{pmatrix} \psi_{m+k+1} \\ \vdots \\ \psi_n \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \phi_{h_1} \\ \vdots \\ \phi_{h_\ell} \end{pmatrix} \quad \text{and} \quad \Upsilon = \begin{pmatrix} \phi_{i_1} \\ \vdots \\ \phi_{i_{m+k-\ell}} \end{pmatrix}. \tag{3.10.20}$$

Since the immersion $\Phi : \bar{M} \to N$ is strongly regular, the admissibility condition (3.4.25) for $V^{\perp}$ is equivalent to

$$\Gamma = -(\hat{A}^{\perp})^{-1}\left(\sum_{j=1}^{m} C_j^{\perp}\, E_j(\Psi) + B^{\perp}\Psi + \tilde{A}^{\perp}\Upsilon\right). \tag{3.10.21}$$

By Theorem 3.10.7 the first variational formula is given by

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} A_d(\Gamma_t(\bar{M})) &= \int_{\bar{M}} \langle V^\perp, \mathbf{H}_d\rangle \\
&= \int_{\bar{M}} \mathbf{H}_d^v \Psi + \mathbf{H}_d^\iota \Upsilon + \mathbf{H}_d^h \Gamma \\
&= \int_{\bar{M}} \mathbf{H}_d^v \Psi + \mathbf{H}_d^\iota \Upsilon - \mathbf{H}_d^h (\hat{A}^\perp)^{-1}\left(\sum_{j=1}^m C_j^\perp E_j(\Psi) + B^\perp \Psi + \tilde{A}^\perp \Upsilon\right) \\
&= \int_{\bar{M}} \left(\mathbf{H}_d^\iota - \mathbf{H}_d^h (\hat{A}^\perp)^{-1} \tilde{A}^\perp\right)\Upsilon + \\
&\quad + \int_{\bar{M}} \left(\mathbf{H}_d^v - \mathbf{H}_d^h (\hat{A}^\perp)^{-1} B^\perp - \sum_{j=1}^m E_j^*\left(\mathbf{H}_d^h (\hat{A}^\perp)^{-1} C_j^\perp\right)\right)\Psi,
\end{aligned}$$

for every $\Psi \in C_0^\infty(W_{\bar{p}}, \mathbb{R}^{n-m-k})$, $\Upsilon \in C_0^\infty(W_{\bar{p}}, \mathbb{R}^{k-\ell})$. By the arbitrariness of Ψ and Υ, the immersion Φ is a critical point of the area A_d if and only if it satisfies equations (3.10.18) and (3.10.19) on $W_{\bar{p}}$. □

Example 3.10.11 (First variation for a hypersurface in a contact manifold)**.** Let (M^{2n+1}, ω) be a contact manifold such that $\mathcal{H} = \ker(\omega)$, see § 3.2.2. Let T be the Reeb vector associated to this contact geometry and g the Riemannian metric on M that extends a given metric on $\mathcal{H}$ and makes T orthonormal to $\mathcal{H}$. Let ∇ be the Riemannian connection associated to g.

Let us consider a C^2 hypersurface Σ immersed in M. As we showed in § 3.2.2, the degree of Σ is maximum and equal to $2n+1$, thus each compactly supported vector field V on Σ is admissible. Following § 3.2.2, we consider the unit normal N to Σ and its horizontal projection N_h, namely

$$N = N_h + \langle N, T\rangle T.$$

As in § 3.2.2, we consider the vector fields

$$\nu_h = \frac{N_h}{|N_h|}, \qquad S = \langle N, T\rangle \nu_h - |N_h| T.$$

Let $e_1, \ldots, e_{2n-1}$ be an orthonormal basis of $T_p\Sigma \cap \mathcal{H}_p$. Letting $e_{2n} = S_p$, we have that $e_1, \ldots, e_{2n}$ is an orthonormal basis of $T_p\Sigma$. Since

$$(e_1 \wedge \ldots \wedge e_{2n})_{2n+1} = -|N_h|\, e_1 \wedge \ldots \wedge e_{2n-1} \wedge T_p,$$

we have

$$\frac{(e_1 \wedge \ldots \wedge e_{2n})_{2n+1}}{|(e_1 \wedge \ldots \wedge e_{2n})_{2n+1}|} = -e_1 \wedge \ldots \wedge e_{2n-1} \wedge T_p.$$

Since Σ has codimension one, N is the only normal vector. Therefore, by the definition of mean curvature $\mathbf{H}_d$ provided in (3.10.12), there follows

$$\mathbf{H}_d = -\operatorname{div}_\Sigma \left(|N_h| N - \sum_{i=1}^{2n} \xi_{i,2n+1} E_i \right) - N(|N_h|) + \sum_{i=1}^{2n} \xi_{i,2n+1} \langle [N, E_i], N \rangle,$$

where, for $i = 1, \ldots, 2n$, the function $\xi_{i,2n+1}$ is given by

$$-\langle e_1 \wedge \ldots \wedge \overset{(i)}{N} \wedge \ldots \wedge e_{2n}, e_1 \wedge \ldots \wedge e_{2n-1} \wedge T \rangle.$$

Notice that N is orthogonal to $T_p\Sigma$, thus we have $\xi_{i,2n+1} = 0$ for all $i = 1, \ldots, 2n-1$. Moreover, we have $\xi_{2n,2n+1} = -\langle N, T \rangle$. Thus

$$|N_h| N - \sum_{i=1}^{2n} \xi_{i,2n+1} E_i = |N_h| N + \langle N, T \rangle S = \nu_h.$$

Now given X, Y vector fields on M, we define the tensor

$$\sigma(X, Y) = \langle \nabla_X T, Y \rangle.$$

Therefore, we have

$$\begin{aligned}
\operatorname{div}_\Sigma(\nu_h) &= \sum_{i=1}^{2n-1} \langle \nabla_{E_i} \nu_h, E_i \rangle + \langle \nabla_S \, \nu_h, S \rangle \\
&= \sum_{i=1}^{2n-1} \langle \nabla_{E_i} \nu_h, E_i \rangle - |N_h| \langle \nabla_S \, \nu_h, T \rangle \\
&= \sum_{i=1}^{2n-1} \langle \nabla_{E_i} \nu_h, E_i \rangle + |N_h| \sigma(S, \nu_h).
\end{aligned}$$

Since $S = |N_h|^{-1}(\langle N, T \rangle N - T)$ and $\langle \nabla_T T, \nu_h \rangle = \langle T, [T, \nu_h] \rangle$ we get

$$|N_h| \sigma(S, \nu_h) = \langle N, T \rangle \sigma(N, \nu_h) + \langle T, [T, \nu_h] \rangle.$$

Therefore we deduce

$$\operatorname{div}_\Sigma(\nu_h) = \sum_{i=1}^{2n-1} \langle \nabla_{E_i} \nu_h, E_i \rangle + \langle N, T \rangle \sigma(N, \nu_h) + \langle T, [T, \nu_h] \rangle. \tag{3.10.22}$$

On the other hand, we have

$$\begin{aligned}
-N(|N_h|)-\langle N,T\rangle\langle[N,S],N\rangle = \\
&= -N(|N_h|)-\langle N,T\rangle\langle\nabla_N S-\nabla_S N,N\rangle \\
&= -N(|N_h|)-\langle N,T\rangle\langle\nabla_N S,N\rangle \\
&= -N(|N_h|)+\langle N,T\rangle\langle S,\nabla_N N\rangle \\
&= -N(|N_h|)+(1-|N_h|^2)\langle\nu_h,\nabla_N N\rangle-\langle N,T\rangle|N_h|\langle T,\nabla_N N\rangle.
\end{aligned} \tag{3.10.23}$$

It can be easily proved, adapting the proof given by Ritoré and Rosales in [89, Lemma 4.2] to this more general setting, that the following relation holds

$$\langle\nu_h,\nabla_N N\rangle = N(|N_h|)+\langle N,T\rangle\langle\nabla_N T,\nu_h\rangle. \tag{3.10.24}$$

Using (3.10.24) we deduce

$$\begin{aligned}
(3.10.23) &= \langle N,T\rangle\langle\nabla_N T,\nu_h\rangle-|N_h|^2\langle\nu_h,\nabla_N N\rangle-\langle N,T\rangle|N_h|\langle T,\nabla_N N\rangle \\
&= \langle N,T\rangle\langle\nabla_N T,\nu_h\rangle-|N_h|\left(\langle|N_h|\nu_h,\nabla_N N\rangle-\langle N,T\rangle|N_h|\langle T,\nabla_N N\rangle\right) \\
&= \langle N,T\rangle\langle\nabla_N T,\nu_h\rangle-|N_h|\left(\langle N,\nabla_N N\rangle\right) \\
&= \langle N,T\rangle\sigma(N,\nu_h).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathbf{H}_d &= -\sum_{i=1}^{2n-1}\langle\nabla_{E_i}\nu_h,E_i\rangle-\langle N,T\rangle\sigma(N,\nu_h)-\langle T,[T,\nu_h]\rangle+\langle N,T\rangle\sigma(N,\nu_h) \\
&= -\operatorname{div}_\Sigma^h(\nu_h)+\langle[\nu_h,T],T\rangle,
\end{aligned} \tag{3.10.25}$$

out from the singular set M_0. When $\langle[\nu_h,T],T\rangle=0$ we obtain well known horizontal divergence of the horizontal normal. This definition of mean curvature for an immersed hypersurface was first given by S.Pauls [84] for graphs over the x,y-plane in $\mathbb{H}^1$, later extended by J.-H. Cheng, J.-F. Hwang, A. Malchiodi and P. Yang in [18] in a 3-dimensional pseudo-hermitian manifold. In a more general setting this formula was deduced in [53, 28]. For more details see also [41, 13, 95, 40, 92, 89].

Example 3.10.12 (First variation for ruled surfaces in an Engel Structure)**.** Here we compute the mean curvature equation for the surface $\Sigma\subset E$ of degree 4 introduced in

Section 3.2.3. In (3.2.16) we determined the tangent adapted basis

$$\begin{aligned}
\tilde{E}_1 &= \cos(\theta)\Phi_x + \sin(\theta)\Phi_y = X_1 + X_1(\kappa)X_2,\\
\tilde{E}_2 &= -\sin(\theta)\Phi_x + \cos(\theta)\Phi_y = X_4 - X_4(\theta)X_3 + X_4(\kappa)X_2
\end{aligned}$$

A basis for the space $(TM)^\perp$ is given by

$$\begin{aligned}
\tilde{N}_3 &= X_4(\theta)X_4 + X_3\\
\tilde{N}_4 &= X_1(\kappa)X_1 - X_2 + X_4(\kappa)X_4
\end{aligned}$$

By the Gram–Schmidt process we obtain an orthonormal basis with respect to the metric g as follows

$$\begin{aligned}
E_1 &= \frac{\tilde{E}_1}{|\tilde{E}_1|} = \frac{1}{\alpha_1}(X_1 + X_1(\kappa)X_2),\\
E_2 &= \frac{1}{\alpha_2}\left(X_4 - X_4(\theta)X_3 + \frac{X_4(\kappa)}{\alpha_1^2}(X_2 - X_1(\kappa)X_1)\right)\\
N_3 &= \frac{1}{\alpha_3}(X_3 + X_4(\theta)X_4)\\
N_4 &= \frac{\alpha_3}{\alpha_2\alpha_1}\left((-X_1(\kappa)X_1 + X_2) + \frac{X_4(\kappa)}{\alpha_3^2}(X_4(\theta)X_3 - X_4)\right)
\end{aligned}$$

where we set

$$\begin{aligned}
\alpha_1 &= \sqrt{1 + X_1(\kappa)^2}, \quad \alpha_3 = \sqrt{1 + X_4(\theta)^2}\\
\alpha_2 &= \sqrt{1 + X_4(\theta)^2 + \frac{X_4(\kappa)^2}{(1 + X_1(\kappa)^2)}} = \frac{\sqrt{\alpha_1^2\alpha_3^2 + X_4(\kappa)^2}}{\alpha_1}
\end{aligned}$$

and

$$N_h = -X_1(\kappa)X_1 + X_2, \quad \nu_h = \frac{1}{\alpha_1}(-X_1(\kappa)X_1 + X_2)$$

Since the degree of Σ is equal to 4 we deduce that

$$(E_1 \wedge E_2)_4 = \frac{1}{\alpha_1\alpha_2}(X_1 \wedge X_4 + X_1(\kappa)X_2 \wedge X_4),$$

then it follows $|(E_1 \wedge E_2)_4| = \alpha_2^{-1}$ and

$$\frac{(E_1 \wedge E_2)_4}{|(E_1 \wedge E_2)_4|} = \frac{1}{\alpha_1}(X_1 \wedge X_4 + X_1(\kappa)X_2 \wedge X_4).$$

A straightforward computation shows that ξ_{i3} for $i=1,2$ defined in (3.10.12) are given by

$$
\begin{aligned}
\xi_{13} &= \langle N_3\wedge E_2, \frac{(E_1\wedge E_2)_4}{|(E_1\wedge E_2)_4|}\rangle = 0,\\
\xi_{23} &= \langle E_1\wedge N_3, \frac{(E_1\wedge E_2)_4}{|(E_1\wedge E_2)_4|}\rangle = \frac{X_4(\theta)}{\alpha_3},\\
\xi_{14} &= \langle N_4\wedge E_2, \frac{(E_1\wedge E_2)_4}{|(E_1\wedge E_2)_4|}\rangle = 0,\\
\xi_{24} &= \langle E_1\wedge N_4, \frac{(E_1\wedge E_2)_4}{|(E_1\wedge E_2)_4|}\rangle = -\frac{X_4(\kappa)}{\alpha_1\alpha_2\alpha_3}
\end{aligned}
$$

Since we have

$$\frac{1}{\alpha_2}N_3 - \frac{X_4(\theta)}{\alpha_3}E_2 = \frac{\alpha_3}{\alpha_2}X_3 - \frac{X_4(\theta)X_4(\kappa)}{\alpha_1\alpha_2\alpha_3}\nu_h.$$

and

$$
\begin{aligned}
\frac{1}{\alpha_2}N_4 + \frac{X_4(\kappa)}{\alpha_1\alpha_2\alpha_3}E_2 &= \frac{1}{\alpha_2^2}\Bigg(\frac{\alpha_3}{\alpha_1}\Bigg(N_h + \frac{X_4(\kappa)}{\alpha_3^2}(X_4(\theta)X_3 - X_4)\Bigg)\\
&\qquad + \frac{X_4(\kappa)}{\alpha_1\alpha_3}\Bigg(-X_4(\theta)X_3 + X_4 - \frac{X_4(\kappa)}{\alpha_1^2}N_h\Bigg)\Bigg)\\
&= \frac{1}{\alpha_2^2\alpha_1}(\alpha_3 N_h + \frac{X_4(\kappa)^2}{\alpha_3\alpha_1^2}N_h) = \frac{1}{\alpha_1\alpha_3}N_h\\
&= \frac{1}{\alpha_3}\nu_h
\end{aligned}
$$

it follows that the third component of $\mathbf{H}_d$ is equal to

$$
\begin{aligned}
H_d^3 =& -\operatorname{div}_M\left(\frac{\alpha_3}{\alpha_2}X_3 - \frac{X_4(\theta)X_4(\kappa)}{\alpha_1\alpha_2\alpha_3}\nu_h\right) - N_3(\alpha_2^{-1})\\
&+ \frac{X_4(\theta)}{\alpha_3}\langle[N_3,E_2],N_3\rangle - \frac{X_4(\kappa)}{\alpha_3\alpha_2\alpha_1}\langle[N_3,E_2],v_4\rangle
\end{aligned}
$$

and the fourth component of $\mathbf{H}_d$ is equal to

$$H_d^4 = -\operatorname{div}_M\left(\frac{\nu_h}{\alpha_3}\right) - N_4(\alpha_2^{-1}) + \frac{X_4(\theta)}{\alpha_3}\langle[N_4,E_2],N_3\rangle - \frac{X_4(\kappa)}{\alpha_3\alpha_2\alpha_1}\langle[N_4,E_2],N_4\rangle.$$

Then first variation formula is given by

$$A_d(\Gamma_t(\Omega)) = \int_\Omega \langle V^\perp, \mathbf{H}_d\rangle = \int_\Omega H_d^3\,\psi_3 + H_d^4\,\psi_4 \tag{3.10.26}$$

for each $\psi_3, \psi_4 \in C_0^\infty$ satisfying (3.5.7). Following Theorem 3.5.2 for each $\psi_3 \in C_0^\infty$ we deduce

$$\psi_4 = -\frac{\bar{X}_1(\psi_3) + b^\perp \psi_3}{a^\perp}, \tag{3.10.27}$$

since $a^\perp > 0$.

Lemma 3.10.13. *Keeping the previous notation. Let $f, g : \Omega \to \mathbb{R}$ be functions in $C_0^1(\Omega)$ and*

$$\begin{aligned}
\bar{X}_1 &= \cos(\theta(x,y))\partial_x + \sin(\theta(x,y))\partial_y, \\
X_4 &= -\sin(\theta(x,y))\partial_x + \cos(\theta(x,y))\partial_y
\end{aligned}$$

Then there holds

$$\int_\Omega g\bar{X}_1(f) + \int_\Omega fg\bar{X}_4(\theta) = -\int_\Omega f\bar{X}_1(g).$$

By Lemma 3.10.13 and the admissibility equation (3.10.27) we deduce that (3.10.26) is equivalent to

$$\int_\Omega \left(H_d^3 - \frac{b^\perp}{a^\perp} H_d^4 + \bar{X}_1\left(\frac{H_d^4}{a^\perp}\right) + X_4(\theta)\frac{H_d^4}{a^\perp}\right)\psi_3,$$

for each $\psi_3 \in C_0^\infty(\Omega)$. Therefore a straightforward computation shows that minimal (θ, κ)-graphs for the area functional A_4 verify the following third order PDE

$$\bar{X}_1(H_d^4) + a^\perp H_d^3 + \left(\frac{X_4(\theta)}{\alpha_3^2}[X_1, X_4](\theta) - \frac{1}{a^\perp}\bar{X}_1\left(a^\perp\right)\right) H_d^4 = 0. \tag{3.10.28}$$

3.11 Calibration for minimal hypersurfaces in the Heisenberg group

In this Section we show a calibration argument for C^2 minimal t-graphs in the Heisenberg group. We prove in Proposition 3.11.3 that a C^2 t-graph solution of the minimal hypersurface equation, deduced by the first variation of the area functional, is a minimum for the area among all possible C^2 t-graphs over the same open set Ω .

Let $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$ be the Heisenberg group, described in Example 2.7.3, we consider the left invariant vector fields

$$X_i = \partial_{x_i} + y_i\partial_t, \quad Y_i = \partial_{y_i} - x_i\partial_t \quad i = 1, \cdots, n$$

and the only non-trivial commutator

$$2T = [X_i, Y_i] = -2\partial_t.$$

Let Ω be an open set of $\mathbb{R}^{2n}$ and $p = (x_1, y_1, \cdots, x_n, y_n)$ be a point in Ω. Then, we consider the C^2 function

$$u : \Omega \to \mathbb{R}$$

and the associated hypersuface

$$\Sigma := \mathrm{Graph}(u) = \{(p,t) \in \mathbb{H}^n \,:\, u(p) = t,\, p \in \Omega\}.$$

Since the Heisenberg group is a nilpotent contact manifold we have that the mean curvature equation (3.10.25) for minimal hypersurfaces is

$$\mathrm{div}^h_\Sigma(\nu_h) = 0, \tag{3.11.1}$$

where $\nu_h = \frac{N_h}{|N_h|}$ and N_h is projection of the unit normal to Σ onto the distribution $\mathcal{H} = \mathrm{span}\{X_i, Y_i\}$. Computing directly the first variation formula of (3.2.2) we realize that for t-graphs

$$\mathrm{div}^h_\Sigma(\nu_h) = \mathrm{div}_{\mathbb{R}^{2n}}\left(\frac{\nabla u - J(p)}{|\nabla u - J(p)|}\right) = 0, \tag{3.11.2}$$

where $J(p) = (y_1, -x_1, \cdots, y_n, -x_n)$, $\nabla u = (u_{x_1}, u_{x_n}, \ldots, u_{x_n}, u_{y_n})$ and $\mathrm{div}_{\mathbb{R}^{2n}}$ is the standard divergence in $\mathbb{R}^{2n}$, for further details see [19]. Let $S = \mathrm{Graph}(u)$ be a minimal hypersurface satisfying equation (3.11.2). Since $S_t = S + (0, t)$ obtained by a vertical translation of S is still a minimal hypersurface we have that $\{S_t\}_{t\in\mathbb{R}}$ is a foliation by minimal hypersurfaces of $\Omega \times \mathbb{R}$.. Therefore, the horizontal normal vector field

$$\nu_h = \frac{\nabla u - J(p)}{|\nabla u - J(p)|} \tag{3.11.3}$$

is defined on the whole cylinder and it is invariant in the T direction. Notice that there exists also the ambient horizontal divergence on the Heisenberg group given by

$$\mathrm{div}_{\mathbb{H}^n}(V_h) = -\sum_i^n X_i(\phi_i) + Y_i(\phi_{i+1})$$

where $V_h = \sum_{i=1}^{n} \phi_i X_i + \phi_{i+1} Y_i$ is a vector field in the horizontal distribution. When vector field V_h do not depend on the t variable we have

$$\mathrm{div}_{\mathbb{H}^n}(V_h) = \mathrm{div}_{\mathbb{R}^{2n}}(V_h). \tag{3.11.4}$$

Since the unit horizontal normal vector field ν_h in (3.11.3) verifies the minimal surface equation (3.11.2) at each $(p, u(p) + t_0) \in S_{t_0}$ and we can extend it to the all cylinder $\Omega \times \mathbb{R}$. Moreover, by equation (3.11.4) we have

$$\mathrm{div}_{\mathbb{R}^{2n}}(\nu_h) = 0.$$

In this case the singular set S_0 is given by

$$S_0 = \{(p,t) \in S \,:\, \nabla u - J(p) = 0\}.$$

Clearly, the tangent vector ν_h is not defined on the singular set S_0 and the area functional has a different formula and a different Hausdorff dimension. In [20, Theorem D] they proved that singular set S_0 of a hypersurface is contained in a submanifold of dimension less than n, here we report the proof of [20, Theorem D].

Theorem 3.11.1. *Let Ω be a open set of $\mathbb{R}^{2n}$. Suppose u in $C^2(\Omega)$ and $S = Graph(u)$. Then for each p in Ω there exists a neighborhood U of p in Ω such that $S_0 \cap U$ is a submanifold of U satisfying*

$$\dim_E(S_0 \cap V) \leqslant n,$$

where $\dim_E$ is the Euclidean dimension.

Proof. First at all we define a function

$$G : \Omega \to \mathbb{R}^{2n}, \quad G(p) = \nabla u(p) - J(p)$$

The differential of G is given by

$$dG(p) = (\partial_j u_i - \partial_j J(p)_i)_{i,j=1,\cdots,2n}$$

where $J(p)_i$ is the i-th component of $J(p)$ and $u_i = \partial_i u$. Notice that if A, B are matrices of dimension m, by elementary linear algebra we know

$$\mathrm{rank}(A) + \mathrm{rank}(B) \geqslant \mathrm{rank}(A + B) \tag{3.11.5}$$

and

$$\mathrm{rank}(A) = \mathrm{rank}(-A) = \mathrm{rank}(-A^T). \tag{3.11.6}$$

Setting $A = dG(p)$ and $B = -dG(p)^T$, if follows by equations (3.11.5) and (3.11.6) that

$$\begin{aligned} 2\,\mathrm{rank}(dG(p)) &= \mathrm{rank}(dG(p)) + \mathrm{rank}(-dG(p)^T) \\ &= \mathrm{rank}(\partial_j u_i - \partial_j J(p)_i) + \mathrm{rank}(-\partial_i u_j + \partial_i J(p)_j) \\ &\geqslant \mathrm{rank}(\partial_j u_i - \partial_j J(p)_i - \partial_i u_j + \partial_i J(p)_j) \\ &= \mathrm{rank}(\partial_i J(p)_j - \partial_j J(p)_i) = 2n. \end{aligned}$$

Indeed, setting $h_{ij} = \partial_i J(p)_j$ where $i, j = 1, \cdots, 2n$, we have

$$(h_{ij}) = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & 0 & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & \ddots & & \\ & & 0 & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}$$

and $\mathrm{rank}(\partial_i J(p)_j - \partial_j J(p)_i) = \mathrm{rank}(2\, h_{ij}) = 2n$. Therefore, we have

$$\dim_E(\mathrm{Ker}(dG(p))) = 2n - \mathrm{rank}(dG(p)) \leqslant n.$$

Then by the Implicit Function Theorem there exists an open neighborhood U of p in Ω such that $G^{-1}(0) \cap U = S_0 \cap U$ is a submanifold of U, having a dimension less that n. □

Lemma 3.11.2. *Let S_0 be the set of singular point of an hypersurface S in $\mathbb{H}^n$, $n \geqslant 2$. Then for any $\varepsilon > 0$ there exist open sets $U' \subset\subset U \subset S$ with $S_0 \subset U'$ and $U \subset \{x : dist(x, S_0) < \varepsilon\}$ and a smooth function $\psi_\varepsilon : S \to [0,1]$ such that $|\nabla \psi_\varepsilon| \leqslant \frac{1}{\varepsilon}$,*

$$\begin{aligned} &\psi_\varepsilon \equiv 0 \quad \textit{in} \quad U', \\ &\psi_\varepsilon \equiv 1 \quad \textit{in} \quad (S \setminus U) \end{aligned} \tag{3.11.7}$$

and

$$\int_S |\nabla \psi_\varepsilon| \xrightarrow{\varepsilon \to 0} 0. \tag{3.11.8}$$

Proof. Here we provide a proof where the singular set S_0 is a compact (paracompact) set following [96, Lemma 2.4]. However, we only know that the singular set S_0 is closed in S, then for a right proof see [78]. By Theorem 3.11.1 we know that the Euclidean dimension of the singular set S_0 is less than or equal to n, thus, if $n \geqslant 1$, we have $H^{2n-1}(S_0) = 0$. Hence we can cover S_0 with (a finite number of) balls $B_i(r_i) = B(p_i, r_i) \cap S$ where p_i is a point in S and $r_i < \varepsilon/2$ such that

$$\sum_i r_i^{2n-1} < \varepsilon.$$

Now, for each i we consider a function ψ_i in $C^\infty(S)$, $0 \leqslant \psi_i \leqslant 1$ such that

$$\psi_i(x) = \begin{cases} 0 & \text{for} \quad x \in B_i(r_i) \\ 1 & \text{for} \quad x \in S \smallsetminus B_i(2r_i) \end{cases}$$

and

$$|\nabla \psi_i(x)| < \frac{1}{r_i} \quad \text{for all} \quad x.$$

Then setting $\tilde{\psi}_\varepsilon(x) = \min_i \psi_i(x)$, we have that $0 \leqslant \tilde{\psi}_\varepsilon \leqslant 1$ and

$$\begin{aligned} \tilde{\psi}_\varepsilon &= 0 \quad \text{on} \quad U' = \bigcup_i B_i(r_i) \\ \tilde{\psi}_\varepsilon &= 1 \quad \text{on} \quad S \smallsetminus U, \end{aligned}$$

where we set

$$U = \bigcup_i B_i(2r_i) \subset \{x : \operatorname{dist}(x, S_0) < \varepsilon\}.$$

Since we consider a minimum the function $\tilde{\psi}_\varepsilon$ is piecewise smooth and we have

$$\begin{aligned} \int_S |\nabla \tilde{\psi}_\varepsilon(x)|^2 \, dH_S^{2n+1}(x) &\leqslant \int_S \sum_i |\nabla \psi_i(x)|^2 \, dH_S^{2n+1}(x) \\ &\leqslant \sum_i \int_{B_i(2r_i) - B_i(r_i)} \frac{1}{r_i^2} \, dH_S^{2n+1}(x) \\ &\leqslant \sum_i \frac{1}{r_i^2} H_S^{2n+1}(B_i(2r_i) - B_i(r_i)) \\ &= C \sum_i r_i^{2n-1} < C\varepsilon. \end{aligned}$$

Finally, in order to obtain ψ_ε in C^∞ we mollify $\tilde{\psi}_\varepsilon$. □

Proposition 3.11.3. *Let E be the subgraph of the minimal surface $S = Graph(u)$ and $\tilde{E}$ be the subgraph of $\tilde{S} = Graph(\tilde{u})$, where $u, \tilde{u} \in C^2(\Omega)$ and u satisfies equation* (3.11.2). *Assume that $K = E \triangle \tilde{E}$ is compact, then we have*

$$A_{2n+1}(S) \leqslant A_{2n+1}(\tilde{S}),$$

where A_{2n+1} denotes the sub-Riemannian area.

Proof. Both E and $\tilde{E}$ have locally finite perimeter, therefore by [64, Lemma 12.22] $K = E \triangle \tilde{E}$ has locally finite perimeter. By assumption K is compact. By Lemma 3.11.2 there exist functions $\psi_\varepsilon : S \to [0,1]$ and $\tilde{\psi}_\varepsilon : \tilde{S} \to [0,1]$ verifying (3.11.7) and (3.11.7). Since both S and $\tilde{S}$ are graphs over Ω we can think that the functions $\psi_\varepsilon, \tilde{\psi}_\varepsilon$ are both defined on Ω where Ω_0 and $\tilde{\Omega}_0$ are the projection onto Ω of the singular set S_0 and respectively $\tilde{S}_0$. Setting $\varphi_\varepsilon : \Omega \times \mathbb{R} \to [0,1]$ equal to $\psi_\varepsilon \cdot \tilde{\psi}_\varepsilon$ and constant in the new variable, we have that there exist open sets $U' \subset\subset U \subset \Omega$ with $\Omega_0 \cup \tilde{\Omega}_0 \subset U'$ and $U \subset \{p : \operatorname{dist}(p, \Omega_0 \cup \tilde{\Omega}_0) < \varepsilon\}$ such that

$$\begin{aligned} \varphi_\varepsilon(p,t) &\equiv 0 \quad \text{for} \quad p \in U', \\ \varphi_\varepsilon(p,t) &\equiv 1 \quad \text{for} \quad p \in (\Omega \setminus U) \end{aligned} \tag{3.11.9}$$

for each $t \in \mathbb{R}$, $|\nabla \varphi_\varepsilon| \leqslant \frac{1}{\varepsilon}$ and

$$\int_K |\nabla \varphi_\varepsilon| \xrightarrow{\varepsilon \to 0} 0.$$

Therefore we have

$$0 \leqslant \left| \int_K \operatorname{div}(\varphi_\varepsilon \nu_h) \right| = \left| \int_K \langle \nabla \varphi_\varepsilon, \nu_h \rangle + \cancel{\int_K \varphi_\varepsilon \operatorname{div}(\nu_h)} \right| \leqslant \int_K |\nabla \varphi_\varepsilon| \xrightarrow{\varepsilon \to 0} 0. \tag{3.11.10}$$

Therefore, by the divergence theorem we have

$$\begin{aligned} \int_K \operatorname{div}(\varphi_\varepsilon \nu_h) &= \int_S \langle \varphi_\varepsilon \nu_h, \nu_h \rangle \, |N_h| \, d\sigma + \int_{\tilde{S}} \langle \varphi_\varepsilon \nu_h, \nu_h^{\tilde{S}} \rangle \, |N_h^{\tilde{S}}| \, d\tilde{\sigma} \\ &\geqslant \int_S \varphi_\varepsilon \, |N_h| \, d\sigma - \int_{\tilde{S}} \varphi_\varepsilon \, |N_h^{\tilde{S}}| \, d\tilde{\sigma}, \end{aligned} \tag{3.11.11}$$

where $N_h^{\tilde{S}}$ is the projection of the unit normal $N^{\tilde{S}}$ to $\tilde{S}$ onto the distribution and

$$\nu_h^{\tilde{S}} := \frac{N_h^{\tilde{S}}}{|N_h^{\tilde{S}}|}.$$

We set $\Phi(p) = (p, u(p))$ and $\tilde{\Phi}(p) = (p, \tilde{u}(p))$ such that $S = \Phi(\Omega)$ and $\tilde{S} = \tilde{\Phi}(\Omega)$. Letting ε tend to zero, we have that the left hand side of (3.11.11) goes to zero by (3.11.10) and form the right hand side we obtain

$$A_{2n+1}(S \smallsetminus \Phi(\Omega_0 \cup \tilde{\Omega}_0)) \leqslant A_{2n+1}(\tilde{S} \smallsetminus \tilde{\Phi}(\Omega_0 \cup \tilde{\Omega}_0)).$$

Since the singular sets do not affect the area functional, we conclude that

$$A_{2n+1}(S) \leqslant A_{2n+1}(\tilde{S}). \qquad \square$$

Remark 3.11.4. Let (p, t) be a point in the Heisenberg group and r_0 be a real number such that $B((p, t), r_0) \cap \partial S = \emptyset$. Then there exist $r_0' < r_0$ and C positive constant such that for each $r < r_0'$ it holds

$$A_{2n+1}(S \cap B((p, t), r)) \leqslant C r^{2n+1}.$$

Index of Symbols

Here is a brief list of notations frequently used in this thesis.

ρ	dimension of the ambient sub-bundle corresponding to $\iota_0(U)$ (page 103).
A	$\ell \times \rho$ matrix (page 104).
B	$\ell \times (n-\rho)$ matrix (page 104).
C_j	$\ell \times (n-\rho)$ matrix for each $j = 1, \dots, m$ (page 104)
k	difference between the dimension of the ambient sub-bundle and the dimension of the tangent induced sub-bundle both of them corresponding to $\iota_0(U)$ (page 110).
H_M^ε	holonomy map for ruled submanifolds (page 132).
$\mathbf{H}_d$	vector field associated to the first variation (page 147).

References

[1] A. A. Agrachev and A. V. Sarychev. Abnormal sub-Riemannian geodesics: Morse index and rigidity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(6):635–690, 1996.

[2] Andrei Agrachev, Davide Barilari, and Ugo Boscain. Introduction to geodesics in sub-Riemannian geometry. In *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. II*, EMS Ser. Lect. Math., pages 1–83. Eur. Math. Soc., Zürich, 2016.

[3] Andrei A. Agrachev and Yuri L. Sachkov. *Control theory from the geometric viewpoint*, volume 87 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. Control Theory and Optimization, II.

[4] Luigi Ambrosio, Francesco Serra Cassano, and Davide Vittone. Intrinsic regular hypersurfaces in Heisenberg groups. *J. Geom. Anal.*, 16(2):187–232, 2006.

[5] Annalisa Baldi and Bruno Franchi. Differential forms in Carnot groups: a Γ-convergence approach. *Calc. Var. Partial Differential Equations*, 43(1-2):211–229, 2012.

[6] Vittorio Barone Adesi, Francesco Serra Cassano, and Davide Vittone. The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations. *Calc. Var. Partial Differential Equations*, 30(1):17–49, 2007.

[7] A. Bellaïche. The tangent space in sub-Riemannian geometry. *J. Math. Sci. (New York)*, 83(4):461–476, 1997. Dynamical systems, 3.

[8] David E. Blair. *Riemannian geometry of contact and symplectic manifolds*, volume 203 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2010.

[9] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[10] Roger W. Brockett. *Finite dimensional linear systems*, volume 74 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015. Reprint of the 1970 original.

[11] Robert L. Bryant and Lucas Hsu. Rigidity of integral curves of rank 2 distributions. *Invent. Math.*, 114(2):435–461, 1993.

[12] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[13] Luca Capogna, Giovanna Citti, and Maria Manfredini. Regularity of mean curvature flow of graphs on Lie groups free up to step 2. *Nonlinear Anal.*, 126:437–450, 2015.

[14] Luca Capogna, Donatella Danielli, Scott D. Pauls, and Jeremy T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2007.

[15] E. Cartan. Sur la representation geometrique des systémes materiels non holonomes. *Proc. Int. Congr. Math.*, vol. 4, Bologna:253–261, 1928.

[16] Jih-Hsin Cheng, Hung-Lin Chiu, Jenn-Fang Hwang, and Paul Yang. Umbilicity and characterization of Pansu spheres in the Heisenberg group. *J. Reine Angew. Math.*, 738:203–235, 2018.

[17] Jih-Hsin Cheng and Jenn-Fang Hwang. Variations of generalized area functionals and p-area minimizers of bounded variation in the Heisenberg group. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 5(4):369–412, 2010.

[18] Jih-Hsin Cheng, Jenn-Fang Hwang, Andrea Malchiodi, and Paul Yang. Minimal surfaces in pseudohermitian geometry. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(1):129–177, 2005.

[19] Jih-Hsin Cheng, Jenn-Fang Hwang, and Paul Yang. Existence and uniqueness for p-area minimizers in the Heisenberg group. *Math. Ann.*, 337(2):253–293, 2007.

[20] Jih-Hsin Cheng, Jenn-Fang Hwang, and Paul Yang. Existence and uniqueness for p-area minimizers in the Heisenberg group. *Math. Ann.*, 337(2):253–293, 2007.

[21] Wei-Liang Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

[22] G. Citti and A. Sarti. A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vision*, 24(3):307–326, 2006.

[23] Giovanna Citti, Gianmarco Giovannardi, and Manuel Ritoré. Variational formulas for curves of fixed degree. *arXiv e-prints*, page arXiv:1902.04015, Feb 2019.

[24] Giovanna Citti, Gianmarco Giovannardi, and Manuel Ritoré. Variational formulas for submanifolds of fixed degree. *arXiv e-prints*, page arXiv:1905.05131, May 2019.

[25] Giovanna Citti and Maria Manfredini. Blow-up in non homogeneous Lie groups and rectifiability. *Houston J. Math.*, 31(2):333–353, 2005.

[26] Giovanna Citti and Alessandro Sarti, editors. *Neuromathematics of vision*. Lecture Notes in Morphogenesis. Springer, Heidelberg, 2014.

[27] Giovanna Citti and Alessandro Sarti. Models of the visual cortex in Lie groups. In *Harmonic and geometric analysis*, Adv. Courses Math. CRM Barcelona, pages 1–55. Birkhäuser/Springer Basel AG, Basel, 2015.

[28] D. Danielli, N. Garofalo, and D. M. Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Adv. Math.*, 215(1):292–378, 2007.

[29] D. Danielli, N. Garofalo, D. M. Nhieu, and S. D. Pauls. Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$. *J. Differential Geom.*, 81(2):251–295, 2009.

[30] Ennio De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[31] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[32] Ermal Feleqi and Franco Rampazzo. Iterated Lie brackets for nonsmooth vector fields. *NoDEA Nonlinear Differential Equations Appl.*, 24(6):Art. 61, 43, 2017.

[33] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[34] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication.

[35] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Rectifiability and perimeter in the Heisenberg group. *Math. Ann.*, 321(3):479–531, 2001.

[36] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups. *Comm. Anal. Geom.*, 11(5):909–944, 2003.

[37] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Intrinsic submanifolds, graphs and currents in Heisenberg groups. In *Lecture notes of Seminario Interdisciplinare di Matematica. Vol. IV*, volume 4 of *Lect. Notes Semin. Interdiscip. Mat.*, pages 23–38. Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2005.

[38] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Regular submanifolds, graphs and area formula in Heisenberg groups. *Adv. Math.*, 211(1):152–203, 2007.

[39] Bruno Franchi, Raul Paolo Serapioni, and Francesco Serra Cassano. Area formula for centered Hausdorff measures in metric spaces. *Nonlinear Anal.*, 126:218–233, 2015.

[40] Matteo Galli. First and second variation formulae for the sub-Riemannian area in three-dimensional pseudo-Hermitian manifolds. *Calc. Var. Partial Differential Equations*, 47(1-2):117–157, 2013.

[41] Matteo Galli and Manuel Ritoré. Existence of isoperimetric regions in contact sub-Riemannian manifolds. *J. Math. Anal. Appl.*, 397(2):697–714, 2013.

[42] Matteo Galli and Manuel Ritoré. Regularity of C^1 surfaces with prescribed mean curvature in three-dimensional contact sub-Riemannian manifolds. *Calc. Var. Partial Differential Equations*, 54(3):2503–2516, 2015.

[43] Nicola Garofalo and Duy-Minh Nhieu. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math.*, 49(10):1081–1144, 1996.

[44] Roberta Ghezzi and Frédéric Jean. Hausdorff measure and dimensions in non equiregular sub-Riemannian manifolds. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 201–218. Springer, Cham, 2014.

[45] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[46] Gianmarco Giovannardi. Geometric properties of 2-dimensional minimal surfaces in a sub-Riemannian manifold which models the Visual Cortex. Master's thesis, University of Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna BO, 2016.

[47] Gianmarco Giovannardi. Higher dimensional holonomy map for ruled submanifolds in graded manifolds. *arXiv e-prints*, page arXiv:1906.05033, Jun 2019.

[48] P. C. Greiner and E. M. Stein. *Estimates for the $\overline{\partial}$-Neumann problem*. Princeton University Press, Princeton, N.J., 1977. Mathematical Notes, No. 19.

[49] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[50] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[51] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[52] Philip Hartman. *Ordinary differential equations*, volume 38 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA; MR0658490 (83e:34002)], With a foreword by Peter Bates.

[53] Robert K. Hladky and Scott D. Pauls. Constant mean curvature surfaces in sub-Riemannian geometry. *J. Differential Geom.*, 79(1):111–139, 2008.

[54] Robert K. Hladky and Scott D. Pauls. Variation of perimeter measure in sub-Riemannian geometry. *Int. Electron. J. Geom.*, 6(1):8–40, 2013.

[55] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[56] Lucas Hsu. Calculus of variations via the Griffiths formalism. *J. Differential Geom.*, 36(3):551–589, 1992.

[57] Ana Hurtado, Manuel Ritoré, and César Rosales. The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$. *Adv. Math.*, 224(2):561–600, 2010.

[58] Thomas A. Ivey and Joseph M. Landsberg. *Cartan for beginners*, volume 175 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016. Differential geometry via moving frames and exterior differential systems, Second edition [of MR2003610].

[59] David S. Jerison. The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. I. *J. Funct. Anal.*, 43(1):97–142, 1981.

[60] David S. Jerison. The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. II. *J. Funct. Anal.*, 43(2):224–257, 1981.

[61] Enrico Le Donne, Gian Paolo Leonardi, Roberto Monti, and Davide Vittone. Extremal curves in nilpotent Lie groups. *Geom. Funct. Anal.*, 23(4):1371–1401, 2013.

[62] Enrico Le Donne and Valentino Magnani. Measure of submanifolds in the Engel group. *Rev. Mat. Iberoam.*, 26(1):333–346, 2010.

[63] Gian Paolo Leonardi and Roberto Monti. End-point equations and regularity of sub-Riemannian geodesics. *Geom. Funct. Anal.*, 18(2):552–582, 2008.

[64] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[65] Valentino Magnani. Characteristic points, rectifiability and perimeter measure on stratified groups. *J. Eur. Math. Soc. (JEMS)*, 8(4):585–609, 2006.

[66] Valentino Magnani. On a measure-theoretic area formula. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(4):885–891, 2015.

[67] Valentino Magnani. Towards a theory of area in homogeneous groups, 2018.

[68] Valentino Magnani, Jeremy T. Tyson, and Davide Vittone. On transversal submanifolds and their measure. *J. Anal. Math.*, 125:319–351, 2015.

[69] Valentino Magnani and Davide Vittone. An intrinsic measure for submanifolds in stratified groups. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2008(619):203–232, 2008.

[70] Sorin Mardare. On systems of first order linear partial differential equations with L^p coefficients. *Adv. Differential Equations*, 12(3):301–360, 2007.

[71] John Mitchell. On Carnot-Carathéodory metrics. *J. Differential Geom.*, 21(1):35–45, 1985.

[72] Francescopaolo Montefalcone. Hypersurfaces and variational formulas in sub-Riemannian Carnot groups. *J. Math. Pures Appl. (9)*, 87(5):453–494, 2007.

[73] Richard Montgomery. Abnormal minimizers. *SIAM J. Control Optim.*, 32(6):1605–1620, 1994.

[74] Richard Montgomery. Singular extremals on Lie groups. *Math. Control Signals Systems*, 7(3):217–234, 1994.

[75] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[76] Roberto Monti. Some properties of Carnot-Carathéodory balls in the Heisenberg group. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 11(3):155–167 (2001), 2000.

[77] Roberto Monti. The regularity problem for sub-Riemannian geodesics. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 313–332. Springer, Cham, 2014.

[78] Frank Morgan and Manuel Ritoré. Isoperimetric regions in cones. *Trans. Amer. Math. Soc.*, 354(6):2327–2339, 2002.

[79] Alexander Nagel and E. M. Stein. *Lectures on pseudodifferential operators: regularity theorems and applications to nonelliptic problems*, volume 24 of *Mathematical Notes*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979.

[80] Alexander Nagel, Elias M. Stein, and Stephen Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.*, 155(1-2):103–147, 1985.

[81] John Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956.

[82] Pierre Pansu. Differential forms and connections adapted to a contact structure, after M. Rumin. In *Symplectic geometry*, volume 192 of *London Math. Soc. Lecture Note Ser.*, pages 183–195. Cambridge Univ. Press, Cambridge, 1993.

[83] Pierre Pansu. Submanifolds and differential forms on Carnot manifolds, after M. Gromov and M. Rumin. *arXiv e-prints*, page arXiv:1604.06333, Apr 2016.

[84] Scott D. Pauls. H-minimal graphs of low regularity in $\mathbb{H}^1$. *Comment. Math. Helv.*, 81(2):337–381, 2006.

[85] Jean Petitot. *Landmarks for Neurogeometry*, pages 1–85. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[86] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The mathematical theory of optimal processes*. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

[87] Petr Konstanovich Rashevskii. About connecting two points of complete non-holonomic space by admissible curve (in russian). *Uch. Zapiski ped. inst. Libknexta*, 2:83–94, 1938.

[88] Ludovic Rifford. *Sub-Riemannian geometry and optimal transport*. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[89] M. Ritoré and C. Rosales. Area-stationary and stable surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$. *Mat. Contemp.*, 35:185–203, 2008.

[90] Manuel Ritoré. Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity. *Calc. Var. Partial Differential Equations*, 34(2):179–192, 2009.

[91] Manuel Ritoré. A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$. *Calc. Var. Partial Differential Equations*, 44(1-2):47–60, 2012.

[92] Manuel Ritoré and César Rosales. Rotationally invariant hypersurfaces with constant mean curvature in the Heisenberg group $\mathbb{H}^n$. *J. Geom. Anal.*, 16(4):703–720, 2006.

[93] Linda Preiss Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(3-4):247–320, 1976.

[94] Michel Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.

[95] Nataliya Shcherbakova. Minimal surfaces in sub-Riemannian manifolds and structure of their singular sets in the $(2,3)$ case. *ESAIM Control Optim. Calc. Var.*, 15(4):839–862, 2009.

[96] Peter Sternberg and Kevin Zumbrun. On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. *Comm. Anal. Geom.*, 7(1):199–220, 1999.

[97] Héctor J. Sussmann. A cornucopia of four-dimensional abnormal sub-Riemannian minimizers. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 341–364. Birkhäuser, Basel, 1996.

[98] Kang Hai Tan and Xiao Ping Yang. Horizontal connection and horizontal mean curvature in Carnot groups. *Acta Math. Sin. (Engl. Ser.)*, 22(3):701–710, 2006.

[99] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.