TEP 190 RESEARCH GROUP

Detailed application of the Linear Acceleration Method for the response of an elasto-plastic SDOF system

Report TEP 190 1-2019

Enrique Hernández Montes, University of Granada

Juan Francisco Carbonell Márquez, University of Córdoba

03/05/2019

Detailed application of the Linear Acceleration Method for the response of an elasto-plastic SDOF system

by

Enrique Hernández-Montes[1]

and

Juan Francisco Carbonell-Márquez[2,*]

Keywords: Linear Acceleration Method, Newmark's Method, SDOF, Elasto-plastic Behavior, Complete Hysteretic Curve, Dynamic Response, Earthquake Excitation

Abstract

The numeric computation procedure for the solution of the equation of motion of a single-degree-of-freedom (SDOF) system subjected to any type of ground acceleration is exhaustively presented. The followed numeric approach is the Linear Acceleration Method, i.e. Newmark's Method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$. The approach allows considering any time of multilinear elasto-plastic behavior. It also allows computing the Complete Hysteretic Curve of the SDOF system.

[1] Professor. Department of Structural Mechanics, University of Granada (UGR). Campus Universitario de Fuentenueva s/n. 18072 Granada, Spain. emontes@ugr.es.

[2] Assistant Professor. Departamento de Mecánica, Universidad de Córdoba. Campus de Rabanales, Edificio Leonardo da Vinci, E-14071 Córdoba, Spain. jcarbonell@uco.es. *Corresponding author

1. Problem statement

Let us consider a single-degree-of-freedom (SDOF) system with some kind of elasto-plastic behavior, $k(u)$, constant mass m, and viscous damping coefficient c, subjected to ground acceleration $\ddot{u}_g(t)$, Fig. 1, the corresponding equation of motion is given by dynamic equilibrium, Eq. (1):

$$m\ddot{u}(t) + c\dot{u}(t) + f_S = -m\ddot{u}_g(t) \tag{1}$$

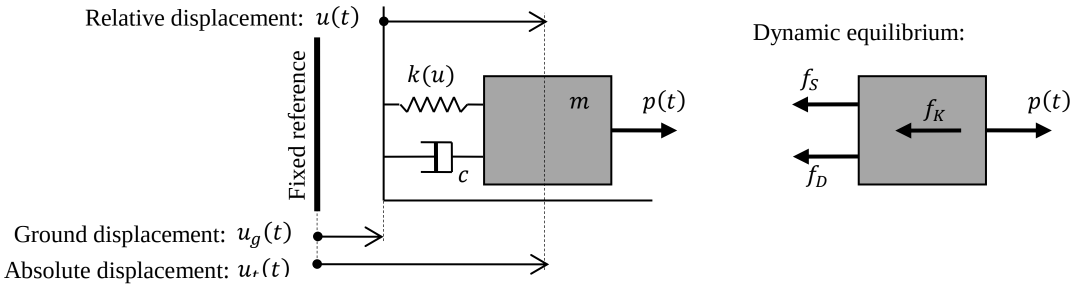

Fig. 1. SDOF system usually employed in earthquake engineering.

Eq. (1) can be numerically solved employing the linear acceleration method (LAM), i.e. Newmark's method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ [1]. Although the method is originally employed to compute the response of the SDOF under the action of an earthquake, $\ddot{u}_g(t)$, it can be also used to perform a "snap-back" analysis, as done by Hernández-Montes et al. [2], in which an initial displacement is imposed and the SDOF system is freely released afterwards.

In what follows, it is assumed that the system's hysteretic model $f_S - u$ is composed by some linear $f_S(u)$ algebraic functions or branches, so that any of them is characterized by a particular stiffness, k, Fig. 2. The values of the different variables involved in the problem, i.e. displacement, velocity, acceleration, spring force, etc. relative to time t_i will be referred to with the subscript i henceforth.

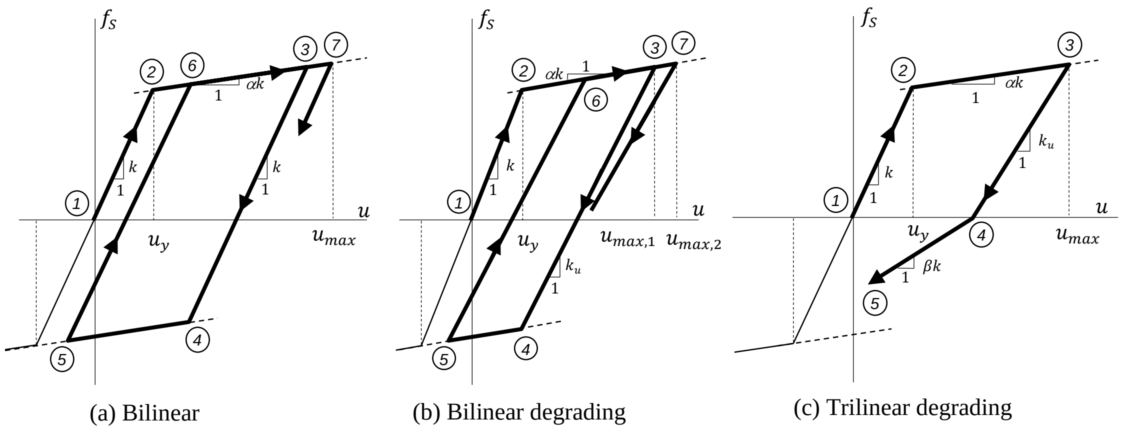

Fig. 2. Some common multi-linear hysteretic models employed in earthquake engineering.

2. Fundamentals of the Linear Acceleration Method

Let us consider the difference between the results of Eq. (1) when it is considered in two very close instants of time t_A and t_B, so that, $\Delta t = t_B - t_A$, assuming that u_A and u_B correspond to the same branch of the $f_S - u$ model, i.e. $f_S = ku$:

$$m[\ddot{u}_B - \ddot{u}_A] + c[\dot{u}_B - \dot{u}_A] + k[u_B - u_A] = -m\left[\ddot{u}_{g_B} - \ddot{u}_{g_A}\right] \tag{2}$$

If the difference $\ddot{u}_B - \ddot{u}_A$ is rewritten as $\Delta\ddot{u}$, and doing the same for velocity, displacement and ground acceleration, Eq. (2) remains:

$$m\Delta\ddot{u} + c\Delta\dot{u} + k\Delta u = -m\Delta\ddot{u}_g \tag{3}$$

Given the mass of the system, m, if its natural circular frequency is written as a function of the stiffness k, $\omega = \sqrt{k/m}$, and its viscous damping coefficient is written as a function of the damping ratio ξ, $c = 2m\omega\xi$, then Eq. (3) can be rewritten as:

$$\Delta\ddot{u} + 2\omega\xi\Delta\dot{u} + \omega^2\Delta u = -\Delta\ddot{u}_g \tag{4}$$

Now, let us focus on the system's acceleration evolution between time instants A and B, $\Delta\ddot{u}$. As A and B are very close, $\Delta\ddot{u}$ can be considered linear, Fig. 3. Therefore, the system's acceleration in a time τ between A and B, i.e. $t_A \leq \tau \leq t_B$, can be written as:

$$\ddot{u}(\tau) = \ddot{u}_A + \frac{\Delta\ddot{u}}{\Delta t}\tau \tag{5}$$

Therefore, to get the velocity and displacement of the system at that time τ Eq. (5) needs to be integrated so that:

$$\dot{u}(\tau) = \dot{u}_A + \ddot{u}_A\tau + \frac{\Delta\ddot{u}}{\Delta t}\frac{\tau^2}{2} \tag{6}$$

$$u(\tau) = u_A + \dot{u}_A\tau + \ddot{u}_A\frac{\tau^2}{2} + \frac{\Delta\ddot{u}}{\Delta t}\frac{\tau^3}{6} \tag{7}$$

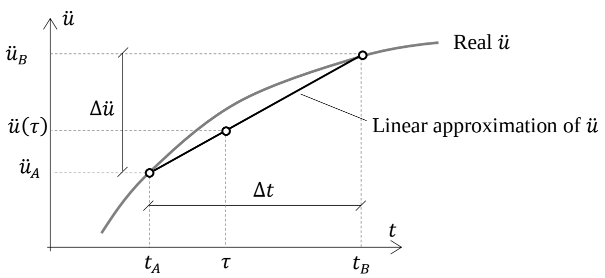

Fig. 3. Linear approximation of the system's acceleration between two very close instants of time.

Evaluating Eqs. (6) and (7) for $\tau = \Delta t$ and expressing the results as increments, i.e $\Delta\dot{u}$ and Δu, remains:

$$\Delta\dot{u} = \left(\ddot{u}_A + \frac{\Delta\ddot{u}}{2}\right)\Delta t \tag{8}$$

$$\Delta u = \dot{u}_A \Delta t + \left(\frac{\ddot{u}_A}{2} + \frac{\Delta\ddot{u}}{6}\right)\Delta t^2 \tag{9}$$

If, now, Eqs. (8) and (9) are introduced in Eq. (4), the increment of acceleration $\Delta\ddot{u}$ remains:

$$\Delta\ddot{u} = -\frac{3\left(2\Delta\ddot{u}_g + 4\ddot{u}_A\Delta t\xi\omega + 2\dot{u}_A\Delta t\omega^2 + u_A\Delta t\omega^2\right)}{6 + 6\Delta t\xi\omega + \Delta t^2\omega^2} \tag{10}$$

Therefore, if the values of displacement and velocity at instant A, u_A and $\dot{u}_A$, are known, Eq. (1) can be rearranged to yield the acceleration at that instant $\ddot{u}_A$:

$$\ddot{u}_A = -\left(\ddot{u}_{g_A} + \omega^2 u_A + 2\omega\xi\dot{u}_A\right) \tag{11}$$

Finally, knowing u_A, $\dot{u}_A$ and $\ddot{u}_A$, these values can be introduced in Eq. (10) and, after this, the increments in velocity $\Delta\dot{u}$ and displacement Δu can be obtained by means of Eqs. (8) and (9). These operations can be repeated to compute the system's time histories for displacement, velocity and acceleration.

3. Numerical algorithm for the equation of motion resolution.

The input for the approach are the assumed constant mass m and fraction of critical damping ξ of the SDOF system, the hysteretic model with known basic rules to compute f_S in terms of u, the time sampling frequency, $1/\Delta t_{general}$ Hz, time when simulation stops, t_{end}, and initial conditions of the system, $u_0 = u(t = 0)$ and $\dot{u}_0 = \dot{u}(t = 0)$. If the system is to be subjected to the action of an earthquake, the samples of ground acceleration need to be presented in a list $\mathbf{\ddot{u}_g}$ so that they have been sampled at the frequency $1/\Delta t_{general}$ Hz. Given that the sampling time step for earthquake records is usually 0.02 s, taking time steps $\Delta t_{general}$ like that ensure stability and low computational errors [3]. The obtained output will consist on system's displacement, velocity, acceleration, spring force and damping force for each time t_i.

The followed algorithm is presented as a flowchart in Fig. 4. In a first iteration ($i = 0$), the branch of the $f_S - u$ model is set, providing the stiffness current branch, k_C, and restoring force, f_{S_0}, of the system. Knowing k_C, the natural circular frequency $\omega_C = \sqrt{k_C/m}$ and the initial damping force, Eq. (12), can be also computed.

$$f_{D_0} = 2m\xi\omega_C\dot{u}_0 \tag{12}$$

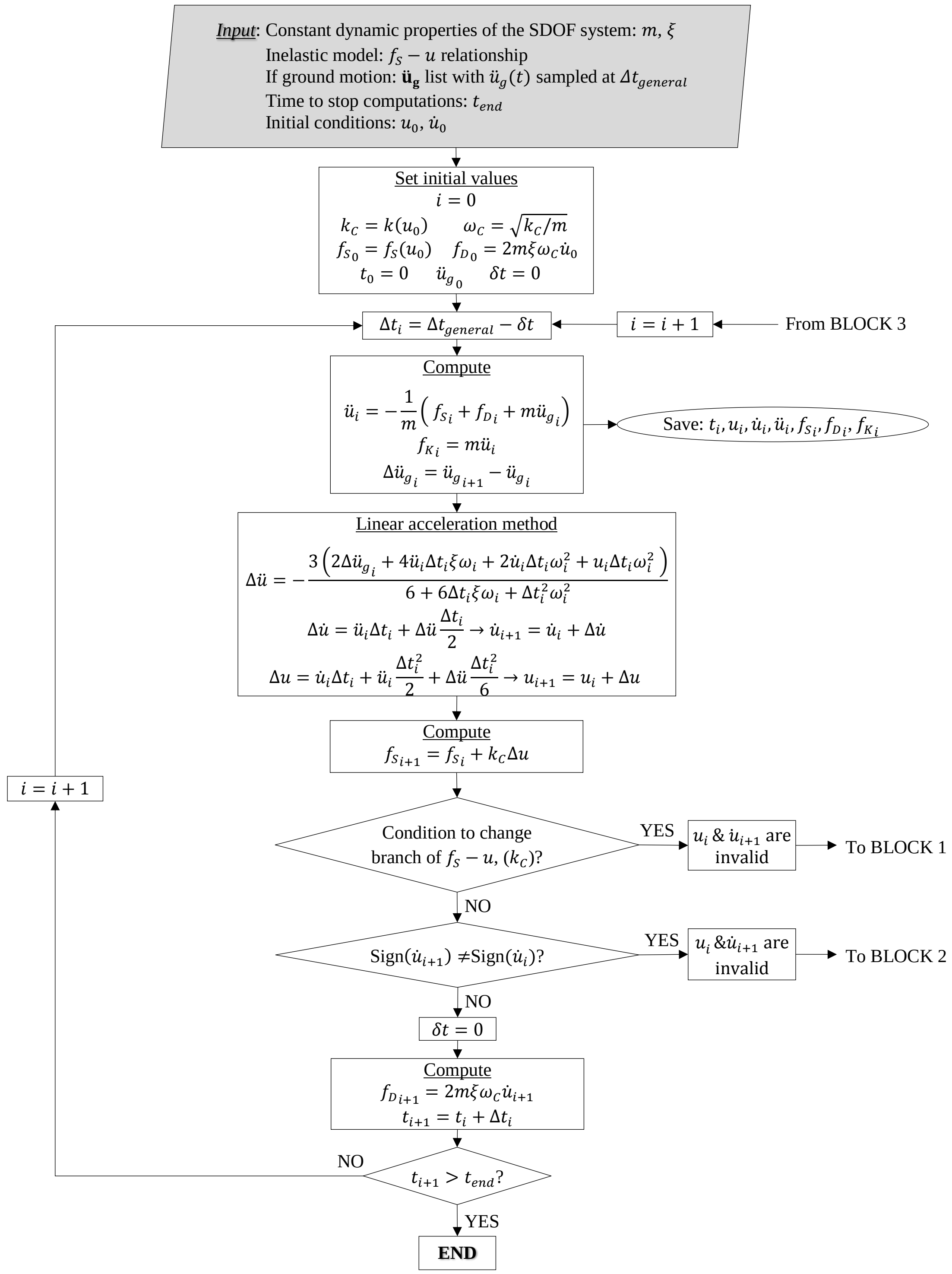

Fig. 4. General flowchart representing the algorithm to compute the response of a SDOF with elasto-plastic behavior.

Therefore, given the system in a particular branch of the $f_S - u$ model (i.e. k_C is fixed) each i iteration starts knowing the corresponding values of time t_i, relative displacement and velocity, u_i and $\dot{u}_i$, restoring force f_{S_i} and damping force f_{D_i} so that the relative acceleration $\ddot{u}_i$ can be computed by means of Eq. (1). The values of u_{i+1} and $\dot{u}_{i+1}$ are then computed by LAM according to:

$$
\begin{aligned}
\Delta\ddot{u} &= -\frac{3\left(2\Delta\ddot{u}_{g_i}+4\ddot{u}_i\Delta t_i\xi\omega_C+2\dot{u}_i\Delta t_i\omega_C^2+u_i\Delta t_i\omega_C^2\right)}{6+6\Delta t_i\xi\omega_C+\Delta t_i^2\omega_C^2} \\
\Delta\dot{u} &= \ddot{u}_i\Delta t_i + \Delta\ddot{u}\frac{\Delta t_i}{2} \\
\Delta u &= \dot{u}_i\Delta t_i + \ddot{u}_i\frac{\Delta t_i^2}{2} + \Delta\ddot{u}\frac{\Delta t_i^2}{6}
\end{aligned} \tag{13}
$$

so that:

$$
\begin{aligned}
\dot{u}_{i+1} &= \dot{u}_i + \Delta\dot{u} \\
u_{i+1} &= u_i + \Delta u
\end{aligned} \tag{14}
$$

Eq. (13) is similar to Eqs. (8) to (10). The only differences are that in Eqs. (8) to (10) Δt is assumed to be constant whereas in Eq. (13) the value of Δt_i can be $\Delta t_{general}$ or a lower value $(\Delta t_{general} - \delta t)$ due to a change of branch in the $f_S - u$ model.

Knowing Δu, the next restoring force is computed as:

$$
f_{S_{i+1}} = f_{S_i} + k_C\Delta u \tag{15}
$$

Finally in this iteration i, next step time t_{i+1} is set and damping force $f_{D_{i+1}}$ is calculated similarly as done in Eq. (12). After this, a new iteration is performed.

However, after LAM and $f_{S_{i+1}}$ computations some checks need to be done. Firstly, it is necessary to verify if any condition to change the branch of the $f_S - u$ model has been met, Fig. 5. If so, the computed value f_{i+1} should lie on the next branch, $f_{S_n}(u)$, instead of remaining on the current one, $f_{S_c}(u)$. Block 1, Fig. 6, computes the value of displacement and time interval to get to the point of intersection of both branches, $f_{S_c}(u)$ and $f_{S_n}(u)$. In this block, knowing $f_{S_c}(u)$ and $f_{S_n}(u)$, the displacement for branch change (BC) u_{BC} can be determined by solving:

$$
f_{S_c}(u_{BC}) = f_{S_n}(u_{BC}) \tag{16}
$$

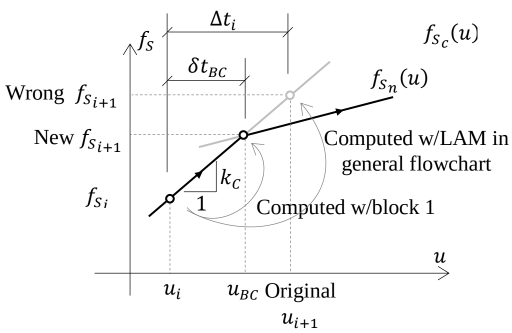

Fig. 5. Condition to branch change in the $f_S - u$ model activating computations in block 1.

Then, using the LAM equations (Eq. (13)), i.e. introducing the value of $\Delta\ddot{u}$ within the equation $\Delta u = u_{BC} - u_i$, the time interval after u_i at which u_{BC} occurs, δt_{BC}, can be determined:

$$u_{BC} = u_i + \dot{u}_i \delta t_{BC} + \ddot{u}_i \frac{\delta t_{BC}^2}{2} - \frac{2\Delta\ddot{u}_{g_i}\frac{\delta t_{BC}}{\Delta t_i} + 4\ddot{u}_i \delta t_{BC}\xi\omega_C + 2\dot{u}_i\delta t_{BC}\omega_C^2 + u_i \delta t_{BC}\omega_C^2}{2\left(6 + 6\delta t_{BC}\xi\omega_C + \delta t_{BC}^2\omega_C^2\right)}\delta t_{BC}^2 \qquad (17)$$

From condition to change branch of $f_S - u$

↓

Compute displacement for branch change (u_{BC})

$f_{S_c}(u_{BC}) = f_{S_n}(u_{BC})$

↓

u_{BC}

↓

Compute (LAM) time interval for branch change (δt_{BC})

$$\left.\begin{aligned} u_{BC} &= u_i + \dot{u}_i \delta t_{BC} + \ddot{u}_i \frac{\delta t_{BC}^2}{2} + \Delta\ddot{u}\frac{\delta t_{BC}^2}{6} \\ \Delta\ddot{u} &= -\frac{3\left(2\Delta\ddot{u}_g \frac{\delta t_{BC}}{\Delta t_i} + 4\ddot{u}_i\delta t_{BC}\xi\omega_C + 2\dot{u}_i\delta t_{BC}\omega_C^2 + u_i\delta t_{BC}\omega_C^2\right)}{6 + 6\delta t_{BC}\xi\omega_C + \delta t_{BC}^2\omega_C^2} \end{aligned}\right\} \rightarrow \delta t_{BC}$$

↓

$\delta t = \delta t_{BC}$

↓

To BLOCK 3

Fig. 6. Block 1 employed to compute displacement and time at which branch in the $f_S - u$ model is switched.

If the SDOF is subjected to an earthquake signal, it must be noticed that ground acceleration increment $\Delta\ddot{u}_{g_i} = \ddot{u}_{g_{i+1}} - \ddot{u}_{g_i}$ employed in the computation of $\Delta\ddot{u}$ needs to be proportional to Δt_i, so that $\ddot{u}_{g_i}$ has been sampled at t_i and $\ddot{u}_{g_{i+1}}$ at t_{i+1}. As Eq. (17) is employed to compute a time interval $\delta t_{BC} < \Delta t_i$, then the used increment of ground acceleration needs to be proportional and $\Delta\ddot{u}_{g_i}\frac{\delta t_{BC}}{\Delta t_i}$ is introduced instead of the original $\Delta\ddot{u}_{g_i}$, Fig. 7. After δt_{BC} is determined, further computations need to be done by means of block 3 prior to coming back to the general flowchart.

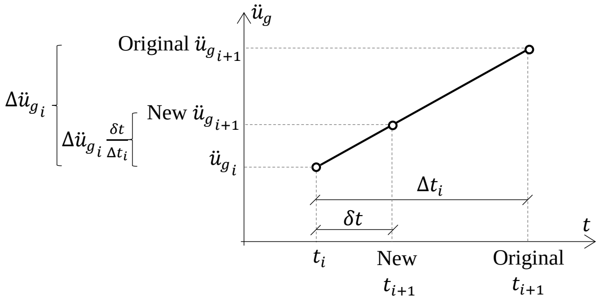

Fig. 7. Correspondence between time intervals and ground acceleration increments to be employed in LAM and computation of new value of $\ddot{u}_g$ to be introduced in $\mathbf{\ddot{u}_g}$.

If after LAM computations within the general flowchart no condition to switch the branch of the $f_S - u$ model has been fulfilled, another check prior to a new iteration is needed to know whether the system has changed the direction of its displacement or not. If so, Sign($\dot{u}_{i+1}$)≠Sign($\dot{u}_i$). Therefore, the computed value f_{i+1} after LAM is wrong again because the system has changed the sense of loading (from loading to unloading or vice versa) and a different branch of the $f_S - u$ model must be adopted for further computations, Fig. 8.

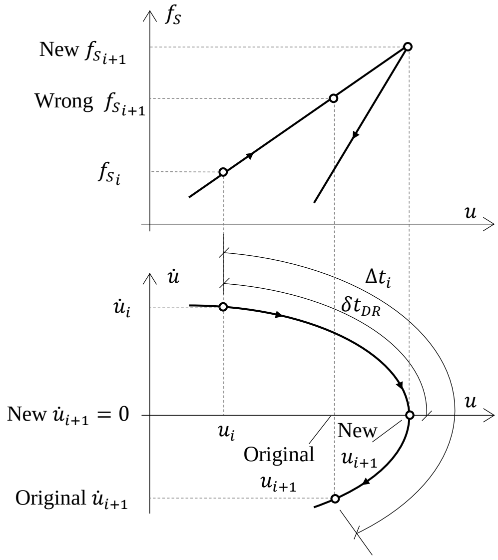

Fig. 8. Computations made in block 2: time interval δt_{DR} after u_i to make velocity $\dot{u}_{i+1} = 0$ is sought.

Block 2, Fig. 9, computes the interval of time after u_i at which displacement reversal (DR) occurs. Therefore, employing LAM equations (Eq. (13)), i.e. introducing the value of $\Delta\ddot{u}$ within the equation $\Delta\dot{u} = 0 - \dot{u}_i$, block 2 computes the interval of time δt_{DR} that makes velocity $\dot{u}_{i+1} = 0$:

$$0 = \dot{u}_i + \ddot{u}_i \delta t_{DR} - \frac{3\left(2\Delta\ddot{u}_g \frac{\delta t_{DR}}{\Delta t_i} + 4\ddot{u}_i \delta t_{DR} \xi \omega_C + 2\dot{u}_i \delta t_{DR} \omega_C^2 + u_i \delta t_{DR} \omega_C^2\right)}{2\left(6 + 6\delta t_{DR} \xi \omega_C + \delta t_{DR}^2 \omega_C^2\right)} \delta t_{DR} \tag{18}$$

From Sign($\dot{u}_{i+1}$) $\neq$ Sign($\dot{u}_i$)

Compute (LAM) time interval for displacement reversal: $\dot{u}_{i+1} = 0$ (δt_{DR})

$$\left.\begin{aligned} 0 &= \dot{u}_i + \ddot{u}_i \delta t_{DR} + \Delta\ddot{u} \frac{\delta t_{DR}}{2} \\ \Delta\ddot{u} &= -\frac{3\left(2\Delta\ddot{u}_g \frac{\delta t_{DR}}{\Delta t_i} + 4\ddot{u}_i \delta t_{DR} \xi \omega_C + 2\dot{u}_i \delta t_{DR} \omega_i^2 + u_i \delta t_{DR} \omega_C^2\right)}{6 + 6\delta t_{DR} \xi \omega_C + \delta t_{DR}^2 \omega_C^2} \end{aligned}\right\} \rightarrow \delta t_{DR}$$

$\boxed{\delta t = \delta t_{DR}}$

To BLOCK 3

Fig. 9. Block 2 employed to calculate time that makes velocity zero and at which displacement reversal occurs.

Right after block 1 or 2, block 3 is employed to compute the valid values of u_{i+1} and $\dot{u}_{i+1}$ needed for the next iteration. In block 1 or 2, a new interval of time δt to perform the LAM computations has been determined. Therefore, block 3 makes use of LAM equations, Eq. (13), with this new δt taking into account that the increment of ground acceleration $\Delta\ddot{u}_{g_i}$ (if present) employed to calculate $\Delta\ddot{u}$ needs to be proportional to that time interval, as explained before, Fig. 7. Consequently, a new sample of ground acceleration needs to be introduced in the list $\ddot{\mathbf{u}}_{\mathbf{g}}$ between $\ddot{u}_{g_i}$ and the original $\ddot{u}_{g_{i+1}}$, now $\ddot{u}_{g_{i+2}}$. Hence, the new value introduced in $\ddot{\mathbf{u}}_{\mathbf{g}}$ is:

$$\ddot{u}_{g_{i+1}} = \ddot{u}_{g_i} + \Delta\ddot{u}_{g_i}\frac{\delta t}{\Delta t_i} \tag{19}$$

Once LAM is performed in block 3, the next iteration values $u_{i+1}, \dot{u}_{i+1}, f_{S_{i+1}}, f_{D_{i+1}}$ and $t_{i+1} = t_i + \delta t$ are computed. Finally, the values of the stiffness and natural circular frequency to be employed in the following iterations, k_C and ω_C respectively, are actualized according to the new branch of the $f_S - u$ driving the process. A new iteration in the general flowchart can be now performed.

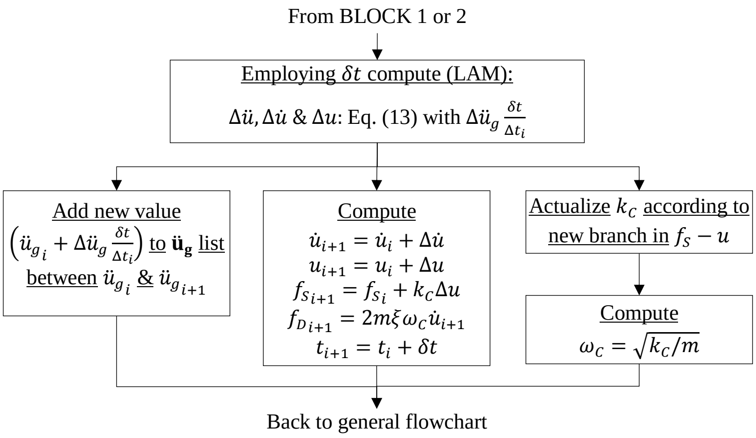

Fig. 10. Block 3 used to recompute the valid values of next iteration variables with the modified time interval δt computed in block 1 or 2.

The above explained approach has been implemented in a Mathematica® notebook that can be downloaded from http://hdl.handle.net/10396/18478 or requested to the corresponding author (jcarbonell@uco.es). It allows to plot the Complete Hysteretic Curve of the system by plotting together the f_S and $\dot{u}$ histories against u.

References

[1] Newmark NM. A Method of Computation for Structural Dynamics. Journal of the Engineering Mechanics Division 1959;85:67–94.

[2] Hernández-Montes E, Aschheim MA, Gil-Martín LM. Energy components in nonlinear dynamic response of SDOF systems. Nonlinear Dynamics 2015;82:933–45. doi:10.1007/s11071-015-2208-9.

[3] Chopra AK. Dynamics of structures: theory and applications to earthquake engineering. 4th ed. Upper Saddle River, New Jersey: Prentice-Hall; 2011.