An Integral Representation of the Massive Dirac Propagator in the Nonextreme Kerr Geometry in Horizon-penetrating Coordinates

By

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ABSTRACT

The main objective of this doctoral thesis is the derivation of an integral spectral representation of the massive Dirac propagator in the nonextreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates. To this end, we divide the doctoral thesis into the following three parts. In the first part, we describe the nonextreme Kerr geometry in the Newman–Penrose formalism by means of a Carter tetrad in advanced Eddington–Finkelstein-type coordinates, which are regular across the event and the Cauchy horizon, respectively, and feature a temporal function for which the level sets are partial Cauchy surfaces. On this background geometry, we define the massive Dirac equation in the Weyl representation in 2-spinor form with a Newman–Penrose dyad basis for the spinor space. We perform Chandrasekhar's mode analysis and thus show the separability of the massive Dirac equation expressed in such horizon-penetrating coordinates into systems of radial and angular ordinary differential equations (ODEs). We compute asymptotic radial solutions at infinity, the event horizon, and the Cauchy horizon, and demonstrate that the corresponding errors have suitable decay. Furthermore, we study specific aspects of the set of eigenfunctions and the eigenvalue spectrum of the angular system. In the second part, we introduce a new method of proof for the essential self-adjointness of the Dirac Hamiltonian for a particular class of nonuniformly elliptic mixed initial-boundary value problems on smooth asymptotically flat Lorentzian manifolds, combining results from the theory of symmetric hyperbolic systems with near-boundary elliptic methods. Finally, in the third part, we present the Hamiltonian formulation of the massive Dirac equation in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates and, within this framework, derive an explicit integral spectral representation of the massive Dirac propagator, which yields the full time-dependent dynamics of massive spin-1/2 fermions outside, across, and inside the event horizon, up to the Cauchy horizon. For the construction of this propagator, we first prove that the Dirac Hamiltonian in the extended Kerr geometry is essentially self-adjoint by employing the method introduced in the second part, and then use the spectral theorem for unbounded self-adjoint operators as well as Stone's formula, which links the spectral measure of the Dirac Hamiltonian to the associated resolvent. We determine the resolvent in a separated form in terms of the projector onto a finite-dimensional invariant spectral eigenspace of the angular operator and the radial Green's matrix both obtained within the mode analysis of the Dirac equation presented in the first part. This propagator may be applied to study the long-time dynamics and the decay rates of massive Dirac fields in a rotating Kerr black hole spacetime. It can furthermore be used in the formulation of an algebraic quantum field theory.

PUBLICATIONS

Thesis-related Publications

The main scientific aspects of this doctoral thesis are contained in the following three publications:

1. RÖKEN, C. *The Massive Dirac Equation in the Kerr Geometry: Separability in Eddington–Finkelstein-type Coordinates and Asymptotics*. General Relativity and Gravitation 49 (2017), pp. 39–62. [Content of Chapter 3]
2. FINSTER, F., AND RÖKEN, C. *Self-adjointness of the Dirac Hamiltonian for a Class of Non-uniformly Elliptic Boundary Value Problems*. Annals of Mathematical Sciences and Applications 1 (2016), pp. 301–320. [Content of Chapter 4]
3. FINSTER, F., AND RÖKEN, C. *An Integral Spectral Representation of the Massive Dirac Propagator in the Kerr Geometry in Eddington–Finkelstein-type Coordinates*. Advances in Theoretical and Mathematical Physics 22 (2018), pp. 47–92. [Content of Chapter 5]

Project-related follow-up Publications

Results related to the scientific content of this doctoral thesis are to be found in the following publication:

4. FINSTER, F., AND RÖKEN, C. *The Fermionic Signature Operator in the Exterior Schwarzschild Geometry*. Annales Henri Poincaré 20 (2019), pp. 3389–3418. [Outline given in Chapter 6]

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CHAPTER 1

INTRODUCTION

In order to determine the dynamics of relativistic massive spin-$1/2$ fermions in a rotating black hole spacetime, the Dirac equation in the nonextreme Kerr geometry was studied extensively over the last five decades employing different approaches. The probably most established approach is Chandrasekhar's mode analysis [18–20], where the massive Dirac equation is separated by means of time and azimuthal angle modes and by a specific product ansatz for the radial and polar angle dependencies in the Dirac spinors, which results in first-order systems of radial and angular ODEs. Within this framework, many physical processes like the emission and absorption of Dirac particles by rotating Kerr black holes, Kerr black hole stability under fermionic field perturbations, and superradiance were investigated [11, 84, 90, 102, 106]. However, it does not give rise to a description of the full time-dependent dynamics of Dirac particles in the Kerr geometry. This was first analyzed in the framework of scattering theory in [5, 17, 26, 54, 59, 60, 80]. More recently, a somewhat different approach to define the full time-dependent dynamics of Dirac particles using an integral spectral representation of the Dirac propagator in the Hamiltonian framework was developed [37–39] (an analogous construction was carried out for the propagator of the scalar wave equation in the Kerr geometry in [40, 41]). This method combines results from both functional analysis and Chandrasekhar's mode analysis, allowing for a derivation of sharp decay rates for Dirac spinors as well as estimates for the probability of a Dirac particle to fall into a Kerr black hole or escape to infinity.

The basis of the mode analysis approach is Chandrasekhar's famous discovery that the massive Dirac equation in the nonextreme Kerr geometry expressed in Boyer–Lindquist coordinates is separable, which was worked out in his original article [18] from 1976 and led to a major breakthrough in the field of black hole physics. At that time, this remarkable result came a bit as a surprise because the Dirac equation was not expected to be separable in the Kerr geometry. Despite the tremendous impact of this discovery, the validity of the associated solutions is naturally restricted to those regions of the Kerr geometry where the Boyer–Lindquist coordinates are well-defined. And as they become singular at the event and the Cauchy horizon, respectively, the dynamics of Dirac particles near and across these horizons cannot be properly described in these coordinates. As a consequence, the above integral spectral representation of the propagator of the massive Dirac equation in the nonextreme Kerr geometry derived in [37–39], which is in part based on results obtained from Chandrasekhar's mode analysis in Boyer–Lindquist coordinates, has the shortcoming that it yields a solution of the Cauchy problem for the Dirac equation only outside the event horizon. Here, we resolve this problem by constructing a generalized integral spectral representation using a specific analytic extension of the Boyer–Lindquist coordinates, the so-called advanced Eddington–Finkelstein-type coordinates, that is regular across the event and the Cauchy horizon, and features a temporal function for which the level sets are partial Cauchy surfaces [28, 98]. More precisely, our generalized horizon-penetrating propagator describes the full time-dependent dynamics of massive Dirac particles outside, across, and inside the event horizon, up to the Cauchy horizon. We point out that since the transformation from Boyer–Lindquist coordinates to advanced Eddington–Finkelstein-type coordinates is singular at the event and the Cauchy horizon, and hence nontrivial, a careful mathematical analysis of this issue is essential. Furthermore, as the mixing of the time and the azimuthal angle variable that arises in this transformation leads to a symmetry breaking of structures inherent to Boyer–Lindquist coordinates, it is a priori not clear that the separation of variables property of the Dirac equation is conserved in advanced Eddington–Finkelstein-type coordinates. We eventually show that this property also holds for the new coordinates. Instead of using advanced Eddington–Finkelstein-type coordinates, one may as well choose different horizon-penetrating coordinates such as Doran coordinates [30] or generalized Kruskal–Szekeres coordinates [19, 71, 97]. These are, however, far more complicated in their handling and result in more intricate computations.

The derivation of the generalized horizon-penetrating integral spectral representation of the Dirac propagator is carried out as follows. Employing the Newman–Penrose formalism, we express the nonextreme Kerr geometry in terms of a regular Carter tetrad in advanced Eddington–Finkelstein-type coordinates (τ, r, θ, ϕ), where $\tau \in \mathbb{R}$, $r \in \mathbb{R}_{>0}$, $\theta \in (0, \pi)$, and $\phi \in [0, 2\pi)$, compute the corresponding spin coefficients by solving the torsion-free first Maurer–Cartan equation of structure, and thus determine the massive Dirac equation in a dyadic Weyl representation. Then, using Chandrasekhar's mode ansatz for the Dirac spinors, we prove the separability of the massive Dirac equation in these horizon-penetrating coordinates. Accordingly, we obtain first-order systems of radial and angular ODEs. We derive asymptotic radial solutions at infinity, the event horizon, and the Cauchy horizon and estimate the decay of the associated errors. Moreover, we show, on the one hand, that by decoupling the angular system, we obtain the usual Chandrasekhar–Page equation, and, on the other hand, that the corresponding set of eigenfunctions is complete, bounded, and smooth, and the spectrum of eigenvalues is discrete, nondegenerate, and ordered. Next, as we apply the spectral theorem in the derivation of the integral representation of the Dirac propagator, we require an essentially self-adjoint Dirac Hamiltonian H. For the proof of the essential self-adjointness, one usually employs methods from standard elliptic theory because this operator is in general elliptic. However, since in the case of the nonextreme Kerr geometry it turns out that the Hamiltonian is not elliptic at the event and the Cauchy horizon, these methods cannot be used, which obliges us to work with an alternative method of proof. Hence, we introduce a new method that combines results from the theory of symmetric hyperbolic systems with near-boundary elliptic methods [47]. To be more precise, we consider the class of smooth oriented and time-oriented asymptotically flat Lorentzian spin manifolds $(\mathrm{M}, \boldsymbol{g})$ of dimension $d \geq 3$ with boundary $\partial\mathrm{M}$, admitting a Killing field that is timelike near – and tangential to – the boundary and having the product structures $\mathrm{M} = \mathbb{R} \times \mathrm{N}$ and $\partial\mathrm{M} = \mathbb{R} \times \partial\mathrm{N}$, where N is a spacelike hypersurface with compact boundary $\partial\mathrm{N}$ (for a complete account of the general framework of such SSTK spacetimes see, e.g., [1, 13, 65, 66]). Furthermore, we choose a spinor bundle $S\mathrm{M}$ with sections $S_p\mathrm{M} \simeq \mathbb{C}^f$, $p \in \mathrm{M}$ and dimension $f = 2^{\lfloor d/2 \rfloor}$. Then, we formulate the nonuniformly elliptic mixed initial-boundary value problem for the Dirac equation in Hamiltonian form

$$
\begin{aligned}
&\mathrm{i}\partial_t \psi = H\psi \\
&\psi_{|\mathrm{N}} = \psi_0 \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \\
&(\not{n} - \mathrm{i})\psi_{|\partial\mathrm{M}} = \mathbf{0}\,,
\end{aligned}
$$

where ψ is a Dirac 4-spinor, H the Dirac Hamiltonian, t a temporal function, $\boldsymbol{n}$ the inner normal on $\partial\mathrm{M}$ with the slash denoting a Clifford contraction, and $C_0^\infty(\mathrm{N}, S\mathrm{M})$ is the class of smooth Dirac 4-spinors with compact support on N. The initial data ψ_0 on $\partial\mathrm{N}$ ($\equiv \{0\} \times \partial\mathrm{N}$) is assumed to be compatible with the boundary condition, i.e.,

$$(\not{n} - \mathrm{i})\psi_{0|\partial\mathrm{N}} = \mathbf{0}\,.$$

We establish the existence of a unique global solution ψ in the class

$$\{\psi \in C_{\mathrm{sc}}^\infty(\mathrm{M}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\partial\mathrm{M}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\}\,,$$

where $C_{\mathrm{sc}}^\infty(\mathrm{M}, S\mathrm{M})$ is the class of smooth Dirac 4-spinors with spacelike compact support on M, i.e., $\operatorname{supp}\psi(t,\,.\,)$ is a compact subset of N for all $t \in \mathbb{R}$, and the stronger set of boundary conditions gives rise to smoothness of the solution in time. To this end, we split the nonuniformly elliptic mixed initial-boundary value problem into two separate ones: in a region near the boundary $\partial\mathrm{M}$, which is sufficiently far away from the points where the Hamiltonian fails to be elliptic, we rewrite the problem in a form for which the standard methods for elliptic systems with boundaries presented in [4] apply, whereas in a region away from the boundary that includes these points, we employ results from the theory of symmetric hyperbolic systems (e.g., [101]). Making essential use of finite propagation speed, we can add the associated solutions, thus finding a unique smooth solution of the nonuniformly elliptic mixed initial-boundary value problem for small times. Iterating this procedure eventually yields a unique global smooth solution. Consequently, we obtain a 1-parameter family of unitary time evolution operators, which makes it possible to apply Chernoff's lemma [21] on the essential self-adjointness of powers of generators of hyperbolic equations in order to prove the following theorem on the essential self-adjointness of the Dirac Hamiltonian.

Main Theorem I. *The Dirac Hamiltonian H with domain of definition*

$$\mathrm{Dom}(H) = \{\psi \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\partial\mathrm{N}} = \mathbf{0} \quad \textit{for all} \quad p \in \mathbb{N}_0\}$$

is essentially self-adjoint.

Subsequently, we rewrite the massive Dirac equation defined in the analytic extension of the nonextreme Kerr geometry in Hamiltonian form

$$\mathrm{i}\partial_\tau\psi(\tau, r, \theta, \phi) = H\psi(\tau, r, \theta, \phi)\,,$$

impose a Dirichlet-type MIT boundary condition on a timelike inner boundary surface placed beyond the Cauchy horizon, and introduce a suitable scalar product on the associated space of solutions. We note that this timelike inner boundary surface corresponds to the above-mentioned boundary $\partial\mathrm{M}$, where the MIT boundary condition prevents Dirac particles from impinging on the curvature singularity without affecting their dynamics in the region outside the Cauchy horizon. Accordingly, the time evolution is unitary. For the proof of the essential self-adjointness of the Dirac Hamiltonian, we use our method for nonuniformly elliptic mixed initial-boundary value problems. We then derive an integral representation of the Dirac propagator via the spectral theorem for unbounded self-adjoint operators

$$\psi = e^{-\mathrm{i}\tau H}\,\psi_0 = \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\,\psi_0\,\mathrm{d}E_\omega\,,$$

where $\mathrm{d}E_\omega$ is the spectral measure of the Hamiltonian, ω the spectral parameter, and $\psi_0 := \psi(\tau = 0, r, \theta, \phi)$ smooth initial data with compact support. Employing Stone's formula [94], we express the spectral measure in terms of the resolvent of the Hamiltonian $\mathrm{Res}(\omega_{\mathrm{c}}; H) := (H - \omega_{\mathrm{c}})^{-1}$, $\omega_{\mathrm{c}} \in \mathbb{C}\backslash\mathbb{R}$, which is determined in terms of specific quantities obtained in the analysis of the radial and angular ODE systems that arise in Chandrasekhar's mode analysis. In more detail, after factoring out the azimuthal angle modes, we project the Dirac Hamiltonian onto a finite-dimensional invariant spectral eigenspace of the angular operator pertaining to the angular ODE system, which leaves us with a matrix-valued first-order ordinary differential operator of the radial variable. The resolvent of this operator can be calculated by means of the Green's matrix of the radial ODE system extended to complex-valued frequencies. For the determination of the Green's matrix, we derive generalized radial Jost-type equations [93] and study specific aspects of their solutions, namely the existence, uniqueness, and boundedness. We moreover use the asymptotic radial solutions at infinity, the event horizon, and the Cauchy horizon for guidance in the implicit construction of the required fundamental radial solutions. Summing over all azimuthal angle modes, we obtain the full resolvent of the Dirac Hamiltonian in separated form. The resulting horizon-penetrating integral spectral representation of the Dirac propagator captures the full time-dependent relativistic dynamics of massive Dirac particles outside and across the event horizon of the nonextreme Kerr geometry, up to the Cauchy horizon [48].

Main Theorem II. *The massive Dirac propagator in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates can be expressed via the integral spectral representation*

$$\begin{aligned}\psi(\tau, r, \theta, \phi) = \frac{1}{2\pi\mathrm{i}} \sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi} \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau} \lim_{\epsilon\searrow 0} \big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big](r, \theta; r', \theta') \\ \times \psi_{0,k}(r', \theta')\, \mathrm{d}\omega\,,\end{aligned}$$

where $\psi_{0,k}$ is smooth initial data for fixed k-modes with compact support and $(H_k - \omega \mp \mathrm{i}\epsilon)^{-1}$ are unique resolvents of the Dirac Hamiltonian H_k for fixed k-modes on the upper and lower complex half-planes.

In a final step, we compute the limit $\epsilon \searrow 0$ of the difference of resolvents leading to a rigorously simplified form of the propagator.

CHAPTER 2

Mathematical Concepts and Physical Foundations

We introduce in a concise form the basic mathematical structures and formalisms relevant for this work. Furthermore, we explain their relations to physics. More precisely, we present the differential geometric framework required to describe rotating black hole spacetimes in general relativity as well as concepts of mathematical analysis, in particular for partial differential equations (PDEs) and spectral theory, to analyze the Dirac equation. For more details on the separate subjects, we refer to the specified literature.

2.1 Differential Geometric Concepts and Setting

We establish the notion of globally hyperbolic Lorentzian manifolds and give a definition of frame fields (with special attention to tetrads), which constitute suitable local bases for the tangent spaces of these manifolds. Subsequently, we specify the Newman–Penrose formalism. We then present the Einstein field equations and outline some basics of the Kerr geometry.

2.1.1 Lorentzian Manifolds

The concept of a (4-dimensional) Lorentzian manifold with indefinite metric is commonly used in general relativity in order to give mathematical meaning to the notion of spacetime and, more importantly, to impose a causal structure allowing tangent vectors to be classified into the categories timelike, spacelike, and null. This cannot be realized using Riemannian manifolds, which are endowed with positive-definite metrics. The Lorentzian type of manifold constitutes a significant subclass of the more general pseudo-Riemannian manifolds.

Definition 2.1.1. A *pseudo-Riemannian manifold* is a differential manifold $\mathfrak{M}$ of dimension n equipped with a nondegenerate metric

$$\boldsymbol{g}\colon \Gamma(T\mathfrak{M}) \times \Gamma(T\mathfrak{M}) \to C^\infty(\mathfrak{M}, \mathbb{R}) \,, \quad (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) \,,$$

where $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(T\mathfrak{M})$ are smooth vector fields, that satisfies for $a, b \in C^\infty(\mathfrak{M}, \mathbb{R})$ the properties

1. $\boldsymbol{g}(a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}) = a\,\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Z}) + b\,\boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{Z})$
2. $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{X})$

3. $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y}) = 0 \quad \text{for all} \quad \boldsymbol{Y} \quad \Rightarrow \quad \boldsymbol{X} = \boldsymbol{0}$

and has the signature (q, r) with $q + r = n$ and both q, r being nonnegative.

Lorentzian manifolds represent the following special case of pseudo-Riemannian manifolds.

Definition 2.1.2. An n-dimensional *Lorentzian manifold* $(\mathfrak{M}, \boldsymbol{g})$ is a pseudo-Riemannian manifold for which the metric signature is $(1, n-1)$.

In order to have a notion of differentiability of vector fields on the tangent bundles of pseudo-Riemannian manifolds, we next define the affine connection.

Definition 2.1.3. An affine connection $\boldsymbol{\nabla}$ on the tangent bundle $T\mathfrak{M}$ of a pseudo-Riemannian manifold $(\mathfrak{M}, \boldsymbol{g})$ is a bilinear mapping

$$\boldsymbol{\nabla}\colon \Gamma(T\mathfrak{M}) \times \Gamma(T\mathfrak{M}) \to \Gamma(T\mathfrak{M}), \quad (\boldsymbol{X},\boldsymbol{Y}) \mapsto \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}$$

such that for all functions $a, b \in C^\infty(\mathfrak{M}, \mathbb{R})$ and all vector fields $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \Gamma(T\mathfrak{M})$ the conditions

1. $\boldsymbol{\nabla}_{a\boldsymbol{X}+b\boldsymbol{Y}}\boldsymbol{Z} = a\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Z} + b\boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{Z}$
2. $\boldsymbol{\nabla}_{\boldsymbol{X}}(\boldsymbol{Y}+\boldsymbol{Z}) = \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} + \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Z}$
3. $\boldsymbol{\nabla}_{\boldsymbol{X}}a = \boldsymbol{X}a$
4. $\boldsymbol{\nabla}_{\boldsymbol{X}}(a\boldsymbol{Y}) = (\boldsymbol{\nabla}_{\boldsymbol{X}}a)\boldsymbol{Y} + a\boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Y}$

are fulfilled.

2.1.2 Globally Hyperbolic Lorentzian Manifolds

Global hyperbolicity is a particular condition on the causal structure of a Lorentzian manifold, which was originally introduced in order to analyze the well-posedness of the Cauchy problems of certain PDE systems. Generally speaking, if a Lorentzian manifold is globally hyperbolic, it can be foliated by Cauchy hypersurfaces. This concept is very useful in the context of the initial value formulation of the Einstein field equations in general relativity. To properly specify the global hyperbolicity condition, we first need to introduce several basic differential geometric notions. In the following, we let $(\mathfrak{M}, \boldsymbol{g})$ always be a smooth and connected Lorentzian manifold without boundary. Furthermore, we call the points of $\mathfrak{M}$ *events*. The causal relations between two such events in the manifold can be defined using certain conditions on the tangent vectors of smooth curves joining pairs of events.

Definition 2.1.4. A tangent vector field $\boldsymbol{X} \in T\mathfrak{M}$ is referred to as

- *timelike* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) > 0$.
- *lightlike* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = 0$ and $\boldsymbol{X} \neq \boldsymbol{0}$.
- *causal* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) \geq 0$ and $\boldsymbol{X} \neq \boldsymbol{0}$.
- *null* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = 0$.
- *spacelike* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) < 0$.
- *nonspacelike* if $\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) \geq 0$.

In order to set up a causal structure on $(\mathfrak{M}, \boldsymbol{g})$, one requires the notion of time orientability.

Definition 2.1.5. A *time orientation at* $p \in \mathfrak{M}$ is a choice of one of the two causal cones in $T_p\mathfrak{M}$, which are formed by the set of all lightlike vectors at that event. This cone is called the *future cone*, whereas the nonchosen cone is called the *past cone*. A smooth choice of time orientations for all $p \in \mathfrak{M}$ is referred to as a *time orientation*. A Lorentzian manifold $(\mathfrak{M}, \boldsymbol{g})$ is then said to be *time-orientable* if such a time orientation exists.

Proposition 2.1.6. *A Lorentzian manifold* $(\mathfrak{M}, \boldsymbol{g})$ *is time-orientable if and only if it admits a globally defined timelike vector field.*

We may then define the causal relations between pairs of events in $\mathfrak{M}$ (see, e.g., [75]).

Definition 2.1.7. In a given time-oriented connected Lorentzian manifold $(\mathfrak{M}, \boldsymbol{g})$, also referred to as *spacetime*, the events p and q are related

- *chronologically* ($p \ll q$) if there is a future-directed timelike curve connecting p with q.
- *strictly causally* ($p < q$) if there is a future-directed causal curve connecting p with q.
- *causally* ($p \leq q$) if there is a future-directed causal or constant curve connecting p with q.
- *horismotically* ($p \to q$) if there is a future-directed causal or constant, but not timelike curve connecting p with q.

The *chronological future*, *causal future*, and *horismos* of an event p are defined as

$$\begin{aligned} I^+(p) &:= \{q \in \mathfrak{M} \,|\, p \ll q\} \\ J^+(p) &:= \{q \in \mathfrak{M} \,|\, p \leq q\} \\ E^+(p) &:= \{q \in \mathfrak{M} \,|\, p \to q\} = J^+(p) \backslash I^+(p)\,. \end{aligned}$$

These definitions carry over to I^-, J^-, and E^- by changing (p, q) into (q, p). Moreover, a subset $\mathfrak{N} \subset \mathfrak{M}$ is called *achronal* if $I^+(\mathfrak{N}) \cap \mathfrak{N} = \emptyset$, i.e., every timelike curve in $\mathfrak{M}$ intersects $\mathfrak{N}$ at most once. Also, for $\mathfrak{N} \subset \mathfrak{M}$ being closed and achronal, the *future/past domain of dependence* $D^{+/-}(\mathfrak{N})$ is the collection of all points $p \in \mathfrak{M}$ such that every past/future inextensible (i.e., devoid of endpoints, or in other words, either infinitely extended or closed) causal curve passing through p intersects $\mathfrak{N}$. The *domain of dependence* of $\mathfrak{N}$ is then defined as the union $D(\mathfrak{N}) = D^+(\mathfrak{N}) \cup D^-(\mathfrak{N})$.

Since there exist several equivalent formulations of the global hyperbolicity condition relying in part on the particular notions of causal simplicity, Cauchy hypersurfaces, and nontotal imprisonment, we next state the corresponding definitions.

Definition 2.1.8. A spacetime $(\mathfrak{M}, \boldsymbol{g})$ is called

- *causal* if it does not contain closed causal curves.
- *strongly causal* if for all $p \in \mathfrak{M}$ and any neighborhood U_p of p there is a causally convex neighborhood $V_p \subset U_p$ of p, where causal convexity means that any causal curve with endpoints in V_p is entirely contained in V_p.
- *nontotal imprisoning* if no inextensible causal curve is contained in a compact set.

Definition 2.1.9. A *Cauchy surface* of a spacetime $(\mathfrak{M}, \boldsymbol{g})$ is any subset $\mathfrak{N} \subset \mathfrak{M}$ that is closed and achronal, and has $D(\mathfrak{N}) = \mathfrak{M}$, i.e., it is intersected by every inextensible timelike curve exactly once. It is therefore a topological hypersurface [82], which can be approximated by a smooth spacelike hypersurface [6]. A *partial Cauchy surface* is a hypersurface, which is intersected by any timelike curve at most once.

We are now in a position to express the four most prominent definitions of global hyperbolicity [6, 56, 61].

Definition 2.1.10. A spacetime $(\mathfrak{M}, \boldsymbol{g})$ is *globally hyperbolic* if and only if

1. it is causal and the space $\mathcal{C}(p, q)$ of continuous future-directed causal curves from p to q is compact in the C^0 topology for all $p, q \in \mathfrak{M}$.
2. it admits a Cauchy hypersurface $\mathfrak{N} \subset \mathfrak{M}$.
3. it is causal and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in \mathfrak{M}$.
4. it is nontotal imprisoning and $J^+(p) \cap J^-(q)$ is contained in a compact set for all $p, q \in \mathfrak{M}$.

We note that the first and third definitions originally used the stronger condition of strong causality (cf. Definition 2.1.8), which was eventually shown in [9] to be relaxable to the weaker condition of causality. For additional relaxations of these definitions under mild overall assumptions see also [64]. Furthermore, in the case that $\mathfrak{M}$ has a conformal boundary, the interior is globally hyperbolic if and only if the boundary is spacelike [51]. Finally, we introduce time-type functions, which are essential in the formulation of any Cauchy problem, and discuss their relation to the notion of global hyperbolicity following [7, 55]. Further background information and results can be found in, e.g., [8, 77, 78].

Definition 2.1.11. We let $(\mathfrak{M}, \boldsymbol{g})$ be a spacetime. A function $t \colon \mathfrak{M} \to \mathbb{R}$ is

1. a *generalized time function* if it is strictly increasing on any future-directed causal curve γ.
2. a *time function* if it is a continuous generalized time function.
3. a *temporal function* if it is a smooth function with past-directed timelike gradient $\boldsymbol{\nabla} t$.

Proposition 2.1.12. *Any globally hyperbolic spacetime $(\mathfrak{M}, \boldsymbol{g})$ contains a Cauchy temporal function t, that is, a temporal function for which the level sets $t^{-1}(\,.\,)$ are smooth spacelike Cauchy hypersurfaces $(\mathfrak{N}_t)_{t\in\mathbb{R}}$ with $\mathfrak{N}_t \subset J^-(\mathfrak{N}_{t'})$ for all $t < t'$. Accordingly, $\mathfrak{M}$ is diffeomorphic to $\mathbb{R} \times \mathfrak{N}_0$.*

One may choose the diffeomorphism to be conformal with parametrized orthogonal product metric on $\mathbb{R} \times \mathfrak{N}_0$. These results can be extended to the case of globally hyperbolic spacetimes with timelike boundary [1].

2.1.3 Frame Fields and the Tetrad Formalism

The notion of a frame field is a generalization of the standard coordinate basis approach to tensor fields on manifolds.

Definition 2.1.13. A *frame field* $\{\boldsymbol{e}_{(1)}, \dots, \boldsymbol{e}_{(n)}\}$ on a general n-dimensional pseudo-Riemannian manifold $(\mathfrak{M}, \boldsymbol{g})$ constitutes a set of n locally defined linearly independent vector fields that form a basis for the tangent space $T_p\mathfrak{M}$ at each point p of their common domain in $\mathfrak{M}$.

Accordingly, a frame field is a section of a linear frame bundle, namely the tangent bundle $T\mathfrak{M}$. The calculus of frame fields is the so-called Cartan formalism, which, on the one hand, is a particularly convenient framework for computing the curvature via two coupled structure equations and, on the other hand, yields a covariant *first-order* action functional for the gravitational field. To be more precise, the Cartan formalism employs the dual coframe field and the so-called Cartan connection as independent fundamental variables. The advantages of this particular local formalism over the standard coordinate-based formulation lie firstly in the adaptivity of the frame field to specific physical aspects as well as underlying symmetries of the manifold, which, as a consequence, leads to drastic simplifications in the computation of the curvature, and secondly in the reduction of the order of the action functional for the gravitational field. We note that the Cartan formalism for the special case of 4-dimensional Lorentzian manifolds, which are the essential geometric structures inherent to general relativity, is commonly known as tetrad formalism. In this regard, we also want to point out that, on a more fundamental level, the Cartan formalism may in principle introduce modifications to general relativity. This occurs in the presence of spinning matter where the torsion becomes dynamical, thus breaking the theory's fundamental assumption of zero torsion. Besides, there exists another crucial reason for the use of local frame fields, namely the proper definition of the Dirac equation in curved spacetimes. This is due to the fact that for the construction of spinors, one requires the restricted Lorentz group $\mathrm{SO}^+(1,3,\mathbb{R})$ – and hence its double covering group $\mathrm{SL}(2,\mathbb{C})$ – to be the local symmetry group, i.e., the local pointwise symmetry in the frame bundle. In the standard metric formulation of general relativity, however, the local symmetry group is the larger diffeomorphism group, which in general does not have a spinor representation. The tetrad formalism on the other hand gives rise to local orthonormal frames with metric signature $(1,3)$, and thus yields a local representation of the restricted Lorentz group.

We now describe the tetrad formalism. To this end, we let $(\mathfrak{M},\boldsymbol{g})$ be a Lorentzian 4-manifold endowed with an affine connection $\boldsymbol{\nabla}$ and a dual pair of coordinate bases $(\boldsymbol{e}_\mu)$ and $(\boldsymbol{e}^\mu)$, $\mu \in \{0,1,2,3\}$, for the tangent and cotangent spaces $T_p\mathfrak{M}$ and $T_p^\star\mathfrak{M}$ at each point $p \in \mathfrak{M}$, respectively, where Greek letters label tensor indices. Moreover, as another dual pair of bases for $T_p\mathfrak{M}$ and $T_p^\star\mathfrak{M}$, we introduce a frame field consisting of four locally defined linearly independent (orthonormal or null) vector fields – the so-called tetrad frame – and its unique metric dual, which are denoted by $(\boldsymbol{e}_{(a)})$ and $(\boldsymbol{e}^{(a)})$, $a \in \{0,1,2,3\}$, with Latin letters enclosed in parenthesis labeling tetrad indices. In terms of the coordinate bases, the tetrad bases can be written as

$$\boldsymbol{e}_{(a)} = e_{(a)}{}^{\mu}\, \boldsymbol{e}_\mu \quad \text{and} \quad \boldsymbol{e}^{(a)} = e^{(a)}{}_{\mu}\, \boldsymbol{e}^\mu \,,$$

where $e_{(a)}{}^{\mu}$ is a linear mapping from $T\mathfrak{M}$ to $T\mathfrak{M}$, namely a 4×4 transition matrix, and $e^{(a)}{}_{\mu}$ is its inverse, that is,

$$e_{(a)}{}^{\mu}\, e^{(b)}{}_{\mu} = \delta^{(b)}_{(a)} \quad \text{and} \quad e_{(a)}{}^{\mu}\, e^{(a)}{}_{\nu} = \delta^{\mu}_{\nu} \,.$$

Conversely, we find

$$\boldsymbol{e}_\mu = e^{(a)}{}_{\mu}\, \boldsymbol{e}_{(a)} \quad \text{and} \quad \boldsymbol{e}^\mu = e_{(a)}{}^{\mu}\, \boldsymbol{e}^{(a)} \,.$$

Usually, one chooses a particular tetrad frame in such a way that the inner product of the tetrad vectors with respect to the prescribed Lorentzian metric $\boldsymbol{g}$ yields a specified constant symmetric matrix $\boldsymbol{\eta}$

$$\boldsymbol{g}(\boldsymbol{e}_{(a)}, \boldsymbol{e}_{(b)}) = e_{(a)}{}^{\mu}\, e_{(b)\,\mu} = \eta_{(a)\,(b)} \,.$$

In the special case of $\boldsymbol{\eta}$ being the Minkowski metric, the tetrad $\boldsymbol{e}_{(a)}$ is orthonormal. Accordingly, tetrad indices are lowered and raised with $\eta_{(a)\,(b)}$ and its inverse $\eta^{(a)\,(b)}$

$$\boldsymbol{e}_{(a)} = \eta_{(a)\,(b)}\, \boldsymbol{e}^{(b)} \quad \text{and} \quad \boldsymbol{e}^{(a)} = \eta^{(a)\,(b)}\, \boldsymbol{e}_{(b)} \,.$$

Furthermore, the Lorentzian metric can be expressed as

$$g_{\mu\nu} = e_{(a)\,\mu}\, e^{(a)}{}_{\nu}\,.$$

Now, given any vector $T_p\mathfrak{M} \ni \boldsymbol{V} = V^{\mu}\,\partial_{\mu}$ represented in terms of a coordinate basis $(\boldsymbol{e}_{\mu})$, we may transform it into a tetrad frame, obtaining the associated components

$$V^{(a)} = \eta^{(a)\,(b)}\, V_{(b)} = e^{(a)}{}_{\mu}\, V^{\mu} = e^{(a)\,\mu}\, V_{\mu}\,,$$

and vice versa

$$V^{\mu} = e_{(a)}{}^{\mu}\, V^{(a)} = e^{(a)\,\mu}\, V_{(a)}\,.$$

For more general tensor fields, there exist similar relations between the components in the coordinate and tetrad bases. The affine connection in the Cartan formalism, which we refer to as Cartan connection and denote by $\boldsymbol{\omega}$, is related to the usual Christoffel symbols $(\Gamma^{\nu}_{\mu\lambda})$ via

$$\omega_{\mu\,(a)\,(b)} = e_{(a)\,\nu}\, e_{(b)}{}^{\lambda}\, \Gamma^{\nu}_{\mu\lambda} - e_{(b)}{}^{\nu}\, \partial_{\mu} e_{(a)\,\nu}\,. \tag{2.1}$$

Moreover, as one has the freedom to perform both local Lorentz transformations at every point on the manifold as well as general coordinate transformations

$$O(1,3) \ni \Lambda^{(a)}{}_{(a')} = \Lambda^{(a)}{}_{(a')}(x^{\mu}) \quad \text{and} \quad \mathrm{Diff}^{\infty}(\mathfrak{M}) \ni \frac{\partial x^{\mu'}}{\partial x^{\mu}}\,,$$

the tetrad and the connection change according to the mixed transformation laws

$$e_{(a)}{}^{\mu} \mapsto e_{(a')}{}^{\mu'} = \Lambda^{(a)}{}_{(a')}\, \frac{\partial x^{\mu'}}{\partial x^{\mu}}\, e_{(a)}{}^{\mu}$$

and

$$\omega_{\mu\,(a)\,(b)} \mapsto \omega_{\mu'\,(a')\,(b')} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \Big[\Lambda_{(a')}{}^{(a)}\, \omega_{\mu\,(a)\,(b)}\, \Lambda^{(b)}{}_{(b')} - \big(\partial_{\mu}\Lambda_{(a')\,(c)}\big)\, \Lambda^{(c)}{}_{(b')}\Big]\,.$$

We next define the two mappings called torsion and Riemann curvature.

Definition 2.1.14. We let $(\mathfrak{M}, \boldsymbol{g})$ be a general pseudo-Riemannian manifold. The *torsion* is an antisymmetric tensor field of type $(1,2)$ given by

$$\boldsymbol{T}\colon \Gamma(T\mathfrak{M}) \times \Gamma(T\mathfrak{M}) \to \Gamma(T\mathfrak{M}) \quad \text{with} \quad \boldsymbol{T}(\boldsymbol{X},\boldsymbol{Y}) := \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} - \boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}]\,, \tag{2.2}$$

whereas the *Riemann curvature* is an antisymmetric tensor field of type $(1,3)$ specified as

$$\boldsymbol{R}\colon \Gamma(T\mathfrak{M}) \times \Gamma(T\mathfrak{M}) \times \Gamma(T\mathfrak{M}) \to \Gamma(T\mathfrak{M}) \quad \text{with} \quad \boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y}) := \boldsymbol{\nabla}_{\boldsymbol{X}}\boldsymbol{\nabla}_{\boldsymbol{Y}} - \boldsymbol{\nabla}_{\boldsymbol{Y}}\boldsymbol{\nabla}_{\boldsymbol{X}} - \boldsymbol{\nabla}_{[\boldsymbol{X},\boldsymbol{Y}]}\,, \tag{2.3}$$

in which $\boldsymbol{\nabla}$ denotes an affine connection and $\boldsymbol{X},\boldsymbol{Y} \in \Gamma(T\mathfrak{M})$.

Choosing the dual pair of tetrad bases $\boldsymbol{e}_{(a)}$ and $\boldsymbol{e}^{(a)}$ for $T_p\mathfrak{M}$ and $T^{\star}_p\mathfrak{M}$, respectively, and the Cartan connection $\boldsymbol{\omega}$ on $T\mathfrak{M}$, one can derive the so-called Maurer–Cartan equations of structure [19] from (2.2) and (2.3).

Corollary 2.1.15. *The torsion $\boldsymbol{T}$ and the Riemann curvature $\boldsymbol{R}$ are related to the tetrad $\mathbf{e}$ and the Cartan connection $\boldsymbol{\omega}$ by means of the Maurer–Cartan equations of structure*

$$\tfrac{1}{2}\,\boldsymbol{T}^{(a)}{}_{(b)\,(c)}\, \boldsymbol{e}^{(b)} \wedge \boldsymbol{e}^{(c)} = \mathrm{d}\boldsymbol{e}^{(a)} + \boldsymbol{\omega}^{(a)}{}_{(b)} \wedge \boldsymbol{e}^{(b)} \tag{2.4}$$

and

$$\tfrac{1}{2}\,\boldsymbol{R}^{(a)}{}_{(b)\,(c)\,(d)}\, \boldsymbol{e}^{(c)} \wedge \boldsymbol{e}^{(d)} = \mathrm{d}\boldsymbol{\omega}^{(a)}{}_{(b)} + \boldsymbol{\omega}^{(a)}{}_{(c)} \wedge \boldsymbol{\omega}^{(c)}{}_{(b)}\,.$$

As general relativity is based on the assumption of vanishing torsion, the first Maurer–Cartan equation (2.4) reduces to

$$\mathrm{d}\boldsymbol{e}^{(a)} = -\boldsymbol{\omega}^{(a)}{}_{(b)} \wedge \boldsymbol{e}^{(b)} \,. \tag{2.5}$$

Hence, in this special case, the Cartan connection, which is now referred to as Levi–Civita connection, becomes a function of the tetrad and ceases to represent an independent variable. Consequently, it is this equation that is fundamental to uniquely determine the connection from a given tetrad.

2.1.4 The Newman–Penrose Formalism

The Newman–Penrose formalism [79] is a specific tetrad formalism, in which spacetime geometries are described in terms of complex-valued null tetrad fields. It is particularly useful for an account of the black hole solutions of general relativity because of their Petrov D-type character and the Goldberg–Sachs theorem [19, 83], that is, one can choose a null tetrad being adapted to the two double principal null directions, and thus to the lightcone structure of the spacetime. It is therefore also well suited for the analysis of radiative transport in black hole spacetimes. As a consequence, this leads to a reduction in the number of conditional equations and to simplified expressions for many primary geometric quantities. The Newman–Penrose formalism may of course be set up to reflect other symmetries of black hole spacetimes, such as the discrete time and angle reversal isometries of the nonextreme Kerr geometry.

Definition 2.1.16. In the *Newman–Penrose formalism*, local tetrad bases are composed of two real-valued null vectors, $\boldsymbol{l} = \boldsymbol{e}_{(0)} = \boldsymbol{e}^{(1)}$ and $\boldsymbol{n} = \boldsymbol{e}_{(1)} = \boldsymbol{e}^{(0)}$, as well as a complex-conjugate pair of null vectors, $\boldsymbol{m} = \boldsymbol{e}_{(2)} = -\boldsymbol{e}^{(3)}$ and $\overline{\boldsymbol{m}} = \boldsymbol{e}_{(3)} = -\boldsymbol{e}^{(2)}$ (following the notation and terminology of [19]). These are required to satisfy the *null* conditions

$$\boldsymbol{l}\cdot\boldsymbol{l} = \boldsymbol{n}\cdot\boldsymbol{n} = \boldsymbol{m}\cdot\boldsymbol{m} = \overline{\boldsymbol{m}}\cdot\overline{\boldsymbol{m}} = 0 \,, \tag{2.6}$$

the *orthogonality conditions*

$$\boldsymbol{l}\cdot\boldsymbol{m} = \boldsymbol{l}\cdot\overline{\boldsymbol{m}} = \boldsymbol{n}\cdot\boldsymbol{m} = \boldsymbol{n}\cdot\overline{\boldsymbol{m}} = 0 \,, \tag{2.7}$$

and the *cross-normalization conditions* (which depend on the signature convention)

$$\boldsymbol{l}\cdot\boldsymbol{n} = -\boldsymbol{m}\cdot\overline{\boldsymbol{m}} = 1 \,. \tag{2.8}$$

The corresponding local metric is nondegenerate and constant, reading

$$\boldsymbol{\eta} = g_{\mu\nu}\, e_{(a)}{}^{\mu}\, e_{(b)}{}^{\nu}\, \boldsymbol{e}^{(a)} \otimes \boldsymbol{e}^{(b)} = \boldsymbol{l}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{l} - \boldsymbol{m}\otimes\overline{\boldsymbol{m}} - \overline{\boldsymbol{m}}\otimes\boldsymbol{m} \,.$$

In order to determine the connection in this formalism, we use the torsion-free first Maurer–Cartan equation of structure (2.5) in the form

$$\mathrm{d}\boldsymbol{e}^{(a)} = \gamma^{(a)}{}_{(b)\,(c)}\, \boldsymbol{e}^{(b)} \wedge \boldsymbol{e}^{(c)} \,, \tag{2.9}$$

in which the Ricci rotation coefficients ($\gamma^{(a)}{}_{(b)(c)}$) are related to the Cartan connection (2.1) via

$$\gamma^{(a)}{}_{(b)(c)}\, \boldsymbol{e}^{(c)} = e_{\mu}{}^{(a)} \big[\mathrm{d}e^{\mu}{}_{(b)} + \omega^{\mu}{}_{(b)(c)}\, \boldsymbol{e}^{(c)}\big] .$$

Inserting the null basis introduced in Definition 2.1.16 into (2.9) leads to the following corollary.

Corollary 2.1.17. *The torsion-free first Maurer–Cartan equation of structure in the Newman–Penrose formalism reads*

$$\begin{aligned}
\mathrm{d}\boldsymbol{l} &= 2\,\mathrm{Re}(\epsilon)\,\boldsymbol{n}\wedge\boldsymbol{l} - 2\,\boldsymbol{n}\wedge\mathrm{Re}(\kappa\,\overline{\boldsymbol{m}}) - 2\,\boldsymbol{l}\wedge\mathrm{Re}\big([\tau-\overline{\alpha}-\beta]\,\overline{\boldsymbol{m}}\big) + 2\mathrm{i}\,\mathrm{Im}(\varrho)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}} \\
\mathrm{d}\boldsymbol{n} &= 2\,\mathrm{Re}(\gamma)\,\boldsymbol{n}\wedge\boldsymbol{l} - 2\,\boldsymbol{n}\wedge\mathrm{Re}\big([\overline{\alpha}+\beta-\overline{\pi}]\,\overline{\boldsymbol{m}}\big) + 2\,\boldsymbol{l}\wedge\mathrm{Re}(\overline{\nu}\,\overline{\boldsymbol{m}}) + 2\mathrm{i}\,\mathrm{Im}(\mu)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}} \qquad (2.10) \\
\mathrm{d}\boldsymbol{m} &= \overline{\mathrm{d}\overline{\boldsymbol{m}}} = (\overline{\pi}+\tau)\,\boldsymbol{n}\wedge\boldsymbol{l} + \big(2\mathrm{i}\,\mathrm{Im}(\epsilon)-\varrho\big)\,\boldsymbol{n}\wedge\boldsymbol{m} - \sigma\,\boldsymbol{n}\wedge\overline{\boldsymbol{m}} + \big(\overline{\mu}+2\mathrm{i}\,\mathrm{Im}(\gamma)\big)\,\boldsymbol{l}\wedge\boldsymbol{m} \\
&\quad + \overline{\lambda}\,\boldsymbol{l}\wedge\overline{\boldsymbol{m}} - (\overline{\alpha}-\beta)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}}\,,
\end{aligned}$$

where the so-called spin coefficients

$$\begin{array}{lll}
\kappa = \gamma_{(2)\,(0)\,(0)} & \varrho = \gamma_{(2)\,(0)\,(3)} & \epsilon = \frac{1}{2}\big(\gamma_{(1)\,(0)\,(0)} + \gamma_{(2)\,(3)\,(0)}\big) \\
\sigma = \gamma_{(2)\,(0)\,(2)} & \mu = \gamma_{(1)\,(3)\,(2)} & \gamma = \frac{1}{2}\big(\gamma_{(1)\,(0)\,(1)} + \gamma_{(2)\,(3)\,(1)}\big) \\
\lambda = \gamma_{(1)\,(3)\,(3)} & \tau = \gamma_{(2)\,(0)\,(1)} & \alpha = \frac{1}{2}\big(\gamma_{(1)\,(0)\,(3)} + \gamma_{(2)\,(3)\,(3)}\big) \\
\nu = \gamma_{(1)\,(3)\,(1)} & \pi = \gamma_{(1)\,(3)\,(0)} & \beta = \frac{1}{2}\big(\gamma_{(1)\,(0)\,(2)} + \gamma_{(2)\,(3)\,(2)}\big)
\end{array}$$

constitute the associated connection representation, thus describing the change in the tetrad from one point to another.

Next, as tetrad frames may be subjected to local Lorentz transformations at every point on the manifold, we briefly address the particular tetrad transformations applied in Section 3.1 and Section 3.2. These transformations are elements of the 2-parameter subgroup of local Lorentz transformations known as class III or spin-boost Lorentz transformations [19, 83], which renormalize the real-valued Newman–Penrose vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, leaving their directions unchanged, and rotate the complex-conjugate pair $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ in the $(\boldsymbol{m},\overline{\boldsymbol{m}})$-plane

$$\boldsymbol{l}\mapsto\boldsymbol{l}' = \varsigma\,\boldsymbol{l}\,, \quad \boldsymbol{n}\mapsto\boldsymbol{n}' = \varsigma^{-1}\boldsymbol{n}\,, \quad \boldsymbol{m}\mapsto\boldsymbol{m}' = e^{\mathrm{i}\psi}\boldsymbol{m}\,, \quad \overline{\boldsymbol{m}}\mapsto\overline{\boldsymbol{m}}' = e^{-\mathrm{i}\psi}\,\overline{\boldsymbol{m}}\,, \qquad (2.11)$$

where the scale factor $\varsigma \in \mathbb{R}\backslash\{0\}$ and the angle $\psi \in \mathbb{R}$ are functions depending on the spacetime coordinates (x^μ). There are various aspects of the Newman–Penrose formalism that are not discussed in this subsection, such as the different classes of local Lorentz transformations or the five complex Weyl scalars and their algebraic classification. These can be found elsewhere in the literature. We refer the interested reader to [83, 85].

2.1.5 The Einstein Field Equations

The Einstein field equations, which are the dynamical equations for the gravitational field in the framework of general relativity, are a nonlinear inhomogeneous second-order PDE system for the metric tensor that describe gravitation as the interaction of geometry with the mass-energy content of spacetime. In order to define these equations, we consider a Lorentzian 4-manifold $(\mathfrak{M},\boldsymbol{g})$ with a metric $\boldsymbol{g}$ of signature $(1,3)$ and the Riemann curvature $\boldsymbol{R}$ as introduced in (2.3).

Definition 2.1.18. The *Einstein field equations* with cosmological constant Λ may be expressed in the form

$$\boldsymbol{G}(\boldsymbol{g}) + \Lambda\,\boldsymbol{g} = 8\pi\,\boldsymbol{T}\,, \qquad (2.12)$$

where

$$\boldsymbol{G}(\boldsymbol{g}) := \mathrm{Ric}(\boldsymbol{g}) - \frac{1}{2}\,\boldsymbol{g}\,\mathrm{Sc}(\boldsymbol{g})$$

is the *Einstein tensor*, in which the *Ricci curvature* Ric is the partial trace and the *scalar curvature* $\mathrm{Sc} = \mathrm{tr}_{\boldsymbol{g}}\,\mathrm{Ric}$ is the full trace of $\boldsymbol{R}$, and $\boldsymbol{T}$ is the *energy-momentum tensor* of matter and fields.

We point out that, as the Einstein tensor is divergence-free and because of metric compatibility, the Einstein field equations imply local conservation of energy and momentum, i.e.,

$$\mathrm{div}\,\boldsymbol{T} = 0\,.$$

Moreover, if both the energy-momentum tensor and the cosmological constant vanish, one obtains the so-called vacuum Einstein field equations

$$\mathrm{Ric}(\boldsymbol{g}) = \boldsymbol{0}\,, \tag{2.13}$$

which entail that the metric is Ricci flat. This particular representation results from the trace-freeness of the Einstein tensor in vacuum $\mathrm{tr}_{\boldsymbol{g}}\,\boldsymbol{G} = -\mathrm{Sc} = 0$. For the construction of the nonextreme Kerr geometry, we are interested in the 2-parameter family of solutions of (2.13) derived under the assumptions of stationarity and axial symmetry.

Also, the vacuum Einstein field equations (2.13) have a well-posed initial value formulation, where one considers a split of spacetime into a family of Cauchy hypersurfaces $(\mathfrak{N}_t)_{t\in\mathbb{R}}$, which are parametrized by a Cauchy temporal function t, are equipped with a smooth Riemannian metric $\boldsymbol{h}$ induced by the spacetime metric $\boldsymbol{g}$, and have an extrinsic curvature $\boldsymbol{K}$ with respect to $(\mathfrak{M}, \boldsymbol{g})$. In addition, one defines a so-called lapse function N and a shift vector $\boldsymbol{N}$ as Lagrange multipliers in order to determine how these Cauchy surfaces are joint together. By using the metric $\boldsymbol{h}$ and the curvature $\boldsymbol{K}$ as dynamical variables, the evolution of the Cauchy hypersurfaces can thus be traced over time. Accordingly, a suitable initial data set for the initial value formulation of the vacuum Einstein field equations consists of a triple $(\mathfrak{N}_0 =: \mathfrak{N}, \boldsymbol{h}, \boldsymbol{K})$ subjected to specific constraint equations. We now state an existence and uniqueness theorem that relates such initial data sets to globally hyperbolic spacetimes.

Theorem 2.1.19. *We let $(\mathfrak{N}, \boldsymbol{h})$ be a smooth 3-dimensional Riemannian manifold and $\boldsymbol{K}$ a smooth second rank symmetric tensor field on $\mathfrak{N}$. Furthermore, we suppose that $\boldsymbol{h}$ and $\boldsymbol{K}$ comply with the vacuum Gauss–Codazzi constraint equations*

$$\mathrm{div}_{\boldsymbol{D}}\boldsymbol{K} - \boldsymbol{D}(\mathrm{tr}_{\boldsymbol{h}}\,\boldsymbol{K}) = \boldsymbol{0}$$

$$\mathrm{Sc}_{\mathfrak{N}}(\boldsymbol{h}) + \mathrm{tr}^2_{\boldsymbol{h}}\,\boldsymbol{K} - \boldsymbol{K}^2 = \boldsymbol{0}\,,$$

where $\boldsymbol{D}$ is the covariant derivative operator associated with $\boldsymbol{h}$ and $\mathrm{Sc}_{\mathfrak{N}}$ is the scalar curvature on $\mathfrak{N}$. Then, there exists a unique smooth spacetime $(\mathfrak{M}, \boldsymbol{g})$ called the maximal Cauchy development of $(\mathfrak{N}, \boldsymbol{h}, \boldsymbol{K})$, which satisfies the following four properties.

(i) $(\mathfrak{M}, \boldsymbol{g})$ is a solution of the vacuum Einstein field equations (2.13).

(ii) $(\mathfrak{M}, \boldsymbol{g})$ is globally hyperbolic with Cauchy surface $\mathfrak{N}$.

(iii) The induced metric and extrinsic curvature of $\mathfrak{N}$ are $\boldsymbol{h}$ and $\boldsymbol{K}$, respectively.

(iv) Every other spacetime satisfying (i)-(iii) can be mapped isometrically into a subset of $(\mathfrak{M}, \boldsymbol{g})$.

Finally, the solution $\boldsymbol{g}$ on $\mathfrak{M}$ depends continuously on the initial data $(\boldsymbol{h}, \boldsymbol{K})$ on $\mathfrak{N}$.

A proof of this theorem can be found in [109]. Furthermore, by taking into account the results obtained in [1], the framework can be extended to globally hyperbolic Lorentzian manifolds $(\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}, \boldsymbol{g})$ with timelike boundary $\partial\mathfrak{M}$, which indeed have a global orthogonal product structure $\mathbb{R} \times \overline{\mathfrak{N}}$, where initial data sets are defined on $\overline{\mathfrak{N}}$ and compatible at $\partial\mathfrak{N}$ with boundary conditions specified on $\partial\mathfrak{M} = \mathbb{R} \times \partial\mathfrak{N}$. Also, the particular case of stationary asymptotically flat Lorentzian manifolds (cf. Subsection 2.1.6) is analyzed in [12] and – with causal or outer trapped boundaries – in [52, 74]. Finally, a discussion of the initial value formulation of the Einstein field equations with matter and energy sources (2.12) can be found in, e.g., [109].

2.1.6 The Kerr Geometry

Prior to introducing the notion of the Kerr geometry, we have to define the concepts of stationarity and axial symmetry.

Definition 2.1.20. A Lorentzian manifold $(\mathfrak{M}, \boldsymbol{g})$ is called *stationary* if it admits a timelike Killing vector field $\boldsymbol{K} \in \Gamma(T\mathfrak{M})$.

We note that in the literature on general relativity, a spacetime featuring a Killing vector field that is only asymptotically timelike might also be called stationary see, e.g., [14, 105]. Assuming the splitting $\mathfrak{M} = \mathbb{R} \times \mathfrak{N}$, in which $\mathfrak{N}$ is a spacelike hypersurface, and working with coordinates $(t, \boldsymbol{x}) \in \mathfrak{M}$, $t \in \mathbb{R}$ being the time coordinate and $\boldsymbol{x} \in \mathfrak{N}$, the metric of a stationary Lorentzian manifold can be written as

$$\boldsymbol{g} = \beta(\boldsymbol{x})\, \mathrm{d}t \otimes \mathrm{d}t - \omega_i\, \mathrm{d}x^i \otimes \mathrm{d}t - \mathrm{d}t \otimes \omega_i\, \mathrm{d}x^i - \boldsymbol{g}_{\mathfrak{N}}\,,$$

where β is a positive smooth function on $\mathfrak{N}$, $\boldsymbol{\omega} \in T^\star_{\boldsymbol{x}}\mathfrak{N}$ a 1-form, and $\boldsymbol{g}_{\mathfrak{N}}$ the induced Riemannian metric on $\mathfrak{N}$. Accordingly, the geometry of a stationary Lorentzian manifold does not change in time, which allows for the local identification of the Killing vector field $\boldsymbol{K}$ with ∂_t. This type of manifold is thus included in the SSTK class [13, 66].

Definition 2.1.21. A Lorentzian manifold $(\mathfrak{M}, \boldsymbol{g})$ is *axially symmetric* if

(i) the special orthogonal group $\mathrm{SO}(2)$ acts as a group of isometries $\mathfrak{I}\colon \mathfrak{M} \times \mathrm{SO}(2) \to \mathfrak{M}$.

(ii) the set of fixed points $\mathfrak{I}\big((t, \boldsymbol{x}), R\big) = (t, \boldsymbol{x})$ for all $R \in \mathrm{SO}(2)$ is a 2-dimensional timelike surface referred to as the *axis of symmetry*.

We are now in a position to give a proper account of the Kerr geometry. In the framework of general relativity, the (final equilibrium state of the) gravitational field of an axially symmetric and rotating black hole without electric charge may be modeled in terms of the Kerr geometry [68], which, up to the event horizon, is a stationary spacetime characterized by only two real-valued parameters, namely the mass of the black hole $M \in \mathbb{R}_{\geq 0}$ as well as its angular momentum per unit mass $\mathbb{R} \ni a := J/M$ defining the rate of rotation. One distinguishes between the nonextreme ("slowly rotating") case $0 < |a| < M$ describing the final equilibrium state of a realistic gravitational collapse, where the rotational energy is less than the total energy and an event horizon that protects the singularity forms, the overextreme ("rapidly rotating") case $0 < M < |a|$ that is considered as unphysical as it allows for closed timelike curves (constituting a global violation of causality), negative black hole masses, as well as naked singularities, and finally the extreme case $0 < |a| = M$, which is regarded as unstable and also not occurring in nature because the absorption of even a single particle will yield a transition to the overextreme case. Furthermore, for

$M > 0$ and $|a| = 0$, the Kerr geometry reduces to the static Schwarzschild geometry, which can be used to model a spherically symmetric uncharged nonrotating black hole, whereas for $M = 0$ and $|a| \geq 0$, it turns into Minkowski space. In the following, we concentrate on the nonextreme case. Then, the Kerr geometry features several physically relevant surfaces [73, 108, 109], the two most important being the (future) event horizon, i.e., an inner null hypersurface $E^+ := \dot{J}^-(\mathcal{I}^+)$, where $\dot{J}^-$ is the boundary of J^- and $\mathcal{I}^+$ is future null infinity, beyond which the escape velocity of particles is greater than the speed of light making it a trapping surface, and the ergosurface located outside the event horizon, which is a static limit surface where the rotational velocity of the spacetime – caused by the dragging of the rotating black hole – is the speed of light. In the region within the event horizon and this ergosurface, the so-called ergosphere, we find that the rotational velocity is larger than the speed of light and, as a consequence, all particles located there are forced to corotate with the black hole independent of their own velocity [76, 103]. A precise picture of this frame dragging effect can actually be obtained by comparison with classical wind Finslerian structures [13, 66]. It may be used to extract rotational energy from a Kerr black hole via the Penrose process [86]. Besides, the Kerr geometry contains, on the one hand, a so-called (future) Cauchy horizon inside the event horizon, being another inner null hypersurface $H^+(S) := \overline{D^+(S)} \setminus I^-\big(D^+(S)\big)$, where S is a closed achronal subset of $\mathfrak{M}$, that defines the domain of validity of the Cauchy problem, and on the other hand, a ring-shaped curvature singularity [23, 89].

Definition 2.1.22. The *Kerr geometry* is a connected orientable and time-orientable smooth asymptotically flat Lorentzian 4-manifold $(\mathfrak{M}, \boldsymbol{g})$, where the differential manifold is of the form $\mathfrak{M} = \mathbb{R} \times \mathbb{R}_{>0} \times S^2$ and the Lorentzian metric $\boldsymbol{g}$ on $\mathfrak{M}$ with signature $(1, 3)$ is stationary up to the event horizon, axially symmetric, parametrized by (M, a), and an exact solution of the vacuum Einstein field equations (2.13).

According to the Carter–Robinson theorem [16, 96], this solution is unique. By endowing the tangent and cotangent bundles $T\mathfrak{M}$ and $T^\star\mathfrak{M}$ with the respective bases $(\boldsymbol{e}_\mu)$ and $(\boldsymbol{e}^\mu)$, $\mu \in \{0, 1, 2, 3\}$, we may choose coordinates $(t, \boldsymbol{x})$ on $\mathfrak{M}$, where $t \in \mathbb{R}$ is the time coordinate and $\boldsymbol{x} = (x^1, x^2, \varphi = x^3)$ are coordinates on the spacelike hypersurface $\mathfrak{N} = \mathbb{R}_{>0} \times S^2$ with $\varphi \in [0, 2\pi)$ being the azimuthal angle about the axis of symmetry.

Proposition 2.1.23. *In the above coordinates, the Kerr metric takes the form*

$$\boldsymbol{g} = g_{tt}\, \mathrm{d}t \otimes \mathrm{d}t + g_{t\varphi}\, (\mathrm{d}t \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}t) - (\boldsymbol{g}_{\mathfrak{N}})_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where the metric coefficients $g_{tt}, g_{t\varphi}$, as well as the induced Riemannian metric $\boldsymbol{g}_{\mathfrak{N}}$ on $\mathfrak{N}$ are in the class $C^\infty\big(\mathbb{R}_{>0} \times (0, \pi), \mathbb{R}\big)$ of smooth functions independent of t and φ, thus yielding Killing vector fields $\boldsymbol{K}_1 = \partial_t$ and $\boldsymbol{K}_2 = \partial_\varphi$ corresponding to the time translational and axial symmetries.

We moreover note that this metric is invariant under the simultaneous discrete time and azimuthal angle isometries $t \mapsto -t$ and $\varphi \mapsto -\varphi$. From an algebraic point of view, the Kerr geometry is of Petrov type D [83], i.e., it possesses two double principal null directions and, hence, a pair of principal ingoing and outgoing null congruences. In the framework of the Newman–Penrose formalism, this amounts to the existence of a tetrad basis in which the Weyl scalars Ψ_0, Ψ_1, Ψ_3, and Ψ_4 as well as the spin coefficients κ, σ, λ and ν vanish.

2.2 Functional Analytic and PDE Concepts and Settings

We briefly review the relevant concepts of spectral theory. We then outline the basics of linear symmetric hyperbolic systems and linear elliptic operators. Furthermore, we define the concept of the Cauchy problem and the notion of spinors. Finally, we formulate and discuss the general relativistic Dirac equation and propagators.

2.2.1 Spectral Theory

Spectral theory is a subdiscipline of functional analysis, which is concerned with the structure and classification of operators on vector spaces. As we aim for the spectral analysis of the Dirac Hamiltonian in the framework of relativistic quantum mechanics, the focus is on the spectral theory of linear unbounded self-adjoint operators on Hilbert spaces. Moreover, we only list the basic definitions and theorems required for the construction of spectral representations of propagators. For more details and proofs see, e.g., [63, 94]. We begin with the definition of bounded and unbounded linear operators.

Definition 2.2.1. A *bounded linear operator* T from a normed linear vector space X to a normed linear vector space Y is a mapping that satisfies the conditions:

1. $T(\alpha v + \beta w) = \alpha T v + \beta T w \quad$ for all $\quad v, w \in X$ and $\alpha, \beta \in \mathbb{C}$.
2. $\|Tv\|_Y \leq M \|v\|_X \quad$ for some constant $\quad M \geq 0$.

In case T does not satisfy the second condition, the operator is called *unbounded*.

Next, we introduce the notions of symmetry and self-adjointness of linear operators.

Definition 2.2.2. We let $\mathcal{H}$ be a complex Hilbert space and $T\colon \mathrm{Dom}(T) \subset \mathcal{H} \to \mathcal{H}$ a linear operator with domain $\mathrm{Dom}(T)$ dense in $\mathcal{H}$. Furthermore, we denote the adjoint of T by $T^\star$, where $\mathrm{Dom}(T^\star)$ is the set of all $\eta \in \mathcal{H}$ such that

$$|\langle T\xi, \eta\rangle| \leq c_\eta \, \|\xi\| \quad \text{for all} \quad \xi \in \mathrm{Dom}(T)\,,$$

where c_η is a constant depending on η.

1. **Symmetry**: The operator T is *symmetric* if $\langle \xi, T\eta\rangle = \langle T\xi, \eta\rangle$ for all $\xi, \eta \in \mathrm{Dom}(T)$. This implies that $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^\star)$.
2. **Self-adjointness**: A symmetric operator T is *self-adjoint* if $\mathrm{Dom}(T) = \mathrm{Dom}(T^\star)$ and $T^\star\xi = T\xi$ for all $\xi \in \mathrm{Dom}(T)$.
3. **Essential self-adjointness**: A self-adjoint operator T' is called a *self-adjoint extension* of a given symmetric operator T if $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T')$ and $T'\xi = T\xi$ for all $\xi \in \mathrm{Dom}(T)$. In case T admits a unique self-adjoint extension, it is referred to as *essentially self-adjoint*.

In the following definition, we specify the resolvent set and the spectrum of a linear operator.

Definition 2.2.3. We let $T\colon \mathfrak{D}_T =: \mathrm{Dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be a linear operator on a complex Hilbert space $\mathcal{H}$. A complex number λ is said to be in the *resolvent set* $\rho(T)$ of T if the operator $T - \lambda\mathbb{1}$ is a bijection with a bounded inverse, i.e., if there exists a bounded operator $S\colon \mathcal{H} \to \mathrm{Dom}(T)$ such that

$$S\,(T - \lambda\mathbb{1}) = \mathbb{1}_{\mathfrak{D}_T} \quad \text{and} \quad (T - \lambda\mathbb{1})\,S = \mathbb{1}_{\mathcal{H}}\,.$$

The operator $\mathrm{Res}(\lambda; T) := (T - \lambda\mathbb{1})^{-1}$ defined for $\lambda \in \rho(T)$ is called the *resolvent* of T at λ. If $\lambda \notin \rho(T)$, it is said to be in the *spectrum* $\sigma(T)$ of T.

Thus, the spectrum of T contains all $\lambda \in \mathbb{C}$ for which the operator $T - \lambda\mathbb{1}$ does not have an inverse that is a bounded linear operator. We distinguish three subsets of the spectrum:

1. If $T - \lambda\mathbb{1}$ is not injective, then λ is said to be in the *point spectrum* of T. In this case, there is a nonzero ζ that satisfies $T\zeta = \lambda\zeta$ called an eigenvector of T, whereas λ is called the corresponding eigenvalue.

2. If $T - \lambda\mathbb{1}$ is injective and $\mathrm{Ran}(T - \lambda\mathbb{1})$ is a proper dense subset of $\mathcal{H}$, then λ is said to be in the *continuous spectrum* of T.

3. If $T - \lambda\mathbb{1}$ is injective and $\mathrm{Ran}(T - \lambda\mathbb{1})$ is not dense, then λ is said to be in the *residual spectrum* of T.

We may now state the spectral theorem for unbounded self-adjoint operators as well as Stone's formula, which relates spectral projections to resolvents [94]. In order to formulate the spectral theorem, we first have to introduce the concept of spectral measure.

Definition 2.2.4. We let X be a set and $\mathcal{A}$ a σ-algebra of X. Furthermore, we let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$, and $P(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is the set of all orthogonal projections

$$P\colon \mathcal{H} \to \mathcal{H} \quad \text{with} \quad P \circ P = P \quad \text{and} \quad \mathrm{Ker}(P) \perp \mathrm{Ran}(P)\,.$$

A *spectral measure* for the triple $(X, \mathcal{A}, \mathcal{H})$ is a mapping $E\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that satisfies the following properties:

1. $E(X) = \mathbb{1}_{\mathcal{H}} \quad \text{and} \quad E(\emptyset) = 0\,.$

2. $E(\Omega) \in P(\mathcal{H}) \quad \text{for all} \quad \Omega \in \mathcal{A}\,.$

3. $E(\Omega_1 \cap \Omega_2) = E(\Omega_1)E(\Omega_2) = E(\Omega_2)E(\Omega_1) \quad \text{for} \quad \Omega_1, \Omega_2 \in \mathcal{A}\,.$

4. For any family of pairwise disjoint sets $(\Omega_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$, one has
$$E\biggl(\bigcup_{i=1}^{\infty} \Omega_i\biggr) = \sum_{i=1}^{\infty} E(\Omega_i)\,.$$

Then, we can write the spectral theorem in the form given below.

Spectral theorem for unbounded operators. *Let T be a densely defined normal operator on a complex Hilbert space $\mathcal{H}$, that is,* $\mathrm{Dom}(T)$ *is dense and $TT^\star = T^\star T$ (where the equality includes the assertion that* $\mathrm{Dom}(TT^\star) = \mathrm{Dom}(T^\star T)$*). There exists a unique spectral measure E on the Borel sets of $\mathbb{C}$ such that*

$$T = \int_{z\in\sigma(T)} z\,\mathrm{d}E(z)\,, \tag{2.14}$$

where $\sigma(T)$ is the spectrum of T.

Moreover, Stone's formula can be represented as:

Stone's formula. *The spectral projection $P_I(T) = \chi_I(T)$, with χ_I being the characteristic function and $I \subset \mathbb{R}$, of a self-adjoint operator T is related to the resolvent by*

$$\mathrm{i}\pi\,[P_{[a,b]}(T) + P_{(a,b)}(T)] = \operatorname*{s-lim}_{\epsilon\searrow 0} \int_a^b \bigl[(T - \lambda - \mathrm{i}\epsilon)^{-1} - (T - \lambda + \mathrm{i}\epsilon)^{-1}\bigr]\mathrm{d}\lambda\,, \tag{2.15}$$

where s-lim denotes the strong limit of operators.

2.2.2 Linear Symmetric Hyperbolic Systems and Linear Elliptic Operators

The basic definitions of linear symmetric hyperbolic systems as well as linear elliptic operators on globally hyperbolic manifolds are outlined from a local point of view, i.e., the definitions are only given for subsets that are equipped with a single coordinate chart. The classes of these systems and operators include the Dirac equation and the Dirac Hamiltonian, respectively. For more details and proofs see, e.g., [33, 67, 95, 99, 100].

Definition 2.2.5. A system of N complex-valued linear first-order PDEs for the N component function $f \in C^1(\mathfrak{M}, \mathbb{C}^N)$ on a subset Ω of an n-dimensional globally hyperbolic manifold $\mathfrak{M} = \mathbb{R} \times \mathfrak{N}$ of the form

$$A^0(t, \boldsymbol{x})\, \partial_t f(t, \boldsymbol{x}) + A^i(t, \boldsymbol{x})\, \partial_i f(t, \boldsymbol{x}) + B(t, \boldsymbol{x}) f(t, \boldsymbol{x}) = w(t, \boldsymbol{x}) \tag{2.16}$$

expressed in coordinates $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathfrak{N}$ and with

$$A^0, A^i, B \in C^1\big(\mathfrak{M}, M_N(\mathbb{C})\big) \quad \text{and} \quad w \in C^1(\mathfrak{M}, \mathbb{C}^N)\,,$$

where $M_N(\mathbb{C})$ denotes the set of all $N \times N$ complex matrices, is called *symmetric hyperbolic* if

1. the matrices A^0 and A^i are Hermitian, that is, $A^{0\dagger} = A^0$ and $A^{i\dagger} = A^i$.
2. the matrix A^0 is uniformly positive definite, i.e., there is a positive constant c such that

$$A^0(t, \boldsymbol{x}) > c\, \mathbb{1}_{\mathbb{C}^N} \quad \text{for all} \quad (t, \boldsymbol{x}) \in \mathfrak{M}\,.$$

Linear PDE systems of higher order may always be reduced to such a first-order system by increasing the number of components, and therefore the dimension of the system.

Definition 2.2.6. A linear differential operator $\mathcal{L}$ of order m defined on a subset Ω of an n-dimensional globally hyperbolic manifold $\mathfrak{M} = \mathbb{R} \times \mathfrak{N}$ given in coordinates $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathfrak{N}$ by

$$\mathcal{L} = \sum_{|\alpha| \le m} a_\alpha(t, \boldsymbol{x})\, \partial^\alpha, \tag{2.17}$$

where $\mathbb{N}_0^n \ni \alpha = (\alpha_0, \dots, \alpha_{n-1})$ is a multi-index of length $|\alpha| := \alpha_0 + \dots + \alpha_{n-1}$ and $\partial^\alpha := \partial_0^{\alpha_0} \dots \partial_{n-1}^{\alpha_{n-1}}$, is called *elliptic* if for every $(t, \boldsymbol{x})$ in Ω and every nonzero vector $\boldsymbol{\xi} \in T_{(t,\boldsymbol{x})}\Omega$ the condition

$$\sum_{|\alpha| = m} a_\alpha(t, \boldsymbol{x})\, \xi^\alpha \neq 0 \tag{2.18}$$

for the highest-order terms holds. In case $\mathcal{L}$ is of degree $m = 2k$ and satisfies the stronger condition

$$(-1)^k \sum_{|\alpha| = 2k} a_\alpha(t, \boldsymbol{x})\, \xi^\alpha > d\, |\boldsymbol{\xi}|^{2k}, \tag{2.19}$$

where d is a positive constant, it is said to be *uniformly elliptic*.

We remark that these definitions can be extended to the global framework, that is, to the entire manifold $\mathfrak{M}$, by first fixing an atlas that consists of all individual coordinate charts, specifying the differential operators in (2.16) and (2.17) for each chart, and imposing compatibility at the intersections of the charts. Then, one requires the conditions 1. and 2. of Definition 2.2.5 as well as the conditions (2.18) and (2.19) of Definition 2.2.6 to hold for all individual charts.

2.2.3 The Cauchy Problem

A Cauchy problem constitutes the class of PDE systems that satisfy certain initial conditions with support on a Cauchy hypersurface in the respective domain. In addition, Cauchy problems may include boundary conditions. The pivotal questions that arise in the study of Cauchy problems refer to the existence and uniqueness of solutions, the properties of general and special solutions, their representations and domains of definition, as well as the dependence of the solutions on the initial data.

Definition 2.2.7. A *Cauchy problem* defined on an n-dimensional globally hyperbolic manifold $\mathfrak{M} = \mathbb{R} \times \mathfrak{N}$ consists of finding the unknown functions $(f_1, \dots, f_N \,|\, N \in \mathbb{N})$ with $f_i \in C^{k_i}(\mathfrak{M}, \mathbb{C})$, $i \in \{1, \dots, N\}$, for the independent variables $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathfrak{N}$ satisfying the system of PDEs

$$F\big(t, \boldsymbol{x}, (\partial^\alpha f_1)_{|\alpha| \le k_1}, \dots, (\partial^\alpha f_N)_{|\alpha| \le k_N}\big) = \mathbf{0}\,,$$

which are subjected to the conditions

$$\partial_t^k f_{i|t=0} = \Phi_i^{(k)}(\boldsymbol{x}) \in C^{k_i - k}(\mathfrak{N}, \mathbb{C}) \quad \text{for all} \quad k \in \{0, \dots, k_i - 1\}$$

on the Cauchy hypersurface $\mathfrak{N} \subset \mathfrak{M}$, where k_i labels the highest-order derivative of the function f_i, $F : \mathrm{Dom}(F) \subset \mathfrak{M} \times J^{k_1}(\mathfrak{M}) \times \dots \times J^{k_N}(\mathfrak{M}) \to C^1(\mathfrak{M}, \mathbb{C}^N)$ is a linear or nonlinear operator with J^{k_i} denoting the k_ith jet space of $\mathfrak{M}$, and α is a multi-index specified in Definition 2.2.6. The set $\{\Phi_i^{(k)} \,|\, i = 1, \dots, N \text{ and } k = 0, \dots, k_i - 1\}$ of given functions defined on the surface $\mathfrak{N}$ is referred to as *Cauchy data*.

If there exist unique solutions $f_i : [0, T) \times \mathfrak{N} \to C^{k_i}(\mathfrak{M}, \mathbb{C})$, where $f_i \in \mathrm{Dom}(F)$ for all times $0 < t < T$ and a given $T > 0$, that depend continuously on the Cauchy data, the Cauchy problem is said to be well-posed.

2.2.4 Spinor Formalism

In physics, all elementary spin-1/2 fermions can be described by the Dirac equation, that is, these fermions exhibit the algebraic and analytic qualities of spinors. Essentially, there exist two interrelated principal frameworks for defining the notion of spinors. On the one hand, there is a very general algebraic topology view on spinors, which employs both Clifford algebras (to account for spin groups, i.e., certain Lie groups that are related to orthogonal groups) as well as the theory of vector bundles on real and complex manifolds (see, e.g., [72, 111]). On the other hand, spinors may be introduced in a simpler, less comprehensive representation theoretic and geometrically inclined way. In the following, since this work focuses on the notion of spinors in the framework of general relativity, we briefly outline the latter approach to viewing spinors, with an emphasis on the more geometric concepts. Moreover, as we thus work only with Lorentzian 4-manifolds, we may consider the group $\mathrm{SL}(2, \mathbb{C})$, which is the double covering group of the restricted Lorentz group $\mathrm{SO}^+(1, 3, \mathbb{R})$ and which is isomorphic to the spin group $\mathrm{Spin}(1, 3)$, as the local spinor group. This group acts naturally on $\mathbb{C}^2$ and preserves its natural 2-dimensional nondegenerate skew-symmetric 2-form $\boldsymbol{\epsilon}$ given by the Levi–Civita symbol. Therefore, we regard $(\mathbb{C}^2, \boldsymbol{\epsilon})$ as our spinor space.

Definition 2.2.8. A *spinor space* $(\mathbb{C}^2, \boldsymbol{\epsilon})$ consists of the 2-dimensional complex vector space $\mathbb{C}^2$ with the $\mathrm{SL}(2, \mathbb{C})$ invariant 2-form $\boldsymbol{\epsilon}$ – the 2-dimensional Levi–Civita symbol – that induces a bilinear nondegenerate skew-symmetric inner product on $\mathbb{C}^2$. Elements $\boldsymbol{\xi}$ of a spinor space are called *spinors*.

Such a notion can of course be extended to more general 2-dimensional complex vector spaces, which then have to be endowed with a 2-dimensional nondegenerate skew-symmetric 2-form. Furthermore, for each spinor $\boldsymbol{\xi} \in \mathbb{C}^2$ there exists a dual with respect to $\boldsymbol{\epsilon}$ that gives rise to the mappings

$$\mathbb{C}^2 \to (\mathbb{C}^2)^\star \quad \text{with} \quad \xi^A \mapsto \xi_A = \epsilon_{BA}\xi^B \quad \text{and} \quad (\mathbb{C}^2)^\star \to \mathbb{C}^2 \quad \text{with} \quad \xi_A \mapsto \xi^A = \epsilon^{AB}\xi_B\,,$$

where ϵ^{AB} is the inverse of ϵ_{AB}, i.e., $\epsilon^{AB}\epsilon_{BC} = \delta^A_C$, and $A, B \in \{0, 1\}$. Accordingly, the 2-form $\boldsymbol{\epsilon}$ is an isomorphism between $\mathbb{C}^2$ and $(\mathbb{C}^2)^\star$ that can be used to raise and lower spinor indices. In addition, since $\mathbb{C}^2$ is naturally endowed with the operation of conjugacy, we can define a complex conjugate, which is usually denoted by a bar over the symbol with simultaneous priming of sub- and superscript letters, and a complex conjugate dual. As we ultimately aim for a study of the Dirac equation in a Lorentzian 4-manifold, we have to consider spinors of the type $\mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$, that is, the solutions of the Dirac equation – the so-called Dirac 4-spinors – are $\mathbb{C}^4$-valued functions defined in terms of the above 2-spinor construction. In this regard, we mention the following theorem by Geroch [55]:

Theorem 2.2.9. *We let $(\mathfrak{M}, \boldsymbol{g})$ be a noncompact time-oriented and connected Lorentzian manifold. Then, it has a spinor structure if and only if there exists a global system of (orthonormal or null) tetrads.*

This theorem is paramount for identifying Lorentzian manifolds that can be used to set up spinors. However, it also allows one to view Dirac 4-spinors as the above global maps from $\mathfrak{M}$ to $\mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$, assuming that such a system of tetrads exists. For a full account on the spinor formalism, more precisely all details of spinor algebra, spinor analysis, and the relation between spinors and tensors, we refer the reader to the standard textbooks [19, 81, 87, 88].

2.2.5 The General Relativistic Dirac Equation and its Hamiltonian Formulation

The general relativistic Dirac equation is a linear first-order PDE system that describes the dynamics of massive spin-1/2 fermions on globally hyperbolic Lorentzian 4-manifolds $(\mathfrak{M} = \mathbb{R} \times \mathfrak{N}, \boldsymbol{g})$. In order to properly define this equation, we first introduce a coordinate system $(t, \boldsymbol{x})$ on $\mathfrak{M}$, where $t \in \mathbb{R}$ is a Cauchy temporal function and $\boldsymbol{x}$ denotes coordinates on the smooth spacelike Cauchy hypersurface $\mathfrak{N}$. Moreover, we assume a spinor bundle $S\mathfrak{M}$ on $\mathfrak{M}$ with fibers $S_p\mathfrak{M} \simeq \mathbb{C}^4$, $p \in \mathfrak{M}$.

Definition 2.2.10. The *general relativistic massive Dirac equation* can be written as [53, 104, 113]

$$[\mathrm{i}\gamma^\mu(t, \boldsymbol{x})\nabla_\mu + \mathscr{B}(t, \boldsymbol{x}) - m]\psi(t, \boldsymbol{x}) = \mathbf{0} \quad \text{with} \quad \mu \in \{0, 1, 2, 3\}\,, \tag{2.20}$$

where (γ^μ) are the *general relativistic Dirac matrices*, which satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{\mathbb{C}^4}\,,$$

$\boldsymbol{\nabla}$ is the metric connection on $S\mathfrak{M}$ given in a coordinate basis (∂_μ) by

$$\nabla_\mu = \partial_\mu + \frac{1}{8}\,\omega_{\mu\alpha\beta}\,[\gamma^\alpha, \gamma^\beta]$$

with $\boldsymbol{\omega}$ being the Levi–Civita spin connection and $[\,.\,,\,.\,]$ the commutator, $\psi \in C^\infty_{\mathrm{sc}}(\mathfrak{M}, \mathbb{C}^4)$ is a Dirac 4-spinor, the quantity $\mathscr{B}$ denotes an external potential, and m is the invariant fermion mass.

The Hamiltonian formulation of (2.20) may be derived simply by separating the t-derivative of the Dirac 4-spinor, yielding

$$\mathrm{i}\partial_t\psi = H\psi\,,$$

where

$$H := -(\gamma^t)^{-1}(\mathrm{i}\gamma^j \nabla_j + \mathscr{B} - m) - \frac{\mathrm{i}}{8}\,\omega_{t\alpha\beta}\,[\gamma^\alpha, \gamma^\beta]$$

is the Dirac Hamiltonian. For the purpose of this work, it is sufficient to consider the case $\mathscr{B} = \mathbf{0}_{\mathbb{C}^4}$, in which no external potential is present. Below, we state the 2-spinor form of the Dirac equation and its representation in the Newman–Penrose formalism.

Corollary 2.2.11. *Using the Weyl representation of the Dirac* 4*-spinors and matrices [54]*

$$\psi = \begin{pmatrix} P^A \\ \overline{Q}_{B'} \end{pmatrix} \quad \text{and} \quad \gamma^\mu = \sqrt{2}\begin{pmatrix} \mathbf{0}_{\mathbb{C}^2} & \sigma^{\mu AB'} \\ \sigma^\mu{}_{AB'} & \mathbf{0}_{\mathbb{C}^2} \end{pmatrix},$$

where P^A and $\overline{Q}_{B'}$ denote 2*-spinors, $(\sigma^\mu{}_{AB'})$ are the Hermitian 2×2 Infeld–van der Waerden symbols [107], and $A \in \{1,2\}$ as well as $B' \in \{1', 2'\}$, we obtain the following* 2*-spinor form of the Dirac equation*

$$\begin{aligned} \nabla_{AB'}P^A + \frac{im}{\sqrt{2}}\,\overline{Q}_{B'} &= \mathbf{0} \\ \nabla_{AB'}Q^A + \frac{im}{\sqrt{2}}\,\overline{P}_{B'} &= \mathbf{0}\,, \end{aligned} \tag{2.21}$$

with $\nabla_{AB'} = \sigma^\mu{}_{AB'}\nabla_\mu$.

In order to express these equations in the framework of the Newman–Penrose formalism for spinors, we now introduce the local dyad basis $(\zeta_{(k)})$, $k \in \{1,2\}$, for the space of Dirac 2-spinors and its dual $(\zeta^{(k)})$ with respect to the 2-dimensional Levi–Civita symbol $\boldsymbol{\epsilon}$. The associated local spinor components $\mathscr{Y}^{(k)}$ and $\mathscr{Y}_{(k)}$ are related to the former 2-spinor components $\mathscr{Y}^A$ and $\mathscr{Y}_A$ via the 2×2 transition matrix $\zeta^{(k)}{}_A$ and its inverse $\zeta_{(k)}{}^A$ in a similar vein as in the tetrad formalism by

$$\mathscr{Y}^{(k)} = \zeta^{(k)}{}_A \mathscr{Y}^A \quad \text{and} \quad \mathscr{Y}_{(k)} = \zeta_{(k)}{}^A\,\mathscr{Y}_A\,.$$

In this representation, the covariant derivative reads

$$\nabla_{(k)(l')}\mathscr{Y}^{(m)} = \zeta^A{}_{(k)}\,\overline{\zeta}^{B'}{}_{(l')}\,\zeta^{(m)}{}_C\,\nabla_{AB'}\mathscr{Y}^C = \partial_{(k)(l')}\mathscr{Y}^{(m)} + \Gamma^{(m)}{}_{(n)(k)(l')}\mathscr{Y}^{(n)}\,, \tag{2.22}$$

where $\partial_{(k)(l')} = \sigma^\mu{}_{(k)(l')}\partial_\mu$ and

$$\begin{aligned} \Gamma^{(m)}{}_{(n)(k)(l')} &= \Gamma^{(m)(o')}{}_{(n)(o')(k)(l')} \\ &= \sqrt{2}\,\epsilon^{(m)(q)}\,\epsilon^{(o')(p')}\,\sigma^\mu{}_{(q)(p')}\,\sigma^\nu{}_{(n)(o')}\,\sigma^\lambda{}_{(k)(l')}\,e_\mu{}^{(a)}\,e_\nu{}^{(b)}\,e_\lambda{}^{(c)}\,\gamma_{(a)(b)(c)}\,. \end{aligned} \tag{2.23}$$

Moreover, the Infeld–van der Waerden symbols take the form

$$\sigma^\mu{}_{(k)(l')} = \begin{pmatrix} l^\mu & m^\mu \\ \overline{m}^\mu & n^\mu \end{pmatrix}. \tag{2.24}$$

Corollary 2.2.12. *Employing (2.22)-(2.24) in (2.21) and using the notation* $\mathscr{F}_1 := P^{(1)}$, $\mathscr{F}_2 := P^{(2)}$, $\mathscr{G}_1 := \overline{Q}^{(2')}$, *as well as* $\mathscr{G}_2 := -\overline{Q}^{(1')}$, *the general relativistic Dirac equation in the Newman–Penrose formalism becomes*

$$\begin{aligned}
(l^\mu \partial_\mu + \varepsilon - \varrho)\mathscr{F}_1 + (\overline{m}^\mu \partial_\mu + \pi - \alpha)\mathscr{F}_2 &= \frac{\mathrm{i}m}{\sqrt{2}}\mathscr{G}_1 \\
(n^\mu \partial_\mu + \mu - \gamma)\mathscr{F}_2 + (m^\mu \partial_\mu + \beta - \tau)\mathscr{F}_1 &= \frac{\mathrm{i}m}{\sqrt{2}}\mathscr{G}_2 \\
(l^\mu \partial_\mu + \overline{\varepsilon} - \overline{\varrho})\mathscr{G}_2 - (m^\mu \partial_\mu + \overline{\pi} - \overline{\alpha})\mathscr{G}_1 &= \frac{\mathrm{i}m}{\sqrt{2}}\mathscr{F}_2 \\
(n^\mu \partial_\mu + \overline{\mu} - \overline{\gamma})\mathscr{G}_1 - (\overline{m}^\mu \partial_\mu + \overline{\beta} - \overline{\tau})\mathscr{G}_2 &= \frac{\mathrm{i}m}{\sqrt{2}}\mathscr{F}_1 \,.
\end{aligned} \tag{2.25}$$

2.2.6 Propagators

In (relativistic) quantum mechanics, the time evolution of a state $\psi(t, \boldsymbol{x})$ is the change brought about by translations of the time parameter t. These translations are associated with a unitary operator U on a Hilbert space, referred to as propagator, which yields a transition from a state for a particle localized at the event $(t, \boldsymbol{x})$ to a consecutive state at the event $(t', \boldsymbol{x}')$

$$\psi(t', \boldsymbol{x}') = U^{t',t}\psi(t, \boldsymbol{x})$$

caused by a specific dynamics.

Definition 2.2.13. A 2-parameter family $(U^{t,s})_{s,t\in\mathbb{R}}$ of operators on a Hilbert space $\mathcal{H}$ is called a *propagator* if

1. $U^{t,s}$ is unitary for all s and t.
2. $U^{t,t} = \mathbb{1}$ for all t.
3. $U^{t,s}\,U^{s,r} = U^{t,r}$ for all r, s, and t.
4. the mapping $(s,t) \mapsto U^{t,s}\psi$ is continuous for all $\psi \in \mathcal{H}$.

The following theorems refer to the existence and the representation of propagators. Their proofs can be found in [92].

Theorem 2.2.14. *We let* $\big(H(t)\big)_{t\in\mathbb{R}}$ *be a family of self-adjoint operators on the common domain*

$$\mathrm{Dom}\big(H(t)\big) =: \mathfrak{D}_H$$

and assume that for any compact interval $I \subset \mathbb{R}$ *the mapping*

$$I \times I \to \mathcal{H}\,, \quad (s,t) \mapsto \frac{1}{t-s}\big[\big(H(t)-z\big)\big(H(s)-z\big)^{-1} - 1\big]\psi \quad \text{with} \quad s<t \quad \text{and} \quad z \in \bigcap_{t\in\mathbb{R}} \rho\big(H(t)\big)$$

is continuous for all $\psi \in \mathcal{H}$ *and extends continuously to* $t = s$. *Then, there exists a propagator* $U^{t,s}$. *Furthermore, if* $\psi \in \mathfrak{D}_H$, *we find that* $U^{t,s}\psi \in \mathfrak{D}_H$ *for all t, being a solution of the equations*

$$\begin{aligned}
\mathrm{i}\partial_t U^{t,s}\psi &= H(t)\,U^{t,s}\psi \\
\mathrm{i}\partial_s U^{t,s}\psi &= -U^{t,s}H(s)\psi \,.
\end{aligned}$$

Theorem 2.2.15. *The assumptions of Theorem 2.2.14 are satisfied for operators*

$$H(t) = H_0 + V(t)$$

if H_0 is self-adjoint and independent of t, $V(t)$ is self-adjoint and bounded for each t, and the mapping $t \mapsto V(t)\psi$ is continuous for all ψ. In this case, we may represent the propagator for the time-ordering $s < t_{n-1} < ... < t_2 < t_1 < t$ in the form

$$\widetilde{U}^{t,\,s} = 1 + \sum_{n=1}^{\infty} (-\mathrm{i})^n \int_s^t \int_s^{t_1} ... \int_s^{t_{n-1}} \widetilde{V}(t_1) \ldots \widetilde{V}(t_n)\, \mathrm{d}t_n \ldots \mathrm{d}t_1\,, \tag{2.26}$$

where

$$\widetilde{U}^{t,\,s} := e^{\mathrm{i}tH_0}\, U^{t,\,s}\, e^{-\mathrm{i}sH_0} \quad \text{and} \quad \widetilde{V}(t) := e^{\mathrm{i}tH_0}\, V(t)\, e^{-\mathrm{i}tH_0}\,.$$

We point out that for a homogeneous time evolution, i.e.,

$$U^{t,\,s} = U^{t-s,\,0} \quad \text{for all} \quad s, t \in \mathbb{R}\,,$$

the family of propagators forms a 1-parameter group of unitary operators. Moreover, if $V(t) = 0$, the propagators are exponentials of the Hamiltonian

$$U^{t,\,s} = e^{-\mathrm{i}(t-s)H_0}\,,$$

which follows directly from (2.26). This will become relevant in the later construction of the Dirac propagator in the nonextreme Kerr geometry, where the Hamiltonian is of the particular form $H = H_0$.

CHAPTER 3

Mode Analysis of the Massive Dirac Equation in the Nonextreme Kerr Geometry in Horizon-penetrating Coordinates

We derive a regular Carter tetrad in horizon-penetrating advanced Eddington–Finkelstein-type coordinates in order to describe the analytic extension of the nonextreme Kerr geometry across the event and the Cauchy horizon. In this background geometry, we define the massive Dirac equation in a dyadic Weyl representation and show its separability into radial and angular ODE systems using Chandrasekhar's mode analysis. We study the asymptotics of the radial solutions at infinity as well as at both horizons, and determine the decay of the associated errors. Moreover, we discuss the nature of the angular eigenfunctions and specific spectral aspects of the corresponding eigenvalues. The chapter concludes with an application of our results to the scattering problem of massive Dirac particles by the gravitational field of a rotating Kerr black hole.

3.1 The Carter Tetrad

For the description of the nonextreme Kerr geometry, we employ a Carter tetrad [15], i.e., a null tetrad frame that is, one the one hand, adapted to the two double principal null directions of the Weyl tensor and, on the other hand, symmetric under the simultaneous action of the fundamental discrete time and azimuthal angle reversal isometries. To this end, we begin by expressing the Kerr geometry in terms of the usual Boyer–Lindquist coordinates [10]

$$(t, r, \theta, \varphi) \quad \text{with} \quad t \in \mathbb{R}\,,\ r \in \mathbb{R}_{>0}\,,\ \theta \in (0, \pi)\,,\ \text{and}\ \ \varphi \in [0, 2\pi)\,,$$

for which the Kerr metric reads

$$\begin{aligned} \boldsymbol{g} = {} & \frac{\Delta}{\Sigma}\big(\mathrm{d}t - a\sin^2(\theta)\,\mathrm{d}\varphi\big) \otimes \big(\mathrm{d}t - a\sin^2(\theta)\,\mathrm{d}\varphi\big) - \frac{\sin^2(\theta)}{\Sigma}\big([r^2 + a^2]\,\mathrm{d}\varphi - a\,\mathrm{d}t\big) \\ & \otimes \big([r^2 + a^2]\,\mathrm{d}\varphi - a\,\mathrm{d}t\big) - \frac{\Sigma}{\Delta}\,\mathrm{d}r \otimes \mathrm{d}r - \Sigma\,\mathrm{d}\theta \otimes \mathrm{d}\theta\,. \end{aligned} \tag{3.1}$$

The horizon function is defined by $\Delta = \Delta(r) := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$, $r_\pm := M \pm \sqrt{M^2 - a^2}$ denote the event and the Cauchy horizon, respectively, M is the mass and aM the angular momentum of the black hole with $0 \le |a| < M$, and $\Sigma = \Sigma(r, \theta) := r^2 + a^2\cos^2(\theta)$.

Furthermore, we assume a Kinnersley tetrad [69], i.e., a Newman–Penrose frame that is already adapted to the two double principal null directions. In this frame, we are presented with the computational advantage that the four spin coefficients κ, σ, λ, and ν vanish and only one Weyl scalar, namely Ψ_2, is nonzero. Accordingly, the congruences formed by the two double principal null directions must be geodesic and shear-free [83]. We construct the Kinnersley tetrad directly from the tangent vectors of the principal null geodesics [19]

$$\frac{\mathrm{d}t}{\mathrm{d}\chi} = \frac{r^2 + a^2}{\Delta}\,E\,, \quad \frac{\mathrm{d}r}{\mathrm{d}\chi} = \pm E\,, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\chi} = 0\,, \quad \text{and} \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\chi} = \frac{a}{\Delta}\,E\,, \tag{3.2}$$

where χ is an affine parameter and E denotes a constant, by aligning the real-valued Newman–Penrose vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ with the associated principal null directions and further by choosing complex-conjugate Newman–Penrose vectors $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ in such a way that they satisfy the conditions (2.6)-(2.8). Thus, we obtain the frame

$$\begin{aligned}
\boldsymbol{l} &= \frac{1}{|\Delta|}\big([r^2 + a^2]\,\partial_t + \Delta\,\partial_r + a\,\partial_\varphi\big) \\
\boldsymbol{n} &= \frac{\operatorname{sign}(\Delta)}{2\,\Sigma}\big([r^2 + a^2]\,\partial_t - \Delta\,\partial_r + a\,\partial_\varphi\big) \\
\boldsymbol{m} &= \frac{1}{\sqrt{2}\big(r + \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\,\partial_t + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big) \\
\overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2}\big(r - \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\,\partial_t - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big)
\end{aligned} \tag{3.3}$$

with the signum function

$$\operatorname{sign}(\Delta) := \begin{cases} +1 & \text{for } \Delta > 0 \\ -1 & \text{for } \Delta < 0\,. \end{cases}$$

We remark that the use of the absolute value of the horizon function in $\boldsymbol{l}$ and the signum function in $\boldsymbol{n}$ leads to a combined representation of the frame for the regions outside the event horizon, inside the event horizon up to the Cauchy horizon, and inside the Cauchy horizon. For the later calculation of the corresponding spin coefficients, i.e., for solving the torsion-free first Maurer–Cartan equation of structure (2.10), one requires the dual cotetrad of (3.3)

$$\begin{aligned}
\boldsymbol{l}_{\mathrm{D}} &= \operatorname{sign}(\Delta)\left(\mathrm{d}t - \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right) \\
\boldsymbol{n}_{\mathrm{D}} &= \frac{|\Delta|}{2\,\Sigma}\left(\mathrm{d}t + \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right) \\
\boldsymbol{m}_{\mathrm{D}} &= \frac{1}{\sqrt{2}\big(r + \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t - \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\varphi\big) \\
\overline{\boldsymbol{m}}_{\mathrm{D}} &= -\frac{1}{\sqrt{2}\big(r - \mathrm{i}a\cos(\theta)\big)}\big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t + \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\varphi\big)\,.
\end{aligned}$$

We now apply a class III local Lorentz transformation (2.11) with parameters of the form

$$\varsigma = \sqrt{\frac{|\Delta|}{2\,\Sigma}} \quad \text{and} \quad e^{\mathrm{i}\psi} = \frac{\sqrt{\Sigma}}{r - \mathrm{i}a\cos(\theta)}$$

to the Kinnersley tetrad (3.3) in order to obtain the so-called Carter tetrad

$$\begin{aligned}
\boldsymbol{l}' &= \frac{1}{\sqrt{2\,\Sigma\,|\Delta|}}\big([r^2+a^2]\,\partial_t + \Delta\,\partial_r + a\,\partial_\varphi\big) \\
\boldsymbol{n}' &= \frac{\operatorname{sign}(\Delta)}{\sqrt{2\,\Sigma\,|\Delta|}}\big([r^2+a^2]\,\partial_t - \Delta\,\partial_r + a\,\partial_\varphi\big) \\
\boldsymbol{m}' &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_t + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big) \\
\overline{\boldsymbol{m}}' &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_t - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big)\,,
\end{aligned} \tag{3.4}$$

for which the dual cotetrad reads

$$\begin{aligned}
\boldsymbol{l}'_{\mathrm{D}} &= \sqrt{\frac{|\Delta|}{2\,\Sigma}}\,\operatorname{sign}(\Delta)\left(\mathrm{d}t - \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right) \\
\boldsymbol{n}'_{\mathrm{D}} &= \sqrt{\frac{|\Delta|}{2\,\Sigma}}\left(\mathrm{d}t + \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right) \\
\boldsymbol{m}'_{\mathrm{D}} &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t - \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2+a^2]\sin(\theta)\,\mathrm{d}\varphi\big) \\
\overline{\boldsymbol{m}}'_{\mathrm{D}} &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t + \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2+a^2]\sin(\theta)\,\mathrm{d}\varphi\big)\,.
\end{aligned} \tag{3.5}$$

This frame transforms under the composition of the discrete time and azimuthal angle reversal isometries $t \mapsto -t$ and $\varphi \mapsto -\varphi$ as

$$\boldsymbol{l}' \mapsto -\operatorname{sign}(\Delta)\,\boldsymbol{n}'\,, \quad \boldsymbol{n}' \mapsto -\operatorname{sign}(\Delta)\,\boldsymbol{l}'\,, \quad \boldsymbol{m}' \mapsto \overline{\boldsymbol{m}}'\,, \quad \overline{\boldsymbol{m}}' \mapsto \boldsymbol{m}'\,,$$

thus giving rise to only six independent spin coefficients

$$\kappa' = -\nu'\,, \quad \pi' = -\tau'\,, \quad \alpha' = -\beta'\,, \quad \sigma' = \operatorname{sign}(\Delta)\,\lambda'\,, \quad \mu' = \operatorname{sign}(\Delta)\,\varrho'\,, \quad \epsilon' = \operatorname{sign}(\Delta)\,\gamma'\,.$$

3.2 Advanced Eddington–Finkelstein-type Coordinates

We require a suitable coordinate system to analyze the propagation of Dirac particles across the event and the Cauchy horizon. Using Boyer–Lindquist coordinates, we have to deal with singularities at the horizons and with a reversal of the roles of space and time in between. These undesirable properties are depicted in the Finkelstein diagram in Figure 3.1, which shows the light cone of a system that approaches – and crosses – the event horizon from outside the black hole. More precisely, approaching the event horizon, the light cone closes up and eventually becomes degenerate. Beyond the event horizon, it tilts over, yielding a reversal of the specific characteristics of the time and the radial variable. These issues preclude a proper study of the propagation of Dirac particles across the horizons. Hence, instead of Boyer–Lindquist coordinates, we employ advanced Eddington–Finkelstein-type coordinates (see [32, 36] for the original advanced Eddington–Finkelstein coordinates), which are regular in the entire black hole spacetime except for the ring singularity at $(r = 0, \theta = \pi/2)$, and therefore allow for well-defined transitions of Dirac particles across the horizons. Furthermore, they feature a temporal function required for the Hamiltonian formulation of the Dirac equation and the corresponding Cauchy problem. We construct the advanced Eddington–Finkelstein-type coordinates from the tangent vectors (3.2) associated with the ingoing principal null geodesics, which lead to the relations

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{r^2 + a^2}{\Delta} \quad \Leftrightarrow \quad t = -\int \frac{r^2 + a^2}{\Delta}\,\mathrm{d}r + c = -r_\star + c \tag{3.6}$$

and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{a}{\Delta} \quad \Leftrightarrow \quad \varphi = -\int \frac{a}{\Delta}\,\mathrm{d}r + c' = -\frac{a}{r_+ - r_-}\ln\left|\frac{r - r_+}{r - r_-}\right| + c',$$

respectively, where

$$r_\star := r + \frac{r_+^2 + a^2}{r_+ - r_-}\ln|r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-}\ln|r - r_-|$$

is the Regge–Wheeler coordinate and c, c' are constants of integration. As the right hand sides of these relations diverge in the horizon limits, which amounts to the aforementioned degeneracy of the light cones at the horizons, one can introduce new variables

$$\mathbb{R} \times \mathbb{R}_{>0} \times (0,\pi) \times [0, 2\pi) \to \mathbb{R} \times \mathbb{R}_{>0} \times (0,\pi) \times [0, 2\pi)\,, \qquad (t, r, \theta, \varphi) \mapsto (\tau, r, \theta, \phi)$$

with

$$\tau := t + r_\star - r = t + \frac{r_+^2 + a^2}{r_+ - r_-}\ln|r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-}\ln|r - r_-| \tag{3.7}$$

$$\phi := \varphi + \frac{a}{r_+ - r_-}\ln\left|\frac{r - r_+}{r - r_-}\right| \tag{3.8}$$

that are constant along the ingoing principle null geodesics, thus removing the previous degeneracy. We point out that in the definition of the time coordinate (3.7), the additional term $-r$ was introduced in order for τ to be a proper temporal function, which is explicitly verified below. These so-called advanced Eddington–Finkelstein-type coordinates are free of singularities at the horizons and their spatio-temporal characteristics across the horizons are conserved. This can be seen in the Finkelstein diagram presented in Figure 3.2, where we again show a system approaching – and crossing – the event horizon from outside the black hole, but now the light cone is nondegenerate at the event horizon and is not tilted over in between the event and the Cauchy horizon. Besides, ingoing light rays are represented simply by straight lines

$$\frac{\mathrm{d}\tau}{\mathrm{d}r}_{|\mathrm{in}} = -1\,,$$

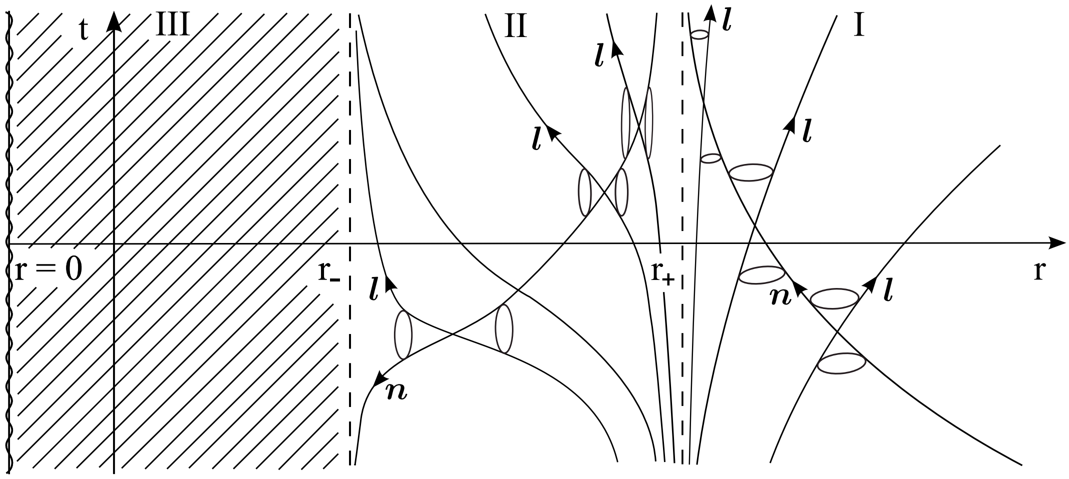

Figure 3.1: Causal structure of the nonextreme Kerr geometry in Boyer–Lindquist coordinates. A projection onto the (t, r)-plane, where every point is a 2-sphere, is presented. The real-valued Newman–Penrose null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, which point along the principal null directions, form the light cones. The light cone of an observer approaching the event horizon from outside the black hole ($r \searrow r_+$) closes up and becomes degenerate. In contrast, it opens up when the observer approaches the event horizon from inside the black hole ($r \nearrow r_+$). This stems from the fact that the roles of space and time are reversed in the black hole interior region II. When $r \to \infty$, the light cone becomes a $45°$-Minkowski light cone because the spacetime is asymptotically flat. We note that all figures are restricted to regions I and II in order to avoid the issues that arise when one considers the ring singularity at $(r = 0, \theta = \pi/2)$ and the maximum analytic extension.

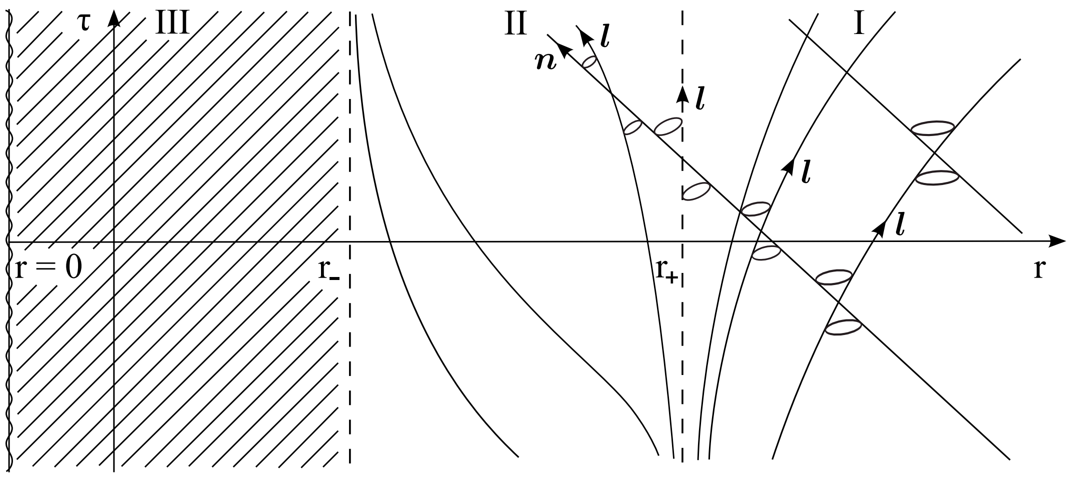

Figure 3.2: Causal structure of the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates. A projection onto the (τ, r)-plane is presented. Ingoing light rays are straight lines pointing in the $\boldsymbol{n}$-direction. The light cone of an observer moving toward the event horizon from outside the black hole turns until – after having crossed the event horizon – its future light cone is completely in the black hole interior. This shows the trapping characteristic of event horizons.

as can be directly inferred from (3.6) by inserting (3.7). The metric (3.1) represented in advanced Eddington–Finkelstein-type coordinates becomes

$$\begin{aligned}\boldsymbol{g} = &\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}\tau\otimes\mathrm{d}\tau-\frac{2Mr}{\Sigma}\big([\mathrm{d}r-a\sin^2(\theta)\,\mathrm{d}\phi]\otimes\mathrm{d}\tau+\mathrm{d}\tau\otimes[\mathrm{d}r-a\sin^2(\theta)\,\mathrm{d}\phi]\big)\\ &-\left(1+\frac{2Mr}{\Sigma}\right)\big(\mathrm{d}r-a\sin^2(\theta)\,\mathrm{d}\phi\big)\otimes\big(\mathrm{d}r-a\sin^2(\theta)\,\mathrm{d}\phi\big)-\Sigma\,\mathrm{d}\theta\otimes\mathrm{d}\theta-\Sigma\sin^2(\theta)\,\mathrm{d}\phi\otimes\mathrm{d}\phi\,.\end{aligned} \tag{3.9}$$

As the associated induced metric tensor on constant-τ hypersurfaces

$$\mathscr{G} := (g_{\mu\nu})_{|\tau=\text{const.}} = \begin{pmatrix} -\left(1+\frac{2Mr}{\Sigma}\right) & 0 & a\sin^2(\theta)\left(1+\frac{2Mr}{\Sigma}\right) \\ 0 & -\Sigma & 0 \\ a\sin^2(\theta)\left(1+\frac{2Mr}{\Sigma}\right) & 0 & -\sin^2(\theta)\left[\Sigma+a^2\sin^2(\theta)\left(1+\frac{2Mr}{\Sigma}\right)\right] \end{pmatrix},$$

with the leading principal minors

$$\begin{aligned}\det(\mathscr{G}_1) &= \mathscr{G}_{11} = -\Sigma^{-1}(\Sigma+2Mr) < 0\\ \det(\mathscr{G}_2) &= \mathscr{G}_{11}\mathscr{G}_{22}-\mathscr{G}_{12}\mathscr{G}_{21} = \Sigma+2Mr > 0\\ \det(\mathscr{G}_3) &= \det(\mathscr{G}) = -\Sigma\,(\Sigma+2Mr)\sin^2(\theta) < 0\,,\end{aligned}$$

is negative definite and therefore Riemannian, we recognize that the coordinate τ is a proper temporal function. In the Carter–Penrose diagrams shown in Figure 3.3, we depict the constant-t and constant-r hypersurfaces in Boyer–Lindquist coordinates (left diagram) and the constant-τ and constant-r hypersurfaces in advanced Eddington–Finkelstein-type coordinates (right diagram). While the constant-t hypersurfaces become timelike inside the black hole in region II, the constant-τ hypersurfaces are always spacelike and smoothly continued across the horizons. We now express the Carter tetrad (3.4) and its metric dual (3.5) in terms of the advanced Eddington–Finkelstein-type coordinates (3.7) and (3.8), yielding

$$\boldsymbol{l}' = \frac{1}{\sqrt{2\,\Sigma\,|\Delta|}}\big([\Delta+4Mr]\,\partial_\tau+\Delta\,\partial_r+2a\,\partial_\phi\big)$$

$$\boldsymbol{n}' = \sqrt{\frac{|\Delta|}{2\,\Sigma}}(\partial_\tau-\partial_r)$$

$$\boldsymbol{m}' = \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau+\partial_\theta+\mathrm{i}\csc(\theta)\,\partial_\phi\big)$$

$$\overline{\boldsymbol{m}}' = -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau-\partial_\theta+\mathrm{i}\csc(\theta)\,\partial_\phi\big)$$

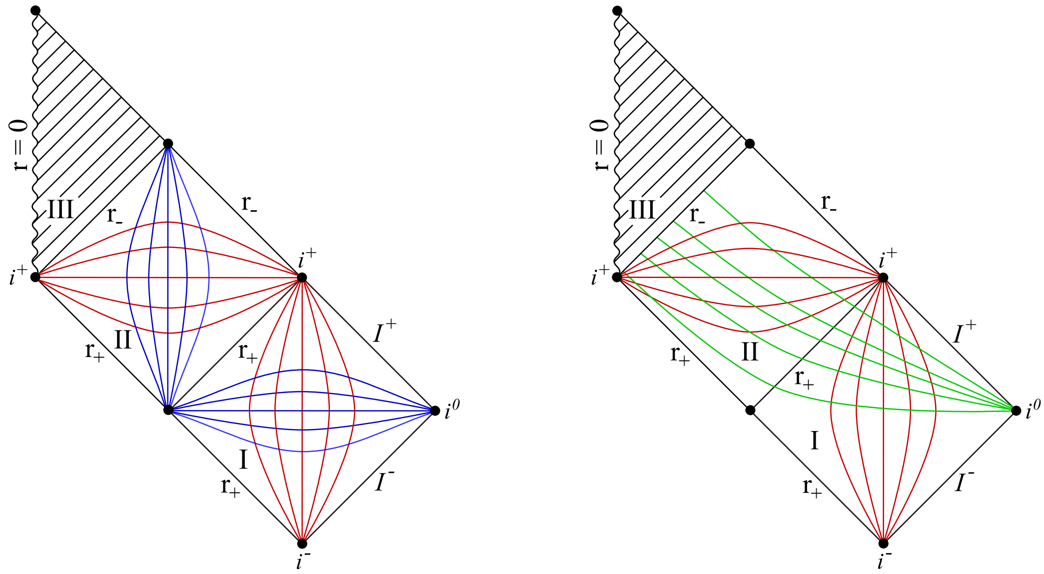

Figure 3.3: Carter–Penrose diagrams for the nonextreme Kerr geometry in Boyer–Lindquist coordinates (left) and advanced Eddington–Finkelstein-type coordinates (right). The blue lines represent constant-t hypersurfaces, the red lines constant-r hypersurfaces, and the green lines constant-τ hypersurfaces. The constant-t and constant-r hypersurfaces are restricted to either the exterior or the interior of the black hole. Their nature changes across the event horizon, i.e., spacelike hypersurfaces become timelike and vice versa. The constant-τ hypersurfaces (cut-off at the Cauchy horizon) are spacelike outside and inside the black hole and smooth across the event horizon.

and

$$\boldsymbol{l}'_{\mathrm{D}} = \sqrt{\frac{|\Delta|}{2\,\Sigma}}\,\mathrm{sign}(\Delta)\left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\right)$$

$$\boldsymbol{n}'_{\mathrm{D}} = \sqrt{\frac{|\Delta|}{2\,\Sigma}}\left(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\right)$$

$$\boldsymbol{m}'_{\mathrm{D}} = \frac{1}{\sqrt{2\,\Sigma}}\left(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\phi\right)$$

$$\overline{\boldsymbol{m}}'_{\mathrm{D}} = -\frac{1}{\sqrt{2\,\Sigma}}\left(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\phi\right).$$

Since the real-valued vector $\boldsymbol{l}'$ and its dual $\boldsymbol{l}'_{\mathrm{D}}$ are both still singular at the horizons, we apply another class III local Lorentz transformation (2.11) with parameters

$$\varsigma = \frac{\sqrt{|\Delta|}}{r_+} \quad \text{and} \quad \psi = 0\,,$$

which leads to the regular Carter tetrad

$$
\begin{aligned}
\boldsymbol{l}'' &= \frac{1}{\sqrt{2\,\Sigma}\,r_+}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi\big) \\
\boldsymbol{n}'' &= \frac{r_+}{\sqrt{2\,\Sigma}}(\partial_\tau - \partial_r) \\
\boldsymbol{m}'' &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big) \\
\overline{\boldsymbol{m}}'' &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big)
\end{aligned}
\tag{3.10}
$$

and to the dual cotetrad

$$
\begin{aligned}
\boldsymbol{l}''_{\mathrm{D}} &= \frac{\Delta}{\sqrt{2\,\Sigma}\,r_+}\left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\right) \\
\boldsymbol{n}''_{\mathrm{D}} &= \frac{r_+}{\sqrt{2\,\Sigma}}\big(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big) \\
\boldsymbol{m}''_{\mathrm{D}} &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\phi\big) \\
\overline{\boldsymbol{m}}''_{\mathrm{D}} &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\,\mathrm{d}\theta - \mathrm{i}\,[r^2 + a^2]\sin(\theta)\,\mathrm{d}\phi\big)\,.
\end{aligned}
$$

Substituting the latter into – and solving – the torsion-free first Maurer–Cartan equation of structure (2.10), we obtain regular spin coefficients for the nonextreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates

$$
\begin{aligned}
&\kappa'' = \sigma'' = \lambda'' = \nu'' = 0\,, \quad \pi'' = -\tau'' = \frac{\mathrm{i}a\sin(\theta)}{\sqrt{2\,\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \quad \mu'' = -\frac{r_+}{\sqrt{2\,\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \\
&\varrho'' = -\frac{\Delta}{\sqrt{2\,\Sigma}\,r_+\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \quad \alpha'' = -\beta'' = -\frac{1}{(2\,\Sigma)^{3/2}}\big([r^2 + a^2]\cot(\theta) - \mathrm{i}ra\sin(\theta)\big)\,, \\
&\gamma'' = -\frac{r_+}{2^{3/2}\sqrt{\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \quad \epsilon'' = \frac{r^2 - a^2 - 2\mathrm{i}a\cos(\theta)\,(r - M)}{2^{3/2}\sqrt{\Sigma}\,r_+\big(r - \mathrm{i}a\cos(\theta)\big)}\,.
\end{aligned}
\tag{3.11}
$$

3.3 The Massive Dirac Equation in the Analytically Extended Kerr Geometry

With the Newman–Penrose functions calculated in the previous section, we determine the explicit form of the massive Dirac equation in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates. Inserting the regular Carter tetrad (3.10) and the associated spin coefficients (3.11) into the PDE system (2.25), and applying the ansatz

$$\mathscr{F}_i(\tau,r,\theta,\phi) = \big(r - \mathrm{i}a\cos(\theta)\big)^{-1/2}\,\mathscr{H}_i(\tau,r,\theta,\phi)$$
$$\mathscr{G}_i(\tau,r,\theta,\phi) = \big(r + \mathrm{i}a\cos(\theta)\big)^{-1/2}\,\mathscr{J}_i(\tau,r,\theta,\phi)$$

for $i \in \{1,2\}$, we find

$$\begin{aligned}
&r_+(\partial_\tau - \partial_r)\mathscr{J}_1 + \big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi - 2^{-1}\cot(\theta)\big)\mathscr{J}_2 = \mathrm{i}m\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_1\\[1em]
&r_+^{-1}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi + r - M\big)\mathscr{J}_2\\
&\quad - \big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi + 2^{-1}\cot(\theta)\big)\mathscr{J}_1 = \mathrm{i}m\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_2\\[1em]
&r_+^{-1}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi + r - M\big)\mathscr{H}_1\\
&\quad - \big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi - 2^{-1}\cot(\theta)\big)\mathscr{H}_2 = \mathrm{i}m\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{J}_1\\[1em]
&r_+(\partial_\tau - \partial_r)\mathscr{H}_2 + \big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi + 2^{-1}\cot(\theta)\big)\mathscr{H}_1 = \mathrm{i}m\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{J}_2\,.
\end{aligned} \tag{3.12}$$

We note in passing that this system corresponds to the transformed Dirac equation

$$-\sqrt{\Sigma}\,\gamma^0\mathscr{P}^\dagger\mathscr{P}^{-1}\big(\gamma'^\mu[\nabla_\mu + \mathscr{P}\,\partial_\mu(\mathscr{P}^{-1})] + \mathrm{i}m\big)\psi' = \mathbf{0}\,, \tag{3.13}$$

in which

$$\psi' = \mathscr{P}\psi = (\mathscr{H}_1, \mathscr{H}_2, -\mathscr{J}_1, -\mathscr{J}_2)^{\mathrm{T}} \quad \text{and} \quad \gamma'^\mu = \mathscr{P}\gamma^\mu\mathscr{P}^{-1}\,, \tag{3.14}$$

where $\psi = (\mathscr{F}_1, \mathscr{F}_2, -\mathscr{G}_1, -\mathscr{G}_2)^{\mathrm{T}}$ and

$$\mathscr{P} := \mathrm{diag}\big(\sqrt{r - \mathrm{i}a\cos(\theta)}, \sqrt{r - \mathrm{i}a\cos(\theta)}, \sqrt{r + \mathrm{i}a\cos(\theta)}, \sqrt{r + \mathrm{i}a\cos(\theta)}\,\big)\,, \tag{3.15}$$

as well as $\gamma^0 := \mathrm{diag}(1,1,-1,-1)$. This will become relevant in the construction of the Hamiltonian formulation.

3.4 Separability of the Massive Dirac Equation in Horizon-penetrating Coordinates

We now show the separability of the system (3.12) employing the method used in [18]. To this end, we substitute the mode ansatz

$$\begin{aligned}
\mathscr{H}_i(\tau,r,\theta,\phi) &= e^{-\mathrm{i}(\omega\tau + k\phi)}\,\mathscr{Y}_i(r,\theta)\\
\mathscr{J}_i(\tau,r,\theta,\phi) &= e^{-\mathrm{i}(\omega\tau + k\phi)}\,\mathscr{Z}_i(r,\theta)\,,
\end{aligned} \tag{3.16}$$

where $\omega \in \mathbb{R}$ is the frequency and $k \in \mathbb{Z} + 1/2$ the wave number, into the Dirac equation (3.12) and obtain the PDE system

$$r_+^{-1}\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,[\Delta + 4Mr] - 2\mathrm{i}ak\big)\mathscr{Y}_1 + \big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{Y}_2$$
$$= \mathrm{i}m\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{X}_1$$

$$r_+(\partial_r + \mathrm{i}\omega)\mathscr{Y}_2 - \big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{Y}_1 = -\mathrm{i}m\big(r - \mathrm{i}a\cos(\theta)\big)\mathscr{X}_2$$

$$r_+^{-1}\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,[\Delta + 4Mr] - 2\mathrm{i}ak\big)\mathscr{X}_2 - \big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{X}_1$$
$$= \mathrm{i}m\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{Y}_2$$

$$r_+(\partial_r + \mathrm{i}\omega)\mathscr{X}_1 + \big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{X}_2 = -\mathrm{i}m\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{Y}_1$$

for the functions $\mathscr{Y}_i$ and $\mathscr{X}_i$, which depends only on the radial and polar angle variables. This system is separable by means of the product ansatz

$$\begin{aligned}
\mathscr{Y}_1(r,\theta) &= \mathscr{R}_+(r)\,\mathscr{T}_+(\theta)\\
\mathscr{Y}_2(r,\theta) &= \mathscr{R}_-(r)\,\mathscr{T}_-(\theta)\\
\mathscr{X}_1(r,\theta) &= \mathscr{R}_-(r)\,\mathscr{T}_+(\theta)\\
\mathscr{X}_2(r,\theta) &= \mathscr{R}_+(r)\,\mathscr{T}_-(\theta)\,,
\end{aligned} \tag{3.17}$$

yielding the first-order radial ODE system

$$\begin{aligned}
&\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,[\Delta + 4Mr] - 2\mathrm{i}ak\big)\,\mathscr{R}_+ = r_+(\xi + \mathrm{i}mr)\,\mathscr{R}_-\\
&r_+(\partial_r + \mathrm{i}\omega)\,\mathscr{R}_- = (\xi - \mathrm{i}mr)\,\mathscr{R}_+
\end{aligned}$$

and the first-order angular ODE system

$$\begin{aligned}
\big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\,\mathscr{T}_- &= -\big(\xi - ma\cos(\theta)\big)\,\mathscr{T}_+\\
\big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\,\mathscr{T}_+ &= \big(\xi + ma\cos(\theta)\big)\,\mathscr{T}_-\,,
\end{aligned}$$

with ξ being the constant of separation. Defining the functions $\widetilde{\mathscr{R}}_+ := \sqrt{|\Delta|}\,\mathscr{R}_+$ and $\widetilde{\mathscr{R}}_- := r_+\,\mathscr{R}_-$, the radial system may be transformed into the more symmetric form

$$\begin{aligned}
&\big(\Delta\,\partial_r - \mathrm{i}\omega\,[\Delta + 4Mr] - 2\mathrm{i}ak\big)\,\widetilde{\mathscr{R}}_+ = \sqrt{|\Delta|}\,(\xi + \mathrm{i}mr)\,\widetilde{\mathscr{R}}_-\\
&\Delta(\partial_r + \mathrm{i}\omega)\,\widetilde{\mathscr{R}}_- = \mathrm{sign}(\Delta)\sqrt{|\Delta|}\,(\xi - \mathrm{i}mr)\,\widetilde{\mathscr{R}}_+\,.
\end{aligned} \tag{3.18}$$

For the analysis of the radial asymptotics at infinity, the event horizon, and the Cauchy horizon, it is, however, advantageous to work with the matrix representation

$$\partial_r\widetilde{\mathscr{R}} = U(r)\,\widetilde{\mathscr{R}}\,, \tag{3.19}$$

where $\widetilde{\mathscr{R}} := (\widetilde{\mathscr{R}}_+, \widetilde{\mathscr{R}}_-)^{\mathrm{T}}$ and

$$U(r) := \frac{1}{\Delta}\begin{pmatrix} \mathrm{i}\big(\omega\,[\Delta + 4Mr] + 2ak\big) & \sqrt{|\Delta|}\,(\xi + \mathrm{i}mr)\\ \mathrm{sign}(\Delta)\sqrt{|\Delta|}\,(\xi - \mathrm{i}mr) & -\mathrm{i}\omega\,\Delta \end{pmatrix}.$$

Furthermore, in this representation, the singular points of the radial system can be calculated straightforwardly as shown in [24]. We find that U has singularities of rank $\mu = 0$ at $r = r_\pm$, that is, the event and the Cauchy horizon are regular singular points of (3.18), even though, there, both the coordinate system and the tetrad frame are nonsingular. As a consequence, the regions outside the event horizon $r_+ < r < \infty$, between the event and the Cauchy horizon $r_- < r < r_+$, and inside the Cauchy horizon $0 < r < r_-$ have to be considered separately. We emphasize that this is a particularity of the specific mode ansatz (3.16). Besides, the matrix representation of the angular system is given by

$$A(\theta)\mathscr{T} = \xi \mathscr{T} \tag{3.20}$$

with $\mathscr{T} := (\mathscr{T}_+, \mathscr{T}_-)^{\mathrm{T}}$ and

$$A(\theta) := \begin{pmatrix} ma\cos(\theta) & -\partial_\theta - 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta) \\ \partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta) & -ma\cos(\theta) \end{pmatrix}. \tag{3.21}$$

3.5 Asymptotic Radial Solutions and Error Estimates

We study the asymptotic solutions of the radial ODE system (3.19) and the decay properties of the corresponding errors at infinity, at the event horizon, and at the Cauchy horizon. In the present work, these results serve as a basis for an elementary description of the scattering process of Dirac particles by the gravitational field of a rotating Kerr black hole on the one hand, and on the other hand for the computation of the resolvent of the Dirac Hamiltonian.

3.5.1 Asymptotic Analysis of the Radial Solution at Infinity

Following the approach of [39], we derive the asymptotic solution of the radial system (3.19) for $r \to \infty$ and examine the decay of the associated error. We begin by expressing (3.19) in terms of the Regge–Wheeler coordinate

$$\partial_{r_\star}\widetilde{\mathscr{R}} = T(r_\star)\,\widetilde{\mathscr{R}}\,, \tag{3.22}$$

where $T(r_\star) := \Delta/(r^2 + a^2)\,U(r_\star)$. Employing the diagonal matrix $S := D^{-1}TD = \mathrm{diag}(\lambda_1, \lambda_2)$, with D being the diagonalization matrix and $\lambda_{1/2}$ the eigenvalues of T, this system can be written in the form

$$\partial_{r_\star}(D^{-1}\widetilde{\mathscr{R}}) = [S - D^{-1}(\partial_{r_\star}D)](D^{-1}\widetilde{\mathscr{R}})\,.$$

Hence, using the ansatz

$$\widetilde{\mathscr{R}}(r_\star) = D(r_\star)\begin{pmatrix} e^{\mathrm{i}\Phi_+(r_\star)}\,\mathfrak{f}_1(r_\star) \\ e^{-\mathrm{i}\Phi_-(r_\star)}\,\mathfrak{f}_2(r_\star) \end{pmatrix},$$

we obtain an ODE system for $\mathfrak{f} := (\mathfrak{f}_1, \mathfrak{f}_2)^{\mathrm{T}}$ given by

$$\partial_{r_\star}\mathfrak{f} = [S - W^{-1}D^{-1}(\partial_{r_\star}D)\,W - W^{-1}\partial_{r_\star}W\,]\,\mathfrak{f}\,,$$

where $W := \mathrm{diag}(e^{\mathrm{i}\Phi_+}, e^{-\mathrm{i}\Phi_-})$. We determine the functions $\Phi_\pm$ by imposing the condition $S = W^{-1}\partial_{r_\star}W$, i.e.,

$$\partial_{r_\star}\Phi_+ = -\mathrm{i}\lambda_1 \quad \text{and} \quad \partial_{r_\star}\Phi_- = \mathrm{i}\lambda_2\,, \tag{3.23}$$

which yields

$$\partial_{r_\star}\mathfrak{f} = -W^{-1}D^{-1}(\partial_{r_\star}D)\,W\,\mathfrak{f}\,. \tag{3.24}$$

Lemma 3.5.1. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (3.22) for $|\omega| > m$ is asymptotically as $r \to \infty$ of the oscillatory form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_\infty(r_\star) + E_\infty(r_\star) = D_\infty \begin{pmatrix} \mathfrak{f}^{(1)}_\infty \, e^{\mathrm{i}\phi_+(r_\star)} \\ \mathfrak{f}^{(2)}_\infty \, e^{-\mathrm{i}\phi_-(r_\star)} \end{pmatrix} + E_\infty(r_\star)\,, \tag{3.25}$$

where

$$D_\infty := \begin{pmatrix} \cosh(\Omega) & \sinh(\Omega) \\ \sinh(\Omega) & \cosh(\Omega) \end{pmatrix} \quad \text{with} \quad \Omega := \frac{1}{4} \ln\left(\frac{\omega - m}{\omega + m}\right), \tag{3.26}$$

the functions

$$\phi_\pm(r_\star) := \operatorname{sign}(\omega)\left[-\sqrt{\omega^2 - m^2}\, r_\star + M\left(\pm 2\omega - \frac{m^2}{\sqrt{\omega^2 - m^2}}\right)\ln(r_\star)\right] \tag{3.27}$$

are the asymptotic phases, and $\mathfrak{f}_\infty := \big(\mathfrak{f}^{(1)}_\infty, \mathfrak{f}^{(2)}_\infty\big)^{\mathrm{T}} \neq \mathbf{0}$ is a vector-valued constant. The error E_∞ has polynomial decay

$$\|E_\infty(r_\star)\| \leq \frac{a}{r_\star} \tag{3.28}$$

for a suitable constant $a \in \mathbb{R}_{>0}$. In the case $|\omega| < m$, the nontrivial solution $\widetilde{\mathscr{R}}$ has both contributions that show exponential decay $\sim e^{-\sqrt{m^2-\omega^2}\, r_\star}$ and exponential growth $\sim e^{\sqrt{m^2-\omega^2}\, r_\star}$.

Proof. In the limit $r \to \infty$, the matrix T defined in (3.22) converges to

$$T_\infty := \lim_{r_\star \to \infty} T = \mathrm{i} \begin{pmatrix} \omega & m \\ -m & -\omega \end{pmatrix}. \tag{3.29}$$

Moreover, it has a regular expansion in powers of $1/r_\star$, i.e., $T = T_\infty + \mathcal{O}(1/r_\star)$. Accordingly, both the diagonal matrix S and the diagonalization matrix D also have regular expansions in powers of $1/r_\star$. The eigenvalues of (3.29) read

$$\lambda^{(0)}_{1/2} = \mp \operatorname{sign}(\omega) \times \begin{cases} \mathrm{i}\sqrt{\omega^2 - m^2} \in \mathbb{C} & \text{for} \quad |\omega| > m \\ \sqrt{m^2 - \omega^2} \in \mathbb{R} & \text{for} \quad |\omega| < m\,. \end{cases} \tag{3.30}$$

For $|\omega| > m$, the diagonalization matrix $D_\infty := \lim_{r_\star \to \infty} D$ associated with (3.29) is given by expression (3.26). This can be easily verified by direct calculation. Furthermore, with the asymptotic first-order eigenvalues of T

$$\lambda^{(1)}_{1/2} = \operatorname{sign}(\omega)\left[\mp \mathrm{i}\sqrt{\omega^2 - m^2} + \frac{\mathrm{i}M}{r_\star}\left(2\omega \mp \frac{m^2}{\sqrt{\omega^2 - m^2}}\right)\right],$$

we can solve (3.23) by simple integration and obtain (3.27) as the asymptotic phases. We point out that for the determination of the asymptotic phases, it is of paramount importance to take the first-order terms of the asymptotic eigenvalues into account because they yield contributions to the radial solution that do not decay at infinity. However, the first-order terms play no role in the computation of D_∞, as the corresponding contributions can be absorbed into the error E_∞. Next, since for $r_\star$ sufficiently close to infinity the Hilbert–Schmidt norms of D^{-1} and $\partial_{r_\star} D$ are bounded from above by

$$\|D^{-1}\|_{\mathrm{HS}} \leq c \quad \text{and} \quad \|\partial_{r_\star} D\|_{\mathrm{HS}} \leq \frac{d}{r_\star^2}\,,$$

where both c and d denote positive constants, the $\mathbb{C}^2$-norm of (3.24) may be estimated by

$$\|\partial_{r_\star}\mathfrak{f}\| \leq 2\,\|D^{-1}\|_{\mathrm{HS}}\cdot\|\partial_{r_\star}D\|_{\mathrm{HS}}\cdot\|\mathfrak{f}\| \leq \frac{2cd}{r_\star^2}\,\|\mathfrak{f}\| \tag{3.31}$$

with $\|W\|_{\mathrm{HS}} = \|W^{-1}\|_{\mathrm{HS}} = \sqrt{2}$. Applying the triangle and the Cauchy–Schwarz inequality, one can derive the following inequality

$$\big|\partial_{r_\star}\|\mathfrak{f}\|\big| = \frac{|\partial_{r_\star}\langle\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|} = \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle + \langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|} \leq \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle| + |\langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|}$$

$$= \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle|}{\|\mathfrak{f}\|} \leq \frac{\|\mathfrak{f}\|\cdot\|\partial_{r_\star}\mathfrak{f}\|}{\|\mathfrak{f}\|} = \|\partial_{r_\star}\mathfrak{f}\|\,.$$

Using this inequality in (3.31), we find

$$\big|\partial_{r_\star}\|\mathfrak{f}\|\big| \leq \frac{2cd}{r_\star^2}\,\|\mathfrak{f}\|\,. \tag{3.32}$$

We note that $\|\mathfrak{f}\| \neq 0$ because $\widetilde{\mathscr{R}}$ has to be nontrivial. Now, integrating (3.32) with respect to the Regge–Wheeler coordinate from r_0 to $r_\star$ and employing the triangle inequality for integrals gives for all $0 < r_0 \leq r_\star$

$$\left|\int_{r_0}^{r_\star}\partial_{r'_\star}\ln\|\mathfrak{f}\|\,\mathrm{d}r'_\star\right| \leq \int_{r_0}^{r_\star}\big|\partial_{r'_\star}\ln\|\mathfrak{f}\|\big|\,\mathrm{d}r'_\star \leq 2cd\int_{r_0}^{r_\star}\frac{\mathrm{d}r'_\star}{r'^2_\star}\,,$$

and hence

$$\left|\ln\|\mathfrak{f}\|\Big|_{r_0}^{r_\star}\right| \leq -\frac{2cd}{r'_\star}\Big|_{r_0}^{r_\star}\,.$$

Consequently, since $0 < 2cd/r'_\star\big|_{r_\star}^{r_0} < \infty$, there exists a constant $N > 0$ such that

$$\frac{1}{N} \leq \|\mathfrak{f}\| \leq N\,. \tag{3.33}$$

Combining this with (3.31), we obtain for sufficiently large values of $r_\star$

$$\|\partial_{r_\star}\mathfrak{f}\| \leq \frac{b}{r_\star^2}\,, \tag{3.34}$$

where $b := 2cdN$, which implies that $\mathfrak{f}$ is integrable and, according to (3.33), has a finite nonzero limit $\mathfrak{f}_\infty := \lim_{r_\star\to\infty}\mathfrak{f}(r_\star) \neq \mathbf{0}$. Integrating (3.34) from $r_\star$ to ∞ and again using the triangle inequality for integrals yields the error estimate

$$\|E_{\mathfrak{f}}\| = \|\mathfrak{f} - \mathfrak{f}_\infty\| = \left\|\int_{r_\star}^{\infty}\partial_{r'_\star}\mathfrak{f}\,\mathrm{d}r'_\star\right\| \leq \int_{r_\star}^{\infty}\|\partial_{r'_\star}\mathfrak{f}\|\,\mathrm{d}r'_\star \leq \frac{b}{r_\star}\,. \tag{3.35}$$

The polynomial $1/r_\star$-decay of the error E_∞ in (3.28) follows directly from the estimate (3.35) and the fact that the matrices T and D – and therefore their eigenvalues – have regular expansions in powers of

$1/r_\star$. This can be verified via the following short computation. Inserting the expansion of D and of the eigenvalues of T into the $\mathbb{C}^2$-norm of the error E_∞, we find

$$\|E_\infty\| = \|\widetilde{\mathscr{R}} - \widetilde{\mathscr{R}}_\infty\| = \left\| D \begin{pmatrix} e^{\mathrm{i}\Phi_+}\, \mathfrak{f}_1 \\ e^{-\mathrm{i}\Phi_-}\, \mathfrak{f}_2 \end{pmatrix} - D_\infty \begin{pmatrix} \mathfrak{f}_\infty^{(1)}\, e^{\mathrm{i}\phi_+} \\ \mathfrak{f}_\infty^{(2)}\, e^{-\mathrm{i}\phi_-} \end{pmatrix} \right\|$$

$$= \left\| \left[D_\infty + \mathcal{O}\left(\frac{1}{r_\star}\right) \right] \begin{pmatrix} e^{\mathrm{i}\phi_+ + \mathcal{O}(1/r_\star)}\, \mathfrak{f}_1 \\ e^{-\mathrm{i}\phi_- + \mathcal{O}(1/r_\star)}\, \mathfrak{f}_2 \end{pmatrix} - D_\infty \begin{pmatrix} \mathfrak{f}_\infty^{(1)}\, e^{\mathrm{i}\phi_+} \\ \mathfrak{f}_\infty^{(2)}\, e^{-\mathrm{i}\phi_-} \end{pmatrix} \right\| .$$

Subsequently, we collect terms of the order $\mathcal{O}(1/r_\star)$ and apply the triangle inequality as well as the submultiplicativity property of operator norms, which results in the estimate

$$\|E_\infty\| = \left\| D_\infty W_\infty \left(\mathfrak{f} - \mathfrak{f}_\infty\right) + \mathcal{O}\left(\frac{1}{r_\star}\right) \right\| \leq \|D_\infty\|_{\text{HS}} \cdot \|W_\infty\|_{\text{HS}} \cdot \|E_{\mathfrak{f}}\| + \left\| \mathcal{O}\left(\frac{1}{r_\star}\right) \right\| ,$$

where $W_\infty := \operatorname{diag}(e^{\mathrm{i}\phi_+}, e^{-\mathrm{i}\phi_-})^{\mathrm{T}}$. Since D_∞ is a constant nonzero matrix, the associated Hilbert–Schmidt norm yields a positive constant. Moreover, one immediately verifies that $\|W_\infty\|_{\text{HS}} = \sqrt{2}$. From this and (3.35), we can thus conclude that the error E_∞ has the polynomial decay $\|E_\infty\| \leq a/r_\star$, with a being a positive constant. We note in passing that in the case $|\omega| < m$, it is obvious from the zero-order eigenvalues (3.30) that the nontrivial solution $\widetilde{\mathscr{R}}$ has contributions with exponential decay and exponential growth. ■

3.5.2 Asymptotic Analysis of the Radial Solution at the Event Horizon

In order to determine the asymptotics of (3.22) at the event horizon, we employ the solution ansatz

$$\widetilde{\mathscr{R}} = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(+)}\right) r_\star}\, \mathfrak{g}_1(r_\star) \\ \mathfrak{g}_2(r_\star) \end{pmatrix},$$

where $\Omega_{\text{Kerr}}^{(+)} := a/(2Mr_+)$ is the angular velocity of the event horizon, and therefore obtain an ODE system for $\mathfrak{g} := (\mathfrak{g}_1, \mathfrak{g}_2)^{\mathrm{T}}$ given by

$$\partial_{r_\star} \mathfrak{g} = -\frac{\mathrm{i}}{r^2 + a^2} \left[2k \left(2M\Omega_{\text{Kerr}}^{(+)}\, r - a\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \operatorname{sign}(\Delta) \sqrt{|\Delta|} \right.$$

$$\left. \times \begin{pmatrix} \sqrt{|\Delta|}\left(\omega + 2k\Omega_{\text{Kerr}}^{(+)}\right) & e^{-2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(+)}\right) r_\star} \operatorname{sign}(\Delta)\, (\mathrm{i}\xi - mr) \\ e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(+)}\right) r_\star} (\mathrm{i}\xi + mr) & \sqrt{|\Delta|}\, \omega \end{pmatrix} \right] \mathfrak{g} \,. \tag{3.36}$$

Since in the limit $r \searrow r_+$ the right hand side of this system vanishes, we find the asymptotic solution $\mathfrak{g}_{r_+} := \lim_{r \searrow r_+} \mathfrak{g} = \text{const.}$

Lemma 3.5.2. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (3.22) is asymptotically as $r \searrow r_+$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_{r_+}(r_\star) + E_{r_+}(r_\star) = \begin{pmatrix} \mathfrak{g}_{r_+}^{(1)}\, e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(+)}\right) r_\star} \\ \mathfrak{g}_{r_+}^{(2)} \end{pmatrix} + E_{r_+}(r_\star)\,, \tag{3.37}$$

with

$$\mathfrak{g}_{r_+} := \left(\mathfrak{g}_{r_+}^{(1)}, \mathfrak{g}_{r_+}^{(2)}\right)^{\mathrm{T}} = \text{const.} \neq \mathbf{0}$$

and an error E_{r_+} that has exponential decay

$$\|E_{r_+}(r_\star)\| \le p\, e^{q r_\star}$$

for r sufficiently close to r_+ and suitable constants $p, q \in \mathbb{R}_{>0}$.

Proof. For $r \searrow r_+$, as $r \sim r_+ + e^{2qr_\star}$ with $q := (r_+ - r_-)/(4Mr_+) \in \mathbb{R}_{>0}$, the right hand side of (3.36) is of the order $\mathcal{O}(e^{qr_\star})$. Hence, there exists a constant $p' \in \mathbb{R}_{>0}$ such that for $r_\star$ sufficiently close to $-\infty$

$$\|\partial_{r_\star}\mathfrak{g}\| \le p' e^{qr_\star}\|\mathfrak{g}\|\,. \tag{3.38}$$

From this inequality, following the method of proof of the previous subsection, we can derive the estimate

$$\left|\ln\|\mathfrak{g}\|\Big|_{r_\star}^{r_0}\right| \le \frac{p'}{q}\, e^{qr'_\star}\Big|_{r_\star}^{r_0}$$

for all $r_\star \le r_0$, and since $e^{qr'_\star}\big|_{r_\star}^{r_0}$ is positive, there is a constant $N' > 0$ such that the $\mathbb{C}^2$-norm of $\mathfrak{g}$ is bounded by

$$\frac{1}{N'} \le \|\mathfrak{g}\| \le N'\,.$$

Combining this bound with inequality (3.38) yields

$$\|\partial_{r_\star}\mathfrak{g}\| \le p\, e^{qr_\star}, \tag{3.39}$$

where $p := p'N'$. Altogether, this implies that $\mathfrak{g}$ is integrable and has a finite nonzero limit for $r_\star \to -\infty$. Integrating (3.39) with respect to the Regge–Wheeler coordinate from $-\infty$ to $r_\star$ and applying the triangle inequality for integrals results in

$$\|E_{\mathfrak{g}}\| = \|\mathfrak{g} - \mathfrak{g}_{r_+}\| = \left\|\int_{-\infty}^{r_\star} \partial_{r'_\star}\mathfrak{g}\, \mathrm{d}r'_\star\right\| \le \int_{-\infty}^{r_\star} \|\partial_{r'_\star}\mathfrak{g}\|\, \mathrm{d}r'_\star \le p\, e^{qr_\star}, \tag{3.40}$$

which proves the exponential decay of the error $E_{\mathfrak{g}}$. The exponential decay of the error E_{r_+} follows directly from (3.40). ■

3.5.3 Asymptotic Analysis of the Radial Solution at the Cauchy Horizon

Similar to the analysis of the radial asymptotics at the event horizon, we begin with a solution ansatz of the form

$$\widetilde{\mathscr{R}} = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(-)}\right) r_\star}\, \mathfrak{h}_1(r_\star) \\ \mathfrak{h}_2(r_\star) \end{pmatrix},$$

where $\Omega_{\text{Kerr}}^{(-)} := a/(2Mr_-)$ is the angular velocity of the Cauchy horizon, and substitute it into the system (3.22). This leads to an ODE system for $\mathfrak{h} := (\mathfrak{h}_1, \mathfrak{h}_2)^{\text{T}}$ reading

$$\partial_{r_\star} \mathfrak{h} = -\frac{\mathrm{i}}{r^2+a^2}\Bigg[2k\left(2M\Omega_{\text{Kerr}}^{(-)} r - a\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \operatorname{sign}(\Delta)\sqrt{|\Delta|}$$

$$\times \begin{pmatrix} \sqrt{|\Delta|}\left(\omega + 2k\Omega_{\text{Kerr}}^{(-)}\right) & e^{-2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(-)}\right) r_\star} \operatorname{sign}(\Delta)\left(\mathrm{i}\xi - mr\right) \\ e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(-)}\right) r_\star}(\mathrm{i}\xi + mr) & \sqrt{|\Delta|}\,\omega \end{pmatrix}\Bigg]\mathfrak{h}\,.$$

In the limit $r \searrow r_-$, the square bracket on the right hand side vanishes, giving rise to the asymptotic radial solution $\mathfrak{h}_{r_-} := \lim_{r \searrow r_-} \mathfrak{h} = \text{const.}$

Lemma 3.5.3. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (3.22) is asymptotically as $r \searrow r_-$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_{r_-}(r_\star) + E_{r_-}(r_\star) = \begin{pmatrix} \mathfrak{h}_{r_-}^{(1)}\, e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(-)}\right) r_\star} \\ \mathfrak{h}_{r_-}^{(2)} \end{pmatrix} + E_{r_-}(r_\star)\,,$$

with

$$\mathfrak{h}_{r_-} := \left(\mathfrak{h}_{r_-}^{(1)}, \mathfrak{h}_{r_-}^{(2)}\right)^{\text{T}} = \text{const.} \neq \mathbf{0}$$

and an error E_{r_-} that has exponential decay

$$\|E_{r_-}(r_\star)\| \leq u\, e^{-v r_\star}$$

for r sufficiently close to r_- and suitable constants $u, v \in \mathbb{R}_{>0}$.

Proof. The proof of this lemma is identical to the proof of Lemma 3.5.2. ■

3.6 Angular Eigenfunctions and Eigenvalues

We discuss the set of eigenfunctions as well as the eigenvalue spectrum of the angular ODE system (3.20), which in its decoupled second-order form is known as the massive Chandrasekhar–Page equation [19]

$$\Bigg[\partial_{\theta\theta} + \left(\cot(\theta) \pm \frac{ma\sin(\theta)}{\xi \pm ma\cos(\theta)}\right)\partial_\theta - \frac{\csc^2(\theta)}{2}\left(1 - \frac{3\cos^2(\theta)}{2}\right)$$

$$-\left(\frac{\cot(\theta)}{2} \mp a\omega\sin(\theta)\right)^2 - \left(\frac{\cot(\theta)}{2} \pm k\csc(\theta)\right)^2 - 2ka\omega + \xi^2$$

$$\pm \frac{ma\sin(\theta)}{\xi \pm ma\cos(\theta)}\left(\frac{\cot(\theta)}{2} \pm a\omega\sin(\theta) \pm k\csc(\theta)\right) - m^2a^2\cos^2(\theta)\Bigg]\mathscr{T}_\pm = 0\,.$$

In the limit $|a| \to 0$, the solutions of this equation reduce to the spin-weighted spherical harmonics for the spin-$1/2$ case [58]. For nonzero angular momenta, the solutions are usually referred to as spin-$1/2$

spheroidal harmonics. A good introduction and a compilation of the basic properties of these functions can be found in the recent paper [29]. For the purposes of the present work and the constructions in [48], it is, however, only of importance that the matrix-valued differential operator on the left hand side of (3.20) has a spectral decomposition with smooth eigenfunctions and discrete, nondegenerate eigenvalues, which was already proven in [37, 39]. In the following, we state the relevant results.

Proposition 3.6.1. *For any $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, the differential operator in* (3.20) *has a complete set of orthonormal eigenfunctions $(\mathscr{T}_n)_{n\in\mathbb{Z}}$ in $L^2\big((0,\pi), \sin(\theta)\,\mathrm{d}\theta\big)^2$ that are bounded and smooth away from the poles. The corresponding eigenvalues ξ_n are real-valued and nondegenerate, and can thus be ordered as $\xi_n < \xi_{n+1}$. Both the eigenfunctions and the eigenvalues depend smoothly on ω.*

3.7 Scattering of Massive Dirac Particles by the Gravitational Field of a Rotating Kerr Black Hole

As an application, we study the scattering of massive Dirac particles by the gravitational field of a rotating Kerr black hole from the point of view of an observer described by horizon-penetrating advanced Eddington–Finkelstein-type coordinates. For this purpose, we consider Dirac particles that emerge from spacelike infinity and propagate toward the black hole's event horizon, where they are either reflected or transmitted. We compute the net current of Dirac particles at infinity as well as at the event horizon, and further derive a condition for the reflection and transmission coefficients. Using the asymptotic radial solutions at infinity (3.25) and the event horizon (3.37), we impose boundary conditions specifying an incident wave of unit amplitude at infinity, a reflected wave of amplitude $A_{\omega,m}$ at infinity, and a transmitted wave of amplitude $B_{\omega,m}$ at the event horizon, yielding

$$\widetilde{\mathscr{R}}_{\text{scat.}}(r \to \infty) \sim D_\infty \begin{pmatrix} e^{\mathrm{i}\phi_+(r_\star)} \\ A_{\omega,m}\, e^{-\mathrm{i}\phi_-(r_\star)} \end{pmatrix} \tag{3.41}$$

and

$$\widetilde{\mathscr{R}}_{\text{scat.}}(r \searrow r_+) \sim \begin{pmatrix} B_{\omega,m}\, e^{2\mathrm{i}\left(\omega + k\Omega^{(+)}_{\text{Kerr}}\right) r_\star} \\ 0 \end{pmatrix}. \tag{3.42}$$

We point out that these boundary conditions are chosen in conformity with the physical requirement that classically no particles can emerge from the event horizon. Besides, only the branch $|\omega| > m$ is considered because free particles at infinity must have energies that exceed – or at least equal – their rest energies. Next, assuming the normalization condition

$$\|\mathscr{T}_+(\theta)\|^2 + \|\mathscr{T}_-(\theta)\|^2 = 1$$

for the angular eigenfunctions (cf. Proposition 3.6.1), the radial Dirac current becomes

$$J^r = \sqrt{2}\,{\sigma^r}_{AB'}\big(P^A\,\overline{P}^{B'} + Q^A\,\overline{Q}^{B'}\big) = \frac{1}{r_+\Sigma}\big(\operatorname{sign}(\Delta)\,\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2\big)$$

with the radial Infeld–van der Waerden symbol

$${\sigma^r}_{AB'} = \begin{pmatrix} l^r & m^r \\ \overline{m}^r & n^r \end{pmatrix}_{AB'} = \frac{1}{\sqrt{2}\,\Sigma\, r_+}\begin{pmatrix} \Delta & 0 \\ 0 & -r_+^2 \end{pmatrix}_{AB'}.$$

As this current has jump discontinuities at the event and the Cauchy horizon, it has to be evaluated separately in each of the respective three regions. For the scattering problem at hand, however, it suffices to consider the exterior region of the black hole $r_+ < r < \infty$, where the radial Dirac current reads

$$J^r = \frac{1}{r_+ \Sigma} \big(\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2 \big) . \tag{3.43}$$

From the radial system (3.19) and its complex conjugate, we obtain the relation

$$\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2 = \text{const.}$$

via simple algebraic manipulations. Substituting this relation into the radial Dirac current (3.43), we derive the conserved net current of Dirac particles

$$\partial_\tau N = \int_0^{2\pi} \int_0^{\pi} J^r \sqrt{|\det(\boldsymbol{g})|} \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4\pi}{r_+} \big(\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2 \big) = \text{const.} \, ,$$

where N is the total number of Dirac particles and $\sqrt{|\det(\boldsymbol{g})|} = \Sigma \sin(\theta)$. Defining the reflection and transmission coefficients

$$R_{\omega,m} := |A_{\omega,m}|^2 \quad \text{and} \quad T_{\omega,m} := |B_{\omega,m}|^2 \, ,$$

and employing the asymptotic radial solutions (3.41) and (3.42), the net current at infinity and at the event horizon results in

$$\partial_\tau N_{|r \to \infty} = \frac{4\pi}{r_+} \left(1 - R_{\omega,m}\right)$$

and

$$\partial_\tau N_{|r \searrow r_+} = \frac{4\pi}{r_+} T_{\omega,m} \, .$$

From the constancy of the net current and these expressions, we infer that

$$R_{\omega,m} + T_{\omega,m} = 1 \, ,$$

which proves that superradiance cannot occur because the reflection coefficient is always less than unity. Furthermore, we have shown that the net current across the event horizon is always positive. These results are in agreement with those found in the accordant analysis of this scattering problem performed in terms of Boyer–Lindquist coordinates (see [19] and references therein).

CHAPTER 4

Essential Self-adjointness of the Dirac Hamiltonian for a Class of Nonuniformly Elliptic Mixed Initial-boundary Value Problems in Lorentzian manifolds

We study the essential self-adjointness of the Dirac Hamiltonian for a particular class of nonuniformly elliptic mixed initial-boundary value problems for the Dirac equation in smooth asymptotically flat Lorentzian manifolds. The results obtained here serve as mathematical tools in the construction of the integral spectral representation of the Dirac propagator in the nonextreme Kerr geometry in horizon-penetrating coordinates presented in the subsequent chapter. In the following, we define the setting.

We let $(\mathrm{M}, \boldsymbol{g})$ be a smooth oriented and time-oriented Lorentzian spin manifold of dimension $d \geq 3$ with boundary $\partial\mathrm{M}$. Moreover, we make the following assumptions:

(i) The manifold $(\mathrm{M}, \boldsymbol{g})$ is asymptotically flat with one asymptotic end.

(ii) There exists a Killing field $\boldsymbol{K}$ that is tangential to – and timelike on – the boundary $\partial\mathrm{M}$. Away from the boundary, this Killing field may change smoothly into being null or spacelike.

(iii) The integral curves γ of $\boldsymbol{K}$, which are defined by the differential equation

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \boldsymbol{K}\big(\gamma(t)\big)\,,$$

exist for all $t \in \mathbb{R}$.

(iv) There exists a spacelike hypersurface N with compact boundary $\partial\mathrm{N}$, which has the property that every integral curve γ in (iii) intersects it exactly once.

These assumptions imply that M and its boundary $\partial\mathrm{M}$ have the respective product structures

$$\mathrm{M} = \mathbb{R} \times \mathrm{N} \quad \text{and} \quad \partial\mathrm{M} = \mathbb{R} \times \partial\mathrm{N}\,. \tag{4.1}$$

They also entail that the metric $\boldsymbol{g}$ is smooth up to the boundary $\partial\mathrm{M}$, thus inducing a $(d-2)$-dimensional Riemannian metric on $\partial\mathrm{N}$. We point out that in the case that $\partial\mathrm{N}$ is empty and N is complete, the product structure (4.1) implies that $(\mathrm{M}, \boldsymbol{g})$ is globally hyperbolic. Furthermore, if $\boldsymbol{K}$ is timelike, the manifold is stationary. In order to get a better geometric understanding of the above setting, we now construct a suitable coordinate system. Choosing the parametrization of each integral curve γ such that $\gamma(0) \in \mathrm{N}$, we obtain a global time function T defined by

$$T \colon \mathrm{M} \to \mathbb{R} \quad \text{with} \quad T\big(\gamma(t)\big) = t\,. \tag{4.2}$$

The level sets of this time function give rise to a foliation of M by spacelike hypersurfaces $\mathrm{N}_t := T^{-1}(t)$ with $\mathrm{N}_0 = \mathrm{N}$. Moreover, the integral curves yield the isometries

$$\Phi_t \colon \mathrm{N} \to \mathrm{N}_t\,, \quad \Phi_t\big(\gamma(0)\big) = \gamma(t)\,.$$

Selecting coordinates $\boldsymbol{x}$ on N, the mapping $\boldsymbol{x} \circ \Phi_t^{-1}$ provides coordinates on N_t. We complement the former coordinate system by the time function t defined in (4.2), finding coordinates $(t, \boldsymbol{x})$ with $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathrm{N}$ such that $\boldsymbol{K} = \partial_t$. In these coordinates, the metric takes the form

$$\boldsymbol{g} = a(\boldsymbol{x})\, \mathrm{d}t \otimes \mathrm{d}t + b_i(\boldsymbol{x})\, \mathrm{d}t \otimes \mathrm{d}x^i - (\boldsymbol{g}_\mathrm{N})_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j\,,$$

where a and b_i are smooth functions, and $\boldsymbol{g}_\mathrm{N}$ is the induced Riemannian metric on N. This coordinate system can be understood as describing an observer that is comoving along the flow lines of the Killing field $\boldsymbol{K}$. We note that in the regions where $\boldsymbol{K}$ is timelike and the metric therefore stationary, the function a is positive. This is the case if $\boldsymbol{x}$ is near the boundary $\partial\mathrm{N}$. However, away from $\partial\mathrm{N}$, the function a may become negative, in which case the metric is no longer stationary.

We next proceed with formulating the Dirac equation. To this end, we choose an arbitrary spin structure and let $S\mathrm{M}$ be the corresponding spinor bundle, that is, a vector bundle with fibers $S_{(t,\boldsymbol{x})}\mathrm{M} \simeq \mathbb{C}^f$, where $(t, \boldsymbol{x}) \in \mathrm{M}$ and $f = 2^{\lfloor d/2 \rfloor}$ is the dimension. The corner brackets $\lfloor\,.\,\rfloor$ denote the floor function. Each fiber is endowed with an indefinite inner product of signature $(f/2, f/2)$

$$\prec .\,|\,. \succ_{(t,\boldsymbol{x})} \colon\; S_{(t,\boldsymbol{x})}\mathrm{M} \times S_{(t,\boldsymbol{x})}\mathrm{M} \to \mathbb{C}\,,$$

referred to as spin scalar product. We consider the Dirac operator

$$\mathcal{D} = \mathrm{i}\gamma^\mu \nabla_\mu + \mathscr{B}$$

with an external smooth matrix-valued potential $\mathscr{B}$, which we assume to be symmetric with respect to the spin scalar product, i.e., $\prec \phi | \mathscr{B}\psi \succ_{(t,\boldsymbol{x})} = \prec \mathscr{B}\phi | \psi \succ_{(t,\boldsymbol{x})}$. The Dirac matrices (γ^μ) are related to the metric by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, \mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}}$$

and $\boldsymbol{\nabla}$ is the metric connection on the spinor bundle. We are interested in solutions ψ of the Dirac equation of mass m

$$(\mathcal{D} - m)\psi = \boldsymbol{0}\,.$$

In order to analyze the dynamics of Dirac particles, it is advantageous to rewrite the Dirac equation in Hamiltonian form

$$\mathrm{i}\partial_t \psi = H\psi\,, \tag{4.3}$$

where H is the Dirac Hamiltonian given by

$$H = -\mathrm{i}\,(\gamma^t)^{-1}\gamma^j \nabla_j + (\text{zero-order terms})\,. \tag{4.4}$$

Taking the domain of definition

$$\mathrm{Dom}(H) = C_0^\infty(\mathrm{N} \backslash \partial\mathrm{N}, S\mathrm{M})\,,$$

this Hamiltonian is indeed symmetric, that is formally self-adjoint, with respect to the scalar product

$$(\psi|\phi)_{\mathrm{N}} = \int_{\mathrm{N}} \prec\!\psi|\not{\nu}\phi\!\succ_{(t=0,\boldsymbol{x})} \mathrm{d}\mu_{\mathrm{N}}\,, \tag{4.5}$$

where $\not{\nu} = \gamma^\mu \nu_\mu$ is the Clifford contraction of the future-directed, timelike normal $\boldsymbol{\nu}$ and $\mathrm{d}\mu_{\mathrm{N}}$ is the volume form on $(\mathrm{N}, \boldsymbol{g}_{\mathrm{N}})$. This can be verified with the following short computation

$$0 = \partial_t(\psi|\phi)_{\mathrm{N}} = (\partial_t\psi|\phi)_{\mathrm{N}} + (\psi|\partial_t\phi)_{\mathrm{N}} = (-\mathrm{i}H\psi|\phi)_{\mathrm{N}} + (\psi| - \mathrm{i}H\phi)_{\mathrm{N}} = \mathrm{i}\,\big((H\psi|\phi)_{\mathrm{N}} - (\psi|H\phi)_{\mathrm{N}}\big)\,,$$

in which we used current conservation together with the fact that the metric coefficients do not depend on the coordinate t.

In order to pose the Cauchy problem for the Dirac equation (4.3), we need to specify initial and boundary conditions. We choose initial data

$$\psi_{|\mathrm{N}} = \psi_0 \in C_0^\infty(\mathrm{N}, S\mathrm{M})$$

that is smooth and compactly supported on N. Moreover, we impose the boundary condition

$$(\not{n} - \mathrm{i})\psi_{|\partial\mathrm{M}} = \mathbf{0}\,, \tag{4.6}$$

where $\boldsymbol{n}$ is the inner normal on $\partial\mathrm{M}$, meaning that for every $p \in \partial\mathrm{M}$ there is a curve

$$c\colon [0,\delta) \to \mathrm{M} \quad \text{with} \quad c(0) = p \quad \text{and} \quad \partial_t c(t)_{|t=0} = n(p)\,.$$

Clearly, the initial data must be compatible with this boundary condition, i.e.,

$$(\not{n} - \mathrm{i})\psi_{0|\partial\mathrm{N}} = \mathbf{0}\,.$$

We point out that the boundary condition (4.6) has the effect that Dirac particles are reflected on $\partial\mathrm{M}$ (cf. Appendix A). Furthermore, it can be seen in analogy to the chiral boundary conditions used in [31, 57]. However, the difference is that, instead of the intrinsic Dirac operator on a hypersurface, we here consider the Hamiltonian obtained from the Dirac operator in the entire spacetime by separating the t-dependence. This gives rise to the additional factor $(\gamma^t)^{-1}$ in (4.4). Our boundary condition (4.6) can thus be understood as an adaptation of the chiral boundary conditions in [31, 57] to the Hamiltonian (4.4). We next incorporate the boundary condition into the functional analytic setting. Accordingly, we extend the domain of definition to

$$\mathrm{Dom}(H) = \{\psi \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})\psi_{|\partial\mathrm{N}} = \mathbf{0}\}\,. \tag{4.7}$$

Then, the operator H is again symmetric with respect to (4.5), as the subsequent consideration shows. First, making use of the Hamiltonian H in the representation (4.4) and employing the relation

$$\not{\nu}(\gamma^t)^{-1} = \frac{\mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}}}{\sqrt{g^{tt}}}\,,$$

we obtain for the scalar product $(\psi|H\phi)_{\mathrm{N}}$ the more convenient form

$$(\psi|H\phi)_{\mathrm{N}} = -\mathrm{i}\int_{\mathrm{N}} \prec\!\psi|\gamma^j\nabla_j\phi\!\succ_{(t=0,\boldsymbol{x})} \frac{1}{\sqrt{g^{tt}}}\,\mathrm{d}\mu_{\mathrm{N}} \;+\; (\text{lower-order terms})\,.$$

Now, computing the boundary terms yields

$$(\psi|H\phi)_{\mathrm{N}} - (H\psi|\phi)_{\mathrm{N}} = \mathrm{i} \int_{\partial\mathrm{N}} \prec\!\psi|\not{n}\phi\!\succ_{(t=0,\boldsymbol{x})} \frac{1}{\sqrt{g^{tt}}} \, \mathrm{d}\mu_{\partial\mathrm{N}} \,. \tag{4.8}$$

We note that the angular derivatives do not give rise to boundary terms because $\partial\mathrm{N}$ is compact without boundary. Applying the boundary condition defined in (4.7), we find

$$\prec\!\psi|\not{n}\phi\!\succ_{(t=0,\boldsymbol{x})} = \prec\!\not{n}\psi|\phi\!\succ_{(t=0,\boldsymbol{x})} = \prec\!\mathrm{i}\psi|\phi\!\succ_{(t=0,\boldsymbol{x})} = -\prec\!\psi|\mathrm{i}\phi\!\succ_{(t=0,\boldsymbol{x})} = -\prec\!\psi|\not{n}\phi\!\succ_{(t=0,\boldsymbol{x})} \tag{4.9}$$

for all $(t=0,\boldsymbol{x}) \in \partial\mathrm{N}$, proving that the boundary values (4.8) indeed vanish. This shows that H with domain (4.7) is symmetric with respect to the scalar product (4.5). In order to solve the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t\psi = H\psi \quad \text{in M} \\ \psi_{|\mathrm{N}} = \psi_0 \in \mathrm{Dom}(H) \\ (\not{n} - \mathrm{i})\psi_{|\partial\mathrm{M}} = \mathbf{0}\,, \end{cases} \tag{4.10}$$

where $\mathrm{Dom}(H)$ is defined in (4.7), it is of paramount importance to construct a unique self-adjoint extension of the Hamiltonian. More precisely, with such a self-adjoint extension at hand, we may apply the spectral theorem for unbounded self-adjoint operators and express the solution of the Cauchy problem (4.10) as

$$\psi(t) = e^{-\mathrm{i}tH}\psi_0 = \int_{\sigma(H)} e^{-\mathrm{i}\omega t}\, \psi_0 \, \mathrm{d}E_\omega$$

(see equation (2.14)). Thus, in Sections 4.1-4.3 below, we prove the following theorem:

Theorem 4.0.1. *The Dirac Hamiltonian* (4.4) *with domain of definition*

$$\mathrm{Dom}(H) = \{\psi \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\partial\mathrm{N}} = \mathbf{0} \quad \textit{for all} \quad p \in \mathbb{N}_0\} \tag{4.11}$$

is essentially self-adjoint.

We remark that the domain (4.11) is smaller than the domain (4.7), which is preferable because we want the Cauchy problem to have a global solution in $C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M})$.

Finally, we explain our method of proof of Theorem 4.0.1. We begin by establishing the existence and uniqueness of a global smooth solution of the mixed initial-boundary value problem (4.10). To this end, we first need to point out the crucial fact that in the situation under consideration here, there is the major complication that the Hamiltonian H is in general not uniformly elliptic. Hence, usual elliptic methods to tackle the Cauchy problem of the Dirac equation, such as the ones presented in [4], no longer apply. The nonuniform ellipticity of the Dirac Hamiltonian can be shown by analyzing its principal symbol [99]. According to (4.4), the principal symbol takes the form

$$P(\boldsymbol{x}, \boldsymbol{\xi}) = -\mathrm{i}\,(\gamma^t)^{-1}\gamma^j\xi_j \,.$$

The ellipticity condition states that it should be invertible for every nonzero $\boldsymbol{\xi} \in T^\star_{\boldsymbol{x}}\mathrm{N}$. In order to verify whether this condition holds, it is most convenient to compute the determinant of the principal symbol, namely

$$\det\big(P(\boldsymbol{x},\boldsymbol{\xi})\big) = \det\big((\gamma^t)^{-1}\big)\det(\gamma^j\xi_j)\,.$$

Using that

$$(\gamma^t)^{-1}(\gamma^t)^{-1} = \frac{\mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}}}{g^{tt}} \quad \text{and} \quad \gamma^i\xi_i\,\gamma^j\xi_j = g^{ij}\xi_i\xi_j\,\mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}}\,,$$

we obtain

$$\det\big(P(\boldsymbol{x},\boldsymbol{\xi})\big) = \left(\frac{g^{ij}\xi_i\xi_j}{g^{tt}}\right)^{f/2} .$$

This computation demonstrates that the Hamiltonian fails to be elliptic if $g^{ij}\xi_i\xi_j = 0$ for every $\boldsymbol{\xi} \neq \boldsymbol{0}$, which then implies that it is also not uniformly elliptic. In the special case of the nonextreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates, we can infer from the dual metric

$$\begin{aligned} \boldsymbol{g} = \Sigma^{-1}[(\Sigma + 2Mr)\,\partial_\tau \otimes \partial_\tau - 2Mr\,(\partial_\tau \otimes \partial_r + \partial_r \otimes \partial_\tau) - \Delta\,\partial_r \otimes \partial_r \\ - a\,(\partial_r \otimes \partial_\phi + \partial_\phi \otimes \partial_r) - \partial_\theta \otimes \partial_\theta - \csc^2(\theta)\,\partial_\phi \otimes \partial_\phi] \end{aligned} \tag{4.12}$$

that this occurs precisely at the event and the Cauchy horizon (for more details, see Section 5.3). We may thus use the points where the Hamiltonian (4.4) fails to be elliptic for the definition of the horizons of our more general spacetime. Our strategy to overcome this problem is to split the solution of the Cauchy problem (4.10) with domain of definition (4.11) up into two separate ones: in a region near the boundary $\partial\mathrm{M}$, which is located sufficiently far beyond the horizons, we rewrite the Cauchy problem in a form where the standard elliptic methods and results derived in [4] apply. However, in a region away from $\partial\mathrm{M}$ that includes the horizons, we employ specific methods and results from the theory of symmetric hyperbolic systems. Then, by making essential use of finite propagation speed, adding the two partial solutions gives rise to a unique smooth solution of the Cauchy problem for small times. Iterating this procedure backwards and forwards in time, we eventually find a global smooth solution. We therefore infer the existence of a 1-parameter family of unitary time evolution operators, which makes it possible to apply Chernoff's lemma on the essential self-adjointness of powers of generators of hyperbolic equations given in [21] in order to prove the essential self-adjointness of the Dirac Hamiltonian.

4.1 Double Boundary Value Problem for the Dirac Equation

As outlined above, we begin the proof of Theorem 4.0.1 by showing that the mixed initial-boundary value problem (4.10) has a unique global smooth solution. For this purpose, it is split up into a Cauchy problem in a region near the boundary $\partial\mathrm{M}$ that does not contain the points where the Hamiltonian fails to be elliptic, and a Cauchy problem in a region away from the boundary $\partial\mathrm{M}$ containing these horizons. As a technical tool for the former, we now prepare a double boundary value problem for the Dirac equation introducing an additional boundary condition on a suitable surface Y placed near $\partial\mathrm{M}$. We work in Gaussian normal coordinates in a tubular neighborhood of $\partial\mathrm{N}$ in N. Thus, for any $p \in \partial\mathrm{N}$, we let $c_p(r)$ for $0 \leq r < r_{\max}(p)$ be the geodesic in N with the initial conditions

$$c_p(0) = p \quad \text{and} \quad \frac{\mathrm{d}c_p(r)}{\mathrm{d}r}_{|r=0} = u\,,$$

where $u \in T_p\mathrm{N}$ is the inner normal to $\partial\mathrm{N}$. Since $\partial\mathrm{N}$ is compact, we can choose $r_{\max}$ independent of p to specify the mapping

$$c\colon [0, r_{\max}) \times \partial\mathrm{N} \to \mathrm{N}\,, \quad c(r,p) = c_p(r)\,.$$

Applying the implicit function theorem, we may arrange that c is a diffeomorphism by suitably decreasing the value of $r_{\max}$. Next, we define the sets $\partial\mathrm{N}(r)$ obtained from $\partial\mathrm{N}$ by the geodesic flow via

$$\partial\mathrm{N}(r) = c(r, \partial\mathrm{N})\,.$$

Subsequently, choosing coordinates $S^{d-2} \ni \Omega = (\vartheta_1, \dots, \vartheta_{d-2})$ on ∂N gives a corresponding coordinate system (r, Ω) on N. In these coordinates, the metric on N takes the form

$$\boldsymbol{g}_{\mathrm{N}} = \mathrm{d}r \otimes \mathrm{d}r + \boldsymbol{g}_{\partial\mathrm{N}} \,, \tag{4.13}$$

and hence

$$(\boldsymbol{g}_{\mathrm{N}})^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & (\boldsymbol{g}_{\partial\mathrm{N}})^{-1} \end{pmatrix} .$$

Using again $t \in \mathbb{R}$ as the time coordinate, we find a coordinate system $(t, \boldsymbol{x}) = (t, r, \Omega)$ of M that describes a neighborhood of ∂M. Then, we let X be the spacetime region

$$X := \{(t, r, \Omega) \,|\, 0 \le r \le r_{\max}/2\} \,.$$

This is a Lorentzian manifold whose boundary ∂X consists of ∂M as well as the $(d-1)$-dimensional surface

$$Y := \{(t, r_{\max}/2, \Omega)\} \,,$$

that is, the boundary of X has the two components $\partial X = \partial \mathrm{M} \cup Y$. By appropriately decreasing the value of $r_{\max}$, we can arrange that the Killing vector field $\boldsymbol{K}$ is timelike in X, implying that Y is a timelike surface. The inner normal on Y is also denoted by $\boldsymbol{n}$. We now consider the initial value problem for the Dirac equation

$$\begin{cases} (\mathcal{D} - m)\psi = \boldsymbol{0} \quad \text{in } X \\ \psi_{|\mathrm{N}} = \psi_0 \in C^\infty(\mathrm{N} \cap X, S\mathrm{M}) \end{cases} \tag{4.14}$$

with the boundary condition

$$(\not{\boldsymbol{n}} - \mathrm{i})\psi_{|\partial X} = \boldsymbol{0} \,. \tag{4.15}$$

It is once more useful to rewrite the Dirac equation in the Hamiltonian form (4.3), in which the Hamiltonian is given by (4.4). In order to take the boundary condition into account, we choose the domain of definition as the Sobolev space

$$\mathrm{Dom}(H) = \{\psi \in W^{1,2}(X \cap \mathrm{N}, S\mathrm{M}) \,|\, (\not{\boldsymbol{n}} - \mathrm{i})\psi_{|\partial X \cap \mathrm{N}} = \boldsymbol{0}\} \,. \tag{4.16}$$

The next proposition yields a spectral decomposition of the Hamiltonian H.

Proposition 4.1.1. *There is a countable orthonormal basis $(\psi_n)_{n \in \mathrm{N}}$ of eigenfunctions of H with $\psi_n \in \mathrm{Dom}(H)$.*

Proof. In order to show the existence of a countable orthonormal basis of eigenfunctions of the Hamiltonian, we apply Theorem 4.1 of [4]. Accordingly, our task is to verify the spectral conditions $(\mathcal{C}0)$-$(\mathcal{C}4)$, which in our setting read:

$(\mathcal{C}0)$ $H \colon \mathrm{Dom}(H) \to L^2(X)$ is linear and bounded in the $W^{1,2}$-topology on $\mathrm{Dom}(H)$.

$(\mathcal{C}1)$ The Gårding inequality holds: There exists a constant C such that for all $\psi \in \mathrm{Dom}(H)$

$$\|\psi\|^2_{W^{1,2}(X \cap \mathrm{N})} \le C \int_{X \cap \mathrm{N}} \big(\prec\! H\psi | \not{\boldsymbol{\nu}} H\psi \!\succ_{(t=0,\boldsymbol{x})} + \prec\! \psi | \not{\boldsymbol{\nu}} \psi \!\succ_{(t=0,\boldsymbol{x})} \big) \, \mathrm{d}\mu_{\mathrm{N}} \,.$$

$(\mathcal{C}2)$ Weak solutions are strong solutions ("elliptic regularity"): If $\phi \in L^2(X \cap \mathrm{N})$ satisfies

$$\int_{X \cap \mathrm{N}} \prec\! H\psi | \not{\boldsymbol{\nu}} \phi \!\succ_{(t=0,\boldsymbol{x})} \, \mathrm{d}\mu_{\mathrm{N}} = 0 \quad \text{for all} \quad \psi \in \mathrm{Dom}(H) \,,$$

then $\phi \in \mathrm{Dom}(H)$.

($\mathcal{C}3$) H is symmetric, i.e., for all $\psi, \phi \in \mathrm{Dom}(H)$

$$\int_{X\cap \mathrm{N}} \prec\!\psi|\slashed{\nu} H\phi\!\succ_{(t=0,\boldsymbol{x})} \, \mathrm{d}\mu_{\mathrm{N}} = \int_{X\cap \mathrm{N}} \prec\! H\psi|\slashed{\nu}\phi\!\succ_{(t=0,\boldsymbol{x})} \, \mathrm{d}\mu_{\mathrm{N}} \,.$$

($\mathcal{C}4$) $\mathrm{Dom}(H)$ is dense in $L^2(X\cap \mathrm{N})$.

In the following, we verify these conditions in ascending order with respect to extent and complexity of proof. We begin with the validity of condition ($\mathcal{C}0$), which is an immediate consequence of the fact that H is a differential operator of first order. The symmetry condition ($\mathcal{C}3$) was already established directly after (4.7). To prove the denseness condition ($\mathcal{C}4$), we make use of the result that on a smooth compact Riemannian manifold, such as $X\cap\mathrm{N}$ with smooth boundary $\partial X\cap \mathrm{N}$, the class C^∞ is dense in the function space L^2 [3]. From this and the fact that $C^\infty(X\cap\mathrm{N}) \subset W^{1,2}(X\cap \mathrm{N})$ [62], it is obvious that also $W^{1,2}(X\cap\mathrm{N})$ is dense in $L^2(X\cap\mathrm{N})$. Next, to verify the Gårding inequality ($\mathcal{C}1$), we exploit the specific form of the Hamiltonian (4.4). Then, the contribution to $\prec\! H\psi|\slashed{\nu} H\psi\!\succ_{(t=0,\boldsymbol{x})}$ involving first-order derivatives squared may be estimated for a suitable constant $c>0$ by

$$\prec\!(\gamma^t)^{-1}\gamma^i\nabla_i\psi|\slashed{\nu}\,(\gamma^t)^{-1}\gamma^j\nabla_j\psi\!\succ_{(t=0,\boldsymbol{x})} \geq c\,(\boldsymbol{g}_{\mathrm{N}})^{ij}\,\|\nabla_i\psi\|\,\|\nabla_j\psi\|\,,$$

where we used that the inner product $\prec\!.\,|\,\slashed{\nu}\,.\!\succ_{(t=0,\boldsymbol{x})}$ is positive definite and that the matrices $(\gamma^t)^{-1}$ and γ^i are uniformly bounded on X. Introducing the notation $\|\nabla\psi\|^2 = (\boldsymbol{g}_{\mathrm{N}})^{ij}\,\|\nabla_i\psi\|\,\|\nabla_j\psi\|$ and estimating the coefficients of the lower-order terms in a similar vein by appropriate constants $d_1, d_2 > 0$, we obtain

$$\prec\! H\psi|\slashed{\nu} H\psi\!\succ_{(t=0,\boldsymbol{x})} \geq c\,\|\nabla\psi\|^2 - d_1\,\|\nabla\psi\|\cdot\|\psi\| - d_2\,\|\psi\|^2 \geq \frac{c}{2}\,\|\nabla\psi\|^2 - \left(\frac{d_1^2}{2c} + d_2\right)\|\psi\|^2\,.$$

Hence, we find the estimate

$$\begin{aligned}
\|\psi\|^2_{W^{1,2}(X\cap\mathrm{N})} &= \int_{X\cap\mathrm{N}} \big(\|\nabla\psi\|^2 + \|\psi\|^2\big)\,\mathrm{d}\mu_{\mathrm{N}} \\
&\leq \frac{2}{c}\int_{X\cap\mathrm{N}} \left(\prec\! H\psi|\slashed{\nu} H\psi\!\succ_{(t=0,\boldsymbol{x})} + \left[\frac{d_1^2}{2c} + d_2 + \frac{c}{2}\right]\|\psi\|^2\right)\mathrm{d}\mu_{\mathrm{N}} \\
&\leq \frac{2}{c}\int_{X\cap\mathrm{N}} \big(\prec\! H\psi|\slashed{\nu} H\psi\!\succ_{(t=0,\boldsymbol{x})} + b\prec\!\psi|\slashed{\nu}\psi\!\succ_{(t=0,\boldsymbol{x})}\big)\,\mathrm{d}\mu_{\mathrm{N}} \\
&\leq \frac{2}{c}\int_{X\cap\mathrm{N}} \big(\prec\! H\psi|\slashed{\nu} H\psi\!\succ_{(t=0,\boldsymbol{x})} + \prec\!\psi|\slashed{\nu}\psi\!\succ_{(t=0,\boldsymbol{x})}\big)\,\mathrm{d}\mu_{\mathrm{N}} \times \begin{cases} 1 & \text{for} \quad b\leq 1 \\ b & \text{for} \quad b>1\,, \end{cases}
\end{aligned}$$

in which b is a positive constant, showing that condition ($\mathcal{C}1$) holds. It remains to derive the regularity condition ($\mathcal{C}2$). This task consists of two parts: one for the interior regularity and one for the regularity at the boundary. For the interior regularity, we need to prove that the operator H is uniformly elliptic (see Theorem 3.7 in [4]). To this end, we make use of the fact that the Killing field $\boldsymbol{K}$ is timelike in X, which allows us to define a norm on the spinors by

$$\|\psi\|^2_{(t,\boldsymbol{x})} := \prec\!\psi|\gamma^t\slashed{K}\gamma^t\psi\!\succ_{(t,\boldsymbol{x})}\,.$$

Applying this norm, we have

$$\|\xi_i(\gamma^t)^{-1}\gamma^i\psi\|^2_{(t,\boldsymbol{x})} = \xi_i\xi_j \prec\!(\gamma^t)^{-1}\gamma^i\psi|\gamma^t \slashed{K}\gamma^t(\gamma^t)^{-1}\gamma^j\psi\!\succ_{(t,\boldsymbol{x})} = \xi_i\xi_j \prec\!\gamma^i\psi|\slashed{K}\gamma^j\psi\!\succ_{(t,\boldsymbol{x})} .$$

Since $\slashed{K} = \gamma_t$, this matrix anticommutes with the Dirac matrices γ^j. Consequently, we obtain

$$\begin{aligned} \|\xi_i(\gamma^t)^{-1}\gamma^i\psi\|^2_{(t,\boldsymbol{x})} &= -\xi_i\xi_j \prec\!\gamma^j\gamma^i\psi|\slashed{K}\psi\!\succ_{(t,\boldsymbol{x})} \\ &= -g^{ij}\,\xi_i\xi_j \prec\!\psi|\slashed{K}\psi\!\succ_{(t,\boldsymbol{x})} = -g^{ij}\,\xi_i\xi_j\,\|(\gamma^t)^{-1}\psi\|^2_{(t,\boldsymbol{x})} , \end{aligned}$$

showing explicitly that H is uniformly elliptic as the matrix $(\gamma^t)^{-1}$ is uniformly bounded. The remaining proof of the boundary regularity is a subtle point, which we now treat in detail. We first note that, by localizing with a test function and using the interior regularity, it suffices to consider weak solutions whose support is in a small neighborhood of $\partial\mathrm{N}$ or $Y \cap \mathrm{N}$. Since both cases can be treated in the same way, we may assume that the solution vanishes identically outside a small neighborhood of $\partial\mathrm{N}$. Our goal is to apply Theorem 5.11 of [4]. Simplifying the statement of this theorem and adapting it to our setting, it gives boundary regularity for boundary value problems of the form

$$\begin{aligned} \mathcal{L}u &= f \\ Pu_{|\partial\mathrm{N}} &= \mathbf{0} . \end{aligned} \tag{4.17}$$

Here, P is a projection operator on $L^2(\partial\mathrm{N})$ and $\mathcal{L}$ is the differential operator

$$\mathcal{L} = \partial_r + A + B ,$$

where the operators A and B are of the following form. The operator $A \colon W^{1,2}(\partial\mathrm{N}) \to L^2(\partial\mathrm{N})$ is an angular differential operator, which is independent of r and satisfies again the above spectral conditions $(\mathcal{C}0)$-$(\mathcal{C}4)$. The operator $B \colon W^{1,2}(X \cap \mathrm{N}) \to L^2(X \cap \mathrm{N})$, on the other hand, should be such that its first-order terms vanish on $\partial\mathrm{N}$. The first step is to rewrite the Dirac equation and the boundary condition in the required form. We suppose that ψ is a weak solution of the inhomogeneous equation $H\psi = f$ satisfying the boundary condition (4.15). Moreover, we assume that ψ and f are supported in a small neighborhood of $\partial\mathrm{N}$. In order to implement the boundary condition on $\partial\mathrm{N}$, we choose the projection operator P as

$$P = \frac{1}{2}(\mathrm{i}\slashed{n} + 1) .$$

Next, using (4.4), we can write the differential equation (4.17) in the form

$$\big(\partial_r + (\gamma^r)^{-1}(\gamma^{\vartheta_1}\,\partial_{\vartheta_1} + \,\dots\, + \gamma^{\vartheta_{d-2}}\,\partial_{\vartheta_{d-2}}) + E\big)\psi = \mathrm{i}\,(\gamma^r)^{-1}\gamma^t f ,$$

where $(\vartheta_a)_{a=1,\dots,d-2}$ are again coordinates on $\partial\mathrm{N}$ and E is a zero-order operator. We also choose

$$\begin{aligned} A &= (\gamma^r)^{-1}(\gamma^{\vartheta_a}\,\partial_{\vartheta_a})_{|\partial\mathrm{N}} + Z \\ B &= (\gamma^r)^{-1}(\gamma^{\vartheta_a}\,\partial_{\vartheta_a}) + E - A \end{aligned} \tag{4.18}$$

with Z being a zero-order operator on $\partial\mathrm{N}$ to be determined below. The crucial point is to show that by a suitable choice of both a scalar product and the zero-order operator Z, we can arrange that the operator A is symmetric. Therefore, we define the scalar product as

$$\langle\,.\,|\,.\,\rangle_{\partial\mathrm{N}} = \int_{\partial\mathrm{N}} \prec\!.\,|\slashed{K}\,.\!\succ_{(t=0,\boldsymbol{x})}\,\mathrm{d}\mu_{\partial\mathrm{N}} . \tag{4.19}$$

We remark that since $\boldsymbol{K}$ is timelike near $\partial\mathrm{N}$, this inner product is indeed positive definite. Using the metric (4.13) in our Gaussian normal coordinate system, the following anticommutation relations hold

$$\{\not{\boldsymbol{K}}, \gamma^r\} = 2g_t{}^r\, \mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}} = 2\delta_t^r\, \mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}} = \boldsymbol{0}\,, \quad \{\not{\boldsymbol{K}}, \gamma^{\vartheta_a}\} = \boldsymbol{0}\,, \quad \{\gamma^r, \gamma^{\vartheta_a}\} = \boldsymbol{0}\,.$$

As a consequence, the matrices $(\gamma^r)^{-1}\gamma^{\vartheta_a}$ are antisymmetric with respect to the scalar product (4.19). Thus, setting

$$Z = -\frac{1}{2}(A_0 - A_0^*) \quad \text{with} \quad A_0 := (\gamma^r)^{-1}(\gamma^{\vartheta_a}\,\partial_{\vartheta_a})_{|\partial\mathrm{N}}\,,$$

where the star denotes the formal adjoint with respect to the scalar product (4.19), the operator Z is truly a multiplication operator. Furthermore, employing the above formulas for Z and A_0 in (4.18), we see that $A = (A_0 + A_0^*)/2$, which is obviously symmetric. Finally, it is clear by construction that the restriction of B to $\partial\mathrm{N}$ is a multiplication operator. Hence, it is obvious that the operator A has the above properties $(\mathcal{C}0)$, $(\mathcal{C}3)$ and $(\mathcal{C}4)$. To prove the Gårding inequality $(\mathcal{C}1)$ and the elliptic regularity condition $(\mathcal{C}2)$, we make use of the anticommutation relations

$$\{(\gamma^r)^{-1}\gamma^{\vartheta_a}, (\gamma^r)^{-1}\gamma^{\vartheta_b}\} = -\frac{1}{g^{rr}}\,\{\gamma^{\vartheta_a}, \gamma^{\vartheta_b}\} = -\frac{2g^{\vartheta_a\vartheta_b}}{g^{rr}}\,\mathbb{1}_{S_{(t,\boldsymbol{x})}\mathrm{M}}\,.$$

Accordingly, the operator A^2 is of the form

$$A^2 = \frac{1}{g^{rr}}\,\Delta_{S^{d-2}} + \text{(lower-order terms)}\,.$$

This is an elliptic operator on a bounded domain. Standard elliptic theory implies $(\mathcal{C}1)$ and $(\mathcal{C}2)$. ∎

The spectral decomposition of Proposition 4.1.1 implies that the mixed initial-boundary value problem (4.14), (4.15) has a unique weak solution in $W^{1,2}(X \cap \mathrm{N}, S\mathrm{M})$ given by

$$\psi(t,\boldsymbol{x}) = \sum_{n=1}^{\infty} c_n e^{-\mathrm{i}\omega_n t}\psi_n(\boldsymbol{x}) \quad \text{with} \quad c_n = \int_{X\cap\mathrm{N}} \prec\!\psi_n|\not{\boldsymbol{\nu}}\psi_0\!\succ_{\boldsymbol{y}} \mathrm{d}\mu_{\mathrm{N}}(\boldsymbol{y})\,, \tag{4.20}$$

where ω_n is the eigenvalue of ψ_n. In order to apply [21], we require a solution that is smooth for all times. We now state the corresponding necessary and sufficient conditions.

Lemma 4.1.2. *Suppose that ψ_0 satisfies the condition*

$$(\not{\boldsymbol{n}} - \mathrm{i})(H^p\psi_0)_{|\partial\mathrm{N}} = \boldsymbol{0} \quad \textit{for all} \quad p \in \mathbb{N}_0\,. \tag{4.21}$$

Then, the solution ψ of the mixed initial-boundary value problem (4.14), (4.15) *is in the class $C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M})$. Conversely, if a solution of the mixed initial-boundary value problem is smooth, then ψ_0 satisfies* (4.21).

Proof. Let ψ be the solution of the mixed initial-boundary value problem (4.14), (4.15) for ψ_0 satisfying (4.21). In order to show that ψ is smooth, it clearly suffices that all time derivatives of ψ exist and are smooth in $\boldsymbol{x}$. To this end, we consider the partial sums of (4.20)

$$\psi^N(t,\boldsymbol{x}) = \sum_{n=1}^{N} c_n e^{-\mathrm{i}\omega_n t}\psi_n(\boldsymbol{x})$$

for given $N \in \mathbb{N}$. Differentiating p times with respect to t yields

$$(\mathrm{i}\partial_t)^p \psi^N(t, \boldsymbol{x}) = \sum_{n=1}^{N} \omega_n^p c_n e^{-\mathrm{i}\omega_n t} \psi_n(\boldsymbol{x}) \,.$$

Moreover,

$$\omega_n^p c_n = \int_{X \cap \mathrm{N}} \prec H^p \psi_n | \not{\nu} \psi_0 \succ_{\boldsymbol{y}} \, \mathrm{d}\mu_{\mathrm{N}}(\boldsymbol{y}) = \int_{X \cap \mathrm{N}} \prec \psi_n | \not{\nu} H^p \psi_0 \succ_{\boldsymbol{y}} \, \mathrm{d}\mu_{\mathrm{N}}(\boldsymbol{y}) \,,$$

where we iteratively integrated by parts and used the boundary conditions (4.21). Since the function $\widetilde{\psi}_0 := H^p \psi_0$ is again in $\mathrm{Dom}(H)$ given by (4.16), we can take the limit $N \to \infty$ to conclude that

$$(\mathrm{i}\partial_t)^p \psi(t, \boldsymbol{x}) = \sum_{n=1}^{\infty} \widetilde{c}_n e^{-\mathrm{i}\omega_n t} \psi_n(\boldsymbol{x}) \quad \text{with} \quad \widetilde{c}_n = \int_{X \cap \mathrm{N}} \prec \psi_n | \not{\nu} \widetilde{\psi}_0 \succ_{\boldsymbol{y}} \, \mathrm{d}\mu_{\mathrm{N}}(\boldsymbol{y}) \,.$$

This proves that ψ is indeed a smooth solution. Assume conversely that ψ is a smooth solution to the mixed initial-boundary value problem (4.14), (4.15). Then, $(\not{n} - \mathrm{i})\psi(t)_{|\partial\mathrm{N}} = \mathbf{0}$ for all t. Differentiating p times with respect to t results in

$$\mathbf{0} = (\mathrm{i}\partial_t)^p \big((\not{n} - \mathrm{i})\psi(t)_{|\partial\mathrm{N}} \big)_{|t=0} = (\not{n} - \mathrm{i})(H^p \psi_0)_{|\partial\mathrm{N}} \,,$$

verifying (4.21). ■

4.2 Solution of the Cauchy Problem for the Dirac Equation

We now return to our original problem of finding a unique global smooth solution of the Dirac equation in Hamiltonian form

$$\mathrm{i}\partial_t \psi = H\psi \quad \text{in } \mathrm{M} \,, \tag{4.22}$$

assuming initial and boundary values

$$\psi_{|\mathrm{N}} = \psi_0 \in \{ \psi \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})(H^p \psi)_{|\partial\mathrm{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0 \} \tag{4.23}$$

and

$$(\not{n} - \mathrm{i})\psi_{|\partial\mathrm{M}} = \mathbf{0} \,, \tag{4.24}$$

respectively. For this purpose, we first prove the following lemma using the results of the previous Section 4.1.

Lemma 4.2.1. *There is an $\varepsilon > 0$ such that the mixed initial-boundary value problem (4.22)-(4.24) has a unique solution ψ in the class*

$$\big\{ \psi \in C_0^\infty\big([0, \varepsilon) \times \mathrm{N}, S\mathrm{M}\big) \,\big|\, (\not{n} - \mathrm{i})(H^p \psi)_{|[0,\varepsilon) \times \partial\mathrm{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0 \big\} \,. \tag{4.25}$$

Proof. We describe the neighborhood of the boundary $\partial\mathrm{N}$ again via the Gaussian normal coordinate system (r, Ω), for which the metric takes the form (4.13). Furthermore, we choose the value of ε so small that the future development $J^\vee$ of initial data sets has the properties

$$J^\vee\big(\{(0, r, \Omega) \,|\, r < r_{\max}/4\}\big) \cap \big(\{\varepsilon\} \times \mathrm{N}\big) \subset \{(\varepsilon, r, \Omega) \,|\, r < r_{\max}/2\} \tag{4.26}$$

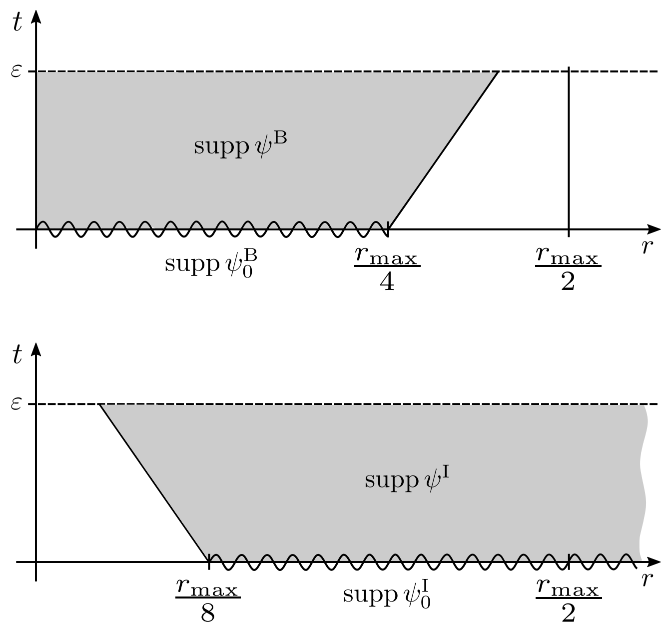

Figure 4.1: Decomposition of the solution of the Cauchy problem into a contribution ψ^{B} with support near the boundary $\partial\mathrm{N}$ (top figure) and a contribution ψ^{I} with support in the interior of N (bottom figure). The gray areas represent the respective future developments of initial data sets given in (4.26) and (4.27).

and

$$J^{\vee}\big(\{(0,r,\Omega)\,|\,r > r_{\max}/8\}\big) \cap (\{\varepsilon\} \times \mathrm{N}) \subset \{(\varepsilon, r, \Omega)\,|\, r > 0\}\,. \tag{4.27}$$

We may then decompose the initial data into a contribution ψ_0^{B} near the boundary $\partial\mathrm{N}$ and a contribution ψ_0^{I} supported in the interior of N, that is,

$$\psi_0 = \psi_0^{\mathrm{B}} + \psi_0^{\mathrm{I}}\,.$$

To this end, we let $\eta \in C_0^\infty\big((-r_{\max}/4, r_{\max}/4), \mathbb{R}\big)$ be a test function with $\eta_{|[0,r_{\max}/8]} \equiv 1$ and set

$$\psi_0^{\mathrm{B}} := \eta(r)\,\psi_0 \quad \text{and} \quad \psi_0^{\mathrm{I}} := \psi_0 - \psi_0^{\mathrm{B}} \tag{4.28}$$

(see Figure 4.1). Accordingly, the desired solution is of the form $\psi = \psi^{\mathrm{B}} + \psi^{\mathrm{I}}$, where ψ^{B} is a solution of the mixed initial-boundary value problem

$$\begin{cases} \mathrm{i}\partial_t \psi^{\mathrm{B}} = H\psi^{\mathrm{B}} \quad \text{in } X \\ \psi^{\mathrm{B}}_{|\mathrm{N}} = \psi_0^{\mathrm{B}} \\ (\not{n} - \mathrm{i})\psi^{\mathrm{B}}_{|\partial\mathrm{M}\cup Y} = \mathbf{0} \end{cases} \tag{4.29}$$

and ψ^{I} a solution of the initial value problem without boundary condition

$$\begin{cases} \mathrm{i}\partial_t \psi^{\mathrm{I}} = H\psi^{\mathrm{I}} \quad \text{in } \mathrm{M}\backslash\partial\mathrm{M} \\ \psi^{\mathrm{I}}_{|\mathrm{N}} = \psi_0^{\mathrm{I}}\,, \end{cases} \tag{4.30}$$

in which the initial values ψ_0^{B} and ψ_0^{I} are given by (4.28). We begin by analyzing the mixed initial-boundary value problem (4.29). Assuming that the initial data ψ_0^{B} satisfies the condition (4.21), we know from Lemma 4.1.2 that (4.29) has a solution ψ^{B} in the class $C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M})$, which satisfies the associated initial and boundary conditions pointwise (this solution even satisfies the stronger boundary conditions in (4.21)). Furthermore, due to finite propagation speed and the specific form of (4.26), we also know that this solution vanishes near the boundary $\{r = r_{\max}/2\}$, i.e.,

$$\operatorname{supp} \psi^{\mathrm{B}}(t, \,.\,) \subset [0, r_{\max}/2) \times \partial\mathrm{N} \quad \text{for all} \quad t \in [0, \varepsilon)\,.$$

Hence, extending ψ^{B} by zero, we obtain a global solution in all M. Next, we evaluate the initial value problem (4.30). Since N is a complete manifold without boundary and the initial data ψ^{I}_0 is smooth, it is well known from the fundamental existence and uniqueness theorems of the theory of symmetric hyperbolic systems [67, 91, 95, 101] that (4.30) has a unique solution ψ^{I} in the class $C^\infty_{\mathrm{sc}}\big([0,\varepsilon) \times \mathrm{N}, S\mathrm{M}\big)$. We point out that this solution vanishes identically near the boundary $\partial\mathrm{M}$ due to finite propagation speed and the specific form of (4.27). Finally, adding the solutions ψ^{B} and ψ^{I} yields a unique solution ψ of the mixed initial-boundary value problem (4.22)-(4.24) in the class (4.25). Its uniqueness follows immediately from standard energy estimates for symmetric hyperbolic systems (see for example [67]). ■

This lemma allows us to prove the existence of a unique global smooth solution of the mixed initial-boundary value problem (4.22)-(4.24) as follows.

Corollary 4.2.2. *The mixed initial-boundary value problem (4.22)-(4.24) has a unique global solution ψ in the class*

$$\{\psi \in C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M}) \,|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\partial\mathrm{M}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\}\,.$$

The resulting time evolution operator

$$\begin{aligned} U^{t',t} \colon &\big\{\psi \in C^\infty_{\mathrm{sc}}(\{t\} \times \mathrm{N}_t, S\mathrm{M}) \,\big|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\{t\}\times\partial\mathrm{N}_t} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\big\} \\ &\to \big\{\psi \in C^\infty_{\mathrm{sc}}(\{t'\} \times \mathrm{N}_{t'}, S\mathrm{M}) \,\big|\, (\not{n} - \mathrm{i})(H^p\psi)_{|\{t'\}\times\partial\mathrm{N}_{t'}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\big\} \end{aligned}$$

is unitary with respect to the scalar product (4.5).

Proof. Since the parameter ε in Lemma 4.2.1 does not depend on the initial data, we may iterate the procedure to obtain smooth solutions for arbitrarily large times. Moreover, we can adapt the procedure in such a way that it yields solutions in the class

$$\big\{\psi \in C^\infty_0\big((-\varepsilon, 0] \times \mathrm{N}, S\mathrm{M}\big) \,\big|\, (\not{n} - \mathrm{i})(H^p\psi)_{|(-\varepsilon,0]\times\partial\mathrm{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\big\}\,,$$

covering time reversals. Then, again by iteration, we find smooth solutions for arbitrarily large negative times. Eventually, this leads to a global smooth solution $\psi \in C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M})$ of the mixed initial-boundary value problem (4.22)-(4.24). Besides, the symmetry of the Dirac Hamiltonian H with respect to the scalar product (4.5), which is shown directly after (4.7), implies that the scalar product is preserved under time evolution. Thus, the time evolution operator $U^{t',t}$ is unitary. ■

4.3 Proof of the Essential Self-adjointness of the Dirac Hamiltonian

Having established the existence of a unique global smooth solution of the mixed initial-boundary value problem (4.22)-(4.24) and therefore the existence of a unitary time evolution operator $U^{t',t}$, we can now give the proof of Theorem 4.0.1 using Chernoff's Lemma on the essential self-adjointness of powers of generators of hyperbolic equations [21]. We begin by stating the original form of this lemma.

Chernoff's Lemma. *Let T be a symmetric operator with dense domain* $\mathrm{Dom}(T) \subset \mathcal{H}$*, a complex Hilbert space. Suppose that T maps* $\mathrm{Dom}(T)$ *into itself. Suppose in addition that there is a* 1*-parameter group V_l of unitary operators on $\mathcal{H}$ such that* $V_l \, \mathrm{Dom}(T) \subset \mathrm{Dom}(T)$*,* $V_l T = T V_l$ *on* $\mathrm{Dom}(T)$*, and* $(\mathrm{d}/\mathrm{d}t) V_l \, u = \mathrm{i} T \, V_l \, u$ *for $u \in$* $\mathrm{Dom}(T)$*. Then, every power of T is essentially self-adjoint.*

Proof of Theorem 4.0.1. We verify the conditions specified in Chernoff's Lemma for our framework, where the operator T corresponds, except for a minus sign, to the Dirac Hamiltonian H with domain $\mathrm{Dom}(H)$ given by (4.11). This operator is, on the one hand, symmetric with respect to the scalar product (4.5) and, on the other hand, the domain $\mathrm{Dom}(H)$ is clearly invariant under its action. Moreover, the unitary time evolution operator $U^{t',t}$ for the mixed initial-boundary value problem (4.22)-(4.24), which is defined in Corollary 4.2.2, is homogeneous, and thus forms a 1-parameter group acting on $\mathrm{Dom}(H)$. The remaining two properties, namely that $U^{t',t} H = H U^{t',t}$ and $\partial_{t'} U^{t',t} \psi = -\mathrm{i} H U^{t',t} \psi$, are trivially satisfied because the former is the commutation relation between $e^{-\mathrm{i}(t'-t)H}$ and H (the exponential representation of the propagator directly follows from the time-independence of the Hamiltonian), whereas the latter is the Dirac equation in Hamiltonian form. Since all conditions are met, we can apply Chernoff's Lemma to conclude that H is essentially self-adjoint on $\mathrm{Dom}(H)$. This completes the proof of Theorem 4.0.1. ∎

CHAPTER 5

An Integral Spectral Representation of the Massive Dirac Propagator in the Nonextreme Kerr Geometry in Horizon-penetrating Coordinates

We construct a generalized horizon-penetrating integral spectral representation of the massive Dirac propagator in the nonextreme Kerr geometry using the results obtained in Chapters 3 and 4. More precisely, we first derive the Hamiltonian formulation of the massive Dirac equation in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates and introduce a suitable scalar product on the associated space of solutions. We subsequently give the proofs of symmetry and essential self-adjointness of the Dirac Hamiltonian. For the proof of the latter, as the Dirac Hamiltonian fails to be elliptic at the event and the Cauchy horizon, we employ the method for nonuniformly elliptic mixed initial-boundary value problems worked out in Chapter 4. We then derive an integral representation of the massive Dirac propagator applying the spectral theorem for unbounded self-adjoint operators. In order to compute the corresponding spectral measure, we use Stone's formula, which yields a relation to the resolvent of the Dirac Hamiltonian. Finally, we determine the resolvent in terms of the angular spectral projector onto a finite-dimensional invariant spectral eigenspace of the angular operator and the radial Green's matrix arising in Chandrasekhar's mode analysis in Chapter 3.

5.1 Hamiltonian Formulation of the Massive Dirac Equation in the Analytically Extended Kerr Geometry

For the derivation of the Hamiltonian formulation of the massive Dirac equation in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates, it is advantageous to first rewrite the system (3.12) in the form

$$(\mathcal{R} + \mathcal{A})\psi' = \mathbf{0}\,,$$

where

$$\mathcal{R} := \begin{pmatrix} -\mathrm{i}mr & 0 & -\mathscr{D}_- & 0 \\ 0 & -\mathrm{i}mr & 0 & -\mathscr{D}_+ \\ \mathscr{D}_+ & 0 & \mathrm{i}mr & 0 \\ 0 & \mathscr{D}_- & 0 & \mathrm{i}mr \end{pmatrix} \tag{5.1}$$

and

$$\mathcal{A} := \begin{pmatrix} ma\cos(\theta) & 0 & 0 & \overline{\mathscr{L}} \\ 0 & ma\cos(\theta) & \mathscr{L} & 0 \\ 0 & \overline{\mathscr{L}} & ma\cos(\theta) & 0 \\ \mathscr{L} & 0 & 0 & ma\cos(\theta) \end{pmatrix} \tag{5.2}$$

are matrix-valued differential operators with

$$\mathscr{D}_+ := r_+^{-1}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi + r - M\big)$$

$$\mathscr{D}_- := r_+(\partial_\tau - \partial_r)$$

$$\mathscr{L} := \mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi + 2^{-1}\cot(\theta)\,.$$

We then separate the τ-derivative and multiply the resulting equation by the inverse of the matrix

$$\widetilde{\gamma}'^{\tau} := -\sqrt{\Sigma}\,\gamma^0\mathscr{P}^\dagger\mathscr{P}^{-1}\gamma'^{\tau} = \begin{pmatrix} 0 & 0 & -r_+ & -\mathrm{i}a\sin(\theta) \\ 0 & 0 & \mathrm{i}a\sin(\theta) & -r_+^{-1}[\Delta+4Mr] \\ r_+^{-1}[\Delta+4Mr] & -\mathrm{i}a\sin(\theta) & 0 & 0 \\ \mathrm{i}a\sin(\theta) & r_+ & 0 & 0 \end{pmatrix} \tag{5.3}$$

(cf. equation (3.13)) as well as by the imaginary unit, which leads to the Schrödinger-type equation

$$\mathrm{i}\partial_\tau\psi' = -\mathrm{i}\,(\widetilde{\gamma}'^{\tau})^{-1}\big(\mathcal{R}^{(3)} + \mathcal{A}^{(3)}\big)\psi' =: H\psi', \tag{5.4}$$

where $\mathcal{R}^{(3)}$ and $\mathcal{A}^{(3)}$ contain the first-order spatial and all zero-order contributions of the operators (5.1) and (5.2), respectively. The Dirac Hamiltonian H may be recast in the more convenient form

$$H = \alpha^j\,\partial_j + \mathscr{V}\,, \quad j \in \{r, \theta, \phi\}\,, \tag{5.5}$$

with the matrix-valued coefficients

$$\alpha^r := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} \mathrm{i}\Delta & r_+ a\sin(\theta) & 0 & 0 \\ r_+^{-1}\Delta\,a\sin(\theta) & -\mathrm{i}\,[\Delta+4Mr] & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,[\Delta+4Mr] & r_+^{-1}\Delta\,a\sin(\theta) \\ 0 & 0 & r_+ a\sin(\theta) & \mathrm{i}\Delta \end{pmatrix} \tag{5.6}$$

$$\alpha^\theta := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} -a\sin(\theta) & \mathrm{i}r_+ & 0 & 0 \\ \mathrm{i}r_+^{-1}[\Delta+4Mr] & a\sin(\theta) & 0 & 0 \\ 0 & 0 & -a\sin(\theta) & -\mathrm{i}r_+^{-1}[\Delta+4Mr] \\ 0 & 0 & -\mathrm{i}r_+ & a\sin(\theta) \end{pmatrix} \tag{5.7}$$

$$\alpha^\phi := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} \mathrm{i}a & r_+\csc(\theta) & 0 & 0 \\ r_+^{-1}\csc(\theta)\,[\Delta-2\Sigma] & -\mathrm{i}a & 0 & 0 \\ 0 & 0 & -\mathrm{i}a & r_+^{-1}\csc(\theta)\,[\Delta-2\Sigma] \\ 0 & 0 & r_+\csc(\theta) & \mathrm{i}a \end{pmatrix} \tag{5.8}$$

and the potential

$$\mathscr{V} := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} \mathscr{B}_1 & \mathscr{B}_2 \\ \mathscr{B}_3 & \mathscr{B}_4 \end{pmatrix}, \tag{5.9}$$

where the quantities $\mathscr{B}_k$, $k \in \{1,2,3,4\}$, are the 2×2 blocks

$$\begin{aligned}
\mathscr{B}_1 &:= \frac{1}{2}\begin{pmatrix} 2\mathrm{i}\,[r-M] - a\cos(\theta) & \mathrm{i}r_+ \cot(\theta) \\ r_+^{-1}\big(2a\sin(\theta)\,[r-M] + \mathrm{i}\,\cot(\theta)\,[\Delta + 4Mr]\big) & a\cos(\theta) \end{pmatrix} \\
\mathscr{B}_2 &:= -m\,\big(r - \mathrm{i}a\cos(\theta)\big)\begin{pmatrix} r_+ & \mathrm{i}a\sin(\theta) \\ -\mathrm{i}a\sin(\theta) & r_+^{-1}[\Delta + 4Mr] \end{pmatrix} \\
\mathscr{B}_3 &:= -m\,\big(r + \mathrm{i}a\cos(\theta)\big)\begin{pmatrix} r_+^{-1}[\Delta + 4Mr] & -\mathrm{i}a\sin(\theta) \\ \mathrm{i}a\sin(\theta) & r_+ \end{pmatrix} \\
\mathscr{B}_4 &:= \frac{1}{2}\begin{pmatrix} -a\cos(\theta) & r_+^{-1}\big(2a\,\sin(\theta)\,[r-M] - \mathrm{i}\,\cot(\theta)\,[\Delta + 4Mr]\big) \\ -\mathrm{i}r_+\cot(\theta) & 2\mathrm{i}\,[r-M] + a\cos(\theta) \end{pmatrix}.
\end{aligned} \tag{5.10}$$

5.2 The Canonical Scalar Product

In order to set up a Hilbert space that contains the solutions of (5.4)

$$\mathcal{H} := \overline{\big(\mathrm{Sol}(H - \mathrm{i}\partial_\tau),\, (\,.\,|\,.\,)\big)} \quad \text{for which} \quad \mathrm{Sol}(H - \mathrm{i}\partial_\tau) = \{\psi' \in L^2(\mathfrak{M}, S\mathfrak{M}) \,|\, (H - \mathrm{i}\partial_\tau)\psi' = \mathbf{0}\}\,,$$

and to establish the symmetry property of the Hamiltonian H

$$(\psi'|H\chi') = (H\psi'|\chi') \quad \text{with} \quad \psi', \chi' \in \mathrm{Sol}(H - \mathrm{i}\partial_\tau)$$

(or its self-adjointness), we require a suitable scalar product $(\,.\,|\,.\,)$. We again work with the scalar product [39]

$$(\psi|\chi)_{\mathfrak{N}} := \int_{\mathfrak{N}} \prec\!\psi|\boldsymbol{\nu\!\!\!/}\chi\!\succ_{(\tau_0,\boldsymbol{x})}\, \mathrm{d}\mu_{\mathfrak{N}} \tag{5.11}$$

defined on the spacelike hypersurface $\mathfrak{N} = \mathbb{R}_{>0} \times S^2$ for a constant time τ_0 with coordinates $\boldsymbol{x} = (r, \theta, \phi)$, where

$$\prec\!.\,|\,.\!\succ_{(\tau,\boldsymbol{x})} : S_{(\tau,\boldsymbol{x})}\mathfrak{M} \times S_{(\tau,\boldsymbol{x})}\mathfrak{M} \to \mathbb{C}\,, \quad (\psi, \chi) \mapsto \psi^\star\chi \tag{5.12}$$

denotes the indefinite spin scalar product of signature $(2,2)$, $\psi^\star := \psi^\dagger\mathscr{S}$ the adjoint Dirac spinor, $\boldsymbol{\nu}$ the future-directed timelike normal on $\mathfrak{N}$, and $\mathrm{d}\mu_{\mathfrak{N}} = \sqrt{|\det(\boldsymbol{g}_{\mathfrak{N}})|}\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r$ is the invariant measure on $\mathfrak{N}$, in which $\boldsymbol{g}_{\mathfrak{N}}$ is the induced Riemannian metric. The matrix $\mathscr{S}$ is defined via the relation [112]

$$\gamma^{\mu\dagger} := \mathscr{S}\gamma^\mu\mathscr{S}^{-1}\,. \tag{5.13}$$

We note that this scalar product is independent of the choice of the specific spacelike hypersurface $\mathfrak{N}$. This can be easily shown via Gauss' theorem and current conservation. In the following, we explicitly compute the above quantities and subsequently derive a more convenient representation for the scalar product. We begin with the calculation of the matrix $\mathscr{S}$. By means of (5.3) and the spinor transformation (3.14), we find

$$\gamma^\mu = -\frac{1}{\sqrt{\Sigma}} (\mathscr{P}^\dagger)^{-1} \gamma^0 \, \tilde{\gamma}'^\mu \mathscr{P} ,$$

which, using the defining equation (5.13), gives rise to the expression

$$\mathscr{S} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Next, we determine the normal vector field $\boldsymbol{\nu}$ by means of the conditions

$$\langle \boldsymbol{\nu} | \partial_r \rangle_{\boldsymbol{g}} = \langle \boldsymbol{\nu} | \partial_\theta \rangle_{\boldsymbol{g}} = \langle \boldsymbol{\nu} | \partial_\phi \rangle_{\boldsymbol{g}} = 0 \quad \text{and} \quad \langle \boldsymbol{\nu} | \boldsymbol{\nu} \rangle_{\boldsymbol{g}} = 1 ,$$

where $\langle\, . \,|\, . \,\rangle_{\boldsymbol{g}} := \boldsymbol{g}(\, . \,, . \,) = g_{\mu\nu} \, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu (\, . \,, . \,)$ is the spacetime inner product on $\mathfrak{M}$. Accordingly, employing (3.9) yields

$$\boldsymbol{\nu} = \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2} \partial_\tau - \frac{2Mr}{\Sigma} \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} \partial_r .$$

The corresponding dual covector reads

$$\boldsymbol{\nu} = \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} \mathrm{d}\tau . \tag{5.14}$$

Moreover, the induced metric $\boldsymbol{g}_{\mathfrak{N}}$ on the hypersurface $\mathfrak{N}$ is simply the restriction of (3.9) to $\mathfrak{N}$ and, thus, we obtain

$$\boldsymbol{g}_{\mathfrak{N}} = -\left(1 + \frac{2Mr}{\Sigma}\right) \big(\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big) \otimes \big(\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big) - \Sigma\,\mathrm{d}\theta \otimes \mathrm{d}\theta - \Sigma \sin^2(\theta)\,\mathrm{d}\phi \otimes \mathrm{d}\phi .$$

The associated Jacobian determinant in the volume measure $\mathrm{d}\mu_{\mathfrak{N}}$ becomes

$$\sqrt{|\det(\boldsymbol{g}_{\mathfrak{N}})|} = \Sigma \sin(\theta) \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2} . \tag{5.15}$$

We now express the scalar product (5.11) in terms of the primed quantities (3.14) used in (5.4), and substitute (5.14) as well as (5.15). This results in

$$(\psi' | \chi')_{\mathfrak{N}} = \iiint \psi'^\dagger \mathscr{S}' \gamma'^\tau \chi' \, \Sigma \sin(\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}r . \tag{5.16}$$

Again employing (5.3), that is with $\gamma'^\tau = -\mathscr{P}(\mathscr{P}^\dagger)^{-1}\gamma^0\,\tilde{\gamma}'^\tau/\sqrt{\Sigma}$, the scalar product (5.16) yields

$$\begin{aligned}
(\psi'|\chi')_{\mathfrak{N}} &= -\iiint \psi'^\dagger \mathscr{S}'\mathscr{P}(\mathscr{P}^\dagger)^{-1}\gamma^0\,\tilde{\gamma}'^\tau\chi'\sqrt{\Sigma}\,\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= -\iiint \psi'^\dagger \mathscr{P}\mathscr{P}^\dagger\mathscr{S}'\mathscr{P}(\mathscr{P}^\dagger)^{-1}\gamma^0\,\tilde{\gamma}'^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= -\iiint \psi'^\dagger \mathscr{P}\mathscr{S}(\mathscr{P}^\dagger)^{-1}\gamma^0\,\tilde{\gamma}'^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \qquad (5.17)\\
&= -\iiint \psi'^\dagger \mathscr{S}\mathscr{P}^\dagger(\mathscr{P}^\dagger)^{-1}\gamma^0\,\tilde{\gamma}'^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \iiint \psi'^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\,,
\end{aligned}$$

where

$$\Gamma^\tau := -\mathscr{S}\gamma^0\,\tilde{\gamma}'^\tau = \begin{pmatrix} r_+^{-1}[\Delta+4Mr] & -\mathrm{i}a\sin(\theta) & 0 & 0 \\ \mathrm{i}a\sin(\theta) & r_+ & 0 & 0 \\ 0 & 0 & r_+ & \mathrm{i}a\sin(\theta) \\ 0 & 0 & -\mathrm{i}a\sin(\theta) & r_+^{-1}[\Delta+4Mr] \end{pmatrix}. \qquad (5.18)$$

We point out that in the above derivation, we have first applied the relation $\sqrt{\Sigma}\,\mathbb{1}_{\mathbb{C}^4} = \mathscr{P}\mathscr{P}^\dagger$, then the transformation law for the matrix $\mathscr{S}'$, namely $\mathscr{S} = \mathscr{P}^\dagger\mathscr{S}'\mathscr{P}$, and finally we have used the fact that both $\mathscr{S}$ and the product $\mathscr{P}\mathscr{S}$ are self-adjoint, which leads to $\mathscr{P}\mathscr{S} = \mathscr{S}\mathscr{P}^\dagger$. Besides, the integration limits and the time restrictions are suppressed for ease of notation. The eigenvalues λ_1, λ_2 of the matrix (5.18) are positive

$$\lambda_1 = \frac{1}{2}\left(r_+ + \frac{\Delta+4Mr}{r_+} + \sqrt{\left(r_+ - \frac{\Delta+4Mr}{r_+}\right)^2 + 4a^2\sin^2(\theta)}\,\right) > 0$$

$$\lambda_2 = \frac{1}{2}\left(r_+ + \frac{\Delta+4Mr}{r_+} - \sqrt{\left(r_+ + \frac{\Delta+4Mr}{r_+}\right)^2 - 4(\Sigma+2Mr)}\,\right) > 0$$

and with algebraic multiplicities $\mu_A(\lambda_1) = \mu_A(\lambda_2) = 2$, demonstrating that (5.17) is indeed positive-definite. The symmetry property of the Dirac Hamiltonian (5.5) with respect to this scalar product is explicitly proven in Appendix A.

5.3 Essential Self-adjointness of the Dirac Hamiltonian

In this section, we show that the Dirac Hamiltonian (5.5) is essentially self-adjoint. The proof involves the technical difficulty that in the nonextreme Kerr geometry the Dirac Hamiltonian is not uniformly elliptic. More precisely, it fails to be elliptic at the event and the Cauchy horizon. This can be easily seen from the evaluation of the determinant of the principal symbol of the Hamiltonian (5.5)

$$P(r,\theta;\boldsymbol{\xi}) = \alpha^j(r,\theta)\,\xi_j\,, \qquad (5.19)$$

where $\boldsymbol{\xi} \in T^{\star}_{\boldsymbol{x}}\mathfrak{N}$. In more detail, we first rewrite the matrices α^j in terms of the original Dirac matrices γ^j

$$\alpha^j = -\mathrm{i}(\widetilde{\gamma}'^{\tau})^{-1}\,\widetilde{\gamma}'^{j} = -\mathrm{i}(\gamma'^{\tau})^{-1}\gamma'^{j} = -\mathrm{i}\mathscr{P}(\gamma^{\tau})^{-1}\gamma^{j}\mathscr{P}^{-1}.$$

Substituting this expression into the principal symbol (5.19) and computing the determinant yields

$$\det\big(P(r,\theta;\boldsymbol{\xi})\big) = \frac{\det(\gamma^j\xi_j)}{\det(\gamma^\tau)}\,.$$

Using the relations

$$(\gamma^\tau)^2 = g^{\tau\tau}\,\mathbb{1}_{S_{(\tau,\boldsymbol{x})}\mathfrak{M}} \quad \text{and} \quad \gamma^i\xi_i\,\gamma^j\xi_j = g^{ij}\xi_i\,\xi_j\,\mathbb{1}_{S_{(\tau,\boldsymbol{x})}\mathfrak{M}}\,,$$

we obtain

$$\det\big(P(r,\theta;\boldsymbol{\xi})\big) = \left(\frac{g^{ij}\xi_i\,\xi_j}{g^{\tau\tau}}\right)^2. \tag{5.20}$$

The Hamiltonian fails to be elliptic if its principal symbol is not invertible, that is, if the determinant (5.20) vanishes. This is the case for

$$g^{ij}\xi_i\xi_j = 0 \quad \text{with} \quad \boldsymbol{\xi} \neq \mathbf{0}\,, \tag{5.21}$$

where the quantities g^{ij} are the spatial components of the inverse of the spacetime metric (3.9) specified in (4.12). Analyzing condition (5.21), we find that the Hamiltonian is not elliptic at the event and the Cauchy horizon. We point out that by using the intrinsic Dirac Hamiltonian on the spacelike hypersurface $\mathfrak{N}$, ellipticity would be conserved even at the horizons, which can be inferred from the analog condition

$$g^{ij}_{\mathfrak{N}}\xi_i\xi_j = 0 \quad \text{with} \quad \boldsymbol{\xi} \neq \mathbf{0}\,,$$

where the $g^{ij}_{\mathfrak{N}}$ denote the components of the inverse of the associated induced Riemannian metric

$$\boldsymbol{g}_{\mathfrak{N}} = -\frac{1}{\Sigma}\left[\frac{(r^2+a^2)^2 - \Delta\,a^2\sin^2(\theta)}{\Sigma + 2Mr}\,\partial_r\otimes\partial_r + a\,(\partial_r\otimes\partial_\phi + \partial_\phi\otimes\partial_r) + \partial_\theta\otimes\partial_\theta + \csc^2(\theta)\,\partial_\phi\otimes\partial_\phi\right].$$

However, as we work with the Hamiltonian obtained from the Dirac operator in the full Kerr geometry, ellipticity is broken. Therefore, we cannot employ standard techniques and results from elliptic theory in order to verify the essential self-adjointness of the Dirac Hamiltonian. Instead, we apply the results derived in Chapter 4, where near-boundary elliptic methods are combined with results from the theory of symmetric hyperbolic systems. In the following, we state the geometric and functional analytic settings for the formulation of the Cauchy problem for the massive Dirac equation in the nonextreme Kerr geometry in horizon-penetrating coordinates in Hamiltonian form, which is used as a technical tool in the proof of the essential self-adjointness of the Dirac Hamiltonian.

We let $(\mathfrak{M},\boldsymbol{g})$ be the nonextreme Kerr geometry with the metric (3.9) in advanced Eddington–Finkelstein-type coordinates and consider the subset

$$\mathrm{M} := \{\tau, r > r_0, \theta, \phi\} \subset \mathfrak{M} \quad \text{for} \quad r_0 < r_-$$

with timelike boundary

$$\partial\mathrm{M} := \{\tau, r = r_0, \theta, \phi\}\,.$$

Furthermore, we introduce the family of spacelike hypersurfaces $(\mathrm{N}_\tau)_{\tau\in\mathbb{R}}$, where

$$\mathrm{N}_\tau := \{\tau = \text{const.}, r > r_0, \theta, \phi\}\,,$$

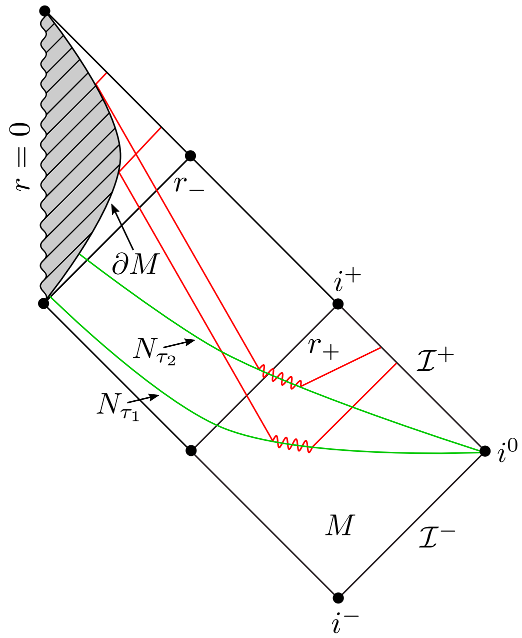

Figure 5.1: Carter–Penrose diagram of the region M of the nonextreme Kerr geometry with constant-τ hypersurfaces N_{τ_1} and N_{τ_2} cut-off at the boundary $\partial\mathrm{M}$. A radial Dirichlet-type boundary condition imposed on $\partial\mathrm{M}$ leads to a reflection of Dirac particles away from the singularity without affecting their dynamics outside the Cauchy horizon. This is represented by Cauchy data propagated in τ-direction.

with boundaries

$$\partial\mathrm{N}_\tau := \partial\mathrm{M} \cap \mathrm{N}_\tau = S^2$$

(see FIG. 5.1). These hypersurfaces constitute a foliation of M. Moreover, we define a Killing vector field that is timelike near $\partial\mathrm{M}$ and spacelike near spatial infinity by [83]

$$\boldsymbol{K} := \partial_\tau + \beta_0\,\partial_\phi \quad \text{with the constant} \quad \beta_0 = \beta_0(r_0) \in \mathbb{R}\backslash\{0\}\,, \tag{5.22}$$

where ∂_τ and ∂_ϕ are the Killing fields related to the time translational and axial symmetries of the Kerr geometry, respectively. We note that this Killing field corresponds to the Killing field $\boldsymbol{K} = \partial_t$ specified in Chapter 4 on page 44, satisfying the assumptions (ii) and (iii) on page 43. We also point out that we cannot employ the simpler Killing field $\boldsymbol{K} = \partial_\tau$ instead of (5.22), as it is not timelike everywhere on $\partial\mathrm{M}$. To be more precise, the condition for ∂_τ not being timelike reads

$$\boldsymbol{g}(\partial_\tau, \partial_\tau) = 1 - \frac{2Mr}{\Sigma} \leq 0\,.$$

This inequality is solved by

$$M - \sqrt{M^2 - a^2\cos^2(\theta)} \leq r \leq M + \sqrt{M^2 - a^2\cos^2(\theta)}\,,$$

which is the ergosphere region. But taking $\boldsymbol{K}$ as the linear combination (5.22), it turns out to be a Killing field that satisfies all the required assumptions. To make the connection between (5.22) and the Killing field $\boldsymbol{K} = \partial_t$ explicit, we use the coordinate transformation

$$\mathbb{R} \times \mathbb{R}_{>0} \times (0,\pi) \times [0,2\pi) \to \mathbb{R} \times \mathbb{R}_{>0} \times (0,\pi) \times [0,2\pi)\,, \qquad (\tau, r, \theta, \phi) \mapsto (t, r, \theta, \Phi)$$

with

$$t = \tau \quad \text{and} \quad \Phi = \phi - \beta_0 \tau \,. \tag{5.23}$$

This transformation can be easily derived from the condition

$$\boldsymbol{K} = \partial_t = \frac{\mathrm{d}\tau}{\mathrm{d}t}\,\partial_\tau + \frac{\mathrm{d}\phi}{\mathrm{d}t}\,\partial_\phi = \partial_\tau + \beta_0\,\partial_\phi \,.$$

Evaluation of the gradient

$$\boldsymbol{\nabla} t = g^{\mu\nu}(\partial_\mu t)\,\partial_\nu = g^{t\nu}\,\partial_\nu$$

and subsequently

$$\boldsymbol{g}(\partial_t, \boldsymbol{\nabla} t) = g^t{}_t = 1 > 0 \quad \text{as well as} \quad \boldsymbol{g}(\boldsymbol{\nabla} t, \boldsymbol{\nabla} t) = g^{tt} = 1 + \frac{2Mr}{\Sigma} > 0$$

demonstrates that $\boldsymbol{\nabla} t$ is future-pointing and timelike and, hence, that the coordinate t is a temporal function as is the original time coordinate τ [98]. Due to the specific form of the transformation (5.23), we find that the induced metric $\boldsymbol{g}_{|t=\text{const.}}$ on the level sets of t is identical to the induced metric

$$\boldsymbol{g}_{|\tau=\text{const.}} = \boldsymbol{g}_{\mathfrak{N}} = -\left(1 + \frac{2Mr}{\Sigma}\right)\mathrm{d}r \otimes \mathrm{d}r + a\sin^2(\theta)\left(1 + \frac{2Mr}{\Sigma}\right)[\mathrm{d}r \otimes \mathrm{d}\phi + \mathrm{d}\phi \otimes \mathrm{d}r]$$

$$- \Sigma\,\mathrm{d}\theta \otimes \mathrm{d}\theta - \sin^2(\theta)\left[\Sigma + a^2\sin^2(\theta)\left(1 + \frac{2Mr}{\Sigma}\right)\right]\mathrm{d}\phi \otimes \mathrm{d}\phi$$

on the level sets of τ for the advanced Eddington–Finkelstein-type coordinates. As a consequence, all the results obtained for the coordinate system employed in Chapter 4 also hold for the advanced Eddington–Finkelstein-type coordinates.

Next, in addition to these geometric structures, we introduce the spinor bundle $S\mathrm{M}$ of M with fibers $S_{(\tau,\boldsymbol{x})}\mathrm{M} \simeq \mathbb{C}^4$, where $(\tau, \boldsymbol{x}) \in \mathrm{M}$. We may then consider the Dirac Hamiltonian given in (5.5)

$$H = \alpha^j \partial_j + \mathscr{V} \quad \text{on } \mathrm{N} := \mathrm{N}_{\tau_0} \,, \tag{5.24}$$

which is a symmetric operator with respect to the scalar product specified in (5.17)

$$(\psi'|\chi')_{\mathrm{N}} = \int_{r_0}^{\infty} \int_{S^2} \psi'^{\dagger}\,\Gamma^\tau \chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \,, \tag{5.25}$$

where the Dirac 4-spinors $\psi', \chi' \in S_{(\tau,\boldsymbol{x})}\mathrm{M}$ and the matrix Γ^τ is defined in (5.18). We note that for H being symmetric with respect to (5.25), we have to impose the radial Dirichlet-type boundary condition (see Appendix A)

$$\big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)\psi'_{|\partial\mathrm{M}} = \boldsymbol{0} \,,$$

in which $\boldsymbol{n}$ is the inner normal to $\partial\mathrm{M}$ and $\mathscr{P}$ is determined by (3.15). This boundary condition has the effect that Dirac particles are reflected at $\partial\mathrm{M}$ away from the singularity such that, without changing their dynamics outside the Cauchy horizon, we obtain a unitary time evolution. The specific domain of the Hamiltonian reads

$$\mathrm{Dom}(H) = \big\{\psi' \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \;\big|\; \big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)\psi'_{|\partial\mathrm{N}} = \boldsymbol{0}\big\} \,.$$

In this setting, we find a unique, global solution of the Cauchy problem for the massive Dirac equation in Hamiltonian form in the class $C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M})$.

Lemma 5.3.1. *The Cauchy problem for the massive Dirac equation in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates*

$$\begin{cases} \mathrm{i}\partial_\tau \psi' = H\psi' \quad \text{in } \mathrm{M} \\ \psi'_{|\tau=\tau_0} = \psi'_0 \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \end{cases} \tag{5.26}$$

with radial Dirichlet-type boundary condition at $\partial\mathrm{M}$

$$\big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)\psi'_{|\partial\mathrm{M}} = \mathbf{0}\,, \tag{5.27}$$

where the initial data ψ'_0 *is smooth compactly supported outside, across, and inside the event horizon, up to the Cauchy horizon, and compatible with the boundary condition, i.e.,*

$$\big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)\psi'_{0|\partial\mathrm{N}} = \mathbf{0}\,,$$

has a unique global solution ψ' *in the class*

$$\big\{\psi' \in C^\infty_{\mathrm{sc}}(\mathrm{M}, S\mathrm{M}) \,\big|\, \big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)(H^p\psi')_{|\partial\mathrm{M}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\big\}\,.$$

The associated time evolution operator is unique and unitary with respect to the scalar product (5.25).

The proof of this lemma is shown in detail in Sections 4.1 and 4.2. The existence of such a unique global smooth solution is imperative for the specific proof of the essential self-adjointness of the Dirac Hamiltonian presented in Section 4.3. Below, we state the result for the Hamiltonian in the nonextreme Kerr geometry in horizon-penetrating coordinates (5.24).

Theorem 5.3.2. *The massive Dirac Hamiltonian* H *in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates with domain of definition*

$$\mathrm{Dom}(H) = \big\{\psi' \in C_0^\infty(\mathrm{N}, S\mathrm{M}) \;\big|\; \big(\not{n} - \mathrm{i}\mathscr{P}(\mathscr{P}^\dagger)^{-1}\big)(H^p\psi')_{|\partial\mathrm{N}} = \mathbf{0} \quad \text{for all} \quad p \in \mathbb{N}_0\big\} \tag{5.28}$$

is essentially self-adjoint.

5.4 Resolvent of the Dirac Hamiltonian and Integral Spectral Representation of the Dirac Propagator

We may now construct an integral spectral representation of the Dirac propagator that yields the relativistic dynamics of massive Dirac particles outside, across, and inside the event horizon, up to the Cauchy horizon. More precisely, we derive an explicit expression for the spectral measure $\mathrm{d}E_\omega$ of the essentially self-adjoint Dirac Hamiltonian (5.24) with domain (5.28), which arises in the formal spectral decomposition of the Dirac propagator

$$\psi'(\tau, r, \theta, \phi) = e^{-\mathrm{i}\tau H}\psi'_0(r, \theta, \phi) = \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\psi'_0(r, \theta, \phi)\,\mathrm{d}E_\omega\,, \tag{5.29}$$

where ψ'_0 is smooth initial data with compact support. To this end, we employ Stone's formula and, thus, express the spectral measure in terms of the resolvent $\mathrm{Res}(\omega_{\mathrm{c}}; H) = (H - \omega_{\mathrm{c}})^{-1}$ of the Hamiltonian. As the spectrum of the Hamiltonian $\sigma(H) \subseteq \mathbb{R}$ is on the real line, this resolvent exists for all $\omega_{\mathrm{c}} \in \mathbb{C}\backslash\mathbb{R}$ with real part $\mathrm{Re}(\omega_{\mathrm{c}}) = \omega \in \sigma(H)$ and is given uniquely. In the computation of the resolvent, we make use of specific quantities obtained in Chandrasekhar's mode analysis, namely the angular projector onto a finite-dimensional invariant eigenspace of the angular operator (3.21) and the Green's matrix of the radial ODE system (3.19) extended to complex-valued frequencies.

Theorem 5.4.1. *The massive Dirac propagator in the nonextreme Kerr geometry in advanced Eddington–Finkelstein-type coordinates can be expressed via the integral spectral representation*

$$\psi'(\tau,r,\theta,\phi)=\frac{1}{2\pi\mathrm{i}}\sum_{k\in\mathbb{Z}}e^{-\mathrm{i}k\phi}\int_{\mathbb{R}}e^{-\mathrm{i}\omega\tau}\lim_{\epsilon\searrow 0}\big[(H_k-\omega-\mathrm{i}\epsilon)^{-1}-(H_k-\omega+\mathrm{i}\epsilon)^{-1}\big](r,\theta;r',\theta')$$
$$\times\,\psi'_{0,k}(r',\theta')\,\mathrm{d}\omega\,,$$

where $\psi'_{0,k}\in C_0^\infty\big((r_0,\infty)\times(0,\pi),S\mathrm{M}\big)$ is initial data for fixed k-modes and $(H_k-\omega\mp\mathrm{i}\epsilon)^{-1}$ are resolvents of the Dirac Hamiltonian for fixed k-modes H_k on the upper and lower complex half-planes. The resolvents are unique and of the form

$$(H_k-\omega\mp\mathrm{i}\epsilon)^{-1}(r,\theta;r',\theta')\,\psi'_{0,k}(r',\theta')=-\sum_{l\in\mathbb{Z}}\int_{-1}^{1}Q_l(\theta;\theta')$$

$$\times\int_{r_0}^{\infty}\mathscr{C}\begin{pmatrix}G(r;r')_{k,l,\omega\pm\mathrm{i}\epsilon} & \mathbf{0}_{\mathbb{C}^2}\\ \mathbf{0}_{\mathbb{C}^2} & G(r;r')_{k,l,\omega\pm\mathrm{i}\epsilon}\end{pmatrix}\mathscr{E}(r',\theta')\,\psi'_{0,k}(r',\theta')\,\mathrm{d}r'\mathrm{d}\big(\cos(\theta')\big)\,,$$

with $Q_l(\,.\,,.\,)$ being the integral kernel of the spectral projector onto a finite-dimensional invariant eigenspace of the angular operator (3.21), which corresponds to the spectral parameter ξ_l, $G(r;r')_{k,l,\omega\pm\mathrm{i}\epsilon}$ is the 2-dimensional Green's matrix of the radial ODE system (3.19) extended to complex-valued frequencies, and

$$\mathscr{C}=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

as well as

$$\mathscr{E}(r',\theta')=-\begin{pmatrix}\mathrm{i}\,[\Delta(r')+4Mr'] & r_+a\sin(\theta') & 0 & 0\\ 0 & 0 & -\mathrm{i}r_+ & a\sin(\theta')\\ 0 & 0 & r_+a\sin(\theta') & \mathrm{i}\,[\Delta(r')+4Mr']\\ a\sin(\theta') & -\mathrm{i}r_+ & 0 & 0\end{pmatrix}.$$

Proof. We first compute the resolvents $(H_k-\omega\mp\mathrm{i}\epsilon)^{-1}$ of the Dirac Hamiltonian for fixed k-modes H_k on the upper and lower complex half-planes, where $\omega\in\mathbb{R}$ and $\epsilon>0$ is sufficiently small so that it can be considered as a slightly non-self-adjoint perturbation. For this purpose, we substitute Chandrasekhar's mode ansatz extended to complex-valued frequencies ω_ϵ with $\mathrm{Re}(\omega_\epsilon)=\omega$ and $\mathrm{Im}(\omega_\epsilon)\in\{-\epsilon,\epsilon\}$

$$\psi'(\tau,r,\theta,\phi)=e^{-\mathrm{i}(\omega_\epsilon\tau+k\phi)}\,\Psi(r,\theta)\,,\quad\text{in which}\quad\Psi\in L^2\big((r_0,\infty)\times(0,\pi),S\mathrm{M}\big)\,,$$

into the Dirac equation specified in (5.26), yielding

$$(H_k-\omega_\epsilon)\Psi=\mathbf{0}\,,\tag{5.30}$$

where

$$H_k=\alpha^r\partial_r+\alpha^\theta\partial_\theta-\mathrm{i}k\,\alpha^\phi+\mathscr{V}$$

with the matrices α^j, $j\in\{r,\theta,\phi\}$, and the potential $\mathscr{V}$ given in (5.6)-(5.9). Furthermore, we define the spectral projector onto the finite-dimensional invariant eigenspace of the angular operator (3.21)

$$Q_l\Psi:=\int_{-1}^{1}Q_l(\theta;\theta')\,\Psi(r,\theta')\,\mathrm{d}\big(\cos(\theta')\big)\,,$$

which corresponds to the spectral parameter ξ_l with $l \in \mathbb{Z}$. This projector is idempotent, i.e.,

$$Q_l^n = Q_l \quad \text{for all} \quad n \in \mathbb{N}\,,$$

and the family $(Q_l)_{l\in\mathbb{Z}}$ is complete, that is,

$$\sum_{l\in\mathbb{Z}} Q_l = \mathbb{1}\,. \tag{5.31}$$

We may therefore express the angular operator (3.21) as

$$A = \sum_{l\in\mathbb{Z}} \xi_l Q_l\,.$$

Applying the completeness constraint (5.31) in equation (5.30) and using this representation of the angular operator, we obtain

$$-(\Sigma + 2Mr)^{-1} \sum_{l\in\mathbb{Z}} \mathcal{M}(\partial_r; r, \theta)_{k,l,\omega_\epsilon}\, Q_l\, \Psi = \mathbf{0}\,, \tag{5.32}$$

where

$$\mathcal{M}(\partial_r; r, \theta)_{k,l,\omega_\epsilon}$$

$$:= \begin{pmatrix} \mathrm{i}O_{k,\omega_\epsilon} & a\sin(\theta)\, U_{\omega_\epsilon} & \mathrm{i}r_+\, S_l & a\sin(\theta)\, \overline{S}_l \\ \dfrac{a\sin(\theta)}{r_+}\, O_{k,\omega_\epsilon} & -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+}\, U_{\omega_\epsilon} & a\sin(\theta)\, S_l & -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+}\, \overline{S}_l \\ -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+}\, \overline{S}_l & a\sin(\theta)\, S_l & -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+}\, U_{\omega_\epsilon} & \dfrac{a\sin(\theta)}{r_+}\, O_{k,\omega_\epsilon} \\ a\sin(\theta)\, \overline{S}_l & \mathrm{i}r_+\, S_l & a\sin(\theta)\, U_{\omega_\epsilon} & \mathrm{i}O_{k,\omega_\epsilon} \end{pmatrix}$$

with the radial differential operators O_{k,ω_ϵ}, U_{ω_ϵ}, and the function S_l defined by

$$\begin{aligned} O_{k,\omega_\epsilon} &:= \Delta\, \partial_r + r - M - \mathrm{i}\omega_\epsilon(\Delta + 4Mr) - 2\mathrm{i}ak \\ U_{\omega_\epsilon} &:= r_+(\partial_r + \mathrm{i}\omega_\epsilon) \\ S_l &:= \xi_l + \mathrm{i}mr\,. \end{aligned}$$

In the following, we show that the computation of the resolvent of the operator $\mathcal{M}(\partial_r; r, \theta)_{k,l,\omega_\epsilon}$ amounts to the determination of the 2-dimensional Green's matrix of the radial ODE system (3.19) extended to complex-valued frequencies. Rewriting the Dirac equation (5.32) in the factorized form

$$-(\Sigma + 2Mr)^{-1}\, \mathcal{B}(r, \theta) \sum_{l\in\mathbb{Z}} \mathcal{R}(\partial_r; r)_{k,l,\omega_\epsilon}\, Q_l\, \Psi = \mathbf{0}\,,$$

where the matrix $\mathcal{B}(r, \theta)$ and the matrix-valued radial differential operator $\mathcal{R}(\partial_r; r)_{k,l,\omega_\epsilon}$ read

$$\mathcal{B}(r, \theta) := \begin{pmatrix} \mathrm{i} & a\sin(\theta) & 0 & 0 \\ \dfrac{a\sin(\theta)}{r_+} & -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+} & 0 & 0 \\ 0 & 0 & -\dfrac{\mathrm{i}\,[\Delta + 4Mr]}{r_+} & \dfrac{a\sin(\theta)}{r_+} \\ 0 & 0 & a\sin(\theta) & \mathrm{i} \end{pmatrix}$$

and

$$\mathcal{R}(\partial_r; r)_{k,l,\omega_\epsilon} := \begin{pmatrix} O_{k,\omega_\epsilon} & 0 & r_+ S_l & 0 \\ 0 & U_{\omega_\epsilon} & 0 & \overline{S}_l \\ \overline{S}_l & 0 & U_{\omega_\epsilon} & 0 \\ 0 & r_+ S_l & 0 & O_{k,\omega_\epsilon} \end{pmatrix},$$

we can easily bring it into the block diagonal representation

$$-\mathscr{E}^{-1}(r,\theta)\sum_{l\in\mathbb{Z}}\begin{pmatrix} \mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon} \end{pmatrix}\mathscr{C}^{-1}Q_l\Psi = \mathbf{0}\,, \tag{5.33}$$

in which

$$\begin{pmatrix} \mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{R}^{2\times 2}(\partial_r; r)_{k,l\omega_\epsilon} \end{pmatrix} = \mathscr{C}^{-1}\,\mathcal{R}(\partial_r; r)_{k,l,\omega_\epsilon}\mathscr{C} \tag{5.34}$$

with

$$\mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon} := \begin{pmatrix} O_{k,\omega_\epsilon} & r_+ S_l \\ \overline{S}_l & U_{\omega_\epsilon} \end{pmatrix}, \tag{5.35}$$

and the matrices $\mathscr{E}^{-1}(r,\theta)$ and $\mathscr{C}$ are defined as

$$\mathscr{E}^{-1}(r,\theta) := (\Sigma + 2Mr)^{-1}\,\mathcal{B}(r,\theta)\,\mathscr{C}$$

and

$$\mathscr{C} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

From the specific form of (5.33), it is obvious that the key quantity in the determination of the resolvent of $\mathcal{M}(\partial_r; r,\theta)_{k,l,\omega_\epsilon}$, and thus of the resolvent $(H_k - \omega_\epsilon)^{-1}$, is the solution $G(r; r')_{k,l,\omega_\epsilon}$ of the distributional equation

$$\mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon}\,G(r; r')_{k,l,\omega_\epsilon} = \delta(r - r')\,\mathbb{1}_{\mathbb{C}^2}\,. \tag{5.36}$$

We point out that since (5.35) is identical to the operator $\mathbb{1}_{\mathbb{C}^2}\partial_r - U(r)$ in the radial ODE system (3.19) extended to complex-valued frequencies, $G(r; r')_{k,l,\omega_\epsilon}$ coincides with the associated Green's matrix. Besides, it is only in the case of complex-valued frequencies that the solution of the radial ODE system may attain an additional dampening contribution guaranteeing that $\Psi(r,\theta)$ is in $L^2\big((r_0,\infty)\times(0,\pi), S\mathrm{M}\big)$. Now, in order to solve the distributional equation (5.36), we first introduce the vector-valued functions

$$\Phi_1(r; r') = \begin{pmatrix} \Phi_{1,1}(r; r') \\ \Phi_{1,2}(r; r') \end{pmatrix} \quad \text{and} \quad \Phi_2(r; r') = \begin{pmatrix} \Phi_{2,1}(r; r') \\ \Phi_{2,2}(r; r') \end{pmatrix}$$

that

- for $r \neq r'$ are linearly independent solutions of the homogeneous equation

$$\mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon}\Phi(r; r') = \mathbf{0}\,,$$

- have jump discontinuities at $r = r'$,
- satisfy the Dirichlet-type boundary condition (5.27) at $r = r_0$,

- and are square-integrable, i.e.,

$$\left\|\Phi_{1/2}(r;r')\right\|_2^2 = \int_{r_0}^{\infty} \left\|\Phi_{1/2}(r;r')\right\|^2 \mathrm{d}r < \infty\,.$$

These fundamental solutions are specified in Appendix B. It turns out that their r'-dependencies can be chosen in such a way that they are solely contained in factors of Heaviside step functions Θ. For clarity, these Heaviside step functions are explicitly stated in what follows, which makes it possible to represent Φ_1 and Φ_2 as functions of only the variable r. Moreover, from the first and the last of the above properties as well as from the complex extensions of Lemma 3.5.1, Lemma 3.5.2, and Lemma 3.5.3, we can infer that they have the specific asymptotics

$$\Phi_{1/2}(r) \sim e^{\mathrm{i}\phi_+(r_\star(r))} \begin{pmatrix} \dfrac{c_{1,\infty}}{\sqrt{\Delta}} \\ c_{2,\infty} \end{pmatrix} \qquad \text{for} \quad r \to \infty \quad \text{and} \quad \begin{cases} \mathrm{Im}(\omega_\epsilon) < 0 & \text{if} \quad |\omega_\epsilon| > m \\ \mathrm{Re}(\omega_\epsilon) \geq 0 & \text{if} \quad |\omega_\epsilon| < m \end{cases}$$

$$\Phi_{1/2}(r) \sim e^{-\mathrm{i}\phi_-(r_\star(r))} \begin{pmatrix} \dfrac{c_{3,\infty}}{\sqrt{\Delta}} \\ c_{4,\infty} \end{pmatrix} \qquad \text{for} \quad r \to \infty \quad \text{and} \quad \begin{cases} \mathrm{Im}(\omega_\epsilon) > 0 & \text{if} \quad |\omega_\epsilon| > m \\ \mathrm{Re}(\omega_\epsilon) < 0 & \text{if} \quad |\omega_\epsilon| < m \end{cases}$$

$$\Phi_{1/2}(r) \sim \begin{pmatrix} \dfrac{c_{1,r_\pm}}{\sqrt{|\Delta|}}\, e^{2\mathrm{i}\left(\omega_\epsilon + k\Omega_{\mathrm{Kerr}}^{(\pm)}\right) r_\star(r)} \\ c_{2,r_\pm} \end{pmatrix} \qquad \text{for} \quad \begin{cases} r \to r_+ & \text{and} \quad \mathrm{Im}(\omega_\epsilon) < 0 \\ r \to r_- & \text{and} \quad \mathrm{Im}(\omega_\epsilon) > 0 \end{cases}$$

$$\Phi_{1/2}(r) \sim c_{3,r_\pm} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{for} \quad \begin{cases} r \to r_+ & \text{and} \quad \mathrm{Im}(\omega_\epsilon) > 0 \\ r \to r_- & \text{and} \quad \mathrm{Im}(\omega_\epsilon) < 0\,, \end{cases}$$

where $c_{n,\infty}$ and $c_{m,r_\pm}$, with $n \in \{1,2,3,4\}$ as well as $m \in \{1,2,3\}$, are scalar constants. We then use the ansatz

$$\begin{aligned} &G(r;r')_{k,l,\omega_\epsilon} \\ &= \begin{cases} \Theta(r-r')\,\Phi_1(r)\,P_1(r') + \Theta(r'-r)\,\Phi_2(r)\,P_2(r') & \text{for} \quad r_+ < r' < \infty \quad \text{and} \quad r_0 \leq r' \leq r_- \\ \Theta(r-r')\,\Phi_1(r)\,P_1(r') + \Theta(r-r')\,\Phi_2(r)\,P_2(r') & \text{for} \quad r_- < r' \leq r_+ \end{cases} \end{aligned} \tag{5.37}$$

for the radial Green's matrix in case

$$|\omega_\epsilon| > m \quad \text{and} \quad \mathrm{Im}(\omega_\epsilon) < 0 \quad \text{or} \quad |\omega_\epsilon| < m \quad \text{and} \quad \mathrm{Re}(\omega_\epsilon) \geq 0\,,$$

whereas in case

$$|\omega_\epsilon| > m \quad \text{and} \quad \mathrm{Im}(\omega_\epsilon) > 0 \quad \text{or} \quad |\omega_\epsilon| < m \quad \text{and} \quad \mathrm{Re}(\omega_\epsilon) < 0\,,$$

we employ the ansatz

$$\begin{aligned} &G(r;r')_{k,l,\omega_\epsilon} \\ &= \begin{cases} \Theta(r-r')\,\Phi_1(r)\,P_1(r') + \Theta(r'-r)\,\Phi_2(r)\,P_2(r') & \text{for} \quad r_+ < r' < \infty \quad \text{and} \quad r_0 \leq r' \leq r_- \\ \Theta(r'-r)\,\Phi_1(r)\,P_1(r') + \Theta(r'-r)\,\Phi_2(r)\,P_2(r') & \text{for} \quad r_- < r' \leq r_+\,, \end{cases} \end{aligned} \tag{5.38}$$

in which P_1 and P_2 are unknowns yet to be determined. Applying the radial operator (5.35) to (5.37) and (5.38), we obtain

$$\begin{aligned}&\mathcal{R}^{2\times 2}(\partial_r; r)_{k,l,\omega_\epsilon}\, G(r;r')_{k,l,\omega_\epsilon}\\ &= \begin{pmatrix} \Delta & 0 \\ 0 & r_+ \end{pmatrix} \delta(r-r') \begin{cases} \left[\ \Phi_1(r')P_1(r') \mp \Phi_2(r')P_2(r')\right] & \text{for } (5.37) \\ \left[\pm\Phi_1(r')P_1(r') - \Phi_2(r')P_2(r')\right] & \text{for } (5.38)\,. \end{cases}\end{aligned}$$

Comparing these equations with (5.36) yields the two systems

$$\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & r_+^{-1} \end{pmatrix} = \begin{cases} \Phi_1(r')P_1(r') \mp \Phi_2(r')P_2(r') & \text{for } (5.37) \\ \pm\Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') & \text{for } (5.38)\,. \end{cases}$$

Their solutions $P_{1/2}(r')$ read

$$P_{1,1}(r') = \frac{\Phi_{2,2}(r')}{\Delta(r')\,W(r')}\,,\ P_{1,2}(r') = -\frac{\Phi_{2,1}(r')}{r_+ W(r')}\,,\ P_{2,1}(r') = \pm\frac{\Phi_{1,2}(r')}{\Delta(r')\,W(r')}\,,\ P_{2,2}(r') = \mp\frac{\Phi_{1,1}(r')}{r_+ W(r')}$$

for the ansatz (5.37) and

$$P_{1,1}(r') = \pm\frac{\Phi_{2,2}(r')}{\Delta(r')\,W(r')}\,,\ P_{1,2}(r') = \mp\frac{\Phi_{2,1}(r')}{r_+ W(r')}\,,\ P_{2,1}(r') = \frac{\Phi_{1,2}(r')}{\Delta(r')\,W(r')}\,,\ P_{2,2}(r') = -\frac{\Phi_{1,1}(r')}{r_+ W(r')}$$

for the ansatz (5.38), where

$$W(r') = W(\Phi_1, \Phi_2)(r') := \Phi_{1,1}(r')\,\Phi_{2,2}(r') - \Phi_{1,2}(r')\,\Phi_{2,1}(r')$$

is the Wronskian. Inserting these expressions into (5.37) and (5.38), respectively, leads in case $r_+ < r' < \infty$ or $r_0 \leq r' \leq r_-$ to the Green's matrix

$$\begin{aligned} G(r;r')_{k,l,\omega_\epsilon} = \frac{1}{W(r')} \Bigg[&\Theta(r-r') \begin{pmatrix} \dfrac{\Phi_{1,1}(r)\Phi_{2,2}(r')}{\Delta(r')} & -\dfrac{\Phi_{1,1}(r)\Phi_{2,1}(r')}{r_+} \\[2ex] \dfrac{\Phi_{1,2}(r)\Phi_{2,2}(r')}{\Delta(r')} & -\dfrac{\Phi_{1,2}(r)\Phi_{2,1}(r')}{r_+} \end{pmatrix}_{k,l,\omega_\epsilon} \\ &+\Theta(r'-r) \begin{pmatrix} \dfrac{\Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & -\dfrac{\Phi_{2,1}(r)\Phi_{1,1}(r')}{r_+} \\[2ex] \dfrac{\Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & -\dfrac{\Phi_{2,2}(r)\Phi_{1,1}(r')}{r_+} \end{pmatrix}_{k,l,\omega_\epsilon} \Bigg] \end{aligned} \tag{5.39}$$

for both (5.37) and (5.38), whereas in case $r_- < r' \leq r_+$ it results in the Green's matrices

$$\begin{aligned} G(r;r')_{k,l,\omega_\epsilon} &= \frac{1}{W(r')} \begin{pmatrix} \dfrac{\Phi_{1,1}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \dfrac{\Phi_{2,1}(r)\Phi_{1,1}(r') - \Phi_{1,1}(r)\Phi_{2,1}(r')}{r_+} \\[2ex] \dfrac{\Phi_{1,2}(r)\Phi_{2,2}(r') - \Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & \dfrac{\Phi_{2,2}(r)\Phi_{1,1}(r') - \Phi_{1,2}(r)\Phi_{2,1}(r')}{r_+} \end{pmatrix}_{k,l,\omega_\epsilon} \\ &\quad \times \begin{cases} \Theta(r-r') & \text{for } (5.37) \\ -\Theta(r'-r) & \text{for } (5.38)\,. \end{cases} \end{aligned} \tag{5.40}$$

Subsequently, we read off the resolvent of the Dirac Hamiltonian for fixed k-modes directly from the block-diagonalized representation (5.33). We thus find

$$(H_k - \omega_\epsilon)^{-1}\Psi = -\sum_{l\in\mathbb{Z}} Q_l \int_{r_0}^{\infty} \mathscr{C} \begin{pmatrix} G(r;r')_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & G(r;r')_{k,l,\omega_\epsilon} \end{pmatrix} \mathscr{E}(r',\theta)\,\Psi(r',\theta)\,\mathrm{d}r', \tag{5.41}$$

where the Green's matrix $G(r;r')_{k,l,\omega_\epsilon}$ is given by (5.39) and (5.40). To show that this expression is indeed the desired resolvent, we verify the identity

$$(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\Psi = \Psi\,. \tag{5.42}$$

Applying the operator defined in (5.33) to (5.41), we obtain in a first step

$$\begin{aligned} &(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\Psi \\ &= \mathscr{E}^{-1}(r,\theta) \sum_{l\in\mathbb{Z}} \begin{pmatrix} \mathcal{R}^{2\times 2}(\partial_r;r)_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{R}^{2\times 2}(\partial_r;r)_{k,l,\omega_\epsilon} \end{pmatrix} \mathscr{C}^{-1} \int_{-1}^{1} Q_l(\theta;\theta') \Bigg\{ \sum_{m\in\mathbb{Z}} \int_{-1}^{1} Q_m(\theta';\theta'') \\ &\quad \times \int_{r_0}^{\infty} \mathscr{C} \begin{pmatrix} G(r;r')_{k,m,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & G(r;r')_{k,m,\omega_\epsilon} \end{pmatrix} \mathscr{E}(r',\theta'')\,\Psi(r',\theta'')\,\mathrm{d}r'\,\mathrm{d}\big(\cos(\theta'')\big) \Bigg\}\,\mathrm{d}\big(\cos(\theta')\big)\,. \end{aligned}$$

Moving the integral kernel $Q_l(\theta;\theta')$ of the spectral projector into the θ''-integral and taking into account that the spectral projectors are idempotent, i.e., their integral kernels satisfy the relation

$$Q_l(\theta;\theta')\,Q_m(\theta';\theta'') = \delta_{lm}\,\delta\big(\cos(\theta) - \cos(\theta')\big)\,Q_m(\theta;\theta'')\,,$$

we infer, after evaluating both the θ'-integral and the sum over all integers m, that

$$\begin{aligned} (H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\Psi &= \mathscr{E}^{-1}(r,\theta) \sum_{l\in\mathbb{Z}} \begin{pmatrix} \mathcal{R}^{2\times 2}(\partial_r;r)_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{R}^{2\times 2}(\partial_r;r)_{k,l,\omega_\epsilon} \end{pmatrix} \mathscr{C}^{-1} \\ &\quad \times \int_{-1}^{1} Q_l(\theta;\theta'') \int_{r_0}^{\infty} \mathscr{C} \begin{pmatrix} G(r;r')_{k,l,\omega_\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & G(r;r')_{k,l,\omega_\epsilon} \end{pmatrix} \mathscr{E}(r',\theta'')\,\Psi(r',\theta'')\,\mathrm{d}r'\,\mathrm{d}\big(\cos(\theta'')\big)\,. \end{aligned}$$

Next, we move the constant matrix $\mathscr{C}^{-1}$ as well as the matrix-valued radial differential operator (5.34) into the θ''- and the r'-integral. Employing (5.36) yields

$$(H_k - \omega_\epsilon)(H_k - \omega_\epsilon)^{-1}\Psi = \mathscr{E}^{-1}(r,\theta)\sum_{l\in\mathbb{Z}} Q_l \int_{r_0}^{\infty} \delta(r - r')\,\mathscr{E}(r',\theta)\,\Psi(r',\theta)\,\mathrm{d}r'.$$

Solving the integral with respect to the variable r' and substituting the completeness constraint for the spectral projectors (5.31), we immediately obtain the identity (5.42).

Having established the explicit form of the resolvent $(H_k - \omega_\epsilon)^{-1}$ in (5.41), we now derive the integral spectral representation of the Dirac propagator. To this end, we express the Dirac spinor ψ' evaluated at an arbitrary time τ via the propagator $U^{\tau,0} = e^{-\mathrm{i}\tau H}$ applied to smooth initial data with compact support ψ'_0 given at time $\tau = 0$ and expand the initial data in terms of k-modes

$$\psi' = e^{-\mathrm{i}\tau H}\psi'_0 = \sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi}\, e^{-\mathrm{i}\tau H_k}\,\psi'_{0,k}\,. \tag{5.43}$$

We furthermore introduce the spectral projector of the Dirac Hamiltonian for fixed k-modes H_k onto the interval $I \subset \mathbb{R}$

$$P_I(H_k) = \chi_I(H_k)\,,$$

where χ_I denotes the characteristic function

$$\chi_I(H_k) := \begin{cases} 1 & \text{for } \omega \in I \\ 0 & \text{for } \omega \notin I \end{cases}$$

with $\omega \in \sigma(H_k)$. Then, making use of the identity relation

$$P_{(-\infty,\infty)}(H_k) = \mathbb{1}\,,$$

we write (5.43) as

$$\psi' = \sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi}\, e^{-\mathrm{i}\tau H_k} \lim_{a\to\infty} P_{(-a,a)}(H_k)\,\psi'_{0,k}$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi} \lim_{a\to\infty} e^{-\mathrm{i}\tau H_k}\big[P_{(-a,a)}(H_k) + P_{[-a,a]}(H_k)\big]\psi'_{0,k}\,.$$

Employing Stone's formula (2.15), which in our framework reads

$$e^{-\mathrm{i}\tau H_k}\big[P_{(-a,a)}(H_k) + P_{[-a,a]}(H_k)\big]\psi'_{0,k}$$

$$= \lim_{\epsilon\searrow 0}\frac{1}{\pi\mathrm{i}}\int_{-a}^{a} e^{-\mathrm{i}\omega\tau}\big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big]\psi'_{0,k}\,\mathrm{d}\omega\,,$$

we obtain

$$\psi' = \frac{1}{2\pi\mathrm{i}}\sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k\phi}\lim_{a\to\infty}\lim_{\epsilon\searrow 0}\int_{-a}^{a} e^{-\mathrm{i}\omega\tau}\big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big]\psi'_{0,k}\,\mathrm{d}\omega\,,$$

where the resolvents are given by (5.41). Since the fundamental radial solutions that occur in the resolvents are bounded for all $\omega \in \mathbb{R}$ and all $\epsilon > 0$ as shown in Appendix B, we are allowed to apply Lebesgue's dominated convergence theorem and commute the ϵ-limit and the integral with respect to ω, yielding

$$\psi'(\tau, r, \theta, \phi) = \frac{1}{2\pi \mathrm{i}} \sum_{k \in \mathbb{Z}} e^{-\mathrm{i}k\phi} \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau} \lim_{\epsilon \searrow 0} \big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big](r, \theta; r', \theta') \\ \times \psi'_{0,k}(r', \theta')\, \mathrm{d}\omega \,. \tag{5.44}$$

Finally, comparing this expression with formula (5.29), we may directly identify the spectral measure $\mathrm{d}E_\omega$ of the Dirac Hamiltonian H. ■

5.5 Simplified Form of the Integral Spectral Representation of the Dirac Propagator

The integral spectral representation (5.44) can be further simplified, on the one hand, by performing the limit $r_0 \nearrow r_-$ and, on the other hand, by computing the difference of the two resolvents $(H_k - \omega - \mathrm{i}\epsilon)^{-1}$ and $(H_k - \omega + \mathrm{i}\epsilon)^{-1}$ for $\epsilon \searrow 0$. Below, we explicitly work out the case $|\omega_\epsilon| > m$. The case $|\omega_\epsilon| < m$ may be treated similarly. As the fundamental solutions $\Phi_1(r; r')$ and $\Phi_2(r; r')$, which constitute the radial Green's matrix $G(r; r')_{k,l,\omega_\epsilon}$ and therefore the resolvent $(H_k - \omega_\epsilon)^{-1}$, are defined piecewise for the domains $r_+ < r' < \infty$, $r_- < r' \leq r_+$, and $r_0 \leq r' \leq r_-$, we begin by splitting the r'-integral in the difference of resolvents in the limit $\epsilon \searrow 0$ into the three associated contributions

$$\lim_{\epsilon \searrow 0} \big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big]\psi'_{0,k} = \lim_{\epsilon \searrow 0} \sum_{l \in \mathbb{Z}} Q_l \bigg(\int_{r_0}^{r_-} + \int_{r_-}^{r_+} + \int_{r_+}^{\infty}\bigg) \mathscr{C}$$

$$\times \begin{pmatrix} G(r; r')_{k,l,\omega - \mathrm{i}\epsilon} - G(r; r')_{k,l,\omega + \mathrm{i}\epsilon} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & G(r; r')_{k,l,\omega - \mathrm{i}\epsilon} - G(r; r')_{k,l,\omega + \mathrm{i}\epsilon} \end{pmatrix} \mathscr{E}(r', \theta)\, \psi'_{0,k}(r', \theta)\, \mathrm{d}r'.$$

Since the integrands – and hence every summand of the sum with respect to l – are bounded for all values of ϵ, r', and θ (see Appendix B and recall that the initial data for fixed k-modes $\psi'_{0,k}$ has compact support), we can again employ Lebesgue's dominated convergence theorem, which allows us to commute the limit $\epsilon \searrow 0$ with the sum over the integers l, the spectral projector Q_l, and the integrals with respect to r'. Then, applying the limit $r_0 \nearrow r_-$ to the integral spectral representation (5.44) and commuting this limit with the sum over the integers k, the integral with respect to ω, and lastly the sum over the integers l as well as the spectral projector Q_l (which are contained in the difference of resolvents) using the same reasoning as before, we obtain the expression

$$\lim_{r_0 \nearrow r_-} \lim_{\epsilon \searrow 0} \big[(H_k - \omega - \mathrm{i}\epsilon)^{-1} - (H_k - \omega + \mathrm{i}\epsilon)^{-1}\big]\psi'_{0,k} = \sum_{l \in \mathbb{Z}} Q_l \bigg(\int_{r_-}^{r_+} + \int_{r_+}^{\infty}\bigg) \mathscr{C}$$

$$\times \begin{pmatrix} \lim\limits_{\epsilon \searrow 0} \big[G(r; r')_{k,l,\omega - \mathrm{i}\epsilon} - G(r; r')_{k,l,\omega + \mathrm{i}\epsilon}\big] & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \lim\limits_{\epsilon \searrow 0} \big[G(r; r')_{k,l,\omega - \mathrm{i}\epsilon} - G(r; r')_{k,l,\omega + \mathrm{i}\epsilon}\big] \end{pmatrix} \tag{5.45}$$

$$\times \mathscr{E}(r', \theta)\, \psi'_{0,k}(r', \theta)\, \mathrm{d}r' .$$

In order to compute the limit $\epsilon \searrow 0$ of the difference of the radial Green's matrices $G(r;r')_{k,l,\omega-\mathrm{i}\epsilon}$ and $G(r;r')_{k,l,\omega+\mathrm{i}\epsilon}$ in the domain $r_+ < r' < \infty$, we introduce the auxiliary functions (cf. Appendix B)

$$\chi_1(r) := \lim_{\epsilon\searrow 0} \check{\Phi}^{(\infty)}(r) \quad \text{and} \quad \chi_2(r) := \lim_{\epsilon\searrow 0} \widehat{\Phi}^{(\infty)}(r)\,,$$

and write the ϵ-limits of the fundamental radial solutions Φ_1 and Φ_2 as

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \Phi_1 = \chi_1 \quad \text{and} \quad \lim_{\epsilon\searrow 0} \Phi_2 = \alpha\chi_1 + \beta\chi_2 \quad \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0 \\ &\lim_{\epsilon\searrow 0} \Phi_1 = \chi_2 \quad \text{and} \quad \lim_{\epsilon\searrow 0} \Phi_2 = \gamma\chi_1 + \delta\chi_2 \quad \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0\,, \end{aligned} \tag{5.46}$$

where α, β, γ, and δ are constants. The corresponding Wronskian yields

$$\lim_{\epsilon\searrow 0} W(\Phi_1,\Phi_2) = \begin{cases} \beta\, W(\chi_1,\chi_2) & \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0 \\ -\gamma\, W(\chi_1,\chi_2) & \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0\,. \end{cases} \tag{5.47}$$

Substituting (5.46) and (5.47) into (5.39) results in

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \left[G(r;r')_{k,l,\omega-\mathrm{i}\epsilon} - G(r;r')_{k,l,\omega+\mathrm{i}\epsilon}\right]_{|r_+<r'<\infty} \\ &= \frac{1}{W(\chi_1,\chi_2)(r')} \sum_{u,v=1}^{2} T_{u,v} \begin{pmatrix} -\dfrac{\chi_{u,1}(r)\,\chi_{v,2}(r')}{\Delta(r')} & \dfrac{\chi_{u,1}(r)\,\chi_{v,1}(r')}{r_+} \\[2ex] -\dfrac{\chi_{u,2}(r)\,\chi_{v,2}(r')}{\Delta(r')} & \dfrac{\chi_{u,2}(r)\,\chi_{v,1}(r')}{r_+} \end{pmatrix}_{k,l,\omega} \end{aligned} \tag{5.48}$$

with the coefficients

$$T_{1,1} = \frac{\alpha}{\beta}\,, \quad T_{1,2} = T_{2,1} = 1\,, \quad \text{and} \quad T_{2,2} = \frac{\delta}{\gamma}\,.$$

For the domain $r_- \le r' \le r_+$ on the other hand, we define the auxiliary functions

$$\widetilde{\chi}_1(r) := \Theta(r_+ - r) \lim_{\epsilon\searrow 0} \widehat{\Phi}^{(-)}(r) + \Theta(r - r_+) \lim_{\epsilon\searrow 0} \widehat{\Phi}^{(\infty)}(r) \quad \text{and} \quad \widetilde{\chi}_2(r) := \Theta(r - r_-) \lim_{\epsilon\searrow 0} \check{\Phi}^{(-)}(r)\,,$$

and express the ϵ-limits of the fundamental radial solutions by

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \Phi_1 = \alpha'\widetilde{\chi}_1 + \beta'\widetilde{\chi}_2 \quad \text{and} \quad \lim_{\epsilon\searrow 0} \Phi_2 = \widetilde{\chi}_2 \quad \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0 \\ &\lim_{\epsilon\searrow 0} \Phi_1 = \widetilde{\chi}_1 \quad \text{and} \quad \lim_{\epsilon\searrow 0} \Phi_2 = \gamma'\widetilde{\chi}_1 + \delta'\widetilde{\chi}_2 \quad \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0\,, \end{aligned} \tag{5.49}$$

where α', β', γ', and δ' are also constants. In this case, the Wronskian becomes

$$\lim_{\epsilon\searrow 0} W(\Phi_1,\Phi_2) = \begin{cases} \alpha'\, W(\widetilde{\chi}_1,\widetilde{\chi}_2) & \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0 \\ \delta'\, W(\widetilde{\chi}_1,\widetilde{\chi}_2) & \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0\,. \end{cases} \tag{5.50}$$

Using (5.49) and (5.50) in (5.40) leads to

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \left[G(r;r')_{k,l,\omega-\mathrm{i}\epsilon} - G(r;r')_{k,l,\omega+\mathrm{i}\epsilon}\right]_{|r_-\le r'\le r_+} \\ &= \frac{1}{W(\widetilde{\chi}_1,\widetilde{\chi}_2)(r')} \sum_{u,v=1}^{2} \widetilde{T}_{u,v} \begin{pmatrix} \dfrac{\widetilde{\chi}_{u,1}(r)\,\widetilde{\chi}_{v,2}(r')}{\Delta(r')} & -\dfrac{\widetilde{\chi}_{u,1}(r)\,\widetilde{\chi}_{v,1}(r')}{r_+} \\[2ex] \dfrac{\widetilde{\chi}_{u,2}(r)\,\widetilde{\chi}_{v,2}(r')}{\Delta(r')} & -\dfrac{\widetilde{\chi}_{u,2}(r)\,\widetilde{\chi}_{v,1}(r')}{r_+} \end{pmatrix}_{k,l,\omega} \end{aligned} \tag{5.51}$$

with the coefficients

$$\widetilde{T}_{1,1} = \widetilde{T}_{2,2} = 0 \quad \text{and} \quad \widetilde{T}_{1,2} = -\widetilde{T}_{2,1} = 1\,.$$

Denoting (5.48) and (5.51) by $\mathcal{G}(r; r_+ < r' < \infty)_{k,l,\omega}$ and $\mathcal{G}(r; r_- \leq r' \leq r_+)_{k,l,\omega}$, respectively, and inserting these quantities into (5.45), the Dirac propagator (5.44) yields

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}} \sum_{k,l\in\mathbb{Z}} e^{-\mathrm{i}k\phi} \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\, Q_l\, \mathscr{C} \Bigg[\int_{r_-}^{r_+} \begin{pmatrix} \mathcal{G}(r; r_- \leq r' \leq r_+)_{k,l,\omega} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{G}(r; r_- \leq r' \leq r_+)_{k,l,\omega} \end{pmatrix} \\
&+ \int_{r_+}^{\infty} \begin{pmatrix} \mathcal{G}(r; r_+ < r' < \infty)_{k,l,\omega} & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathcal{G}(r; r_+ < r' < \infty)_{k,l,\omega} \end{pmatrix}\Bigg] \mathscr{E}(r', \theta)\, \psi'_{0,k}(r', \theta)\, \mathrm{d}r'\, \mathrm{d}\omega \\
&= \frac{1}{2\pi\mathrm{i}} \sum_{k,l\in\mathbb{Z}} e^{-\mathrm{i}k\phi} \int_{\mathbb{R}} e^{-\mathrm{i}\omega\tau}\, Q_l\, \mathscr{C} \Bigg(\mathbb{1}_{\mathbb{C}^2} \otimes \Bigg[\int_{r_-}^{r_+} \mathcal{G}(r; r_- \leq r' \leq r_+)_{k,l,\omega} \\
&+ \int_{r_+}^{\infty} \mathcal{G}(r; r_+ < r' < \infty)_{k,l,\omega}\Bigg]\Bigg) \mathscr{E}(r', \theta)\, \psi'_{0,k}(r', \theta)\, \mathrm{d}r'\, \mathrm{d}\omega = \psi'(\tau, r, \theta, \phi)\,.
\end{aligned} \tag{5.52}$$

Given in this form, our generalized horizon-penetrating integral spectral representation resembles the one derived in [39], which is limited to the region outside the event horizon.

CHAPTER 6

OUTLOOK

The primary objective of this doctoral thesis was to derive a generalized integral spectral representation of the massive Dirac propagator in the nonextreme Kerr geometry in terms of horizon-penetrating advanced Eddington–Finkelstein-type coordinates by combining functional analytic concepts and methods with results from Chandrasekhar's mode analysis. Such an integral spectral representation may be used to properly study the long-time dynamics and the decay rates of relativistic spin-$1/2$ fermions in a rotating black hole spacetime. This was, however, already worked out in detail in [37–39], where an integral spectral representation of the massive Dirac propagator expressed in Boyer–Lindquist coordinates, which is restricted to the exterior of the nonextreme Kerr geometry, was employed. Reproducing these computations with the generalized propagator yields the same results. New applications arise in situations that also include the interior Kerr geometry and require regularity at the horizons. One of these is the formulation of an algebraic quantum field theory for the Dirac field in a rotating black hole spacetime. In this regard, we intend to determine the so-called fermionic signature operator introduced in [44, 45], which is a symmetric operator on the solution space of the massive Dirac equation in globally hyperbolic spacetimes, both in the exterior of the nonextreme Kerr geometry as well as in its analytic extension across the event horizon, up to the Cauchy horizon. This operator gives rise to pure quasi-free fermionic Fock ground states for Dirac fields [34, 35, 42].

In more detail, on solutions $\psi_m, \chi_m \in C^\infty_{\text{sc}}(\mathfrak{M}, S\mathfrak{M})$ of the Dirac equation for a fixed mass parameter $m \in \mathbb{R}$ in a globally hyperbolic Lorentzian 4-manifold $(\mathfrak{M} = \mathbb{R} \times \mathfrak{N}, \boldsymbol{g})$ with spinor bundle $S\mathfrak{M}$, we can define the following two inner products: the first one is the Lorentz invariant inner product obtained by integrating the pointwise indefinite spin scalar product $\prec . \,|\, . \succ_{(t,\boldsymbol{x})}$ of Dirac spinors on fibers $S_{(t,\boldsymbol{x})}\mathfrak{M}$, $(t, \boldsymbol{x}) \in \mathfrak{M}$, specified in (5.12) over the entire manifold

$$\langle \psi_m | \chi_m \rangle := \int_{\mathfrak{M}} \prec \psi_m | \chi_m \succ_{(t,\boldsymbol{x})} \mathrm{d}\mu_{\mathfrak{M}} \,, \tag{6.1}$$

whereas the second one is the scalar product obtained by integrating the polarized probability density over the Cauchy hypersurface $\mathfrak{N}$ (cf. (5.11))

$$(\psi_m | \chi_m)_m := \int_{\mathfrak{N}} \prec \psi_m | \boldsymbol{\nu}\!\!\!/ \chi_m \succ_{(t=0,\boldsymbol{x})} \mathrm{d}\mu_{\mathfrak{N}} \,. \tag{6.2}$$

We note that the latter can be used to give the solution space of the Dirac equation the structure of a Hilbert space $(\mathcal{H}_m, (. \,|\, .)_m)$ with corresponding norm $\| . \|_m$. The fermionic signature operator S_m

arises when representing the inner product (6.1) in terms of the scalar product (6.2). In spacetimes of finite lifetime [44], this amounts to a relation of the form

$$\langle \psi_m | \chi_m \rangle = (\psi_m | S_m \chi_m)_m \quad \text{for all} \quad \psi_m, \chi_m \in \mathcal{H}_m \,, \tag{6.3}$$

which uniquely defines S_m as a symmetric bounded self-adjoint operator on $\mathcal{H}_m$ with $\|S_m\| \leq 2$. In spacetimes of infinite lifetime [45], however, such a relation is in general ill-defined on solutions of the Dirac equation, as the time integration in (6.1) may diverge. A method to overcome this problem is to make use of so-called mass oscillations. That is, instead of analyzing solutions for a fixed mass parameter m, one considers families $\psi := (\psi_m)_{m \in I}$ of Dirac solutions for m varying in an interval $I := (m_{\mathrm{L}}, m_{\mathrm{R}}) \backslash \{0\}$. Then, integrating over the mass parameter

$$\mathfrak{p}\psi := \int_I \psi_m \, \mathrm{d}m \,,$$

we obtain a superposition of waves oscillating with different frequencies, which leads to destructive interference and thus yields the desired decay of the Dirac spinors for large times. This makes it possible to replace (6.3) by the condition

$$\langle \mathfrak{p}\psi | \mathfrak{p}\chi \rangle = \int_I (\psi_m | S_m \chi_m)_m \, \mathrm{d}m \tag{6.4}$$

that has to hold for all families of solutions $(\psi_m)_{m \in I}$ and $(\chi_m)_{m \in I}$ in a suitably chosen dense subspace $\mathcal{H}^\infty$ of the Hilbert space of families of solutions, referred to as the domain for the mass oscillations. This construction gives rise to a uniquely defined bounded linear operator S_m on $\mathcal{H}_m$ for every $m \in I$. We remark that the conditions required for the construction to work are subsumed in various notions of mass oscillation properties [44, 45]. Furthermore, considering spacetimes that include either horizons or curvature singularities makes it necessary to introduce considerable modifications to (6.4). To give an example, in the exterior Schwarzschild geometry, the main complication is that part of the Dirac wave may cross the event horizon and disappear into the black hole. As a consequence, the mass oscillation properties no longer hold, and a representation of the form (6.4) no longer exists. Instead, we employ a so-called mass decomposition

$$\langle \mathfrak{p}\psi | \mathfrak{p}\chi \rangle = \int_I (\psi_m | S_m \chi_m)_m \, \mathrm{d}m + \frac{\mathrm{i}}{\pi} \int_I \int_I \frac{\mathrm{PP}}{m - m'} \, \mathfrak{B}(\psi_m, \chi_{m'}) \, \mathrm{d}m' \, \mathrm{d}m \,, \tag{6.5}$$

where $\mathfrak{B}(\psi_m, \chi_{m'})$ is a smooth function in m and m', and PP denotes the principal value. The first term on the right hand side again uniquely defines the fermionic signature operator S_m for any fixed $m \in I$, whereas the second term gives a contribution for pairs of solutions ψ_m and $\chi_{m'}$ of the Dirac equation with different masses $m \neq m'$, which can be associated to the flux of the current $J^\mu(t, \boldsymbol{x}) = \prec \psi_m | \gamma^\mu \chi_{m'} \succ_{(t, \boldsymbol{x})}$ through the event horizon of the black hole. Since the positive and negative spectral subspaces of S_m can be used in order to decompose the solution space of the Dirac equation as

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^- \quad \text{with} \quad \mathcal{H}_m^+ := \chi_{(0,\infty)}(S_m) \, \mathcal{H}_m \quad \text{and} \quad \mathcal{H}_m^- := \chi_{(-\infty,0)}(S_m) \, \mathcal{H}_m \,,$$

one may set up a Fock space $\mathcal{F}_m$, in which for the ground state all 1-particle states of $\mathcal{H}_m^-$ are occupied. This ground state can be obtained by applying Araki's construction in [2] to the projection operator onto the negative spectral subspace of S_m. Moreover, it is physically sensible, provided it is of Hadamard form [110]. Specifying anticommutation relations for creation and annihilation operators using the above frequency splitting, one can apply the canonical Fock space construction and analyze the resulting many-particle quantum states similar to the preparatory works [25, 42–46].

So far, we have derived the fermionic signature operator in the exterior Schwarzschild geometry employing the integral spectral representation of the Dirac propagator constructed in [39], computed its spectrum, and analyzed the associated pure quasi-free fermionic Fock ground state for the Dirac field [49]. Besides, we quantified in the mass decomposition (6.5) the boundary contributions at the event horizon. We found that this ground state coincides with the Hadamard state obtained by the usual frequency splitting for an observer in a rest frame at spacelike infinity. Hence, $\mathcal{H}_m^+$ is spanned by all fundamental solutions of the form $e^{-\mathrm{i}\omega t}$ with positive frequency, whereas $\mathcal{H}_m^-$ is spanned by all such solutions with negative frequency. We furthermore aim at deriving and analyzing the fermionic signature operator in the analytic extension of the Schwarzschild geometry across the event horizon, working with the integral spectral representation (5.52) in the limit $|a| \to 0$. Here, the main obstacle is the evaluation of the boundary term at the curvature singularity of the black hole at $r = 0$ in the mass decomposition (6.5). We expect that in this analytic extension, the fermionic signature operator no longer yields the desired frequency splitting. Instead, we presume to find a thermal Hawking–Unruh state, up to corrections due to the mutual couplings of spin and gravity. Eventually, we plan to generalize our constructions to the nonextreme Kerr geometry.

APPENDIX

A

Symmetry of the Dirac Hamiltonian and Dirichlet-type Boundary Condition

We show the symmetry of the Dirac Hamiltonian H with respect to the canonical scalar product $(\,.\,|\,.\,)_{\mathfrak{N}}$ by direct computation. Furthermore, we introduce and discuss the radial Dirichlet-type boundary condition imposed on the Dirac spinors.

Lemma A.0.1. *The Dirac Hamiltonian (5.5) is symmetric with respect to the scalar product (5.17).*

Proof. To establish the symmetry, namely that

$$(\psi'|H\chi')_{\mathfrak{N}} = (H\psi'|\chi')_{\mathfrak{N}}\,,$$

we begin by splitting the potential $\mathscr{V}$ given in (5.9) into mass-independent and mass-dependent parts

$$\mathscr{V} = \mathscr{V}_0 + \mathscr{V}_m\,,$$

where

$$\mathscr{V}_0 := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} \mathscr{B}_1 & \mathbf{0}_{\mathbb{C}^2} \\ \mathbf{0}_{\mathbb{C}^2} & \mathscr{B}_4 \end{pmatrix} \quad \text{and} \quad \mathscr{V}_m := -\frac{1}{\Sigma + 2Mr}\begin{pmatrix} \mathbf{0}_{\mathbb{C}^2} & \mathscr{B}_2 \\ \mathscr{B}_3 & \mathbf{0}_{\mathbb{C}^2} \end{pmatrix}.$$

The 2×2 blocks $\mathscr{B}_k$, with $k \in \{1, 2, 3, 4\}$, are specified in (5.10). This splitting has the advantage of yielding anti-self-adjoint and self-adjoint matrices

$$\Gamma^\tau \mathscr{V}_0 = -\mathscr{V}_0^\dagger \Gamma^\tau \quad \text{and} \quad \Gamma^\tau \mathscr{V}_m = \mathscr{V}_m^\dagger \Gamma^\tau\,, \tag{A.1}$$

for which $\Gamma^\tau = \Gamma^{\tau\dagger}$ is defined in (5.18). We may then write

$$\begin{aligned}(\psi'|H\chi')_{\mathfrak{N}} &= \iiint \psi'^\dagger\, \Gamma^\tau H\, \chi' \sin(\theta)\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r \\ &= \iiint \psi'^\dagger\, \Gamma^\tau \alpha^j \partial_j(\chi') \sin(\theta)\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r + \iiint \psi'^\dagger\, \Gamma^\tau \mathscr{V}_0\, \chi' \sin(\theta)\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r \\ &\quad + \iiint \psi'^\dagger\, \Gamma^\tau \mathscr{V}_m\, \chi' \sin(\theta)\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r\,.\end{aligned}$$

Integration by parts of the first triple integral in the second line and substitution of the relations (A.1) into the remaining two triple integrals results in

$$
\begin{aligned}
(\psi'|H\chi')_{\mathfrak{N}} &= -\iiint \partial_j\big(\psi'^{\dagger}\,\Gamma^{\tau}\alpha^j \sin(\theta)\big)\,\chi'\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r - \iiint \psi'^{\dagger}\,\mathscr{V}_0^{\dagger}\,\Gamma^{\tau}\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&\quad + \iiint \psi'^{\dagger}\,\mathscr{V}_m^{\dagger}\,\Gamma^{\tau}\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= -\iiint \partial_j(\psi'^{\dagger})\,\Gamma^{\tau}\alpha^j\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&\quad - \iiint \psi'^{\dagger}\,[\partial_j(\Gamma^{\tau})\,\alpha^j + \Gamma^{\tau}\partial_j(\alpha^j) + \Gamma^{\tau}\alpha^{\theta}\cot(\theta)]\,\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&\quad - \iiint (\mathscr{V}_0\,\psi')^{\dagger}\,\Gamma^{\tau}\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r + \iiint (\mathscr{V}_m\,\psi')^{\dagger}\,\Gamma^{\tau}\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\,.
\end{aligned}
\tag{A.2}
$$

We remark that in the integration by parts, the angular derivatives do not give rise to boundary terms because the 2-dimensional submanifold S^2 of $\mathfrak{N}$ is compact without boundary. The radial derivative, on the other hand, yields a boundary term that only vanishes because we impose a suitable Dirichlet-type boundary condition on the Dirac spinors. More precisely, with

$$
\Gamma^{\tau}\alpha^r = \mathrm{i}\,\mathrm{diag}\bigg(-\frac{\Delta}{r_+}, r_+, r_+, -\frac{\Delta}{r_+}\bigg)\,,
$$

the radial boundary term becomes

$$
\begin{aligned}
&\iint_{S^2} \psi'^{\dagger}\,\Gamma^{\tau}\alpha^r\chi' \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\bigg|_{r_1}^{r_2} \\
&\qquad = \mathrm{i}r_+ \iint_{S^2} \bigg(-\frac{\Delta}{r_+^2}\,\overline{\psi'}_1\,\chi'_1 + \overline{\psi'}_2\,\chi'_2 + \overline{\psi'}_3\,\chi'_3 - \frac{\Delta}{r_+^2}\,\overline{\psi'}_4\,\chi'_4\bigg) \sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\bigg|_{r_1}^{r_2}\,.
\end{aligned}
$$

In order for this term to vanish, we impose the radial Dirichlet-type boundary condition

$$
\sum_{i=1}^{2} (-1)^i \bigg(-\frac{\Delta}{r_+^2}\,\overline{\psi'}_1\,\chi'_1 + \overline{\psi'}_2\,\chi'_2 + \overline{\psi'}_3\,\chi'_3 - \frac{\Delta}{r_+^2}\,\overline{\psi'}_4\,\chi'_4\bigg)_{|r=r_i} = 0\,. \tag{A.3}
$$

Taking into account that in the construction of the integral spectral representation of the Dirac propagator presented in Chapter 5 we consider only Dirac spinors with spatial support extending from a specific timelike inner boundary surface at $r = r_0 < r_-$ beyond the Cauchy horizon up to infinity, i.e.,

$$
\mathrm{supp}\,\chi'(\tau,\,.\,) = (r_0, \infty) \times S^2\,,
$$

and moreover require that the Dirac spinors are in $L^2\big(\mathbb{R} \times (r_0, \infty) \times S^2, S\mathfrak{M}\big)$, which implies proper decay at infinity, (A.3) reduces to a condition for the timelike inner boundary at $r = r_0$

$$
\bigg(-\frac{\Delta}{r_+^2}\,\overline{\psi'}_1\,\chi'_1 + \overline{\psi'}_2\,\chi'_2 + \overline{\psi'}_3\,\chi'_3 - \frac{\Delta}{r_+^2}\,\overline{\psi'}_4\,\chi'_4\bigg)_{|r=r_0} = 0\,. \tag{A.4}
$$

This condition can be brought into a more convenient form as follows. By means of the spin scalar product (5.12) and the relation $\mathscr{S}'\gamma'^r = \mathrm{i}\,\Gamma^\tau\alpha^r/\Sigma$, we may represent (A.4) as

$$\prec\!\psi'|\gamma'^r\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0} = 0\,. \tag{A.5}$$

Now, introducing $\boldsymbol{n}$ as the unit normal to the hypersurface $\partial\mathfrak{M}$, we can write (A.5) in the form

$$\prec\!\psi'|\not{n}\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0} = 0 \quad \Leftarrow \quad (\not{n} - \mathrm{i}\,\mathcal{K})\psi'_{|r=r_0} = \mathbf{0}\,, \tag{A.6}$$

where $\mathcal{K}$ is an arbitrary matrix with the property $\mathcal{K} = (\mathscr{S}')^{-1}\mathcal{K}^\dagger\mathscr{S}'$. Similar to (4.9), the implication can be easily verified via the calculation

$$\begin{aligned}\prec\!\psi'|\not{n}\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0} &= \prec\!\not{n}\psi'|\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0} = \prec\!\mathrm{i}\mathcal{K}\psi'|\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0}\\ &= -\prec\!\psi'|\mathrm{i}\mathcal{K}\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0} = -\prec\!\psi'|\not{n}\chi'\!\succ_{(\tau_0,\boldsymbol{x})|r=r_0}\,.\end{aligned}$$

To allow for the boundary condition given on the right hand side of (A.6) to be compatible with a potential product representation of the Dirac 4-spinors, in which the dependencies on the variables τ, r, θ, and ϕ are separated (such as in Chandrasekhar's separation ansatz (3.17)), we choose

$$\mathcal{K} = \mathscr{P}(\mathscr{P}^\dagger)^{-1}\,,$$

where $\mathscr{P}$ is defined in (3.15). We note in passing that this Dirichlet-type boundary condition is a so-called MIT-type boundary condition for Dirac fields [22] that leads to a total reflection at the respective boundary surface. Continuing the proof of symmetry, the explicit computation of the square bracket in the fourth line of (A.2) yields the result

$$\partial_j(\Gamma^\tau)\,\alpha^j + \Gamma^\tau\partial_j(\alpha^j) + \Gamma^\tau\alpha^\theta\cot(\theta) = -2\mathscr{V}_0^\dagger\Gamma^\tau\,.$$

Besides, all three matrix products $\Gamma^\tau\alpha^j$, with $j \in \{r, \theta, \phi\}$, are anti-self-adjoint

$$\Gamma^\tau\alpha^j = -\alpha^{j\dagger}\Gamma^{\tau\dagger} = -\alpha^{j\dagger}\Gamma^\tau\,.$$

Therefore, we immediately find that

$$\begin{aligned}(\psi'|H\chi')_{\mathfrak{N}} = &\iiint \partial_j(\psi'^\dagger)\,\alpha^{j\dagger}\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r + 2\iiint(\mathscr{V}_0\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\\ &- \iiint(\mathscr{V}_0\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r + \iiint(\mathscr{V}_m\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\\ = &\iiint(\alpha^j\partial_j\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r + \iiint(\mathscr{V}_0\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\\ &+ \iiint(\mathscr{V}_m\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r\\ = &\iiint(H\psi')^\dagger\,\Gamma^\tau\chi'\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r = (H\psi'|\chi')_{\mathfrak{N}}\,.\end{aligned}$$

■

Appendix B

Fundamental Solutions for the Construction of the Radial Green's Matrix

For the purpose of determining the fundamental solutions $\Phi_1(r;r')$ and $\Phi_2(r;r')$ of the radial first-order ODE system (3.19) extended to complex-valued frequencies, which are used in the construction of the Green's matrix defined via equation (5.36), we first study certain aspects of the associated Jost-type solutions [27, 93]. In more detail, we derive radial Jost-type equations that yield solutions with asymptotic behaviors near infinity, the event horizon, and the Cauchy horizon similar to those given in Lemmas 3.5.1, 3.5.2, and 3.5.3, and briefly discuss the existence, uniqueness, and boundedness of these solutions. Since we apply Lebesgue's dominated convergence theorem to simplify our generalized horizon-penetrating integral spectral representation of the Dirac propagator, the boundedness aspect becomes also relevant for the commutation of specific limits, sums, and integrals. In order to derive the Jost-type equations, we decouple the radial system (3.19) into two second-order scalar equations, which in terms of the Regge–Wheeler coordinate $r_\star$ and the function $\widetilde{\mathscr{R}} = (\widetilde{\mathscr{R}}_+, \widetilde{\mathscr{R}}_-)^{\mathrm{T}} = (\sqrt{|\Delta|}\,\mathscr{R}_+, r_+\,\mathscr{R}_-)^{\mathrm{T}}$ read

$$\big[\partial_{r_\star r_\star} + \mathfrak{J}_{\pm\xi,k,\omega}(r)\,\partial_{r_\star} + \mathfrak{K}^{\pm}_{\xi,k,\omega}(r)\big]\widetilde{\mathscr{R}}_\pm = 0\,, \tag{B.1}$$

where

$$\mathfrak{J}_{\xi,k,\omega}(r) := \frac{1}{r^2+a^2}\left[r - \frac{M\,(3r^2-a^2)}{r^2+a^2} - 4\mathrm{i}\omega Mr - 2\mathrm{i}ka - \frac{\mathrm{i}m\Delta}{\xi+\mathrm{i}mr}\right]$$

$$\mathfrak{K}^{+}_{\xi,k,\omega}(r) := \frac{\Delta}{(r^2+a^2)^2}\left[\big(\mathrm{i}\omega\,[\Delta+4Mr]+2\mathrm{i}ka\big)\left(-\mathrm{i}\omega + \frac{r-M}{\Delta} + \frac{\mathrm{i}m}{\xi+\mathrm{i}mr}\right)\right.$$
$$\left. - 2\mathrm{i}\omega\,[r+M] - m^2r^2 - \xi^2\right]$$

$$\mathfrak{K}^{-}_{\xi,k,\omega}(r) := \frac{\Delta}{(r^2+a^2)^2}\left[\mathrm{i}\omega\left(r - M - 4\mathrm{i}\omega Mr - 2\mathrm{i}ka + \frac{\mathrm{i}m\Delta}{\xi-\mathrm{i}mr}\right) + \omega^2\Delta - m^2r^2 - \xi^2\right].$$

Employing the ansatzes

$$\widetilde{\mathscr{R}}_\pm(r_\star) = \exp\left(-\frac{1}{2}\int \mathfrak{K}^\pm_{\xi,k,\omega}(r)\,\mathrm{d}r_\star\right)\mathscr{Y}_\pm(r_\star)\,,$$

we may transform (B.1) into the Schrödinger-type equations

$$\left[\partial_{r_\star r_\star} + \mathscr{V}^\pm_{\xi,k,\omega}(r)\right]\mathscr{Y}_\pm = 0 \tag{B.2}$$

with the potentials

$$\mathscr{V}^\pm_{\xi,k,\omega}(r) := \mathfrak{K}^\pm_{\xi,k,\omega}(r) - \frac{\mathfrak{I}^2_{\pm\xi,k,\omega}(r)}{4} - \frac{\partial_{r_\star}\mathfrak{I}_{\pm\xi,k,\omega}(r)}{2}\,.$$

To obtain Jost-type equations with boundary conditions prescribed at infinity, we split these potentials into an asymptotic contribution effective at infinity and otherwise regular contributions

$$\mathscr{V}^\pm_{\xi,k,\omega} = \mathscr{V}_\infty + \mathscr{V}^\pm_{\text{reg.}}\,, \tag{B.3}$$

where the asymptotic contribution is given by the expression

$$\mathscr{V}_\infty = \mathscr{V}_\infty(r_\star) := \omega^2 - m^2 + \frac{2Mm^2}{r_\star} \tag{B.4}$$

and the regular contributions are on the order of $\mathscr{V}^\pm_{\text{reg.}} = \mathcal{O}(1/r_\star^2)$ satisfying the condition

$$\int_{r_\star}^\infty |\mathscr{V}^\pm_{\text{reg.}}(y)|\,\mathrm{d}y < \infty \quad \text{for all} \quad r_\star > 0\,.$$

We remark that the asymptotic potential (B.4) corresponds to the equation

$$[\partial_{r_\star r_\star} + \mathscr{V}_\infty(r_\star)]\mathscr{Y}_\infty = 0\,, \tag{B.5}$$

which has the solution [114]

$$\mathscr{Y}_\infty = \mathcal{Z}_1 W_{-\alpha,\frac{1}{2}}\big(2\mathrm{i}\,\mathrm{sign}(\omega)\sqrt{\omega^2-m^2}\,r_\star\big) + \mathcal{Z}_2 W_{+\alpha,\frac{1}{2}}\big(-2\mathrm{i}\,\mathrm{sign}(\omega)\sqrt{\omega^2-m^2}\,r_\star\big),$$

where $W_{\pm\alpha,\frac{1}{2}}(\,.\,)$ are Whittaker functions with $\alpha := \mathrm{i}\,\mathrm{sign}(\omega)\,Mm^2/\sqrt{\omega^2-m^2}$ and $\mathcal{Z}_{1/2}$ denote constants. In case $|\omega| > m$, the asymptotics of this solution at infinity is given by

$$\begin{aligned}\mathscr{Y}_\infty \sim\ & \mathcal{Z}_1' \exp\left(\mathrm{i}\,\mathrm{sign}(\omega)\left[\sqrt{\omega^2-m^2}\,r_\star + \frac{Mm^2}{\sqrt{\omega^2-m^2}}\ln(r_\star)\right]\right)\\ &+ \mathcal{Z}_2' \exp\left(-\mathrm{i}\,\mathrm{sign}(\omega)\left[\sqrt{\omega^2-m^2}\,r_\star + \frac{Mm^2}{\sqrt{\omega^2-m^2}}\ln(r_\star)\right]\right),\end{aligned} \tag{B.6}$$

whereas for $|\omega| < m$, it yields

$$\begin{aligned}\mathscr{Y}_\infty \sim\ & \mathcal{Z}_1' \exp\left(\mathrm{sign}(\omega)\left[\sqrt{m^2-\omega^2}\,r_\star + \frac{Mm^2}{\sqrt{m^2-\omega^2}}\ln(r_\star)\right]\right)\\ &+ \mathcal{Z}_2' \exp\left(-\mathrm{sign}(\omega)\left[\sqrt{m^2-\omega^2}\,r_\star + \frac{Mm^2}{\sqrt{m^2-\omega^2}}\ln(r_\star)\right]\right)\end{aligned}$$

with $\mathscr{Z}'_{1/2}$ also denoting constants (cf. Lemma 3.5.1). In the following, we restrict our attention to the case $|\omega| > m$. The case $|\omega| < m$ may be treated accordingly. As in the usual study of Jost equations and their solutions, we complexify the Schrödinger-type equations (B.2) via the analytic continuation $\omega \to \omega_{\mathrm{c}} \in \mathbb{C}$ of the frequency. Then, by means of the above splittings of the potentials (B.3) and the specific form of the asymptotic solution (B.6), we can express the Jost-type equation representation of (B.2) as

$$\begin{aligned} \mathscr{Y}_{\pm}(r_\star) = \exp\Bigg(\mathrm{i}\,\mathrm{sign}\big(\mathrm{Im}(\omega_{\mathrm{c}})\big)\,\mathrm{sign}(\omega_{\mathrm{c}})\Bigg[\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}\; r_\star + \frac{Mm^2}{\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}} \ln{(r_\star)}\Bigg]\Bigg) \\ + \int_{r_\star}^{\infty} \frac{\sin\left(\sqrt{\mathscr{V}_\infty(y)}\,[r_\star - y]\right)}{\sqrt{\mathscr{V}_\infty(y)}}\, \mathscr{V}^{\pm}_{\text{reg.}}(y)\, \mathscr{Y}_{\pm}(y)\,\mathrm{d}y\,. \end{aligned} \tag{B.7}$$

We note that the proper complexification of the asymptotic Schrödinger-type equation (B.5), which is in accordance with the particular representation (B.6) of the asymptotic solutions containing signum functions, is obtained by first rewriting the potential $\mathscr{V}_\infty$ defined in (B.4) in the form

$$\mathscr{V}_\infty = \mathrm{sign}^2(\omega)\Bigg(|\omega|^2 - m^2 + \frac{2Mm^2}{r_\star}\Bigg),$$

and only afterwards extending the frequency ω to complex values. This is relevant for the derivation of the exponential term in (B.7). Applying the series ansatzes

$$\mathscr{Y}_{\pm}(r_\star) = \sum_{n=0}^{\infty} \mathscr{Y}_{\pm,n}(r_\star)\,,$$

where the zeroth-order terms are given by

$$\mathscr{Y}_{\pm,0}(r_\star) = \exp\Bigg(\mathrm{i}\,\mathrm{sign}\big(\mathrm{Im}(\omega_{\mathrm{c}})\big)\,\mathrm{sign}(\omega_{\mathrm{c}})\Bigg[\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}} \ln{(r_\star)}\Bigg]\Bigg),$$

yields the recurrence relations

$$\mathscr{Y}_{\pm,n}(r_\star) = \int_{r_\star}^{\infty} \frac{\sin\left(\sqrt{\mathscr{V}_\infty(y)}\,[r_\star - y]\right)}{\sqrt{\mathscr{V}_\infty(y)}}\, \mathscr{V}^{\pm}_{\text{reg.}}(y)\, \mathscr{Y}_{\pm,n-1}(r_\star)\,\mathrm{d}y \quad \text{for} \quad n \geq 1\,.$$

In the theorem below, we discuss the relevant results pertaining to the existence, uniqueness, and boundedness of such solutions for the case $\mathrm{Im}(\omega_{\mathrm{c}}) < 0$. Detailed proofs are worked out explicitly in, e.g., [41, 50, 70, 93]. The results and proofs for the case $\mathrm{Im}(\omega_{\mathrm{c}}) > 0$ are in essence identical.

Theorem B.0.1. *For each $\omega_{\mathrm{c}} \in \mathbb{C}$ with $\omega_{\mathrm{c}} \neq 0$ and $\mathrm{Im}(\omega_{\mathrm{c}}) < 0$, the Jost-type equations (B.7) have unique solutions $\mathscr{Y}_{\pm}(r_\star)$ obeying*

$$\lim_{r_\star \to \infty} \left| \exp\Bigg(\mathrm{i}\,\mathrm{sign}(\omega_{\mathrm{c}})\Bigg[\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}} \ln{(r_\star)}\Bigg]\Bigg)\mathscr{Y}_{\pm}(r_\star)\right| < \infty\,.$$

These solutions are moreover continuously differentiable in $r_\star$ on the interval $(0,\infty)$ with

$$\lim_{r_\star \to \infty} \Bigg[\exp\Bigg(\mathrm{i}\,\mathrm{sign}(\omega_{\mathrm{c}})\Bigg[\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_{\mathrm{c}}|^2 - m^2}} \ln{(r_\star)}\Bigg]\Bigg)\mathscr{Y}_{\pm}(r_\star)\Bigg] = 1$$

and

$$\lim_{r_\star \to \infty} \left[\exp\left(\mathrm{i}\,\mathrm{sign}(\omega_\mathrm{c}) \left[\sqrt{|\omega_\mathrm{c}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_\mathrm{c}|^2 - m^2}} \ln(r_\star) \right] \right) \partial_{r_\star} \mathscr{Y}_\pm(r_\star) \right]$$

$$= -\mathrm{i}\,\mathrm{sign}(\omega_\mathrm{c}) \sqrt{|\omega_\mathrm{c}|^2 - m^2}\,.$$

For each fixed value of $r_\star$*,* $\mathscr{Y}_\pm(r_\star)$ *and* $\partial_{r_\star}\mathscr{Y}_\pm(r_\star)$ *are functions that are analytic in* $\{\omega_\mathrm{c} \,|\, \mathrm{Im}(\omega_\mathrm{c}) < 0\}$*, continuous in* $\{\omega_\mathrm{c} \,|\, \omega_\mathrm{c} \neq 0 \;\text{ and }\; \mathrm{Im}(\omega_\mathrm{c}) < 0\}$*, and satisfy the bound*

$$\left| \mathscr{Y}_\pm(r_\star) - \exp\left(-\mathrm{i}\,\mathrm{sign}(\omega_\mathrm{c}) \left[\sqrt{|\omega_\mathrm{c}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_\mathrm{c}|^2 - m^2}} \ln(r_\star) \right] \right) \right|$$

$$\leq \exp\Big(\mathrm{Im}\big(\sqrt{\mathscr{V}_\infty(r_\star)}\;\big)\, r_\star \Big) \left| \exp\big(\mathfrak{Q}^\pm(r_\star)\big) - 1 \right|$$

as well as

$$\left| \partial_{r_\star}\mathscr{Y}_\pm(r_\star) + \exp\left(-\mathrm{i}\,\mathrm{sign}(\omega_\mathrm{c}) \left[\sqrt{|\omega_\mathrm{c}|^2 - m^2}\, r_\star + \frac{Mm^2}{\sqrt{|\omega_\mathrm{c}|^2 - m^2}} \ln(r_\star) \right] \right) \mathrm{i}\,\mathrm{sign}(\omega_\mathrm{c}) \right.$$

$$\left. \times \left(\sqrt{|\omega_\mathrm{c}|^2 - m^2} + \frac{Mm^2}{\sqrt{|\omega_\mathrm{c}|^2 - m^2}\, r_\star} \right) \right| \leq \exp\Big(\mathrm{Im}\big(\sqrt{\mathscr{V}_\infty(r_\star)}\;\big)\, r_\star + \mathfrak{Q}^\pm(r_\star) \Big) \int_{r_\star}^\infty \left| \mathscr{V}^\pm_{\text{reg.}}(y) \right| \mathrm{d}y\,,$$

where

$$\mathfrak{Q}^\pm(r_\star) := \int_{r_\star}^\infty \frac{4y\,\left|\mathscr{V}^\pm_{\text{reg.}}(y)\right|}{1 + y\,\left|\sqrt{\mathscr{V}_\infty(y)}\,\right|} \exp\Big(\big[\mathrm{Im}\big(\sqrt{\mathscr{V}_\infty(y)}\;\big) + \big|\mathrm{Im}\big(\sqrt{\mathscr{V}_\infty(y)}\;\big)\big| \big] y \Big)\, \mathrm{d}y\,.$$

It remains to determine the Jost-type equations with boundary conditions prescribed at the event horizon and at the Cauchy horizon. This can be accomplished using a similar approach as in the above case. For details, we again refer to [41, 50].

We now specify the fundamental solutions $\Phi_1(r; r')$ and $\Phi_2(r; r')$ of the radial system (3.19) extended to complex-valued frequencies. Due to the high degree of complexity of this ODE system, explicit analytical expressions for the fundamental solutions are not known. Thus, we give a description in terms of suitable asymptotic expansions. To this end, we define, on the one hand, auxiliary functions that have the proper decay at infinity in accordance with Lemma 3.5.1

$$\widehat{\Phi}^{(\infty)}(r) := \frac{1}{\sqrt{|\Delta|}} \left[d_{1,\infty}\, e^{\mathrm{i}\phi_+(r_\star(r))} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r_\star(r)} \right) \right] \qquad \text{for} \quad \begin{cases} \mathrm{Im}(\omega_\epsilon) < 0 & \text{if} \quad |\omega_\epsilon| > m \\ \mathrm{Re}(\omega_\epsilon) > 0 & \text{if} \quad |\omega_\epsilon| < m \end{cases}$$

$$\check{\Phi}^{(\infty)}(r) := d_{2,\infty}\, e^{-\mathrm{i}\phi_-(r_\star(r))} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r_\star(r)} \right) \qquad \text{for} \quad \begin{cases} \mathrm{Im}(\omega_\epsilon) > 0 & \text{if} \quad |\omega_\epsilon| > m \\ \mathrm{Re}(\omega_\epsilon) < 0 & \text{if} \quad |\omega_\epsilon| < m\,, \end{cases}$$

where the quantities $d_{1/2,\infty}$ are scalar constants and the functions $\phi_\pm$ are specified in (3.27), but with frequencies $\omega_\epsilon \in \{\omega + \mathrm{i}\epsilon, \omega - \mathrm{i}\epsilon\}$, for which $\epsilon > 0$, and with the substitution $\sqrt{\omega^2 - m^2} \to \sqrt{|\omega_\epsilon|^2 - m^2}$. On the other hand, we use auxiliary functions that are regular at the event and the Cauchy horizon, and further comply with the associated asymptotics stated in Lemmas 3.5.2 and 3.5.3

$$\widehat{\Phi}^{(+)}(r) := \frac{1}{\sqrt{|\Delta|}}\Big[d_{1,r_+}\, e^{2\mathrm{i}\left(\omega_\epsilon + k\Omega_{\text{Kerr}}^{(+)}\right) r_\star(r)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}\big(e^{q r_\star(r)}\big)\Big] \qquad \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0$$

$$\check{\Phi}^{(+)}(r) := d_{2,r_+} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}\big(e^{q r_\star(r)}\big) \qquad \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0$$

$$\widehat{\Phi}^{(-)}(r) := d_{1,r_-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}\big(e^{-q r_\star(r)}\big) \qquad \text{for} \quad \mathrm{Im}(\omega_\epsilon) < 0$$

$$\check{\Phi}^{(-)}(r) := \frac{1}{\sqrt{|\Delta|}}\Big[d_{2,r_-}\, e^{2\mathrm{i}\left(\omega_\epsilon + k\Omega_{\text{Kerr}}^{(-)}\right) r_\star(r)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}\big(e^{-q r_\star(r)}\big)\Big] \qquad \text{for} \quad \mathrm{Im}(\omega_\epsilon) > 0\,,$$

where $d_{1/2,r_\pm}$ are scalar constants as well. To clarify the notation, we point out that the superscripts (∞), $(+)$, and $(-)$ designate asymptotic expansions at infinity, the event horizon, and the Cauchy horizon, respectively. Besides, we remark that the existence and uniqueness of the fundamental solutions of the complexified radial system (3.19), determined in terms of these particular asymptotic expansions, follows directly from the above study of the radial Jost-type equations. Moreover, asymptotic expansions of this form ensure the square-integrability of the fundamental solutions. For example, as the Regge–Wheeler coordinate $r_\star$ tends to minus infinity at the event horizon, the exponential factor in the auxiliary function $\widehat{\Phi}^{(+)}$ tends to zero because $\mathrm{Im}(\omega_\epsilon) < 0$. However, this exponential factor would not be square-integrable if $\mathrm{Im}(\omega_\epsilon) > 0$. Lastly, we introduce an auxiliary function that satisfies the Dirichlet-type boundary condition (5.27)

$$\Phi^{(\partial \mathrm{M})}(r) := \Xi(r) \begin{pmatrix} 1 \\ \mathrm{i}\sqrt{|\Delta|}\,/\,r_+ \end{pmatrix},$$

in which Ξ is an arbitrary square-integrable function. Then, in case

$$|\omega_\epsilon| > m \;\text{ and }\; \mathrm{Im}(\omega_\epsilon) < 0 \quad \text{or} \quad |\omega_\epsilon| < m\,,\; \mathrm{Im}(\omega_\epsilon) < 0\,,\; \text{ and }\; \mathrm{Re}(\omega_\epsilon) > 0\,,$$

the fundamental radial solutions Φ_1 and Φ_2 are given by

$$\begin{aligned}
&\Phi_1(r; r_+ < r' < \infty) = \Theta(r - r')\, \widehat{\Phi}^{(\infty)}(r) \\
&\Phi_2(r; r_+ < r' < \infty) = \Theta(r' - r)\, \Theta(r - r_+)\, \widehat{\Phi}^{(+)}(r) \\
&\\
&\Phi_1(r; r_- < r' \leq r_+) = \Theta(r - r')\big[\Theta(r_+ - r)\, \widehat{\Phi}^{(-)}(r) + \Theta(r - r_+)\, \widehat{\Phi}^{(\infty)}(r)\big] \\
&\Phi_2(r; r_- < r' \leq r_+) = \Theta(r - r')\, \Theta(r_+ - r)\, \widehat{\Phi}^{(+)}(r) \\
&\\
&\Phi_1(r; r_0 \leq r' \leq r_-) = \Theta(r - r')\big[\Theta(r_+ - r)\, \widehat{\Phi}^{(-)}(r) + \Theta(r - r_+)\, \widehat{\Phi}^{(\infty)}(r)\big] \\
&\Phi_2(r; r_0 \leq r' \leq r_-) = \Theta(r' - r)\, \Phi^{(\partial \mathrm{M})}(r)\,,
\end{aligned} \tag{B.8}$$

whereas for

$$|\omega_\epsilon| > m \;\text{ and }\; \mathrm{Im}(\omega_\epsilon) > 0 \quad \text{or} \quad |\omega_\epsilon| < m\,,\; \mathrm{Im}(\omega_\epsilon) > 0\,,\; \text{ and }\; \mathrm{Re}(\omega_\epsilon) < 0\,,$$

they read

$$
\begin{aligned}
\Phi_1(r;r_+ < r' < \infty) &= \Theta(r-r')\,\check{\Phi}^{(\infty)}(r) \\
\Phi_2(r;r_+ < r' < \infty) &= \Theta(r'-r)\big[\Theta(r-r_-)\,\check{\Phi}^{(+)}(r) + \Theta(r_- - r)\,\Phi^{(\partial\mathrm{M})}(r)\big] \\[1em]
\Phi_1(r;r_- < r' \le r_+) &= \Theta(r'-r)\big[\Theta(r-r_-)\,\check{\Phi}^{(+)}(r) + \Theta(r_- - r)\,\Phi^{(\partial\mathrm{M})}(r)\big] \\
\Phi_2(r;r_- < r' \le r_+) &= \Theta(r'-r)\,\Theta(r-r_-)\,\check{\Phi}^{(-)}(r) \\[1em]
\Phi_1(r;r_0 \le r' \le r_-) &= \Theta(r-r')\,\Theta(r_- - r)\,\check{\Phi}^{(-)}(r) \\
\Phi_2(r;r_0 \le r' \le r_-) &= \Theta(r'-r)\,\Phi^{(\partial\mathrm{M})}(r)\,.
\end{aligned}
\tag{B.9}
$$

In the remaining cases

$$|\omega_\epsilon| < m\,,\ \ \mathrm{Im}(\omega_\epsilon) < 0\,,\ \ \text{and}\ \ \mathrm{Re}(\omega_\epsilon) < 0 \quad \text{or} \quad |\omega_\epsilon| < m\,,\ \ \mathrm{Im}(\omega_\epsilon) > 0\,,\ \ \text{and}\ \ \mathrm{Re}(\omega_\epsilon) > 0\,,$$

we also obtain fundamental solutions of the form (B.8) or (B.9), respectively, however now with the auxiliary functions $\widehat{\Phi}^{(\infty)}$ and $\check{\Phi}^{(\infty)}$ interchanged. These representations of the fundamental radial solutions and their uniqueness can be determined case by case via the conditions and asymptotics listed in the proof of Theorem 5.4.1.

Bibliography

[1] Aké Hau, L., Flores, J. L., and Sánchez, M. Structure of Globally Hyperbolic Spacetimes with Timelike Boundary. *arXiv:1808.04412 [gr-qc]* (2018).

[2] Araki, H. On Quasifree States of CAR and Bogoliubov Automorphisms. *Publications of the Research Institute for Mathematical Sciences 6* (1971), pp. 385–442.

[3] Aubin, T. *Nonlinear Analysis on Manifolds. Monge–Ampère Equations*. Springer-Verlag, 1982.

[4] Bartnik, R. A., and Chruściel, P. T. Boundary Value Problems for Dirac-type Equations. *Journal für Reine und Angewandte Mathematik 579* (2005), pp. 13–73.

[5] Batic, D. Scattering for Massive Dirac Fields on the Kerr Metric. *Journal of Mathematical Physics 48* (2007), id. 022502.

[6] Bernal, A. N., and Sánchez, M. On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem. *Communications in Mathematical Physics 243* (2003), pp. 461–470.

[7] Bernal, A. N., and Sánchez, M. Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes. *Communications in Mathematical Physics 257* (2005), pp. 43–50.

[8] Bernal, A. N., and Sánchez, M. Further Results on the Smoothability of Cauchy Hypersurfaces and Cauchy Time Functions. *Letters in Mathematical Physics 77* (2006), pp. 183–197.

[9] Bernal, A. N., and Sánchez, M. Globally Hyperbolic Spacetimes can be defined as "Causal" instead of "Strongly Causal". *Classical and Quantum Gravity 24* (2007), pp. 745–750.

[10] Boyer, R. H., and Lindquist, R. W. Maximal Analytic Extension of the Kerr Metric. *Journal of Mathematical Physics 8* (1967), pp. 265–281.

[11] Brill, D. R., and Wheeler, J. A. Interaction of Neutrinos and Gravitational Fields. *Review of Modern Physics 29* (1957), pp. 465–479.

[12] Caponio, E., Javaloyes, M. A., and Sánchez, M. On the Interplay between Lorentzian Causality and Finsler Metrics of Randers Type. *Revista Matemática Iberoamericana 27* (2011), pp. 919–952.

[13] Caponio, E., Javaloyes, M. A., and Sánchez, M. Wind Finslerian Structures: From Zermelo's Navigation to the Causality of Spacetimes. *arXiv:1407.5494 [math.DG]* (2014).

[14] Carroll, S. M. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019.

[15] CARTER, B. Black Hole Equilibrium States. In: DeWitt, C., and DeWitt, B. S. Black holes/Les astres occlus. Gordon and Breach, 1973.

[16] CARTER, B. Axisymmetric Black Hole has only two Degrees of Freedom. *Physical Review Letters 26* (1971), pp. 331–333.

[17] CHAKRABARTI, S. K., AND MUKHOPADHYAY, B. Scattering of Dirac Waves off Kerr Black Holes. *Monthly Notices of the Royal Astronomical Society 317* (2000), pp. 979–984.

[18] CHANDRASEKHAR, S. The Solution of Dirac's Equation in Kerr Geometry. *Proceedings of the Royal Society London 349* (1976), pp. 571–575.

[19] CHANDRASEKHAR, S. *The Mathematical Theory of Black Holes*. Oxford University Press, 1983.

[20] CHANDRASEKHAR, S., AND DETWEILER, S. On the Reflexion and Transmission of Neutrino Waves by a Kerr Black Hole. *Proceedings of the Royal Society London 352* (1977), pp. 325–338.

[21] CHERNOFF, P. R. Essential Self-adjointness of Powers of Generators of Hyperbolic Equations. *Journal of Functional Analysis 12* (1973), pp. 401–414.

[22] CHODOS, A., JAFFE, R. L., JOHNSON, K., THORN, C. B., AND WEISSKOPF, V. F. New Extended Model of Hadrons. *Physical Review D 9* (1974), pp. 3471–3495.

[23] CHOQUET–BRUHAT, Y. *General Relativity and the Einstein Equations*. Oxford University Press, 2009.

[24] CODDINGTON, E. A., AND LEVINSON, N. *Theory of Ordinary Differential Equations*. Tata McGraw–Hill Publishing Company Limited, 1972.

[25] DAPPIAGGI, C., FINSTER, F., MURRO, S., AND RADICI, E. The Fermionic Signature Operator in De Sitter Spacetime. *arXiv:1902.09144 [math-ph]* (2019).

[26] DAUDÉ, T. Propagation Estimates for Dirac Operators and Application to Scattering Theory. *Annales de l'Institut Fourier 54* (2004), pp. 2021–2083.

[27] DE ALFARO, V., AND REGGE, T. *Potential Scattering*. North–Holland Publishing Company, 1965.

[28] DOLAN, S. R., AND DEMPSEY, D. Bound States of the Dirac Equation on Kerr Spacetime. *Classical and Quantum Gravity 32* (2015), id. 184001.

[29] DOLAN, S. R., AND GAIR, J. R. The Massive Dirac Field on a Rotating Black Hole Spacetime: Angular Solutions. *Classical and Quantum Gravity 26* (2009), id. 175020.

[30] DORAN, C. New Form of the Kerr Solution. *Physical Review D 61* (2000), id. 067503.

[31] DOUGAN, A. J., AND MASON, L. J. Quasilocal Mass Constructions with Positive Energy. *Physical Review Letters 67* (1991), pp. 2119–2122.

[32] EDDINGTON, A. S. A Comparison of Whitehead's and Einstein's Formulae. *Nature 113* (1924), p. 192.

[33] EVANS, L. C. *Partial Differential Equations*. Oxford University Press, 1998.

[34] FEWSTER, C. J., AND LANG, B. Pure Quasifree States of the Dirac Field from the Fermionic Projector. *Classical and Quantum Gravity 32* (2015), id. 095001.

[35] FEWSTER, C. J., AND VERCH, R. On a Recent Construction of 'Vacuum-like' Quantum Field States in Curved Spacetime. *Classical and Quantum Gravity 29* (2012), id. 205017.

[36] FINKELSTEIN, D. Past-future Asymmetry of the Gravitational Field of a Point Particle. *Physical Review 110* (1958), pp. 965–967.

[37] FINSTER, F., KAMRAN, N., SMOLLER, J., AND YAU, S.-T. Non-existence of Time-periodic Solutions of the Dirac Equation in an Axisymmetric Black Hole Geometry. *Communications on Pure and Applied Mathematics 53* (2000), pp. 902–929.

[38] FINSTER, F., KAMRAN, N., SMOLLER, J., AND YAU, S.-T. Decay Rates and Probability Estimates for Massive Dirac Particles in the Kerr–Newman Black Hole Geometry. *Communications in Mathematical Physics 230* (2002), pp. 201–244.

[39] FINSTER, F., KAMRAN, N., SMOLLER, J., AND YAU, S.-T. The Long-time Dynamics of Dirac Particles in the Kerr–Newman Black Hole Geometry. *Advances in Theoretical and Mathematical Physics 7* (2003), pp. 25–52.

[40] FINSTER, F., KAMRAN, N., SMOLLER, J., AND YAU, S.-T. An Integral Spectral Representation of the Propagator for the Wave Equation in the Kerr Geometry. *Communications in Mathematical Physics 260* (2005), pp. 257–298.

[41] FINSTER, F., KAMRAN, N., SMOLLER, J., AND YAU, S.-T. Decay of Solutions of the Wave Equation in the Kerr Geometry. *Communications in Mathematical Physics 264* (2006), pp. 465–503.

[42] FINSTER, F., MURRO, S., AND RÖKEN, C. The Fermionic Projector in a Time-dependent External Potential: Mass Oscillation Property and Hadamard States. *Journal of Mathematical Physics 57* (2016), id. 072303.

[43] FINSTER, F., MURRO, S., AND RÖKEN, C. The Fermionic Signature Operator and Quantum States in Rindler Space-time. *Journal of Mathematical Analysis and Applications 454* (2017), pp. 385–411.

[44] FINSTER, F., AND REINTJES, M. A Non-perturbative Construction of the Fermionic Projector on Globally Hyperbolic Manifolds I: Space-times of Finite Lifetime. *Advances in Theoretical and Mathematical Physics 19* (2015), pp. 761–803.

[45] FINSTER, F., AND REINTJES, M. A Non-perturbative Construction of the Fermionic Projector on Globally Hyperbolic Manifolds II: Space-times of Infinite Lifetime. *Advances in Theoretical and Mathematical Physics 20* (2016), pp. 1007–1048.

[46] FINSTER, F., AND REINTJES, M. The Fermionic Signature Operator and Hadamard States in the Presence of a Plane Electromagnetic Wave. *Annales Henri Poincaré 18* (2017), pp. 1671–1701.

[47] FINSTER, F., AND RÖKEN, C. Self-adjointness of the Dirac Hamiltonian for a Class of Non-uniformly Elliptic Boundary Value Problems. *Annals of Mathematical Sciences and Applications 1* (2016), pp. 301–320.

[48] FINSTER, F., AND RÖKEN, C. An Integral Spectral Representation of the Massive Dirac Propagator in the Kerr Geometry in Eddington–Finkelstein-type Coordinates. *Advances in Theoretical and Mathematical Physics 22* (2018), pp. 47–92.

[49] FINSTER, F., AND RÖKEN, C. The Fermionic Signature Operator in the Exterior Schwarzschild Geometry. *Annales Henri Poincaré 20* (2019), pp. 3389–3418.

[50] FINSTER, F., AND SMOLLER, J. Decay of Solutions of the Teukolsky Equation for Higher Spin in the Schwarzschild Geometry. *Advances in Theoretical and Mathematical Physics 13* (2009), pp. 71–110.

[51] FLORES, J. L., HERRERA, J., AND SÁNCHEZ, M. On the Final Definition of the Causal Boundary and its Relation with the Conformal Boundary. *Advances in Theoretical and Mathematical Physics 15* (2011), pp. 991–1057.

[52] FLORES, J. L., HERRERA, J., AND SÁNCHEZ, M. Gromov, Cauchy and Causal Boundaries for Riemannian, Finslerian and Lorentzian Manifolds. *Memoirs of the American Mathematical Society 226* (2013), 72 pp.

[53] FOCK, V. Geometrisierung der Diracschen Theorie des Elektrons. *Zeitschrift für Physik 57* (1929), pp. 261–277.

[54] FUTTERMAN, J. A. H., HANDLER, F. A., AND MATZNER, R. A. *Scattering from Black Holes*. Cambridge University Press, 1988.

[55] GEROCH, R. Spinor Structure of Space-times in General Relativity. I. *Journal of Mathematical Physics 9* (1968), pp. 1739–1744.

[56] GEROCH, R. Domain of Dependence. *Journal of Mathematical Physics 11* (1970), pp. 437–449.

[57] GIBBONS, G. W., HAWKING, S. W., HOROWITZ, G. T., AND PERRY, M. J. Positive Mass Theorems for Black Holes. *Communications in Mathematical Physics 88* (1983), pp. 295–308.

[58] GOLDBERG, J. N., MACFARLANE, A. J., NEWMAN, E. T., ROHRLICH, F., AND SUDARSHAN, E. C. G. Spin-s Spherical Harmonics and $\eth$. *Journal of Mathematical Physics 8* (1967), pp. 2155–2161.

[59] HÄFNER, D. Creation of Fermions by Rotating Charged Black-holes. *arXiv:math/0612501 [math.AP]* (2006).

[60] HÄFNER, D., AND NICOLAS, J.-P. Scattering of Massless Dirac Fields by a Kerr Black Hole. *Reviews in Mathematical Physics 16* (2004), pp. 29–123.

[61] HAWKING, S. W., AND ELLIS, G. F. R. *The Large Scale Structure of Space-time*. Cambridge University Press, 1973.

[62] HEBEY, E. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*. American Mathematical Society, 2000.

[63] HEUSER, H. *Funktionalanalysis: Theorie und Anwendung*. Vieweg + Teubner Verlag, 2006.

[64] HOUNNONKPE, R. A., AND MINGUZZI, E. Globally Hyperbolic Spacetimes can be defined without the 'Causal' Condition. *Classical and Quantum Gravity 36* (2019), id. 197001.

[65] JAVALOYES, M. A., AND SÁNCHEZ, M. A Note on the Existence of Standard Splittings for Conformally Stationary Spacetimes. *Classical and Quantum Gravity 25* (2008), id. 168001.

[66] JAVALOYES, M. A., AND SÁNCHEZ, M. Some Criteria for Wind Riemannian Completeness and Existence of Cauchy Hypersurfaces. *Proceedings of the 8th International Meeting on Lorentzian Geometry 211* (2017), pp. 117–151.

[67] JOHN, F. *Partial Differential Equations*. Springer-Verlag, 1991.

[68] KERR, R. P. Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. *Physical Review Letters 11* (1963), pp. 237–238.

[69] KINNERSLEY, W. Type D Vacuum Metrics. *Journal of Mathematical Physics 10* (1969), pp. 1195–1203.

[70] KRONTHALER, J. The Cauchy Problem for the Wave Equation in the Schwarzschild Geometry. *Journal of Mathematical Physics 47* (2006), id. 042501.

[71] KRUSKAL, M. D. Maximal Extension of Schwarzschild Metric. *Physical Review 119* (1960), pp. 1743–1745.

[72] LAWSON, H. B., AND MICHELSOHN, M.-L. *Spin Geometry*. Princeton University Press, 1989.

[73] LUDVIGSEN, M. *General Relativity: A Geometric Approach*. Cambridge University Press, 2004.

[74] MARS, M., AND REIRIS, M. Global and Uniqueness Properties of Stationary and Static Spacetimes with Outer Trapped Surfaces. *Communications in Mathematical Physics 322* (2013), pp. 633–666.

[75] MINGUZZI, E., AND SÁNCHEZ, M. The Causal Hierarchy of Spacetimes. *Recent developments in pseudo-Riemannian geometry, ESI Lectures in Mathematics and Physics* (2008), pp. 299–358.

[76] MISNER, C. W., THORNE, K. S., AND WHEELER, J. A. *Gravitation*. W. H. Freeman and Company, 1973.

[77] MÜLLER, O. Special Temporal Functions on Globally Hyperbolic Manifolds. *Letters in Mathematical Physics 103* (2013), pp. 285–297.

[78] MÜLLER, O., AND SÁNCHEZ, M. Lorentzian Manifolds Isometrically Embeddable in $\mathbb{L}^N$. *Transactions of the American Mathematical Society 363* (2011), pp. 5367–5379.

[79] NEWMAN, E. T., AND PENROSE, R. An Approach to Gravitational Radiation by a Method of Spin Coefficients. *Journal of Mathematical Physics 3* (1962), pp. 566–578.

[80] NICOLAS, J.-P. Dirac Fields on Asymptotically Simple Space-times. *Jean Leray '99 Conference Proceedings 24* (2003), pp. 205–217.

[81] O'DONNELL, P. *Introduction to 2-Spinors in General Relativity*. World Scientific, 2003.

[82] O'NEILL, B. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, 1983.

[83] O'NEILL, B. *The Geometry of Kerr Black Holes*. Dover Publications, 2014.

[84] PAGE, D. N. Dirac Equation Around a Charged, Rotating Black Hole. *Physical Review D 14* (1976), pp. 1509–1510.

[85] PENROSE, R. A Spinor Approach to General Relativity. *Annals of Physics 10* (1960), pp. 171–201.

[86] PENROSE, R., AND FLOYD, R. M. Extraction of Rotational Energy from a Black Hole. *Nature Physical Science 229* (1971), pp. 177–179.

[87] PENROSE, R., AND RINDLER, W. *Spinors and Space-time I: Two-spinor Calculus and Relativistic Fields*. Cambridge University Press, 1984.

[88] PENROSE, R., AND RINDLER, W. *Spinors and Space-time II: Spinor and Twistor Methods in Space-time Geometry*. Cambridge University Press, 1986.

[89] POISSON, E. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. Cambridge University Press, 2008.

[90] PRESS, W. H., AND TEUKOLSKY, S. A. Perturbations of a Rotating Black Hole. II. Dynamical Stability of the Kerr Metric. *Astrophysical Journal 185* (1973), pp. 649–674.

[91] RAUCH, J. *Hyperbolic Partial Differential Equations and Geometric Optics*. American Mathematical Society, 2012.

[92] REED, M., AND SIMON, B. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press, 1975.

[93] REED, M., AND SIMON, B. *Methods of Modern Mathematical Physics III: Scattering Theory*. Academic Press, 1979.

[94] REED, M., AND SIMON, B. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, 1980.

[95] RINGSTRÖM, H. *The Cauchy Problem in General Relativity*. European Mathematical Society Publishing House, 2009.

[96] ROBINSON, D. C. Uniqueness of the Kerr Black Hole. *Physical Review Letters 34* (1975), pp. 905–906.

[97] RÖKEN, C. Kerr Isolated Horizons in Ashtekar and Ashtekar–Barbero Connection Variables. *General Relativity and Gravitation 49* (2017), id. 114.

[98] RÖKEN, C. The Massive Dirac Equation in the Kerr Geometry: Separability in Eddington–Finkelstein-type Coordinates and Asymptotics. *General Relativity and Gravitation 49* (2017), pp. 39–62.

[99] TAYLOR, M. E. *Partial Differential Equations I*. Springer-Verlag, 1996.

[100] TAYLOR, M. E. *Partial Differential Equations II*. Springer-Verlag, 1997.

[101] TAYLOR, M. E. *Partial Differential Equations III*. Springer-Verlag, 1997.

[102] TEUKOLSKY, S. A. Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-field Perturbations. *Astrophysical Journal 185* (1973), pp. 635–648.

[103] TEUKOLSKY, S. A. The Kerr Metric. *Classical and Quantum Gravity 32* (2015), id. 124006.

[104] THALLER, B. *The Dirac Equation*. Springer-Verlag, 1992.

[105] TOWNSEND, P. K. Black Holes. *arXiv:gr-qc/9707012* (1997).

[106] UNRUH, W. G. Separability of the Neutrino Equations in a Kerr Background. *Physical Review Letters 31* (1973), pp. 1265–1267.

[107] VAN DER WAERDEN, B. L. Spinoranalyse. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (1929), pp. 100–110.

[108] VISSER, M. The Kerr Spacetime: A Brief Introduction. *arXiv:0706.0622 [gr-qc]* (2007).

[109] WALD, R. M. *General Relativity*. University of Chicago Press, 1984.

[110] WALD, R. M. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. University of Chicago Press, 1994.

[111] WARD, R. S., AND WELLS, R. O. *Twistor Geometry and Field Theory*. Cambridge University Press, 1990.

[112] WELDON, H. A. Fermions without Vierbeins in Curved Space-time. *Physical Review D 63* (2001), id. 104010.

[113] WEYL, H. Elektron und Gravitation. *Zeitschrift für Physik 56* (1929), pp. 330–352.

[114] WHITTAKER, E. T., AND WATSON, G. N. *A Course of Modern Analysis*. Cambridge University Press, 1927.