Purely quasilinear elliptic problems with singularities

TESIS DOCTORAL

por

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A mis padres, María Ángeles y Salvador,
quienes siempre han creído en mí.
Por ellos, hoy también yo creo en mí mismo.

Contents

How to read this thesis

In the preface (chapter 1) we gather the common features about the research contained in this thesis. Within this chapter, the introduction (section 1.1) represents a global overview of the problems that the present thesis is concerned with. We also include in this chapter the motivations of the problems into consideration (section 1.2), the specific objectives of the research (section 1.3), a summary of the results that are proven (section 1.3) and, finally, some aspects about the methodology that is employed (section 1.4).

Next five chapters constitute the core of the thesis. They contain the precise statements of the results that have been obtained together with their proofs, among further comments and consequences. This thesis adjusts to the compendium form, so the mentioned chapters are, indeed, independent papers written by the author of the thesis and collaborators. Namely, in chapter 2 we include a joint work with J. Carmona, T. Leonori and P.J. Martínez-Aparicio [38]; chapter 3 corresponds to a paper in collaboration with J. Carmona and P.J. Martínez-Aparicio [39]; in chapter 4 we reproduce an article of single author [103], and chapters 5 and 6 correspond to two preprints, the first one of single author [104] and the second one in collaboration with A. Molino [105]. The enumerated chapters have been included in chronological order of appearance of the work that they refer to. The only exceptions are chapters 5 and 6, which are the last works of the author but have been handled in inverse order; we have permuted them for the sake of a more coherent presentation. Naturally, these central chapters are self-contained and they may present slightly different notations due to the nature of the compendium form. Thus, they must be read as independent entities.

After presenting the main results and proofs, we gather the conclusions of the research in chapter 7. Finally, a summary in Spanish language of the whole thesis, as well as the list of references, can be found at the end of the document.

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Chapter 1

Preface

1.1 Introduction

This PhD thesis provides contributions to the field of Nonlinear Partial Differential Equations. More specifically, it is concerned with the existence and qualitative properties of solutions to nonlinear elliptic boundary value problems of Dirichlet type. The general model, from which many interesting particular cases arise, is the following:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = F(x,u,\nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{$\wp$}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth[1] boundary $\partial\Omega$, where $N \in \mathbb{N}$ ($N \geq 3$) stands for the dimension; A is an $N \times N$ matrix whose coefficients are bounded functions in Ω, and $F : \Omega \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, i.e., $F(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$ and $F(\cdot,s,\xi)$ is measurable for every $s \neq 0$ and $\xi \in \mathbb{R}^N$. Note that $\lim_{s\to 0} F(x,s,\xi)$ need not exist. The implications of this peculiarity are remarkable and will be commented below.

The fact that the second order term in the equation is given in divergence form allows us to consider solutions only weakly differentiable that satisfy ($\wp$) in the distributional sense. To be more precise, if we denote $\{u \neq 0\} := \{x \in \Omega : u(x) \neq 0 \text{ a.e. } x \in \Omega\}$, then a *weak solution* (or simply a solution) to ($\wp$) is a function $u \in H^1_{\mathrm{loc}}(\Omega)$ such that $|u|^\gamma \in H^1_0(\Omega)$ for some $\gamma > 0$, $F(x,u,\nabla u) \in L^1_{\mathrm{loc}}(\{u \neq 0\})$ and the following equality holds:

[1] We will specify the required smoothness for each result in section 1.3.

$$\int_\Omega A(x)\nabla u \nabla \phi = \int_\Omega F(x,u,\nabla u)\phi \quad \forall \phi \in C_c^1(\Omega). \tag{1.1}$$

The reader may have noticed that, if $\lim_{s\to 0} F(x,s,\xi)$ does not exist, then the right hand side of (1.1) is not well defined in principle, as one would have to give a sense to $\int_{\{u=0\}} F(x,u,\nabla u)\phi$. In order to overcome this issue, it is usual in the literature to look only for *positive solutions*, i.e., functions u satisfying (1.1) and, in addition, $u(x) \geq c$ for a.e. $x \in \omega$ for every domain ω compactly contained in Ω, where $c > 0$ is a constant that depends on ω. However, there are functions F for which *sign-changing solutions* appear in a natural way. We will show in chapter 3 that general sign-changing solutions can be rigorously defined in a way that generalizes the usual concept of weak solution for functions F which are continuous at $s = 0$.

Throughout the thesis, some specific conditions on A, F will be assumed, and specially the ones for F will determine the particular nature of the problem. On the one hand, the following condition on A implies that the principal operator $-\mathrm{div}(A(x)\nabla)$ is *uniformly elliptic*:

$$\exists \eta > 0: \quad A(x)\xi\xi \geq \eta|\xi|^2 \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N. \tag{1.2}$$

On the other hand, we will consider lower order terms with *natural growth* in the third variable. That is to say, for some Carathéodory functions $f, g : \Omega \times (\mathbb{R} \setminus \{0\}) \to [0, +\infty)$, the following estimate holds:

$$|F(x,s,\xi)| \leq g(x,s)|\xi|^2 + f(x,s) \quad \text{a.e. } x \in \Omega,\ \forall (s,\xi) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{R}^N. \tag{1.3}$$

Last condition allows F, in particular, to be nonlinear with respect to the gradient variable. We will focus precisely in the case in which F presents this kind of nonlinearity, i.e., the equation becomes *quasilinear*.

Several difficulties appear due to this kind of lower order terms. The most evident is that problem ($\wp$) is *not variational*. Thus, variational techniques are not applicable to our problem, at least not in a direct way. Another handicap is the *lack of regularity* of the solutions. Note that, due to hypothesis (1.3), $\mathrm{div}(A(x)\nabla u)$ belongs at most to $L^1_{\mathrm{loc}}(\Omega)$ for any solution $u \in H^1_{\mathrm{loc}}(\Omega)$, even if $g(\cdot,u), f(\cdot,u) \in L^\infty(\Omega)$. Hence, classical elliptic regularity theory does not yield any information. Going further, not even the *maximum principle* holds unless F enjoys nice properties. The challenges that problems with natural growth present will become more clear later when analyzing more specific situations.

As the variational theory seems not to be useful for our purposes, we deal with problem ($\wp$) by means of *topological methods*. It is well-known that, when employing such methods, probably the main issue is the proof of the existence of *a priori estimates* on the solutions. Particularly in our weak framework, this represents a nontrivial task due to the low regularity of the data and the quadratic growth in the gradient. The classical references for the existence of solution to natural growth problems through topological approaches are [4, 27–32, 91]. Specially the works by Boccardo, Murat and Puel [27–32] are remarkable because of the method that they developed for finding the estimates, which requires minimal conditions on the regularity of the data. These works signified a starting point of a new area of research which has become very prolific and remains an active field in the present as many open problems are still unsolved. Nowadays, the literature concerning natural growth problems is vast. Without the aim of being exhaustive, the interested reader is referred to the bibliography of this thesis for further references.

In the quoted classical references [4,27–32,91], it is always assumed that the lower order term F is continuous at $s = 0$. However, as we announced, we will consider problems for which $\lim_{s\to 0} F(x,s,\xi)$ may not exist for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$. In other words, the nonlinear term is possibly *singular* near $s = 0$. This fact adds a further difficulty to the problem since the solutions vanish (at least in a weak sense) on the boundary, so the nonlinear term may blow up as $x \to \partial\Omega$. Even more, observe that, in principle, the solutions could vanish in the interior of Ω as well, in which case the blow-up could happen also far from the boundary and problems of definition of the solutions appear, as we commented above.

Elliptic equations with lower order terms singular as $s \to 0$ have been widely studied since the seminal papers [51, 124] (see also [61, 97]). The present thesis will be mainly concerned with singularities that appear in combination with the gradient term. To be more precise, the lower order terms that we consider will take the form

$$F(x,s,\xi) = g(x,s)|\xi|^q + f(x,s) \quad \text{a.e. } x \in \Omega,\ \forall (s,\xi) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{R}^N, \tag{1.4}$$

where $1 < q \le 2$ and f, g are Carathéodory functions such that f is continuous at $s = 0$, while g is possibly singular at $s = 0$. The study of elliptic problems involving this kind of nonlinearities was initiated roughly a decade ago with the works [6, 8, 14, 24, 76] and remains as an active field nowadays.

Summarizing, the main features of the nonlinearities under consideration that we have presented so far are the natural growth with respect to the gradient variable and the singularity as $s \to 0$. It remains to introduce a more specific peculiarity which represents one of the principal motivations of the research contained here. In order to present this new feature, we will describe in a formal way a usual procedure in the study of equations with natural growth which has its origin in [91].

Indeed, let us assume that A is the $N\times N$ identity matrix, and suppose also that there exist a continuous function $g:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ and a Carathéodory function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ such that

$$F(x,s,\xi)=g(s)|\xi|^2+f(x,s)\quad \text{a.e. } x\in\Omega,\ \forall(s,\xi)\in(\mathbb{R}\setminus\{0\})\times\mathbb{R}^N. \tag{1.5}$$

Thus, problem ($\wp$) becomes

$$\begin{cases}-\Delta u=g(u)|\nabla u|^2+f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases} \tag{1.6}$$

Observe that F satisfies (1.3) and g may be singular as $s\to 0$, but the latter does not depend on x. We will assume in addition that

$$\left|\int_0^s e^{\int_1^t g(r)dr}dt\right|<+\infty\quad \forall s\in(-1,1)\setminus\{0\}. \tag{1.7}$$

Note that the last condition holds whenever g is continuous at $s=0$, but it does not, for instance, if there exists a constant $k\geq 1$ such that $g(s)\leq -\frac{k}{s}$ for $s>0$ near zero. Also assumption (1.7) allows us to consider the function $\Psi:\mathbb{R}\to\mathbb{R}$ defined by

$$\Psi(s)=\int_0^s e^{\int_1^t g(r)dr}dt\quad \forall s\in\mathbb{R}.$$

Since $\Psi'>0$, we may also consider the inverse function $\Psi^{-1}:\mathrm{Im}(\Psi)\to\mathbb{R}$.

In this setting, if there exists a solution u to (1.6), then the function $v=\Psi(u)$ satisfies

$$\begin{cases}-\Delta v=\Psi'(\Psi^{-1}(v))f(x,\Psi^{-1}(v)), & x\in\Omega,\\ v=0, & x\in\partial\Omega.\end{cases} \tag{1.8}$$

Reciprocally, if v is a solution to (1.8), then $v(x)\in\mathrm{Im}(\Psi)$ for almost every $x\in\Omega$ and $u=\Psi^{-1}(v)$ is a solution to (1.6). Hence, problem (1.6) can be handled through the associated problem (1.8). The advantage of this approach is that the lower order term in (1.8) does not depend on the gradient any more, i.e., the equation is *semilinear*. In particular, variational techniques and regularity theory are available tools in general. Without being exhaustive, some examples of works in which this approach is strongly followed are [1, 15, 16, 59, 90, 111].

In view of last references, it seems apparent that the change of variable represents a very useful tool for studying natural growth problems. However, it works only for very specific nonlinearities of type $g(u)|\nabla u|^2$. On the contrary, just to give some simple examples, last approach does not work for lower order terms of type $g(u)|\nabla u|^q$ with $q < 2$, nor for $g(x,u)|\nabla u|^2$. Thus, we will be mainly interested in problem ($\wp$), for F satisfying (1.3), such that the equation is not convertible into a semilinear one. We will refer to such problems as *purely quasilinear*. The difficulties that purely quasilinear problems present should be now clear: the techniques that apply for semilinear equations do not work in this setting. Consequently, this kind of problems forces to develop new techniques that are compatible with the presence of a gradient term in the equation.

Of course, the study of purely quasilinear problems is not new. Actually, the pioneering works by Boccardo, Murat and Puel themselves consider general equations which are not reducible to semilinear ones. Nevertheless, the more complex the equation into consideration is, the poorer and less precise the results that one can prove are. For instance, problems like determining the specific structure of the set of solutions, showing uniqueness and/or multiplicity of solutions, etc., are specially hard to deal with in the general context of purely quasilinear problems. Our aim is to overcome some difficulties that purely quasiliear problems present in order to prove accurate results, as an attempt of getting closer to the sharpness that characterizes the semilinear theory.

In conclusion, as the title of the thesis reads, our goal is to study purely quasilinear problems with natural growth and singularities and prove results about them as complete as possible in this general framework. For this purpose, we have to derive new tools that work for quasilinear equations in the sense that they do not get rid of the gradient term.

The next section is devoted to explaining in detail some known results about problem ($\wp$) for particular choices of F. These results give rise to open problems that will mean the specific motivations to this thesis. As a matter of fact, we will also present some open questions concerning a semilinear problem. This problem may be considered independent of the rest which are included in this thesis, though it represents a future challenge to be explained from a purely quasilinear point of view. As the problem remains within the elliptic framework and combines simplicity and interest, we consider that it deserves to be included in this report.

1.2 Motivation

1.2.1 Problems with a linear zero order term

Let us consider the following problem

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

where $0 \lneq \mu \in L^\infty(\Omega)$, $0 \lneq f \in L^p(\Omega)$ for some $p > \frac{N}{2}$, $1 < q \leq 2$ and $\lambda \in \mathbb{R}$. Of course, this corresponds to the choices $A = I$ and $F(x,s,\xi) = \lambda s + \mu(x)|\xi|^q + f(x)$ in ($\wp$). The existence of solution to this problem was first proved in [32], even though the condition $\lambda < 0$ is required by the authors in order to provide the problem with enough coercivity. Thus, the study of (1.9) for $\lambda \geq 0$ arises in a natural way. In this regard, the case $\lambda = 0$ has been shown to be of special interest itself. In fact, it is proved in [69] that there exists a solution to (1.9) for $\lambda = 0$ provided $\|\mu\|_{L^\infty(\Omega)}$ and $\|f\|_{L^p(\Omega)}$ are small enough (see also [68] for a similar result where the hypothesis on the summability of f is relaxed). This existence result for $\lambda = 0$ and small data, as well as the previous one for $\lambda < 0$ and without size condition on the data, have been improved in [85] in the case $q < 2$ by requiring less summability on f. As far as the uniqueness of solution is concerned, it was first proved in [18] and improved in several directions in [11, 17, 19].

Furthermore, it has been also proved that the smallness assumptions on the data in the case $\lambda = 0$ are not just technical. Indeed, in [3] (see also [1,87]) it is shown that there are smallness conditions on f and μ that are necessary in order problem (1.9) to admit a solution for $\lambda = 0$. It follows that existence of solutions for $\lambda = 0$ is not guaranteed.

The case of nonexistence for $\lambda = 0$ has been deeply analyzed in [113]. Here, it is proved that the solutions to (1.9) blow up pointwise in Ω as $\lambda \to 0$. Moreover, the rate of explosion is studied. Finally, the author gives a necessary and sufficient condition for the existence of solution to (1.9) in terms of the sign of the ergodic constant to the associated ergodic problem.

The case $\lambda > 0$ has been much less studied. The reason is that problem (1.9) becomes highly non-coercive, so a priori estimates (and, in turn, existence results) are hard to obtain. The first results in this direction appeared in [90], where a multiplicity result is proven for $\lambda > 0$. Roughly speaking, the authors prove that, if $q = 2$ and $\mu \equiv$ constant, and μf is small enough in some sense, then

there exists $\lambda_0 > 0$ *such that* (1.9) *admits at least two solutions for all* $0 < \lambda \leq \lambda_0$. (1.10)

We emphasize that the authors of [90] are restricted to the case $q = 2$ and $\mu \equiv$ constant. These choices allow them to perform the change of variable explained in the previous section and handle problem (1.9) via the associated semilinear problem by means of variational methods.

An outstanding improvement of [90] can be found in the recent work [12]. Here, the authors prove a result of type (1.10) under more general hypotheses. Namely, they allow μ to be x dependent, though they require the existence of two positive constants $0 < \mu_1 \leq \mu_2$ such that

$$\mu_1 \leq \mu(x) \leq \mu_2 \quad \text{a.e. } x \in \Omega. \tag{1.11}$$

Furthermore, they do not impose smallness conditions on f nor μ at the expense of assuming the existence of solution to (1.9) for $\lambda = 0$. However, they do not relax the growth condition on the gradient, i.e., $q = 2$ is needed. The proof of the multiplicity result relies on an a priori estimate that they show to hold for $\lambda > \lambda_0$, for any fixed $\lambda_0 > 0$. The proof of such an estimate is based in turn on a sort of "double change of variables" in the spirit of the one introduced in the previous section. Indeed, using that $q = 2$ and $\mu \geq \mu_1$ (resp. $\mu \leq \mu_2$), one can check that every solution to (1.9) is a subsolution (resp. supersolution) to a certain semilinear problem. Thus, the authors develop a technique, inspired by the approach due to [35] for semilinear equations, which leads to proving the estimate. The proof of (1.10) works also thanks to the fact (which they also prove to hold) that the set of pairs (λ, u), where $\lambda > 0$ and u satisfy (1.9), constitutes an unbounded continuum emanating from $(0, u_0)$, being u_0 the unique solution to (1.9) for $\lambda = 0$. Thus, the a priori estimate for $\lambda > \lambda_0$ implies that $\lambda = 0$ is a bifurcation point from infinity to the right. This phenomenon encloses an interesting consequence, which is that $\lambda = 0$ is *always* a bifurcation point from infinity, no matter that there exists a solution to (1.9) for $\lambda = 0$ or not.

Shortly after [12], the very related paper [120] was published. Here, the same approach based on bifurcation and an a priori estimate for $\lambda > \lambda_0$ is employed to prove (1.10) for $q = 2$ and μ nonconstant. The difference is that μ is allowed to vanish in subsets of $\overline{\Omega}$. This implies that the double change of variables is useful only in the set where μ is positive. The loss of information implies that the approach in [120] works only for low dimension N.

Several other works proving (1.10) for $q = 2$, and under various conditions involving new terms in the equation, have appeared very recently. For instance, in [89] the authors consider the equation in (1.9) with $\lambda = \lambda(x)$ possibly changing sign, while in [56] any coefficient in the equation is allowed to change sign. We remark [55] as well, where (1.10) has been proven for a more general problem involving the $p-$Laplace operator (of course, in this case the condition $q = 2$ becomes $q = p$). Furthermore, in [57] it is proven that the structure of the set of solutions to (1.9) for $q = 2$ and $\lambda > 0$ is even richer by assuming different sign conditions on f. However, to the best of our knowledge, the case $1 < q < 2$ has not been considered in the literature. Thus, proving (1.10) for

$1 < q < 2$ (under suitable conditions on μ, f, N, Ω) remains as an open problem.

Moving forward to further motivations, we introduce now a problem related to the previous one:

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where $0 \lneq \mu \in L^\infty(\Omega)$, $0 \lneq f \in L^p(\Omega)$ for some $p > \frac{N}{2}$, $1 < q \leq 2$, $\lambda \in \mathbb{R}$ and $\alpha > 0$. The novelty with respect to problem (1.9) is that the gradient term is now singular as $x \to \partial\Omega$. The existence of solution to elliptic problems involving singular lower order terms depending on the gradient was first studied in [6,8,14,24,76] and it is currently an active research topic. Some more recent references are [25,36,42], among many others that will appear throughout the thesis. The literature for the uniqueness of solution is more limited, we quote [10,15,16,37].

Concerning problem (1.12) specifically, existence of solution has been proven in [76] for $\lambda < 0$, $q = 2$, $\alpha \in (0,1)$ and $f \in L^\infty(\Omega)$. Regarding the case $\lambda = 0$, it has been studied in [7,77,78] for $q = 2$, and in [2] for $1 < q \leq 2$, each of them under several conditions on μ, f, α. The first work in the literature, as far as we know, dealing with the noncoercive case $\lambda > 0$ is the recent paper [15]. Here, the authors set $q = 2$ and $\alpha = 1$. With this choice, they prove that there exists at least a solution to (1.12) for every $\lambda < \frac{\lambda_1}{\|\mu\|_{L^\infty(\Omega)}+1}$, where λ_1 denotes the principal eigenvalue of the operator $-\Delta$ in Ω with zero Dirichlet boundary conditions. Moreover, if $\mu \equiv$ constant $\in (0,1)$, they prove that the solution is unique, that the condition $\lambda < \frac{\lambda_1}{\mu+1}$ is necessary for the existence of solution, and also that $\frac{\lambda_1}{\mu+1}$ is a bifurcation point from infinity. Thus, it is shown that the presence of the singular term $|\nabla u|^2/u$ produces a structural change in the set of solutions with respect to the nonsingular case $|\nabla u|^2$. Indeed, the singular problem behaves like the well-known linear one corresponding to $\mu \equiv 0$. However, the proof of this linear-like result for $\mu \equiv$ constant $\in (0,1)$ is based one more time on the mentioned change of variable. The question of whether an optimal existence and uniqueness result holds in the singular case also for μ nonconstant was left unanswered in [15].

In any case, it is noteworthy that the singular term has *linear homogeneity* in the sense that $|\nabla(tu)|^2/(tu) = t|\nabla u|^2/u$ for all $t > 0$, while the nonsingular one enjoys *superlinear homogeneity*, i.e., $|\nabla(tu)|^2 = t^2|\nabla u|^2$ for all $t > 0$, and this is irrespective of μ being constant or not. Hence, one may think that a linear homogeneity of the lower order term yields a linear-like behavior of the solutions. Moreover, in the known linear-like result for $q = 2$, $\alpha = 1$ and $\mu \equiv$ constant $\in (0,1)$, the critical value $\frac{\lambda_1}{\mu+1}$ may be seen as the *principal eigenvalue* of the operator $u \mapsto -\Delta u - \mu|\nabla u|^2/u$. That is to say, if $q = 2$, $\alpha = 1$, $\mu \equiv$ constant $\in (0,1)$ and $f \equiv 0$, then it is easy to prove (via the change of variable) that problem (1.12) admits a solution if, and only if, $\lambda = \frac{\lambda_1}{\mu+1}$. Thus, also in the purely quasilinear case $1 < q \leq 2$, $\alpha = q-1$ and $0 \lneq \mu \in L^\infty(\Omega)$, it is natural to expect the existence of a principal eigenvalue of the nonlinear 1-homogeneous operator $u \mapsto -\Delta u - \mu(x)|\nabla u|^q/|u|^{q-1}$. Therefore, it seems reasonable to deal with problem (1.12) (in the 1-homogeneous case $\alpha = q-1$) by means of nonlinear eigenvalue theory.

Up to our knowledge, the first work dealing with nonvariational (though linear) eigenvalue problems is [21]. In this paper, the authors characterize the principal eigenvalue, say λ^*, associated to the linear operator that they consider, say L, by using only supersolutions and avoiding any variational argument. Formally, the authors defined λ^* as follows:

$$\lambda^* = \sup\{\lambda \in \mathbb{R} : \exists v > 0 \text{ in } \Omega \text{ satisfying } -Lv \geq \lambda v \text{ in } \Omega\}. \tag{1.13}$$

Such a characterization has been proven to be valid even for the principal eigenvalue to fully nonlinear elliptic problems (see [22] and references therein). It is surprising that the so-called *ergodic constant*, which appears when studying large solutions to ergodic problems (see [95, 98]), can be characterized in a similar way, as was shown in [113]. Actually, the connection between principal eigenvalues and ergodic constants was already observed in [95]. Our claim is that a characterization of type (1.13) applies also in our quasilinear and singular framework in the 1-homogeneous case $\alpha = q-1$.

In the general situation of problem (1.12) for $1 < q \leq 2$ and $\alpha \in (0,q)$, the lower order term presents homogeneity $q-\alpha > 0$. The previous discussion motivates the study of problem (1.12) focusing specially on the different possible homogeneities of the lower order term, i.e., either $q-\alpha \in (0,1)$, or $q-\alpha = 1$, or else, $q-\alpha > 1$. In each case one expects a different structure of the set of solutions for $\lambda > 0$. However, the fact that μ is allowed to be nonconstant and q might be different from 2 implies that the usual change of unknown is not useful. Therefore, a purely quasilinear approach must be derived.

So far we have observed that the interaction between the exponents q, α might determine the structure of the solutions to problem (1.12). We want now to turn the attention to a different aspect of (1.12), which is the singularity as such. We have already stressed that, since the solutions vanish on $\partial\Omega$ in some sense, then the lower order term may blow up as $x \to \partial\Omega$. On the contrary, when dealing with singular problems it is usual to look for *positive solutions* in the interior, as in (1.12), so that the singular term remains locally bounded away from zero, and thus, it is well-defined in Ω. Nevertheless, if $\alpha \in (0,q)$, the equation in (1.12) can be easily rewritten in such a way that solutions which are not positive are perfectly meaningful:

$$\begin{cases} -\Delta u = \lambda u + \hat{\mu}(x) \left|\nabla |u|^{1-\frac{\alpha}{q}}\right|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.14}$$

where $\hat{\mu} = \left(\frac{q}{q-\alpha}\right)^q \mu$. Written this way, the nonlinear term in (1.14) may still blow up whenever $u = 0$, even though it remains bounded if, for instance, $|u|^{1-\frac{\alpha}{q}} \in C^1(\overline{\Omega})$.

Anyway, it is typical in the literature to impose conditions on the data in order to assure that every solution to (1.14) is positive, solving in turn (1.12). In fact, the conditions $\lambda < \lambda_1, \mu \geq 0$ and $f \gneq 0$ are sufficient to apply the strong maximum principle, which guarantees the positivity of the solutions to (1.14). However, there are examples of sign-changing data f for which there exists at least a solution to (1.14) which vanishes in subsets of Ω. Indeed, let $u \in C^\infty(\overline{B_2(0)})$ be defined by

$$u(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & x \in B_1(0), \\ 0, & x \in \overline{B_2(0)} \setminus B_1(0). \end{cases}$$

Then, it is straightforward to verify that u satisfies (1.14) for $\lambda = 0, \mu \equiv 1, q = 2, \alpha = 1$ and

$$f(x) = \begin{cases} 2e^{\frac{1}{|x|^2-1}} \dfrac{N - 2N|x|^2 + (N-4)|x|^4}{(1-|x|^2)^4}, & x \in B_1(0), \\ 0, & x \in B_2(0) \setminus B_1(0). \end{cases}$$

It can be checked that $f(x) > 0$ for $|x| \approx 0$ while $f(x) < 0$ for $|x| < 1, |x| \approx 1$.

There are few works dealing with the existence of solution to quasilinear singular problems admitting sign-changing data. Up to our knowledge, the first publication in this topic is [77] (see also [36, 78]). In this work, the authors set a suitable concept of solution to (1.12) (in the spirit of the rewritten version (1.14)) and they prove the existence of at least a solution provided $\lambda = 0, q = 2, \alpha \in (0,1), 0 \lneq \mu \in L^\infty(\Omega)$ and $f \in L^{\frac{N}{2}}(\Omega)$. The main difficulty that they have to overcome relies on the lack of local a priori estimates from below of the following type:

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad u \geq c_\omega \text{ in } \omega \tag{1.15}$$

for every solution u to the problem into consideration (we clarify that c_ω does not depend on u). Whenever it holds true, the estimate (1.15) keeps the singularity controlled far from $\partial\Omega$ and permits to pass to the limit in suitable approximated problems (see [8], where this issue is specially highlighted, and see also reference therein). Of course, (1.15) cannot hold for sign-changing data, as the previous explicit example shows. Therefore, for passing to the limit the authors of [77] argue in a different way, namely, by proving a *global estimate* of the nonlinearity in $L^1(\Omega)$ which eventually leads to the existence of a solution. As far as we know, the question of whether there exists a solution to problem (1.14) for sign-changing datum f and general λ, q, α has not been solved.

In relation to this last question, we finally introduce a *homogenization* problem. Namely, consider an $\varepsilon-$family of bounded smooth domains $\{\Omega^\varepsilon\}$ and assume that, for every $\varepsilon > 0$, there exists a solution u^ε to (1.12) replacing Ω with Ω^ε. Assume also that $\Omega^\varepsilon \subset \Omega$ for every $\varepsilon > 0$ and for a fixed bounded smooth domain Ω. Then, we may define the function $\widetilde{u^\varepsilon} : \Omega \to \mathbb{R}$ as follows:

$$\widetilde{u^\varepsilon}(x) = \begin{cases} u^\varepsilon(x), & x \in \Omega^\varepsilon, \\ 0, & x \in \Omega \setminus \Omega^\varepsilon. \end{cases}$$

The homogenization problem consists of finding estimates on $\widetilde{u^\varepsilon}$, independent of ε, which yield the existence of some limit $u : \Omega \to \mathbb{R}$ of a subsequence of $\{\widetilde{u^\varepsilon}\}$ and, if this is the case, of characterizing u as a solution to a certain boundary value problem. Intuitively, such a problem would be close to (1.12), though need not be the same.

The main reference for linear elliptic homogenization problems (with zero Dirichlet boundary conditions) is [49]. Here, the authors show that a particular choice of sequences $\{\Omega^\varepsilon\}$ yields the appearance of a so-called "strange term" in the limit equation. Thus, the limit equation indeed defers from the original one. Regarding nonlinear elliptic homogenization problems that involve a gradient term (with or without singularity), we quote [41, 45, 46].

As we announced above, the homogenization problem that we consider is somehow related to the existence of solution to (1.14) for sign-changing data. The reason is that the sequence $\{\widetilde{u^\varepsilon}\}$ does not satisfy a local estimate of type (1.15) because each $\widetilde{u^\varepsilon}$ vanishes in $\Omega \setminus \Omega^\varepsilon$. As explained before, this fact makes difficult to prove that the limit of the sequence satisfies a singular boundary value problem, as the terms $|\nabla \widetilde{u^\varepsilon}|^q / (\widetilde{u^\varepsilon})^\alpha$, in principle, are not well defined in $\Omega \setminus \Omega_\varepsilon$ and, in any case, might not be controlled by appropriate estimates independent of ε. This is why a suitable strategy for passing to the limit is necessary. The challenges that the singular homogenization problem presents motivates a thorough analysis.

1.2.2 Problems with a superlinear zero order term

We introduce now a different type of problem:

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{(u+\delta)^\gamma} = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.16}$$

where $\mu \in L^\infty(\Omega)$, $\delta > 0$, ≥ 0, $\gamma, \lambda > 0$, $p > 1$. Thus, we have chosen $A = I$ and $F(x,s,\xi) = \lambda s^p - \mu(x)\frac{|\xi|^2}{(s+\delta)^\gamma}$ in ($\wp$). Note that, if $\delta > 0$, then $u \equiv 0$ satisfies the equation in (1.16) as well as the boundary condition. Conversely, if $\delta = 0$, then the gradient term presents a singularity as $x \to \partial\Omega$. In both cases, it is natural to look for *positive solutions*. We stress also that this problem presents a new difficulty apart from the the ones coming from the gradient term and the possible singularity ($\delta = 0$), which is the zero order term on the right hand side. Of course, such a term is nonlinear too and, since $p > 1$, it has *superlinear growth*.

For $\mu \equiv 0$, the semilinear problem with superlinear growth that we obtain is classical. Indeed, as the problem becomes variational, then the celebrated Mountain Pass Theorem in [5] yields the existence of at least a (positive) solution provided $p < 2^* - 1$. Topological methods have also being developed in order to study more general nonvariational semilinear problems with superlinear growth. To this respect, the classical references are [60, 82], where it is proved in particular that there exists a solution to (1.16) for every $\lambda > 0$ provided $\mu \equiv 0$ and $p \in (1, 2^* - 1)$. It is also worth to mention the work [35], in which the authors prove a similar existence result only for $p \in \left(1, \frac{N+1}{N-1}\right)$, though the technique is original and has been shown to be very useful in deferent contexts (see [12] for instance). On the other hand, as a consequence of the well-known Pohozaev Identity in [112], it follows that there exists no solution to problem (1.16) for $\mu \equiv 0$ provided $p \geq 2^* - 1$ and Ω is starshaped. Therefore, it was shown that $2^* - 1$ is a natural critical value for the existence of solutions.

As far the case $\mu \not\equiv 0$ is concerned, the first work in considering problem (1.16) is [111]. To be more precise, the authors study the following problem:

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.17}$$

where $g : [0, +\infty) \to [0, +\infty)$ is a general continuous function. Observe that g is neither allowed to depend on x, nor to be singular as $s \to 0$. Among the several results that the authors of [111] prove, we remark the following one:

Result 1 (Theorem 1.2 in [111])**.** *If* $1 < p < 2^* - 1$ *and* $g \in L^1((0,+\infty))$*, then there exits a solution to* (1.17) *for all* $\lambda > 0$*.*

Thus, the authors show that an integrable coefficient of the gradient term yields the same behavior as the classical semilinear case. In particular, they prove existence of solution to (1.16) for all $\lambda > 0$ in the case $\mu \equiv$ constant > 0, $\delta > 0$ and $\gamma > 1$. The key point in the proof is the change of variable explained in the introduction (section 1.1) above, which permits the authors to apply the Mountain Pass Theorem on the resulting semilinear problem.

Note that problem (1.16) with $\gamma \leq 1$ is not covered by Result 1, not even for $\delta > 0$ and $\mu \equiv$ constant, since $g(s) = \frac{\mu}{(s+\delta)^\gamma}$ is not integrable at $+\infty$. Nonetheless, nonintegrable functions g are also considered in [111]. For such functions, the authors show that the nature of problem (1.17) differs from the semilinear case. In particular, regarding problem (1.16), they prove the following

Result 2 (Theorems 1.5 and 1.8 in [111])**.** *Let* $\mu \equiv$ *constant* > 0*,* $\delta > 0$*,* $\gamma \geq 1$ *and* $1 < p < 2^* - 1$*. If* $\gamma = 1$*, assume also that* $\mu > p$*. Then problem* (1.16) *admits at least a solution* u_λ *for* $\lambda > 0$ *large, while admits no solution for* $\lambda > 0$ *small. Moreover,* $\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$*.*

As the authors themselves point out, they do not handle the possibility $\gamma = 1$ and $\mu \equiv$ constant $\in (0, p]$. To this respect, a first improvement of Result 2 can be found in [9], where the authors extend the nonexistence statement for $\lambda > 0$ small to the case $\gamma = 1$ and $\mu \equiv p$. Regarding the existence of solution, the next remarkable advance appears almost simultaneously in [100]. There, the authors study a general quasilinear problem with superlinear growth in u and natural growth in ∇u, so that problem (1.16) is included in their framework as long as $\gamma = 1$, $\mu \equiv$ constant and $\delta > 0$. What they prove is gathered in the following

Result 3 (Theorem 4.1 in [100])**.** *Let* $p > 1$*,* $\mu \equiv$ *constant* $< \frac{2^*-1-p}{2^*-2}$*,* $\delta > 0$ *and* $\gamma = 1$*. Then, problem* (1.16) *admits at least a solution for every* $\lambda > 0$*.*

Observe that the result allows p to be supercritical (i.e., $p \geq 2^* - 1$), even though in such a case μ is forced to be negative. It is also interesting that the existence statement for $\gamma = 1$ in Result 2 is improved in [100] since $\frac{2^*-1-p}{2^*-2} < 1 < p$. The apparently strange critical value $\frac{2^*-1-p}{2^*-2}$ shows up in a natural way when one performs the change of variable

$$v=\int_0^u e^{-\int_0^t \frac{\mu}{r+\delta}dr}dt=\delta^\mu\,\frac{(u+\delta)^{1-\mu}-\delta^{1-\mu}}{1-\mu},$$

where $\mu\equiv$ constant $\in(0,1)$ and $\delta>0$. Then, problem (1.17) with $\gamma=1$, $\delta>0$ and $\mu\equiv$ constant $\in(0,1)$ is equivalent to

$$\begin{cases}-\Delta v=h(v), & x\in\Omega,\\ v>0, & x\in\Omega,\\ v=0, & x\in\partial\Omega,\end{cases}\tag{1.18}$$

where

$$h(s)=\frac{\lambda\left[((1-\mu)s+\delta)^{\frac{1}{1-\mu}}-\delta^{\frac{1}{1-\mu}}\right]^p}{\delta^{\frac{\mu(p-1)}{1-\mu}}((1-\mu)s+\delta)^{\frac{\mu}{1-\mu}}}\quad \forall s\geq 0.$$

It is clear that $\lim_{s\to+\infty}s^{-\frac{p-\mu}{1-\mu}}h(s)=$ constant >0. Then, [82] implies that there exists at least a solution to (1.18), and thus, to (1.16), provided $\frac{p-\mu}{1-\mu}<2^*-1$. This last condition is equivalent to $\mu<\frac{2^*-1-p}{2^*-2}$.

The approach of [100] lies, in fact, on the previous transformation. However, as their original problem is rather general, the resulting problem after the transformation is not semilinear, so not directly solvable in principle. Actually, the change of variable removes a quadratic term of the form $g(u)|\nabla u|^2$ from the equation that they consider, though a lower order term still depending on ∇v remains after the transformation. The advantage is that its growth with respect to the gradient is *less than quadratic*. This fact allows the authors to employ *blow-up techniques* to derive a priori estimates of the solutions. Nevertheless, a lower order term of the form $g(x,u)|\nabla u|^2$ cannot be handled with their approach since it remains in the equation after performing whichever change of unknown of the type $v=\psi(u)$. At this point, a natural question arises: can a priori estimates (and, in turn, solutions) be found for problem (1.16) if $\delta>0$, $\gamma=1$ and μ depends on x? As far as we know, the question is still unanswered.

On the other hand, focusing again on problem (1.16) for $p\in(1,2^*-1)$, $\delta>0$, $\gamma=1$ and $\mu\equiv$ constant, the range $\mu\in\left(\frac{2^*-1-p}{2^*-2},p\right)$ has not been studied in the literature, to the best of our knowledge. Nevertheless, some direct conclusions can be deduced by looking at the associated semilinear problem (1.18). Indeed, the behavior of h at infinity varies in terms of the size of μ. Namely, it has superlinear and *supercritical growth* for $\mu\in\left(\frac{2^*-1-p}{2^*-2},1\right)$, *exponential growth* for $\mu=1$, and has an *asymptote* at $s=\frac{1}{\mu-1}$

for[2] $\mu \in (1, p)$. In any case, formally speaking, $h(s)$ grows faster than s^{2^*-1} at infinity, so the classical results for semilinear problems with superlinear growth do not apply. However, it is immediate to check that $\lim_{s\to 0} s^{-p}h(s) = \text{constant} > 0$, no matter the size of μ. This means that problem (1.18) is *subcritical at zero* for $p \in (1, 2^* - 1)$ and $\mu \in \left(\frac{2^*-1-p}{2^*-2}, p\right)$. For this kind of problems, it has been proved in [13], by means of variational methods, that there exists at least a solution to (1.18) for $\lambda > 0$ large enough. To be more precise, from the results in that paper we deduce the following straightforward

Result 4 (corollary of Theorem 8 in [13])**.** *Let $p \in (1, 2^* - 1)$, $\delta > 0$, $\gamma = 1$ and $\mu \equiv \text{constant} \in \left(\frac{2^*-1-p}{2^*-2}, p\right)$. Then, there exists $\lambda_0 > 0$ such that problem* (1.16) *admits at least a solution u_λ for every $\lambda > \lambda_0$. Moreover,* $\lim_{\lambda\to+\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.

In particular, Results 2, 3 and 4 imply that, if $p \in (1, 2^* - 1), \delta > 0, \gamma = 1$ and $\mu \equiv \text{constant} > 0$, then problem (1.17) admits at least a solution for every $\lambda > 0$ large enough, *no matter the size of* μ. An interesting open problem consists of showing whether a similar statement holds true in the general case $\mu = \mu(x)$ with no assumptions on its size. As usual, in this case the change of unknown does not turn the quasilinear equation into a semilinear one, so variational methods are not allowed and a specific approach for quasilinear problems must be derived.

Just to finish with problem (1.16), we consider the singular case $\delta = 0$. As far as we know, the only reference dealing with the existence of solution to problem (1.16) in the singular case is [43] (see also [33, 44] for related singular problems involving a nontrivial source term). The authors consider again only the case $\mu \equiv \text{constant} > 0$ and they prove an existence result for $\lambda > 0$ large, as well as a nonexistence result for $\lambda > 0$ small, provided the singularity is somehow *mild* and $p > 1$ is small. Thus, the situation reminds of the nonsingular case, as Result 4 reads. However, they show that the singular problem has a special behavior in the sense that, if the singularity is too *strong*, then there is no solution to problem (1.16) for *any* $\lambda > 0$. To be precise, they prove the following

Result 5 (Theorem 1.3 in [43])**.** *Let $p > 1$, $\delta = 0$, $\gamma > 0$ and $\mu \equiv \text{constant} > 0$. On the one hand, if $\gamma < 1$ and $p < 2 - \gamma$, then there exists $\lambda_0 > 0$ such that problem* (1.16) *admits at least a solution for every $\lambda \geq \lambda_0$, while it admits no solution for any $\lambda \in (0, \lambda_0)$. On*

[2]Actually, in this last case one cannot talk about "behavior at infinity" since $h(s)$ is not defined for $s \geq \frac{1}{\mu-1}$. However, as $\lim_{s\to\frac{1}{\mu-1}} h(s) = +\infty$, one can formally extend h by continuity to the whole real line and think of $h(s)$ to be *identically* $+\infty$ for $s \geq \frac{1}{\mu-1}$. In these terms, its "growth at infinity" would be higher than exponential.

the other hand, if either $\gamma > 1$, or $\gamma = 1$ and $\mu > 1$, then there exits no solution to (1.16) *for any $\lambda > 0$.*

As before, the case μ depending on x is not handled, neither the case $\gamma = 1$ and $\mu \equiv$ constant $\in (0,1]$, nor the case $\gamma < 1$ and $p \geq 2-\gamma$. Thus, several open questions arise in the singular framework which deserve a careful study.

1.2.3 A semilinear problem

Let us consider now the following simple problem:

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.19}$$

where $f : \Omega \to \mathbb{R}$ is a continuous function. This corresponds to the choice $A \equiv I$ and $F \equiv f$ in ($\wp$). Since f does not depend on ∇u, the equation is *semilinear*. The literature about existence results for problem (1.19), under diverse hypotheses on f, is huge. The interested reader is referred to the review [101] and references therein.

We will focus here on finding general necessary conditions for the existence of solution to (1.19) or, from another point of view, on proving nonexistence results. Actually, a first nonexistence result can be proved by means of a simple computation. Indeed, assume that there exists a solution u to (1.19) and let us take it as test function in the weak formulation of (1.19). Then we obtain

$$\int_\Omega |\nabla u|^2 = \int_\Omega f(u)u.$$

Therefore, if

$$f(s)s \leq 0 \quad \forall s \in \mathbb{R}, \tag{1.20}$$

then there exists no solution to (1.19) but the trivial one $u \equiv 0$. Observe that, if f is continuous and satisfies (1.20), then $f(0) = 0$, so $u \equiv 0$ is always a solution to (1.19). Anyway, last statement represents a nonexistence result of nontrivial solutions.

The well-known Pohozaev Identity (see [112]) generalizes last result, though only for equations posed in domains with a special geometry. To be more precise, every solution u to (1.19) satisfies Pohozaev identity:

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu + \frac{N-2}{2}\int_\Omega |\nabla u|^2 = N\int_\Omega F(u),$$

where $\nu = \nu(x)$ denotes the unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$ pointing outward, and $F(s) = \int_0^s f(t)dt$ for all $s \in \mathbb{R}$. It is clear that, if Ω is starshaped with respect to the origin, then $x \cdot \nu(x) \geq 0$ for all $x \in \partial\Omega$. Therefore, if in addition

$$F(s) \leq 0 \quad \forall s \in \mathbb{R}, \tag{1.21}$$

then there exist no nontrivial solutions to (1.19). It is easy to see that (1.20) implies (1.21), but not conversely. For instance, a counterexample is $f(s) = \lambda \sin(s)$ for any $\lambda < 0$. It is natural now to wonder whether the previous geometrical assumption is essential or just technical. In other words, are there nontrivial solutions to (1.19) for some non-starshaped Ω and for some f satisfying (1.21) but not (1.20)?

It is remarkable that the geometry of Ω does determine the existence or nonexistence of nontrivial solutions in the specific case $f(s) = |s|^{p-1}s$, where $p \in (1, 2^* - 1)$. In this case, it is easy to derive also from Pohozaev identity that, if Ω is starshaped with respect to the origin, then there are no nontrivial solutions to (1.19). On the contrary, there are non-starshaped domains for which there exist nontrivial solutions to (1.19) (see [62,92]).

Much less is known about the influence of the geometry of Ω on the existence of nontrivial solutions to (1.19) if (1.21) holds. The scarce literature contains only partial nonexistence results (see [66, 84, 116]). We consider that this problems deserves a thorough study.

1.3 Objectives and results

In view of the motivations that we presented above, we set next the goals for this thesis and the main results that have been obtained. Their proofs and a deeper analysis can be found in the subsequent chapters.

1.3.1 Study of (1.12)

The first target that we propose is the study of problem (1.12) for $0 \lneq \mu \in L^\infty(\Omega)$, $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}, 1 < q \leq 2, \alpha \geq 0$ and $\lambda \in \mathbb{R}$, specially in the non-coercive case $\lambda > 0$. Specifically, we want to describe in a precise way the set

$$\Sigma = \{(\lambda, u) : u \text{ is a solution to } (1.12)\}.$$

Thus, we will consider λ to be a free parameter and we will work to determine the bifurcation diagram of (1.12). Therefore, we aim not only to prove existence, nonexistence, uniqueness and multiplicity results for problem (1.12), but also to analyze the dependence of the solutions with respect to λ and to find bifurcation points. We will put the focus on the effect produced by the different homogeneities of the gradient term, coming in turn from the different choices of the exponents q, α.

Some specific goals are listed here:

1. We will consider the nonsingular case $\alpha = 0$ and try to prove a multiplicity result for $\lambda > 0$ and $1 < q < 2$, extending thus the known results for $\lambda > 0$ and $q = 2$.

2. We will analyze the particular case $q = 2$ and $\alpha = 1$ with the aim of proving existence, nonexistence and uniqueness results for $\lambda > 0$ and μ nonconstant. Such results would improve the ones which are known for $\mu \equiv$ constant.

3. If we relax the hypotheses on f by letting it change sign, a specific goal will be to prove the existence of a sign-changing solution to (1.14) for some choices of the parameters.

4. We also want to deal with the homogenization problem associated to problem (1.12) for particular choices of the parameters.

All those objectives led to proving several results which we present here. First of all, we consider the following hypotheses:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain of class } \mathscr{C}^2, \\ \mu \in L^\infty(\Omega) \text{ satisfies that } \mu \geq \mu_0 \text{ in } \Omega \text{ for some constant } \mu_0 > 0, \\ 0 \lneq f \in L^p(\Omega) \text{ for some } p > \frac{N}{2}, \\ q \in (1,2), \\ \alpha \in [0, q-1). \end{cases} \tag{H1}$$

Observe that $q - \alpha > 1$, so we are placed in the superlinear homogeneity range. The main result in the superlinear case implies what we claimed in section 1.2: multiplicity of solutions for $\lambda > 0$ small. It reads as follows:

Theorem 1.3.1. *Assume that* (H1) *holds and that* (1.12) *admits a solution for* $\lambda = 0$. *If* $q > \frac{N}{N-1}$, *suppose also that*

$$\frac{q-1-\alpha}{q-2\alpha} \leq \frac{q-\alpha}{N-q+1}. \tag{1.22}$$

Then, there exists $\bar{\lambda} \in (0, \lambda_1)$ *such that problem* (1.12) *admits at least two different solutions for all* $\lambda \in (0, \bar{\lambda}]$. *Moreover, zero is the unique bifurcation point from infinity to problem* (1.12).

The proof of the theorem can be found in chapter 4. Just to sketch the general ideas in the proof, it is based in a bifurcation argument from [12]. The estimates required by such an argument are obtained by combining some techniques from [120], that take advantage of weighted Hardy-Sobolev inequalities, with a bootstrap argument inspired by [85].

In order to understand condition (1.22) in a better way, we derive the following two corollaries as direct consequences of Theorem 1.3.1 in which stronger conditions implying (1.22) are imposed. The first of them reads as follows:

Corollary 1.3.2. *Assume that* (H1) *holds with* $q \in (1, Q_N] \setminus \{2\}$, *where*

$$Q_N = \begin{cases} 2 & \forall N \leq 4, \\ \dfrac{N+2-\sqrt{N^2-4N-4}}{4} & \forall N \geq 5. \end{cases}$$

Assume also that there exists a solution to (1.12) *for* $\lambda = 0$. *Then, the conclusions of Theorem 1.3.1 hold true.*

Observe that $Q_n > 1$ but $\lim_{n\to\infty} Q_n = 1$. This means that, if N is large, then q has to be chosen close to 1. The advantage is that the previous corollary allows to choose whichever $\alpha \in [0, q-1)$. In particular, for $\alpha = 0$ we extend the known multiplicity results in the literature which are valid only for $q = 2$.

The following corollary adopts a different point of view: it permits to take whichever $q \in (1,2)$ at the expense of restricting α to be far from zero.

Corollary 1.3.3. *Assume that* (H1) *holds and that there exists a solution to* (1.12) *for* $\lambda = 0$. *If* $q > \frac{N}{N-1}$, *suppose also that* $\alpha \geq \left(q - \frac{N}{N-1}\right)\frac{N-1}{N-2}$. *Then, the conclusions of Theorem 1.3.1 hold true.*

The following results will show that, in Theorem 1.3.1, the condition $q - \alpha > 1$ is necessary in order to have multiplicity of solution for $\lambda > 0$ small. In other words, next results imply uniqueness of solution for $\lambda > 0$ small enough provided $q - \alpha \leq 1$.

We first focus on the sublinear homogeneity range, i.e., $q - \alpha < 1$, in which the following (weak) hypothesis on the domain will be used:

$$\begin{cases} \text{There exist } r_0, \theta_0 > 0 \text{ such that, if } x \in \partial\Omega \text{ and } 0 < r < r_0, \text{ then} \\ |\Omega_r| \leq (1-\theta_0)|B_r(x)| \text{ for every connected component } \Omega_r \text{ of } \Omega \cap B_r(x). \end{cases} \tag{A}$$

We will also need the following set of hypotheses:

$$
\begin{cases}
\Omega \subset \mathbb{R}^N \text{ is a bounded domain satisfying condition (A)}, \\
0 \lneq \mu \in L^\infty(\Omega), \\
0 \lneq f \in L^p(\Omega) \text{ for some } p > \frac{N}{2}, \\
q \in (1,2), \\
q-1 < \alpha \leq 1.
\end{cases} \tag{H2}
$$

Next theorem is the main result in the sublinear case.

Theorem 1.3.4. *Assume that* (H2) *holds. Then, there exists a solution to* (1.12) *for all* $\lambda < \lambda_1$*, and there exists no solution to* (1.12) *for all* $\lambda \geq \lambda_1$*. Moreover, the solution is unique for all* $\lambda \leq 0$*. Finally, if* f *satisfies that*

$$
\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad f(x) \geq c_\omega \quad a.e.\ x \in \omega,
$$

then the solution is unique for all $\lambda < \lambda_1$ *and* λ_1 *is the unique bifurcation point from infinity to problem* (1.12).

The proof of Theorem 1.3.4 is contained in chapter 4. The main issue in the proof is to develop a comparison principle valid for singular problems with gradient terms.

It is left to deal with the critical case $\alpha = q-1$, which corresponds to a linear homogeneity of the lower order term. To this aim, we establish the following hypotheses:

$$
\begin{cases}
\Omega \subset \mathbb{R}^N \text{ is a bounded domain of class } \mathscr{C}^{1,1}, \\
q \in (1,2], \\
\alpha = q-1, \\
0 \lneq \mu \in L^\infty(\Omega) \text{ with } \|\mu\|_{L^\infty(\Omega)} < 1 \text{ if } q = 2, \\
f \in L^p(\Omega) \text{ for some } p > \frac{N}{2}.
\end{cases} \tag{H3}
$$

This new case is special since the linear homogeneity allows to look at (1.12) as a nonlinear eigenvalue problem. Thus, for $f \equiv 0$, one expects to prove an eigenvalue-type result. Indeed, the following real number will play the role of a principal eigenvalue:

$$
\lambda^* = \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \exists v \in H^1(\Omega) \cap L^\infty(\Omega),\ \exists c > 0: \\ v(x) \geq c,\ -\Delta v \geq \lambda v + \mu(x) \dfrac{|\nabla v|^q}{v^{q-1}}, \text{ a.e. } x \in \Omega. \end{array} \right\} \tag{1.23}
$$

The main result in the linear homogeneity range for $f \equiv 0$ is the following:

Theorem 1.3.5. *Assume that* (H3) *holds for* $f \equiv 0$*. Then* $\lambda^* \in (0, \lambda_1]$ *and problem* (1.12) *admits a solution if, and only if,* $\lambda = \lambda^*$*. Moreover such a solution is unique up to multiplication by positive constants.*

The previous eigenvalue-type result in proved in chapter 2. It turns out to be an essential tool to prove the main theorem for $f \gneq 0$. Actually, for some parts of that theorem we will need to impose stronger positivity conditions on f, such as

$$\exists \gamma \in \left(\frac{1}{2}, 1\right), \exists c > 0: \quad f(x) \geq c\varphi_1(x)^\gamma \quad \text{a.e. } x \in \Omega; \tag{f_1}$$

$$\exists c > 0: \quad f(x) \geq c\varphi_1(x)^\gamma \text{ in } \Omega, \text{ where } \gamma = \frac{1}{1 + \|\mu\|_{L^\infty(\Omega)}}, \tag{f_2}$$

where $\varphi_1 > 0$ denotes the principal eigenfunction corresponding to λ_1 normalized as $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. Here we state the main result for $f \gneq 0$ and $\alpha = q - 1$ (for the proof, see chapter 2):

Theorem 1.3.6. *Assume that* (H3) *holds for* $f \gneq 0$. *Then,* (1.12) *has a unique solution if* $\lambda \leq 0$, *has at least a solution if* $\lambda < \lambda^*$, *and has no solution if* $\lambda > \lambda^*$. *If, in addition,* f *satisfies that*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad f(x) \geq c_\omega \quad \text{a.e. } x \in \omega,$$

then (1.12) *has a unique solution for every* $\lambda < \lambda^*$. *Finally, if* f *satisfies condition* (f_1) *for* $1 < q < 2$ *and* (f_2) *for* $q = 2$, *then* (1.12) *has no solution for any* $\lambda \geq \lambda^*$ *and, moreover,* λ^* *is the unique bifurcation point from infinity to problem* (1.12).

Notice that the case $q = 2$ is included in last theorem, so it improves some results in [15] in the sense that Theorem 1.3.6 allows to take μ nonconstant and yet it implies an optimal existence and uniqueness result. It is also remarkable that, in this critical case $\alpha = q - 1$, the structure of the set of solutions differs again from the previous two cases. Actually, it shares with the sublinear case the uniqueness of solution for $\lambda > 0$ small. Nevertheless, the critical value for the existence of solution and the bifurcation point, namely λ^*, depends on μ, q and, in fact, $\lambda^* < \lambda_1$ if $\mu(x) > 0$ for almost every $x \in \Omega$ (see Remark 2.6.3 in chapter 2). This was not the case when $q - \alpha < 1$ since, recall, the bifurcation point was λ_1, which obviously does not depend on μ, q.

Going further, if we drop the positivity conditions on f and allow it to change sign, we can still prove an existence result for problem (1.14); recall that (1.14) is a generalized version of problem (1.12) which includes solutions that change sing. Actually, such an existence result will be valid for a more general problem (see chapter 3), even though we present it here in a simpler version for the sake of clarity:

Theorem 1.3.7. *Assume that* (H3) *holds for* $1 < q < 2$. *Then,* (1.14) *has at least a solution provided* $\lambda < \lambda^*$.

Just to finish with the results related to problem (1.12), we handle the homogenization problems that we described in section 1.2. Thus, we prove a general homogenizaton result which, again, we introduce here in a simplified version. The general statement and the proof can be found in chapter 3.

Theorem 1.3.8. *Assume that* (H3) *holds for* $1 < q < 2$, *and assume that* $\lambda < \lambda^*$, *where* λ^* *is given by* (1.23). *Let* $\{\Omega^\varepsilon\}$ *be an* $\varepsilon-$*family of bounded domains contained in* Ω, *and let* $\{u^\varepsilon\}$ *be a sequence of solutions to* (1.14), *replacing* Ω *with* Ω^ε. *Then, there are conditions on* $\{\Omega^\varepsilon\}$ *(to be specified in chapter 3) that assure the existence of a distribution* $\sigma \in H^{-1}(\Omega)$ *such that a subsequence of* $\{\widetilde{u^\varepsilon}\}$ *weakly converges in* $H_0^1(\Omega)$ *to a solution* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *to the following problem:*

$$\begin{cases} -\Delta u + \sigma u = \lambda u + \hat{\mu}(x) \left| \nabla |u|^{1-\frac{\alpha}{q}} \right|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\hat{\mu} = \left(\frac{q}{q-\alpha}\right)^q \mu$. *More precisely, u satisfies that* $\left|\nabla |u|^{1-\frac{\alpha}{q}}\right|^q \in L^1(\{|u| > 0\})$ *and*

$$\int_\Omega \nabla u \nabla \phi + \langle \sigma, u\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} \mu(x) \frac{|\nabla u|^q}{|u|^{q-1}} \phi + \int_\Omega f(x)\phi$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Thus, in Theorems 1.3.7 and 1.3.8 we overcome the difficulty of the absence of local positive a priori estimates from below. In fact, the proofs rely on alternative global estimates on the lower order term. More details can be found in chapter 3.

We point out that the results we have introduced for $\alpha = q - 1$ are also contained in the review [40] with unified proofs and explanations.

1.3.2 Study of (1.24)

The next goal of the thesis is the study of the following problem:

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.24}$$

where $p > 1, \lambda > 0$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ is a general Carathéodory function satisfying certain conditions that will be specified below. As before, we aim to describe the set $\Sigma = \{(\lambda, u) : u \text{ is a solution to (1.24)}\}$ and determine the dependence of the solutions with respect to λ. We want (1.24) to embrace the following particular cases:

1. We attempt to deal with the case $g(x,s) = \frac{\mu(x)}{s+\delta}$ for some $0 \lneq \mu \in L^\infty(\Omega)$ and $\delta > 0$. Our purpose is to prove an existence result for *every* $\lambda > 0$, assuming if necessary a smallness condition on μ. On the other hand, we expect to find solutions to (1.24) for every $\lambda > 0$ *large enough*, irrespective of the size of μ. Such results would generalize the ones which are known for $\mu \equiv$ constant.

2. Furthermore, we will consider the case $g(x,s) = \frac{\mu(x)}{s^\gamma}$ for some $0 \lneq \mu \in L^\infty(\Omega)$ and $\gamma > 0$. In view of Result 5 in the previous section, we expect to prove nonexistence results for $\gamma > 1$, as well as for $\gamma = 1$ and μ large in some sense. On the contrary, if either $\gamma < 1$, or $\gamma = 1$ and μ is sufficiently small, solutions are expected to exist at least for $\lambda > 0$ large.

We state now the main results that we have proved concerning problem (1.24) in order to accomplish the previous objectives. Let us consider the following hypotheses:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain of class } \mathscr{C}^2, \\ \tau, \sigma \geq 0 : \tau \leq \sigma < 1,\ \sigma - \tau < 1 - \sigma, \\ \delta \in [0,+\infty),\ M \in (0,+\infty], \\ g : \Omega \times (0,+\infty) \to [0,+\infty) \text{ is a Carathódory function such that} \\ \quad \bullet\ \exists \lim_{s\to 0} sg(x,s) \text{ for a.e. } x \in \Omega \text{ and} \\ \quad \bullet\ \tau \leq (s+\delta)g(x,s) \leq \sigma \text{ for a.e. } x \in \Omega,\ \forall s \in (0,M). \end{cases} \tag{H4}$$

Moreover, for g as in (H4) and for $p \in (1, 2^*-1)$, we will need the following condition:

$$\begin{cases} \omega \subset\subset \Omega,\ \mu \in C(\overline{\Omega \setminus \omega}) : \\ \|\mu\|_{L^\infty(\Omega\setminus\omega)} < \frac{2^*-1-p}{2^*-2}, \\ \lim_{s\to+\infty} \|sg(\cdot,s) - \mu\|_{L^\infty(\Omega\setminus\omega)} = 0. \end{cases} \tag{H5}$$

We point out that, in last condition, ω is a possibly empty bounded domain.

The main existence result about problem (1.24) reads as follows:

Theorem 1.3.9. *Let* $p \in (1, 2^*-1)$. *Assume that* (H4) *is satisfied for* $M = +\infty$ *and also that one of the following possibilities holds true:*

1. (H5) *is satisfied with* $\omega = \emptyset$.
2. (H5) *is satisfied with* $\omega \neq \emptyset$ *and* $p < \frac{N}{N-2}, \sigma \leq \frac{N-(N-2)p}{2}$.
3. $p < \frac{N+1}{N-1}, \sigma \leq \frac{N+1-(N-1)p}{2}$.

Then, there exists at least a solution u_λ *to* (1.24) *for every* $\lambda > 0$. *Moreover, in case item 3 holds, then* $\lim_{\lambda\to+\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.

In particular, the previous theorem shows that, if $p \in (1, 2^* - 1)$ and $g(x,s) = \frac{\mu(x)}{s+\delta}$, with $\delta \geq 0$ and $0 \lneq \mu \in C(\overline{\Omega})$ satisfying certain size conditions, then there exists a solution to (1.24) for every $\lambda > 0$. This fact was somehow expected, as we claimed above, and it generalizes the known results for $\mu \equiv$ constant.

On the other hand, relaxing hypothesis (H4) at infinity, i.e., taking $M < +\infty$, we will be able also to prove an existence result but only for $\lambda > 0$ large enough. The statement of the result is the following.

Theorem 1.3.10. *Let $p \in \left(1, \frac{N+1}{N-1}\right)$. Assume that* (H4) *is satisfied for $M < +\infty$ and $\sigma \leq \frac{N+1-(N-1)p}{2}$. Then, there exists $\lambda_0 > 0$ such that there exists at least a solution u_λ to* (1.24) *for every $\lambda > \lambda_0$ and, moreover, $\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.*

This result includes the cases $g(x,s) = \frac{\mu(x)}{s+\delta}$ with $\delta > 0$ and $g(x,s) = \frac{\mu(x)}{s^\gamma}$ with $\gamma \in (0,1)$. The proofs of the previous two theorems rely on a priori estimates that are obtained via a blow-up method, following [82]. It is remarkable that, due to the presence of the gradient term, the adaptation of such a method to problem (1.24) is not trivial. The details are gathered in chapter 5.

We present also a general nonexistence result about problem (1.24). It asserts the following:

Theorem 1.3.11. *Consider a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$. Let $p \geq 1, \lambda > 0$ and let $g : \Omega \times (0, +\infty) \to \mathbb{R}$ be a Carathéodory function satisfying that there exist a domain $\omega \subset\subset \Omega$ and constants $\tau > 1$, $s_0 \in (0,1)$ such that*

$$sg(x,s) \geq \tau \quad \text{a.e. } x \in \Omega \setminus \omega,\ \forall s \in (0, s_0).$$

Then, there exists no solution to (1.24).

Of course, the previous theorem includes the cases $g(x,s) = \frac{\mu(x)}{s^{1+\varepsilon}}$ with either $\varepsilon > 0$ and $\mu(x) > 0$ for a.e. x near $\partial\Omega$, or $\varepsilon = 0$ and $\mu(x) > 1$ for a.e. x near $\partial\Omega$. This shows that strong singularities are impediments for the existence of solution to (1.24). The proof follows some ideas in [43] and can be found in chapter 5.

1.3.3 Study of (1.19)

Our last target is the study of problem (1.19) provided (1.21) is satisfied, but (1.20) is not. In this setting, there are two possibilities: either there exist non-starshaped domains Ω for which (1.19) admits a nontrivial solution, or it admits no nontrivial solutions for any Ω, irrespective of its shape. In other words, we aim to determine whether the geometry of Ω has any influence on the existence of nontrivial solution to (1.19). The answer to this question is contained in the following result.

Theorem 1.3.12. *Let $N \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathscr{C}^{1,1}$. Let also $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, and assume that $\int_0^s f(t)dt \leq 0$ for all $s \in \mathbb{R}$. Then, there exists no nontrivial solution to* (1.19).

In conclusion, last theorem shows that, if any primitive of f is nonpositive, then there is *never* a nontrivial solution to (1.19), irrespective of the geometry of the domain Ω. For a more extensive exposition we refer to chapter 6.

1.4 General aspects in the proofs

As we announced in section 1.1, the existence results (and even the multiplicity results) that we prove in this thesis rely on topological methods and, in consequence, on the existence of a priori estimates. The way of proving the existence of such estimates depends on each specific problem. Just to give some examples, for problem (1.24) it is appropriate to employ a blow-up method for deriving the estimates, while for problem (1.12) with $\alpha = q - 1$ the key point is the characterization of the principal eigenvalue λ^* together with a suitable comparison principle. We will show the specific methods in the subsequent chapters.

In contrast, there are some common features in terms of the uniqueness and regularity of the solutions that deserve comments in this section. First of all, regarding the regularity of the solutions, we observed in section 1.1 that the classical Calderon-Zygmund theory is not valid for natural growth problems. Nevertheless, in spite of the presence of a gradient term and a singularity, the solutions to the problems that we handle belong to $C^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$. The proof of the Hölder regularity is performed by following the methods developed in [94], while the local regularity of the gradients is actually a consequence of the Hölder regularity and follows from a bootstrap argument. More details about the proof can be found in the Appendix of chapter 2.

The mentioned regularity is evidently important by itself, though it implies side consequences as well. For instance, one can prove not only Hölder regularity with these techniques. Actually, an a priori bound in $L^\infty(\Omega)$ yields an a priori bound in $C^{0,\alpha}(\overline{\Omega})$ (see the Appendix in chapter 2). On the other hand, it is also remarkable that the joint regularity $C^{0,\alpha}(\overline{\Omega})\cap W^{1,N}_{\text{loc}}(\Omega)$ implies uniqueness of solutions to some of the problems under consideration. To this respect, the other essential ingredients are the *comparison principles* that we prove in several papers contained in this report. Next three results constitute a representative sample of such comparison principles. The proofs follow essentially [11] and they can be found in chapter 4 (see also chapters 2, 3 and 5).

Roughly speaking, the first of the results allows to compare positive sub and super-solutions to the model equation

$$-\Delta u = \lambda u + \mu(x)\frac{|\nabla u|^q}{u^\alpha} + f(x),\ x\in\Omega \tag{1.25}$$

provided they are well-ordered on the boundary. In (1.25), one may consider any $0\lneq \mu\in L^\infty_{\text{loc}}(\Omega)$, $f\in L^1_{\text{loc}}(\Omega)$, $1<q\le 2$ and $\alpha>0$, even though λ must be non-positive. The statement reads as follows:

Theorem 1.4.1. *Let $1<q\le 2$, $\lambda\le 0$, $f\in L^1_{loc}(\Omega)$ and $g:\Omega\times(0,+\infty)\to[0,+\infty)$ satisfying*

$$\begin{aligned} s\mapsto g(x,s)\ &\textit{is nonincreasing for a.e. } x\in\Omega,\\ x\mapsto g(x,s)\ &\textit{is locally essentially bounded for all } s>0. \end{aligned}$$

Let $u,v\in C(\Omega)\cap W^{1,N}_{loc}(\Omega)$, with $u,v>0$ in Ω, be such that

$$\int_\Omega \nabla u\nabla\phi \le \lambda\int_\Omega u\phi + \int_\Omega g(x,u)|\nabla u|^q\phi + \int_\Omega f(x)\phi \quad \textit{and} \tag{1.26}$$

$$\int_\Omega \nabla v\nabla\phi \ge \lambda\int_\Omega v\phi + \int_\Omega g(x,v)|\nabla v|^q\phi + \int_\Omega f(x)\phi \tag{1.27}$$

for every $0\le\phi\in H^1_0(\Omega)\cap L^\infty(\Omega)$ with compact support. Suppose also that the following boundary condition holds:

$$\limsup_{x\to x_0}(u(x)-v(x))\le 0 \quad \forall x_0\in\partial\Omega.$$

Then, $u\le v$ in Ω.

The second theorem is valid also for equation (1.25). The advantage is that it allows to choose $\lambda>0$ at the expense of taking $\alpha\ge q-1$ and f locally bounded away from zero.

Theorem 1.4.2. *Let $1<q\le 2$, $\lambda\in\mathbb{R}$, $0\le f\in L^1_{loc}(\Omega)$ and $g:\Omega\times(0,+\infty)\to[0,+\infty)$ satisfying*

$$\begin{aligned} s\mapsto s^{q-1}g(x,s)\quad &\textit{is nonincreasing for a.e. } x\in\Omega,\\ x\mapsto g(x,s)\quad &\textit{is locally essentially bounded for all } s>0. \end{aligned}$$

If $\lambda > 0$, assume also that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f \geq c_\omega \text{ in } \omega.$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, satisfying respectively (1.26) *and* (1.27) *for every $0 \leq \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that, for every $\varepsilon > 0$, the following boundary condition holds:*

$$\limsup_{x \to x_0} \left(\frac{u(x)}{v(x) + \varepsilon} \right) \leq 1 \quad \forall x_0 \in \partial\Omega.$$

Then, $u \leq v$ in Ω.

Finally, next result can be applied also to (1.25), where μ is allowed to change sign and $f \gneq 0$ may vanish in subsets of Ω. The counterpart is that one has to impose $q = 2$, $\alpha = 1$ and $\inf_{x \in \Omega} \mu(x) > -1$.

Theorem 1.4.3. *Let $0 \lneq f \in L^1_{loc}(\Omega)$ and let $g : \Omega \times (0, +\infty) \to \mathbb{R}$ be a Carathéodory function. Assume that there exist a continuous function $h : (0, +\infty) \to [0, +\infty)$ and a constant $\sigma \in (0, 1)$ such that*

$$-h(s) \leq g(x, s) \leq \frac{\sigma}{s} \quad \text{a.e. } x \in \Omega,\ \forall s > 0.$$

Assume in addition that the function $s \mapsto s g(x, s)$ is nondecreasing for a.e. $x \in \Omega$. Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, be such that

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega g(x, u) |\nabla u|^2 \phi \leq \int_\Omega f(x) \phi \quad \text{and}$$
$$\int_\Omega \nabla v \nabla \phi + \int_\Omega g(x, v) |\nabla v|^2 \phi \geq \int_\Omega f(x) \phi$$

for every $0 \leq \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that the following boundary condition holds:

$$\limsup_{x \to x_0} (u(x)^{1-\sigma} - v(x)^{1-\sigma}) \leq 0 \quad \forall x_0 \in \partial\Omega.$$

Then, $u \leq v$ in Ω.

Chapter 2

Quasilinear elliptic problems with singular and homogeneous lower order terms

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Abstract. We deal with singular quasilinear elliptic equations, namely

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$, $\lambda \in \mathbb{R}$, $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$ and $0 \leq f \in L^p(\Omega)$ for some $p > \frac{N}{2}$. We completely describe the set of values of the parameter λ for which the problem admits solution. Thus, we study existence, nonexistence and uniqueness of bounded weak solutions in both cases $f \gneq 0$ and $f \equiv 0$.

2.1 Introduction

The present paper is devoted to the study of the following quasilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary (say, of class $C^{1,1}$), $\lambda \in \mathbb{R}$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \leq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $1 < q \leq 2$.

Problem (P_λ) is a particular case of the following general model

$$\begin{cases} -\Delta u = \lambda u + \mu(x) g(u)|\nabla u|^q + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

for some nontrivial real function g. We first observe that, for $\mu \equiv 0$, the equation above becomes linear. In fact, it is an eigenvalue problem if $f \equiv 0$ which admits solution if and only if $\lambda = \Lambda$ (the principal eigenvalue of the Laplacian in Ω with zero Dirichlet boundary condition), while if $f \not\equiv 0$, it is well known that there exists a solution to (2.1) for any f if and only if $\lambda < \Lambda$ (and in such a case, the solution is also unique).

The picture changes drastically if $\mu \not\equiv 0$. Indeed, in such a case the equation becomes quasilinear and the above results are no longer true. In fact, when the gradient term is considered, existence and/or uniqueness of solutions may fail. For instance, the model problem, with $\mu \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, $p > N/2$,

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^2 + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

can be studied, for $\mu(x) \equiv \mu \in \mathbb{R}^+$, through the Hopf-Cole transformation, and it is turned ($v = \frac{e^{\mu u}-1}{\mu}$) into the following semilinear problem

$$\begin{cases} -\Delta v = (\mu v + 1)\left(f(x) + \dfrac{\lambda}{\mu}\log(1+\mu v)\right), & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Thus it is clear that the existence and (possibly) the uniqueness of a solution depends on the sizes of μ and f. Furthermore the nature of the problem is essentially different from the one of the linear problem. Indeed, it has been recently proved in [12] (see also [90]) that if problem (2.2) with $\mu(x) \geq \mu_0 > 0$ admits a solution with $\lambda = 0$, then there exist at least two different solutions to (2.1) for $0 < \lambda < \lambda^*$, for a suitable value $0 < \lambda^* < \Lambda$.

Our idea is that the threshold value λ^* is associated, in some sense, to the principal eigenvalue of the nonlinear differential operator that appears in the equation in (2.1) (with $f(x) \equiv 0$). Thus the lower order term has necessarily to satisfy a 1-homogeneous condition, that leads to the choice of a singular term of the form $g(u) = 1/u^{q-1}$ in (2.1) (see (P_λ)).

The study of boundary value problems of Dirichlet type with singular gradient terms having quadratic ($q = 2$) growth in the gradient has raised considerable interest in recent years. Let us quote the main references [2, 6–8, 24, 43, 76–78], among others, dealing with existence (and nonexistence) results for equations with singular lower order terms, while we mention [10, 16, 37] for uniqueness results on this type of problems.

In contrast with the results of [12], when one considers a singular function $g(u) = \frac{1}{u}$ in problem (2.1) with $q = 2$ and $f \gneq 0$, in [15] the authors prove the existence of solution for every $\lambda < \frac{\Lambda}{\|\mu\|_{L^\infty(\Omega)}+1}$. Moreover, if $\mu(x) \equiv \mu \in (0,1)$, they prove that there exists a solution if and only if $\lambda < \frac{\Lambda}{\mu+1}$, and in such a case, the solution is unique and $\frac{\Lambda}{\mu+1}$ is a bifurcation point from infinity.

Surprisingly, this phenomenon, analogous to the one observed in the the linear case, is not only due to the presence of a singularity at $u = 0$. Actually, the technique developed in [12] applies (with some small changes) to problem (2.1) with $\mu(x) \geq \mu_0 > 0$, $q = 2$, $g(s) = 1/s^\theta$, $\theta \in (0,1)$ and consequently if there exists a solution with $\lambda = 0$, then multiplicity occurs for $\lambda > 0$ small enough (as in the nonsingular case $g(s) \equiv 1$).

In the present paper we aim to provide a general method to deal with problem (P_λ) in the general framework $1 < q \leq 2$, depending only on the quasilinear nature of the problem and allowing the complete description of the set of values of the parameter λ such that (P_λ) admits a solution. Of course, the main difficulties in order to study such a problem are due to the superlinearity of the lower order term and the singularity as u approaches 0. In fact, we will notice that the key point is not (only) the singularity by itself, but the homogeneity that the singularity gives to the lower order term which allows us to look at (P_λ) through the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{E_λ}$$

We provide a kind of eigenvalue existence result for problem (E_λ) making use of a precise characterization of the principal eigenvalue:

$$\lambda^* = \sup\left\{ \lambda \in \mathbb{R} \,\middle|\, \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\} \tag{2.3}$$

(the precise meaning of supersolution used in (2.3) is specified in Section 2.2 below). This characterization has been inspired by the seminal paper [21] and allows us to study the (nonvariational) eigenvalue problem (E_λ) since it requires only working with supersolutions, but does not involve any variational structure of the problem.

However, the definition (2.3) will be useful only if we can compare subsolutions and supersolutions to problem (E_λ) that are appropriately ordered on the boundary of the domain. Indeed, we will be able to derive the required Comparison Principle (see Theorem 2.3.1 below) by adapting the ideas contained in [11].

Let us stress that, at least formally, the change of unknown $v = -\log(u)$ turns the solutions to (E_λ) into solutions to

$$\begin{cases} -\Delta v + |\nabla v|^2 + \mu(x)|\nabla v|^q + \lambda = 0, & x \in \Omega, \\ v \to +\infty, & x \to \partial\Omega. \end{cases} \tag{$\widetilde{E}_\lambda$}$$

Quasilinear problems whose solutions blow up at the boundary of the domain (known in literature as *large solutions*) have been widely studied (see for instance [95, 98, 113]). A particular feature of $(\widetilde{E}_\lambda)$ is that it is invariant under transformations of the type $v \mapsto v + t$ for all $t \in \mathbb{R}$. For problems of this class, it has been proven in [95, 98] that there is a unique value of the parameter λ (the so called *ergodic constant*) for which the problem into consideration admits a (unique, up to additive constants) large solution.

We state now our first theorem about problem (E_λ), in which we show by an approximation and compactness argument that, in fact, λ^* is the principal eigenvalue to (E_λ).

Theorem 2.1.1. *Assume that $1 < q \leq 2$ and $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$. Then $\lambda^* \in (0, \Lambda]$ and problem (E_λ) admits a solution if and only if $\lambda = \lambda^*$. Moreover such a solution is unique up to multiplication by positive constants.*

As far as (P_λ) is concerned, some parts of the main result will require stronger hypotheses on the datum f, that we list here:

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f(x) \geq c_\omega \quad \text{a.e. } x \in \omega; \tag{f_0}$$

$$\exists \gamma \in \left(\frac{1}{2}, 1\right), C_1 > 0 : \quad f \geq C_1 \varphi_1^\gamma \text{ in } \Omega; \tag{f_1}$$

$$\exists C_1 > 0 : \quad f \geq C_1 \varphi_1^\gamma \text{ in } \Omega, \text{ where } \gamma = \frac{1}{1 + \|\mu\|_{L^\infty(\Omega)}}. \tag{f_2}$$

Now we state our main result.

Theorem 2.1.2. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. Then (P_λ) has a unique solution if $\lambda \leq 0$, has at least a solution if $\lambda < \lambda^*$, and has no solution if $\lambda > \lambda^*$. If, in addition, f satisfies condition (f_0), then (P_λ) has a unique solution for every $\lambda < \lambda^*$. Finally, if f satisfies condition (f_1) for $1 < q < 2$ and (f_2) for $q = 2$, then (P_λ) has no solution for any $\lambda \geq \lambda^*$ and moreover the set $\Sigma := \{(\lambda, u_\lambda) : u_\lambda$ is a solution to $(P_\lambda)\}$ is an unbounded continuum in $\mathbb{R} \times C(\overline{\Omega})$ which bifurcates from infinity at λ^* to the left.*

We stress that the previous theorem improves the existence result contained in [15] for μ nonconstant and $q = 2$. In fact, we determine that the set of $\lambda \in \mathbb{R}$ where problem (P_λ) admits a solution is either $(-\infty, \lambda^*)$ or possibly its closure. Moreover, we consider the whole range $1 < q \leq 2$. The critical problem corresponding to $\lambda = \lambda^*$, and also the uniqueness for $\lambda > 0$, exhibit some difficulties. Nonetheless, we overcome them by imposing stronger hypothesis on f. Doing so, we prove that the interval $(-\infty, \lambda^*)$ is optimal for the existence of solution, and we even prove uniqueness in this interval.

It is worth to stress that one of the main contributions of this paper is the comparison principle. In fact it is not obvious, an indeed the literature on this topic is extremely poor, that a comparison principle holds true when we deal with positive values of λ in (2.2).

The plan of the paper is the following: we devote Section 2.2 to introduce the definitions of solution, supersolution and bifurcation point from infinity, and we also prove some regularity properties of the solutions; in Section 2.3 we state and prove some comparison principles and a uniqueness result to problem (P_λ); Section 2.4 is devoted to proving that λ^* is well defined and positive, to give some alternative characterizations of it, and to prove some nonexistence results; in Section 2.5 we introduce the approximate problems, we prove some a priori estimates and a compactness result, and we give several existence and bifurcation results, and in Section 2.6 we collect the proofs of Theorem 2.1.1 and Theorem 2.1.2. Finally, in Appendix A we show that problem (P_λ) possesses two equivalent formulations and we also prove the regularity of the solution.

2.2 Definitions and preliminary results

In this section we make precise some definitions and we prove some results that we will use in the rest of the paper.

First of all we specify the meaning of solution to problem (P_λ), as well as the concept of supersolution used in (2.3).

Definition 2.2.1. For every $\lambda \in \mathbb{R}$, we say that $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a *solution* to (P_λ) if $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\text{loc}}(\Omega)$ and it holds

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi \quad \forall \phi \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{2.4}$$

Similarly, we say that $u \in H^1(\Omega)\cap L^\infty(\Omega)$ is a *supersolution* to (P_λ) if $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\text{loc}}(\Omega)$ and the following inequality holds

$$\int_\Omega \nabla u \nabla \phi \geq \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi \quad \forall \phi \in H_0^1(\Omega)\cap L^\infty(\Omega), \quad \phi \geq 0. \tag{2.5}$$

Some remarks on about the formulation are in order.

Remark 2.2.2. Let us observe that, since the lower order term is only locally integrable in Ω, there is a term above that, a priori, might not make sense. Actually, applying some density arguments we can show that, in spite of the presence of a singular lower order term, the above formulations are equivalent to the ones in which the test functions belong to $C_c^1(\Omega)$ both in (2.4) and (2.5). We collect the proof of such an equivalence in the Appendix.

Remark 2.2.3. In the model case $q = 2$ and μ constant, it is clear that the condition $\mu < 1$ is in fact necessary for the existence of solutions to problem (P_λ) with $\lambda > 0$. Indeed we can use u as test function in (2.4), so that we obtain $\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 + \mu \int_\Omega |\nabla u|^2 + \int_\Omega f(x)u$. Therefore, since $\lambda u^2 > 0$ in Ω, we have that $(1-\mu)\int_\Omega |\nabla u|^2 > 0$.

The following three lemmata provide some properties of the solutions to (P_λ) which will be useful later.

Lemma 2.2.4. *Let $\lambda \in \mathbb{R}$, $1 < q \leq 2$ and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. If $q = 2$, assume that $\|\mu\|_{L^\infty(\Omega)} < 1$. Then, every solution to (P_λ) belongs to $\in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,2p}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.*

The proof of the above lemma is given in its details in the Appendix. Anyway, we observe here that for the interior regularity we exploit that the solutions are strictly positive, as a consequence of the Strong Maximum Principle.

As far as the Hölder continuity up to the boundary is concerned, we need to strongly use the techniques developed in [94]: let us observe that since the singularity has the order of $1/u^{q-1}$ with $q < 2$ (in the case $q = 2$ it is also used that $\mu(x)$ is small), then it represents, in some sense, a "mild" singularity.

The Sobolev interior regularity is proved via an interpolation and bootstrap argument.

Remark 2.2.5. Notice that Lemma 2.2.4 provides as much information about the regularity of the solutions to (P_λ) as the knowledge that one has about the regularity of the data. For instance, under the hypotheses of Lemma 2.2.4, we have in particular that any solution u to (E_λ) satisfies that $-\Delta u \in L^r_{\text{loc}}(\Omega)$ for any $r < \infty$. Hence, $u \in W^{2,r}_{\text{loc}}(\Omega)$ for any $r < \infty$. Even more, if $\mu \in W^{1,\infty}_{\text{loc}}(\Omega)$, we easily deduce that $-\Delta u \in W^{1,r}_{\text{loc}}(\Omega)$ for any $r < \infty$, and thus, $u \in W^{3,r}_{\text{loc}}(\Omega)$ for any $r < \infty$ (see [83, Theorem 9.19]). We may continue the bootstrap in this way so that, if $\mu \in W^{k,\infty}_{\text{loc}}(\Omega)$ for some $k \geq 1$, then $u \in W^{k+2,r}_{\text{loc}}(\Omega)$ for any $r < \infty$. Thus, if $\mu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Lemma 2.2.6. *Let $\lambda \in \mathbb{R}$, $1 < q \leq 2$ and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$. If $q = 2$, assume that $\|\mu\|_{L^\infty(\Omega)} < 1$. Then, every solution u to (P_λ) satisfies that $u^\gamma \in H^1_0(\Omega)$ for every $\gamma > \gamma_0(q)$, given by*

$$\gamma_0(q) = \begin{cases} \dfrac{1}{2} & \text{if } 1 < q < 2, \\[2ex] \dfrac{1 + \|\mu\|_{L^\infty(\Omega)}}{2} & \text{if } q = 2. \end{cases}$$

Proof. We follow here the arguments of Theorem 3.1 in [15], which in turn come from the ideas of [7]. We claim that

$$\int_\Omega \frac{|\nabla u|^2}{u^{1-\beta}} < \infty, \qquad \forall \beta \in (\beta_0(q), 1], \tag{2.6}$$

where

$$\beta_0(q) = \begin{cases} 0 & \text{if } 1 < q < 2, \\ \|\mu\|_{L^\infty(\Omega)} & \text{if } q = 2. \end{cases}$$

Indeed, given $\beta \in (\beta_0(q),1]$, observe that the function $(u+\varepsilon)^\beta - \varepsilon^\beta \in H_0^1(\Omega)\cap L^\infty(\Omega)$ for any $\varepsilon \in (0,1]$. Using it as test function in (P_λ) we obtain that

$$\beta \int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C + \|\mu\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla u|^q}{u^{q-1}}((u+\varepsilon)^\beta - \varepsilon^\beta)$$

for some constant $C>0$, independent of ε, whose value may vary from line to line. Next, in the case $1<q<2$, using Young's inequality conveniently we easily derive that

$$\frac{\beta}{2}\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C\left(1+\int_\Omega \left((u+\varepsilon)^{(1-\beta)\frac{q}{2}}\frac{(u+\varepsilon)^\beta-\varepsilon^\beta}{u^{q-1}}\right)^{\frac{2}{2-q}}\right).$$

It is straightforward to check that $(s,t)\mapsto (s+t)^{(1-\beta)\frac{q}{2}}\frac{(s+t)^\beta - t^\beta}{s^{q-1}}$ is a continuous function in $[0,\|u\|_{L^\infty(\Omega)}]\times[0,1]$, which implies that

$$\frac{\beta}{2}\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C. \tag{2.7}$$

On the other hand, if $q=2$ and $\|\mu\|_{L^\infty(\Omega)}<1$, we observe that

$$\begin{aligned}\frac{|\nabla u|^q}{u^{q-1}}((u+\varepsilon)^\beta-\varepsilon^\beta) &= \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\,\frac{(u+\varepsilon)^{1-\beta}((u+\varepsilon)^\beta-\varepsilon^\beta)}{u}\\ &= \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\left(1+\varepsilon^\beta\,\frac{\varepsilon^{1-\beta}-(u+\varepsilon)^{1-\beta}}{u}\right)\le \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}}\end{aligned}$$

in Ω for any $\varepsilon\in(0,1]$. Hence, we deduce that

$$(\beta - \|\mu\|_{L^\infty(\Omega)})\int_\Omega \frac{|\nabla u|^2}{(u+\varepsilon)^{1-\beta}} \le C. \tag{2.8}$$

Finally, we apply Fatou's Lemma with respect to ε in (2.7) and in (2.8) to obtain (2.6). The Lemma follows by choosing $\gamma = \frac{\beta+1}{2}$. □

Lemma 2.2.7. *Let $\lambda\in\mathbb{R}$, $1<q\le 2$, $0\le\mu\in L^\infty(\Omega)$ and $0\lneq f\in L^r(\Omega)$ with $r>N$. If $1<q<2$, then every solution u to (P_λ) satisfies that*

$$\forall\gamma\in\left(\frac{1}{2},1\right)\quad \exists C>0:\quad u\le C\varphi_1^\gamma \quad \text{in } \Omega.$$

Moreover, if $q=2$ and $\|\mu\|_{L^\infty(\Omega)}<1$, then every solution u to (P_λ) satisfies that

$$\exists C>0:\quad u\le C\varphi_1^\gamma\quad \text{in } \Omega,\qquad \text{where } \gamma=\frac{1}{1+\|\mu\|_{L^\infty(\Omega)}}\in\left(\frac{1}{2},1\right).$$

Proof. Let $\gamma \in \left(\frac{1}{2}, 1\right)$. First of all observe that, if $q < 2$, we can use Young's inequality in such a way that

$$-\Delta u \le \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla u|^2}{u} + (C_\gamma + \lambda) u + f(x) \tag{2.9}$$

for some $C_\gamma > 0$ large enough. If $q = 2$, we arrive to the same inequality directly with $\gamma = \frac{1}{1+\|\mu\|_{L^\infty(\Omega)}}$ and $C_\gamma = 0$.

Let now $g \equiv \frac{1}{\gamma}\left((C_\gamma + \lambda) u + f\right)$. Clearly, $0 \lneq g \in L^r(\Omega)$, so there exists a solution $0 < z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to

$$\begin{cases} -\Delta z = g(x), & x \in \Omega, \\ z = 0, & x \in \partial\Omega. \end{cases}$$

Since $r > N$, it is well-known that $z \in C^1(\overline{\Omega})$. This implies, by using Hopf's Lemma, that there is a constant $C > 0$ such that

$$z \le C\varphi_1 \quad \text{in } \Omega.$$

On the other hand, for every $k > 0$, the function $v = (kz)^\gamma \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$-\Delta v = \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + \frac{\gamma k g(x)}{(kz)^{1-\gamma}}.$$

If we choose $k = \|z\|_{L^\infty(\Omega)}^{\frac{1}{\gamma}-1}$, then

$$-\Delta v \ge \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + \gamma g(x) = \left(\frac{1}{\gamma} - 1\right) \frac{|\nabla v|^2}{v} + (C_\gamma + \lambda) u + f(x). \tag{2.10}$$

Therefore, by (2.9) and (2.10), we can use Theorem 2.3.2 (see next section) and conclude that

$$u \le v = (kz)^\gamma \le C\varphi_1^\gamma.$$

□

We conclude this section by recalling the concept of bifurcation point from infinity.

Definition 2.2.8. A *bifurcation point from infinity* to problem (P_λ) is said to be a real number $\bar{\lambda}$ for which there exists a sequence $\{(\lambda_n, u_n)\}_{n \in \mathbb{N}}$ contained in the set

$$\Sigma := \{(\lambda, u) : u \text{ is a solution to } (P_\lambda)\},$$

such that $\lambda_n \to \bar{\lambda}$ and $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$.

We say that the bifurcation occurs *to the left* if there exist $\varepsilon > 0$ and $M > 0$ such that for any $(\lambda, u) \in \Sigma$ with $\lambda \in (\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon)$ and $\|u\|_{L^\infty(\Omega)} \ge M$, it holds that $\lambda < \bar{\lambda}$.

2.3 Comparison principles

In this section we prove a Comparison Principle which allows us to compare suitable subsolutions and supersolutions to the equation

$$-\Delta u = \lambda u + g(x)\frac{|\nabla u|^q}{u^{q-1}} + h(x) \quad \text{in } \Omega$$

that are well ordered on the boundary.

Theorem 2.3.1. *Let $1 < q \leq 2$, $\lambda \in \mathbb{R}$, $g \in L^\infty(\Omega)$, $0 \leq h \in L^1_{loc}(\Omega)$ and assume that $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$ are such that $u, v > 0$ in Ω and they satisfy*

$$\limsup_{x \to x_0} \frac{u(x)}{v(x)} \leq 1 \quad \forall x_0 \in \partial\Omega. \tag{2.11}$$

Assume also that, for every $0 \leq \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support, the following inequalities hold:

$$\int_\Omega \nabla u \nabla \phi \leq \lambda \int_\Omega u\phi + \int_\Omega g(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega h(x)\phi \tag{2.12}$$

and

$$\int_\Omega \nabla v \nabla \phi \geq \lambda \int_\Omega v\phi + \int_\Omega g(x)\frac{|\nabla v|^q}{v^{q-1}}\phi + \int_\Omega h(x)\phi. \tag{2.13}$$

Then $u \leq v$ in Ω.

Proof. We follow the ideas contained in [11, Lemma 2.2] (see the references therein as well). Let $u_1 = \log(u)$, $v_1 = \log(v)$, and denote $w = u_1 - v_1$. Arguing by contradiction, assume that $w^+ \not\equiv 0$. Then, it is clear that $(w-k)^+ \not\equiv 0$ for every $k \in [0, \|w^+\|_{L^\infty(\Omega)})$. Observe that, using (2.11), we have that for every $k \in (0, \|w^+\|_{L^\infty(\Omega)})$, the function $(w-k)^+$ has compact support in Ω and, in consequence, it belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$. This fact, together with the continuity of u and v (which implies that u and v are locally bounded away from zero), allows us to take $\frac{(w-k)^+}{u}$ as test function in (2.12) and $\frac{(w-k)^+}{v}$ in (2.13), obtaining

$$\begin{aligned} -\int_\Omega \frac{|\nabla u|^2}{u^2}(w-k)^+ + \int_\Omega \frac{\nabla u}{u}\nabla(w-k)^+ \leq \lambda \int_\Omega (w-k)^+ \\ + \int_\Omega g(x)\frac{|\nabla u|^q}{u^q}(w-k)^+ + \int_\Omega \frac{h(x)}{u}(w-k)^+ \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} -\int_\Omega \frac{|\nabla v|^2}{v^2}(w-k)^+ + \int_\Omega \frac{\nabla v}{v}\nabla(w-k)^+ \geq \lambda \int_\Omega (w-k)^+ \\ + \int_\Omega g(x)\frac{|\nabla v|^q}{v^q}(w-k)^+ + \int_\Omega \frac{h(x)}{v}(w-k)^+. \end{aligned} \tag{2.15}$$

Consider now the set

$$A_k = \{x \in \Omega : w(x) \geq k\} = \{x \in \Omega : u(x) \geq e^k v(x)\}.$$

Notice that $\operatorname{supp}(w-k)^+ \subseteq A_k$ and $h\left(\frac{1}{u} - \frac{1}{v}\right) \leq 0$ in A_k. Hence, subtracting (2.14) from (2.15) and using the definition of u_1, v_1 we have that

$$\int_\Omega \nabla w \nabla (w-k)^+ \leq \int_{A_k} \left(g(x)\left(|\nabla u_1|^q - |\nabla v_1|^q\right) + |\nabla u_1|^2 - |\nabla v_1|^2\right)(w-k)^+. \quad (2.16)$$

For every $j \in \mathbb{R}$, let us denote $\Omega_j = \{x \in \Omega : |w(x)| = j\}$, and consider also the set $J = \{j \in \mathbb{R} : |\Omega_j| \neq 0\}$. Since $|\Omega| < \infty$, then J is at most countable, which implies that the set $\bigcup_{j\in J} \Omega_j$ is measurable, and we also have that

$$\nabla w = 0 \quad \text{in } \bigcup_{j\in J} \Omega_j \implies |\nabla u_1| = |\nabla v_1| \quad \text{in } \bigcup_{j\in J} \Omega_j.$$

Hence, if we define the set $Z = \Omega \setminus \bigcup_{j\in J} \Omega_j$ and denote $\xi_t = t\nabla u_1 + (1-t)\nabla v_1$, with $0 < t < 1$, we deduce from (2.16) that

$$\begin{aligned}
\int_\Omega \nabla w \nabla (w-k)^+ &\leq \int_{A_k\cap Z} \left(g(x)\left(|\nabla u_1|^q - |\nabla v_1|^q\right) + |\nabla u_1|^2 - |\nabla v_1|^2\right)(w-k)^+ \\
&= \int_{A_k\cap Z} \left(\int_0^1 \frac{d}{dt}(g(x)|\xi_t|^q + |\xi_t|^2)dt\right)(w-k)^+. \qquad (2.17)
\end{aligned}$$

Taking into account that $u_1, v_1 \in W^{1,N}_{\text{loc}}(\Omega)$ and $A_k \subset\subset \Omega$, we have that

$$|\xi_t| \leq |\nabla u_1| + |\nabla v_1| + 1 \equiv \eta \in L^N(A_k \cap Z).$$

Hence, from (2.17) we derive that

$$\begin{aligned}
\|(w-k)^+\|^2_{H^1_0(\Omega)} &\leq \int_{A_k\cap Z} \left(\int_0^1 (g(x)q|\xi_t|^{q-2}\xi_t + 2\xi_t)\nabla w dt\right)(w-k)^+ \\
&\leq \int_{A_k\cap Z} \left(\|g\|_{L^\infty(\Omega)} q\eta^{q-1} + 2\eta\right)|\nabla w|(w-k)^+ \\
&\leq C\int_{A_k\cap Z} \eta |\nabla (w-k)^+|(w-k)^+ \qquad (2.18) \\
&\leq C\|\eta\|_{L^N(A_k\cap Z)} \|(w-k)^+\|_{H^1_0(\Omega)} \|(w-k)^+\|_{L^{2^*}(\Omega)} \\
&\leq C\|\eta\|_{L^N(A_k\cap Z)} \|(w-k)^+\|^2_{H^1_0(\Omega)}.
\end{aligned}$$

For some $k_0 \in (0, \|w^+\|_{L^\infty(\Omega)})$, let us define the function $F : [k_0, \|w^+\|_{L^\infty(\Omega)}] \to \mathbb{R}$ by

$$F(k) = \|\eta\|_{L^N(A_k\cap Z)} = \||\nabla u_1| + |\nabla v_1| + 1\|_{L^N(A_k\cap Z)}, \quad \forall k \in [k_0, \|w^+\|_{L^\infty(\Omega)}),$$

and $F(\|w^+\|_{L^\infty(\Omega)}) = 0$. It is clear that F is nonincreasing and continuous in the interval $[0, \|w^+\|_{L^\infty(\Omega)}]$. Thus, choosing k close enough to $\|w^+\|_{L^\infty(\Omega)}$, we deduce from (2.18) that $(w-k)^+ \equiv 0$, a contradiction.

In conclusion, we have proved that $w^+ \equiv 0$, i.e., $u \le v$ in Ω. $\square$

The previous comparison principle does not guarantee uniqueness of solution to (P_λ) in the space $C(\Omega) \cap W^{1,N}_{\text{loc}}(\Omega)$ unless it is assured that any pair of such solutions satisfy (2.11). However, stronger hypotheses on h and g allow us to weaken (2.11) and derive another comparison result that provides uniqueness for (P_λ).

Theorem 2.3.2. *Let $1 < q \le 2$, $\lambda \in \mathbb{R}$, $0 \le g \in L^\infty(\Omega)$ and $0 \le h \in L^1_{loc}(\Omega)$. Assume that $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, and satisfy* (2.12) *and* (2.13) *respectively. Suppose also that, for every $\varepsilon > 0$, the following boundary condition holds:*

$$\limsup_{x \to x_0} \left(\frac{u(x)}{v(x) + \varepsilon} \right) \le 1 \quad \forall x_0 \in \partial\Omega. \tag{2.19}$$

Furthermore, if $\lambda > 0$, assume also that h satisfies condition (f_0). Then, $u \le v$ in Ω.

Proof. For every $\varepsilon > 0$, let us consider the function

$$w_\varepsilon = \log\left(\frac{u}{v + \varepsilon} \right).$$

We claim that $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that $w_{\varepsilon_0}^+ \not\equiv 0$. Let us fix $k_0 \in \left(0, \|w_{\varepsilon_0}^+\|_{L^\infty(\Omega)}\right)$ and $\varepsilon \in (0, \varepsilon_0)$, the latter to be chosen small enough later. It is clear that $w_{\varepsilon_0} \le w_\varepsilon$ in Ω, so $w_\varepsilon^+ \not\equiv 0$, too.

For $k \in [k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}]$, let us denote

$$A_k = \{x \in \Omega : w_\varepsilon(x) \ge k\} = \{x \in \Omega : u(x) \ge e^k(v(x) + \varepsilon)\}.$$

From (2.19), we deduce that $\operatorname{supp}(w-k)^+ \subset A_k \subset\subset \Omega$. Then, the function $(w_\varepsilon - k)^+$ has compact support, and in particular, $(w_\varepsilon - k)^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)$. Therefore, we may take $\frac{(w_\varepsilon - k)^+}{u}$ as test function in (2.12), and $\frac{(w_\varepsilon - k)^+}{v+\varepsilon}$ in (2.13), obtaining

$$\begin{aligned} \int_\Omega \frac{\nabla u}{u} \nabla (w_\varepsilon - k)^+ \le & \int_\Omega \frac{|\nabla u|^2}{u^2} (w_\varepsilon - k)^+ + \lambda \int_\Omega (w_\varepsilon - k)^+ \\ & + \int_\Omega g(x) \frac{|\nabla u|^q}{u^q} (w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{u} (w_\varepsilon - k)^+ \end{aligned} \tag{2.20}$$

and, using that $g \geq 0$,

$$\begin{aligned}\int_\Omega \frac{\nabla v}{v+\varepsilon}\nabla(w_\varepsilon-k)^+ &\geq \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon-k)^+ + \lambda\int_\Omega \frac{v}{v+\varepsilon}(w_\varepsilon-k)^+ \\ &\quad + \int_\Omega g(x)\frac{|\nabla v|^q}{v^{q-1}(v+\varepsilon)}(w_\varepsilon-k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon-k)^+ \\ &\geq \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2}(w_\varepsilon-k)^+ + \lambda\int_\Omega (w_\varepsilon-k)^+ - \int_\Omega \frac{\lambda\varepsilon}{v+\varepsilon}(w_\varepsilon-k)^+ \\ &\quad + \int_\Omega g(x)\frac{|\nabla v|^q}{(v+\varepsilon)^q}(w_\varepsilon-k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon}(w_\varepsilon-k)^+.\end{aligned} \tag{2.21}$$

Moreover, it is clear that

$$h\left(\frac{1}{u}-\frac{1}{v+\varepsilon}\right)+\frac{\lambda\varepsilon}{v+\varepsilon} \leq 0 \quad \text{in } A_k \text{ for every } k\in[k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}] \tag{2.22}$$

whenever $\lambda \leq 0$. We claim that this is also true if $\lambda > 0$, h satisfies (f_0) and ε is small enough.

Indeed, let $\omega \subset\subset \Omega$ be an open set such that $A_{k_0} \subset \omega$. Since $A_k \subset A_{k_0}$ for all $k \geq k_0$, there exists $c_\omega > 0$ such that $h \geq c_\omega$ in A_k for all $k \geq k_0$. If we choose now

$$\varepsilon < \min\left\{\varepsilon_0, \frac{1-e^{-k_0}}{\lambda}\, c_\omega\right\},$$

we deduce easily that (2.22) holds.

Therefore, subtracting (2.20) and (2.21), and taking into account that $u, v \in W^{1,N}_{\mathrm{loc}}(\Omega)$ and also (2.22), we may argue as in the proof of Theorem 2.3.1 and achieve a contradiction taking k close enough to $\|w_\varepsilon^+\|_{L^\infty(\Omega)}$.

In conclusion, necessarily $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$, i.e., $u \leq v+\varepsilon$ in Ω for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ it follows that $u \leq v$ in Ω. □

2.4 The principal eigenvalue and nonexistence results

We devote this section to giving some properties of λ^* defined by (2.3). In particular, we show that λ^* is the only possible value of the parameter λ for which (E_λ) admits a solution. This is a crucial fact on which are based the proofs of our main results, that exploit the existence of the principal eigenvalue associated to the nonlinear operator $-\Delta u - \mu(x)\frac{|\nabla u|^q}{u^{q-1}}$ (see Theorem 2.1.1). For the sake of clarity we collected such proofs in the last section.

Let us recall that $\lambda^* = \sup I^*$, where

$$I^* = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\}.$$

Firstly we point out some useful characterizations of λ^* as the supremum of the following sets:

$$I_1 = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v - c \in H_0^1(\Omega) \text{ for some } c > 0 \end{array} \right\},$$

and

$$I_2 = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v^\gamma \in H^1(\Omega)\ \forall \gamma > \gamma_0 \text{ for some } \gamma_0 < 1 \end{array} \right\}.$$

Proposition 2.4.1. *Assume that $1 < q \leq 2$ and $0 \leq \mu \in L^\infty(\Omega)$. Then, the sets I^*, I_1 and I_2 are nonempty intervals, unbounded from below and they satisfy*

$$I^* = I_1, \tag{2.23}$$

$$\lambda^* = \sup I_2. \tag{2.24}$$

Moreover, $\lambda^* > 0$ and we have that $\lambda^* \leq \Lambda \equiv \inf\limits_{w \in H_0^1(\Omega)\setminus\{0\}} \dfrac{\|w\|^2_{H_0^1(\Omega)}}{\|w\|^2_{L^2(\Omega)}}.$

Proof. We first observe that the sets under consideration are intervals. Moreover, if in the definition of the sets I^*, I_1 and I_2 we take $\varphi \equiv c$ for any constant $c > 0$, then we deduce that $(-\infty, 0] \subset I^* \cap I_1 \cap I_2$.

We split the rest of the proof into several steps.

Step 1. We first prove (2.23). In order to prove that $I_1 \subseteq I^*$ we take $\lambda \in I_1$ and assume, without loss of generality, that $\lambda > 0$. Hence, there exist $0 < \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $c > 0$ with $\varphi > 0$ in Ω, $\varphi - c \in H_0^1(\Omega)$, and

$$-\Delta(\varphi - c) = -\Delta\varphi \geq \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} \geq 0, \quad x \in \Omega.$$

Therefore, the maximum principle yields to $\varphi \geq c$ in Ω, and so $\lambda \in I^*$, too.

Reciprocally, assume that $0 < \lambda \in I^*$, and let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $c > 0$ with $\varphi \geq c$ and $-\Delta\varphi \geq \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$ in Ω. Clearly, thanks to Remark 2.2.2 we have that $\overline{\psi} = \varphi - c \geq 0$ is an $H^1(\Omega) \cap L^\infty(\Omega)$ supersolution to the nonsingular problem

$$\begin{cases} -\Delta\psi = \lambda\psi + \mu(x)\dfrac{|\nabla\psi|^q}{|\psi + c|^{q-1}} + \lambda c, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.25}$$

On the other hand, $\underline{\psi} \equiv 0$ is obviously a subsolution. Therefore, [30, Théorème 3.1] (see also [88]) implies that there exists a solution $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (2.25) satisfying that $0 \leq \psi \leq \varphi - c$ in Ω. Thus, the function $\psi + c$ satisfies: $(\psi + c) \in H^1(\Omega) \cap L^\infty(\Omega)$, $\psi + c > 0$ in Ω, $(\psi + c) - c \in H_0^1(\Omega)$ and

$$-\Delta(\psi + c) = \lambda(\psi + c) + \mu(x)\frac{|\nabla(\psi + c)|^q}{(\psi + c)^{q-1}}, \quad x \in \Omega.$$

This proves that $\lambda \in I_1$.

Step 2. We deduce now (2.24). Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq I_2$ be an increasing sequence of real numbers such that $\lambda_n < \sup I_2$ for any n, satisfying $\lambda_n \to \sup I_2$. In particular, for every n there exists $u_n \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\widetilde{\gamma}_n \in (0,1)$ satisfying

$$u_n > 0,\ x \in \Omega, \quad -\Delta u_n \geq \lambda_n u_n + \mu(x)\frac{|\nabla u_n|^q}{u_n^{q-1}},\ x \in \Omega, \quad u_n^\gamma \in H^1(\Omega) \quad \forall \gamma > \widetilde{\gamma}_n.$$

Let $\varphi_1 > 0$ be the principal eigenfunction (normalized in $L^\infty(\Omega)$) to the $-\Delta$ operator in Ω with zero Dirichlet boundary conditions. Let us fix $n > 1$, and consider $\varepsilon = \varepsilon_n > 0$ (to be chosen small enough later) and $\gamma = \gamma_n \in \left(\max\left\{\frac{1}{2}, \widetilde{\gamma}_n, \frac{\lambda_{n-1}}{\lambda_n}\right\}, 1\right)$. Since $\gamma > \frac{1}{2}$ and $\gamma > \widetilde{\gamma}_n$, we have, using Lemma 2.2.6, that the function

$$\psi_n = \varepsilon(\varphi_1^\gamma + 1) + u_n^\gamma \in H^1(\Omega) \cap L^\infty(\Omega)$$

and, clearly, $\psi_n \geq \varepsilon$ in Ω.

Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\phi \geq 0$ in Ω and has compact support. Observe that the function $\gamma\varphi_1^{\gamma-1}\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so it may be chosen as test function in

$$\begin{cases} -\Delta\varphi_1 = \Lambda\varphi_1, & x \in \Omega, \\ \varphi_1 = 0, & x \in \partial\Omega. \end{cases}$$

Similarly, $\gamma u_n^{\gamma-1}\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and has compact support, so it may be taken as test function in the inequality satisfied by u_n. Therefore, direct computations yield to

$$\int_\Omega \nabla\varphi_1^\gamma \nabla\phi = \gamma(1-\gamma)\int_\Omega \frac{|\nabla\varphi_1|^2}{\varphi_1^{2-\gamma}}\phi + \gamma\Lambda\int_\Omega \varphi_1^\gamma\phi \tag{2.26}$$

and

$$\int_\Omega \nabla u_n^\gamma \nabla\phi \geq \gamma(1-\gamma)\int_\Omega \frac{|\nabla u_n|^2}{u_n^{2-\gamma}}\phi + \gamma\lambda_n\int_\Omega u_n^\gamma\phi + \gamma\int_\Omega \mu(x)\frac{|\nabla u_n|^q}{u_n^{q-\gamma}}\phi. \tag{2.27}$$

Recalling that

$$\int_\Omega \nabla\psi_n \nabla\phi = \varepsilon\int_\Omega \nabla\varphi_1^\gamma\nabla\phi + \int_\Omega \nabla u_n^\gamma\nabla\phi\,,$$

using both (2.26) and (2.27) we easily deduce that

$$\begin{aligned}
&\int_\Omega \left(-\nabla \psi_n \nabla \phi + \lambda_{n-1} \psi_n \phi + \mu(x) \frac{|\nabla \psi_n|^q}{\psi_n^{q-1}} \phi \right) \\
&\leq \varepsilon \int_\Omega \left(-\gamma(1-\gamma) \frac{|\nabla \varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1} - \gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} \right) \phi + \\
&\int_\Omega \left(-\gamma(1-\gamma) \frac{|\nabla u_n|^2}{u_n^{2-\gamma}} - (\gamma\lambda_n - \lambda_{n-1}) u_n^\gamma - \gamma\mu(x) \frac{|\nabla u_n|^q}{u_n^{q-\gamma}} + \mu(x) \frac{|\nabla \psi_n|^q}{\psi_n^{q-1}} \right) \phi.
\end{aligned} \tag{2.28}$$

Since $\gamma < 1 < q$, there exists a constant $C_1 > 0$ (that depends only on q and γ) such that

$$\begin{aligned}
\frac{|\nabla \psi_n|^q}{\psi_n^{q-1}} &\leq \frac{1}{\psi_n^{q-1}} \left(C_1 |\nabla(\varepsilon\varphi_1^\gamma)|^q + \frac{1}{\gamma^{q-1}} |\nabla(u_n^\gamma)|^q \right) \\
&\leq C_1 \frac{|\nabla(\varepsilon\varphi_1^\gamma)|^q}{\varepsilon^{q-1}} + \frac{|\nabla(u_n^\gamma)|^q}{(\gamma u_n^\gamma)^{(q-1)}} = C_1 \varepsilon \frac{|\nabla \varphi_1|^q}{\varphi_1^{q(1-\gamma)}} + \gamma \frac{|\nabla u_n|^q}{u_n^{q-\gamma}}
\end{aligned} \tag{2.29}$$

in Ω. Hence, combining (2.28) and (2.29) we deduce that

$$\begin{aligned}
&\int_\Omega \left(-\nabla \psi_n \nabla \phi + \lambda_{n-1} \psi_n \phi + \mu(x) \frac{|\nabla \psi_n|^q}{\psi_n^{q-1}} \phi \right) \leq -(\gamma\lambda_n - \lambda_{n-1}) \int_\Omega u_n^\gamma \phi + \\
&\varepsilon \int_\Omega \left(-\gamma(1-\gamma) \frac{|\nabla \varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1} - \gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)} C_1 \frac{|\nabla \varphi_1|^q}{\varphi_1^{q(1-\gamma)}} \right) \phi.
\end{aligned} \tag{2.30}$$

Denoting $d(x) = \text{dist}(x, \partial\Omega)$, since $\varphi_1 \in C^1(\overline{\Omega})$, Hopf's Lemma yields that there exist two constants $\delta_0, C_2 > 0$ such that $|\nabla \varphi_1|^2 \geq C_2$ in the set $\Omega_\delta = \{x \in \Omega : d(x) \leq \delta\}$ for every $\delta \in (0, \delta_0)$. Hence, using now that $\varphi_1 \in C(\overline{\Omega})$ and $\varphi_1 = 0$ on $\partial\Omega$, we have that, for every $\kappa > 0$, there exists $\delta \in (0, \delta_0)$ such that $\frac{|\nabla \varphi_1|^2}{\varphi_1^{2-\gamma}} \geq \kappa$ in Ω_δ. Using also that $\gamma\lambda_n - \lambda_{n-1} > 0$ and $q(1-\gamma) < 2-\gamma$, we choose δ sufficiently small, but independent of ε, such that

$$\varepsilon \left(-\gamma(1-\gamma) \frac{|\nabla \varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1} - \gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)} C_1 \frac{|\nabla \varphi_1|^q}{\varphi_1^{q(1-\gamma)}} \right) \leq 0 \tag{2.31}$$

in Ω_δ. Consequently, we take ε small enough in order to have

$$\begin{aligned}
&\varepsilon \left(-\gamma(1-\gamma) \frac{|\nabla \varphi_1|^2}{\varphi_1^{2-\gamma}} + (\lambda_{n-1} - \gamma\Lambda)\varphi_1^\gamma + \lambda_{n-1} + \|\mu\|_{L^\infty(\Omega)} C_1 \frac{|\nabla \varphi_1|^q}{\varphi_1^{q(1-\gamma)}} \right) \\
&\qquad \leq \varepsilon C_3 \leq (\gamma\lambda_n - \lambda_{n-1}) \inf_{\Omega \setminus \Omega_\delta} (u_n^\gamma) \leq (\gamma\lambda_n - \lambda_{n-1}) u_n^\gamma
\end{aligned} \tag{2.32}$$

in $\Omega \setminus \Omega_\delta$, where $C_3 > 0$ is a constant independent of ε. Gathering (2.30), (2.31) and (2.32) together we conclude that

$$\int_\Omega \nabla \psi_n \nabla \phi \geq \lambda_{n-1} \int_\Omega \psi_n \phi + \int_\Omega \mu(x) \frac{|\nabla \psi_n|^q}{\psi_n^{q-1}} \phi.$$

In short, we have proved that $\lambda_{n-1} \in I^*$ for any $n > 1$, and thus, $\lambda_{n-1} \leq \lambda^*$ for any $n > 1$. Therefore, letting $n \to \infty$ we get that $\sup I_2 \leq \lambda^*$. Finally, the reverse inequality is trivial since $I^* \subseteq I_2$.

Step 3. We show now that $\lambda^* > 0$. Indeed, given the constants $c, \delta >$, let us consider the problem

$$\begin{cases} -\Delta u = \dfrac{\mu(x)}{c^{q-1}}|\nabla u|^q + \delta, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.33}$$

If $q < 2$, by using Young's inequality, we obtain that

$$\frac{\mu(x)}{c^{q-1}}|\xi|^q + \delta \leq \mu(x)|\xi|^2 + \left(1 - \frac{q}{2}\right)\left(\frac{q}{2}\right)^{\frac{q}{2-q}} \frac{\|\mu\|_{L^\infty(\Omega)}}{c^{\frac{2(q-1)}{2-q}}} + \delta$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. Then, taking c large enough and δ small enough, [69, Theorem 3.4] implies that there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (2.33). If $q = 2$, then the same result provides a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ if δ is small enough. In both cases, by the Maximum Principle, $u \geq 0$ in Ω.

Let $v = u + c \in H^1(\Omega) \cap L^\infty(\Omega)$. It is clear that $v \geq c$ in Ω and, for some $\lambda > 0$,

$$\begin{aligned} -\Delta v = -\Delta u &= \frac{\mu(x)}{c^{q-1}}|\nabla u|^q + \delta \geq \mu(x)\frac{|\nabla u|^q}{(u+c)^{q-1}} + \delta \\ &= \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + \lambda v + (\delta - \lambda v) \geq \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + \lambda v + (\delta - \lambda\|v\|_{L^\infty(\Omega)}). \end{aligned}$$

Taking now λ sufficiently small, we conclude that v is a supersolution to (E_λ). This means that $0 < \lambda \leq \lambda^*$, as we wished to prove.

Step 4 We prove here that $\lambda^* \leq \Lambda$. Let $0 < \lambda \in I^* = I_1$. We know from Step 1 that there exists a solution $\psi \geq 0$ to (2.25) for some $c > 0$. Then, taking φ_1 as test function in (2.25) we have

$$\Lambda \int_\Omega \varphi_1 \psi = \int_\Omega \nabla\varphi_1 \nabla\psi = \lambda \int_\Omega \psi\varphi_1 + \int_\Omega \mu(x)\frac{|\nabla\psi|^q}{(\psi+c)^{q-1}}\varphi_1 + \lambda c \int_\Omega \varphi_1.$$

In particular

$$(\Lambda - \lambda)\int_\Omega \psi\varphi_1 = \int_\Omega \mu(x)\frac{|\nabla\psi|^q}{(\psi+c)^{q-1}}\varphi_1 + c\lambda\int_\Omega \varphi_1 > 0.$$

Thus, necessarily $\lambda < \Lambda$, which implies that $\lambda^* \leq \Lambda$. □

Remark 2.4.2. We point out that in Step 1 of the previous proof it has been shown that one can equivalently define I_1 in terms of solutions instead of supersolutions. That is to say,

$$I_1 = \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a solution } v \text{ to the equation in } (E_\lambda) \\ \text{such that } v - c \in H_0^1(\Omega) \text{ for some } c > 0 \end{array} \right\}.$$

In order to prove that λ^* is the only possible eigenvalue to (E_λ) we need to use the comparison principle proved in the previous section. Indeed, it allows us to prove nonexistence of solutions to (E_λ) when $\lambda < \lambda^*$. On the other hand, we use the characterization of λ^* given by (2.24) to prove nonexistence for $\lambda > \lambda^*$; this latter nonexistence result is valid for (P_λ), even with $f \gneq 0$. Summarizing, we have the following result.

Proposition 2.4.3. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$. *Then, there is no solution to* (P_λ) *for any* $\lambda > \lambda^*$. *Moreover, there is no solution to* (E_λ) *for any* $\lambda \neq \lambda^*$.

Proof. Arguing by contradiction, assume that there exists a solution u to (P_λ) for some $\lambda > \lambda^*$. Then, it is in particular a supersolution to (E_λ), and Lemma 2.2.6 implies that $u^\gamma \in H^1(\Omega)$ for every $\gamma > \gamma_0(q)$. Since $\gamma_0(q) < 1$, then this contradicts (2.24) in Proposition 2.4.1. In conclusion, there is no solution to (P_λ) for any $\lambda > \lambda^*$. Observe that, in particular, we have nonexistence of solutions to (E_λ) for $\lambda > \lambda^*$.

On the other hand, assume now that there exists a solution u to problem (E_λ) for some $\lambda < \lambda^*$. By virtue of Proposition 2.4.1 we have that $\lambda \in I_1$, so there exist $c > 0$ and $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfying that $\varphi - c \in H_0^1(\Omega)$ and (see also Remark 2.4.2) $-\Delta\varphi = \lambda\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$ in Ω. Hence, arguing as in the proof of Lemma 2.2.4 we deduce that $\varphi \in C(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$.

Observe that tu is also a solution to (E_λ) for every $t > 0$. Then, Lemma 2.2.4 implies that $tu \in C(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$, and in particular,

$$\limsup_{x \to x_0} \frac{tu(x)}{\varphi(x)} = \lim_{x \to x_0} \frac{tu(x)}{\varphi(x)} = 0 \leq 1 \quad \forall x_0 \in \partial\Omega, \forall t \geq 0.$$

Consequently, using also that tu and φ satisfy respectively (2.12) and (2.13) with the choices $g \equiv \mu$ and $h \equiv 0$, an application of Theorem 2.3.1 gives that $tu \leq \varphi$ in Ω. But this is impossible if t is taken large enough. Therefore, there is no solution to (E_λ) for any $\lambda < \lambda^*$. □

2.5 Existence and bifurcation results

We turn now to the problem of finding sufficient conditions on λ for the existence of solutions to (P_λ). The proofs of our results are based on an approximation procedure and make use of the main results of the previous sections.

Consider for every $n \in \mathbb{N}$ the family of approximate problems

$$\begin{cases} -\Delta u_n = \lambda u_n + \mu(x) \dfrac{|\nabla u_n|^q}{\left(|u_n| + \frac{1}{n}\right)^{q-1}} + T_n(f(x)), & x \in \Omega, \\ u_n = 0 & x \in \partial\Omega, \end{cases} \tag{Q_n}$$

where $T_n(s) = \min\{n, \max\{-n, s\}\}$ for $s \in \mathbb{R}$. The following result is devoted to show that, below λ^*, the approximate problems (Q_n) admit a positive solution for any n. We also prove that such a sequence of solutions is locally bounded away from zero. Finally, we prove that an a priori bound in $L^\infty(\Omega)$ implies compactness of the approximate sequence.

Lemma 2.5.1. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)} < 1$ if $q = 2$, and $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and let $\lambda < \lambda^*$. Then, there exists at least a solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (Q_n) for any n. In addition, the following local lower bound, uniform with respect to n, holds:*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 :\ u_n \geq c_\omega \ \text{ in } \ \omega, \quad \forall n. \tag{2.34}$$

Moreover, if the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ in Ω and, passing to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ uniformly in $\overline{\Omega}$.

Proof. Let us fix $n \in \mathbb{N}$, and let $\bar{\lambda} \in I^*$ be such that $\lambda < \bar{\lambda} < \lambda^*$. Then, there exist a constant $c > 0$ and a function $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfying in Ω that $\varphi \geq c$ and $-\Delta\varphi \geq \bar{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}$. Taking $M > 0$ large enough, the function $\overline{\psi} := M\varphi$ turns out to be an $H^1(\Omega) \cap L^\infty(\Omega)$ supersolution for (Q_n), since

$$\Delta\overline{\psi} + \lambda\overline{\psi} + \mu(x)\frac{|\nabla\overline{\psi}|^q}{\left(|\overline{\psi}| + \frac{1}{n}\right)^{q-1}} + T_n(f(x)) \leq n - Mc(\bar{\lambda} - \lambda) < 0, \quad x \in\ \Omega.$$

Clearly, $\underline{\psi} \equiv 0$ is a subsolution to (Q_n) and $\underline{\psi} \equiv 0 \leq \overline{\psi}$ in Ω. Therefore, Théorème 3.1 in [30] (see also [88]) implies that there exists a solution u_n to (Q_n) such that $0 \leq u_n \leq \overline{\psi}$ in Ω.

In order to prove (2.34), we use an argument by comparison. Indeed, we first observe that

$$\begin{cases} -\Delta u_n \geq \lambda u_n + T_1(f), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases}$$

Since $\lambda < \lambda^* \leq \Lambda$, then the operator $-(\Delta+\lambda)$ verifies the maximum principle, so that we compare u_n with the solution ζ to the problem

$$\begin{cases} -\Delta\zeta = \lambda\zeta + T_1(f), & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases}$$

and thus, we obtain that $u_n \geq \zeta$ in Ω. Now, since $f \gneq 0$ in Ω, the strong maximum principle (which holds since Ω is connected and, again, because $\lambda < \Lambda$, see [83]) implies that ζ satisfies (2.34), and then, so does u_n.

In order to prove the compactness of the sequence $\{u_n\}_{n\in\mathbb{N}}$, we choose now u_n as test function in (Q_n), and using that $T_n(f) \leq f$ in Ω for any n together with the $L^\infty(\Omega)$ bound, we easily deduce that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. This implies that there exists a function $u \in H_0^1(\Omega)$ such that, passing to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Using that, in particular, $u_n \to u$ a.e. in Ω, we deduce that $u > 0$ and $u \in L^\infty(\Omega)$.

On the other hand, Lemma 2.6.14 implies that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Hence, the compact embedding $C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ yields that $u_n \to u$ uniformly in $\overline{\Omega}$. □

Using the compactness provided by the previous result, we prove now the existence of a solution to (P_λ) for $f \gneq 0$ and for every $\lambda < \lambda^*$.

Proposition 2.5.2. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$. *Then, there exists a solution to* (P_λ) *for every* $\lambda < \lambda^*$.

Proof. Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to (Q_n) given by Lemma 2.5.1. We claim that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Indeed, arguing by contradiction, assume that it is unbounded in $L^\infty(\Omega)$, and take a (not relabelled) subsequence with $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Then, we have that the function $z_n \equiv \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies, for every n, that

$$\begin{cases} -\Delta z_n = \lambda z_n + \mu(x) \dfrac{|\nabla z_n|^q}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}} + \dfrac{T_n(f(x))}{\|u_n\|_{L^\infty(\Omega)}}, & x \in \Omega, \\ z_n = 0, & x \in \partial\Omega. \end{cases} \tag{2.35}$$

Since $\|z_n\|_{L^\infty(\Omega)} = 1$ for any n, then $\{z_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, so following the arguments of the proof of Lemma 2.5.1 we deduce that there exists $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $z \geq 0$ in Ω and, passing to a subsequence, $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and $z_n \to z$ uniformly in $\overline{\Omega}$. However, we can not argue as in Lemma 2.5.1 to prove neither the local lower bound to the sequence $\{z_n\}_{n\in\mathbb{N}}$, nor that the limit $z>0$ in Ω, since one does not have a lower bound for $\left\{\frac{T_n(f(x))}{\|u_n\|_{L^\infty(\Omega)}}\right\}_{n\in\mathbb{N}}$ independent of n. Hence, we need to use a different argument.

Indeed, observe first that the uniform convergence implies that $\|z\|_{L^\infty(\Omega)}=1$, so $z\gneq 0$ in Ω. Fix now $0\le\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$. We know by the weak $H_0^1(\Omega)$ convergence that

$$\lim_{n\to+\infty}\int_\Omega \nabla z_n\nabla\phi-\lambda\int_\Omega z_n\phi=\int_\Omega\nabla z\nabla\phi-\lambda\int_\Omega z\phi$$

and, since $\int_\Omega \nabla z_n\nabla\phi-\lambda\int_\Omega z_n\phi\ge 0$ for any n, we have that $\int_\Omega\nabla z\nabla\phi-\lambda\int_\Omega z\phi\ge 0$, too.

On the other hand, since $\lambda<\lambda^*\le\Lambda$, the strong maximum principle holds for the operator $-(\Delta+\lambda)$. Thus, since Ω is connected and z is not constant (the only constant in $H_0^1(\Omega)$ is the null function), the strong maximum principle implies that, for every $\omega\subset\subset\Omega$, there exists a constant $\widetilde{c}_\omega>0$ such that $z\ge\widetilde{c}_\omega$ in ω for any n (in particular, $z>0$ in Ω and $\frac{|\nabla z|^q}{z^{q-1}}\in L^1_{\text{loc}}(\Omega)$). Furthermore, the uniform convergence yields that z_n satisfies $z_n\ge c_\omega>0$, $\forall\omega\subset\subset\Omega$, $\forall n\in\mathbb{N}$. This implies that $\{-\Delta z_n\}_{n\in\mathbb{N}}$ is bounded in $L^1_{\text{loc}}(\Omega)$, that combined with the H^1 bound implies that

$$\nabla z_n\to\nabla z\ \text{ strongly in } L^r(\Omega)^N \text{ for any } r<2$$

(see [26]). The local lower bound and the convergence of the gradients will allow us to pass to the limit in (2.35).

Indeed, assume first that $1<q<2$, and let $\phi\in C_c^1(\Omega)$. We know that there exists a function $h\in L^1(\Omega)$ such that, passing to a subsequence if necessary, $|\nabla z_n|^q\le h(x)$ in Ω, and we also have that $|\nabla z_n|^q\to|\nabla z|^q$ a.e. in Ω. Therefore, choosing an open set $\omega\subset\subset\Omega$ such that $\operatorname{supp}(\phi)\subset\omega$, we have that

$$\frac{\mu(x)|\nabla z_n|^q\phi}{\left(z_n+\frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}}\le\frac{\mu(x)h(x)\phi}{\widetilde{c}_\omega^{\,q-1}},\quad x\in\Omega,$$

where $\dfrac{\mu h\phi}{\widetilde{c}_\omega^{\,q-1}}\in L^1(\Omega)$. On the other hand, we also have that

$$\frac{\mu(x)|\nabla z_n|^q\phi}{\left(z_n+\frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)^{q-1}}\to\frac{\mu(x)|\nabla z|^q\phi}{z^{q-1}},\quad \text{a.e. } x\in\Omega,\quad \text{as } n\to+\infty.$$

Hence, we may pass to the limit in the second term of the right hand side of the equation in (2.35). It is straightforward to verify that the rest of the terms also converge, so that we conclude that z is a solution (see Lemma 2.6.11 in the Appendix below) to (E_λ), but this is a contradiction with Theorem 2.4.3 since $\lambda < \lambda^*$.

On the other hand, assume now that $q = 2$ and $\|\mu\|_{L^\infty(\Omega)} < 1$, and let $\phi \in C_c^1(\Omega)$. We may assume without loss of generality that $\phi \geq 0$ in Ω. In this case we argue as in [7] (see also [15]). Thus, using that

$$\begin{aligned} z_n \to z \quad &\text{a.e. in } \Omega, \text{ weakly in } H_0^1(\Omega), \text{ strongly in } L^2(\Omega), \\ \nabla z_n \to \nabla z \quad &\text{a.e. in } \Omega, \end{aligned}$$

we obtain, by virtue of Fatou's Lemma, the inequality

$$\int_\Omega \nabla z \nabla \phi \geq \lambda \int_\Omega z\phi + \int_\Omega \mu(x) \frac{|\nabla z|^2}{z}\phi.$$

In order to prove the reverse inequality, let us take $\frac{z_n}{z}\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (2.35). It follows that

$$\begin{aligned} &\int_\Omega \frac{|\nabla z_n|^2\phi}{z} - \int_\Omega \frac{z_n\phi}{z^2}\nabla z_n \nabla z + \int_\Omega \frac{z_n}{z}\nabla z_n \nabla \phi \\ &\qquad = \lambda \int_\Omega \frac{z_n^2\phi}{z} + \int_\Omega \frac{\mu(x)|\nabla z_n|^2 z_n \phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)z} + \int_\Omega \frac{T_n(f(x))z_n\phi}{\|u_n\|_{L^\infty(\Omega)}z}. \end{aligned}$$

Since $\|\mu\|_{L^\infty(\Omega)} < 1$, we deduce that

$$\frac{|\nabla z_n|^2\phi}{z} - \frac{\mu(x)|\nabla z_n|^2 z_n \phi}{\left(z_n + \frac{1}{n\|u_n\|_{L^\infty(\Omega)}}\right)z} \geq 0 \quad \text{in } \Omega,$$

therefore, Fatou's Lemma yields to

$$\begin{aligned} &\int_\Omega \frac{|\nabla z|^2\phi}{z} - \int_\Omega \frac{\mu(x)|\nabla z|^2\phi}{z} \\ &\qquad \leq \liminf_{n\to\infty}\left(\int_\Omega \frac{z_n\phi}{z^2}\nabla z_n\nabla z - \int_\Omega \frac{z_n}{z}\nabla z_n\nabla\phi + \lambda\int_\Omega \frac{z_n^2\phi}{z} + \int_\Omega \frac{T_n(f(x))z_n\phi}{\|u_n\|_{L^\infty(\Omega)}z}\right). \end{aligned}$$

Finally, using that $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and $z_n \to z$ strongly in $L^2(\Omega)$, we obtain

$$\int_\Omega \nabla z \nabla \phi \leq \lambda \int_\Omega z\phi + \int_\Omega \mu(x) \frac{|\nabla z|^2}{z}\phi.$$

In conclusion, z is a solution (see again Lemma 2.6.11) to problem (E_λ), so that we get again a contradiction.

Thus, we have proved that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. We conclude the proof of the result by applying Lemma 2.5.1 and passing to the limit in (Q_n) as we did for $\{z_n\}_{n\in\mathbb{N}}$, with the only difference that this time the local lower bound for $\{u_n\}_{n\in\mathbb{N}}$ is provided directly by (2.34) in Lemma 2.5.1. □

We are ready now to prove our bifurcation result.

Proposition 2.5.3. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, *with* $\|\mu\|_{L^\infty(\Omega)} < 1$ *if* $q = 2$, *and* $0 \lneq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$. *Then, if* $\bar{\lambda} \in \mathbb{R}$ *is a bifurcation point from infinity of* (P_λ), *necessarily* $\bar{\lambda} = \lambda^*$. *Moreover, if* f *satisfies condition* (f_0), *then the set*

$$\Sigma := \{(\lambda, u_\lambda) \in \mathbb{R} \times C(\overline{\Omega}) : u_\lambda \text{ is a solution to } (P_\lambda)\}$$

is a continuum.

If in addition (P_λ) *has no solution for* $\lambda = \lambda^*$, *then the continuum is unbounded and it bifurcates from infinity at* λ^* *to the left.*

Proof. Assume that $\bar{\lambda} \in \mathbb{R}$ is a bifurcation point from infinity to (P_λ), i.e., there exists a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to \bar{\lambda}$ such that there exists a solution u_n to (P_{λ_n}) for any n satisfying $\|u_n\|_{L^\infty(\Omega)} \to \infty$. Proceeding as for Proposition 2.5.2, we may pass to the limit in the equation satisfied by $z_n \equiv \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}$, so that we obtain a solution z to problem (E_λ). Therefore, by virtue of Proposition 2.4.3, we have necessarily that $\lambda = \lambda^*$.

Assume now that f satisfies (f_0). We will prove that the set Σ is a continuum. In other words, we will show that the function

$$\begin{aligned} (-\infty, \lambda^*) &\to C(\overline{\Omega}) \\ \lambda &\mapsto u_\lambda \end{aligned}$$

is continuous, where u_λ denotes the unique solution to (P_λ) given by Proposition 2.5.2 (the uniqueness of $u_\lambda \in C(\overline{\Omega})$ follows from Lemma 2.2.4 and Theorem 2.3.2). Indeed, let us fix $\lambda \in (-\infty, \lambda^*)$, and choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, \lambda^*)$ such that $\lambda_n \to \lambda$ as n diverges. Arguing again as in the proof of Proposition 2.5.2, if one assumes that $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is unbounded in $L^\infty(\Omega)$, then a solution to problem (E_λ) can be found, but this is impossible because $\lambda < \lambda^*$. Thus, necessarily $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, so that we deduce as in Lemma 2.5.1 that $u_{\lambda_n} \to u_\lambda$ uniformly in $\overline{\Omega}$, i.e., in the space $C(\overline{\Omega})$.

To conclude we prove that the continuum is unbounded by showing that λ^* is a bifurcation point from infinity to the left of the axis $\lambda = \lambda^*$. Indeed, assuming that $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ for some sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, \lambda^*)$ with $\lambda_n \to \lambda^*$ as n diverges, we can pass to the limit in (P_{λ_n}) and we find a solution to (P_{λ^*}), but this is a contradiction.

In conclusion, λ^* is a bifurcation point from infinity to the left of $\lambda = \lambda^*$. $\square$

2.6 Proofs of the main results and final remarks

Proof of Theorem 2.1.2. We deduce from Proposition 2.5.2 the existence of at least one solution to (P_λ) if $\lambda < \lambda^*$. Moreover, the nonexistence for $\lambda > \lambda^*$ in deduced by Proposition 2.4.3.

As far as uniqueness is concerned, we observe that if u, v are two solutions to (P_λ), then Lemma 2.2.4 implies that $u, v \in C(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$. In particular, using the continuity up to the boundary of u, v and the fact that $u(x_0) = 0$ for any $x_0 \in \partial\Omega$, we have that u, v satisfy (2.19) for any $\varepsilon > 0$. Moreover, they obviously satisfy (2.12) and (2.13) respectively. Therefore, Theorem 2.3.2 implies that $u \leq v$ in Ω. The reverse inequality follows by interchanging the roles of u and v.

We give now a proof for the nonexistence of solutions to (P_{λ^*}). Thus, assume by contradiction that there exists a solution u to (P_{λ^*}). Then, we can find a solution v to

$$\begin{cases} -\Delta v = \lambda^* v + \mu(x)\dfrac{|\nabla v|^q}{v^{q-1}} + C\varphi_1^\gamma, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.36}$$

The proof of this fact follows basically the same steps as Proposition 2.5.2: the only difference is the way of proving the L^∞ estimate, which does not work in this case. However, since we are assuming that there is a solution u to (P_{λ^*}), then, by comparison, any solution to the approximate problems (Q_n) is smaller than $\|u\|_{L^\infty(\Omega)}$, which gives the a priori estimate.

Furthermore, using Lemma 2.2.7 we deduce that for $\varepsilon > 0$ small enough, the following holds:

$$C_1\varphi_1^\gamma - \varepsilon v \geq (C_1 - \varepsilon C))\varphi_1^\gamma \geq 0.$$

Therefore,

$$-\Delta v = (\lambda^* + \varepsilon)v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}} + (C_1\varphi_1^\gamma - \varepsilon v) \geq (\lambda^* + \varepsilon)v + \mu(x)\frac{|\nabla v|^q}{v^{q-1}}.$$

That is to say, v is a supersolution to $(E_{\lambda^*+\varepsilon})$. Moreover, Proposition 2.2.6 implies that $v^\eta \in H_0^1(\Omega)$ for every $\eta \in (\eta_0, 1)$ for some $\eta_0 < 1$. This is a contradiction with the characterization (2.24) in Proposition 2.4.1. So we have proved the nonexistence result.

Finally, the claim about bifurcation follows from Proposition 2.5.3. $\square$

Proof of Theorem 2.1.1. We have shown in Proposition 2.4.1 that $\lambda^* \in (0, \Lambda]$. Moreover, as a consequence of Proposition 2.4.3, if (E_λ) admits a solution then $\lambda = \lambda^*$.

Thus, for the first part of the theorem it only remains to prove the existence of solution to (E_λ) for $\lambda = \lambda^*$. In order to do that, by virtue of Proposition 2.5.3, we may choose $\lambda_n \to \lambda^*$ such that $\|u_{\lambda_n}\|_{L^\infty(\Omega)} \to \infty$, where u_{λ_n} denotes, for any n, the unique solution to the problem

$$\begin{cases} -\Delta u = \lambda_n u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + 1, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Hence, arguing again as in Proposition 2.5.2, we may pass to the limit in the equation satisfied by $z_n \equiv \dfrac{u_{\lambda_n}}{\|u_{\lambda_n}\|_{L^\infty(\Omega)}}$ using that $\|z_n\|_{L^\infty(\Omega)} = 1$ for any n, concluding that the limit z is a solution to (E_{λ^*}).

Regarding the uniqueness of the solution up to multiplicative constants, it follows by adapting the uniqueness result proved in [95] to $v_i = \log(u_i)$, $i = 1, 2$, being u_1 and u_2 two solutions to (E_{λ^*}). $\square$

We conclude the section with some remarks concerning the principal eigenvalue and some possible extensions of our results.

Remark 2.6.1. Let us remark that the global behavior of the continuum considered in Proposition 2.5.3 corresponds to that obtained in [15] for $q = 2$. That is to say, $\lambda^* > 0$ is the only possible bifurcation point from infinity. However, as it was pointed out in the introduction, there are similar singular problems that exhibit a completely different behavior. For instance, whenever bifurcation occurs for the problem

$$\begin{cases} -\Delta u = \lambda u + \mu\dfrac{|\nabla u|^2}{u^\theta} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

with $\theta \in [0, 1)$, it is possible to prove (see [12]) that $\lambda = 0$ is the only possible bifurcation point from infinity. Hence the qualitative behavior of the continuum of solutions is different if the problem above possesses or not a solution for $\lambda = 0$.

Remark 2.6.2. We show here that, in the case $q = 2$, there exists $0 \le \mu \in L^\infty(\Omega)$ with $\|\mu\|_{L^\infty(\Omega)} < 1$ such that $\lambda^* > \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)} + 1}$. This proves that, if μ is not a constant, then the condition $\lambda < \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)} + 1}$ is not necessary in general for the existence of solutions to (P_λ).

Indeed, by contradiction, assume that $\lambda^* = \dfrac{\Lambda}{\|\mu\|_{L^\infty(\Omega)}+1}$ for any $0 \le \mu \in L^\infty(\Omega)$ with $\|\mu\|_{L^\infty(\Omega)} < 1$. Fix $x_0 \in \Omega$ and consider a sequence of balls $\{B_{\frac{1}{n}}(x_0)\}_{n\in\mathbb{N}} \subset \Omega$. For any n, let us define in Ω the functions

$$\mu_n(x) = \frac{1}{2}\chi_{B_{\frac{1}{n}}(x_0)}.$$

Since $\|\mu_n\|_{L^\infty(\Omega)} = 1/2 < 1$ for any n, we may consider a solution u_n to

$$\begin{cases} -\Delta u_n = \dfrac{\Lambda}{\|\mu_n\|_{L^\infty(\Omega)}+1}u_n + \mu_n(x)\dfrac{|\nabla u_n|^2}{u_n,} & x \in \Omega, \\ u_n > 0, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases}$$

Observe that $\dfrac{\Lambda}{\|\mu_n\|_{L^\infty(\Omega)}+1} = \dfrac{2\Lambda}{3}$ for any n, and that $\mu_n \to 0$ a.e. in Ω.

If we choose u_n so that $\|u_n\|_{L^\infty(\Omega)} = 1$ for any n, then, arguing as in the proof of Proposition 2.5.2, we may pass to the limit and find a solution u to

$$\begin{cases} -\Delta u = \dfrac{2\Lambda}{3}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

But this is a contradiction since $\dfrac{2\Lambda}{3} < \Lambda$.

Remark 2.6.3. It is worth to highlight that we can prove the strict inequality $\lambda^* < \Lambda$ provided $\mu > 0$ in Ω. Indeed, let u be a solution to (E_{λ^*}). Then, taking φ_1 as test function in (E_{λ^*}), we obtain

$$(\Lambda - \lambda^*)\int_\Omega u\varphi_1 = \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\varphi_1 > 0,$$

which clearly implies what we claimed.

Remark 2.6.4. We also point out that Theorem 2.1.1 yields to

$$I^* = I_1 = (-\infty, \lambda^*) \quad \text{and} \quad I_2 = (-\infty, \lambda^*].$$

Indeed, if u is a solution to (E_{λ^*}), then it follows trivially from Lemma 2.2.6 that $\lambda^* \in I_2$. On the other hand, assume by contradiction that $\lambda^* \in I^*$. Then, there exists a supersolution $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ to (E_{λ^*}) with $\varphi \ge c$ in Ω for some $c > 0$. Hence, we may argue as in Proposition 2.4.3 to obtain that $tu \le \varphi$ in Ω for every $t > 0$, which is obviously impossible.

Remark 2.6.5. In the whole paper we have confined ourselves to the case $q > 1$ in order to deal with the singular problems (P_λ) and (E_λ). Nevertheless, our results hold true also for $q = 1$ by following the same approach (with some small difference in the proof of the positivity of λ^*).

Remark 2.6.6. The hypotheses made on the smoothness of $\partial\Omega$ deserve also some comments. For the sake of clarity we have assumed in the whole paper that the boundary is of class $C^{1,1}$. Actually, such a regularity of the boundary is needed only in order to obtain $C^1(\overline{\Omega})$ regularity of solutions to linear problems (which is provided by Calderon-Zygmund regularity theory). Apart from those results, it suffices to impose a weaker regularity assumption on $\partial\Omega$ in order to prove the rest of our results. Indeed, one needs to suppose that Ω satisfies the following condition:

$$\begin{gathered}\text{Let } \Omega \subset \mathbb{R} \text{ be an open set and suppose that there exist } r_0,\ \theta_0 > 0\\ \text{such that, if } x \in \partial\Omega \text{ and } 0 < r < r_0, \text{ then } |\Omega_r| \leq (1-\theta_0)\ |B_r(x)|\\ \text{for every connected component } \Omega_r \text{ of } \Omega \cap B_r(x).\end{gathered} \tag{2.37}$$

Such a condition is specifically needed to prove $C^{0,\alpha}(\overline{\Omega})$ regularity (and also uniform estimates in this space) of the solutions.

Appendix

Wider class of test functions

This subsection consists of five lemmata that prove that the formulation given in Definition 2.2.1 is totally meaningful and actually can be changed into an equivalent one in which the test functions have compact support.

Lemma 2.6.7. *Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that* $\mathrm{supp}(\phi) \subset \Omega$*. Then, there exist an open set $\omega \subset\subset \Omega$ and a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$ such that it is bounded in $L^\infty(\Omega)$,* $\mathrm{supp}(\phi_n) \subset \omega$ *for any n, and $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$.*

Proof. Take an open set ω such that $\mathrm{supp}(\phi) \subset \omega \subset\subset \Omega$. Then, $\phi \in H_0^1(\omega) \cap L^\infty(\omega)$. Let $\psi_n \in C_c^1(\omega)$ be such that $\psi_n \to \phi$ strongly in $H_0^1(\omega)$, then

$$\psi_n \to \phi \quad \text{a.e. in } \Omega, \qquad \nabla\psi_n \to \nabla\phi \quad \text{a.e. in } \Omega.$$

Consider now a function $G : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) $G \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$,

(ii) $G(s) = s \quad \forall s \in \left[-\|\phi\|_{L^\infty(\omega)}, \|\phi\|_{L^\infty(\omega)}\right]$.

Clearly, we have that $\{G(\psi_n)\}_{n\in\mathbb{N}} \subset C_c^1(\omega)$, it is bounded in $L^\infty(\omega)$ and, in addition,

$$\nabla G(\psi_n) = G'(\psi_n)\nabla\psi_n \to G'(\phi)\nabla\phi = \nabla G(\phi) = \nabla\phi \quad \text{a.e. in } \omega.$$

Moreover,

$$|\nabla G(\psi_n) - \nabla\phi|^2 \le |\nabla G(\psi_n)|^2 + |\nabla\phi|^2 + 2|\nabla G(\psi_n)||\nabla\phi|$$

and therefore, the Vitali's Theorem yields that $G(\psi_n) \to \phi$ strongly in $H_0^1(\omega)$.

For any n, let us define the function ϕ_n in Ω by

$$\phi_n = \begin{cases} G(\psi_n) & \text{in } \omega, \\ 0 & \text{in } \Omega \setminus \omega. \end{cases}$$

Thus, we have that $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$, it is bounded in $L^\infty(\Omega)$, $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$ and, in addition, $\operatorname{supp}(\phi_n) \subset \omega$ for any n. □

Lemma 2.6.8. *Let $\phi \in H_0^1(\Omega)$ be such that $\phi \ge 0$ a.e. in Ω, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $C_c^1(\Omega)$ such that $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. Then, $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$.*

Proof. We know, passing to a subsequence, that

$$\phi_n \to \phi \quad \text{a.e. in } \Omega, \quad \text{and } \nabla\phi_n \to \nabla\phi \quad \text{a.e. in } \Omega.$$

Observe that

$$\begin{aligned} \int_\Omega |\nabla(\phi_n^+ - \phi)|^2 &= \int_\Omega |\nabla(\phi_n - \phi)|^2 + 2\int_\Omega \nabla\phi_n^-\nabla\phi - \int_\Omega |\nabla\phi_n^-|^2 \\ &\le \int_\Omega |\nabla(\phi_n - \phi)|^2 + 2\int_\Omega \nabla\phi_n^-\nabla\phi \end{aligned} \tag{2.38}$$

for any n.

Now, by continuity we deduce that $\phi_n^- \to \phi^- = 0$ a.e. in Ω. Therefore, passing to a new subsequence, we infer that $\{\phi_n^-\}_{n\in\mathbb{N}}$ weakly converges in $H_0^1(\Omega)$ to some limit ψ, and then $\phi_n^- \to \psi$ a.e. in Ω. Hence, necessarily $\psi \equiv 0$, that is to say, $\phi_n^- \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

Finally, from (2.38) we conclude that $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$, where $\{\phi_n\}_{n\in\mathbb{N}}$ is a subsequence of the original sequence. Actually we have proved that any subsequence of $\{\phi_n\}_{n\in\mathbb{N}}$ admits a subsequence such that $\phi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$. Then we have necessarily that, indeed, the positive part of the original sequence strongly converges to ϕ in $H_0^1(\Omega)$. $\square$

Lemma 2.6.9. *Let $\phi \in H_0^1(\Omega)$ be such that $\phi \geq 0$ a.e. in Ω. Then, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $\operatorname{supp}(\phi_n) \subset \Omega$ for any n, $0 \leq \phi_n \leq \phi$ a.e. in Ω for any n, and $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. In particular, if $\phi \in L^\infty(\Omega)$, then $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$.*

Proof. Let $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^1(\Omega)$ be such that $\psi_n \to \phi$ strongly in $H_0^1(\Omega)$. By virtue of Lemma 2.6.8, we have that $\psi_n^+ \to \phi$ strongly in $H_0^1(\Omega)$. For any n, let us define now the function $\phi_n = \phi + (\psi_n^+ - \phi)^-$ in Ω. Clearly, $\phi_n \in H_0^1(\Omega)$ and $0 \leq \phi_n \leq \phi$ a.e. in Ω for any n. Observe also that, for any n, it holds that $\phi_n \leq \psi_n^+$ a.e. in Ω and $\psi_n^+ = 0$ a.e. in $\Omega\backslash\operatorname{supp}(\psi_n^+)$, so $\phi_n = 0$ a.e. in $\Omega\backslash\operatorname{supp}(\psi_n^+)$. Hence, by the definition of the essential support of ϕ_n, we have that $\Omega\backslash\operatorname{supp}(\psi_n^+) \subset \Omega\backslash\operatorname{supp}(\phi_n)$, and in consequence, $\operatorname{supp}(\phi_n) \subset \operatorname{supp}(\psi_n^+) \subset \Omega$.

Finally, we have that

$$\int_\Omega |\nabla(\phi_n - \phi)|^2 = \int_\Omega |\nabla(\psi_n^+ - \phi)^-|^2 \leq \int_\Omega |\nabla(\psi_n^+ - \phi)|^2,$$

and therefore, $\phi_n \to \phi$ strongly in $H_0^1(\Omega)$. $\square$

Lemma 2.6.10. *Assume that $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \leq f \in L^p(\Omega)$ with $p > \frac{N}{2}$ and $\lambda \in \mathbb{R}$. Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $u > 0$ a.e. in Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ and satisfies*

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega f(x)\phi$$

for any $\phi \in C_c^1(\Omega)$. Then, the same equality holds for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with compact support.

Proof. Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $\operatorname{supp}(\phi) \subset \Omega$, and let ω and $\{\phi_n\}_{n\in\mathbb{N}}$ be the open set and the sequence given by Lemma 2.6.7, respectively. This lemma gives also that

$$\mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n \leq \|\mu\|_{L^\infty(\Omega)}\frac{|\nabla u|^q}{u^{q-1}}\|\phi_n\|_{L^\infty(\Omega)} \leq C\frac{|\nabla u|^q}{u^{q-1}}$$

a.e. in Ω, where $C > 0$ is a constant independent of n. Therefore, since $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{\text{loc}}(\Omega)$, we can use the Lebesgue's Theorem to derive

$$\int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n = \int_\omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi_n \to \int_\omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi = \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi.$$

The conclusion of the lemma is now straightforward. $\square$

Lemma 2.6.11. *Assume that* $1 < q \leq 2$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \leq f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$ *and* $\lambda \in \mathbb{R}$. *Let* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *be such that* $u > 0$ *a.e. in* Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ *and satisfies*

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega f(x)\phi$$

for any $\phi \in C_c^1(\Omega)$. *Then,* u *is a solution to* (P_λ) *in the sense of Definition 2.2.1.*

Similarly, if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ *is such that* $u > 0$ *a.e. in* Ω, $\frac{|\nabla u|^q}{u^{q-1}} \in L^1_{loc}(\Omega)$ *and satisfies*

$$\int_\Omega \nabla u \nabla \phi \geq \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi + \int_\Omega f(x)\phi$$

for any $0 \leq \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *with compact support, then* u *is a supersolution to* (P_λ) *in the sense of Definition 2.2.1.*

Proof. Let $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and let $\{\phi_n\}_{n\in\mathbb{N}}$ be the sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$ given by Lemma 2.6.9 such that $\phi_n \to \phi^+$ strongly in $H_0^1(\Omega)$. We have, by virtue of Lemma 2.6.10, that

$$\int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi_n = \int_\Omega \nabla u \nabla \phi_n - \lambda \int_\Omega u\phi_n - \int_\Omega f(x)\phi_n \quad \forall n \in \mathbb{N}. \tag{2.39}$$

Hence, by Fatou's Lemma and by the weak convergence $\phi_n \rightharpoonup \phi^+$ in $H_0^1(\Omega)$, we obtain that

$$\int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi^+ \leq \int_\Omega \nabla u \nabla \phi^+ - \lambda \int_\Omega u\phi^+ - \int_\Omega f(x)\phi^+.$$

That means that $\frac{|\nabla u|^q}{u^{q-1}} \phi^+ \in L^1(\Omega)$. This fact allows us to pass to the limit in (2.39) by using the Lebesgue's Theorem for the left hand side (recall that $\phi_n \leq \phi^+$ for any n), and again the weak convergence for the right hand side, so that we obtain

$$\int_\Omega \mu(x) \frac{|\nabla u|^q}{u^{q-1}} \phi^+ = \int_\Omega \nabla u \nabla \phi^+ - \lambda \int_\Omega u\phi^+ - \int_\Omega f(x)\phi^+.$$

An analogous procedure provides us the same identity but replacing ϕ^+ by ϕ^-. The proof of the first part of the lemma concludes by simply adding both identities.

The last part of the lemma about supersolutions can be proved in a similar way. □

Regularity of the solutions

Proof of Lemma 2.2.4: Hölder regularity.

Here we prove that any $H_0^1(\Omega)\cap L^\infty(\Omega)$ solution to (P_λ) actually belongs to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$. For this purpose, we make use of the regularity theory developed by Ladyzhenskaya and Ural'tseva in [94].

We denote an open ball with radius $\rho>0$ as B_ρ, and for $v:\Omega\to\mathbb{R}$, $k\in\mathbb{R}$, we also write

$$A_{k,\rho}(v)=\{x\in\Omega\cap B_\rho : v(x)\geq k\}.$$

Definition 2.6.12 ([94], p. 90)**.** Let $M,\gamma,\delta>0$, $m>1$, $r\in(N,+\infty]$, and consider an open domain $\Omega\subset\mathbb{R}^N$. We say that a function $u:\Omega\to\mathbb{R}$ belongs to the class $\mathscr{B}_m\left(\overline{\Omega},M,\gamma,\delta,\frac{1}{r}\right)$ if $u\in W^{1,m}(\Omega)\cap L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)}\leq M$, and the following holds for $v=u$ and also for $v=-u$:

$$\int_{A_{k,\rho-\sigma\rho}(v)}|\nabla v|^m\leq\gamma\left(\frac{1}{\sigma^m\rho^{m\left(1-\frac{N}{r}\right)}}\|v-k\|^m_{L^\infty(A_{k,\rho}(v))}+1\right)|A_{k,\rho}(v)|^{1-\frac{m}{r}}, \tag{2.40}$$

for any $\rho>0$ and all B_ρ such that $\Omega\cap B_\rho\neq\emptyset$, for all $\sigma\in(0,1)$ and for all $k\geq k_\rho$, where $k_\rho=\max\left\{\sup_{\Omega\cap B_\rho}(v)-\delta,\ \sup_{\partial\Omega\cap B_\rho}(v)\right\}$ if $\partial\Omega\cap B_\rho\neq\emptyset$, while $k_\rho=\sup_{B_\rho}(v)-\delta$ otherwise.

We will use the following result (see [94, Theorem 7.1, p. 91]).

Theorem 2.6.13. *Let $\Omega\subset\mathbb{R}^N$ be a $C^{1,1}$ domain and let $M,\gamma,\delta>0$, $r\in(N,+\infty]$, $m>1$, $\beta\in(0,1)$, $L>0$. Then, there exist $\alpha\in(0,1),K>0$ such that, if $u\in\mathscr{B}_m\left(\overline{\Omega},M,\gamma,\delta,\frac{1}{r}\right)$ satisfies*

$$\sup_{\partial\Omega\cap B_\rho}(u)-\inf_{\partial\Omega\cap B_\rho}(u)\leq L\rho^\beta \tag{2.41}$$

for every ball B_ρ centered at $\partial\Omega$ with $0<\rho<a_0$, then $u\in C^{0,\alpha}(\overline{\Omega})$ and, moreover, $\|u\|_{C^{0,\alpha}(\overline{\Omega})}\leq K$.

Thus, the core of our result is proving that any solution to (P_λ) belongs to the class $\mathscr{B}_m\left(\overline{\Omega},M,\gamma,\delta,\frac{1}{r}\right)$.

Lemma 2.6.14. *Let $\Omega\subset\mathbb{R}^N$ be a $C^{1,1}$ domain, $\lambda\in\mathbb{R}$, $1<q\leq 2$, $0\leq\mu\in L^\infty(\Omega)$, with $\|\mu\|_{L^\infty(\Omega)}<1$ if $q=2$, and $0\lneq f\in L^p(\Omega)$ for $p>\frac{N}{2}$. Then, for every $M>0$, there exist $\alpha\in(0,1)$ and $K>0$ such that every solution u to (P_λ) satisfying $\|u\|_{L^\infty(\Omega)}\leq M$ belongs to $C^{0,\alpha}(\overline{\Omega})$ with $\|u\|_{C^{0,\alpha}(\overline{\Omega})}\leq K$.*

Remark 2.6.15. The condition on the regularity of the boundary can be relaxed, assuming that $\partial\Omega$ satisfies (2.37).

Proof of Lemma 2.6.14. Let $M > 0$ and let u be a solution to (P_λ) with $\|u\|_{L^\infty(\Omega)} \le M$. We will show that $u \in \mathscr{B}_2\left(\overline{\Omega}, M, \gamma, M, \frac{1}{2p}\right)$ for some $\gamma > 0$.

Indeed, fix $\rho > 0$ and B_ρ such that $\Omega \cap B_\rho \neq \emptyset$, fix also $\sigma \in (0,1)$, and consider a function $\zeta \in C_c^\infty(\mathbb{R}^N)$, compactly supported in B_ρ, satisfying that $0 \le \zeta \le 1$ in B_ρ, $\zeta \equiv 1$ in the concentric ball $B_{\rho-\sigma\rho}$, and $|\nabla\zeta| < \frac{a}{\sigma\rho}$ in B_ρ for some constant $a > 0$ independent of ρ, σ. We start by showing that inequality (2.40) is satisfied for $v = u$.

Thus, let $k \ge k_\rho$. If $\partial\Omega \cap B_\rho \neq \emptyset$, then $k_\rho = 0$ (since $v = 0$ on $\partial\Omega$). Consequently, $(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and in particular $\zeta^2(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. On the other hand, if $B_\rho \subset \Omega$, then $\zeta \in C_c^\infty(\Omega)$, so $\zeta^2(v-k^+)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In both cases, we can take $\zeta^2(v-k^+)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (P_λ), so that we obtain

$$\int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^2 \le 2\int_{A_{k,\rho}(v)} \zeta(v-k^+)|\nabla\zeta||\nabla v| + |\lambda| \int_{A_{k,\rho}(v)} v(v-k^+)$$
$$+ \|\mu\|_{L^\infty(\Omega)} \int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^q \frac{v-k^+}{|v|^{q-1}} + \int_{A_{k,\rho}(v)} f(x)(v-k^+). \tag{2.42}$$

Notice that $A_{k,\rho} = A_{k^+,\rho}$ since $v > 0$. Using now that $v \le M$ and $k^+ = \max\{0,k\}$, we deduce that

$$\int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^2 \le 2\int_{A_{k,\rho}(v)} \zeta(v-k)|\nabla\zeta||\nabla v| + |\lambda| M^2 |A_{k,\rho}(v)|$$
$$+ \|\mu\|_{L^\infty(\Omega)} M^{2-q} \int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^q + M\int_{A_{k,\rho}(v)} f(x).$$

If $q < 2$ we use Young's inequality conveniently in the first and the third terms of the right hand side of the last inequality, so we derive

$$\frac{1}{C}\int_{A_{k,\rho}(v)} \zeta^2 |\nabla v|^2 \le (|\lambda| + C) M^2 |A_{k,\rho}(v)|$$
$$+ M\|f\|_{L^p(\Omega)} |A_{k,\rho}(v)|^{\frac{1}{p'}} + C\int_{A_{k,\rho}(v)} (v-k)^2 |\nabla\zeta|^2,$$

for some $C = C(q,\mu) > 0$ large enough. Similarly, if $q = 2$ and $\|\mu\|_{L^\infty(\Omega)} < 1$, we arrive at the same inequality in a similar way, but Young's inequality is not needed in the second term of the right hand side.

Noticing that

$$|A_{k,\rho}(v)| = |A_{k,\rho}(v)|^{\frac{1}{p'}} |A_{k,\rho}(v)|^{\frac{1}{p}} \le |A_{k,\rho}(v)|^{\frac{1}{p'}} C(N,p) \rho^{\frac{N}{p}},$$

and recalling the properties of ζ, we finally arrive at

$$\int_{A_{k,\rho-\sigma\rho}(v)} |\nabla v|^2 \le \gamma \left(\frac{1}{\sigma^2 \rho^{2-\frac{N}{p}}} \|v-k\|^2_{L^\infty(A_{k,\rho}(v))} + 1 \right) |A_{k,\rho}(v)|^{\frac{1}{p'}}, \tag{2.43}$$

for some $\gamma > 0$ which depends on M but not on v, k, ρ, σ.

Let us now prove that (2.40) holds for $v = -u$. First of all, notice that v satisfies

$$\begin{cases} -\Delta v \leq \lambda v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.44}$$

Let $k \geq k_\rho$; if, on the contrary, $\partial\Omega \cap B_\rho \neq \emptyset$, then $k_\rho = 0$, so $A_{k,\rho}(v) = \emptyset$ and (2.40) is trivially satisfied.

On the other hand, if $B_\rho \subset \Omega$, then $\zeta^2(v-k)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and it can be used as test function in (2.44). In particular, (2.42) also holds, so the same computations above can be reproduced up to (2.43), and the proof of our claim is done.

In conclusion, we have proved that $u \in \mathscr{B}_2\left(\overline{\Omega}, M, \gamma, M, \frac{1}{2p}\right)$. Since (2.41) is satisfied being $u = 0$ on $\partial\Omega$, then Theorem 2.6.13 implies that $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, and, in addition, $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq K$, where K, α are positive constants independent of u. □

Proof of Lemma 2.2.4: Local Sobolev regularity. As far as the local Sobolev regularity is concerned, we use a classical bootstrap argument (see [63] for more details). We first observe that since $u > 0$, by virtue of the strong maximum principle we have that $u \geq \inf_\omega(u) > 0$ in ω for every smooth open set $\omega \subset\subset \Omega$. Hence, we may apply [80, Chapter V, Proposition 2.1] to derive that $u \in W^{1,t_0}(\omega)$ for some $t_0 > 2$, and by standard regularity theory we have that $u \in W^{2,\frac{t_0}{2}}(\omega)$. Now, since $u \in C^{0,\alpha}(\omega) \cap W^{2,\frac{t_0}{2}}(\omega)$, then [108, Teorema IV] implies that $u \in W^{1,t_1}(\omega)$, where $t_1 = \frac{\frac{t_0}{2}(2-\alpha)-\alpha}{1-\alpha} > t_0$.

We may continue the bootstrap argument as in the proof of [11, Lemma 2.1] to obtain that $u \in W^{1,t_n}(\omega)$ as long as $t_{n-1} < 2p$, where

$$t_n = \frac{\frac{t_{n-1}}{2}(2-\alpha)-\alpha}{1-\alpha} \quad \forall n \in \mathbb{N}.$$

Observe that the sequence $\{t_n\}_{n\in\mathbb{N}}$ is increasing. Assume by contradiction that $t_n < 2p$ for any n. Then the sequence $\{t_n\}_{n\in\mathbb{N}}$ is convergent, and $t_n \to 2$. But this contradicts the fact that $t_0 > 2$. Hence, necessarily $t_n \geq 2p$ for some n, and the proof is done. □

Chapter 3

Singular quasilinear elliptic problems with changing sign datum: Existence and homogenization

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Abstract. We study singular quasilinear elliptic equations whose model is

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ ($N \geq 3$), $\lambda \in \mathbb{R}$, $1 < q < 2$, $0 \leq \mu \in L^\infty(\Omega)$ and the datum $f \in L^p(\Omega)$, for some $p > \frac{N}{2}$, may change sign. We prove existence of solution and we deal with the homogenization problem posed in a sequence of domains Ω^ε obtained by removing many small holes from a fixed domain Ω.

3.1 Introduction

In this paper we consider the following boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \lambda u + g(x,u)|\nabla u|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain satisfying the boundary condition (A) below. Here, $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$ and no assumption on its sign is imposed. Moreover $M(x)$ is an $N \times N$ matrix satisfying

$$\begin{cases} M \in (L^\infty(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega))^{N\times N} \text{ and for some } \eta > 0, \\ \eta|\xi|^2 \leq M(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega. \end{cases} \tag{M_1}$$

We consider $1 < q < 2$, $\lambda \in \mathbb{R}$ and a Carathéodory function $g : \Omega \times \mathbb{R} \setminus \{0\} \to [0, +\infty)$ satisfying that,

$$\begin{cases} \text{for a.e. } x \in \Omega, \text{ the function } s \mapsto g(x,s)|s|^{q-1} \text{ is bounded} \\ \text{and } \mu(x) \equiv \sup\limits_{s \in \mathbb{R}\setminus\{0\}} g(x,s)|s|^{q-1} \in L^\infty(\Omega). \end{cases} \tag{g_1}$$

Observe that hypothesis (g_1) includes the case in which g, at $s = 0$, admits a continuous extension but also the case in which the lower order term may have a singularity.

Our first goal is to study the existence of solution to problem (P_λ) under the previous hypotheses. Since the function $g(x,s)$ may be defined only for $|s| > 0$, having in mind the model problem where $g(x,s) = \frac{\mu(x)}{|s|^{q-1}}$, we have to clarify the meaning of solution. We say that a solution to problem (P_λ) is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $g(x,u)|\nabla u|^q \in L^1(\{|u| > 0\})$ and

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi + \int_\Omega f(x)\phi,$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let us note that, due to Stampacchia's Theorem, $\nabla u \equiv 0$ in the set $\{|u| = 0\}$, so this concept of solution coincides with the usual one when $g(x,s)$ is continuous[1] at $s = 0$ or just g is bounded at $s = 0$. In the case g unbounded at $s = 0$ we remark that integrating in the set $\{|u| > 0\}$ does not avoid the singularity, to the contrary, the integrand is singular on $\partial\{|u| > 0\}$ and this set is nonempty if u is nontrivial.

Observe also that, if $f \gneq 0$, then the strong maximum principle implies that $u > 0$ a.e. in Ω for every solution u to problem (P_λ) with $\lambda < \lambda_1(M) \equiv \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega M(x)\nabla v \cdot \nabla v}{\int_\Omega v^2}$, and therefore, $\{|u| > 0\} = \Omega$. This framework with nonnegative datum f and positive solutions is usual for this kind of singular problems (see [7, 15, 38] and references

[1] Observe that, even though g is not continuous at $s = 0$, one can define, for instance, $g(x,0) = 1$ a.e. $x \in \Omega$. Therefore, Stampacchia's Theorem implies that $\int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi = \int_\Omega g(x,u)|\nabla u|^q\phi$ for every solution u to (P_λ). In particular, every solution to (P_λ) satisfies (1.1) in the Introduction (chapter 1).

therein). In fact, it was adopted in [38], where the authors studied problem (P_λ) in the model case where M is the identity matrix and $g(x,s) = \frac{\mu(x)}{|s|^{q-1}}$.

Up to our knowledge, the first time a sign changing datum was considered in a singular quasilinear equation was in [77] (see also [78,79]). In [77] the authors studied a general problem whose simplest model is

$$\begin{cases} -\Delta u = \dfrac{|\nabla u|^2}{|u|^\theta} + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\theta \in (0,1)$ and $f \in L^p(\Omega)$, with $p \geq \frac{N}{2}$. With a concept of solution very close to the one that we established above, they proved the existence of at least a solution to the problem.

In the present work, we aim to improve the existence results contained in [38] in several directions (in Section 2 we describe some concepts and results). On the one hand, we generalize the principal operator of the equation and the nonlinear term by imposing conditions (M_1) and (g_1) respectively. On the other hand, we will allow f to change sign. Hence, the solutions may vanish in a set of positive measure, in fact in Remark 3.3.2 we include two examples for which this actually happens.

Concerning the techniques that we use, we approximate the singular problem by a sequence of nonsingular ones. We prove that there exists a solution u_n to the approximated problems using the sub-supersolution method in [30, Théorème 3.1]. Then, we prove that, passing to a subsequence, u_n converges to u strongly in $H_0^1(\Omega)$ and also in $L^r(\Omega)$ for all $r \in [1,\infty)$ in order to pass to the limit in the approximated problems and to obtain a solution u to problem (P_λ).

The main interest of our proof by approximation lies on the way that the a priori estimates, needed for the compactness of the sequence $\{u_n\}$, are obtained. The greatest difficulty comes from the fact that f changes sign because, in this case, such a sequence is not uniformly bounded away from zero. This lower estimate represents a usual tool for proving, for instance, that the lower order term is bounded in $L^1_{\text{loc}}(\Omega)$. However, we will be able to prove a global L^1 estimate even if the lower estimate does not hold true (see Lemma 3.3.5 and Remark 3.3.6 below). It is also remarkable that an L^∞ estimate for $\{u_n\}$ can be obtained by using carefully the Comparison Principle [38, Theorem 3.2] (see also next section).

Another goal of the paper is the following homogenization problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u^\varepsilon) = \lambda u^\varepsilon + g(x,u^\varepsilon)|\nabla u^\varepsilon|^q + f(x), & x \in \Omega^\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega^\varepsilon, \end{cases} \tag{3.1}$$

where Ω^ε is a sequence of open sets which are included in a fixed bounded open set Ω of $\mathbb{R}^N$, $M(x)$ is an $N \times N$ matrix satisfying (M_1), g satisfies (g_1), $1 < q < 2$, $\lambda \in \mathbb{R}$ and $f \in L^p(\Omega)$, $p > \frac{N}{2}$.

More precisely, we study the asymptotic behavior, as ε goes to zero, of a sequence of solutions to these problems posed in domains Ω^ε obtained by removing many small holes from a fixed domain Ω, following the framework of [49]. In such a paper it has been considered the linear homogenization problem

$$\begin{cases} -\Delta u^\varepsilon = f(x), & x \in \Omega^\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega^\varepsilon, \end{cases} \tag{3.2}$$

with $f \in L^2(\Omega)$ (see also [52], where this homogenization problem is studied in a more general framework). It is well known that (3.2) has a unique solution $u^\varepsilon \in H_0^1(\Omega)$. In [49] the authors showed that, if the holes satisfy certain hypotheses on their size and distribution, and if we denote as $\widetilde{u^\varepsilon}$ the extension of u^ε by zero in $\Omega \setminus \Omega^\varepsilon$, then $\widetilde{u^\varepsilon} \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution to

$$\begin{cases} -\Delta u + \sigma u = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

with σ a positive constant. In fact, this case of σ constant is only a model example, but the hypotheses on the holes imposed in [49] are more general and σ can be proved to be, in the general framework, only a nonnegative finite Radon measure. It is widely remarked the presence of the "strange term" σu (which is the "asymptotic memory of the fact that $\widetilde{u^\varepsilon}$ was zero on the holes") appearing in the limit equation (3.3).

In [49], the authors proved also a corrector result, that is to say, a representation of $\nabla\widetilde{u^\varepsilon}$ in the strong topology of $L^2(\Omega)^N$. They showed that the corrector for the linear homogenization problem depends on the holes, and also depends on the limit u in a linear way.

In [46] the author studied the quasilinear homogenization problem

$$\begin{cases} -\Delta u^\varepsilon + \lambda u^\varepsilon = \gamma|\nabla u^\varepsilon|^2 + f(x), & x \in \Omega^\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega^\varepsilon, \end{cases}$$

where γ is a real constant, $\lambda > 0$ and $f \in L^\infty(\Omega)$. He used a suitable change of unknown function that turns the equation into a semilinear one, a careful analysis of this semilinear homogenization problem allowed the author to pass to the limit as in the linear case. Undoing the change of variables, he proved that the limit u satisfies that

$$\begin{cases} -\Delta u + \lambda u + \dfrac{\sigma(e^{\gamma u}-1)}{\gamma e^{\gamma u}} = \gamma|\nabla u|^2 + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

As in the linear case, a new term appears in the equation that satisfies u, but in this case the new term is nonlinear (σ is the same constant that appears in the linear problem). As the author remarked, this means that the perturbation of the linear problem (3.2) by a nonlinear term, namely $\gamma|\nabla u^\varepsilon|^2$, changes the structure of the new term in the limit equation. This perturbation affects also the corrector corresponding to this problem, since it turns out to be nonlinear in u as well. Similar results were proved in [45] in which the nonlinear perturbation of (3.2) is a general function of the form $H(x,u,\nabla u)$, where H has (at most) natural growth in the gradient.

We remark that in all the previous cases the lower order term is locally bounded with respect to u. Up to our knowledge, the first time it was considered a singular term was in [41]. The authors studied the homogenization of the model problem

$$\begin{cases} -\Delta u^\varepsilon + \dfrac{|\nabla u^\varepsilon|^2}{|u^\varepsilon|^\theta} = f(x), & x \in \Omega^\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega^\varepsilon, \end{cases} \tag{3.4}$$

with $\theta \in (0,1)$ and f a nonnegative datum (positive solutions) in a suitable space of Lebesgue. There, since the lower order term is positive, following [99] and [122], it is easy to prove that $\widetilde{u^\varepsilon}$ is bounded in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$ respectively, thus the main difficulty resides in avoiding the singularity when passing to the limit. Their main result, written here only in the case σ constant, is that for every $f \in L^{\frac{2N}{N+2}}(\Omega)$, $f \gneq 0$, the unique solution u^ε to problem (3.4) satisfies $\widetilde{u^\varepsilon} \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution to problem

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 + \sigma\Psi(u)e^{G(u)} = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

and $G(s) = \int_1^s g(t)dt$, $\Psi(s) = \int_0^s e^{-G(t)}dt$ for every $s > 0$. Thus, the strange term turns out to be again nonlinear in u, as well as the corrector, as it is shown in [41].

The above results describe the general questions we are concerned with. We will prove that, also for our problem (3.1) there is a limit u which is a solution to a new problem. We will show that, unlike in the cases mentioned above, the strange term is linear even if the equation is not, and the Radon measure depends only on the holes. Furthermore, the corrector is also linear. The reason for this unexpected phenomenon to occur is that the lower order term is bounded in $L^1(\Omega)$, so it represents a *mild* perturbation for the linear equation. As for the existence result explained above, the proof of this estimate is not trivial since the functions $\widetilde{u^\varepsilon}$ vanish on the holes, so the usual local lower estimate does not hold true neither in this case. The L^1 estimate will allow us to prove that $\{\widetilde{u^\varepsilon}\}$ converges strongly in $W_0^{1,r}(\Omega)$ for all $r \in [1,2)$, which is essential for passing to the limit in the equation.

The plan of the paper is the following. We collect some preliminary results in the second section. We prove that the problem (P_λ) has solution in a suitable sense in Section 3. We dedicate Section 4 to the homogenization of problem (3.1). In Subsection 4.1 we give the precise assumptions of the perforated domains, following the framework of [49]. In Subsection 4.2 we prove the existence of solution to problem (3.1). We enunciate our homogenization result in Subsection 4.3. In Subsection 4.4 we prove the main tool in order to pass to the limit as ε tends to zero, the L^r-strong convergence of the gradients for $r < 2$. Our homogenization result for the singular quasilinear problem (3.1) is studied in Subsection 4.5 and a corrector result is proved in Subsection 4.6.

3.2 Preliminary results

As we announced, in this paper we will improve some existence results contained in [38]. In order to make a simpler exposition, we will include in this section some concepts and results from [38] that we will need in our proofs.

Recall that in such a paper the authors studied the singular problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^{q-1}} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.5}$$

in the special case in which $M(x)$ is the identity matrix and $f \gneqq 0$. They dealt with the problem by taking advantage of the homogeneous structure of the equation. Thus, they studied first the eigenvalue problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{E_λ}$$

again in the case $M = I$. The authors proved the existence and main properties of the principal eigenvalue, that can be characterized by

$$\lambda^* = \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exists a supersolution } v \text{ to } (E_\lambda) \\ \text{such that } v \geq c \text{ in } \Omega \text{ for some } c > 0 \end{array} \right\}. \tag{3.6}$$

If necessary, we will write $\lambda^*(\Omega)$ to make explicit the dependence on the domain.

Arguing as in [38] without relevant changes, it is possible to prove that, assuming (M_1), then $\lambda^* \in (0, \lambda_1(M)]$ and, if $\partial\Omega$ is smooth enough, problem (E_λ) admits a positive solution if and only if $\lambda = \lambda^*$.

Concerning the smoothness of the domain, we introduce the following definition.

Definition 3.2.1. Let $D \subset \mathbb{R}^N$ be an open set. We say that D satisfies condition (A) if there exist $r_0, \theta_0 > 0$ such that, if $x \in \partial D$ and $0 < r < r_0$, then

$$|D_r| \leq (1 - \theta_0)|B_r(x)|$$

for every connected component D_r of $D \cap B_r(x)$, where $B_r(x)$ denotes the ball centered at x with radius r.

We remark that a sufficient condition for Ω to satisfy condition (A) is that $\partial\Omega$ is Lipschitz (see [11]).

The existence result from [38], adapted to our needs, reads as follows.

Theorem 3.2.2. *Let $1 < q < 2$, $0 \leq \mu \in L^\infty(\Omega)$, $0 \lneq f \in L^p(\Omega)$ with $p > \frac{N}{2}$, and assume that M and Ω satisfy conditions (M_1) and (A) respectively. Then, there exists at least a solution to* (3.5) *for every $\lambda < \lambda^*$, where λ^* is given by* (3.6)*.*

We will also use the following comparison principle proved as in [38].

Theorem 3.2.3. *Let $1 < q < 2$, $\lambda \in \mathbb{R}$, $0 \leq \mu \in L^\infty(\Omega)$ and $0 \leq h \in L^1_{\mathrm{loc}}(\Omega)$. Assume that $u, v \in C(\Omega) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ are such that $u, v > 0$ in Ω and satisfy*

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi \leq \lambda \int_\Omega u\phi + \int_\Omega \mu(x)\frac{|\nabla u|^q}{u^{q-1}}\phi + \int_\Omega h(x)\phi, \tag{3.7}$$

and

$$\int_\Omega M(x)\nabla v\cdot\nabla\phi \geq \lambda\int_\Omega v\phi + \int_\Omega \mu(x)\frac{|\nabla v|^q}{v^{q-1}}\phi + \int_\Omega h(x)\phi, \tag{3.8}$$

for all $0\leq\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ *with compact support. Suppose also that, for every* $\varepsilon>0$*, the following boundary condition holds*

$$\limsup_{x\to x_0}\left(\frac{u(x)}{v(x)+\varepsilon}\right)\leq 1 \quad \forall x_0\in\partial\Omega. \tag{3.9}$$

Furthermore, if $\lambda>0$*, assume also that* $\inf_\Omega(h)>0$*. Then,* $u\leq v$ *in* Ω*.*

Some remarks are now in order.

Remark 3.2.4. Even though the principal operator considered in [38] is the Laplacian, we observe here that a perturbation with a bounded elliptic matrix $M(x)$ satisfying (M_1) does not involve any additional difficulty in the proofs of the previous results. We remark that the fact that the coefficients of $M(x)$ are locally Lipschitz is needed in order to apply elliptic regularity (see [125, Theorem 3.8] and also problem 3.3, p. 202, in that book).

Remark 3.2.5. Originally, in [38] condition (A) is replaced by a more restrictive hypothesis, i.e., it is imposed that $\partial\Omega$ is of class $C^{1,1}$. This last smoothness condition is used only for proving a nonexistence result for $\lambda>\lambda^*$. In our context, we do not expect that a similar nonexistence result holds true because our solutions are not necessarily positive. Hence, condition (A) is enough for our purposes since suffices to prove that the solutions are Hölder continuous up the boundary.

3.3 Existence of solution for the quasilinear problem

In this section we will prove existence of solution to (P_λ) for every $\lambda<\lambda^*$, generalizing thus Theorem 3.2.2 above. As was pointed out at the Introduction, our concept of solution is the following.

Definition 3.3.1. We say that $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a solution to problem (P_λ) if $g(x,u)|\nabla u|^q\in L^1(\{|u|>0\})$ and

$$\int_\Omega M(x)\nabla u\cdot\nabla\phi = \lambda\int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi + \int_\Omega f(x)\phi,$$

for every $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Remark 3.3.2. The following examples show that, when f changes sign, a solution to (P_λ) may vanish in a set of positive measure either in a neighborhood of the boundary or even far away from the boundary. Indeed, standard computations show that, for convenient data f_1 and f_2, the functions

$$u_1(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| \le 1, \\ 0, & 1 < |x| \le 2, \end{cases}$$

and

$$u_2(x) = \begin{cases} 0, & |x| < 1, \\ (|x|-1)^2(2-|x|)^2, & 1 \le |x| < 2, \end{cases}$$

satisfy $-\Delta u_i = \frac{|\nabla u_i|^q}{|u_i|^{q-1}} + f_i(x)$ in $B_2(0)$.

The statement of the main result of this section is as follows.

Theorem 3.3.3. *Assume that Ω satisfies condition (A), $1 < q < 2$, $f \in L^p(\Omega)$ for some $p > \frac{N}{2}$ and conditions (M_1) and (g_1) are satisfied. Then, there exists at least a solution to problem (P_λ) for all $\lambda < \lambda^*$.*

We will find the solution of Theorem 3.3.3 as the limit of a sequence of solutions to nonsingular problems that approximate (P_λ). More precisely, we consider, for every $n \in \mathbb{N}$, the following problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \lambda u_n + g_n(x,u_n)|\nabla u_n|^q + f_n(x), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{Q_n}$$

where

$$g_n(x,s) = \begin{cases} g(x,s), & |s| \ge \dfrac{1}{n}, \\ g(x,s)|s|^q n^q, & 0 < |s| \le \dfrac{1}{n}, \\ 0, & s = 0, \end{cases}$$

and $f_n(x) = \max\{-n, \min\{f(x), n\}\}$. Observe that $g_n : \Omega \times \mathbb{R} \to [0,+\infty)$ is continuous in the second variable and

$$g_n(x,s) \le g(x,s) \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R} \setminus \{0\}.$$

In the next lemma we prove the existence of solution to (Q_n) by means of the subsolution and supersolution method in [30].

Lemma 3.3.4. *Let $1 < q < 2$, $\lambda < \lambda^*$, $f \in L^1(\Omega)$, and assume that conditions (M_1) and (g_1) are satisfied. Then, there exists a solution u_n to problem (Q_n) for all n.*

Proof. Let $\overline{\lambda} \in (\lambda, \lambda^*)$, and let $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ be such that

$$\varphi \geq c > 0 \quad \text{and} \quad -\operatorname{div}(M(x)\nabla\varphi) \geq \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} \quad \text{in } \Omega.$$

For some constant $k > 0$, let $\overline{\psi} = k\varphi$. Then,

$$\begin{aligned}
&\operatorname{div}(M(x)\nabla\overline{\psi}) + \lambda\overline{\psi} + g_n(x,\overline{\psi})|\nabla\overline{\psi}|^q + f_n(x) \\
&\quad \leq k\left(\operatorname{div}(M(x)\nabla\varphi) + \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}\right) + n - (\overline{\lambda} - \lambda)kc \leq 0,
\end{aligned}$$

if k is chosen large enough.

On the other hand, let $\underline{\psi} = -k\varphi$. Then,

$$\begin{aligned}
&\operatorname{div}(M(x)\nabla\underline{\psi}) + \lambda\underline{\psi} + g_n(x,\underline{\psi})|\nabla\underline{\psi}|^q + f_n(x) \\
&\quad \geq \operatorname{div}(M(x)\nabla\underline{\psi}) + \lambda\underline{\psi} - k\mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}} + f_n(x) \\
&\quad \geq -k\left(\operatorname{div}(M(x)\nabla\varphi) + \overline{\lambda}\varphi + \mu(x)\frac{|\nabla\varphi|^q}{\varphi^{q-1}}\right) + (\overline{\lambda} - \lambda)kc - n \geq 0.
\end{aligned}$$

Obviously, $\underline{\psi} \leq 0 \leq \overline{\psi}$ in $\overline{\Omega}$. Therefore, by virtue of [30, Théorème 3.1], there exists a solution u_n to problem (Q_n) such that $\underline{\psi} \leq u_n \leq \overline{\psi}$. □

In the following lemma we prove the a priori estimates and the compactness needed for passing to the limit.

Lemma 3.3.5. *Let $1 < q < 2$, $\lambda \in \mathbb{R}$, $f \in L^1(\Omega)$ and assume that conditions (M_1) and (g_1) are satisfied. Assume also that $\{u_n\}$ is a sequence of solutions to problem (Q_n) bounded in $L^\infty(\Omega)$. Then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that*

i) up to a subsequence, $u_n \to u$ strongly in $L^r(\Omega)$, $r \in [1,\infty)$,

ii) $\{|u_n|^\alpha\}$ is bounded in $H_0^1(\Omega)$ for all $\alpha > \frac{1}{2}$,

iii) up to a subsequence, $u_n \to u$ strongly in $H_0^1(\Omega)$.

Remark 3.3.6. Observe that, for any $u \in W^{1,1}_{\text{loc}}(\Omega)$ and any $\delta > 0$, the chain rule for weak derivatives implies that

$$\frac{|\nabla u|^q}{|u|^{q-1}} = q^q \left|\nabla|u|^{\frac{1}{q}}\right|^q$$

in the set $\{|u| > \delta\}$. In particular, the equality holds in $\{|u| > 0\} = \bigcap_{\delta>0}\{|u| > \delta\}$. Thus, by (g_1),

$$g(x,u)|\nabla u|^q \leq \mu(x)q^q \left|\nabla |u|^{\frac{1}{q}}\right|^q$$

in the set $\{|u| > 0\}$. Therefore, if $|u|^{\frac{1}{q}} \in W^{1,q}(\Omega)$ (which is precisely a consequence of Lemma 3.3.5) one has that $g(x,u)|\nabla u|^q \in L^1(\{|u| > 0\})$.

Proof of Lemma 3.3.5. Let us take u_n as test function in (Q_n). Then, using the $L^\infty(\Omega)$ bound we immediately obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, passing to a subsequence, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a.e. in Ω. Furthermore, again the $L^\infty(\Omega)$ estimate clearly implies that $u \in L^\infty(\Omega)$ and also that $u_n \to u$ strongly in $L^r(\Omega)$ for all $r \in [1,\infty)$. This completes the proof of item i).

Now we deal with the proof of item ii) which is straightforward, using the $L^\infty(\Omega)$ estimate, in the case $\alpha \geq 1$. When $\frac{1}{2} < \alpha < 1$ we take $\beta = 2\alpha - 1 \in (0,1)$ and we first prove a uniform bound in $L^1(\Omega)$ for $\frac{|\nabla u_n|^2}{(|u_n|+\delta)^{1-\beta}}$ with $n \in \mathbb{N}$, $\delta \in (0,1)$. Then, passing to the limit as $\delta \to 0$ we show that $|u_n|^{\frac{\beta+1}{2}} = |u_n|^\alpha$ is bounded in $H_0^1(\Omega)$.

It is clear that $v_{n,\delta} = (-u_n^- + \delta)^\beta - \delta^\beta \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where $u_n^- = \min\{u_n, 0\}$. Therefore, it can be taken as test function in (Q_n), and using (M_1), the $L^\infty(\Omega)$ bound and the fact that g_n is nonnegative, we obtain that

$$\begin{aligned}
-\beta\eta \int_{\{u_n \leq 0\}} \frac{|\nabla u_n|^2}{(-u_n+\delta)^{1-\beta}} &\geq \int_\Omega M(x)\nabla u_n \cdot \nabla v_{n,\delta} \\
&= \lambda \int_\Omega u_n v_{n,\delta} + \int_\Omega g_n(x,u_n)|\nabla u_n|^q v_{n,\delta} + \int_\Omega f_n(x) v_{n,\delta} \\
&\geq \lambda \int_\Omega u_n v_{n,\delta} + \int_\Omega f_n(x) v_{n,\delta} \geq -C.
\end{aligned} \tag{3.10}$$

On the other hand, $w_{n,\delta} = (u_n^+ + \delta)^\beta - \delta^\beta \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so we can use it as test function in (Q_n). Hence, using the $L^\infty(\Omega)$ bound and Young inequality conveniently we deduce that

$$\begin{aligned}
&\beta\eta \int_{\{u_n > 0\}} \frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}} \leq \int_\Omega M(x)\nabla u_n \cdot \nabla w_{n,\delta} \\
&= \lambda \int_\Omega u_n w_{n,\delta} + \int_\Omega g_n(x,u_n)|\nabla u_n|^q w_{n,\delta} + \int_\Omega f_n(x) w_{n,\delta} \\
&\leq C + \int_{\{u_n > 0\}} \mu(x) \frac{|\nabla u_n|^q}{u_n^{q-1}} [(u_n+\delta)^\beta - \delta^\beta] \\
&\leq C + \frac{\beta\eta}{2} \int_{\{u_n > 0\}} \frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}} + C\int_\Omega \left[\frac{(u_n+\delta)^\beta - \delta^\beta}{u_n^{q-1}} (u_n+\delta)^{(1-\beta)\frac{q}{2}}\right]^{\frac{2}{2-q}}.
\end{aligned}$$

It is straightforward to prove that

$$(s,t) \mapsto \left[\frac{(s+t)^\beta - t^\beta}{s^{q-1}} (s+t)^{(1-\beta)\frac{q}{2}} \right]^{\frac{2}{2-q}}, \ (0,t) \mapsto 0$$

is a continuous function in $[0,B] \times [0,1]$ for any $B > 0$. Thus, choosing $B > 0$ such that $\|u_n\|_{L^\infty(\Omega)} \leq B$ for all n, we deduce that

$$\int_{\{u_n>0\}} \frac{|\nabla u_n|^2}{(u_n+\delta)^{1-\beta}} \leq C, \tag{3.11}$$

where $C > 0$ is independent of n and δ.

From (3.10) and (3.11) we conclude that

$$\int_\Omega \frac{|\nabla u_n|^2}{(|u_n|+\delta)^{1-\beta}} \leq C.$$

In other words,

$$\frac{4}{(1+\beta)^2} \int_\Omega \left| \nabla \left[(|u_n|+\delta)^{\frac{\beta+1}{2}} - \delta^{\frac{\beta+1}{2}} \right] \right|^2 \leq C. \tag{3.12}$$

Denoting $z_{n,\delta} = (|u_n|+\delta)^{\frac{\beta+1}{2}} - \delta^{\frac{\beta+1}{2}}$, we have proved that there exists $C > 0$ such that $\|z_{n,\delta}\|_{H_0^1(\Omega)} \leq C$ for all $\delta > 0$. Hence, there exists $z_n \in H_0^1(\Omega)$ such that, passing to a subsequence, $z_{n,\delta} \rightharpoonup z_n$ weakly in $H_0^1(\Omega)$ as $\delta \to 0$. On the other hand, $z_{n,\delta} \to |u_n|^{\frac{\beta+1}{2}}$ a.e. in Ω as $\delta \to 0$. This implies that $z_n = |u_n|^{\frac{\beta+1}{2}}$, so $|u_n|^{\frac{\beta+1}{2}} \in H_0^1(\Omega)$.

Since the $H_0^1(\Omega)$ norm is weakly lower semicontinuous, the inequality (3.12) yields

$$\int_\Omega |\nabla |u_n|^\alpha|^2 = \int_\Omega \left| \nabla |u_n|^{\frac{1+\beta}{2}} \right|^2 \leq C,$$

for $C > 0$ independent of n. This concludes the proof of item ii).

Regarding item iii) let us take $u_n - u$ as test function in (Q_n). We obtain that

$$\begin{aligned} \int_\Omega M(x)\nabla u_n \cdot \nabla(u_n - u) &= \lambda \int_\Omega u_n(u_n-u) \\ &\quad + \int_\Omega g_n(x,u_n)|\nabla u_n|^q (u_n-u) + \int_\Omega f_n(x)(u_n-u). \end{aligned}$$

It is clear that the first and the third terms on the right hand side of the last equality converge to zero as n tends to infinity. Concerning the nonlinear term, observe first that we can use item ii) with $\alpha = 1/q$ so that $\{|u_n|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)$. Hence, using

Remark 3.3.6 together with the facts that $g_n(x,0)=0$ and $g_n(x,s)\le g(x,s)$, we deduce that

$$\begin{aligned}\left|\int_\Omega g_n(x,u_n)|\nabla u_n|^q(u_n-u)\right| &\le C\int_\Omega \left|\nabla |u_n|^{\frac{1}{q}}\right|^q |u_n-u|\\ &\le C\left(\int_\Omega \left|\nabla |u_n|^{\frac{1}{q}}\right|^2\right)^{\frac{q}{2}}\left(\int_\Omega |u_n-u|^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}}\\ &\le C\left(\int_\Omega |u_n-u|^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}}.\end{aligned}$$

This sequence converges to zero, using item i) with $r=2/(2-q)$. Thus it is clear now that

$$\int_\Omega M(x)\nabla u_n\cdot\nabla(u_n-u)\to 0.$$

Therefore, the weak convergence in $H_0^1(\Omega)$ yields

$$\begin{aligned}\eta\int_\Omega |\nabla(u_n-u)|^2 &\le \int_\Omega M(x)\nabla(u_n-u)\cdot\nabla(u_n-u)\\ &=\int_\Omega M(x)\nabla u_n\cdot\nabla(u_n-u)-\int_\Omega M(x)\nabla u\cdot\nabla(u_n-u)\to 0,\end{aligned}$$

finishing the proof of item iii). □

Proof of Theorem 3.3.3. Let $\{u_n\}$ be the sequence of solutions to problems (Q_n) given by Lemma 3.3.4, i.e. given $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega M(x)\nabla u_n\cdot\nabla\phi=\lambda\int_\Omega u_n\phi+\int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi+\int_\Omega f_n(x)\phi. \tag{3.13}$$

We will obtain a solution to problem (P_λ) as a limit of this sequence. We divide the proof into two steps, in the first one we prove a uniform $L^\infty(\Omega)$ estimate which allows us to take limits easily in all the terms of the previous equality, except the nonlinear one which will be treated in the second step.

First of all observe that we can argue as in [94, Theorem 1.1] at Section 4 (p. 249-251) to deduce, thanks to condition (A), that $u_n\in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$ (see also [38, Appendix]).

Step 1. $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and pass to the limit in some terms of (3.13).

In order to find a uniform upper bound on u_n we observe that it is immediately deduced if the open set

$$\omega_n=\{x\in\Omega : u_n(x)>0\}$$

is empty. Assuming that ω_n is not empty, u_n satisfies that

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) = \lambda u_n + g_n(x,u_n)|\nabla u_n|^q + f_n(x), & x \in \omega_n, \\ u_n > 0, & x \in \omega_n, \\ u_n = 0, & x \in \partial\omega_n. \end{cases}$$

Now, since $u_n \in C^{0,\alpha}(\omega_n)$, then we deduce that $u_n \in W^{1,N}_{\mathrm{loc}}(\omega_n)$ arguing as in [38, Lemma 2.4] and using condition (M_1) for the elliptic regularity (see [125, Theorem 3.8] and also problem 3.3, p. 202, in that book). Moreover, u_n is a subsolution to the following problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla \zeta) = \lambda \zeta + \mu(x)\dfrac{|\nabla \zeta|^q}{\zeta^{q-1}} + |f(x)| + 1, & x \in \omega_n, \\ \zeta > 0, & x \in \omega_n, \\ \zeta = 0, & x \in \partial\omega_n, \end{cases} \tag{3.14}$$

in the sense that $0 < u_n \in C(\omega_n) \cap W^{1,N}_{\mathrm{loc}}(\omega_n)$ satisfies (3.7) with $h = |f| + 1$.

On the other hand, Theorem 3.2.2 implies that there exists a solution v to

$$\begin{cases} -\mathrm{div}(M(x)\nabla v) = \lambda v + \mu(x)\dfrac{|\nabla v|^q}{v^{q-1}} + |f(x)| + 1, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Moreover, $v \in C(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ reasoning as before. Then, v is a supersolution to (3.14) in the sense it satisfies (3.8) with $h = |f| + 1$. Furthermore, condition (3.9) is clearly satisfied on $\partial\omega_n$.

Therefore, applying Theorem 3.2.3 we deduce that

$$u_n \le v \le \|v\|_{L^\infty(\Omega)} \quad \text{in } \omega_n.$$

Thus, $u_n \le \|v\|_{L^\infty(\Omega)}$ in Ω, and this is a uniform upper bound on u_n.

A similar (actually simpler) argument by comparison provides us an analogue lower bound. Indeed, as was pointed out in Section 2, we know that $\lambda < \lambda^* \le \lambda_1(M)$, and we have from the maximum principle that $u_n \ge z$ with $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $-\mathrm{div}(M(x)\nabla z) = \lambda z - |f(x)|$ in Ω. In conclusion, $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and the proof of Step 1 is concluded.

Our aim now is to use this a priori estimate to pass to the limit in (Q_n). In order to do so, recall that Lemma 3.3.5 implies that there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u_n \to u$ strongly in $H_0^1(\Omega)$ and in $L^r(\Omega)$ for all $r \in [1,\infty)$. Hence, given $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega g_n(x,u_n)|\nabla u_n|^q \phi = \lim_{n\to\infty} \bigg(\int_\Omega M(x)\nabla u_n \cdot \nabla \phi - \lambda \int_\Omega u_n \phi \\ - \int_\Omega f_n(x)\phi \bigg) = \int_\Omega M(x)\nabla u \cdot \nabla \phi - \lambda \int_\Omega u\phi - \int_\Omega f(x)\phi.$$

Step 2. $\displaystyle\lim_{n\to\infty} \int_\Omega g_n(x,u_n)|\nabla u_n|^q \phi = \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi.$

Let us fix a decreasing sequence (to be specified later) $\{\delta_m\}$ of positive real numbers such that $\delta_m \to 0$. We have that

$$\int_\Omega g_n(x,u_n)|\nabla u_n|^q \phi = \int_{\{|u_n|>\delta_m\}} g_n(x,u_n)|\nabla u_n|^q \phi \\ + \int_{\{|u_n|\le\delta_m\}} g_n(x,u_n)|\nabla u_n|^q \phi. \tag{3.15}$$

We will pass to the limit in both terms, first with respect to n, and after that with respect to m.

Concerning the first term, we know that there exists $h \in L^1(\Omega)$ such that, passing to a subsequence, $|\nabla u_n|^q \le h$ in Ω, for all n. Then,

$$|g_n(x,u_n)|\nabla u_n|^q \phi \chi_{\{|u_n|>\delta_m\}}| \le \frac{Ch}{\delta_m^{q-1}} \in L^1(\Omega),$$

so we have domination.

In order to prove the almost everywhere convergence, consider the set

$$\mathcal{N} = \{\delta \ge 0 : |\{x \in \Omega : |u(x)| = \delta\}| > 0\}.$$

It is well known that $\mathcal{N}$ is countable, so the sequence $\{\delta_m\}$ can be chosen in $\mathbb{R} \setminus \mathcal{N}$. Thus, since $u_n \to u$ a.e. in Ω, it is straightforward to check that $\chi_{\{|u_n|>\delta_m\}} \to \chi_{\{|u|>\delta_m\}}$ a.e. in Ω as $n \to \infty$.

On the other hand, if $|u_n| > \delta_m$, we can take n large enough such that $|u_n| \ge \frac{1}{n}$, so $g_n(x,u_n) = g(x,u_n)$. Hence, using the continuity of $g(x,\cdot)$ in the set $(-\infty,-\delta_m] \cup [\delta_m,\infty)$ and the fact that $u_n \to u$ a.e. in Ω, we deduce that

$$g_n(x,u_n) \to g(x,u) \text{ a.e. } x \in \Omega \text{ as } n \to \infty.$$

In sum, using also that $\nabla u_n \to \nabla u$ a.e. in Ω, we obtain that

$$g_n(x,u_n)|\nabla u_n|^q \phi \chi_{\{|u_n|>\delta_m\}} \to g(x,u)|\nabla u|^q \phi \chi_{\{|u|>\delta_m\}} \text{ a.e. } x \in \Omega \text{ as } n \to \infty.$$

Therefore, the Dominated Convergence Theorem implies that

$$\int_{\{|u_n|>\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi \to \int_{\{|u|>\delta_m\}} g(x,u)|\nabla u|^q\phi \quad \text{as } n\to\infty.$$

In order to pass to the limit with respect to m we recall that Lemma 3.3.5 gives also that $\{|u_n|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)$. This, in particular, implies that $|u|^{\frac{1}{q}}\in H_0^1(\Omega)$. Now, taking into account Remark 3.3.6, we have a uniform domination with respect to m:

$$\begin{aligned}\left|g(x,u)|\nabla u|^q\phi\chi_{\{|u|>\delta_m\}}\right| \leq& C\frac{|\nabla u|^q}{|u|^{q-1}}\chi_{\{|u|>0\}}\\ =&C|\nabla |u|^{\frac{1}{q}}|^q\in L^1(\{|u|>0\}),\end{aligned}$$

and also almost everywhere convergence

$$g(x,u)|\nabla u|^q\phi\chi_{\{|u|>\delta_m\}}\to g(x,u)|\nabla u|^q\phi \quad \text{a.e. in } \{|u|>0\}.$$

Therefore, the Dominated Convergence Theorem yields

$$\int_{\{|u|>\delta_m\}} g(x,u)|\nabla u|^q\phi \to \int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi.$$

To conclude the proof, we will show that the last term in (3.15) vanishes as n and m tend to infinity. Notice first that such a term has a limit with respect to n (because the remaining two terms do). Furthermore, by virtue of Lemma 3.3.5 we derive, taking $\varepsilon>0$ with $\frac{1-\varepsilon}{q}>\frac{1}{2}$, that

$$\begin{aligned}&\left|\int_{\{|u_n|\leq\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi\right| \leq C\int_{\{0<|u_n|\leq\delta_m\}}\frac{|\nabla u_n|^q}{|u_n|^{q-1}}\\ &=C\int_{\{0<|u_n|\leq\delta_m\}}\frac{|\nabla u_n|^q}{|u_n|^{q+\varepsilon-1}}|u_n|^\varepsilon = C\int_{\{0<|u_n|\leq\delta_m\}}\left(\frac{|\nabla u_n|}{|u_n|^{1+\frac{\varepsilon-1}{q}}}\right)^q|u_n|^\varepsilon\\ &\leq C\delta_m^\varepsilon\int_\Omega\left|\nabla|u_n|^{\frac{1-\varepsilon}{q}}\right|^q\leq C\delta_m^\varepsilon\left(\int_\Omega\left|\nabla|u_n|^{\frac{1-\varepsilon}{q}}\right|^2\right)^{\frac{q}{2}}\leq C\delta_m^\varepsilon.\end{aligned}$$

In consequence,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\int_{\{|u_n|\leq\delta_m\}} g_n(x,u_n)|\nabla u_n|^q\phi\right)=0.$$

In conclusion, we have proved that

$$\int_\Omega g_n(x,u_n)|\nabla u_n|^q\phi\to\int_{\{|u|>0\}} g(x,u)|\nabla u|^q\phi.$$

That is to say, u is a solution to (P_λ). □

3.4 Homogenization of problem (3.1)

The existence result of the previous section allows us to consider the homogenization problem associated to (3.1).

3.4.1 The perforated domains

In this subsection, following [49], we describe the geometry of the domains Ω^ε in which we study our homogenization result.

Consider for every $\varepsilon > 0$ a finite number, $n(\varepsilon) \in \mathbb{N}$, of closed subsets $T_i^\varepsilon \subset \mathbb{R}^N$, $1 \leq i \leq n(\varepsilon)$, which are the holes. Let us denote $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon = \Omega \cap D^\varepsilon.$$

Hypotheses on the holes.

We suppose that the sequence of domains Ω^ε is such that there exist a sequence of functions $\{w^\varepsilon\}$ and $\sigma \in H^{-1}(\Omega)$ such that

$$w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega), \tag{3.16}$$

$$0 \leq w^\varepsilon \leq 1 \text{ a.e. } x \in \Omega, \tag{3.17}$$

$$w^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)\ \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.18}$$

$$w^\varepsilon \rightharpoonup 1 \text{ weakly in } H^1(\Omega), \tag{3.19}$$

and given $z^\varepsilon, \phi, z \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $z^\varepsilon \phi \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ and $z^\varepsilon \rightharpoonup z$ weakly in $H^1(\Omega)$ it is satisfied that

$$\int_\Omega M(x)^T \nabla w^\varepsilon \cdot \nabla(z^\varepsilon \phi) \to \langle \sigma, z\phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{3.20}$$

The model example for Ω^ε

The prototype of the examples where assumptions (3.16), (3.17), (3.18), (3.19) and (3.20) are satisfied is the case where the matrix $M(x)$ is the identity (and where therefore the operator is the Laplace's operator $-div(M(x)\nabla) = -\Delta$), where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and where the holes T_i^ε are balls of radius r^ε given by

$$\begin{cases} r^\varepsilon = C_0 \varepsilon^{N/(N-2)} \text{ if } N \geq 3, \\ \varepsilon^2 \log r^\varepsilon \to -C_0 \text{ if } N = 2, \end{cases}$$

for some $C_0 > 0$ (taking $r^\varepsilon = \exp(-C_0/\varepsilon^2)$ is the model case for $N = 2$) which are periodically distributed at the vertices of an N-dimensional lattice of cubes of size 2ε; in this case the measure σ is given by

$$\begin{cases} \sigma = \dfrac{S_{N-1}(N-2)}{2^N} C_0^{N-2} & \text{if } N \geq 3, \\ \sigma = \dfrac{2\pi}{4} \dfrac{1}{C_0} & \text{if } N = 2, \end{cases}$$

where S_{N-1} is the surface of the unit sphere in $\mathbb{R}^{N-1}$, see e.g. [49, 106] for more details, and for other examples, in particular for the case where the holes have a different shape and/or are distributed on a manifold.

Remark 3.4.1. In dimension $N = 1$, there is no sequence w^ε which satisfies (3.18) and (3.19) whenever for every ε there exists at least one hole $T^\varepsilon_{i_\varepsilon}$ with $T^\varepsilon_{i_\varepsilon} \cap \overline{\Omega} \neq \emptyset$, see Remark 5.1 of [74] for more details. □

3.4.2 Existence of solution to problem (3.1)

We study in this subsection the existence of solution to problem (3.1) in order to deal with the homogenization result.

Proposition 3.4.2. *Let* $1 < q < 2$, $\lambda < \lambda^*(\Omega)$, $f \in L^p(\Omega)$ *with* $p > \frac{N}{2}$ *and assume that conditions* (M_1) *and* (g_1) *are satisfied. Consider the open set* $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$ *for all* $\varepsilon > 0$. *If both* Ω *and* D^ε *satisfy condition (A) from Definition 3.2.1, then there exists a solution to* (3.1) *for all* $\varepsilon > 0$.

Proof. Let us fix $\varepsilon > 0$. First of all notice that $\lambda < \lambda^*(\Omega) \leq \lambda^*(\Omega^\varepsilon)$. Thus, if Ω^ε satisfies condition (A), then this result is a mere consequence of Theorem 3.3.3. Let us show that, in fact, Ω^ε has the required regularity. Let $r_0, \theta_0 > 0$ be small enough so that they correspond to condition (A) for both Ω and D^ε. Fix $x \in \partial\Omega^\varepsilon$, and assume first that $x \in \partial\Omega \cap \partial\Omega^\varepsilon$. For $0 < r < r_0$, let Ω_r^ε be any connected component of $\Omega^\varepsilon \cap B_r(x)$, and let Ω_r be the connected component of $\Omega \cap B_r(x)$ which contains Ω_r^ε. Then,

$$|\Omega_r^\varepsilon| \leq |\Omega_r| \leq (1-\theta_0)|B_r|.$$

The same idea is valid if $x \in \partial D^\varepsilon \cap \partial\Omega^\varepsilon$. Therefore, Ω^ε satisfies condition (A) with parameters r_0, θ_0, and the proof is concluded. □

3.4.3 The homogenization result

Now, we can state our homogenization result.

Theorem 3.4.3. *Assume that the sequence of perforated domains Ω^ε satisfies* (3.16)*,* (3.17)*,* (3.18)*,* (3.19) *and* (3.20)*. Suppose also that conditions* (M_1) *and* (g_1) *are satisfied for* $1 < q < 2$*, that* $f \in L^p(\Omega)$ *for some* $p > \frac{N}{2}$*, that* $\lambda < \lambda^*(\Omega)$ *and that both* Ω *and* D^ε *satisfy condition (A), where* $D^\varepsilon = \mathbb{R}^N \setminus \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon$*. Then, there exists a sequence* $\{u^\varepsilon\}$ *of solutions to problem* (3.1) *such that* $\{\widetilde{u^\varepsilon}\}$ *is bounded in* $L^\infty(\Omega)$ *and* $\widetilde{u^\varepsilon} \rightharpoonup u$ *weakly in* $H_0^1(\Omega)$*, being* u *a solution to*

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + \sigma u = \lambda u + g(x,u)|\nabla u|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.21}$$

in the sense that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$*,* $g(x,u)|\nabla u|^q \in L^1(\{|u| > 0\})$ *and*

$$\int_\Omega M(x)\nabla u \cdot \nabla \phi + \langle \sigma, u\phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \lambda \int_\Omega u\phi + \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi + \int_\Omega f(x)\phi \tag{3.22}$$

for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3.4.4. Under the hypotheses of Theorem 3.4.3, assume also that f satisfies that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 : \quad f \geq c_\omega \text{ in } \omega.$$

Then, it is easy to prove that every solution u to problem (3.21) satisfies that $u \geq 0$. If we further assume that the holes are "good enough" so that $\sigma \in L^r(\Omega)$ for some $r > \frac{N}{2}$, then the strong maximum principle holds (see [126, Corollary 5.1]), so that $u > 0$. With this hypothesis, it can also be proven, following the arguments in [38], that $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{\text{loc}}(\Omega)$. Having this regularity and the strict positivity, the proof of the comparison principle [38, Theorem 3.2] can be reproduced with no relevant changes. In conclusion, we have uniqueness of solution to problem (3.21) for a right choice of the holes. However, if σ is a general measure, the strong maximum principle does not hold in general. In [75] the authors have given two counterexamples which prove it. Hence, the uniqueness of solution to problem (3.21) is still open in the general case.

3.4.4 Strong convergence of the gradients

In the present subsection we prove some properties of the solutions u^ε to the problems (3.1) which allow us to prove our homogenization result Theorem 3.4.3 and a corrector result Theorem 3.4.6.

Lemma 3.4.5. *Let $1 < q < 2$, assume that conditions (M_1) and (g_1) are satisfied, and that $f \in L^1(\Omega)$. Let $\{\Omega^\varepsilon\}$ be any sequence of domains such that $\Omega^\varepsilon \subset \Omega$ for all $\varepsilon > 0$, and let $\{u^\varepsilon\}$ be a sequence of solutions to problem* (3.1) *with $\{\widetilde{u^\varepsilon}\}$ bounded in $L^\infty(\Omega)$. Then, $\{|\widetilde{u^\varepsilon}|^\alpha\}$ is bounded in $H_0^1(\Omega)$ for all $\alpha > \frac{1}{2}$. Moreover, there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, passing to a subsequence, $\widetilde{u^\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\widetilde{u^\varepsilon} \to u$ strongly in $W_0^{1,r}(\Omega) \cap L^s(\Omega)$ for all $r \in [1,2)$ and $s \in [1,\infty)$.*

Proof. The proof of this result is analogous to the one of Lemma 3.3.5, except for the strong convergence in $W_0^{1,r}(\Omega)$. In this case, it is not possible to take $u^\varepsilon - u$ as test function in (3.1) since in general $u^\varepsilon - u \notin H_0^1(\Omega^\varepsilon)$. Thus, the proof for the strong convergence in $H_0^1(\Omega)$ does not work here. However, [47, Lemma 4.8] can be applied to obtain that $\widetilde{u^\varepsilon} \to u$ strongly in $W_0^{1,r}(\Omega)$ for all $r \in [1,2)$. Since the proof of this fact is simple in our context, we include it here for completeness.

Indeed, for given $\delta > 0$ observe that

$$
\begin{aligned}
\int_\Omega |\nabla(\widetilde{u^\varepsilon} - u)|^r &= \int_{\{|\tilde{u^\varepsilon} - u| \geq \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^r + \int_{\{|\tilde{u^\varepsilon} - u| < \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^r \\
&\leq |\{|\widetilde{u^\varepsilon} - u| \geq \delta\}|^{1-\frac{r}{2}} \left(\int_\Omega |\nabla(\widetilde{u^\varepsilon} - u)|^2\right)^{\frac{r}{2}} \\
&\quad + |\Omega|^{1-\frac{r}{2}} \left(\int_{\{|\tilde{u^\varepsilon} - u| < \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^2\right)^{\frac{r}{2}}.
\end{aligned}
\tag{3.23}
$$

Clearly, $|\{|\widetilde{u^\varepsilon} - u| \geq \delta\}| \to 0$ as $\varepsilon \to 0$, and $\int_\Omega |\nabla(\widetilde{u^\varepsilon} - u)|^2$ is bounded uniformly in ε. Hence, the first term of (3.23) vanishes as $\varepsilon \to 0$. Let us focus on the second term.

Let us define $f^\varepsilon : \Omega \to \mathbb{R}$ by $f^\varepsilon(x) = \lambda u^\varepsilon + g(x, u^\varepsilon)|\nabla u^\varepsilon|^q + f(x)$ if $x \in \{|u^\varepsilon| > 0\}$ and $f^\varepsilon(x) = f(x)$ otherwise. Consider also $T_\delta(t) = \max\{-\delta, \min\{\delta, t\}\}$ for all $t \in \mathbb{R}$, $\delta > 0$.

Using the function $T_\delta(\widetilde{u^\varepsilon} - u) + T_\delta(u) \in H_0^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ as test in the weak formulation of (3.1), we obtain that

$$
\begin{aligned}
\int_\Omega f^\varepsilon(x)(T_\delta(\widetilde{u^\varepsilon} - u) + T_\delta(u)) &= \int_\Omega M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla(T_\delta(\widetilde{u^\varepsilon} - u) + T_\delta(u)) \\
&\geq \eta \int_{\{|\tilde{u^\varepsilon} - u| < \delta\}} |\nabla(\widetilde{u^\varepsilon} - u)|^2 + \int_{\{|\tilde{u^\varepsilon} - u| < \delta\}} M(x)\nabla u \cdot \nabla(\widetilde{u^\varepsilon} - u) \\
&\quad + \int_{\{|u| < \delta\}} M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla u.
\end{aligned}
$$

And this implies that

$$C\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}}|\nabla(\widetilde{u^\varepsilon}-u)|^2\le\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|$$
$$+\int_{\{|u|<\delta\}}|\nabla\widetilde{u^\varepsilon}||\nabla u|+\left|\int_\Omega M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-u)\right| \tag{3.24}$$

for some constant $C>0$ dependent on η and M but independent of ε.

On the one hand, since $\{|\widetilde{u^\varepsilon}|^{\frac{1}{q}}\}$ is bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$, we have that $\{f^\varepsilon\}$ is bounded in $L^1(\Omega)$. Then, we deduce that

$$\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|\le C\delta,$$

for another constant $C>0$ independent of ε. Therefore,

$$\lim_{\delta\to0}\left(\limsup_{\varepsilon\to0}\int_\Omega|f^\varepsilon(x)||T_\delta(\widetilde{u^\varepsilon}-u)+T_\delta(u)|\right)=0. \tag{3.25}$$

On the other hand, since $\{\widetilde{u^\varepsilon}\}$ is bounded in $H_0^1(\Omega)$, we have that

$$\int_{\{|u|<\delta\}}|\nabla\widetilde{u^\varepsilon}||\nabla u|\le\|\widetilde{u^\varepsilon}\|_{H_0^1(\Omega)}\left(\int_{\{|u|<\delta\}}|\nabla u|^2\right)^{\frac12}$$
$$\le C\left(\int_\Omega|\nabla u|^2\chi_{\{|u|<\delta\}}\right)^{\frac12},$$

again for a constant $C>0$ independent of ε. Thus,

$$\lim_{\delta\to0}\left(\limsup_{\varepsilon\to0}\int_{\{|u|<\delta\}}|\nabla\widetilde{u^\varepsilon}||\nabla u|\right)=0. \tag{3.26}$$

Finally, the weak convergence in $H_0^1(\Omega)$ yields to

$$\left|\int_\Omega M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon}-u)\right|\to0\quad\text{as }\varepsilon\to0. \tag{3.27}$$

In conclusion, from (3.24), (3.25), (3.26) and (3.27), we deduce that

$$\lim_{\delta\to0}\left(\limsup_{\varepsilon\to0}\int_{\{|\widetilde{u^\varepsilon}-u|<\delta\}}|\nabla(\widetilde{u^\varepsilon}-u)|^2\right)=0,$$

and the proof finishes by applying this last convergence to (3.23). $\square$

3.4.5 Proof of Theorem 3.4.3

We dedicate this subsection to proving Theorem 3.4.3 in two steps.

Step 1. $\{\widetilde{u^\varepsilon}\}$ is bounded in $L^\infty(\Omega)$.

Let $\{u^\varepsilon\}$ be the sequence of solutions to (3.1) given by Proposition 3.4.2. One can easily follow the arguments by comparison in the proof of Theorem 3.3.3 to deduce that $\{\widetilde{u^\varepsilon}\}$ is bounded in $L^\infty(\Omega)$. Therefore, Lemma 3.4.5 implies the existence of a function $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that $\widetilde{u^\varepsilon} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\widetilde{u^\varepsilon} \to u$ strongly in $W_0^{1,r}(\Omega)\cap L^s(\Omega)$ for all $r\in[1,2)$ and $s\in[1,\infty)$.

Step 2. $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is solution of (3.21) and $g(x,u)|\nabla u|^q \in L^1(\{|u|>0\})$.

The idea is to take $w^\varepsilon\phi$ as test function in (3.1) for some $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, and then pass to the limit as $\varepsilon\to 0$ (observe that, thanks to (3.18) $w^\varepsilon\phi \in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon)$ for every $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$).

Taking $w^\varepsilon\phi$ as test function in (3.1) we have, using (3.16),

$$\begin{aligned}&\int_{\Omega^\varepsilon}(M(x)\nabla u^\varepsilon\cdot\nabla\phi)w^\varepsilon+\int_{\Omega^\varepsilon}(M(x)\nabla u^\varepsilon\cdot\nabla w^\varepsilon)\phi\\ &\qquad=\lambda\int_{\Omega^\varepsilon}u^\varepsilon w^\varepsilon\phi+\int_{\{|u^\varepsilon|>0\}}g(x,u^\varepsilon)|\nabla u^\varepsilon|^q w^\varepsilon\phi+\int_{\Omega^\varepsilon}f(x)w^\varepsilon\phi\end{aligned}$$

or equivalently

$$\begin{aligned}&\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla\phi)w^\varepsilon+\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla w^\varepsilon)\phi\\ &\qquad=\lambda\int_{\Omega}\widetilde{u^\varepsilon}w^\varepsilon\phi+\int_{\{|\widetilde{u^\varepsilon}|>0\}}g(x,\widetilde{u^\varepsilon})|\nabla\widetilde{u^\varepsilon}|^q w^\varepsilon\phi+\int_{\Omega}f(x)w^\varepsilon\phi.\end{aligned}\tag{3.28}$$

Now we pass to the limit as $\varepsilon\to 0$ in each term of the previous equality. For the first term of the left hand side we use (3.19) and that $\widetilde{u^\varepsilon}\rightharpoonup u$ weakly in $H_0^1(\Omega)$ to obtain that

$$\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla\phi)w^\varepsilon\to\int_{\Omega}M(x)\nabla u\cdot\nabla\phi.$$

In order to pass to the limit as $\varepsilon\to 0$ in the second term of the left hand side of (3.28) we use (3.18), (3.19) and (3.20) and we get

$$\begin{aligned}\int_{\Omega}(M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla w^\varepsilon)\phi&=\int_{\Omega}M(x)\nabla(\widetilde{u^\varepsilon}\phi)\cdot\nabla w^\varepsilon\\ -\int_{\Omega}(M(x)\nabla\phi\cdot\nabla w^\varepsilon)\widetilde{u^\varepsilon}&=\int_{\Omega}M(x)^T\nabla w^\varepsilon\cdot\nabla(\widetilde{u^\varepsilon}\phi)\\ -\int_{\Omega}(M(x)\nabla\phi\cdot\nabla w^\varepsilon)\widetilde{u^\varepsilon}&\to\langle\sigma,u\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}$$

With respect to the first and third terms of the right hand side of (3.28), we use the estimate in the Sobolev space and (3.19). Passing to the limit as $\varepsilon \to 0$ we get

$$\lambda \int_\Omega \widetilde{u^\varepsilon} w^\varepsilon \phi \to \lambda \int_\Omega u\phi$$

and

$$\int_\Omega f(x) w^\varepsilon \phi \to \int_\Omega f(x)\phi.$$

Passing to the limit in the second term of the right hand side of (3.28) is more delicate. Nevertheless, one can argue as in the proof of Theorem 3.3.3 to prove that $g(x,u)|\nabla u|^q \in L^1(\{|u|>0\})$ and

$$\int_{\{|\tilde{u^\varepsilon}|>0\}} g(x,\widetilde{u^\varepsilon})|\nabla \widetilde{u^\varepsilon}|^q w^\varepsilon \phi \to \int_{\{|u|>0\}} g(x,u)|\nabla u|^q \phi.$$

Therefore u satisfies (3.22) and we conclude the proof. □

3.4.6 The corrector result

Finally, in this subsection we prove a corrector result.

Theorem 3.4.6. *Assume the hypotheses of Theorem 3.4.3, and suppose also that the matrix M is symmetric. Let $\{u^\varepsilon\}$ and u be the sequence of solutions to* (3.1) *and its limit, respectively, given by Theorem 3.4.3. Then,*

$$\widetilde{u^\varepsilon} - uw^\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega).$$

Remark 3.4.7. In [45] it is proven that, for general problems with gradient-dependent lower order terms, the simple representation given by Theorem 3.4.6 does not hold in general. However, the nature of our problem allows us to prove a corrector result analogous to the linear case in spite of the presence of the gradient term.

Proof of Theorem 3.4.6. First of all observe that

$$\begin{aligned}\eta \int_\Omega |\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)|^2 &\le \int_\Omega M(x)\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) \\ &= \int_\Omega M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) - \int_\Omega M(x)\nabla(uw^\varepsilon)\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon). \end{aligned} \tag{3.29}$$

We will now pass to the limit in each term of the right hand side of the equality (3.29).

Indeed, for the first one, we take $\widetilde{u^\varepsilon} - uw^\varepsilon$ as test function in (3.1) and obtain that

$$\begin{aligned}\int_\Omega M(x)\nabla\widetilde{u^\varepsilon}\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) &= \lambda \int_\Omega \widetilde{u^\varepsilon}(\widetilde{u^\varepsilon} - uw^\varepsilon) \\ &+ \int_{\{|\widetilde{u^\varepsilon}|>0\}} g(x,\widetilde{u^\varepsilon})|\nabla\widetilde{u^\varepsilon}|^q(\widetilde{u^\varepsilon} - uw^\varepsilon) + \int_\Omega f(x)(\widetilde{u^\varepsilon} - uw^\varepsilon).\end{aligned} \tag{3.30}$$

By virtue of Theorem 3.4.3, Lemma 3.4.5 and also (3.17) and (3.19), we know that $\widetilde{u^\varepsilon} - uw^\varepsilon \to 0$ strongly in $L^s(\Omega)$ for all $s \in [1,\infty)$. Moreover, Lemma 3.4.5 also implies that $\{|u^\varepsilon|^{\frac{1}{q}}\}$ is bounded in $W_0^{1,q}(\Omega)$. Therefore, we can pass to the limit in (3.30) arguing as in the proof of the strong convergence in Lemma 3.3.5. In sum, we deduce that

$$\int_\Omega M(x)\nabla\widetilde{u^\varepsilon} \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) \to 0.$$

Concerning the second term of (3.29), we derive, using the symmetry of M, that

$$\begin{aligned}&\int_\Omega M(x)\nabla(uw^\varepsilon)\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) = \int_\Omega uM(x)\nabla w^\varepsilon \cdot \nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)\\ &+\int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon) = \int_\Omega M(x)^T\nabla w^\varepsilon\cdot\nabla(u(\widetilde{u^\varepsilon} - uw^\varepsilon))\\ &-\int_\Omega(\widetilde{u^\varepsilon} - uw^\varepsilon)M(x)\nabla w^\varepsilon\cdot\nabla u + \int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)\end{aligned}$$

Observe now that (3.20) implies that

$$\int_\Omega M(x)^T\nabla w^\varepsilon\cdot\nabla(u(\widetilde{u^\varepsilon} - uw^\varepsilon)) \to 0.$$

Moreover, the remaining terms

$$\int_\Omega(\widetilde{u^\varepsilon} - uw^\varepsilon)M(x)\nabla w^\varepsilon\cdot\nabla u, \quad \int_\Omega w^\varepsilon M(x)\nabla u\cdot\nabla(\widetilde{u^\varepsilon} - uw^\varepsilon)$$

are both products in $L^2(\Omega)^N$ of a strongly convergent sequence times a weakly convergent one. Therefore both terms converge and the limits are clearly zero.

In conclusion, we have proved that we can pass to the limit in (3.29), and the limit is zero. The proof of the result is now finished. □

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Chapter 4

A singularity as a break point for the multiplicity of solutions to quasilinear elliptic problems

S. López-Martínez, *A singularity as a breakpoint for the multiplicity of solutions to quasilinear elliptic problems.* To appear in *Adv. Nonlinear Anal.*

Abstract. In this paper we deal with the elliptic problem

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \lneqq \mu \in L^\infty(\Omega)$, $0 \lneqq f \in L^{p_0}(\Omega)$ for some $p_0 > \frac{N}{2}$, $1 < q < 2$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We establish existence and multiplicity results for $\lambda > 0$ and $\alpha < q-1$, including the non-singular case $\alpha = 0$. In contrast, we also derive existence and uniqueness results for $\lambda > 0$ and $q-1 < \alpha \leq 1$. We thus complement the results in [38, 39], which are concerned with $\alpha = q-1$, and show that the value $\alpha = q-1$ plays the role of a break point for the multiplicity/uniqueness of solution.

4.1 Introduction

In this paper we deal with the following boundary value problem:

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{P_λ}$$

Here, Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 3$) with boundary $\partial\Omega$ smooth enough, $\lambda \in \mathbb{R}$, $0 \lneqq \mu \in L^\infty(\Omega)$, $0 \lneqq f \in L^{p_0}(\Omega)$ for some $p_0 > \frac{N}{2}$, $1 < q < 2$ and $0 \leq \alpha \leq 1$. A solution to (P_λ) is a function $0 < u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which satisfies the equation in (P_λ) in the usual weak sense (we will be more precise about the concept of solution in Definition 4.3.1 below). Observe that, if $\alpha > 0$, then the lower order term presents a singularity as u approaches zero, i.e., as x approaches $\partial\Omega$. Our goal is to study the existence, nonexistence, uniqueness and multiplicity of solutions to (P_λ), specially for $\lambda > 0$.

The first motivation for dealing with this problem comes from the non-singular case $\alpha = 0$, i.e.,

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{R_λ}$$

Non-singular problems with lower order terms having natural growth in the gradient have been extensively studied since the pioneering works by Boccardo, Murat and Puel in the '80s and '90s (see [28, 30, 32] and references therein) and, in particular, problem (R_λ) is very well understood for $1 < q \leq 2$ and $\lambda \leq 0$. Indeed, it is well-known from classical results (see [28, 32]) that problem (R_λ) admits at least one solution for all $\lambda < 0$. Concerning the uniqueness of solution, it was first dealt with in [18], and their results have been improved in several directions since then (see [11] and references therein). In particular, it has been recently proved in [11] that uniqueness holds for all $\lambda \leq 0$. However, the existence of solution for $\lambda = 0$ is not always guaranteed. Roughly speaking, if $\|f\|_{L^{p_0}(\Omega)}$ or $\|\mu\|_{L^\infty(\Omega)}$ are small enough (if $1 < q < 2$, then one needs to ask both to be small, see Proposition 4.3.4 below), then there exists a unique solution to (R_0), as it is shown for instance in [69] (see also [85] and references therein). Conversely, it is proved in [3] (see also [87]) that, if f or μ are large in some sense, there exists no solution to (R_0); in consequence, $\lambda = 0$ is a bifurcation point from infinity. Concerning this last case, a very precise description of the blow-up of the solutions at

$\lambda = 0$, and also a necessary and sufficient condition for the existence of solution to (R_0) in terms of the corresponding ergodic problem, are given in [113] under slightly stronger hypotheses on f and μ.

The scenario in which (R_0) has a solution is not so well understood and has risen interest in the recent years. In this case one expects to find solutions to (R_λ) for small $\lambda > 0$ by a continuation argument. However, the uniqueness and multiplicity problems are harder to deal with for $\lambda > 0$, and very few results are known in this direction. In fact, up to our knowledge, the literature contains results concerning only the quadratic case $q = 2$. In this regard, the first advances can be found in [90] for $\mu > 0$ constant. Shortly after that, some improvements appeared in [89], where $\lambda = \lambda(x)$ is allowed to change sign but μ is still constant. These two works employ variational techniques. Going further, topological degree and bifurcation are used in [12] to handle problem (R_λ) with $\lambda > 0$ and $\mu \in L^\infty(\Omega)$ such that $\mu_1 \leq \mu \leq \mu_2$ for some constants $\mu_2 > \mu_1 > 0$. We also quote [120], where functions $0 \lneqq \mu \in L^\infty(\Omega)$ vanishing on $\partial\Omega$, and even with compact support, are permitted at the expense of imposing $N \leq 3$ (the cases $N = 4, 5$ are also handled provided $\lambda = \lambda(x)$ satisfies extra hypotheses). Very recently, a similar problem to (R_λ) with the p -Laplacian as principal operator has been considered in [55], while sign-changing coefficients (including μ) are allowed in [56].

In all these works, the authors prove that, if there is a solution to (R_0), then problem (R_λ) admits at least two different solutions for all $\lambda > 0$ small enough, and it was first shown in [12] that the branch of positive solutions bifurcates from infinity to the right of the axis $\lambda = 0$ (see [57] for a more complete picture when different sign conditions on f are imposed). We stress again that all the mentioned papers have in common the assumption $q = 2$. Indeed, the techniques employed for $q = 2$ usually involve exponential test functions which somehow remove the dependence on the gradient in the equation. For instance, this idea allows the authors of [90] to study the problem variationally, while in [12] it is essential in order to find a priori estimates for $\lambda > 0$. However, this idea fails for $1 < q < 2$ as the gradient term can not be removed when one looks for a priori estimates satisfied by supersolutions to (R_λ). Up to our knowledge, the multiplicity or uniqueness of solutions for $\lambda > 0$ is an open problem if $1 < q < 2$.

Turning back to (P_λ), another motivation for studying this problem comes from the very recent paper [38]. In such a work, problem (P_λ) is studied in the singular case $\alpha > 0$, mostly in the special case $\alpha = q - 1$. Elliptic problems with singularities at $u = 0$ have become of remarkable interest since the seminal papers [51, 97, 124]. Without the aim of being exhaustive, some related references dealing with this kind of singularities (with or without lower order terms with natural growth in the gradient) are [6–8, 15, 24, 48, 64, 65, 72, 73, 76–78]. The interested reader is referred to [65] and references therein, where a rather complete background on singular problems can be found.

Focusing specifically on problem (P_λ), in Remark 6.1 of [38] the authors observe that, if $q = 2$ and $0 < \alpha < q-1 = 1$, the techniques in [12] can be adapted to derive again a multiplicity result for $\lambda > 0$. Hence, roughly speaking, mild singularities at zero do not alter the behavior of the solutions, as far as the multiplicity for $\lambda > 0$ is concerned. Nonetheless, the main result in that paper shows that multiplicity fails for $1 < q \le 2$ and $\alpha = q-1$ (see [15] for $q = 2$ and μ constant). To be precise, the authors prove under natural hypotheses on μ and f that, if $\alpha = q-1$, there exists $\lambda^* \in (0,\lambda_1]$ (where $\lambda_1 = \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \int_\Omega |\nabla v|^2 / \int_\Omega v^2$) such that problem (P_λ) has a solution if and only if $\lambda < \lambda^*$, and in this case, the solution is unique (see also [39] for a similar existence result when f and u may change sign). In particular, one has existence and uniqueness for $\lambda > 0$ small. Since this result is true for $1 < q \le 2$, it is natural to wonder whether $\alpha = q-1$ is a break point for the multiplicity of solutions not only in the case $q = 2$, but also for $1 < q < 2$.

In the present work we contribute to these topics by proving that, if there is a solution to (P_0), then there are at least two different solutions to (P_λ) for all $\lambda > 0$ small enough provided q and α satisfy certain relations involving also the dimension N. We prove also that the branch of positive solutions bifurcates from infinity to the right of the axis $\lambda = 0$.

To be more precise, we consider the following set of hypotheses:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain of class } \mathscr{C}^2, \\ \mu \in L^\infty(\Omega) \text{ satisfies that } \mu(x) \ge \mu_0 \text{ a.e. } x \in \Omega \text{ for some constant } \mu_0 > 0, \\ 0 \lneqq f \in L^{p_0}(\Omega) \text{ for some } p_0 > \frac{N}{2}, \\ q \in (1,2), \\ \alpha \in [0, q-1). \end{cases} \tag{H1}$$

Observe that μ is bounded away from zero but not necessarily constant. We introduce here the main result of this paper:

Theorem 4.1.1. *Assume that* (H1) *holds and that* (P_0) *admits a solution* u_0*. If* $q > \frac{N}{N-1}$*, suppose also that*

$$\frac{q-1-\alpha}{q-2\alpha} \le \frac{q-\alpha}{N-q+1}. \tag{4.1}$$

Then, there exists $\bar{\lambda} \in (0,\lambda_1)$ *such that problem* (P_λ) *admits at least two different solutions for all* $\lambda \in (0,\bar{\lambda}]$*. Moreover, zero is the unique bifurcation point from infinity to problem* (P_λ)*.*

Even though this result deals only with the range $\lambda > 0$, in order to make a more complete picture we will gather and prove in Section 4.3 some existence, nonexistence and uniqueness results about problem (P_λ) for $\lambda \leq 0$. We stress that the uniqueness result for $\lambda \leq 0$, apart from being new in the literature, shows that $\lambda = 0$ is a critical point beyond which the nature of the problem changes drastically, as in the well-known case $q = 2$ and $\alpha = 0$.

Concerning the proof of Theorem 4.1.1, the idea is to derive a priori estimates of the solutions to (P_λ) for all $\lambda > \lambda_0$ which are independent of $\lambda > 0$. This idea first appeared in [12] for $q = 2$ and $\alpha = 0$, but the approach for deriving the estimates does not work in our framework. For our purposes, it is more convenient to use the arguments developed in [120], which allow us to find L^p estimates of supersolutions. After that, we establish a bootstrap argument, which works thanks to some results in [85], that yields an L^∞ estimate. Actually, these results are valid only in the nonsingular case $\alpha = 0$, so we will extend some parts of them to our singular framework. After writing the present work, it came to the author's knowledge that similar results extending [85] to a more general setting have been recently obtained in [96].

Hypothesis (4.1) in Theorem 4.1.1 deserves some comments. It appears in the proof as a result of the combination of the mentioned techniques from [120] and the bootstrap from [85]. However, we presume that this is a technical assumption forced by the tools we employed, so the theorem might admit some improvements. In order to clarify the meaning of this condition, we derive two corollaries below in which simpler conditions assuring (4.1) are imposed. For instance, if we consider the sequence

$$Q_n = \begin{cases} 2 & \forall n \leq 4, \\ \dfrac{n+2-\sqrt{n^2-4n-4}}{4} & \forall n \geq 5, \end{cases} \tag{4.2}$$

then $q \in (1, Q_N] \setminus \{2\}$ implies (4.1), with no extra hypotheses on α but $0 \leq \alpha < q-1$ (see Corollary 4.3.17). Observe that $Q_n > 1$ but $\lim_{n\to\infty} Q_n = 1$. This means that, if N is large, then q has to be chosen close to 1. However, one would expect a multiplicity result for any $q \in (1,2)$ and any N. This still remains as an open problem. In any case, Corollary 4.3.17 represents a remarkable advance, in particular, about the nonsingular problem (R_λ). Changing the point of view, we give in Corollary 4.3.18 below a condition on α that is sufficient for applying Theorem 4.1.1 even for q close to 2 and for N large.

With the aim of having a deeper insight into problem (P_λ), we also consider in this work the case $q-1 < \alpha \leq 1$. In contrast to the previous situation ($0 \leq \alpha < q-1$), we will prove that existence and *uniqueness* hold for $\lambda > 0$ small enough. For this purpose, we will need the following assumption on Ω:

$$\begin{cases} \text{There exist } r_0, \theta_0 > 0 \text{ such that, if } x \in \partial\Omega \text{ and } 0 < r < r_0, \text{ then} \\ |\Omega_r| \leq (1-\theta_0)|B_r(x)| \text{ for every connected component } \Omega_r \text{ of } \Omega \cap B_r(x). \end{cases} \tag{A}$$

Note that, if $\partial\Omega$ is Lipschitz, then Ω satisfies (A) (see [11]), so this represents only a mild restriction. The precise hypotheses that we need are gathered here:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ is a bounded domain satisfying condition (A)}, \\ 0 \lneqq \mu \in L^\infty(\Omega), \\ 0 \lneqq f \in L^{p_0}(\Omega) \text{ for } p_0 > \frac{N}{2}, \\ q \in (1,2), \\ q-1 < \alpha \leq 1. \end{cases} \tag{H2}$$

We emphasize that μ is allowed to vanish in subsets of Ω with nonzero measure.

The statement of the main result in the $q-1 < \alpha \leq 1$ case is the following:

Theorem 4.1.2. *Assume that* (H2) *holds. Then there exists a solution to* (P_λ) *for all* $\lambda < \lambda_1$*, and there exists no solution to* (P_λ) *for all* $\lambda \geq \lambda_1$*. Moreover, the solution is unique for all* $\lambda \leq 0$*. Finally, if f satisfies that*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f(x) \geq c_\omega \quad \text{a.e. } x \in \omega,$$

then the solution is unique for all $\lambda < \lambda_1$ *and* λ_1 *is the unique bifurcation point from infinity to problem* (P_λ)*.*

Even though we are specially interested in the uniqueness part, the existence statement in Theorem 4.1.2 deserves also attention. Observe that one has existence of solution if and only if $\lambda < \lambda_1$. This suggests that the nonlinear term does not play an essential role in this case, since the situation is analogous to the linear problem ($\mu \equiv 0$). Recall that this is not the case when $\alpha = q-1$, for which one has existence if and only if $\lambda < \lambda^*$, where $\lambda^* < \lambda_1$ provided $\mu > 0$ (see [38, Remark 6.3]).

The proof of the existence of solution in Theorem 4.1.2 is performed by passing to the limit in certain family of approximate nonsingular problems. We will derive Hölder continuous a priori estimates on the solutions to such a family, which will allow us to pass to the limit. For proving such estimates, the assumption $\alpha \leq 1$ is essential (see Remark 4.3.3 below). Moreover, the continuity of the solutions is also essential to prove their uniqueness. Indeed, we state and prove in Section 4.2 two comparison principles valid for continuous lower and upper solutions to singular equations. As far as we know, these two results are new, and they are interesting by themselves as only few uniqueness results for singular equations are known (see [10, 15, 16, 37, 38]). We follow in their proofs the arguments in [11] and [38].

As a summary, our results contribute to the theory of equations with subquadatic growth in the gradient, extending what it is known about the multiplicity of solutions in the quadratic case. On the other hand, they can be seen as a link between the singular and nonsingular theory, in the sense that they show that the presence or not of a singularity is determining only if it is strong enough. Finally, new existence and uniqueness results are given for strong singularities, where the uniqueness part is specially remarkable.

We organize the paper as follows: in Section 4.2 we deal with the mentioned comparison principles; we devote Section 4.3 to prove Theorem 4.1.1 as well as some auxiliary results and some consequences of the mentioned theorem; Section 4.4 contains the proof of Theorem 4.1.2, and Section 4.5 is an appendix where we prove a continuation result needed in the proof of Theorem 4.1.1.

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Notation

- For every $x \in \mathbb{R}^N$, the distance from x to $\partial\Omega$ will be denoted as $\delta(x)$. Furthermore, for $p \geq 1$ we will denote as $L^p(\Omega, \delta)$ the space of measurable functions $u : \Omega \to \mathbb{R}$ such that
$$\|u\|_{L^p(\Omega,\delta)} := \left(\int_\Omega |u(x)|^p \delta(x) dx \right)^{\frac{1}{p}} < +\infty,$$
identifying functions equal up to a set of zero measure.

- For $p \geq 1$, we will denote the usual Marcinkiewicz space as $\mathcal{M}^p(\Omega)$, i.e., the space of measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $c > 0$ such that $|\{|u| > k\}| k^p \leq c$ for all $k > 0$. In this case, we denote
$$\|u\|_{\mathcal{M}^p(\Omega)} := \left(\inf\{c > 0 : |\{|u| > k\}| k^p \leq c \text{ for all } k > 0\right)^{\frac{1}{p}}.$$

- For $k \geq 0$, we will write $T_k(s) = \max\{-k, \min\{s, k\}\}$ and $G_k(s) = s - T_k(s)$ for all $s \in \mathbb{R}$.

- The principal eigenvalue of the $-\Delta$ operator in Ω under zero Dirichlet boundary conditions will be denoted as λ_1. In other words, λ_1 is the unique real number satisfying that the equation $-\Delta\varphi = \lambda_1\varphi$ has a solution $0 < \varphi \in H_0^1(\Omega)$. We will write φ_1 for the positive eigenfunction associated with λ_1 such that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$.

4.2 Comparison principles

We start with a comparison principle valid for singular equations. The proof basically follows the steps of a similar result in [11]. However, up to our knowledge this is the first time that a comparison result has been proved including a general positive singular lower order term on the right hand side of the equation (see the comparison results in [38], where a specific 1-homogeneous singular term is considered).

Theorem 4.2.1. *Let $1 < q \leq 2$, $\lambda \leq 0$, $h \in L^1_{loc}(\Omega)$ and $g : \Omega \times (0,+\infty) \to [0,+\infty)$ satisfying*

$$s \mapsto g(x,s) \ \textit{is nonincreasing for a.e.}\ x \in \Omega,$$
$$x \mapsto g(x,s) \ \textit{is locally essentially bounded for all}\ s > 0.$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, be such that

$$\int_\Omega \nabla u \nabla\phi \leq \lambda\int_\Omega u\phi + \int_\Omega g(x,u)|\nabla u|^q\phi + \int_\Omega h(x)\phi \quad \textit{and} \tag{4.3}$$
$$\int_\Omega \nabla v \nabla\phi \geq \lambda\int_\Omega v\phi + \int_\Omega g(x,v)|\nabla v|^q\phi + \int_\Omega h(x)\phi \tag{4.4}$$

for every $0 \leq \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that the following boundary condition holds:

$$\limsup_{x\to x_0}(u(x) - v(x)) \leq 0 \quad \forall x_0 \in \partial\Omega. \tag{4.5}$$

Then, $u \leq v$ in Ω.

Remark 4.2.2. Theorem 4.2.1 is valid for a wide class of lower order terms. For instance, the model example is

$$g(x,s) = \frac{\mu(x)}{s^\alpha} \quad \text{a.e. } x \in \Omega,\ \forall s > 0,$$

for any $\alpha > 0$ and $0 \leq \mu \in L^\infty_{\text{loc}}(\Omega)$. In particular, the growth of the singularity is irrelevant in the proof. Nonetheless, the comparison principle does not work for $\lambda > 0$. Indeed, as we pointed out in the Introduction, if the singularity is mild enough in some sense, then a multiplicity phenomenon appears for $\lambda > 0$. Thus, for the model case, the comparison result is sharp in terms of the sign of λ.

Remark 4.2.3. In Theorem 4.2.1, $u,v \in C(\Omega)$ are not assumed to be continuous up to $\partial\Omega$, so a suitable ordering condition on the boundary is given by (4.5). However, if $u,v \in C(\overline{\Omega})$, then hypothesis (4.5) is equivalent to the usual and more natural condition $u(x_0) \leq v(x_0)$ for all $x_0 \in \partial\Omega$.

Proof of Theorem 4.2.1. Let us denote $w = u - v$. For $k > 0$, we consider the function $\phi = (w-k)^+$, and we also denote

$$A_k = \{x \in \Omega : w(x) \geq k\}.$$

Arguing by contradiction, assume that $w^+ \not\equiv 0$. Notice that, in consequence, $A_k \neq \emptyset$ for every $k \in \big(0, \|w^+\|_{L^\infty(\Omega)}\big)$. Moreover, condition (4.5) implies that $A_k \subset\subset \Omega$. Therefore, ϕ has compact support in Ω. In particular, $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so it can be taken as test function in (4.3) and (4.4), obtaining that

$$\int_\Omega \nabla u \nabla (w-k)^+ \leq \lambda \int_\Omega u(w-k)^+ + \int_\Omega g(x,u)|\nabla u|^q (w-k)^+ + \int_\Omega h(x)(w-k)^+ \quad (4.6)$$

and

$$\int_\Omega \nabla v \nabla (w-k)^+ \geq \lambda \int_\Omega v(w-k)^+ + \int_\Omega g(x,v)|\nabla v|^q (w-k)^+ + \int_\Omega h(x)(w-k)^+. \quad (4.7)$$

Subtracting (4.7) from (4.6) we get

$$\begin{aligned}\int_\Omega |\nabla (w-k)^+|^2 \leq \lambda &\int_\Omega ((w-k)^+)^2 \\ &+ \lambda k \int_\Omega (w-k)^+ + \int_\Omega (g(x,u)|\nabla u|^q - g(x,v)|\nabla v|^q)(w-k)^+.\end{aligned}$$

Since $\lambda \leq 0$, we deduce that

$$\int_\Omega |\nabla (w-k)^+|^2 \leq \int_\Omega (g(x,u)|\nabla u|^q - g(x,v)|\nabla v|^q)(w-k)^+. \quad (4.8)$$

Let $k_0 \in \big(0, \|w^+\|_{L^\infty(\Omega)}\big)$ and let $\omega \subset\subset \Omega$ be an open set such that $A_{k_0} \subset \omega$. Observe that $A_k \subset A_{k_0}$ for all $k \geq k_0$. Then, using the properties of g, it is clear that

$$g(x,u) \leq g(x,v) \leq g(x, \inf_\omega(v)) \leq \Big\| g(\cdot, \inf_\omega(v)) \Big\|_{L^\infty(\omega)}$$

in A_k for every $k \in \big[k_0, \|w^+\|_{L^\infty(\Omega)}\big)$. Therefore, from (4.8) we deduce that

$$\begin{aligned}\int_\Omega |\nabla (w-k)^+|^2 &\leq \int_\Omega g(x,v)||\nabla u|^q - |\nabla v|^q|(w-k)^+ \quad (4.9)\\ &\leq \Big\| g(\cdot, \inf_\omega(v)) \Big\|_{L^\infty(\omega)} \int_{A_k} ||\nabla u|^q - |\nabla v|^q|(w-k)^+\end{aligned}$$

for every $k \in \left[k_0, \|w^+\|_{L^\infty(\Omega)}\right)$.

For every $j \in \mathbb{R}$, let us denote $\Omega_j = \{x \in \Omega : |w(x)| = j\}$, and consider also the set $J = \{j \in \mathbb{R} : |\Omega_j| \neq 0\}$. Since $|\Omega| < \infty$, then J is at most countable, which implies that the set $\bigcup_{j \in J} \Omega_j$ is measurable, and we also have that

$$\nabla w = 0 \quad \text{in } \bigcup_{j \in J} \Omega_j \implies |\nabla u_1| = |\nabla v_1| \quad \text{in } \bigcup_{j \in J} \Omega_j.$$

Hence, if we define the set $Z = \Omega \setminus \bigcup_{j \in J} \Omega_j$, we deduce from (4.9) that

$$\begin{aligned}\int_\Omega |\nabla (w-k)^+|^2 &\leq \left\| g(\cdot, \inf_\omega(v)) \right\|_{L^\infty(\omega)} \int_{A_k \cap Z} \left(\int_0^1 \frac{d}{dt}(|t\nabla u + (1-t)\nabla v|^q)dt \right)(w-k)^+ \\ &= C \int_{A_k \cap Z} \left(\int_0^1 |t\nabla u + (1-t)\nabla v|^{q-2}(t\nabla u + (1-t)\nabla v)\nabla w dt \right)(w-k)^+. \end{aligned} \tag{4.10}$$

Taking into account that $u, v \in W^{1,N}_{\text{loc}}(\Omega)$ and $A_k \subset\subset \Omega$, we have that

$$|t\nabla u + (1-t)\nabla v| \leq |\nabla u| + |\nabla v| + 1 \equiv \eta \in L^N(A_k \cap Z).$$

Hence, from (4.10) we derive that

$$\begin{aligned}\|(w-k)^+\|^2_{H^1_0(\Omega)} &\leq C \int_{A_k \cap Z} \eta^{q-1} |\nabla w|(w-k)^+ \leq C \int_{A_k \cap Z} \eta |\nabla (w-k)^+|(w-k)^+ \\ &\leq C\|\eta\|_{L^N(A_k \cap Z)} \|(w-k)^+\|_{H^1_0(\Omega)} \|(w-k)^+\|_{L^{2^*}(\Omega)} \\ &\leq C\|\eta\|_{L^N(A_k \cap Z)} \|(w-k)^+\|^2_{H^1_0(\Omega)}. \end{aligned} \tag{4.11}$$

Let us now define the function $F : \left[k_0, \|w^+\|_{L^\infty(\Omega)}\right] \to \mathbb{R}$ by

$$F(k) = \|\eta\|_{L^N(A_k \cap Z)} = \||\nabla u| + |\nabla v| + 1\|_{L^N(A_k \cap Z)} \quad \forall k \in \left[k_0, \|w^+\|_{L^\infty(\Omega)}\right),$$

and $F(\|w^+\|_{L^\infty(\Omega)}) = 0$. It is clear that F is nonincreasing and continuous. Thus, choosing k close enough to $\|w^+\|_{L^\infty(\Omega)}$, we deduce from (4.11) that $(w-k)^+ \equiv 0$. That is to say, $w \leq k$ in Ω. But this is not possible since $k < \|w^+\|_{L^\infty(\Omega)} = \sup_\Omega(w)$.

In conclusion, we have proved that $w^+ \equiv 0$, i.e., $u \leq v$ in Ω. □

Next theorem is another comparison principle which works for $\lambda > 0$. In turn, one has to impose stronger hypotheses on g and h. The proof is similar to the one above combined with some ideas in [38].

Theorem 4.2.4. *Let $1 < q \leq 2$, $\lambda \in \mathbb{R}$, $0 \leq h \in L^1_{loc}(\Omega)$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ satisfying*

$$\begin{aligned} &s \mapsto s^{q-1} g(x,s) \quad \textit{is nonincreasing for a.e. } x \in \Omega, \\ &x \mapsto g(x,s) \qquad \textit{is locally essentially bounded for all } s > 0. \end{aligned}$$

If $\lambda > 0$, assume also that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad h(x) \geq c_\omega \quad a.e.\ x \in \omega. \tag{4.12}$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, satisfying respectively (4.3) *and* (4.4) *for every $0 \leq \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that, for every $\varepsilon > 0$, the following boundary condition holds:*

$$\limsup_{x \to x_0} \left(\frac{u(x)}{v(x) + \varepsilon} \right) \leq 1 \quad \forall x_0 \in \partial\Omega. \tag{4.13}$$

Then, $u \leq v$ in Ω.

Remark 4.2.5. The observation made in Remark 4.2.3 is valid also for Theorem 4.2.4 substituting condition (4.5) with (4.13).

Proof of Theorem 4.2.4. For every $\varepsilon > 0$, let us consider the function

$$w_\varepsilon = \log\left(\frac{u}{v + \varepsilon} \right).$$

We claim that $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that $w_{\varepsilon_0}^+ \not\equiv 0$. Let us fix $k_0 \in \big(0, \|w_{\varepsilon_0}^+\|_{L^\infty(\Omega)}\big)$ and $\varepsilon \in (0, \varepsilon_0)$, the latter to be chosen small enough later. It is clear that $w_{\varepsilon_0} \leq w_\varepsilon$ in Ω, so $w_\varepsilon^+ \not\equiv 0$.

For $k \in \big[k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}\big)$, let us denote

$$A_k = \{x \in \Omega : w_\varepsilon(x) \geq k\} = \{x \in \Omega : u(x) \geq e^k (v(x) + \varepsilon)\}.$$

It is clear that $A_k \neq \emptyset$. Moreover, from (4.13) we easily deduce that $A_k \subset\subset \Omega$. Then, the function $(w_\varepsilon - k)^+$ has compact support and, in particular, $(w_\varepsilon - k)^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)$. Therefore, we may take $\frac{(w_\varepsilon - k)^+}{u}$ as test function in (4.3), and $\frac{(w_\varepsilon - k)^+}{v+\varepsilon}$ in (4.4), obtaining

$$\begin{aligned} \int_\Omega \frac{\nabla u}{u} \nabla (w_\varepsilon - k)^+ \leq \int_\Omega \frac{|\nabla u|^2}{u^2} (w_\varepsilon - k)^+ + \lambda \int_\Omega (w_\varepsilon - k)^+ \\ + \int_\Omega u^{q-1} g(x,u) \frac{|\nabla u|^q}{u^q} (w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{u} (w_\varepsilon - k)^+ \end{aligned} \tag{4.14}$$

and, using that $g \geq 0$,

$$\begin{aligned} \int_\Omega \frac{\nabla v}{v + \varepsilon} \nabla (w_\varepsilon - k)^+ &\geq \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2} (w_\varepsilon - k)^+ + \lambda \int_\Omega \frac{v}{v+\varepsilon} (w_\varepsilon - k)^+ \\ &\quad + \int_\Omega v^{q-1} g(x,v) \frac{|\nabla v|^q}{v^{q-1}(v+\varepsilon)} (w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon} (w_\varepsilon - k)^+ \\ &\geq \int_\Omega \frac{|\nabla v|^2}{(v+\varepsilon)^2} (w_\varepsilon - k)^+ + \lambda \int_\Omega (w_\varepsilon - k)^+ - \int_\Omega \frac{\lambda \varepsilon}{v+\varepsilon} (w_\varepsilon - k)^+ \\ &\quad + \int_\Omega v^{q-1} g(x,v) \frac{|\nabla v|^q}{(v+\varepsilon)^q} (w_\varepsilon - k)^+ + \int_\Omega \frac{h(x)}{v+\varepsilon} (w_\varepsilon - k)^+. \end{aligned} \tag{4.15}$$

Let $\omega \subset\subset \Omega$ be an open set such that $A_{k_0} \subset \omega$. Observe that $A_k \subset A_{k_0}$ for all $k \geq k_0$. Then, it is clear that

$$u^{q-1}g(x,u) \leq v^{q-1}g(x,v) \leq \sup_{\omega}(v)^{q-1}g(x,\inf_{\omega}(v)) \leq \sup_{\omega}(v)^{q-1}\left\|g(\cdot,\inf_{\omega}(v))\right\|_{L^\infty(\omega)}$$

in A_k for every $k \in \left[k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}\right)$. Therefore,

$$\begin{aligned}&\int_\Omega \left(u^{q-1}g(x,u)\frac{|\nabla u|^q}{u^q} - v^{q-1}g(x,v)\frac{|\nabla v|^q}{(v+\varepsilon)^q}\right)(w_\varepsilon - k)^+ \\ &\leq \sup_{\omega}(v)^{q-1}\left\|g(\cdot,\inf_{\omega}(v))\right\|_{L^\infty(\omega)}\int_\Omega \left|\frac{|\nabla u|^q}{u^q} - \frac{|\nabla v|^q}{(v+\varepsilon)^q}\right|(w_\varepsilon - k)^+.\end{aligned}$$

Moreover, we have that

$$h\left(\frac{1}{u} - \frac{1}{v+\varepsilon}\right) + \frac{\lambda\varepsilon}{v+\varepsilon} \leq 0 \quad \text{in } A_k \text{ for every } k \in \left[k_0, \|w_\varepsilon^+\|_{L^\infty(\Omega)}\right) \tag{4.16}$$

whenever $\lambda \leq 0$. On the other hand, if $\lambda > 0$, let us take

$$\varepsilon < \min\left\{\varepsilon_0, \frac{1-e^{-k_0}}{\lambda}\, c_\omega\right\},$$

where c_ω is the constant given by (4.12). With this choice, it is straightforward to deduce that (4.16) holds again.

Therefore, subtracting (4.14) and (4.15), and taking into account that $u, v \in W^{1,N}_{\text{loc}}(\Omega)$ and also (4.16), we may argue as in the proof of [38, Theorem 3.2] and achieve a contradiction taking k close enough to $\|w_\varepsilon^+\|_{L^\infty(\Omega)}$.

In conclusion, necessarily $w_\varepsilon^+ \equiv 0$ for any $\varepsilon > 0$, i.e., $u \leq v + \varepsilon$ in Ω for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ it follows that $u \leq v$ in Ω. □

4.3 Multiplicity for $0 \leq \alpha < q-1$

In this section we will study problem (P_λ) under condition (H1). In this case observe that, if $0 < u \in W^{1,1}_{\text{loc}}(\Omega)$ and $t > 0$, then

$$\frac{|\nabla tu|^q}{(tu)^\alpha} = t^{q-\alpha}\frac{|\nabla u|^q}{u^\alpha}.$$

Since $\alpha < q-1$, then $q - \alpha > 1$. That is to say, the lower order term has *superlinear homogeneity*.

The concept of solution we will adopt is gathered in the following definition.

Definition 4.3.1. Given $\lambda \in \mathbb{R}$, a subsolution to (P_λ) is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ a.e. in Ω, $\mu \frac{|\nabla u|^q}{u^\alpha} \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi \le \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^\alpha}\phi + \int_\Omega f(x)\phi \quad \forall 0 \le \phi \in C_c^1(\Omega).$$

Reciprocally, a supersolution to (P_λ) is a function $u \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ a.e. in Ω, $\mu \frac{|\nabla u|^q}{u^\alpha} \in L^1_{\text{loc}}(\Omega)$ and satisfies the reverse inequality. Finally, a solution to (P_λ) is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which is both a subsolution and a supersolution to (P_λ).

Remark 4.3.2. Arguing as in [38, Appendix], it can be proved that, for $\lambda \in \mathbb{R}$, every solution u to (P_λ) satisfies that $\mu \frac{|\nabla u|^q}{u^\alpha}\phi \in L^1(\Omega)$ for all $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega u\phi + \int_\Omega \mu(x) \frac{|\nabla u|^q}{u^\alpha}\phi + \int_\Omega f(x)\phi \quad \forall \phi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

This fact allows us, in particular, to take u itself as test function.

Remark 4.3.3. Assume that (H1) holds. By taking φ_1 as test function in the weak formulation of (P_λ) one easily deduces that, if u is a solution to (P_λ), then $\lambda < \lambda_1$. Furthermore, since $\alpha \in [0,1]$, it can be proven as in [38, Appendix], which follows the ideas in [94], that every solution u to (P_λ), for any $\lambda < \lambda_1$, satisfies that $u \in C^{0,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Finally, since the solutions to (P_λ) are positive in compact subsets of Ω, then it can be seen again as in the mentioned appendix that $u \in W^{1,N}_{\text{loc}}(\Omega)$ for every solution to (P_λ) for any $\lambda < \lambda_1$.

Our first result is concerned with the existence and uniqueness of solution to (P_λ) for $\lambda \le 0$. The existence is well-known from the works that are quoted in the proof below. However, a precise statement for unbounded datum f is required for our purposes. In any case, the uniqueness is new up to our knowledge.

Proposition 4.3.4. *Assume that* (H1) *holds. Then, problem* (P_λ) *has a unique solution for all* $\lambda < 0$. *Moreover, assume additionally that either* $\alpha > 0$ *or the following smallness condition holds:*

$$a\left(b + \|f\|_{L^{p_0}(\Omega)}\right) < \left(\frac{2}{N} - \frac{1}{p_0}\right) \frac{N^2 |B_1(0)|^{\frac{2}{N}}}{|\Omega|^{\frac{2}{N} - \frac{1}{p_0}}},$$

where $B_1(0)$ *denotes the unit ball in* $\mathbb{R}^N$, *and* $a, b > 0$ *are such that*

$$\|\mu\|_{L^\infty(\Omega)} |s|^q \le a|s|^2 + b \quad \forall s \in \mathbb{R}.$$

Then (P_0) *has a unique solution.*

Remark 4.3.5. For the sake of simplicity, we have assumed (H1) in Proposition 4.3.4. Nevertheless, as it will be shown in the proof, the condition $\mu \geq \mu_0$ is not needed, only $\mu \gneqq 0$ is sufficient.

Proof of Proposition 4.3.4. The result for $\alpha = 0$ and $\lambda \leq 0$ is well-known. Indeed, the existence of solution for $\alpha = 0$ and $\lambda < 0$ is proven in [28, 32], the existence for $\alpha = \lambda = 0$ under the smallness condition is proven in [69], and the uniqueness for $\alpha = 0$ and $\lambda \leq 0$, in [11]. Thus, we assume that $\alpha \in (0, q-1)$.

Observe now that, by Young's inequality, there exist $C_1, C_2 > 0$ such that

$$0 \leq \mu(x) \frac{|\xi|^q}{|s|^\alpha} \leq C_1 \frac{|\xi|^2}{|s|^{\frac{2\alpha}{q}}} + C_2 \tag{4.17}$$

for all $\xi \in \mathbb{R}^N$, for all $s \in \mathbb{R} \setminus \{0\}$ and for a.e. $x \in \Omega$, where

$$\frac{2\alpha}{q} < \frac{2(q-1)}{q} = 2 - \frac{2}{q} < 1. \tag{4.18}$$

Then, the hypotheses of [77, Proposition 4.1] are fulfilled, so there exists a solution $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (P_0) in some weaker sense than Definition 4.3.1. Nonetheless, since $f \gneqq 0$ in Ω, then the strong maximum principle implies that $u_0 > 0$ in Ω, so u_0 is in fact a solution to (P_λ) in the sense of Definition 4.3.1.

Concerning the existence for $\lambda < 0$, we argue by approximation as follows. For all $n \in \mathbb{N}$, let us consider the problem

$$\begin{cases} -\Delta u_n = \lambda u_n + \mu(x) \dfrac{|\nabla u_n|^q}{u_n^\alpha} + T_n(f(x)), & x \in \Omega, \\ u_n > 0, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{4.19}$$

Since (4.17) and (4.18) are satisfied, we know from [76] that there exists a solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (4.19) for all n. Notice now that

$$-\Delta u_n \leq \mu(x) \frac{|\nabla u_n|^q}{u_n^\alpha} + f(x), \quad x \in \Omega.$$

Hence, Theorem 4.2.1 applies (see Remark 4.3.3) and yields

$$u_n \leq u_0 \leq \|u_0\|_{L^\infty(\Omega)} \text{ a.e. } x \in \Omega.$$

In other words, $\{u_n\}$ is bounded in $L^\infty(\Omega)$. By taking u_n as test function in the weak formulation of (4.19), we immediately deduce that $\{u_n\}$ is also bounded in $H_0^1(\Omega)$.

Hence, there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that, passing to a subseqence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ strongly in $L^p(\Omega)$ for any $p \in [1, \infty)$.

Observe also that, again by comparison, $u_n \geq z$ for all n, where $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to

$$\begin{cases} -\Delta z = \lambda z + T_1(f(x)), & x \in \Omega, \\ z = 0, & x \in \partial\Omega. \end{cases}$$

Now, the strong maximum principle applied on z implies that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad u_n \geq c_\omega \quad \text{a.e. in } \omega,\ \forall n.$$

Therefore, $\{-\Delta u_n\}$ is bounded in $L^1_{\text{loc}}(\Omega)$. Thus, by virtue of [26, Theorem 2.1], it follows that $\nabla u_n \to \nabla u$ strongly in $L^q(\Omega)^N$, up to a subsequence. The convergences we have proved about $\{u_n\}$ and $\{\nabla u_n\}$ are enough to pass to the limit in (4.19). The proof is standard, we refer to the proof of [38, Proposition 5.2] for further details. In sum, u is a solution to (P_λ).

The uniqueness of u is a direct consequence of Theorem 4.2.1 and Remark 4.3.3. $\square$

Next result shows that, if $\alpha = 0$, then the existence of solution to (P_0) may fail if f or μ are too large in some sense, in contrast to the case $\alpha > 0$. Thus, the smallness assumption in Proposition 4.3.4 is justified. In fact, this result is basically contained in [3, Theorem 2.1]. We include the statement and proof in our context for completeness.

Proposition 4.3.6. *Assume that* (H1) *holds with $\alpha = 0$, and suppose that (P_λ) admits a solution for some $\lambda \geq 0$. Then,*

$$\int_\Omega f(x)\phi^{q'} \leq \int_\Omega \frac{|\nabla\phi|^{q'}}{((q-1)\mu(x))^{\frac{1}{q-1}}} \quad \forall 0 \leq \phi \in W_0^{1,q'}(\Omega) \cap L^\infty(\Omega).$$

Proof. Let u be a solution to (P_λ), and let $0 \leq \phi \in W_0^{1,q'}(\Omega) \cap L^\infty(\Omega)$. Since $q' > 2$, then $\phi^{q'} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so it can be taken as test function in the weak formulation of (P_λ) to obtain, after using Young's inequality, that

$$\begin{aligned} \int_\Omega (\lambda u + \mu(x)|\nabla u|^q + f(x))\,\phi^{q'} &= \int_\Omega \nabla u \nabla(\phi^{q'}) = q' \int_\Omega \phi^{q'-1} \nabla u \nabla \phi \\ &\leq \int_\Omega \mu(x)|\nabla u|^q \phi^{q'} + \int_\Omega \frac{|\nabla\phi|^{q'}}{((q-1)\mu(x))^{\frac{1}{q-1}}}. \end{aligned}$$

Hence, it is now clear that the result follows. $\square$

Our aim in the next two subsections is to prove, for a fixed $\lambda_0 > 0$, an L^∞ estimate for the solutions to (P_λ) for all $\lambda > \lambda_0$. Such an estimate implies that zero is the only possible bifurcation point from infinity to problem (P_λ). This fact will be the key to prove multiplicity of solutions to (P_λ) for $\lambda > 0$ small enough.

4.3.1 A priori L^p estimates

This subsection is devoted to proving an L^p estimate on the supersolutions to (P_λ) for $\lambda > 0$. The techniques employed here have been taken from [120].

The first result of the subsection provides an apparently weak local estimate on the solutions to (P_λ). Notwithstanding, this is the starting point for proving the L^∞ estimate we are aiming at. Concerning the proof, we will argue similarly as in Proposition 4.3.6.

Lemma 4.3.7. *Assume that* (H1) *holds. Then, for every $\lambda_0 > 0$ and $\omega \subset\subset \Omega$ there exists $C > 0$ such that*

$$\int_\omega u \leq C. \tag{4.20}$$

for every supersolution u to (P_λ) with $\lambda > \lambda_0$.

Proof. Let $\phi \in C_c^1(\Omega)$ be such that $\omega \subset\subset \mathrm{supp}(\phi)$, $0 \leq \phi \leq 1$ in Ω and $\phi = 1$ in ω. Taking $\phi^\beta \in C_c^1(\Omega)$ for some $\beta > 1$ as test function in (P_λ) and using Young's inequality twice we obtain that

$$\begin{aligned}
\int_\Omega \left(\lambda u + \mu(x)\frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\phi^\beta &\leq \int_\Omega \nabla u \nabla(\phi^\beta) = \beta\int_\Omega \phi^{\beta-1}\nabla u \nabla\phi \\
&\leq \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + C\int_\Omega \frac{|\nabla(\phi^\beta)|^{q'}}{\phi^{\beta(q'-1)}} u^{\frac{\alpha}{q-1}} \\
&\leq \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + \frac{\lambda_0}{2}\int_\Omega u\phi^\beta + C\int_\Omega \left(\frac{|\nabla\phi|}{\phi}\right)^{\frac{q}{q-1-\alpha}}\phi^\beta.
\end{aligned}$$

Taking $\beta = \frac{q}{q-1-\alpha}$, the last term in the previous inequality is bounded. Therefore,

$$\int_\Omega \left(\lambda_0 u + \mu_0\frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\phi^\beta \leq \frac{\mu_0}{2}\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\phi^\beta + \frac{\lambda_0}{2}\int_\Omega u\phi^\beta + C,$$

so (4.20) follows by taking into account that $\phi = 1$ in ω. $\square$

The following is a slightly more general version of [34, Lemma 3.2].

Lemma 4.3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class $\mathscr{C}^2$, and let $0 \leq h \in L^1(\Omega)$ and $v \in H^1(\Omega)$ be such that $v^- \in H_0^1(\Omega)$ and $-\Delta v \geq h$ in Ω. Then, there exists a constant $C > 0$ depending only on Ω such that*

$$\frac{v}{\delta} \geq C \int_\Omega \delta h \quad \text{a.e. } x \in \Omega.$$

Proof. Let us consider the following problem for all $n \in \mathbb{N}$:

$$\begin{cases} -\Delta v_n = T_n(h(x)), & x \in \Omega, \\ v_n = 0, & x \in \partial\Omega, \end{cases}$$

It is well-known that it has a unique solution $v_n \in C_0^{1,\nu}(\overline{\Omega})$ for all $\nu \in (0,1)$. Moreover, [34, Lemma 3.2] implies that

$$v_n(x) \geq C\delta(x) \int_\Omega \delta T_n(h) \quad \forall x \in \Omega,$$

for some $C > 0$ depending only on Ω. In particular, it does not depend on n.

On the other hand, by comparison, it is clear that $v_n \leq v$ a.e. in Ω, so

$$v \geq C\delta \int_\Omega \delta T_n(h) \quad \text{a.e. } x \in \Omega.$$

We conclude the proof by letting n tend to infinity. $\square$

Next lemma is an immediate consequence of Lemma 4.3.8.

Lemma 4.3.9. *Assume that* (H1) *holds. Then, there exists $C > 0$ such that*

$$u(x) \geq C\delta(x) \int_\Omega \left(\lambda u + \mu(y) \frac{|\nabla u|^q}{u^\alpha} + f(y) \right) \delta(y) dy \quad \text{a.e. } x \in \Omega, \tag{4.21}$$

for every supersolution u to (P_λ) with $\lambda > 0$.

Combining Lemmas 4.3.7 and 4.3.9 we obtain in the following result some estimates in weighted Lebesgue spaces.

Lemma 4.3.10. *Assume that* (H1) *holds. Then, for every $\lambda_0 > 0$ there exists $C > 0$ such that*

1. $\|u\|_{L^p(\Omega,\delta)} \leq C \quad \forall p \in \left[1, \frac{N+1}{N-1}\right)$,
2. $\left\| \frac{|\nabla u|^q}{u^\alpha} \right\|_{L^1(\Omega,\delta)} = C \| |\nabla u^{1-\frac{\alpha}{q}}| \|_{L^q(\Omega,\delta)} \leq C$,

for every supersolution u to (P_λ) *with* $\lambda > \lambda_0$.

Proof. Integrating both sides of inequality (4.21) over any open set $\omega \subset\subset \Omega$ and using the estimate (4.20) we deduce that

$$\int_\Omega (-\Delta u)\delta = \int_\Omega \left(\lambda u + \mu(x)\frac{|\nabla u|^q}{u^\alpha} + f(x)\right)\delta \le C\left(\int_\omega u\right)\left(\int_\omega \delta\right)^{-1} \le C.$$

In particular,

$$\int_\Omega \frac{|\nabla u|^q}{u^\alpha}\delta \le C,$$

and this is equivalent to item 2. Regarding item 1, observe that

$$\|\Delta u\|_{L^1(\Omega,\delta)} \le C.$$

Hence, by [70, Proposition 2.2] we obtain directly item 1. □

We finish the subsection with the best L^p estimate for supersolutions that we obtain with these techniques.

Lemma 4.3.11. *Assume that* (H1) *holds. Then, for every* $\lambda_0 > 0$ *there exists* $C > 0$ *such that*

$$\|u\|_{L^m(\Omega)} \le C \tag{4.22}$$

for every supersolution u to (P_λ) *with* $\lambda > \lambda_0$*, where* $m = \frac{(q-\alpha)N}{N-q+1} \in (q-\alpha, (q-\alpha)^*)$.

Proof. Let us denote $v = u^{1-\frac{\alpha}{q}}$. Since $1 - \frac{\alpha}{q} > \frac{1}{2}$, we can argue as in [38, Lemma 2.6] to prove that $v \in H_0^1(\Omega)$. Then, [120, Proposition 2] implies that

$$\int_\Omega v^q \delta^{-(q-1)} \le C\left(\int_\Omega v\delta\right)^q + C\left(\int_\Omega |\nabla v|^q \delta\right),$$

and

$$\left(\int_\Omega v^{q^*} \delta^{\frac{N}{N-q}}\right)^{q/q^*} \le C\left(\int_\Omega v\delta\right)^q + C\left(\int_\Omega |\nabla v|^q \delta\right).$$

Hence, by Lemma 4.3.10 we derive that

$$\int_\Omega v^q \delta^{-(q-1)} \le C \quad \text{and} \quad \int_\Omega v^{q^*} \delta^{\frac{N}{N-q}} \le C. \tag{4.23}$$

Now, [120, Lemma 3] implies that

$$\int_\Omega v^b \delta^\gamma \le C\left(\int_\Omega v^q \delta^{-(q-1)}\right)^\theta \left(\int_\Omega v^{q^*} \delta^{\frac{N}{N-q}}\right)^{1-\theta}, \tag{4.24}$$

where

$$b = \frac{qN}{N-q+1}, \quad \theta = \frac{q^*-b}{q^*-q} \in (0,1) \quad \text{and} \quad \gamma = \frac{N}{N-q} - \frac{(q^*-b)(q-1+\frac{N}{N-q})}{q^*-q}.$$

It is easy to check that, in fact, $\gamma = 0$. Therefore, recalling that $m = b\left(1-\frac{\alpha}{q}\right)$, by (4.24) and (4.23) we conclude that

$$\int_\Omega v^b = \int_\Omega u^m \leq C,$$

and the result holds true. $\square$

4.3.2 A priori L^∞ estimates

In this subsection we will show how to obtain L^∞ estimates on the solutions to (P_λ) for $\lambda > 0$ by combining the L^p estimate given by Lemma 4.3.11 and a bootstrapp argument. We will make use of several results in [85]. In fact, the ideas in such a paper will be used also to derive some new results which provide analogous estimates in our singular framework.

We start the subsection with the easier case $\alpha = 0$, which is interesting itself; we will deal with the singular case $\alpha \in (0, q-1)$ later. Thus we state and prove the following

Proposition 4.3.12. *Assume that* (H1) *holds with* $\alpha = 0$*, and consider the sequence* $\{Q_n\}$ *defined by* (4.2)*, i.e.,*

$$Q_n = \begin{cases} 2 & \forall n \leq 4, \\ \dfrac{n+2-\sqrt{n^2-4n-4}}{4} & \forall n \geq 5. \end{cases}$$

Then, for every $q \in (1, Q_N] \setminus \{2\}$ *and every* $\lambda_0 > 0$*, there exists* $C > 0$ *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \tag{4.25}$$

for every solution u *to* (P_λ) *with* $\lambda > \lambda_0$*.*

Proof. In this proof, C denotes a positive constant independent of u and λ whose value may vary from line to line.

We start by assuming that $1 < q < \frac{N}{N-1}$. Observe that $\frac{N}{N-1} < Q_N$, so $q \leq Q_N$ is not a restriction in this case.

Let us denote $h(x) = (\lambda+1)u + f(x)$. Then, u satisfies

$$\begin{cases} u - \Delta u = \mu(x)|\nabla u|^q + h(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

We know from Lemma 4.3.11 that $\|u\|_{L^m(\Omega)} \leq C$, where $m = \frac{qN}{N-q+1}$. Consequently, $\|h\|_{L^p(\Omega)} \leq C$, where $p = \min\{m, p_0\}$. If $m > \frac{N}{2}$, and taking into account that $p_0 > \frac{N}{2}$, then [85, Theorem 5.8, item (i)] implies that $\|u\|_{L^\infty(\Omega)} \leq C$.

Let us assume now that $m = \frac{N}{2}$. Then, [85, Theorem 5.8, item (ii)] implies that $\|u\|_{L^p(\Omega)} \leq C$ for all $p < \infty$. In particular, $\|h\|_{L^{p_0}(\Omega)} \leq C$. Since $p_0 > \frac{N}{2}$, then again item (i) of the same mentioned theorem yields the L^∞ estimate.

Suppose now that $(2^*)' < m < \frac{N}{2}$. Let us define the sequence $\{m_n\}$ inductively as

$$m_n = m_{n-1}^{**} = \frac{Nm_{n-1}}{N-2m_{n-1}} \quad \forall n \in \mathbb{N},$$

where $m_0 = m$. This is clearly an increasing sequence. Moreover, using one more time [85, Theorem 5.8, item (iii)], it is easy to see that $\|u\|_{L^{m_n}(\Omega)} \leq C$ for $n \in \mathbb{N}$ as long as $m_n < \frac{N}{2}$. In particular, the same holds for h.

Assume by contradiction that $m_n < \frac{N}{2}$ for all $n \in \mathbb{N}$. Since $\{m_n\}$ is increasing and bounded from above, there exists $l \leq \frac{N}{2}$ such that, passing to a not relabeled subsequence, $m_n \to l$. Consequently,

$$l = \frac{Nl}{N-2l}.$$

From this equality we deduce that $l = 0$. But this is a contradiction because $m_0 > 0$ and the sequence is increasing. Therefore, $m_n \geq \frac{N}{2}$ for some $n \in \mathbb{N}$, so the previous cases imply that $\|u\|_{L^\infty(\Omega)} \leq C$.

It only remains to consider the case $1 < m \leq (2^*)'$. Now, item (iv) of the same theorem implies that

$$\|(1+u)^{\tau-1}u\|_{L^{2^*}(\Omega)} \leq C, \quad \text{where} \quad \tau = \frac{m(N-2)}{2(N-2m)} = \frac{m^{**}}{2^*} \leq 1.$$

On the other hand, it is straightforward to prove that, for any $a \in (0,1)$, there exists a constant $b > 0$ such that

$$as^\tau \leq \frac{s}{(1+s)^{1-\tau}} + b \quad \forall s \geq 0.$$

Then, with $m_n = m_{n-1}^{**}$ and $m_0 = m$, as before,

$$\|u\|_{L^{m_1}(\Omega)} = \|u\|_{L^{2^*\tau}(\Omega)} \leq C(\|(1+u)^{\tau-1}u\|_{L^{2^*}(\Omega)} + 1) \leq C.$$

In particular, $\|h\|_{L^{m_1}(\Omega)} \leq C$. It can be proved inductively that $\|u\|_{L^{m_n}(\Omega)} \leq C$ as long as $m_n \leq (2^*)'$. Arguing as above, we deduce that $\{m_n\}$ is increasing and divergent. Hence, $m_n > (2^*)'$ for some $n \in \mathbb{N}$, and the proof concludes using the previous cases.

We now turn to the range $\frac{N}{N-1} < q < 2$. The procedure is the same as above. However, in this case, instead of Theorem 5.8, one has to apply (a finite number of times) either [85, Theorem 4.9] or [85, Theorem 3.8], depending on the value of q. In both cases, one has to verify in the first step of the bootstrap that $h \in L^{\frac{(q-1)N}{q}}(\Omega)$ so that the hypotheses of both theorems are satisfied. We know by virtue of Lemma 4.3.11 that $h \in L^m(\Omega)$, so we have to impose that

$$\frac{N(q-1)}{q} \leq \frac{qN}{N-q+1}.$$

One can easily check that the previous inequality is satisfied if and only if $q \leq Q_N$.

It is left to consider the case $q = \frac{N}{N-1}$. Since $\frac{N}{N-1} < Q_N$, we can take $\varepsilon > 0$ small enough so that

$$\frac{N}{N-1} < q + \varepsilon < Q_N.$$

Moreover, we have by Young's inequality that

$$\mu(x)|\xi|^q + h(x) \leq \mu(x)|\xi|^{q+\varepsilon} + h_\varepsilon(x) \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega,$$

where $h(x) = (\lambda+1)u + f(x)$ and $h_\varepsilon(x) = h(x) + C_\varepsilon$ for some $C_\varepsilon > 0$. Therefore, the previous case can be applied and the proof concludes. $\square$

We deal now with the singular case. For this purpose, it is necessary to derive results similar to those from [85] mentioned in the previous proof, but valid for singular equations. Even though our results are not proper extensions in the whole generality (as in [85] the solutions are weaker than ours and the terms in their equation are not explicit and only satisfy growth restrictions), they are new in considering singular terms.

The mentioned results will be concerned with the following auxiliary problem:

$$\begin{cases} \beta u - \Delta u = \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + h(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.26}$$

where the parameters satisfy

$$1 < q < 2, \quad \alpha \in [0, q-1), \quad \beta > 0, \quad 0 \lneq \mu \in L^\infty(\Omega). \tag{4.27}$$

For any $p \in \left(1, \frac{N}{2}\right)$, let us denote

$$\sigma = \frac{(N-2)p}{2(N-2p)} \in \left(\frac{1}{2}, +\infty\right). \tag{4.28}$$

The following result provides estimates on solutions to (4.26) when q is large and h has enough summability.

Proposition 4.3.13. *Assume that q,α,β,μ satisfy* (4.27)*, and assume in addition that*

$$q > \frac{N}{N-1}.$$

Then, for all $M > 0$ and $p \geq \frac{N(q-1-\alpha)}{q-2\alpha}$, $p > 1$, there exists $C > 0$ such that, for any $h \in L^p(\Omega)$ with $\|h\|_{L^p(\Omega)} \leq M$ and for any solution u to problem (4.26)*, the following holds:*

1. *If $p < \frac{2N}{N+2}$, then $\sigma \in \left(\frac{1}{2},1\right)$ and $\|u(u+1)^{\sigma-1}\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (4.28)*;*
2. *if $\frac{2N}{N+2} \leq p < \frac{N}{2}$, then $\sigma \geq 1$ and $\|u\|_{H_0^1(\Omega)} + \|u^\sigma\|_{H_0^1(\Omega)} \leq C$, where σ is defined by* (4.28)*;*
3. *if $p = \frac{N}{2}$, then $\|u^\tau\|_{H_0^1(\Omega)} \leq C$ for all $\tau < \infty$, and*
4. *if $p > \frac{N}{2}$, then $\|u\|_{L^\infty(\Omega)} \leq C$.*

Proof. **Proof of 1.**

First of all, note that $\sigma \in \left(\frac{1}{2},1\right)$ if and only if $p \in \left(1,\frac{2N}{N+2}\right)$.

Observe also that, if $1+\frac{2}{N}+\frac{N-2}{N}\alpha \leq q < 2$, then $\frac{N(q-1-\alpha)}{q-2\alpha} \geq \frac{2N}{N+2}$, so the condition in item 1 may be fulfilled only if

$$\frac{N}{N-1} < q < 1+\frac{2}{N}+\frac{N-2}{N}\alpha.$$

We will assume consequently that q belongs to such an interval. In fact, we will divide the proof of this item into several steps, considering different ranges for p and q. It can be easily checked that each of these ranges is nonempty.

Case 1: $\frac{N}{N-1} < q < 1+\frac{2}{N}$ and $\frac{N(q-1-\alpha)}{q-2\alpha} \leq p \leq \frac{N(q-1)}{q}$, $p > 1$.

In this case, there exists $\theta \in \left[0,\frac{N-1}{N-2}\left(q-\frac{N}{N-1}\right)\right) \cap [0,\alpha]$ such that $p = \frac{N(q-1-\theta)}{q-2\theta}$. Then, it is clear that the following relation is satisfied:

$$\frac{2}{2-q}(2\sigma-1-\theta-q(\sigma-1)) = 2^*\sigma. \tag{4.29}$$

Let us now consider the following functions defined for every $t \geq 0$:

$$
\begin{aligned}
\phi(t) &= \frac{1}{(\zeta+t)^{1-\sigma}}\left(\frac{t}{\zeta+t}\right)^{\frac{1}{2}},\\
\Phi_1(t) &= \int_0^t \phi(s)ds,\\
\Phi_2(t) &= \int_0^t \phi(s)^2 ds,
\end{aligned}
$$

where $\zeta > 0$ will be fixed later. First of all observe that

$$
\nabla v \nabla \Phi_2(v) = |\nabla \Phi_1(v)|^2
$$

for any $v \in H_0^1(\Omega)$. Moreover, using (4.29) and also that $2\sigma - 1 = \frac{2^*\sigma}{p'}$, it can be proved respectively that

$$
\left(t^{-\theta}\phi(t)^{-q}\Phi_2(t)\right)^{\frac{2}{2-q}} \leq C\left(\Phi_1(t)^{2^*} + \zeta^{2^*\sigma}\right) \quad \forall t \geq 0. \tag{4.30}
$$

and

$$
\Phi_2(t) \leq C\Phi_1(t)^{\frac{2^*}{p'}} \quad \forall t \geq 0. \tag{4.31}
$$

For $k > 0$, let us take $\Phi_2(G_k(u)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (4.26), so that we obtain

$$
\beta \int_\Omega u\Phi_2(G_k(u)) + \int_\Omega |\nabla \Phi_1(G_k(u))|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha} + h(x)\right)\Phi_2(G_k(u)). \tag{4.32}
$$

Let us now estimate the nonlinear term. Thanks to (4.30) we derive that

$$
\begin{aligned}
\int_\Omega \mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) \leq & \frac{\|\mu\|_{L^\infty(\Omega)}}{k^{\alpha-\theta}}\int_{\{u\geq k\}} |\nabla \Phi_1(G_k(u))|^q \frac{\Phi_2(G_k(u))}{G_k(u)^\theta \phi(G_k(u))^q}\\
\leq & C\left(\int_\Omega |\nabla \Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\int_{\{u\geq k\}}\left(\frac{\Phi_2(G_k(u))}{G_k(u)^\theta \phi(G_k(u))^q}\right)^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}}\\
\leq & C\left(\int_\Omega |\nabla \Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\int_\Omega \left(\Phi_1(G_k(u))^{2^*} + \zeta^{2^*\sigma}\right)\right)^{1-\frac{q}{2}}\\
\leq & C\left(\int_\Omega |\nabla \Phi_1(G_k(u))|^2\right)^{\frac{q}{2}}\left(\left(\int_\Omega |\nabla \Phi_1(G_k(u))|^2\right)^{\frac{2^*}{2}\left(1-\frac{q}{2}\right)} + \zeta^{2^*\sigma\left(1-\frac{q}{2}\right)}\right).
\end{aligned}
$$

We now focus on the last term in (4.32). Using (4.31) we deduce that

$$\begin{aligned}
\int_\Omega |h(x)|\Phi_2(G_k(u)) &= \int_{\{|h(x)|\le\beta u\}} |h(x)|\Phi_2(G_k(u)) + \int_{\{|h(x)|>\beta u\}} |h(x)|\Phi_2(G_k(u)) \\
&\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\int_{\{|h(x)|>\beta k\}} |h(x)|\Phi_1(G_k(u))^{\frac{2^*}{p'}} \\
&\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\left(\int_{\{|h(x)|\ge\beta k\}} |h(x)|^p\right)^{\frac{1}{p}}\left(\int_\Omega \Phi_1(G_k(u))^{2^*}\right)^{\frac{1}{p'}} \\
&\le \beta\int_\Omega u\Phi_2(G_k(u)) + C\left(\int_{\{|h(x)|\ge\beta k\}} |h(x)|^p\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla\Phi_1(G_k(u)|^2\right)^{\frac{2^*}{2p'}}.
\end{aligned}$$

If we denote $Y_k = \|\Phi_1(G_k(u))\|_{H_0^1(\Omega)}$, we have proved so far that

$$Y_k^2 \le CY_k^q\left(Y_k^{2^*\left(1-\frac{q}{2}\right)} + \zeta^{2^*\sigma\left(1-\frac{q}{2}\right)}\right) + C\|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}Y_k^{\frac{2^*}{p'}}.$$

Hence, using Young's inequality we obtain that

$$\frac{1}{2}Y_k^2 \le CY_k^{q+2^*\left(1-\frac{q}{2}\right)} + C\zeta^{2^*\sigma} + C\|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma},$$

or equivalently,

$$C_1Y_k^2 - C_2Y_k^{q+2^*\left(1-\frac{q}{2}\right)} \le \zeta^{2^*\sigma} + \|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma}, \tag{4.33}$$

for some $C_1, C_2 > 0$ independent of k and ζ.

Let us define the function $F : [0,+\infty) \to \mathbb{R}$ by

$$F(Y) = C_1Y^2 - C_2Y^{q+2^*\left(1-\frac{q}{2}\right)} \quad \forall Y \ge 0.$$

Since $q < 2$, it easy to see that

$$2 < q + 2^*\left(1-\frac{q}{2}\right).$$

This means that F is positive near zero, negative far from zero, and has a unique maximum $F^* > 0$ with a corresponding unique maximizer $Z^* > 0$.

We now choose $\zeta = \min\left\{1, \left(\frac{F^*}{2}\right)^{\frac{1}{2^*\sigma}}\right\}$. Thus,

$$\max_{Y\ge 0}(F(Y) - \zeta^{2^*\sigma}) = F^* - \zeta^{2^*\sigma} \ge \frac{F^*}{2} > 0.$$

Let us now consider

$$k^* = \inf\left\{k > 0 : \|h\chi_{\{|h(x)|\ge\beta k\}}\|_{L^p(\Omega)}^{2\sigma} < F^* - \zeta^{2^*\sigma}\right\}.$$

Hence, for any $\rho > 0$, the equation $F(Y) = \zeta^{2^*\sigma} + \|h\chi_{\{h(x)\ge\beta(k^*+\rho)\}}\|^{2\sigma}_{L^p(\Omega)}$ has two roots Z_1 and Z_2 such that $Z_1 < Z^* < Z_2$. By virtue of inequality (4.33), it holds that for every $k \ge k^* + \rho$, either $Y_k \le Z_1$ or $Y_k \ge Z_2$. But the function $k \mapsto Y_k$ is continuous and tends to zero as k tends to infinity. Therefore,

$$Y_{k^*+\rho} \le Z_1 < Z^*.$$

If we let now ρ tend to zero, we obtain that

$$Y_{k^*} = \|\Phi_1(G_{k^*}(u))\|_{H_0^1(\Omega)} \le Z^*.$$

Notice that

$$\begin{aligned}
\|\Phi_1(G_k(u))\|^2_{H_0^1(\Omega)} &= \int_\Omega \frac{|\nabla G_k(u)|^2 G_k(u)}{(\zeta + G_k(u))^{2(1-\sigma)+1}} \ge \int_\Omega \frac{|\nabla G_k(u)|^2 G_k(u)}{(1 + G_k(u))^{2(1-\sigma)+1}} \\
&\ge \int_\Omega \frac{|\nabla u|^2 (u-k)}{(1+u-k)^{2(1-\sigma)+1}} \chi_{\{u \ge k+1\}} \\
&\ge \frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{(1+u-k)^{2(1-\sigma)}} \chi_{\{u \ge k+1\}} \\
&\ge \frac{1}{2^{2(1-\sigma)+1}} \int_\Omega \frac{|\nabla G_{k+1}(u)|^2}{(G_{k+1}(u)+1)^{2(1-\sigma)}}.
\end{aligned}$$

Hence, we have that

$$\int_\Omega \frac{|\nabla G_k(u)|^2}{(G_k(u)+1)^{2(1-\sigma)}} \le C \quad \forall k \ge k^* + 1. \tag{4.34}$$

For $k \ge k^* + 1$, estimate (4.34) implies that

$$\begin{aligned}
\left\| \frac{u}{(1+u)^{1-\sigma}} \right\|^2_{H_0^1(\Omega)} &= \int_\Omega \frac{|\nabla u|^2}{(1+u)^{2(1-\sigma)}} \left(\frac{1+\sigma u}{1+u} \right)^2 \\
&= \int_\Omega \frac{|\nabla G_k(u)|^2}{(1+u)^{2(1-\sigma)}} \left(\frac{1+\sigma u}{1+u} \right)^2 + \int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}} \left(\frac{1+\sigma u}{1+u} \right)^2 \\
&\le C + \int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}} \left(\frac{1+\sigma u}{1+u} \right)^2.
\end{aligned}$$

We claim now that

$$\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+u)^{2(1-\sigma)}} \left(\frac{1+\sigma u}{1+u} \right)^2 \le C. \tag{4.35}$$

Indeed, let us define the real functions for all $t \ge 0$:

$$\begin{aligned}
z(t) &= \frac{1}{(1+t)^{2(1-\sigma)}} \left(\frac{1+\sigma t}{1+t} \right)^2, \\
y(t) &= \frac{1}{t} \int_0^t z(s)\,ds.
\end{aligned}$$

It is easy to see that
$$ty'(t)+y(t)=z(t)\quad \forall t\geq 0,$$
and also that
$$y(t)\leq Cz(t)\quad \forall t\geq 0, \text{ for some } C>0.$$
Now we take $T_k(u)y(u)$ as test function in the weak formulation of (4.26) and get
$$\int_\Omega y(u)|\nabla T_k(u)|^2+\int_\Omega T_k(u)y'(u)|\nabla u|^2=\int_\Omega\left(\mu(x)\frac{|\nabla u|^q}{u^\alpha}+h(x)-\beta u\right)T_k(u)y(u). \tag{4.36}$$
Concerning the left hand side of (4.36), observe that
$$\int_\Omega y(u)|\nabla T_k(u)|^2+\int_\Omega T_k(u)y'(u)|\nabla u|^2=\int_\Omega z(u)|\nabla T_k(u)|^2+k\int_\Omega y'(u)|\nabla G_k(u)|^2, \tag{4.37}$$
where, by virtue of (4.34),
$$-k\int_\Omega y'(u)|\nabla G_k(u)|^2\leq\int_\Omega\frac{ky(u)}{u}|\nabla G_k(u)|^2\leq C\int_\Omega z(u)|\nabla G_k(u)|^2\leq C. \tag{4.38}$$
Gathering (4.36), (4.37) and (4.38) together we deduce that
$$\begin{aligned}\int_\Omega z(u)|\nabla T_k(u)|^2&\leq C(k^*+1)\left(\int_\Omega y(u)|\nabla u|^q+1\right) \\ &\leq C(k^*+1)\left(\int_\Omega z(u)|\nabla T_k(u)|^q+\int_\Omega z(u)|\nabla G_k(u)|^q+1\right).\end{aligned} \tag{4.39}$$

We will show now that there exists $k_0>0$ independent of $\|h\|_{L^p(\Omega)}$ such that $k^*\leq k_0$. Indeed, the absolute continuity of the integral implies that there exists $\rho>0$ such that, if $|\{|h(x)|\geq\beta k_0\}|<\rho$ for some $k_0>0$, then$\|h\chi_{\{|h(x)|\geq\beta k_0\}}\|_{L^p(\Omega)}^{2\sigma}<F^*-\zeta^{2^*\sigma}$, i.e., $k^*\leq k_0$. Observe that, if $k_0>\frac{M|\Omega|^{\frac{1}{p'}}}{\beta\rho}$, where $\|h\|_{L^p(\Omega)}\leq M$, then $|\{|h(x)|\geq\beta k_0\}|<\rho$ and k_0 does not depend on $\|h\|_{L^p(\Omega)}$, as we wanted to show.

Therefore, we can estimate k^* in (4.39) and, by virtue of (4.34), we obtain that
$$\int_\Omega z(u)|\nabla T_k(u)|^2\leq C\left(\int_\Omega z(u)|\nabla T_k(u)|^q+1\right).$$
We finally arrive at (4.35) by using Young's inequality and by the fact that z is a bounded function. This concludes the proof of Case 1.

Case 2: $\frac{N}{N-1}<q<1+\frac{2}{N}$ and $\frac{N(q-1)}{q}<p<\frac{2N}{N+2}$.

Observe that, in this range, one has in particular that $\|h\|_{L^r(\Omega)}\leq|\Omega|^{\frac{p-r}{p}}M$, where $r=\frac{N(q-1)}{q}$. Then, Case 1 can be applied for $\theta=0$. We will use this fact later. Let us also denote $\sigma_r=\frac{(N-2)r}{2(N-2r)}=\frac{(N-2)(q-1)}{2(2-q)}\in\left(\frac{1}{2},1\right)$.

Recalling the definitions of ϕ, Φ_1 and Φ_2 in the previous case, for some $k>0$ we take $\Phi_2(G_k(u))$ as test function in the weak formulation of (4.26), so that we obtain

$$\int_\Omega |\nabla \Phi_1(G_k(u))|^2 \le C \int_\Omega \left(\frac{|\nabla G_k(u)|^q}{u^\alpha} + |h(x)| \right) \Phi_2(G_k(u)). \tag{4.40}$$

It can be easily proved that

$$\Phi_2(t) \le C\phi(t)\Phi_1(t) \quad \forall t \ge 0,$$

for some $C>0$. Thus, using this inequality in the singular term of (4.40), we deduce that

$$\begin{aligned}
\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) &\le C\int_\Omega |\nabla G_k(u)|^{q-1}|\nabla G_k(u)|\phi(G_k(u))\Phi_1(G_k(u)) \\
&\le C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} \left(\int_\Omega |\nabla G_k(u)|^2\phi(G_k(u))^2\right)^{\frac{1}{2}} \left(\int_\Omega \Phi_1(G_k(u))^{2^*}\right)^{\frac{1}{2^*}} \\
&\le C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} \int_\Omega |\nabla \Phi_1(G_k(u))|^2.
\end{aligned} \tag{4.41}$$

Now we claim that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C$$

for some $k>0$ large enough. Indeed, since $q < 1+\frac{2}{N}$, we can apply Hölder's inequality with exponent $\frac{2}{N(q-1)} > 1$ and obtain that, for any $k>0$,

$$\begin{aligned}
&\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C\int_\Omega \left|\nabla \frac{G_k(u)}{(1+G_k(u))^{1-\sigma_r}}\right|^{N(q-1)} (1+G_k(u))^{(1-\sigma_r)N(q-1)} \\
&\le C\left(\int_\Omega \left|\nabla \frac{G_k(u)}{(1+G_k(u))^{1-\sigma_r}}\right|^2\right)^{\frac{N(q-1)}{2}} \left(\int_\Omega (1+G_k(u))^{\left(\frac{2}{N(q-1)}\right)'(1-\sigma_r)N(q-1)}\right)^{1-\frac{N(q-1)}{2}}.
\end{aligned}$$

Therefore, by Case 1 and Sobolev's inequality,

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C + C\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{1-\sigma_r}.$$

Hence, the fact that $\sigma_r < 1$ implies that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \quad \forall k \ge k^*+1,$$

and the proof of the claim is done. As a consequence, it can be shown, again by virtue of the absolute continuity of the integral, that the limit

$$\lim_{k\to\infty}\left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} = 0,$$

is uniform in u. Hence, from (4.41) we deduce that there exists $k_0 > 0$ independent of u such that

$$\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha}\Phi_2(G_k(u)) \le \frac{1}{2}\int_\Omega |\nabla \Phi_1(G_k(u))|^2 \quad \forall k \ge k_0.$$

Then, we derive from (4.40) that

$$\int_\Omega |\nabla \Phi_1(G_k(u))|^2 \le C\int_\Omega |h(x)|\Phi_2(G_k(u)) \quad \forall k \ge k_0.$$

By virtue of (4.31) we immediately obtain the estimate

$$\int_\Omega |\nabla \Phi_1(G_{k_0}(u))|^2 \le C.$$

We conclude this case similarly as Case 1.

Case 3: $1+\frac{2}{N} \le q < 1+\frac{2}{N}+\frac{N-2}{N}\alpha$ and $\frac{N(q-1-\alpha)}{q-2\alpha} \le p \le \frac{2N}{N-2}$, $p>1$.

In this case, it is clear that $\frac{2N}{N+2} \le \frac{N(q-1)}{q}$. Thus, the proof of Case 1 can be reproduced here.

We conclude this way the proof of item 1.

Proof of 2.

Case 1: $1+\frac{2}{N} \le q < 2$ and $\frac{2N}{N+2} \le p \le \frac{N(q-1)}{q}$.

In this case, there exists $\theta \in \left[0, \left(q-1-\frac{2}{N}\right)\frac{N}{N-2}\right] \cap [0,\alpha]$ such that $p = \frac{N(q-1-\theta)}{q-2\theta}$. Then, (4.29) holds.

Now, for $k>0$, let us take $G_k(u)^{2\sigma-1}$ as test function in the weak formulation of (4.26). Notice that this choice is valid since $\sigma > 1$. Then, following the arguments of the proof of Case 1 of item 1 we obtain that

$$\|G_{k^*}(u)^\sigma\|_{H_0^1(\Omega)} \le C,$$

where $k^* = \inf\left\{k>0 : \|h\chi_{\{|h(x)|\ge \beta k\}}\|_{L^p(\Omega)} < F^*\right\}$ and $F^*>0$ is the unique maximum of the function $F(Y) = CY^{2-\frac{2^*}{p'}} - Y^{q+2^*\left(1+\frac{q}{2}\right)-\frac{2^*}{p'}}$, $Y \ge 0$, for some $C>0$.

Observe that

$$G_{k^*}(u) = u - k^* \ge 1 \quad \text{in the set } \{u \ge k^*+1\}.$$

Therefore,

$$\begin{aligned}\int_\Omega |\nabla G_{k^*+1}(u)|^2 &= \int_\Omega \chi_{\{u\ge k^*+1\}}|\nabla u|^2 \le \int_\Omega \chi_{\{u\ge k^*+1\}}|\nabla u|^2 G_{k^*}(u)^{2(\sigma-1)} \\ &\le \int_\Omega \chi_{\{u\ge k^*\}}|\nabla u|^2 G_{k^*}(u)^{2(\sigma-1)} = \frac{1}{\sigma^2}\int_\Omega |\nabla G_{k^*}(u)^\sigma|^2 \le C.\end{aligned}$$

Now we take $T_{k^*+1}(u)$ as test function in the weak formulation of (4.26) so we get

$$\begin{aligned}\int_\Omega |\nabla T_{k^*+1}(u)|^2 &= \int_\Omega \left(\mu(x)\frac{|\nabla u|^q}{u^\alpha} + h(x) - \beta u\right) T_{k^*+1}(u)\\ &\leq C(k^*+1)^{1-\alpha}\int_\Omega |\nabla u|^q + (k^*+1)\int_\Omega |h(x)|\\ &\leq C(k^*+1)\left(\int_\Omega |\nabla T_{k^*+1}(u)|^q + \int_\Omega |\nabla G_{k^*+1}(u)|^q + 1\right)\\ &\leq C(k^*+1)\left(\int_\Omega |\nabla T_{k^*+1}(u)|^q + 1\right).\end{aligned}$$

Again, the absolute continuity of the integral implies that $k^* \leq k_0$ for some $k_0 > 0$ independent of $\|h\|_{L^p(\Omega)}$. Thus we can estimate k^* in the last inequality and, using Young's inequality, deduce that

$$\int_\Omega |\nabla T_{k^*+1}(u)|^2 \leq C.$$

Summarizing, $\int_\Omega |\nabla u|^2 \leq C$, which proves the first part of item 2. Moreover,

$$\begin{aligned}\int_\Omega |\nabla u^\sigma|^2 &= \int_\Omega |\nabla G_{k^*}(u)^\sigma|^2 + \int_\Omega |\nabla T_{k^*}(u)^\sigma|^2\\ &\leq C + \sigma^2\int_\Omega T_{k^*}(u)^{2(\sigma-1)}|\nabla T_{k^*}(u)|^2 \leq C + \sigma^2 (k^*)^{2(\sigma-1)}\int_\Omega |\nabla u|^2 \leq C.\end{aligned}$$

Thus, the proof of Case 1 is concluded.

Case 2: $1+\frac{2}{N} \leq q < 2$ and $\frac{N(q-1)}{q} < p < \frac{N}{2}$.

Let us denote, as above, $r = \frac{N(q-1)}{q}$ and $\sigma_r = \frac{(N-2)(q-1)}{2(2-q)} \geq 1$. It is easy to see that $\|h\|_{L^r(\Omega)} \leq |\Omega|^{\frac{p-r}{p}} M$ and $\sigma_r = \frac{(N-2)r}{2(N-2r)}$, so Case 1 of item 2 can be applied.

For some $k > 0$, we take $G_k(u)^{2\sigma-1}$ as test function in the weak formulation of (4.26), so we obtain

$$\beta\int_\Omega u G_k(u)^{2\sigma-1} + \frac{2\sigma-1}{\sigma^2}\int_\Omega |\nabla G_k(u)^\sigma|^2 = \int_\Omega \left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^\alpha} + h(x)\right) G_k(u)^{2\sigma-1}. \tag{4.42}$$

In order to estimate the nonlinear term, notice that

$$\frac{q}{2} + \frac{2-q}{2^*} + \frac{2-q}{N} = 1.$$

Hence, we can use Hölder inequality with those three exponents, and we deduce that

$$\begin{aligned}\int_\Omega \frac{|\nabla G_k(u)|^q}{u^\alpha} G_k(u)^{2\sigma-1} &\leq C\int_\Omega |\nabla G_k(u)|^q G_k(u)^{(\sigma-1)q} G_k(u)^{(2-q)\sigma} G_k(u)^{q-1}\\ &\leq C\left(\int_\Omega |\nabla G_k(u)^\sigma|^2\right)^{\frac{q}{2}}\left(\int_\Omega G_k(u)^{\sigma 2^*}\right)^{\frac{2-q}{2^*}}\left(\int_\Omega G_k(u)^{2^*\sigma_r}\right)^{\frac{2-q}{N}}\\ &\leq C\|G_k(u)\|^{q-1}_{L^{2^*\sigma_r}(\Omega)}\int_\Omega |\nabla G_k(u)^\sigma|^2.\end{aligned}$$

Now, thanks to Case 1 of item 2 and the absolute continuity of the integral, there exists $k_0 > 0$ independent of u such that

$$C\|G_k(u)\|^{q-1}_{L^{2^*\sigma r}(\Omega)} < \frac{2\sigma - 1}{\sigma^2} \quad \forall k \geq k_0.$$

Then, from (4.42) we derive that

$$C\int_\Omega |\nabla G_k(u)^\sigma|^2 \leq \|h\|_{L^p(\Omega)} \left(\int_\Omega G_k(u)^{(2\sigma-1)p'}\right)^{\frac{1}{p'}} \quad \forall k \geq k_0.$$

Since $(2\sigma - 1)p' = 2^*\sigma$, we conclude that

$$C\int_\Omega |\nabla G_k(u)^\sigma|^2 \leq \left(\int_\Omega |\nabla G_k(u)^\sigma|^2\right)^{\frac{2^*}{2p'}} \quad \forall k \geq k_0.$$

Clearly, $\frac{2^*}{2p'} = \frac{2\sigma-1}{2\sigma} < 1$, so we deduce that

$$\int_\Omega |\nabla G_{k_0}(u)^\sigma|^2 \leq C.$$

Finally, using that u is bounded in $H_0^1(\Omega)$ (from Case 1), we obtain that

$$\begin{aligned}\int_\Omega |\nabla u^\sigma|^2 &= \int_\Omega |\nabla G_{k_0}(u)^\sigma|^2 + \int_\Omega |\nabla T_{k_0}(u)^\sigma|^2 \\ &\leq C + \sigma^2 \int_\Omega T_{k_0}(u)^{2(\sigma-1)} |\nabla T_{k_0}(u)|^2 \leq C + \sigma^2 k_0^{2(\sigma-1)} \int_\Omega |\nabla u|^2 \leq C.\end{aligned}$$

This proves Case 2.

Case 3: $\frac{N}{N-1} < q < 1 + \frac{2}{N}$ and $\frac{2N}{N+2} \leq p < \frac{N}{2}$.

Here one can argue as in Case 2 of the proof of item 1, but considering this time $\phi(s) = s^{\sigma-1}$ for all $s \geq 0$.

Proof of 3.

Since $\sigma = \frac{(N-2)p}{2(N-2p)} \to +\infty$ as $p \to \frac{N}{2}$, item 3 is a clear consequence of item 2.

Proof of 4.

Let us take $G_k(u)$ as test function in the weak formulation of (4.26) for some $k > 0$, so we obtain this time, removing the term with β,

$$\int_\Omega |\nabla G_k(u)|^2 \leq C\int_\Omega |\nabla G_k(u)|^q G_k(u) + \int_\Omega |h(x)| G_k(u). \tag{4.43}$$

We consider now two different cases.

Case 1: $1 + \frac{2}{N} \leq q < 2$.

In this case, we have that $r = \frac{N(q-1)}{q} \in \left[\frac{2N}{N+2}, \frac{N}{2}\right)$, so $\sigma_r = \frac{(N-2)r}{2(N-2r)} \geq 1$. On the other hand, it can be checked that

$$\left(1 - \frac{2}{N}\right) 2^* + \frac{2}{N} 2^* \sigma_r = \frac{2}{2-q}.$$

Then, we can use Hölder's inequality in such a way that

$$\begin{aligned}
\int_\Omega |\nabla G_k(u)|^q G_k(u) &\leq \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{q}{2}} \left(\int_\Omega G_k(u)^{\frac{2}{2-q}}\right)^{1-\frac{q}{2}} \\
&\leq \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{q}{2}} \left(\int_\Omega G_k(u)^{2^*}\right)^{\frac{2-q}{2^*}} \|G_k(u)\|^{q-1}_{L^{2^*\sigma_r}(\Omega)} \\
&\leq C \|G_k(u)\|^{q-1}_{L^{2^*\sigma_r}(\Omega)} \int_\Omega |\nabla G_k(u)|^2.
\end{aligned}$$

Next, by item 2 we can take $k \geq k_0$, with k_0 independent of u, so that $\|G_k(u)\|^{q-1}_{L^{2^*\sigma_r}(\Omega)}$ is small enough. Then, from (4.43) we deduce that

$$C \int_\Omega |\nabla G_k(u)|^2 \leq \int_\Omega |h(x)| G_k(u).$$

We conclude by using the Stampacchia's method in a direct way.

Case 2: $\frac{N}{N-1} < q < 1 + \frac{2}{N}$.

In this case, Hölder's inequality yields

$$\int_\Omega |\nabla G_k(u)|^q G_k(u) \leq \left(\int_\Omega |\nabla G_k(u)|^{N(q-1)}\right)^{\frac{1}{N}} \left(\int_\Omega |\nabla G_k(u)|^2\right)^{\frac{1}{2}} \left(\int_\Omega G_k(u)^{2^*}\right)^{\frac{1}{2^*}}.$$

By virtue of Case 2 of item (2), we can take $k \geq k_0$, with k_0 independent of u, such that $\int_\Omega |\nabla G_k(u)|^{N(q-1)}$ is small enough. Then, from (4.43) we deduce that

$$C \int_\Omega |\nabla G_k(u)|^2 \leq \int_\Omega |h(x)| G_k(u),$$

and we can apply again Stampacchia's method.

The proof is now concluded. □

We prove now a result analogous to Proposition 4.3.13 for q small.

Proposition 4.3.14. *Assume that q, α, β, μ satisfy* (4.27)*, and assume in addition that*

$$q < \frac{N}{N-1}.$$

Then, for all $M > 0$ and $p \geq 1$, there exists $C > 0$ such that, for any $h \in L^p(\Omega)$ with $\|h\|_{L^p(\Omega)} \leq M$ and for any solution u to problem (4.26)*, the following holds:*

1. *If* $p=1$, *then* $\|u\|_{\mathscr{M}^{\frac{N}{N-2}}(\Omega)}+\||\nabla u|\|_{\mathscr{M}^{\frac{N}{N-1}}(\Omega)}\leq C$;
2. *if* $1<p<\frac{2N}{N+2}$, *then* $\|u(1+u)^{\sigma-1}\|_{H_0^1(\Omega)}\leq C$, *where* σ *is defined by* (4.28);
3. *if* $\frac{2N}{N+2}\leq p<\frac{N}{2}$, *then* $\|u^{\sigma}\|_{H_0^1(\Omega)}\leq C$, *where* σ *is defined by* (4.28);
4. *if* $p=\frac{N}{2}$, *then* $\|u^{\tau}\|_{H_0^1(\Omega)}\leq C$ *for all* $\tau<\infty$, *and*
5. *if* $p>\frac{N}{2}$, *then* $\|u\|_{L^{\infty}(\Omega)}\leq C$.

Proof. We will prove first item 1. Thus, for $j,k>0$, let us take $T_j(G_k(u))$ as test function in the weak formulation of (4.26), so we obtain

$$\beta\int_{\Omega}uT_j(G_k(u))+\int_{\Omega}\nabla u\nabla T_j(G_k(u))=\int_{\Omega}\left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^{\alpha}}+|h(x)|\right)T_j(G_k(u)). \tag{4.44}$$

On the one hand, it is clear that

$$\int_{\Omega}\nabla u\nabla T_j(G_k(u))=\int_{\Omega}\left|\nabla T_j(G_k(u))\right|^2.$$

On the other hand, concerning the right hand side of (4.44), we obtain that

$$\int_{\Omega}\left(\mu(x)\frac{|\nabla G_k(u)|^q}{u^{\alpha}}+|h(x)|\right)T_j(G_k(u))\leq jC\left(\int_{\Omega}|\nabla G_k(u)|^q+\int_{\{|h(x)|\geq\beta k\}}|h(x)|\right)+\beta\int_{\Omega}uT_j(G_k(u)).$$

In sum, we deduce that

$$\int_{\Omega}|\nabla T_j(G_k(u))|^2\leq jC\left(\int_{\Omega}|\nabla G_k(u)|^q+\int_{\{|h(x)|\geq\beta k\}}|h(x)|\right).$$

Then, we apply [20, Lemma 4.2], so that we deduce that

$$\|\nabla G_k(u)\|_{\mathscr{M}^{\frac{N}{N-1}}(\Omega)}\leq C\left(\int_{\Omega}|\nabla G_k(u)|^q+\int_{\{|h(x)|\geq\beta k\}}|h(x)|\right).$$

Since $q<\frac{N}{N-1}$, we have the immersions

$$\mathscr{M}^{\frac{N}{N-1}}(\Omega)\subset L^{\frac{N}{N-1}}(\Omega)\subset L^q(\Omega).$$

Therefore,

$$C\|\nabla G_k(u)\|_{L^q(\Omega)}\leq\int_{\Omega}|\nabla G_k(u)|^q+\int_{\{|h(x)|\geq\beta k\}}|h(x)|.$$

We now consider the function $F : [0,\infty) \to \mathbb{R}$ defined as

$$F(Y) = CY - Y^q \quad \forall Y \ge 0,$$

and we denote

$$Y_k = \|\nabla G_k(u)\|_{L^q(\Omega)}.$$

Thus we have proved that

$$F(Y_k) \le \|h\chi_{\{|h(x)|\ge \beta k\}}\|_{L^1(\Omega)}.$$

The proof of this part concludes as in the previous proposition.

The proofs of the rest of the items follow the same arguments of Proposition 4.3.13. We only stress that the estimate

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \quad \forall k \ge k_0$$

is proved in a different way. Indeed, since $q < \frac{N}{N-1}$, then $N(q-1) < \frac{N}{N-1}$, so we deduce that

$$\int_\Omega |\nabla G_k(u)|^{N(q-1)} \le C \left(\int_\Omega |\nabla G_k(u)|^{\frac{N}{N-1}} \right)^{(N-1)(q-1)}.$$

Therefore, the estimate holds by virtue of item 1. □

The same arguments we have employed in the proof of Proposition 4.3.12 (but using Propositions 4.3.13 and 4.3.14 instead of the results in [85]) are valid also for proving the main result of this subsection.

Proposition 4.3.15. *Assume that* (H1) *holds. If $q > \frac{N}{N-1}$, suppose also that* (4.1) *is satisfied. Then, for every $\lambda_0 > 0$, there exists $C > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C$$

for every solution u to (P_λ) with $\lambda > \lambda_0$.

Remark 4.3.16. Notice that, in principle, one cannot apply Propositions 4.3.13 nor 4.3.14 to prove Proposition 4.3.15 in the case $q = \frac{N}{N-1}$. However, for $\varepsilon > 0$ small, we have that $\frac{N}{N-1} + \varepsilon < 1 + \frac{2}{N}$ and

$$\frac{|\nabla u|^{\frac{N}{N-1}}}{u^\alpha} \chi_{\{u\ge k\}} \le \frac{|\nabla u|^{\frac{N}{N-1}+\varepsilon}}{u^\alpha} \chi_{\{u\ge k\}} + C_\varepsilon$$

for any $k > 0$ and any solution u to (P_λ). Hence, the conclusions of Proposition 4.3.13 hold for $q = \frac{N}{N-1} + \varepsilon$.

4.3.3 Proof of the main result and consequences

We prove now the main result of the paper.

Proof of Theorem 4.1.1. Since there is a solution u_0 to (P_0), then Proposition 4.5.2 (see also Remark 4.5.3) implies that there exists an unbounded connected set Σ^+ such that

$$(0,u_0) \in \Sigma^+ \subset ([0,+\infty) \times L^\infty(\Omega)) \cap \Sigma,$$

where

$$\Sigma = \{(\lambda,u) \in \mathbb{R} \times L^\infty(\Omega) : u \text{ is a solution to } (P_\lambda)\}.$$

We claim that Σ^+ bifurcates from infinity to the right of the axis $\lambda = 0$. Indeed, since (P_λ) does not have any solution for $\lambda \geq \lambda_1$, then $\Sigma^+ \subset ([0,\lambda_1) \times L^\infty(\Omega)) \cap \Sigma$. Therefore, since Σ^+ is unbounded, then its projection onto $L^\infty(\Omega)$ is unbounded. On the other hand, Proposition 4.3.15 implies that $\Sigma^+ \cap ((\lambda_0,\lambda_1) \times L^\infty(\Omega))$ is bounded for all $\lambda_0 \in (0,\lambda_1)$. That is to say, $\Sigma^+ \cap ((0,\lambda_0) \times L^\infty(\Omega))$ is unbounded for all $\lambda_0 > 0$, and our claim is true.

We have proved that there exists a sequence $\{(\lambda_n,u_n)\} \subset \Sigma^+$ such that $\lambda_n \to 0$ and $\|u_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$. We will show now that this fact and the connection of Σ^+ are enough to proof multiplicity of solutions for all $\lambda > 0$ small enough. Indeed, assume by contradiction that there exists another sequence $\{(\mu_n,v_n)\} \subset \Sigma^+$ such that $\mu_n \to 0$ as $n \to \infty$ and (P_{μ_n}) admits no other solution but v_n for all n. On the other hand, using that $(0,u_0) \in \Sigma^+$ and Σ^+ is connected, it is clear that $\Sigma^+ \cap B_r((0,u_0)) \setminus \{(0,u_0)\} \neq \emptyset$ for all $r > 0$, where $B_r((0,u_0))$ denotes the open ball in $\mathbb{R} \times L^\infty(\Omega)$ centered at $(0,u_0)$ with radius r. Hence, since v_n is unique and $\mu_n \to 0$, we have that, for all $r > 0$, there exists $n_r \in \mathbb{N}$ such that, if $n \geq n_r$, then $(\mu_n,v_n) \in \Sigma^+ \cap B_r((0,u_0)) \setminus \{(0,u_0)\}$. In other words, $v_n \to u_0$ in $L^\infty(\Omega)$ as $n \to +\infty$. Let us now take a not relabeled subsequence $\{(\mu_n,v_n)\}$ such that $\mu_{n+1} < \lambda_n < \mu_n$ for all n. Let us also fix $\eta > \|u_0\|_{L^\infty(\Omega)}$, and take n large enough so that $\max\{\|v_n\|_{L^\infty(\Omega)}, \|v_{n+1}\|_{L^\infty(\Omega)}\} < \eta < \|u_n\|_{L^\infty(\Omega)}$. We claim that there exists $(\nu_n,w_n) \in \Sigma^+$ such that $\nu_n \in (\mu_{n+1},\mu_n)$ and $\|w_n\|_{L^\infty(\Omega)} = \eta$.

Indeed, let us consider the set

$$A_{n,\eta} = \{(\lambda,u) \in \Sigma : \lambda \in (\mu_{n+1},\mu_n), \|u\|_{L^\infty(\Omega)} = \eta\}.$$

Arguing by contradiction, assume that $\Sigma^+ \cap A_{n,\eta} = \emptyset$. Let us define also

$$B_{n,\eta} = \{(\lambda,u) \in \Sigma : \lambda \in \{\mu_{n+1},\mu_n\}, \|u\|_{L^\infty(\Omega)} > \eta\}.$$

On the one hand, the fact that $\max\{\|v_n\|_{L^\infty(\Omega)}, \|v_{n+1}\|_{L^\infty(\Omega)}\} < \eta$ and also the uniqueness of v_n imply that $\Sigma^+ \cap B_{n,\eta} = \emptyset$. On the other hand, if we consider the set

$$U_{n,\eta} = \{(\lambda,u) \in \Sigma^+ : \lambda \in (\mu_{n+1},\mu_n), \|u\|_{L^\infty(\Omega)} > \eta\},$$

then it is clear that $U_{n,\eta}$ is open in Σ^+, $(\lambda_n, u_n) \in U_{n,\eta}$ and $\partial U_{n,\eta} = A_{n,\eta} \cup B_{n,\eta}$. Hence, if we denote $V_{n,\eta} = \Sigma^+ \setminus \overline{U_{n,\eta}}$, we deduce that $V_{n,\eta}$ is also nonempty and open in Σ^+, $U_{n,\eta} \cap V_{n,\eta} = \emptyset$ and $\Sigma^+ = U_{n,\eta} \cup V_{n,\eta}$. This contradicts that Σ^+ is connected.

Therefore, we have found a sequence $\{(\nu_n, w_n)\} \subset \Sigma^+$ such that $\nu_n \to 0$ as $n \to +\infty$ and $\|w_n\|_{L^\infty(\Omega)} = \eta$ for all n large enough. In particular, $\{w_n\}$ is bounded in $L^\infty(\Omega)$. Then, we can argue as in the proof of Proposition 4.3.4 in order to pass to the limit in (P_{ν_n}). Thus, there exists $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $w_n \rightharpoonup w$ weakly in $H_0^1(\Omega)$, $w_n \to w$ strongly in $L^\infty(\Omega)$ and w is a solution to (P_0). But $\|w\|_{L^\infty(\Omega)} = \eta > \|u_0\|_{L^\infty(\Omega)}$. This is a contradiction, as u_0 is unique by virtue of Theorem 4.2.1 and Remark 4.3.3. The proof in now concluded. □

We conclude the section by stating and proving two corollaries of Theorem 4.1.1. The first one provides multiplicity of solutions for q small, but for any $\alpha \in [0, q-1)$.

Corollary 4.3.17. *Assume that* (H1) *holds with $q \in (1, Q_N] \setminus \{2\}$, where Q_N is defined in* (4.2). *Assume in addition that there exists a solution to* (P_0). *Then, the conclusions of Theorem 4.1.1 hold true.*

Proof. Consider the function $z : [0, q-1) \to \mathbb{R}$ given by

$$z(s) = \frac{q-s}{N-q+1} - \frac{q-1-s}{q-2s} \quad \forall s \in [0, q-1).$$

It can be proven that z is increasing. Indeed, for any $s \in [0, q-1)$, we deduce that

$$\begin{aligned} Nz'(s) &= -\frac{1}{N-q+1} + \frac{2-q}{(q-2s)^2} \\ &= \tilde{z}(s) \left(s - \frac{q - \sqrt{(2-q)(N+1-q)}}{2} \right) \left(\frac{q + \sqrt{(2-q)(N+1-q)}}{2} - s \right), \end{aligned}$$

where $\tilde{z}(s) = \frac{4}{(N-q+1)(q-2s)^2} > 0$ for all $s \in [0, q-1)$. Using that $N \ge 3$ and $q < 2$, it is straightforward to deduce that

$$\frac{q - \sqrt{(2-q)(N+1-q)}}{2} < 0 \quad \text{and} \quad \frac{q + \sqrt{(2-q)(N+1-q)}}{2} > q-1,$$

which means that $z'(s) > 0$ for all $s \in [0, q-1)$. Moreover, since $q \le Q_N$, then $z(0) \ge 0$ (see Proposition 4.3.12). Thus, $z(\alpha) \ge 0$, or equivalently, condition (4.1) holds and Theorem 4.1.1 can be applied. □

The second corollary gives multiplicity of solutions for all $q \in (1, 2)$ at the expense of taking α close to $q-1$.

Corollary 4.3.18. *Assume that* (H1) *holds and that* (P_0) *admits a solution. If* $q > \frac{N}{N-1}$, *suppose in addition that* $\alpha \geq \left(q - \frac{N}{N-1}\right)\frac{N-1}{N-2}$. *Then, the conclusions of Theorem 4.1.1 hold true.*

Proof. One only has to notice that, if $\alpha \geq \left(q - \frac{N}{N-1}\right)\frac{N-1}{N-2}$, then $\frac{N(q-1-\alpha)}{q-2\alpha} \leq 1$. But $\frac{(q-\alpha)N}{N-q+1} > 1$, that is to say, (4.1) holds and Theorem 4.1.1 can be applied. □

4.4 Uniqueness for $q-1 < \alpha \leq 1$

We will consider in this section problem (P_λ) under condition (H2). Observe that if $0 < u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $t > 0$, then

$$\frac{|\nabla tu|^q}{(tu)^\alpha} = t^{q-\alpha}\frac{|\nabla u|^q}{u^\alpha}.$$

In this case, $\alpha > q-1$, so $q - \alpha < 1$. That is to say, the lower order term has *sublinear homogeneity*.

Remark 4.4.1. The conclusions of Remark 4.3.3 are valid also under hypothesis (H2).

We will prove the existence of solution to (P_λ) after deriving certain a priori estimates on an approximate problem and passing eventually to the limit, in a way that such a limit will be the solution we look for. Thus, consider the following approximate problem:

$$\begin{cases} -\Delta u_n = \lambda u_n + \mu(x)\dfrac{T_n(|\nabla u_n|^q)}{\left(|u_n| + \frac{1}{n}\right)^\alpha} + T_n(f(x)), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{4.45}$$

In the next lemma we show that problem (4.45) admits a solution.

Lemma 4.4.2. *Assume that* (H2) *holds and let* $\lambda < \lambda_1$. *Then there exists a solution* $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *to problem* (4.45) *for all* $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$. Then, the following linear problem

$$\begin{cases} -\Delta u = \lambda u + n^{1+\alpha}\mu(x) + n, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

has a solution $0 < \overline{\psi} \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Clearly, $\overline{\psi}$ is a supersolution to (4.45). Moreover, $\underline{\psi} = 0$ is a subsolution to (4.45). Since $\underline{\psi} \leq \overline{\psi}$, then there exists a solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (4.45) (see [30]). □

We prove now the key estimates for proving the existence of solution to problem (P_λ).

Proposition 4.4.3. *Assume that* (H2) *holds, and let $\lambda<\lambda_1$. Then there exist $\eta\in(0,1)$ and $C>0$ such that*

$$\|u_n\|_{H_0^1(\Omega)}+\|u_n\|_{C^{0,\eta}(\overline{\Omega})}\leq C$$

for every solution u_n to (4.45) *and for every n.*

Proof. **Step 1: H_0^1 estimate.**

Let us take u_n as test function in the weak formulation of (4.45). Then we obtain by using Poincaré's and Hölder's inequalities that

$$\begin{aligned}\int_\Omega|\nabla u_n|^2&\leq\lambda\int_\Omega u_n^2+\|\mu\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_n|^q u_n^{1-\alpha}+\int_\Omega f(x)u_n\\&\leq\frac{\lambda}{\lambda_1}\int_\Omega|\nabla u_n|^2+C\left(\int_\Omega|\nabla u_n|^2\right)^{\frac{q}{2}}\left(\int_\Omega u_n^{\frac{2(1-\alpha)}{2-q}}\right)^{1-\frac{q}{2}}+C\left(\int_\Omega u_n^{2^*}\right)^{\frac{1}{2^*}}.\end{aligned}$$

Now, since $\alpha>q-1$, then $\frac{2(1-\alpha)}{2-q}<2<2^*$. Hence, we can apply Sobolev's inequality to get that

$$\left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega|\nabla u_n|^2\leq C\left(\int_\Omega|\nabla u_n|^2\right)^{\frac{q+1-\alpha}{2}}+C\left(\int_\Omega|\nabla u_n|^2\right)^{\frac{1}{2}}.$$

Observe now that $\frac{q+1-\alpha}{2}<1$. Therefore, we deduce that $\|u_n\|_{H_0^1(\Omega)}\leq C$.

Step 2: L^∞ estimate.

Assume now, in order to achieve a contradiction, that $\{\|u_n\|_{L^\infty(\Omega)}\}$ is unbounded, and choose a not relabeled divergent subsequence. Then, the function $v_n=\frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}$ satisfies

$$\begin{cases}-\Delta v_n=\lambda v_n+\dfrac{\mu(x)T_n(|\nabla u_n|^q)}{\|u_n\|_{L^\infty(\Omega)}\left(u_n+\frac{1}{n}\right)^{q-1+\alpha}}+\dfrac{f(x)}{\|u_n\|_{L^\infty(\Omega)}}, & x\in\Omega,\\ v_n>0, & x\in\Omega,\\ v_n=0, & x\in\partial\Omega.\end{cases}\tag{4.46}$$

Notice that $\|v_n\|_{L^\infty(\Omega)}=1$ for all n, and also that

$$0\leq\frac{\mu(x)T_n(|\nabla u_n|^q)}{\|u_n\|_{L^\infty(\Omega)}\left(u_n+\frac{1}{n}\right)^{q-1+\alpha}}\leq\frac{\|\mu\|_{L^\infty(\Omega)}|\nabla v_n|^q}{\|u_n\|^\alpha_{L^\infty(\Omega)}v_n^{q-1+\alpha}}.\tag{4.47}$$

Then, it is standard to prove that $\|v_n\|_{C^{0,\eta}(\overline{\Omega})} \le C$ for all n and for some $\eta \in (0,1)$ independent of n following the arguments in [94] (see [38, Appendix]). Hence, by Arzelà-Ascoli theorem, there exists $v \in C(\overline{\Omega})$ such that, up to a subsequence, $v_n \to v$ uniformly in $\overline{\Omega}$. Necessarily, $\|v\|_{L^\infty(\Omega)} = 1$, so $v \not\equiv 0$. Moreover, by using the strong maximum principle conveniently, $v > 0$ in Ω. This last fact combined with the uniform convergence implies that,

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 :\ v_n(x) \ge c_\omega \text{ a.e. } x \in \omega.$$

See the proof of [38, Proposition 5.2] for more details.

Let now $\phi \in C_c^1(\Omega)$ be such that $\mathrm{supp}(\phi) \subset \omega$ for some open set $\omega \subset\subset \Omega$. Then, from (4.47) we deduce that

$$\left| \int_\Omega \frac{\mu(x) T_n(|\nabla u_n|^q)\phi}{\|u_n\|_{L^\infty(\Omega)} \left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} \right| \le \frac{\|\mu\phi\|_{L^\infty(\Omega)}}{\|u_n\|^\alpha_{L^\infty(\Omega)} c_\omega^{q-1+\alpha}} \int_\omega |\nabla v_n|^q.$$

Using now that $\{v_n\}$ is bounded in $H_0^1(\Omega)$, we conclude that

$$\left| \int_\Omega \frac{\mu(x) T_n(|\nabla u_n|^q)\phi}{\|u_n\|_{L^\infty(\Omega)} \left(u_n + \frac{1}{n}\right)^{q-1+\alpha}} \right| \to 0$$

as $n \to \infty$.

Finally, we pass to the limit in (4.46) and obtain that

$$\begin{cases} -\Delta v = \lambda v, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

This contradicts the fact that $\lambda < \lambda_1$.

Step 3: Hölder estimate.

Using that $\alpha \le 1$ and the previous step, one can easily prove as in [38, Appendix] that $\|u_n\|_{C^{0,\eta}(\overline{\Omega})} \le C$ for all n and some $C > 0, \eta \in (0,1)$. □

We are ready now to prove the main theorem of this section.

Proof of Theorem 4.1.2. Concerning the existence of solution, one has only to pass the limit in (4.45) using the a priori estimates in Proposition 4.4.3. The proof is similar to the one of Proposition 4.3.4. The nonexistence of solution comes from Remark 4.3.3.

On the other hand, the uniqueness of solution is a consequence of Theorem 4.2.4 and Remark 4.3.3.

Finally, similar arguments as in the proof of Step 2 in Proposition 4.4.3 can be used to prove that λ_1 is the only possible bifurcation point from infinity. Actually, reasoning by contradiction and using that there is no solution to (P_{λ_1}), it is also standard to prove that λ_1 is, indeed, a bifurcation point from infinity. $\square$

4.5 Appendix: Existence of an unbounded continuum

For every $w \in L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$, let us consider the following problem:

$$\begin{cases} -\Delta u + u = \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) + (\lambda^+ + 1)w^+, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4.48}$$

If (H1) is satisfied, it is clear from Proposition 4.3.4 that there exists a unique solution $u_{\lambda,w} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (4.48). Hence, we are allowed to define the map

$$K : \mathbb{R} \times L^\infty(\Omega) \to L^\infty(\Omega), \quad (\lambda, w) \mapsto K(\lambda, w) = u_{\lambda,w}.$$

We will prove next that K is a completely continuous operator, i.e., it is continuous and maps bounded sets to relatively compact sets.

Proposition 4.5.1. *Assume that* (H1) *holds. Then, K is a completely continuous operator.*

Proof. We first prove that K is continuous. Indeed, let $\{(\lambda_n, w_n)\}$ be a sequence in $\mathbb{R} \times L^\infty(\Omega)$ such that $(\lambda_n, w_n) \to (\lambda, w)$ for some $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$. Let us denote $u_n = K(\lambda_n, w_n)$, and let $B > 0$ be such that $(\lambda_n^+ + 1)w_n^+ \leq B$. From Proposition 4.3.4, it follows that there exists $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta v + v = \mu(x)\dfrac{|\nabla v|^q}{v^\alpha} + f(x) + B, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

Hence, by virtue of Theorem 4.2.1 (see also Remark 4.3.3), we deduce that

$$u_n \leq v \leq \|v\|_{L^\infty(\Omega)}.$$

In particular, $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

Now we can argue as in [38, Appendix] to prove that $\{u_n\}$ is, in fact, bounded in $C^{0,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Therefore, Arzelà-Ascoli theorem implies that $\{u_n\}$ admits a uniformly convergent subsequence. Say, up to a not relabeled subsequence, $u_n \to u$ uniformly in $\overline{\Omega}$ for some $u \in C(\overline{\Omega})$.

On the other hand, taking u_n as test function in the weak formulation of (4.48) yields

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 = \int_\Omega \mu(x)|\nabla u_n|^q u_n^{1-\alpha} + \int_\Omega (f(x) + (\lambda_n^+ + 1)w_n^+.$$

Using that $\{u_n\}$ and $\{(\lambda_n, w_n)\}$ are bounded in $L^\infty(\Omega)$ and in $\mathbb{R} \times L^\infty(\Omega)$, and also that $\alpha < q-1 < 1$, the previous equality clearly implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then, $u \in H_0^1(\Omega)$ and, up to a new subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Moreover, by [26], $\nabla u_n \to \nabla u$ strongly in $L^q(\Omega)^N$. Furthermore, a lower local estimate on $\{u_n\}$ can be derived by comparison in the usual way. With all these estimates and convergences, the passing to the limit in (4.48) is standard.

Therefore, $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to (4.48). This means that $K(\lambda, w) = u$. Thus, we have proved that, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$ strongly in $L^\infty(\Omega)$. Actually, since (λ, w) was fixed from the beginning, the whole sequence, and not just a subseqence, converges to (λ, w). That is to say, K is continuous.

It is left to prove that K maps bounded sets to relatively compact sets. In other words, we claim that for every sequence $\{(\lambda_n, w_n)\}$ bounded in $\mathbb{R} \times L^\infty(\Omega)$, there exists $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$ such that, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$ strongly in $L^\infty(\Omega)$. Indeed, it is well-known that, up to a subsequence, $\lambda_n \to \lambda$ in $\mathbb{R}$ and $w_n \to w$ weakly* in $L^\infty(\Omega)$ for some $(\lambda, w) \in \mathbb{R} \times L^\infty(\Omega)$. This convergence is enough to pass to the limit in the term with w_n. In the rest of the terms, we pass to limit arguing as above. Thus, up to a subsequence, $K(\lambda_n, w_n) \to K(\lambda, w)$, and the proof is finished. □

Let us define $\Phi(\lambda, u) = u - K(\lambda, u)$, and

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times L^\infty(\Omega) : \Phi(\lambda, u) = 0\}.$$

For any $\lambda_0 \in \mathbb{R}$ and any isolated solution $u_0 \in L^\infty(\Omega)$ to the equation $\Phi(\lambda_0, u) = 0$, the Leray-Schauder degree $\deg(\Phi(\lambda_0, \cdot), B_r(u_0), 0)$ is well defined and is constant for $r > 0$ small enough. Thus it is possible to define the so called *index* as

$$i(\Phi(\lambda_0, \cdot), u_0) = \lim_{r \to 0} \deg(\Phi(\lambda_0, \cdot), B_r(u_0), 0).$$

Proposition 4.5.2. *Assume that* (H1) *holds, and suppose also that* (P_0) *has a solution* u_0. *Then, there exist two unbounded connected sets* $\Sigma^-, \Sigma^+ \subset \Sigma$ *such that* $(0,u_0) \in \Sigma^- \cap \Sigma^+$, $\Sigma^- \subset (-\infty,0] \times L^\infty(\Omega)$ *and* $\Sigma^+ \subset [0,\infty) \times L^\infty(\Omega)$.

Remark 4.5.3. Observe that, if $\lambda \geq 0$, solving the equation $\Phi(\lambda,u) = 0$ is equivalent to finding a solution to (P_λ). In particular, the projection of Σ^+ onto $L^\infty(\Omega)$ is actually made of solutions to (P_λ).

Proof of Proposition 4.5.2. By virtue of Proposition 4.5.1, K is completely continuous. Moreover, since (P_0) admits at most one solution (by virtue of [11]), then u_0 is the unique solution to $\Phi(0,u) = 0$ (see Remark 4.5.3). In particular, it is isolated. We will prove now that $i(\Phi(0,\cdot),u_0) \neq 0$ by using the properties of the Leray-Schauder degree.

Indeed, let us define the operator $T : [0,1] \times L^\infty(\Omega) \to L^\infty(\Omega)$ by $T(t,w) = u$, where $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the unique solution to the problem

$$\begin{cases} -\Delta u + u = (1-t)\mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x) + w^+, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

It is easy to prove that T is continuous and $T(t,\cdot) : L^\infty(\Omega) \to L^\infty(\Omega)$ is completely continuous arguing as in the proof of Proposition 4.5.1. Moreover, for any $t \in [0,1]$, the unique solution $u_t \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to $T(t,u_t) = u_t$ satisfies, thanks to Theorem 4.2.1 (see also Remark 4.3.3), that $u_t \leq u_0 \leq \|u_0\|_{L^\infty(\Omega)}$. Hence, if we set $\Psi_t(u) = u - T(t,u)$ and $R = 2\|u_0\|_{L^\infty(\Omega)}$, then $\Psi_t(u) \neq 0$ for all $u \in \partial B_R(0) = \partial\{v \in L^\infty(\Omega) : \|v\|_{L^\infty(\Omega)} < R\}$ and for all $t \in [0,1]$. Therefore, the homotopy property of the degree shows that

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_1, B_R(0), 0) \neq 0.$$

On the other hand, let $r > 0$ be small enough so that $B_r(u_0) \subset\subset B_R(0)$. Let us denote the following open, bounded and disjoint subsets of $B_R(0)$ as $A_1 = B_r(u_0)$ and $A_2 = B_R(0) \setminus \overline{B_r(u_0)}$. Since u_0 is unique, then $\Psi_0(u) \neq 0$ for all $u \in \overline{B_R(0)} \setminus (A_1 \cup A_2)$ i.e., for all $u \in \partial B_R(0) \cup \partial B_r(u_0)$. Then, the additivity property of the degree implies that

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_0, A_1, 0) + \deg(\Psi_0, A_2, 0).$$

Now, again by the uniqueness of u_0, we have that $\Psi_0(u) \neq 0$ for all $u \in A_2$. Thus the solution property of the degree says that $\deg(\Psi_0, A_2, 0) = 0$. That is to say,

$$\deg(\Psi_0, B_R(0), 0) = \deg(\Psi_0, B_r(u_0), 0).$$

Putting all together, we have proved that

$$i(\Phi(0,\cdot),u_0) = \deg(\Phi(0,\cdot),B_r(u_0),0) = \deg(\Psi_0,B_r(u_0),0) \neq 0.$$

We can now apply [12, Theorem 2.2], which is essentially [115, Theorem 3.2], and the proof is finished. $\square$

Chapter 5

A blow-up approach for singular elliptic problems with natural growth

S. López-Martínez, *A blow-up approach for singular elliptic problems with natural growth. Preprint.*

Abstract. We prove existence results concerning elliptic problems whose basic model is

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{(u+\delta)^\gamma} = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\lambda > 0$, $p \in (1, 2^* - 1)$, $\delta \geq 0$, $\gamma > 0$ and $0 \lneq \mu \in L^\infty(\Omega)$. Observe that, if $\delta = 0$, the lower order term is singular as $x \to \partial\Omega$. We thus generalize some known results for non-singular [111] and singular [43] problems (see also [100]).

Our approach is based on fixed point theory. With the aim of applying it, a previous analysis on a related non-homogeneous problem is carried out. Moreover, the required a priori estimates are proven via a blow-up method.

5.1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain of class $\mathscr{C}^2$, let $g : \Omega \times (0, +\infty) \to [0, +\infty)$ be a Carathéodory function, and let us also consider the real numbers $\lambda > 0$, $p \in (1, 2^* - 1)$, where $2^* = \frac{2N}{N-2}$. In this work we will study elliptic problems of the following form:

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{P_λ}$$

The function g will satisfy some regularity and growth restrictions which will be shown later. For the sake of a clear presentation, we consider for now the model problem

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{(u+\delta)^\gamma} = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $\delta \geq 0$, $\gamma > 0$ and $0 \lneq \mu \in L^\infty(\Omega)$. Each result concerning problem (5.1) has its corresponding general version regarding problem (P_λ). They can be found in the subsequent sections.

For $\mu \equiv 0$, problem (5.1) is classical. Indeed, both variational (in [5]) and topological (in [60, 82]) methods can be used to prove the existence of a solution to (5.1) for all $\lambda > 0$ provided $\mu \equiv 0$, $p \in (1, 2^* - 1)$. In such a result, the restriction $p < 2^* - 1$ is, in fact, necessary for the existence of solution if the domain is starshaped, as Pohozaev's identity shows (see [112]). The study of problem (5.1) for a nontrivial μ was initiated in [111]. There, the authors considered $\delta > 0$ and $\mu > 0$ constant, and they proved existence, nonexistence and multiplicity results. If $\gamma = 1$, the problem has been shown to be more delicate as some restrictions involving μ and p appear (see [9, 100, 111]). On the other hand, the singular case $\delta = 0$ has been dealt with in [43] (see also [33, 44] for similar singular problems that involve a nonzero source term). However, as far as we know, non-constant bounded functions μ in problem (P_λ) have not been considered in the literature and, moreover, many questions are still open even for μ constant, specially in the singular case $\delta = 0$.

In the present work we aim to develop an approach that permits to deal with non-constant μ in problem (5.1) and also with singular lower order terms, i.e., $\delta = 0$. In order to do so, we will employ topologycal methods. More precisely, we will find solutions to (5.1) as fixed points of certain compact operator that will be defined in Section 5.4. The well-definition of such an operator will require the well-posedness of the following problem:

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{u} = f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

where $0 \lneq f \in L^q(\Omega)$ for some $q > \frac{N}{2}$.

Singular problems of this kind have risen interest in the recent years. The reader is referred to [65], and references therein, for a rather complete overview in its introduction. Paying attention specifically on problem (5.2), the existence of solution with $\|\mu\|_{L^\infty(\Omega)} < \frac{1}{2}$ has been proven in [24], and extended to $\|\mu\|_{L^\infty(\Omega)} < 1$ in [107]. As far as the uniqueness of solution is concerned, some results are known for problems similar to (5.2), even though they require either the singularity to be milder or μ to be constant (see [10, 16, 37]). On the other hand, existence and uniqueness results for problem (5.2) are also known for $\|\mu\|_{L^\infty(\Omega)} \geq 1$ provided f is locally bounded away from zero (see [8] for the existence and [37] for the uniqueness). However, for general non-negative f and for $\|\mu\|_{L^\infty(\Omega)} \geq 1$ one can find a nonexistence result in [37], even though only for dimension $N = 1$. In next result we prove that uniqueness for problem (5.2) holds provided $\|\mu\|_{L^\infty(\Omega)} < 1$, while nonexistence holds provided $\mu > 1$ and $f \equiv 0$ in a neighborhood of $\partial\Omega$ (for any dimension N). The proof of the uniqueness relies on a comparison principle that we also prove in Section 5.2, while the nonexistence part follows the ideas in [43, Lemma 2.5].

Theorem 5.1.1. *Let $0 \lneq f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and let $0 \lneq \mu \in L^\infty(\Omega)$. The following statements hold true:*

1. *If $\|\mu\|_{L^\infty(\Omega)} < 1$, then there exists a unique solution to problem* (5.2).
2. *If there exist a domain $\omega \subset\subset \Omega$ and a constant $\tau > 1$ such that $\mu(x) \geq \tau$ and $f(x) = 0$, both for a.e. $x \in \Omega \setminus \omega$, then problem* (5.2) *admits no solution.*

Once we have shown that problem (5.2) is well-posed, we will be able to define a compact operator whose fixed points are solutions to (P_λ) (see Section 5.4). A version of a result in [93] (see [60]) will assure the existence of a fixed point of the operator.

As it is mandatory for fixed point theorems, we will prove the existence of a priori estimates on the solutions to a problem related to (P_λ). To this task, we will adapt the blow-up method due to [82]. Roughly speaking, this technique consists of assuming by contradiction that there exists a sequence of solutions whose norms blow up as n tends to infinity. The conclusion follows by passing to the limit in a problem satisfied by a certain normalized sequence. In fact, the limit function is a solution to a problem which, however, does not admit any solution by virtue of some Liouville type result. Therefore, one gets a contradiction, so any sequence of solutions must be bounded. In this sense, the difficulties that we find are twofold. Firstly, the normalized sequence, say $\{v_n\}$, satisfies an equation with a lower order term of type $\frac{|\nabla v_n|^2}{v_n+\delta_n}$, where $0 \leq \delta_n \to 0$ as $n \to +\infty$. If we aim to pass to the limit, then we need to find positive lower bounds on $\{v_n\}$, otherwise the lower order term may blow up as $n \to +\infty$. And lastly, we arrive to a limit problem, having a quadratic gradient lower order term like $\frac{|\nabla v|^2}{v}$, for which nonexistence Liouville type results are not known in the literature (some references for Liouville type results about problems depending on the gradient are [67, 71, 102, 110, 114, 118, 119], among others).

We overcome the first of the difficulties by proving Hölder estimates in spite of the singular quadratic term. The proof follows the ideas of [94], which have been widely used for singular problems (see [8,15,38,39,65,103,107], among others). We will show that these estimates yield in turn positive lower estimates from below and this will be enough to pass to the limit. Regarding the second difficulty, we observe that the limit equation admits a convenient change of unknown which gets rid of the gradient term, so that we may apply classical Liouville type results (see Section 5.2 below).

We state here the main existence result for problem (5.1) in the case $\gamma = 1$.

Theorem 5.1.2. *Let $p \in (1, 2^* - 1)$, $\gamma = 1$, $\delta \geq 0$ and $0 \lneq \mu \in L^\infty(\Omega)$. The following statements hold true:*

1. *If $\mu \in C(\overline{\Omega})$ and there exist two real numbers τ, σ such that $0 \leq \tau \leq \sigma < \frac{2^*-1-p}{2^*-2}$, $\sigma - \tau < 1 - \sigma$ and $\tau \leq \mu(x) \leq \sigma$ for all $x \in \overline{\Omega}$, then there exists at least a solution to* (5.1) *for every $\lambda > 0$.*

2. *If $\delta = 0$ and there exist a domain $\omega \subset\subset \Omega$ and a constant $\tau > 1$ such that $\mu(x) \geq \tau$ for a.e. $x \in \Omega \setminus \omega$, then problem* (5.1) *admits no solution for any $\lambda > 0$.*

We point out that the smallness condition on σ in Theorem 5.1.2 appears naturally in the simpler case $\mu \equiv$ constant. In fact, a straightforward argument shows that problem (5.1) has no bounded solutions provided $\gamma = 1$, $\delta = 0$, $\mu \equiv$ constant $\in \left[\frac{2^*-1-p}{2^*-2}, 1\right)$ and Ω is starshaped (see Remark 5.4.2 below). Moreover, we will show later that, strengthening the smallness condition conveniently (in terms of p, N), one may assume μ to be continuous only in a neighborhood of $\partial\Omega$. Furthermore, taking σ even smaller, the continuity assumption can be completely removed so that functions μ which are merely bounded in Ω may be considered. On the other hand, concerning also the existence part of the theorem, we stress that we need to control μ from below in order to prove the Hölder estimates that we mentioned. However, if μ is constant, i.e., $\sigma = \tau$, then the condition $\sigma - \tau < 1 - \sigma$ becomes $\sigma = \tau < 1$, which means no restriction since $\sigma < \frac{2^*-1-p}{2^*-2}$. On the other hand, concerning the nonexistence statement for the case $\delta = 0$, we stress that the case $\mu \equiv 1$ remains unsolved, i.e., there are neither existence nor non-existence results about problem (5.1) for $\gamma = 1, \delta = 0, \mu \equiv 1, p > 1$ in the literature.

Next result shows that our approach allows also to prove existence for $\gamma > 1$ and for all $\lambda > 0$.

Theorem 5.1.3. *Let $p \in (1, 2^* - 1)$, $\gamma > 1$, $\delta \geq 0$ and $0 \lneq \mu \in L^\infty(\Omega)$. The following statements hold true:*

1. *If $\delta > 0$ and $\|\mu\|_{L^\infty(\Omega)} < \frac{\delta^{\gamma-1}}{2}$, then there exists at least a solution to* (5.1) *for every $\lambda > 0$.*
2. *If $\delta = 0$ and there exist a domain $\omega \subset\subset \Omega$ and a constant $\tau > 0$ such that $\mu(x) \geq \tau$ for a.e. $x \in \Omega \setminus \omega$, then problem* (5.1) *admits no solution for any $\lambda > 0$.*

Unlike the case $\gamma = 1$ in Theorem 5.1.2, we presume that the smallness assumption on $\|\mu\|_{L^\infty(\Omega)}$ in Theorem 5.1.3 is technical as it is not needed if μ is constant (see [111]). However, we cannot avoid it since it is used to prove the Hölder estimates mentioned above. Concerning the nonexistence statement for the singular case $\delta = 0$, it is proven again following closely [43, Lemma 2.5]. We stress that the fact that the singularity is strong, namely $\gamma > 1$, allows to take $\mu > 0$ near $\partial\Omega$ (in contrast to the case $\gamma = 1$, for which $\mu > 1$ near the boundary was needed).

We will be able go beyond Theorems 5.1.2 and 5.1.3 and deal with any $\gamma > 0$. Indeed, for $\gamma \in (0, 1)$ and $\delta \geq 0$, we will show that existence of solution holds for every $\lambda > 0$ large enough. We will also deal with $\gamma \geq 1, \delta > 0$ and μ a bounded function with arbitrary size, i.e., we remove the restrictions on μ from above and from below in Theorems 5.1.2 and 5.1.3 as long as $\delta > 0$. However, λ will have to be taken large enough again. The statement of the result is the following.

Theorem 5.1.4. *Let $p \in \left(1, \frac{N+1}{N-1}\right)$, $\gamma > 0$, $\delta \geq 0$ and $0 \lneq \mu \in L^\infty(\Omega)$. If $\gamma \geq 1$, assume in addition that $\delta > 0$. Then, there exists $\lambda_0 > 0$ such that there exists at least a solution u_λ to* (5.1) *for every $\lambda > \lambda_0$. Moreover, $\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.*

If $\gamma \in (0, 1)$, last result is consistent with the known results for $\mu > 0$ constant which assure nonexistence for $\lambda > 0$ small (see [111] for $\delta > 0$ and [43] for $\delta = 0$). On the contrary, as we pointed out above, for $\gamma > 1$ we would expect an existence result for every $\lambda > 0$ without size restrictions. As far as the case $\gamma = 1$ and $\delta > 0$ is concerned, we ignore whether an existence result for $\lambda > 0$ small and general μ should be expected or not. Two exceptions are the ranges $\mu < \frac{2^*-1-p}{2^*-2}$ and $\mu \geq p$, for which existence (see Theorem 5.1.2 above and [100]) and nonexistence (see [10, 111]) for $\lambda > 0$ small are known respectively. In other words, the existence of solution to (5.1) for $\gamma = 1$, $\delta > 0$, $\frac{2^*-1-p}{2^*-2} \leq \mu < p$ and $\lambda > 0$ small remains as an open problem, even for $\mu > 0$ constant.

We organize the paper as follows. In Section 5.2 we state and prove several preliminary results which will be used later, among which we remark a comparison principle (Theorem 5.2.6), a nonexistence result (Theorem 5.2.7) and a Liouville type result (Lemma 5.2.9); also in Section 5.2 we prove a slightly more general version of Theorem 5.1.1. We devote Section 5.3 to proving the a priori estimates using the blow-up method. Section 5.4 contains the main existence results of the paper and also the proofs of Theorems 5.1.3 and 5.1.4 (Theorem 5.1.2 will follow from a general result in a straightforward way, so we will skip its proof). Section 5.5 includes a list of open problems derived from the results in this work. Finally, in the Appendix we prove two technical results required by the blow-up method.

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5.2 Preliminary results

Let us consider the following problem:

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.3}$$

where $0 \lneq f \in L^q(\Omega)$ for some $q > \frac{N}{2}$ and $g : \Omega \times (0,+\infty) \to [0,+\infty)$ is a Carathéodory function satisfying that

$$\exists h \in C((0,+\infty)) : g(x,s) \leq h(s) \quad \text{a.e. } x \in \Omega,\ \forall s > 0. \tag{5.4}$$

This hypothesis is essentially the minimal condition that g must satisfy so that the weak formulation of (5.3) is well defined:

Definition 5.2.1. Let $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying (5.4) and let $f \in L^1(\Omega)$. A solution to (5.3) is a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} \forall \omega \subset\subset \Omega,\ \exists c > 0 : \quad u(x) \geq c \text{ a.e. } x \in \omega, \text{ and} \\ \displaystyle\int_\Omega \nabla u \nabla \phi + \int_\Omega g(x,u)|\nabla u|^2 \phi = \int_\Omega f(x)\phi \quad \forall \phi \in C_c^1(\Omega). \end{cases}$$

The solutions to (P_λ) are defined in the same way by changing f with λu^p.

Remark 5.2.2. It can be proved by following [38, Appendix] that, in the previous definition, one can take test functions $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Next result assures that any solution to (5.3) is Hölder continuous up to the boundary. Its proof relies on some results in [94]. More details can be found in [38, Appendix].

Proposition 5.2.3. *Let $f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and let $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying* (5.4). *Then, every solution to* (5.3) *belongs to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

Once the solutions are Hölder continuous, since they are also locally bounded away from zero, then one can prove that their gradients enjoy more local summability. This fact is contained in the result below. The proof is based on a standard bootstrap argument, see [38, Appendix] for further details.

Proposition 5.2.4. *Let $f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and let $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying* (5.4). *Then, every solution to* (5.3) *belongs to $W^{1,2p}_{loc}(\Omega)$.*

Remark 5.2.5. Combining Proposition 5.2.4 with the classical regularity theory of Calderon-Zygmund one can easily prove that, if $f \in L^\infty(\Omega)$, then every solution to (5.3) belongs to $W^{2,q}_{\mathrm{loc}}(\Omega)$ for every $q < \infty$.

The regularity results above will allow us to apply the following comparison principle. Observe that g is allowed to change sign, so it may be applied in more general settings.

Theorem 5.2.6. *Let $0 \lneq f \in L^1_{loc}(\Omega)$ and let $g : \Omega \times (0,+\infty) \to \mathbb{R}$ be a Carathéodory function satisfying* (5.4). *Assume that there exists $\sigma \in (0,1)$ such that*

$$sg(x,s) \le \sigma \quad \text{a.e. } x \in \Omega,\ \forall s > 0. \tag{5.5}$$

Assume in addition that

$$s \mapsto sg(x,s) \ \text{ is nondecreasing for a.e. } x \in \Omega. \tag{5.6}$$

Let $u, v \in C(\Omega) \cap W^{1,N}_{loc}(\Omega)$, with $u, v > 0$ in Ω, be such that

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega g(x,u)|\nabla u|^2 \phi \le \int_\Omega f(x)\phi \quad \text{ and} \tag{5.7}$$

$$\int_\Omega \nabla v \nabla \phi + \int_\Omega g(x,v)|\nabla v|^2 \phi \ge \int_\Omega f(x)\phi \tag{5.8}$$

for every $0 \le \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support. Suppose also that the following boundary condition holds:

$$\limsup_{x \to x_0} (u(x)^{1-\sigma} - v(x)^{1-\sigma}) \le 0 \quad \forall x_0 \in \partial\Omega.$$

Then, $u \le v$ in Ω.

Proof. Let us first define the function $\tilde{g} : \Omega \times (0,+\infty) \to [0,+\infty)$ by

$$\tilde{g}(x,s) = \frac{\sigma - s^{\frac{1}{1-\sigma}} g\left(x, s^{\frac{1}{1-\sigma}}\right)}{(1-\sigma)s}, \quad \text{a.e. } x \in \Omega,\ \forall s > 0.$$

Thanks to (5.5) and (5.6), it is clear that $\tilde{g}$ is nonincreasing in the s variable. Moreover,

$$|\tilde{g}(s,x)| \le \frac{\sigma + h\left(s^{\frac{1}{1-\sigma}}\right)}{(1-\sigma)s}, \quad \text{a.e. } x \in \Omega,\ \forall s > 0,$$

so it is a bounded function in the x variable.

For some $0 \le \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with compact support, let us take $\frac{(1-\sigma)\phi}{u^\sigma}$ as test function in (5.7). Then, if we denote $\tilde{u} = u^{1-\sigma}$, we deduce that

$$\int_\Omega \nabla \tilde{u} \nabla \phi \le \int_\Omega \tilde{g}(x,\tilde{u})|\nabla \tilde{u}|^2 \phi + \int_\Omega \frac{(1-\sigma)f(x)}{\tilde{u}^{\frac{\sigma}{1-\sigma}}}\phi.$$

Arguing similarly, $\tilde{v} = v^{1-\sigma}$ satisfies

$$\int_\Omega \nabla \tilde{v} \nabla \phi \ge \int_\Omega \tilde{g}(x,\tilde{v})|\nabla \tilde{v}|^2 \phi + \int_\Omega \frac{(1-\sigma)f(x)}{\tilde{v}^{\frac{\sigma}{1-\sigma}}}\phi.$$

Moreover, $\tilde{u}, \tilde{v} \in C(\Omega) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ and they satisfy $\limsup_{x \to x_0}(\tilde{u}(x) - \tilde{v}(x)) \le 0$ for all $x_0 \in \partial\Omega$.

At this point one can reproduce the proof of [103, Theorem 2.1] without relevant changes and conclude that $\tilde{u} \le \tilde{v}$ in Ω. Equivalently, $u \le v$ in Ω. □

Next result shows, roughly speaking, that $sg(x,s)$ cannot be too large neither for s near 0 nor for x near $\partial\Omega$ if one expects to find solutions to (5.3) or (P_λ). It is stated again for sign-changing g for more generality. The proof follows the ideas in [43, Lemma 2.5].

Theorem 5.2.7. *Let $f \in L^1(\Omega)$ and let $g : \Omega \times (0,+\infty) \to \mathbb{R}$ be a Carathéodory function satisfying* (5.4). *Assume that there exist a domain $\omega \subset\subset \Omega$ and two constants $\tau > 1$, $s_0 \in (0,1)$ such that*

$$sg(x,s) \geq \tau \quad a.e.\ x \in \Omega \setminus \omega,\ \forall s \in (0,s_0). \tag{5.9}$$

Then, every solution u to (5.3) *satisfies*

$$\int_\Omega \frac{|f(x)|}{u} = +\infty.$$

In particular, if f has compact support in Ω, or $f = \lambda u^p$ for some $p \geq 1$ and $\lambda > 0$, then there exists no solution to problem (5.3).

Proof. Let u be a solution to (5.3). Recall that the definition of solution implies that there exists $c > 0$ such that $u \geq c$ in ω. For every $\varepsilon \in (0, \min\{s_0, c\})$, let us define the function $\varphi_\varepsilon : (0,+\infty) \to (0,+\infty)$ by

$$\varphi_\varepsilon(s) = \begin{cases} \dfrac{1}{s} + \dfrac{s^{\tau-1} - \varepsilon^{\tau-1}}{(\tau-1)s^\tau}, & s \geq \varepsilon, \\ \dfrac{s}{\varepsilon^2}, & 0 < s < \varepsilon. \end{cases}$$

Clearly, $\varphi_\varepsilon(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Hence, taking $\varphi_\varepsilon(u)$ as test function in the weak formulation of (5.3) (this can be done thanks to Remark 5.2.2) we obtain

$$\int_\Omega \varphi'_\varepsilon(u)|\nabla u|^2 + \int_\Omega g(x,u)|\nabla u|^2 \varphi_\varepsilon(u) = \int_\Omega f(x)\varphi_\varepsilon(u). \tag{5.10}$$

Observe that, from (5.9), it follows that

$$\begin{aligned} \int_\Omega g(x,u)|\nabla u|^2\varphi_\varepsilon(u) &= \int_{\omega\cup\{u\geq s_0\}} g(x,u)|\nabla u|^2\varphi_\varepsilon(u) + \int_{(\Omega\setminus\omega)\cap\{u<s_0\}} g(x,u)|\nabla u|^2\varphi_\varepsilon(u) \\ &\geq \int_{(\Omega\setminus\omega)\cap\{\varepsilon<u<s_0\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u} + \frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right) - C. \end{aligned} \tag{5.11}$$

On the other hand, it is clear that

$$\int_\Omega \varphi'_\varepsilon(u)|\nabla u|^2 = \int_{\{u>\varepsilon\}} \frac{\tau|\nabla u|^2}{u^2} - \int_{\{u>\varepsilon\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u} + \frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right) + \int_{\{u\leq\varepsilon\}} \frac{|\nabla u|^2}{\varepsilon^2}. \tag{5.12}$$

We will now absorb the negative term in (5.12) with the last integral term in (5.11). Indeed,

$$
\begin{aligned}
\int_{(\Omega\setminus\omega)\cap\{\varepsilon<u<s_0\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u}+\frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right) - \int_{\{u>\varepsilon\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u}+\frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right)&\\
= -\int_{[\{\varepsilon<u<s_0\}\cap\omega]\cup\{u\geq s_0\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u}+\frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right)&\\
\geq -\int_{\{u\geq \min\{s_0,c\}\}} \frac{\tau|\nabla u|^2}{u}\left(\frac{1}{u}+\frac{u^{\tau-1}-\varepsilon^{\tau-1}}{(\tau-1)u^\tau}\right) \geq -C.&
\end{aligned}
$$

In conclusion, from (5.10), (5.12) and from the previous discussion it follows that

$$
\int_{\{u>\varepsilon\}} \frac{\tau|\nabla u|^2}{u^2} \leq \int_\Omega f(x)\varphi_\varepsilon(u)+C.
$$

Since $\varphi_\varepsilon(s)\leq \frac{\tau}{(\tau-1)s}$ for all $s>0$, we finally deduce that

$$
\int_{\{u>\varepsilon\}} \frac{\tau|\nabla u|^2}{u^2} \leq C\left(\int_\Omega \frac{|f(x)|}{u}+1\right).
$$

Therefore, we let ε tend to zero and by virtue of Fatou lemma we obtain that

$$
\int_\Omega \frac{|\nabla u|^2}{u^2} \leq C\left(\int_\Omega \frac{|f(x)|}{u}+1\right).
$$

Now, in [127] it is proved that $\int_\Omega \frac{|\nabla u|^2}{u^2}=+\infty$. Therefore, $\int_\Omega \frac{|f(x)|}{u}=+\infty$ and the proof is finished. □

The results that we have presented in this section imply, in particular, a rather complete result about the model problem (5.3), in which the uniqueness and nonexistence parts are new. Such a result reads as follows.

Theorem 5.2.8. *Let $0\lneq f\in L^q(\Omega)$ for some $q>\frac{N}{2}$ and let $g:\Omega\times(0,+\infty)\to[0,+\infty)$ be a Carathédory function satisfying* (5.4) *and* (5.5) *for some $\sigma\in(0,1)$. Then, there exists at least a solution to* (5.3)*. Moreover, if* (5.6) *holds, the solution is unique. On the contrary, if there exist a domain $\omega\subset\subset\Omega$ and constants $\tau>1, s_0\in(0,1)$ such that* (5.9) *holds and $f(x)=0$ a.e. $x\in\Omega\setminus\omega$, then problem* (5.3) *admits no solutions.*

Proof. The existence of a solution $u\in H_0^1(\Omega)$ (not necessarily bounded) is a consequence of [107, Theorem 3.1]. Moreover, since $q>\frac{N}{2}$, the well-known Stampacchia's Lemma (see [121]) implies that $u\in L^\infty(\Omega)$. The uniqueness of solution follows from Proposition 5.2.3, Proposition 5.2.4 and Theorem 5.2.6. Finally, Theorem 5.2.7 implies the nonexistence part. □

We conclude the section with a Liouville type result which will be the key point for proving a priori estimates.

Lemma 5.2.9. *Let $p \in (1, 2^* - 1)$ and $\sigma \in \left(0, \frac{2^*-1-p}{2^*-2}\right)$. Then, the following problem*

$$\begin{cases} -\Delta u + \sigma \dfrac{|\nabla u|^2}{u} = u^p, & x \in X, \\ u > 0, & x \in X, \\ u = 0, & x \in \partial X, \end{cases} \tag{5.13}$$

admits no solutions in $H^1_{loc}(X) \cap C(\overline{X})$, where X denotes either $\mathbb{R}^N$ or $\mathbb{R}^N_+$. Moreover, if we assume additionally that $\sigma \leq \frac{N-(N-2)p}{2}$, then problem (5.13) *with $X = \mathbb{R}^N$ admits no supersolutions in $H^1_{loc}(X) \cap C(\overline{X})$. Finally, if we further assume that $\sigma \leq \frac{N+1-(N-1)p}{2}$, then problem* (5.13) *with $X = \mathbb{R}^N_+$ admits no supersolutions in $H^1_{loc}(X) \cap C(\overline{X})$.*

Remark 5.2.10. Note that $\frac{N+1-(N-1)p}{2} < \frac{N-(N-2)p}{2} < \frac{2^*-1-p}{2^*-2} < 1$, so the smallness conditions on σ in Lemma 5.2.9 are gradually less restrictive. We also stress that such conditions on σ are sharp. Indeed, if $\sigma > \frac{N-(N-2)p}{2}$ (resp. $\sigma > \frac{N+1-(N-1)p}{2}$), then one can find explicit supersolutions to (5.13) for $X = \mathbb{R}^N$, see [109] (resp. $X = \mathbb{R}^N_+$, see [23]).

Proof of Lemma 5.2.9. Assume, in order to achieve a contradiction, that there exists $u \in H^1_{loc}(X) \cap C(\overline{X})$ a solution to (5.13). Then, for some constant $c > 0$, the function $v = cu^{1-\sigma} \in H^1_{loc}(X) \cap C(\overline{X})$ is a solution to

$$\begin{cases} -\Delta v = v^{\frac{p-\sigma}{1-\sigma}}, & x \in X, \\ v > 0, & x \in X, \\ v = 0, & x \in \partial X. \end{cases} \tag{5.14}$$

Hence, if $\sigma < \frac{2^*-1-p}{2^*-2}$, then $\frac{p-\sigma}{1-\sigma} \leq 2^* - 1$. This is a contraction with [81, Theorem 1.1] if $X = \mathbb{R}^N$ and with [82, Theorem 1.3] if $X = \mathbb{R}^N_+$.

On the other hand, arguing again by contradiction, assume that there exists a supersolution $u \in H^1_{loc}(X) \cap C(\overline{X})$ to (5.13). Then, there is a constant $c > 0$ such that $v = cu^{1-\sigma} \in H^1_{loc}(X) \cap C(\overline{X})$ is a supersolution to (5.14). Therefore, if $\sigma \leq \frac{N-(N-2)p}{2}$, then $\frac{p-\sigma}{1-\sigma} \leq \frac{N}{N-2}$, so in case $X = \mathbb{R}^N$ we have a contraction with [109, Theorem 2.1]. Finally, if $\sigma < \frac{N+1-(N-1)p}{2}$, then $\frac{p-\sigma}{1-\sigma} < \frac{N+1}{N-1}$, so we arrive again to a contradiction with [23, Theorem 3.1]. $\square$

5.3 A priori estimates

Let $t \geq 0$, $\lambda > 0$, $p \in (1, 2^* - 1)$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ be a Carathéodory function. In this section, we will impose a condition on g stronger than (5.4) that will allow us to prove certain Hölder estimates (see the Appendix below). Namely, we assume that there exist three real numbers $\delta, \tau, \sigma \geq 0$ such that

$$\begin{cases} \tau \leq \sigma < 1, \\ \sigma - \tau < 1 - \sigma, \\ \tau \leq (s + \delta) g(x, s) \leq \sigma \quad \text{a.e. } x \in \Omega, \ \forall s > 0. \end{cases} \tag{5.15}$$

Let us consider the following auxiliary problem:

$$\begin{cases} -\Delta u + g(x, u) |\nabla u|^2 = \lambda u^p + t u^\sigma, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{P^t}$$

Note that, in the term tu^σ, the exponent σ is the same number that appears in condition (5.15). We will derive a priori estimates on the solutions to (P^t) which will provide the existence of solution to (P_λ).

Next proposition gives an a priori estimate on the parameter t in problem (P^t). For this result we will not need (5.15) but only (5.5).

Proposition 5.3.1. *Let $\lambda > 0$, $p > 1$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ be a Carathéodory function satisfying* (5.4) *and* (5.5) *for some $\sigma \in (0, 1)$. Then, there exists $t_0 > 0$ such that problem (P^t) admits no solution for any $t > t_0$.*

Proof. Let u be a solution to (P^t) for some $t > 0$. For a fixed smooth open set $\omega \subset\subset \Omega$, let λ_1 be the principal eigenvalue to the homogeneous Dirichlet eigenvalue problem in ω, and let φ_1 be any positive associated eigenfunction, i.e., λ_1 and φ_1 satisfy

$$\begin{cases} -\Delta \varphi_1 = \lambda_1 \varphi_1, & x \in \omega, \\ \varphi_1 > 0, & x \in \omega \\ \varphi_1 = 0, & x \in \partial\omega. \end{cases}$$

If we extend $\varphi_1 \equiv 0$ in $\Omega \setminus \omega$, then the function $\phi = \frac{\varphi_1}{u^\sigma}$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Taking ϕ as test function in (P^t) we obtain

$$\int_\Omega \frac{\nabla u \nabla \varphi_1}{u^\sigma} - \sigma \int_\Omega \frac{\varphi_1 |\nabla u|^2}{u^{\sigma+1}} + \int_\Omega g(x, u) \frac{\varphi_1 |\nabla u|^2}{u^\sigma} = \lambda \int_\Omega u^{p-\sigma} \varphi_1 + t \int_\Omega \varphi_1. \tag{5.16}$$

On the one hand, it is clear by (5.15) that

$$-\sigma\int_\Omega \frac{\varphi_1|\nabla u|^2}{u^{\sigma+1}}+\int_\Omega g(x,u)\frac{\varphi_1|\nabla u|^2}{u^\sigma}\leq 0.$$

On the other hand, let us denote as ν the exterior normal unit vector to $\partial\omega$. Then, Hopf's lemma implies that $\nu\nabla\varphi_1<0$ on $\partial\omega$. Hence, integration by parts and Young's inequality yield

$$\int_\Omega \frac{\nabla u\nabla\varphi_1}{u^\sigma}=\lambda_1\int_\omega \frac{\varphi_1 u^{1-\sigma}}{1-\sigma}+\int_{\partial\omega}\frac{u^{1-\sigma}}{1-\sigma}\nu\nabla\varphi_1<\lambda_1\int_\omega \frac{\varphi_1 u^{1-\sigma}}{1-\sigma}\leq\frac{\lambda}{2}\int_\Omega u^{p-\sigma}\varphi_1+C.$$

In sum, from (5.16) we deduce that $t\leq t_0$ for some $t_0>0$, as we wanted to prove. □

In the following result we prove a priori estimates on the solutions to (P^t) if g satisfies (5.15) and $sg(\cdot,s)$ has a uniform continuous limit at infinity which is small in some sense. In the proof we exploit the blow-up method due to [82].

Proposition 5.3.2. *Let $\lambda>0$, $p\in(1,2^*-1)$ and $g:\Omega\times(0,+\infty)\to[0,+\infty)$ be a Carathédory function satisfying* (5.15) *for some $\delta,\tau,\sigma\geq 0$. Assume also that*

$$\begin{cases} \exists\mu\in C(\overline{\Omega}): \|\mu\|_{L^\infty(\Omega)}<\frac{2^*-1-p}{2^*-2}, \\ \lim_{s\to+\infty}\|sg(\cdot,s)-\mu\|_{L^\infty(\Omega)}=0. \end{cases} \tag{5.17}$$

Then, there exists $C>0$ such that $\|u\|_{L^\infty(\Omega)}\leq C$ for every solution u to (P^t) for all $t\in[0,t_0]$, where $t_0>0$ is given by Proposition 5.3.1.

Proof. Let us assume by contradiction that there exist two sequences $\{t_n\}\subset[0,t_0]$ and $\{u_n\}$ such that u_n is a solution to (P^{t_n}) for all n and $\|u_n\|_{L^\infty(\Omega)}\to+\infty$ as $n\to+\infty$. Let us consider a sequence $\{x_n\}\subset\Omega$ satisfying

$$\|u_n\|_{L^\infty(\Omega)}=u_n(x_n)\quad\forall n,\quad x_n\to x_0\in\overline{\Omega}, \text{ up to a subsequence.}$$

We divide the rest of the proof into two parts. In the first of them we consider the case $x_0\in\Omega$, while the second one is devoted to the case $x_0\in\partial\Omega$.

Case 1: $x_0\in\Omega$.

Let us denote $d=\text{dist}(x_0,\partial\Omega)/2>0$ and $\eta_n=\|u_n\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}$. We define the function $v_n:B_{d/\eta_n}(0)\to[0,+\infty)$ by

$$v_n(y)=\eta_n^{\frac{2}{p-1}}u_n(x_n+\eta_n y)\quad\forall y\in B_{d/\eta_n}(0).$$

Therefore, $v_n \in H^1(B_{d/\eta_n}(0)) \cap L^\infty(B_{d/\eta_n}(0))$ and satisfies the equation

$$-\Delta v_n + u_n g_n(y,u_n)\frac{|\nabla v_n|^2}{v_n} = \lambda v_n^p + t_n \eta_n^{\frac{2(p-\sigma)}{p-1}} v_n^\sigma, \quad y \in B_{d/\eta_n}(0), \tag{5.18}$$

where $g_n(y,s) = g(x_n + \eta_n y, s)$ for every $y \in B_{d/\eta_n}(0)$ and $s > 0$. Moreover, it is clear that $\|v_n\|_{L^\infty(B_{d/\eta_n}(0))} = v_n(0) = 1$. Our aim now is to pass to the limit in (5.18). In the sequel we will prove the a priori estimates that will provide such a limit.

Indeed, let us fix $R > 0$ and denote $\omega = B_R(0)$. It is clear that $\omega \subset B_{d/\eta_n}(0)$ for every n sufficiently large, so v_n satisfies the equation (5.18) in ω and $\|v_n\|_{L^\infty(\omega)} = 1$ for n large. Of course, the same thing happens in $B_{2R}(0)$. Therefore, using condition (5.15) one can argue as in the Appendix below to prove that there exist $C > 0$, $\alpha \in (0,1)$ such that $\|v_n\|_{C^{0,\alpha}(B_{2R}(0))} \le C$, and this implies that

$$\|v_n\|_{C^{0,\alpha}(\overline{\omega})} \le C$$

for every n large enough. As a consequence, there exists $v \in C(\overline{\omega})$ such that, up to a subsequence,

$$v_n \to v \text{ uniformly in } \overline{\omega}.$$

Observe that $\|v\|_{L^\infty(\omega)} = 1$ so, in particular, $v \not\equiv 0$.

On the other hand, let us consider a function $\varphi \in C_c^1(B_{2R}(0))$ such that $0 \le \varphi \le 1$ in $B_{2R}(0)$ and $\varphi \equiv 1$ in ω. Now we multiply both sides of the equation (5.18) by $v_n\varphi^2 \in H_0^1(B_{2R}(0)) \cap L^\infty(B_{2R}(0))$ and integrate by parts, obtaining

$$\int_{B_{2R}(0)} |\nabla v_n|^2 \varphi^2 + 2\int_{B_{2R}(0)} v_n \varphi \nabla v_n \nabla \varphi \le C,$$

where we have used that $\|v_n\|_{L^\infty(B_{2R}(0))} = 1$. Hence, by Young's inequality we easily deduce that

$$\int_\omega |\nabla v_n|^2 \le C\left(\int_{B_{2R}(0)} |\nabla \varphi|^2 v_n^2 + 1\right) \le C.$$

That is to say, $\|v_n\|_{H^1(\omega)} \le C$, and then, up to a subsequence,

$$v_n \rightharpoonup v \text{ weakly in } H^1(\omega).$$

We will prove next that, for all $\omega_0 \subset\subset \omega$, v_n is bounded from below in $\overline{\omega_0}$ by a positive constant independent of n. The approach by comparison due to [107] is valid here. Indeed, it is straightforward to see that the function $w_n = \frac{v_n^{1-\sigma}}{1-\sigma} \in H^1(\omega) \cap L^\infty(\omega)$ satisfies

$$\int_\omega \nabla w_n \nabla \phi = \lambda \int_\omega v_n^{p-\sigma}\phi + \int_\omega (\sigma - u_n g_n(y,u_n))\frac{|\nabla v_n|^2 \phi}{v_n^{\sigma+1}} + t_n \eta_n^{\frac{2(p-\sigma)}{p-1}} \int_\omega \phi$$

for all $\phi \in C_c^\infty(\omega)$. Therefore,

$$\int_\omega \nabla w_n \nabla \phi \geq \lambda \int_\omega v_n^{p-\sigma} \phi$$

for all $0 \leq \phi \in C_c^1(\omega)$. Since $v_n \to v$ in $C(\overline{\omega})$, then $v_n(x) \geq \frac{1}{2} v(x)$ for all n large enough and for all $x \in \overline{\omega}$. Hence,

$$\int_\omega \nabla w_n \nabla \phi \geq \frac{\lambda}{2} \int_\omega v^{p-\sigma} \phi$$

for all $0 \leq \phi \in C_c^1(\omega)$ and for all n large enough. Let $z \in H_0^1(\omega) \cap C(\overline{\omega})$ be the unique solution to

$$\begin{cases} -\Delta z = \dfrac{\lambda}{2} v^{p-\sigma}, & x \in \omega, \\ z = 0, & x \in \partial\omega. \end{cases}$$

On the one hand, the strong maximum principle implies that $z > 0$ in ω. On the other hand, by comparison, $w_n \geq z$. As a consequence,

$$\forall \omega_0 \subset\subset \omega,\ \exists c_{\omega_0} > 0 : \quad v_n \geq c_{\omega_0} \text{ in } \omega_0 \quad \forall n \text{ large.} \tag{5.19}$$

By using the previous estimates, it is straightforward to prove that $\{\Delta v_n\}$ is bounded in $L^1_{\text{loc}}(\omega)$. Then, [26] implies that, passing to a subsequence,

$$\nabla v_n \to \nabla v \text{ a.e. in } \omega.$$

We are ready now to pass to the limit in (5.18). Indeed, let us take $\phi \in C_c^1(\omega)$ such that $\phi \geq 0$ as test function in the weak formulation of (5.18). We already know that $v_n \geq c_{\omega_0} > 0$ in $\omega_0 \subset\subset \omega$. Moreover, $\|u_n\|_{L^\infty(\omega)} \to +\infty$. Therefore, $u_n \to +\infty$ locally uniformly in ω. Then, by (5.17) we deduce that $|u_n g_n(y, u_n) - \mu(x_n + \eta_n y)| \to 0$ locally uniformly in ω. In consequence, the continuity of μ yields

$$u_n g_n(y, u_n) \to \mu(x_0) \quad \text{locally uniformly in } \omega.$$

In sum, we have that

$$u_n g_n(y, u_n) \frac{|\nabla v_n|^2}{v_n} \to \mu(x_0) \frac{|\nabla v|^2}{v} \quad \text{pointwise in } \omega.$$

By virtue of Fatou lemma and using the convergences that we have proved, it is immediate to show that

$$\int_\omega \nabla v \nabla \phi + \mu(x_0) \int_\omega \frac{|\nabla v|^2}{v} \phi \leq \lambda \int_\omega v^p \phi.$$

If we take now $\frac{v\phi}{v_n} \in H_0^1(\omega) \cap L^\infty(\omega)$ as test function in (5.18), we obtain

$$\int_\omega \frac{\nabla v_n \nabla v}{v_n}\phi - \int_\omega \frac{v|\nabla v_n|^2}{v_n^2}\phi + \int_\omega \frac{v}{v_n}\nabla v_n \nabla \phi + \int_\omega u_n g_n(y,u_n)\frac{v|\nabla v_n|^2}{v_n^2}\phi$$
$$= \lambda \int_\omega v_n^{p-1} v\phi + t_n \eta_n^{\frac{2(p-\sigma)}{p-1}} \int_\omega \frac{v\phi}{v_n^{1-\sigma}}.$$

Observe that

$$0 \le (1-\sigma)\frac{v|\nabla v_n|^2\phi}{v_n^2} \le (1-u_n g_n(y,u_n))\frac{v|\nabla v_n|^2\phi}{v_n^2} \to (1-\mu(x_0))\frac{|\nabla v|^2}{v}\phi.$$

Then, again by Fatou lemma we derive

$$\int_\omega \nabla v \nabla \phi + \mu(x_0)\int_\omega \frac{|\nabla v|^2}{v}\phi \ge \lambda \int_\omega v^p \phi.$$

That is to say, $v \in H^1(\omega) \cap C(\overline{\omega})$ satisfies

$$-\Delta v + \mu(x_0)\frac{|\nabla v|^2}{v} = \lambda v^p \quad \text{in } \omega.$$

But $\omega = B_R(0)$ for arbitrary $R > 0$, so v is actually well-defined in $\mathbb{R}^N$, it belongs to $H^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and it satisfies

$$-\Delta v + \mu(x_0)\frac{|\nabla v|^2}{v} = \lambda v^p \quad \text{in } \mathbb{R}^N.$$

Furthermore, it is straightforward to check that the function $w = \lambda^{\frac{1}{p-1}} v$ satisfies

$$-\Delta w + \mu(x_0)\frac{|\nabla w|^2}{w} = w^p \quad \text{in } \mathbb{R}^N.$$

This is impossible by virtue of Lemma 5.2.9.

Case 2: $x_0 \in \partial\Omega$.

Recall that we are assuming that there exist three sequences $\{t_n\} \subset [0,t_0]$, $\{u_n\}$ and $\{x_n\} \subset \Omega$ such that u_n is a solution to (P^{t_n}) for all n, $\|u_n\|_{L^\infty(\Omega)} = u_n(x_n) \to +\infty$ as $n \to +\infty$ and $x_n \to x_0$ for some $x_0 \in \partial\Omega$. Taking advantage of the smoothness of $\partial\Omega$, we are allowed to perform a convenient change of coordinates in such a way that u_n is a solution to a similar problem except that $\partial\Omega$ becomes flat near x_0 (see Lemma 5.6.1 in the Appendix below for the detailed proof). In other words, we may assume without loss of generality that u_n is a solution to (R^{t_n}) for all n, where

$$\begin{cases} -\text{div}(M(x)\nabla v) + b(x)\nabla v + j(x,v)M(x)\nabla v \nabla v = \lambda v^p + t v^\sigma, & x \in \Omega, \\ v > 0, & x \in \Omega, \\ v = 0, & x \in \Gamma, \end{cases} \qquad (R^t)$$

with $\Omega \subset \mathbb{R}^N_+$, $\emptyset \neq \Gamma \subset \partial\Omega \cap \partial\mathbb{R}^N_+$ and it is connected, and M, b, j are as in Lemma 5.6.1.

It is clear that $d_n = \operatorname{dist}(x_n, \partial\Omega) = \operatorname{dist}(x_n, \Gamma) = x_{n,N}$ for all n large enough. Arguing as in the previous case, we define

$$v_n(y) = \eta_n^{\frac{2}{p-1}} u_n(x_n + \eta_n y) \quad \forall y \in \Omega_n \cup \Gamma_n,$$

where $\eta_n^{-\frac{2}{p-1}} = \|u_n\|_{L^\infty(\Omega)}$, $0 \in \Omega_n = B_{d/\eta_n}(0) \cap \{y_N > -d_n/\eta_n\}$ and $\Gamma_n = B_{d/\eta_n}(0) \cap \{y_N = -d_n/\eta_n\}$ for some $d > 0$. It is easy to see that v_n is well-defined for all n large enough and it satisfies

$$\begin{aligned} -\operatorname{div}(M_n(y)\nabla v_n) + \eta_n b_n(y)\nabla v_n + j_n(y, u_n)u_n \frac{M_n(y)\nabla v_n \nabla v_n}{v_n} \\ = \lambda v_n^p + \eta_n^{\frac{2(p-\sigma)}{p-1}} t_n v_n^\sigma, \quad y \in \Omega_n, \end{aligned}$$

where $M_n(y) = M(x_n + \eta_n y)$, $b_n(y) = b(x_n + \eta_n y)$ and $j_n(y, \cdot) = j(x_n + \eta_n y, \cdot)$.

Now, if $\{d_n/\eta_n\}$ is unbounded, we can extract a subsequence such that $d_n/\eta_n \to +\infty$ as $n \to +\infty$. In this case, $\bigcup_{n\in\mathbb{N}} \Omega_n = \mathbb{R}^N$, so we can argue as in the case $x_0 \in \Omega$ without relevant changes and arrive to a contradiction.

Let us assume now that $\{d_n/\eta_n\}$ is bounded. Then, up to a (not relabeled) subsequence, $d_n/\eta_n \to \kappa$ for some $\kappa \geq 0$. Thus, $\bigcup_{n\in\mathbb{N}} \Omega_n = \{y_N > \kappa\}$. We assume without loss of generality that $\{d_n/\eta_n\}$ is monotone and we distinguish two cases: either it is increasing or decreasing.

First, assume that $\{d_n/\eta_n\}$ is increasing. Since $d_n/\eta_n \leq \kappa$ for all n, it is clear that $\kappa > 0$. We fix now $R > \kappa$ and consider the open set

$$\omega = B_R(0) \cap \{y_N > -\kappa\}.$$

Observe that, since $\kappa > 0$, then $0 \in \omega$. Also, $\omega = B_R(0) \cap (\Omega_n \cup \{-\kappa < y_N \leq -d_n/\eta_n\})$ for n large enough. Thus we may define $\tilde{v}_n : \overline{\omega} \to [0, +\infty)$ by

$$\tilde{v}_n(y) = \begin{cases} v_n(y) & \text{if } y \in \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{v}_n \in H^1(\omega) \cap C(\overline{\omega})$, $\tilde{v}_n = 0$ on $\partial\omega \cap \{y_N = -\kappa\}$ and $\|\tilde{v}_n\|_{L^\infty(\omega)} = \tilde{v}_n(0) = 1$. In fact, $\tilde{v}_n$ is well-defined in $\omega_1 = B_{2R}(0) \cap \{y_N > -\kappa\}$ as well. Let us now denote $\beta_1 = B_{2R}(0) \cap \{y_N = -\kappa\}$. Thus, following again the Appendix and using condition (5.15), it can be proved that there exist $C > 0, \alpha \in (0,1)$ such that $\|\tilde{v}_n\|_{C^{0,\alpha}(\omega_1 \cup \beta_1)} \leq C$ for all n large enough. Now, since $\overline{\omega} \subset \omega_1 \cup \beta_1$, then

$$\|\tilde{v}_n\|_{C^{0,\alpha}(\overline{\omega})} \leq C$$

for every n large enough. As a consequence, there exists $v \in C(\overline{\omega})$ such that $\tilde{v}_n \to v$ uniformly in $\overline{\omega}$.

Next, an estimate on $\{\tilde{v}_n\}$ in $H^1_{\mathrm{loc}}(\omega)$ can be proved as in the previous case $x_0 \in \Omega$, so that $\tilde{v}_n \rightharpoonup v$ weakly in $H^1_{\mathrm{loc}}(\omega)$, up to a subsequence. Moreover, the same arguments are valid to prove that $\{\tilde{v}_n\}$ satisfies also (5.19) and, furthermore, that $\nabla\tilde{v}_n \to \nabla v$ a.e. in ω. Thus, we can now pass to the limit as in the case $x_0 \in \Omega$ and deduce that

$$-\mathrm{div}(M(x_0)\nabla v) + \mu(x_0)\frac{M(x_0)\nabla v \nabla v}{v} = \lambda v^p, \quad y \in \omega.$$

We want to prove now that $v(y) = 0$ on $\partial\omega \cap \{y_N = -\kappa\}$. In order to do so, let $y = (y', -\kappa) \in \partial\omega \cap \{y_N = -\kappa\}$. Then, the sequence $\{y_n\} = \{(y', -d_n/\eta_n)\} \subset \omega$ and $y_n \to y$. Hence, the uniform convergence yields $v_n(y_n) \to v(y)$. On the other hand, recall that $v_n(y_n) = 0$ for all n. In consequence, $v(y) = 0$.

In sum, we have proved that $v \in H^1_{\mathrm{loc}}(\omega) \cap C(\overline{\omega})$ is a solution to the problem:

$$\begin{cases} -\mathrm{div}(M(x_0)\nabla v) + \mu(x_0)\dfrac{M(x_0)\nabla v \nabla v}{v} = \lambda v^p, & y \in \omega, \\ v > 0, & y \in \omega, \\ v = 0, & y \in \omega \cap \{y_N = -\kappa\}. \end{cases}$$

Actually, taking R arbitrarily large we deduce that $v \in H^1_{\mathrm{loc}}(\{y_N > -\kappa\}) \cap C(\{y_N \geq -\kappa\})$ and it satisfies

$$\begin{cases} -\mathrm{div}(M(x_0)\nabla v) + \mu(x_0)\dfrac{M(x_0)\nabla v \nabla v}{v} = \lambda v^p, & y \in \{y_N > -\kappa\}, \\ v > 0, & y \in \{y_N > -\kappa\}, \\ v = 0, & y \in \{y_N = -\kappa\}. \end{cases}$$

Observe that, if we denote $M(x_0) = (m_{ij})$ for $i, j = 1, \ldots, N$, then the previous equation may be written as

$$\sum_{i,j=1}^{N} m_{ij}\frac{\partial^2 v}{\partial y_i \partial y_j} + \frac{\mu(x_0)}{v}\sum_{i,j=1}^{N} m_{ij}\frac{\partial v}{\partial y_i}\frac{\partial v}{\partial y_j} = \lambda v^p, \quad y \in \{y_N > -\kappa\}.$$

Since both $\frac{\partial^2 v}{\partial y_i \partial y_j}$ and $\frac{\partial v}{\partial y_i}\frac{\partial v}{\partial y_j}$ commute in i, j, then a simple change of coordinates (see the conclusion of Case 1 in Section 2 of [82] for the details) leads to finding a solution $w \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ to

$$\begin{cases} -\Delta w + \mu(x_0)\dfrac{|\nabla w|^2}{w} = w^p, & y \in \mathbb{R}^N_+, \\ w > 0, & y \in \mathbb{R}^N_+, \\ w = 0, & y \in \partial\mathbb{R}^N_+. \end{cases}$$

This contradicts Lemma 5.2.9.

It is left to consider the case $\{d_n/\eta_n\}$ decreasing (observe that, in principle, κ might equal 0). Let us take $R > \kappa$ and denote $\omega = B_R(0)$. We define the function $\tilde{v}_n : \overline{\omega} \to \mathbb{R}$ by

$$\tilde{v}_n(y) = \begin{cases} v_n(y) & \text{if } y \in \Omega_n, \\ 0 & \text{otherwise .} \end{cases}$$

It is clear that $\overline{\omega} \cap \Omega_n = \overline{\omega} \cap \{y_N > -d_n/\eta_n\} \neq \emptyset$ for all n large enough, so $\tilde{v}_n$ is well-defined. Arguing as above, it can be proved that $\|\tilde{v}_n\|_{C^{0,\alpha}(\overline{\omega})} \leq C$.

Furthermore, $\tilde{v}_n(0) = v_n(0) = 1$. Thus, for $y = (0, \cdots, 0, -d_n/\eta_n) \in B_R(0) \cap \Gamma_n$, the Hölder estimate yields

$$1 = |\tilde{v}_n(y) - \tilde{v}_n(0)| \leq C|y|^\alpha = C\,(d_n/\eta_n)^\alpha\,.$$

This implies that $\kappa > 0$.

Therefore, arguing as above, there exists $0 < v \in H^1_{\text{loc}}(\omega) \cap C(\overline{\omega})$ such that, up to subsequences, $\tilde{v}_n \to v$ uniformly in $\overline{\omega}$ and weakly in $H^1_{\text{loc}}(\omega)$. Observe that, in particular, $v(y) = 0$ for all $y \in \overline{\omega} \cap \{y_N = -\kappa\}$. Moreover, we can pass to the limit as before and we obtain that v satisfies

$$\begin{cases} -\operatorname{div}(M(x_0)\nabla v) + \mu(x_0)\dfrac{M(x_0)\nabla v \nabla v}{v} = \lambda v^p, & y \in \omega \cap \{y_N > -\kappa\}, \\ v > 0, & y \in \omega \cap \{y_N > -\kappa\}, \\ v = 0, & y \in \omega \cap \{y_N = -\kappa\}. \end{cases}$$

We arrive to a contradiction similarly as in the case $\{d_n/\eta_n\}$ increasing. The proof is concluded. □

The following result, as Proposition 5.3.2, provides a priori estimates on the solutions to (P^t). The difference lies on the fact that we do not impose the limit condition at infinity at the expense of making a stronger restriction on the sizes of g, p.

Proposition 5.3.3. *Let $\lambda > 0$, $p \in \left(1, \frac{N+1}{N-1}\right)$ and $g : \Omega \times (0, +\infty) \to [0, +\infty)$ be a Carathéodory function satisfying* (5.15) *for some $\delta, \tau, \sigma \geq 0$. Assume also that $\sigma \leq \frac{N+1-(N-1)p}{2}$. Then, there exists $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for every solution u to (P^t) for all $t \in [0, t_0]$, where $t_0 > 0$ is given by Proposition 5.3.1.*

Proof. The proof is very similar to the proof of Proposition 5.3.2. Here we give only a sketch.

Arguing by contradiction, we assume that there exist two sequences $\{t_n\} \subset [0,t_0]$ and $\{u_n\}$ such that u_n is a solution to (P^{t_n}) for all n and $\|u_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$. We also consider a sequence $\{x_n\} \subset \Omega$ satisfying

$$\|u_n\|_{L^\infty(\Omega)} = u_n(x_n) \quad \forall n, \quad x_n \to x_0 \in \overline{\Omega}, \text{ up to a subsequence.}$$

We denote $\eta_n = \|u_n\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}$. Let us assume that $x_0 \in \partial\Omega$ (we will omit the simpler case $x_0 \in \Omega$). Arguing as in the proof of Proposition 5.3.2, we assume without loss of generality that u_n is a solution to (R^{t_n}) for all n, with $\Omega = V \subset \mathbb{R}^N_+$ and $\Gamma \subset \partial\Omega \cap \partial\mathbb{R}^N_+$. Thus, there exists a sequence of bounded domains $\{\Omega_n\}$ satisfying, for every n, that $0 \in \Omega_n$, $x_n + \eta_n y \in \Omega$ for all $y \in \Omega_n$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = X$, where X may be either $\mathbb{R}^N$ or $\{y_n > \kappa\}$ for some $\kappa \geq 0$.

In any case, we define $v_n : \Omega_n \to \mathbb{R}$ by

$$v_n(y) = \eta_n^{\frac{2}{p-1}} u_n(x_n + \eta_n y) \quad \forall y \in \Omega_n.$$

It is easy to prove that v_n satisfies the equation

$$\begin{aligned} -\mathrm{div}(M_n(y)\nabla v_n) + \eta_n b_n(y)\nabla v_n + u_n j_n(y,u_n)\frac{M_n(y)\nabla v_n \nabla v_n}{v_n} \\ = \lambda v_n^p + \eta_n^{\frac{2(p-\sigma)}{p-1}} t_n v_n^\sigma, \quad y \in \Omega_n, \end{aligned}$$

where $M_n(y) = M(x_n + \eta_n y)$, $b_n(y) = b(x_n + \eta_n y)$ and $j_n(y,\cdot) = j(x_n + \eta_n y, \cdot)$. Since $s j_n(x,s) \leq \sigma$, we deduce that

$$-\mathrm{div}(M_n(y)\nabla v_n) + \eta_n b_n(y)\nabla v_n + \sigma\frac{M_n(y)\nabla v_n \nabla v_n}{v_n} \geq \lambda v_n^p, \quad y \in \Omega_n.$$

Passing to the limit as in Proposition 5.3.2 and applying after that a convenient change of coordinates, we obtain a supersolution $v \in H^1_{\mathrm{loc}}(X) \cap C(\overline{X})$ to (5.13), where either $X = \mathbb{R}^N$ or $X = \mathbb{R}^N_+$. This is a contradiction with Lemma 5.2.9. □

Next proposition provides similar estimates as Propositions 5.3.2 and 5.3.3. The novelty is that we assume the limit condition at infinity, as well as the continuity of the limit, only in a neighborhood of $\partial\Omega$. This means that we need to impose stronger size restrictions on g, p than in Proposition 5.3.2, but milder than in Proposition 5.3.3.

Proposition 5.3.4. *Let $\lambda > 0$, $p \in \left(1, \frac{N}{N-2}\right)$ and $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying* (5.15) *for some $\delta, \tau, \sigma \geq 0$. Assume also that $\sigma \leq \frac{N-(N-2)p}{2}$ and*

$$\begin{cases} \exists \omega \subset\subset \Omega,\ \mu \in C(\overline{\Omega \setminus \omega}) : \|\mu\|_{L^\infty(\Omega\setminus\omega)} < \frac{2^*-1-p}{2^*-2}, \\ \lim_{s\to+\infty} \|sg(\cdot,s) - \mu\|_{L^\infty(\Omega\setminus\omega)} = 0. \end{cases} \tag{5.20}$$

Then, there exists $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \le C$ for every solution u to (P^t) for all $t \in [0,t_0]$, where $t_0 > 0$ is given by Proposition 5.3.1.

Proof. We argue similarly as for Propositions 5.3.2 and 5.3.3, so we give a sketch of the proof.

Assume by contradiction that there exist two sequences $\{t_n\} \subset [0,t_0]$ and $\{u_n\}$ such that u_n is a solution to (P^{t_n}) for all n and $\|u_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$. We also consider a sequence $\{x_n\} \subset \Omega$ satisfying

$$\|u_n\|_{L^\infty(\Omega)} = u_n(x_n) \quad \forall n, \quad x_n \to x_0 \in \overline{\Omega} \text{ up to a subsequence.}$$

Suppose that $x_0 \in \Omega$. Since x_0 might belong to ω (where the asymptotic behavior of g at infinity is unknown), then we cannot proceed as in the proof of Proposition 5.3.2. Nevertheless, rescaling u_n conveniently and using (5.15) we may argue as in the proof of Proposition 5.3.3 to find a supersolution $0 < v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ to

$$-\Delta v + \sigma \frac{|\nabla v|^2}{v} = v^p, \quad y \in \mathbb{R}^N.$$

This is a contradiction with Lemma 5.2.9.

On the other hand, if $x_0 \in \partial\Omega$, then we may take advantage of the continuity of μ given by (5.20) to obtain, arguing as in Proposition 5.3.2, a solution $v \in H^1_{\mathrm{loc}}(X) \cap C(\overline{X})$ to (5.13) for $\sigma = \mu(x_0)$ and either $X = \mathbb{R}^N$ or $X = \mathbb{R}^N_+$. This contradicts Lemma 5.2.9. $\square$

In the last result of this section we find an estimate for the solutions to problem (P_λ) whose dependence on λ is explicit. As a consequence, it is shown that the norm of the solutions to problem (P_λ), if they exist, becomes arbitrarily small as λ tends to infinity.

Proposition 5.3.5. *Let $p \in \left(1, \frac{N+1}{N-1}\right)$ and $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying (5.15) for some $\delta, \tau, \sigma \ge 0$. Assume in addition that $\sigma \le \frac{N+1-(N-1)p}{2}$. Then, there exists $C > 0$ such that*

$$\lambda^{\frac{1}{p-1}} \|u\|_{L^\infty(\Omega)} \le C$$

for every solution u to (P_λ) for all $\lambda > 0$.

Proof. Arguing again as in the proof of Proposition 5.3.2, assume that there exist two sequences $\{\lambda_n\} \subset [0,+\infty)$ and $\{u_n\}$ such that u_n is a solution to (P_{λ_n}) for all n and $\|z_n\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$, where $z_n = \lambda_n^{\frac{1}{p-1}} u_n$. It is easy to see that z_n satisfies

$$-\Delta z_n + \lambda_n^{-\frac{1}{p-1}} g\left(x, \lambda_n^{-\frac{1}{p-1}} z_n\right) |\nabla z_n|^2 = z_n^p, \quad x \in \Omega.$$

Let $\{x_n\} \subset \Omega$ be such that $z_n(x_n) = \|z_n\|_{L^\infty(\Omega)}$ for all n, and let $x_0 \in \overline{\Omega}$ be such that, passing to a subsequence if necessary, $x_n \to x_0 \in \partial\Omega$ (the case $x_0 \in \Omega$ is analogous, so we omit it). Let us consider the function $v_n(y) = \eta_n^{\frac{2}{p-1}} z_n(x_n + \eta_n y)$ defined for all $y \in \Omega_n$, with $\Omega_n \subset \mathbb{R}^N$ as in the proof of Proposition 5.3.2 (Case 2) and $\eta_n = \|z_n\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}$. Then, thanks to Lemma 5.6.1 in the Appendix below, we may assume without loss of generality that $v_n \in H^1(\Omega_n) \cap L^\infty(\Omega_n)$ satisfies

$$-\mathrm{div}(M_n(y)\nabla v_n) + \eta_n b_n(y)\nabla v_n + \frac{v_n}{L_n} j_n\left(y, \frac{v_n}{L_n}\right) \frac{M_n(y)\nabla v_n \nabla v_n}{v_n} = v_n^p, \quad y \in \Omega_n, \tag{5.21}$$

where $L_n^{p-1} = \lambda_n \eta_n^2$ and M_n, b_n, j_n are as in the proof of Proposition 5.3.2 (Case 2). From (5.21), and using condition (5.15) (which is obviously satisfied by j_n), one can prove the same estimates on $\{v_n\}$ as in the proof of Proposition 5.3.2. Moreover, again by (5.15) we deduce that

$$-\mathrm{div}(M_n(y)\nabla v_n) + \eta_n b_n(y)\nabla v_n + \sigma \frac{M_n(y)\nabla v_n \nabla v_n}{v_n} \geq v_n^p, \quad y \in \Omega_n.$$

Now we pass to the limit as usual and obtain a supersolution $v \in H^1_{\mathrm{loc}}(X) \cap C(\overline{X})$ to (5.13), where either $X = \mathbb{R}^N$ or $X = \mathbb{R}^N_+$. This is a contradiction with Lemma 5.2.9. □

5.4 Main existence results

The following is the main existence result of the paper. There we prove that there exists at least a solution to problem (P_λ) for all $\lambda > 0$ provided g is small in some sense.

Theorem 5.4.1. *Assume that the hypotheses of either Proposition 5.3.2, or Proposition 5.3.3, or else Proposition 5.3.4 are satisfied, and assume also that there exists* $\lim_{s\to 0} sg(x,s)$ *for a.e.* $x \in \Omega$. *Then, there exists at least a solution to* (P_λ) *for all* $\lambda > 0$. *Moreover, in case the hypotheses of Proposition 5.3.4 are satisfied, then there exists* $C > 0$ *such that*

$$\lambda^{\frac{1}{p-1}} \|u\|_{L^\infty(\Omega)} \leq C$$

for every solution u *to* (P_λ) *and for every* $\lambda > 0$.

Remark 5.4.2. We point out that the smallness condition $\sigma < \frac{2^*-1-p}{2^*-2}$ required by Theorem 5.4.1 is necessary for the existence of solutions to (P_λ), at least in the particular case of Ω a satarshaped domain and $sg(x,s) \equiv \sigma$ for some constant $\sigma < 1$. Indeed, assume by contradiction that $\sigma \in \left[\frac{2^*-1-p}{2^*-2}, 1\right)$ and that $0 < u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies $-\Delta u + \sigma \frac{|\nabla u|^2}{u} = \lambda u^p$ in Ω. Then, $v = cu^{1-\sigma}$ satisfies $-\Delta v = v^{\frac{p-\sigma}{1-\sigma}}$ in Ω for some $c > 0$.

Therefore, since $\frac{p-\sigma}{1-\sigma} \geq 2^* - 1$ and Ω is starshaped, the well-known Pohozaev's identity (see [112]) yields a contradiction.

Remark 5.4.3. Apart from the model examples in the Introduction above, Theorem 5.4.1 allows us to handle many others. For instance, a curious one corresponds to the choice $g(x,s) = \frac{\mu(x)}{s^\alpha + s^\beta}$ with $0 \leq \alpha \leq 1 \leq \beta$ and $0 \lneq \mu \in L^\infty(\Omega)$ is small enough.

Proof. For every $0 \lneq v \in C(\overline{\Omega})$, $t \geq 0$, we consider the following problem:

$$\begin{cases} -\Delta u + vg(x,v)\dfrac{|\nabla u|^2}{u} = \lambda v^p + tv^\sigma, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{Q^t}$$

Since $\sigma \in (0,1)$, then Theorem 5.2.8 implies that there exists a unique solution u to (Q^t). Moreover, $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{1,N}_{\mathrm{loc}}(\Omega)$ by virtue of Propositions 5.2.3 and 5.2.4.

Let $X = \{w \in C(\overline{\Omega}) : w(x) \geq 0\, \forall x \in \overline{\Omega}\}$. We now define $K : X \times [0,+\infty) \to X$ by $K(v,t) = u$ for all $v \in X \setminus \{0\}$ and $t \geq 0$, where u is the unique solution to (Q^t), while $K(0,t) = 0$ for any $t \geq 0$. We aim to prove that there exists $0 \lneq u \in X$ such that $K(u,0) = u$, which is equivalent to finding a solution to (P_λ). In order to do so, we first prove that K is continuous. Indeed, let $\{v_n\} \subset X$ and $\{t_n\} \subset [0,+\infty)$ be such that $v_n \to v$ in $C(\overline{\Omega})$ for some $v \in X$ and $t_n \to t$ for some $t \geq 0$. Let us denote $u_n = K(v_n,t_n)$. Then, noticing that $-\Delta u_n \leq C$ for some $C > 0$, it is straightforward to prove that $\|u_n\|_{H^1_0(\Omega)} \leq C$ for all n. Hence, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega)$ for some $0 \leq u \in H^1_0(\Omega)$. Furthermore, $-\Delta u_n \leq C$ also implies, thanks to the well-known Stampacchia's Lemma (see [121]), that $\|u_n\|_{L^\infty(\Omega)} \leq C$ for all n. Therefore, condition (5.15) and the arguments in the Appendix below imply that there exist $\alpha \in (0,1), C > 0$ such that $\|u_n\|_{C^{0,\alpha}(\overline{\Omega})} \leq C$ for all n. In consequence, $u_n \to u$ strongly in $C(\overline{\Omega})$. In particular, $u \in X$.

Assume now that $v \gneq 0$. Then, it can be proven as in [107] that $u > 0$ in Ω. Hence, one can pass to the limit in a standard way (see [38], for instance) so that u is the unique solution to (Q^t), i.e., $u = K(v,t)$. The uniqueness implies also that the original sequence $\{K(v_n,t_n)\}$, and not merely a subsequence, converges itself to $K(v,t)$ strongly in $C(\overline{\Omega})$. That is to say, K is continuous in $X \setminus \{0\} \times [0,+\infty)$.

Suppose now that $v \equiv 0$. Observe that

$$\begin{cases} -\Delta u_n \leq \lambda v_n^p + t_n v_n^\sigma, & x \in \Omega, \\ u_n > 0, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases}$$

Hence, passing to the limit,

$$\begin{cases} -\Delta u \leq 0, & x \in \Omega, \\ u \geq 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Therefore, the maximum principle implies that $u = 0 = K(0,t)$. One more time, in fact the original sequence $\{K(v_n,t_n)\}$ converges itself to $K(0,t)$ strongly in $C(\overline{\Omega})$. This proves that K is continuous in $X \times [0,+\infty)$.

It remains to prove that K is compact, i.e., it maps bounded sets to relatively compact sets. Indeed, let $\{v_n\} \subset X$ and $\{t_n\} \subset [0,+\infty)$ be bounded sequences. Taking subsequences, $v_n \to v$ in the weak-$\star$ topology of $L^\infty(\Omega)$ for some $v \in L^\infty(\Omega)$, while $t_n \to t$ for some $t \geq 0$. This is enough to pass to the limit in the equations as above. We conclude that, up to subsequences, $K(v_n,t_n) \to K(v,t)$ strongly in $C(\overline{\Omega})$. This proves that K is compact.

We will prove next that there exist $0 < r < R$ and $t_1 \geq 0$ such that

1. $u \neq sK(u,0)$ $\forall s \in [0,1]$, $\forall u \in X$ with $\|u\|_{L^\infty(\Omega)} = r$,
2. $u \neq K(u,t)$ $\forall t \geq 0$, $\forall u \in X$ with $\|u\|_{L^\infty(\Omega)} = R$,
3. $u \neq K(u,t)$ $\forall t \geq t_1$, $\forall u \in X$ with $\|u\|_{L^\infty(\Omega)} \leq R$.

In order to prove item 1, let us assume by contradiction that, for all $r > 0$, there exist $s \in [0,1]$ and $u \in X$ with $\|u\|_{L^\infty(\Omega)} = r$ such that $u = sK(u,0)$. In particular, $u > 0$ in Ω and

$$\begin{cases} -\Delta u \leq \lambda s u^p, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Since $p > 1$, we can choose $r > 0$ such that $\lambda a^p \leq \frac{\lambda_1}{2} a$ for all $a \in [0,r]$, where λ_1 stands for the principal eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions. Furthermore, $u \leq r$ in Ω, so we have that

$$\begin{cases} -\Delta u \leq \dfrac{\lambda_1}{2} u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

This is a clear contradiction with the definition of λ_1.

On the other hand, if we take $R > C$, where $\|u\|_{L^\infty(\Omega)} \le C$ (see Propositions 5.3.2, 5.3.3 and 5.3.4) then it is clear that item 2 holds. Moreover, if we take $t_1 > t_0$, where $t_0 > 0$ is given by Proposition 5.3.1, then item 3 also holds.

In conclusion, [60, Proposition 2.1 and Remark 2.1] can be applied and in consequence we obtain a positive fixed point of K_0, i.e., a solution to (P_λ). Finally, the last statement of the result is trivial from Proposition 5.3.5. The proof is finished. □

Now we present the last existence result of the paper. It provides the existence of solution to (P_λ) for $\lambda > 0$ large, although g may be very general for s large, as only conditions for s near zero are needed.

Theorem 5.4.4. *Let $p \in \left(1, \frac{N+1}{N-1}\right)$ and $g : \Omega \times (0,+\infty) \to [0,+\infty)$ be a Carathéodory function satisfying that there exist $\delta, \tau, \sigma \ge 0$ and $s_0 > 0$ such that $\tau \le \sigma \le \frac{N+1-(N-1)p}{2}$, $\sigma - \tau < 1 - \sigma$ and*

$$\tau \le (s+\delta)g(x,s) \le \sigma \quad \text{a.e. } x \in \Omega,\ \forall s \in (0,s_0]. \tag{5.22}$$

Assume in addition that there exists $\lim_{s\to 0} sg(x,s)$ for a.e. $x \in \Omega$. Then, there exists $\lambda_0 > 0$ such that problem (P_λ) admits at least a solution u_λ for all $\lambda > \lambda_0$. Moreover, there exists $C > 0$, independent of λ, such that

$$\lambda^{\frac{1}{p-1}} \|u_\lambda\|_{L^\infty(\Omega)} \le C \quad \forall \lambda > \lambda_0.$$

Remark 5.4.5. We point out that Theorem 5.4.4 is valid for a very wide class of nonlinearities. For instance, if

$$g(x,s) = \mu(x)h(s) \quad \text{a.e. } x \in \Omega,\ \forall s \ge 0,$$

where $0 \lneq \mu \in L^\infty(\Omega)$ and $h : [0,+\infty) \to [0,+\infty)$ is continuous (also at $s = 0$), then the hypotheses of Theorem 5.4.4 are fulfilled; roughly speaking, the idea is to take $\delta = \tau = 0$ and $\sigma < \min\left\{\frac{1}{2}, \frac{N+1-p(N-1)}{2}\right\}$ in (5.22), and choose $s_0 > 0$ small enough. On the other hand, a prototypical example of function g singular at $s = 0$ satisfying the conditions in Theorem 5.4.4 is

$$g(x,s) = \frac{\mu(x)}{s^\gamma} \quad \text{a.e. } x \in \Omega,\ \forall s \in (0,s_0),$$

where $\gamma \in (0,1)$, $0 \lneq \mu \in L^\infty(\Omega)$ and, for $s > s_0$, g is a Carathéodory function with arbitrary growth.

Proof of Theorem 5.4.4. Consider the Carathéodory function $\bar{g} : \Omega \times (0,+\infty) \to [0,+\infty)$ defined by

$$\bar{g}(x,s) = \begin{cases} g(x,s) & \text{if } x \in \Omega,\ s \in (0,s_0), \\ \dfrac{(s_0+\delta)g(x,s_0)}{s+\delta} & \text{if } x \in \Omega,\ s \ge s_0. \end{cases}$$

It is clear that $\bar{g}$ satisfies the hypotheses of Proposition 5.3.5. Thus, if we denote problem (P_λ) with $\bar{g}$ instead of g as $(\bar{P}_\lambda)$, Theorem 5.4.1 implies that there exists a positive solution u_λ to $(\bar{P}_\lambda)$ for all $\lambda > 0$ that satisfies

$$\lambda^{\frac{1}{p-1}} \|u\|_{L^\infty(\Omega)} \leq C \quad \forall \lambda > 0.$$

In particular, there exists $\lambda_0 > 0$ such that $\|u_\lambda\|_{L^\infty(\Omega)} < s_0$ for every $\lambda \geq \lambda_0$. Hence, u_λ is, in fact, a solution to (P_λ) for every $\lambda \geq \lambda_0$. The proof is finished. $\square$

We conclude the section by proving Theorems 5.1.3 and 5.1.4 as direct consequences of Theorems 5.4.1, 5.4.4 and 5.2.7. We omit the proof of Theorem 5.1.2 because it follows after applying *literally* Theorems 5.4.1 and 5.2.7.

Proof of Theorem 5.1.3. Let us denote $g(x,s) = \frac{\mu(x)}{(s+\delta)^\gamma}$ and $h(x,s) = (s+\delta)g(x,s)$. In order to prove the first statement, we assume that $\delta > 0$ and $\|\mu\|_{L^\infty(\Omega)} < \frac{\delta^{\gamma-1}}{2}$. Observe that

$$\inf_{(x,s)\in\Omega\times[0,+\infty)} h(x,s) = 0, \qquad \sup_{(x,s)\in\Omega\times[0,+\infty)} h(x,s) = \frac{\|\mu\|_{L^\infty(\Omega)}}{\delta^{\gamma-1}}.$$

Therefore, $\tau \leq h(x,s) \leq \sigma$ for $\tau = 0$ and $\sigma = \frac{\|\mu\|_{L^\infty(\Omega)}}{\delta^{\gamma-1}}$. Note that $\sigma - \tau < 1 - \sigma$ since $2\sigma < 1$. Hence, (5.15) is satisfied. On the other hand, it is clear that $h(x,s) \to 0$ as $s \to +\infty$ uniformly for a.e. $x \in \Omega$. Thus, (5.17) also holds. In conclusion, Theorem 5.4.1 implies that there exists a solution to (5.1) for any $\lambda > 0$.

On the contrary, let us assume now that $\delta = 0$ and $\mu(x) \geq \tau > 0$ for a.e. $x \in \Omega \setminus \omega$. Then, $h(x,s) = \frac{\mu(x)}{s^{\gamma-1}}$. In consequence, there exists $s_0 > 0$ small enough such that (5.9) holds. Thus, Theorem 5.2.7 implies that problem (5.1) admits no solution for any $\lambda > 0$. $\square$

Proof of Theorem 5.1.4. Let us denote $g(x,s) = \frac{\mu(x)}{(s+\delta)^\gamma}$ and $h(x,s) = sg(x,s) = \frac{\mu(x)s}{(s+\delta)^\gamma}$. It is clear that, for any $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that

$$0 \leq h(x,s) \leq \|\mu\|_{L^\infty(\Omega)} \frac{s}{(s+\delta)^\gamma} < \varepsilon \quad \text{a.e. } x \in \Omega,\ \forall s \in (0, s_\varepsilon].$$

Therefore, if $\varepsilon < \min\left\{\frac{1}{2}, \frac{N+1-p(N-1)}{2}\right\}$, then the hypotheses of Theorem 5.4.4 are fulfilled. This completes the proof of the result. $\square$

5.5 Open problems

After the analysis carried out in this work, many problems remain still unsolved. We list some of them here:

1. As we pointed out in the Introduction, the existence of solution to problem (5.1) for $p \in (1, 2^* - 1)$, $\gamma = 1$, $\delta > 0$, $\frac{2^*-1-p}{2^*-2} \leq \mu < p$ and $\lambda > 0$ small is unknown, even for $\mu > 0$ constant. This represents a challenge also in the theory of semilinear equations. Indeed, it is well-known that, if μ is constant, then there is a change of variable that turns the equation in (5.1) into a semilinear one. Simple computations show that the nonlinearity obtained after performing the change of unknown is superlinear and subcritical at zero, but supercritical at infinity. This behavior makes the problem hard to deal with for $\lambda > 0$ small.

2. Concerning also problem (5.1) for $p \in (1, 2^* - 1)$ and $\gamma = 1$, some questions remain unanswered in the singular case $\delta = 0$. Indeed, nonexistence of solution is known if $\mu \equiv \text{constant} \in \left[\frac{2^*-1-p}{2^*-2}, 1\right)$ and Ω is starshaped (see Remark 5.4.2 above). However, we are not aware of any existence or nonexistence result for μ non-constant in that range. Furthermore, if $\mu > 1$ near $\partial\Omega$, then nonexistence is known (see Theorem 5.2.7 above), but for $\mu \equiv 1$ we do not have an answer. To this respect, the only related result that we know is a nonexistence result about problem (5.3) for $\mu \equiv 1$, $f \equiv 0$ near $\partial\Omega$ and $N = 1$ (see [37]). An extension of such a result to any dimension N would be remarkable too.

3. In Theorem 5.2.8, no restriction on μ from below (apart from $\mu \gneqq 0$) is needed. In view of this fact, condition (5.15) does not seem to be completely natural. We presume that it appears by imposition of the techniques used to prove the Hölder estimates (see the Appendix below). We suspect that, by using only $\|\mu\|_{L^\infty(\Omega)} < 1$, it must be possible to prove positive local estimates from below. This would be enough to prove the Hölder estimates and to pass to the limit in the results of Section 5.3.

4. In Theorem 5.1.3, the smallness condition on μ is obviously unnatural. We presume that a different approach must lead to the existence of solution to (5.1) for every $\lambda > 0$ provided $p \in (1, 2^* - 1), \gamma > 1, \delta > 0$ and $0 \lneqq \mu \in L^\infty(\Omega)$ without size conditions.

5. In [100] it is proved, among other results, that there exists at least a solution to (5.1) for every $\lambda > 0$ provided that $p > 1, \gamma = 1, \delta > 0$ and $\mu \in C(\overline{\Omega})$ with $\max_{x \in \overline{\Omega}} \mu(x) < \frac{2^*-1-p}{2^*-2}$. Observe that p is allowed to be supercritical and μ is allowed to change sign; actually, it is forced to be negative if $p \geq 2^* - 1$. We think that the restrictions $p < 2^* - 1$ and $g \geq 0$ are also unnecessary for μ non-constant.

6. Finally, we point out that the condition $p < \frac{N+1}{N-1}$ in Theorem 5.1.4 does not seem to be sharp in view of Theorems 5.1.2 and 5.1.3. In fact, we expect a similar result to hold, at least, for every $p < 2^* - 1$.

5.6 Appendix

In this section we prove some technical results that are required by the blow-up method.

5.6.1 Technical lemma

Lemma 5.6.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with boundary of class $\mathscr{C}^2$, $g : \Omega \times (0, +\infty) \to \mathbb{R}$ be a Carathéodory function, and $f : [0, +\infty) \to \mathbb{R}$ be a continuous function. Then, for every $x_0 \in \partial\Omega$, there exist $U \subset \mathbb{R}^N$ a neighborhood of x_0 and an injective and C^2 map $y : U \to \mathbb{R}^N$, with C^2 inverse, such that $V = y(U \cap \Omega) \subset \mathbb{R}^N_+$, $\Gamma = y(U \cap \partial\Omega) \subset \partial V \cap \partial\mathbb{R}^N_+$ and, if u is a solution to*

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.23}$$

then the function $v = u \circ y^{-1} : V \to (0, +\infty)$ is a solution to

$$\begin{cases} -\mathrm{div}(M(y)\nabla v) + b(y)\nabla v + j(y,v)M(y)\nabla v \nabla v = f(v), & y \in V, \\ v > 0, & y \in V, \\ v = 0, & y \in \Gamma, \end{cases} \tag{5.24}$$

where $M \in C^1(\overline{V})^{N \times N}$ is uniformly elliptic, $b \in C(\overline{V})^N$ and $j(\cdot,\cdot) = g(y^{-1}(\cdot),\cdot)$, being M, b and j independent of u.

Proof. Since $\partial\Omega$ is of class $\mathscr{C}^2$, there exist $U \subset \mathbb{R}^N$ a neighborhood of x_0 and a C^2 function $\psi : U' \to \mathbb{R}$, where $U' = \{x' \in \mathbb{R}^{N-1} : \exists x_N \in \mathbb{R}, (x', x_N) \in U\}$, such that

$$\psi(x') < x_N \quad \forall (x', x_N) \in U \cap \Omega,$$
$$\psi(x') = x_N \quad \forall (x', x_N) \in U \cap \partial\Omega.$$

Let us define the change of variables $y : U \to \mathbb{R}^N$ by

$$y(x) = (x', x_N - \psi(x')) \quad \forall x \in U.$$

It is clear that

$$
\begin{aligned}
y(x) \in \mathbb{R}^N_+ \quad & \forall (x', x_N) \in U \cap \Omega, \\
y(x) \in \partial\mathbb{R}^N_+ \quad & \forall (x', x_N) \in U \cap \partial\Omega.
\end{aligned}
$$

This proves that $V \subset \mathbb{R}^N_+$ and $\Gamma \subset \partial\mathbb{R}^N_+$. Moreover, it is a simple exercise to prove that $\Gamma \subset \partial V$. It is also easy to see that the function $y^{-1} : y(U) \to U$ given by

$$
y^{-1}(z) = (z', z_N + \psi(z')) \quad \forall z \in y(U)
$$

is the inverse function of y. Note that y^{-1} is well defined since $z' \in U'$ for every $z \in y(U)$.

Let us now define $v : V \cup \Gamma \to \mathbb{R}$ by

$$
v(z) = u(y^{-1}(z)) \quad \forall z \in V \cup \Gamma.
$$

Observe that $v = 0$ on Γ and $u(x) = v(y(x))$ for all $x \in U \cap \Omega$. We will show next that v satisfies an equation in V.

Now we compute the derivatives that we need. We emphasize that such derivatives can be understood in a pointwise sense due to Remark 5.2.5 and to the C^2 regularity of ψ. We stress also that, as ψ does not depend on x_N, it will be understood that $\frac{\partial\psi}{\partial x_N}(x) = 0$ for $x \in U$.

a) $$
Dy(x) = \left(\begin{array}{ccccc|c}
1 & 0 & \cdots & & 0 & 0 \\
0 & 1 & & & \vdots & \vdots \\
\vdots & & & \ddots & 0 & \vdots \\
0 & \cdots & & 0 & 1 & 0 \\
\hline
-\frac{\partial\psi}{\partial x_1}(x') & \cdots & & \cdots & -\frac{\partial\psi}{\partial x_{N-1}}(x') & 1
\end{array}\right),
$$

b) $$
\begin{cases}
\dfrac{\partial u}{\partial x_i}(x) = \dfrac{\partial v}{\partial z_i}(y(x)) - \dfrac{\partial v}{\partial z_N}(y(x))\dfrac{\partial\psi}{\partial x_i}(x'), \quad i = 1, \cdots, N-1, \\[2ex]
\dfrac{\partial u}{\partial x_N}(x) = \dfrac{\partial v}{\partial z_N}(y(x)),
\end{cases}
$$

c) $|\nabla u(x)|^2 = |\nabla v(y(x))|^2 + \left(\frac{\partial v}{\partial z_N}(y(x))\right)^2 |\nabla\psi(x')|^2 - 2\frac{\partial v}{\partial z_N}(y(x))\nabla v(y(x))\nabla\psi(x')$, d)

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_i^2}(x) &= \frac{\partial^2 v}{\partial z_1 \partial z_i}(y(x))\frac{\partial y_1}{\partial x_i}(x) + \cdots + \frac{\partial^2 v}{\partial z_N \partial z_i}(y(x))\frac{\partial y_N}{\partial x_i}(x) \\
&- \frac{\partial v}{\partial z_N}(y(x))\frac{\partial^2 \psi}{\partial x_i^2}(x') \\
&- \left[\frac{\partial^2 v}{\partial z_1 \partial z_N}(y(x))\frac{\partial y_1}{\partial x_i}(x) + \cdots + \frac{\partial^2 v}{\partial z_N^2}(y(x))\frac{\partial y_N}{\partial x_i}(x)\right]\frac{\partial \psi}{\partial x_i}(x') \\
&= \frac{\partial^2 v}{\partial z_i^2}(y(x)) - 2\frac{\partial^2 v}{\partial z_i \partial z_N}(y(x))\frac{\partial \psi}{\partial x_i}(x') + \frac{\partial^2 v}{\partial z_N^2}(y(x))\left(\frac{\partial \psi}{\partial x_i}(x')\right)^2 \\
&- \frac{\partial v}{\partial z_N}(y(x))\frac{\partial^2 \psi}{\partial x_i^2}(x'), \quad i = 1, \cdots, N-1,
\end{aligned}$$

e) $\Delta u = \Delta v - 2\nabla \dfrac{\partial v}{\partial z_N}\nabla \psi + \dfrac{\partial^2 v}{\partial z_N^2}|\nabla \psi|^2 - \dfrac{\partial v}{\partial z_N}\Delta \psi.$

Let us denote $j(z,s) = g(y^{-1}(z),s)$ for a.e. $z \in V$ and for all $s > 0$. Thus, $v = v(y)$ (from this point y will simply denote variable in V) satisfies the equation

$$\begin{aligned}
&-\Delta v - \frac{\partial^2 v}{\partial y_N^2}|\nabla \psi|^2 + 2\nabla \frac{\partial v}{\partial y_N}\nabla \psi + \frac{\partial v}{\partial y_N}\Delta \psi \\
&\quad = f(v) - j(y,v)\left(|\nabla v|^2 + \left(\frac{\partial v}{\partial y_N}\right)^2 |\nabla \psi|^2 - 2\frac{\partial v}{\partial y_N}\nabla v \nabla \psi\right), \quad y \in V. \qquad (5.25)
\end{aligned}$$

Let us define the matrix

$$M(y) = \left(\begin{array}{cccc|c}
1 & 0 & \cdots & 0 & -2\frac{\partial \psi}{\partial x_1}(y') \\
0 & 1 & & \vdots & \vdots \\
\vdots & & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & -2\frac{\partial \psi}{\partial x_{N-1}}(y') \\
\hline
0 & \cdots & \cdots & 0 & 1 + |\nabla \psi(y')|^2
\end{array}\right), \quad y \in V,$$

and also the vector $b(y) = (0, \cdots, 0, -\Delta \psi(y'))$, $y \in V$. Then, one can check that v is a solution to (5.24).

Moreover, let $a > 1$ be such that $(a-1)\|\nabla \psi\|^2_{L^\infty(V)} < 1$. Then, by Cauchy-Schwarz's and Young's inequalities, the following holds for every $\xi \in \mathbb{R}^N$:

$$\begin{aligned}
M(y)\xi\xi &= |\xi|^2 - 2\xi_N \nabla\psi\xi' + \xi_N^2|\nabla\psi|^2 \\
&\geq |\xi|^2 - (a-1)\xi_N^2|\nabla\psi|^2 - \frac{1}{a}|\xi'|^2 \\
&= \left(1-\frac{1}{a}\right)|\xi'|^2 + (1-(a-1)|\nabla\psi|^2)\xi_N^2 \\
&\geq \left(1-\frac{1}{a}\right)|\xi'|^2 + (1-(a-1)\|\nabla\psi\|^2_{L^\infty(V)})\xi_N^2 \\
&\geq \min\left\{1-\frac{1}{a}, 1-(a-1)\|\nabla\psi\|^2_{L^\infty(V)}\right\}|\xi|^2.
\end{aligned}$$

Then, M is uniformly elliptic. The proof is finished. $\square$

5.6.2 Hölder estimates

Let $\omega \subset \mathbb{R}^N$ be a smooth bounded domain and let $\{v_n\} \subset H^1(\omega) \cap L^\infty(\omega)$ be such that $0 < v_n \leq C$ in ω for all n and for some $C > 0$. Assume that v_n satisfies

$$-\mathrm{div}(M_n(y)\nabla v_n) + b_n(y)\nabla v_n + \frac{1}{\varepsilon_n} g_n\left(y, \frac{v_n}{\varepsilon_n}\right) M_n(y)\nabla v_n \nabla v_n = f_n(v_n), \quad y \in \omega,$$

where, for all n, $\varepsilon_n > 0$; $M_n \in L^\infty(\omega)^{N\times N}$ is uniformly elliptic (with elliptic constant independent of n) and bounded uniformly in n; $b_n \in L^\infty(\omega)^N$ is bounded uniformly in n; $f_n(s) = \lambda_n s^p + t_n s^q$ for all $s \geq 0$ and for some $p, q > 0$ and some bounded sequences $\{\lambda_n\}, \{t_n\} \subset [0, C]$, and $g_n : \omega \times (0, +\infty) \to [0, +\infty)$ is a Carathédory function that satisfies (5.15) for some $\delta, \tau, \sigma \geq 0$ independent of n. Let us also assume that, for some connected (possibly empty) set $\Gamma \subset \partial\omega$,

$$v_n = 0, \quad y \in \Gamma.$$

This is precisely the situation in several proofs of Section 5.4.

Let us denote $\delta_n = \varepsilon_n \delta$. Simple computations show that, for any $\gamma > 0$, the function $u_n = (v_n + \delta_n)^\gamma - \delta_n^\gamma$ satisfies

$$-\mathrm{div}(M_n(y)\nabla u_n) + b_n(y)\nabla u_n = \tilde{g}_n(y, u_n) M_n(y)\nabla u_n \nabla u_n + \tilde{f}_n(u_n), \quad y \in \omega, \qquad (5.26)$$

where

$$\tilde{g}_n(y,s) = \frac{1 - \gamma - \frac{(s+\delta_n^\gamma)^{\frac{1}{\gamma}}}{\varepsilon_n} g_n\left(y, \frac{(s+\delta_n^\gamma)^{\frac{1}{\gamma}}}{\varepsilon_n} - \delta\right)}{\gamma(s+\delta_n^\gamma)} \quad \text{a.e. } y \in \omega,\ \forall s > 0,\ \forall n,$$

$$\tilde{f}_n(s) = \gamma(s+\delta_n^\gamma)^{\frac{\gamma-1}{\gamma}} f_n\left((s+\delta_n^\gamma)^{\frac{1}{\gamma}} - \delta_n\right) \quad \forall s > 0,\ \forall n.$$

It is clear that $\tilde{f}_n(u_n)$ is bounded uniformly in n. We will now choose $\gamma > 0$ in such a way that

$$0 \le s\tilde{g}_n(y,s) \le c \quad \text{a.e. } y \in \omega,\ \forall s > 0,\ \forall n, \tag{5.27}$$

for some $c \in (0,1)$. Indeed, it is clear that

$$\gamma(s+\delta_n^\gamma)\tilde{g}_n(y,s) \ge 1-\gamma-\sigma.$$

Thus, we choose $\gamma \le 1-\sigma$. On the other hand, if $\gamma > \frac{1-\tau}{2}$, then

$$s\tilde{g}_n(y,s) \le \frac{1-\gamma-\tau}{\gamma} < 1.$$

In sum, (5.27) is satisfied if $\gamma \in \left(\frac{1-\tau}{2}, 1-\sigma\right]$. Notice that this interval is nonempty thanks to the condition $\sigma-\tau < 1-\sigma$.

Next, using (5.27) we may apply the arguments in [38, Appendix] without relevant changes to prove that $\|u_n\|_{C^{0,\alpha}(\omega\cup\Gamma)} \le C$ for some $C > 0, \alpha \in (0,1)$. In particular, using that the function $s \mapsto s^{\frac{1}{\gamma}}$ is locally Lipschitz for $s \ge 0$, we have that

$$\begin{aligned}
|v_n(x)-v_n(y)| &= |v_n(x)+\delta_n - v_n(y) - \delta_n| \\
&\le C|(v_n(x)+\delta_n)^\gamma - (v_n(y)+\delta_n)^\gamma| \\
&= C|u_n(x)-u_n(y)| \le C|x-y|^\alpha \quad \forall x,y \in \omega\cup\Gamma.
\end{aligned}$$

In conclusion, $\|v_n\|_{C^{0,\alpha}(\omega\cup\Gamma)} \le C$, as we wanted to prove.

Chapter 6

Nonexistence result for a semilinear elliptic problem

S. López-Martínez, A. Molino, *Nonexistence result for a semilinear elliptic problem*, arXiv: https://arxiv.org/abs/1902.08800.

Abstract. In this paper we prove the nonexistence of nontrivial solution to

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

being $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) a bounded domain and f locally Lispchitz with non-positive primitive.

6.1 Introduction

Problems of partial differential equations are extensively studied at present, mainly motivated by their applications in fields of physics, biology and engineering among others. One of the simplest models of nonlinear elliptic differential equations is the following

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

being $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) a bounded domain with boundary of class $\mathscr{C}^{1,1}$ and the source term $f : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function.

Along this note, a *classical solution* to (P) (*solution* from now on) will be a function $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfying (P) pointwise. Observe that, by regularity results, every bounded weak solution is a solution to this problem (see e.g. [123]).

When studying any kind of problem involving differential equations, it is always useful to know necessary conditions for the existence of solution. For instance, it follows immediately that a necessary condition for the existence of a solution u to (P) is that u must satisfy the equality

$$\int_\Omega |\nabla u|^2 = \int_\Omega f(u)u. \tag{6.1}$$

In consequence, a straightforward nonexistence result for problem (P) states that if

$$f(s)s \leq 0, \quad \textit{for all } s \in \mathbb{R}, \tag{6.2}$$

there exists no nontrivial solution to (P). In addition, the well-known Pohozaev identity ([112]) yields a sort of generalization of this simple result. To be more precise, every solution u to (P) must satisfy the following equality:

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u(x)|^2 x\cdot \nu(x)dx + \frac{N-2}{2}\int_\Omega |\nabla u(x)|^2 dx = N\int_\Omega F(u(x))dx, \tag{6.3}$$

where $F(s) = \int_0^s f(t)dt$ for all $s \in \mathbb{R}$ and ν denotes the unit outward normal to $\partial\Omega$ vector. Observe that if Ω is starshaped with respect to 0 (i.e., $x\cdot\nu(x) > 0$ on $\partial\Omega$) and $N \geq 3$, the left hand side of (6.3) is non-negative. Therefore, if

$$F(s) \leq 0, \quad \textit{for all } s \in \mathbb{R}, \tag{6.4}$$

there exists no nontrivial solution to (P) whenever Ω is starshaped. Keep in mind that condition (6.4) implies that $f(0) = 0$. Thus, zero is always a solution.

Condition $sf(s) \leq 0$ clearly guarantees $F(s) \leq 0$, but not conversely. A simple example is $f(s) = \lambda \sin s$, being $\lambda < 0$. Where, to our knowledge, the existence of a nontrivial solution until now is unknown. Instead, existence of solutions for $\lambda > 0$ were established in [58]. In this way, a natural question is whether the condition Ω is starshaped is essential for the nonexistence of nontrivial solution to (P), for any bounded domain Ω and f satisfying (6.4).

A similar situation arises when one analyzes the well-known supercritical case result, also derived from (6.3). Concretely, if $f(s) = \lambda|s|^{p-2}s$, for $\lambda > 0$ and $p \geq 2^*$, there exists no nontrivial solution to (P) provided $N \geq 3$ and Ω is starshaped. However, it is surprising the existence of nontrivial solutions for $p \geq 2^*$ when the domain is not starshaped. For instance, positive solutions have been found when the domain is an annulus (see the seminal paper [92] and references therein) or for domains with small holes ([62]).

But nevertheless, much less is known about the influence of the geometry of Ω in the existence of solution to problem (P) in the case $F(s) \le 0$ and the literature contains only partial nonexistence results. Observe that for functions f globally Lipschitz, with $L-$Lipschitz constant, it follows that $|f(s)| \le L|s|$. Thus, applying Poincaré inequality in (6.1), we obtain

$$\lambda_1 \int_\Omega u^2 \le \int_\Omega |\nabla u|^2 = \int_\Omega f(u)u \le L \int_\Omega u^2.$$

Therefore, this simple computation gives the nonexistence of nontrivial solutions as long as $L < \lambda_1$, being λ_1 the first eingenvalue for the Laplacian operator in Ω with zero Dirichlet boundary conditions. In this line, in [66, 116] the authors prove the nonexistence provided that $L \le 3\lambda_1$ $(N \ge 2)$. Recently, in [84], the nonexistence of nontrivial solutions is shown if either $\partial\Omega$ has non-negative mean curvature or Ω is an annulus, also for functions f globally Lipschitz and $N \ge 2$. On the other hand, in [50] (see also [54]), a condition similar to $F(s) \le 0$ is imposed, and the authors prove the nonexistence of positive solutions which satisfy a certain extra property; no geometric condition on Ω is assumed.

In the present paper, inspired by the results in [50], we prove that there is no nontrivial solution to problem (P) provided $F(s) \le 0$, being f a locally Lispchitz function. Here, no additional hypotheses on Ω, N nor f are imposed. This exposes the unexpected fact that there is no geometric assumption on Ω that gives a nontrivial solution.

6.2 Main result

Theorem 6.2.1. *If $F(s) \le 0$ for all $s \in \mathbb{R}$, there exists no nontrivial solution to* (P)*.*

Proof. Clearly, zero is a solution. We argue by contradiction and assume that there exists a nontrivial solution u to (P). First of all, notice that $-u$ is a solution to

$$\begin{cases} -\Delta u = -f(-u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Since the function $-f(-s)$ is under the hypotheses of the theorem, there is no loss of generality in assuming that $u_\infty := \max_{x\in\overline{\Omega}} u(x) > 0$. On the other hand, since f is locally Lipschitz and the value of $f(s)$ for $s > u_\infty$ is irrelevant, we can also assume that f is globally Lipschitz, with Lipschitz constant $L > 0$, and that $\lim_{s\to+\infty} f(s) = -\infty$.

It is easy to check that $f(u_\infty) > 0$. Indeed, arguing by contradiction, assume that $f(u_\infty) \le 0$. Then,

$$-\Delta u_\infty + Lu_\infty \ge f(u_\infty) + Lu_\infty, \quad x \in \Omega. \tag{6.5}$$

Moreover, we have proved that

$$-\Delta u + Lu = f(u) + Lu, \quad x \in \Omega. \tag{6.6}$$

Subtracting (6.6) from (6.5), and using that $f(s) + Ls$ is non-decreasing, we obtain

$$-\Delta(u_\infty - u) + L(u_\infty - u) \ge f(u_\infty) + Lu_\infty - f(u) - Lu \ge 0, \quad x \in \Omega.$$

Since $u_\infty > u$ on $\partial\Omega$, the strong maximum principle implies that $u_\infty > u$ in Ω, which is a contradiction.

Thus, the fact that $f(u_\infty) > 0$ implies that there are $s_1, s_2 > 0$ such that $s_1 < u_\infty < s_2$ and

$$f(s) > 0 \quad \forall s \in (s_1, s_2). \tag{6.7}$$

Moreover, since $F(s) \le 0$ and $\lim_{s\to+\infty} f(s) = -\infty$, we can choose respectively s_1 and s_2 such that $f(s_1) = f(s_2) = 0$. Further, we can assume that $F(s_2) < 0$ since, otherwise (i.e., if $F(s_2) = 0$), we can modify f so that we obtain another L-Lipschitz function f^* such that $f(s) > f^*(s) > 0$ for $s \in (u_\infty, s_2)$ and $f = f^*$ elsewhere. In this way, u is still a solution to (*P*), but now $F(s_2) < 0$.

Now we will find a family of supersolutions to (*P*) which will lead to a contradiction by comparison with u. For this purpose, we follow the original reasoning in [50], which in principle is performed for $f \in C^1(\mathbb{R})$. Here we adapt the proof to our setting and check that it also works for Lipschitz functions f.

Indeed, consider the following initial value problem

$$\begin{cases} -w''(r) = f(w(r)), & \forall r > 0, \\ w(0) = s_2, \\ w'(0) = -\sqrt{-F(s_2)}. \end{cases}$$

Since f is Lipschitz there is a unique solution $w \in C^2([0,+\infty))$. Multiplying the equation by $w'(r)$ and integrating, we obtain

$$\begin{aligned} (w'(r))^2 &= -F(s_2) + 2\int_{w(r)}^{s_2} f(s)ds \\ &= F(s_2) - 2F(w(r)). \end{aligned} \tag{6.8}$$

Thus, using (6.7) we get that

$$(w'(r))^2 > 0 \text{ for } w(r) \in [s_1, s_2]. \tag{6.9}$$

Now, since $w(0) = s_2$ and $w'(0) < 0$, we deduce easily that $w(r) \in (s_1, s_2)$ for all $r > 0$ small enough. We claim now that there exists $r_0 > 0$ such that $w(r_0) = s_1$. Indeed, assume by contradiction that $w(r) > s_1$ for all $r > 0$. Then, by (6.9) we have that w is decreasing in $(0, +\infty)$. Hence, there exists $s_3 \in [s_1, s_2)$ such that $\lim_{r\to+\infty} w(s) = s_3$. But this is impossible as $w''(r) = -f(w(r)) < 0$ for all $r > 0$, i.e., w is concave.

In consequence, since $w(r_0) = s_1$ and $w'(r_0) < 0$, we deduce that $\inf_{r\geq 0} w(r) < s_1$. Moreover, it is easy to show that $\inf_{r\geq 0} w(r) > 0$. Indeed, assuming otherwise, there exists a sequence $\{r_n\} \subset [0, +\infty)$ such that $\lim_{n\to\infty} w(r_n) = 0$. Then, for n large enough, we deduce from (6.8) that $(w'(r_n))^2 < \frac{F(s_2)}{2} < 0$, a contradiction.

Thus, we have proved that

$$0 < \inf w < s_1. \tag{6.10}$$

Next, we define

$$W(r) = \begin{cases} s_2, & r \in (-\infty, 0], \\ \min\{w(r), s_2\}, & r \in (0, \infty). \end{cases}$$

Since we can assume that $f(s) < 0$ for $s > s_2$, it follows that w is convex if $w(r) > s_2$. This implies that, if $w(r_2) = s_2$ for some $r_2 > 0$, then $W(r) = s_2$ for all $r \geq r_2$. Otherwise, $w(r) < s_2$ for all $r > 0$, so $W(r) = w(r)$ for all $r > 0$.

For every $t \in \mathbb{R}$, consider the family of parametric functions $v_t(x) = W(x_1 - t)$ for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. We will prove now that $u(x) \leq v_t(x)$ for all $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}$ using the sweeping principle of Serrin. Indeed, let

$$U = \{t \in \mathbb{R} : u(x) \leq v_t(x) \text{ for all } x \in \overline{\Omega}\}.$$

Note that $v_t = s_2$ for t large enough, and $u < s_2$ in $\overline{\Omega}$, so U is nonempty. Notice also that W is a globally Lipschitz function, so the function $t \mapsto v_t(x)$ is continuous uniformly in x. In particular, U is closed.

Let us now take $t \in U$. Observe that $v_t \in W^{1,\infty}(\Omega)$ and $-\Delta v_t \geq f(v_t)$ in Ω (in the weak sense). Then, since $s \mapsto f(s) + Ls$ is non-decreasing and $u \leq v_t$ in $\overline{\Omega}$, we have that $-\Delta(v_t - u) + L(v_t - u) \geq 0$ in Ω. Notice that

$$u(x) = 0 < \inf w \leq v_t(x) \quad \forall x \in \partial\Omega,$$

so $v_t \not\equiv u$. Then, the strong maximum principle implies that $u(x) < v_t(x)$ for all $x \in \overline{\Omega}$. Therefore, the uniform continuity of $s \mapsto v_s$ implies that there exits $T > 0$, independent of x, such that $u(x) < v_s(x)$ for all $x \in \overline{\Omega}$ and for all $s \in (t-T, t+T)$. That is to say, $(t-T, t+T) \subset U$, so U is open. In conclusion, $U = \mathbb{R}$, and thus, $u \leq v_t$ for all $t \in \mathbb{R}$. In consequence,

$$u(x) \leq \inf_{t \in \mathbb{R}} v_t(x) = \inf_{r>0} w(r) < s_1, \quad \forall x \in \Omega,$$

which is a contradiction with the fact that $u_\infty \in (s_1, s_2)$. $\square$

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Chapter 7

Conclusions

In light of the results (section 1.3 in chapter 1) that have been obtained throughout the doctoral period of the author, some conclusions can be drawn. Many of them have been already exhibited in the previous chapters 2, 3, 4, 5 and 6. In the present chapter we aim to give a general overview relating the different results together.

Let us start with the most relevant contributions of the thesis, namely, the results concerning problem (1.12) that has been introduced in the preface (chapter 1). In the first place, in the superlinear range $\alpha \in [0, q-1)$ and under some further conditions, Theorem 1.3.1 asserts two remarkable facts about problem (1.12). Namely, that the existence of a solution for $\lambda = 0$ implies the existence of at least two solutions for $\lambda > 0$ small enough, and that $\lambda = 0$ is the unique bifurcation point from infinity. The first statement shows that $\lambda = 0$ acts as a critical value beyond which the structure of the problem changes drastically, as for $\lambda \leq 0$ one has uniqueness of solution (see Proposition 4.3.4 in chapter 4). On the other hand, if there exists no solution to (1.12) for $\lambda = 0$, then it is easy to deduce that $\lambda = 0$ is a bifurcation point from infinity (otherwise, one would find a solution to (1.12) for $\lambda = 0$ by approximation, letting $\lambda \to 0^-$). The strength of the second statement of Theorem 1.3.1 is that it shows that $\lambda = 0$ is *always* a bifurcation point from infinity, even if there exists a solution to (1.12) for $\lambda = 0$. This last fact has a natural explanation if one looks at $\lambda u + f$ as a unity in (1.12). Indeed, Proposition 4.3.6 in chapter 4 shows that, if u is a solution to (1.12) for some $\lambda > 0$ then, roughly speaking, $\lambda u + f$ cannot be too large in some sense. Thus, if there is a sequence of solutions u_n to problem (1.12) with $\lambda = \lambda_n > 0$ such that some norm of u_n diverges, then it is reasonable that λ_n tends to zero (as Theorem 1.3.1 asserts) in such a way that $\lambda_n u_n + f$ does not become too large. In any case, it seems apparent that $\lambda = 0$ is a fundamental value which defines the structure of the set of solutions in the superlinear range.

In view of the previous discussion, the restriction (1.22) in Theorem 1.3.1 seems not to be natural since it does not affect the structural properties of problem (1.12) for $\lambda \leq 0$, it is only used due to the fact that $\lambda > 0$. In other words, we expect a similar multiplicity result to hold true for every $\alpha \in [0, q-1)$ without any further restriction on α, q. Nevertheless, the information that the theorem provides is remarkable as it extends some known results concerned with the nonsingular quadratic case $\alpha = 0$ and $q = 2$. To be precise, Corollary 1.3.2 states in particular that, if $\alpha = 0$ and $q > 1$ is close enough to 1, and if there exists a solution to (1.12), then multiplicity occurs for $\lambda > 0$ small. Such a result was already known for $q = 2$. We suspect that a similar result must be true for any $q \in (1,2)$, though the problem remains open.

On the other hand, Corollary 1.3.3 shows that multiplicity for $\lambda > 0$ small does happen for any $q \in (1,2)$ provided $\alpha > 0$ is chosen close enough to $q-1$. At this point, we observe that one cannot reach $\alpha = q-1$ since it corresponds to the linear homogeneity range, for which Theorem 1.3.6 assures that uniqueness holds for $\lambda > 0$ small enough. Moreover, if $\alpha > q-1$ we arrive to the sublinear range, where we also have uniqueness for $\lambda \geq 0$ small enough by virtue of Theorem 1.3.4. Thus, as the title of chapter 4 reads, one may provide the exponent $\alpha = q-1$ with the structural meaning of being a break point between uniqueness and multiplicity phenomena.

Even though both Theorems 1.3.6 and 1.3.4 yield uniqueness of solution for $\lambda > 0$ small, they present a subtle but fundamental difference. Namely, the critical value for the existence of solution, which coincides in turn with the unique bifurcation point, is λ_1 in the sublinear case, while it is λ^* in the linear one. It is clear that λ_1 depends only on Ω, while λ^* depends on μ, q as well. In fact, if $\mu \equiv$ constant $\in (0,1)$ and $q = 2$, then the dependence on μ becomes explicit since $\lambda^* = \frac{\lambda_1}{\mu+1}$. This shows that, if $\alpha > q-1$, the lower order term in the equation signifies only a mild perturbation of the principal Laplacian operator. Actually, Theorem 1.3.4 shows that the nature of problem (1.12) in the sublinear range is very close to the well-known linear problem corresponding to $\mu \equiv 0$. On the contrary, if $\alpha = q-1$, Theorem 1.3.6 also reminds of the classical one for the linear equation, though in this case the lower order term plays a significant role. Indeed, the principal term and the lower order term have the same homogeneities so that, roughly speaking, the contribution of both terms is comparable. In chapter 2 we have seen that a suitable way of handling the problem is looking at both terms as a single 1-homogeneous differential operator $u \mapsto -\Delta u + \mu(x)|\nabla u|^q/|u|^{q-1}$ and to prove that it admits a conveniently defined principal eigenvalue. This last fact is contained in Theorem 1.3.5.

The linear-like nature of problem (1.12) for $\alpha = q-1$ has been brought to light also by the homogenization Theorem 1.3.8. Indeed, the strange term that appears in the limit equation coincides with that for the linear problem, as it is shown in [49]. It is also worth to mention that the techniques we have developed allowed us to deal with the singularity in spite of the fact that the functions into consideration vanish in the "holes", which have nonzero measure. Similar difficulties appear in the proof of the existence Theorem 1.3.7, where the fact that f may change sign implies that the solutions do not satisfy a positive local lower bound from below.

Summarizing, regarding problem (1.12), we have found several structural properties which show that the problem is very rich as presents many different phenomena. We consider that, after the work in this thesis, many previous results about singular problems with natural growth in the gradient can be understood in a better way as part of a unified and organized framework. It is also remarkable that all the proofs are compatible with the presence of a gradient term. Therefore, we underline the new methods that have been developed for proving a priori estimates, comparison principles, etc. We hope that this unified perspective, as well as the techniques employed, are useful for the future work in this topic.

Without going any further, many of the ideas developed for studying problem (1.12) have been essential for dealing with problem (1.24). To this respect, the achievements are twofold. On the one hand, in the proofs of Theorems 1.3.9 and 1.3.10 we have developed a blow-up method (adapting [82]) which provides a priori estimates despite the gradient term and the superlinear zero order term. On the other hand, and as a consequence of the blow-up method, we have been able to solve a number of open problems for general functions $g(x,s)$ either singular or non-singular as $s \to 0$. For this second aspect, also the nonexistence Theorem 1.3.11 represents an interesting contribution. An outstanding implication of the mentioned three theorems is that, in particular, problem (1.16) is now much better understood. However, many problems still remain unsolved and others have emerged after the work carried out so far (see section 5.5 of chapter 5). At the present, we consider that the topic is far from being fully understood. This is why it constitutes one of the main interests in the current research of the author.

Something similar can be said about problem (1.19). To be precise, Theorem 1.3.12 represents an important contribution by itself, but it means also a motivation for facing more general problems. Indeed, there exist suitable versions of Pohozaev's identity for the quasilinear equation $-\Delta_p u = f(u)$ with $p > 1$ (see [86]), as well as for the nonlocal one $(-\Delta)^s u = f(u)$ with $s \in (0,1)$ (see [117]) ; here, Δ_p stands for the $p-$Laplace operator, while $(-\Delta)^s$ denotes the fractional Laplace operator. Therefore, nonexistence results for data f with nonpositive primitive and for starshaped bounded domains Ω are straightforwardly deduced. The question is whether such a nonexistence result can be generalized or not to any bounded domain Ω without geometric assumptions. These open problems are planned to be analyzed by the author in the near future.

Resumen

Esta tesis doctoral aporta contribuciones al campo de las Ecuaciones en Derivadas Parciales No Lineales. Concretamente, se centra en estudiar la existencia de solución de problemas de contorno elípticos no lineales de tipo Dirichlet, así como en determinar propiedades cualitativas de las soluciones como su unicidad, multiplicidad, regularidad, etc. Un modelo representativo de las ecuaciones que se tratan en la tesis es el siguiente:

$$\begin{cases} -\Delta u = g(x,u)|\nabla u|^q + f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7.1}$$

donde Ω es un dominio acotado de $\mathbb{R}^N$ ($N \geq 3$) con frontera $\partial\Omega$ de clase $\mathscr{C}^2$, $0 < q \leq 2$, y $f : \Omega \times \mathbb{R} \to \mathbb{R}$ y $g : \Omega \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ son funciones de Carathéodory, es decir, son medibles en la primera variable fijada la segunda, y continuas en la segunda variable fijada la primera. Un marco apropiado para estudiar (7.1) es el de las *soluciones débiles*. Así, una solución de (7.1) será una función $u \in H_0^1(\Omega)$ tal que $f(\cdot,u) \in L^1_{\text{loc}}(\Omega)$, $g(\cdot,u)|\nabla u|^q \in L^1_{\text{loc}}(\{u \neq 0\})$ y

$$\int_\Omega \nabla u \nabla \phi = \int_{\{u \neq 0\}} g(x,u)|\nabla u|^q \phi + \int_\Omega f(x,u)\phi \quad \forall \phi \in C_c^1(\Omega).$$

Es habitual en la literatura establecer hipótesis sobre f, g para que toda solución no trivial de (7.1) sea positiva. De este modo, $\{u \neq 0\} = \Omega$ y se recupera la formulación débil usual. De hecho, en esta memoria consideraremos soluciones positivas en la mayor parte de los casos, pero habrá excepciones que destacaremos más abajo.

La característica fundamental de la ecuación en (7.1) es la presencia del término de orden uno $g(x,u)|\nabla u|^q$, el cual la convierte en una ecuación *casilineal*. Además, el hecho de ser $q \leq 2$ conlleva que el término de orden inferior presenta un *crecimiento natural* en el gradiente. En contraposición con el caso semilineal $g \equiv 0$, algunas dificultades fundamentales que acarrea la presencia de un término no lineal de primer orden son la ausencia de una teoría general de regularidad de soluciones y, sobre todo, la falta de una estructura variacional (haremos hincapié en esta dimensión no variacional más adelante). Por tanto, para abordar estos problemas son más apropiados los *métodos topológicos*: teoremas de punto fijo, sub y supersoluciones, aproximación y compacidad, etc. Las dificultades mencionadas son ejemplos que ponen de manifiesto el interés matemático que encierran los problemas con crecimiento natural en el gradiente. Su estudio se inició con los trabajos pioneros [4, 27–32, 91] y representa un área de investigación activa y de impacto en la actualidad, como muestra la lista (no exhaustiva) de referencias de esta tesis.

Los problemas que consideraremos en esta memoria presentan una dificultad adicional. A saber, la función g puede ser *singular* cuando $s \to 0$, es decir, admitimos la posibilidad de que $\lim_{s\to 0} g(x,s)$ no exista para ningún $x \in \Omega$, o de que no sea finito. Obsérvese que, si $u \in C(\overline{\Omega})$ es solución de (7.1) y $g(\cdot,u) \in C(\Omega)$, entonces $g(x,u(x))$ podría divergir cuando $x \to \partial\Omega$, ya que $u(x) = 0$ si $x \in \partial\Omega$. Esto dificulta considerablemente la aplicación de los métodos topológicos clásicos, especialmente cuando no es posible controlar la singularidad ni siquiera en el interior de Ω, es decir, cuando no hay cotas a priori por abajo que permitan asegurar que toda solución es positiva en Ω. El estudio de problemas elípticos con singularidades en el término de gradiente tuvo su origen hace algo más de una década con los trabajos [6, 8, 14, 24, 76]. Actualmente, este área sigue gozando de interés ya que muchas cuestiones siguen sin resolver.

Introducimos en este punto una tercera dificultad que supone una de las motivaciones principales de la investigación contenida en esta tesis. Para ello, describiremos de manera formal un procedimiento usual en el estudio de ecuaciones con crecimiento natural que se originó a partir de [91]. En efecto, supongamos que $q = 2$ y que g no depende de x, es decir, $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ es una función continua. Supongamos además que

$$\left| \int_0^s e^{\int_1^t g(r)dr} dt \right| < +\infty \quad \forall s \in (-1,1) \setminus \{0\}.$$

La anterior condición nos permite definir la función $\Psi : \mathbb{R} \to \mathbb{R}$ como

$$\Psi(s) = \int_0^s e^{\int_1^t g(r)dr} dt \quad \forall s \in \mathbb{R}.$$

Como $\Psi' > 0$, podemos considerar también la función inversa $\Psi^{-1} : \mathrm{Im}(\Psi) \to \mathbb{R}$. De esta manera, es fácil comprobar que, si u es solución de (7.1), entonces $v = \Psi(u)$ satisface

$$\begin{cases} -\Delta v = \Psi'(\Psi^{-1}(v)) f(x, \Psi^{-1}(v)), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{7.2}$$

Recíprocamente, el cambio de variable inverso permite pasar de (7.2) a (7.1).

Con este procedimiento, se reduce el estudio del problema casilineal (7.1) al del semilineal (7.2). La ventaja es evidente ya que, en general, se puede recurrir al Cálculo de Variaciones o a otras técnicas propias de la teoría semilineal para estudiar (7.1) (a través de (7.2)). Esta útil estrategia se ha empleado en numerosos trabajos, como [1, 15, 16, 59, 90, 111]. Sin embargo, su utilidad depende de que el término de primer orden en (7.1) tenga la forma concreta de $g(u)|\nabla u|^2$. Así pues, en el caso más general $g(x,u)|\nabla u|^q$ con $q \leq 2$, que será el que concierna a los problemas de esta memoria, el término de gradiente permanece en la ecuación tras efectuar cualquier cambio de variable de la forma $v = \Psi(u)$. Esta es precisamente la tercera dificultad que anunciamos.

En definitiva, en esta tesis estaremos interesados en estudiar problemas singulares con crecimiento natural en el gradiente que no son reducibles a problemas semilineales vía cambios de variable. Este marco obliga a desarrollar métodos que sean compatibles con la presencia de un término de gradiente. A pesar del marco general, el objetivo es demostrar resultados completos sobre la estructura del conjunto de soluciones de (7.1). Esto es, entre otras cuestiones, trataremos de demostrar resultados óptimos de existencia de solución, pondremos el foco en la unicidad o multiplicidad de solución y analizaremos la dependencia continua y el comportamiento asintótico de las soluciones con respecto a parámetros de la ecuación.

A continuación pasamos a desarrollar con más detalle los problemas concretos que se abordan en esta tesis.

Problemas con un término lineal de orden cero

En primer lugar, consideremos el siguiente caso particular de (7.1):

$$\begin{cases} -\Delta u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{u^\alpha} + f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7.3}$$

donde $\lambda \in \mathbb{R}$, $\alpha \geq 0$, $1 < q \leq 2$, $0 \lneq \mu \in L^\infty(\Omega)$, $0 \lneq f \in L^p(\Omega)$ para algún $p > \frac{N}{2}$. Los primeros antecedentes del problema (7.3) se remontan a los trabajos [27–32], que conciernen al caso no singular $\alpha = 0$. Entre los resultados que se demuestran, destacamos la existencia de solución para todo $\lambda < 0$. El caso $\lambda = 0$ es especial porque hay condiciones sobre f, μ, q que aseguran la existencia (ver [68, 69]) o la no existencia (ver [1, 3, 87]) de solución. En relación con esta segunda posibilidad, se ha probado en [113], asumiendo $f \in L^\infty(\Omega)$, que la familia de soluciones de (7.3) para $\lambda < 0$ diverge localmente uniformemente cuando λ tiende a cero por la izquierda. En particular, $\lambda = 0$ es punto de bifurcación desde infinito. Además, el autor estableció una condición necesaria y suficiente para existencia de solución de (7.3) para $\lambda = 0$ que está relacionada con el problema ergódico asociado. Por otra parte, también se ha probado la unicidad de solución para $\lambda \leq 0$ en [11, 17–19]. En definitiva, el caso no singular $\alpha = 0$ ha sido ampliamente estudiado para $\lambda \leq 0$.

Sin embargo, para $\lambda > 0$ y $\alpha = 0$ se conocen resultados solamente para el caso cuadrático $q = 2$. El primer trabajo en esa dirección es [90], donde se considera μ constante. Los autores prueban que, si $q = 2$, $\alpha = 0$, μ es constante y μf es suficientemente pequeño, entonces existen al menos dos soluciones de (7.3) para todo $\lambda > 0$ suficientemente pequño. Más tarde, este resultado se mejoró en [12] permitiendo μ no constante tal que $\mu_1 \leq \mu(x) \leq \mu_2$ para $x \in \Omega$ y para ciertas constantes $\mu_2 \geq \mu_1 > 0$. También en este trabajo se eliminó la hipótesis de tamaño de μf a expensas de asumir la existencia de solución de (7.3) para $\lambda = 0$. De hecho, los autores probaron que, desde esta solución correspondiente a $\lambda = 0$, emana un continuo de soluciones para $\lambda > 0$ que bifurca desde infinito a la derecha de $\lambda = 0$; como consecuencia se obtiene el mencionado resultado de multiplicidad. Destacamos también el trabajo [120], que se basa también en esta técnica topológica de bifurcación. El autor prueba un resultado análogo al anterior permitiendo que μ se anule en subconjuntos de $\overline{\Omega}$, aunque su estrategia es válida solo para dimensión N pequeña. Otros resultados más recientes relacionados son [55–57, 89].

Como se ha indicado, los anteriores resultados usan fuertemente que $q = 2$. En efecto, las pruebas involucran en mayor o menor medida el cambio de variable descrito arriba que permite “eliminar” el término de gradiente de la ecuación. En realidad, dicho cambio de variable no genera una ecuación semilineal si μ depende de x, pero sí que aporta información sobre las subsoluciones o las supersoluciones de ciertos problemas semilineales, lo cual se aprovecha ingeniosamente en [12, 120]. Sin embargo, con $q < 2$, las pruebas conocidas no funcionan. En definitiva, la existencia y multiplicidad de solución del problema no singular ($\alpha = 0$) y subcuadrático ($q < 2$) para $\lambda > 0$ son problemas abiertos que requieren del desarrollo de nuevos métodos compatibles con el término de gradiente.

Volviendo al problema (7.3), nos centraremos ahora en el caso singular $\alpha > 0$. En el paper [76] se estudia, bajo ciertas condiciones de los datos, la existencia de solución para $\lambda > 0$, mientras que en [2, 7, 77, 78] se analiza el caso $\lambda = 0$. La literatura sobre la unicidad de solución en problemas singulares es más limitada, citamos [10, 15, 16, 37]. Los resultados de unicidad contenidos en estos artículos requieren, entre otras hipótesis, que $\mu \leq 0$, así que no son aplicables en nuestro marco. Tampoco son abundantes las referencias sobre el problema singular con $\lambda > 0$. Los únicos resultados que conocemos se publicaron recientemente en [15], aunque conciernen únicamente al caso particular $q = 2$ y $\alpha = 1$. En este contexto, los autores prueban que existe al menos una solución de (7.3) para todo $\lambda < \frac{\lambda_1}{\|\mu\|_{L^\infty(\Omega)}+1}$, donde $\lambda_1 > 0$ denota al autovalor principal del operador $-\Delta$ en Ω con condición de borde $u = 0$. Además, si $\mu \equiv$ constante $\in (0,1)$, también prueban que la solución es única, que la condición $\lambda < \frac{\lambda_1}{\mu+1}$ es necesaria para la existencia de solución y que $\frac{\lambda_1}{\mu+1}$ es punto de bifurcación desde infinito. De esta manera, los autores muestran que la presencia de un término singular $|\nabla u|^2/u$ produce un importante cambio estructural en el conjunto de soluciones del problema (7.3) en comparación con el caso no singular $|\nabla u|^2$. En efecto, el resultado óptimo sobre el problema singular con $\mu \equiv$ constante $\in (0,1)$ recuerda al conocido caso lineal $\mu \equiv 0$. Sin embargo, la prueba recurre una vez más al ya comentado cambio de variable. La pregunta natural de si un resultado óptimo similar es también válido para μ no constante se quedó abierta en [15]. En cualquier caso, vale la pena observar que, independientemente de si μ es o no constante, el término singular presenta una homogeneidad lineal, en el sentido de que $|\nabla(tu)|^2/(tu) = t|\nabla u|^2/u$ para todo $t > 0$, mientras que el no singular tiene homogeneidad superlineal, esto es, $|\nabla(tu)|^2 = t^2|\nabla u|^2$ para todo $t > 0$. Esto hace pensar que la homogeneidad del término de orden inferior podría determinar la estructura del conjunto de soluciones de (7.3).

Así pues, el primer objetivo principal de esta tesis consiste en el estudio del problema general (7.3), especialmente para $\lambda > 0$, poniendo el foco en las distintas homogeneidades que se pueden dar en función de las elecciones de q, α. Así pues, consideraremos el rango de homogeneidad superlineal $q - \alpha > 1$, el de homogeneidad sublineal $q - \alpha < 1$, y el caso crítico de homogeneidad lineal $q - \alpha = 1$.

El siguiente es uno de los resultados más destacados de la tesis. El mismo muestra que, en efecto, en el rango superlineal $q - \alpha > 1$ se produce un fenómeno de multiplicidad de solución para $\lambda > 0$ pequeño, al igual que sucedía en el caso conocido $q = 2, \alpha = 0$. Las condiciones que se imponen sobre los datos se recogen a continuación:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ es un dominio acotado de clase } \mathscr{C}^2, \\ \mu \in L^\infty(\Omega) \text{ satisface que } \mu \geq \mu_0 \text{ en } \Omega \text{ para alguna constante } \mu_0 > 0, \\ 0 \lneq f \in L^p(\Omega) \text{ para algún } p > \frac{N}{2}, \\ q \in (1,2), \\ \alpha \in [0, q-1). \end{cases} \tag{H1}$$

El enunciado del mencionado resultado es el siguiente:

Teorema 1. *Supongamos que* (H1) *se satisface y que* (7.3) *tiene solución para* $\lambda = 0$. *Si* $q > \frac{N}{N-1}$, *supongamos también que*

$$\frac{q-1-\alpha}{q-2\alpha} \leq \frac{q-\alpha}{N-q+1}. \tag{7.4}$$

Entonces, existe $\bar{\lambda} \in (0,\lambda_1)$ *tal que el problema* (7.3) *admite al menos dos soluciones para todo* $\lambda \in (0,\bar{\lambda}]$. *Además, cero es el único punto de bifurcación desde infinito del problema* (7.3).

La prueba se puede consultar en el capítulo 4 de esta tesis (ver también [103]). Ésta se basa en el mismo argumento de bifurcación introducido en [12], aunque la prueba de la existencia de cotas a priori, que es la pieza fundamental de dicho argumento de bifurcación, se realiza de forma diferente. A grandes rasgos, las estimas se derivan de algunas ideas de [120] combinadas con un argumento iterativo inspirado en [85]. Como resultado de esta combinación aparece en la prueba la restricción (7.4). Para $\alpha = 0$, esa condición equivale a $q \leq Q_N$, donde

$$Q_N = \begin{cases} 2 & \forall N \leq 4, \\ \dfrac{N+2-\sqrt{N^2-4N-4}}{4} & \forall N \geq 5. \end{cases}$$

Esto nos hace pensar que el Teorema 1 no es óptimo, ya que se esperaría un resultado de multiplicidad para cualquier $q \in (1,2)$. Así, la restricción (7.4) se debe probablemente a las técnicas empleadas. No obstante, el Teorema 1 supone un primer paso significativo en el estudio del caso no singular $\alpha = 0$. Más aún, el Teorema 1 también aporta información relevante sobre el caso singular. En efecto, si $q \leq Q_N$ entonces la condición (7.4) se verifica en realidad para todo $\alpha \in [0,q-1)$. Además, para cualquier $q \in (1,2)$, la condición (7.4) también se cumple si $\alpha \in [c_{N,q}, q-1)$ para cierta constante $c_{N,q} \in [0,q-1)$.

Los dos resultados que presentaremos a continuación mostrarán que la condición $\alpha < q-1$ en el Teorema 1 es necesaria. De hecho, veremos que si $\alpha \geq q-1$ se produce un fenómeno de unicidad de solución para $\lambda > 0$ pequeño, de manera que la naturaleza del problema cambia radicalmente.

Empezaremos considerando el rango de homogeneidad sublineal $\alpha > q-1$. Las condiciones precisas que se impondrán son las siguientes:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ es un dominio acotado con frontera Lipschitz,} \\ 0 \lneq \mu \in L^\infty(\Omega), \\ 0 \lneq f \in L^p(\Omega) \text{ para algún } p > \frac{N}{2}, \\ q \in (1,2), \\ q-1 < \alpha \leq 1. \end{cases} \tag{H2}$$

El teorema relativo al rango de homogeneidad sublineal afirma lo siguiente:

Teorema 2. *Supongamos que* (H2) *se verifica. Entonces, existe al menos una solución de* (7.3) *para todo $\lambda < \lambda_1$, mientras que no existe solución para ningún $\lambda \geq \lambda_1$. Además, la solución es única para todo $\lambda \leq 0$. Finalmente, si f satisface que*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad f(x) \geq c_\omega \quad \textit{para casi todo } x \in \omega,$$

entonces la solución es única para todo $\lambda < \lambda_1$ y λ_1 es el único punto de bifurcación desde infinito del problema (7.3).

La prueba del teorema se encuentra en el capítulo 4 (ver también [103]). En concreto, la demostración del enunciado sobre existencia de solución se basa en la aproximación mediante problemas no singulares que verifican las cotas a priori necesarias para pasar al límite, siendo dicho límite solución del problema original (7.3). Es especialmente interesante la parte del teorema sobre unicidad de solución, ya que se trata de uno de los primeros y más completos resultados de este tipo referentes a ecuaciones singulares con crecimiento natural en el gradiente. La unicidad se basa en un principio de comparación que también se encuentra en el capítulo 4.

Falta abordar el caso crítico $\alpha = q-1$ correspondiente a la homogeneidad lineal del término no lineal. Para ello, establecemos las siguientes hipótesis:

$$\begin{cases} \Omega \subset \mathbb{R}^N \text{ es un dominio acotado de clase } \mathscr{C}^{1,1}, \\ q \in (1,2], \\ \alpha = q-1, \\ 0 \lneq \mu \in L^\infty(\Omega) \text{ con } \|\mu\|_{L^\infty(\Omega)} < 1 \text{ si } q = 2, \\ f \in L^p(\Omega) \text{ para algún } p > \frac{N}{2}. \end{cases} \tag{H3}$$

Este nuevo caso es especial porque la homogeneidad lineal permite ver el problem (7.3) como un problema no lineal de autovalores. Así, para $f \equiv 0$, se espera la existencia de un autovalor principal. El candidato a dicho autovalor se define como

$$\lambda^* = \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \exists v \in H^1(\Omega) \cap L^\infty(\Omega),\ \exists c > 0 : \\ v(x) \geq c,\ -\Delta v \geq \lambda v + \mu(x) \dfrac{|\nabla v|^q}{v^{q-1}}, \text{ para casi todo } x \in \Omega. \end{array} \right\}. \tag{7.5}$$

En efecto, el resultado principal en el caso de homogeneidad lineal con $f \equiv 0$ es el siguiente:

Teorema 3. *Supongamos que* (H3) *se satisface para* $f \equiv 0$. *Entonces,* $\lambda^* \in (0, \lambda_1]$ *y el problema* (7.3) *admite al menos una solución si, y solo si,* $\lambda = \lambda^*$. *Además, la solución es única salvo multiplicación por constantes positivas.*

La prueba del Teorema 3 se encuentra en el capítulo 2 (ver también [38]). Dicho resultado sobre el autovalor principal λ^* es la herramienta clave para probar el teorema principal para $f \gneqq 0$. En realidad, algunas partes de dicho teorema requerirán de condiciones más fuertes que $f \gneqq 0$. A saber, denotando por $\varphi_1 > 0$ la autofunción principal asociada a λ_1 tal que $\|\varphi_1\|_{L^\infty(\Omega)} = 1$, consideramos las siguientes condiciones:

$$\exists \gamma \in \left(\frac{1}{2}, 1\right),\ \exists c > 0: \quad f(x) \geq c\varphi_1(x)^\gamma \quad \text{para casi todo } x \in \Omega; \tag{f_1}$$

$$\exists c > 0: \quad f(x) \geq c\varphi_1(x)^\gamma \text{ para casi todo } \Omega, \text{ donde } \gamma = \frac{1}{1 + \|\mu\|_{L^\infty(\Omega)}}. \tag{f_2}$$

Enunciamos seguidamente el resultado principal para $f \gneqq 0$ y $\alpha = q - 1$. Su demostración se puede consultar también en el capítulo 2 (ver [38]).

Teorema 4. *Supongamos que* (H3) *se satisface para* $f \gneqq 0$. *Entonces,* (7.3) *tiene una única solución si* $\lambda \leq 0$, *tiene al menos una solución si* $\lambda < \lambda^*$, *y no tiene solución si* $\lambda > \lambda^*$. *Si, además,* f *satisface que*

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0: \quad f(x) \geq c_\omega \quad \textit{para casi todo } x \in \omega,$$

entonces (7.3) *tiene una única solución para todo* $\lambda < \lambda^*$. *Finalmente, si* f *satisface* (f_1) *para* $1 < q < 2$ *y* (f_2) *para* $q = 2$, *entonces* (7.3) *no tiene solución para ningún* $\lambda \geq \lambda^*$ *y, además,* λ^* *es el único punto de bifurcación desde infinito del problema* (7.3).

Obsérvese que el caso $q = 2$ está incluido en el teorema, de forma que se mejoran algunos resultados de [15] que eran válidos solo para μ constante. Destacamos también que este caso $\alpha = q - 1$ es similar al caso $\alpha > q - 1$ en el sentido de que ambos implican unicidad de solución para $\lambda > 0$ pequeño, aunque existe una importante diferencia. A saber, λ^* depende de q, μ; de hecho, si $q = 2$ y $\mu \equiv$ constante $\in (0, 1)$, entonces la dependencia en μ es explícita ya que $\lambda^* = \frac{\lambda_1}{\mu + 1}$. Por el contrario, λ_1 obviamente no depende de q, μ. Esto pone de manifiesto que la influencia del término no lineal en la estructura del conjunto de soluciones en el caso $\alpha = q - 1$ es relevante, mientras que en el caso $\alpha > q-1$ dicho término representa más bien una perturbación débil del término principal Δu. En cuanto a la demostración del teorema, sigue en líneas generales los argumentos comentados para el Teorema 2, con la salvedad de que λ^* juega en este caso un papel fundamental en la existencia de cotas a priori de los problemas aproximantes no singulares.

En los Teoremas 1, 2 y 4, la hipótesis $f \gneq 0$ es fundamental para probar que las soluciones de los problemas aproximantes u_n verifican que,

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0: \quad u_n \geq c_\omega \quad \forall n. \tag{7.6}$$

La ventaja de la anterior estima es que, localmente, la sucesión se queda alejada de cero. De esta forma, se evita la singularidad del término de orden inferior y es posible pasar al límite. Por el contrario, hay ejemplos de funciones f que cambian de signo para las cuales el problema (7.3) no puede admitir sucesiones aproximantes cumpliendo (7.6). Esto es, para ciertas f que cambian de signo, existen soluciones de ecuaciones singulares (que son casos particulares de la ecuación en (7.3)) que no verifican que $u > 0$ en Ω, sino que se anulan en subconjuntos $\overline{\Omega}$ de medida no nula (ver [77]). Así, debe emplearse una estrategia diferente que permita encontrar soluciones no necesariamente positivas. Naturalmente, este escenario plantea la dificultad añadida de trabajar con un concepto de solución apropiado que dote de sentido al término singular en el conjunto donde la solución vale cero.

El primer trabajo en el que se estudiaron ecuaciones singulares con crecimiento natural en el gradiente y con datos cambiando de signo es [77] (ver también [36, 78]). En dicho artículo, la clave de las demostraciones reside en un concepto apropiado de solución y en una estima global del término de orden inferior que en cierto modo suple la carencia de una estima local del tipo (7.6). En esta tesis seguimos de cerca esas ideas y las combinamos con otras herramientas que hemos desarrollado y ya hemos comentado, como la caracterización del autovalor principal λ^* (dado por (7.5)) y un principio de comparación. Como resultado demostramos el siguiente teorema, que generaliza la afirmación sobre existencia de solución en el Teorema 4 permitiendo que f cambie de signo. La demostración se encuentra en el capítulo 3 (ver [39]).

Teorema 5. *Supongamos que* (H3) *se satisface con* $1 < q < 2$, *y sea* $\lambda < \lambda^*$, *donde* λ^* *viene dado por* (7.5). *Entonces, existe al menos una solución* u *(convenientemente definida en el capítulo 3) de*

$$-\Delta u = \lambda u + \mu(x)\frac{|\nabla u|^q}{|u|^{q-1}} + f(x) \quad en\ \Omega$$

tal que $u = 0$ *en* $\partial\Omega$.

Aclaramos que en el capítulo 3 demostramos que el anterior resultado es válido para una clase más amplia de problemas singulares.

Otros problemas en los que la estima (7.6) no se verifica son los llamados problemas de *homogeneización*. Se trata de estudiar el comportamiento asintótico de sucesiones $\{u_n\}$ de soluciones de (7.3) en dominios acotados $\Omega_n \subset \Omega$, siendo Ω un dominio acotado fijo. Se ha demostrado en [49], en el caso lineal $\mu \equiv 0$ y no coercivo $\lambda = 0$, y bajo ciertas condiciones concretas sobre Ω_n, que la sucesión $\{\widetilde{u_n}\}$, definida por

$$\widetilde{u_n}(x) = \begin{cases} u_n(x), & x \in \Omega_n, \\ 0, & x \in \Omega \setminus \Omega_n, \end{cases}$$

converge a una solución de u de $-\Delta u = \sigma u + f(x)$ en Ω tal que $u = 0$ en $\partial\Omega$, donde $\sigma \in H^{-1}(\Omega)$ depende de $\{\Omega_n\}$. Es sorprendente la aparición del llamado "término extraño" σ, que no formaba parte de la ecuación original.

Obviamente, la sucesión $\{\widetilde{u_n}\}$ no verifica la estima (7.6) ya que $\widetilde{u_n} \equiv 0$ en $\Omega \setminus \Omega_n$. Por tanto, el problema de homogeneización asociado a (7.3) con $\mu \gneq 0$ se dificulta debido a que no hay forma de evitar la singularidad localmente. No obstante, aprovechando las ideas comentadas anteriormente relativas al Teorema 5, es posible demostrar un resultado similar al de [49] en nuestro caso singular. Mostramos aquí una versión simplificada del mismo en favor de una exposición más clara. El enunciado general y la demostración se pueden consultar en el capítulo 3 (ver [39]).

Teorema 6. *Supongamos que* (H3) *se satisface con* $1 < q < 2$*, y sea* $\lambda < \lambda^*$*, donde* λ^* *viene dado por* (7.5)*. Sea* $\{\Omega_n\}$ *una sucesión de dominios contenidos en* Ω*, y sea* $\{u_n\}$ *una sucesión de soluciones de* (7.3)*, reemplazando* Ω *por* Ω_n*. Entonces, hay condiciones sobre* Ω_n *(que se especifican en el capítulo 3) que aseguran que existe una distribución* $\sigma \in H^{-1}(\Omega)$ *tal que una subsucesión de* $\{\widetilde{u_n}\}$ *converge débilmente en* $H_0^1(\Omega)$ *a una solución (convenientemente definida en [39]) del problema*

$$\begin{cases} -\Delta u + \sigma u = \lambda u + \mu(x)\dfrac{|\nabla u|^q}{|u|^{q-1}} + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Problemas con un término superlineal de orden cero

Consideramos ahora otro caso particular del problema (7.1), a saber,

$$\begin{cases} -\Delta u + \mu(x)\dfrac{|\nabla u|^2}{(u+\delta)^\gamma} = \lambda u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7.7}$$

donde $\delta \geq 0, \gamma > 0, p > 1, \lambda > 0$ y $0 \lneq \mu \in L^\infty(\Omega)$. El estudio de (7.7) tiene su origen en el caso semilineal correspondiente a $\mu \equiv 0$, que ha sido ampliamente estudiado desde los trabajos pioneros [5,60,82], donde se demuestra existencia de solución de (7.7) para todo $\lambda > 0$ si $p < 2^* - 1 = \frac{N+2}{N-2}$. También destacamos el trabajo clásico [112], donde se prueba que, si $\mu \equiv 0$, $p \geq 2^* - 1$ y Ω es un dominio estrellado, entonces no existe solución de (7.7) para ningún $\lambda > 0$.

El caso casilineal $\mu \not\equiv 0$ se estudió por primera vez en [111]. Aquí se prueban, principalmente, dos tipos de resultados, asumiendo siempre $\mu > 0$ constante y $\delta > 0$: por un lado, los que dan lugar a fenómenos similares a los del caso semilineal, y por otro, los que presentan diferencias, como no existencia de solución para $\lambda > 0$ pequeño o multiplicidad de solución para $\lambda > 0$ grande. El primero de los casos corresponde a la elección de $\gamma > 1$, mientras que el segundo corresponde a $\gamma \in (0,1)$, o bien $\gamma = 1$ y $\mu > p$. Es especialmente interesante esta segunda opción, donde la condición $\mu > p$ pone de manifiesto que, en el caso $\gamma = 1$, la interacción entre los dos términos no lineales de (7.7) juega un papel importante. A este respecto, algunos resultados sobre el caso $\gamma = 1$ (siempre con $\delta > 0$ y $\mu > 0$ constante) que mejoran los de [111] se encuentran en [9,100]. En el primero, los autores demuestran no existencia de solución para $\lambda > 0$ suficientemente pequeño si $\mu \geq p$, mientras que en el segundo se prueba existencia de solución para todo $\lambda > 0$ si $\mu \in \left(0, \frac{2^*-1-p}{2^*-2}\right)$.

Sin embargo, para $\delta > 0$ y $\mu \equiv \text{constante} \in \left[\frac{2^*-1-p}{2^*-2}, p\right)$ no conocemos resultados en la literatura. Aun así, en este caso concreto es fácil comprobar que el cambio de variable explicado arriba transforma la ecuación de (7.7) en una semilineal donde la no linealidad tiene crecimiento supercrítico en infinito y subcrítico en cero, de manera que [13, Theorem 8] implica que existe al menos una solución de (7.7) para todo $\lambda > 0$ suficientemente grande. En cualquier caso, tanto este resultado (inmediato) como los ya citados de [100, 111] se basan en gran medida en el cambio de variable que elimina el término de gradiente. En definitiva, los resultados conocidos sobre el problema (7.7) con $\delta > 0$ no son aplicables si μ es una función no constante.

Por otra parte, el caso singular $\delta = 0$ ha sido abordado más recientemente en [43], también con $\mu > 0$ constante (ver también [33, 44]). Los autores de [43] muestran que, si $\gamma \in (0,1)$, entonces la situación es parecida a la del caso no singular $\delta > 0$. En efecto, demuestran un resultado de no existencia de solución para $\lambda > 0$ pequeño y de existencia para $\lambda > 0$ grande. Por el contrario, para $\gamma \geq 1$ prueban resultados de no existencia para todo $\lambda > 0$ que ponen de manifiesto la influencia de una singularidad fuerte en la ecuación. De nuevo, estos resultados son válidos si $\mu > 0$ es constante. El caso no constante no ha sido estudiado.

En definitiva, nuestro objetivo es estudiar el problema (7.7) para una función μ general no constante, tanto en el caso singular ($\delta = 0$) como en el no singular ($\delta > 0$). A continuación enunciamos los resultados más relevantes que probamos en esta tesis en relación con el problema (7.7), aunque puntualizamos que dichos resultados son válidos para problemas más generales donde el término de gradiente es de la forma $g(x,u)|\nabla u|^2$. Los enunciados en su forma general y las demostraciones se pueden consultar en el capítulo 5 (ver [104]).

Comenzamos presentando un resultado concerniente al caso $\gamma = 1$ que generaliza el resultado de existencia en [100], así como uno de no existencia en [43], ambos válidos para μ constante.

Teorema 7. *Sean $p \in (1, 2^* - 1), \gamma = 1, \delta \geq 0$ y $0 \lneq \mu \in L^\infty(\Omega)$. Se verifican las siguientes afirmaciones:*

1. *Si $\mu \in C(\overline{\Omega})$ y existen dos números reales τ, σ tales que $0 \leq \tau \leq \sigma < \frac{2^*-1-p}{2^*-2}$, $\sigma - \tau < 1 - \sigma$ y $\tau \leq \mu(x) \leq \sigma$ para todo $x \in \overline{\Omega}$, entonces existe al menos una solución de* (7.7) *para todo $\lambda > 0$.*

2. *Si $\delta = 0$ y existen un dominio $\omega \subset\subset \Omega$ y una constante $\tau > 1$ tales que $\mu(x) \geq \tau$ para casi todo $x \in \Omega \setminus \omega$, entonces el problema* (7.7) *no tiene solución para ningún $\lambda > 0$.*

El siguiente resultado aborda el rango $\gamma > 1$. En este caso no obtenemos una generalización del resultado de existencia correspondiente para μ constante en [111], ya que necesitamos imponer una condición extra sobre el tamaño de μ. No obstante, es el primer resultado que conocemos en este rango que permite que μ sea no constante.

Teorema 8. *Sean $p \in (1, 2^* - 1), \gamma > 1, \delta \geq 0$ y $0 \lneq \mu \in L^\infty(\Omega)$. Se verifican las siguientes afirmaciones:*

1. *Si $\delta > 0$ y $\|\mu\|_{L^\infty(\Omega)} < \frac{\delta^{\gamma-1}}{2}$, entonces existe al menos una solución de* (7.7) *para todo $\lambda > 0$.*

2. *Si $\delta = 0$ y existen un dominio $\omega \subset\subset \Omega$ y una constante $\tau > 0$ tales que $\mu(x) \geq \tau$ para casi todo $x \in \Omega \setminus \omega$, entonces no existe solución de* (7.7) *para ningún $\lambda > 0$.*

Finalmente, enunciamos un resultado en el que consideramos $\gamma \in (0,1)$. Además, también consideramos $\delta > 0$ y $\gamma \geq 1$ sin las hipótesis de tamaño sobre μ que se imponían en los Teoremas 7 y 8. Como contrapartida, el siguiente teorema requiere tomar $\lambda > 0$ suficientemente grande.

Teorema 9. *Sean $p \in \left(1, \frac{N+1}{N-1}\right), \gamma > 0, \delta \geq 0$ y $0 \lneq \mu \in L^\infty(\Omega)$. Si $\gamma \geq 1$, supongamos además que $\delta > 0$. Entonces, existe $\lambda_0 > 0$ tal que existe al menos una solución u_λ de* (7.7) *para todo $\lambda > \lambda_0$. Además, $\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0$.*

Destacamos que en las pruebas de los Teoremas 7, 8 y 9 es fundamental el conocido método de “blow-up”, debido a [82], para demostrar la existencia de estimas a priori. Dicho método se ha adaptado de forma no trivial al problema (7.7), haciéndose compatible con la presencia del término de gradiente.

Un problema semilineal

Consideramos finalmente el problema semilineal

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7.8}$$

donde $f : \mathbb{R} \to \mathbb{R}$ es una función lipschitziana. La existencia de solución de (7.8) bajo distintas condiciones para f es un problema clásico. Para una visión general remitimos al lector o la lectora a [101] y a las referencias que contiene. Nosotros nos centramos en estudiar resultados de no existencia de solución de (7.8).

Un primer resultado sencillo de no existencia de solución se obtiene trivialmente multiplicando la ecuación por una solución dada u e integrando por partes. De esta manera, obtenemos que

$$\int_\Omega |\nabla u|^2 = \int_\Omega f(u)u.$$

Por tanto, es claro que si

$$f(s)s \leq 0 \quad \forall s \in \mathbb{R}, \tag{7.9}$$

entonces no pueden existir soluciones no triviales de (7.8). Obsérvese que (7.9) implica que $f(0)=0$, así que $u=0$ es siempre solución (trivial) de (7.8).

El anterior resultado se puede extender a funciones f más generales a costa de imponer condiciones a la geometría del dominio Ω. La idea es clásica y se basa en la conocida identidad de Pohozaev (ver [112]) que verifica toda solución u de (7.8):

$$\frac{1}{2}\int_{\partial\Omega}|\nabla u(x)|^2 x\cdot\nu(x)dS+\frac{N-2}{2}\int_{\Omega}|\nabla u(x)|^2dx=N\int_{\Omega}F(u(x))dx, \tag{7.10}$$

donde $F(s)=\int_0^s f(t)dt$ para todo $s\in\mathbb{R}$ y ν denota el vector unitario normal a $\partial\Omega$ con orientación exterior. Nótese ahora que, si Ω es un dominio estrellado con respecto al origen (es decir, $x\cdot\nu(x)>0$ para todo $x\in\partial\Omega$) y $N\geq 2$, entonces la parte izquierda de (7.10) es no negativa. Por tanto, si

$$F(s)\leq 0 \quad \forall s\in\mathbb{R}, \tag{7.11}$$

entonces no existe solución no trivial de (7.8) siempre que Ω sea estrellado.

Es claro que (7.9) implica (7.11), pero el contraejemplo $f(s)=-\sin(s)$ muestra que la implicación recíproca no es cierta. La pregunta natural que surge es la siguiente: ¿Es determinante la geometría de Ω para la no existencia de solución no trivial de (7.8) con f satisfaciendo (7.11) pero no (7.9)? En otras palabras, ¿existen dominios Ω no estrellados y funciones f satisfaciendo (7.11), pero no (7.9), de manera que existe solución no trivial de (7.8)?

Por lo que respecta a las anteriores preguntas, solo se conocen en la literatura resultados de no existencia parciales. A saber, en [66,116] se prueba no existencia imponiendo condiciones adicionales sobre la constante de Lipschitz de f, mientras que en [84] se consideran dominios no necesariamente estrellados pero cumpliendo otras condiciones geométricas que también implican no existencia de solución. Por otra parte, en [50] se demuestra que, si $f\in C^1(\mathbb{R})$ y cumple que $f(s)>0$ para todo $s\in[s_1,s_2]$, siendo $s_1<s_2$ dos ceros de f con $s_2>0$, entonces una condición menos restrictiva que (7.8) implica que no existe ninguna solución positiva u de (7.8) que verifique que $\max_{\overline{\Omega}}u\in(s_1,s_2)$. Aunque este último resultado no asume condiciones sobre la geometría del dominio, el marco en el que se consideran f y u parece demasiado restrictivo en vista de los anteriores resultados de no existencia que hemos deducido de forma sencilla más arriba en un ambiente en el que f y u son generales.

En esta tesis se demuestra a través del siguiente teorema que la respuesta a las anteriores preguntas es negativa. La prueba emplea algunas ideas de [50] y se puede encontrar en el capítulo 6 (ver [97]).

Teorema 10. *Si $f:\mathbb{R}\to\mathbb{R}$ es una función lipschitziana satisfaciendo* (7.11)*, entonces no existe solución no trivial de* (7.8)*.*

En definitiva, el teorema muestra que la geometría del dominio es irrelevante para la no existencia de solución de (7.8) si se cumple (7.9). Este fenómeno es sorprendente en tanto que es contrario a lo que sucede para otro tipo de no linealidades. Así, si $f(s)=\lambda|s|^{p-1}s$ con $p\geq 2^*-1$ y $\lambda>0$, entonces es sabido que la identidad de Pohozaev también implica en este caso que no existe solución no trivial de (7.8) si Ω es estrellado. Por contra, existen dominios estrellados tales que (7.8) admite solución no trivial para $f(s)=\lambda|s|^{p-1}s$ con $p\geq 2^*-1$ y $\lambda>0$ (ver [62, 92]).

Una implicación importante del Teorema 10 es que puede dar lugar a nuevas vías de investigación. En concreto, sería interesante estudiar la influencia de la geometría de Ω en ecuaciones más generales que (7.8) que verifiquen una identidad tipo Pohozaev. Dos ejemplos que la cumplen son la ecuación casilineal $-\Delta_p u=f(u)$ con $p>1$ (ver [86]) y la no local $(-\Delta)^s u=f(u)$ con $s\in(0,1)$ (ver [117]), donde $-\Delta_p$ denota al operador $p-$laplaciano, y $(-\Delta)^s$, al laplaciano fraccionario.

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