



Matching high-energy electroweak fermion loops onto the Fermi theory without higher dimensional operators

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Abstract

We derive the conditions for matching high-energy renormalizable Quantum Field Theories onto low-energy nonrenormalizable ones by means of the FDR approach described in [1]. Our procedure works order-by-order in the loop expansion and avoids the addition of higher dimensional interactions into the non-renormalizable Lagrangian. To illustrate our strategy, we match the high-energy fermion-loop corrections computed in the complete electroweak theory onto the nonrenormalizable four-fermion Fermi model. As a result, the Fermi Lagrangian can be used without modifications to reproduce, at arbitrary loop orders and energies, the exact electroweak interactions between two massless fermion lines induced by one-fermion-loop resummed gauge boson propagators.

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1. Introduction

Renormalizable Quantum Field Theories (QFT) are the commonly used language to describe high-energy interactions in particle physics. They are considered as fundamental theories, in the sense that predictions can be obtained, at any desired perturbative order and scale, by consistently reabsorbing the ultraviolet (UV) infinities appearing in the intermediate stages of the calculation in the set $\{p_i\}$, $i = 1 \div m$, of the free parameters of the Lagrangian

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$$\mathcal{L}(p_1, \dots, p_m). \quad (1)$$

On the other hand, nonrenormalizable QFTs belong to a larger class of theories, namely the effective QFTs (EFT), and are extensively employed in cases when the fundamental renormalizable model is unknown, or not easily calculable. The problem of computing high-energy loop corrections in EFTs is usually dealt with by using the seminal Weinberg's approach [2], in which higher dimensional operators O_i , compatible with the symmetries of the theory, are added to the lowest order Lagrangian \mathcal{L} to reabsorb the UV infinities which remain after fixing the parameters of the model,

$$\mathcal{L} \rightarrow \mathcal{L} + \sum_i C_i O_i := \mathcal{L} + \mathcal{L}_{\text{HD}}.$$

By doing so order-by-order in the loop expansion, EFTs can be treated as ordinary renormalizable QFTs at the price of introducing a large set of Wilson coefficients C_i (possibly, an infinite one) to be fixed by experiment. Of course, not all the C_i are relevant at the energy scale under study. As a matter of fact, if N is the number of independent kinematic invariants s_n , one organizes the EFT as a perturbative expansion in the ratios

$$\lambda_n = s_n/M_n^2, \quad n = 1 \div N, \quad (2)$$

where the M_n are mass scales parameterizing the range of validity of the effective description [3,4]. In this way, physical predictions can be obtained, order-by-order in the λ_n , in terms of a finite set of measurements.

In [1] a different way to include high-energy loop corrections in nonrenormalizable QFTs is presented based on FDR [5]. In FDR UV divergences are eliminated by way of a redefinition of the loop integration that does not rely on an order-by-order renormalization. Hence, UV finite quantities are directly computed without adding \mathcal{L}_{HD} to \mathcal{L} . The price of this is the appearance of an arbitrary renormalization scale μ_R . In the case of renormalizable models, the dependence on μ_R disappears from physical predictions $\mathcal{O}^{\text{TH},\ell\text{-loop}}$,

$$\frac{d\mathcal{O}^{\text{TH},\ell\text{-loop}}(\tilde{p}_1(\mu_R), \dots, \tilde{p}_m(\mu_R), \mu_R)}{d\mu_R} = 0, \quad (3)$$

when they are expressed in terms of the set of parameters $\{\tilde{p}_i(\mu_R)\}$ fixed by m experiments $\mathcal{O}_i^{\text{EXP}}$ determined up to the same perturbative order ℓ one is working,

$$\tilde{p}_i(\mu_R) := p_i^{\text{TH},\ell\text{-loop}}(\mathcal{O}_1^{\text{EXP}}, \dots, \mathcal{O}_m^{\text{EXP}}, \mu_R), \quad i = 1 \div m. \quad (4)$$

On the contrary, (3) is not fulfilled, in general, by nonrenormalizable QFTs. However, in the procedure of [1] μ_R is an adjustable parameter rather than a UV cutoff,¹ so that an additional measurement $\mathcal{O}_{m+1}^{\text{EXP}}$ can be used to fix it by imposing

$$\mathcal{O}_{m+1}^{\text{TH},\ell\text{-loop}}(\tilde{p}_1(\mu'_R), \dots, \tilde{p}_m(\mu'_R), \mu'_R) = \mathcal{O}_{m+1}^{\text{EXP}}. \quad (5)$$

After this is done, observables different from those used to determine the model,

$$\mathcal{O}_i^{\text{TH},\ell\text{-loop}}(\tilde{p}_1(\mu'_R), \dots, \tilde{p}_m(\mu'_R), \mu'_R), \quad i > m + 1, \quad (6)$$

¹ This means that, at any fixed value of μ_R , the nonrenormalizable Lagrangian \mathcal{L} describes a legitimate effective theory, even without adding \mathcal{L}_{HD} to it.

can be predicted and tested experimentally. If in a given range of energy

$$\mathcal{O}_i^{\text{TH},\ell\text{-loop}}(\tilde{p}_1(\mu'_R), \dots, \tilde{p}_m(\mu'_R), \mu'_R) = \mathcal{O}_i^{\text{EXP}} \quad (7)$$

for a large class of observables $i > m + 1$, the nonrenormalizable QFT can be used as a plausible effective model.

In this work we study under which conditions a known renormalizable theory can be matched onto a low-energy nonrenormalizable effective model by means of the FDR approach. In this case, the matching condition (5) is replaced by²

$$B_{m+1}^{\ell\text{-loop}}(\lambda, \alpha, \mu'_R) = A_{m+1}^{\ell\text{-loop}}(\lambda, \alpha), \quad (8)$$

where B_{m+1} and A_{m+1} are amplitudes computed up to the ℓ th order in the coupling constant α within the nonrenormalizable and renormalizable QFT, respectively, and λ stands for all the N ratios in (2). In particular, we derive the conditions to be obeyed by the coefficients of the perturbative expansion of equation (8) for ensuring the independence of μ'_R from kinematics. In addition, we conjecture that, when such a μ'_R exists, additional independent amplitudes can be matched at $\lambda \neq 0$,

$$B_i^{\ell\text{-loop}}(\lambda, \alpha, \mu'_R) = A_i^{\ell\text{-loop}}(\lambda, \alpha), \quad i > m + 1, \quad (9)$$

if they coincide at $\lambda = 0$,

$$B_i^{\ell\text{-loop}}(0, \alpha, \mu'_R) = A_i^{\ell\text{-loop}}(0, \alpha), \quad i > m + 1. \quad (10)$$

At the present stage of our investigation we cannot prove this in general. However, it holds true when the A_i are resummed one-fermion-loop amplitudes computed in the full electroweak theory and the B_i are calculated in the four-fermion Fermi model. In such a case, if μ'_R is fixed once for all as in (8), the Fermi theory reproduces, at any loop order, all the exact amplitudes describing any process involving fermion-loop mediated interactions between two massless fermions at arbitrary energy scales. This demonstrates that realistic low-energy nonrenormalizable QFTs exist that can be consistently uplifted to higher energies by FDR without modifying their Lagrangian, at least under special classes of loop corrections. Conversely, if nonrenormalizable and renormalizable amplitudes can be matched with a μ'_R independent of kinematics, the coefficients of their expansions necessarily obey the same conditions which ensure the validity of (8).

The structure of the paper is as follows. In section 2 we recall the essential principles of FDR. The conditions for the matching in (8) are derived in section 3. Section 4 describes the one-fermion-loop matching of the high-energy electroweak corrections onto the Fermi model. Finally, the last section includes a comparison between our procedure and a customary EFT approach.

2. FDR integration and loop functions

Here we sketch out the basic axioms of FDR with the help of a simple one-dimensional example. The interested reader can find more details in the relevant literature [5–10].

² Here and in the following, amplitudes used to fix μ_R are denoted by the subscript $m + 1$, while the label $i > m + 1$ refers to processes different from those employed to determine the Lagrangian's parameters and the renormalization scale.

Let's assume one needs to define the UV divergent integral

$$I = \lim_{\Lambda \rightarrow \infty} \int_0^\Lambda dx \frac{x}{x + P}, \tag{11}$$

where P stands for a physical energy scale. FDR identifies the UV divergent pieces in terms of integrands independent of P , dubbed FDR vacua, and rewrites

$$\frac{x}{x + P} = 1 - \frac{P}{x} + \frac{P^2}{x(x + P)}. \tag{12}$$

Thus, the first term in the r.h.s. of (12) is the vacuum responsible for the linear UV divergence, while $1/x$ generates the $\ln \Lambda$ behavior. By definition, the linearly divergent contribution is subtracted from (11) over the full integration domain $[0, \Lambda]$, while the logarithmic divergence over the interval $[\mu_R, \Lambda]$ only. The arbitrary separation scale $\mu_R \neq 0$ is needed to keep a-dimensional and finite the arguments of the logarithms appearing in the subtracted and finite parts. Thus,

$$I_{\text{FDR}} := I - \lim_{\Lambda \rightarrow \infty} \left(\int_0^\Lambda dx - \int_{\mu_R}^\Lambda dx \frac{P}{x} \right) = P \ln \frac{P}{\mu_R}. \tag{13}$$

The advantage of this definition is twofold. Firstly, the UV cutoff Λ is traded for μ_R , which is interpreted as the renormalization scale. Secondly, other than logarithmic UV divergences do not contribute. The explicit appearance of μ_R in the interval of integration makes the use of (13) inconvenient in practical calculations. An equivalent definition is obtained by adding an auxiliary unphysical scale μ to x , $x \rightarrow \bar{x} := x + \mu$,³ and introducing an integral operator $\int_0^\infty [dx]$ which annihilates the FDR vacua before integration. Hence,

$$I_{\text{FDR}} = \int_0^\infty [dx] \frac{\bar{x}}{\bar{x} + P} := \lim_{\mu \rightarrow 0} \int_0^\infty dx \frac{P^2}{\bar{x}(\bar{x} + P)} \Big|_{\mu=\mu_R},$$

where $\mu \rightarrow 0$ is an asymptotic limit.

This strategy can be extended to more dimensions and to rational integrands depending on any number of variables, as those appearing in ℓ -loop integrals I_{FDR}^ℓ . They are polynomials of degree ℓ in $\ln \mu_R^2$, [8]

$$I_{\text{FDR}}^\ell = \sum_{k=0}^\ell c_k L_R^k, \quad L_R := \ln(\mu_R^2). \tag{14}$$

For instance, at one loop one has

$$\begin{aligned} & \int [d^4 q] \frac{1}{(\bar{q}^2 - m^2)(\bar{q}^2 + p^2 + 2q \cdot p - m_1^2)} \\ &= I_{\text{FDR}}^1(p^2, m^2, m_1^2) = -i\pi^2 \int_0^1 dy \ln \frac{\chi}{\mu_R^2}, \end{aligned} \tag{15a}$$

³ This replacement must be performed in both numerators and denominators of the integrated functions.

$$\int [d^4 q] \frac{q^\alpha}{(\bar{q}^2 - m^2)(\bar{q}^2 + p^2 + 2q \cdot p - m_1^2)}$$

$$= i\pi^2 p^\alpha \int_0^1 dy y \ln \frac{\chi}{\mu_R^2}, \tag{15b}$$

$$\int [d^4 q] \frac{q^\alpha q^\beta}{(\bar{q}^2 - m^2)(\bar{q}^2 + p^2 + 2q \cdot p - m_1^2)}$$

$$= \frac{i\pi^2}{2} g^{\alpha\beta} \int_0^1 dy \chi \left(1 - \ln \frac{\chi}{\mu_R^2} \right) + \mathcal{O}(p^\alpha p^\beta), \tag{15c}$$

with $\bar{q}^2 := q^2 - \mu^2$ and $\chi := m^2 y + m_1^2(1 - y) - p^2 y(1 - y)$. Finally, it is important to realize that internal consistency requires μ_R to be independent of kinematics and identical in all loop functions. This guarantees correct cancellations when combining integrals.⁴

3. The conditions for matching two amplitudes

Our aim is determining the renormalization scale μ'_R in (8). The all-order expansions of A_{m+1} and B_{m+1} read

$$A_{m+1}(\lambda, \alpha) = K(\alpha) + K(\alpha) \sum_{j=1}^{\infty} A_{0j}^{\{m_j\}} \lambda^{\{m_j\}} + K(\alpha) \sum_{i,j=1}^{\infty} A_{ij}^{\{m_j\}} \alpha^i \lambda^{\{m_j\}}, \tag{16a}$$

$$B_{m+1}(\lambda, \alpha, \mu_R) = K(\alpha) + K(\alpha) \sum_{\substack{i,j=1 \\ 0 \leq k \leq i}}^{\infty} B_{ijk}^{\{m_j\}} \alpha^i \lambda^{\{m_j\}} L_R^k, \tag{16b}$$

where $K(\alpha)$ is defined by the constraint

$$B_{m+1}(0, \alpha, \mu_R) = A_{m+1}(0, \alpha) = K(\alpha), \tag{17}$$

which states that the amplitudes computed in the exact theory and the effective model coincide when $\lambda \rightarrow 0$. $A_{0j}^{\{m_j\}}$, $A_{ij}^{\{m_j\}}$, $B_{ijk}^{\{m_j\}}$ are perturbative coefficients, in which i refers to the α expansion, whereas j denotes the power degree of the products of λ_n multiplying the coefficients. The notation

$$\{m_j\} := (m_{j1}, m_{j2}, \dots, m_{jN})$$

symbolizes an assignment of integer numbers $m_{jn} \geq 0$ fulfilling

$$\sum_{n=1}^N m_{jn} = j, \tag{18}$$

⁴ For example, the UV finite combination $I_{\text{FDR}}^1(p_1^2, m^2, m_1^2) - I_{\text{FDR}}^1(p_2^2, m^2, m_1^2)$ is equal to the right result,

$$\int d^4 q \frac{2q \cdot (p_2 - p_1) + p_2^2 - p_1^2}{(q^2 - m^2)((q + p_1)^2 - m_1^2)((q + p_2)^2 - m_1^2)},$$

only if μ_R^2 in (15a) takes the same constant value in both $I_{\text{FDR}}^1(p_1^2, m^2, m_1^2)$ and $I_{\text{FDR}}^1(p_2^2, m^2, m_1^2)$.

and a sum over all possible assignments is understood when contracting with⁵

$$\lambda^{\{m_j\}} := \prod_{n=1}^N \lambda_n^{m_j n}. \quad (19)$$

The coefficients in (16) may involve functions of s_n singular at $\lambda = 0$, such as $\ln s_n$ or $s_n^{-\frac{1}{2}}$,⁶ but (17) requires

$$A_{0j}^{\{m_j\}} \lambda^{\{m_j\}} \rightarrow 0, \quad A_{ij}^{\{m_j\}} \lambda^{\{m_j\}} \rightarrow 0, \quad B_{ijk}^{\{m_j\}} \lambda^{\{m_j\}} \rightarrow 0$$

when $\lambda \rightarrow 0$. Furthermore, B_{m+1} in (16b) depends on λ only through loop corrections, unlike A_{m+1} . Typically, the second term in the r.h.s. of (16a) is generated by Taylor expanding the tree-level propagators $1/(s_n - M_n^2)$ of the exact theory, that are absent in the effective model, whose natural expansion parameters are, instead, dimensionful couplings of the type $\alpha^a/(M_n^2)^b$ with $a, b > 0$. Note also that the dependence upon μ_R is driven by (14).

Solutions to (8) are found by replacing its two sides by (16a) computed with ($i \leq \ell, j \leq \ell$) and (16b) truncated at ($i \leq \ell + 1, j \leq \ell, k \geq i - \ell$), and allowing L_R in (16b) to mix different perturbative orders,

$$L_R = \sum_{i=-1}^{\ell-1} X_i \alpha^i. \quad (20)$$

Equating the powers of α and $\lambda^{\{m_j\}}$ gives a system of equations to be fulfilled by the unknown coefficients X_i . We are interested in constant solutions,

$$L'_R := \ln(\mu_R'^2) = \sum_{i=-1}^{\ell-1} X'_i \alpha^i, \quad (21)$$

in which the X'_i are independent of both the λ_n and the s_n contained in $A_{0j}^{\{m_j\}}, A_{ij}^{\{m_j\}}, B_{ijk}^{\{m_j\}}$. This requirement determines the conditions to be fulfilled by the coefficients of the two series in (16) to be compatible with the FDR treatment of the loop integrals outlined in section 2. In what follows, we discuss the first two perturbative orders and delineate the structure of the general ℓ -loop case.

When $\ell = 1$, $L_R = X_{-1}/\alpha + X_0$ and the system reads

$$\begin{cases} A_{01}^{\{m_1\}} - B_{111}^{\{m_1\}} X_{-1} - B_{212}^{\{m_1\}} X_{-1}^2 = 0, \\ A_{11}^{\{m_1\}} - B_{110}^{\{m_1\}} - B_{111}^{\{m_1\}} X_0 - B_{211}^{\{m_1\}} X_{-1} - 2B_{212}^{\{m_1\}} X_{-1} X_0 = 0, \quad \forall \{m_1\}. \end{cases} \quad (22)$$

If $N = 1$, only one assignment is possible, $\{m_1\} = (1)$, and a solution compatible with (22) can always be found for nonexceptional values of the coefficients,

$$\begin{aligned} \hat{X}_{-1}^2 B_{212}^{(1)} + \hat{X}_{-1} B_{111}^{(1)} - A_{01}^{(1)} &= 0, \\ \hat{X}_0 &= \frac{A_{11}^{(1)} - B_{110}^{(1)} - B_{211}^{(1)} \hat{X}_{-1}}{B_{111}^{(1)} + 2B_{211}^{(1)} \hat{X}_{-1}}. \end{aligned} \quad (23)$$

⁵ For instance, if $N = 2$, $A_{02}^{\{m_2\}} \lambda^{\{m_2\}} = A_{02}^{(2,0)} \lambda_1^2 + A_{02}^{(0,2)} \lambda_2^2 + A_{02}^{(1,1)} \lambda_1 \lambda_2$.

⁶ For example, if the $\lambda_n \rightarrow 0$ asymptotic expansion of the loop functions produces a $\sqrt{\lambda_n}$, it is rewritten as $\sqrt{\lambda_n} = \lambda_n \left(M_n s_n^{-\frac{1}{2}} \right)$ in (16).

If, in addition, this solution is such that

$$\frac{\partial \hat{X}_i}{\partial s_n} = 0 \quad \forall n, \quad i = -1, 0, \tag{24}$$

then

$$X'_i = \hat{X}_i, \quad i = -1, 0. \tag{25}$$

With N invariants, there are N possible assignments,

$$\{m_1\} = (1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1),$$

so that (22) is a system of $2N$ equations and two unknowns, that admits solutions only if relations exist among the coefficients. In practice, one determines \hat{X}_{-1} and \hat{X}_0 for a particular assignment and checks whether this solution obeys (22) $\forall \{m_1\}$. After that, one also verifies the validity of (24). Thus, (22) and (24) give $4N$ conditions. If they are all obeyed, the matching

$$B_{m+1}^{1\text{-loop}}(\lambda, \alpha, \mu'_R) = A_{m+1}^{1\text{-loop}}(\lambda, \alpha) \tag{26}$$

is realized by inserting (25) in (21) with $\ell = 1$.

If $\ell = 2$, $L_R = X_{-1}/\alpha + X_0 + X_1\alpha$, and

$$\left\{ \begin{array}{l} A_{0j}^{\{m_j\}} - B_{1j1}^{\{m_j\}} X_{-1} - B_{2j2}^{\{m_j\}} X_{-1}^2 \\ - B_{3j3}^{\{m_j\}} X_{-1}^3 = 0, \\ A_{1j}^{\{m_j\}} - B_{1j0}^{\{m_j\}} - B_{1j1}^{\{m_j\}} X_0 - B_{2j1}^{\{m_j\}} X_{-1} \\ - 2B_{2j2}^{\{m_j\}} X_{-1} X_0 - B_{3j2}^{\{m_j\}} X_{-1}^2 \\ - 3B_{3j3}^{\{m_j\}} X_{-1}^2 X_0 = 0, \\ A_{2j}^{\{m_j\}} - B_{2j0}^{\{m_j\}} - B_{1j1}^{\{m_j\}} X_1 - B_{2j1}^{\{m_j\}} X_0 \\ - B_{2j2}^{\{m_j\}} (X_0^2 + 2X_{-1} X_1) \\ - B_{3j1}^{\{m_j\}} X_{-1} - 2B_{3j2}^{\{m_j\}} X_{-1} X_0 \\ - 3B_{3j3}^{\{m_j\}} (X_{-1} X_0^2 + X_{-1}^2 X_1) = 0, \\ \forall \{m_j\}, \text{ with } j = 1 \div 2. \end{array} \right. \tag{27a}$$

$$\tag{27b}$$

Values of \hat{X}_{-1} , \hat{X}_0 and \hat{X}_1 fulfilling (27a) can in general be found for a particular assignment. Subsequently, one checks if

$$\frac{\partial \hat{X}_i}{\partial s_n} = 0 \quad \forall n, \quad i = -1 \div 1, \tag{28}$$

and whether this very same solution holds for all the remaining assignments of (27b). Therefore, (27) and (28) give the conditions for the matching

$$B_{m+1}^{2\text{-loop}}(\lambda, \alpha, \mu'_R) = A_{m+1}^{2\text{-loop}}(\lambda, \alpha). \tag{29}$$

If they are met, (29) is obeyed by setting $\ell = 2$ and $X'_i = \hat{X}_i$ in (21).

At ℓ loops and fixed assignment, \hat{X}_{-1} is a solution of an algebraic equation of degree $(\ell + 1)$. Once \hat{X}_{-1} is known, the rest of the system is linear and triangular, so that the remaining coefficients \hat{X}_i , $i = 0 \div (\ell - 1)$, can be easily determined. After that, one checks the validity of this solution for all the other assignments. If, in addition,

$$\frac{\partial \hat{X}_i}{\partial s_n} = 0 \quad \forall n, \quad i = -1 \div (\ell - 1), \quad (30)$$

the matching is achieved by choosing $X'_i = \hat{X}_i$ in (21).

4. An effective model for the high-energy electroweak fermion loops

When the constraints derived in the previous section are fulfilled, the result predicted by A_{m+1} is reproduced, order by order in α and λ , by the effective nonrenormalizable amplitude B_{m+1} . This allows one to determine μ'_R from (8) and use it in further amplitudes B_i computed within the effective model. If, after fixing the Lagrangian's parameters as in (4), the B_i obey (10), we argue that they can be matched as in (9). Here we prove this in the case of the electroweak Fermi model when the coupling constant expansion is in terms of resummed one-fermion-loop corrections. In section 4.1 we detail the nonrenormalizable and renormalizable theories to be matched and the radiative corrections involved. The fitting procedure of (4) is discussed in section 4.2 and the matching implied by (8) and (9) is the subject of section 4.3.

4.1. The models and the loop corrections

Our renormalizable theory is defined by the fermionic sector of the electroweak standard model interaction Lagrangian, namely

$$\mathcal{L}_{\text{INT}}^{\text{SM}} = \mathcal{L}_{\text{INT}}^{\text{QED}} + \mathcal{L}_{\text{INT}}^{\text{ZW}}, \quad (31)$$

with

$$\mathcal{L}_{\text{INT}}^{\text{QED}} = -g s_\theta A_\alpha \sum_f Q_f \bar{f}_j \gamma^\alpha f_j \quad (32)$$

and

$$\begin{aligned} \mathcal{L}_{\text{INT}}^{\text{ZW}} = & -\frac{g}{2c_\theta} Z_\alpha \sum_f \bar{f}_j \gamma^\alpha (v_f + a_f \gamma_5) f_j \\ & -\frac{g}{2\sqrt{2}} W_\alpha^+ \sum_f \frac{2I_{3f} + 1}{2} \bar{f}_j \gamma^\alpha (1 - \gamma_5) f'_j \\ & -\frac{g}{2\sqrt{2}} W_\alpha^- \sum_f \frac{1 - 2I_{3f}}{2} \bar{f}_j \gamma^\alpha (1 - \gamma_5) f'_j. \end{aligned} \quad (33)$$

The photon and the massive gauge boson fields are denoted by A_α , Z_α and W_α^\pm , respectively. The spinor associated with a fermion f with color j is denoted by f_j , with the convention that $j = 1 \div 3$ for quarks and $j = 1$ for leptons. The sum runs over all fermions and f' is the isospin partner of f in the limit of diagonal CKM quark-mixing matrix. The vector and axial couplings are

$$v_f = I_{3f} - 2s_\theta^2 Q_f, \quad a_f = -I_{3f}, \quad (34)$$

where I_{3f} is the third isospin component, Q_f the electric charge and s_θ (c_θ) is the sine (cosine) of the weak mixing angle. The Feynman gauge is used, hence the gauge boson propagators read

$$\begin{aligned} P_A^{\alpha\beta}(p^2) &= -ig^{\alpha\beta} \frac{1}{p^2}, & P_W^{\alpha\beta}(p^2) &= -ig^{\alpha\beta} \frac{1}{p^2 - M^2}, \\ P_Z^{\alpha\beta}(p^2) &= -ig^{\alpha\beta} \frac{1}{p^2 - M^2/c_\theta^2}. \end{aligned} \quad (35)$$

Our effective nonrenormalizable interaction Lagrangian reads

$$\mathcal{L}_{\text{INT}}^{\text{EFF}} = \mathcal{L}_{\text{INT}}^{\text{QED}} + \mathcal{L}^{\text{FERMI}}, \quad (36)$$

with

$$\mathcal{L}^{\text{FERMI}} = -\frac{g^2}{8M^2} J_{c\alpha}^\dagger J_c^\alpha - \frac{g^2}{8M^2} J_{n\alpha} J_n^\alpha, \quad (37)$$

where the charged and neutral currents are given by

$$\begin{aligned} J_c^\alpha &= \sum_f \frac{2I_{3f} + 1}{2} \bar{f}_j \gamma^\alpha (1 - \gamma_5) f'_j, \\ J_n^\alpha &= \sum_f \bar{f}_j \gamma^\alpha (v_f + a_f \gamma_5) f_j. \end{aligned} \quad (38)$$

In (37) the four-fermion coupling between currents is written in a form which reproduces the tree-level low-energy result obtained with $\mathcal{L}_{\text{INT}}^{\text{SM}}$ when using $P_{W,Z}^{\alpha\beta}(0)$. Massive gauge boson propagators are absent in the effective theory, while the photon propagator is as in (35).

The main objects entering our calculation are the truncated one-fermion-loop contributions depicted in Fig. 1. Fermion masses are neglected, when possible, except in the case of the top quark, for which the leading m_t^2 contribution is also included. The $p^\alpha p^\beta$ parts are omitted, because they do not contribute on-shell. An FDR computation of the form factors requires the integrals in (15). The result is

$$\begin{aligned} \Sigma_Z(p^2) &= -\frac{p^2}{\pi^2} \left(1 - 2s_\theta^2 + \frac{8}{3}s_\theta^4 \right) \left(L_R - L + \frac{5}{3} \right) + \frac{3m_t^2}{8\pi^2} \left(L_R - \ln m_t^2 \right), \\ \Sigma_W(p^2) &= -\frac{p^2}{\pi^2} \left(L_R - L + \frac{5}{3} \right) + \frac{3m_t^2}{8\pi^2} \left(L_R - \ln m_t^2 + \frac{1}{2} \right), \\ \Pi_A(p^2) &= -\frac{2}{3\pi^2} \left(L_R - L + \frac{5}{3} \right), \\ \Pi_A(0) &= -\frac{2}{3\pi^2} (L_R - K_2), \\ \Pi_{ZA}(p^2) &= -\frac{1}{\pi^2} \left(\frac{1}{4} - \frac{2}{3}s_\theta^2 \right) \left(L_R - L + \frac{5}{3} \right) \\ &= \Pi_{AZ}(p^2), \\ \Pi_{ZA}(0) &= -s_\theta^2 \Pi_A(0) - \frac{1}{4\pi^2} (L_R - K_1) \\ &= \Pi_{AZ}(0), \end{aligned} \quad (39)$$

$$\begin{aligned}
 \sum_f \alpha \frac{p}{Z} & \begin{array}{c} \xrightarrow{f_j} \\ \circlearrowleft \\ \xleftarrow{\bar{f}_j} \end{array} \begin{array}{c} \xrightarrow{\beta} \\ \circlearrowright \\ \xleftarrow{\beta} \end{array} = ig^{\alpha\beta} \frac{g^2}{4c_\theta^2} \Sigma_Z(p^2), \\
 \sum_f \frac{2I_{3f} + 1}{2} \alpha \frac{p}{W} & \begin{array}{c} \xrightarrow{f_j} \\ \circlearrowleft \\ \xleftarrow{\bar{f}'_j} \end{array} \begin{array}{c} \xrightarrow{\beta} \\ \circlearrowright \\ \xleftarrow{\beta} \end{array} = ig^{\alpha\beta} \frac{g^2}{4} \Sigma_W(p^2), \\
 \sum_f \alpha \frac{p}{Z} & \begin{array}{c} \xrightarrow{f_j} \\ \circlearrowleft \\ \xleftarrow{\bar{f}_j} \end{array} \begin{array}{c} \xrightarrow{\beta} \\ \circlearrowright \\ \xleftarrow{A} \end{array} = ig^{\alpha\beta} \frac{g^2 s_\theta}{c_\theta} p^2 \Pi_{ZA}(p^2), \\
 \sum_f \alpha \frac{p}{A} & \begin{array}{c} \xrightarrow{f_j} \\ \circlearrowleft \\ \xleftarrow{\bar{f}_j} \end{array} \begin{array}{c} \xrightarrow{\beta} \\ \circlearrowright \\ \xleftarrow{A} \end{array} = ig^{\alpha\beta} g^2 s_\theta^2 p^2 \Pi_A(p^2).
 \end{aligned}$$

Fig. 1. The parts of the truncated one-fermion-loop diagrams proportional to the metric tensor. The dots in the vertices denote that the external propagators are not included.

with $L := \ln(-p^2 - i\epsilon)$. Furthermore

$$\begin{aligned}
 K_1 & := \frac{1}{2} + \frac{\ln m_e^2 + \ln m_\mu^2 + \ln m_\tau^2}{12} + \frac{\ln m_u^2 + \ln m_c^2 + \ln m_t^2}{6} \\
 & \quad + \frac{\ln m_d^2 + \ln m_s^2 + \ln m_b^2}{12}, \\
 K_2 & := \frac{1}{2} + \frac{\ln m_e^2 + \ln m_\mu^2 + \ln m_\tau^2}{8} + \frac{\ln m_u^2 + \ln m_c^2 + \ln m_t^2}{6} \\
 & \quad + \frac{\ln m_d^2 + \ln m_s^2 + \ln m_b^2}{24},
 \end{aligned} \tag{40}$$

where the light quark masses have to be considered as effective parameters adjusted to fit the dispersion integral defining the hadronic contribution to the vacuum polarization.

4.2. Fixing the free parameters of the models

Both Lagrangians in (31) and (36) depend on the set of bare parameters $\{g^2, M^2, s_\theta^2\}$, which need to be fixed by experiment. As input data we choose the fine structure constant α_{EM} , measured in the Thomson limit of the Compton scattering, the muon decay constant G_F , extracted from the muon lifetime, and the ratio R_{ev} between the total $e^- \nu_\mu$ and $e^- \bar{\nu}_\mu$ elastic cross sections at zero momentum transfer. In the following, we determine and solve the fitting equations [11,12] linking $\{\alpha_{EM}, G_F, R_{ev}\}$ to $\{g^2, M^2, s_\theta^2\}$ in both renormalizable and nonrenormalizable models.

In the renormalizable theory one constructs the fermion-loop dressed propagators,

$$D_V^{\alpha\beta}(p^2) = -i g^{\alpha\beta} \Delta_V(p^2), \quad V = W, Z, A, ZA, AZ, \tag{41}$$

by Dyson resumming to all orders the self-energy contributions of Fig. 1. The result reads

$$\begin{aligned} \Delta_W(p^2) &= \frac{1}{g^2} \frac{1}{P_W(p^2)}, \\ \Delta_Z(p^2) &= \frac{1}{g^2} \frac{1}{P_Z(p^2)} \frac{1}{\mathcal{Z}(p^2)}, \\ p^2 \Delta_A(p^2) &= \frac{1}{P_A(p^2) \mathcal{Z}(p^2)}, \\ \Delta_{ZA}(p^2) &= g^2 \frac{s_\theta}{c_\theta} \frac{\Pi_{ZA}(p^2)}{P_A(p^2)} \Delta_Z(p^2) = \Delta_{AZ}(p^2), \end{aligned} \tag{42}$$

with

$$\begin{aligned} P_W(p^2) &= \frac{p^2}{g^2} - \frac{M^2}{g^2} - \frac{\Sigma_W(p^2)}{4}, \\ P_Z(p^2) &= \frac{p^2}{g^2} - \frac{M^2}{g^2 c_\theta^2} - \frac{\Sigma_Z(p^2)}{4c_\theta^2}, \\ P_A(p^2) &= 1 - g^2 s_\theta^2 \Pi_A(p^2), \\ \mathcal{Z}(p^2) &= 1 - p^2 g^2 \frac{s_\theta^2}{c_\theta^2} \frac{\Pi_{ZA}^2(p^2)}{P_A(p^2) P_Z(p^2)}. \end{aligned} \tag{43}$$

Using the propagators in (41) to compute the Thomson scattering, the muon lifetime and R_{ev} , gives the fitting equations

$$4\pi \alpha_{EM} = \frac{g^2 s_\theta^2}{1 - g^2 s_\theta^2 \Pi_A(0)}, \tag{44a}$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 \left[M^2 + \frac{g^2}{4} \Sigma_W(0) \right]}, \tag{44b}$$

$$R_{ev} = \frac{16S^4 - 12S^2 + 3}{16S^4 - 4S^2 + 1}, \tag{44c}$$

where

$$S^2 := s_\theta^2 \left\{ 1 - \frac{g^2 \Pi_{ZA}(0)}{1 - g^2 s_\theta^2 \Pi_A(0)} \right\}.$$

In the case of the nonrenormalizable model, it is easy to prove that

Theorem 1. *Computing $\{\alpha_{EM}, G_F, R_{ev}\}$ in terms of $\{g^2, M^2, s_\theta^2\}$ produces the same fitting equations (44) of the renormalizable theory.*

Proof. When resumming to all orders the interactions mediated by the fermion loops, one arrives at results which have the same form of transitions induced by the dressed propagators of (42) computed at $p^2 = 0$. Since the observables used as input data only involve zero momentum transfer, the equations (44) are also valid in the nonrenormalizable theory. \square

$$\rightarrow^p \alpha \circlearrowleft \beta = -2ig^{\alpha\beta} \Sigma_W(p^2)$$

Fig. 2. The diagram mediating fermion-loop induced interactions between charged currents in the nonrenormalizable theory.

$$p^2 = 0 \rightarrow \text{tree} + \text{loop} + \dots = A_W^{\text{EFF}}(0)$$

Fig. 3. The amplitude describing the muon decay in the nonrenormalizable theory. The interaction of Fig. 2 is evaluated at $p^2 = 0$ and resummed to all orders.

As an example, the diagram relevant in the case of charged currents is given in Fig. 2. That modifies the muon decay amplitude as depicted in Fig. 3. One computes

$$A_W^{\text{EFF}}(0) = -\frac{i\Gamma}{8} \frac{g^2}{M^2} \frac{1}{1 + \frac{g^2}{4M^2} \Sigma_W(0)} = \frac{i\Gamma}{8} g^2 \Delta_W(0), \tag{45}$$

where Γ is the result of the contraction of the two charged currents $\Gamma := \gamma_\alpha(1 - \gamma_5) \otimes \gamma^\alpha(1 - \gamma_5)$, in which the symbol \otimes understands multiplication by the relevant external spinors. Using (45) to define the combination g^2/M^2 leads to (44b).

Finally, to solve the fitting equations we first introduce the tree-level solution to (44c), namely \hat{s}_θ such that

$$R_{ev} = \frac{16\hat{s}_\theta^4 - 12\hat{s}_\theta^2 + 3}{16\hat{s}_\theta^4 - 4\hat{s}_\theta^2 + 1}. \tag{46}$$

Radiative corrections do not change R_{ev} when $S^2 = \hat{s}_\theta^2$, that gives

$$s_\theta^2 = \hat{s}_\theta^2 \frac{F_1}{F_2}, \tag{47a}$$

$$g^2 = \frac{4\pi\alpha_{EM}}{\hat{s}_\theta^2 F_1}, \tag{47b}$$

$$M^2 = \frac{\hat{M}^2}{F_1} \left(1 - \sqrt{2} G_F \Sigma_W(0) \right), \tag{47c}$$

with

$$\begin{aligned} \hat{M}^2 &:= \frac{\pi\alpha_{EM}}{\sqrt{2}G_F\hat{s}_\theta^2}, & F_1 &:= 1 - \frac{\alpha_{EM}}{\pi\hat{s}_\theta^2} (L_R - K_1), \\ F_2 &:= 1 - \frac{8\alpha_{EM}}{3\pi} (L_R - K_2). \end{aligned} \tag{48}$$

4.3. Matching the exact theory onto the nonrenormalizable model

The high-energy fermion-loop corrections computed with $\mathcal{L}_{\text{INT}}^{\text{SM}}$ are matched onto $\mathcal{L}_{\text{INT}}^{\text{EFF}}$ by comparing amplitudes induced by charged currents of virtuality p^2 . In the renormalizable theory one has

$$\begin{aligned}
 A_W^{\text{SM}}(p^2) &= \frac{i\Gamma}{8} g^2 \Delta_W(p^2) \\
 &= \frac{i\Gamma}{8} \left\{ \frac{p^2}{g^2} - \frac{M^2}{g^2} - \frac{\Sigma_W(p^2)}{4} \right\}^{-1},
 \end{aligned} \tag{49}$$

while resumming the interaction as in Fig. 3, but with $p^2 \neq 0$, gives

$$A_W^{\text{EFF}}(p^2, L_R) = \frac{i\Gamma}{8} \left\{ -\frac{M^2}{g^2} - \frac{\Sigma_W(p^2)}{4} \right\}^{-1}. \tag{50}$$

Equations (49) and (50) differ by the term p^2/g^2 , so that inserting the solution (47) produces a result independent of L_R for A_W^{SM} , whilst A_W^{EFF} still depends on L_R ,

$$\frac{A_W^{\text{SM}}(p^2)}{K(\alpha_{EM})} = \left\{ 1 - \frac{p^2}{\hat{M}^2} - \frac{\alpha_{EM}}{\pi \hat{s}_\theta^2 \hat{M}^2} p^2 (K_1 - L + 5/3) \right\}^{-1}, \tag{51}$$

$$\frac{A_W^{\text{EFF}}(p^2, L_R)}{K(\alpha_{EM})} = \left\{ 1 - \frac{\alpha_{EM}}{\pi \hat{s}_\theta^2 \hat{M}^2} p^2 (L_R - L + 5/3) \right\}^{-1} \tag{52}$$

$$K(\alpha_{EM}) = -\frac{i\Gamma}{2} \frac{\pi \alpha_{EM}}{\hat{s}_\theta^2 \hat{M}^2}.$$

At fixed ℓ , the amplitudes in (51) and (52) are the right and left sides of the matching equation (8) needed to determine μ'_R . For instance, the conditions ensuring the validity of (29) can be verified by expanding up to the second order in $\lambda = p^2/\hat{M}^2$,

$$\begin{aligned}
 \frac{A_W^{\text{SM}}(p^2)}{K(\alpha)} &= 1 + \lambda \left(1 + \frac{\alpha}{\pi \hat{s}_\theta^2} (K_1 - L + 5/3) \right) \\
 &\quad + \lambda^2 \left(1 + \frac{\alpha}{\pi \hat{s}_\theta^2} (K_1 - L + 5/3) \right)^2 + \mathcal{O}(\lambda^3), \\
 \frac{A_W^{\text{EFF}}(p^2, L_R)}{K(\alpha)} &= 1 + \frac{\alpha \lambda}{\pi \hat{s}_\theta^2} (L_R - L + 5/3) + \frac{\alpha^2 \lambda^2}{\pi^2 \hat{s}_\theta^4} (L_R - L + 5/3)^2 + \mathcal{O}(\lambda^3),
 \end{aligned} \tag{53}$$

where $\alpha = \alpha_{EM}$. From (53) one reads off the nonzero coefficients⁷

$$\begin{aligned}
 A_{01}^{(1)} &= 1, & A_{02}^{(2)} &= 1, & A_{11}^{(1)} &= \frac{5/3 - L + K_1}{\pi \hat{s}_\theta^2}, \\
 A_{12}^{(2)} &= 2A_{11}^{(1)}, & A_{22}^{(2)} &= \left(A_{11}^{(1)} \right)^2, & B_{110}^{(1)} &= \frac{5/3 - L}{\pi \hat{s}_\theta^2}, \\
 B_{220}^{(2)} &= \left(B_{110}^{(1)} \right)^2, & B_{111}^{(1)} &= \frac{1}{\pi \hat{s}_\theta^2}, & B_{221}^{(2)} &= \frac{2}{\pi \hat{s}_\theta^2} B_{110}^{(1)}, \\
 B_{222}^{(2)} &= \left(B_{111}^{(1)} \right)^2,
 \end{aligned} \tag{54}$$

and the solution $X'_{-1} = \pi \hat{s}_\theta^2$, $X'_0 = K_1$, $X'_1 = 0$, namely

⁷ Since $N = 1$, $\{m_j\} = (j)$.

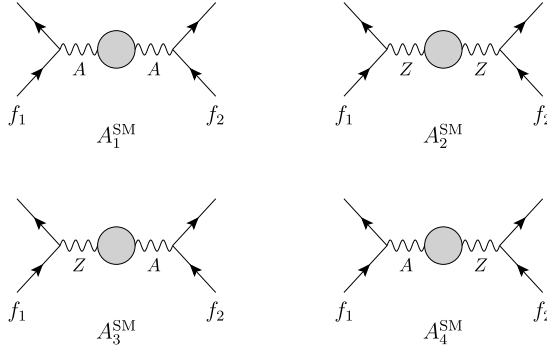


Fig. 4. The four sub-amplitudes in (57) induced by the fermion-loop dressed propagators of (41). The external fermions are massless, so that diagrams involving the exchange of neutral scalars are absent.

$$L'_R = \frac{\pi \hat{s}_\theta^2}{\alpha_{EM}} + K_1, \tag{55}$$

fulfills, for any value of j , all conditions stated by (27a) and (28). As a matter of fact, $L_R = L'_R$ solves (8) to all orders. In fact, this is the value for which the resummed amplitudes of (51) and (52) coincide. Hence, choosing the renormalization scale as in (55) reproduces the effect of interchanging a one-fermion-loop dressed W boson of arbitrary virtuality p^2 .

Now we consider a further amplitude A_Z^{EFF} obtained by contracting two neutral currents. It obeys (10) by construction and

Theorem 2. *When computed at $L_R = L'_R$, any effective amplitude involving two massless neutral currents reproduces, at any value of p^2 , the exact all-order result predicted by \mathcal{L}_{INT}^{SM} .*

So that, A_Z^{EFF} fulfills (9) at any ℓ .

Proof. Consider the full amplitude

$$A_Z^{SM}(p^2) = \sum_{k=1}^4 A_k^{SM}(p^2, L_R) \tag{56}$$

describing the interaction between two massless fermions f_1 and f_2 in the renormalizable theory. A computation of the sub-amplitudes in Fig. 4 gives

$$\begin{aligned} A_1^{SM}(p^2, L_R) &= ig^2 s_\theta^2 Q_{f_1} Q_{f_2} \Delta_A(p^2) \gamma_\alpha \otimes \gamma^\alpha, \\ A_2^{SM}(p^2, L_R) &= ig^2 \frac{1}{4c_\theta^2} \Delta_Z(p^2) \gamma_\alpha (v_{f_1} + a_{f_1} \gamma_5) \otimes \gamma^\alpha (v_{f_2} + a_{f_2} \gamma_5), \\ A_3^{SM}(p^2, L_R) &= ig^2 \frac{s_\theta Q_{f_2}}{2c_\theta} \Delta_{ZA}(p^2) \gamma_\alpha (v_{f_1} + a_{f_1} \gamma_5) \otimes \gamma^\alpha, \\ A_4^{SM}(p^2, L_R) &= ig^2 \frac{s_\theta Q_{f_1}}{2c_\theta} \Delta_{ZA}(p^2) \gamma_\alpha \otimes \gamma^\alpha (v_{f_2} + a_{f_2} \gamma_5). \end{aligned} \tag{57}$$

Since \mathcal{L}_{INT}^{SM} is renormalizable, $A_Z^{SM}(p^2)$ does not depend on L_R . Therefore, one is allowed to choose $L_R = L'_R$ in each of the four sub-amplitudes. But this implies $F_1 = 0$ in (48), which means

$p^2/g^2 = 0$ inside the function $P_Z(p^2)$ contained in the definition of the dressed propagators $\Delta_A(p^2)$, $\Delta_Z(p^2)$ and $\Delta_{ZA}(p^2)$ in (57). Since this is the only difference between the results computed within the nonrenormalizable and renormalizable models, one obtains

$$A_k^{\text{EFF}}(p^2, L'_R) = A_k^{\text{SM}}(p^2, L'_R) \quad \forall k. \tag{58}$$

Thus,

$$\begin{aligned} A_Z^{\text{EFF}}(p^2, L'_R) &= \sum_{k=1}^4 A_k^{\text{EFF}}(p^2, L'_R) \\ &= \sum_{k=1}^4 A_k^{\text{SM}}(p^2, L'_R) = A_Z^{\text{SM}}(p^2). \quad \square \end{aligned} \tag{59}$$

An interesting consequence is

Corollary 1. *In the renormalizable theory of (31) it is possible to rearrange the fermion-loop corrections in such a way that all fermions couple to Z and W bosons with the same V-A interaction.*

Proof. This is again obtained by choosing μ_R in (57) as in (55), that implies $s_{\theta}^2 = 0$ in (47a) and $v_f = -a_f = I_{3f}$ in (34). \square

To summarize, any exact amplitude, in which two massless fermion lines are connected by a one-fermion-loop dressed W , Z or γ propagator of arbitrary virtuality, is reproduced by $\mathcal{L}_{\text{INT}}^{\text{EFF}}$ if the solution in (55) is used for the renormalization scale.

Finally, it should be explicitly noticed that the choice of the interactions included in (36) is ultimately driven by the requirement that the effective and the exact model coincide, when $\lambda \rightarrow 0$, for the class of processes and corrections under study. For example, $\mathcal{L}_{\text{INT}}^{\text{EFF}}$ is too poor to accommodate contributions not induced by fermion loops, e.g. the amplitudes B_i in the l.h.s. of (10) would not match the A_i if the latter would involve three-gauge-boson vertices.

5. Comparing with customary calculations

In what follows, we use the model of (36) to compare our treatment with a more standard order-by-order renormalization approach based on Dimensional Regularization (DReg). Our formulae are converted to DReg by replacing [13]

$$L_R \rightarrow L_R + \frac{1}{\epsilon_{\text{UV}}}, \tag{60}$$

where

$$\frac{1}{\epsilon_{\text{UV}}} := \frac{2}{4-d} - \gamma_E - \ln \pi \quad \text{with } d \rightarrow 4. \tag{61}$$

Upon this substitution, the effective amplitudes in (52) and (58) develop a dependence on the UV cutoff $1/\epsilon_{\text{UV}}$. To cancel it in the Weinberg's way, one adds to the effective Lagrangian interactions induced by higher dimensional operators,

$$\mathcal{L}_{\text{HD}} = -c_w \frac{g^4}{32M^4} (\partial_\nu J_{c\alpha})^\dagger (\partial^\nu J_c^\alpha) - c_z \frac{g^4 c_\theta^2}{32M^4} (\partial_\nu J_{n\alpha}) (\partial^\nu J_n^\alpha). \quad (62)$$

Matching the exact results of (51) and (56) onto a computation performed with $\mathcal{L}_{\text{INT}}^{\text{EFF}} + \mathcal{L}_{\text{HD}}$ fixes the unknown coefficients,

$$c_w(L_R) = c_z(L_R) = \frac{\hat{s}_\theta^2}{\pi\alpha_{EM}} + \frac{1}{\pi^2} \left(K_1 - L_R - \frac{1}{\epsilon_{\text{UV}}} \right).$$

Even when choosing μ_R as in (55) only the finite parts of $c_{w,z}$ are removed,

$$c_{w,z}(L'_R) = -\frac{1}{\pi^2\epsilon_{\text{UV}}}, \quad (63)$$

hence adding \mathcal{L}_{HD} to $\mathcal{L}_{\text{INT}}^{\text{EFF}}$ is necessary to compensate the UV poles contained in the DReg variant of the one-loop functions of (15). Such poles are absent when defining UV divergent integrals as in (13). This explains why FDR circumvents the introduction of the counterterm Lagrangian \mathcal{L}_{HD} , which is instead needed in the standard method.⁸ It is also interesting to speculate about the FDR matching of (55) from the point of view of the sole EFT. In particular, would it be possible to guess the “right” value of μ_R without knowing $\mathcal{L}_{\text{INT}}^{\text{SM}}$? Requiring that $\mathcal{L}_{\text{INT}}^{\text{EFF}}$ describe as many processes as possible leads to the universal V-A interaction realized by the value $s_\theta^2 = 0$ implied by (55), as noted in Corollary 1. More than that, choosing $s_\theta^2 = 0$ effectively reduces from three to two the number of free parameters in (47). In summary, minimality could be used as a criterion to fix μ_R in nonrenormalizable QFTs whose UV completion is unknown. Note that, in any standard procedure based on DReg, s_θ^2 would be a bare parameter containing $1/\epsilon_{\text{UV}}$ poles, which cannot be compensated by any finite value of μ_R . Thus, setting $s_\theta^2 = 0$ directly in (47) would not be possible.

In the rest of this section we briefly outline the steps towards a possible generalization of our approach beyond the simple model of (36). Given the current interest in precise EFT analyses of collider data, we directly focus on a phenomenologically relevant problem by studying how new physics effects could be parameterized within the FDR framework at the NLO accuracy.⁹ To be definite, we consider the Lagrangian

$$\mathcal{L}_{\text{NP}} = \mathcal{L}_{\text{SM}}^{(4)} + \frac{g^2}{\Lambda^2} \mathcal{L}^{(6)}, \quad (64)$$

where $\mathcal{L}_{\text{SM}}^{(4)}$ is the full standard model bare Lagrangian and g is the $\text{SU}(2)_L$ coupling constant. $\mathcal{L}^{(6)}$ contains a set of gauge invariant dimension-six operators, multiplied by Wilson coefficients, which we want to determine, and Λ is the new physics scale, with which all the M_n in (2) are identified. The reader should be aware of the fact that a systematic and detailed treatment of

⁸ Note that FDR is not equivalent to DReg in which the loop integrals are redefined by dropping $1/\epsilon_{\text{UV}}$ terms. For instance, [8,14] when $\ell > 1$

$$\text{Finite Part} \left\{ \int d^d q \frac{\mu_R^{(4-d)}}{(q^2 - m^2)((q+p)^2 - m_1^2)} \right\}^\ell \neq \left(I_{\text{FDR}}^1(p^2, m^2, m_1^2) \right)^\ell,$$

with $I_{\text{FDR}}^1(p^2, m^2, m_1^2)$ given in (15a). In DReg this mismatch is cured by the $1/\epsilon_{\text{UV}}$ pole contained in \mathcal{L}_{HD} . Hence, setting $\mathcal{L}_{\text{HD}} = 0$ would give a wrong DReg result for the resummed propagators of (42).

⁹ This means including all corrections $\mathcal{O}(g^2)$, $\mathcal{O}(\lambda_n)$ and $\mathcal{O}(g^2\lambda_n)$ with respect to the lowest order standard model predictions.

this problem is far beyond our scope. Here we simply want to point out the general qualitative differences with respect to more standard approaches.

Equation (64) very much resembles the customary SMEFT [15] dimension-six parameterization. However, in our case the operators in $\mathcal{L}^{(6)}$ are not necessarily closed under renormalization. For instance, they could be a sub-set of the operators of the Warsaw basis [16]. Furthermore, \mathcal{L}_{NP} remains the same at all loop orders (see footnote 1). Before starting the calculation, one needs to expand $\mathcal{L}^{(6)}$ around the Higgs vacuum expectation value v . This gives rise to powers of v/Λ that modify the relations connecting weak eigenstates to mass eigenstates and alter the gauge fixing needed to quantize \mathcal{L}_{NP} . An analogous problem is encountered in the SMEFT, and can be solved, for instance, as described in [17].¹⁰ A difference arises when the v/Λ terms generate contact interactions not present in $\mathcal{L}_{\text{SM}}^{(4)}$. In this case they should be included in the factor $K(\alpha)$ of (17). This is due to the fact that the expansion in (16b) is in terms of the λ_n .

The starting point to determine the Wilson coefficients and Λ is a set of observables \mathcal{O}_i , $i \geq m + 1$, for which there is an experimental agreement, when all $\lambda_n \rightarrow 0$, with the theoretical predictions obtained with \mathcal{L}_{NP} .¹¹ This may require to fit different compositions of the dimension-six operators in $\mathcal{L}^{(6)}$ until this agreement is reached. After this is achieved, one measures one of the observables, say \mathcal{O}_{m+1} , at small values of the λ_n and tries to determine $X'_{-1,0}$ in (21) such that the agreement persists also when $\lambda_n \neq 0$. Note that, when several λ_n are involved, this may require measuring \mathcal{O}_{m+1} in different phase-space regions. If the $\lambda_n \neq 0$ agreement is not reached, one is led to reconsider once again the combination of dimension-six operators in $\mathcal{L}^{(6)}$. When X'_{-1} and X'_0 can be found, the theory is fixed and our conjecture states that all the other observables \mathcal{O}_i , $i > m + 1$, are also reproduced by \mathcal{L}_{NP} . If necessary, this can be checked experimentally.

6. Conclusion

We have derived the order-by-order conditions which have to be fulfilled by effective amplitudes computed in FDR to reproduce exact high-energy predictions. In our procedure the Lagrangian of the effective model is not modified by the inclusion of higher dimensional operators. At the core of our analysis lies an expansion of the renormalization scale μ_R that mixes different perturbative orders.

We have postulated that if there exist classes of amplitudes for which the effective and the exact theory coincide at low energies, and if a value of μ_R can be found, for one of them, that matches at higher energies the exact result onto the effective one, all the other effective amplitudes computed at μ_R reproduce the exact high-energy predictions.

We have proven this explicitly to all loop orders by matching onto the Fermi model electroweak processes induced by the exchange of a one-fermion-loop dressed W , Z or γ propagator of arbitrary virtuality. In such a situation our approach is more direct than a standard EFT calculation, and gives some hints on how to handle nonrenormalizable models when more fundamental theories are not known.

We plan to corroborate our conjecture by considering further classes of theories and corrections in future investigations.

¹⁰ Alternatively, since our matching conditions only involve physical amplitudes, one can use any gauge expressed in terms of the bare fields in $\mathcal{L}_{\text{SM}}^{(4)}$, at the price of correcting the external particle wave functions such that propagators have residue one at their poles [18].

¹¹ Adding real corrections might be needed at this stage to define infrared safe quantities.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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