CHARACTERIZING PROJECTIONS AMONG POSITIVE **OPERATORS IN THE UNIT SPHERE**

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ABSTRACT. Let E and P be subsets of a Banach space X, and let us define the unit sphere around E in P as the set

 $Sph(E; P) := \{x \in P : ||x - b|| = 1 \text{ for all } b \in E\}.$

Given a C*-algebra A, and a subset $E \subset A$, we shall write $Sph^+(E)$ or $Sph^+_A(E)$ for the set $Sph(E; S(A^+))$, where $S(A^+)$ stands for the set of all positive operators in the unit sphere of A. We prove that, for an arbitrary complex Hilbert space H, the following statements are equivalent for every positive element ain the unit sphere of B(H):

(a) a is a projection;

(b) $Sph_{B(H)}^{+}(Sph_{B(H)}^{+}(\{a\})) = \{a\}.$ We also prove that the equivalence remains true when B(H) is replaced with an atomic von Neumann algebra or with $K(H_2)$, where H_2 is an infinitedimensional and separable complex Hilbert space. In the setting of compact operators we prove a stronger conclusion by showing that the identity

$$Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \left\{b \in S(K(H_2)^+) : \frac{s_{K(H_2)}(a) \le s_{K(H_2)}(b)}{1 - r_{B(H_2)}(a) \le 1 - r_{B(H_2)}(b)}\right\},\$$

holds for every a in the unit sphere of $K(H_2)^+$, where $r_{B(H_2)}(a)$ and $s_{K(H_2)}(a)$ stand for the range and support projections of a in $B(H_2)$ and $K(H_2)$, respectively.

1. INTRODUCTION

In a recent attempt to solve a variant of Tingley's problem for surjective isometries of the set formed by all positive operators in the unit sphere of $M_n(\mathbb{C})$, the space of all $n \times n$ complex matrices endowed with the spectral norm, G. Nagy has established an interesting characterization of those positive norm-one elements in $M_n(\mathbb{C})$ which are projections (see the final paragraph in the proof of [11, Claim 1). Motivated by the terminology employed by Nagy in the just quoted paper, we introduce here the notion of *unit sphere around a subset* in a Banach space. Let E and P be subsets of a Banach space X. We define the unit sphere around E in P as the set

 $Sph(E; P) := \{x \in P : ||x - b|| = 1 \text{ for all } b \in E\}.$

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If x is an element in X, we write Sph(x; P) for $Sph(\{x\}; P)$. Henceforth, given a Banach space X, let S(X) denote the unit sphere of X. The cone of positive elements in a C^{*}-algebra A will be denoted by A^+ . If M is a subset of X, we shall write S(M) for $M \cap S(X)$. To simplify the notation, given a C^{*}-algebra A, and a subset $E \subset A$, we shall write $Sph^+(E)$ or $Sph^+_A(E)$ for the set $Sph(E; S(A^+))$. For each element a in A, we shall write $Sph^+(a)$ instead of $Sph^+(\{a\})$.

Let a be a positive norm-one element in $B(\ell_2^n) = M_n(\mathbb{C})$. The commented characterization established by Nagy proves that the following two statements are equivalent:

(i)
$$a$$
 is a projection
(ii) $Sph^{+}_{M_{n}(\mathbb{C})}\left(Sph^{+}_{M_{n}(\mathbb{C})}(a)\right) = \{a\},$ (1.1)

(see the final paragraph in the proof of [11, Claim 1]). As remarked by G. Nagy in [11, §3], the previous characterization (and the whole statement in [11, Claim 1]) remains as an open problem when H is an arbitrary complex Hilbert space. This is an interesting problem to be considered in operator theory, and in the wider setting of general C^{*}-algebras.

In this note we extend the characterization in (1.1) to the case in which H is an arbitrary complex Hilbert space. In a first result we prove that, for any positive element a in the unit sphere of a C*-algebra A, the equality $Sph_A^+(Sph_A^+(a)) = \{a\}$ is a sufficient condition to guarantee that a is a projection in A (cf. Proposition 2.2). In Theorem 2.3 we extend Nagy's characterization to the setting of atomic von Neumann algebras by showing that the following statements are equivalent for every positive norm-one element a in an atomic von Neumann algebra M (in particular when M = B(H), where H is an arbitrary complex Hilbert space):

- (a) a is a projection;
- (b) $Sph_M^+(Sph_M^+(a)) = \{a\}.$

We shall also explore whether the above characterization also holds when M is replaced with K(H), the space of all compact operators on a complex Hilbert space H. Our conclusion in this case is the following: Let H_2 be a separable complex Hilbert space, and suppose that a is a positive norm-one element in $K(H_2)$. Then the following statements are equivalent:

- (a) a is a projection;
- (b) $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}.$

When H is a finite-dimensional complex Hilbert space Nagy computed in [11] the second unit sphere around a positive element in the unit sphere of $B(H)^+$, and showed that the identity

$$Sph_{B(H)}^{+}\left(Sph_{B(H)}^{+}(a)\right) = \left\{b \in S(B(H)^{+}): \begin{array}{c} \operatorname{Fix}(a) \subseteq \operatorname{Fix}(b), \\ \operatorname{and} \operatorname{ker}(a) \subseteq \operatorname{ker}(b)\end{array}\right\}$$

holds for every element a in $S(B(H)^+)$, where for each a in $S(B(H)^+)$ we set $Fix(a) = \{\xi \in H : a(\xi) = \xi\}$, (see the beginning of the proof of [11, Claim 1]). In Theorem 2.8 we establish a generalization of this fact to be setting of compact

operators. We prove that if H_2 is a separable infinite-dimensional complex Hilbert space, then the identity

$$Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \left\{b \in S(K(H_2)^+) : \frac{s_{K(H_2)}(a) \le s_{K(H_2)}(b)}{1 - r_{B(H_2)}(a) \le 1 - r_{B(H_2)}(b)}\right\},\$$

holds for every a in the unit sphere of $K(H_2)^+$, where $r_{B(H_2)}(a)$ and $s_{K(H_2)}(a)$ stand for the range and support projections of a in $B(H_2)$ and $K(H_2)$, respectively.

As we have already commented at the beginning of this introduction, the characterization obtained by Nagy in (1.1) is one of the key results to establish that every surjective isometry $\Delta : S(M_n(\mathbb{C})^+) \to S(M_n(\mathbb{C})^+)$ admits an extension to a surjective real linear or complex linear isometry on $M_n(\mathbb{C})$ (see [11, Theorem]). Another related results are known when $M_n(\mathbb{C}) = B(\ell_2^n)$ is replaced with the space $(C_p(H), \|\cdot\|_p)$ of all *p*-Schatten-von Neumann operators on a complex Hilbert space *H*, with $1 \leq p < \infty$. L. Molnár and W. Timmermann proved that for every complex Hilbert space *H*, every surjective isometry $\Delta : S(C_1(H)^+) \to S(C_1(H)^+)$ can be extended to a surjective complex linear isometry on $C_1(H)$. Nagy shows in [10, Theorem 1] that the same conclusion remains true for every 1 .

The results commented in the previous paragraph are subtle variants of the so-called Tingley's problem. This problem asks whether every surjective isometry between the unit spheres of two Banach spaces X and Y admits an extension to a surjective real linear isometry from X onto Y. Tingley's problem remains open after thirty years. However, in what concerns operator algebras, certain positive solutions to this problem have been recently established in the setting of finite-dimensional C*-algebras and finite von Neumann algebras [17, 18], spaces of compact linear operators and compact C*-algebras [14], B(H) spaces [4], von Neumann algebras [6], spaces of trace class operators [1], preduals of von Neumann algebras [9], and spaces of p-Schatten von Neumann operators with 2 [2]. The reader is referred to the survey [13] for additional details.

After completing the description of all surjective isometries on $S(M_n(\mathbb{C})^+)$, Nagy conjectured that a similar result should also hold for surjective surjective isometries on $S(B(H)^+)$, where H is an arbitrary complex Hilbert space (see [11, §3]). The results presented in this note are a first step towards a proof of Nagy's conjecture.

2. The results

Let us fix some notation. Along the paper, the closed unit ball and the dual space of a Banach space X will be denoted by \mathcal{B}_X and X^* , respectively. Given a subset $B \subset X$, we shall write \mathcal{B}_B for $\mathcal{B}_X \cap B$.

The cone of positive elements in a C^{*}-algebra A will be denoted by A^+ , while the symbol $(A^*)^+$ will stand for the set of positive functionals on A. A state of Ais a positive functional in $S(A^*)$. The set of states of A will be denoted by \mathcal{S}_A . It is well known that $\mathcal{B}_{(A^*)^+} = \mathcal{B}_{A^*} \cap (A^*)^+$ is a weak*-closed convex subset of \mathcal{B}_{A^*} . The set of *pure states* of A is precisely the set $\partial_e(\mathcal{B}_{(A^*)^+})$ of all extreme points of $\mathcal{B}_{(A^*)^+}$ (see [12, §3.2]).

Suppose a is a positive element in the unit sphere of a von Neumann algebra M. The range projection of a in M (denoted by r(a)) is the smallest projection p in M satisfying ap = a. It is known that the sequence $((1/n\mathbf{1} + a)^{-1}a)_n$ is monotone increasing to r(a), and hence it converges to r(a) in the weak*-topology of M. Actually, r(a) also coincides with the weak*-limit of the sequence $(a^{1/n})_n$ in M (see [12, 2.2.7]). It is also known that the sequence $(a^n)_n$ converges to a projection $s(a) = s_M(a)$ in M, which is called the support projection of a in M. Unfortunately, the support projection of a norm-one element in M might be zero. For example, let $\{\xi_n : n \in \mathbb{N}\}$ denote an orthonormal basis of ℓ_2 , and let a be the positive operator in $B(\ell_2)$ given by $a = \sum_{m=1}^{\infty} \frac{m-1}{m} p_m$, where, for each m, p_m is the rank one projection $\xi_m \otimes \xi_m$. It is not hard to check that $s_{B(\ell_2)}(a) = 0$.

Elements a, b in a C*-algebra A are called orthogonal (written $a \perp b$) if $ab^* = b^*a = 0$. It is known that $||a + b|| = \max\{||a||, ||b||\}$, for every $a, b \in A$ with $a \perp b$. Clearly, self-adjoint elements $a, b \in A$ are orthogonal if and only if ab = 0.

We recall some geometric properties of C^{*}-algebras. Let p be a projection in a unital C^{*}-algebra A. Suppose that $x \in S(A)$ satisfies pxp = p, then

$$x = p + (1 - p)x(1 - p),$$
(2.1)

(see, for example, [5, Lemma 3.1]). Another property needed later reads as follows: Suppose that $b \in A^+$ satisfies pbp = 0, then

$$pb = bp = 0$$
, equivalently, $p \perp b$. (2.2)

To see this property let us take a positive $c \in A$ satisfying $c^2 = b$. The identity $0 \leq (pc)(pc)^* = pc^2p = pbp = 0$ and the Gelfand-Naimark axiom imply that pc = cp = 0, and hence $pb = pc^2 = 0 = c^2p = bp$.

A non-zero projection p in a C*-algebra A is called minimal if $pAp = \mathbb{C}p$. A von Neumann algebra M is called atomic if it coincides with the weak* closure of the linear span of its minimal projections. It is known from the structure theory of von Neumann algebras that every atomic von Neumann algebra M can be written in the form $M = \bigoplus_{j=1}^{\ell_{\infty}} B(H_j)$, where each H_j is a complex Hilbert space (compare [15, §2.2] or [16, §V.1]).

Let p be a non-zero projection in an atomic von Neumann algebra $M = \bigoplus_{j=1}^{\ell_{\infty}} B(H_j)$. In this case we can always find a family (q_{λ}) of mutually orthogonal minimal projections in M such that $p = w^* - \sum_{\lambda} q_{\lambda}$ (compare [15, Definition 1.13.4]). Furthermore, p is the least upper bound of the set of all minimal projections in M which are smaller than or equal to p.

The bidual, A^{**} , of a C^{*}-algebra A is a von Neumann algebra whose predual contains an abundant collection of pure states of A. This geometric advantage

implies that the support projection in A^{**} of every element in $S(A^+)$ is a nonzero projection. Namely, if a lies in $S(A^+)$ it is well known that we can find a pure state $\phi \in \partial_e(\mathcal{B}_{(A^*)^+})$ satisfying $\phi(a) = 1$. Pure states in A^* are in one-toone correspondence with minimal projections in A^{**} , more concretely, for each $\phi \in \partial_e(\mathcal{B}_{(A^*)^+})$ there exists a unique minimal partial isometry $p_{\phi} \in A^{**}$ satisfying $\phi(p_{\phi}) = 1$, and $p_{\phi}xp_{\phi} = \phi(x)p_{\phi}$ for all $x \in M$ (see [12, Proposition 3.13.6]). The projection p_{ϕ} is called the *support projection* of ϕ . Since A is weak*-dense in A^{**} , and the product of the latter von Neumann algebra is separately weak*continuous (see [12, Proposition 3.6.2 and Remark 3.6.5] or [15, Theorem 1.7.8]), it can be easily seen that every minimal projection in A is minimal in A^{**} .

Let *a* be a positive norm-one element in a C*-algebra *A*. Let us take an state $\phi \in S_A$ satisfying $\phi(a) = 1$ (compare [15, Proposition 1.5.4 and its proof]). The set $\{\psi \in \mathcal{B}_{(A^*)^+} : \psi(a) = 1\}$ is a non-empty weak* closed convex subset of \mathcal{B}_{A^*} . By the Krein-Milman theorem there exists $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ belonging to the previous set, and hence $\varphi(a) = 1$. We consider the support projection p_{φ} of φ in A^{**} , which is a minimal projection. The condition $\varphi(a) = 1$ implies $p_{\varphi} = p_{\varphi} a p_{\varphi}$, and (2.1) assures that $a = p_{\varphi} + (1 - p_{\varphi})a(1 - p_{\varphi})$, and thus $0 \neq p_{\varphi} \leq s_{A^{**}}(a)$. We can therefore deduce that

$$s_{A^{**}}(a) \neq 0$$
, for all $a \in S(A^+)$. (2.3)

In order to recall the connections with Nagy's paper, we observe that, given a norm-one positive operator a in B(H), we denote $\operatorname{Fix}(a) = \{\xi \in H : a(\xi) = \xi\}$, and we write p_a for the projection of H onto $\operatorname{Fix}(a)$. Since $a = p_a + (\mathbf{1} - p_a)a(\mathbf{1} - p_a)$, it follows that p_a is smaller than or equal to the support projection of a in $B(H)^{**}$. In some cases, p_a may be zero while $s_{B(H)^{**}}(a) \neq 0$. When H is finite dimensional p_a and s(a) coincide. If we take a positive norm-one element in the space K(H) of all compact operators on H, the element $s_{B(H)}(a) = s_{K(H)^{**}}(a) = p_a$ is a (non-zero) finite rank projection and lies in K(H). We shall write $s_{K(H)}(a)$.

If p is a non-zero projection in a C*-algebra A then

for each
$$a$$
 in $S(A^+)$ such that $p \le a$, we have $a = p + (\mathbf{1} - p)a(\mathbf{1} - p)$. (2.4)

Namely, under the above hypothesis, we also have $p \leq a$ in the von Neumann algebra A^{**} . It follows that $p \leq s_{A^{**}}(a) \leq a$, and hence $s_{A^{**}}(a) - p$ is a projection in A^{**} which is orthogonal to p. Since $a = s_{A^{**}}(a) + (\mathbf{1} - s_{A^{**}}(a))a(\mathbf{1} - s_{A^{**}}(a))$, we have $pap = ps_{A^{**}}(a)p = p$, and thus $a = p + (\mathbf{1} - p)a(\mathbf{1} - p)$ (compare (2.1)).

It is part of the folklore in the theory of C*-algebras that the distance between two positive elements a, b in the closed unit ball of a C*-algebra A is bounded by one. Namely, since $-\mathbf{1} \leq -b \leq a - b \leq a \leq \mathbf{1}$, we deduce that $||a - b|| \leq 1$.

In our first result, which is an infinite-dimensional version of [11, Corollary], we establish a precise description of those pairs of elements in $S(A^+)$ whose distance is exactly one.

Lemma 2.1. Let A be a C*-algebra, and let a, b be elements in $S(A^+)$. Then ||a - b|| = 1 if and only if there exists a minimal projection e in A^{**} such that one of the following statements holds:

(a) $e \leq a$ and $e \perp b$ in A^{**} ; (b) $e \leq b$ and $e \perp a$ in A^{**} .

 $(0) \ c \le 0 \ ana \ c \pm a \ m m m$

Proof. Let us first assume that ||a - b|| = 1. Arguing as in the proof of (2.3), we can find $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ such that $|\varphi(a - b)| = 1$. Since $0 \leq \varphi(a), \varphi(b) \leq 1$, we can deduce that precisely one of the following holds:

- (a) $\varphi(a) = 1$ and $\varphi(b) = 0$;
- (b) $\varphi(b) = 1$ and $\varphi(a) = 0$.

Let $e = p_{\varphi}$ be the minimal projection in A^{**} associated to the pure state φ . In case (a) we know that eae = e and ebe = 0. Thus, by (2.1) and (2.2) it follows that $a = e + (\mathbf{1} - e)a(\mathbf{1} - e) \ge e$ and $b \perp e$ in A^{**} . Similar arguments show that in case (b) we get $e \le b$ and $e \perp a$ in A^{**} .

Suppose now that we can find a minimal projection e in A^{**} satisfying (a) or (b) in the statement of the lemma. We shall only consider the case in which statement (a) holds, the other case is identical. Let φ be the pure state in A^* associated with e. Since $a = e + (\mathbf{1} - e)a(\mathbf{1} - e)$ and $b = (\mathbf{1} - e)b(\mathbf{1} - e)$ in A^{**} we obtain $\varphi(a - b) = \varphi(e) = 1 \le ||a - b|| \le 1$.

We are now in position to establish a sufficient condition in terms of the set $Sph_A^+(Sph_A^+(a))$, to guarantee that a positive norm-one element a in a C*-algebra A is a projection.

Proposition 2.2. Let A be a C^{*}-algebra, and let a be a positive norm-one element in A. Suppose $Sph_{A}^{+}(Sph_{A}^{+}(a)) = \{a\}$. Then a is a projection.

Proof. Let $\sigma(a)$ denote the spectrum of a. We identify the C*-subalgebra of A generated by a with the commutative C*-algebra $C_0(\sigma(a))$ of all continuous functions on $\sigma(a) \cup \{0\}$ vanishing at 0. Fix an arbitrary function $c \in C_0(\sigma(a))$ with $0 \le c \le 1$, c(0) = 0 and c(1) = 1. We claim that any such element c satisfies the following properties:

(P1) If q is a minimal projection in A^{**} with $q \leq a$, then $q \leq c$ in A^{**} ; (P2) If q is a projection in A^{**} , with $q \perp a = 0$ then qc = 0.

We shall next prove the claim. (P1) Let q be a minimal projection in A^{**} with $q \leq a$. Let $\varphi \in \partial_e(\mathcal{B}_{(A^*)^+})$ be a pure state of A satisfying $\varphi(q) = 1$. In this case a = q + (1-q)a(1-q) in A^{**} . This proves that $s_{A^{**}}(a) = q + s_{A^{**}}((1-q)a(1-q)) \geq q$ in A^{**} . The element c has been defined to satisfy $s_{C_0(\sigma(a))^{**}}(a) \leq s_{C_0(\sigma(a))^{**}}(c)$. Since $C_0(\sigma(a))^{**}$ can be identified with the weak* closure of $C_0(\sigma(a))^{**}$ in A^{**} , we can actually conclude that $q \leq s_{A^{**}}(a) = s_{C_0(\sigma(a))^{**}}(a) \leq s_{C_0(\sigma(a))^{**}}(c) = s_{A^{**}}(c)$. This implies that $\varphi(c) = 1$ and hence $q \leq c$ in A^{**} .

(P2) Any element in A^{**} which is orthogonal to a must be orthogonal to every element in $C_0(\sigma(a))$, because the latter is the C*-subalgebra of A generated by a. This finishes the proof of the claim.

By Lemma 2.1, an element x lies in $Sph_A^+(a)$ if and only if there exists a minimal projection e in A^{**} such that one of the following statements holds:

(a) e < a and $e \perp x$ in A^{**} ;

(b) $e \leq x$ and $e \perp a$ in A^{**} .

In case (a), $e \perp x$ and $e \leq c$ by (P1), and Lemma 2.1 implies that ||x - c|| = 1. In case (b), $e \leq x$ and $e \perp a$, and hence $e \perp c$ by (P2). Lemma 2.1 implies that ||x - c|| = 1.

We have proved that, any function $c \in C_0(\sigma(a))$ with $0 \le c \le 1$, c(0) = 0 and c(1) = 1 belongs to $Sph_A^+(Sph_A^+(a)) = \{a\}$, which forces to $\sigma(a) = \{0, 1\}$, and hence a is a projection.

The promised characterization of non-zero projections in an atomic von Neumann algebra is established next.

Theorem 2.3. Let M be an atomic von Neumann algebra, and let a be a positive norm-one element in M. Then the following statements are equivalent:

- (a) a is a projection;
- (b) $Sph_{M}^{+}(Sph_{M}^{+}(a)) = \{a\}.$

Proof. $(a) \Rightarrow (b)$ Suppose a = p is a projection. Clearly

$$\{p\} \subseteq Sph_M^+ \left(Sph_M^+(p)\right).$$

Let us take b in the set $Sph_M^+(Sph_M^+(p))$. We shall first prove that $1-p \perp b$. If 1-p=0 there is nothing to prove. Otherwise, let e be a minimal projection in M with $e \leq 1-p$. Since $||e+\frac{1}{2}(1-e)-p|| = 1$, we deduce that $||e+\frac{1}{2}(1-e)-b|| = 1$.

Lemma 2.1 proves the existence of a minimal projection $q \in M^{**}$ such that one of the next statements holds:

- (1) $q \le e + \frac{1}{2}(1-e)$ and $q \perp b$ in M^{**} ; (2) $q \le b$ and $q \perp e + \frac{1}{2}(1-e)$ in M^{**} .

We claim that case (2) is impossible. Indeed, $q \perp e + \frac{1}{2}(1-e)$ is equivalent to $q \perp r_{M^{**}}(e + \frac{1}{2}(1-e)) = 1$, which is impossible. Therefore, only case (1) holds, and thus $q \leq e$. Since e also is a minimal projection in M^{**} , we deduce from the minimality of q that $e = q \perp b$.

We have shown that for every minimal projection e in M with $e \leq 1 - p$ we have $e \perp b$. Since 1 - p is the least upper bound of all minimal projections q in M with $q \leq 1 - p$ (actually $1 - p = \sum_{i} e_{j}$ where $\{e_{j}\}$ is a family of mutually orthogonal minimal projections in M), it follows that $1 - p \perp b$ (equivalently,

pb = bp = b).

We shall next prove that b is a projection and p = b. Let $\sigma(b)$ be the spectrum of b, let \mathcal{C} denote the C^{*}-subalgebra of M generated by b and p, and let us identify \mathcal{C} with $C(\sigma(b))$, b with the function $t \mapsto t$, and p with the unit of \mathcal{C} . We shall distinguish two cases:

(i) $0 \notin \sigma(b)$ (that is, b is invertible in \mathcal{C});

(*ii*) $0 \in \sigma(b)$ (that is, b is not invertible in C).

We deal first with case (i). If $0 \notin \sigma(b)$, let m_0 be the minimum of $\sigma(b)$. If $0 < m_0 < 1$, we consider the function $d \in \mathcal{C} \equiv C(\sigma(b))$ defined by $d(t) = \frac{1}{1-m_0}(t-m_0)$, $(t \in \sigma(b))$. It is not hard to check that $0 \leq ||b - d|| = m_0 < 1$ and ||p - d|| = 1, which contradicts that $b \in Sph_M^+(Sph_M^+(p))$. Therefore $m_0 = 1$, and hence b is invertible with $\sigma(b) = \{1\}$, witnessing that $\mathbf{1} = b \leq p \leq \mathbf{1}$. We have proved that $b = p = \mathbf{1}$.

In case (*ii*), $0 \in \sigma(b)$. If there exists $t_0 \in \sigma(b) \cap (0, 1)$, the function

$$c(t) = \begin{cases} 0 & \text{if } t \in \sigma(b) \cap [0, t_0];\\ \frac{1+t_0}{1-t_0}(t-t_0) & \text{if } t \in \sigma(b) \cap [t_0, \frac{1+t_0}{2}];\\ t & \text{if } t \in \sigma(b) \cap [\frac{1+t_0}{2}, 1], \end{cases}$$
(2.5)

defines a positive norm-one element in $c \in C(\sigma(b))$ such that ||p - c|| = 1, and $||b - c|| = t_0 < 1$. This contradicts that $b \in Sph_M^+(Sph_M^+(p))$. Therefore, $\sigma(b) \subseteq \{0, 1\}$, and hence b is a projection. If b < p, we get ||b - b|| = 0 and ||p - b|| = 1, contradicting that $b \in Sph_M^+(Sph_M^+(p))$. Therefore p = b.

We have shown that $Sph_M^+(Sph_M^+(p)) = \{p\}.$

The implication $(b) \Rightarrow (a)$ follows from Proposition 2.2.

The next result is a clear consequence of our previous theorem and extends the characterization of projections in $M_n(\mathbb{C})$ established by G. Nagy in the final paragraph of the proof of [11, Claim 1] (compare (1.1)).

Corollary 2.4. Let H be an arbitrary complex Hilbert space, and let a be a positive norm-one element in B(H). Then the following statements are equivalent:

(a) a is a projection; (b) $Sph^+_{B(H)}\left(Sph^+_{B(H)}(a)\right) = \{a\}.$

It seems natural to ask whether the above corollary remains true if B(H) is replaced with K(H). For an infinite-dimensional separable complex Hilbert space H_2 , the conclusion of Theorem 2.3 and Corollary 2.4 can be also extended to projections in the space $K(H_2)$. The arguments in the proof of Theorem 2.3 actually require a subtle adaptation.

Theorem 2.5. Let a be a positive norm-one element in $K(H_2)$, where H_2 is a separable complex Hilbert space. Then the following statements are equivalent:

- (a) a is a projection;
- (b) $Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right) = \{a\}.$

Proof. When H_2 is finite-dimensional the equivalence is proved in [11, final paragraph of the proof of Claim 1]. We can therefore assume that H_2 is infinitedimensional.

 $(a) \Rightarrow (b)$ We assume first that $a = p \in K(H_2)$ is a projection. We can find a family $\{q_1, \ldots, q_n\}$ of mutually orthogonal minimal projections in K(H) such

that
$$p = \sum_{j=1}^{n} q_j$$
. As before, the inclusion

$$\{p\} \subseteq Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(p)\right)$$

always holds. Let us take b in the set $Sph_{K(H_2)}^+(Sph_{K(H_2)}^+(p))$. Clearly $0 \neq \mathbf{1} - p \notin K(H_2)$. Let e be a minimal projection in $K(H_2)$ with $e \leq \mathbf{1} - p$ in $B(H_2)$. Since H_2 is separable, we can pick a maximal family $\{v_n : n \in \mathbb{N}\}$ of mutually orthogonal minimal projections in $(\mathbf{1} - e)K(H_2)(\mathbf{1} - e)$ with $\mathbf{1} - e = \sum_{n=1}^{\infty} v_n$.

The element
$$e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$$
 lies in $S(K(H_2)^+)$ and $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n - p \right\| = 1$, thus,

the hypothesis on *b* implies that $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n - b \right\| = 1$. Lemma 2.1 proves the existence of a minimal projection $q \in K(H_2)^{**} = B(H_2)$ such that one of the next statements holds:

(1)
$$q \leq e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$$
 and $q \perp b$ in $K(H_2)^{**} = B(H_2)$;
(2) $q \leq b$ and $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ in $K(H_2)^{**} = B(H_2)$.
In case (2), $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ and hence $q \perp e, v_n$ for all n , which proves that
 $q \perp e + \sum_{n=1}^{\infty} v_n = \mathbf{1}$ in $B(H_2)$, which is impossible. Therefore, case (1) holds, and
thus $q \leq e$. Since e is a minimal projection in $K(H_2)^{**} = B(H_2)$, we deduce from

thus $q \leq e$. Since e is a minimal projection in $K(H_2)^{**} = B(H_2)$, we deduce from the minimality of q that $e = q \perp b$.

We have shown that for every minimal projection e in $B(H_2)$ with $e \leq 1 - p$ we have $e \perp b$, and then $1 - p \perp b$ (equivalently, pb = bp = b).

The above arguments show that $b, p \in pK(H_2)p \cong M_n(\mathbb{C})$. Furthermore, every $x \in Sph_{pK(H_2)p}^+(a)$ lies in $Sph_{K(H_2)}^+(a)$ and hence ||b - x|| = 1, therefore b lies in $Sph_{pK(H_2)p}^+(Sph_{pK(H_2)p}^+(p))$. It follows from [11, final paragraph of the proof of Claim 1] (see also (1.1)) that $Sph_{pK(H_2)p}^+(Sph_{pK(H_2)p}^+(p)) = \{p\}$, and hence b = p. Therefore, $Sph_{K(H_2)}^+(Sph_{K(H_2)}^+(p)) = \{p\}$.

The implication $(b) \Rightarrow (a)$ follows from Proposition 2.2.

Many consequences can be expected from the characterizations established in Theorem 2.3 and Corollary 2.4. We shall conclude this note with a first application. For a C^{*}-algebra A, let $\mathcal{P}roj(A)^*$ denote the set of all non-zero projections in A. The next result is an infinite-dimensional version of [11, Claim 1] which proves one of the conjectures posed at the end of the just quoted paper.

Corollary 2.6. Let $\Delta : S(M^+) \to S(N^+)$ be a surjective isometry, where M and N are atomic von Neumann algebras. Then Δ maps $\operatorname{Proj}(M)^*$ onto $\operatorname{Proj}(N)^*$, and the restriction $\Delta|_{\operatorname{Proj}(M)^*} : \operatorname{Proj}(M)^* \to \operatorname{Proj}(N)^*$ is a surjective isometry.

Proof. Let p be a non-zero projection in M. Applying Theorem 2.3 we have $Sph_M^+(Sph_M^+(p)) = \{p\}$. Since Δ is a surjective isometry, the sphere around a set $E \subset S(M^+)$, $Sph_M^+(E)$, is always preserved by Δ , that is, $\Delta(Sph_M^+(E)) = Sph_N^+(\Delta(E))$. We consequently have

$$\{\Delta(p)\} = \Delta(\{p\}) = \Delta\left(Sph_M^+\left(Sph_M^+(p)\right)\right) = Sph_N^+\left(Sph_N^+(\Delta(p))\right),$$

and a new application of Theorem 2.3 assures that $\Delta(p)$ is a projection in N.

We have shown that $\Delta(\mathcal{P}roj(M)^*) \subseteq \mathcal{P}roj(N)^*$. Since Δ^{-1} also is a surjective isometry, we get $\Delta(\mathcal{P}roj(M)^*) = \mathcal{P}roj(N)^*$. Clearly $\Delta|_{\mathcal{P}roj(M)^*} : \mathcal{P}roj(M)^* \to \mathcal{P}roj(N)^*$ is a surjective isometry. \Box

When in the previous proof we replace Theorem 2.3 with Theorem 2.5 the same arguments are valid to prove the following:

Corollary 2.7. Let H_2 and H_3 be separable complex Hilbert spaces, and let us assume that $\Delta : S(K(H_2)^+) \to S(K(H_3)^+)$ is a surjective isometry. Then Δ maps $\mathcal{P}roj(K(H_2))^*$ to $\mathcal{P}roj(K(H_3))^*$, and the restriction

$$\Delta|_{\mathcal{P}roj(K(H_2))^*}: \mathcal{P}roj(K(H_2))^* \to \mathcal{P}roj(K(H_3))^*$$

is a surjective isometry.

Another result established by G. Nagy in [11] asserts that for a finite-dimensional complex Hilbert space H, the equality

$$Sph_{B(H)}^{+}\left(Sph_{B(H)}^{+}(a)\right) = \left\{b \in S(B(H)^{+}): \begin{array}{c} \operatorname{Fix}(a) \subseteq \operatorname{Fix}(b), \\ \operatorname{and} \ker(a) \subseteq \ker(b) \end{array}\right\}$$

holds for every element a in $S(B(H)^+)$ (see the beginning of the proof of [11, Claim 1]). Our next result is an abstract version of Nagy's result to the space of compact operators.

Theorem 2.8. Let H_2 be a separable infinite-dimensional complex Hilbert space. Then the identity

$$Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \left\{b \in S(K(H_2)^+): \begin{array}{c} s_{K(H_2)}(a) \leq s_{K(H_2)}(b), \text{ and} \\ \mathbf{1} - r_{B(H_2)}(a) \leq \mathbf{1} - r_{B(H_2)}(b)\end{array}\right\},$$

holds for every a in the unit sphere of $K(H_2)^+$.

Proof. (\supseteq) We recall that, for each $b \in S(K(H_2)^+)$ we have $s_{K(H_2)}(b) = p_b \in K(H_2)$. Let $b \in S(K(H_2)^+)$ with $s_{K(H_2)}(a) \leq s_{K(H_2)}(b)$, and $1 - r_{B(H_2)}(a) \leq 1 - r_{B(H_2)}(b)$. We pick an arbitrary $c \in Sph^+_{K(H_2)}(a)$. Since ||a - c|| = 1, Lemma 2.1 implies the existence of a minimal projection e in $B(H_2)$ such that one of the following statements holds:

- (a) $e \le a$ and $e \perp c$ in $K(H_2)^{**} = B(H_2);$
- (b) $e \le c$ and $e \perp a$ in $K(H_2)^{**} = B(H_2)$.

In case (a), we have $e \leq s_{K(H_2)}(a) \leq s_{K(H_2)}(b)$ and $e \perp c$. Lemma 2.1 implies that ||c - b|| = 1.

In case (b), the condition $e \perp a$ implies that $e \leq \mathbf{1} - r_{B(H_2)}(a) \leq \mathbf{1} - r_{B(H_2)}(b)$, and thus $e \perp b$. Since $e \leq c$, Lemma 2.1 assures that ||c - b|| = 1.

We have shown that ||c - b|| = 1 for all $c \in Sph^+_{K(H_2)}(a)$, and thus b lies in $Sph^+_{K(H_2)}(Sph^+_{K(H_2)}(a))$.

 (\subseteq) Let us take $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$.

We shall first prove that $\mathbf{1} - r_{B(H_2)}(a) \leq \mathbf{1} - r_{B(H_2)}(b)$. If $\mathbf{1} - r_{B(H_2)}(a) = 0$ there is nothing to prove. Otherwise, let e be a minimal projection in $K(H_2)$ with $e \leq \mathbf{1} - r_{B(H_2)}(a)$. Let (e_n) be a maximal family of mutually orthogonal minimal projections in $K(H_2)$ such that $\mathbf{1} - e = \sum_{n=1}^{\infty} e_n$ (here we apply that H_2 is separable). Since $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n - a \right\| = 1$, and $e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n \in K(H_2)$, we deduce that $\left\| e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n - b \right\| = 1$. Lemma 2.1 proves the existence of a minimal projection $q \in B(H_2)$ such that one of the next statements holds: (a) $q \leq e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n$ and $q \perp b$ in $B(H_2)$; (b) $q \leq b$ and $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n$ in $B(H_2)$.

We claim that case (b) is impossible. Indeed, $q \perp e + \sum_{n=1}^{\infty} \frac{1}{2n} e_n$ is equivalent to

 $q \perp r_{B(H_2)}\left(e + \sum_{n=1}^{\infty} \frac{1}{2n}e_n\right) = \mathbf{1}$, which is impossible. Therefore, only case (a) holds, and by the minimality of q, q coincides with e, and $e = q \perp b$ assures that $q = e \leq \mathbf{1} - r_{B(H_2)}(b)$.

We have shown that for every minimal projection e in $B(H_2)$ with $e \leq 1 - r_{B(H_2)}(a)$ we have $q \leq 1 - r_{B(H_2)}(b)$. Since in $B(H_2)$ every projection is the least upper bound of all minimal projections smaller than or equal to it, we deduce that

$$1 - r_{B(H_2)}(a) \le 1 - r_{B(H_2)}(b).$$

Our next goal is to show that $s_{\kappa(H_2)}(a) \leq s_{\kappa(H_2)}(b)$. If $r_{B(H_2)}(a) - s_{B(H_2)}(a) = 0$, we have $s_{\kappa(H_2)}(a) = a = r_{B(H_2)}(a) \geq r_{B(H_2)}(b) \geq s_{B(H_2)}(b)$. In particular, a is a projection in $K(H_2)$. We shall prove that b is a projection and a = b. Let $\sigma(b)$ be the spectrum of b, let C denote the C^{*}-subalgebra of $K(H_2)$ generated by b and $a = r_{\kappa(H_2)}(a)$, and let us identify C with $C(\sigma(b))$ and b with the identity function on $\sigma(b)$. If there exists $t_0 \in \sigma(b) \cap (0, 1)$, then the function

$$c(t) = \begin{cases} 0 & \text{if } t \in \sigma(b) \cap [0, t_0];\\ \frac{1+t_0}{1-t_0}(t-t_0) & \text{if } t \in \sigma(b) \cap [0, t_0];\\ t & \text{if } t \in \sigma(b) \cap [\frac{1+t_0}{2}, 1], \end{cases}$$
(2.6)

defines a positive, norm-one element in $c \in C(\sigma(b)) \subset K(H_2)$ such that ||a-c|| = 1 and ||b-c|| < 1. This contradicts that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore, $\sigma(b) \subseteq \{0,1\}$, and hence b is a projection. If $s_{B(H_2)}(b) = b < s_{K(H_2)}(a) = a$, we get $||b-s_{K(H_2)}(b)|| = 0$, and $||a-b|| = ||a-s_{K(H_2)}(b)|| = ||s_{K(H_2)}(a) - s_{K(H_2)}(b)|| = 1$, contradicting that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore a = b is a projection and $s_{K(H_2)}(b) = b = a = s_{K(H_2)}(a)$.

We assume next that $r_{B(H_2)}(a) - s_{K(H_2)}(a) \neq 0$. We first prove the following **Property** $(\checkmark .1)$: for each pair of minimal projections $v, q \in B(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $q \leq r_{B(H_2)}(a) - s_{K(H_2)}(a)$ one of the following statements holds: (1) $q \perp b$, or equivalently, $q \leq 1 - r_{B(H_2)}(b)$; (2) $v \leq s_{B(H_2)}(b) \leq b$.

To prove the property, we consider a family (v_n) of mutually orthogonal minimal projections in $K(H_2)$ satisfying $1 - v - q = \sum_{n=1}^{\infty} v_n$, and the element $q + \infty$

$$\sum_{n=1}^{\infty} \frac{1}{2n} v_n \in S(K(H_2)^+).$$
 Clearly, v is a minimal projection in $B(H_2)$ satisfy-

ing $v \leq a$ and $v \perp q, 1 - v$, and hence $v \perp q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$. Lemma 2.1 assures that $\left\| a - \left(q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n \right) \right\| = 1$, and by hypothesis $\left\| b - \left(q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n \right) \right\| = 1$. A new application of Lemma 2.1 assures the existence of a minimal projection

A new application of Lemma 2.1 assures the existence of a minimal projection $e \in B(H_2)$ such that one of the following statements holds:

(a)
$$e \le b$$
 and $e \perp q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ in $B(H_2)$;
(b) $e \le q + \sum_{n=1}^{\infty} \frac{1}{2n} v_n$ and $e \perp b$ in $B(H_2)$.

In the second case $e = q \perp b$, equivalently, $q \leq \mathbf{1} - r_{B(H_2)}(b)$. In the first case $e \leq b \leq r_{B(H_2)}(b) \leq r_{B(H_2)}(a)$, and $e \perp q, \mathbf{1} - v$. Since $e \leq r_{B(H_2)}(a)$ and $r_{B(H_2)}(a) = (r_{B(H_2)}(a) - v) + v$, we deduce that $e \leq v$. The minimality of e and v proves that $e = v \leq b$, and thus $v \leq s_{B(H_2)}(b) \leq b$. This finishes the proof of *Property* (\checkmark .1).

We discuss now the following dichotomy:

• There exists a minimal projection v in $B(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $v \not\leq s_{K(H_2)}(b)$;

• For every minimal projection v in $B(H_2)$ with $v \leq s_{\kappa(H_2)}(a)$ we have $v \leq s_{\kappa(H_2)}(b)$.

In the first case, let v be a minimal projection in $K(H_2)$ with $v \leq s_{K(H_2)}(a)$ and $v \not\leq s_{K(H_2)}(b)$. Property $(\checkmark .1)$ implies that for every minimal projection $q \in B(H_2)$ with $q \leq r_{B(H_2)}(a) - s_{K(H_2)}(a)$ we have $q \leq 1 - r_{B(H_2)}(b)$. This proves that

$$r_{B(H_2)}(a) - s_{K(H_2)}(a) \le \mathbf{1} - r_{B(H_2)}(b).$$

We have therefore shown that

$$\mathbf{1} - s_{_{K(H_2)}}(a) = (\mathbf{1} - r_{_{B(H_2)}}(a)) + (r_{_{B(H_2)}}(a) - s_{_{K(H_2)}}(a)) \le \mathbf{1} - r_{_{B(H_2)}}(b),$$

and thus $r_{B(H_2)}(b) \leq s_{K(H_2)}(a)$. In this case we have $0 \leq b \leq r_{B(H_2)}(b) \leq s_{K(H_2)}(a)$, and then ab = ba = b. If $\sigma(b) \cap (0,1) \neq \emptyset$, by considering the C*-subalgebra of $K(H_2)$ generated by b, and the definition in (2.6), we can find an element c in $S(K(H_2)^+)$ such that ||a - c|| = 1 and ||b - c|| < 1, contradicting that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. Therefore $\sigma(b) \subseteq \{0,1\}$, and hence b is a projection with $b \leq s_{K(H_2)}(a)$. If $b < s_{K(H_2)}(a)$, we have ||b - b|| = 0 and ||a - b|| = 1contradicting, again, that $b \in Sph^+_{K(H_2)}\left(Sph^+_{K(H_2)}(a)\right)$. We have shown that in this case $b = s_{K(H_2)}(b) = s_{K(H_2)}(a)$.

In the second case of the above dichotomy, having in mind that $s_{_{K(H_2)}}(a)$ can be written as a finite sum of mutually orthogonal minimal projections in $K(H_2)$, we have $s_{_{K(H_2)}}(a) \leq s_{_{K(H_2)}}(b)$ as desired.

Remark 2.9. Let us remark that Theorem 2.5 can be derived as a straight consequence of our previous Theorem 2.8. Namely, let H_2 be a separable complex Hilbert space, and let a be an element in $S(K(H_2)^+)$. Applying Theorem 2.8 we get

$$Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \left\{b \in S(K(H_2)^+) : \begin{array}{c} s_{K(H_2)}(a) \le s_{K(H_2)}(b), \text{ and} \\ \mathbf{1} - r_{B(H_2)}(a) \le \mathbf{1} - r_{B(H_2)}(b)\end{array}\right\}$$

$$Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \{a\}$$

If, on the other hand, $Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \{a\}$, having in mind that $s_{_{K(H_2)}}(a)$ belongs to $S(K(H_2)^+)$, and $s_{_{K(H_2)}}(a) \leq r_{_{B(H_2)}}(a)$, we deduce that $s_{_{K(H_2)}}(a)$ lies in the set $Sph_{K(H_2)}^+\left(Sph_{K(H_2)}^+(a)\right) = \{a\}$, and hence $s_{_{K(H_2)}}(a) = a$ is a projection.

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