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Fuzzy data approximation using smoothing methods by multivariate splines and radial function basis spaces. Similarity and error analysis.

by

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Resumen en español

El objetivo de esta tesis doctoral es la aproximación de funciones fuzzy bivariantes y datos fuzzy 3D usando métodos de interpolación y de ajuste suavizado mediante splines variacionales multivariantes y fuciones de base radial, realizando el análisis del error y la similitud correspondientes en cada caso.

La tesis está estructurada en cuatro capítulos, ya que este primero no es más que un amplio resumen de la misma en castellano. El resto de capítulos han sido redactados directamente en inglés, que es el idioma en el que se han confeccionado los artículos extraidos de este trabajo para ser publicados en revistas internacionales (incluidas en el listado JCR), aparte de ser el idioma vehicular habitual en el que tanto el doctorando como los directores de tesis se han podido comunicar más fluidamente, la mayor parte del tiempo.

Los artículos referidos anteriormente son:

- González, P., Idais, H., Pasadas, M., Yasin, M., Approximation of fuzzy functions by fuzzy interpolating bicubic splines, Journal of Mathematical Chemistry 57(5), 2019, 1252-1267.
- González, P., Idais, H., Pasadas, M., Yasin, M., 3D fuzzy data approximation by fuzzy smoothing bicubic splines, Mathematics and Computers in Simulation, to appear, DOI: 10.1016/j.matcom.2018.10.005

Capítulo 1

El Capítulo 1 es una mera Presentación en inglés de la tesis y una pequeña introducción a los conceptos fundamentales para seguir el resto del trabajo de investigación realizado. Así pues, el objetivo primordial del mismo es presentar las principales definiciones y conceptos preliminares necesarios para las técnicas de interpolación o aproximación empleadas y que harán uso de B-splines, o bien funciones radiales, junto con un breve repaso de ciertas cuestiones teóricas generales requeridas en nuestra investigación, y referidas fundamentalmente a los conjuntos y números fuzzy, en particular los trapezoidales, y las operaciones fundamentales usuales entre ellos.

En concreto, en la sección 1.3 se introducen algunas de las notaciones necesarias en el espacio real Euclídeo $\mathbb{R}^n$, y algunas definiciones importantes que usaremos más tarde.

La sección 1.4 está dedicada a los conceptos relacionados con convergencia en espacios de Hilbert, así como algunos resultados sobre compacidad e inyección en estos espacios. Al final de la sección introducimos algunos teoremas importantes que son de vital importancia en el estudio de las soluciones de los problemas de aproximación que se tratan en la tesis.

La sección 1.5 trata exclusivamente de la presentación de los espacios de funciones en los que la teoría se enmarca: espacios de funciones continuas de clase C^k, espacios de funciones λ-hölderianas y espacios de Sobolev.

La sección 1.6 introduce brevemente las funciones B-splines, su construcción y las propiedades principales, tanto de estas funciones como de la interpolación mediante las mismas.

En la seccción 1.7 presentamos las ideas básicas en la tería de conjuntos fuzzy, definiciones, propiedades y operaciones entre los mismos.

La sección 1.8 introduce la definición de números fuzzy y los tipos más populares de los mismos: números fuzzy triangulares y trapezoidales. También se incluye la definición más adecuada de función fuzzy para el desarrollo de la teoría.

La sección 1.9 introduce los espacios generados por funciones de base radial y el método de interpolación de datos y funciones en estos espacios.

Finalmente, la sección 1.10 trata sobre los métodos de interpolación y ajuste suavizado mediante los splines variacionales tanto continuos como discretos.

Capítulo 2

La aproximación e interpolación de funciones son problemas esenciales en casi todos los campos científicos. Dado un conjunto de datos de entrada múltiples con un solo dato de salida cada uno, el principal objetivo de la aproximación funcional es obtener un modelo para aproximar los datos de salida, que constituyen la variable dependiente, a partir de los datos de entrada, que constituyen la variable independiente, siendo los datos conjuntos de números reales.

El problema de interpolación de datos fuzzy fue introducido por Zadeh [70] y se puede formular de la siguiente manera: dados $n+1$ puntos $x_0, ..., x_n \in \mathbb{R}$ y para cada uno de estos puntos un número fuzzy en $\mathbb{R}$, entonces la pregunta sería "si es posible construir alguna función sobre $\mathbb{R}$, de la misma forma que en el caso de números crisp se puede definir alguna clase de función suave en $\mathbb{R}$ con los $n+1$ puntos dados". Lowen en [45] da un teorema de interpolación de Lagrange Fuzzy, y Kaleva presenta algunas propiedades para la interpolación Fuzzy de Lagrange y spline cúbica en [35].

Abbasbandy et al. presentan in [2] un método de aproximaión numérica de funciones fuzzy mediante polinomios fuzzy y encuentran la mejor aproximación para funciones fuzzy mediante optimización para obtener un polinomio fuzzy. Un nuevo método para la modelización de datos difusos mediante B-splines fuzzy aparece en Anile et al. [6]. En [4] se propone la interpolación de datos difusos usando splines fuzzy con el fin de obtener un nuevo conjunto de funciones splines llamado Fuzzzy Splines para interpolar los datos difusos. Valenzuela and Pasadas en [66] definen nuevos índices de error y similitud para determinar la bondad de la interpolación de datos difusos mediante funciones spline cúbicas.

En este capítulo se presenta un nuevo método de interpolación de datos fuzzy mediante spline bicúbicos fuzzy como un método de aproximación de funciones fuzzy bivariantes.

Este método de interpolación podría ser utilizado para resolver diferentes problemas prácticos en diferentes campos de la química analítica, como, por ejemplo: Búsqueda de bibliotecas en el rango espectral infrarrojo y ultravioleta, análisis cromatográfico de muestras de orina para la clasificación de nefritis, clasificación de gasolinas basada en cromatografía capilar de gases, para la calibración de dependencias de concentración de señales lineales y no lineales, para el análisis espectrofotométrico multicomponente, y muchos otros.

El capítulo está organizado como sigue: después de esta introducción, en la sección 2.2 recordamos brevemente algunas notaciones preliminares y algunos resultados. En la sección 2.3, desarrollamos los métodos de interpolación spline bicúbica. La sección 2.4 presenta brevemente algunas definiciones y fundamentos básicos de los números fuzzy; en la sección 2.5, explicamos la metodología propuesta de splines bicúbicos fuzzy de interpolación. La convergencia del método se establece en la sección 2.6 y la sección 2.7 introduce algunas de las medidas de similitud de números fuzzy frecuentemente usadas en el campo de los datos difusos. En la seccción 2.8 se realizan diferentes simulaciones y los resultados muestran el buen comportamiento de los índices de error y similitud propuestos. En la sectción 2.9 exponemos el análisis estadístico de los índices de error de interpolación y similitud propuestos. Finalmente, presentamos algunas conclusiones en la sección 2.10.

Capítulo 3

Este capítulo, como continuación del anterior, trata sobre la aproximación de datos difusos mediante un método de ajuste suavizado en un espacio de funciones splines fuzzy bicúbicas, se trata por tanto de un método de ajuste de datos difusos mediante splines fuzzy de ajuste que se obtienen mediante la minimización de un funcional cuadrático en dicho espacio funcional.

Como ya hemos comentado, existen diferentes metodologías para la aproximación de datos. Por ejemplo, en [47, 55] dan un método para aproximar o ajustar superficies a un conjunto de datos dado, usando splines de ajuste. Abbasbandy et al. [2] dan un método para encontrar la mejor aproximación entre funciones fuzzy a un conjunto de puntos dado. En [57] se determina un nuevo método para encontrar la mejor aproximación para funciones usando números fuzzy trapezoidales.

En [24] los autores proponen un nuevo conjunto de funciones spline para ajustar datos dispersos dados. La aproximación de datos dispersos o fuzzy aparece en diferentes áreas de la investigación. Por ejemplo, en [17] se construyen aproximaciones para comprimir una sucesión de números fuzzy usando la F-transformada y operadores de Bernstein.

En la literatura, existen diferentes metodologías para el ajuste de funciones o datos utilizando algoritmos de evolución multiobjetivo [49]. En [65, 66] se presentan ejemplos numéricos para ilustrar una nueva metodología para la aproximación de números fuzzy mediante splines fuzzy de ajuste cúbicos.

En este capítulo se presenta un nuevo método de aproximación de datos fuzzy o funciones fuzzy bivariantes mediante splines bicúbicos fuzzy de ajuste.

El capítulo está organizado como sigue: Después de esta introducción, la sección 3.2 presenta algunas notaciones y preliminares acerca de los espacios de B-splines C^2-cúbicos y bicúbicos. La sección 3.3 está dedicada a la definición, cálculo y resultados de convergencia de los splines bicúbicos variacionales de ajuste. En la sección 3.4 presentamos la definición básica de los números fuzzy y algunas de las medidas de similitud de números fuzzy presentes en la bibliografía. En la sección 3.5 presentamos la metodología que proponemos. En la sección 3.6 analizamos mediante varios experimentos el comportamiento del método de aproximación propuesto y los resultados muestran el buen comportamiento de los índices de error y similitud propuestos, y exponemos el análisis estadístico de los mismos. Finalmente, realizamos algunas conclusiones en la sección 3.7.

Capítulo 4

En este capítulo presentamos una nueva metodología para la aproximación de datos fuzzy mediante funciones de base radial (RBFs).

La interpolación y aproximación mediante funciones de base radial ha sido revisada en varios artículos (véase [10]-[53] entre otros), y es suficiente para nuestro objetivo comentar como se realiza. En [65] los autores definen unos nuevos índices de error y similitud para determinar la exactitud de la apoximación de datos difusos mediante funciones spline cúbicas.

En este capítulo un nuevo método de aproximación basado en (RBFs) para la aproximación de datos fuzzy generalmente de dos variables. Las funciones de base radial constituyen una herramienta extensamente usada para la aproximación de funciones (no lineales) que es un tema central en el análisis y reconocimiento de formas [54]-[51]; véase también [30] para un resumen reciente y para obtener referencias adicionales.

La relación entrada-salida basada en (RBFs) (a menudo vista como una red neuronal [30]) tiene la forma

$$y = g(x) = \sum_{i=1}^{n} \alpha_i \, \phi(\|x - t_i\|)$$

donde $x = [x_1, ..., x_d]^T$ es la entrada, los α_i son pesos, y la función radial ϕ en $\mathbb{R}^d$ está definida a tavés de una función $\phi : [0, \infty] \longrightarrow \mathbb{R}$ de tal forma que $\phi(x) = \phi(\|x\|^2)$ donde $\|.\|$ es la norma euclídea usual en $\mathbb{R}^d$, y los puntos t_i son llamados los centros.

El resto de este capítulo está organizado como sigue: En la sección 4.2 se propone una nueva metodología para la aproximación de ajuste suavizado con funciones de base radial de números difusos, y se definen tres índices de errror y similitud para diferetes conjuntos de datos difusos. Varios resultados de simulaciones se llevan a cabo en la sección 4.3 para verificar los buenos resultados del método propuesto. En la sección 4.4 analizamos el comportamiento individual y global de los diferentes índices de error y similitud definidos. Finalmente, las conclusiones son expuestas en la sección 4.5.

Chapter 1

Introduction and Preliminaries

1.1 Thesis Presentation

One of the most important and widespread problems in different scientific-technical fields (such as image processing, visualization of computational graphics, geometric modeling and design, etc.) is the approximation and adjustment of curves and surfaces (that will be given of functions from one or several variables). In the last few decades, many of researcher were developed a theoretical study or build algorithms with different splines and radial basis function.

On the other hand, function approximation and interpolation for B-spline or radial basis functions plays an important role in applied science for the industry standard shape modelling as found in the constructions of care bodies, airplane fuselage and many others.

Given a set of data, obtained from multiple input values and a (dependent) output variable, the main objective of this problem is to find a surface curve that fits well enough to that data. This proposal focuses on the two-dimensional case. In the abundant literature on this subject, different methodologies for obtaining the surface that best adapts to these data are shown, satisfying also some other restrictions, such as some regularity and form conditioning.

A very frequent technique is the use of interpolate polynomials, which in general have the disadvantage of presenting significant fluctuations when the number of data is large, or in regions where there is less information available. The piecewise polynomial functions of spline type, give rise to a better behavior in that sense. However, the use of B-splines bases for the approximation of functions also possesses the advantages of a great stability and simplicity in the calculation, so that this methodology has been extended in many areas of application, as shown in the bibliography in this regard.

Likewise, the use of bases of radial functions is becoming a very popular tool in approximation theory, especially if they have compact support, so that in the last few years an abundant bibliography has appeared on the subject.

Less literature is found when we speak of fuzzy functions of real or vectorial variable. The approximation of this type of functions, which so many applications have in economics and engineering in general, is an underdeveloped subject and that is currently considered essential. We only find some examples of Polynomial interpolation with splines in one variable as it could be observed in the bibliography that is contributed to the project.

One of the great difficult of this type of approximation is the measure of the goodness of the same, so that it is necessary to resort to measures of similarity of fuzzy numbers and, from these, to determine the error of the different approximations in one or in several variables, and both in B-spline base spline spaces and in spaces generated by radial functions of compact support.

This thesis is organized as follows:
after this introduction, Chapter 1 presents some basic fundamentals and definitions used throughout this document.

The interpolation error of fuzzy data in 3D using similarity measures of fuzzy numbers is addressed in Chapter 2. This involves the definition of the error of fuzzy data and similarity measures to determine the accuracy of interpolation of fuzzy data by using a basis of functions of two-dimensional cubic splines "bicubic splines".

Chapter 3 gives a detailed treatment of approximation errors of 3D fuzzy data by using smoothing bicubic splines. This involves for a given fuzzy data set several approximation errors using similarity measures for fuzzy numbers.

Finally, a new methodology to consider the error is approximately of 3D fuzzy data by using radial basis function is presented in Chapter **??**. This methodology uses radial basis function for defining the error and similarity indices to determine and compare the accuracy of the approximation of fuzzy data.

1.2 Introduction

This chapter aims to introduce both the notations and some of the preliminary results necessary for the development of the theories which are studied throughout this thesis.

The present chapter is structured as follows: in section 1.3, some of the precise notations in the real Euclidean space $\mathbb{R}^n$, and some important definitions that we will use later.

The section 1.4 deals with the different concepts related with convergence in Hilbert spaces, as well as some results on compactness and injection in these spaces. At the end of this section we introduce some important theorems, which are of vital importance in the study of the solutions of approximation problems to be dealt in this thesis.

The section 1.5 deals exclusively with the presentation of the function at spaces in which the theory is framed: spaces of continuous functions of class C^k, spaces λ-hölderianas functions and of Sobolev spaces.

Section 1.6 briefly introduces the B-splines functions, their construction and its the main properties. In section 1.7 we present the basic ideas and entities in fuzzy set theory and introduce the definitions and properties on fuzzy. Section 1.8 introduces definition of fuzzy numbers and we will provide the most popular types of fuzzy numbers (trapezoidal and triangular fuzzy numbers) and the best definition of fuzzy function will be included.

Section 1.9 we introduces the radial basis functions method, the popular radial basis function and the definition of the radial basis function interpolation.

Finally, in section 1.10 we will introduce the definitions of the variational spline theory and to describe properties of interpolation and smoothing variational splines.

1.3 Notations

Suppose we have the usual symbols $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ representing respectively, the set of real numbers, integers, naturals. Let's say we have positive sets $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

If $n \in \mathbb{N}^*$. Some, but not all, norms are based on inner products. The most basic example is the familiar *dot product*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_n = \boldsymbol{x}.\boldsymbol{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i \, y_i \tag{1.1}$$

between vectors $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, lying in the Euclidean space $\mathbb{R}^n$. The Euclidean norm, or length of vector, is found by taking the square root

$$\forall \, \boldsymbol{x} \in \mathbb{R}^n, \; \| \, \boldsymbol{x} \, \|_n = \langle \boldsymbol{x}, \boldsymbol{x} \rangle_n^{\frac{1}{2}} . \tag{1.2}$$

Recall that the set $\mathbb{R}^n$ has structure Hilbert space with this Euclidean scalar product.

Let $\boldsymbol{x} \in \mathbb{R}^n$, the *Open ball* centered at a with radius $\lambda > 0$, denoted $B(x, \lambda)$, is defined as the set of points $y \in \mathbb{R}^n$ such that $\|\boldsymbol{x} - \boldsymbol{y}\|_n < \lambda$, that is,

$$B(\boldsymbol{x}, \lambda) = \{ \, \boldsymbol{y} \in \mathbb{R}^n \; : \; \|\boldsymbol{x} - \boldsymbol{y}\|_n < \lambda \, \} \, , \tag{1.3}$$

We can also define closed balls in $\mathbb{R}^n$ too. The *closed ball* centered at a with radius $\lambda > 0$ denoted $\overline{B}(x, \lambda)$ is defined as the set of points $y \in \mathbb{R}^n$ such that $\langle \boldsymbol{x} - \boldsymbol{y} \rangle_n \leq \lambda$, that is,

$$\overline{B}(\boldsymbol{x}, \lambda) = \{ \, \boldsymbol{y} \in \mathbb{R}^n \; : \; \langle \boldsymbol{x} - \boldsymbol{y} \rangle_n \leq \lambda \, \} \, . \tag{1.4}$$

Definition 1.3.1. *Let E' denote the set of limit points of E, then* the closure *of E, denoted by $\overline{E}$, is the set of $E \cup E'$.*

Definition 1.3.2. *A set E in $\mathbb{R}^n$ is bounded if and only if there exists a positive number r such that $E \subset \{ \, \boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n, \|\boldsymbol{x}\| \leq r \, \},$.*

If $n \in \mathbb{N}^*$ and $E \subset \mathbb{R}^n$, we will represent by $\langle E \rangle$ the vector subspace of $\mathbb{R}^n$ generated by the subset of E.
Given $E \subset \mathbb{R}^n$. The *diameter* of E, denoted by $\operatorname{diam}(E)$, is defined as follows:
If E is the empty set, the n $\operatorname{diam}(E) = 0$;
If E is a non-empty bounded set, then $\operatorname{diam}(E) = \sup\{\|\boldsymbol{x} - \boldsymbol{y}\|_n : \boldsymbol{x}, \boldsymbol{y} \in E\}$;
If E is unbounded, then $\operatorname{diam}(E) = \infty$.
Given a function $\boldsymbol{u}$ with domain E, the support of $\boldsymbol{u}$ is define as the closure of the subset of E

$$\sup \boldsymbol{u} = \overline{\{ \, \boldsymbol{x} \in E : \; \boldsymbol{u}(\boldsymbol{x}) \neq \boldsymbol{0} \, \}} \, . \tag{1.5}$$

The space of matrices of real numbers with m rows and n columns, for every $m, n \in \mathbb{N}^*$, it is denoted by $\mathcal{M}_{m,n}$.

Suppose $A = (a_{ij}),\ a_{ij} \in \mathbb{R},\ 1 \leq i \leq m,\ 1 \leq j \leq n$ and $A \in \mathcal{M}_{m,n}$. We will define in this space the following scaler product

$$\forall A = (a_{ij}),\ B = (b_{ij}) \in \mathcal{M}_{m,n},\ \langle A, B\rangle_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}\ ,$$

and the corresponding matrix norm

$$\forall A \in \mathcal{M}_{m,n},\ \langle A\rangle_{m,n} = \langle A, A\rangle_{m,n}^{1/2}\ .$$

is called *the Hilbert-Schmidt* norm.

1.4 Metric spaces. Hilbert Spaces

In this section we will continue to introduce definitions and results of Functional Analysis to be used, to view and expand more see K. Atkinson [5], W. Cheney [15], H. Brezis [9] and P. Yosida [69].

Definition 1.4.1. *Let X be a normed linear space, and let $x_n, x \in X, \forall n \in \mathbb{N}$.*

1. *We say that x_n* converges strongly, *or* converges in norm *to x, and write $x_n \longrightarrow x$, if*
$$\lim_{n\to+\infty} \|x_n - x\| = 0.$$
2. *We say that x_n* converges weakly *to x, and write $x_n \longrightarrow^w x$, if*
$$\forall\, y \in X, \quad \lim_{n\to+\infty} \langle x_n, y\rangle = \langle x, y\rangle.$$

Definition 1.4.2. *A linear space with a scalar product that is complete with respect to the induced norm is called a Hilbert space.*

Definition 1.4.3. *A metric space is a couple (X, d) where X is a set and d is a metric (or a distance) on X, that is a function $d : X \times X \longrightarrow \mathbb{R}^+$ such that*

1. *$d(x, y) > 0$ (non-negativity)*

2. $d(x, y) = 0$ *if and only if* $x = y$ *(identity)*

3. $d(x, y) = d(y, x)$ *(symmetry)*

4. $d(x, z) < d(x, y) + d(y, z)$ *(triangle inequality)*

Definition 1.4.4. *Let X be a metric space with metric d and let $A \subset X$. We say that A is a compact subset if the metric space A with the inherited metric d is compact.*

Definition 1.4.5. *A subset A of X is relatively compact if the closure $\overline{A} \subset X$ is a compact subset of X.*

Definition 1.4.6. *A metric space is called sequentially compact if every sequence in X has a convergent subsequence.*

Definition 1.4.7. *Let X and Y be two normed linear spaces and $T : X \longrightarrow Y$ a linear map between X and Y. T is called a compact operator if for all bounded sets $E \subset X$, $T(E)$ is relatively compact in Y.*

Theorem 1.4.8. *Let X and Y be two normed linear spaces; suppose $T : X \longrightarrow Y$, is a linear operator. Then the following are equivalent.*

1. *T is compact.*

2. *The image of the open unit ball under T is relatively compact in Y.*

3. *For any bounded sequence $\{x_n\}$ in X, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in Y.*

Theorem 1.4.9. ***(Lax-Milgram Lemma).*** *Let H be a Hilbert space, and $B(x, y)$ a function of two vectors with the following properties:*

1. *$B(x, y)$ is for fixed y a linear function of x, for fixed x a skewlinear function of y.*

2. *B is bounded: there is a constant c so that for all x and y in H*

$$|B(x,y)| \leq c\|x\|\|y\| \tag{1.6}$$

3. *There is a positive constant b such that*

$$|B(y,y)| \geq b\|y\|^2 \tag{1.7}$$

for all y in H.

Definition 1.4.10. *A bilinear functional ϕ on a normed space E is called coercive (or sometimes elliptic) if there exists a positive constant K such that*

$$\phi(x,x) \geq K\|x\|^2 \tag{1.8}$$

for all $x \in E$.

Theorem 1.4.11. ***(Stampacchia).*** *Let a be a continuous coercive bilinear form on a Hilbert space H, T be a continuous linear form on H, and K be a non-empty closed convex subset of H. Then there is a unique vector u in K such that*

$$a(u, u-v) \geq T(u-v) \quad \forall v \in K. \tag{1.9}$$

Moreover, if a is symmetric, then u is characterized by the property:

$$\begin{cases} u \in K, \\ \dfrac{1}{2}a(u,u) - \phi(u) = \min\limits_{v \in K} \left\{ \dfrac{1}{2}a(v,v) - \phi(v) \right\}. \end{cases}$$

Theorem 1.4.12. ***(Riesz-Frechet representation theorem).*** *Let H be a Hilbert space and let $\phi \in H'$. Then there is a unique $f \in H$ such that*

$$\varphi(v) = (f,v)_H, \quad \forall v \in H. \tag{1.10}$$

Moreover

$$\|f\|_H = \|\varphi\|_{H'}. \tag{1.11}$$

1.5 Functional spaces

In this section we will introduce some the functional spaces that are used later in a numerical analysis. The main references that have been considered for the development of this section are [5], [31], [21] and [28].

Let Ω be an open set in $\mathbb{R}^n$. We denote the space of continuous functions $u : \Omega \longrightarrow \mathbb{R}$ by $C(\Omega)$; the space of functions with continuous partial derivatives in Ω of order less than or equal to $k \in \mathbb{N}$ by $C^k(\Omega)$; and the space of functions with continuous derivatives of all orders by $C^\infty(\Omega)$. Functions in these spaces need not be bounded even if Ω is bounded.

Let $p, n \in \mathbb{N}^*$ and Ω be a non-empty open of $\mathbb{R}^p$.

Definition 1.5.1. *For each $k \in \mathbb{N}$, it is denoted by $C^k(\Omega; \mathbb{R}^n)$ (or $C^k(\Omega)$ if $n = 1$) the space of functions $\boldsymbol{\varphi}$ defined in Ω with values in $\mathbb{R}^n$ that are continuous in Ω, as well as all its partial derivatives $\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}$ with orders $|\boldsymbol{\alpha}| \leq k$.*

Recall that, for each $i = 1, \dots, p$, $p_i \circ \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi} = \partial^{\boldsymbol{\alpha}}(p_i \circ \boldsymbol{\varphi})$, that is, to derive $\boldsymbol{\varphi}$ it suffices to derive each of its component functions.

Definition 1.5.2. *For each $k \in \mathbb{N}$, it is indicated $C^k(\overline{\Omega}; \mathbb{R}^n)$ (or $C^k(\overline{\Omega})$ if $n = 1$) the function space $\boldsymbol{\varphi} \in C^k(\Omega; \mathbb{R}^n)$ such that, for all $|\boldsymbol{\alpha}| \leq k$, $\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}$ is bounded and uniformly continuous.*

If $\boldsymbol{\varphi} \in C^k(\overline{\Omega}; \mathbb{R}^n)$, then, for all $|\boldsymbol{\alpha}| \leq k$, there exists a single extension continuous of $\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}$ to $\overline{\Omega}$. If Ω is bounded, the reciprocal is true. Hereafter we will identify $\boldsymbol{\varphi}$ and its partial derivatives with their respective extensions continuous.

It is verified that $C^k(\overline{\Omega}; \mathbb{R}^n)$ is a Banach space with the norm

$$\|\boldsymbol{\varphi}\|_{C^k(\overline{\Omega};\mathbb{R}^n)} = \max_{|\boldsymbol{\alpha}|\leq k} \sup_{\boldsymbol{x}\in\Omega} \|\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{x})\|_n .$$

Note 1.5.3. *If $k = 0$ we will use the notation $\|\cdot\|_{\infty,\Omega}$ rather than $\|\cdot\|_{C^0(\overline{\Omega};\mathbb{R}^n)}$ and it is usually referred to as the Tchebycheff rule.*

Definition 1.5.4. *Let $k \in \mathbb{N}$ and $\lambda \in (0, 1]$. We will denote by $C^{k,\lambda}(\overline{\Omega}; \mathbb{R}^n)$ (or $C^{k,\lambda}(\overline{\Omega})$ if $n = 1$) the space of functions $\boldsymbol{\varphi} \in C^k(\overline{\Omega}; \mathbb{R}^n)$ that are λ-*

hölderians, as well as all its partial derivatives $\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}$ *of orders* $|\boldsymbol{\alpha}| \leq k$, *i.e,. such that*

$$\exists\, C > 0,\ \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega,\ \forall\, |\boldsymbol{\alpha}| \leq k, \|\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{x}) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{y})\|_n \leq C\|\boldsymbol{x} - \boldsymbol{y}\|_p^{\lambda}.$$

In addition $C^{k,\lambda}(\overline{\Omega};\mathbb{R}^n)$ is a Banach space with the norm

$$\|\boldsymbol{\varphi}\|_{C^{k,\lambda}(\overline{\Omega};\mathbb{R}^n)} = \|\boldsymbol{\varphi}\|_{C^{k}(\overline{\Omega};\mathbb{R}^n)} + \max_{|\boldsymbol{\alpha}|\leq k} \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Omega \\ \boldsymbol{x}\neq\boldsymbol{y}}} \frac{\|\partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{x}) - \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{y})\|_n}{\|\boldsymbol{x} - \boldsymbol{y}\|_p^{\lambda}}.$$

Some of the properties of the spaces that are introduced below require some degree of regularity of the boundary $\partial\Omega$ of the open subset Ω.

Definition 1.5.5. *Let* Ω *be open and bounded in* $\mathbb{R}^p$, *and let* V *be a space of functions defined in* $\mathbb{R}^{p-1}$. *We say that* $\partial\Omega$ *is of class* V *if for each point* $\boldsymbol{x}_0 \in \partial\Omega$, *exists* $r > 0$ *and a function* $g \in V$ *such that*

$$\Omega \cap B(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, r) \,:\, x_p > g(x_1, \ldots, x_{p-1})\}$$

except for a change of the coordinate system by means of a transformation, if it is necessary. In particular, if V *is a space of Lipschitz functions, we will say that* Ω *is a Lipschitz domain. When* V *is a space of functions* $C^{k,\alpha}, 0 < \alpha \leq 1$, *we say that* Ω *is a Hölder domain of class* $C^{k,\alpha}$.

Definition 1.5.6. *It is denoted by* $L^2(\Omega;\mathbb{R}^n)$ *the space of (classes of) measurable functions (Lebesgue)* $\boldsymbol{u}$ *defined in* Ω *and with values in* $\mathbb{R}^n$ *such that*

$$\int_{\Omega} \|\boldsymbol{u}(\boldsymbol{x})\|_n^2 \, d\boldsymbol{x} < +\infty.$$

Definition 1.5.7. *For each* $m \in \mathbb{N}$, *we denote by* $H^m(\Omega;\mathbb{R}^n)$ *(or* $H^m(\Omega)$ *if* $n = 1$*) the Sobolev space of order* m *of (classes of) functions* $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^n)$ *such that, for all* $|\boldsymbol{\alpha}| \leq m$, *the partial derivative* $\partial^{\boldsymbol{\alpha}}\boldsymbol{u}$, *in the sense of the distributions, belongs to* $L^2(\Omega;\mathbb{R}^n)$.

Note 1.5.8. *We recall that if $\boldsymbol{u} \in H^m(\Omega;\mathbb{R}^n)$, for each $|\boldsymbol{\alpha}| \le m$, $\partial^{\boldsymbol{\alpha}}\boldsymbol{u}$ is the unique function of $L^2(\Omega;\mathbb{R}^n)$, such that*

$$\forall \boldsymbol{\varphi} \in \mathcal{D}(\Omega;\mathbb{R}^n), \ \int_\Omega \langle \boldsymbol{u}(\boldsymbol{x}), \partial^{\boldsymbol{\alpha}}\boldsymbol{\varphi}(\boldsymbol{x})\rangle_n \, d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega \langle \partial^{\boldsymbol{\alpha}}\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_n \, d\boldsymbol{x},$$

where $\mathcal{D}(\Omega;\mathbb{R}^n)$ designates the space of functions with values in $\mathbb{R}^n$ infinitely differentiables in Ω and with compact support contained in Ω. We also observe that $H^0(\Omega;\mathbb{R}^n) = L^2(\Omega;\mathbb{R}^n)$.

It the space $H^m(\Omega;\mathbb{R}^n)$ with the scalar product is endowed

$$((\boldsymbol{u},\boldsymbol{v}))_{m,I,\mathbb{R}^n} = \sum_{|\boldsymbol{\alpha}|\le m} \int_\Omega \langle \partial^{\boldsymbol{\alpha}}\boldsymbol{u}(\boldsymbol{x}), \partial^{\boldsymbol{\alpha}}\boldsymbol{v}(\boldsymbol{x})\rangle_n \, d\boldsymbol{x}, \tag{1.12}$$

and is denoted by

$$\|\boldsymbol{u}\|_{m,I,\mathbb{R}^n} = ((\boldsymbol{u},\boldsymbol{u}))^{\frac{1}{2}}_{m,I,\mathbb{R}^n}$$

the corresponding standard.
Also are

$$(\boldsymbol{u},\boldsymbol{v})_{j,\Omega,\mathbb{R}^n} = \sum_{|\boldsymbol{\alpha}|=j} \int_\Omega \langle \partial^{\boldsymbol{\alpha}}\boldsymbol{u}(\boldsymbol{x}), \partial^{\boldsymbol{\alpha}}\boldsymbol{u}(\boldsymbol{x})\rangle_n \, dx\,, \quad 0 \le j \le m\,,$$

defined as scalar semiproducts and induced seminormas

$$|\boldsymbol{u}|_{j,\Omega,\mathbb{R}^n} = (\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}_{j,\Omega,\mathbb{R}^n}\,, \quad 0 \le j \le m\,.$$

Note 1.5.9. *Note that $\boldsymbol{u} \in H^m(\Omega;\mathbb{R}^n)$ if and only if $p_i \circ \boldsymbol{u} \in H^m(\Omega)$, for all $i = 1,\dots,n$. Also,*

$$\|\boldsymbol{u}\|_{m,\Omega,\mathbb{R}^n} = \left(\sum_{i=1}^n \|p_i \circ \boldsymbol{u}\|^2_{m,\Omega}\right)^{\frac{1}{2}},$$
$$|\boldsymbol{u}|_{j,\Omega,\mathbb{R}^n} = \left(\sum_{i=1}^n |p_i \circ \boldsymbol{u}|^2_{j,\Omega}\right)^{\frac{1}{2}}, \quad 0 \le j \le m\,.$$

Theorem 1.5.10. *$H^m(\Omega;\mathbb{R}^n)$ is a Hilbert space with the scalar product defined in (1.12).*

Proof. See Atkinson[5, Theorem 6.2.3] and apply note 1.5.9. □

1.6 B-splines functions

1.6.1 Introduction

B-Splines functions play an important role in the approximation theory (multi-resolution approximation, . . .). In this section, we shall give a definition of the B-Spline functions. Then, we will present some of the most important properties, at least needed when using uniform B-Splines in a Finite Elements method, Finally the interpolation and approximation splines.

The theory of B-splines functions in the way in which to be presented, has been developed among many references and is shown, for example, in [29], [50], [22], [16], [62] and [61].

1.6.2 Definition and properties

The B-splines are a type of functions that adapt very well to the various numerical theories of interpolation, data adjustment, etc.

If we have an infinite set of nodes

$$\ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 \ldots, \tag{1.13}$$

and

$$\lim_{i\to+\infty} t_{-i} = -\infty, \qquad \lim_{i\to+\infty} t_i = +\infty.$$

From this partition we can introduce different families of B-splines.

Definition 1.6.1. *The formal definition of the B-spline of zero degree, denoted by $B_i^0(x)$, is the characteristic function over the interval [0,1], i.e.,*

$$\forall i \in \mathbb{Z} \qquad B_i^0(x) = \begin{cases} 1, & \text{if} \quad t_i \le x < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The following sentences are the most important properties of the B-Spline of degree zero.

i) $B_i^0(x) = 0$, $\forall x \notin [t_i, t_{i+1})$.

ii) $B_i^0(x) \ge 0$, $\forall x \in \mathbb{R}$, $\forall i \in \mathbb{Z}$.

iii) $B_i^0(x)$ is continuous on the right across the real line.

iv) $\sum_{i=-\infty}^{+\infty} B_i^0(x) = 1$, $\forall x \in \mathbb{R}$.

A B-Spline of degree k, $k \in \mathbb{N}$, denoted by B_i^k, is defined by convolution as

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x), \quad k \geq 1. \quad (1.14)$$

We will include some of the most interesting properties of the different families of B-splines of degree k, which are but generalizations of the statements for the B-splines of degree 0.

i) $B_i^k(x) = 0$, $\forall x \notin [t_i, t_{i+k+1})$.

ii) $B_i^k(x) > 0$, $\forall x \in (t_i, t_{i+k+1})$.

iii) $\sum_{i=-\infty}^{+\infty} B_i^k(x) = 1, \quad \forall x \in \mathbb{R}$.

iv) For $k \geq 2$,

$$\frac{d}{dx} B_i^k = \left(\frac{k}{t_{i+k} - t_i}\right) B_i^{k-1} - \left(\frac{k}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}.$$

v) For $k \geq 1$, B_i^k belongs to $C^{k-1}(\mathbb{R})$.

vi) For every k the following identity is verified

$$\int_{\infty}^{x} B_i^k(s) ds = \left(\frac{t_{i+k+1} - t_i}{k+1}\right) \sum_{j=i}^{\infty} B_j^{k+1}(x).$$

vii) The set of B-splines $B_i^k, B_{i+1}^k, \ldots, B_{i+k}^k$ is linearly independent in (t_{i+k}, t_{i+k+1}).

1.6.3 Interpolation and approximation splines

This section is dedicated to briefly review how spline functions can be used in interpolation and approximation.

Let $I = (\alpha, \beta)$ and $\Delta_p = \{\alpha = t_0 < t_1 < \ldots < t_p = \beta\}$ be a subset of distinct points of $\overline{I}$. Let $S(k, r; \Delta_p)$ the vector space of global class functions r and that are polynomials of degree $\leq k$ in each of the p intervals of the partition Δ_p, i.e

$$S(k, r; \Delta_p) = \{s \in C^r(\overline{I}) : s|_{[t_{i-1}, t_i]} \in \mathbb{P}^k[t_{i-1}, t_i], i = 1, \ldots, p\}.$$

Then we have the following theorem

Theorem 1.6.2. *A basis for vector space* $S(k, k-1; \Delta_p)$ *is*

$$\{B_i^k|_{\overline{I}} : -k \leq i \leq p-1\}.$$

As a consequence, the dimension of $S(k, k-1; \Delta_p)$ *is* $p+k$.

Proposition 1.6.3. *The following interpolation schemes are unisolvent*

i) Interpolation with Hermite conditions.

The problem that arise is: Given $f \in C^k(\overline{I})$, *find* $s \in S(2k-1, 2k-2; \Delta_p)$ *such that*

- $s(t_i) = f(t_i) \quad i = 0, \ldots, p,$
- $s^{(\mu)}(\alpha) = f^{(\mu)}(\alpha) \quad y \quad s^{(\mu)}(\beta) = f^{(\mu)}(\beta), \quad \mu = 1, \ldots, k-1.$

ii) Interpolation with natural conditions.

The problem that arise is: Given $f \in C^m(\overline{I}), \quad 2 \leq k \leq p-1$ *find* $s \in S(2k-1, 2k-2; \Delta_p)$ *such that*

- $s(t_i) = f(t_i), \quad i = 0, \ldots, p,$
- $s^{(\mu)}(\alpha) = s^{(\mu)}(\beta) = 0, \quad \mu = k, \ldots, 2k-2.$

iii) Interpolation with periodic conditions.

Given $f \in C^k(\overline{I})$, *with* $f^{(\kappa)}(\alpha) = f^{(\kappa)}(\beta)$ *for* $\kappa = 0, \ldots, k-1$, *find* $s \in S(2k-1, 2k-2; \Delta_p)$ *such that*

- $s(t_i) = f(t_i) \quad i = 0, \ldots, p,$
- $s^{(\mu)}(\alpha) = s^{(\mu)}(\beta), \quad \mu = 1, \ldots, 2k-2.$

Theorem 1.6.4. *The nodes are considered* $t_{-p}, \ldots, t_{p+k}$. *Then there is a unique spline* $s \in S(k, k-1; \Delta_p)$ *which interpolates the values* $y_1, \ldots, y_{p+k}$ *in the points* $\zeta_1 < \ldots < \zeta_{p+k}$ *If and only if*

$$B_{j-p-1}^k(\zeta_j) \neq 0, \quad 1 \leq j \leq p+k.$$

If we want to interpolate the values of a known function, $f \in C[t_0, t_p]$, en $x_1, \ldots, x_n \in [t_0, t_p]$ we can define the Lagrange interpolation operator associated with the function f as

$$\mathcal{L}f(x) = \sum_{i=1}^{p+k} f(x_i) B_i^k(x), \quad \forall x \in [t_0, t_p]. \tag{1.15}$$

Theorem 1.6.5. *Let $I = (\alpha, \beta)$ and $\Delta_p = \{\alpha = t_0 < t_1 < \ldots < t_p = \beta\}$. Consider s the cubic spline with Hermite conditions that interpolates $f \in C^2(\overline{I})$ in the nodes of the considered partition. Then we have*

$$|f(x) - s(x)| \leq h^{\frac{3}{2}} \left(\int_I (f''(t))^2 dt \right)^{\frac{1}{2}}, \quad \text{where} \quad h = \max_{0 \leq j \leq p-1} (t_{j+1} - t_j).$$

Theorem 1.6.6. *If $I = (\alpha, \beta)$ and Δ_p a uniform partition of step h of $\overline{I}$. Consider s the cubic spline with Hermite conditions that interpolates $f \in C^4(\overline{I})$ in the nodes of the considered partition. Then we have*

$$|f - s|_{l, \overline{I}} \leq K_l h^{4-l}, \quad l = 0, 1, 2, 3,$$

where K_l, $l = 0, 1, 2, 3$, depends only of f and $h = \max_{0 \leq j \leq p-1} (t_{j+1} - t_j)$.

Theorem 1.6.7. *Let $I = (\alpha, \beta)$, $h = \dfrac{\beta - \alpha}{p}$ and consider a uniform step partition h of $\overline{I}$. consider also the natural cubic spline s that interpolates $f \in C^4(\overline{I})$ in the nodes of the partition considered. So we have*

$$\|f - s\|_{\infty, I} \leq \frac{5}{384} h^4 \|f^{(4)}\|_{\infty, I}.$$

Finally, we enunciate a theorem that gives us information about the error for the three cases of interpolation presented in Proposition 1.6.3, which although it offers us information on the convergence of the derivatives of the spline, is not optimal in terms of the error.

Theorem 1.6.8. *Let $I = (\alpha, \beta)$ and $\Delta_p = \{\alpha = t_0 < t_1 < \ldots < t_p = \beta\}$. Let's consider $s \in S(2k-1, 2k-2; \Delta_p)$ the spline that interpolates to $f \in C^k(\overline{I})$, $k \geq 2$ in Δ_p. Then, for $0 \leq l \leq k-1$ We have*

$$\|f^{(l)} - s^{(l)}\|_{\infty, I} \leq \frac{k!}{l! \sqrt{k}} h^{k-l-\frac{1}{2}} \, \|f^{(k)}\|_{2, I}, \quad h = \max_{0 \leq j \leq p-1} (t_{j+1} - t_j).$$

1.7 Fuzzy sets

Since its inception in 1965, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines, prof. Lotfi A. Zadeh [70] was the first one to analyze the need of "mathematics of fuzzy," which led to the publication of "Fuzzy Sets"; applications of this theory can be found in almost every scientific field. In this section we will review some notation for sets, and then view the concept of fuzzy sets and some important definitions are also introduced.

An universal set X is defined as the largest set within a given universe of discourse and it includes all possible elements. The following notation will be used extensively.

$A \subseteq X$ means that a set A is included in the universal set X.
$A \not\subset X$ means that a set A is not included in the universal set X.
$x \in A$ means that an element x is included in the set A.
$x \notin A$ means that an element x is not included in the set A.

Definition 1.7.1. ***(Membership function)*** *For a set* A, *we define a membership function* μ_A *such as*

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ \\ 0, & x \notin A. \end{cases}$$

We can say that the function μ_A *maps the elements of the universal set* X *to the set* $\{0, 1\}$.

$$\mu_A : X \longrightarrow \{0, 1\}$$

Definition 1.7.2. ***(Convex set)*** *The term convex is applicable to a set* A *in* $\mathbb{R}^n$ *(n-dimensional Euclidian vector space) if the followings are satisfied.*

1. *Two arbitrary points* $s, r \in A$.

$$r = (r_i | i \in \mathbb{N}_n), \quad s = (s_i | i \in \mathbb{N}_n)$$

($\mathbb{N}$ *is a set of positive integers.)*

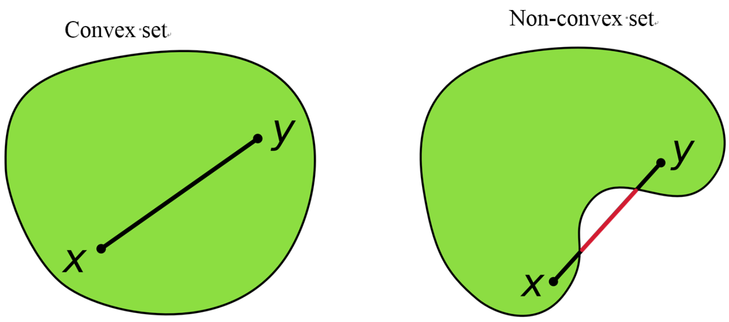

Figure 1.7.1: Examples of convex and non convex set

2. *For arbitrary real number λ between 0 and 1, we have $t \in A$*

   Where t is

$$t = (\lambda r_i + (1 - \lambda)s_i | i \in \mathbb{N}_n).$$

In other words, if every point on the line connecting two points s and r in A is also on A. Figure 1.7.1 shows examples of convex and non convex sets.

Definition 1.7.3. *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x) \ / \ x \in X)\}$$

$\mu_A(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space μ, with the following properties:

i) $\mu_{X-A}(x) = 1 - \mu_A(x)$ *for any* $x \in X$.

ii) $\mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x))$ *for any* $x \in X$.

iii) $\mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x))$ *for any* $x \in X$.

As can be observed by these properties, the union and intersection of fuzzy sets can be determined by their membership functions of the constituent sets.

Definition 1.7.4. *The support of a fuzzy set A, $Supp(A)$ is the crisp set of all $x \in X$ such that $\mu_A > 0$. The (crisp) set of elements that belong to the fuzzy set A at least to the degree α is called the α-level set or α-cut:*

$$A_\alpha = \{x \in X \ / \ \mu_A \geq \alpha\}$$

Definition 1.7.5. *A fuzzy set A in a universe of discourse X is called convex (convex fuzzy set), if all the α- cut sets are convex, the fuzzy set with these α- cut sets is convex; in other words, if a relation*

$$\mu_A(t) \geq min[\mu_A(r), \mu_A(s)]$$

where

$$t = \lambda r + (1-\lambda)s \quad r, s \in \mathbb{A}, \quad \lambda \in [0,1]$$

holds, the fuzzy set A is convex.

Definition 1.7.6. *The height of a fuzzy set A in a universe of discourse X is defined by:*

$$height(A) = sup\{\mu_A(x), x \in X\}.$$

Definition 1.7.7. *A fuzzy set A in a universe of discourse X is called normal if there exists an $x \in A$ such that $\mu_A(x) = 1$. Otherwise A is subnormal.*

1.7.1 Operations with Fuzzy sets

In his first publication, Zadeh in [70], defined the following operations with fuzzy sets as generalizations of crisp sets and of crisp statements (the reader should realize that the theoretic operations intersection, union and complement correspond to the logical operators and, inclusive or and negation, respectively):

Definition 1.7.8. *Intersection (logical and): the membership function of the intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as:*

$$\mu_{\tilde{A}\cap\tilde{B}}(X) = Min(\mu_{\tilde{A}}(X), \mu_{\tilde{B}}(X)), \forall x \in X \qquad (1.16)$$

Definition 1.7.9. *Union (exclusive or): the membership function of the union is defined as:*

$$\mu_{\tilde{A}\cup\tilde{B}}(X) = Max(\mu_{\tilde{A}}(X), \mu_{\tilde{B}}(X)), \forall x \in X \tag{1.17}$$

Definition 1.7.10. *Complement (negation): the membership function of the complement is defined as:*

$$\bar{\mu}_{\tilde{A}}(X) = 1 - \mu_{\tilde{A}}(X), \forall x \in X \tag{1.18}$$

1.8 Fuzzy numbers

A fuzzy number is simply an ordinary number whose precise value has some uncertainty. Fuzzy numbers are expressed as fuzzy sets defining a fuzzy interval in the real line $\mathbb{R}$.

Definition 1.8.1. *A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0, 1]$ with the following properties (see [36]).*

i) u is an upper semi-continuous function on $\mathbb{R}$.

ii) $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.

iii) There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with

a) $u(x)$ a monotonic increasing function on $[a_1, a_2]$,

b) $u(x)$ a monotonic decreasing function on $[a_3, a_4]$,

c) $u(x) = 1$, for all $x \in [a_2, a_3]$.

The set of all fuzzy numbers is denoted by ζ (see Figure 1.8.2).

If a fuzzy set is convex and normalized, and its membership function is defined in R and piecewise continuous, it is called as fuzzy number. So fuzzy numbers (fuzzy set) represent a real number interval whose boundary is fuzzy.

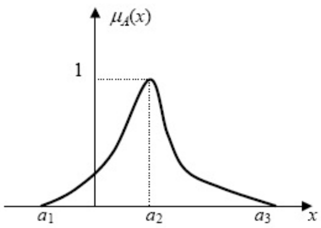

Figure 1.8.2: Fuzzy number $A = (a_1, a_2, a_3)$

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, that can be defined as $A = (a_1, a_2, a_3, a_4)$, and their membership function has the form (see Figure 1.8.3)

$$\mu(A) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases}$$

If $a_1 = a_2 = a_3 = a_4$, then the real number is represented by a. If $a_1 = a_2$

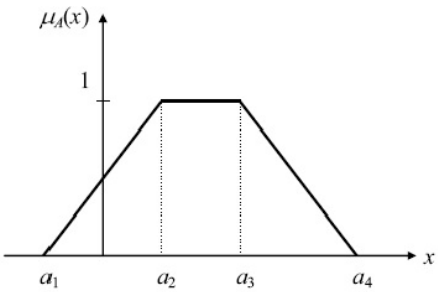

Figure 1.8.3: Trapezoidal Fuzzy numbers

and $a_3 = a_4$, then A is called a crisp interval.
Note that a triangular fuzzy number is obtained when $a_2 = a_3$, in which case triangular fuzzy numbers can be defined by $A = (a_1, a_2, a_3)$, and their

membership function is defined by, (see Figure 1.8.4)

$$\mu(A) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3, \\ 0, & otherwise. \end{cases}$$

In classical set theory, equality relation between two points of a crisp set

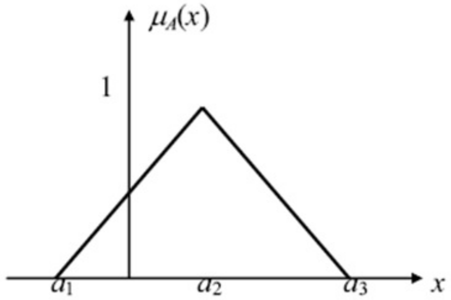

Figure 1.8.4: Triangular Fuzzy numbers

has the following well known properties:

1. $x = x, \ \forall x \in X,$
2. $x = y \Leftrightarrow y = x, \ \forall x, \forall y \in X,$
3. $x = y$ and $y = z \Rightarrow x = z, \ \forall x, \forall y, \forall z \in X.$

Equality of two points in the classical sense can be graded by the function $E_X : X \times X \longrightarrow \{0, 1\}$ defined by

$$E_X(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \ne y. \end{cases} \forall x, \forall y \in X$$

For two crisp sets X and Y, a function f is considered as a relation on $X \times Y$ with the characteristic function $\chi f : X \times X \longrightarrow \{0, 1\}$ which satisfies the following two conditions [58]

i) $\forall a \in X, \ \exists c \in Y$ such that $\chi f(a, c) = 1,$

ii) $\forall a, \forall b \in X, \quad \forall c, \forall d \in Y, \quad \chi f(a, c) = 1$ and $\chi f(b, d) = 1$ and $a = b \Rightarrow c = d$.

We shall define a fuzzy function as a special fuzzy relation.
Let X and Y be sets. A function $f : X \longrightarrow Y$ is considered as a relation $f \subseteq X \times Y$ with the characteristic function $\chi_f : X \times Y \longrightarrow \{0, 1\}$ which satisfies

1. $\forall a \in X, \exists c \in Y$ such that $\chi_f(a, c) = 1$.
2. $\chi_f(a, c) = 1$ and $\chi_f(b, d) = 1$ and $a = b \Rightarrow c = d$.

1.8.1 Operations on Fuzzy numbers

Operations on fuzzy numbers can be generalized from that on crisp intervals. Let's have a look at the operations on intervals. $\forall a_1, a_3, b_1, b_3 \in \mathbb{R}$

$$A = [a_1, a_3], \quad B = [b_1, b_3]$$

Assuming A and B as numbers expressed as intervals, the main operations on interval are:

1) Addition $[a_1, a_3](+)[b_1, b_3] = [a_1 + b_1, a_3 + b_3]$.

2) Subtraction $[a_1, a_3](-)[b_1, b_3] = [a_1 - b_1, a_3 - b_3]$.

3) Multiplication $[a_1, a_3](\bullet)[b_1, b_3] = [a_1 \bullet b_1 \wedge a_1 \bullet b_3 \wedge a_3 \bullet b_1 \wedge a_3 \bullet b_1, a_1 \bullet b_1 \vee a_1 \bullet b_3 \vee a_3 \bullet b_1 \vee a_3 \bullet b_3]$.

4) Division $[a_1, a_3](/)[b_1, b_3] = [a_1/b_1 \wedge a_1/b_3 \wedge a_3/b_1 \wedge a_3/b_1, a_1/b_1 \vee a_1/b_3 \vee a_3/b_1 \vee a_3/b_3]$, excluding the case $b_1 = 0$ or $b_3 = 0$.

5) Inverse interval $[a_1, a_3]^{-1} = [1/a_1 \wedge 1/a_3, 1/a_1 \vee 1/a_3]$, excluding the case $a_1 = 0$ or $a_3 = 0$.

6) Multiplication by a scalar $a[a_1, a_3] = [a \bullet a_1 \wedge a \bullet a_3, a \bullet a_1 \vee a \bullet a_3] \forall a \in \mathbb{R}$.

When previous sets A and B is defined in the positive real number $\mathbb{R}^+$, the operations of multiplication, division, and inverse intervals written as,

3*) Multiplication $[a_1, a_3](\bullet)[b_1, b_3] = [a_1 \bullet b_1, a_3 \bullet b_3]$.

4*) Division $[a_1, a_3](/)[b_1, b_3] = [a_1/b_1, a_2/b_3]$.

5*) Inverse interval $[a_1, a_3]^{-1} = [1/a_3, 1/a_1]$.

If $\tilde{M}$ and $\tilde{N}$ are fuzzy numbers, membership of $\tilde{M}(*)\tilde{N}$ is defined as follow:

$$\mu_{\tilde{M}(*)\tilde{N}} = \vee_{z=x*y}(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{N}}(y)) \tag{1.19}$$

Where $*$ stands for any of the four arithmetic operators.

1) $\mu_{\tilde{M}(+)\tilde{N}} = \vee_{z=x+y}(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{N}}(y))$.
2) $\mu_{\tilde{M}(-)\tilde{N}} = \vee_{z=x-y}(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{N}}(y))$.
3) $\mu_{\tilde{M}(\bullet)\tilde{N}} = \vee_{z=x\bullet y}(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{N}}(y))$.
4) $\mu_{\tilde{M}(/)\tilde{N}} = \vee_{z=x/y}(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{N}}(y))$.

The procedure of addition or substraction is simple, but the procedure of multiplication or division is more complicated.

1.8.2 Operation with Triangular Fuzzy Numbers

Some important properties of the operations on triangular fuzzy numbers are summarized

1. The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.
2. The result from multiplication or division are not triangular fuzzy number.
3. Max or Min operation does not give triangular fuzzy numbers.

Suppose triangular fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3)$$

i) Addition
$A(+)B = (a_1, a_2, a_3)(+)(b_1, b_2, b_3)$
$= (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

ii) Subtraction
$A(-)B = (a_1, a_2, a_3)(-)(b_1, b_2, b_3)$
$= (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

iii) Symmetric image
$-(A) = -(a_1, a_2, a_3)$
$= (-a_3, -a_2, -a_1)$

1.8.3 Operation with Trapezoidal Fuzzy Numbers

Some important properties of operations on trapezoidal fuzzy numbers are as in the triangular fuzzy number the case,

1. Addition and subtraction between fuzzy numbers become trapezoidal fuzzy numbers.
2. Multiplication and division and inverse need not be trapezoidal fuzzy numbers.
3. Max and Min of fuzzy number is not always in the form of a trapezoidal fuzzy number.

Suppose trapezoidal fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3, a_4), \quad B = (b_1, b_2, b_3, b_4)$$

i) Addition
$A(+)B = (a_1, a_2, a_3, a_4)(+)(b_1, b_2, b_3, b_4)$
$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_{4+b_4})$

ii) Substraction
$A(-)B = (a_1, a_2, a_3, a_4)(-)(b_1, b_2, b_3, b_4)$
$= (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

iii) Symmetric image
$-(A) = -(a_1, a_2, a_3, a_4)$
$= (-a_4, -a_3, -a_2, -a_1)$

1.9 Radial basis functions

Radial basis function (**RBF**) are a special class of function. There characteristic feature is that there response decreases (or increases) monotonically with distance from a central point.

that is a radial basis function is real-valued function whose value depends only on the distance from the origin, so that $\phi(x) = \phi(\|x\|)$; or alternatively on the distance from other point c, called a center, so that $\phi(x) = \phi(\|x - c\|)$. Any function ϕ that satisfies the property $\phi(x) = \phi(\|x\|)$ is a radial function, the norm is usually Euclidean distance. Formulate for commonly used types of radial basis functions are given in the Table (1.1).

1.9.1 Radial basis function interpolation

Let $\underline{x}, \underline{x}_i \in \mathbb{R}^d, i = 1, 2, \ldots, n$, and d is some positive integer, f_i be a scalar value, and $\| \cdot \|$ denote the Euclidean norm.

The basis **RBF** method is define as follows:

Definition 1.9.1. *Given a set of n distinct date points $\{x_i\}_{i=1}^n$, the basis **RBF** interpolant is given by*

$$S(\underline{x}) = \sum_{i=1}^{n} \lambda \phi(\|\underline{x} - \underline{x}_i\|) \tag{1.20}$$

Where $\phi(r), r \geq 0$, is some radial function. The expansion coefficients λ_i are determined so that $S(\underline{x}_i) = f_i$, $i = 1, \ldots, n$, which leads to the following symmetric linear system:

$$A\lambda = f, \tag{1.21}$$

Where

$$A = \left(\phi(\|\underline{x}_i - \underline{x}_j\|)\right)_{i,j=1}^n \tag{1.22}$$

The primary choice in the implementation is what function $\phi(r)$ to use. For the piecewise smooth cases, n is usually selected as $n = 1$ or $n = 2$; whereas the best choice for the shape parameter ε in the smooth cases has been the subject of extensive studies. (e.g. [12], [26] and [56]).

In this definition, H can be any Hilbert space isomorphic to $\ell^2(\mathbb{Z})$.

Definition 1.9.2. *A function $f : [0, \infty) \longrightarrow \mathbb{R}$ will be called positive definite on Hilbert space (HPD) if the matrix*

$$\left(f \parallel x_j - x_k \parallel^2\right)_{j,k=1}^n \tag{1.23}$$

is non-negative definite for every positive integer n and any points $x_1, ..., x_n \in H$. We shall call any matrix of the form (1.23) a distance matrix.

The classical theory of positive definite functions provides the well-known inequality

$$\mid f(t) \mid \leq f(0),$$

but a strong bound holds for HPD functions.

Type of basis function	$\phi(r), (r \geq 0)$
Gaussian (GA)	$\phi(r) = e^{-(\varepsilon r)^2}$
Inverse quadratic (IQ)	$\phi(r) = \frac{1}{1+(\varepsilon r)^2}$
Inverse multiquadric (IM)	$\phi(r) = \frac{1}{\sqrt{1+(\varepsilon r)^2}}$
Multiquadric (MQ)	$\phi(r) = \sqrt{1+(\varepsilon r)^2}$
Polyharmonic spline (PS)	$\phi(r) = r^k, k = 1, 3, 5, ...$
	$\phi(r) = r^k ln(k), k = 2, 4, 6, ...$

Table 1.1 : Some commonly used radial basis function.

Definition 1.9.3. *(Completely Monotone Function) A function $\phi : [0,\infty) \longrightarrow \mathbb{R}$ which is in $C[0,\infty) \cap C^{\infty}(0,\infty)$ and which satisfies*

$$(-1)^{\ell}\phi^{(}\ell)(r) \geq 0,\ \ r > 0,\ \ell = 0, 1, ...,$$

is called completely monotone on $[0,\infty)$

Theorem 1.9.4. *A function ϕ is completely monotone on $[0,\infty)$ if and only if $\Phi = \phi(\| \cdot \|^2)$ is positive definite and radial on $\mathbb{R}^s$ for all s.*

Schoenberg introduced the following interpolation problem in [59].

Theorem 1.9.5. *A function $f : [0,\infty) \longrightarrow \mathbb{R}$ is completely monotone but not constant if and only if $\phi(\| \cdot \|^2)$ is strictly positive definite and radial on $\mathbb{R}^s$ for any s.*

The Schoenberg interpolation theorem tells us that

Theorem 1.9.6. ***(Schoenberg)*** *If $\phi(r) = \phi(\sqrt{r})$ is completely monotone but not constant on $[0,\infty)$, then for any set of n distinct points $\{x_{j}{}_{j=1}^{n}\}$, then the $n \times n$ matrix A with entries $a_{j,k} = \phi(\| \underline{x}_j - \underline{x}_k \|)$ is positive definite (and therefore non-singular)*

We can conclude, for example, that the basis **RBF** method (1.20) is uniquely solvable for the first three **RBFs** in table (1.1) since $\ell = 0, 1, \cdots$, and $r > 0$,

$$GA \Rightarrow \phi(r) = \phi(\sqrt{r}) = e^{-(\varepsilon^2 r)} \Rightarrow (-1)^{\ell}\phi^{(\ell)}(r) = \varepsilon^{2\ell}e^{-\varepsilon^2 r} > 0$$

$$IQ \Rightarrow \phi(r) = \phi(\sqrt{r}) = \frac{1}{1+(\varepsilon^2 r)} \Rightarrow (-1)^{\ell}\phi^{(\ell)}(r) = \frac{\ell!\varepsilon^{2\ell}}{(1+\varepsilon^2 r)^{\ell+1}} > 0$$

$$IM \Rightarrow \phi(r) = \phi(\sqrt{r}) = \frac{1}{\sqrt{1+(\varepsilon^2 r)}} \Rightarrow (-1)^{\ell}\phi^{(\ell)}(r) = \frac{\Gamma(\ell+\frac{1}{2})\varepsilon^{2\ell}}{\sqrt{\pi}(1+\varepsilon^2 r)^{\ell+\frac{1}{2}}} > 0$$

This same conclusion cannot be made for the MQ and linear **RBFs** since in both cases $\phi(r) > 0$ and $\acute{\phi}(r) > 0$ for $r > 0$. However, by Micchelli [46] elegant extension of theorem (1.9.6) this conclusion does follow,

Theorem 1.9.7. ***(Micchelli)*** *Suppose ϕ is strictly conditionally positive definite of order one and that $\phi(0) < 0$. Then for any distinct points $\{x_{j}{}_{j=1}^{n}\}$, the $n \times n$ matrix A with entries $a_{j,k} = \phi(\| \underline{x}_j - \underline{x}_k \|)$ has $N-1$ positive and one negative eigenvalue, and is therefore non-singular.*

Clearly, the MQ and linear **RBFs** satisfy the continuity and positivity requirements of the theorem. In addition, for both functions the first derivative of their corresponding $\phi(r)$ function is completely monotone and not constant on $(0, \infty)$ since

$$Mq \Rightarrow \phi(r) = \phi(\sqrt{r}) = \sqrt{1 + (\varepsilon^2 r)} \Rightarrow (-1)^{\ell-1}\phi^{(\ell)}(r) = \frac{\Gamma(\ell - \frac{1}{2})\varepsilon^{2\ell}}{2\sqrt{\pi}(1+\varepsilon^2 r)^{\ell-\frac{1}{2}}} > 0$$

$$Linear \Rightarrow \phi(r) = \phi(\sqrt{r}) = r^{\frac{1}{2}} \Rightarrow (-1)^{\ell-1}\phi^{(\ell)}(r) = \frac{\Gamma(\ell - \frac{1}{2})}{2\sqrt{\pi} r^{\ell-\frac{1}{2}}} > 0$$

for $\ell = 1, 2, \cdots$, and $r > 0$. Thus, the basis **RBFs** method (1.20) is uniquely solvable for the MQ and linear radial functions.

1.10 Variational splines

Variational spline methods are a family of techniques for approximating, the aim of this section is to introduce the definitions of the variational spline theory and to describe properties of interpolation and smoothing variational splines.

We will need some notation. Let $p, n, m \in \mathbb{N}^*$ and $N, \mu \in \mathbb{N}$ such that

$$m > \frac{p}{2} + \mu. \tag{1.24}$$

We denote by $\langle \cdot \rangle_{\mathbb{R}^n}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$. Let Ω be a non-empty open bounded set of $\mathbb{R}^p$ with a Lipschitz-continuous boundary (see [5]). We will use the usual Sobolev space $H^m(\Omega; \mathbb{R}^n)$ of (classes of) functions $\boldsymbol{u} \in L^2(\Omega; \mathbb{R}^n)$, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}} \boldsymbol{u}$, in the sense of the distributions, of order $\mid \boldsymbol{\alpha} \mid \leq m$, where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_p) \in \mathbb{N}^p$ and $\mid \boldsymbol{\alpha} \mid = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_p$, (see Section 1.5).

We designate by $\mathbb{R}^{N,n}$ the space of real matrices with N rows and n columns, with the inner product $\langle A, B \rangle_{N,n} = \sum_{i,j=1}^{N,n} a_{ij} b_{ij}$, and the corresponding norm $\langle A \rangle_{N,n} = (\langle A, A \rangle)^{\frac{1}{2}}$.

1.10.1 Interpolating variational spline

To find the suitable function that interpolates data point , many researchers such as Cauchy, Lagrange and Hermit work on developing the classic approach through different techniques and methodologies to find the best polynomial, (to view and expand more see [27],[23],[22],[41]).

Kouibia and his co-author in [38] proposed an interpolation variational spline. Suppose we are given:

- an finite subset A of distinct points of $\bar{\Omega}$;
- an finite set of linear applications $\sum$ of the type

$$\Phi : \upsilon \longrightarrow \partial^{\gamma}\upsilon(a), \quad | \gamma | \leq \mu \quad with \gamma \in \mathbb{N}^p, \tag{1.25}$$

  with $a \in A$ such that each point of A is associated with at least one element of Σ;
- an finite set Θ of continuous inner semi-products defined in the Sobolev space $H^m(\Omega;\mathbb{R}^n)$.

Let $N_1 = \text{card } \Sigma, N_2 = \text{card } \Theta$ and $\boldsymbol{\alpha}$ be a data matrix of $\mathbb{R}^{N_1,n}$, and let us consider the set

$$K = \{\upsilon \in H^m(\Omega;\mathbb{R}^n) \backslash L\upsilon = \boldsymbol{\alpha}\}.$$

Proposition 1.10.1. *K is a non-empty closed and convex subset of $H^m(\Omega;\mathbb{R}^n)$. Furthermore, it is an affine variety associated with the vector space*

$$K_0 = \{\upsilon \in H^m(\Omega;\mathbb{R}^n) \backslash L_\upsilon = 0\}.$$

Proof. See Kouibia [38] Proposition 1. □

We get the following minimization problem to find σ such that

$$\begin{cases} \sigma \in K, \\ \forall \upsilon \in K, \quad J(\sigma) \leq J(\upsilon). \end{cases} \tag{1.26}$$

J being the functional defined, for all $\upsilon \in H^m(\Omega;\mathbb{R}^n)$, by

$$J(\upsilon) = \langle \tau, \beta(\upsilon,\upsilon) \rangle_{\mathbb{R}^{N_2}} + \varepsilon \mid \upsilon \mid^2_{m,\Omega,\mathbb{R}^n},$$

where $\tau = (\tau_1, \ldots, \tau_{N_2}) \in \mathbb{R}^{N_2}$, with $\tau \geq 0$, for all $i = 1, \ldots, N_2$, and $\varepsilon > 0$.

Definition 1.10.2. *The solution of (1.26), if exists, is called the interpolating variational spline in* Ω, *relative to* $A, L, \beta, \boldsymbol{\alpha}, \tau$ *and* ε.

Lemma 1.10.3. *The application* $(((\cdot,\cdot))) : H^m(\Omega;\mathbb{R}^n) \times H^m(\Omega;\mathbb{R}^n) \to \mathbb{R}$ *defined by*

$$(((\nu,\upsilon))) = \langle L\nu, L\upsilon\rangle_{N_{1,n}} + \langle \tau, \beta(\nu,\upsilon)\rangle_{\mathbb{R}^{N_2}} + \varepsilon(\nu,\upsilon)_{m,\Omega,\mathbb{R}^n} \tag{1.27}$$

is an inner product in $H^m(\Omega;\mathbb{R}^n)$ *and its associated norm given by* $[[\upsilon]] = (((\upsilon,\upsilon)))^{\frac{1}{2}}$ *is equivalent to the norm of Sobolev* $\|\cdot\|_{m,\Omega,\mathbb{R}^n}$.

Theorem 1.10.4. *Proplem (1.26) has a unique solution which is the unique solution of the following variational problem:*

$$\begin{cases} \sigma \in K, \\ \forall \upsilon \in K_0, \quad \langle \tau, \beta(\sigma,\upsilon)\rangle_{\mathbb{R}^{N_2}} + \varepsilon(\sigma,\upsilon)_{m,\Omega,\mathbb{R}^n} = 0. \end{cases} \tag{1.28}$$

Theorem 1.10.5. *There exist a unique element* $(\sigma,\lambda) \in H^m(\Omega;\mathbb{R}^n) \times \mathbb{R}^{N_{1,n}}$ *such that*

$$\begin{cases} \sigma \in K, \quad \forall \upsilon \in H^m(\Omega;\mathbb{R}^n), \\ \langle \tau, \beta(\sigma,\upsilon)\rangle_{\mathbb{R}^{N_2}} + \varepsilon(\sigma,\upsilon)_{m,\Omega,\mathbb{R}^n} + \langle \lambda, L\upsilon\rangle_{\mathbb{R}^{N_{1,n}}} = 0. \end{cases} \tag{1.29}$$

1.10.2 Smoothing variational splines

Definition 1.4.2.discrete smoothing variational spline is the solution of approximating curve or surface problem by minimizing the quadratic function in a parametric finite element space.
Before we talk about smoothing variational spline, its necessary to define $D \in \mathbb{R}^r$ as a non empty open set, where $r, n, s \in \mathbb{N}^*$. Let $\langle .,. \rangle_{\mathbb{R}^n}$, $\langle . \rangle_{\mathbb{R}^n}$ be the the inner product and the Euclidean norm in $\mathbb{R}^n$ respectively, where $s > \frac{r}{2} + \mu$, and $H^s(D;\mathbb{R}^n)$ is the Sobolev space of function v back to $L^2(D;\mathbb{R}^n)$, while retaining all partial derivatives of order $|\alpha| \leq s$, while $\alpha = (\alpha_1, \ldots, \alpha_r)$, where

$$|\alpha| = \sum_{1}^{r} \alpha_i.$$

Which gives

$$(v,w)_{l,D,\mathbb{R}^n} = \sum_{|\alpha|=l} \int_D \langle \partial^\alpha v(x), \partial^\alpha w(x)\rangle_{\mathbb{R}^n} dx, \quad 0 \le l \le s$$

.
However it is important to define the space of all polynomial of degree $\le n$ by ρ_n and $\mathbb{R}^{N,n}$ the space of all real matrices, where the inner product is as follows:

$$\langle A, C\rangle_{N,n} = \sum_{i,j=1}^{N,n} a_{ij}c_{ij}$$

with

$$\langle A\rangle_{N,n} = (\langle A, A\rangle_{N,n})^{\frac{1}{2}}$$

Then from [42] we have that the linear system

$$B\alpha = d$$

with

$$d_j = \langle L^b g, L^b u_j\rangle_{M_1,n}$$

$$b_{ij} = \langle L^b u_i, L^b u_j\rangle_{M_1,n} + \langle \theta, \beta^b(u_i,u_j),\rangle_{\mathbb{R}^{M_2}} + \varepsilon(u_i,u_j)_{s,D,\mathbb{R}^n}$$

Where

$$L^b = C^\mu(\bar{D}, \mathbb{R}^n) \to \mathbb{R}^{M_1,n},$$

$M_1 =$ card $\sum^b$ and $M2 = card(\theta^b)$, θ^b is a finite set in $H^s(D, \mathbb{R}^n)$.
But the minimization problem is to find σ such that:

$$\begin{cases} \sigma_{\varepsilon,\theta} \in H^s(D,\mathbb{R}^n), \\ u \in H^s(D,\mathbb{R}^n) \end{cases}$$

and

$$J_{\varepsilon,\theta}(\sigma_{\varepsilon,\theta}) \le J_{\varepsilon,\theta}(u)$$

Theorem 1.10.6. *This problem has a unique solution called the smoothing variational spline with $A^b, L^b, \beta^b, L^b g$, and ε, which is also the unique solution of the following variational problem:*

$$\begin{cases} \sigma_{\varepsilon,\theta} \in H^s(D,\mathbb{R}^n), \\ u \in H^s(D,\mathbb{R}^n), \end{cases}$$

and

$$\langle L\sigma_{\varepsilon,\theta}, Lu\rangle_{M_1,n} + \langle \theta, \beta(\sigma_{\varepsilon,\theta}, u)\rangle_{\mathbb{R}^n} + \varepsilon(\sigma_{\varepsilon,\theta})_{s,D,\mathbb{R}^n} = \langle Lg, Lv\rangle_{M_1,n}$$

Chapter 2

Interpolation of bicubic fuzzy functions.

2.1 Introduction

Function approximation and interpolation are essential problems in almost every scientific field. Given a set of multiple input single output data, with input data $X = \{x_0, ..., x_n\}$, and output data Y, the main goal of function approximation is obtaining a model to approximate the dependent variable Y, given the input variable X, being X and Y real number sets. The interpolation problem of fuzzy data was first introduced by Zadeh [70] and can be formulated as "suppose we are given $n+1$ points $x_0, ..., x_n \in \mathbb{R}$ and for each of these points a Fuzzy Value in $\mathbb{R}$; then, it is possible to construct some function on $\mathbb{R}$, rather than a crisp one to define some kind of smooth function in $\mathbb{R}$ with the given $n+1$ points?". Lowen in [45] gave a fuzzy Lagrange interpolation theorem, then Kaleva presented some properties of Lagrange and cubic spline interpolation in [35]. Abbasbandy et al. presented in [2] a numerical approximation of fuzzy functions by fuzzy polynomials and found the best approximation for fuzzy functions by optimization to obtain a fuzzy polynomial. A novel methodology for modeling uncertain data with fuzzy B-splines is presented in Anile et al. [6]. In [4] interpolation of fuzzy data by using fuzzy splines is proposed in order to find new set of spline functions called Fuzzy Splines to interpolate fuzzy data. Valenzuela and Pasadas in [66] define new error and similarity indices to determine the accuracy of interpolation of fuzzy data by cubic spline functions.

In this chapter a new interpolation method of fuzzy data by fuzzy bicubic splines is presented as an approximation method of bivariate fuzzy functions. As already mentioned, this fuzzy interpolation procedure could be useful and worhtwile for posing and solving different practical problems in different fields of analytical chemistry, as for example: library searching in the infrared and ultraviolet spectral range, chromatographic analysis of urine samples to nephritis classification, gasolines classification based on capillary gas chromatography, for calibration of linear and non-linear signal concentration dependencies, for spectrophotometric multicomponent analysis, and many others.

In fact, many times the chemist becomes aware of the fact that he has to deal with many types of vague, incomplete or inexact data or information, and that the uncertainty of those cannot always be described by means of statistical terms, but can be taken into account by means of the fuzzy theory introduced by Zadeh [70] in 1965. Nowadays, this mathematical theory is a very mature and important branch of Mathematics and Computer Science, with a wide ranging collection of concepts, techniques and applications in almost all branches of general Science and Engineering, and in particular in Analytical and Formal Chemistry. Sometimes a molecule may be regarded as a graph, and all the intrinsecal uncertainty of its formal description can be included in a rigorous framework, via the fuzzy numbers and their corresponding arithmetics and logic. In this new context, fuzzy radius and bond length for atoms become familiar and completely feasible within this theory. In this way, the analytical chemist is able to consider and solve more and more complex problems and answer questions about his own research studies and those raised for the industry and/or society interests.

In principle, some of these procedures are not always limited to the consideration of a single variable but can be also used in the several variables framework. For instance, in chromatography, the retention position and the signal of a peak could be used for classifying unknown samples with a fuzzy method. In such cases, a two-dimensional membership function would be needed (based on a circle, elipse or some trapezoidal piramide). Over these domains of influence (or fuzzy supports) the membership functions are specified as surfaces of suitable structure. Further applications of comparing fuzzy functions are known for peak tracking in high-performance liquid chromatographic separation and for depth-profiling in secondary ion mass spetrometry, see [48] and the references therein.

The chapter is organized as follows: after this introduction, in Section 3.4 we briefly recall some preliminary notation and results. In Section 2.3, we explain the bicubic spline interpolation methods. Section 2.4 briefly presents some basic fundamentals and definitions of fuzzy numbers; in Section 2.5, we explain the proposed methodology of fuzzy interpolating bicubic splines. The convergence of the method is established in Section 2.6 and Section 2.7 introduces some of the similarity measures of fuzzy numbers frequently used in the field of fuzzy data. In Section 3.6, different simulation results are carried out showing the good performance of the proposed error and similarity indices. In section 2.9, we explain the statistical analysis of the proposed interpolating error and similarity indices Finally, the conclusions are discussed in Section 3.7.

2.2 Preliminaries

We denote by $< \cdot >_n$ and $< \cdot, \cdot >_n$, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$. For any real intervals (a,b) and (c,d), with $a < b$ and $c < d$, we consider the rectangle $R = (a, b) \times (c, d)$ and let $H^3(R)$ be the usual Sobolev space of (classes of) functions u belonging to $L^2(R)$, together with all their partial derivatives $D^{\boldsymbol{\beta}}(u)$ with $\boldsymbol{\beta} = (\beta_1, \beta_2)$, in the distribution sense, of order $|\boldsymbol{\beta}| = \beta_1 + \beta_2 \leq 3$. this space is equipped with the norm

$$\|u\| = \left(\sum_{|\boldsymbol{\beta}| \leq 3} \int_R (D^{\boldsymbol{\beta}} u(p))^2 dp \right)^{\frac{1}{2}},$$

the seminorms

$$|u|_\ell = \left(\sum_{|\boldsymbol{\beta}| = \ell} \int_R (D^{\boldsymbol{\beta}} u(p))^2 dp \right)^{\frac{1}{2}}, \quad 0 \leq \ell \leq 3.$$

and the corresponding inner semiproducts

$$(u, v)_\ell = \sum_{|\boldsymbol{\beta}| = \ell} \int_R D^{\boldsymbol{\beta}} u(p) D^{\boldsymbol{\beta}} v(p) dp, \quad 0 \leq \ell \leq 3.$$

Moreover, for $n, m \in \mathbb{N}^*$, let $T_n = \{x_0, \ldots, x_n\}$, $T_m = \{y_0, \ldots, y_m\}$ be some subsets of distinct points of $[a, b]$ and $[c, d]$, with $a = x_0 < x_1 < \ldots < x_n = b$ and $c = y_0 < y_1 < \ldots < y_m = d$. We denote by $S_3(T_n)$ and $S_3(T_m)$ the spaces of cubic splines of class $\mathcal{C}^2$ given by

$$S_3(T_n) = \{s \in \mathcal{C}^2[a, b] : \; s|_{[x_{i-1}, x_i]} \in \mathbb{P}_3[x_{i-1}, x_i], \; i = 1, \ldots, n\},$$

$$S_3(T_m) = \{s \in \mathcal{C}^2[c, d] : \; s|_{[y_{j-1}, y_j]} \in \mathbb{P}_3[y_{j-1}, y_j], \; j = 1, \ldots, m\},$$

where $\mathbb{P}_3[x_{i-1}, x_i]$ ($\mathbb{P}_3[y_{j-1}, y_j]$) is the restriction on $[x_{i-1}, x_i]$ ($[y_{j-1}, y_j]$) of the linear space of real polynomials with total degree less than or equal to 3. It is known that dim $S_3(T_n) = n+3$ (dim $S_3(T_m) = m+3$). Let $\{\phi_1, \dots, \phi_{n+3}\}$ and $\{\psi_1, \dots, \psi_{m+3}\}$ be bases of functions with local support of $S_3(T_n)$ and $S_3(T_m)$ respectively, and consider the space $S_3(T_n \times T_m)$ of bicubic spline functions of class $\mathcal{C}^2$ given by

$$S_3(T_n \times T_m) = span\ \{\phi_1, \dots, \phi_{n+3}\} \otimes\ span\ \{\psi_1, \dots, \psi_{m+3}\}$$

This space is a Hilbert subspace of $H^3(R)$ equipped with the same norm, semi-norms and inner semi-products of such space, and verifies

$$S_3(T_n \times T_m) \subset H^3(R) \cap \mathcal{C}^2(R). \tag{2.1}$$

Particulary, let

$$\{B_0^3(x), \dots, B_{n+2}^3(x)\}\ \left(\{B_0^3(y), \dots, B_{m+2}^3(y)\}\right)$$

be the $\mathcal{C}^2-$cubic B-splines basis of $S_3(T_n)$ ($S_3(T_m)$), then

$$\{B_r^3(x)B_s^3(y),\ r = 0, \dots, n+2,\ s = 0, \dots, m+2\}$$

is the $\mathcal{C}^2-$bicubic B-splines basis of $S_3(T_n \times T_m)$, then $dim\ S_3(T_n \times T_m) = (n+3)(m+3)$ and we can define

$$B_k(x,y) = B_r^3(x)B_s^3(y),\ (x,y) \in R,$$

for $r = 0, \dots, n+2$, $s = 0, \dots, m+2$, $k = (m+3)r + s + 1$. Then $1 \leq k \leq (n+3)(m+3)$ and if we denote $M = (n+3)(m+3)$, we have that

$$B_1(x,y), \dots, B_M(x,y)$$

is the $\mathcal{C}^2-$bicubic B-splines basis of $S_3(T_n \times T_m)$.

2.3 Interpolating bicubic splines

Let $A^N = \{(x_i, y_j) \in T_n \times T_m,\ i = 0, \dots, n,\ j = 0, \dots, m\}$, with $N = (n+1)(m+1)$ and suppose that

$$\underset{p \in R}{sup}\ \underset{\boldsymbol{a} \in A^N}{min} < \boldsymbol{p} - \boldsymbol{a} >_2 = O\left(\frac{1}{N}\right),\ N \longrightarrow +\infty. \tag{2.2}$$

From (2.2) we deduce that $n \longrightarrow +\infty$ and $m \longrightarrow +\infty$ when $N \longrightarrow +\infty$. Let L_1^N be a Lagrangian operator defined from $H^3(R)$ into $\mathbb{R}^N$ given by

$$L_1^N v = (v(\boldsymbol{a}))_{\boldsymbol{a}\in A^N} \tag{2.3}$$

and $L_2^N : H^3(R) \longrightarrow \mathbb{R}^{2n+2m+8}$ given by.

$$L_2^N v = (\mathcal{L}_\ell v)_{\ell=1,\dots,2n+2m+8}, \tag{2.4}$$

where

$$\mathcal{L}_\ell v = \begin{cases} \dfrac{\partial^2 v}{\partial y^2}(x_{\ell-1}, c), & \ell = 1,\dots,n+1, \\[2ex] \dfrac{\partial^2 v}{\partial y^2}(x_{\ell-n-2}, d), & \ell = n+2,\dots,2n+2, \\[2ex] \dfrac{\partial^2 v}{\partial x^2}(a, y_{\ell-2n-3}), & \ell = 2n+3,\dots,2n+m+3, \\[2ex] \dfrac{\partial^2 v}{\partial x^2}(b, y_{\ell-2n-m-4}), & \ell = 2n+m+4,\dots,2n+2m+4, \\[2ex] \dfrac{\partial^4 v}{\partial x^2 \partial y^2}(x_{in}, y_{jm}), & i=0,1,\ j=0,1,\ \ell = 2n+2m+4+2i+j+1, \end{cases}$$

Let $B^N = \{u_\ell,\ \ell = 1,\dots,N\} \subset \mathbb{R}$. It is easy to prove the following result.

Theorem 2.3.1. *There exists a unique $S_N \in S_3(T_n \times T_m)$ such that*

$$L_1^N S_N = (u_\ell)_{\ell=1,\dots,N},$$

$$L_2^N S_N = \mathbf{0} \in \mathbb{R}^{2n+2m+8}$$

called the interpolating natural $\mathcal{C}^2-$bicubic spline associated with A^N and B^N.

Thus $\mathcal{C}^2$-bicubic spline verifies that

$$S_N(x,y) = \sum_{k=1}^{M} \alpha_k B_k(x,y),\ (x,y) \in R,$$

where $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_M)^\top \in \mathbb{R}^M$ is the solution of the linear system

$$\boldsymbol{A}\boldsymbol{\alpha} = \boldsymbol{b}, \tag{2.5}$$

with $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \end{pmatrix}$, where

$$\boldsymbol{A_i} = \left(L_i^N B_k\right)_{k=1,\ldots,M}, \ (i = 1, 2) \tag{2.6}$$

$$\boldsymbol{b_1} = (u_\ell)_{\ell=1,\ldots,N}, \tag{2.7}$$

$$\boldsymbol{b_2} = (0)_{\ell=1,\ldots,M-N}. \tag{2.8}$$

Following the techniques used in [50] it follows the following result.

Theorem 2.3.2. *Let $f \in \mathcal{C}^4(R)$ and let S_N be the interpolating natural $\mathcal{C}^2-$bicubic spline associated with A^N and $L_1^N f$, then there exists a constant $C > 0$ such that*

$$| f - S_N |_\ell \leq Ch^{4-\ell}, \ \ \ell = 0, 1, 2, 3, \ N \to +\infty, \tag{2.9}$$

where $h = max\left\{\frac{b-a}{n}, \frac{d-c}{m}\right\}$. Hence

$$\lim_{N\to+\infty} \|f - S_N\| = 0. \tag{2.10}$$

2.4 Fuzzy numbers

When a spectroscopic line has to be identified in order to specify a functional group in infrared spectroscopy or to decide on the presence of an element in atomic spectroscopy, it is necessary to compare the line position with other lines appearing in certain library of reference lines (see [8], [48] and references therein). As the experimentally obtained line will surely not match exactly any of the tabulated ones, usually an interval around the reference lines has to be considered in order to decide whether the experimental line coincides with the reference candidate or not. In such a way that a value of 1 is assigned to a line that matches the interval around this reference line and a value of 0 is assigned to lines outside this reference interval. So, only a yes/no answer would be obtained with this crisp (1/0) procedure, and no difference could be made with respect to a line that comes close to the borders of the interval compared with a line that matches exactly the reference line!

With the concept of fuzzy numbers, as we will see in the definitions below, the coincidence or not of these experimental and reference lines can be described in a much more convenient way, and the advantages of fuzzy modelling and fuzzy pattern recognition have demostrated many worthwhile theoretical and practical applications in Formal and Analytical Chemistry.

Definition 2.4.1. *A fuzzy number is a mapping* $u : \mathbb{R} \longrightarrow [0,1]$ *with the following properties (see [36]).*

i) u is an upper semi-continuous function on $\mathbb{R}$.

ii) $u(x) = 0$ *outside some interval* $[a_1, a_4] \subset \mathbb{R}$.

iii) There exist real numbers a_2 *and* a_3 *such that* $a_1 \leq a_2 \leq a_3 \leq a_4$ *with*

a) $u(x)$ *is a monotonic increasing function on* $[a_1, a_2]$,

b) $u(x)$ *is a monotonic decreasing function on* $[a_3, a_4]$,

c) $u(x) = 1$, *for all* $x \in [a_2, a_3]$.

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, (see fig.2.8.3), that can be defined as $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$, and their membership function is defined by

$$\mu(\boldsymbol{a}) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & \textit{otherwise}. \end{cases}$$

If $a_1 = a_2 = a_3 = a_4 = a$ then the real number is presented by a. If $a_1 = a_2$ and $a_3 = a_4$, then $\boldsymbol{a}$ is called a crisp interval. Note that a triangular fuzzy number is obtained when $a_2 = a_3$, (see Figure 2.8.3), in which case triangular fuzzy numbers can be defined by $\boldsymbol{a} = (a_1, a_2, a_3)$ and their membership

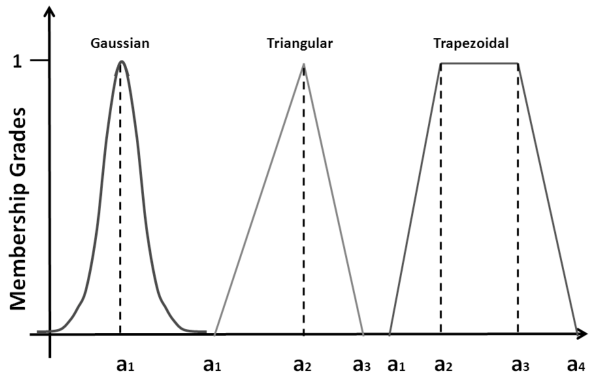

Figure 2.4.1: Examples of gaussian, triangular and trapezoidal fuzzy numbers

function is defined by

$$\mu(\boldsymbol{a}) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & otherwise. \end{cases}$$

Definition 2.4.2. *Let* $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \in \mathbb{TFN}$ *and* $0 < \alpha \leq 1$, *then it is called* $\alpha-$*cut of* $\boldsymbol{u}$ *the set*

$$[\boldsymbol{u}]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

It is defined the $0-$*cut of* u *as its support, i.e.,*

$$[\boldsymbol{u}]^0 = \bigcup_{0<\alpha\leq 1} [\boldsymbol{u}]^\alpha = [u_1, u_4]$$

Remark 2.4.3. *An equivalent definition of a trapezoidal fuzzy number* $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ *is a function* $u : [0,1] \longrightarrow I$ *given by*

$$u(\alpha) = [\underline{u}(\alpha), \bar{u}(\alpha)],$$

with

$$\begin{aligned} \underline{u}(\alpha) &= u_1 + (u_2 - u_1)\alpha, \\ \bar{u}(\alpha) &= u_4 + (u_3 - u_4)\alpha, \end{aligned} \tag{2.11}$$

where I is the set of the all real closed intervals. Obviously we have that $u(\alpha) = [\boldsymbol{u}]^\alpha$, for any $0 \leq \alpha \leq 1$. □

For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{TFN}$, $\lambda \in \mathbb{R}$, the sum $\boldsymbol{u} + \boldsymbol{v}$ and the product $\lambda \boldsymbol{u}$ are defined by

$$[\boldsymbol{u} + \boldsymbol{v}]^\alpha = [\boldsymbol{u}]^\alpha + [\boldsymbol{v}]^\alpha, \ [\lambda \boldsymbol{u}]^\alpha = \lambda [\boldsymbol{u}]^\alpha,$$

for all $\alpha \in [0, 1]$, $\lambda > 0$, taking into account that

$$\lambda[\underline{u}(\alpha), \bar{u}(\alpha)] = \begin{cases} [\lambda \underline{u}(\alpha), \lambda \bar{u}(\alpha)], & \lambda \geq 0, \\ [\lambda \bar{u}(\alpha), \lambda \underline{u}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 2.4.4. *For any $u, v \in \mathbb{TFN}$, it is defined the Haussdorf distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ as the quantity*

$$d(\boldsymbol{u}, \boldsymbol{v}) \equiv S_d(\boldsymbol{u}, \boldsymbol{v}) = \sup_{\alpha \in (0,1]} max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|\}. \qquad (2.12)$$

Definition 2.4.5. *A fuzzy bivariate function defined on the set $R \subset \mathbb{R}^2$ is an application $\boldsymbol{f} : R \longrightarrow \mathbb{TFN}$ such that $\boldsymbol{f} = (f_1, f_2, f_3, f_4)$, where f_i is a real function defined on R and $\boldsymbol{f}(x, y) \in \mathbb{TFN}$, for any $(x, y) \in R$.*

2.5 Fuzzy interpolating bicubic splines

Suppose given two partitions $T_n = \{a = x_0 < x_1, \ldots, < x_n = b\}$ and $T_m = \{c = y_0 < y_1, \ldots, < y_m = d\}$ of $[a, b], [c, d] \subset \mathbb{R}$, respectively, and let $S_3(T_n \times T_m)$ be the corresponding $\mathcal{C}^2-$bicubic spline space and $R = [a, b] \times [c, d]$. Let $N = (n+1)(m+1)$ and $M = (n+3)(m+3)$ and let $\{B_1(x, y), \ldots, B_M(x, y)\}$ be the $\mathcal{C}^2-$bicubic B-spline basis of $S_3(T_n \times T_m)$.

Definition 2.5.1. *The fuzzy bicubic splines space constructed on the partition $T_n \times T_m$ is the set of fuzzy functions*

$$S_3(T_n \times T_m; \mathbb{TFN}) \equiv$$

$$\{\boldsymbol{s} : R \longrightarrow \mathbb{TFN}, \quad \boldsymbol{s} = \sum_{\ell=1}^{M} \alpha_\ell B_\ell, \ \alpha_\ell \in \mathbb{TFN}, \ \ell = 1, \ldots, M\}.$$

Now, we consider the following interpolation problem:
Given $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\} \subset \mathbb{TFN}$, find a fuzzy function $\boldsymbol{s} \equiv (s_1, s_2, s_3, s_4) \in S_3(T_n \times T_m; \mathbb{TFN})$ such that

$$\begin{cases} \boldsymbol{s}(x_i, y_j) = \boldsymbol{u}_\ell, & \ell = (n+1)j + i + 1, \\ & i = 0, \ldots, n, \; j = 0, \ldots, m, \\ L_2^N s_k = 0, & k = 1, 2, 3, 4, \end{cases} \tag{2.13}$$

Theorem 2.5.2. *Problem (2.13) has a unique solution* $\boldsymbol{\sigma} \in S_3(T_n, T_m; \mathbb{TFN})$ *given by*

$$\boldsymbol{\sigma}(x, y) = \sum_{i=1}^{M} \alpha_i B_i(x, y), \; (x, y) \in R,$$

where $\boldsymbol{\Lambda} \equiv (\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_M}) \in \mathbb{TFN}$ *are the solution of the linear systems* $\boldsymbol{A}\boldsymbol{\Lambda} = \boldsymbol{b}$, *where* $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{pmatrix}$ *is given by (2.6) and* $\boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$, *with* $\boldsymbol{b}_1 = (\boldsymbol{u}_\ell)_{\ell=1,\ldots,N}$ *and* $\boldsymbol{b}_2 = (\boldsymbol{0})_{\ell=1,\ldots,M-N}$, *being* $\boldsymbol{0} \equiv (0, 0, 0, 0)$ *and* $\boldsymbol{u}_\ell = (u_{\ell 1}, u_{\ell 2}, u_{\ell 3}, u_{\ell 4})$, $\ell = 1, \ldots, N$.

Proof. For any $k = 1, 2, 3, 4$, taking into account Theorem 1, there exists a unique $\sigma_k \in S_3(T_n \times T_m)$ such that

$$\begin{cases} \sigma_k(x_i, y_j) = u_{\ell k}, & \ell = (n+1)j + i + 1, \\ & i = 0, \ldots, n, \; j = 0, \ldots, m \\ L_2^N \sigma_k = 0, & \end{cases} \tag{2.14}$$

Let $\boldsymbol{\sigma} : R \longrightarrow \mathbb{R}^4$ given by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, then $\boldsymbol{\sigma} \in (S_3(T_n \times T_m))^4$. Also, taking into account Theorem 1 for $\ell = 1, \ldots, N$, there exists a unique $L_\ell \in S_3(T_n \times T_m)$ such that

$$\begin{cases} L_\ell(\boldsymbol{a}_r) = \delta_{\ell,r} = \begin{cases} 1, & \text{if } \ell = r, \\ 0, & \text{otherwise,} \end{cases} & r = 1, \ldots, N, \\ L_2^N(L_\ell) = 0, & \end{cases}$$

being $\boldsymbol{a}_r = (x_i, y_j)$, for any $i = 0, \ldots, n,\ j = 0, \ldots, m$, with $r = (n+1)j+i+1$. Then

$$\sigma_k(x,y) = \sum_{1}^{N} u_{\ell k}\, L_\ell(x,y),\ k = 1,2,3,4. \tag{2.15}$$

From $u_{\ell 1} \leq u_{\ell 2} \leq u_{\ell 3} \leq u_{\ell 4},\ \ell = 1, \ldots, N$ and (2.15) we obtain that

$$\sigma_1(x,y) \leq \sigma_2(x,y) \leq \sigma_3(x,y) \leq \sigma_4(x,y), \quad (x,y) \in R.$$

Then $\boldsymbol{\sigma}(x,y) \in \mathbb{TFN}$, for any $(x,y) \in R$, and hence $\boldsymbol{\sigma} \in S_3(T_n, T_m; \mathbb{TFN})$. $\square$

2.6 Convergence result

Let $\boldsymbol{f} : R \longrightarrow \mathbb{TFN}$ be a fuzzy function

$$\boldsymbol{f}(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y)),\ \text{for all } (x,y) \in R,$$

being $f_i \in \mathcal{C}^4(R),\ i = 1,2,3,4$. Let $\boldsymbol{\sigma} \in S_3(T_n \times T_m; \mathbb{TFN})$ the fuzzy bicubic spline verifying (2.13) for $\boldsymbol{u}_\ell = \boldsymbol{f}(\boldsymbol{a}_\ell),\ \ell = 1, \ldots, N$, and $\boldsymbol{a}_\ell = (x_i, y_j),\ i = 0, \ldots, n,\ j = 0, \ldots, m,\ \ell = (n+1)j + i + 1$. Let $h = max\left\{\dfrac{b-a}{n}, \dfrac{d-c}{m}\right\}$. From (2.2), we also have

$$h = \mathcal{O}(\frac{1}{N}),\ N \to +\infty \tag{2.16}$$

Theorem 2.6.1. *Suppose hypothesis* (2) *holds. Then, for any* $(x,y) \in R$

$$d(\boldsymbol{\sigma}(x,y), \boldsymbol{f}(x,y)) = \mathcal{O}(h^4),\ N \to +\infty$$

and thus

$$\lim_{N \to +\infty} d(\boldsymbol{\sigma}(x,y), \boldsymbol{f}(x,y)) = 0.$$

Proof. From Definition 3, we have

$$d(\boldsymbol{u}, \boldsymbol{v}) = \underset{i=1,2,3,4}{max} |u_i - v_i|, \tag{2.17}$$

for all $\boldsymbol{u} = (u_1, u_2, u_3, u_4),\ \boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{TFN}$. From Theorem 2 and (2.2) we obtain that

$$|f_i(x,y) - \sigma_i(x,y)| = \mathcal{O}(h^4), \ N \to +\infty \tag{2.18}$$

for all $i = 1, 2, 3, 4$ and $(x, y) \in R$. From (3.10) and (2.18) we deduce that

$$d(\boldsymbol{\sigma}(x,y), \boldsymbol{f}(x,y)| = \mathcal{O}(h^4), \ N \to +\infty,$$

and thus

$$\lim_{N\to+\infty} d(\boldsymbol{\sigma}(x,y), \boldsymbol{f}(x,y)) = 0,$$

for any $(x, y) \in R$.

□

2.7 Similarity measures of fuzzy numbers

The concept of similarity of fuzzy numbers is fundamental in the field of fuzzy decision making [33], fuzzy risk and safety analysis [60], pipping risk assesment, batch crystallizer, combustion processes, food production, fluidized catalytic cracking units and chemical separation processes in general; see [37], [44], [48] and references therein.

In this section, we consider some existing similarity measures of fuzzy numbers. If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the degree of similarity $S(A, B)$ between the trapezoidal fuzzy numbers A and B is defined by Chen [13] as follows:

$$S_{CHEN}(A,B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \in [0,1]$$

where $\mid a \mid$ is the absolute value of the real number a.
In [43] Lee proposed another similarity measure as follows :

$$S(A,B) = 1 - \frac{\parallel A - B \parallel_{l_p}}{\parallel U \parallel} \times 4^{-\frac{1}{p}},$$

Where U is the universe of discourse (the range of all possible values for an input to a fuzzy variable or system)

$$\parallel A - B \parallel_{l_p} = \left(\sum_{i=1}^{4} |a_i - b_i|^p \right)^{\frac{1}{p}}.$$

and $\| U \| = max(U) - min(U)$.
In [32], Hsieh et al. proposed a similarity measure using the *graded mean integration-representation distance* where the degree of similarity $S(A,B)$ between the fuzzy numbers A and B is calculated as follows:

$$S_{HSIEH}(A,B) = \frac{1}{1+d(A,B)},$$

where $d(A,B) = \mid P(A) - P(B) \mid$, and $P(A)$, $P(B)$ are the graded mean integration representations of A and B, respectively. If A and B are trapezoidal fuzzy numbers, with $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$, then the graded mean integration of these fuzzy numbers is defined as:

$$P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},$$

$$P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

In [14] Chen & Chen presented another similarity measure between generalized trapezoidal fuzzy numbers. They presented the (simple center of gravity method) denoted as SCGM to calculate the center of gravity points (x_A^*, y_A^*) and (x_B^*, y_B^*) of the generalized trapezoidal fuzzy number A and B respectively. $A = (a_1, a_2, a_3, a_4)$, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, and $B = (b_1, b_2, b_3, b_4)$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$. Then the degree of similarity $S(A,B)$ between the trapezoidal fuzzy numbers A and B, using the SCGM methodology, is calculated as follows :

$$\begin{aligned} S_{SCGM}(A,B) = \; & 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \times \\ & (1 - \mid x_A^* - x_B^* \mid)^{B(S_A, S_B)} \times \\ & \frac{min(y_A^*, y_B^*)}{max(y_A^*, y_B^*)}, \end{aligned}$$

where $S(A,B) \in [0,1]$, and

$$x_A^* = \frac{y_A^*(a_3 + a_2) + (a_4 + a_1)(1 - y_A^*)}{2},$$

$$y_A^* = \begin{cases} \frac{1}{2}, & if \quad a_1 = a_4, \\ \frac{1}{6}\left(\frac{a_3 - a_2}{a_4 - a_1} + 2\right), & if \quad a_1 \neq a_4, \end{cases}$$

and

$$B(S_A, S_B) = \begin{cases} 1, & if \quad S_A + S_B > 0, \\ 0, & if \quad S_A + S_B = 0, \end{cases}$$

where S_A and S_B are the lengths of the bases of trapezoidal fuzzy numbers A and B, respectively, and defined by:

$$S_A = a_4 - a_1,$$
$$S_B = b_4 - b_1.$$

2.8 Simulation results

In this section, different interpolation error and similarity estimations are proposed in order to analyze the presented fuzzy interpolation method. The definition of these estimations is

$$\overline{S} = \frac{1}{Z}\sum_{i=1}^{Z} S(f(\xi_i), \sigma(\xi_i)), \tag{2.19}$$

where $\{\xi_1, \ldots, \xi_Z\} \subset R$ is a set of Z random points in the domain R and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}), the Chen & Chen index (S_{SCGM}), defined in Section 2.7, or the Haussdorf distance (S_d) given in Definition 2.4.4 . From Theorem 4, it should be verified that $\overline{S}$ tends to 1 as $N \to +\infty$, for $S = S_{CHEN}, S_{HSIEH}, S_{SCGM}$, and $\overline{S}$ tends to 0 as $N \to +\infty$, for $S = S_d$.

To test our method we consider two examples for two fuzzy functions and for different partitions of its domains.

Example 1: $\boldsymbol{f}_1 : [0,\pi] \times [0,\pi] \longrightarrow \mathbb{TFN}$,

$$\begin{aligned}
&\boldsymbol{f}_1(x,y) = \\
&(0.5\sin(\tfrac{\pi}{2} - x)^2\cos(\tfrac{\pi}{2} - y)^2 - 0.2\sin(\tfrac{\pi}{2} - x)^2, \\
&0.5\sin(\tfrac{\pi}{2} - x)^2\cos(\tfrac{\pi}{2} - y)^2 + 0.2\sin(\tfrac{\pi}{2} - x)^2 + 0.2, \\
&0.4\sin(\tfrac{\pi}{2} - x)^2\cos(\tfrac{\pi}{2} - y)^2 + 0.3\sin(\tfrac{\pi}{2} - x)^2 + 0.4, \\
&-0.2\sin(\tfrac{\pi}{2} - x)^2\cos(\tfrac{\pi}{2} - y)^2 + 0.5\sin(\tfrac{\pi}{2} - x)^2 + 0.5).
\end{aligned}$$

Example 2: $\boldsymbol{f}_2 : [0,4\pi] \times [0,4\pi] \longrightarrow \mathbb{TFN}$,

$$\begin{aligned}
&\boldsymbol{f}_2(x,y) = \\
&(0.01(x-2\pi)^2 + 0.4\sin(0.2(x-2\pi)^2)e^{-0.1(y-2\pi)^2} - 0.04, \\
&0.01(x-2\pi)^2 + 0.2\cos(0.1(y-2\pi)^2)e^{-0.3(x-2\pi)^2} + 0.5, \\
&0.01(x-2\pi)^2 + 0.3\sin(0.2(x-2\pi)^2)e^{-0.05(y-2\pi)^2} + 1, \\
&0.01(x-2\pi)^2 + 0.3\cos(0.2(x-2\pi)^2)e^{-0.05(y-2\pi)^2} + 1.6).
\end{aligned}$$

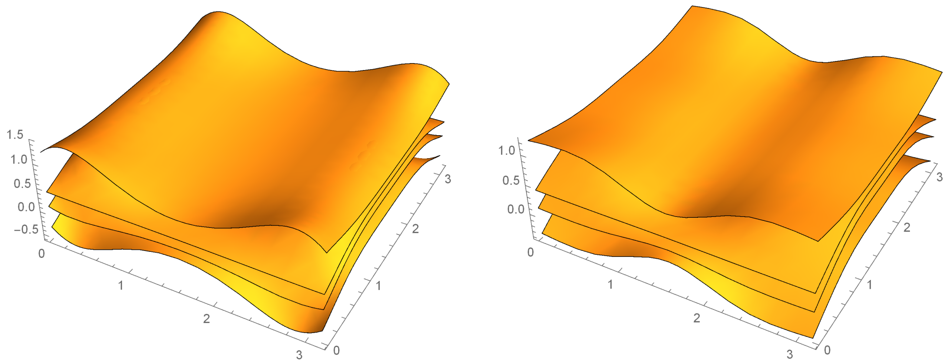

Figure 2.8.2: *Example 1. From top to bottom, graph of the fuzzy function* $\boldsymbol{f}_1$ *and its fuzzy interpolating bicubic spline from a partition of the domain in* 4×4 *equal squares* ($n = m = 4$)*. The error estimation is* $\overline{S}_d = 7.8692\times 10^{-2}$.

For the simulations presented in this section, the number of points to compute the estimation $\overline{S}$ given by (3.13) is $Z = 500$ in all cases.

Figures 2 and 3 show the graph of the fuzzy functions f_1 and f_2 and its fuzzy interpolating bicubic splines for $n = m = 4$ in the first case, and $n = m = 6$ in the second case. The error estimations are $S_d = 7.8692 \times 10^{-2}$ and $S_d = 1.0454 \times 10^{-1}$ respectively.

Tables 1 and 2 illustrate, for Examples 1 and 2, the performance of the fuzzy interpolating bicubic spline for different values of the knot numbers n and m, taking $n = m$ in all cases. The proposed simulations show the influence and relative importance of the knot number in the effectiveness of the approximation. Specifically, the error estimation $\overline{S}_d$ decreases to 0 as $N = (n + 1) \times (m + 1)$ tends to $+\infty$ and the similarity index estimations increases to 1 as $N \to +\infty$.

To verify the performance of the new estimation presented in this section, another two different examples of two dimensional fuzzy functions will be considered for several values of knot numbers.

Example 3: $\quad \boldsymbol{f}_3 : [0,\pi] \times [0,\pi] \longrightarrow \mathbb{TFN},$

$$\boldsymbol{f}_3(x,y) = \\ (0.5\sin x\cos y - 0.1\sin x, 0.5\sin 5x\cos 3y + 0.3\cos y, \\ 0.5\sin xe^{-0.5y} + 0.28\cos y, 0.5\sin 5x\cos 5y + 0.04\cos y).$$

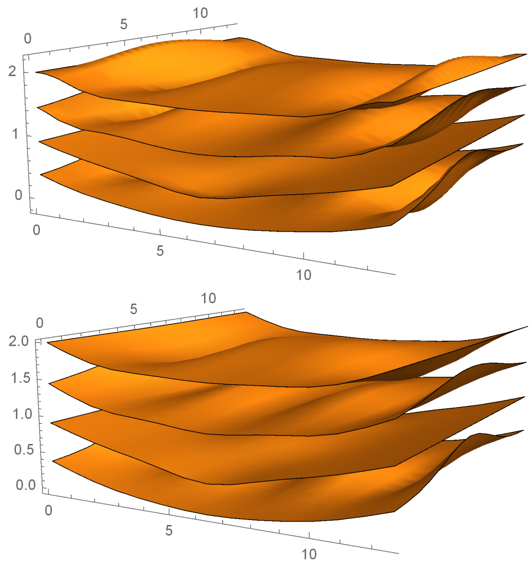

Figure 2.8.3: *Example 2. From top to bottom, graph of the fuzzy function* $\boldsymbol{f}_2$ *and its fuzzy interpolating bicubic spline from a partition of the domain in* 6×6 *equal squares* ($n = m = 6$)*. The error estimation is* $\overline{S}_d = 1.0454\times10^{-1}$.

Table 2.1 : Example 1. Error and similarity indices estimates for different knot numbers.

Values of $n = m$	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
4	7.8692×10^{-2}	0.949328	0.955893	0.955190
6	2.7178×10^{-2}	0.981314	0.982188	0.982048
8	9.8117×10^{-3}	0.992527	0.993364	0.993119
10	5.4005×10^{-3}	0.995724	0.996427	0.996109
12	3.0842×10^{-3}	0.997768	0.997906	0.997895
20	6.9539×10^{-4}	0.999533	0.999617	0.999539

Table 2.2 : Example 2. Error and similarity indices estimates for different knot numbers.

Values of $n = m$	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
4	1.5776×10^{-1}	0.926303	0.945866	0.942131
6	1.0454×10^{-1}	0.950933	0.956212	0.954863
8	5.4174×10^{-2}	0.979548	0.979488	0.981600
10	2.5779×10^{-2}	0.988891	0.987013	0.987599
12	1.6717×10^{-2}	0.989741	0.991822	0.992621
20	2.3462×10^{-3}	0.998460	0.998929	0.998613

Example 4: $\boldsymbol{f}_4 : [0, 4\pi] \times [0, 4\pi] \longrightarrow \mathbb{TFN}$,

$$\boldsymbol{f}_4(x, y) = (0.2 \sin 5xe^{-0.5y} - 0.04, 0.5e^{-0.5x} \sin 5y + 0.3, 0.5e^{-0.5y} \sin 3x + 0.4, 0.1e^{-0.6y} \cos 5y + 0.1e^{-0.5y} \sin x + 0.5).$$

For the simulations presented in this section, the number of points to compute the estimation $\overline{S}$ given by (3.13) is $Z = 500$ in all cases.

In order to compare the behavior of the fuzzy spline interpolation method, different interpolation error and similarity indices are used for several simulations in both examples 3 and 4 (in each simulation, the value of the elements and the number of knots are modified with values: 4, 8, 12 and 20). The different interpolation error and similarity indices proposed in this chapter, using several similarity measures, present a good performance, being an enough criterion for comparing interpolations of fuzzy data by fuzzy bicubic spline functions from different interpolation parameters.

Table 2.3 illustrated the performance of the fuzzy interpolating bicubic spline, taking into account all the interpolating error and similarity indices defined (the mean, standard deviation, maximum and minimum values for the 500 simulations of each interpolating error and similarity indices) for Examples 3 and 4, respectively. For example 3 and 4, Fig. 2.8.4 and 2.8.5 respectively show the $\bar{S}_{SCGM}$ evolution using the interpolating error and similarity indices proposed in this chapter, running 500 simulation for 4, 8, 12 and 20 numbers knot.

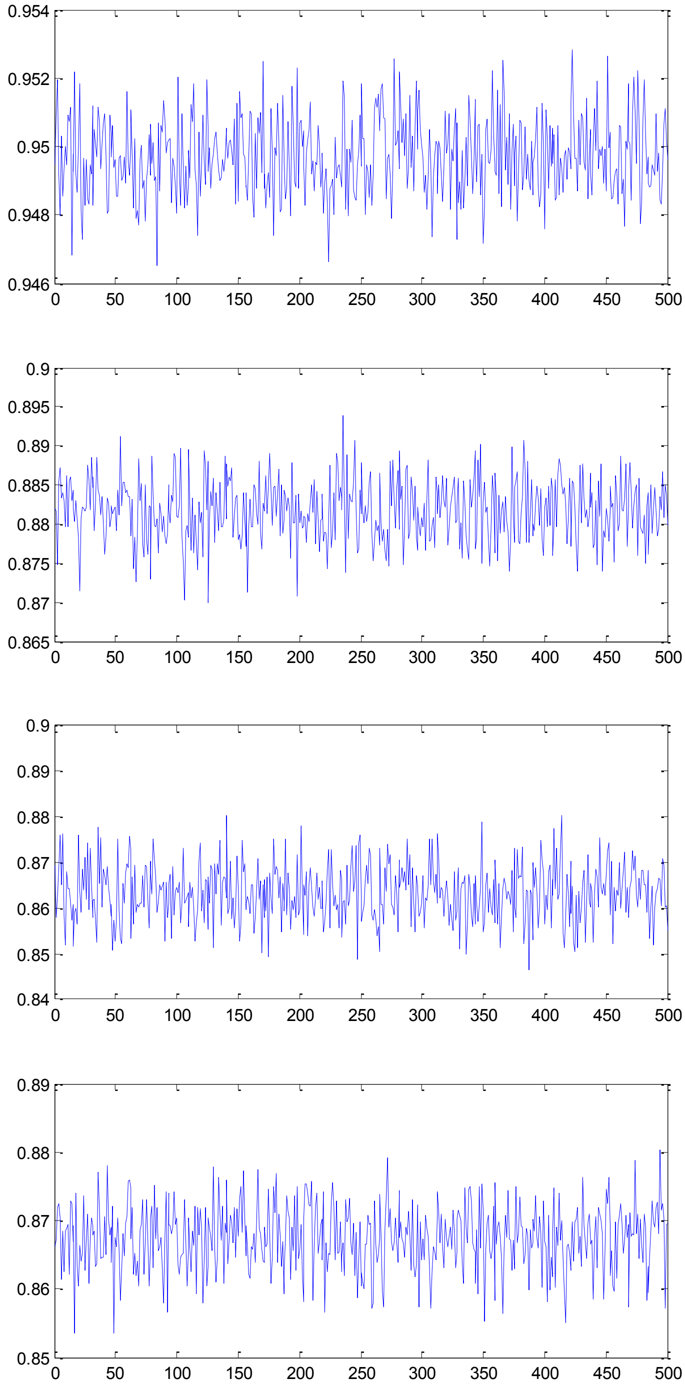

Figure 2.8.4: $\bar{S}_{SCGM}$ evolution of the mean error index for example 3

Table 2.3 : Comparison of the proposed error and similarity indexes for Example 3 and 4, using different knot numbers.

		Example 3			Example 4		
Values of $n = m$	Index	$\overline{S}_{SCGM}$	$\overline{S}_{HSIEH}$	$\overline{S}_{CHEN}$	$\overline{S}_{SCGM}$	$\overline{S}_{HSIEH}$	$\overline{S}_{CHEN}$
4	Mean	0.867470	0.840446	0.771953	0.882781	0.843301	0.712834
	Std	0.004692	0.002787	0.004404	0.016889	0.007183	0.017919
	Max	0.880308	0.848219	0.784822	0.938465	0.864046	0.756069
	Min	0.853461	0.831331	0.758369	0.846258	0.816674	0.650042
8	Mean	0.863352	0.839728	0.758032	0.823719	0.890256	0.805510
	Std	0.006163	0.004113	0.006681	0.003025	0.002819	0.003516
	Max	0.880413	0.851310	0.774986	0.832356	0.900044	0.815093
	Min	0.846512	0.826102	0.736783	0.813802	0.879973	0.794517
12	Mean	0.881757	0.898143	0.875938	0.859597	0.878151	0.794964
	Std	0.003865	0.002768	0.003235	0.001494	0.003108	0.001766
	Max	0.893942	0.906942	0.886265	0.864884	0.887585	0.800662
	Min	0.869893	0.890009	0.866054	0.853517	0.868141	0.789427
20	Mean	0.949716	0.937761	0.915731	0.858117	0.875988	0.837100
	Std	0.001113	0.001467	0.001337	0.002546	0.002641	0.002191
	Max	0.952832	0.942108	0.920103	0.865160	0.883508	0.844105
	Min	0.946523	0.933766	0.911855	0.849287	0.868993	0.829623

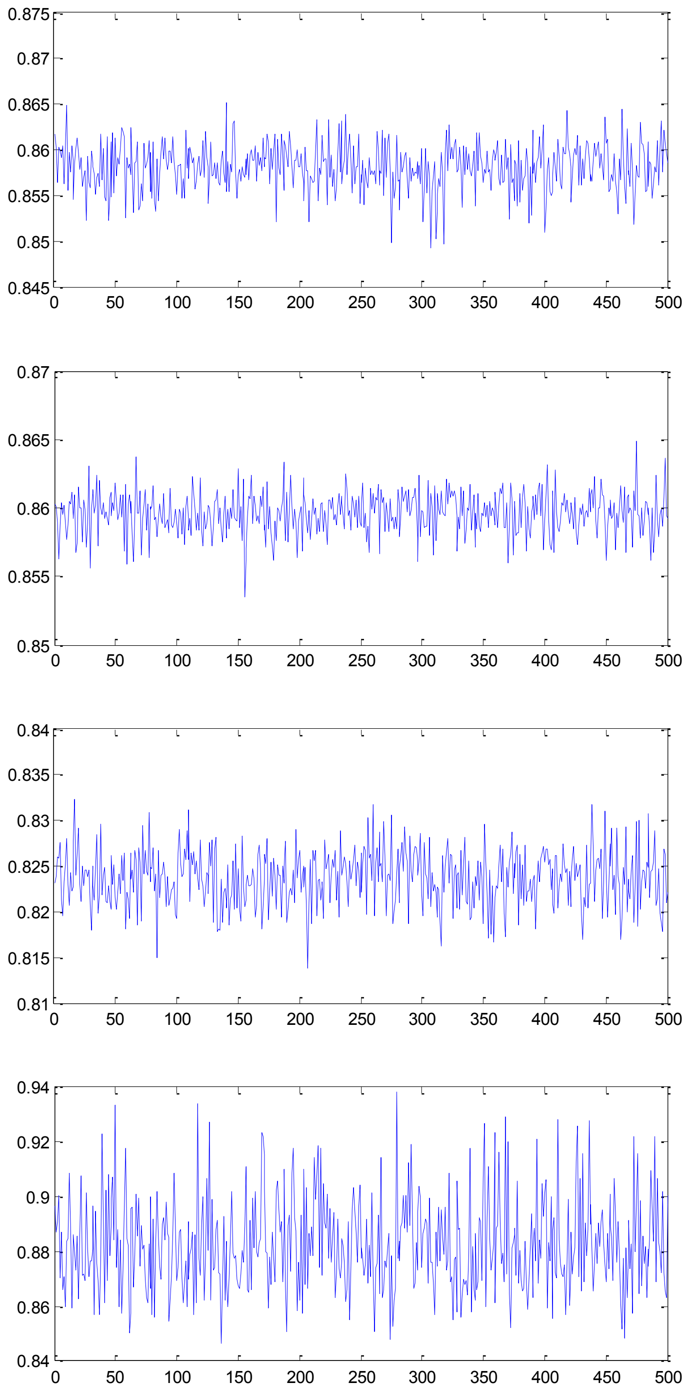

Figure 2.8.5: $\bar{S}_{SCGM}$ evolution of the mean error index for example 4

2.9 Statistical analysis of the proposed interpolating error and similarity indices.

It is very important to analyze individual and global behavior for the different proposed error and similarity indices. In order to perform this analysis, we will study whether different error and similarity indices provide a very similar criterion to decide the degree of accuracy for fuzzy interpolating data set.

In performing this analysis, the correlation matrix between the different

Table 2.4 : Correlation matrix R for the error and similarity indexes proposed for example 3, using different values of the number of Knots in. (a total of 2000 random simulations were used for the computation of the matrix R).

R	$\overline{S}_{SCGM}$	$\overline{S}_{HSIEH}$	$\overline{S}_{CHEN}$
$\overline{S}_{SCGM}$	1.000000	0.907381	0.846211
$\overline{S}_{HSIEH}$	0.907381	1.000000	0.988015
$\overline{S}_{CHEN}$	0.846211	0.988015	1.000000

error and similarity indices for each of the examples 3 and 4 used will be constructed. For each example, the linear relationships among the error and similarity indices will be tested with 2000 samples. The matrix R of the correlation coefficients is calculated for each of their components by means of the following equation:

$$R(i,j) = \frac{C(x_i, x_j)}{\sqrt{C(x_i, x_i)C(x_j, x_j)}}$$

where $C(x_i, x_j)$ is the covariance matrix between the input variable x_i and the x_j input variable, obtained as:

$$C(x_i, x_j) = E\left[(x_i - \mu_i)((x_j - \mu_j))\right]$$

where E is the mathematical expectation and $\mu_i = E[x_i]$. In our case, $x_i, i = 1, \ldots, 3$ are the different error and similarity indices presented in Tables 2.3 . Taking into account simulations from Example 3 and Example 4, the matrix R is obtained (Table 2.4 and 2.5), with its corresponding P matrix. P is a matrix of p-values for testing the hypothesis of no correlation. Each p-value is the probability of getting a correlation as large as the value observed by random chance, when the true correlation is zero. If $P(i,j)$ is small, say less than 0.05, then the correlation $R(i,j)$ is significant. The

Table 2.5 : Correlation matrix R for the error and similarity indexes proposed for example 4, using different several values of the number of Knots (a total of 2000 random simulations were used for the computation of the matrix R).

R	$\overline{S}_{SCGM}$	$\overline{S}_{HSIEH}$	$\overline{S}_{CHEN}$
$\overline{S}_{SCGM}$	1.000000	-0,821920	-0,613498
$\overline{S}_{HSIEH}$	-0,821920	1.000000	0,844894
$\overline{S}_{CHEN}$	-0,613498	0,844894	1.000000

Table 2.6 : Example 1: matrix of p-values for testing the hypothesis of no correlation associated with R matrix of table 2.4 and 2.5 .

P	$\overline{S}_{SCGM}$	$\overline{S}_{HSIEH}$	$\overline{S}_{CHEN}$
$\overline{S}_{SCGM}$	1.000	0.000	0.000
$\overline{S}_{HSIEH}$	0.000	1.000	0.000
$\overline{S}_{CHEN}$	0.000	0.000	1.000

matrix P or matrix of p-values for testing the hypothesis of no correlation is, in this examples, the identity matrix as seen in Table 2.6 .
As observed for matrix P, all the correlations are statistically significant (to be expected due to the fact that all the error and similarity indices are based on similarity measures). It is also important to underline the negative correlation coefficients. This negative correlation could be expected by analyzing the equation of the error and similarity indices presented in Section 2.7. the correlation coefficient among error and similarity indices $\bar{S}_{SCGM}$, $\bar{S}_{HSICH}$ and $\bar{S}_{CHEN}$, are positive and closed to one. Analyzing the results presented in both examples 3 and 4, it can be concluded that the proposed error and similarity indices illustrated a homogeneous behavior, having a statistically significant linear correlation between them.

2.10 Conclusions

Fuzzy theory based methods (see for example [7]) can help to solve, among many others, the following problems of contemporary analytical chemistry:

- handling uncertain and incomplete data sets or identifying blurred spectra in infrared/ultraviolet spectroscopy or in chromatography,

- modelling data in cases where the assumed model is not exactly valid,
- incorporating and managing, in a rigorous and consistent way, uncertain, inconsistent and/or incomplete information in modern automatic and expert analytical systems.

In this way, by using the fuzzy approach, the error in observations is modelled by the concept of membership to the set of possible and predicted concentration values by means of an appropriate membership function, without the necessity of introducing probability based assumptions.

Also, reviewing references concerning the interpolation and approximation of fuzzy data, there is a significant lack of development of an interpolation method for a 3D fuzzy data set or a fuzzy bivariate function.

In this chapter, we present a fuzzy interpolation method of 3D fuzzy data or fuzzy bivariate functions. We study the solution of this problem and we establish a convergence result about the presented method.

Then, we develop a similar methodology as in [66] to define and use error and similarity indices suitable for the 3D interpolation problem of fuzzy data by means of fuzzy bivariate spline functions.

Two different examples with two-variables fuzzy functions have been presented in order to analyze the behavior of these indices.

Analyzing the results presented in Section 3.6 it can be concluded that the proposed error and similarity indices estimations confirm the effectiveness of this method and the convenience of using it in all kind of chemical and other similar engineering situations.

Chapter 3

Smoothing fuzzy bicubic splines.

3.1 Introduction

One of the most interesting and extended problems in applied mathematics is function surface fitting approximation and interpolation. The following interpolation problem of fuzzy data was first proposed by Zadeh, see [70]. Suppose that we have $n+1$ different points in $\mathbb{R}$, and for each of these numbers a fuzzy value in $\mathbb{R}$, u_i, $i = 0, 1, \cdots, n$. Zadeh gave the question whether it is possible to define some kind of smooth function on $\mathbb{R}$ to the $n+1$ points. Lowen investigated a fuzzy lagrange interpolation in [45]. Later, Kaleva [35] proposed some properties of Lagrange interpolation by using cubic spline approximation. Wang and Li gave the definition of simple fuzzy numbers and the expressions of their membership functions in [67]. There exist different methodologies for approximation to data. For example, the authors in [47, 55] gave a method for approximation or fitting surfaces to data, using smoothing splines. Abbasbandy et al. [2] gave a method for finding the best approximation of fuzzy function on a set of points. In [57] a new method for finding best approximation for function by using trapezoidal fuzzy numbers if given.

The authors in [24] proposed a new set of spline functions to interpolate given fuzzy data. Approximation of fuzzy data can be obtained in different research areas. For example, the authors in [17] constructed approximations that comprised a sequence of fuzzy numbers by using the F-transform and

the max-product Bernstein operators. Later, Huang and his co-authors [34] considered how to smooth fuzzy numbers and construct smooth approximations for fuzzy numbers by using the convolution method. In [63] Shu and Wu introduced a new methodology to analyzed the quality-based supplier selection and evaluation using fuzzy data from light emitting diodes. In [4], a new set of spline functions denoted as "fuzzy splines" to interpolate fuzzy data is defined.
In the literature, there exist different methodologies for fitness approximation techniques, using multi-objective evolutionary algorithms [49]. Numerical examples are presented to illustrate a new methodology for approximation and interpolation of fuzzy numbers by cubic smoothing spline presented in [65, 66].
In this paper a new approximation method of fuzzy data by fuzzy smoothing bicubic splines is presented as an approximation of fuzzy bivariate functions. The paper is organized as follows: After this introduction, Section 2 present some notations and preliminaries about C^2-cubic and bicubic B-splines spaces. Section 3 is devoted the definition, computations and convergence results of the smoothing variational bicubic splines. In Section 4 we present the basic definition of the fuzzy numbers and some of the existing similarity measures of fuzzy numbers presented in the bibliography [14]–[32]. In Section 5 the proposed methodology for fuzzy approximation of fuzzy numbers by fuzzy smoothing bicubic spline functions is presented in detail and a convergence result is established. Several experiments are carried out in Section 6 to analyze de behavior of the approximation of fuzzy numbers, performed by fuzzy smoothing bicubic spline functions using different configurations (number of knots, different numbers of approximation points and several values of parameter ε). Then, to analyze individual and global behavior for the different proposed error and similarity indices, statistical analyses are shown. Finally, some conclusions are realized in Section 7.

3.2 Preliminaries

We denote by $< \cdot >_k$ and $< \cdot, \cdot >_k$, respectively, the Euclidean norm and inner product in $\mathbb{R}^k$. For any real intervals (a, b) and (c, d), with $a < b$ and $c < d$, we consider the rectangle $\Omega = (a, b) \times (c, d)$ and let $H^3(\Omega; \mathbb{R}^k)$ be the usual Sobolev space of (classes of) function u belong to $L^2(\Omega; \mathbb{R}^k)$, together with all their partial derivatives $D^\beta(u)$ with $\beta = (\beta_1, \beta_2)$, in the distribution sense, of order $|\beta| = \beta_1 + \beta_2 \leq 3$. For $k = 1$ we denote $H^3(\Omega; \mathbb{R}^k)$ by $H^3(\Omega)$.

The Sobolev space $H^3(\Omega; \mathbb{R}^k)$ is equipped with the norm

$$\|u\| = \left(\sum_{|\beta| \le 3} \int_\Omega \langle D^\beta u(p) \rangle_k^2 dp \right)^{\frac{1}{2}},$$

the semi-norms

$$|u|_\ell = \left(\sum_{|\beta| = \ell} \int_\Omega \langle D^\beta u(p) \rangle_k^2 dp \right)^{\frac{1}{2}}, \quad 0 \le \ell \le 3.$$

and the corresponding inner semi-products

$$(u, v)_\ell = \sum_{|\beta| = \ell} \int_\Omega \langle D^\beta u(p), D^\beta v(p) \rangle_k dp, \quad 0 \le \ell \le 3.$$

Moreover, for $n, m \in \mathbb{N}^*$, let $\Delta_n = \{x_0, \cdots, x_n\}$, $\Delta_m = \{y_0, \cdots, y_m\}$ be some subsets of distinct points of $[a, b]$ and $[c, d]$, with $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_0 < y_1 < \cdots < y_m = d$. We denote by $S_3(\Delta_n)$ and $S_3(\Delta_m)$ the spaces of cubic splines of class C^2 given by

$$S_3(\Delta_n) = \{s \in C^2[a, b] : \ s|_{[x_{i-1}, x_i]} \in \mathbb{P}_3[x_{i-1}, x_i], \ i = 1, \cdots, n\},$$

$$S_3(\Delta_m) = \{s \in C^2[c, d] : \ s|_{[y_{j-1}, y_j]} \in \mathbb{P}_3[y_{j-1}, y_j], \ j = 1, \cdots, m\},$$

where $\mathbb{P}_3[x_{i-1}, x_i]$ ($\mathbb{P}_3[y_{j-1}, y_j]$) is the restriction on $[x_{i-1}, x_i]$ ($[y_{j-1}, y_j]$) of the linear space of real polynomials with total degree less than or equal to 3. It is known that dim $S_3(\Delta_n) = n + 3$ (dim $S_3(\Delta_m) = m + 3$). Let $\{\phi_1, \cdots, \phi_{n+3}\}$ and $\{\psi_1, \cdots, \psi_{m+3}\}$ be bases of functions with local support of $S_3(\Delta_n)$ and $S_3(\Delta_m)$, respectively and consider the space $S_3(\Delta_n \times \Delta_m)$ of bicubic splines functions of class C^2 given by

$$S_3(\Delta_n \times \Delta_m) = span\ \{\phi_1, \cdots, \phi_{n+3}\} \otimes \ span\ \{\psi_1, \cdots, \psi_{m+3}\}$$

This space is a Hilbert subspace of $H^3(\Omega)$ equipped with the same norm, semi-norms and inner semi-products of such space and verifies

$$S_3(\Delta_n \times \Delta_m) \subset H^3(\Omega) \cap C^2(\Omega). \tag{3.1}$$

Particulary, let

$$\{B_0^3(x), \cdots, B_{n+2}^3(x)\} \ \left(\{B_0^3(y), \cdots, B_{m+2}^3(y)\}\right)$$

be the C^2-cubic B-splines basis of $S_3(\Delta_n)$ $(S_3(\Delta_m))$, then

$$\{B_r^3(x)B_s^3(y),\ r=0,\cdots,n+2,\ s=0,\cdots,m+2\}$$

is the C^2-bicubic B-splines basis of $S_3(\Delta_n\times\Delta_m)$, then $dim\ S_3(\Delta_n\times\Delta_m)=(n+3)(m+3)$ and we can define

$$B_k(x,y)=B_r^3(x)B_s^3(y),\ (x,y)\in\Omega,$$

for $r=0,\cdots,n+2,\ s=0,\cdots,m+2,\ k=(m+3)r+s+1.$ Then $1\le k\le(n+3)(m+3)$ and if we denote $M=(n+3)(m+3)$, we have that

$$B_1(x,y),\cdots,B_M(x,y)$$

is the C^2-bicubic B-splines basis of $S_3(\Delta_n\times\Delta_m)$.

3.3 Smoothing bicubic splines

Let $N\in\mathbb{N}$, $A^N=\{a_1,\ldots,a_N\}\subset\Omega$ and suppose that A^N contains a $\mathbb{P}_2$-unisolvent subset and

$$\sup_{p\in\Omega}\min_{a\in A^N}\langle p-a\rangle_2=O(\frac{1}{N}),\ N\to+\infty. \tag{3.2}$$

For $k\in\mathbb{N}^*$ let $B^N=\{b_1,\ldots,b_N\}\subset\mathbb{R}^k$, $\varepsilon>0$ and let J be the real functional defined on $H^3(\Omega;\mathbb{R}^k)$ by

$$J(v)=\sum_{i=1}^N\langle v(a_i)-b_i\rangle_k^2+\varepsilon|v|_3^2. \tag{3.3}$$

Then, we consider the following minimization problem: Find $\sigma^N\in(S_3(\Delta_n\times\Delta_m))^k$ such that

$$J(\sigma^N)\le J(v),\ \forall\ v\in(S_3(\Delta_n\times\Delta_m))^k. \tag{3.4}$$

Theorem 3.3.1. *Problem (3.4) has a unique solution, called the smoothing variational bicubic spline associated with A^N, B^N and ε, which is also the unique solution of the following variational problem: Find $\sigma^N\in(S_3(\Delta_n\times\Delta_m))^k$ such that*

$$\sum_{i=1}^N\langle\sigma^N(a_i),v(a_i)\rangle_k+\varepsilon(\sigma^N,v)_3=\sum_{i=1}^N\langle b_i,v(a_i)\rangle_k,\forall\ v\in(S_3(\Delta_n\times\Delta_m))^k. \tag{3.5}$$

Proof. It is similar to the proof of Theorem 3.1 in [41]. □

Applying (3.5) and taking into account that $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^k$ we can write

$$\sigma^N(x,y) = \sum_{i=1}^{M} \alpha_i B_i(x,y), \ \forall\ (x,y) \in \Omega,$$

where $\alpha_i \in \mathbb{R}^k$, $i = 1, \ldots, M$, and $\alpha = (\alpha_1, \ldots, \alpha_M)^t$ is the solution of the linear system

$$(AA^t + \varepsilon R)\alpha = Ab,$$

being $A = (B_i(a_j))_{\substack{i=1,\ldots,M,\\ j=1,\ldots,N}}$, $R = ((B_i, B_j)_3)_{i,j=1,\ldots,M}$ and $b = (b_i)_{i=1,\ldots,N}$.

Let $h = \max\{\dfrac{b-a}{n}, \dfrac{d-c}{m}\}$.

Theorem 3.3.2. *Let $f \in (C^4(\Omega))^k$. Suppose that the hypothesis (3.2) hold and that*

$$\varepsilon = o(N^2), \ N \to +\infty, \tag{3.6}$$

$$\frac{Nh^4}{\varepsilon^{\frac{1}{2}}} = o(1), \ N \to +\infty. \tag{3.7}$$

Then, one has

$$\lim_{N \to +\infty} \| f - \sigma^N \| = 0. \tag{3.8}$$

Proof. It is analogous to the proof of Theorem 5.3 of [41] taking into account that from (3.7), $h \to 0$ as $N \to +\infty$. □

3.4 Basic definitions about fuzzy numbers

Definition 3.4.1. *A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0,1]$ with the following properties (see [36]).*

i) u is an upper semi-continuous function on $\mathbb{R}$.

ii) $u(x) = 0$ outside some interval $[a_1, a_4] \subset \mathbb{R}$.

iii) There exist real numbers a_2 and a_3 such that $a_1 \leq a_2 \leq a_3 \leq a_4$ with *a) $u(x)$ is a monotonic increasing function on $[a_1, a_2]$,*

b) $u(x)$ is a monotonic decreasing function on $[a_3, a_4]$,

c) $u(x) = 1$, for all $x \in [a_2, a_3]$.

A popular type of fuzzy number is the set of trapezoidal fuzzy numbers, $\mathbb{TFN}$, (see Figure 2.8.3), that can be defined as $A = (a_1, a_2, a_3, a_4)$, and their membership function is defined by

$$\mu(A) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 0, & otherwise. \end{cases}$$

If $a_1 = a_2 = a_3 = a_4$, then the real number is represented by a. If $a_1 = a_2$ and $a_3 = a_4$, then A is called a crisp interval. Note that a triangular fuzzy number is obtained when $a_2 = a_3$, (see Figure 2.8.3), in which case triangular fuzzy numbers can be defined by $A = (a_1, a_2, a_3)$.

Definition 3.4.2. *Let $u = (u_1, u_2, u_3, u_4) \in \mathbb{TFN}$ and $0 < \alpha \leq 1$, then it is called $\alpha-$cut of u the set*

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}.$$

It is defined the $0-$cut of u as its support, i.e.,

$$[u]^0 = \bigcup_{0<\alpha\leq 1} [u]^\alpha = [u_1, u_4]$$

Definition 3.4.3. *An equivalent definition of a trapezoidal fuzzy number $u = (u_1, u_2, u_3, u_4)$ is a function $u : [0, 1] \longrightarrow I$ given by*

$$u(\alpha) = [\underline{u}(\alpha), \bar{u}(\alpha)],$$

with

$$\begin{aligned} \underline{u}(\alpha) &= u_1 + (u_2 - u_1)\alpha, \\ \bar{u}(\alpha) &= u_4 + (u_3 - u_4)\alpha, \end{aligned} \tag{3.9}$$

where I is the set of the all real closed intervals. Obviously we have that $u(\alpha) = [u]^\alpha$, for any $0 \leq \alpha \leq 1$. □

For any $u, v \in \mathbb{TFN}$, $\lambda \in \mathbb{R}$, the sum $u+v$ and the product λu are defined by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \ [\lambda u]^\alpha = \lambda [u]^\alpha,$$

for all $\alpha \in [0,1]$, $\lambda > 0$, taking into account that

$$\lambda[\underline{u}(\alpha), \bar{u}(\alpha)] = \begin{cases} [\lambda \underline{u}(\alpha), \lambda \bar{u}(\alpha)], & \lambda \geq 0, \\ [\lambda \bar{u}(\alpha), \lambda \underline{u}(\alpha)], & \lambda < 0. \end{cases}$$

Definition 3.4.4. *For any $u, v \in \mathbb{TFN}$, it is defined the Haussdorf distance between u and v as the quantity*

$$d(u,v) = \underset{\alpha \in (0,1]}{sup}\, max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\bar{u}(\alpha) - \bar{v}(\alpha)|\}.$$

Definition 3.4.5. *From Definition 6, we have*

$$d(u,v) = \underset{i=1,2,3,4}{max} |u_i - v_i|, \tag{3.10}$$

The concept of similarity or dissimilarity between two data sets is fundamental in almost every scientific field. The analysis of similarity or dissimilarity measures between fuzzy sets has gained importance due to the widespread of applications in diverse fields, including fuzzy risk analysis problems [60], decision making [33] and function approximation [14]. In this section, we introduce some existing similarity measures of fuzzy numbers.
Let A and B be two trapezoidal fuzzy numbers, where $A = (a_1, a_2, a_3, a_4)$, $a_1 \leq a_2 \leq a_3 \leq a_4$, and $B = (b_1, b_2, b_3, b_4)$, $b_1 \leq b_2 \leq b_3 \leq b_4$. Then the degree of similarity $S(A,B)$ between the trapezoidal fuzzy numbers A and B is defined in Chen [**?**] as follows

$$S(A,B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4},$$

where $\mid a \mid$ is the absolute value of the real number a.
Hsieh and Chen [32] proposed a similarity measure using the *graded mean integration-representation distance* where the degree of similarity $S(A,B)$ between the fuzzy numbers A and B is calculated as follows:

$$S(A,B) = \frac{1}{1 + d(A,B)}$$

where $d(A,B) = \mid P(A) - P(B) \mid$; $P(A)$ and $P(B)$ are the graded mean integration representations of A and B, respectively. If A and B are trapezoidal fuzzy numbers, then the graded mean integration of these fuzzy numbers is defined as:

$$P(A) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6},$$
$$P(B) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

In [14], Chen et al. introduced a new method called the simple center of gravity method (denoted as SCGM) to calculate the center of gravity points (x_A^*, y_A^*) and (x_B^*, y_B^*) of generalized fuzzy numbers A and B respectively. If A and B are two trapezoidal fuzzy numbers, the degree of similarity $S(A,B)$ between these numbers can be calculated as follows:

$$S(A,B) = 1 - \frac{\sum_{i=1}^{4} \mid a_i - b_i \mid}{4} \times (1- \mid x_A^* - x_B^* \mid)^{B(S_A,S_B)} \times \frac{min(y_A^*, y_B^*)}{max(y_A^*, y_B^*)},$$

where $S(A,B) \in [0,1]$, and

$$x_A^* = \frac{y_A^*(a_3 + a_2) + (a_4 + a_1)(1 - y_A^*)}{2},$$

$$y_A^* = \begin{cases} \frac{1}{2}, & if \quad a_1 = a_4, \\ \frac{1}{6}(\frac{a_3 - a_2}{a_4 - a_1} + 2), & if \quad a_1 \neq a_4, \end{cases}$$

and

$$B(S_A, S_B) = \begin{cases} 1, & if \quad S_A + S_B > 0, \\ 0, & if \quad S_A + S_B = 0, \end{cases}$$

where S_A and S_B are the lengths of the bases of trapezoidal fuzzy numbers A and B, respectively, and defined by:

$$S_A = a_4 - a_1,$$
$$S_B = b_4 - b_1.$$

In [71], to make the similarity well distributed, a new method SIAM (Shape's Indifferent Area and Midpoint) to measure triangular fuzzy number is put forward, which takes the shape's indifferent area and midpoint of two triangular fuzzy numbers into consideration.

The goal of the present paper is to show the effectiveness of a new fuzzy numbers approximation method and not the similarity indexes so the choice of these is not essential in our work.

Definition 3.4.6. *A fuzzy function defined on* $\Omega \subset \mathbb{R}^2$ *to the trapezoidal fuzzy number set* $\mathbb{TFN}$ *is an application* $f: \ \Omega \to \mathbb{TFN}$ *such that* $f = (f_1, f_2, f_3, f_4)$*, where* f_i *is a real function defined on* Ω*, for* $i = 1, 2, 3, 4$*, and* $f(x, y) \in \mathbb{TFN}$*, for any* $(x, y) \in \Omega$*.*

Definition 3.4.7. *The fuzzy bicubic spline space constructed on the partition*

$$\Delta_n \times \Delta_m$$

of Ω *is the set of the fuzzy functions*

$$S_3(\Delta_n \times \Delta_m; \mathbb{TFN}) = \{s: \ \Omega \to \mathbb{TFN} \,|\, s = \sum_{i=1}^{M} \alpha_i B_i, \ \alpha_i \in \mathbb{TFN}, \ i = 1, \ldots, M\}.$$

3.5 Fuzzy smoothing bicubic splines

Now, we consider the following approximation problem: Given $U^N = \{u_1, \ldots, u_N\} \subset \mathbb{TFN}$ find a fuzzy function $s \in S_3(\Delta_n \times \Delta_m; \mathbb{TFN})$ such that $s(a_\ell) \approx u_\ell$, for any $\ell = 1, \ldots, N$.

We can consider that U^N is a subset of $\mathbb{R}^4$ since $u_\ell = (u_{\ell 1}, u_{\ell 2}, u_{\ell 3}, u_{\ell 4}) \in \mathbb{R}^4$, for any $\ell = 1, \ldots, N$.

Let σ^N be the smoothing bicubic variational spline associated with A^N, U^N and $\varepsilon > 0$ defined in Theorem 3.3.1. Then $\sigma^N \in (S_3(\Delta_n \times \Delta_m))^4$ and thus there exist $\alpha_1, \ldots, \alpha_M \in \mathbb{R}^4$ such that

$$\sigma^N(x, y) = \sum_{i=1}^{M} \alpha_i B_i(x, y), \ \forall \, (x, y) \in \Omega.$$

Consider a fuzzy function $f: \Omega \to \mathbb{TFN}$ and $U^N = f(A^N) = \{f(a_\ell) : \ell = 1, \ldots N\}$. For all $i = 1, \ldots, M$, let $\overline{\alpha}_i \in \mathbb{R}^4$ such that its components are the same as those of α_i ordered from lowest to highest, i.e., if $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4})$ and $\overline{\alpha}_i = (\overline{\alpha}_{i1}, \overline{\alpha}_{i2}, \overline{\alpha}_{i3}, \overline{\alpha}_{i4})$ then

$$\overline{\alpha}_{ij} = \alpha_{i\gamma(j)}, \ \jmath = 1, 2, 3, 4,$$

being $\gamma: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ the permutation such that

$$\overline{\alpha}_{i1} \leq \overline{\alpha}_{i2} \leq \overline{\alpha}_{i3} \leq \overline{\alpha}_{i4}.$$

Then $\overline{\alpha}_i \in \mathbb{TFN}$, $i = 1, \ldots, M$ and thus the function s^N given by

$$s^N(x, y) = \sum_{i=1}^{M} \overline{\alpha}_i B_i(x, y), \ \forall\, (x, y) \in \Omega,$$

verifies that $s^N \in S_3(\Delta_n \times \Delta_m; \mathbb{TFN})$.

This fuzzy function is called the fuzzy smoothing bicubic spline associated with A^N, $f(A^N)$ and ε.

Theorem 3.5.1. *Consider a fuzzy function $f : \Omega \to \mathbb{TFN}$ such that $f = (f_1, f_2, f_3, f_4)$ and $f_i \in C^4(\Omega)$ and suppose the hypotheses (3.2), (3.6) and (3.7) hold. Then, one has*

$$\lim_{N \to +\infty} \overline{S}(f, s^N) = 1, \tag{3.11}$$

being $\overline{S}(f, s^N) = \dfrac{1}{Z} \displaystyle\sum_{i=1}^{Z} S(f(\xi_i), s^N(\xi_i))$, where $\{\xi_1, \ldots, \xi_Z\} \subset \Omega$ is a set of Z random points of the domain Ω and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}), the Chen & Chen index (S_{SCGM}), defined in Section 4. Moreover

$$\lim_{N \to +\infty} d(f(p), s^N(p)) = 0, \ \forall\, p \in \Omega. \tag{3.12}$$

Proof. For any $N \in \mathbb{N}$ let s^N be the fuzzy smoothing bicubic spline associated with A^N, $f(A^N)$ and ε. Let d_N the real number given by

$$d_N = \max_{p \in \Omega} \max_{i=1,2,3,4} f_{i+1}(p) - f_i(p).$$

Let σ^N be the smoothing bicubic variational spline associated with A^N, $f(A^N) \subset \mathbb{R}^4$ and ε, considering $f(a_\ell)$ as an element of $\mathbb{R}^4$, for any $\ell = 1, \ldots, N$.

From Theorem 3.3.2 we can deduce that there exists $N_0 \in \mathbb{N}$ such that for $N > N_0$, $i = 1, 2, 3, 4$ and $p \in \Omega$ we have

$$|f_i(p) - \sigma_i^N(p)| < \frac{d_N}{2}.$$

Thus, for $i = 1, 2, 3$, we obtain that

$$\sigma_i^N(p) - \sigma_{i+1}^N(p) = \sigma_i^N(p) - f_i(p) + f_i(p) - f_{i+1}(p) + f_{i+1}(p) - \sigma_{i+1}^N(p) \leq 0.$$

Then, for $N > N_0$, $\sigma_1^N(p) \leq \sigma_2^N(p) \leq \sigma_3^N(p) \leq \sigma_4^N(p)$ and thus $\sigma^N(p) \in \mathbb{TFN}$, for any $p \in \Omega$. Hence $s^N = \sigma^N$ and from Theorem 3.3.2

$$\lim_{N \to +\infty} |f_i(p) - s_i^N(p)| = 0, \ \forall\, p \in \Omega, \ i = 1, 2, 3, 4,$$

and we can confirm that (3.11)and (3.12) hold. □

3.6 Numerical examples

In this section, different interpolation error and similarity estimations are proposed in order to analyze the presented fuzzy interpolation method. The definition of these estimations is

$$\overline{S} = \frac{1}{Z} \sum_{i=1}^{Z} S(f(\xi_i), \sigma(\xi_i)), \tag{3.13}$$

where $\{\xi_1, \dots, \xi_Z\} \subset R$ is a set of Z random points in the domain R and S is the Chen (S_{CHEN}) index, the Hsieh index (S_{HSIEH}), the Chen & Chen index (S_{SCGM}), defined in Section 4, or the Haussdorf distance (d) given in Definition 6. From Theorem 10, it should be verified that $\overline{S}$ tends to 1 as $N \to +\infty$, for $S = S_{CHEN}, S_{HSIEH}, S_{SCGM}$, and $\overline{S}$ tends to 0 as $N \to +\infty$, for $S = d$.

To test our method we consider an examples for different partitions of the domain, different numbers of knots and different values of smoothing parameter ε for the function $f : [0, \pi] \times [0, \pi] \longrightarrow \mathbb{TFN}$ given by

$$\begin{aligned} f(x, y) = &(0.2 \exp(-0.5y) \sin(5x) + 0.5, 0.1 \exp(-0.5x) \sin(5y) + 0.3, \\ &0.2 \exp(-0.5y) \sin(3x), 0.1 \exp(-0.5y) \sin(x) + 0.3 \exp(-0.6y) \cos(5y) - 0.4). \end{aligned}$$

For the simulations presented in this section, the number of points to compute the estimation $\overline{S}$ given by (3.13) is $Z = 500$ in all cases.

Table 1 illustrates the performance of the fuzzy smoothing bicubic spline for different values of the knot numbers n and m, taking $n = m$ in all cases, the approximation points number and the smoothing parameter ε. The proposed simulations show the influence and relative importance of these parameter values in the effectiveness of the approximation. Specifically, under the hypotheses of Theorem 10, the error estimation $\overline{S}_d$ decreases to 0 and the similarity index estimations increases to 1 as $N \to +\infty$. Finally, we observe that if M and N are fixed then the error and similarity estimations are not monotones respect to ε; then we can surmise the existence of an optimal value of ε.

3.7 Conclusion

In this paper, we have extended the methodology in the case of univariate cubic spline presented in [65] for approximation error of 3D fuzzy data by using fuzzy smoothing bicubic splines.
Through various experiments with a two-dimensional fuzzy function, we have modified the number of knots $((n+1) \times (m+1))$, the number of points used for the smoothing spline system (M) and the parameter ε used for the trade-off between precision and smoothness. The developed numerical examples illustrate the relevance of the approximation method presented in this work and it confirms the established convergence result.

Table 3.1 : Function $f(x, y)$. Error and similarity indices estimates for different values of the approximation method parameters.

$n = m$	N	ε	$\overline{S}_d$	$\overline{S}_{CHEN}$	$\overline{S}_{HSIEH}$	$\overline{S}_{SCGM}$
5	50	10^{-5}	7.4742×10^{-2}	0.9626	0.9845	0.9579
5	50	10^{-7}	2.4238×10^{-2}	0.9846	0.9866	0.9598
5	50	10^{-9}	3.7776×10^{-2}	0.9883	0.9568	0.9300
5	100	10^{-5}	9.5961×10^{-2}	0.9946	0.9919	0.9764
5	100	10^{-7}	4.7167×10^{-2}	0.9956	0.9921	0.9783
5	100	10^{-9}	4.8663×10^{-2}	0.9839	0.9850	0.9710
5	500	10^{-5}	6.7796×10^{-2}	0.9682	0.9940	0.9845
5	500	10^{-7}	9.6577×10^{-3}	0.9939	0.9943	0.9838
5	500	10^{-9}	2.9117×10^{-2}	0.9894	0.9941	0.9821
9	50	10^{-5}	2.7781×10^{-2}	0.9883	0.9910	0.9753
9	50	10^{-7}	2.0309×10^{-2}	0.9907	0.9921	0.9785
9	50	10^{-9}	3.7296×10^{-2}	0.9898	0.9917	0.9818
9	100	10^{-5}	3.2391×10^{-2}	0.9826	0.9860	0.9900
9	100	10^{-7}	1.3013×10^{-2}	0.9955	0.9918	0.9925
9	100	10^{-9}	6.0140×10^{-2}	0.9818	0.9841	0.9890
9	500	10^{-5}	4.1107×10^{-3}	0.9985	0.9994	0.9984
9	500	10^{-7}	2.1545×10^{-3}	0.9986	0.9996	0.9990
9	500	10^{-9}	2.8174×10^{-3}	0.9985	0.9995	0.9898
17	50	10^{-5}	7.4268×10^{-2}	0.9679	0.9898	0.9720
17	50	10^{-7}	1.5358×10^{-2}	0.9920	0.9913	0.9743
17	50	10^{-9}	2.2139×10^{-2}	0.9894	0.9912	0.9713
17	100	10^{-5}	6.7632×10^{-3}	0.9962	0.9976	0.9931
17	100	10^{-7}	1.8586×10^{-3}	0.9988	0.9983	0.9957
17	100	10^{-9}	8.0263×10^{-3}	0.9977	0.9982	0.9956
17	500	10^{-5}	3.7018×10^{-4}	0.9998	0.9998	0.9960
17	500	10^{-7}	8.2134×10^{-5}	0.99997	0.99991	0.99983
17	500	10^{-9}	7.1242×10^{-5}	0.99998	0.99992	0.99985

Chapter 4

Approximation error of 3D fuzzy data using radial basis functions

4.1 Introduction

One of the most interesting and important problem in various scientific fields is approximation. In this chapter, we present a new methodology to approximate a trapezoidal fuzzy numbers set by using radial basis functions ($RBFs$), the methodology use different similarity indices to determine and compare the accuracy of the approximation of the fuzzy data. The method uses radial basis functions for defining some examples of two-dimension to compare its behavior by using the indices proposed for different configurations of the smoothing radial basis functions.

Multivariate interpolation and approximation with radial basis functions have been reviewed in several papers (see [10]-[53] among others), and it is sufficient here to mention how it works. In [65] the author define a new error and similarity indices to determine the accuracy of approximation of fuzzy data by cubic spline functions.

In this chapter we describe a novel approach based on ($RBFs$) for fuzzy data approximation generally of two variables. Radial basis functions constitute a widely used and researched tool for (nonlinear) function approximation, which is a central theme in pattern analysis and recognition [54]-[51]; see also [30] for a recent and comprehensive overview and further references.

The $(RBF)-$based (often seen as a neural network [30]) input-output relation has the form

$$y = g(x) = \sum_{i=1}^{n} \alpha_i \, \phi(\|x - t_i\|)$$

where $x = [x_1, ..., x_d]^T$ is the input, the α_i are weights, and the radial function ϕ on $\mathbb{R}^d$ is defined through a univariate function $\phi : [0, \infty] \longrightarrow \mathbb{R}$ in such a way that $\phi(x) = \phi(\|x\|^2)$ where $\|.\|$ is the usual Euclidean norm in $\mathbb{R}^d$, and the t_i are called the centers.

The rest of this chapter is organized as follows: In the next Section 4.2, a new methodology for smoothing approximation with radial basis functions of fuzzy numbers is proposed, and three error and similarity indices for different fuzzy data sets are defined. Several simulation results are carried out in Section 4.3 to verify the good performance of the proposed method. In Section 4.4, we analyze individual and global behavior of the different error and similarity indices. Finally the conclusions are discussed in Section 4.5.

4.2 Proposed methodology

Let Ω be an open bounded connected nonempty subset of $\mathbb{R}^2$ with Lipschitz-continuous boundary. We will use the classical notation $H^k(\Omega)$ to denote the usual Sobolev space of all distribution u whose all derivative up to and including order k are in the classical Lebesgue space $L^2(\Omega)$.

The Sobolev space $H^k(\Omega)$ is a Hilbert space equipped with the inner semi-products given by

$$(u, v)_l = \sum_{|\alpha|=l} \int_{\Omega} \partial^{\alpha} u(x) \partial^{\alpha} v(x) dx, \quad 0 \leq l \leq k$$

the semi-norms given by $|u|_l = (u, u)_l^{\frac{1}{2}}$, for all $l = 0, \cdots, k$, and the norm $\|u\|_k = \left(\sum_{l \leq k} |u|_l^2 \right)^{\frac{1}{2}}$, where for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $|\alpha| = \alpha_1 + \alpha_2$.

4.2.1 Radial spaces of class C^2

Let $m > 1$ be a positive integer and let $\Pi_{m-1}(\mathbb{R}^2)$ denote the space of polynomials on $\mathbb{R}^2$ of degree at most $m-1$ whose dimension is denoted $d(m)$, where $d(m) = \frac{m(m+1)}{2}$ and $\left\{q_1, \cdots, q_{d(m)}\right\}$ be the standard basis of $\Pi_{m-1}(\mathbb{R}^2)$. Let us give an arbitrary finite set $\{b_1, \cdots, b_M\} \subset \mathbb{R}^2$ of M distinct approximation points, $b_i = (x_i, y_i) \in \Omega$, $i = 1, \ldots, M$, and a set of fuzzy numbers $U = \{u_1, \cdots, u_M\}$ such as $u_i = (u_{i1}, u_{i2}, u_{i3}, u_{i4})$ is a trapezoidal fuzzy number, with $i = 1, \cdots, M$, i.e., $U \subset \mathbb{TFN}$, where $\mathbb{TFN}$ is the set of the trapezoidal fuzzy numbers. We need a center points set $\{a_1, \cdots, a_N\}$ and for each $i = 1, \ldots, N$, a radial function $\phi(\cdot - a_i)$.

The main goal of this section is to solve if it is possible to approximate the points $\{(b_1, u_1), \cdots, (b_M, u_M)\} \subset \mathbb{R}^2 \times \mathbb{TFN}$.

To conclude this section we define the radial basis functions. We consider the following function:

$$\phi_\varepsilon(t) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon\sqrt{t}} + \varepsilon\sqrt{t}\right), \quad \varepsilon \in \mathbb{R}^+, \ t \geq 0 \tag{4.1}$$

and the following radial function

$$\phi_\varepsilon(x) = \phi_\varepsilon(< x >_2^2) = -\frac{1}{2\varepsilon^3}\left(e^{-\varepsilon < x >_2} + \varepsilon < x >_2\right), \quad \varepsilon \in \mathbb{R}^+, \ x \in \mathbb{R}^2$$

where $< \cdot >_k$ be the Euclidean norm on $\mathbb{R}^k$. Let H the functional space generated by

$$\left\{q_1, \cdots, q_{d(m)}, \phi_\epsilon(\cdot - a_1), \cdots, \phi_\epsilon(\cdot - a_N)\right\}$$

4.2.2 Fuzzy smoothing radial basis functions

In this case, the problem to be resolved can be formulated as follows: Let U be a set of fuzzy numbers $U = \{u_i : i = 1, \cdots, M\}$ such that every $u_i = (u_{i1}, u_{i2}, u_{i3}, u_{i4})$, $i = 1, \cdots, M$, is a trapezoidal fuzzy number, i.e., $U \subset \mathbb{TFN}$, where $\mathbb{TFN}$ is the set of the trapezoidal fuzzy numbers.
Let $H^{\mathbb{T}}$ the set of functions $S : \Omega \to \mathbb{TFN}$ such that

$$S(x, y) = \sum_{l=1}^{N+d(m)} \alpha_l w_l(x, y), \quad (x, y) \in \Omega,$$

where $\alpha_1, \cdots, \alpha_{N+d(m)} \in \mathbb{TFN}$ and

$$w_l = \left\{ \begin{array}{ll} \phi_\varepsilon(\cdot - a_l), & l = 1, \cdots, N \\ q_{l-N}, & l = N+1, \cdots, N+d(m), \end{array} \right\}.$$

Given the approximation data set $\{(b_i, u_i) : i = 0, \cdots, M\} \in \mathbb{R}^2 \times \mathbb{TFN}$, we want to obtain a fuzzy function $S \in H^{\mathbb{T}}$ such that

$$S(x_i, y_i) \approx u_i, \quad i = 1, \cdots, M.$$

For this we consider the following minimization problem [39, 40]: Given $\tau \in (0, \infty)$, find $\sigma \in H^4$ such that:

$$J(\sigma) \leq J(\upsilon), \quad \forall \upsilon \in H^4, \tag{4.2}$$

Where

$$J(\upsilon) = \sum_{i=1}^{M} < u_i - \upsilon(x_i, y_i) >_4^2 + \tau|\upsilon|_2^2,$$

$$|\upsilon|_2^2 = \sum_{i=1}^{4} \sum_{|\alpha|=2} \int \int_{\Omega} (\partial^\alpha \upsilon_i(x, y)\,)^2 dxdy$$

being $\upsilon = (\upsilon_1, \ldots, \upsilon_4) \in H^4$.

Theorem 4.2.1. *The minimization problem (4.2) has an unique solution that is the unique solution of the following variational problem: Find* $\sigma = (\sigma_1, \ldots, \sigma_4) \in H^4$ *such that for all* $\upsilon \in H^4$:

$$\sum_{i=1}^{M} < \sigma(x_i, y_i), \upsilon(x_i, y_i) >_4 + \tau(\sigma, \upsilon)_2 = \sum_{i=1}^{M} < \upsilon(x_i, y_i), u_i >_4, \tag{4.3}$$

being $(\sigma, \upsilon)_2 = \sum_{i=1}^{4} (\sigma_i, \upsilon_i)_2$.

Proof. The expression

$$[[\upsilon]] := \left(\sum_{i=1}^{M} < \upsilon >_4^2 + \tau|\upsilon|_2^2 \right)^{\frac{1}{2}}$$

constitutes a norm on H^4 equivalent to the usual Sobolev norm $||.||$. As a consequence, the continuous symmetric bilinear form

$$a : \; H^4 \times H^4 \longrightarrow \mathbb{R}$$

defined by

$$a(u,v)=\sum_{i=1}^{M} < u(x_i,y_i), v(x_i,y_i) >_4 + \tau(u,v)_2$$

is H^4-elliptic. Besides, let $\psi: \ H^4 \longrightarrow \mathbb{R}$ the linear application defined by

$$\psi(v):=\sum_{i=1}^{M} < u_i, v(x_i,y_i) >_4$$

Applying the Lax-Milgram Lemma [5], there exists a unique function $\sigma \in H^4$ such that

$$a(\sigma,v)=\psi(v), \ \ \forall v \in H^4$$

that is, (4.3) holds. Moreover σ is the unique function of H^4 that minimizes the functional

$$\widetilde{J}(v)=2\left(\frac{1}{2}a(v,v)-\psi(v)\right)+\sum_{i=1}^{M} < u_i >_4^2,$$

we conclude that S minimizes the functional J. □

By linearity we can reduce the problem (4.3) to the following linear system:

$$\left(AA^T+\tau R\right)\alpha = A^T b$$

where $A=(w_i(b_j)), \ \ i=1,\cdots,M, \ \ j=1,\cdots,N.$

$$R=\left(\sum_{|\alpha|=2}\int\int_{\Omega}\partial^{\alpha}w_i\partial^{\alpha}w_j dxdy\right)_{i,j=1,\cdots,N+d(m)}$$

$$\alpha=\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_{N+d(m)}\end{pmatrix}\in\left(\mathbb{R}^4\right)^{N+d(m)}, \ b=(u_{i1},u_{i2},u_{i3},u_{i4})_{i=1,\cdots,M}$$

We will consider that $\alpha_{i1} \leq \alpha_{i2} \leq \alpha_{i3} \leq \alpha_{i4}, \ \ i=1,\cdots,N+d(m)$; in any other case, we will sort the components of the numbers α_i from lowest to highest.

In this way $\sigma \in H^{\mathbb{T}}$.

In order to analyze the fuzzy data approximation using smoothing radial basis functions we consider the following three error and similarity indices:

$$\bar{S} = \frac{1}{ntest} \sum_{i=0}^{ntest} S(u_i^{Test}, \hat{u}_i^{Test}), \tag{4.4}$$

$$E_s = \frac{1}{ntest} \left(\sum_{i=0}^{ntest} (1 - S(u_i^{Test}, \hat{u}_i^{Test}))^2 \right)^{\frac{1}{2}}, \tag{4.5}$$

$$I_s = \frac{1}{ntest} \left(\int_{i=0}^{ntest} (1 - S(u_i^{Test}, \hat{u}_i^{Test}))^2 \right)^{\frac{1}{2}}, \tag{4.6}$$

where S is any of the indexes of similarity proposed in section 2.7.

In order to verify the ability of generalization of the fuzzy radial basis function approximation, we will illustrate the different phases carried out for radial basis approximation of fuzzy data. The steps to be undertaken in the simulation process are in follows.

i) Let TDS be a test data set $X^{test} = [(x_1, y_1)^{Test}, ..., (x_{ntest}, y_{ntest})^{Test}]$ to verify the ability of the fuzzy approximation method and its corresponding output fuzzy numbers, i.e.,$TDS = [X^{Test}; U^{Test}]$. With the data of TDS, it is possible to obtain the output fuzzy data approximation using the smoothing radial basis methodology presented.

ii) This output set is termed as $\hat{U}^{Test} = [\hat{u}_1^{Test}, ..., \hat{u}_{ntest}^{Test}]$, with $ntest$ being the number of nodes in the set X^{Test}.

iii) Measure the similarity between the fuzzy numbers u_i^{Test} and $\hat{u}_i^{Test}$, for any $i = 1, ..., ntest$ (the similarity measure is defined as $S(u_i^{Test}, \hat{u}_i^{Test})$; see Section 3).

4.3 Numerical examples

In this section, to analyze the behavior of the approximation of fuzzy numbers, performed by smoothing radial basis functions, various simulations have been carried out, in which four important factors can be modified:

- The number of center points to build the radial basis functions, that will be denoted as $(nctrs)$.

- The number of approximation points used denoted by (*npunt*) the smoothing with radial basis functions.

- Many radial basis functions, have a variable ε in their definitions. This variable ε is called the shape parameter.

- The parameter τ, that reflects the relative importance which we give to the two conflicting objectives: accuracy in approximation (remaining close to the data), and obtaining a smooth curve. It is a tradeoff between precision and smoothness.

In order to show the performance of the method, we will take two different examples of two dimensional fuzzy functions to analyze the behavior of the approximation with radial basis functions for various values of $nctrs$, $npunt$, ε and τ.

$$
\begin{aligned}
\text{Example 1:} \quad & f_1 : [0,1] \times [0,1] \longrightarrow \mathbb{TFN}, \\
f_1(x,y) = \quad & (a_1(x,y), a_2(x,y), a_3(x,y), a_4(x,y)) = \\
& (0.75e^{(-((9x-2)^2+(9y-2)^{\frac{2}{4}}))}, 0.75e^{(-((9x+1)^2+(9y+1)^{\frac{2}{10}}))} \\
& 0.5e^{(-((9x-7)^2+(9y-3)^{\frac{2}{4}}))}, 0.2e^{(-((9x-4)^2+(9y-7)^2))}). \\
\text{Example 2:} \quad & f_2 : [0,1] \times [0,1] \longrightarrow \mathbb{TFN}, \\
f_2(x,y) = \quad & (a_1(x,y), a_2(x,y), a_3(x,y), a_4(x,y)) = \\
& (0.5\sin x \cos y - 0.1 \sin x, 0.5 \sin 5x \cos 3y + 0.3 \cos y, \\
& 0.5 \sin x e^{-0.5y} + 0.28 \cos y, 0.5 \sin 5x \cos 5y + 0.04 \cos y).
\end{aligned}
$$

In order to compare the behavior of the fuzzy spline approximation, different error and similarity indices are used for several simulations in both examples, several values of the parameter ε and τ have been used to analyze the behavior of the error and similarity indices proposed. Table 4.1 shows the behavior of the proposed indices for smoothing radial basis approximations of functions f_1 and f_2 respectively.

The value of the error and similarity index proposed will depend largely on the parameters selected for smoothing (i.e. the parameters $nctrs$, $npunt$, τ and ε). Tables (4.2 - 4.13) illustrate the performance of the approximation error and similarity indices, when analyzing the different values of *nnod*, *npunt*, τ and ε, running 100 simulations for each configuration defined (the

Table 4.1 : Simulation summary of the error and similarity indices for both examples, for fixed position of the knots (linear spacing distributed in the domain of function), for several values of parameter $nctrs$, ε, τ.

$nctrs$	$npunt$	ε	τ	$\overline{S}_{SCGM}$	$std(S_{SCGM})$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	$std(S_{HSIEH})$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	$std(S_{CHEN})$	E_{CHEN}	I_{CHEN}
Example 1															
100	400	1.0e -3	0.4	0.8883	0.0361	0.1117	0.1103	0.9546	0.0492	0.0454	0.0452	0.7686	0.0805	0.2314	0.2286
100	400	1.0e -3	3	0.8893	0.0456	0.1107	0.1093	0.9591	0.0488	0.0409	0.0405	0.7813	0.0948	0.2187	0.2157
150	400	1.0e -5	0.4	0.8885	0.0431	0.1115	0.1107	0.9550	0.0483	0.0450	0.0449	0.7741	0.0925	0.2259	0.2244
150	400	1.0e -5	3	0.8878	0.0382	0.1122	0.1110	0.9586	0.0445	0.0414	0.0413	0.7597	0.0868	0.2403	0.2377
200	400	1.0e -8	0.4	0.8880	0.0346	0.1120	0.1110	0.9558	0.0470	0.0442	0.0437	0.7623	0.0797	0.2377	0.2356
200	400	1.0e -8	3	0.8878	0.0416	0.1122	0.1107	0.9579	0.0449	0.0421	0.0418	0.7637	0.0922	0.2363	0.2333
Example 2															
100	400	1.0e -3	0.4	0.8373	0.0530	0.1627	0.1608	0.7563	0.1164	0.2437	0.2404	0.5943	0.1321	0.4057	0.4010
100	400	1.0e -3	3	0.8366	0.0566	0.1634	0.1621	0.7598	0.1140	0.2402	0.2392	0.6064	0.1364	0.3936	0.3904
150	400	1.0e -5	0.4	0.8355	0.0535	0.1645	0.1629	0.7859	0.1159	0.2141	0.2116	0.6149	0.1251	0.3851	0.3813
150	400	1.0e -5	3	0.8127	0.0626	0.1873	0.1851	0.7901	0.1290	0.2099	0.2066	0.6012	0.1334	0.3988	0.3942
200	400	1.0e -8	0.4	0,8391	0,0542	0,1609	0,1588	0,7761	0,1153	0,2239	0,2212	0,6114	0,1309	0,3886	0,3837
200	400	1.0e -8	3	0.8400	0.0523	0.1600	0.1586	0.7801	0.1236	0.2199	0.2194	0.6025	0.1300	0.3975	0.3939

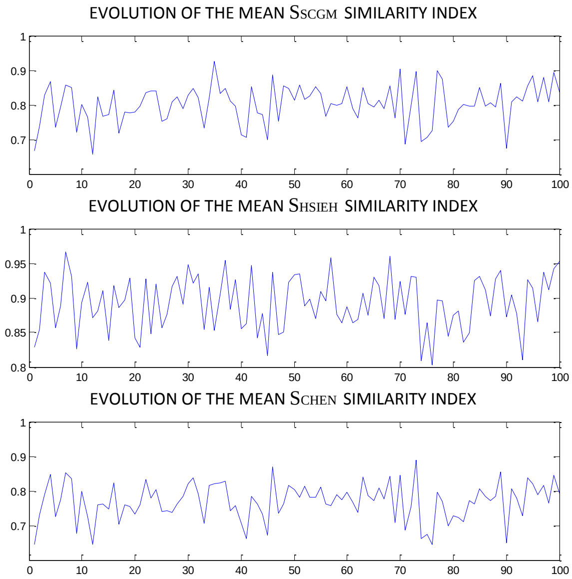

Figure 4.3.1: Function 1. $nctrs = 200$, $npunt = 900, \varepsilon = 1e - 8$ and $\tau = 3$. Evolution of the similarity indices obtained

mean, standard deviation, maximum and minimum values for each configuration) for Examples 1 and 2 respectively. For Examples 1 and 2, Fig. 4.3.1 and 4.3.2 respectively show the $\bar{S}_{SCGM}$, $\bar{S}_{HSIEH}$ and $\bar{S}_{CHEN}$ evolution using the error and similarity indices proposed in this paper to determine the correctness of the approximation, running 100 simulations for $nctrs = 200$, $npunt = 900$, $\varepsilon = 1e - 8$ and $\tau = 3$.

4.4 Statistical Results

It is very important to analyze both individual and global behavior for the different error and similarity indices proposed. In order to perform this analysis, we will study whether different error and similarity indices provide a very similar criterion to decide the degree of accuracy of a smooth approximation of fuzzy data. The two methods that were applied to performing this analysis are:

Table 4.2 : Simulation summary for function 1, using several values of the number of $npunt$, τ, $\varepsilon = 0.4$ and $nctrs = 100$ (in all tables (4.2 -4.11), 100 simulations were carried out to obtain the statistical measures).

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,665210	0,334789	0,331426	0,834950	0,165049	0,163407	0,594203	0,405796	0,401714
		Std	0,100082	0,100082	0,099189	0,067849	0,067849	0,067160	0,105669	0,105669	0,104675
		Max	0,901348	0,641085	0,635074	0,956969	0,340041	0,336480	0,806768	0,665934	0,659690
		Min	0,358914	0,098651	0,097861	0,659958	0,043030	0,042620	0,334065	0,193231	0,190516
100	1e -5	Mean	0,655121	0,344878	0,341429	0,830635	0,169364	0,167767	0,581229	0,418770	0,414601
		Std	0,097854	0,097854	0,096876	0,054038	0,054038	0,053566	0,097946	0,097946	0,097052
		Max	0,863120	0,593362	0,589720	0,949053	0,289959	0,288886	0,817524	0,689616	0,684377
		Min	0,406637	0,136879	0,135846	0,710040	0,050946	0,050573	0,310383	0,182475	0,180524
100	1e -8	Mean	0,664044	0,335955	0,332557	0,823095	0,176904	0,175108	0,578713	0,421286	0,417041
		Std	0,132384	0,132384	0,131086	0,057585	0,057585	0,057006	0,122732	0,122732	0,121587
		Max	1,049348	0,784609	0,776903	0,941166	0,344662	0,340354	0,805416	0,867037	0,858522
		Min	0,215390	0,049348	0,048808	0,655337	0,058833	0,058282	0,132962	0,194583	0,193045
400	1e -3	Mean	0,674456	0,325543	0,322361	0,834919	0,165080	0,163320	0,604460	0,395539	0,391670
		Std	0,097417	0,097417	0,096449	0,057299	0,057299	0,056643	0,098215	0,098215	0,097194
		Max	0,864781	0,608301	0,605631	0,942668	0,312522	0,309532	0,816766	0,626240	0,623491
		Min	0,391698	0,135218	0,134111	0,687477	0,057331	0,056622	0,373759	0,183233	0,181520
400	1e -5	Mean	0,674702	0,325297	0,322053	0,834221	0,165778	0,164182	0,609551	0,390448	0,386591
		Std	0,100704	0,100704	0,099597	0,050255	0,050255	0,049777	0,096415	0,096415	0,095487
		Max	0,891960	0,601482	0,594081	0,932556	0,280536	0,276971	0,764644	0,635025	0,627212
		Min	0,398517	0,108039	0,107298	0,719463	0,067443	0,066362	0,364974	0,235355	0,231678
400	1e -8	Mean	0,669517	0,330482	0,327117	0,831120	0,168879	0,167070	0,595212	0,404787	0,400660
		Std	0,109202	0,109202	0,108135	0,060596	0,060596	0,059890	0,111285	0,111285	0,110294
		Max	0,912816	0,664880	0,656507	0,947649	0,305232	0,300983	0,803450	0,806241	0,802628
		Min	0,335119	0,087183	0,086117	0,694767	0,052350	0,051621	0,193758	0,196549	0,194208
900	1e -3	Mean	0,651619	0,348380	0,344881	0,825159	0,174840	0,173004	0,591388	0,408611	0,404500
		Std	0,096702	0,096702	0,095762	0,056143	0,056143	0,055764	0,096676	0,096676	0,095726
		Max	0,866833	0,650121	0,643954	0,948857	0,350289	0,346821	0,792295	0,724338	0,717032
		Min	0,349878	0,133166	0,131609	0,649710	0,051142	0,050774	0,275661	0,207704	0,204881
900	1e -5	Mean	0,669234	0,330765	0,327375	0,828017	0,171982	0,170243	0,592089	0,407910	0,403746
		Std	0,104012	0,104012	0,102890	0,060688	0,060688	0,060066	0,107175	0,107175	0,106055
		Max	0,908523	0,599158	0,589648	0,940278	0,343982	0,339031	0,791652	0,757224	0,745206
		Min	0,400841	0,091476	0,090357	0,656017	0,059721	0,059545	0,242775	0,208347	0,206260
900	1e -8	Mean	0,672546	0,327453	0,324174	0,837673	0,162326	0,160659	0,603272	0,396727	0,392754
		Std	0,110166	0,110166	0,109216	0,050471	0,050471	0,050044	0,097270	0,097270	0,096495
		Max	1,006169	0,650176	0,644232	0,941433	0,277646	0,274437	0,783449	0,671113	0,666331
		Min	0,349823	0,006169	0,006126	0,722353	0,058566	0,057936	0,328886	0,216550	0,215269

Table 4.3 : Simulation summary for function 1, using several values of the number of *npunt*, τ, $\varepsilon = 0.4$ and $nctrs = 150$.

npunt	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,650009	0,349990	0,347755	0,842131	0,157868	0,156886	0,575879	0,424120	0,421402
		Std	0,096738	0,096738	0,096188	0,051253	0,051253	0,050866	0,100754	0,100754	0,100171
		Max	0,842854	0,592439	0,589193	0,953128	0,305716	0,303731	0,779556	0,714829	0,708892
		Min	0,407560	0,157145	0,156693	0,694283	0,046871	0,046632	0,285170	0,220443	0,219057
100	1e -5	Mean	0,664309	0,335690	0,333522	0,838036	0,161963	0,160923	0,596428	0,403571	0,400977
		Std	0,097676	0,097676	0,097108	0,050453	0,050453	0,050155	0,105672	0,105672	0,105087
		Max	0,888154	0,551867	0,548462	0,962357	0,280919	0,279004	0,828632	0,660406	0,656332
		Min	0,448132	0,111845	0,111386	0,719080	0,037642	0,037104	0,339593	0,171367	0,170035
100	1e -8	Mean	0,637954	0,362045	0,359597	0,826106	0,173893	0,172727	0,571867	0,428132	0,425222
		Std	0,110974	0,110974	0,110276	0,059157	0,059157	0,058743	0,102460	0,102460	0,101768
		Max	0,843359	0,702492	0,699186	0,948223	0,317238	0,315169	0,801600	0,737906	0,734433
		Min	0,297507	0,156640	0,155469	0,682761	0,051776	0,051431	0,262093	0,198399	0,197862
400	1e -3	Mean	0,650009	0,349990	0,347755	0,842131	0,157868	0,156886	0,575879	0,424120	0,421402
		Std	0,096738	0,096738	0,096188	0,051253	0,051253	0,050866	0,100754	0,100754	0,100171
		Max	0,842854	0,592439	0,589193	0,953128	0,305716	0,303731	0,779556	0,714829	0,708892
		Min	0,407560	0,157145	0,156693	0,694283	0,046871	0,046632	0,285170	0,220443	0,219057
400	1e -5	Mean	0,664309	0,335690	0,333522	0,838036	0,161963	0,160923	0,596428	0,403571	0,400977
		Std	0,097676	0,097676	0,097108	0,050453	0,050453	0,050155	0,105672	0,105672	0,105087
		Max	0,888154	0,551867	0,548462	0,962357	0,280919	0,279004	0,828632	0,660406	0,656332
		Min	0,448132	0,111845	0,111386	0,719080	0,037642	0,037104	0,339593	0,171367	0,170035
400	1e -8	Mean	0,654841	0,345158	0,342887	0,829627	0,170372	0,169246	0,584736	0,415263	0,412523
		Std	0,114015	0,114015	0,113392	0,053307	0,053307	0,052980	0,113057	0,113057	0,112486
		Max	0,868636	0,674833	0,673855	0,942874	0,315411	0,313134	0,788708	0,764032	0,762925
		Min	0,325166	0,131363	0,130710	0,684588	0,057125	0,056984	0,235967	0,211291	0,209271
900	1e -3	Mean	0,650009	0,349990	0,347755	0,842131	0,157868	0,156886	0,575879	0,424120	0,421402
		Std	0,096738	0,096738	0,096188	0,051253	0,051253	0,050866	0,100754	0,100754	0,100171
		Max	0,842854	0,592439	0,589193	0,953128	0,305716	0,303731	0,779556	0,714829	0,708892
		Min	0,407560	0,157145	0,156693	0,694283	0,046871	0,046632	0,285170	0,220443	0,219057
900	1e -5	Mean	0,664309	0,335690	0,333522	0,838036	0,161963	0,160923	0,596428	0,403571	0,400977
		Std	0,097677	0,097676	0,097108	0,050453	0,050453	0,050155	0,105672	0,105672	0,105087
		Max	0,888154	0,551867	0,548462	0,962357	0,280919	0,279004	0,828632	0,660406	0,656332
		Min	0,448132	0,111845	0,111386	0,719080	0,037642	0,037104	0,339593	0,171367	0,170035
900	1e -8	Mean	0,637954	0,362045	0,359597	0,826106	0,173893	0,172727	0,571867	0,428132	0,425222
		Std	0,110974	0,110974	0,110276	0,059157	0,059157	0,058743	0,102460	0,102460	0,101768
		Max	0,843359	0,702492	0,699186	0,948223	0,317238	0,315169	0,801600	0,737906	0,734433
		Min	0,297507	0,156640	0,155469	0,682761	0,051776	0,051431	0,262093	0,198399	0,197862

Table 4.4 : Simulation summary for function 1, using several values of the number of $npunt$, τ, $\varepsilon = 0.4$ and $nctrs = 200$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,662984	0,337015	0,335307	0,837335	0,162664	0,161848	0,592903	0,407096	0,405042
		Std	0,105090	0,105090	0,104561	0,058141	0,058141	0,057812	0,093932	0,093932	0,093423
		Max	0,976269	0,583851	0,581629	0,938199	0,321923	0,319679	0,820099	0,616139	0,613232
		Min	0,416148	0,097626	0,097437	0,678076	0,061800	0,061477	0,383860	0,179900	0,179457
100	1e -5	Mean	0,678767	0,321232	0,319591	0,849735	0,150264	0,149550	0,612240	0,387759	0,385765
		Std	0,092434	0,092434	0,091924	0,049554	0,049554	0,049310	0,095352	0,095352	0,094760
		Max	0,873753	0,585930	0,583150	0,944850	0,266733	0,265240	0,797146	0,687970	0,684069
		Min	0,414069	0,126246	0,125711	0,733266	0,055149	0,054771	0,312029	0,202853	0,201623
100	1e -8	Mean	0,648514	0,351485	0,349803	0,832636	0,167363	0,166528	0,590074	0,409925	0,407965
		Std	0,114204	0,114204	0,113698	0,051156	0,051156	0,050912	0,094488	0,094488	0,094087
		Max	0,961587	0,672724	0,671239	0,934070	0,275514	0,273771	0,754647	0,743195	0,741554
		Min	0,327275	0,038412	0,038258	0,724485	0,065929	0,065598	0,256804	0,245352	0,243766
400	1e -3	Mean	0,662984	0,337015	0,335307	0,837335	0,162664	0,161848	0,592903	0,407096	0,405042
		Std	0,105090	0,105090	0,104561	0,058141	0,058141	0,057812	0,093932	0,093932	0,093423
		Max	0,976269	0,583851	0,581629	0,938199	0,321923	0,319679	0,820099	0,616139	0,613232
		Min	0,416148	0,097626	0,097437	0,678076	0,061800	0,061477	0,383860	0,179900	0,179457
400	1e -5	Mean	0,669194	0,330805	0,329151	0,841519	0,158480	0,157671	0,604506	0,395493	0,393507
		Std	0,108213	0,108213	0,107675	0,057310	0,057310	0,056993	0,102537	0,102537	0,102001
		Max	0,857664	0,672724	0,671239	0,944850	0,269559	0,267420	0,764475	0,743195	0,741554
		Min	0,327275	0,142335	0,141411	0,730440	0,055149	0,054771	0,256804	0,235524	0,233960
400	1e -8	Mean	0,678767	0,321232	0,319591	0,849735	0,150264	0,149550	0,612240	0,387759	0,385765
		Std	0,092434	0,092434	0,091924	0,049554	0,049554	0,049310	0,095352	0,095352	0,094760
		Max	0,873754	0,585930	0,583150	0,944850	0,266733	0,265240	0,797146	0,687969	0,684069
		Min	0,414069	0,126245	0,125711	0,733266	0,055149	0,054771	0,312030	0,202853	0,201623
900	1e -3	Mean	0,662984	0,337015	0,335307	0,837335	0,162664	0,161848	0,592903	0,407096	0,405042
		Std	0,105090	0,105090	0,104561	0,058141	0,058141	0,057812	0,093932	0,093932	0,093423
		Max	0,976269	0,583851	0,581629	0,938199	0,321923	0,319679	0,820099	0,616139	0,613232
		Min	0,416148	0,097626	0,097437	0,678076	0,061800	0,061477	0,383860	0,179900	0,179457
900	1e -5	Mean	0,678767	0,321232	0,319591	0,849735	0,150264	0,149550	0,612240	0,387759	0,385765
		Std	0,092434	0,092434	0,091924	0,049554	0,049554	0,049310	0,095352	0,095352	0,094760
		Max	0,873753	0,585930	0,583150	0,944850	0,266733	0,265240	0,797146	0,687970	0,684069
		Min	0,414069	0,126246	0,125711	0,733266	0,055149	0,054771	0,312029	0,202853	0,201623
900	1e -8	Mean	0,671397	0,328602	0,327049	0,837744	0,162255	0,161503	0,600645	0,399354	0,397456
		Std	0,098485	0,098485	0,098031	0,054415	0,054415	0,054163	0,096056	0,096056	0,095589
		Max	0,925279	0,625039	0,623132	0,940475	0,324611	0,322521	0,802811	0,667777	0,664302
		Min	0,374960	0,074720	0,074339	0,675388	0,059524	0,059366	0,332222	0,197188	0,196227

Table 4.5 : Simulation summary for function 1, using several values of the number of $npunt$, τ, $\varepsilon = 3$ and $nctrs = 100$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,784418	0,215581	0,213447	0,878524	0,121475	0,120252	0,747893	0,252106	0,249617
		Std	0,061616	0,061616	0,061104	0,038870	0,038870	0,038504	0,057257	0,057257	0,056833
		Max	0,935502	0,442789	0,438972	0,967780	0,199028	0,196783	0,843111	0,461849	0,457867
		Min	0,557210	0,064497	0,064000	0,800971	0,032219	0,032045	0,538150	0,156888	0,155677
100	1e -5	Mean	0,788420	0,211579	0,209456	0,876947	0,123052	0,121838	0,741297	0,258702	0,256108
		Std	0,053822	0,053822	0,053350	0,039359	0,039359	0,039029	0,055477	0,055477	0,055014
		Max	0,893790	0,384050	0,382529	0,953787	0,247666	0,246836	0,841720	0,479068	0,477171
		Min	0,615949	0,106209	0,104923	0,752333	0,046212	0,045971	0,520931	0,158279	0,156438
100	1e -8	Mean	0,791958	0,208041	0,205926	0,886857	0,113142	0,111992	0,749165	0,250834	0,248279
		Std	0,058050	0,058050	0,057540	0,047192	0,047192	0,046711	0,060436	0,060436	0,059881
		Max	0,931831	0,397949	0,394198	0,961191	0,241472	0,238972	0,871996	0,415915	0,411995
		Min	0,602050	0,068168	0,067611	0,758527	0,038808	0,038339	0,584084	0,128003	0,126299
400	1e -3	Mean	0,781464	0,218535	0,216366	0,882608	0,117391	0,116169	0,745605	0,254394	0,251864
		Std	0,058820	0,058820	0,058323	0,040716	0,040716	0,040259	0,055310	0,055310	0,054848
		Max	0,927017	0,442789	0,438972	0,967780	0,204649	0,202044	0,855583	0,461849	0,457867
		Min	0,557210	0,072982	0,072330	0,795350	0,032219	0,032045	0,538150	0,144416	0,143258
400	1e -5	Mean	0,798686	0,201313	0,199271	0,887870	0,112129	0,111024	0,760112	0,239887	0,237481
		Std	0,058637	0,058637	0,057944	0,035090	0,035090	0,034731	0,056433	0,056433	0,055832
		Max	0,936352	0,379083	0,374275	0,954357	0,202107	0,199517	0,849251	0,390531	0,385578
		Min	0,620916	0,063647	0,063155	0,797892	0,045642	0,044974	0,609468	0,150748	0,148311
400	1e -8	Mean	0,790367	0,209632	0,207494	0,883869	0,116130	0,114874	0,747950	0,252049	0,249470
		Std	0,066409	0,066409	0,065781	0,042519	0,042519	0,041986	0,063838	0,063838	0,063256
		Max	1,001439	0,403349	0,398580	0,960339	0,248219	0,244856	0,863065	0,439075	0,434132
		Min	0,596650	0,001439	0,001421	0,751780	0,039660	0,039310	0,560924	0,136934	0,135377
900	1e -3	Mean	0,788392	0,211607	0,209483	0,876940	0,123059	0,121846	0,741272	0,258727	0,256132
		Std	0,053824	0,053824	0,053351	0,039374	0,039374	0,039044	0,055530	0,055530	0,055066
		Max	0,893681	0,384426	0,382909	0,953810	0,247788	0,246957	0,841526	0,479458	0,477564
		Min	0,615573	0,106318	0,105031	0,752211	0,046189	0,045948	0,520541	0,158473	0,156627
900	1e -5	Mean	0,788671	0,211328	0,209136	0,878601	0,121398	0,120107	0,744164	0,255835	0,253189
		Std	0,060355	0,060355	0,059637	0,043642	0,043642	0,043215	0,059387	0,059387	0,058646
		Max	0,925623	0,364412	0,361427	0,956982	0,250170	0,246483	0,872647	0,453772	0,446581
		Min	0,635587	0,074376	0,073547	0,749829	0,043017	0,042755	0,546227	0,127352	0,126043
900	1e -8	Mean	0,788339	0,211660	0,209567	0,883680	0,116319	0,115091	0,751739	0,248260	0,245808
		Std	0,055608	0,055608	0,055032	0,036877	0,036877	0,036523	0,055431	0,055431	0,054850
		Max	0,905627	0,379032	0,375519	0,950664	0,246478	0,244167	0,874281	0,431097	0,426874
		Min	0,620967	0,094372	0,093543	0,753521	0,049335	0,049011	0,568902	0,125718	0,124668

Table 4.6 : Simulation summary for function 1, using several values of the number of $npunt$, τ, $\varepsilon = 3$ and $nctrs = 150$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,778142	0,221857	0,220373	0,889224	0,110775	0,110031	0,739421	0,260578	0,258835
		Std	0,062804	0,062804	0,062379	0,034569	0,034569	0,034314	0,060511	0,060511	0,060090
		Max	0,905929	0,391362	0,388786	0,964141	0,193069	0,191828	0,833204	0,440780	0,437879
		Min	0,608637	0,094070	0,093513	0,806930	0,035858	0,035527	0,559219	0,166795	0,165744
100	1e -5	Mean	0,790261	0,209738	0,208363	0,883789	0,116210	0,115467	0,748695	0,251304	0,249667
		Std	0,057159	0,057159	0,056689	0,038501	0,038501	0,038257	0,059507	0,059507	0,059059
		Max	0,893202	0,427012	0,423002	0,954407	0,220018	0,218332	0,865754	0,469262	0,464856
		Min	0,572987	0,106797	0,106048	0,779981	0,045592	0,045368	0,530737	0,134245	0,133562
100	1e -8	Mean	0,781428	0,218571	0,217152	0,889492	0,110507	0,109814	0,744840	0,255159	0,253501
		Std	0,050383	0,050383	0,050031	0,033724	0,033724	0,033468	0,052673	0,052673	0,052280
		Max	0,881498	0,354956	0,352767	0,954638	0,190668	0,189335	0,847108	0,383894	0,380651
		Min	0,645043	0,118501	0,118055	0,809331	0,045361	0,045098	0,616105	0,152891	0,151782
400	1e -3	Mean	0,778142	0,221857	0,220373	0,889224	0,110775	0,110031	0,739421	0,260578	0,258835
		Std	0,062804	0,062804	0,062379	0,034569	0,034569	0,034314	0,060511	0,060511	0,060090
		Max	0,905929	0,391362	0,388786	0,964141	0,193069	0,191828	0,833204	0,440780	0,437879
		Min	0,608637	0,094070	0,093513	0,806930	0,035858	0,035527	0,559219	0,166795	0,165744
400	1e -5	Mean	0,778036	0,221963	0,220477	0,889072	0,110927	0,110181	0,739149	0,260850	0,259103
		Std	0,063614	0,063614	0,063185	0,034561	0,034561	0,034307	0,060650	0,060650	0,060230
		Max	0,905990	0,390285	0,387721	0,965059	0,192329	0,191076	0,834063	0,439900	0,437010
		Min	0,609714	0,094009	0,093452	0,807670	0,034940	0,034593	0,560099	0,165936	0,164330
400	1e -8	Mean	0,781428	0,218571	0,217152	0,889492	0,110507	0,109814	0,744840	0,255159	0,253501
		Std	0,050383	0,050383	0,050031	0,033724	0,033724	0,033468	0,052673	0,052673	0,052280
		Max	0,881498	0,354956	0,352767	0,954638	0,190668	0,189335	0,847108	0,383894	0,380651
		Min	0,645043	0,118501	0,118055	0,809331	0,045361	0,045098	0,616105	0,152891	0,151782
900	1e -3	Mean	0,781395	0,218604	0,217148	0,884959	0,115040	0,114242	0,740670	0,259329	0,257586
		Std	0,064525	0,064525	0,064216	0,039455	0,039455	0,039240	0,062501	0,062501	0,062197
		Max	0,922899	0,378299	0,376318	0,950892	0,228943	0,227737	0,861151	0,439417	0,437116
		Min	0,621700	0,077100	0,076726	0,771056	0,049107	0,048705	0,560582	0,138848	0,138174
900	1e -5	Mean	0,772711	0,227288	0,225744	0,881139	0,118860	0,118016	0,731648	0,268351	0,266519
		Std	0,059010	0,059010	0,058547	0,041420	0,041420	0,041093	0,058826	0,058826	0,058317
		Max	0,919491	0,366771	0,363930	0,953047	0,252165	0,250538	0,883417	0,465096	0,461985
		Min	0,633228	0,080508	0,079982	0,747834	0,046952	0,046361	0,534903	0,116582	0,115820
900	1e -8	Mean	0,787410	0,212589	0,211231	0,893402	0,106597	0,105931	0,743134	0,256865	0,255209
		Std	0,056615	0,056615	0,056293	0,033711	0,033711	0,033453	0,056983	0,056983	0,056648
		Max	0,976532	0,341547	0,339688	0,961262	0,212100	0,210716	0,874005	0,401047	0,397883
		Min	0,658452	0,023467	0,023201	0,787899	0,038737	0,038621	0,598952	0,125994	0,124952

Table 4.7 : Simulation summary for function 1, using several values of the number of $npunt$, τ, $\varepsilon = 3$ and $nctrs = 200$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,791556	0,208443	0,207392	0,890686	0,109313	0,108773	0,751406	0,248593	0,247343
		Std	0,057322	0,057322	0,057021	0,039678	0,039678	0,039455	0,053696	0,053696	0,053400
		Max	0,996014	0,344981	0,343645	0,961901	0,225593	0,223967	0,875572	0,371797	0,370111
		Min	0,655018	0,003985	0,003976	0,774406	0,038098	0,037903	0,628202	0,124427	0,124141
100	1e -5	Mean	0,785554	0,214445	0,213423	0,887670	0,112329	0,111796	0,753902	0,246097	0,244925
		Std	0,060970	0,060970	0,060670	0,035764	0,035764	0,035608	0,052346	0,052346	0,052082
		Max	0,948939	0,349679	0,348042	0,966808	0,186698	0,185730	0,852589	0,362485	0,360435
		Min	0,650320	0,051060	0,050858	0,813301	0,033191	0,032963	0,637514	0,147410	0,146471
100	1e -8	Mean	0,785584	0,214415	0,213375	0,888661	0,111338	0,110759	0,752977	0,247022	0,245825
		Std	0,062161	0,062161	0,061861	0,036901	0,036901	0,036716	0,055043	0,055043	0,054786
		Max	0,907092	0,404794	0,403742	0,961380	0,186697	0,185730	0,842901	0,442812	0,441661
		Min	0,595205	0,092907	0,092310	0,813302	0,038619	0,038475	0,557187	0,157098	0,156254
400	1e -3	Mean	0,791556	0,208443	0,207392	0,890686	0,109313	0,108773	0,751406	0,248593	0,247343
		Std	0,057322	0,057322	0,057021	0,039678	0,039678	0,039455	0,053696	0,053696	0,053400
		Max	0,996014	0,344981	0,343645	0,961901	0,225593	0,223967	0,875572	0,371797	0,370111
		Min	0,655018	0,003985	0,003976	0,774406	0,038098	0,037903	0,628202	0,124427	0,124141
400	1e -5	Mean	0,791559	0,208440	0,207389	0,890685	0,109314	0,108774	0,751401	0,248598	0,247348
		Std	0,057318	0,057318	0,057017	0,039678	0,039678	0,039455	0,053696	0,053696	0,053401
		Max	0,995754	0,345146	0,343810	0,961893	0,225591	0,223965	0,875598	0,371560	0,369874
		Min	0,654853	0,004245	0,004236	0,774408	0,038106	0,037912	0,628439	0,124401	0,124115
400	1e -8	Mean	0,785584	0,214415	0,213375	0,888661	0,111338	0,110759	0,752977	0,247022	0,245825
		Std	0,062161	0,062161	0,061861	0,036901	0,036901	0,036716	0,055043	0,055043	0,054786
		Max	0,907092	0,404794	0,403742	0,961380	0,186697	0,185730	0,842901	0,442812	0,441661
		Min	0,595205	0,092907	0,092310	0,813302	0,038619	0,038475	0,557187	0,157098	0,156254
900	1e -3	Mean	0,789289	0,210710	0,209682	0,888223	0,111776	0,111218	0,749187	0,250812	0,249592
		Std	0,056491	0,056491	0,056162	0,037773	0,037773	0,037572	0,059824	0,059824	0,059464
		Max	0,901501	0,412221	0,409551	0,957113	0,207809	0,206800	0,867410	0,446078	0,443189
		Min	0,587778	0,098498	0,098008	0,792190	0,042886	0,042629	0,553921	0,132589	0,131888
900	1e -5	Mean	0,797821	0,202178	0,201141	0,893563	0,106436	0,105911	0,766842	0,233157	0,231960
		Std	0,059444	0,059444	0,059110	0,036959	0,036959	0,036769	0,053477	0,053477	0,053148
		Max	0,948939	0,341585	0,339737	0,966808	0,190456	0,189073	0,888343	0,355223	0,353696
		Min	0,658414	0,051060	0,050858	0,809543	0,033191	0,032963	0,644776	0,111656	0,111025
900	1e -8	Mean	0,801961	0,198038	0,197029	0,892654	0,107345	0,106805	0,769085	0,230914	0,229736
		Std	0,055859	0,055859	0,055517	0,038778	0,038778	0,038558	0,053831	0,053831	0,053474
		Max	0,924568	0,341589	0,339742	0,966808	0,196734	0,195034	0,888343	0,355221	0,353695
		Min	0,658410	0,075431	0,074957	0,803265	0,033191	0,032963	0,644778	0,111656	0,111025

Table 4.8 : Simulation summary for function 2, using several values of the number of $npunt$, τ, $\varepsilon = 0.4$ and $nctrs = 100$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,817597	0,182402	0,180673	0,858534	0,141465	0,140093	0,740236	0,259763	0,257268
		Std	0,073039	0,073039	0,072421	0,033515	0,033515	0,033239	0,051656	0,051656	0,051193
		Max	1,069806	0,438013	0,435490	0,939862	0,240446	0,239158	0,846018	0,443085	0,440533
		Min	0,561986	0,069806	0,069005	0,759553	0,060137	0,059719	0,556914	0,153981	0,152731
100	1e -5	Mean	0,814918	0,185081	0,183307	0,857961	0,142038	0,140723	0,729902	0,270097	0,267489
		Std	0,083825	0,083825	0,083071	0,031091	0,031091	0,030758	0,052990	0,052990	0,052419
		Max	1,271015	0,359766	0,357888	0,930903	0,225038	0,222607	0,837764	0,419739	0,417548
		Min	0,640233	0,271015	0,268711	0,774961	0,069096	0,069019	0,580260	0,162235	0,160994
100	1e -8	Mean	0,801039	0,198960	0,196987	0,852833	0,147166	0,145700	0,731178	0,268821	0,266183
		Std	0,068249	0,068249	0,067545	0,035401	0,035401	0,035094	0,065830	0,065830	0,065211
		Max	0,966503	0,357631	0,355748	0,911572	0,219665	0,216291	0,867148	0,427976	0,425723
		Min	0,642368	0,033496	0,033141	0,780334	0,088427	0,087903	0,572023	0,132851	0,131982
400	1e -3	Mean	0,803309	0,196690	0,194741	0,857709	0,142290	0,140841	0,723812	0,276187	0,273452
		Std	0,059662	0,059662	0,059088	0,033238	0,033238	0,032869	0,054274	0,054274	0,053767
		Max	0,952233	0,359335	0,356538	0,915104	0,241835	0,239636	0,833996	0,433243	0,429206
		Min	0,640664	0,047766	0,047189	0,758164	0,084895	0,084111	0,566756	0,166003	0,164023
400	1e -5	Mean	0,803309	0,196690	0,194741	0,857709	0,142290	0,140841	0,723812	0,276187	0,273452
		Std	0,059662	0,059662	0,059088	0,033238	0,033238	0,032869	0,054274	0,054274	0,053767
		Max	0,952233	0,359335	0,356538	0,915104	0,241835	0,239636	0,833996	0,433243	0,429206
		Min	0,640664	0,047766	0,047189	0,758164	0,084895	0,084111	0,566756	0,166003	0,164023
400	1e -8	Mean	0,810772	0,189227	0,187190	0,861561	0,138438	0,136918	0,734267	0,265732	0,262884
		Std	0,055705	0,055705	0,055023	0,033168	0,033168	0,032707	0,055361	0,055361	0,054689
		Max	0,913286	0,349663	0,345799	0,920147	0,240450	0,237905	0,858089	0,411994	0,406832
		Min	0,650336	0,086713	0,085457	0,759549	0,079852	0,078414	0,588005	0,141910	0,139966
900	1e -3	Mean	0,802787	0,197212	0,195190	0,856593	0,143406	0,141984	0,720223	0,279776	0,276915
		Std	0,060910	0,060910	0,060284	0,035351	0,035351	0,035043	0,062198	0,062198	0,061552
		Max	0,931596	0,344988	0,341612	0,930848	0,262581	0,259981	0,821485	0,486152	0,481235
		Min	0,655011	0,068403	0,067347	0,737418	0,069151	0,067570	0,513847	0,178514	0,175733
900	1e -5	Mean	0,814918	0,185081	0,183307	0,857961	0,142038	0,140723	0,729902	0,270097	0,267489
		Std	0,083825	0,083825	0,083071	0,031091	0,031091	0,030758	0,052990	0,052990	0,052419
		Max	1,271015	0,359766	0,357888	0,930903	0,225038	0,222607	0,837764	0,419739	0,417548
		Min	0,640233	0,271015	0,268711	0,774961	0,069096	0,069019	0,580260	0,162235	0,160994
900	1e -8	Mean	0,812959	0,187040	0,185121	0,857007	0,142992	0,141597	0,738976	0,261023	0,258299
		Std	0,062916	0,062916	0,062253	0,031735	0,031735	0,031381	0,048324	0,048324	0,047742
		Max	1,071151	0,326164	0,322705	0,909257	0,233803	0,232059	0,833808	0,376109	0,371431
		Min	0,673835	0,071151	0,070007	0,766196	0,090742	0,090217	0,623890	0,166191	0,164716

Table 4.9 : Simulation summary for function 2, using several values of the number of $npunt$, τ, $\varepsilon = 0.4$ and $nctrs = 150$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,805677	0,194322	0,192956	0,856517	0,143482	0,142458	0,744235	0,255764	0,253938
		Std	0,063280	0,063280	0,062877	0,035115	0,035115	0,034898	0,052401	0,052401	0,052052
		Max	1,071764	0,363789	0,362064	0,931204	0,245575	0,244343	0,840131	0,442423	0,440582
		Min	0,636210	0,071764	0,070981	0,754424	0,068795	0,068189	0,557576	0,159868	0,159087
100	1e -5	Mean	0,811554	0,188445	0,187205	0,856201	0,143798	0,142840	0,737310	0,262689	0,260962
		Std	0,058336	0,058336	0,057909	0,031544	0,031544	0,031301	0,052160	0,052160	0,051789
		Max	1,015265	0,312438	0,309302	0,917129	0,231228	0,229709	0,841540	0,409243	0,407559
		Min	0,687561	0,015265	0,015122	0,768771	0,082870	0,082520	0,590756	0,158459	0,157390
100	1e -8	Mean	0,822503	0,177496	0,176343	0,858916	0,141083	0,140165	0,745175	0,254824	0,253194
		Std	0,069954	0,069954	0,069456	0,029952	0,029952	0,029760	0,047172	0,047172	0,046909
		Max	1,105375	0,321602	0,319357	0,924388	0,227321	0,225974	0,851483	0,380240	0,377352
		Min	0,678397	0,105375	0,104740	0,772678	0,075611	0,075242	0,619759	0,148516	0,147499
400	1e -3	Mean	0,800638	0,199361	0,197979	0,854960	0,145039	0,144062	0,735280	0,264719	0,262883
		Std	0,068849	0,068849	0,068392	0,034281	0,034281	0,034051	0,054722	0,054722	0,054377
		Max	0,954185	0,434588	0,431004	0,915252	0,244043	0,242866	0,832906	0,453070	0,449120
		Min	0,565411	0,045814	0,045584	0,755956	0,084747	0,083864	0,546929	0,167093	0,166044
400	1e -5	Mean	0,800638	0,199361	0,197979	0,854960	0,145039	0,144062	0,735280	0,264719	0,262883
		Std	0,068849	0,068849	0,068392	0,034281	0,034281	0,034051	0,054722	0,054722	0,054377
		Max	0,954185	0,434588	0,431004	0,915252	0,244043	0,242866	0,832906	0,453070	0,449120
		Min	0,565411	0,045814	0,045584	0,755956	0,084747	0,083864	0,546929	0,167093	0,166044
400	1e -8	Mean	0,795503	0,204496	0,203132	0,852517	0,147482	0,146497	0,730552	0,269447	0,267619
		Std	0,067884	0,067884	0,067458	0,035835	0,035835	0,035580	0,057641	0,057641	0,057187
		Max	0,935884	0,424825	0,421780	0,915457	0,246420	0,244909	0,839312	0,430559	0,427473
		Min	0,575174	0,064115	0,063589	0,753579	0,084542	0,083524	0,569440	0,160687	0,159761
900	1e -3	Mean	0,813155	0,186844	0,185611	0,860968	0,139031	0,138087	0,748873	0,251126	0,249475
		Std	0,051814	0,051814	0,051449	0,031373	0,031373	0,031211	0,052214	0,052214	0,051824
		Max	0,925979	0,312348	0,310786	0,928723	0,210712	0,209869	0,855270	0,393376	0,389087
		Min	0,687651	0,074020	0,073599	0,789287	0,071276	0,070819	0,606623	0,144729	0,143797
900	1e -5	Mean	0,811554	0,188445	0,187205	0,856201	0,143798	0,142840	0,737310	0,262689	0,260962
		Std	0,058336	0,058336	0,057909	0,031544	0,031544	0,031301	0,052160	0,052160	0,051789
		Max	1,015265	0,312438	0,309302	0,917129	0,231228	0,229709	0,841540	0,409243	0,407559
		Min	0,687561	0,015265	0,015122	0,768771	0,082870	0,082520	0,590756	0,158459	0,157390
900	1e -8	Mean	0,817989	0,182010	0,180807	0,862317	0,137682	0,136757	0,746281	0,253718	0,252038
		Std	0,069341	0,069341	0,068863	0,033552	0,033552	0,033345	0,053599	0,053599	0,053150
		Max	1,205926	0,312347	0,310785	0,928724	0,210712	0,208481	0,855270	0,393376	0,389087
		Min	0,687652	0,205926	0,204533	0,789287	0,071275	0,070819	0,606623	0,144729	0,143797

Table 4.10 : Simulation summary for function 2, using several values of the number of $npunt$, τ, $\varepsilon = 0.4$ and $nctrs = 200$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,825844	0,174155	0,173262	0,857001	0,142998	0,142286	0,740309	0,259690	0,258396
		Std	0,088647	0,088647	0,088234	0,035846	0,035846	0,035740	0,056190	0,056190	0,055909
		Max	1,227944	0,460820	0,458001	0,922577	0,248434	0,247282	0,835606	0,467480	0,464620
		Min	0,539179	0,227944	0,227455	0,751565	0,077422	0,076886	0,532519	0,164393	0,163377
100	1e -5	Mean	0,820130	0,179869	0,178962	0,859388	0,140611	0,139926	0,743065	0,256934	0,255636
		Std	0,062485	0,062485	0,062159	0,031672	0,031672	0,031491	0,049402	0,049402	0,049111
		Max	1,023883	0,368054	0,366080	0,915178	0,237112	0,235489	0,848169	0,378250	0,376525
		Min	0,631945	0,023883	0,023770	0,762887	0,084821	0,084214	0,621749	0,151830	0,151102
100	1e -8	Mean	0,818969	0,181030	0,180101	0,856881	0,143118	0,142381	0,735787	0,264212	0,262851
		Std	0,066955	0,066955	0,066587	0,033550	0,033550	0,033312	0,054806	0,054806	0,054483
		Max	1,063439	0,368054	0,366080	0,919030	0,263363	0,261910	0,852961	0,420451	0,418315
		Min	0,631945	0,063439	0,063117	0,736636	0,080969	0,080653	0,579548	0,147038	0,146436
400	1e -3	Mean	0,831347	0,168652	0,167753	0,860390	0,139609	0,138913	0,744294	0,255705	0,254385
		Std	0,084300	0,084300	0,083879	0,033965	0,033965	0,033802	0,058093	0,058093	0,057751
		Max	1,227944	0,394493	0,391021	0,920439	0,242554	0,240522	0,861351	0,456544	0,452526
		Min	0,605506	0,227944	0,227455	0,757445	0,079560	0,078992	0,543455	0,138648	0,137908
400	1e -5	Mean	0,831347	0,168652	0,167753	0,860390	0,139609	0,138913	0,744294	0,255705	0,254385
		Std	0,084300	0,084300	0,083879	0,033965	0,033965	0,033802	0,058093	0,058093	0,057751
		Max	1,227943	0,394493	0,391021	0,920439	0,242554	0,240522	0,861351	0,456544	0,452526
		Min	0,605506	0,227943	0,227455	0,757445	0,079560	0,078992	0,543455	0,138648	0,137908
400	1e -8	Mean	0,822883	0,177116	0,176227	0,857431	0,142568	0,141836	0,738702	0,261297	0,259981
		Std	0,067997	0,067997	0,067660	0,034327	0,034327	0,034129	0,057580	0,057580	0,057306
		Max	1,063439	0,341827	0,340856	0,924346	0,263363	0,261910	0,878097	0,420451	0,418315
		Min	0,658172	0,063439	0,063117	0,736636	0,075653	0,075215	0,579548	0,121902	0,121076
900	1e -3	Mean	0,811274	0,188725	0,187779	0,857974	0,142025	0,141331	0,738997	0,261002	0,259693
		Std	0,058661	0,058661	0,058394	0,032671	0,032671	0,032503	0,055202	0,055202	0,054953
		Max	0,974553	0,342979	0,341577	0,923323	0,231133	0,230097	0,841059	0,392220	0,390905
		Min	0,657020	0,025446	0,025366	0,768866	0,076676	0,076124	0,607779	0,158940	0,158029
900	1e -5	Mean	0,820130	0,179869	0,178962	0,859388	0,140611	0,139926	0,743065	0,256934	0,255636
		Std	0,062485	0,062485	0,062159	0,031672	0,031672	0,031491	0,049402	0,049402	0,049111
		Max	1,023883	0,368054	0,366080	0,915178	0,237112	0,235489	0,848169	0,378250	0,376525
		Min	0,631945	0,023883	0,023770	0,762887	0,084821	0,084214	0,621749	0,151830	0,151102
900	1e -8	Mean	0,816287	0,183712	0,182761	0,855594	0,144405	0,143661	0,739842	0,260157	0,258817
		Std	0,058353	0,058353	0,058038	0,031391	0,031391	0,031246	0,053037	0,053037	0,052793
		Max	0,930959	0,366784	0,364412	0,916723	0,243921	0,242376	0,835930	0,404512	0,401896
		Min	0,633215	0,069040	0,068712	0,756078	0,083276	0,082798	0,595487	0,164069	0,163175

Table 4.11 : Simulation summary for function 2, using several values of the number of *npunt*, τ, $\varepsilon = 3$ and $nctrs = 100$.

npunt	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,838560	0,161439	0,159845	0,875683	0,124316	0,123034	0,775698	0,224301	0,222086
		Std	0,043490	0,043490	0,043061	0,023678	0,023678	0,023446	0,032428	0,032428	0,032107
		Max	0,995686	0,261088	0,258474	0,921692	0,188905	0,187282	0,844784	0,333009	0,329675
		Min	0,738911	0,004313	0,004266	0,811094	0,078307	0,077405	0,666990	0,155215	0,153517
100	1e -5	Mean	0,839230	0,160769	0,159155	0,874391	0,125608	0,124322	0,782773	0,217226	0,215055
		Std	0,049768	0,049768	0,049283	0,023570	0,023570	0,023363	0,040985	0,040985	0,040643
		Max	1,005234	0,279427	0,277835	0,918686	0,180636	0,178813	0,859591	0,326994	0,325130
		Min	0,720572	0,005234	0,005170	0,819363	0,081313	0,080362	0,673005	0,140408	0,139250
100	1e -8	Mean	0,841679	0,158320	0,156649	0,879704	0,120295	0,118999	0,785350	0,214649	0,212384
		Std	0,037942	0,037942	0,037497	0,021941	0,021941	0,021686	0,034968	0,034968	0,034522
		Max	0,916745	0,280156	0,277142	0,933274	0,214025	0,211663	0,853955	0,329296	0,325753
		Min	0,719843	0,083254	0,082299	0,785974	0,066725	0,065202	0,670703	0,146044	0,144671
400	1e -3	Mean	0,843494	0,156505	0,154899	0,878000	0,121999	0,120756	0,780587	0,219412	0,217171
		Std	0,044900	0,044900	0,044492	0,023696	0,023696	0,023475	0,031965	0,031965	0,031686
		Max	1,035923	0,298512	0,296387	0,937149	0,208713	0,205811	0,843251	0,336289	0,333895
		Min	0,701487	0,035923	0,035743	0,791286	0,062850	0,062451	0,663710	0,156748	0,154369
400	1e -5	Mean	0,839995	0,160004	0,158389	0,877351	0,122648	0,121363	0,775708	0,224291	0,222037
		Std	0,042338	0,042338	0,041857	0,025609	0,025609	0,025342	0,036232	0,036232	0,035860
		Max	0,930082	0,265194	0,262189	0,922919	0,202810	0,201308	0,845170	0,322336	0,320110
		Min	0,734805	0,069917	0,068911	0,797189	0,077080	0,076872	0,677663	0,154829	0,152766
400	1e -8	Mean	0,843503	0,156496	0,155030	0,878136	0,121863	0,120800	0,783611	0,216388	0,214342
		Std	0,041577	0,041577	0,041235	0,022389	0,022389	0,022143	0,032038	0,032038	0,031702
		Max	0,947625	0,256436	0,254738	0,929412	0,187750	0,186459	0,844621	0,306885	0,304167
		Min	0,743563	0,052374	0,051726	0,812249	0,070587	0,069956	0,693114	0,155378	0,153852
900	1e -3	Mean	0,841699	0,158300	0,156724	0,879456	0,120543	0,119360	0,784921	0,215078	0,212933
		Std	0,048368	0,048368	0,047910	0,024251	0,024251	0,024058	0,037143	0,037143	0,036774
		Max	0,959336	0,290461	0,287818	0,923084	0,212270	0,210517	0,862498	0,351861	0,349037
		Min	0,709538	0,040663	0,040287	0,787729	0,076915	0,076662	0,648138	0,137501	0,136730
900	1e -5	Mean	0,839230	0,160769	0,159155	0,874391	0,125608	0,124322	0,782773	0,217226	0,215055
		Std	0,049768	0,049768	0,049283	0,023570	0,023570	0,023363	0,040985	0,040985	0,040643
		Max	1,005234	0,279427	0,277835	0,918686	0,180636	0,178813	0,859591	0,326994	0,325130
		Min	0,720572	0,005234	0,005170	0,819363	0,081313	0,080362	0,673005	0,140408	0,139250
900	1e -8	Mean	0,845815	0,154184	0,152641	0,877329	0,122670	0,121386	0,778726	0,221273	0,219049
		Std	0,065025	0,065025	0,064395	0,024623	0,024623	0,024411	0,036964	0,036964	0,036578
		Max	1,082924	0,304704	0,302288	0,917408	0,201163	0,200102	0,848592	0,335980	0,333316
		Min	0,695295	0,082924	0,081986	0,798836	0,082591	0,081665	0,664019	0,151407	0,150283

Table 4.12 : Simulation summary for function 2, using several values of the number of $npunt$, τ, $\varepsilon = 3$ and $nctrs = 150$.

$npunt$	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,856430	0,143569	0,142631	0,875415	0,124584	0,123730	0,788959	0,211040	0,209644
		Std	0,051421	0,051421	0,051108	0,024201	0,024201	0,024100	0,030885	0,030885	0,030750
		Max	1,090984	0,250879	0,249728	0,926611	0,210521	0,209740	0,846150	0,348532	0,346933
		Min	0,749120	0,090984	0,090119	0,789478	0,073388	0,073101	0,651467	0,153849	0,153040
100	1e -5	Mean	0,838724	0,161275	0,160191	0,879136	0,120863	0,120067	0,790139	0,209860	0,208444
		Std	0,042490	0,042490	0,042242	0,020458	0,020458	0,020285	0,034143	0,034143	0,033945
		Max	0,938370	0,289489	0,286446	0,919789	0,184430	0,182377	0,856531	0,331500	0,328015
		Min	0,710510	0,061629	0,061076	0,815569	0,080210	0,079874	0,668499	0,143468	0,142141
100	1e -8	Mean	0,836561	0,163438	0,162348	0,874978	0,125021	0,124171	0,782909	0,217090	0,215653
		Std	0,048481	0,048481	0,048104	0,022269	0,022269	0,022116	0,033124	0,033124	0,032876
		Max	1,007543	0,291371	0,289717	0,932256	0,186119	0,184681	0,891554	0,299581	0,297880
		Min	0,708628	0,007543	0,007475	0,813880	0,067743	0,067717	0,700418	0,108445	0,108124
400	1e -3	Mean	0,848745	0,151254	0,150217	0,876894	0,123105	0,122292	0,791998	0,208001	0,206582
		Std	0,050749	0,050749	0,050392	0,022951	0,022951	0,022789	0,031063	0,031063	0,030820
		Max	1,015467	0,303902	0,300815	0,914216	0,207090	0,204964	0,859311	0,316846	0,315392
		Min	0,696097	0,015467	0,015369	0,792909	0,085783	0,085517	0,683153	0,140688	0,139830
400	1e -5	Mean	0,847273	0,152726	0,151723	0,874699	0,125300	0,124470	0,786118	0,213881	0,212475
		Std	0,057410	0,057410	0,057070	0,023346	0,023346	0,023210	0,030858	0,030858	0,030608
		Max	1,073079	0,269455	0,268443	0,920595	0,196765	0,196253	0,860653	0,283295	0,279841
		Min	0,730544	0,073079	0,072763	0,803234	0,079404	0,079304	0,716704	0,139346	0,138754
400	1e -8	Mean	0,840916	0,159083	0,158060	0,874568	0,125431	0,124614	0,786619	0,213380	0,212018
		Std	0,051789	0,051789	0,051482	0,023402	0,023402	0,023303	0,038272	0,038272	0,038079
		Max	1,011863	0,293138	0,291809	0,918667	0,188483	0,187025	0,860137	0,307493	0,306098
		Min	0,706861	0,011863	0,011798	0,811516	0,081332	0,080563	0,692506	0,139862	0,138698
900	1e -3	Mean	0,845831	0,154168	0,153148	0,878782	0,121217	0,120429	0,790870	0,209129	0,207748
		Std	0,049799	0,049799	0,049473	0,024322	0,024322	0,024205	0,031676	0,031676	0,031470
		Max	1,078578	0,274673	0,272246	0,929363	0,182538	0,182259	0,867038	0,278784	0,277104
		Min	0,725326	0,078578	0,078115	0,817461	0,070636	0,070175	0,721215	0,132961	0,132248
900	1e -5	Mean	0,838724	0,161275	0,160191	0,879136	0,120863	0,120067	0,790139	0,209860	0,208444
		Std	0,042490	0,042490	0,042242	0,020458	0,020458	0,020285	0,034143	0,034143	0,033945
		Max	0,938370	0,289489	0,286446	0,919789	0,184430	0,182377	0,856531	0,331500	0,328015
		Min	0,710510	0,061629	0,061076	0,815569	0,080210	0,079874	0,668499	0,143468	0,142141
900	1e -8	Mean	0,845389	0,154610	0,153579	0,879099	0,120900	0,120113	0,793931	0,206068	0,204696
		Std	0,042190	0,042190	0,041930	0,022965	0,022965	0,022852	0,030775	0,030775	0,030617
		Max	0,968331	0,273925	0,271506	0,928416	0,182072	0,181802	0,865946	0,292379	0,290019
		Min	0,726074	0,031668	0,031411	0,817927	0,071583	0,070655	0,707620	0,134053	0,132676

Table 4.13 : Simulation summary for function 2, using several values of the number of *npunt*, τ, $\varepsilon = 3$ and $nctrs = 200$.

npunt	τ	Index	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
100	1e -3	Mean	0,852487	0,147512	0,146771	0,879637	0,120362	0,119795	0,790653	0,209346	0,208289
		Std	0,044132	0,044132	0,043921	0,020988	0,020988	0,020935	0,036492	0,036492	0,036284
		Max	0,938449	0,276802	0,275837	0,929196	0,180708	0,179613	0,857693	0,314828	0,311950
		Min	0,723197	0,061550	0,061395	0,819291	0,070803	0,070166	0,685171	0,142306	0,141502
100	1e -5	Mean	0,852205	0,147794	0,147053	0,876351	0,123648	0,123022	0,787664	0,212335	0,211267
		Std	0,051045	0,051045	0,050792	0,023896	0,023896	0,023757	0,034902	0,034902	0,034743
		Max	1,024664	0,262833	0,262003	0,921315	0,206671	0,205459	0,863145	0,314408	0,312754
		Min	0,737166	0,024664	0,024504	0,793328	0,078684	0,078296	0,685591	0,136854	0,136083
100	1e -8	Mean	0,849236	0,150763	0,150052	0,877831	0,122168	0,121614	0,792974	0,207025	0,206040
		Std	0,050346	0,050346	0,050113	0,024193	0,024193	0,024120	0,035612	0,035612	0,035452
		Max	1,024800	0,289074	0,287812	0,923264	0,200038	0,199246	0,863137	0,301168	0,299853
		Min	0,710925	0,024800	0,024639	0,799961	0,076735	0,075979	0,698831	0,136862	0,136091
400	1e -3	Mean	0,848920	0,151079	0,150318	0,877963	0,122036	0,121433	0,789300	0,210699	0,209645
		Std	0,040894	0,040894	0,040677	0,019826	0,019826	0,019767	0,032626	0,032626	0,032474
		Max	0,950718	0,275256	0,273684	0,929196	0,180708	0,179613	0,850257	0,304579	0,303079
		Min	0,724743	0,049281	0,049029	0,819291	0,070803	0,070166	0,695420	0,149742	0,148955
400	1e -5	Mean	0,846644	0,153355	0,152579	0,878010	0,121989	0,121366	0,790531	0,209468	0,208414
		Std	0,045735	0,045735	0,045504	0,023039	0,023039	0,022899	0,030021	0,030021	0,029897
		Max	0,986009	0,281290	0,279354	0,927877	0,180576	0,179601	0,850839	0,325700	0,324057
		Min	0,718709	0,013990	0,013944	0,819423	0,072122	0,071921	0,674299	0,149160	0,148285
400	1e -8	Mean	0,843225	0,156774	0,156026	0,876170	0,123829	0,123226	0,786959	0,213040	0,212027
		Std	0,053239	0,053239	0,052970	0,022547	0,022547	0,022493	0,035566	0,035566	0,035395
		Max	0,952836	0,328819	0,326810	0,927415	0,200038	0,199246	0,862867	0,333454	0,331416
		Min	0,671180	0,047163	0,046896	0,799961	0,072584	0,072085	0,666545	0,137132	0,136150
900	1e -3	Mean	0,852431	0,147568	0,146850	0,877180	0,122819	0,122202	0,792728	0,207271	0,206251
		Std	0,044232	0,044232	0,044015	0,021683	0,021683	0,021538	0,033577	0,033577	0,033367
		Max	0,971233	0,270462	0,269795	0,915893	0,182839	0,181429	0,846055	0,295307	0,293848
		Min	0,729537	0,028766	0,028660	0,817160	0,084106	0,083858	0,704692	0,153944	0,153613
900	1e -5	Mean	0,852205	0,147794	0,147053	0,876351	0,123648	0,123022	0,787664	0,212335	0,211267
		Std	0,051045	0,051045	0,050792	0,023896	0,023896	0,023757	0,034902	0,034902	0,034743
		Max	1,024664	0,262833	0,262003	0,921315	0,206671	0,205459	0,863145	0,314408	0,312754
		Min	0,737166	0,024664	0,024504	0,793328	0,078684	0,078296	0,685591	0,136854	0,136083
900	1e -8	Mean	0,850481	0,149518	0,148771	0,878016	0,121983	0,121359	0,790319	0,209680	0,208644
		Std	0,049261	0,049261	0,048985	0,023353	0,023353	0,023183	0,033883	0,033883	0,033681
		Max	1,050201	0,273349	0,272166	0,921023	0,185980	0,185157	0,853438	0,288597	0,286159
		Min	0,726650	0,050201	0,049992	0,814019	0,078976	0,078171	0,711402	0,146561	0,145912

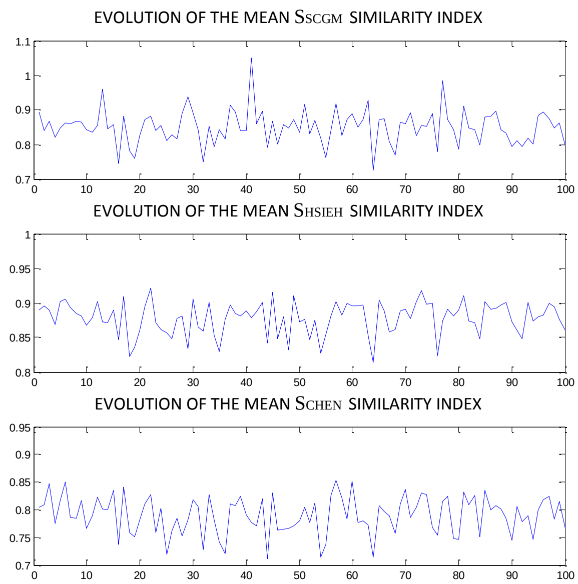

Figure 4.3.2: Function 2. $nctrs = 200$, $npunt = 900$, $\varepsilon = 1e-8$ and $\tau = 3$. Evolution of the similarity indices obtained

1. Analysis of the correlation matrix between the different error and similarity indices

2. Determining the Principal Component Analysis of the different error and similarity indices

Pasadas and his co-authors in [65] proposed the same methods to analyze the behavior for different error and similarity indices in the case of univariate cubic splines. This chapter is an extension of their work by using bicubic splines. In these two methods, we are going to take into consideration all the simulations run in this chapter.

In the first method, to construct the correlation matrix R for each of the components by means we use the following equations:

$$R(i,j) = \frac{C(x_i, x_j)}{\sqrt{C(x_i, x_i) C(x_j, x_j)}}$$

where $C(x_i, x_j)$ is the covariance matrix between the input variable x_i and the x_j input variable, the linear relationship among the error indices will be tested with approximately 5400 samples in each example (in each of the Tables 4.2 -4.13 of simulation, there are 9 different configurations of the parameters, and each configuration entails 100 executions).

Table 4.14 : Correlation matrix R for the error and similarity indices proposed for example 1, using several values for *npunt*, τ, ε and *nctrs* (a total of 5400 random simulations were used to compute of the matrix R).

R	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
$\overline{S}_{SCGM}$	1,000000	-1,000000	-0,999965	0,541247	-0,541247	-0,541457	0,893827	-0,893827	-0,893905
E_{SCM}	-1,000000	1,000000	0,999965	-0,541247	0,541247	0,541457	-0,893827	0,893827	0,893905
I_{SCM}	-0,999965	0,999965	1,000000	-0,540789	0,540789	0,541057	-0,893686	0,893686	0,893839
$\overline{S}_{HSIEH}$	0,541247	-0,541247	-0,540789	1,000000	-1,000000	-0,999950	0,677476	-0,677476	-0,676894
E_{HSIEH}	-0,541247	0,541247	0,540789	-1,000000	1,000000	0,999950	-0,677476	0,677476	0,676894
I_{HSIEH}	-0,541457	0,541457	0,541057	-0,999950	0,999950	1,000000	-0,677592	0,677592	0,677077
$\overline{S}_{CHEN}$	0,893827	-0,893827	-0,893686	0,677476	-0,677476	-0,677592	1,000000	-1,000000	-0,999958
E_{CHEN}	-0,893827	0,893827	0,893686	-0,677476	0,677476	0,677592	-1,000000	1,000000	0,999958
I_{CHEN}	-0,893905	0,893905	0,893839	-0,676894	0,676894	0,677077	-0,999958	0,999958	1,000000

Table 4.15 : Correlation matrix R for the error and similarity indices proposed for example 2, using several values for *npunt*, τ, ε and *nctrs* (a total of 5400 random simulations were used to compute of the matrix R).

R	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
$\overline{S}_{SCGM}$	1,000000	-1,000000	-0,999958	0,531546	-0,531546	-0,530760	0,614224	-0,614224	-0,614034
E_{SCM}	-1,000000	1,000000	0,999958	-0,531546	0,531546	0,530760	-0,614224	0,614224	0,614034
I_{SCM}	-0,999958	0,999958	1,000000	-0,531548	0,531548	0,530876	-0,613736	0,613736	0,613671
$\overline{S}_{HSIEH}$	0,531546	-0,531546	-0,531548	1,000000	-1,000000	-0,999853	0,727499	-0,727499	-0,728080
E_{HSIEH}	-0,531546	0,531546	0,531548	-1,000000	1,000000	0,999853	-0,727499	0,727499	0,728080
I_{HSIEH}	-0,530760	0,530760	0,530876	-0,999853	0,999853	1,000000	-0,726751	0,726751	0,727518
$\overline{S}_{CHEN}$	0,614224	-0,614224	-0,613736	0,727499	-0,727499	-0,726751	1,000000	-1,000000	-0,999896
E_{CHEN}	-0,614224	0,614224	0,613736	-0,727499	0,727499	0,726751	-1,000000	1,000000	0,999896
I_{CHEN}	-0,614034	0,614034	0,613671	-0,728080	0,728080	0,727518	-0,999896	0,999896	1,000000

In the first strategy, analysis of the correlation matrix between the different error and similarity indices, we will take into account simulations from Example 1 and Example 2, the matrix R is obtained (Tables 4.14 and 4.15), with its corre-

Table 4.16 : Example 1: matrix of p-values for testing the hypothesis of no correlation associated with R matrix of tables 4.14 -4.15 .

P	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
$\overline{S}_{SCGM}$	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
E_{SCM}	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
I_{SCM}	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000
$\overline{S}_{HSIEH}$	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000
E_{HSIEH}	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000
I_{HSIEH}	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000
$\overline{S}_{CHEN}$	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000
E_{CHEN}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000
I_{CHEN}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000

Table 4.17 : Matrix $Coef$ of the coefficient of the Principal Component.

Coef	$\overline{S}_{SCGM}$	E_{SCM}	I_{SCM}	$\overline{S}_{HSIEH}$	E_{HSIEH}	I_{HSIEH}	$\overline{S}_{CHEN}$	E_{CHEN}	I_{CHEN}
Princ.Comp. 1	0,530221	-0,310289	0,407932	-0,112991	0,016478	-0,169585	0,284683	-0,380025	-0,434642
Princ.Comp. 2	0,149702	0,437131	-0,340427	0,248375	-0,400892	0,047020	-0,335656	-0,380025	-0,434642
Princ.Comp. 3	0,148620	0,433987	-0,338415	0,246405	0,420119	-0,218085	0,624710	-1,28e-13	-3,41e-14
Princ.Comp. 4	0,588447	-0,004621	-0,316712	-0,331592	-0,101111	0,316053	0,000771	-0,186398	0,546432
Princ.Comp. 5	0,091477	0,131463	0,384218	0,466977	-0,283301	-0,438619	-0,051743	-0,186398	0,546432
Princ.Comp. 6	0,090814	0,130461	0,381283	0,463478	0,184525	0,760073	0,052747	7,38e-14	5,66e-15
Princ.Comp. 7	0,488908	-0,316306	-0,218486	0,412023	0,084552	-0,146483	-0,285445	0,566424	-0,111790
Princ.Comp. 8	0,191015	0,443147	0,285992	-0,276638	-0,468965	0,023918	0,234473	0,566424	-0,111790
Princ.Comp. 9	0,189634	0,439928	0,283441	-0,275138	0,557585	-0,171612	-0,523554	3,95e-14	1,94e-14

Table 4.18 : Vector of the variability explained by each principal component obtained by PCA

Princ.Comp.	σ_{PC_i}	V_{PC_i}	Princ.Comp.	σ_{PC_i}	V_{PC_i}
PC1	152.7712	80.4656	PC6	0.0284	0.0054
PC2	24.5625	12.9372	PC7	0.0103	0.0054
PC3	7.3621	3.8776	PC8	0.0000	0.0000
PC4	5.0341	2.6515	PC9	0.0000	0.0000
PC5	0.0906	0.0150			

sponding P matrix. The matrix P or matrix of p-values for testing the hypothesis of no correlation, is in this examples, the identity matrix as seen in Table 4.16

In the second method the Principal Components Analysis (PCA) finds low dimensional approximations to the data by projecting the data onto linear subspaces. Let A be a real $m \times n$ matrix. The Singular Value Decomposition (SVD) of an $m \times n$ matrix A expresses the matrix as the product of three "simple" matrices

$$A = U\Sigma V^T$$

To better see how the SVD expresses A, check that the factorization A is equivalent to the expression

$$A = \sum_{i=1}^{p} E_i$$

where the subindex $p = min(m, n)$. The component matrices E_i are rank outer products: $E_i = \sigma_i u_i v_i^T$, each column of E_i being a multiple of u_i, the ith column of U, and each row being a multiple of v_i^T, the transpose of the ith column of V. The coefficients of the linear combinations of the original variables that generate the principal components, are presented in the matrix $Coef$ of the Table 4.17 .

Finally, it is very important to calculate the variance explained by the corresponding principal component. This variance will be denoted as σ_{PC_i}, PC_i being the Principal Component, with $i = 1, \ldots, n$, and n is the number of variables (in our case the number of error and similarity indices presented, 9 in this case). Furthermore, it is very important to calculate the percentage of the total variability explained by each principal component (denoted V_{PC_i}), obtained as

$$V_{PC_i} = 100 \frac{\sigma_{PC_i}}{\sum_{j=1}^{n} \sigma_{PC_j}},$$

In the simulations carried out in this chapter, the variance explained and the percentage of the total variability explained are presented in Table 4.18 , and V_{PC_i} is graphically shown in Figure 4.4.3. We can conclude from analyzing the results obtained of the two statistical methodologies, that the error and similarity indices presented illustrate a homogeneous behavior, in which the linear correlation between them is statistically significant, the first two components of the principal component analysis having explained of 93.4% of the variance.

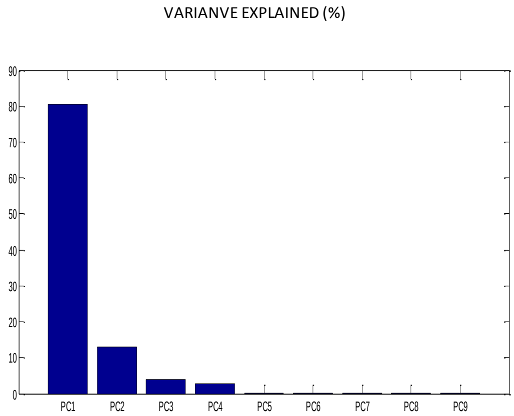

Figure 4.4.3: Variability explained for each principal component.

4.5 Conclusions

A conclusion can be drawn after the analysis of the results presented in Section 4.3, that the presented error indices how a logical behavior in order to evaluate the accuracy of the approximation obtained from fuzzy data. For Example, when using various numbers of nodes ($npunt$ and ($nctrs$)) in example 1 the Tables 4.2 to 4.7 illustrates an improvement in the correctness or accuracy of the approximation. As presented in these tables we would obtain a better approximation when the value in the error index $\bar{S}$ is closest to the unit (according to different similarity measures used such as S_{SCGM}, S_{HSIEH} and S_{CHEN}). Similar conclusions can be achieved for Example 2, from Tables 4.8 to 4.13 .

We analyzed the excellent homogeneity of these error and similarity indices and compared the approximation results carried out by a fuzzy data set from different simulations, using two statistical methodologies, the first one is analysis of the correlation and P-values matrix between the different error and similarity indices presented in Tables 4.14 , 4.15 and 4.16 , is that any of these indices can be used independently to measure and compare the accuracy of an approximation using fuzzy data (in absolute value, the elements of the correlation matrix is very close to unity), the same conclusion can be obtained in the second methodology which is principal component analysis of the different error and similarity indices, where the first two components of the component analysis having explained the 93.4% of the variability; the variability explained for each principal component is presented in Table 4.18 and is graphically shown in Figure 4.4.3.

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