DISPERSION VS DIFFUSION IN TRANSPORT PARTIAL DIFFERENTIAL EQUATIONS

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La presente memoria, titulada “Dispersion vs diffusion in transport partial differential equations”, ha sido realizada bajo la dirección de los doctores Juan S. Soler Vizcaíno y Oscar Sánchez Romero del Departamento de Matemática Aplicada de la Universidad de Granada, para obtener el título de Doctor en Matemáticas por la Universidad de Granada. Esta tesis fué defendida el día 26 de Marzo de 2010 ante el tribunal constituido por los Doctores: Vicent Caselles (presidente), Pierre–Emmanuel Jabin (vocal), Francois Golse (vocal), José M. Mazón (vocal) y Simone Carmelo Calogero (secretario), obteniendo la máxima calificación: Sobresaliente *Cum Laude* por unanimidad.

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Contents

Chapter 1

Introduction and results

This Thesis is articulated around the analysis of some mathematical models based on partial differential equations (PDEs) of kinetic type arising in various phenomena related to Astrophysics and Biology.

What does a kinetic PDE mean? Kinetic models provide statistical descriptions of systems constituted by a large number of interacting particles. The aim is to incorporate at the transport PDE level of description the microscopic properties of interaction between particles, which must be deduced from first principles. At this point there is also an interesting controversy between models supported on basic interaction laws and phenomenological models (usually describing macroscopic quantities). In this sense, a kinetic description provides an intermediate scale between the microscopic and the macroscopic pictures. The range of applicability is quite wide and flexible enough to admit many different entities to be qualified as particles, ranging from the biggest systems that we can imagine (galaxies and the Universe itself) to very small ones in which quantum effects may become important, like systems composed by atomic particles. Also, the word 'particle' includes individuals that have the capacity of taking decisions: vehicles in traffic flow or cells in biology. On the other hand, their versatility allows to describe a number of different interactions with the PDE system, as, for instance, long range interactions like gravitational or electrostatic potentials, short range interactions like aggregation or coagulation processes, or even diffusive effects. Another remarkable aspect is the possibility of deriving new models by performing hydrodynamic macroscopic limits from the kinetic PDE, either of diffusive parabolic type (low–field regime) or of hyperbolic type (high–field regimes), that incorporate the properties of the microscopic level —see for example [104, 105, 167, 178, 179]. The effects of one or other choice are fundamental for the qualitative properties of the transport by (parabolic or hyperbolic) fluxes of geometrical structures, fronts, patterns, etc. In this context a natural question arises: What kind of description is better suited to study a concrete physical reality? This might be regarded as one of the leading ideas of this Memory.

One of the main conceptual aspects concerning this Thesis is the discussion on modeling small fluctuations, stemming from the interactions not included initially in the system, without having the dynamics of the system (patterns, fronts, special configurations, singularities, etc) destroyed (vanishing). Parabolic and hyperbolic macroscopic limits are the two apparently dissociated ways dealing with this problem in the literature. These approaches are associated with the concepts of diffusion and dispersion respectively. The Thesis also concerns with a kind of intermediate situation: the flux–limited corrections, that induce a behavior closer to hyperbolic than to usual linear parabolic (Fokker–Planck) effects. Then the solutions involved in systems with this kind of terms contain features from both approaches such as finite velocity of propagation or regularizing effects in the interior of their supports.

This Memory has two well differentiated parts. In the first one we analyze the various concepts of dispersion in the framework of the dynamics of classical or relativistic Vlasov matter, giving examples and necessary and sufficient conditions for dispersion, coagulation or the apparition of stationary spatial configurations or breathing modes. This part connects a variety of concepts and is much less technical that the second one concerned with the analysis of a flux–limited system motivated by the transport of morphogens. The reader can freely choose the starting point of the reading without worrying to be lost, as from both a mathematical and a descriptive point of view the two parts are quite independent. The Thesis schedule is based on the journey connecting the microscopic to the macroscopic (mathematical) levels of description.

Let us briefly describe the precise contents of this Memory. The first part is concerned with kinetic descriptions including only long range interactions (save the last chapter of this part, which adds also some short-range interactions to the models). We will see that this scenario allows for a wealth of possibilities and consequently for a very rich dynamics. Depending on the macroscopic parameters of the initial conditions, a wide range of configurations such as spatial patterns, breathing modes, dispersive dynamics and even more complex phenomena may show up. We will provide explicit examples of all these as soon as we proceed with the concrete systems. These structures may even be robust (like stable stationary configurations or solitons) and need not be still. Such a feature must not be overlooked, as we have lots of examples in the real world that match this description: tumor dynamics, galactic dynamics or dark matter halos, to name only a few that will attract our attention in this Memory.

Therefore it is important to keep in mind that we can model a wide spectrum of dynamics using hyperbolic equations that are able to describe dispersive behavior in some regimes. We may enlarge our possibilities if we allow for other types of interaction in our kinetic hyperbolic models; as an example, we will deal in the last chapter of the first part of this Memory with the case of hyperbolic systems that include coagulation (formation of aggregates) mechanisms too.

A very important ingredient concerns the use of diffusive terms in our equations. These terms are related to some form of stochasticity, either because an explicit model is unknown or because there are too many factors to be taken into account that would make the model much more complex. The point being that, if we know that these uncontrollable variables have a small influence on the real phenomena, we want it to be the same for our equations.

Not any form of stochasticity is valid for any problem, as a matter of fact this choice is a crucial question. We are well aware that usual (linear) diffusion related to Brownian motion (white noise) will smooth everything out and this sort of uniformization mechanism could destroy in many cases most of the structure we would be interested in for the type of problems mentioned above. This is the case if we add a Fokker-Planck type term to the kinetic equations that describe the evolution of self-gravitating systems under long range interactions. Here, a recurrent theme arises: the confrontation of dispersion against diffusion, which lies in the heart of many of the topics that we are going to deal with. Dispersive behavior seems to be compatible with other complex structures that kinetic descriptions can bear: the invariances (conservation laws) of the system are preserved and can coexist with the presence of steady configurations, for instance. This need not be the case for standard (linear Fokker–Planck) diffusive mechanisms, that do not preserve the physically meaningful quantities and tend to spoil the whole dynamics, no matter the smallness of their contribution. In fact, the tails of the Gaussian distribution are the only structures that survive to the dynamics even if being initially very small.

As we will require our models to be able to preserve macroscopic (indeed physically observable) structures, this motivates the search for alternative diffusion mechanisms. Such diffusions have to be necessarily nonlinear. Many of these have been investigated in the physical and mathematical literature; we will be interested in the class of diffusion mechanisms that enjoy the additional property of providing finite speed of propagation (as in the case of kinetic transport equations). Porous media type equations provide this kind of descriptions, although the speed of propagation is not intrinsic to the general laws governing the observable phenomena at a microscopic level of description, rather it depends on the initial configuration. An approach that fulfills these requirements is that of flux limitation, which will be the subject of the second part of this Memory. These models allow for robust structures like propagating fronts and introduce new phenomena such as singular traveling waves; we will see some of these features in action in the form of macroscopic models. In fact, the results contained in this Thesis together with those of [200] prove that the application of these arguments to model the transport of morphogens implies that the unphysical diffusion is eliminated and that we can induce the preservation of the dynamical structures such as propagating fronts or biological responses to them, that are in perfect qualitative agreement with experimental results. On the other hand, there are various recent efforts trying to deduce flux–limited terms from first microscopic principles, from which we can mention here the hyperbolic limits of a kinetic system for the case of a flux–limited chemotaxis system [40] or the diffusion arising from stochastic processes related with mean curvature fluxes. The qualitative differences between a linear diffusion equation and a flux-limited one allowing propagation of fronts can be seen on Figure 1.1.

To summarize, on one hand we will have transport equations to study dispersive behavior and steady states in Astrophysics, that we will also use to study the long time dynamics of populations; their character is mostly hyperbolic. On the other hand we meet the flux limited equations, exhibiting a mixture of parabolic and hyperbolic behavior, that will be used to study the transport of morphogenes in the embryo. These equations can also have a role in Astrophysics, as an alternative to Fokker–Planck equations.

In the following we detail the sort of problems to be studied, the mathematical models to be used and the obtained results.

1.1.1 Self-gravitating systems

In this memory, several kinetic models that are used to describe self-gravitating systems are studied. Needless to say, in so doing we will always work in three spatial dimensions. We focus on the long time behavior of their solutions as well as on certain properties of their steady state solutions. This will have several applications in the field of galactic

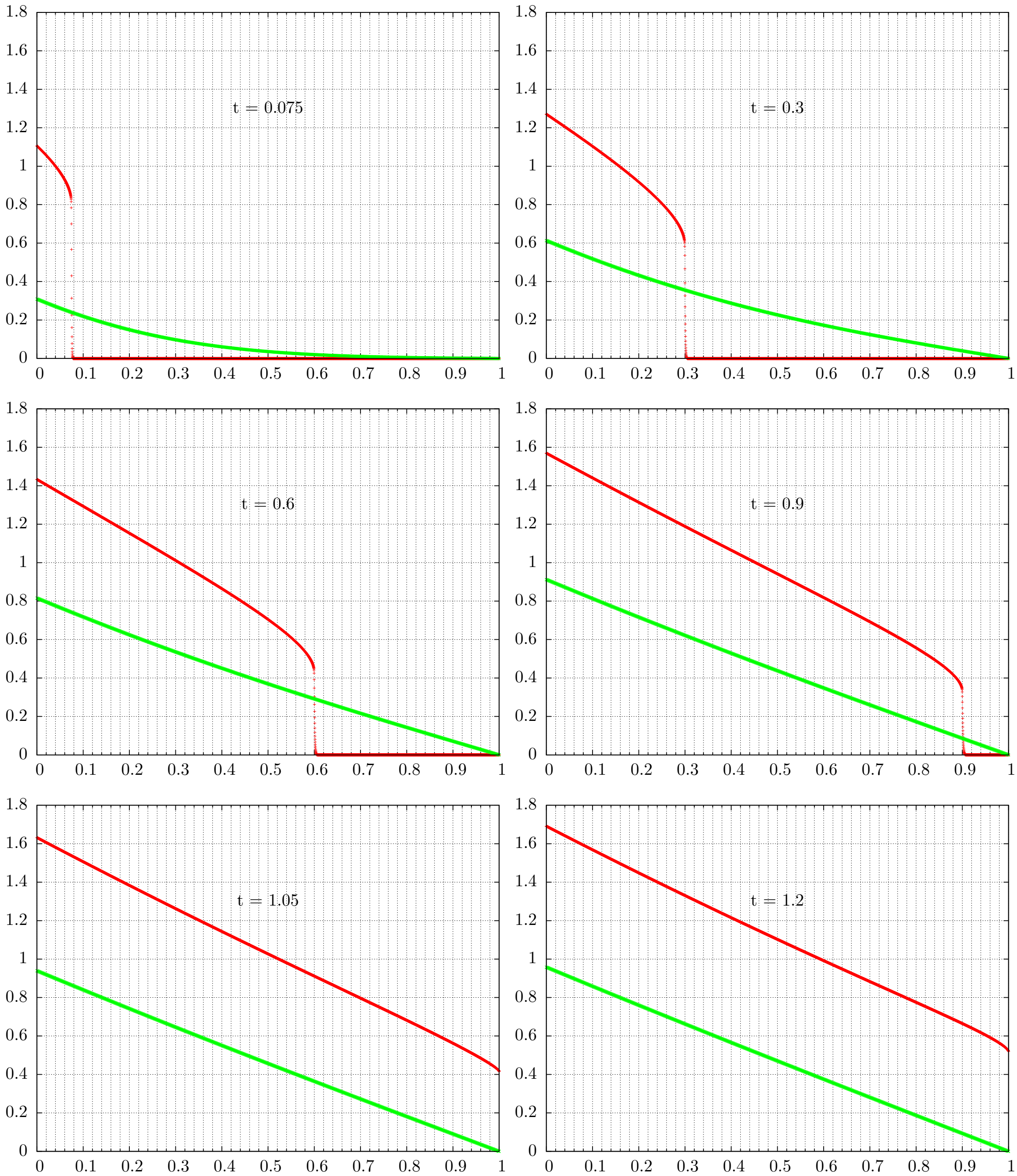

Figure 1.1: Evolution for a mixed problem on heat conduction. Non-zero Neumann boundary conditions are imposed on the left end and “zero Dirichlet” conditions are imposed on the right end; the initial data is zero in both cases. All physical constants have been set to one. The green plot corresponds to the solution associated with the usual heat equation, the red one to the solution associated with the relativistic heat equation (see Chapter 6 for an explanation on the model and the boundary conditions). These graphics will be included in [70].

dynamics. Indeed, we present a direct application of some of these ideas to one of the hottest topics in Astrophysics nowadays: the modeling of dark matter halos.

In this paragraph we are going to introduce several mathematical models that are widely used to describe self-gravitating systems. The type of objects that we have in mind are galaxies, big clusters of galaxies and even dark matter halos. What these systems share in common is that they are composed of a large number of individual entities, or particles (say the stars of a galaxy, galaxies itself when regarded in a cluster of galaxies and so on and so forth), evolving under gravitational interactions. It turns out that these general features are also shared by some other, very important physical systems, like gases or plasmas; the only difference being that the laws of interaction are of a different nature. We shall see that kinetic theory provides a common framework in which all these systems can be studied on an equal footing.

It is reasonable to assume that galaxies are approximately in a steady state at the present time, meaning that actually they vary so slowly that in our time scale we might regard them as static objects (physicists call this metastable states or metastable equilibria). Thus a representation in terms of static models seems a coherent approach to their study. This encourages us to find and analyze stationary solutions for our (kinetic) models. It is also interesting to study how do dynamical solutions evolve into such states and to study if a given configuration can indeed wander around or evolve into such states, or show on the contrary a very different behavior. Once we have presented the models we will focus on the long time behavior for their solutions and on certain properties of their steady states solutions.

A first naive approximation would be to model self-gravitating systems as clusters of point masses evolving under gravitational interactions. That is, an N-body problem. It is widely known that this mathematical model cannot be solved analytically, but there is a more important difficulty, more of a practical than of a theoretical nature. Typically the number of particles composing these systems is enormous, so there is no way and certainly no desire to keep track of this overwhelming quantity of information.

This first attempt is abandoned in favor of more sophisticated models and less ambitious goals: the description of the gross dynamical behavior might be enough for most purposes. To make such theoretical predictions continuous models are better suited than discrete ones; in particular the most widely used in this setting are kinetic models.

Indeed, the goal of kinetic theory in its general setting is the description of gases at an intermediate scale between the microscopic and the hydrodynamical ones. These cover a broad range of interesting applications, for gases are, loosely speaking, systems with a large number of particles which are described at a statistical level. For these problems a description of the position and the velocity of each particle is irrelevant, but the description of the system itself cannot be reduced to the computation of an average velocity at a given position in a concrete time instant (as it would be with fluid-type models). We want to take into account more than one possible velocity at each point and thus the description has to be done at the level of phase space.

So now we look for a statistical description of our system, in terms of a so-called distribution function, which depends on the time t and the coordinates of phase space (x, v) —for the case of relativistic models is better to shift to (x, p), being p the momentum. The actual value of this function on a particular point accounts for the probable number of particles in an infinitesimal volume around that point in phase space. Keep in mind that macroscopic information in enclosed in this mathematical object, despite how complicated it might be. We only have to know how to extract it and this will be detailed for each of the models that we are going to consider.

As we already said, there are a number of kinetic descriptions of self-gravitating systems, which obviously depend on the type of effects that we want to take into account and the ones that we prefer to neglect. But all of them share some common principles. First of all, the main object is the distribution function $f(t,x,v)$ which describes the statistical evolution of the ensemble of particles. Thus, two basic requirements are that this function should be non-negative and locally integrable over phase space in order that its physical interpretation given in the previous paragraph be meaningful. A second central issue is to give a law for the evolution of the distribution function. Here is where differences between models come into play, but all the equations encoding the laws of evolution for f stem from the same principle: the so-called Vlasov's equation. It states that the material (total) derivative of f equals the rate of change along particle paths in phase space. We shall denote this rate of change by $C(f)$. Such a change is related to short range interactions. Here we can include collisions between particles —in a broad sense— or coagulation among them, which give rise typically to bilinear terms; we can also consider fragmentation effects, which can be encoded using linear terms.

Now let us compute the material derivative of f. Under the domain of classical mechanics Newton's law states that for particle trajectories the derivative of the position is the velocity and the derivative of the velocity equals the force that is being exerted on the particle, let's say F —all this has to be modified in a suitable way if we want to consider relativistic models. Then the material derivative of f can be written as

$$\frac{Df}{Dt} = \frac{df}{dt} + v \cdot \nabla_x f + F \cdot \nabla_v f$$

and thus Vlasov's equation reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = C(f).$$

Depending on the type of long range interactions F and on the type of right hand sides $C(f)$ we obtain different models; we are going to consider some of these in turn.

1.1.2 The classical case

We study first the classical Vlasov–Poisson system, which describes in an statistical way a big cluster of collisionless particles evolving via self-generated gravitational potential $\phi(t,x)$ according to Newton's law of gravitation. The distribution function $f(t,x,v)$ of the ensemble and its density function

$$\rho(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\,dv$$

satisfy (with $G = 1$)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ \Delta_x \phi = 4\pi\rho. \end{cases}$$

The dynamics of solutions to this system are studied in Chapter 2, focusing mainly on dispersive behavior. This is done through a detailed study of the long time behavior of the density function. There are several ways in which a solution can exhibit such dispersive behavior and we focus on strong or normed dispersion, what we call total and partial dispersion —two notions that quantify the quantity of mass that a solution looses to infinity— and statistical dispersion (unlimited growth of the variance for the density function). It turns out that the occurrence of any of them is closely tied to certain macroscopic parameters of the systems under study: the mass M, the linear momentum Q and the energy H, defined by

$$M = \int_{\mathbb{R}^6} f\, dvdx, \quad Q = \int_{\mathbb{R}^6} v\, f\, dvdx,$$

and

$$H = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f\, dvdx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla_x \phi|^2 dx = E_{\text{kin}} - E_{\text{pot}},$$

which are conserved quantities. We give several examples of each dispersive behavior throughout Chapter 2, using some constructions of outgoing shells of matter and suitable modifications. The relation between these concepts is also clarified. We have the following result:

Proposition 1 *Let f be a regular solution of the Vlasov–Poisson system. Then the following assertions are equivalent:*

1. *f is strongly dispersive.*
2. *f is totally dispersive.*
3. *the potential energy vanishes as $t \to \infty$.*

Moreover, if any of the above holds then f satisfies the inequality

$$H \geq \frac{Q^2}{2M}\,.$$

Besides, if f is totally or partially dispersive then it is statistically dispersive too.

Next we study the fastest rates that are allowed for strong dispersion, recovering (with different proofs) and extending the results in [87] about this issue. The improved result claims that (Proposition 2.3.3)

$$\|\rho(t)\|_p \geq C(1+t)^{-\frac{3(p-1)}{p}} \quad \text{for } t \gg 1, \quad p \in]1,\infty].$$

We also complement the results in [87] about the occurrence of statistical dispersion whenever $H > \frac{Q^2}{2M}$ by analyzing the limiting case $H = \frac{Q^2}{2M}$ that was not dealt with there. We conclude that statistical dispersion also turns up in this case, with a rate which is generically at least linear in time (Proposition 2.3.8). Examples showing that there do exist statistically dispersive solutions having $H < \frac{Q^2}{2M}$ are also given; as in this regime there do exist non-dispersive solutions like steady states for instance, the dynamics under $H < \frac{Q^2}{2M}$ seems to be much more intricate and deserves lot of future work. As an example of this, we construct solutions that while being dispersive remain in the stability basin of stable steady states.

Then we test all this machinery with Kurth's solutions [136], which are the most famous examples of time-dependent solutions for the Vlasov–Poisson system, as they can be described almost explicitly. Finally we treat also two other classes of solutions that exhibit interesting dynamical behavior: time-periodic solutions (for which we show the relation $H < -\frac{Q^2}{2M}$ among their macroscopic parameters —Proposition 2.3.12) and virialized solutions. For the latter we generalize the virial theorem for N-body systems [177] to the continuous setting in Lemma 2.3.13, relating the virial identity to a certain growth condition on the system (namely, that its spatial variance grows in time strictly slower than t^2).

1.1.3 Relativistic generalizations

When relativistic effects become important the Vlasov–Poisson system ceases to be a reliable model; then other models have to be used. The model which is currently accepted to be the right generalization of the Vlasov–Poisson system is the Einstein–Vlasov system, where the Poisson law is replaced with a coupling to Einstein's equations of General Relativity. The resulting system is far from being completely well understood, and consequently what is commonly done for the sake of analyzing it is to reduce it to situations with symmetry (or to deal with simpler relativistic generalizations).

The spherically symmetric Einstein–Vlasov system in Schwarzschild coordinates takes the form (in units G=c=1)

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\sqrt{1+|v|^2}}\cdot\nabla_x f - \left(\lambda_t\frac{x\cdot v}{r} + e^{\mu-\lambda}\mu_r\sqrt{1+|v|^2}\right)\frac{x}{r}\cdot\nabla_v f = 0,$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 h\,,$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p^{\mathrm{rad}}\,,$$

being

$$p^{\mathrm{rad}}(t,r) = \int_{\mathbb{R}^3}\left(\frac{x\cdot v}{r}\right)^2 f\frac{dv}{\sqrt{1+|v|^2}}$$

the radial pressure,

$$h(t,r) = \int_{\mathbb{R}^3}\sqrt{1+|v|^2}f\,dv$$

the energy density and λ, μ the metric potentials —in the sense that the metric is completely determined by these two functions. This formulation is studied in Chapter 3.

The relevant macroscopic parameters of a solution to the spherically symmetric Einstein–Vlasov system are the ADM mass (or energy) H and the total rest mass M, defined by

$$H = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sqrt{1+|v|^2}\,f\,dvdx, \quad M = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\lambda}f\,dvdx.$$

These are constant for regular solutions. Another quantity that turns up to be important is the central redshift, defined by $Z_c := e^{-\mu(0)} - 1$. It is the redshift of a photon emitted from the center of the galaxy, and does not need to remain constant during evolution.

We prove a very general virial identity for solutions to this system (Lemma 3.4.1); then we particularize it to steady states and obtain an identity connecting some of their macroscopic parameters. The result is the following.

Proposition 2 *Let f be a static, compactly supported solution of the spherically symmetric Einstein–Vlasov system with ADM mass H, rest mass M and central redshift Z_c. Then the following inequality holds true*

$$Z_c \geq \left| \frac{M}{H} - 1 \right| .$$

We also study two particular classes of steady states. For Jeans type steady states (static solutions depending on conserved quantities along geodesics, see Chapter 3 for a precise definition) with radius R we show the inequality

$$e^{\mu(0)} \leq \min\left\{1, \frac{M}{H}\right\} \sqrt{1 - \frac{2H}{R}},$$

while for static shells with inner radius R_1 we show that

$$R_1 \leq \frac{18H}{\ln\left(\left|\frac{M}{H} - 1\right| + 1\right)}.$$

Another relativistic generalization of the Vlasov–Poisson system that we tackle in this memory is the Norsdtröm–Vlasov system, which constitutes an unphysical model that nevertheless incorporates some features of General Relativity Theory (by means of an scalar theory of gravitation) and is more tractable than the full Einstein–Vlasov system. Thus it constitutes a nice laboratory system. We write it as

$$\partial_t f + \frac{p}{\sqrt{e^{2\phi} + |p|^2}} \cdot \nabla_x f - \nabla_x \left(\sqrt{e^{2\phi} + |p|^2} \right) \cdot \nabla_p f = 0,$$

$$\partial_t^2 \phi - \Delta_x \phi = -e^{2\phi} \int_{\mathbb{R}^3} f \, \frac{dp}{\sqrt{e^{2\phi} + |p|^2}} \, .$$

The function ϕ determines the metric of the underlying spacetime and thus can be thought as a potential. The local energy and momentum of a solution (f, ϕ) are defined respectively as $(i = 1, 2, 3)$

$$h(t, x) = \int_{\mathbb{R}^3} \sqrt{e^{2\phi} + |p|^2} \, f \, dp + \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} |\nabla_x \phi|^2,$$

$$q_i(t, x) = \int_{\mathbb{R}^3} p_i f \, dp - \partial_t \phi \, \partial_i \phi \, ,$$

where ∂_i denotes the partial derivative along x^i. The total energy and the total momentum

$$H = \int_{\mathbb{R}^3} h(t, x) \, dx, \quad Q = \int_{\mathbb{R}^3} q(t, x) \, dx$$

are conserved quantities during evolution. Moreover, solutions of the Nordström–Vlasov system satisfy the conservation of the total rest mass

$$M = \int_{\mathbb{R}^3} \rho(t, x) \, dx = \int_{\mathbb{R}^6} f(t, x, v) \, dx dp.$$

We study several features of this system in Chapters 2 and 3. In the former we prove a dispersion estimate in terms of the local energy h:

Proposition 3 *Let (f, ϕ) be a solution of the Norsdtröm–Vlasov system with mass M, energy H and momentum Q. Assume that*

$$H^2 - HM - |Q|^2 > 0.$$

Then, there exist a time instant t_0 and positive constants $0 < C_1 < C_2$ such that the spatial variance $\Delta_x(t)$ of the unitary energy density function,

$$\Delta_x(t) = \int_{\mathbb{R}^3} |x - \bar{h}(t)|^2 \frac{h(t,x)}{H}\, dx, \quad \textit{where} \quad \bar{h}(t) = \int_{\mathbb{R}^3} x \frac{h(t,x)}{H}\, dx,$$

satisfies

$$C_1 t^2 \leq \Delta_x(t) \leq C_2 t^2 \qquad \forall t > t_0\,.$$

In the latter we establish an identity (Lemma 3.2.1) that holds for dynamical solutions of this system and then we restrict it to steady states, obtaining that the energy of regular steady states is bounded by their mass (Theorem 3.2.2), a property that makes a clear parallel with the fact that $H < 0$ for static solutions to the Vlasov–Poisson system. In fact, this result motivated the one in Proposition 2.

1.1.4 Study of dark matter halos

The first conceptual block of this memory (almost all the first part), dedicated to the modeling of self-gravitating systems, concludes with an application of the Vlasov–Poisson system to the modeling of dark matter halos, in Chapter 4. These are spherical structures surrounding each galaxy made up of some kind of exotic matter that cannot be detected by direct measurements. We only feel their presence by their gravitational effects, which are indeed very strong, for it is believed that they contribute with 9 out of 10 parts to the overall mass of the configuration. Although this paradigm has several detractors it is a mainstream in present day Astrophysics and lots of efforts have been dedicated to the construction of models for the density profiles of these objects. These come usually under two major trends: phenomenological models (fitting to data) and numerical simulations. The Navarro–Frenk–White density profile [164] is the most popular among those originated by the latter, while the Isothermal [109] and Burkert [58] profiles are good representatives of the former. None of these models has finite radius, which is not physically reasonable. Even more controversial is the fact that models originated by numerical simulations predict an infinite value for the density profiles at the center.

We propose to generate density profiles for dark matter halos using a three-parametric family of static solutions to the Vlasov–Poisson system: the isotropic polytropes. These models have a strong theoretical background supporting them (ranging from the very time-honored equation that originates them, passing through fine scaling properties, to thermodynamical theories that hold in a much broader setting than ours —but still under debate in the physical community) and are also relatively easy to handle. We compare them with the models that we commented on earlier using a least square criterion. A very good agreement is obtained once the parameters are tuned properly: we obtain errors of the order of 3%. Moreover, the analytical foundation of our models allows us to perform an expansion around the origin and suggests a formula (equation (6.4)) for dark matter density profiles in the very center, which cures the unphysical divergence of the numerical simulations in the core region.

1.1.5 Asymptotic behavior of a coagulation model

The first part of this memory concludes with Chapter 5, where we will be concerned with a kinetic coagulation model describing two species of particles (typically molecules or cells). The particles can be in two states: a "free" state where they simply move with a given velocity or an aggregated state where they do not move anymore. The sticking together, coagulation, aggregation or adhesion into a cluster, of particles, whether they are cells, lipid droplets, proteins, etc, is of fundamental importance in biological and biotechnological processes, see [1, 92, 100, 217] for instance. This is the primary motivation for the model that we study.

The distribution function for the free particles is denoted by f and the density function of stuck particles is represented by ρ. As a model for the above situation in d dimensions we consider the following system of equations:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' - \beta(v)\rho(t,x) f(t,x,v) \qquad (1.1)$$

$$\frac{\partial \rho}{\partial t} = \int_{\mathbb{R}^{2d}} \alpha(v,v') f(t,x,v') f(t,x,v)\, dv' dv + \rho(t,x) \int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv \qquad (1.2)$$

supplied with initial data $0 \le f^0(x,v) \in L^1(\mathbb{R}^{2d})$ and $0 \le \rho^0(x) \in L^1(\mathbb{R}^d_x)$.

The functions $\alpha(v,v')$ and $\beta(v)$ are collision or coagulation kernels and give the probability that two free particles with velocities v and v' will coagulate (for α) or one free particle with velocity v will coagulate with a stuck particle (for β). These collision kernels will be nonnegative and they will satisfy a domination property, which is motivated by physical considerations (this is explained in Chapter 5): there exists a constant $C > 0$ such that

$$\alpha(v,v') \le C|v-v'|^a, \;\; \beta(v) \le C|v|^a, \quad \text{for some } a \in \mathbb{R}. \qquad (1.3)$$

We focus mainly on the study of the large time asymptotics for this model. It is obvious from the equations that the mass related to the population of free particles may only decrease and the mass of the population related to coagulated particles can only increase. Hence the main issue as $t \to +\infty$ is whether all free particles will finally coagulate or if some of them remain free. We show that this depends only on the strength of the interactions (i.e. the value of a). The analysis is based on precise dispersion estimates for kinetic equations. We also show that the distribution of free particles exhibits self-similar behavior for long times. Our results are summarized in the following

Theorem 1 *Assume that the kernels* α, β *are non-negative, satisfy* (1.3) *and* $a+d > 0$. *For any* $0 \le f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ *and* $0 \le \rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ *such that for some* $\eta > 0$ *there holds that*

$$f^0(x,v) \le \frac{C}{1+|v|^{\max\{a,0\}+d+\eta}}, \; \textit{for a.e. } (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v,$$

there exists a weak solution of the system (1.1)–(1.2) *with initial data* $f(0,x,v) = f^0$ *and* $\rho(0,x) = \rho^0$. *If this weak solution can be approximated strongly in* $L^\infty(0,T,L^1(\mathbb{R}^d_x \times$

$\mathbb{R}^d_v) \times L^1(\mathbb{R}^d_x))$ *by a sequence of smooth solutions then this weak solution is unique. Moreover, there exists a function* $g_\infty(x,v)$ *such that*

$$\left\| f(t,x,v) - g_\infty\left(\frac{x}{t}, t(v-\frac{x}{t})\right)\right\| \to 0 \quad \text{as } t \to \infty$$

in the norm of $W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$. *Furthermore,*

- *if* $a > 1-d$ *(or* $a > 1$ *if* $d = 1$*) and* f^0 *and* ρ^0 *are compactly supported in* x*, the amount of mass* $\int_{\mathbb{R}^{2d}} f(t,x,v)\,dxdv$ *is bounded from below by a positive constant independently of time.*
- *if* $-d < a \leq 1-d$*, the amount of mass* $\int_{\mathbb{R}^{2d}} f(t,x,v)\,dxdv$ *is strictly positive for all times but converges to zero as* t *goes to infinity.*

1.1.6 A flux-limited model to the transport of morphogenes

The last part of this memory is devoted to the study of the transport of morphogenes in biological systems. This is a classical problem since the pioneering works by Turing [225], Meinhardt [101, 157], Wolpert [234] and Lander [139], who focus the question as a main problem in the understanding of the transport of proteins via signaling pathways: Do morphogen gradients arise by diffusion?

We focus on a more concrete problem: the study of the dynamics of the Sonic Hedgehog (Shh) morphogenetic function, which plays a very important role in the evolution of some transcription factors and in cellular differentiation in the embryonic neural tube. These phenomena are of capital importance in Developmental Biology. For instance, within the central nervous system the development of the early vertebrate ventral neural tube [130] and of the brain [198] depend on Shh signaling. Shh signaling has also an important role in tumor formation: the deregulation of the Shh pathway leads to the development of various tumors, including those in skin, prostate and brain [196].

There do exist mathematical models to study this problem [202], but from our point of view their recourse to diffusion mechanisms in this context is not realistic. We propose, as a suitable modification to solve this caveat, to suppress the diffusive mechanisms and to use a flux limiter instead; all of this is explained in detail in Chapter 6.

Then, our purpose in the second part of this memory is to analyze a mixed initial-boundary value problem associated with a nonlinear flux–limited reaction–diffusion system for the concentration of Shh, $u(t,x)$, given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u,u_x))_x - f(t-\tau, u(t,x))\,u(t,x) + g(t,u(t,x)), & \text{in} \quad]0,T[\times]0,L[\\ -\mathbf{a}(u(t,0),u_x(t,0)) = \beta > 0 \;\; \text{and} \;\; u(t,L) = 0 & \text{on} \quad t \in]0,T[\\ u(0,x) = u_0(x) & \text{in} \quad x \in]0,L[\,, \end{cases}$$

with

$$\mathbf{a}(z,\xi) := \nu \frac{|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}}\,,$$

where f stands for the concentration of transmembrane receptors in the cells, g represents the concentration of the complex binding the morphogen to the receptor, and the dependence on u is given through a coupling with a system of seven ODEs modeling the rates of change of the concentrations of the proteins participating in the signaling pathway coming from the biochemical cascade inside the cells, see [200]. The meaning of the physical constants c, ν, τ is explained in Chapter 6.

We try to give some insight on this biological problem, and to proceed we study as a first step a simplified model without source terms. This is done in Chapter 6 using nonlinear semigroup theory. Our simplified model reads

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u, u_x))_x & \text{in} \quad]0,T[\times]0,L[\,, \\ -\mathbf{a}(u(t,0), u_x(t,0)) = \beta > 0 \text{ and } u(t,L) = 0 & \text{on} \quad t \in]0,T[\,, \\ u(0,x) = u_0(x) & \text{in} \quad x \in]0,L[\,. \end{cases} \tag{1.4}$$

Nevertheless, it is a simple matter to realize (this is also supported by numerical simulations) that the Dirichlet boundary condition has to be weakened to a boundary condition resembling that of the obstacle problem. This reflects the fact that traveling fronts will eventually hit the boundary and persist after that, see Figure 1.1. The precise formulation of these facts can be found in chapter 6.

The use of semigroup theory would yield mild solutions for our problem, but our concern here is to characterize them in more operative terms. We will use the theory of nonlinear semigroups to provide a suitable approximating scheme to the parabolic problem. In this setting, we analyze the related elliptic problems and construct the associated semigroup so as to generate a sequence of approximate solutions that are seen to converge to a reasonable solution of the parabolic equation. We are also able to prove uniqueness of "reasonable" (we call them bounded entropy) solutions to the parabolic equation. Namely we prove much more, as a contraction principle holds; this is obtained using a suitable adapted variant of Kruzkov's doubling variables technique. These results are summarized in the following

Theorem 2 *For any initial datum $0 \le u_0 \in L^\infty(]0,L[)$, there exists a unique bounded entropy solution u of (1.4) in $Q_T =]0,T[\times]0,L[$ for every $T > 0$. Moreover, if $u(t)$, $\overline{u}(t)$ are bounded entropy solutions of (1.4) in $Q_T =]0,T[\times]0,L[$ corresponding to initial data u_0, $\overline{u}_0 \in L^\infty(]0,L[)^+$ respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \le \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all} \quad t \ge 0.$$

In particular, we have uniqueness of bounded entropy solutions for (1.4).

In the course of this study we have also determined a stationary profile (Proposition 6.6.2) to which all solutions of this model seem to converge. The speed of propagating fronts is also analyzed, and we find that the incoming signal propagates precisely with speed c.

1.2 About notation

We hope that most of our notation is standard, anyway we explain it in detail here. Our function spaces concern always objects with range in the real numbers. Let Ω be a domain. $L^p(\Omega)$ stands for the usual Lebesgue spaces, with norms $\|\cdot\|_{L^p(\Omega)}$, or $\|\cdot\|_p$ when the domain is clear from the context; p' denotes the exponent which is conjugate to p. $W^{1,p}(\Omega)$ are the standard Sobolev spaces, sometimes we use $W^{1,2} = H^1$. Concerning spaces of classical functions we have $C_c^k(\Omega)$, $0 \le k < \infty$ for k-times compactly supported differentiable functions with continuous derivatives, $C_b^k(\Omega)$ for continuous and bounded derivatives, $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact support and $\mathcal{D}'(\Omega)$ the space of associated distributions. Partial derivatives are often indicated by ∂_x or $(\cdot)_x$. Finally, $\mathcal{M}(\Omega)$ denotes the space of bounded Radon measures. Conventions like $L^1(\Omega)^+$ mean that the members of the corresponding space are non-negative in the sense provided by that space. Topological duals are denoted as $L^1(\Omega)^*$, except for the cases of distributions and Sobolev spaces (here the standard notation $W^{-1,p}$ is used). Duality pairings are indicated by $\langle\, ,\rangle$.

The N-dimensional Lebesgue measure is either denoted by $\mathcal{L}^N$ or dx depending on the context. Dirac measures concentrated on a point x are always represented by $\delta(x)$. The remaining notational conventions related to measure theory that we shall use are detailed in the corresponding Appendix.

About elementary functions: ln denotes the usual logarithm in base e, $\ln_{10}$ will be used for logarithms in base 10, the hyperbolic sine and cosine will be written as sh and ch. The sing function $sign_0(x)$ values -1 if $x < 0$, 0 if $x = 0$ and 1 if $x > 0$. The positive part of a function is $f^+(x) = \max\{f(x), 0\}$; sometimes we use f_+ for the same purpose.

The symbols H, M, Q will be used in the first part of the memory to represent macroscopic parameters. We warn the reader that the meaning of these symbols is different for each of the models that we are going to consider.

In the context of the geometry of $\mathbb{R}^3$, $|\cdot|$ is the usual modulus of a vector. Given two vectors x, y the notation $x \wedge y$ represents the usual vector product in $\mathbb{R}^3$ and $x \cdot y$ stands for their scalar product (sometimes we even write simply $x\,y$). $SO(3)$ denotes the group of rotations in $\mathbb{R}^3$.

When dealing with special and general relativity we stick to Einstein's convention: the sum over repeated indexes is always understood. Greek indices always run from 0 to 3; Latin indices run form 1 to 3. A Lorentzian metric is usually denoted by g, but η stands for the Minkowsky metric. We follow the $(-+++)$ convention on signature.

We also have some miscellaneous notations: $\uparrow$ and $\downarrow$ for monotone sequences, $*$ for convolution products, ∂ for the boundary operator; sup and inf are for the usual supremum and infimum. Open and closed intervals are denoted as $]a, b[$ and $[a, b]$ respectively. The characteristic function of a set Ω is denoted by χ_Ω. In particular, $sign_0^+(x) = \chi_{]0,+\infty[}$. Notations like $[u \ge a]$ and the obvious variants indicate level sets like $\{x \in \Omega/u(x) \ge a\}$. The essential support of a set Ω is indicated by $\operatorname{supp}\Omega$. Composition of mappings is denoted by $f \circ g$. We will use the notation $f \sim g$ to mean that there exist two positive constants c_1, c_2 such that $c_1 g \le f \le c_2 g$.

Some physical conventions are also used. G is the gravitational constant and c the speed of light in vacuum unless otherwise specified. We use kpc and Mpc to refer to kilo and mega parsecs; $\mathrm{M}_\odot$ is the mass of our Sun.

The symbol C is always used to denote generic constants. These may change form line to line, or even within the same line. A notation like $C(a, b, \ldots)$ indicates that the constant C depends on the specified quantities.

Chapter 2

Dynamical behavior for gravitational kinetic models

In this chapter we shall study several dynamical features that solutions to kinetic models exhibit. We will focus for the most part of it on solutions that for large times exhibit some sort of dispersive behavior. The examples considered in the following sections show that solutions of the Vlasov–Poisson system may disperse in several different ways. Another goal of the present chapter is to study the relations between these various concepts of dispersion. The corresponding program for the relativistic models is much less developed, nevertheless we are able to give some hints on the long time behavior for solutions to the Nordström–Vlasov system. The results of this chapter are contained in [61] and [67].

This chapter is organized as follows. First we describe in Section 2.1 the Vlasov–Poisson system and review the theory about it that has been developed up to date. In Section 2.2 we introduce several concepts of dispersion for a mass distribution—not necessarily originated by the Vlasov–Poisson system— and give some examples. In Section 2.3 we specialize our study to the case of the Vlasov–Poisson system. We give necessary or sufficient conditions for the existence of various types of dispersive solutions. Then we test all these material with explicit solutions in Paragraph 2.3.3. For completeness we also briefly discuss in Paragraph 2.3.5 a related notion, namely that of virialized solutions. Time periodic solutions are also discussed. Paragraph 2.3.6 contains some concluding remarks, open problems and interpretations of our results. One of the main ideas is that although the solutions to the Vlasov–Poisson system could have unbounded velocities, the system is able to reproduce the physically correct rates of dispersion when regarded in the appropriate way. Next we pass to a handy relativistic generalization of the Vlasov–Poisson system: the Nordström–Vlasov system. We introduce it in detail in Section 2.4. The last section is a gathering of what is known up to date about the dynamical behavior for solutions to the Nordström–Vlasov system. The study of conserved quantities and behavior under physically meaningful transformations is more or less standard, although it has been never performed explicitly; our results about dispersive behavior are of a more tentative nature and much more effort has to be done in this direction to obtain plenty satisfactory results.

2.1 The Vlasov-Poisson system

2.1.1 Description of the model

The Vlasov–Poisson system (VP for short) is widely used in Astrophysics [48, 80] to model gravitational ensembles composed by a large number of particles (the stars of a galaxy for instance). The assumptions on the ensemble are that collisions and external forces are negligible and no relativistic effects are present. Later on we shall consider some models that take into account relativistic effects; as regards to collisions, it can be justified that these play no role whenever dealing with astrophysical systems like galaxies (note however that they become important for some types of astrophysical systems, like globular clusters or rich clusters of galaxies [48]). Indeed, the word collision in this setting refers more to close encounters than to actual collisions. That is, an event causing a significant deviation of the stellar trajectories during an encounter. Estimating the so-called relaxation time it can be deduced that collisions in the above sense can be neglected when describing a galaxy [48, 80]. To simplify the exposition we will always refer to galaxies hereafter but keep in mind that we can consider also other types of self-gravitating systems; we will come back to this in Chapter 4.

In this framework the dynamics of the galaxy is described by the distribution function in phase space $f = f(t, x, v)$, which gives the probability density to find a star at time $t \in \mathbb{R}$ in the position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$. The mass density $\rho = \rho(t, x)$ of the ensemble is given by

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv. \tag{1.1}$$

We denote by L^q either $L^q(\mathbb{R}^6)$ or $L^q(\mathbb{R}^3)$ in what follows, depending on the function under consideration, *e.g.* $f(t) \equiv f(t,\cdot,\cdot)$ or $\rho(t) \equiv \rho(t,\cdot)$. For notational simplicity we assume that the stars have all the same mass m and fixed units such that $m = G = 1$. The gravitational potential $\phi = \phi(t, x)$ generated by the galaxy solves the Poisson equation

$$\Delta_x \phi = 4\pi\gamma\rho, \qquad \lim_{|x|\to\infty} \phi = 0,\ \forall\, t \in \mathbb{R}, \tag{1.2}$$

where the boundary condition at infinity means that the system is isolated, and $\gamma = 1$ for the moment. A few words are in order in connection with the use of Poisson's law: it implies that we are dealing with a mean-field theory. Meaning that fluctuations in the gravitational potential, which is created by the stars collectively, are smoothed out. Our way of describing the system allows us to cope with this fact, for the importance of these fluctuations for the motion of a given star (due mostly to neighboring stars) diminishes as the number of distant stars the potential is averaged over increases. That is, stars move under the influence of the mean potential generated by all the stars of the galaxy.

The assumption that the stars interact only by gravity leads to the Vlasov equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0. \tag{1.3}$$

The system (1.1)–(1.3), together with a function $f^0(x, v)$ representing the initial configuration, is the Vlasov–Poisson system. The solution of (1.2) is given by the formula

$$\phi(t,x) = -\frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|}\,dy, \tag{1.4}$$

whence the Vlasov–Poisson system is equivalent to the non-linear Vlasov equation obtained by replacing the formula for ϕ in (1.3).

This system of equations has become quite a standard to model galaxies. If we want to model an ensemble of point particles interacting via self-generated repulsive electric potential we change to $\gamma = -1$. The comparison of both versions is often fruitful, and several features are common to both systems, as we are going to see.

Note that $\mathrm{div}(v,\ \nabla_x \phi(t,x)) = 0$ formally in phase space, so that the associated flux in phase space is measure-preserving, if we are able to give it a sense. This is, in case that the following system, known as the characteristic system,

$$\begin{cases} \frac{d}{dt}X(t;s,x,v) = & V(t;s,x,v) \\ X(s;s,x,v) = & x \\ \frac{d}{dt}V(t;s,x,v) = & -\nabla_x \phi(t, X(t;s,x,v)) \\ V(s;s,x,v) = & v \end{cases}$$

has a (unique) solution. In that case we can define a solution of Vlasov's equation by means of

$$f(t,x,v) = f^0\left(X(0;t,x,v), V(0;t,x,v)\right).$$

Such formula —pure transport in phase space— has several important consequences.

2.1.2 Conserved quantities and invariances

Since the characteristic flow of the Vlasov equation preserves the Lebesgue measure, then all L^q norms of f are preserved:

$$\|f(t)\|_{L^q} = \text{constant}\,, \quad \text{for all } 1 \le q \le \infty\,. \tag{1.5}$$

In particular, the mass M of a solution

$$M = \int_{\mathbb{R}^6} f\, dvdx$$

is conserved during evolution. Another very important macroscopic quantity is the energy of a solution, given by

$$\begin{aligned} H &= \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f\, dvdx - \frac{\gamma}{8\pi}\int_{\mathbb{R}^3} |\nabla_x \phi|^2 dx \\ &= E_{\text{kin}} - \gamma E_{\text{pot}}. \end{aligned}$$

It can be showed that this is a conserved quantity at least for smooth solutions. Likewise, the total linear momentum Q and angular momentum Ω,

$$Q = \int_{\mathbb{R}^6} v\, f\, dvdx, \qquad \Omega = \int_{\mathbb{R}^6} x \wedge v\, f\, dvdx, \tag{1.6}$$

are conserved quantities.

This system is invariant under temporal and spatial translations. It is also invariant with respect to the action of a given rotation in position and velocity space simultaneously. The invariance of Vlasov–Poisson by (time dependent) Galilean transformations is the property that, given $u \in \mathbb{R}^3$ and the transformation of coordinates

$$\mathcal{G}_u: \quad t' = t, \quad x' = x - ut, \quad v' = v - u, \tag{1.7}$$

then $f_u(t,x,v) = f(t',x',v')$ and $\phi_u(t,x) = \phi(t',x')$ solve the system (1.1)–(1.3) if and only if (f,ϕ) does. The physical interpretation of this fact is that all inertial observers/reference frames are equivalent; the Galilean transformations give the rules to carry all the relevant information from a given frame to any other frame. Note that Q can be made to vanish with a suitable Galilean transformation; the resulting reference frame is at rest with respect to the center of mass of the distribution, which is defined as

$$c_\rho(t) = M^{-1} \int_{\mathbb{R}^3} x\,\rho\,dx. \tag{1.8}$$

This particular invariance of the system plays a very important role: whenever we are to define dynamical concepts, if we want them to be physically reasonable we have to do it in such a way that all the inertial observers/reference frames agree when deciding if these features are present or not. We don't want to include spurious effects (instability, dispersion) that are mere artifacts of a description under a particular frame and can be cured when passing to an appropriate reference frame.

An important role in our discussion is played by spherically symmetric solutions of the Vlasov–Poisson system. A solution of Vlasov–Poisson is spherically symmetric if $f(t,Ax,Av) = f(t,x,v)$ for all rotations $A \in \mathrm{SO}(3)$. If this happens to be so for f^0 then this property is preserved during evolution. The potential induced by a spherically symmetric solution is a function of the radial variable $r = |x|$ only and indeed we have the representation formula

$$\partial_r \phi = \frac{4\pi}{r^2} \int_0^r \lambda^2 \rho(t,\lambda)\,d\lambda. \tag{1.9}$$

Clearly, the center of mass of spherically symmetric solutions is at $r = 0$.

Another class of symmetries that has been exploited sometimes is that of cylindrically symmetric solutions. These are such that $f(t,Ax,Av) = f(t,x,v)$ for all rotations $A \in \mathrm{SO}(3)$ that leave the (say) third axis fixed. Again, if we start with a cylindrically symmetric initial data this property is preserved during evolution.

2.1.3 The Cauchy problem

The issue of existence, uniqueness and regularity of solutions to this system has been extensively studied in the mathematical literature during the last decades and this problem is by now well understood. For the sake of completeness we will review here all the steps that were performed up to this date. The first local existence result is that of [137]. A local existence and uniqueness result for smooth, compactly supported initial data, together with a continuation criterion were contained in [32], see [180] for a clearer exposition. His results state that a solution ceases to exist essentially as soon as its velocity support becomes unbounded. That is, the solution exists as long as

$$\sup\{|v|/(x,v) \in \operatorname{supp} f(s),\ 0 \le s < t\} < \infty \tag{1.10}$$

Some years passed before a general global existence result was obtained, but in the meantime some breakthroughs and a better understanding of the problem were achieved. Namely, [119, 121] obtained several equivalent forms of the continuation criterion that would become important later. Several global existence results for particular types of special initial data were also obtained during that years: first for spherically symmetric solutions by [32], later for cylindrically symmetric solutions by [120] and also for small initial data by [31]. Moreover, the case of nearly symmetric plasmas was treated by [208]. These results remain important nowadays as they give more information on the solution than the general existence theory.

In the spherically symmetric case it can be shown that the increase in velocity, that is, the quantity

$$\sup\{|V(s,0,x,v)-v|,\, 0\le s\le t,\ x,v\in\mathbb{R}^3\}$$

is bounded by a constant which depends only on the initial datum. For the case of cylindrically symmetric data that quantity is bounded by a constant times t. This issue is much more involved in the general case. For small data solutions, there hold the estimates

$$\|\rho(t)\|_\infty \le C(1+t)^{-3}, \quad \|\nabla_x\phi\|_\infty \le C(1+t)^{-2}.$$

These bounds were extended to the derivatives of the density in [124]. Here it is stated that

$$\|D^k\rho(t)\|_\infty \le C(1+t)^{-3-k}.$$

For the case of nearly spherically symmetric plasmas we have the following decay rates [208]

$$\|\rho(t)\|_\infty,\ \|\nabla_x\rho(t)\|_\infty \le C(1+t)^{-3}, \quad \|\nabla_x\phi\|_\infty \le C(1+t)^{-2},$$

that refine the ones given in [122] for symmetric plasmas. It should be noted that the results in [208] do not imply that the solutions remain nearly symmetric during evolution.

Another important direction was explored by [123], who succeeded in showing the existence of global weak solutions for the VP system under some integrability assumptions and the finiteness of the kinetic energy. Their result has several drawbacks, being the most important the fact that they are not able to guarantee uniqueness nor conservation of energy. The existence of weak solutions for the repulsive case was discovered ten years before [28] (again, uniqueness and conservation of energy cannot be assured, only boundness for the initially finite energy).

Finally a global classical theory (meaning data and solutions in $C_c^1(\mathbb{R}^6)$) was obtained in 1989. There were two equivalent approaches to this question that coexisted in time. The first one is due to [152]; the authors ensure global existence by means of the control of high velocity moments of the solution. The authors give also some sufficient conditions to guarantee uniqueness, which hold in a larger class than that of classical solutions. Estimates on velocity moments have been improved later in [99]; spatial moments are treated in [73]. The second approach to the global classical theory is that of controlling the size of the velocity support of a solution launched by a smooth initial datum. This problem was first solved in [174] and subsequently sharpened and clarified in [209], [233] and finally [118]. The upshot is that for any smooth, compactly supported initial datum, the radius of the velocity support grows at most with a linear rate (well, this is not completely true, but is almost true: the best rate available today is that of [118], which is $t\ \ln^{\frac{11}{14}} t$).

After these results a suitable theory for the Cauchy problem with smooth, compactly supported initial data is founded, but still there are some caveats. Some partial refinements have been obtained later: uniqueness for compactly supported solutions with integrability conditions [192], uniqueness for solutions whose density is bounded

[153] and existence and uniqueness of a weak solution for bounded and compactly supported initial data [240].

2.1.4 Long time behavior

Very little is known on the time asymptotics of the Vlasov–Poisson system in the gravitational case ($\gamma = 1$). On the contrary, the large time behavior of solutions to the Vlasov–Poisson system is relatively simple in the case of electrostatic interaction among the particles ($\gamma = -1$). All solutions exhibit a (strong or L^q-norm) dispersive character [125, 172], as there holds the estimate

$$\|\rho(t)\|_{5/3} \leq C(1+t)^{-1/2}.$$

A slightly different statement concerning dispersive behavior is given in [102]. The results in [73] about allowed dispersion rates mustn't be overlooked.

In the gravitational case the dynamics is more intricate: there exist (stable or unstable) steady states, periodic solutions (breathers) and (fully or partially) dispersive solutions. Most of these possibilities were already exemplified by the explicit solutions found in [136], some have been discovered later. For the applications in Astrophysics it would be desirable to have a classification of the possible asymptotic behavior of solutions in terms of relations between quantities preserved by the evolution (such as the energy and the mass). This is clearly a very difficult—may be impossible—task, but in this chapter we show that partial answers in this direction can be given for the Vlasov–Poisson system.

One of the first works in this direction was [33], dealing with spherically symmetric systems. More recently we have the results of [87], in which a dispersion regime for the solutions is identified. This constitutes one of the starting points for the present investigation. The issue of self-similar solutions is treated in [163], through no completely satisfactory answer is reached. The self-similar behavior for small data solution on the long time run is investigated in [124].

2.1.5 About steady states

There are two general strategies to construct stationary solutions for the Vlasov–Poisson system. The first one is to choose a plausible distribution function and then solve Poisson's equation to determine the spatial structure of the associated model. The second is to proceed the other way around: start with a specified density function and try to figure out the distribution function (Eddington's inversion formula and related procedures [48]). We won't discuss the second method here.

The construction of static solutions for the Vlasov–Poisson system following the first idea can be done in two ways. Firstly, by choosing a particle density f which depends only on quantities that are conserved along the characteristics of the time independent Vlasov equation; with this choice the Vlasov equation is automatically satisfied and the problem is reduced to that of proving an existence theorem for the non-linear elliptic equation obtained by replacing f in the Poisson equation. So far this 'direct' method was used mostly in the spherically symmetric case (see however [212]), where by the Jeans theorem [35] all solutions of the time independent Vlasov equation can be expressed in terms of the particles energy and angular momentum. More precisely:

Theorem 2.1.1 (Jeans theorem) *Any stationary solution of the Vlasov–Poisson system depends on the phase space coordinates only through integrals of motion.*

Concerning the use of this result, let us recall the following classical integrals of motion (conserved quantities along particle trajectories):

- the local energy $E = \frac{1}{2}|v|^2 + \phi(x)$, in the case of a time independent potential ϕ.
- the components of the angular momentum $L = x \wedge v$, in the case of a radial potential ϕ. Let us denote $F = |L|^2$.
- the third component of the angular momentum if the potential ϕ is invariant under rotations about the third axis, and so forth.
- the Jacobi integral $E_J = \frac{1}{2}|v|^2 + \phi(x) - \frac{1}{2}|\omega \wedge x|$ for steadily rotating potentials ϕ with angular momentum $\omega = (0, 0, w)$.

A second method to construct steady states of the Vlasov–Poisson system is by minimizing the energy (or a related) functional subject to suitable constraints. The choice of the functional and/or the constraints selects the type of steady state to be constructed. The advantage of the variational method on the direct method is that the former automatically proves a stability property for the steady state. We refer to [111, 180, 184, 188, 206] and references therein for several works on the construction and stability of steady states of the Vlasov–Poisson system.

2.2 Dispersive behavior

In this section we introduce several concepts of dispersion for regular mass distributions. By a *regular mass distribution* of total mass M we mean a non-negative C^1 function $\rho(t, x)$ such that $\rho(t, \cdot)$ has compact support and $\|\rho(t)\|_1 = M$ (independent of time t). This terminology is consistent with the one used for solutions of the Vlasov–Poisson system: the mass distribution ρ defined by (1.1) is regular whenever f is regular.

These definitions work nicely in Newtonian settings, as they are closely tied to the Galilean invariance. For relativistic systems things are not that easy since Lorentz invariance mixes up space and time; other ways of defining dispersion have to be found.

2.2.1 Strong dispersion

Definition 2.2.1 *A regular mass distribution ρ is said to be* strongly *dispersive if there exists $q > 1$ such that the limit*

$$\lim_{t\to\infty} \|\rho(t)\|_{L^q} \text{ exists and is zero.} \tag{2.11}$$

Obviously, strong dispersion is a Galilean invariant concept. For the Vlasov–Poisson system in the plasma physics case, that is, $\gamma = -1$, it was proved in [125, 172] (see also [102]) that all solutions are strongly dispersive, with (2.11) being verified for $q \in]1, 5/3]$. Namely,

$$\|\rho(t)\|_{5/3} \le C(1 + t)^{-1/2}$$

and the claimed decay for the rest of the norms is obtained by interpolation. In the gravitational case, examples of strongly dispersive solutions are those constructed in [31] for small initial data, see also [124]. For these solutions there holds the estimate

$$\rho(t,x) \leq C(1+|t|+|x|)^{-3} \,,$$

for a positive constant C, which clearly implies strong dispersion.

2.2.2 Total and partial dispersion

The next types of dispersive solution that we are going to discuss use the notion of "concentration function of a measure" introduced by Lévy [146] and applied by P.-L. Lions in the proof of the concentration-compactness Lemma [151].

Definition 2.2.2 *A regular mass distribution ρ is said to be* totally*, respectively* partially *dispersive, if and only if the limit*

$$\mathcal{M}(R) = \lim_{t\to\infty} \sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|<R} \rho(t,x)\,dx, \tag{2.12}$$

exists and

$$\mathcal{M}_\infty = \lim_{R\to\infty} \mathcal{M}(R) \tag{2.13}$$

satisfies $\mathcal{M}_\infty = 0$, respectively $\mathcal{M}_\infty \in]0, M[$.

Remark 2.2.3 Of course, it is possible that $\mathcal{M}(R)$ could not be well defined for all R (*e.g.* when $\rho(t)$ is time periodic). Whenever it exists, $\mathcal{M}(R)$ is a bounded non-decreasing function and therefore the limit (2.13) is well defined. Moreover $\mathcal{M}_\infty \in [0, M]$.

It is clear that strong dispersion implies total dispersion. Moreover, total dispersion is equivalent to the vanishing property in the concentration-compactness theory, see [151]; precisely, a mass distribution ρ is totally dispersive if and only if the limit

$$\lim_{t\to\infty} \sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|<R} \rho(t,x)\,dx \text{ exists and is zero, } \forall\, R>0. \tag{2.14}$$

An important physical property of (2.14) is that it is invariant by Galilean transformations, unlike the decay of the mass (or energy) over a ball with arbitrary radius,

$$\int_{|x|\leq R} \rho\,dx \to 0, \quad \text{as} \;\; t\to\infty\,, \quad \forall R>0, \tag{2.15}$$

which has also been used as definition of dispersion for evolution type equations (including non-linear Vlasov equations), see [102, 218] for instance. Thus for example, according to our definition of total dispersion, a static (i.e., time independent) solution which is "put in motion" by a Galilean transformation is not to be regarded as a dispersive solution (whereas it would be so according to (2.15)).

Example 2.2.4 As an example of totally dispersive solution of the Vlasov–Poisson system, consider a spherically symmetric shell of matter with internal radius $R_1(t)$ and—possibly infinite—external radius $R_2(t)$ (this example was first introduced in [10]). Let $r = |x|$ and $w = x{\cdot}v/r$, the radial velocity variable. Now suppose that in the support of f^0 it is verified that

$$\inf\{w, w \in \text{ supp } f^0\} > \sqrt{\frac{2M}{R_1(0)}}\,, \tag{2.16}$$

where M is the total mass, i.e., initially the particles are moving outwardly with sufficiently high speed. Using that in spherical symmetry the maximal force experienced by a particle is bounded by M/r^2, see (1.9), we find that, along the characteristics of the Vlasov equation,

$$\frac{d}{dt}\left(\frac{1}{2}w^2 - \frac{M}{r}\right) = w\,\dot{w} + \frac{M}{r^2}\,\dot{r} = w\left(\dot{w} + \frac{M}{r^2}\right),$$

which is positive in the time interval $[0, T[$ in which $w > 0$, i.e., as long as the shell keeps moving outwardly. It follows that

$$w(t)^2 > w(0)^2 - \frac{2M}{r(0)} > \inf_{\text{supp } f^0} w^2 - \frac{2M}{R_1(0)} := W > 0,$$

where for the last inequality we use (2.16). This implies that $T = \infty$, that is, the shell moves outwardly for all future times. Moreover, $W > 0$ is a uniform lower bound on the radial momentum, which entails

$$R_2(t) > R_1(t) > R_1(0) + Wt. \tag{2.17}$$

We claim that, because of (2.17), the solution under consideration is totally dispersive. We shall achieve this by proving that the potential energy vanishes in the limit $t \to \infty$, which for a solution of Vlasov–Poisson is equivalent to total dispersion, see Proposition 2.3.1 in the next section. Thanks to the rotational symmetry we have the representation (1.9), which allows to estimate the potential energy as

$$\begin{aligned}
E_{\text{pot}}(t) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x \phi|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3}\left(\frac{4\pi}{r^2}\int_0^r \lambda^2 \rho(t,\lambda)\, d\lambda\right)^2 dx \\
&= 32\pi^3 \int_{R_1(t)}^{R_2(t)} \frac{1}{r^2}\left(\int_0^r \lambda^2 \rho(t,\lambda)\, d\lambda\right)^2 dr \\
&\le 2\pi \int_{R_1(t)}^{R_2(t)} \frac{M^2}{r^2}\, dr \le \frac{2\pi M^2}{R_1(t)}
\end{aligned}$$

and the claim follows.

For the next result we denote by $d(A, B)$ the distance of the sets $A, B \subset \mathbb{R}^3$:

$$d(A, B) = \inf\{|x - y|\,, x \in A\,, y \in B\}.$$

Lemma 2.2.5 *Let ρ be a partially dispersive regular mass distribution. Then, for any given $\varepsilon > 0$ there exist $t_n \xrightarrow{n} \infty$ and two sequences of non-negative L^1 functions $\rho_1^n, \rho_2^n : \mathbb{R}^3 \to \mathbb{R}^+$, such that $\rho(t_n) \ge \rho_1^n + \rho_2^n$ and* *a)* $\|\rho(t_n) - (\rho_1^n + \rho_2^n)\|_1 \le \varepsilon$;

b) $|\|\rho_1^n\|_1 - \mathcal{M}_\infty| \le \varepsilon\,, |\|\rho_2^n\|_1 - (M - \mathcal{M}_\infty)| \le \varepsilon$;

c) $d(\text{supp}\rho_1^n, \text{supp}\rho_2^n) \to \infty$, *as* $n \to \infty$.

d) There exists a sequence of vectors $y^n \in \mathbb{R}^3$ *and* $0 < R^*_{(\varepsilon)}$ *such that* $\rho_1^n = 0$, *for* $|x - y^n| > R^*_{(\varepsilon)}$.

Remark 2.2.6 The conditions a)-c) define the *dichotomy* property of the mass distribution ρ in the concentration-compactness Lemma, see [151]. Condition *d)* is also consequence of the same result, although this was not pointed out in [151]; for the sake of completeness we shall give here the proof of Lemma 2.2.5. In our context, the relevance of the extra condition *d)* arises from the fact that it prevents the system from being strongly dispersive, as we will see in Section 2.3.3.

Proof. The following proof is adapted from [151]. Owing to (2.14), along any sequence $t_n \overset{n}{\to} \infty$ we have

$$\lim_{n\to\infty} \sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|<R} \rho(t_n, x)\, dx = \mathcal{M}(R).$$

Since $\mathcal{M}(R) \to \mathcal{M}_\infty \in]0, M[$, for all $\varepsilon > 0$ we can find $R^* = R^*_{(\varepsilon)}$ such that, for all n sufficiently large,

$$\sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|<R*} \rho(t_n, x)\, dx \in]\mathcal{M}_\infty - \varepsilon, \mathcal{M}_\infty + \varepsilon[\,.$$

Moreover there exists $y^n \in \mathbb{R}^3$ such that

$$\int_{|x-y^n|<R^*} \rho(t_n, x)\, dx \in]\mathcal{M}_\infty - \varepsilon, \mathcal{M}_\infty + \varepsilon[\,.$$

Finally, we can find a sequence $R_n \overset{n}{\to} \infty$ and a subsequence of times—still denoted t_n—such that

$$\sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|<R_n} \rho(t_n, x)\, dx \in]\mathcal{M}_\infty - \varepsilon, \mathcal{M}_\infty + \varepsilon[\,;$$

The functions $\rho_1^n = \rho(t_n)\chi_{\{B_{y^n}(R^*)\}}$ and $\rho_2^n = \rho(t_n)\chi_{\{\mathbb{R}^3\setminus B_{y^n}(R_n)\}}$ are easily seen to satisfy the properties a)-d). □

2.2.3 Statistical dispersion

We shall now discuss another Galilean invariant concept of dispersion, which was introduced in [87]. Define the statistical dispersion operator in space by

$$\langle(\Delta x)^2\rangle := \frac{1}{M}\left[\int_{\mathbb{R}^3} |x|^2 \rho(t, x)\, dx - \frac{1}{M}\left(\int_{\mathbb{R}^3} x\rho(t, x)\, dx\right)^2\right].$$

Up to a mass normalization, the statistical dispersion operator coincides with the statistical variance of the density mass function and, consequently, it is a measure of the dispersion of such distribution. Note also that

$$\langle(\Delta x)^2\rangle = \int_{\mathbb{R}^3} |x - c_\rho(t)|^2\, \frac{\rho}{M}\, dx$$

and therefore the statistical dispersion operator coincides with the moment of inertia of the mass distribution with respect to the center of mass.

Definition 2.2.7 *A regular mass distribution ρ is said to be* statistically *dispersive if and only if*

$$\sup_{t>0} \langle (\Delta x)^2 \rangle = +\infty.$$

Remark 2.2.8 The above definition differs slightly from that given in [87], where statistical dispersion is defined by the condition $\lim_{t\to\infty} \langle (\Delta x)^2 \rangle = +\infty$.

Statistical dispersion is the weakest concept of dispersion among those introduced so far.

Proposition 2.2.9 *If a regular mass distribution is totally or partially dispersive, then it is statistically dispersive. In particular, total dispersion implies that* $\lim_{t\to\infty} \langle (\Delta x)^2 \rangle = +\infty$.

Proof. We prove first that total dispersion implies $\lim_{t\to\infty} \langle (\Delta x)^2 \rangle = +\infty$. Fix $R > 0$ arbitrarily and write

$$\begin{aligned} M &= \int_{|x-c_\rho(t)|\le R} \rho\, dx + \int_{|x-c_\rho(t)|>R} \rho\, dx \\ &\le \sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|\le R} \rho\, dx + \int_{|x-c_\rho(t)|>R} \rho\, dx. \end{aligned}$$

Assume the solution is totally dispersive. Then by (2.14) there exists $t_0 = t_0(R)$ such that, for all $t > t_0$,

$$\sup_{x_0\in\mathbb{R}^3} \int_{|x-x_0|\le R} \rho\, dx < \frac{M}{2}\, .$$

Thus for $t > t_0$,

$$\langle (\Delta x)^2 \rangle \ge \frac{R^2}{M} \int_{|x-c_\rho(t)|>R} \rho\, dx \ge \frac{R^2}{2}\, ,$$

which yields the claim. To prove that partial dispersion implies statistical dispersion we use the dichotomy property of partially dispersive solutions, see Lemma 2.2.5. Let t_n, ρ_1^n, ρ_2^n satisfy the properties of Lemma 2.2.5. Then

$$\begin{aligned} \langle (\Delta x)^2 \rangle (t_n) &\ge \int_{\mathrm{supp}\rho_1^n} |x - c_\rho(t_n)|^2 \rho_1^n\, dx + \int_{\mathrm{supp}\rho_2^n} |x - c_\rho(t_n)|^2 \rho_2^n\, dx \\ &\ge d(c_\rho(t_n), \mathrm{supp}\rho_1^n)^2 \|\rho_1^n\|_1 + d(c_\rho(t_n), \mathrm{supp}\rho_2^n)^2 \|\rho_2^n\|_1 \\ &\ge C \left[d(c_\rho(t_n), \mathrm{supp}\rho_1^n)^2 + d(c_\rho(t_n), \mathrm{supp}\rho_2^n)^2 \right]. \end{aligned}$$

Finally, by the triangle inequality,

$$\langle (\Delta x)^2 \rangle (t_n) \ge d(\mathrm{supp}\rho_1^n, \mathrm{supp}\rho_2 n)^2 \to \infty, \text{ as } t_n \to \infty.$$

□

Example 2.2.10 We give now an example of solution to the Vlasov–Poisson system which is partially (and therefore statistically) dispersive but not strongly dispersive. This example is a modification of the fully dispersive shell considered before, in which a static, spherically symmetric configuration with given mass M_0 is located in the interior of a shell with mass m (alternatively, the interior part may consist of a static shell [183], leaving a neighborhood of the origin empty, or a spherically symmetric periodic solution, such as the one found by Kurth [136], see also Paragraph 2.3.3). Since the potential inside the shell is constant, the static configuration in the interior will persist as long as the shell is moving outwardly. This again will be verified under condition (2.16), which now reads

$$\inf\{w, w \in \text{ supp } f^0_{\text{shell}}\} > \sqrt{\frac{2(M_0+m)}{R_1(0)}}\,.$$

Then we have

$$(M_0+m)\langle(\Delta x)^2\rangle = \int_{\{R_1(t)>|x|\}} |x|^2\rho\,dx + \int_{\{R_1(t)\le|x|\}} |x|^2\rho\,dx \ge R_1(t)^2 m.$$

By (2.17), this gives a growth of the spatial variance of order t^2. Partial dispersion also follows immediately by (2.17).

2.3 Dynamical behavior for the Vlasov–Poisson system

Hereafter we assume that f is a non-trivial global classical solution of the Vlasov–Poisson system such that, at any fixed time t, f has compact support in the variables (x,v) (however this assumption can be substituted by suitable decay conditions on the variables (x,v) or by requiring only that f has bounded moments in these variables up to a sufficiently high order). We shall refer to these solutions as *regular solutions.* It is well known that for any initial datum $0 \le f^0 = f_{|t=0} \in C^1_c(\mathbb{R}^6)$, there exists a unique global regular solution of the Vlasov–Poisson system, see Paragraph 2.1.3 in the present chapter.

2.3.1 Strong and total dispersion

Recall that strong dispersion implies total dispersion. Indeed we shall now prove that these two concepts of dispersion are equivalent for the Vlasov–Poisson system. Let

$$E_{\text{kin}} = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f\,dvdx, \quad E_{\text{pot}} = \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla_x\phi|^2 dx$$

and note that

$$E_{\text{kin}} - \frac{Q^2}{2M} = \frac{1}{2}\int_{\mathbb{R}^6} |v - Q/M|^2 f\,dvdx > 0. \tag{3.18}$$

The main result relating strong and total dispersion in the attractive case is the following.

Proposition 2.3.1 *Let f be a regular solution of the Vlasov–Poisson system ($\gamma = \pm 1$). Then the following assertions are equivalent:*

1. *f is strongly dispersive.*

2. *f is totally dispersive.*

3. *$E_{\text{pot}} \to 0$, as $t \to \infty$.*

Moreover, if any of the above holds then f satisfies the inequality

$$H \geq \frac{Q^2}{2M}\,.$$

Remark 2.3.2 The same result holds for time sequences; that is, along any sequence $t_n \to \infty$ the property

$$\lim_{n\to\infty} \|\rho(t_n)\|_p = 0$$

is equivalent to

$$\lim_{R\to\infty}\lim_{n\to\infty}\sup_{x_0\in\mathbb{R}^3}\int_{|x-x_0|<R}\rho(t_n,x)\,dx = 0$$

and this is in turn equivalent to

$$\lim_{n\to\infty} E_{\text{pot}}(t_n) = 0.$$

Finally if this holds then necessarily the energy of the initial datum is greater or equal to $\frac{Q^2}{2M}$.

Proof. We only prove the result for $\gamma = +1$, later we comment on $\gamma = -1$. By (3.18),

$$H > \frac{Q^2}{2M} - E_{\text{pot}}\,,$$

and the last claim follows by letting $t \to \infty$. Let us prove the equivalence of the three statements:

1. $\Longrightarrow$ *2.* Clear.

2. $\Longrightarrow$ *3.* Fix $R > 0$ and rewrite the potential energy as $2E_{\text{pot}} = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \int\int_{|x-y|\leq 1/R} \frac{\rho(t,x)\rho(t,y)}{|x-y|}\,dxdy,\\
I_2 &= \int\int_{1/R<|x-y|\leq R} \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy,\\
I_3 &= \int\int_{|x-y|>R} \frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy.
\end{aligned}$$

Using the Young inequality, see [147], the first integral is bounded as

$$I_1 \leq C\|\rho(t)\|_{5/3}^2\left(\int_{|x|\leq R^{-1}}|x|^{-5/4}dx\right)^{4/5} \leq CR^{-7/5}\,;$$

we recall that $\|\rho(t)\|_{5/3} \leq C$ for regular solutions of the Vlasov–Poisson system, see for instance [180]. For I_3 we use that

$$I_3 \leq \frac{M^2}{R}\,.$$

Finally

$$I_2 \leq R \int_{|x-y|\leq R} \rho(t,x)\rho(t,y)\,dxdy \leq MR \sup_{y\in\mathbb{R}^3} \int_{|x-y|\leq R} \rho(t,x)\,dx = R\,\varepsilon_R(t),$$

where, by (2.14), $\varepsilon_R(t) \to 0$, as $t \to \infty$, for all $R > 0$. Collecting,

$$2E_{\text{pot}} \leq C\left(R^{-1} + R^{-7/5}\right) + R\,\varepsilon_R(t).$$

Taking the limit $t \to \infty$ and then $R \to \infty$ concludes the proof.

3. $\Longrightarrow$ *1.* We recall from [125] the following interpolation inequality

$$\|\rho\|_{5/3}^{5/3} \leq Ct^{-2}\left(\int_{\mathbb{R}^6} |x-tv|^2 f(t,x,v)\,dxdv\right) \tag{3.19}$$

and the pseudoconformal law for the attractive case

$$\frac{d}{dt}\int_{\mathbb{R}^6} |x-tv|^2 f(t,x,v)\,dxdv = \frac{d}{dt}\left(\frac{t^2}{4\pi}\|\nabla_x\phi(t)\|_2^2\right) - \frac{t}{4\pi}\|\nabla_x\phi(t)\|_2^2.$$

Integrating we get

$$\int_{\mathbb{R}^6} |x-tv|^2 f\,dxdv - \int_{\mathbb{R}^6} |x|^2 f^0\,dxdv = \frac{t^2}{4\pi}\|\nabla_x\phi(t)\|_2^2 - \int_0^t \frac{s}{4\pi}\|\nabla_x\phi(s)\|_2^2\,ds, \tag{3.20}$$

so that

$$0 \leq t^{-2}\int_{\mathbb{R}^6} |x-tv|^2 f\,dxdv \leq t^{-2}\int_{\mathbb{R}^6} |x|^2 f^0\,dxdv + \frac{1}{4\pi}\|\nabla_x\phi(t)\|_2^2\,, \tag{3.21}$$

and the r.h.s. converges to zero by hypothesis, which in combination with (3.19) concludes the proof. □

The first part in Proposition 2.3.1 was already known in the electrostatic case, as it was proved in [125] and [172] (see also [102]) that all solutions are strongly dispersive. More precisely, recall that

$$\|\rho(t)\|_{5/3} \leq C(1+t)^{-1/2}$$

and we also have

$$E_{\text{pot}}(t) \leq C(1+t)^{-1}.$$

Being all solutions strongly dispersive, it does not make any sense to study partial dispersion in this setting.

In the gravitational case not all solutions are strongly dispersive, as there exist static solutions. Examples of strongly dispersive solutions are those constructed in [31] for small initial data. As we have seen, these solutions verify the estimate

$$\rho(t,x) \leq C(1+|t|+|x|)^{-3}\,,$$

for a positive constant C, which clearly implies strong dispersion. Apart from this result, no sufficient condition on the initial datum is known which ensures that a solution will exhibit strongly dispersive behavior.

Moreover, it was proved in [73] that for the electrostatic case no solution can violate the following bounds on dispersion rates:

$$\|\rho(t)\|_p \geq C(1+t)^{-\frac{3(p-1)}{p}}, \quad p \in]1,\infty].$$

This was first generalized (with the same bounds) to the attractive setting in [87], but neither the case $p = \infty$ nor the case $H = 0$ were covered. We consider these limiting cases in the following proposition, recovering also the results in [87] with a different proof.

Proposition 2.3.3 *The following statements hold for regular solutions to the attractive Vlasov–Poisson system:*

1. *Strongly dispersive solutions verify that there exists some $R > 0$ large enough such that*
$$\liminf_{t\to\infty} \int_{|x|\leq Rt} \rho(t)\ dx > 0. \tag{3.22}$$
2. *For any strongly dispersive solution we can find a $t^* > 0$ such that for $t > t^*$ the following inequality is satisfied:*
$$\|\rho(t)\|_p \geq C(1+t)^{-\frac{3(p-1)}{p}}, \quad p \in]1,\infty].$$
3. *For any solution that is not strongly dispersive there exist $C > 0$ and a sequence of times $t_n \to \infty$ such that*
$$\|\rho(t_n)\|_p \geq C,\ \forall n \in \mathbb{N}, \quad p \in]1,\infty].$$
Furthermore, for any solution with $H < 0$ (and thus not strongly dispersive) we have that
$$\|\rho(t)\|_p \geq C,\ \forall t \geq 0, \quad p \in]1,\infty].$$

Remark 2.3.4 Although the first point is included here only to help in the proof of the second one, it has some interest on its own. It ensures that at least some part of the system won't spread faster than linearly in time. We will come back to these ideas in Section 2.3.6.

Proof. It follows essentially the lines of that in [73]. The first statement is proved by contradiction, i.e., assume that there exists a sequence $t_n \uparrow \infty$ such that

$$\int_{|x|\leq Rt_n} \rho(t_n)\ dx \longrightarrow 0. \tag{3.23}$$

Now we notice that

$$\begin{aligned}
&\int_{|x|\leq Rt_n} v^2 f(t_n,x,v)\ dxdv \\
&\leq 2\int_{|x|\leq Rt_n} \left(v - \frac{x}{t_n}\right)^2 f(t_n,x,v)\ dxdv + 2\int_{|x|\leq Rt_n} \left(\frac{x}{t_n}\right)^2 f(t_n,x,v)\ dxdv \\
&\leq \frac{2}{t_n^2}\int_{|x|\leq Rt_n} (t_n v - x)^2 f(t_n,x,v)\ dxdv + 2R^2\int_{|x|\leq Rt_n} \rho(t_n)\ dx \\
&\leq 2\left(\frac{1}{t_n^2}\int_{\mathbb{R}^6} x^2 f^0(x,v)\ dvdx + 2E_{\text{pot}}(t_n)\right) + 2R^2\int_{|x|\leq Rt_n} \rho(t_n)\ dx,
\end{aligned}$$

where last estimate is due to the pseudoconformal law (3.20). By strong dispersion, $\lim_{t\to\infty} E_{\text{pot}}(t) = 0$ and so, using (3.23), we get that

$$\int_{|x|\leq Rt_n} v^2 f(t_n,x,v)\ dxdv \to 0 \quad \text{as } n\to\infty.$$

So that

$$\lim_{n\to\infty}\int_{|x|\geq Rt_n} v^2 f(t_n,x,v)\ dxdv = \lim_{n\to\infty}\int_{\mathbb{R}^6} v^2 f(t_n,x,v)\ dxdv$$
$$= \lim_{n\to\infty}\left[\int_{\mathbb{R}^6} v^2 f^0(x,v)\ dxdv - \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla_x\phi(0,x)|^2\ dx + \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla_x\phi(t_n,x)|^2\ dx\right]$$

thanks to the conservation of energy. As the potential energy vanishes in the limit we have

$$\lim_{n\to\infty}\int_{|x|>Rt_n} v^2 f(t_n,x,v)\ dxdv = \int_{\mathbb{R}^6} v^2 f^0(x,v)\ dxdv - \frac{1}{4\pi}\int_{\mathbb{R}^3}|\nabla_x\phi(0,x)|^2\ dx = 2H.$$

Also thanks to (3.23)

$$\int_{|x|>Rt_n}\rho(t_n)\ dx \to \int_{\mathbb{R}^6} f^0(x,v)\ dxdv = M \quad \text{as } n\to\infty.$$

Now, estimating in the opposite direction,

$$\left(2\int_{|x|>Rt_n} v^2 f(t_n,x,v)\ dxdv\right)^{\frac12}$$
$$\geq \left(\int_{|x|>Rt_n}\left(\frac{x}{t_n}\right)^2 f(t_n,x,v)\ dxdv\right)^{\frac12} - \left(2\int_{|x|>Rt_n}\left(\frac{x}{t_n}-v\right)^2 f(t_n,x,v)\ dxdv\right)^{\frac12}$$
$$\geq R\left(\int_{|x|>Rt_n} f(t_n,x,v)\ dxdv\right)^{\frac12} - \left(\frac{2}{t_n^2}\int_{\mathbb{R}^6} x^2 f^0(x,v)\ dvdx + 4E_{\text{pot}}(t_n)\right)^{\frac12}.$$

Taking limits when $n\to\infty$ yields

$$2\sqrt{H} \geq R\left(\int_{\mathbb{R}^6} f^0(x,v)\ dxdv\right)^{\frac12}$$

which is a contradiction if $R>0$ is chosen big enough.

To prove the second point we combine the first one with the following inequality

$$\int_{|x|\leq Rt}\rho(t)\ dx \leq \|\rho(t)\|_p (Rt)^{\frac{3(p-1)}{p}}.$$

The result is readily obtained.

The third statement is essentially that of Proposition 2 of [87]. □

Remark 2.3.5 The second point above holds also for time sequences: if there is a sequence $t_n\to\infty$ such that $\|\rho(t_n)\|_p$ converges to zero, we can repeat the proof above to show that $\|\rho(t_n)\|_p \geq C(1+t_n)^{-\frac{3(p-1)}{p}}$ for n large enough.

2.3.2 Statistical dispersion

In [87] a sufficient condition for statistical dispersion was established (see also [27]). The proof uses

$$M\frac{d^2}{dt^2}\langle(\Delta x)^2\rangle = 2H + 2E_{\rm kin} - 2\frac{Q^2}{M}\,, \tag{3.24}$$

which is proved by direct calculation and holds for both values of γ. The precise result is the following:

Proposition 2.3.6 *Regular solutions of the attractive Vlasov–Poisson system which satisfy the condition* $H > \frac{Q^2}{2M}$ *are statistically dispersive. Moreover, there exist constants* $C_1, C_2 > 0$ *such that, for all sufficiently large times,*

$$C_1 t^2 \le \langle(\Delta x)^2\rangle \le C_2 t^2\,.$$

Proof. First we rewrite (3.24) as

$$M\frac{d^2}{dt^2}\langle(\Delta x)^2\rangle = 4H - 2\frac{Q^2}{M} + 2E_{\rm pot}\,, \tag{3.25}$$

Using that $E_{\rm pot} \ge 0$ and integrating in time twice we get

$$\langle(\Delta x)^2\rangle(t) \ge \langle(\Delta x)^2\rangle(0) + \left[\frac{d}{dt}\langle(\Delta x)^2\rangle\right]_{t=0} t + \frac{2}{M}\left(H - \frac{Q^2}{2M}\right)t^2\,,$$

where

$$\left[\frac{d}{dt}\langle(\Delta x)^2\rangle\right]_{t=0} = \frac{2}{M}\int_{\mathbb{R}^6} x\cdot(v - M^{-1}Q)f^0(x,v)\,dxdv. \tag{3.26}$$

The bound from below follows immediately. To prove the upper bound, we recall—see [87, 180] for instance—that the potential energy satisfies the bound

$$E_{\rm pot} \le C\sqrt{E_{\rm kin}}\,,$$

where the positive constant C depends only on $M = \|f(t)\|_1$ and $\|f(t)\|_\infty = \|f^0\|_\infty$. Thus

$$E_{\rm kin} - C\sqrt{E_{\rm kin}} - H \le 0,$$

which in the case of non-negative total energy H gives immediately a uniform upper bound on the kinetic energy:

$$E_{\rm kin} \le \left(\frac{1}{2}\left(C + \sqrt{C^2 + 4H}\right)\right)^2 \tag{3.27}$$

and therefore the potential energy is uniformly bounded as well. Using this in (3.24) it follows that $\langle(\Delta x)^2\rangle \le C_2 t^2$ and the proof is complete. □

As in the repulsive case the energy is always nonnegative, the following is an immediate corollary of (3.24), although never before stated explicitly. Obviously the first part is implied by the results in [125], [172].

Proposition 2.3.7 *Any regular solution to the repulsive Vlasov–Poisson system is statistically dispersive. Moreover, there exists constants* $C_1, C_2 > 0$ *such that, for all sufficiently large times,*

$$C_1 t^2 \le \langle(\Delta x)^2\rangle \le C_2 t^2\,.$$

We point out that, for the case of the linear transport equation, $\langle(\Delta x)^2\rangle$ behaves like t^2 for every solution, see the Appendix.

The threshold $Q^2/(2M)$ that appears in Propositions 2.3.1 and 2.3.7 represents the kinetic energy of a point mass at the center of mass having the same mass of the whole system. Next we are going to give some insight into the dispersion rates for the case $H = Q^2/(2M)$.

Proposition 2.3.8 *Regular solutions with $H = \frac{Q^2}{2M}$ in the attractive case disperse statistically. Furthermore:*

1. *If there is some $t_* > 0$ such that*

$$\int_0^{t_*} \left(E_{\text{kin}}(\tau) - \frac{Q^2}{2M} \right) d\tau + \int_{\mathbb{R}^6} \left(x \cdot v - \frac{Q}{M} \right) f^0 \, dv \, dx > 0$$

(in particular if $\int_{\mathbb{R}^6} (x \cdot v - Q/M) f^0 \, dv \, dx$ is positive) then there exists some $C > 0$ such that

$$\langle(\Delta x)^2\rangle \geq Ct \text{ for } t > t_*.$$

2. *Otherwise, along any time sequence $t_n \to +\infty$ such that $E_{pot}(t_n) \to 0$ we have the estimate*

$$\langle(\Delta x)^2\rangle(t_n) \geq C \left(t_n^{-2} \int_{\mathbb{R}^6} |x|^2 f^0 \; dv \, dx + 2E_{\text{pot}}(t_n) \right)^{-1},$$

where the constant $C > 0$ depends only on the initial datum.

Proof. Integrating once in (3.24) we get that

$$\frac{M}{2} \frac{d}{dt} \langle(\Delta x)^2\rangle = \int_{\mathbb{R}^6} \left(x \cdot v - \frac{Q}{M} \right) f^0 \; dx \, dv + \int_0^t \left(E_{\text{kin}}(\tau) - \frac{Q^2}{2M} \right) \; d\tau. \tag{3.28}$$

Note that $E_{\text{kin}}(t) > Q^2/2M$ and then the first point is proved. Thus the only case left to study is

$$0 < \int_0^{\infty} \left(E_{\text{kin}}(\tau) - \frac{Q^2}{2M} \right) \; d\tau \leq - \int_{\mathbb{R}^6} \left(x \cdot v - \frac{Q}{M} \right) f^0 \; dx \, dv < +\infty.$$

We are going to show that the solution is also dispersive under this assumption.

The integrability of $E_{\text{kin}} - \frac{Q^2}{2M}$ as a function of time assures that we have lots of sequences $t_n \to \infty$ such that

$$\lim_{n \to \infty} E_{\text{kin}}(t_n) = \frac{Q^2}{2M}$$

and so

$$\lim_{n \to \infty} E_{\text{pot}}(t_n) = 0$$

as the energy of the system is precisely $Q^2/2M$.

We recall from [125] that

$$\|\rho(t)\|_{5/3}^{5/3} \leq Ct^{-2} \left(\int_{\mathbb{R}^6} |x - tv|^2 f(t, x, v) \; dv \, dx \right).$$

Using the pseudoconformal law we show —by (3.21)— that

$$\|\rho(t_n)\|_{5/3}^{5/3} \leq C\left(t_n^{-2}\int_{\mathbb{R}^6} |x|^2 f^0 \; dv\, dx + 2E_{\text{pot}}(t_n)\right) := \varphi(t_n).$$

Thus, we get that

$$\int_{|x-c_p(t_n)|\leq R(t_n)} \rho(t_n)\; dx \leq (\frac{4\pi}{3})^{\frac{2}{5}} R(t_n)^{\frac{6}{5}} \|\rho(t_n)\|_{\frac{5}{3}} \leq (\frac{4\pi}{3})^{\frac{2}{5}} R(t_n)^{\frac{6}{5}} \varphi(t_n)^{\frac{3}{5}}$$

for any function $R(t)$. Choose $R(t_n) = \frac{3^{1/3}M^{5/6}}{\pi^{1/3}2^{3/2}}\varphi(t_n)^{-\frac{1}{2}}$ and so

$$\int_{|x-c_p(t_n)|\leq R(t_n)} \rho(t_n)\; dx \leq \frac{M}{2}.$$

Then we can decompose

$$\begin{aligned} M &= \int_{|x-c_p(t_n)|\leq R(t_n)} \rho(t_n)\; dx + \int_{|x-c_p(t_n)|> R(t_n)} \rho(t_n)\; dx \\ &\leq \frac{M}{2} + \int_{|x-c_p(t_n)|> R(t_n)} \rho(t_n)\; dx, \end{aligned}$$

so that

$$\langle(\Delta x)^2\rangle \geq \frac{R(t_n)^2}{M}\int_{|x-c_p(t_n)|\geq R(t_n)} \rho(t_n)\; dx \geq \frac{R(t_n)^2}{2} \to +\infty.$$

□

Remark 2.3.9 The previous result can be regarded as the fact that if the behavior of the kinetic energy is not too chaotic then the spatial variance grows linearly in time. To illustrate this, consider for simplicity the case $Q = 0$, in which the total energy is zero. Whenever the kinetic energy is integrable in time, it is very likely —but not always true, as some pathological counterexamples can be constructed with the aid of an enumeration of $\mathbb{Q}$— that for some sequence t_n that is linear as a function of n we could have

$$\lim_{n\to\infty} \frac{E_{\text{kin}}(t_n)}{1/n} = 0$$

and then, following the ideas of the previous proof, a statement like

$$\lim_{n\to\infty} \frac{1}{n}\sup_{t\leq n}\langle(\Delta x)^2\rangle > 0,$$

would be true.

The next result about the borderline case says that, if not all the mass of the system is lost to infinity then dispersion takes place at the highest possible rate. The proof is straightforward once a proper way of stating that the system is not strongly dispersive is found.

Proposition 2.3.10 *Consider a regular solution of the gravitational Vlasov–Poisson system with $H = \frac{Q^2}{2M}$. Then, if for a.e. $\epsilon \in [0, 1[$ there exist constants $c_\epsilon > 0$ such that*

$$\liminf_{n\to\infty} E_{\text{pot}}(n+\epsilon) > c_\epsilon,$$

we have that $\langle(\Delta x)^2\rangle \sim t^2$.

Proof. First reduce the situation to the case $Q = 0$, no generality is lost. As $H = 0$ in this setting, the lower bound for the potential energy assures that

$$\int_0^t E_{\text{kin}}(\tau)\, d\tau \geq [t] \int_0^1 c_\epsilon \, d\epsilon,$$

being $[t]$ the integer part of t, and inserting this into (3.28) concludes the proof. □

The conclusion of Proposition 2.3.10 can be observed in the following example, considering an outgoing shell surrounding a spherically symmetric steady state with zero total energy.

Example 2.3.11 At this point it is interesting to reconsider the example of the shell surrounding a static configuration introduced at the end of Section 2.2. Say that the inner part of the solution has mass M_0 and that the surrounding shell has mass m and initial inner radius R_1. Our previous computations show that the escape threshold associated with this configuration is

$$\sqrt{\frac{2(M_0+m)}{R_1}}\,,$$

see (2.16), so that any particle with initial radial momentum greater than this threshold will escape towards infinity. We shall now prove that it is possible to obtain an escaping shell even when the total energy of the system is negative. Note that the total energy H consists of the energy E_0 of the interior part plus the kinetic energy of the shell minus the potential energy of the shell (the interaction energy has negative sign and as a consequence the term contributing to the potential energy can be disregarded). Neglecting the last negative term and estimating above the kinetic energy of the shell we get

$$H < E_0 + \frac{1}{2} m \sup_{\text{shell}} |v|^2\,,$$

where the supremum is taken in the support of the shell at time $t = 0$. The interior part has energy $E_0 < 0$, since it is static (cf. (1.1) in Chapter 3); thus in order to have the whole shell escaping to infinity while the total energy of the system remains negative we must choose the initial radial velocities for all the particles in the shell according to

$$\frac{2(M_0+m)}{R_1} < \inf_{\text{shell}} w^2 < \inf_{\text{shell}} |v|^2 < \sup_{\text{shell}} |v|^2 < \frac{-2E_0}{m}.$$

This can be done if m is strictly contained in the interval between zero and the value $\frac{1}{2}[-M_0 + \sqrt{M_0^2 - 4E_0R_1}]$. For bigger values of m we have no guarantee that the total energy can be kept negative.

The previous example shows us that:

- There are solutions that are partially (therefore statistically) dispersive with $H < Q^2/2M$, so that there is no simple way to extend the results of Propositions 2.3.6 and 2.3.8. By Proposition 2.3.1, these solutions cannot be totally dispersive.

- We know that spherically symmetric solutions with $H > 0$ disperse statistically with a dispersion rate of t^2 (equivalently their spatial support spreads with a velocity of order t). We will see in Paragraph 2.3.3 that for solutions with $H = 0$ this needs not to be the case. On the other hand the previous example shows that there are solutions with $H \leq 0$ which also statistically disperse with a rate t^2. So there is no evident relation between the admissible rates of dispersion and the sign of the energy.

- There is no lower limit for the fraction of total mass of the system that is lost to infinity for a partially dispersive system.

- Using these ideas we can also show that there exist dispersive solutions as "close" as desired to a stable steady state. More precisely, given a polytropic steady state (see [206] for details and notation) with mass M, polytropic index μ and $L^{1+\mu^{-1}}$-norm J, which is stable in the sense of [206], we can find solutions as described above that are partially dispersive and that remain in the stability region of the polytrope. (This is not a contradiction since the quantity of mass that is lost to infinity is almost negligible.) These solutions correspond to initial data of the following type. Starting from the given polytrope we construct — using scaling techniques— a second polytrope having mass $M - m$ and $L^{1+\mu^{-1}}$-norm $J - j$, for m, j positive and small. Then we add an outer shell of mass m and $L^{1+\mu^{-1}}$-norm less or equal to j. We choose m, j in order that all the computations in Example 2.3.11 remain valid and that the total energy of the solution is as close as desired to the energy of the original polytrope. Then we can invoke the stability criterion in [206]. Using ODE's terminology, we have proved that these steady states are stable but not asymptotically stable.

2.3.3 Kurth's solution

Here we shall exemplify the findings of the previous paragraph by considering an explicit class of spherically symmetric solutions to the Vlasov–Poisson system found by Kurth in [136] (see also [33, 123, 180]). The main idea of this model is to find a solution whose associated density is of the form

$$\rho(t, x) = (4\pi/3)^{-1} \varphi(t)^{-3} \chi_{\{|x| < \varphi(t)\}} \tag{3.29}$$

where the function φ is interpreted as the radius of the system. This function φ solves the following ODE

$$\varphi^3 \varphi'' + \varphi = 1,$$

subject to the initial condition $\varphi(0) = 1$. We get solutions to the Vlasov–Poisson system exhibiting different types of behavior depending on the prescribed value of $\varphi'(0)$:

- If $\varphi'(0) = 0$, the solution is static;

- If $0 < |\varphi'(0)| < 1$, the solution is periodic in time;

- If $|\varphi'(0)| \geq 1$, the solution is strongly dispersive.

Let us relate the above classification in terms of $\varphi'(0)$ with the values of the energy H. Note first that the associated distribution function f can be chosen to be spherically symmetric so that $Q = 0$. Doing so, we have

$$f(x,v) = \frac{3}{4\pi^3}\left(1 - \left|\frac{x}{\varphi(t)}\right|^2 - |\varphi(t)v - \varphi'(t)x|^2 + |x \wedge v|^2\right)_+^{-1/2} \cdot \chi_{\{|x\wedge v|<1\}} \cdot$$

The energy of Kurth's solutions is given by, see [123],

$$H = \frac{3}{5}\left((\varphi')^2 + \varphi^{-2} - 2\varphi^{-1}\right) = \frac{3}{5}(\varphi'(0)^2 - 1).$$

Moreover $M = 1$, which follows integrating (3.29). Thus

- If $H = -3/5$ ($\Leftrightarrow \varphi'(0) = 0$), the solution is static (this steady state is studied in [35]).
- If $-3/5 < H < 0$ ($\Leftrightarrow 0 < |\varphi'(0)| < 1$), the solution is time periodic.
- If $H \geq 0$ ($\Leftrightarrow |\varphi'(0)| \geq 1$), the radius of the system goes to infinity and, by (3.29), $\rho \to 0$ in L^q, for all $q > 1$, i.e., the solution is strongly (and therefore also totally) dispersive. When $H = 0$ ($\Leftrightarrow |\varphi'(0)| = 1$), we have $\langle(\Delta x)^2\rangle \sim t^{4/3}$. When $H > 0$ ($\Leftrightarrow |\varphi'(0)| > 1$), we get $\langle(\Delta x)^2\rangle \sim t^2$, in agreement with Proposition 2.3.6.

Let us prove the latter claim. First we show that $\langle(\Delta x)^2\rangle \sim \varphi(t)^2$. Since the solution under study is spherically symmetric, we have

$$\langle(\Delta x)^2\rangle = \int_{\mathbb{R}^3} |x|^2\rho(x)\,dx = 4\pi\left(\frac{4\pi}{3}\right)^{-1}\varphi(t)^{-3}\int_0^{\varphi(t)} r^4\,dr = \frac{3}{5}\varphi(t)^2.$$

Thus we reduce the problem to find out the large time behavior of the function φ. Following Kurth [136], if $|\varphi'(0)| = 1$ we have that

$$\varphi(t) = \frac{1}{2}(1 + v(t)^2)\,,$$

where $v(t)$ solves

$$v(t) + \frac{1}{3}v(t)^3 = 2\left(t + \frac{2}{3}\right).$$

For t big enough the term $v(t)^3$ dominates and then $\varphi(t) \sim t^{2/3}$. If $|\varphi'(0)| > 1$, we have that

$$\varphi(t) = \frac{|\varphi'(0)|\mathrm{ch}(v(t)) - 1}{|\varphi'(0)|^2 - 1}\,,$$

where $v(t)$ solves

$$v(t) - |\varphi'(0)|\mathrm{sh}(v(t)) = (|\varphi'(0)| - 1)^{3/2}(t - t_0)$$

(t_0 depends on $|\varphi'(0)|$). For t big enough $|\mathrm{sh}(v(t))|$ dominates $|v(t)|$, and we infer that $|v(t)| \sim \ln t$, which entails $\varphi(t) \sim t$.

Note that the solution with $H = 0$ (there are two of them actually) is the only known example of statistically dispersive solution for which statistical dispersion takes place at a slower rate than t^2 and for which the support spreads slower than t. Also the rate for strong dispersion is slower than for the other examples considered in this chapter. Taking a closer look to the trajectories reveals that these are strongly oscillatory (like in a forced harmonic oscillator).

This special solution highlights also the role of condition *d*) in Lemma 2.2.5. For we can surround this "slowly dispersing" solution with a strongly dispersive shell configuration, and the resulting solution verifies *a*), *b*) and *c*) but not *d*), and happens to be totally but not partially dispersive, see previous Remark 2.2.6.

2.3.4 Time periodic solutions

Not so much is known about time periodic solutions to the Vlasov–Poisson system. The first example of such solutions is that of Kurth (see the previous paragraph). Later several steadily rotating solutions have been constructed. First we have those on [184], which are cylindrically symmetric and indeed static (the mean velocity field is non-vanishing but thanks to the symmetry the density profile remains static), obtained as deformations via the implicit function theorem of spherically symmetric steady states. Then came those in [212], which are also axially symmetric but obtained using Jeans' theorem and the Jacobi integral. We have also the examples of time periodic solutions given in [34]; many of them are not even cylindrically symmetric, opposed to the ones that we already commented on. The reason that they are not so interesting is that these models represent systems with infinite mass.

We can give a fairly easy condition on macroscopic parameters for a time periodic solution to exist:

Proposition 2.3.12 *Time periodic solutions of the Vlasov–Poisson system satisfy* $H < -\frac{Q^2}{2M}$.

Proof. The result follows from the dilation identity

$$\frac{d}{dt}\int_{\mathbb{R}^6} x\cdot v f(t)\,dxdv = H + E_{\text{kin}}$$

which can be proved by direct computation. We integrate it over a period T to get

$$0 = HT + \int_0^T E_{\text{kin}}\,dt.$$

Using (3.18) we conclude the proof. □

2.3.5 Virialized solutions

It is a classical result in Astrophysics that bounded systems of self-gravitating particles roughly in equilibrium verify that the time average of the kinetic energy of the ensemble equals twice the time average of its potential energy. This statement and some of its variants and particularizations go under the name of "virial theorems" (cf. [48, 177, 207], for instance), and are common tools in Astrophysics. We shall comment here on the connection between the notion of *virialized* solutions of the Vlasov–Poisson system and our preceding results.

In this paragraph we shall only consider solutions such that $Q = 0$, i.e., the reference frame is chosen at rest with respect to the center of mass of the system, as we did in the proof of Proposition 2.3.8. Following [177] we shall say that a solution of the Vlasov–Poisson system is *virialized* if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t H + E_{\rm kin}(\tau)\, d\tau = 0. \tag{3.30}$$

Note that solutions with $H > 0$ cannot be virialized in this sense. In fact, it is a straightforward consequence of the inequality (3.18) that virialized solutions of the Vlasov–Poisson system must necessarily satisfy $H \leq 0$. Examples of virialized solutions are Kurth's solutions with energy $H \leq 0$. The notion of virialized system is usually applied in Astrophysics only in the case of bounded N-body systems, but, as pointed out in [177], strict boundedness is not necessary (Kurth's solution with energy $H = 0$ shows that also in the case of Vlasov–Poisson, the support of a virialized solution can spread out to infinity). Then we regard virialized systems as apparently bounded systems, i.e., systems that disperse so slowly that in our time scale they appear to be bounded and in equilibrium.

The following proposition extends the result in [177], which is valid for N-body systems, to the continuous setting; note that the diameter of the N-body system used in [177], is replaced by the statistical dispersion operator $\langle(\Delta x)^2\rangle$.

Lemma 2.3.13 *Let $f(t)$ be a given solution of the Vlasov–Poisson system with $Q = 0$. Then the following statements hold true:*

1. *If $f(t)$ is virialized then* $\lim_{t\to\infty} \frac{\langle(\Delta x)^2\rangle}{t^2} = 0$.
2. *If* $\lim_{t\to\infty} \frac{\langle(\Delta x)^2\rangle}{t^2} = 0$ *and* $\lim_{t\to\infty} \frac{1}{t}\int_0^t H + E_{\rm kin}(\tau)\, d\tau$ *exits, then $f(t)$ is virialized.*

Proof. The proof is a straightforward application of L'Hôpital's rule (as formulated in [195]) and of(3.24). □

The existence of the limit in (3.30) can be guaranteed under some of the most frequent situations: spherical symmetry [33] — his result in that setting gives more information than ours—, periodicity in time (including static solutions of course) and whenever $E_{\rm kin}(t)$ has a limit as $t \to \infty$.

2.3.6 Summary and open problems

The main results of this chapter and the following one concerning the Vlasov–Poisson system are summarized in Table 2.1. Completing the entries marked with a question mark would lead to a considerable extension of the results presented here and of our understanding of the large time behavior of the Vlasov–Poisson system in the gravitational case. For the sake of completeness we gather here the dispersion rates which are known for examples of solutions to the Vlasov–Poisson system.

- Small data solutions in the attractive case [31] have $\|\nabla_x \phi\|_\infty \sim t^{-2}$ and $\|\rho(t)\|_\infty \sim t^{-3}$.
- All the examples of outgoing shells and modifications of them considered in this section have $\langle(\Delta x)^2\rangle \sim t^2$. The radius of the system grows linearly in time.

Dispersive behavior	**Necessary**	**Sufficient**	**Example**
Strong Dispersion	$H \geq Q^2/2M$	?	Example 2.2.4
(VP)$\uparrow \downarrow$			
Total Dispersion	$H \geq Q^2/2M$	?	Kurth's $H \geq 0$
$\downarrow$			
Statistical Dispersion	?	$H \geq Q^2/2M$	
$\uparrow$ (VP) $\downarrow$?			
Partial Dispersion	?	?	Example 2.2.10
Other solutions			
Static Solutions	$H < 0$	–	Kurth's $H = -\frac{3}{5}$
Periodic Solutions	$H < -Q^2/2M$	–	Kurth's $H \in]-\frac{3}{5}, 0[$
Virialized Solutions	$H \leq 0$	–	Kurth's $H = 0$

Table 2.1: Main results proved for the Vlasov–Poisson system and open problems.

- Kurth solutions [136] with $H > 0$ verify $E_{\text{pot}} \sim t^{-1}$, $\|\rho(t)\|_\infty \sim t^{-3}$ and $\langle(\Delta x)^2\rangle \sim t^2$. The radius of the system grows linearly in time.
- Kurth solutions [136] with $H = 0$ have $E_{\text{pot}} \sim t^{-2/3}$, $\|\rho(t)\|_\infty \sim t^{-2}$ and $\langle(\Delta x)^2\rangle \sim t^{4/3}$. The radius of the system grows like $t^{2/3}$.
- In the electrostatic case, $E_{\text{pot}} \sim t^{-1}$ and $\|\rho(t)\|_{5/3} \sim t^{-3/5}$ [125, 172].

The Vlasov–Poisson system is based on classical (i.e., non-relativistic) mechanics. Thus, it allows in principle that particle velocities become unbounded (and this is a paramount issue when stating existence results, as we commented on paragraph 2.1.3 on the present chapter). In relativistic mechanics particles cannot travel faster than the speed of light. As a consequence, a self-gravitating system cannot expand faster than linearly in time. Although the Vlasov–Poisson system does not take into account any relativistic effect, it is able to reproduce the same spreading rates if these features are averaged in a suitable sense.

A first result supporting these ideas is (3.22). It says that if we look at the natural spreading scale we will find that at least part of the system is trapped within the physically relevant region in position space.

The lower bounds for strong dispersion given in Proposition 2.3.3 are also consistent with a linear rate of spreading. Consider for instance a system which has a constant density function which is supported in a ball of linearly expanding radius; conservation of mass forces the system to satisfy $\|\rho(t)\|_\infty \sim t^{-d}$, being $d = 3$ the dimension of position space. We actually have some examples of such systems: the Kurth solutions [136] with positive energy. For other norms of the density function, the reasonable fastest rates of strong dispersion should never surpass the ones for the linear transport equation (see the Appendix). These rates can be obtained by interpolation with $\|\rho(t)\|_1 = M$, compare with [73]. They coincide with the ones given in Proposition 2.3.3.

The strongest evidence supporting the idea that the Vlasov–Poisson system is able to reproduce the correct spreading rates is the result in Proposition 2.3.6. It shows that the spatial variance is able to rule out the adequate outliers —like particles with unbounded velocities— so that in an averaged sense spreading systems expand generically linearly in time and never faster than that.

Finally, Proposition 2.3.8 shows that apparently there is no sub-diffusive regime for expanding systems (the case of strictly negative energy is still open).

All this evidence supports also the following claim: it seems very convenient in most scenarios to demand finite spatial variance from the initial datum. All our results require this hypothesis, but this fact is hidden in our definition of "regular solution". Thus, a physically meaningful class in which the initial value problem for Vlasov–Poisson system could be studied is that of smooth initial data with finite kinetic energy and finite spatial variance.

Next we want to comment on one of the open problems stated in Table 2.1. It seems natural that if a solution disperses statistically, then this solution is loosing some mass to infinity, which causes the increase of the variance of the system. If this were true then the concepts of partial and statistical dispersion would be equivalent. But maybe this is not the case.

The following shows that this problem is not that easy. There is an infinite family of steady states with finite mass, infinite radius and unbounded spatial moments. These steady states are polytropes of the form $f(x,v) = (-E)_+^{3k+7/2} F^k$; here E is the particle energy and F the particle angular momentum squared (see Chapter 3 for a detailed explanation). For fixed k the resulting steady state has unbounded spatial moments of order greater or equal to $2+4k$. Moreover, the Plummer/Schuster model, corresponding to $k=0$, is stable [110]. Then, it could happen that a solution starting close to this steady state qualifies as statistically dispersive without loosing mass to infinity. Nowadays we are unable to rule out this possibility.

2.4 The Nordström–Vlasov system

2.4.1 Description of the model

When relativistic effects become important the Vlasov–Poisson no longer provides an adequate description of self-gravitating matter; its role is taken by the Einstein–Vlasov system. No wonder that the Einstein–Vlasov system is much more complicated than Vlasov–Poisson's. Serious difficulties arise from the character of the Einstein equations, which are essentially hyperbolic and highly non-linear even in the absence of sources, from the equivalence of all the coordinate systems in General Relativity and from the fact that in the Einstein theory of gravitation, the space-time is not given in advance but is itself part of the solution of the equations. All these features make it extremely difficult to analyze the solutions of the Einstein equations coupled to any kind of matter model. Besides, the usual reduction to spherical symmetry rules out one of the most interesting new features of General Relativity, which is the propagation of gravitational waves, since this restriction forces the spacetime to be static outside the support of the matter.

In this section we are going to propose a simpler —although unphysical— model in which the dynamics of the matter is still described by the Vlasov equation but where the gravitational forces between the particles are now supposed to be mediated by a scalar field. These has some advantages; the resulting model is more affordable from the mathematical point of view and it can be treated without recourse to symmetry assumptions, so that it constitutes a good toy model. Also the coupling of the kinetic equation to a scalar gravitation theory allows for the propagation of gravitational waves, a feature that we cannot afford nowadays for the full Einstein–Vlasov system. For this reason, this model may represent a comparatively easier framework where to study the effects of the gravitational radiation on the dynamics of a many particles system.

We will refer to this scalar model as the Nordström–Vlasov system (NV for short), since the scalar gravitation theory which we use to describe the interaction among the particles reduces, in a particular frame, to the one introduced by Nordström in [168]. The model will describe the collective motion of collisionless particles interacting by means of their own self-generated gravitational forces, under the condition that the dynamics of the gravitational field is described in accordance to a simple scalar gravitation metric theory.

By "scalar gravitation metric theory" we mean a theory in which the gravitational forces are mediated by a scalar field ϕ and the effect of such forces is to induce a curvature in the space-time. It is also assumed that the scalar field modifies the otherwise flat metric only by a rescaling. Therefore the metric in this theory will be conformally flat, that is given by

$$g_{\mu\nu} = A^2(\phi)\eta_{\mu\nu}, \tag{4.31}$$

where A is a strictly positive function and, adopting Cartesian coordinates, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. As we already know, the condition that the particles make up a collisionless ensemble in the space-time is carried out by requiring that the particle density be a solution of the Vlasov equation on a curved spacetime (we will elaborate this further in Chapter 3 when describing the Einstein–Vlasov system). This equation is equivalent to postulate that the function f is constant on the geodesic flow of the metric (4.31).

In [168], the Finnish physicist Gunnar Nordström proposed a relativistic scalar theory of gravitation which was based on a nonlinear wave equation for ϕ as field equation. Following that spirit, we write down what we call the Nordström–Vlasov system, presented in the formulation used in [66]:

$$\partial_t f + \frac{p}{\sqrt{e^{2\phi} + |p|^2}} \cdot \nabla_x f - \nabla_x \left(\sqrt{e^{2\phi} + |p|^2} \right) \cdot \nabla_p f = 0, \tag{4.32a}$$

$$\partial_t^2 \phi - \Delta_x \phi = -e^{2\phi} \int_{\mathbb{R}^3} f \, \frac{dp}{\sqrt{e^{2\phi} + |p|^2}} \,. \tag{4.32b}$$

Here $f = f(t, x, p) \geq 0$ and $\phi = \phi(t, x)$. The physical interpretation of a solution (f, ϕ) is the following: the spacetime is the Lorentzian manifold $(\mathbb{R}^4, g = e^{2\phi}\eta)$, whereas f is the kinetic distribution function of particles (here we are thinking of the stars of a galaxy) moving along the geodesic curves of the metric g. The motion along geodesics reflects the condition that gravity is the only interaction among the particles. The system has been written in units such that $4\pi G = c = 1$, where G is Newton's gravitational constant and c the speed of light. The mass of particles has also been normalized to 1. For further details about the deduction of the model, reader is urged to consult [59, 63].

2.4.2 The Cauchy problem and related results

It should be noted that not only an initial datum f^0 for the kinetic equation is required, we have to prescribe also initial data ϕ_0 and $\phi_1 = \partial_t \phi_0$ for the wave equation. It is known that $f^0 \in C^1_c(\mathbb{R}^3)$, $\phi_0 \in (C^3_b \cap H^1)(\mathbb{R}^3)$ and $\phi_1 \in (C^2_b \cap L^2)(\mathbb{R}^3)$ suffices, but let us review the results concerning the study of the Cauchy problem.

The model under consideration was introduced in [59], where the existence of a wide class of spherically symmetric steady states of finite radius was proved. The Cauchy problem for this system was first considered in [64]; the authors state a local existence result for regular initial data together with a continuation criterion in the same spirit of that for the Vlasov–Poisson system (1.10). This criterion was improved in [65], as a part of the proof of existence of weak solutions that conserve mass. Meanwhile, this criterion was further improved in [63]. The purpose of that paper was to study the limit $c \to \infty$; an expansion for $c \gg 1$ was performed and as a result the fact that in the limit $c \to \infty$ one recovers the Vlasov–Poisson system in the gravitational case was obtained. The issue of classical solutions was taken up again in [9], which establishes global existence for spherically symmetric initial data such that a lower cutoff in the modulus of the angular momentum is introduced. We have also the results in [98] for the case of small initial data, assuring global existence and decay rates for large t. The definitive answer can be found in [60], where an existence and uniqueness result for compactly supported initial data was proved.

As regards the long time behavior we recall the results in [98] and the dispersive estimate contained in [66] concerning the conformal energy. Static solutions for this model are studied in [59] and [66]. In the second reference we can find an orbital stability result for a certain class of (polytropic) steady states, together with a related virial theorem.

2.5 Dynamical behavior for the Nordström–Vlasov system

This section gathers most of what is known nowadays about the dynamical behavior of solutions to the Nordström–Vlasov system. We begin with the study of conserved quantities and the set of transformations that leave the system invariant. We are very interested in describing how does the system behave under the action of Lorentz transformations, whose role in Special Relativity is parallel to that of the Galilean transformations in the classical setting: again, these are the mathematical realization of the fact that all inertial observers are equivalent. The section concludes with a tentative and non-optimal dispersion result for the solutions to the Nordström–Vlasov system.

2.5.1 Conserved quantities and Lorentz transforms

The local energy, momentum and stress tensor of a solution (f,ϕ) of (4.32) are defined respectively as $(i,j=1,2,3)$

$$h(t,x)=\int_{\mathbb{R}^3}\sqrt{e^{2\phi}+|p|^2}\,f\,dp+\frac{1}{2}(\partial_t\phi)^2+\frac{1}{2}|\nabla_x\phi|^2,$$
$$q_i(t,x)=\int_{\mathbb{R}^3}p_i f\,dp-\partial_t\phi\,\partial_i\phi\,,$$
$$\tau_{ij}(t,x)=\int_{\mathbb{R}^3}\frac{p_i\,p_j}{\sqrt{e^{2\phi}+|p|^2}}\,f\,dp+\partial_i\phi\,\partial_j\phi+\frac{1}{2}\delta_{ij}\left[(\partial_t\phi)^2-|\nabla_x\phi|^2\right],$$

where ∂_i denotes the partial derivative along x^i. These quantities are related by the conservation laws

$$\partial_t h+\nabla_x\cdot q=0,\quad \partial_t q_i+\partial_j\tau_{ij}=0, \tag{5.34}$$

Upon integration, the previous identities lead to the conservation of the total energy and of the total momentum:

$$H(t)=\int_{\mathbb{R}^3}h(t,x)\,dx=constant,\quad Q(t)=\int_{\mathbb{R}^3}q(t,x)\,dx=constant.$$

Moreover, solutions of the Nordström–Vlasov system satisfy the conservation of the total rest mass:

$$M(t)=\int_{\mathbb{R}^3}\rho(t,x)\,dx=constant,$$

which is obtained by integrating the local rest mass conservation law

$$\partial_t\rho+\nabla_x\cdot j=0,\quad \rho=\int_{\mathbb{R}^3}f\,dp,\quad j=\int_{\mathbb{R}^3}\frac{p}{\sqrt{e^{2\phi}+|p|^2}}\,f\,dp. \tag{5.35}$$

The system (4.32) satisfies the fundamental property of Lorentz invariance. Precisely, let (t',x') be a system of coordinates in Minkowski space obtained from (t,x) by a Lorentz boost, that is

$$t'=u_0t-u\cdot x,\quad x'=x-u\,t+\frac{u_0-1}{|u|^2}(u\cdot x)u,$$

where u is a fixed vector in $\mathbb{R}^3$ and $u_0=\sqrt{1+|u|^2}$. The inverse Lorentz transformation is obtained by exchanging u with $-u$, that is

$$t=u_0t'+u\cdot x',\quad x=x'+u\,t'+\frac{u_0-1}{|u|^2}(u\cdot x')u, \tag{5.36}$$

which we shorten by $(t,x)=L_u(t',x')$. Define the field ϕ_u in the new coordinates by

$$\phi_u(t',x')=\phi\circ L_u\,(t',x').$$

Introduce the new momentum variable

$$p'=p-u\sqrt{e^{2\phi(t,x)}+|p|^2}+\frac{u_0-1}{|u|^2}(u\cdot p)u$$

or, inverting,

$$p = p' + u\sqrt{e^{2\phi(t,x)} + |p'|^2} + \frac{u_0 - 1}{|u|^2}(u \cdot p')u. \tag{5.37}$$

We shall write $(t, x, p) = \mathcal{L}_u(t', x', p')$ to shorten the set of transformations (5.36)-(5.37). Finally, define the distribution function in the new variables as

$$f_u(t', x', p') = f \circ \mathcal{L}_u\,(t', x', p').$$

In the language of special relativity, one says that f and ϕ transform like scalar functions under Lorentz transformations. The Lorentz invariance of the Nordström–Vlasov system means that the pair (f, ϕ) solves the system (4.32) in the coordinates (t, x, p) if and only if (f_u, ϕ_u) satisfies the same system in the coordinates (t', x', p'). Thus, in particular, also the mass-energy-momentum of (f_u, ϕ_u) is conserved along the time evolution,

$$M[f_u] = constant, \quad H[f_u, \phi_u] = constant, \quad Q[f_u, \phi_u] = constant.$$

The conservation laws (5.34) can be expressed in a more concise form as

$$\partial_\mu T^\mu_{\ \nu} = 0, \qquad \mu, \nu = 0, \ldots, 3,\ x^0 = t, \tag{5.38}$$

where $T_{\mu\nu}$ is the stress-energy tensor, whose components are given by

$$T_{00} = -h,\ T_{0i} = -q_i\ ,\ T_{ij} = \tau_{ij}\ .$$

Indexes are raised and lowered with Minkowski's metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. In particular $x^i = x_i$ and $x^0 = -x_0$. Upon multiplying the conservation law (5.38) by a vector field $\xi^\mu = \xi^\mu(t, x)$, integrating on a compact spacetime region Ω with piecewise differentiable boundary $\partial\Omega$ and applying the divergence theorem we obtain the integral identity

$$\int_{\partial\Omega} T^\mu_{\ \nu}\xi^\nu n_\mu d\sigma = \int_\Omega T^\mu_{\ \nu}\partial_\mu\xi^\nu dtdx, \tag{5.39}$$

where n_μ denotes the exterior normal vector field to the boundary $\partial\Omega$ and $d\sigma$ the invariant volume element thereon. Let

$$\Omega = [0, T] \times B_R(0), \quad R, T > 0,\ B_R(0) = \{x \in \mathbb{R}^3 : |x| \le R\}, \quad S_R = \{x : |x| = R\}$$

and dS_R the invariant volume measure on S_R. Under suitable decay conditions at infinity, the integral in the right hand side of (5.39) converges when $R \to \infty$, which gives the identity

$$\left[\int_{\mathbb{R}^3} T^0_{\ \nu}\xi^\nu\, dx\right]_0^T = \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} T^{\mu\nu}\mathcal{L}_\xi\eta_{\mu\nu}\, dxdt, \tag{5.40}$$

where $[g(t)]_0^T = g(T) - g(0)$, for all functions of t, $\mathcal{L}_\xi$ denotes the Lie derivative operator along the vector field ξ and

$$\mathcal{L}_\xi\eta_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\ .$$

This provides a complementary approach to the study of conserved quantities for the system, that can be used to give some insight on dynamical properties of its solutions too. Conserved quantities are recovered using vector fields ξ^μ that induce isometries. That is, $\mathcal{L}_\xi\eta_{\mu\nu} = 0$, i.e., ξ is a Killing vector field of Minkowski spacetime. This choice yields the conservation of the integral quantity in the left hand side of (5.40). Minkowski's spacetime admits ten independent Killing vectors:

- Time translations have as infinitesimal generator $\xi^0 = 1$, $\xi^i = 0$ and lead to the conservation of energy, under the decay condition

$$\lim_{R\to\infty} \int_{[0,T]\times S(R)} q_i \frac{x^i}{R}\, dtdS_R = 0.$$

- Spatial translations are obtained with the choice $\xi^0 = 0$, $\xi^i = 1$ for a given $i \in \{1,2,3\}$ and $\xi^j = 0$ for $j \neq i$.This yields invariance of linear momenta $\int_{\mathbb{R}^3} q_i\, dx$ if the decay condition

$$\lim_{R\to\infty} \int_{[0,T]\times S(R)} x^j \tau_{ji}\, dtdS_R = 0$$

is satisfied. All together assures the conservation of the total momentum Q.

- Spatial rotations are related to the infinitesimal generators $\xi^0 = 0, \xi^i = 0, \xi^j = -x^k, \xi^k = x^j$ for $(i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1))\}$. For instance, the vector field $\xi^0 = 0$, $\xi^1 = -x_2$, $\xi^2 = x_1$, $\xi^3 = 0$ gives the conservation of

$$\int_{\mathbb{R}^3} x_1 q_2 - x_2 q_1\, dx$$

under the decay condition

$$\lim_{R\to\infty} \int_{[0,T]\times S(R)} -\frac{x_2}{R} x^i \tau_{i1} + \frac{x_1}{R} x^i \tau_{i2}\, dtdS_R = 0.$$

This leads to the conservation of the angular momentum vector

$$\Omega_i = \int_{R^3} (x_j q_k - x_k q_j)\, dx = constant,\ (i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1))\}$$

- Lorentz boosts have $\xi^0 = x^i$, $\xi^i = -t$ for a given $i \in \{1,2,3\}$ and $\xi^j = 0$ for $j \neq i$ as generators, which lead to the conservation of the total spin vector

$$S_i = \int_{\mathbb{R}^3} (h\, x_i - q_i t)\, dx = constant,\ \forall i = 1,2,3\,.$$

We can use also vector fields that are not Killing in (5.40) and obtain other types of information. For instance, the action under spatial homoteties (generated by $\xi^0 = 0$, $\xi^i = x^i$ for a given $i \in \{1,2,3\}$ and $\xi^j = 0$ for $j \neq i$) will be studied as a particular case in Chapter 3.

For future usage we shall need the relation between the mass-energy-momentum of (f,ϕ) and of (f_u,ϕ_u), which is derived in the following Lemma. In the language of Special Relativity, it states that M transforms like a scalar function, whereas the quadruple (H,Q) transforms like a four-vector under Lorentz transformations.

Lemma 2.5.1 *For all $u \in \mathbb{R}^3$,*

$$M[f_u] = M[f]\,, \tag{5.41a}$$

$$H[f_u,\phi_u] = \sqrt{1+|u|^2}\, H[f,\phi] - Q[f,\phi]\cdot u, \tag{5.41b}$$

$$Q[f_u,\phi_u] = Q[f,\phi] - H[f,\phi]u + \frac{u_0 - 1}{|u|^2}(u\cdot Q[f,\phi])u. \tag{5.41c}$$

Proof. Since the mass-energy-momentum of both pairs (f, ϕ) and (f_u, ϕ_u) is conserved, it is sufficient to prove the relations (5.41) for the initial value of $M(u) := M[f_u]$, $H(u) := H[f_u, \phi_u]$ and $Q(u) := Q[f_u, \phi_u]$. We restrict ourselves to prove the invariance of the total mass, the proof for the other transformations being similar. We shall need that, by (5.37),

$$\sqrt{e^{2\phi(t,x)} + |p'|^2} = u_0\sqrt{e^{2\phi(t,x)} + |p|^2} - u \cdot p$$

or, inverting,

$$\sqrt{e^{2\phi(t,x)} + |p|^2} = u_0\sqrt{e^{2\phi(t,x)} + |p'|^2} + u \cdot p'. \tag{5.42}$$

In order to prove (5.41a) we write

$$\begin{aligned} M(u) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_u(0, x', p')\, dx'dp' = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f \circ \mathcal{L}_u(0, x', p')\, dx'dp' \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\left(u \cdot x', x' + \frac{u_0 - 1}{|u|^2}(u \cdot x')u, p'\right. \\ &\qquad \left. + u\sqrt{e^{2\phi_u(0,x')} + |p'|^2} + \frac{u_0 - 1}{|u|^2}(u \cdot p')u\right) dx'dp'. \end{aligned}$$

Next we make the change of variable

$$x = x' + \frac{u_0 - 1}{|u|^2}(u \cdot x')u, \quad p = p' + u\sqrt{e^{2\phi_u(0,x')} + |p'|^2} + \frac{u_0 - 1}{|u|^2}(u \cdot p')u.$$

The Jacobian of this transformation is given by

$$J = \frac{\widehat{u} \cdot p' + \sqrt{e^{2\phi_u(0,x')} + |p'|^2}}{\sqrt{e^{2\phi_u(0,x')} + |p'|^2}}\ ,$$

where $\widehat{u} = u/u_0$. Using (5.37) and (5.42) we obtain

$$J = \left(1 - \frac{\widehat{u} \cdot p}{\sqrt{e^{2\phi_u(0,x')} + |p|^2}}\right)^{-1}.$$

Since the volume measure transforms as $dx'dp' = J^{-1}dxdp$, we obtain

$$\begin{aligned} M(u) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\widehat{u} \cdot x, x, p)\left(1 - \frac{\widehat{u} \cdot p}{\sqrt{e^{2\phi(\widehat{u}\cdot x,x)} + |p|^2}}\right) dxdp \\ &= \int_{\mathbb{R}^3} (\rho(\widehat{u} \cdot x, x) - \widehat{u} \cdot j(\widehat{u} \cdot x, x))\, dx, \end{aligned}$$

where ρ and j are defined by (5.35). Taking the partial derivative ∂_{u_i} of the previous expression we get

$$\begin{aligned} \partial_{u_i} M(u) &= \int_{\mathbb{R}^3} (\partial_t\rho\, \partial_{u_i}(\widehat{u} \cdot x) - (\partial_{u_i}\widehat{u}_k)j_k - \widehat{u}_k\partial_t j_k \partial_{u_i}(\widehat{u} \cdot x))\,(\widehat{u} \cdot x, x)\, dx \\ &= \int_{\mathbb{R}^3} (-\partial_{u_i}(\widehat{u} \cdot x)(\partial_{x_k} j_k + \widehat{u}_k\partial_t j_k) - (\partial_{u_i}\widehat{u}_k)j_k)\,(\widehat{u} \cdot x, x)\, dx \\ &= -\int_{\mathbb{R}^3} (\partial_{u_i}(\widehat{u} \cdot x)\partial_{x_k}[j_k(\widehat{u} \cdot x, x)] + (\partial_{u_i}\widehat{u}_k)j_k(\widehat{u} \cdot x, x))\, dx \\ &= 0, \end{aligned}$$

where we used the continuity equation

$$\partial_t \rho + \nabla_x \cdot j = 0 \tag{5.43}$$

to pass from the first to the second line and integration by parts to pass from the third to the last line. Thus we obtained that $\nabla_u M(u) = 0$, i.e., $M(u) = M(0)$, which yields the claim on the invariance of the total rest mass. □

2.5.2 A dispersion estimate

Nowadays we lack of a widely accepted definition of dispersion in relativistic settings; this is partially due to the fact that the concepts that work finely in the classical setting are not Lorentz invariant when translated into this new situation. Thus, the search for a physically meaningful concept of dispersion for relativistic models like the NV system remains as a very challenging problem. Meanwhile classical ideas can be used to try to give some insight into this problem, and this is the route that we follow in this paragraph. Our plan is to borrow in some way the concept of statistical dispersion and to be able to give sufficient conditions for its occurrence, following the strategy that we used with the Vlasov–Poisson system. As a first step in so doing, we define the conformal energy as

$$\mathcal{E}_C(t) = \int_{\mathbb{R}^3} |x|^2 h(t,x)\, dx$$

for classical solutions. Making a parallel with the case of the Vlasov–Poisson system, we prove the following result (we stress again the fact that this is a partial and non-optimal result —we will explain later—, that could nevertheless constitute a first step to study this problem).

Theorem 2.5.2 *There exists a constant $t_0 > 0$, depending only on bounds on the initial data, such that the following holds: if the initial data are regular and satisfy $H > M$ then*

$$\mathcal{E}_C(t) \geq (H - M)\, t^2,$$

for all $t > t_0$. If $H = M$, then

$$\mathcal{E}_C(t) \geq 2Q_0\, t,$$

again for $t > t_0$, provided $Q_0 > 0$, where

$$Q_0 = \int_{\mathbb{R}^3} (x \cdot q(0,x) - \phi_0 \phi_1)\, dx.$$

Proof. This was already treated in [66] but we include the proof for completeness.

By the first of (5.34) we have

$$\frac{d}{dt}\mathcal{E}_C = 2\int_{\mathbb{R}^3} x \cdot q\, dx, \tag{5.44}$$

whence, using the second equation in (5.34),

$$\frac{d^2}{dt^2}\mathcal{E}_C = 2\int_{\mathbb{R}^3} \mathrm{Tr}(\tau_{ij})\, dx. \tag{5.45}$$

Here $\mathrm{Tr}(\tau_{ij})$ denotes the trace of the tensor τ_{ij} which is given by

$$\mathrm{Tr}(\tau_{ij}) = \int_{\mathbb{R}^3} \frac{|p|^2}{\sqrt{e^{2\phi}+|p|^2}} f\, dp - \frac{1}{2}(\nabla_x\phi)^2 + \frac{3}{2}(\partial_t\phi)^2.$$

It follows that one can rewrite (5.45) as

$$\frac{d^2}{dt^2}\mathcal{E}_C = 2H + 2\mathcal{Q}(\partial_t\phi, \nabla_x\phi) - 2\int_{\mathbb{R}^3} \mu(t,x)\, dx, \tag{5.46}$$

where μ is minus the right hand side of

$$\partial_t^2\phi - \Delta_x\phi = -e^{2\phi}\int_{\mathbb{R}^3} f\, \frac{dp}{\sqrt{e^{2\phi}+|p|^2}}\,. \tag{5.47}$$

and $\mathcal{Q}$ is the quadratic operator

$$\mathcal{Q}(\partial_t\phi, \nabla_x\phi) = \int_{\mathbb{R}^3} \left((\partial_t\phi)^2 - (\nabla_x\phi)^2\right)\, dx.$$

By means of the identity $(\partial_t\phi)^2 = \partial_t(\phi\partial_t\phi) - \phi\partial_t^2\phi$ and using (5.47) we have

$$\int_0^t \mathcal{Q}(\partial_t\phi, \nabla_x\phi)\, ds = \int_0^t\int_{\mathbb{R}^3} \mu\,\phi\; dx\, ds + \frac{1}{2}\partial_t\int_{\mathbb{R}^3}\phi^2\, dx - \int_{\mathbb{R}^3}\phi_0\phi_1\, dx. \tag{5.48}$$

From (5.44), (5.46) and (5.48) we obtain

$$\begin{aligned}\mathcal{E}_C(t) \;&=\; \mathcal{E}_C(0) - \int_{\mathbb{R}^3}\phi_0^2\, dx + \int_{\mathbb{R}^3}\phi^2\, dx + 2Q_0t + Ht^2\\ &\quad +2\int_0^t\int_0^s\int_{\mathbb{R}^3}\mu\,(\phi-1)\, dx d\tau ds.\end{aligned} \tag{5.49}$$

Using the simple lower bound $\xi - 1 \geq -e^{-\xi}$, which holds for all $\xi \in \mathbb{R}$, the last term in (5.49) is bounded from below by

$$-2\int_0^t\int_0^s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f\, e^{\phi}}{\sqrt{e^{2\phi}+|p|^2}}\, dp dx d\tau ds \geq -Mt^2. \tag{5.50}$$

Substituting into (5.49) we finally obtain

$$\mathcal{E}_C(t) \geq \mathcal{E}_C(0) - \|\phi_0\|_{L^2}^2 + 2Q_0t + (H-M)t^2,$$

which yields the claim.

□

At this point, we propose as a dispersion measure the spatial variance of the unitary energy density function, $h(t,x)/H$, i.e.

$$\Delta_x(t) = \int_{\mathbb{R}^3} |x - \bar{h}(t)|^2\frac{h(t,x)}{H}\, dx \quad \text{where} \quad \bar{h}(t) = \int_{\mathbb{R}^3} x\frac{h(t,x)}{H}\, dx$$

Let us observe that this coincides with the statistical variance of a probability function, a well known dispersion tester (but obviously not Lorentz invariant in this setting, so that our results are of a totally tentative nature). The next result proves a time growth estimate for large time of this quantity under appropriate condition between mass, energy and momentum.

Proposition 2.5.3 *Let (f, ϕ) be a regular solution with mass M, energy H and momentum Q. Assume that*

$$H^2 - HM - |Q|^2 > 0. \tag{5.51}$$

Then, there exist a time instant $t_0 > 0$ and positive constants $0 < C_1 < C_2$ such that

$$C_1 t^2 \le \Delta_x(t) \le C_2 t^2 \qquad \forall t > t_0 \,.$$

Proof. Thanks to the conservation laws (5.34) we can prove that

$$\bar{h}(t) = \bar{h}(0) + t\frac{Q}{H} \,,$$

which allow us to compute the variance in an equivalent way

$$\Delta_x(t) = \int_{\mathbb{R}^3} x^2 \frac{h}{H}\, dx - |\bar{h}|^2 = \int_{\mathbb{R}^3} x^2 \frac{h}{H}\, dx - \frac{|Q|^2}{H^2} t^2 + 2\bar{h}(0) \cdot \frac{Q}{H} t - |\bar{h}(0)|^2 \,.$$

Using Theorem 2.5.2 we can deduce that

$$\Delta_x(t) \ge \frac{H - M}{H} t^2 - \frac{|Q|^2}{H^2} t^2 + 2\bar{h}(0) \cdot \frac{Q}{H} t - |\bar{h}(0)|^2 \,.$$

which concludes the proof. □

We argue that condition (5.51) is not optimal —that threshold is not Lorentz invariant— and maybe it could be possible to change it by:

$$H^2 - M^2 - |Q|^2 > 0, \tag{5.52}$$

which is a Lorentz invariant condition. According to the transformation law of the total momentum Q (Lemma 2.5.1), the Lorentz transformation that makes Q to vanish, i.e. that moves the reference frame to the center of mass system[1], is the transformation $\mathcal{L}_u$ with $u = Q/\sqrt{H^2 - |Q|^2}$. The energy of the transformed solution is minimal and values $\sqrt{H^2 - |Q|^2}$. If we apply Theorem 2.2.2 to the new solution (f_u, ϕ_u) the dispersion condition (5.51) reduces to:

$$H[f_u, \phi_u] - M[f_u] > 0,$$

that written in terms of the original solution f, ϕ becomes into (5.52) thanks to the fact that $|Q[f, \phi]| \le H[f, \phi]$. Yet we are not able to show that, if (f_u, ϕ_u) is a dispersive solution in the sense of Proposition 2.5.3, then any Lorentz Transformation of this one would also be dispersive, as it is the case for the Vlasov–Poisson system.

[1]To be more precise, this is called the *center of momentum system*. In Relativity there is no general acceptance on the concept of center of mass.

Chapter 3

Properties of static solutions

3.1 Introduction and main results

In this chapter we shall investigate certain conditions (called mass-energy bounds or virial inequalities) required for the existence of steady states to the relativistic models so far considered. It is well known in fact that static solutions of the Vlasov–Poisson system, which correspond to equilibrium configurations of a galaxy, have negative energy. The same property cannot of course be true for the relativistic models, since the energy in the latter case is always positive. We seek for extensions of this fact to the relativistic setting.

Now we give a brief explanation on how we prove our main results. As a first step we employ the vector fields multipliers method to the local conservation laws for the Nordström–Vlasov and the Einstein–Vlasov system to establish a virial identity which has to be satisfied by all time dependent solutions. These identities are of independent interest and could be useful to derive space-time (Morawetz type) estimates for the evolution problem. The virial identities restricted to time independent solutions give rise, after applying some simple bounds on the moments of the distribution f, to the virial inequalities (2.3) and (4.29). The results of this chapter can be found in [62]. Before considering the relativistic models in detail we present the role of the virial identities in the case of steady states to the Vlasov–Poisson system.

3.1.1 The classical case: Vlasov–Poisson system

A galaxy in equilibrium can be assumed to be described by steady states solutions of the Vlasov–Poisson system. We distinguish between two types of steady states: static solutions and traveling steady states. The formers are defined as time independent solutions of the Vlasov–Poisson system and have total momentum $Q = 0$. A solution f is a *traveling steady state* (with total momentum $Q \neq 0$) if $f \circ \mathcal{G}_u$, where $u = Q/M$, is a time independent solution of the Vlasov–Poisson system (i.e., a static solution). Our interest on traveling steady states is motivated by the fact that their energy provides a lower limit for the energy of totally dispersive solutions, as stated in Proposition 2.3.1 of Chapter 2. Moreover, the non-linear stability theorems proved for the Vlasov–Poisson system consider the traveling steady states as possible perturbations of a static equilibria, see [66, 180, 206] and references therein.

A fundamental property shared by all static solutions of the Vlasov–Poisson system is that of having negative energy. The proof goes as follows. Any sufficiently regular solution of the Vlasov–Poisson system satisfies the dilation identity:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} x\cdot v\, f\, dvdx = H + E_{\text{kin}}\,,$$

as it follows by direct computation. If f is a static solution, then the previous identity implies the *virial* relation $H = -E_{\text{kin}}$, which yields that

$$H < 0, \quad \text{for static solutions of the Vlasov–Poisson system.} \tag{1.1}$$

For traveling steady states, we just apply to (1.1) a Galilean transformation with $u = Q/M$ and we obtain

$$H < \frac{|Q|^2}{2M}\,, \quad \text{for traveling steady states of the Vlasov–Poisson system.} \tag{1.2}$$

Our purpose in this chapter is to extend these fundamental inequalities to the relativistic case.

3.2 The Nordström–Vlasov case

In the case of the Nordström–Vlasov system, the generalization of (1.1) is that the energy of regular steady states is bounded by their mass, i.e.

$$H \leq M. \tag{2.3}$$

Furthermore, the counterpart to (1.2) for traveling steady states is

$$\sqrt{H^2 - |Q|^2} \leq M.$$

Moreover the equality sign could only hold for steady states with unbounded support. We also remark that the bound $H < M$, which holds for all regular and compactly supported static solutions of the Nordström–Vlasov system, is crucial in the proof of orbital stability of the polytropic steady states established in [66].

These results will be proved in paragraph 3.2.2, after a suitable virial identity for time-dependent solutions is established.

3.2.1 Virial identities for time dependent solutions

We pointed out in Chapter 2 that the local conservation laws for the Nordström–Vlasov can be used to deduce the following integral identity:

$$\int_{\partial\Omega} T^{\mu}_{\ \nu}\xi^{\nu} n_{\mu} d\sigma = \int_{\Omega} T^{\mu}_{\ \nu}\partial_{\mu}\xi^{\nu} dtdx, \tag{2.4}$$

being $\xi^{\mu} = \xi^{\mu}(t,x)$ a vector field, Ω a compact spacetime region with piecewise differentiable boundary $\partial\Omega$, n_{μ} denoting the exterior normal vector field to the boundary and $d\sigma$ the invariant volume measure thereon. We claimed there that this identity provides a mean not only to study the conserved quantities of the system, but to deduce some other properties of its solutions; here we will pursue such an application.

The identities obtained from (2.4) upon a specific choice of the vector field multiplier go under the general name of *virial identities*. We prove here one that applies to *regular asymptotically flat solutions*. By this we mean that $f \in C^1([0,+\infty[\times\mathbb{R}^6)$, $\phi \in C^2([0,+\infty[\times\mathbb{R}^3) \cap L^\infty_{loc}([0,+\infty, L^2(\mathbb{R}^3))$, the mass and energy are finite and

$$\lim_{R\to\infty}\int_{S_R} h(t,x)dS_R = 0,\ \forall\, t\in\mathbb{R}. \tag{2.5}$$

By ω we shall denote the outward unit normal to $S_R = \{x : |x| = R\}$, and dS_R stands for the invariant volume measure on S_R. Moreover we denote by $\chi(r)$, $r > 0$, a function that satisfies:

$$\chi \in C^2\,,\quad \chi' \in L^\infty\,,\quad \frac{\chi}{r} \in C^2 \cap L^\infty\,. \tag{2.6}$$

Lemma 3.2.1 *Let*

$$\mathcal{I}(t) = \int_{\mathbb{R}^3} \chi(r)\left(q\cdot\omega - r^{-1}\phi\,\partial_t\phi\right)dx,\ r = |x|\,.$$

For all regular asymptotically flat solutions of (4.32) *the following identity holds:*

$$\begin{aligned}\frac{d\mathcal{I}}{dt} =& \int_{\mathbb{R}^3}\chi'\,h\,dx + \int_{\mathbb{R}^3}\frac{\chi}{r}e^{2\phi}(\phi-1)\int_{\mathbb{R}^3}\frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dpdx\\ &+\int_{\mathbb{R}^3}\left(\frac{\chi}{r}-\chi'\right)\left[|\omega\wedge\nabla_x\phi|^2 + \int_{\mathbb{R}^3}\frac{|\omega\wedge p|^2+e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\,f\,dp\right]dx\\ &-\frac{1}{2}\int_{\mathbb{R}^3}\frac{\chi''}{r}\phi^2dx.\end{aligned} \tag{2.7}$$

Proof. In (2.4) we use $\Omega = [0,T]\times B(R)$, where $B(R) = \{x : |x| \le R\}$ and

$$\xi^\mu\ :\ \xi^0 = 0,\ \xi^i = \chi(r)\omega^i\,.$$

We obtain

$$\begin{aligned}\left[\int_{B(R)}\chi(r)q\cdot\omega\,dx\right]_0^T =& \int_0^T\int_{S(R)}\chi(r)\tau_{ij}\omega^i\omega^j dS_R dt\\ &+\int_0^T\int_{B(R)}\left[\left(\chi'-\frac{\chi}{r}\right)\tau_{ij}\omega^i\omega^j + \frac{\chi}{r}\delta^{ij}\tau_{ij}\right]dx\,dt,\end{aligned} \tag{2.8}$$

where for any function $g(t)$ we denote $[g(t)]_0^T = g(T) - g(0)$. Using the bound $|\tau_{ij}\omega^i\omega^j| \le 3h$ and (2.5) we get

$$\left|\int_{S(R)}\chi(r)\tau_{ij}\omega^i\omega^j dS_R\right| \le 3\|\chi\|_\infty\int_{S(R)}h\,dS_R \to 0,\ R\to\infty.$$

Then, letting $R\to\infty$ in (2.8) we obtain

$$\left[\int_{\mathbb{R}^3}\chi(r)q\cdot\omega\,dx\right]_0^T = \int_0^T\int_{\mathbb{R}^3}\left[\left(\chi'-\frac{\chi}{r}\right)\tau_{ij}\omega^i\omega^j + \frac{\chi}{r}\delta^{ij}\tau_{ij}\right]dx\,dt,$$

whence

$$\frac{d}{dt}\int_{\mathbb{R}^3}\chi(r)q\cdot\omega\,dx=\int_{\mathbb{R}^3}\left[\left(\chi'-\frac{\chi}{r}\right)\tau_{ij}\omega^i\omega^j+\frac{\chi}{r}\delta^{ij}\tau_{ij}\right]dx.$$

We compute

$$\delta^{ij}\tau_{ij}=\int_{\mathbb{R}^3}\frac{|p|^2 f\,dp}{\sqrt{e^{2\phi}+|p|^2}}+\frac{3}{2}(\partial_t\phi)^2-\frac{1}{2}|\nabla_x\phi|^2$$

and

$$\tau_{ij}\omega^i\omega^j=\int_{\mathbb{R}^3}\frac{(\omega\cdot p)^2 f\,dp}{\sqrt{e^{2\phi}+|p|^2}}+(\omega\cdot\nabla_x\phi)^2+\frac{1}{2}[(\partial_t\phi)^2-|\nabla_x\phi|^2]\,.$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^3}\chi(r)q\cdot\omega\,dx=\int_{\mathbb{R}^3}\frac{\chi}{r}\left[\int_{\mathbb{R}^3}\frac{|p|^2 f\,dp}{\sqrt{e^{2\phi}+|p|^2}}+\frac{3}{2}(\partial_t\phi)^2-\frac{1}{2}|\nabla_x\phi|^2\right]dx$$
$$+\int_{\mathbb{R}^3}\left(\chi'-\frac{\chi}{r}\right)\left[\int_{\mathbb{R}^3}\frac{(\omega\cdot p)^2 f\,dp}{\sqrt{e^{2\phi}+|p|^2}}+(\omega\cdot\nabla_x\phi)^2+\frac{1}{2}[(\partial_t\phi)^2-|\nabla_x\phi|^2)\right]dx.$$

Using that $|\omega\wedge y|^2=|y|^2-|\omega\cdot y|^2$ holds for all vectors $y\in\mathbb{R}^3$, we can rewrite the previous equation as

$$\frac{d}{dt}\int_{\mathbb{R}^3}\chi(r)q\cdot\omega\,dx=\int_{\mathbb{R}^3}\frac{\chi}{r}((\partial_t\phi)^2-|\nabla_x\phi|^2)\,dx-\int_{\mathbb{R}^3}\chi'\int_{\mathbb{R}^3}\frac{e^{2\phi}f\,dp}{\sqrt{e^{2\phi}+|p|^2}}\,dx$$
$$+\int_{\mathbb{R}^3}\chi' h\,dx+\int_{\mathbb{R}^3}\left(\frac{\chi}{r}-\chi'\right)\left(\int_{\mathbb{R}^3}\frac{|\omega\wedge p|^2 f\,dp}{\sqrt{e^{2\phi}+|p|^2}}+|\omega\wedge\nabla_x\phi|^2\right)dx. \tag{2.9}$$

Moreover, using the field equation (4.32b) for ϕ and integrating by parts twice, we find

$$\frac{d}{dt}\int_{B(R)}\frac{\chi}{r}\,\phi\,\partial_t\phi\,dx=\int_{B(R)}\frac{\chi}{r}\left((\partial_t\phi)^2-|\nabla_x\phi|^2-\phi\int_{\mathbb{R}^3}\frac{e^{2\phi}f\,dp}{\sqrt{e^{2\phi}+|p|^2}}\right)dx$$
$$+\frac{1}{2}\int_{B(R)}\Delta\left(\frac{\chi}{r}\right)\phi^2\,dx+\int_{S(R)}\frac{\chi}{r}\phi\,\omega\cdot\nabla_x\phi\,dS_R$$
$$-\frac{1}{2}\int_{S(R)}\omega\cdot\nabla\left(\frac{\chi}{r}\right)\phi^2 dS_R\,. \tag{2.10}$$

Applying the Cauchy-Schwartz inequality, the regularity of the solution and the assumptions on χ, we obtain

$$\left|\int_{S(R)}\frac{\chi}{r}\phi\,\omega\cdot\nabla_x\phi\,dS_R\right|\leq C\|\phi\|_{L^2(S(R))}\|\nabla_x\phi\|_{L^2(S(R))}$$
$$\leq C\sqrt{\int_{S(R)}h(t,x)\,dS_R}\to 0,\ R\to\infty,$$

and

$$\left|\int_{S(R)}\omega\cdot\nabla\left(\frac{\chi}{r}\right)\phi^2 dS_R\right|=\frac{1}{R}\int_{S(R)}\left|\chi'-\frac{\chi}{r}\right|\phi^2 dS_R\leq\frac{C}{R}\,,$$

where C is a constant independent from R. Thus taking the limit $R \to \infty$ in (2.10) we get

$$-\frac{d}{dt}\int_{\mathbb{R}^3}\frac{\chi}{r}\,\phi\,\partial_t\phi\,dx = -\int_{\mathbb{R}^3}\frac{\chi}{r}\left((\partial_t\phi)^2 - |\nabla_x\phi|^2 - \phi\int_{\mathbb{R}^3}\frac{e^{2\phi}f\,dp}{\sqrt{e^{2\phi}+|p|^2}}\right)dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^3}\Delta\left(\frac{\chi}{r}\right)\phi^2\,dx. \tag{2.11}$$

The quantity $\Delta\left(\frac{\chi}{r}\right)$ is nothing but $\frac{\chi''}{r}$. The sum of (2.9) and (2.11) yields the desired result. □

3.2.2 Virial inequalities for steady states

As in the Vlasov–Poisson case, we distinguish between two types of steady states. Static solutions, which are defined as time independent solutions of the Nordström–Vlasov system, and traveling steady states, which are defined as solutions $f(t,x,p)$ such that $f \circ \mathcal{L}_u$, where $u = Q/\sqrt{H^2 - |Q|^2}$, is a time independent solution of the Nordström–Vlasov system (i.e., a static solution). For static solutions one has $Q = 0$, whereas $Q \neq 0$ for traveling steady states. Note that for static solutions (that vanish at infinity) the field is determined by f through a non-linear Poisson equation. Thus when we refer to a steady state solution we mean simply the distribution function f. The main goal of this paragraph is to prove the following property of steady states to the NV system.

Theorem 3.2.2 *Let f be a static regular asymptotically flat solution of the NV system. Then*

$$H \le M. \tag{2.12}$$

Traveling steady states satisfy $\sqrt{H^2 - |Q|^2} \le M$. Moreover, equality in (2.12) *implies that the support of the static solution is unbounded.*

Remark 3.2.3 Note that in the case of the Vlasov–Poisson system the supremum of the steady states energy coincides with the infimum energy of totally dispersive time dependent solutions. The analogous statement for the Nordström–Vlasov system is currently not known, due to the difficulties in defining a Lorentz invariant concept of total dispersion.

Proof. The statement on traveling steady states follows by applying the Lorentz transformation $\mathcal{L}_u$ with $u = Q/\sqrt{H^2 - |Q|^2}$ to the inequality for static solutions, thus it suffices to prove the latter. To this purpose consider a function χ that, in addition to (2.6), satisfies

$$\frac{\chi}{r} - \chi' \ge 0, \quad \chi'' \le 0. \tag{2.13}$$

Next we observe the simple inequality $y - 1 \ge -e^{-y}$, with equality if and only if $y = 0$. Using

$$\phi - 1 \ge -e^{-\phi} \tag{2.14}$$

in the identity (2.7) we obtain

$$\frac{d\mathcal{I}}{dt} \ge \int_{\mathbb{R}^3}\left(\chi' h - \frac{\chi}{r}\rho\right)dx.$$

In particular, for time independent solutions we have

$$\int_{\mathbb{R}^3} \Big(\chi' h - \frac{\chi}{r}\rho\Big)\, dx \leq 0. \tag{2.15}$$

Let $R > 0$ and consider the function $\chi(r) = \chi_R(r)$ given by

$$\chi(r) = \begin{cases} r & \text{for } r \leqslant R\,, \\ 3R - \frac{3R^2}{r} + \frac{R^3}{r^2} & \text{for } r > R\,. \end{cases}$$

This function satisfies the properties (2.6) and (2.13). The left hand side of (2.15) becomes

$$\begin{aligned} \int_{\mathbb{R}^3} \Big(\chi' h - \frac{\chi}{r}\rho\Big)\, dx &= \int_{B(R)} (h-\rho)\, dx + \int_{B(R)^c} \Big(\chi' h - \frac{\chi}{r}\rho\Big) \\ &\geq \int_{B(R)} (h-\rho)\, dx - C\left(\int_{B(R)^c} h\, dx + \int_{B(R)^c} \rho\, dx\right) \\ &= \int_{B(R)} (h-\rho)\, dx + \varepsilon(R), \end{aligned} \tag{2.16}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$. Thus, assuming $H > M$, there exists $R_0 > 0$ such that $\varepsilon(R) < (H - M)/4$ and $\int_{B(R)}(h-\rho)\, dx > (H-M)/2$, for all $R > R_0$, whence

$$\int_{\mathbb{R}^3} \Big(\chi' h - \frac{\chi}{r}\rho\Big)\, dx > \frac{1}{4}(H-M) > 0,$$

which contradicts (2.15). This concludes the proof of (2.12). To prove that the strict inequality holds for static solutions with compact support, we observe that in the latter case the field ϕ never vanishes in the support of f (it is strictly negative) and thus the stronger inequality $\phi - 1 > -e^{-\phi}$ holds instead of (2.14). Thus also the inequality in (2.15) is strict. Since the last member of (2.16) goes to zero for $R \to \infty$ when $H = M$, the claim follows. □

Theorem 3.2.2 improves a similar result proved in [66] in two aspects. Firstly, in [66] the fact that the strict inequality holds for compactly supported steady states was overlooked. Secondly the result presented here requires less decay than the inequality proved in [66] and therefore applies to more general steady states. In particular, this result allows to remove some technical hypothesis in the stability result obtained in [66].

3.3 The spherically symmetric Einstein–Vlasov system

We remark that the Vlasov–Poisson system ceases to be valid as a physical model when the particles (stars) move with large velocities (of the order of the speed of light) or in the presence of very massive galaxies, since then relativistic effects become important. Typical relativistic effects are the redshift of the luminous signals emitted by a galaxy and the formation of black holes. The model which is currently accepted to represent the physically correct relativistic generalization of the Vlasov–Poisson system is the Einstein–Vlasov system, which we shall introduce now, where Poisson's equation is substituted by Einstein's equations of General Relativity. The Vlasov–Poisson system is recovered in the limit $c \to \infty$ [186]. As compared to the Vlasov–Poisson system, the Einstein–Vlasov system is far more complicated and less understood.

3.3.1 Description of the model

Let us first formulate the Einstein–Vlasov system (EV for short) in general local coordinates; for more details, consult [181], a detailed derivation of this system can be found in [90]. Let M be a four-dimensional spacetime manifold with local coordinates x^α. On M a Lorentz metric is given so that the four-dimensional line element is given by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta.$$

The metric determines the Christoffel symbols by means of

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_{x^\beta} g_{\gamma\delta} + \partial_{x^\gamma} g_{\beta\delta} - \partial_{x^\delta} g_{\beta\gamma} \right)$$

and given these we can write down the geodesics equations

$$\frac{dX^\alpha}{d\tau} = P^\alpha, \quad \frac{dP^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} P^\beta P^\gamma.$$

These are the general relativistic counterpart of Newton's equations of motion and in particular describe the motion of particles and light rays which are subject only to the effects of gravity, as represented by the metric $g_{\alpha\beta}$. Here τ is an affine parameter of the geodesics corresponding to proper time. The phase space on which the particle density is to be defined is an appropriate subset of the tangent bundle TM, coordinated by (x^α, p^α), where p^α denotes the coordinate basis components of tangent vectors. Then the Vlasov equation reads

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0.$$

It is easily seen that the quantity $g_{\alpha\beta} p^\alpha p^\beta$ is conserved along solutions of the geodesics equations. In case of timelike geodesics this corresponds to the conservation of the rest mass of the particles; we will consider only particles with rest mass 1. To do so choose coordinates such that

$$ds^2 = g_{00}(dx^0)^2 + g_{ab} dx^a dx^b.$$

The requirement that our particles have rest mass 1 restricts the distribution function f to

$$PM = \{g_{\alpha\beta} p^\alpha p^\beta = -1, \ p^0 > 0\}, \tag{3.17}$$

a seven-dimensional submanifold of the tangent bundle. It can be coordinated by (x^0, x^a, p^b), the component p^0 depending on the others via

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ab} p^a p^b}.$$

The coordinate x^0 can be thought of as a timelike coordinate; we rename it $t := x^0$. In terms of coordinate time, the geodesic equations transform to

$$\frac{dX^a}{dt} = \frac{P^a}{P^0}, \quad \frac{dP^a}{dt} = -\frac{1}{P^0} \Gamma^\alpha_{\beta\gamma} P^\beta P^\gamma.$$

In this way Vlasov's equation for f, now defined on PM and written as a function of (t, x^a, p^b), reads

$$\frac{\partial f}{\partial t} + \frac{p^a}{p^0} \partial_{x^a} f - \frac{1}{p^0} \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^a} f = 0. \tag{3.18}$$

As in the Newtonian case now we are to couple Vlasov's equation to the field equation, here Einstein's equation

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where the Einstein tensor $G_{\alpha\beta}$ is a second-order differential expression in the metric $g_{\alpha\beta}$ and the energy momentum tensor represents the matter content of the spacetime and is determined by f. More precisely, for our collisionless kinetic matter model we have

$$T_{\alpha\beta} = \int p_\alpha p_\beta f |g|^{1/2} \frac{dp^1 dp^2 dp^3}{-p^0},$$

$|g|$ denotes the determinant of the metric; indices are lowered and raised using the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. The coupling of Einstein's equation to the Vlasov equation is referred to as the Einstein–Vlasov system.

This model is very interesting from the physical point of view. One of the main reasons for this interest is concerned with the issue of the choice of the matter model in General Relativity. We could describe matter as a perfect fluid or as dust, for instance, but these matter models suffer a serious drawback: they develop singularities (even in the Newtonian case) that have nothing to do with General Relativity —like formation of shocks. This breakdown of the matter model might prevent the extension of the solution, maybe up to genuine spacetime singularities, that are the ones we are interested in. Opposed to this, the current results tend to indicate that for the collisionless gas matter model these nasty features won't turn up, so that any singularity in the solutions must be due to truly relativistic effects.

Present-day mathematical techniques do not allow to cope with the mathematical complexity of the Einstein–Vlasov system as it stands. Imposing some kind of symmetry to the solutions reduces the difficulty of the problem and for the spherically symmetric case the resulting system is considerably simpler, although still being quite complicated.

We are going to introduce the Einstein–Vlasov system in this simplified setting. To obtain an even simpler formulation we will pass to certain non-canonical coordinates on momentum space (to an ortonormal frame instead of a reference frame). We define

$$v^a = p^a + (e^\lambda - 1)\frac{x\cdot p}{r}\frac{x^a}{r},$$

the inverse transform being

$$p^a = v^a + (e^\lambda - 1)\frac{x\cdot v}{r}\frac{x^a}{r}.$$

The function f will be spherically symmetric in the sense that $f(t,x,v) = f(t,Ax,Av)$, for all $A \in SO(3)$. Then, the spherically symmetric Einstein–Vlasov system in Schwarzschild coordinates is given by the following set of equations (in units $G = c = 1$, which is the standard choice in this field):

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\sqrt{1+|v|^2}}\cdot\nabla_x f - \left(\lambda_t \frac{x\cdot v}{r} + e^{\mu-\lambda}\mu_r\sqrt{1+|v|^2}\right)\frac{x}{r}\cdot\nabla_v f = 0, \qquad (3.19)$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 h, \qquad (3.20a)$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p^{\text{rad}}\ , \qquad (3.20b)$$

$$\lambda_t = -4\pi r e^{\lambda+\mu} q, \tag{3.20c}$$

$$e^{-2\lambda}\left(\mu_{rr} + (\mu_r - \lambda_r)(\mu_r + \frac{1}{r})\right) - e^{-2\mu}\left(\lambda_{tt} + \lambda_t(\lambda_t - \mu_t)\right) = 4\pi p^{\tan}, \tag{3.20d}$$

where

$$h(t,r) = \int_{\mathbb{R}^3} \sqrt{1+|v|^2} f dv, \quad p^{\rm rad}(t,r) = \int_{\mathbb{R}^3} \left(\frac{x\cdot v}{r}\right)^2 f \frac{dv}{\sqrt{1+|v|^2}},$$
$$q(t,r) = \int_{\mathbb{R}^3} \frac{x\cdot v}{r} f dv, \quad p^{\tan}(t,r) = \int_{\mathbb{R}^3} \left|\frac{x\wedge v}{r}\right|^2 f \frac{dv}{\sqrt{1+|v|^2}}.$$

The functions $p^{\rm rad}$ and $p^{\tan}$ are the radial and tangential pressure; h is the energy density and q the local momentum density[1]. As usual, $f \geq 0$ is the distribution function of particles (stars) in the phase space in the coordinates $t\in\mathbb{R}$, $x\in\mathbb{R}^3$, $v\in\mathbb{R}^3$. We saw that the variable v is not the canonical momentum of the particles, the latter being denoted by p in the previous sections. For a function $g = g(t,r)$, $r = |x|$, we denote by g_t and g_r the time and radial derivative, respectively. By abuse of notation, $g(t,r) = g(t,x)$ for any spherically symmetric function. The functions λ, μ determine the metric of the space-time according to

$$ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2 d\omega^2, \tag{3.21}$$

where $d\omega^2$ is the standard line element on the unit sphere. The system is supplied with the boundary conditions

$$\lim_{r\to\infty}\lambda(t,r) = \lim_{r\to\infty}\mu(t,r) = \lambda(t,0) = 0, \tag{3.22}$$

which define the asymptotically flat solutions (meaning that we are considering isolated systems) with a regular center, and the initial condition

$$0 \leq f(0,x,v) = f^0(x,v),\ f^0(Ax,Av) = f^0(x,v),\ \forall A\in SO(3).$$

We also remark that the equation

$$\lambda_r + \mu_r = 4\pi r e^{2\lambda}(h + p^{\rm rad}), \tag{3.23}$$

follows by (3.20a)-(3.20b); by (3.23) we have $\lambda_r + \mu_r \geqslant 0$ and so, by (3.22),

$$0 \geqslant \lambda + \mu \geqslant \mu(0,t). \tag{3.24}$$

The ADM mass (or energy) H and the total rest mass M of a solution to the spherically symmetric Einstein–Vlasov system are defined by

$$H = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sqrt{1+|v|^2}\, f\, dvdx, \quad M = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\lambda} f\, dvdx \tag{3.25}$$

and are constant for regular solutions (the other two conserved quantities that were considered in the previous section, the linear momentum Q and angular momentum

[1]This differs from the standard notation for the Einstein–Vlasov system.

Ω, are identically zero in the present context by spherical symmetry). Related to the ADM mass we have the quasi-local mass, defined by

$$m(t,r) = 4\pi \int_0^r s^2 h(t,s)\, ds = \frac{r}{2}\left(1 - e^{-2\lambda}\right), \tag{3.26}$$

where we used (3.20a). Thus $\lim_{r\to\infty} m(t,r) = H$.

For later convenience, we recall that the non-zero Christoffel symbols for the metric (3.21) are given by

$$\Gamma^0{}_{00} = \mu_t\,, \quad \Gamma^0{}_{0a} = \mu_r \frac{x_a}{r}\,, \quad \Gamma^0{}_{ab} = e^{2(\lambda-\mu)}\lambda_t \frac{x_a x_b}{r^2}\,,$$

$$\Gamma^a{}_{00} = e^{-2(\lambda-\mu)}\mu_r \frac{x^a}{r}\,, \quad \Gamma^a{}_{0b} = \lambda_t \frac{x^a x_b}{r^2}\,,$$

$$\Gamma^c{}_{ab} = \lambda_r \frac{x^c x_b x_a}{r^3} + \frac{1 - e^{-2\lambda}}{r}\left(\delta^c_b - \frac{x_b x^c}{r^2}\right)\frac{x_a}{r}\,.$$

Note also that $|g| = e^{2\lambda+2\mu}$ is the determinant of the metric.

The stress-energy tensor $T^{\mu\nu}$ for Vlasov matter in spherical symmetry is given by

$$T^{00} = e^{-2\mu} h, \quad T^{0a} = e^{-\lambda-\mu} q\, \frac{x^a}{r}, \tag{3.27a}$$

$$T^{ab} = e^{-2\lambda} p^{\mathrm{rad}} \frac{x^a x^b}{r^2} + \frac{1}{2} p^{\tan}\left(\delta^{ab} - \frac{x^a x^b}{r^2}\right) \tag{3.27b}$$

and satisfies the conservation law

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.28}$$

Here ∇_μ stands for the covariant derivative. These identities (3.28) are a consequence of the Vlasov equation alone, see [90].

3.3.2 The Cauchy problem and related results

The first thing that we want to state clearly is the fact that the existence and uniqueness of global regular solutions to the Cauchy problem for the system (3.19)–(3.20) is open for general initial data. (The very definition of what do we mean by Cauchy problem in the general setting is fairly complicated, see [5] for instance; the difficulties come from the fact that the spacetime is not given in advance but is itself part of the solution. Once we stick to a fixed set of coordinates the Cauchy problem can be understood in the usual way.) Let us trace back the developments around this question.

Of paramount importance is the result of [79] stating local solvability without any symmetry assumption, but a clear mechanism for the continuation or breakdown of the solution is lacking. The next important result for our model is that of [185], ensuring local existence of classical solutions in Schwarzschild coordinates, a continuation criterion analogous to that for the Vlasov–Poisson system (1.10) and global existence of solutions related to small initial data, which exhibit dispersive behavior. The mechanism that prevents a solution from existing globally in Schwarzschild coordinates was better understood thanks to the results in [189], which entail that this failure must be due to the apparition of a singularity at the very center of the system. Thus global existence can be ensured as long as the distribution of matter stays away from the origin. This spirit was captured in [10], where global existence for outgoing shells of matter was proved for the spherically symmetric Einstein–Vlasov system in maximal areal coordinates (which coincide with Schwarzschild coordinates for static solutions). The results in [8] try also to give some insight into this very complicated question, that is not completely understood yet.

The theory about steady states is under development still, too. Several families of steady states were constructed in [182, 187] relying mostly on the polytropic ansatz (this will be explained in Section 3.5). Later these results were extended to include also a family of static shell solutions [183]. The next interesting result was that in [188], giving a sufficient condition for a steady state solution to be compactly supported. Meanwhile, the first preliminary studies for the stability issue were performed by [231]; this question has not improved much since and remains one of the most challenging open problems for the Einstein–Vlasov system, which is of capital importance from the physical point of view. Another striking result obtained by the time is that of [210], which entails the failure of the natural analog of Jeans' theorem in the Einstein–Vlasov setting. Heuristics here tell that we can construct some strange steady states that are not globally functions of invariants of motion starting with a static configuration constituted by concentric and disjoint shells, removing some of the outer rings of matter and matching the result with the exterior Schwarzschild solution. These multi-peaked shells consisting on various disjoint rings of matter were further investigated in [12], together with a series of related topics such as the Buchdahl inequality and radius-mass spiral diagrams for steady states.

Partially due to the fact that the mathematical understanding of this system seems out of our reach, at least with the present-day techniques, a series of numerical studies have been devised. Let us mention here that in [11] concerning the stability of steady states and the investigation of critical collapse performed in [190]. A classical work about numerical relativity for kinetic models is [214].

3.4 The Einstein–Vlasov case

Throughout this section we assume that f is a regular solution of (3.19)-(3.20) in the sense defined in [185]. In particular, $f(t, x, v)$ is C^1 and has compact support in (x, v), for $t \in [0, T]$, and for any $T > 0$. For regular solutions, the metric coefficients are C^2 functions of their arguments.

We will derive an inequality that involves not only the energy (ADM mass, H) and the mass (rest mass, M) of the steady state, but also the central redshift Z_c:

$$Z_c \geq \left| \frac{M}{H} - 1 \right| . \tag{4.29}$$

Let us comment on the connections of this result with the theory so far developed for the Einstein–Vlasov system. The metric of the space-time for spherically symmetric static solutions of the Einstein–Vlasov system is determined, following the notation in Paragraph 3.3.1 of the present chapter, by two functions $\lambda(r) \geq 0$ and $\mu(r) \leq 0$ of the radial variable. The central redshift is defined in terms of the second one at the origin by $Z_c := e^{-\mu(0)} - 1$. It is the redshift of a photon emitted from the center of the galaxy. The estimate (4.29) can thus be seen as an upper bound for $\mu(0)$. Similarly, the celebrated Buchdahl's inequality in General Relativity [228] can be seen as an upper bound on the metric component $\lambda(r)$ for spherically symmetric steady states of the Einstein-matter equations. A quite general version of the Buchdahl inequality was proved recently in [6] and reads

$$\sup_{r\geq 0}\left(1-e^{-2\lambda(r)}\right)\leq\frac{8}{9}\,,\ \text{or equivalently}\ \sup_{r\geq 0}\lambda(r)\leq \ln 3\,. \tag{4.30}$$

For static shells the Buchdahl inequality is equivalent to a lower bound for the external radius. We will show that estimate (4.29) leads to an upper bound on the internal radius. We refer to [12] for an analytical/numerical investigation of the Buchdahl inequality in the context of the spherically symmetric Einstein–Vlasov system.

We do not know whether, as for the Vlasov–Poisson and the Nordström–Vlasov system, the inequality (4.29) could also be related to the problem of stability of spherically symmetric static solutions. This is a difficult question to answer, since the stability problem for the Einstein–Vlasov system is still poorly understood. However it is worth noticing that heuristic and numerical studies [214, 238, 239] indicate that the regime of stability of compact galaxies is indeed characterized by the central redshift and the fractional binding energy (defined as $1-H/M$). Moreover it was conjectured that the binding energy maximum along a steady state sequence signals the onset of instability. There are several numerical studies on the problem of stability for the spherically symmetric Einstein–Vlasov system; we refer to [11, 12, 190].

A last basic comment on (4.29) is that, as opposed to the inequalities that hold for steady states of the Vlasov–Poisson and the Nordström–Vlasov system, the bound (4.29) contains a quantity, the central redshift, which is not preserved along time dependent solutions. It is therefore not clear whether one can interpret (4.29) as the exact analog of the mass-energy inequalities for the steady states of the Vlasov–Poisson and Nordström–Vlasov system.

3.4.1 Virial identities for time dependent solutions

To begin with we derive an integral identity for the spherically symmetric Einstein–Vlasov system as we did in Lemma 3.2.1 for the Nordström–Vlasov system, i.e., using the vector fields multipliers method. Actually, the identity in Lemma 3.4.1 below is valid not only for the Einstein–Vlasov system, but for all matter models in spherical symmetry. This is due to the fact that equation

$$\nabla_\mu T^{\mu\nu}=0, \tag{4.31}$$

which is the starting point for deriving the integral identity, must be satisfied by all matter models for compatibility with the Einstein equations.

Multiplying the conservation law (4.31) by a vector field ξ^μ, integrating on a compact spacetime region Ω with piecewise differentiable boundary $\partial\Omega$ and applying the divergence theorem we obtain the integral identity

$$\int_{\partial\Omega}J^\mu\eta_\mu\,d\sigma_g=\int_\Omega T^{\mu\nu}\nabla_\mu\xi_\nu\,dg, \tag{4.32}$$

where η_μ is the normal covector related to the boundary, $J^\mu=T^\mu{}_\nu\xi^\nu=T^{\mu\nu}\xi_\nu$ is the current associated to the vector field ξ^μ and $\nabla_\mu\xi_\nu=\partial_\mu\xi_\nu-\Gamma^\sigma_{\mu\nu}\xi_\sigma$ is the covariant derivative of the vector field. Moreover dg is the invariant volume element on the spacetime and $d\sigma_g$ the invariant volume element induced on $\partial\Omega$.

Lemma 3.4.1 *Assume that* $(h, q, p^{\rm rad}, p^{\rm tan})$ *satisfy the compatibility condition*[2] (4.31), *where* $T_{\mu\nu}$ *is the stress-energy tensor* (3.27). *In addition, we assume that* $h(t,\cdot), q(t,\cdot)$, $p^{\rm rad}(t,\cdot)$, $p^{\rm tan}(t,\cdot)$, *have compact support. Given any smooth function* $\chi(t,r)$ *in* $W^{1,\infty}_{\rm loc}$ *and any solution of* (3.20) *define*

$$\mathcal{I}(t) = \int_{\mathbb{R}^3} \chi\, q(t,r)\, dx.$$

Then the following integral identity is verified:

$$\frac{d\mathcal{I}}{dt} = \int_{\mathbb{R}^3} \left[e^{\mu-\lambda} p^{\rm rad} \frac{\partial\chi}{\partial r} - e^{\mu-\lambda}\chi \left(h\mu_r + p^{\rm rad}\lambda_r - \frac{p^{\rm tan}}{r} \right) + q\left(\frac{\partial\chi}{\partial t} - 2\chi\lambda_t \right) \right] dx. \quad (4.33)$$

Proof. In (4.32) we use

$$\xi_0 = 0,$$
$$\xi_i = \chi(t,r)\frac{x_i}{r}.$$

After a long but straightforward computation we obtain

$$T^{\mu\nu}\nabla_\mu\xi_\nu = e^{-2\lambda} p^{\rm rad}\frac{\partial\chi}{\partial r} + e^{-\lambda-\mu} q \frac{\partial\chi}{\partial t} - \chi\left[e^{-2\lambda} h\mu_r + 2q\lambda_t e^{-\lambda-\mu} + e^{-2\lambda} p^{\rm rad}\lambda_r - p^{\rm tan}\frac{e^{-2\lambda}}{r} \right].$$

We will choose Ω to be the coordinate image of a cylinder $[0,T]\times B(R)$. In this fashion, we have that

$$\int_\Omega T^{\mu\nu}\nabla_\mu\xi_\nu\, dg = \int_0^T \int_{|x|\le R} \left[e^{\mu-\lambda} p^{\rm rad}\frac{\partial\chi}{\partial r} - e^{\mu-\lambda}\chi\left(h\mu_r + p^{\rm rad}\lambda_r - \frac{p^{\rm tan}}{r}\right) + q\left(\frac{\partial\chi}{\partial t} - 2\chi\lambda_t\right)\right] dx dt. \quad (4.34)$$

Now we compute the corresponding boundary integral in (4.32). First, the current reads

$$J^0 = q\chi e^{-\lambda-\mu},$$
$$J^a = e^{-2\lambda} p^{\rm rad}\chi(r)\frac{x^a}{r}.$$

Next we write $\partial\Omega = A_1 \cup A_2 \cup A_3$, where

- $A_1 = \{t = T, |x| \le R\}$. The outer unit normal is $e^{\mu(r,T)}dt$; the induced metric is $e^{2\lambda(r,T)}dr^2 + r^2 d\omega^2$, the volume element is $e^{2\lambda(r,T)}\, dx$.
- $A_2 = \{t = 0, |x| \le R\}$. The outer unit normal is $-e^{\mu(r,0)}dt$; the induced metric is $e^{2\lambda(r,0)}dr^2 + r^2 d\omega^2$, the volume element is $e^{2\lambda(r,0)}\, dx$.

[2] In the case of a perfect fluid, the compatibility condition (4.31) is the system of Euler equations.

- $A_3 = \{0 < t < T,\, |x| = R\}$. The outer unit normal has the form $-e^{\lambda(t,R)}\frac{x^i}{R}dx_i$. The metric is $ds^2 = -e^{2\mu(t,R)}dt^2 + R^2 d\omega^2$, the volume element is $e^{2\mu(t,R)}dS_R dt$, where dS_R is the surface element on the sphere of radius R.

Summing up we get

$$\int_{\partial\Omega} J^\mu \eta_\mu \, d\sigma_g = \int_{|x|\le R} q(T,r)\chi(T,r)\,dx - \int_{|x|\le R} q(0,r)\chi(0,r)\,dx \\ - \int_0^T \int_{|x|=R} p^{\tan}(t,R)\chi(t,R)e^{\mu(t,R)-\lambda(t,R)}\,dS_R\,dt. \tag{4.35}$$

Having assumed that the matter quantities are compactly supported in the variable r, the boundary integral vanishes in the limit $R \to \infty$, whereas the other integrals remain bounded[3]. Thus in the limit we obtain

$$\left[\int_{\mathbb{R}^3} q\chi\,dx\right]_0^T = \int_0^T \int_{\mathbb{R}^3} \left[e^{\mu-\lambda}p^{\text{rad}}\frac{\partial\chi}{\partial r} \right. \\ \left. -e^{\mu-\lambda}\chi\left(h\mu_r + p^{\text{rad}}\lambda_r - \frac{p^{\tan}}{r}\right) + q\left(\frac{\partial\chi}{\partial t} - 2\chi\lambda_t\right)\right] dxdt,$$

which is the integral version of (4.33). □

We shall now derive a particular case of the identity (4.33). First let us choose

$$\chi = e^{2\lambda}F(r),$$

for a smooth function F. We have $\partial_t\chi = 2\lambda_t e^{2\lambda}F(r)$ and then $\partial_t\chi - 2\chi\lambda_t = 0$. In this way equation (4.33) implies —recall the notation $[g]_0^T = g(T) - g(0)$—

$$\left[\int_{\mathbb{R}^3} qe^{2\lambda}F\,dx\right]_0^T = \int_0^T \int_{\mathbb{R}^3} e^{\mu+\lambda}\left[p^{\text{rad}}F' + F\left(\lambda_r p^{\text{rad}} - h\mu_r + \frac{p^{\tan}}{r}\right)\right] dxdt. \tag{4.36}$$

Note now that, using (3.23), (3.20b) and (3.26),

$$\lambda_r p^{\text{rad}} - h\mu_r = p^{\text{rad}}(\lambda_r + \mu_r) - \mu_r(p^{\text{rad}} + h) = (\lambda_r + \mu_r)\left(p^{\text{rad}} - \frac{\mu_r e^{-2\lambda}}{4\pi r}\right) \\ = -\frac{m}{4\pi r^3}(\lambda_r + \mu_r).$$

Then (4.36) becomes

$$\left[\int_{\mathbb{R}^3} qe^{2\lambda}F\,dx\right]_0^T = \int_0^T \int_{\mathbb{R}^3} e^{\mu+\lambda}\left[p^{\text{rad}}F' + p^{\tan}\frac{F}{r} - \frac{F}{r}\frac{m(\lambda_r + \mu_r)}{4\pi r^2}\right] dxdt. \tag{4.37}$$

For the integral in the right hand side we use that

$$-\int_0^T \int_{\mathbb{R}^3} e^{\mu+\lambda}\frac{F}{r}\frac{(\lambda_r + \mu_r)m}{4\pi r^2}\,dxdt = -\int_0^T \int_0^\infty \frac{de^{\lambda+\mu}}{dr}\frac{F}{r}m\,drdt \\ = \int_0^T \int_0^\infty \frac{d}{dr}\left(\frac{Fm}{r}\right)e^{\lambda+\mu}\,drdt - HT\left(\lim_{r\to\infty}\frac{F(r)}{r}\right).$$

[3]Of course the compact support condition can be replaced by a suitable decay assumption.

This leads to

$$\left[\int_{\mathbb{R}^3} q e^{2\lambda} F\, dx\right]_0^T = -HT\left(\lim_{r\to\infty}\frac{F(r)}{r}\right) + \int_0^T\int_{\mathbb{R}^3} e^{\lambda+\mu}\left[p^{\text{rad}}F' + p^{\tan}\frac{F}{r} + h\frac{F}{r} + \frac{m}{4\pi r^2}\frac{d}{dr}\left(\frac{F}{r}\right)\right] dxdt. \quad (4.38)$$

Finally for $F(r) = r$ we obtain

$$\left[\int_{\mathbb{R}^3} q e^{2\lambda} r\, dx\right]_0^T = -HT + \int_0^T\int_{\mathbb{R}^3} e^{\lambda+\mu}\left(p^{\text{rad}} + p^{\tan} + h\right) dxdt. \quad (4.39)$$

3.4.2 Virial inequalities for steady states

The existence of steady states solutions to the Einstein–Vlasov system is well understood, we refer to [12, 95] and the references therein. The identity (4.39) restricted to steady states implies

$$H = \int_{\mathbb{R}^3} e^{\lambda+\mu}(p^{\tan} + p^{\text{rad}} + h)\, dx. \quad (4.40)$$

The fundamental identity (4.40) can be proved directly using the Einstein equations for static spherically symmetric spacetimes, see [6, 7]. Our derivation has two advantages. Firstly, we obtained (4.40) as a special case of a more general identity which holds for time dependent solutions, see Lemma 3.4.1. Secondly, the technique of the vector fields multipliers, which we used to derive (4.40), can also be used on spacetimes which are not spherically symmetric and therefore our argument could be useful to prove generalizations of (4.40) for solutions with less symmetry. This identity leads naturally to a bound on the *central redshift*

$$Z_c = e^{-\mu(0)} - 1 \in [0, +\infty[$$

in terms of the mass-energy of the static solution. We consider only static solutions of the spherically symmetric Einstein–Vlasov system.

Proposition 3.4.2 *Let f be a static solution of the spherically symmetric Einstein–Vlasov system with compact support. Then the following inequality holds true*

$$e^{\mu(0)} \le \begin{cases} \frac{H}{M} & \text{if } H \le M \\ \frac{H}{2H-M} & \text{if } H \ge M \end{cases} \quad i.e \quad Z_c \ge \left|\frac{M}{H} - 1\right|. \quad (4.41)$$

Proof. Since μ is increasing, $\mu(r) \ge \mu(0)$ and so

$$\int_{\mathbb{R}^3} e^{\lambda+\mu}(p^{\text{rad}} + p^{\tan} + h)\, dx \ge e^{\mu(0)}\int_{\mathbb{R}^3} e^{\lambda} h\, dx \ge M e^{\mu(0)}\,.$$

Using this in (4.40) gives

$$e^{\mu(0)} \le \frac{H}{M}\,. \quad (4.42)$$

Moreover

$$p^{\text{rad}} + p^{\tan} + h = h + \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r}\right)^2 f \frac{dv}{\sqrt{1+|v|^2}} + \int_{\mathbb{R}^3} \left|\frac{x \wedge v}{r}\right|^2 f \frac{dv}{\sqrt{1+|v|^2}}$$
$$= 2h + \int_{\mathbb{R}^3} \left(\frac{|v|^2}{\sqrt{1+|v|^2}} - \sqrt{1+|v|^2}\right) f\, dv = 2h - \int_{\mathbb{R}^3} f \frac{dv}{\sqrt{1+|v|^2}}$$

Thus, since $\lambda + \mu \geqslant \mu(0)$ and $e^{\mu} \leq 1 \leq e^{\lambda}$,

$$\int_{\mathbb{R}^3} e^{\lambda+\mu}(p^{\text{rad}} + p^{\tan} + h)\, dx \geq e^{\mu(0)}(2H - M)$$

and so by (4.40),

$$e^{\mu(0)} \leq \frac{H}{2H - M}, \quad \text{when} \quad \frac{H}{M} > \frac{1}{2}. \tag{4.43}$$

The result follows from (4.42) and (4.43) and taking into account that

$$\frac{H}{M} \leq \frac{H}{2H - M}$$

is satisfied in the case $\frac{1}{2} < \frac{H}{M} \leq 1$. □

3.4.3 Shells and Jeans' type steady states

Let R be the radius support of the steady state. Using that μ is negative and increasing and that the steady state matches the Schwarzschild solution at $r = R$ we obtain the bound

$$e^{\mu(0)} \leq e^{\mu(R)} = \sqrt{1 - \frac{2H}{R}}. \tag{4.44}$$

The inequality (4.44) can be combined with (4.41) to obtain an upper bound on $e^{\mu(0)}$ in terms of R, H and M.

Now, we consider briefly an important class of steady states, namely the Jeans type steady states (see [231]). For these steady states the distribution function f has the form

$$f(x, v) = \psi(E, F), \text{ where } E = e^{\mu}\sqrt{1+|v|^2},\ F = |x \wedge v|^2. \tag{4.45}$$

Since the particles energy E and the angular momentum F are conserved quantities, the particle density (4.45) is automatically a solution of the Vlasov equation. The existence of Jeans type steady states is then obtained by replacing the ansatz $f = \psi(E, F)$ into the (time independent) Einstein equations and proving existence of global solutions for the resulting system of ODEs. We refer to [184] where this procedure is carried out for a large class of profiles ψ; moreover the Jeans type steady states constructed in [184] all have compact support and satisfy that

$$\exists\, E_0 \in]0, 1[\text{ such that } \psi = 0, \text{ for } E \geq E_0. \tag{4.46}$$

Thus E_0 is the maximum particle energy in the ensemble. The property (4.46) is necessary in order that the distribution function (4.45) be asymptotically flat and with finite energy. For Jeans type steady states one obtains a new estimate on $e^{\mu(0)}$ in a straightforward way:

$$H = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sqrt{1+|v|^2} f\, dvdx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\lambda-\mu} e^{\mu} \sqrt{1+|v|^2} e^{\lambda} f\, dvdx \le \frac{E_0}{e^{\mu(0)}} M,$$

whence

$$e^{\mu(0)} \le E_0 \frac{M}{H}\,. \tag{4.47}$$

In fact, for Jeans' type solutions we have [184]

$$E_0 = \sqrt{1-\frac{2H}{R}}$$

and thus, combining (4.47) with (4.44) we conclude that for Jeans' type steady states the inequality

$$e^{\mu(0)} \le \min\left\{1, \frac{M}{H}\right\} \sqrt{1-\frac{2H}{R}}$$

hods.

Consider now the case of a static shell [183]. Let f be a static shell solution of the spherically symmetric Einstein–Vlasov system with inner radius R_1 and outer radius R_2. Using (3.20b) we can write $\mu(0)$ as follows

$$\begin{aligned} \mu(0) &= -\int_0^\infty e^{2\lambda}\left(\frac{m}{r^2} + 4\pi r p^{\rm rad}\right) dr \\ &= -\int_0^\infty \frac{1}{1-2m/r}\left(\frac{m}{r^2} + 4\pi r p^{\rm rad}\right) dr. \end{aligned}$$

By Buchdahl's inequality (4.30), the identity (3.26) and the bound $p^{\rm rad} \le h$, we obtain

$$\begin{aligned} \mu(0) &\ge -9\int_{R_1}^\infty \left(\frac{m}{r^2} + 4\pi r p^{\rm rad}\right) dr \\ &\ge -9\int_{R_1}^\infty \left(\frac{H}{r^2} + 4\pi r h\right) dr \\ &\ge -\frac{9H}{R_1} - \frac{9}{R_1}\int_{R_1}^\infty 4\pi r^2 h\, dr \\ &= -\frac{18H}{R_1}. \end{aligned}$$

Now we use the upper estimates on $\mu(0)$ of the Proposition 3.4.2 to find

$$\mu(0) = \ln\left(\frac{1}{Z_c+1}\right) \le \ln\left(\frac{1}{\left|\frac{M}{H}-1\right|+1}\right).$$

Combining both estimates we obtain

$$R_1 \le \frac{18H}{\ln\left(\left|\frac{M}{H}-1\right|+1\right)}\,,$$

i.e., the inner radius of a static shell with given ADM energy and rest mass cannot be arbitrarily large.

3.5 The polytropic models

We close the chapter introducing explicit examples of static solutions to the models so far considered, to which the theory that we developed in the previous paragraphs applies. This story originates with the Vlasov–Poisson system, as several ideas from celestial mechanics concerning integrals of motion had a suitable translation that has proven very fruitful to the construction of steady configurations. Later these mechanisms were brought to the relativistic domain.

The Vlasov–Poisson system admits lots of static solutions, so there is chance to balance the advantages and disadvantages of each over the overwhelming catalog at our disposal [180]. Maybe the most popular family of such steady states is that of spherical polytropes. They are reasonable simple solutions and easy to deal with, yet they are realistic enough to give good approximations to certain self-gravitating systems. Not only in these aspects lies their importance, as most of the mathematical theory for static solutions has been developed around the findings that were made for these special solutions. Besides, they have provided the first insight into the topic of static solutions in kinetic relativistic models, a departure point from where more general approaches try to develop themselves.

Needless to say, the knowledge about polytropic solutions is much more mature in the Vlasov–Poisson system than in the relativistic models. We will start our discussion precisely there.

3.5.1 The classical case

The polytropic family of stationary solutions to the Vlasov–Poisson system can be obtained either by the Jeans theorem or by variational arguments. In the notation stated after Jeans theorem (Theorem 2.1.1 in Chapter 2), they are given by the formula

$$f(x,v) = \varphi(E,F) = c(E_0 - E)_+^{\mu} F^k \tag{5.48}$$

The exponents have to verify $\mu, k > -1$ and $\mu + k + \frac{3}{2} \geq 0$ in order that the models be locally integrable. Obviously $c > 0$ for the sake of non-negativity. Finally, $E_0 \leq 0$ is a cut-off energy. This fairly easy structure can be brought in an straightforward manner to the relativistic models, and there is some chance that this ansatz produces static solutions for these models also; we will comment on this later. Returning to the Vlasov–Poisson case, let us point out that this solutions are spherically symmetric and induce radial density profiles. They are (locally) isotropic if $k = 0$. Their density functions vanish at the origin if $k \neq 0$. They have finite mass if $\mu \leq 3k + 7/2$ and finite radius if $\mu < 3k + 7/2$; we will consider models of finite mass only.

These models are solutions to the Vlasov–Poisson system in the sense that they verify the Jeans theorem, thus the distribution function is constant along the characteristic curves. To see to what extent does the theory of existence of weak solutions in [123] apply, we study the integral regularity and the finiteness of the kinetic energy of the polytropic distribution functions. We begin first with the simpler case in which $k = 0$.

For $\mu \in [0, 7/2]$ the distribution functions belong to $(L^1 \cap L^\infty)(\mathbb{R}^6)$ and have finite kinetic energy. If $\mu < 0$ then the distribution functions belong to $L^p(\mathbb{R}^6)$ for $1 \leq p < -1/\mu$; they have finite kinetic energy if $\mu > -1$. Thus the theory in [123] does not apply for $\mu \le \frac{-11}{12+3\sqrt{5}} \cong -0.7007$. For the general case, the distribution function belongs for sure to all $L^p(\mathbb{R}^6)$ spaces such that $2kp > -1$ and $\mu p > -1$, being a sufficient condition for the finiteness of the kinetic energy that $2k > -1$ and $\mu > -1$. Whenever these estimates are enough to define a weak solution we get a posteriori that these models launch a unique solution [153] as their density profiles are bounded (this follows from the analysis of Emden–Fowler's equation, see below).

We will focus mainly in the classical range $\mu \in]-1, 7/2]$, $k = 0$. Thus our pretended models conform a three-parametric family (depending upon c, E_0 and μ). This includes the well-known Plummer/Schuster model [48]

$$\rho(r) = \left(1 + \frac{r^2}{3}\right)^{-\frac{5}{2}}$$

which corresponds to the borderline case (with respect to finiteness of mass and radius) $\mu = 7/2$.

Now we pass to different considerations, namely the stability of all these solutions. This problem is important, because reasonable physical models for stellar equilibria must not be unstable, as the smallest deviation from the mathematical model would cause the system to evolve away from the model to some quite different configuration. There are numerical evidences that whenever $k \neq 0$ some polytropes may be unstable [114]. On the contrary, several theoretical studies on this matter have been developed for the case $k = 0$ and the upshot is that these polytropes are stable in several senses:

- spectral/linear stability for $\mu \ge 0$ [25].
- stable for the mass function distance against spherically symmetric perturbations, $0 < \mu \le 3/2$ [232].
- stable for the energy-Casimir distance in the range $0 \le \mu \le 7/2$ [87, 110].
- stable for the L^1 distance in the range $0 < \mu < 7/2$ [206].

The mathematical methods here involved point to nonstandard thermodynamics. As an aside, these polytropic solutions are prime examples of the application of a paradigm called "q-statistics". We shall elaborate on this, but we warn the reader that these topics are pretty controversial and there is a very strong debate in the physical community about the validity and applicability of them.

As we have justified, present day galaxies can be assumed to be described by stationary solutions of the Vlasov–Poisson system. A natural question that arises then is the following: Can we derive the structure of galaxies from the principle of maximum (or minimum) entropy? Here we enter into very subtle issues —we urge the reader to consult [48]— that are better discussed introducing the notion of coarse-grained distribution function, denoted by $\overline{f}$. Its value at any phase-space point (x, v) is the average value of f in some specified small volume centered on (x, v). This function $\overline{f}$ does not satisfy any interesting equation but it is empirically measurable and we expect that it resembles f when the averages are made over sufficiently small volumes.

Being f constant along trajectories it is fairly obvious that its Boltzmann entropy (and in fact any functional depending on f, these are sometimes called Casimirs if convex) remains constant during evolution. But this is not so for $\overline{f}$! Physicist have taken advantage on this fact to develop a theory of stellar relaxation to an equilibrium configuration. Under that theories, such configurations are usually assumed to be the result of appropriate relaxation processes (phase mixing, violent relaxation [150]). These processes are characterized by a loss of memory about the initial conditions of the system (we stress again the fact that we are taking about $\overline{f}$, this loss of memory cannot affect the solutions of the Vlasov–Poisson system, in virtue of the representation formula for the solutions). The usefulness of this theoretical construction is to justify that the maximization-of-the-entropy approach is worth to be used to give some insight into the complicated question consisting on deciding to which equilibrium state will settle down a given galactic system. But when we are to perform actual computations we cannot afford to do it with $\overline{f}$ and we are forced to use f instead.

The first attempt is somehow disappointing: maximization of the Boltzmann entropy $-f \log f$ under mass and energy constrains leads to the distribution function of the isothermal sphere [150], a model with infinite mass and energy. Indeed, we can always increase the Boltzmann entropy of a self-gravitating system by increasing its degree of central concentration (see [48]). We might conclude that Boltzmann's entropy is not the appropriate tool to deal with this problem.

A suitable generalization of the Boltzmann entropy is nowadays available [224]. A whole nonextensive statistics can be build upon this objects, which share with Boltzmann's entropy the fundamental properties of positivity, equiprobability and irreversibility. The main theorems of Maxwell–Boltzmann statistics —the central limit theorem in particular— admit deep generalizations in this landscape. Its applications concern systems that, in one way or another, exhibit scale invariance.

How does this class of new entropies, denoted by S_q, appear? Consider two subsystems A and B such that

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B),$$

being q a real parameter measuring the degree of nonextensivity. Nonlocality or long-range interactions are introduced by the multiplicative term, accounting for correlations between the subsystems A and B. This behavior can be obtained by setting

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1},$$

W being the number of microstates, p_i the probability of each and k the Boltzmann constant (Boltzmann's entropy is recovered for $q=1$). In our kinetic framework we have

$$S_q = \frac{k}{q-1}\int f - f^q \, dxdv$$

Note that, mathematically speaking, this is just a standard L^q-norm! Anyway, [175] showed that the polytropic distribution functions (case of polytropic index $k=0$ only) are obtained maximizing such entropies under mass and energy constraints, being $\mu = (q-1)^{-1}$. Independently, during the last years, [87, 110, 206, 232] derived the nonlinear stability of polytropes by obtaining them as minima of variational problems involving in one way or another the functional $\int f^{1+\frac{1}{\mu}} \, dxdv$; the form of the functional to be minimized in order to obtain the polytropic profiles was also known for the mathematical community.

This game can be played even further and further. From the mathematical point of view, technique is more or less mature so that we have a wide catalog relating what functional to minimize in order to obtain a certain prefixed steady state [110, 180]. From the physicists' side, recourse to all kind of convex functionals as exotic entropies seems to be justified [222].

All these q-thermodynamical issues have to do with the concepts of scale invariance and self-similarity. The polytropic profiles have a very convenient behavior under rescaling, that we detail now. Set $\alpha = E_0 - \phi(0)$. Successive application of the transformations $y(r) = E_0 - \phi\left((c\bar{c}_\mu)^{-1/2} r\right)$ and $\overline{y}(r) = \frac{1}{\alpha} y\left(\alpha^{-1/4-\mu/2} r\right)$ lead us to the fact that all density profiles induced by an isotropic polytropic model can be reduced to an universal expression of the form

$$\rho(r) = \rho_0 \overline{y}(sr)^{\mu+\frac{3}{2}}, \tag{5.49}$$

where $\overline{y}$ solves the normalized Emden–Fowler's equation

$$\begin{cases} \frac{1}{r^2}(r^2 \overline{y}'(r))' = -\left(\overline{y}(r)\right)_+^{\mu+\frac{3}{2}} \\ \overline{y}(0) = 1 \\ \overline{y}'(0) = 0 \end{cases}$$

being ρ_0 and s free parameters in correspondence with α and c.

The following discussion constitutes itself a proof for this statement; for future usage we include here also the gravitational constant G. Although we won't require it we discuss also the case $k \neq 0$ since it poses no additional complications and might be useful elsewhere.

Note that our choice of f —the polytropic ansatz— only ensures that it satisfies the kinetic equation and it remains to check that the spatial density associated with the model solves indeed the remaining semilinear Poisson's equation which specifies the potential ϕ. We detail here how is this coupling performed. In fact, integrating (5.48) over velocity space we find

$$\int_{\mathbb{R}^3_v} f(x,v)\, dv = \rho(r) = \frac{c\, \bar{c}_\mu}{4\pi\, G} r^{2k} \left(E_0 - \phi(r)\right)_+^{\mu+k+\frac{3}{2}},$$

where $\bar{c}_\mu = 2^{k+\frac{7}{2}} \pi^2 G \beta(\mu+1, k+\frac{3}{2}) \beta(\frac{1}{2}, k+1)$ and β is the usual beta function. For this ansatz, Poisson's equation reads

$$\frac{1}{r^{2+2k}} (r^2 \phi')' = c\, \bar{c}_\mu \left(E_0 - \phi(r)\right)_+^{\mu+k+\frac{3}{2}}$$

with initial conditions $\phi(0) = \phi_0$, $\phi'(0) = 0$. This is what we call Lane–Emden–Fowler's equation. The transformation $y(r) = E_0 - \phi((c\, \bar{c}_\mu)^{-\frac{1}{2+2k}} r)$ leads us to the initial value problem

$$\begin{cases} \frac{1}{r^{2+2k}} (r^2 y'(r))' = -(y(r))_+^{\mu+k+\frac{3}{2}} \\ y(0) = \alpha = E_0 - \phi(0) > 0\,, \quad y'(0) = 0. \end{cases}$$

We can rescale once more by means of $\overline{y}(r) = \frac{1}{\alpha} y(\alpha^{\frac{-\frac{1}{2}-\mu-k}{2k+2}} r)$ so as to get the normalized equation

$$\begin{cases} \frac{1}{r^{2+2k}} (r^2 \overline{y}'(r))' = -(\overline{y}(r))_+^{\mu+k+\frac{3}{2}} \\ \overline{y}(0) = 1\,, \quad \overline{y}'(0) = 0. \end{cases} \tag{5.50}$$

Thus, $E_0 - \phi(r) = \alpha\,\overline{y}\left(\alpha^{\frac{\frac{1}{2}+\mu+k}{2k+2}}(c\,\overline{c}_\mu)^{\frac{1}{2+2k}}r\right)$.

3.5.2 The relativistic case

The polytropic ansatz is also used to construct steady solutions to kinetic relativistic models. For the case of scalar gravity it was shown in [59] that the natural generalization of Jeans' theorem holds. Namely, any spherically symmetric solution depends on the coordinates through the local energy and the modulus of the angular momentum squared. This gives great support for the use of the polytropic ansatz; in this way we define the family of polytropes as

$$f(x,p) = \left(\frac{E_0 - E}{c}\right)_+^\mu F^k, \quad E = \sqrt{e^{2\phi(x)} + |p|^2}, \quad F = |x \wedge p|^2.$$

Here $\mu > -1$, $k > -1/2$, $c > 0$ and $1 > E_0 > 0$ are constants. E is the local or particle energy and as usual $(\cdot)_+$ stands for the positive part. These solutions were shown to have finite radius [59] for $\mu < k + 3/2$. In the isotropic case $k = 0$ they are orbitally stable for $0 < \mu < 2$ [66].

Although Jeans' theorem does not hold for the Einstein–Vlasov system [210], the polytropic ansatz provides still a handy family of static solutions. The quantities $E = e^{\mu(r)}\sqrt{1 + |v|^2}$ (particle energy) and $F = |x \wedge v|^2$ (angular momentum squared) are conserved along characteristics. Then we let

$$f(x,v) = (E - E_0)_+^k F^l$$

for $E_0 \geq 0$ and $l > -1/2,\ k > -1$. That they do define actual static solutions for the Einstein–Vlasov system was first proved in the case of isotropic pressure $l = 0$ [187], then extended to the general case in [182]. They are for sure compactly supported whenever $k < 3l + 7/2$. The stability theory for these models is not yet satisfactory (see however [231]).

Chapter 4

Application to the modeling of dark matter halos

4.1 The problem of missing matter

There are several empirical evidences pointing out the fact that, either present day galaxies must be much more massive than what we think from the estimates available from light emissions, or that the widely accepted laws of the physics of gravitation must have some strange caveat. During the last decades the majority of the Astrophysical community took party for the first alternative as an explanation to this series of paradoxes; the problem seems to be that there are lots of matter in each galaxy that cannot be detected directly by means of any known measurement technique nowadays available.

Maybe the most quoted evidence of the above-mentioned phenomenon is that given by the rotation curves of spiral galaxies. These are functions relating the velocity of the stars to their distance to the galactic center; reviews on rotation curves are given in [36, 215]. The fact here is that observations contradict the Keplerian prediction for the shape of rotation curves of galaxies: instead of decreasing asymptotically to zero as the effect of gravity wanes, these curves remains flat, showing the same velocity at increasing distances from the bulge. Astronomers call this phenomenon the "flattening of galaxies' rotation curve". Some possible explanations that have been suggested are:

- Presence of dark matter
- Modified Newtonian Dynamics (MOND theory)
- Influence of the magnetic field.

The present-day paradigm sticks to the first explanation, which is the most accepted interpretation of the flat rotation curve of spiral galaxies, even if other scenarios cannot be disregarded, see [36] for an extensive review. For instance, a combination of some of the above possibilities and even the use of General Relativity (see for instance [155], although the present paradigm states that dark matter halos are basically Newtonian structures; particles conforming them are slow and hence "cold") cannot be completely neglected. We use the term *dark matter* (maybe a more illustrative name could have been "hidden matter") to denote any form of matter whose existence is inferred solely from its gravitational effects. It is believed to be set in spherical structures surrounding each galaxy, conforming 9 out of 10 parts of the total mass of the system (as inferred from the available rotation curves), that are given the name Cold Dark Matter halos (CDM for short).

The rotation curves paradox is related to the use of the virial theorem. It is more or less customary to assume that a gravitationally bounded system is roughly in equilibrium. In this way the time averages of kinetic and potential energy can be assumed to be close to their current values. If we are able to measure the speed of a representative set of objects of our system we can give a tentative value for its kinetic energy, a value that we assumed to be close to the time average of the kinetic energy. Then we use the virial theorem to compute the potential energy of our system; if we get strong discrepancies with the sum of the contributions of the masses of visible particles to the overall potential we can claim that we are missing something (and this "something" we call dark matter). There are even more paradoxes that point to the existence of dark matter halos. The reader can consult for instance [48] for other contradictory facts.

It is widely accepted that CDM halos exist. This was already found in the early analytical calculations focused on the scale free nature of the gravitational collapse, see for example [109, 229]. Cosmological N-body simulations (being the seminal works by Navarro, Frenk and White —abridged in the sequel as NFW— [164, 165] the most representative) also have given very strong support to the fact that galaxies could be surrounded by an extended massive dark matter halo. The existence of dark matter in the Universe has been supported by the space mission WMAP observing CMB (Cosmic Microwave Background) [115]. The so called NFW profiles, giving the density function of these halos as a function of the distance to the center of the visible galaxy, are found to be universal, which means that they hold for a very large span of scales of dark halos, ranging from dwarf galaxies (few kiloparsecs) to rich cluster of galaxies (several megaparsecs). See for instance the results of the "Millenium" simulation with 10 billion particles [46]. Other possibilities are the so called "Isothermal non-singular" [109] profile and Burkert's [58] profile, among many others. These three profiles will be addressed in this chapter, as they are completely representative for the plethora of models that have been proposed up to this time. The results of this chapter are contained in [68].

4.2 Some existing models for spherical halos

There are mainly two sources for the different proposals of CDM density profiles that have been made up to date. These are, first, the numerical simulations of N-body problems, and second, fittings to data —like rotation curves of some representative galaxies.

The most popular model coming from N-body simulations is the Navarro–Frenk–White (NFW) density profile [164], given by

$$\rho(r) = \frac{\rho_0(c)}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2}.$$

Here c is a concentration parameter, $\rho_0(c)$ is given by

$$\rho_0(c) = \frac{100H^2}{4\pi G} \frac{c^3}{\ln(1+c) - \frac{c}{1+c}}$$

—being H the Hubble constant—, $r_s = \frac{r_v}{c}$ is a scale radius (gives the scale in which the profile changes shape) and r_v the "virial radius". This virial radius is defined as the radius of a sphere such that the portion of the system there enclosed satisfies the virial theorem; it is approximated by the following fix-up convention: $r_v \simeq r_{200}$, where r_{200} is the radius of the sphere such that the mean density of the enclosed mass equals 200 times the critical density of the Universe $\rho_{crit} = \frac{3H^2}{8\pi G}$. With this fix-up, we have

$$\frac{d\ln(\rho)}{d\ln r}(r = r_s) = -2.$$

We will regard this profile in an equivalent way, as

$$\rho_{\mathrm{NFW}}(r) = \frac{\rho_0}{\frac{r}{R_0}\left(1 + \frac{r}{R_0}\right)^2}, \tag{2.1}$$

where ρ_0 and R_0 are constants. A bunch of related models consisting mainly on slight modifications of this power law have been published since its apparition (see [159] for instance).

Concerning models for CDM halos coming from observations, we have for instance the Isothermal non singular [109]

$$\rho_{\mathrm{I}}(r) = \frac{\rho_0}{\left(1 + \frac{r^2}{R_0^2}\right)} \tag{2.2}$$

(where R_0 is a constant —defining the core radius— and ρ_0 is the density at the center) and the Burkert profile [58]

$$\rho_{\mathrm{B}}(r) = \frac{\rho_0}{\left(1 + \frac{r}{R_0}\right)\left(1 + \left(\frac{r}{R_0}\right)^2\right)}. \tag{2.3}$$

Here R_0 and ρ_0 have the same meaning as in the Isothermal profile.

NFW profiles seem to explain very reasonably the dark matter distribution in clusters as demonstrated by weak lensing observations (e.g. [117]) and by X-ray observations (e.g. [211]). In galaxies, the agreement between simulations and observations is still under debate. The rotation curve of spiral galaxies is the main observational tool to look for this agreement. CDM models do not satisfactorily explain the Tully–Fisher relation [166] and even have difficulties to produce large disks [97]. Probably, the explanation of rotation curves without considering magnetic fields could be unrealistic [37, 38]. On the other hand, as discussed below, the steep rise of rotation velocity in the central part has been claimed to be incompatible with the simulation outputs.

The above models have some obvious drawbacks: infinite radius, infinite mass and infinite density at the center. It could be argued that the first two are not that serious, as we can introduce an artificial cutoff for the density function —this could be even something to be encouraged when dealing with all the phenomenological models above, anyway we will come back to this later. But the third point is much more delicate and deserves some discussion.

It is a fact that the resolution of numerical simulations avoids giving firm predictions in the central parts of galaxies (≥ 1 kpc) and these simulations have a complex dynamics there. In particular, if we define the slope in a log-log plot of density versus radius as $\gamma = -\frac{d\ln\rho}{d\ln r}$, the NFW profile gives $\gamma_0 = \gamma(r = 0) = 1$, while the Isothermal and Burkert profiles give $\gamma_0 = 0$. Observations also seem to suggest lower or vanishing values. Values of $\gamma_0 > 0$ correspond to "cuspy" halos, as the density becomes infinite for $r = 0$. Values of $\gamma_0 = 0$ correspond to halos with core, being the core a region in which the density is nearly constant.

An infinite value for the density seems to be unphysical, which implies $\gamma_0 \leq 0$, and continuity arguments (in the first derivative) clearly indicate $\gamma_0 = 0$ at the very center. Near this inner region with size less than the spatial resolution of the simulations, the NFW profiles must break down and converge to a function with $d\rho/dr(r = 0) = 0$.

Of particular interest are the low surface brightness galaxies (LSB) as the contribution of the stellar component is so low than these galaxies are assumed to be dark matter dominated. In [50, 51, 52] it has been found $\gamma_0 = 0.2 \pm 0.2$ for LSB galaxies and shown that even considering the influence of non-circular motions, asymmetries and off sets between optical and dynamical centers, the values close to vanishing are incompatible with NFW halos. These effects can indeed be very large. Non-circular motions are very large even in normal non-active galaxies (e.g. [74]) as shown by bidimensional spectroscopy.

For high surface brightness, the results in [203, 204] also found this incompatibility between rotation velocities and cuspy simulated halos. In our galaxy, [47] also concluded that cuspy halos are inconsistent with observational data. It is worthy to be reminded that the early interpretation of rotation curves in terms of dark matter determinations adopted the hypothesis of maximum disc, see [39], i.e. assumed that the contribution of dark matter was negligible in the center, and the results were considered as acceptable. More recently, [171] found that the dynamic of bars also favors the maximum disc hypothesis, with no need of dark matter in the very center. Clearly, these holes in the center are inconsistent with cusps. However, the present situation remains unclear. [112] found γ_0 between 1 (as in NFW) and zero, see [159, 160]. Observational studies like that in [219] and others showed that rotation curves are not manifestly inconsistent with $\gamma_0 = 1$. Another indirect interesting way was shown by [116], introducing constraints arising from the radial velocity dispersion and obtained $\gamma_0 \leq 0.58$. Therefore, there are at present large discrepancies about if the central dynamics of spirals are in contradiction with cuspy halos: Cusps or cores?

Numerical codes based on cold dark matter produce central cuspy density profiles of the type $\rho(r)$ proportional to r^α. The value of α varies from $\alpha = -1$ (for NFW profiles [164, 165]) to $\alpha = -1.5$ [57, 159, 160]. This in clear contrast with observations based on the rotation curve of spiral galaxies, which require a core with a constant density or slightly decreasing with r, i.e. with α close to zero. This is particularly clear in the case of dwarf and low surface brightness spiral galaxies, see for example [49, 51, 52, 154, 156], but the discrepancy is as well observed in normal spirals [203, 204]. Cusps are also inconsistent with observations in our galaxy [47]. We refer to [51] for a detailed discussion about this discrepancy. The numerically obtained cuspy profiles could be considered unacceptable from the physical point of view, as the density becomes infinite at the very center, even if the mass converges. This infinite value is not directly obtained, but is the result of fitting as the resolution in the numerical outputs is not large enough. This fact is discussed later as our approach does not have this limitation.

Another unphysical output from NFW halos is that they have no end. NFW halos even give $\rho \sim r^{-3}$ for large radii, which means that the halo mass does not converge and becomes indeterminate. The same can be said for Burkert's halos. Isothermal halos give $\rho \sim r^{-2}$ for $r \to \infty$, thus the same kind of problems persist. On the contrary, in the enormous family of steady state solutions to the Vlasov–Poisson system we can find models with a wide variety of large but finite radii. Certainly, halos are not ideal isolated systems. [149] considered DM halos inhabiting high density environments and provided strong confirmation of tidal truncations. Tidal stripping of halos in clusters has been studied by [29]. Weak lensing observations provide typical truncation radius of 185 kpc, see [117]. But on theoretical grounds, also isolated halos should have finite size and finite mass. In [205] it has been provided values of the order of 250 kpc from observational rotation curves. The highest values are provided by analysis of satellite galaxies of around 400 kpc [236]. We will adopt, as a tentative value, 300 kpc.

4.3 A kinetic theory approach

Our aim here is to give some insight into the controversy between N-body simulations and observational models of cold dark matter halos by considering polytropic gas spheres associated to the Vlasov–Poisson system. The polytropic profiles have been proposed to model a wide variety of concrete applications in different fields, see for example [77, 223, 227]. Nevertheless, the use of polytropes is under discussion in some areas, even in the context of dark matter halos [94]. Concerning this point, we will compare our results here with those of [94] at the end of the chapter.

Our study confirms the results obtained by simulations in a very wide range of galactocentric radii. We provide density CDM profiles in very good agreement with both numerical (NFW) and other observational results with errors less than 3%. Then, polytropic density profiles provide a complementary scenario where simulations and observations might be unified. With this approach our resulting polytropic model can then be used to make predictions on the behavior of the CDM halos in those regions in which the N-body simulations models cannot produce detailed results, i.e. near the center ≤ 1 kpc (due to resolution limitations) and at the rim (as halos cannot have infinite extent). This provides complementary information where other models present difficulties to make predictions. Our opinion is that with the use of polytropic models in the study of DM a complementary tool is at our disposal for considering more detailed and realistic halos by using recent and future observations.

This approach is by no means new, not even in the field of dark matter halo modeling. [155] indeed showed that the radial pressure and mass density approximately satisfy a polytropic relation, as it is the case for different classes of stars, from main sequences to relativistic and non relativistic degenerate stars. [175, 220, 221] applied Tsallis Statistical Mechanics to self-gravitating systems. [76] considered the gravitational stability of polytropic spheres and studied the particular case of stars. The polytropic distributions can be obtained (as we saw in Chapter 3) from the constrained optimization of Tsallis's q-entropy under physically meaningful constraints, see for example

[175, 206]. We also refer to [176], a review on astrophysical applications of the q-thermostatistical formalism to self-gravitating many-body models. [237] showed that this formalism is favored over gaseous spheres. [135] also points out the possibility of employing polytropic systems to model CDM, but their study focuses mainly on thermodynamical aspects.

Last but not least, as CDM particles are collisionless, CDM halos constitute ideal systems for the generalized kinetic theory to be applied, with a highly abstract mathematical handling for which the polytropic approach can provide useful results. Keep in mind also that the contribution of the dark matter to the overall structure is believed to be about $90-95\%$ of the total matter content, while visible baryonic matter (stars and gas) is clustered in galactic discs. Then, as a first approximation it is reasonable to consider the gravitational field of a galaxy as that of its dark matter halo, while visible matter can be though of as 'test particles' in this field. That's why we shall model dark matter only.

We claim that the strength of our approach lies in its flexibility. Here we have restricted ourselves to the simplest case —namely that of spherically symmetric halos—, but our theoretical framework can be extended to cover many more cases, such as elliptical configurations, which are also in good agreement with recent observations [113]. Once the appropriate solution of the Vlasov–Poisson system is chosen, the required numerical calculations are much more affordable than an N-body simulation, and we have several tools to discuss all relevant stability issues. Using a power-law ansatz, this approach leads to the Lane–Emden–Fowler's equation and subsequently to polytropic systems. It is the balance between their good properties (such as their stability, the thermodynamical theories supporting them —still under debate, so this has to be taken carefully— or their behavior under rescaling) and their relative simplicity what makes them a convenient choice to fit the observational data.

Let us remind here that the applications of q-statistics concern systems that, in one way or another, exhibit scale invariance (a feature which is closely related to the universality of NFW profiles [165]). Recall also that equation (5.49) in Chapter 3 shows the crucial fact that every isotropic polytropic density profile can be reduced to an easy and useful canonical form,

$$\rho_{poly}(r) = \rho_0 \overline{y}(sr)^{\mu+\frac{3}{2}},$$

where $\overline{y}$ is the solution to the normalized Emden–Fowler's equation (5.50) of exponent μ and ρ_0, s are free parameters in correspondence with c, $\alpha = E_0 - \phi(0)$ by means of

$$\rho_0 = \frac{c\overline{c}_\mu}{4\pi G}\alpha^{\mu+\frac{3}{2}}, \quad s = (c\overline{c}_\mu)^{\frac{1}{2}}\alpha^{\frac{1}{4}+\frac{\mu}{2}}\,.$$

This canonical form of the polytropic densities justifies even more the polytropic choice because the above self-similarity property of the density profiles includes the idea of universality of CDM halos in the sense explicitly expressed in [165].

The polytropic ansatz can be extended to a wider class of solutions such that $\rho(x) = h(E_0 - \phi(r))$, where the function h might not be a power law [35]; this is in accordance with Jeans' theorem. To perform the previous scalings we must be able to compare $h(y(r))$ with $h(\lambda y(r))$ for any positive real number λ. Thus we must postulate an homogeneity property of the form $h(\lambda y(r)) = g(\lambda)h(y(r))$ for $\lambda \geq 0$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$. If h is assumed to be regular, then this relation implies that h is a power law. As a consequence, only power laws of the particle energy can be normalized to have a density profile that fulfills the conditions $\rho(0) = 1$ and $\rho'(0) = 0$ (so that they relate the parameters ρ_0, R_0 in the same way as the observational models, Isothermal and Burkert's).

A final comment would be that we restricted ourselves to use the polytropic models with $k = 0$. This was mainly justified by the fact that the models with $k \neq 0$ have zero density at the origin. But they could be useful in a number of situations, as there are some cases (LSB) in which an absence of dark matter at the center is claimed. The properties under rescaling and the theory that we are going to develop can be generalized easily to this case (see Chapter 3).

4.4 Generic fitting procedure

Note that many —if not all— of the models for density profiles of CDM halos proposed in the literature can be written as $r \mapsto a\overline{\rho}(br)$, where a and b are constants with precise dimensions and $\overline{\rho}$ is an universal function of the radius that can be regarded as a normalized density profile. If we try to recover this property for a density $\rho(x) = h(E_0 - \phi(r))$ we should be able to perform the previous scalings; in particular we must be able to compare profiles with densities induced by polytropical models, i.e., with expressions as $c\rho(dr)$, being ρ the normalized profile arising from the integration of the normalized Emden–Fowler's equation (μ held fixed at this stage) and c, d our free parameters. Given that we want to focus on a range of radii $I = [r_0, r_\infty]$, the comparison criterion (least squares) will be given by

$$\left\| a\overline{\rho}(br) - c\rho(dr) \right\|^2_{L^2(I)} = \frac{a^2}{b} \int_{b\cdot r_0}^{b\cdot r_\infty} \left[\overline{\rho}(y) - \frac{c}{a}\rho\left(\frac{d}{b}y\right)\right]^2 dy.$$

Introduce the new fitting parameters $h = \frac{c}{a}, k = \frac{d}{b}$ to reformulate our problem as

$$\min_{h,k>0} \int_{b\cdot r_0}^{b\cdot r_\infty} \left(\overline{\rho}(y) - h\rho(ky)\right)^2 dy.$$

Expanding and minimizing in h we find that our problem reduces then to

$$\min_{k>0} \left\{ \int_{b\cdot r_0}^{b\cdot r_\infty} \overline{\rho}(y)^2 \, dy - \frac{\left(\int_{b\cdot r_0}^{b\cdot r_\infty} \overline{\rho}(y)\rho(ky)\, dy\right)^2}{\int_{b\cdot r_0}^{b\cdot r_\infty} \rho(ky)^2 \, dy} \right\}.$$

To search for optimal k, we differentiate and impose numerically equality to zero.

Due to the great size of the constants above in the NFW case, it is more convenient to search for the relative error, which we will take as the following adimensional quantity

$$\frac{\left\| a\overline{\rho}(br) - c\rho(dr) \right\|^2_{L^2(I)}}{\left\| a\overline{\rho}(br) \right\|^2_{L^2(I)}}.$$

Up to this point μ was kept fixed, and this recipe produces the best polytrope among the class of exponent μ. To conclude, we must check which exponent μ in the admissible range $]-1, 7/2]$ yields the best results.

Table 4.1: Unrestricted fits to NFW's profile

Virial radius (kpc)	$\overline{c}$	Virial mass ($\times 10^{12} M_\odot$)	Relative error	c_{poly} (1)
177	19.230	0.319	0.0206624	4.03295
172	17.543	0.293	0.0219044	4.45049
193	12.195	0.414	0.0275006	6.49123
209	21.739	0.525	0.0191086	3.52726
348	16.666	2.425	0.0226281	4.69939
342	14.084	2.301	0.0251543	5.60226
394	14.705	3.519	0.0244842	5.35748
354	8.064	2.552	0.0351619	9.78656

(1) Height at the origin is given by $c_{poly} \frac{100H^2}{4\pi G} \frac{\overline{c}^3}{\ln(1+\overline{c}) - \frac{\overline{c}}{1+\overline{c}}}$.

4.5 Numerical results

To perform our fittings we have only considered profiles with $\mu \leq 7/2$. Values higher than 7/2 could produce better fittings but the halo mass becomes infinity.

1. *Fits to NFW profiles in a wide range of radii*: we pick up the first eight profiles (out of 19, these happen to be those which give rise to the least massive halos of the sample) that adjust the N-body simulations of [164] to compare them against the whole, three-parametric, family of polytropes in the range comprised between the virial radius over 100 and the virial radius itself. Here

$$\overline{\rho}(r) = \frac{1}{r(1+r)^2}, \quad a = \frac{100H^2}{4\pi G} \frac{\overline{c}^3}{\ln(1+\overline{c}) - \frac{\overline{c}}{1+\overline{c}}}, \quad b = \frac{\overline{c}}{r_v},$$

   H stands for a Hubble constant of 50 km s^{-1} Mpc^{-1}, r_v is the corresponding virial radius, and $\overline{c}$ the fitting parameter (we changed slightly their notation in order to avoid confusion with computations performed in previous paragraphs). We observe that the higher the exponent μ, the smaller the relative error (Table 4.1). Thus, the best fitting is fulfilled with $\mu = 7/2$, which corresponds to the Plummer/Schuster model. The relative errors are of orders comprised between 2% and 3.5% (see Fig. 4.1) .

2. *Fits to NFW profiles assuming bounded halos*: as before, but we impose an upper bound on the radius of 300 kpc, which rules out Plummer/Schuster's profile. We see that for μ held fixed the obtained profile reaches the maximum radius prefixed (Table 4.2). This fact can be used to obtain numerically the best exponents. These are around $\mu = 3.2$ for our sample; in any case, the relative error is of order 2% or 3%, and differs from those given by the Plummer/Schuster model only in the third decimal (see Fig. 4.2).

   The predictions at the origin arise from the numerically determined height at the origin combined with the expansion around zero of the solutions to the normalized

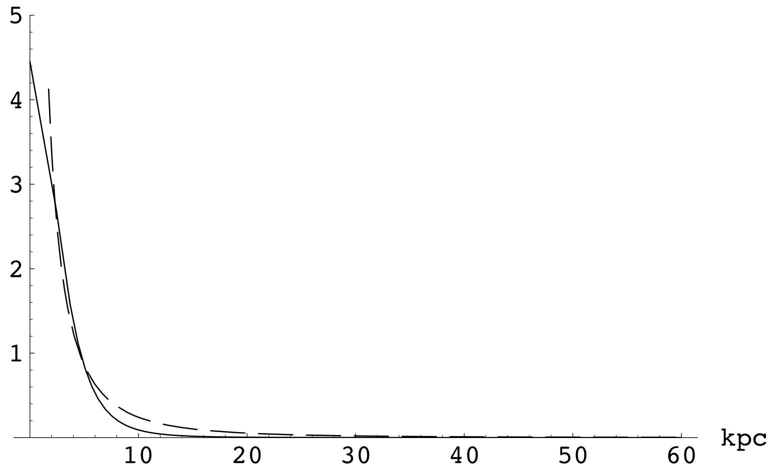

Figure 4.1: NFW's profile corresponding to a virial radius of 172 kpc and a virial mass of $29.3 \cdot 10^{10}$ $M_\odot$ (dashed line) plotted against Plummer/Schuster's density profile (bold line), normalized scale in y-axis. NFW's density profile is not represented in the central region (from the origin to the virial radius over 100).

Table 4.2: Fits to NFW's profile with truncation radius of 300 kpc

Virial radius (kpc)	$\overline{c}$	Virial mass ($\times 10^{12} M_\odot$)	Best exponent	Relative error	c_{poly} (2)
177	19.230	0.319	3.33428	0.021223	4.01223
172	17.543	0.293	3.33413	0.022476	4.42845
193	12.195	0.414	3.29355	0.028275	6.45623
209	21.739	0.525	3.31307	0.019728	3.50559
348	16.666	2.425	3.17289	0.023811	4.65328
342	14.084	2.301	3.16183	0.026423	5.54897
394	14.705	3.519	3.11952	0.025913	5.29882
354	8.064	2.552	3.09269	0.036896	9.69161

(2) Height at the origin is given by $c_{poly} \frac{100H^2}{4\pi G} \frac{\overline{c}^3}{\ln(1+\overline{c}) - \frac{\overline{c}}{1+\overline{c}}}$.

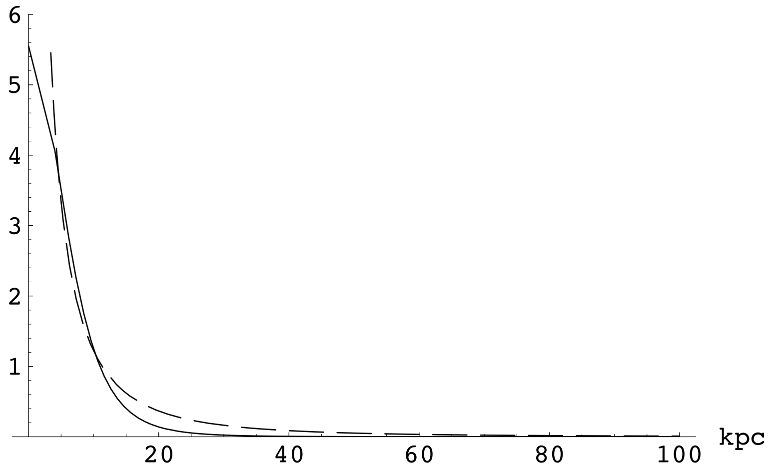

Figure 4.2: NFW's profile corresponding to a virial radius of 342 kpc and a virial mass of $230.1 \cdot 10^{10}$ $M_\odot$ (dashed line) plotted against the polytropic model corresponding to $\mu = 3.16183$ (bold line), normalized scale in y-axis. NFW's density profile is not represented in the central region (from the origin to the virial radius over 100).

Emden–Fowler's equation (cf [83]):

$$\overline{y}_\mu(r) \sim 1 - \frac{r^2}{3!} + (\mu + 3/2)\frac{r^4}{5!} + \big(5(\mu + 3/2) - 8(\mu + 3/2)^2\big)\frac{r^6}{3 \cdot 7!},$$

and thus for the normalized density profile we have that $\ln \rho \sim (\mu+3/2)(1-r^2/3!)$, so that $\gamma(r) \sim \frac{\mu+3/2}{3} r$ close to the origin —the notation $\sim$ in this paragraph is not that explained in the introduction but the usual one pertaining two functions that are asymptotic.

3. *Isothermal*: we compare the normalized ($a = b = 1$) Isothermal density profile with ours for $r \in [0, \infty]$ and again we find that the error diminishes for increasing exponent μ (Table 4.3), so that the Plummer/Schuster model is the one that gives the best fitting with finite mass. The corresponding relative error is of order 1% (see Fig. 4.3).

4. *Burkert*: in the same vein as before, we compare the normalized ($a = b = 1$) Burkert's density profile with ours for $r \in [0, \infty]$ and the behavior is the same: as the exponent grows the error diminishes (Table 4.3). Again Plummer/Schuster's profile is the bounded mass profile that yields the best fitting, with a relative error of order 1% (see Fig. 4.4).

4.6 Conclussions

We have shown that a unified theory of cold dark matter halos based on collisionless polytropes is a powerful complementary method for studying galactic and cluster halos. We confirm the results obtained by numerical simulations (NFW universal profiles) but other profiles (Isothermal and Burkert) cannot be disregarded. Once the agreement between our results and the N-body simulations is established, we are able to explore

Table 4.3: Fits to phenomenological models

Isothermal		Burkert	
Exponent	Relative error	Exponent	Relative error
3.5	0.011816	3.5	0.010159
3.4	0.012010	3.4	0.010323
3.3	0.012213	3.3	0.010494
3.2	0.012426	3.2	0.010675
3.1	0.012650	3.1	0.010865
3.0	0.012886	3.0	0.011065
2.9	0.013131	2.9	0.011276
2.8	0.013390	2.8	0.011150

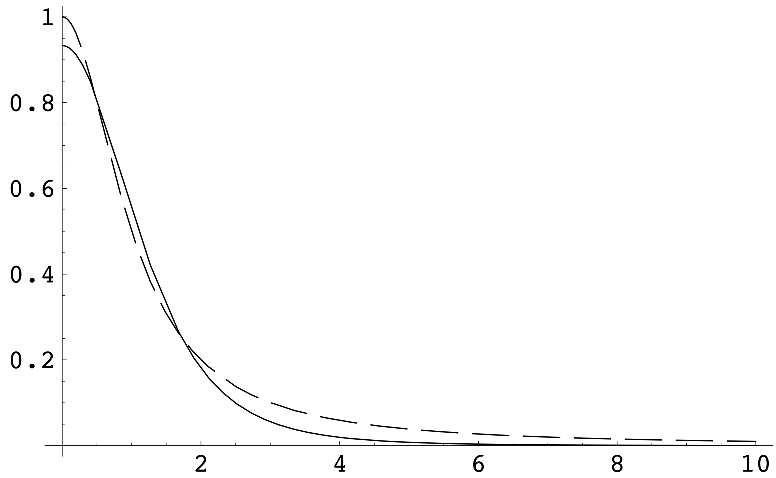

Figure 4.3: Isothermal density profile (dashed line) plotted against the density profile arising from Plummer/Schuster's model (bold line). Both profiles are normalized in density and radius.

complementary problems out of the range of validity of simulations. We are able to investigate the polytropic solutions for finite mass and size halos and, mainly, we are able to find the profiles in the inner region where simulations are limited by resolution problems. We therefore provide the shape of the profile in the ≤ 1 kpc-region to which NFW-profiles must converge.

There is a very wide family of solutions of the Vlasov–Poisson system and here we have restricted ourselves to consider solutions for steady-state spherical systems in which the phase space distribution function does not depend on the angular momentum. In convenient cases this distribution function depends on the power μ of the energy of the particles, and this exponent characterizes the member of the family which is used as a free parameter.

For finite mass but unbounded halos it is found that the exponent μ that better reproduces the NFW-profiles has the value $\mu = 7/2$ which exactly corresponds to Plummer/Schuster profiles. Values lower than 7/2 give higher errors. Values higher than 7/2 do not provide finite total mass. Of particular interest is the case in which we establish a maximum radius for the halo. Several observations can be used to determine that halos have finite size and radius, in particular gravitational lenses, x-ray observations and those based in interactions of halos belonging to galaxy clusters. We have adopted a tentative value of 300 kpc for this radius. In this case the profile does not match the Plummer/Schuster profile but we obtain $\mu \simeq 3.2$ for the best agreement with the NFW profiles.

In this case, we are able to deal with realistic halos that a) have finite size and volume, b) have finite density at r=0, c) have a vanishing slope at r=0, and d) coincide with NFW-profiles for r higher than the resolution length in the simulation.

We reproduce the plot of this profile for this unexplored region and expand the density as a function of the exponent μ and of the galactocentric radius. A very simple formula for the profile of the very inner region, ignoring third order terms, would be

$$\rho(r) \sim \rho_0 \left(1 - \left(\frac{r}{R_0}\right)^2\right)^{\mu+3/2}, \tag{6.4}$$

R_0 and ρ_0 being constants (here $\sim$ means again "being asymptotic to").

Previous discussions on the inner region density profile have concentrated on the function γ defined as $\gamma = -d \ln \rho / d \ln r$. We obtain for this function $\gamma \sim \frac{\mu+3/2}{3} r$. Therefore, $\gamma_o = \gamma(r = 0) = 0$ as should be expected from continuity arguments in the very center.

One of the major interests of using this mathematical technique is that we can provide results in the inner region ≤ 1 kpc region (see Fig. 4.5). This is important because a) we complement the well known and widely accepted universal profiles at larger radii and b) we explore a region in which a comparison can be made with the rotation curve of spiral galaxies. For this comparison much further effort must be done in the future as it is necessary to introduce realistic baryonic disk and bulge, galactic components of great importance to establish the rotation curve in the inner galaxy.

The conclusion in [94] assuring that polytropes do not properly describe the inner part of simulated halos is based on the approximation of the NFW profile by polytropes fixing in a concrete point the value of the density together with its derivative (obtained from the NFW numerical results). Then, going back towards the inner part of the halo an error of one order of magnitude is obtained, which induces the authors to justify their conclusion. In spite of the experimental data do not corroborate the behavior of the NFW results near the centre of the halo, we can modify the criteria of the approximation given in [94] by the (least squares) L^2–approach, performed in the region where they focus, and we obtain for their sample of experiments a very good fitting in the inner part of the halo, see Fig. 4.6. This shows that the fittings presented in [94] can be improved if the fitting criterium is chosen carefully enough, thus calling into question partially the conclusion there. Note that Figure 4.6 is plotted in logarithmic scale which distorts small differences at large radii. This logarithmic scale choice differs from the one used in the other figures in this chapter and this is motivated by what was done in [94].

The polytropic solutions that have provided a large insight in the knowledge of stellar interiors can be applied to dark matter halos even more realistically as they perfectly match the condition of collisionless particles.

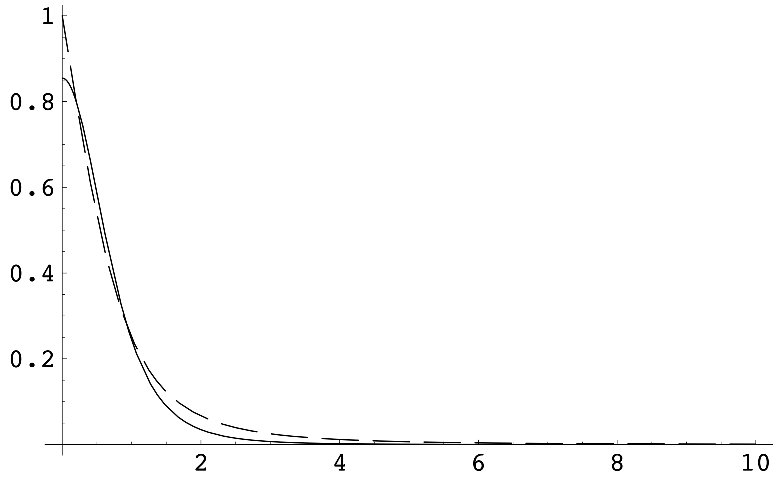

Figure 4.4: Burkert's density profile (dashed line) plotted against the density profile arising from Plummer/Schuster's model (bold line). Both profiles are normalized in density and radius.

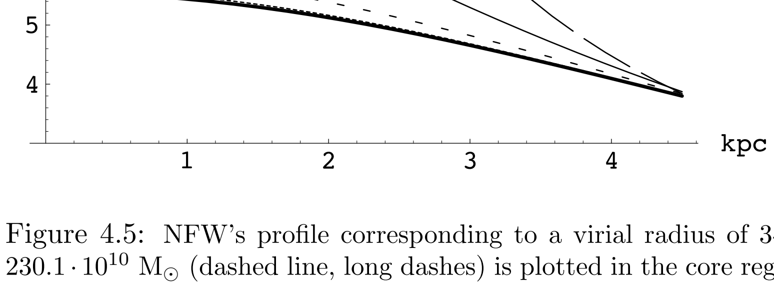

Figure 4.5: NFW's profile corresponding to a virial radius of 342 kpc and a virial mass of $230.1 \cdot 10^{10}$ $M_{\odot}$ (dashed line, long dashes) is plotted in the core region against the least squares fits corresponding to the following families: 1) the polytropic model with $\mu = 3.16183$ (bold thick line), 2) the Plummer/Schuster model (dotted line), 3) the Isothermal profile (dashed line, short dashes) and 4) Burkert's profile (bold thin line). Normalized scale in densities is used.

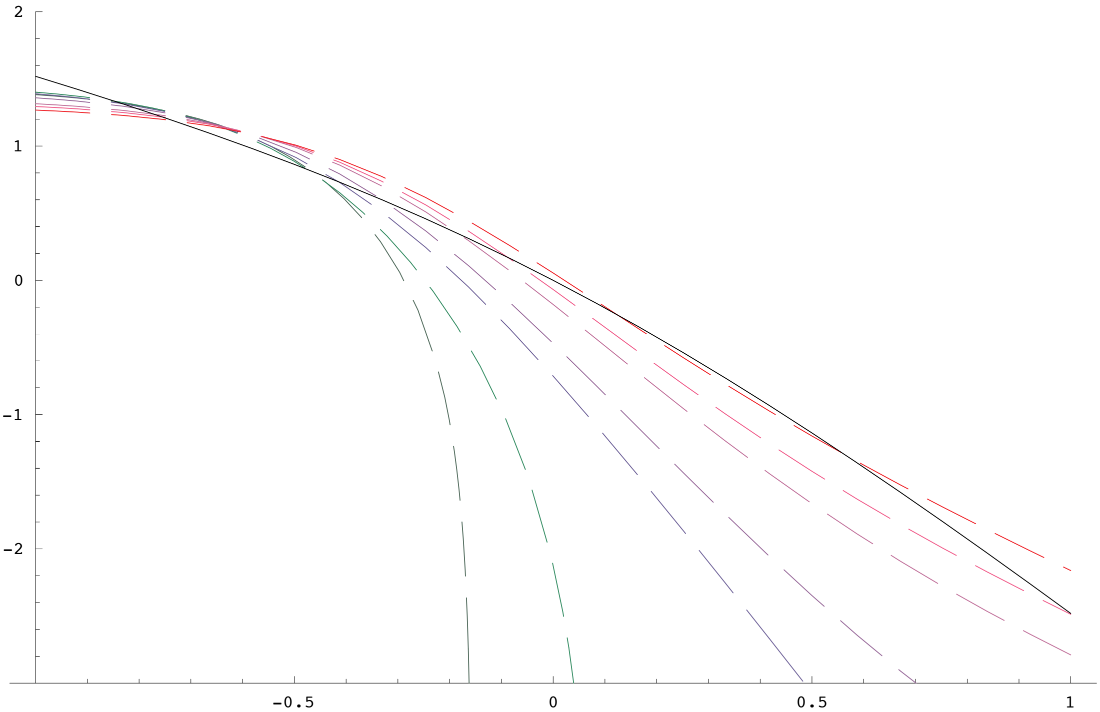

Figure 4.6: $\ln_{10}(\rho/\rho(r_s))$ plotted against $\ln_{10}(r/r_s)$ in the range $[0.1r_s, 10r_s]$ —recall that $r_s = r_v/\overline{c}$ for the NFW model— for the NFW profile (solid line) and the least square fits by polytropic profiles of exponents 1/2 (grey), 3/2 (green), 7/2 (blue), 9/2 (purple), 17/2 (violet), 15 (pink) and 498.5 (red). Notice that as the exponent increases so do the fitting curves in the "outer" region: for instance the Plummer/Schuster model corresponds to the third exponent of the sample and to the curve that crosses the axis close to 0.5.

Chapter 5

Asymptotic behavior of a kinetic coagulation model

5.1 Introduction

In this chapter we will be concerned with a kinetic coagulation model describing two species of particles (molecules or cells for instance); one of them remains still and thus constitutes a short of background density, while the other species moves freely and interacts with itself and the stuck species. As an outcome of this new aggregates may be formed. Thus, the kinetic model that we present here constitutes an example in which short-range interactions are included successfully in the equations to give an account for phenomena others than the ones considered in previous chapters.

The models representing coagulation phenomena can be classified according to the chosen scale of description. Microscopic descriptions try to represent the evolution of a finite set of individual particles, the Smoluchowski–type models are typical examples in this context, see [91] and references therein. Mean-field (mesoscopic) models are concerned with the evolution of the number of particles of each possible size, and not that of the individual particles; these descriptions are valid when the number of particles is sufficiently high. Mesoscopic models may or may not include the spatial distribution of the particles, [2, 141]. On the other hand, macroscopic models describe the evolution of some macroscopic quantities, which represent some kind of average of the microscopic properties of the system (such as the mean cluster size), [142, 144].

We study in this chapter a kinetic model. Those approaches to the phenomenon of coagulation and fragmentation, take into account the effects of the movement and trajectories of the particles; see for example [4, 30, 89, 128, 129] for other studies of kinetic models for coagulation or fragmentation.

Many physical phenomena consist of a great number of small particles that can stick together in some way to form aggregates or new particles of larger size. At the same time big particles could split into smaller ones. This occurs in multiphase fluids, in many examples of phase change, in the behavior of aerosols with liquid or solid particles suspended in a gas and in crystallization in colloids, among other examples in this context, for example [1, 88, 100, 148, 201, 230].

The sticking together, coagulation, aggregation or adhesion into a cluster, of particles, whether they are cells, lipid droplets, proteins, etc., is of fundamental importance in biological and biotechnological processes. This is the primary motivation for the model that we study.

For example, in animals, small cells called platelets cluster at the site of an injury to the skin or blood vessels. Also, during the development of an embryo, space between aggregated cells decreases and cell-to-cell contact increases. Other example of this process can be found in cell aggregation by Chemotaxis or in flocculation of sticky phytoplankton cells into rapidly sinking aggregates, which has been invoked as a mechanism explaining mass sedimentation of phytoplankton blooms in the ocean. In the biological explanation of this context appear surface ligands that mediate cell-to-cell adhesion or any molecule involved in cellular adhesive phenomena such as in liver cell adhesion molecule and neural cell adhesion molecule. Experimental observations show that cell aggregation in suspension promoted cell survival and proliferation, in particular it has been demonstrated a correlation between tumor cell aggregation in suspension and cell growth.

The interaction forces between particles ultimately determine the stability and rheological properties of any system, and in many biological cases the principal adsorbed component that mediates these interaction forces is a protein.

Coagulation and aggregation phenomena have been the object of many studies in the recent years, both in physics and mathematics [88, 143]. For applications in physics, one typically assumes that the coagulation of two particles preserves the total mass and total momentum. However there are cases where the last is not true. The prime example that we have in mind is the dynamics of some cells in biology; for instance endothelial cells, but it is a very common phenomenon in biology. Those cells may move freely when they are alone. However they may also join with other cells of the same kind and then may not move any more. In particular endothelial cells compose blood vessels, once they are combined with other endothelial cells, and hence do not move [92, 217].

The aim of this chapter is to propose a suitably modified coagulation model, taking distinct states for the particles or cells into account. A first state corresponds to free particles that, consequently, have velocities v and will require to consider the density in the phase space. A second state represents the coagulated particles that are fixed and thus only have a density in the physical space. We introduce the two corresponding densities

$$f : [0,T] \times \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R}^+ \quad \text{representing free particles}$$

$$\rho : [0,T] \times \mathbb{R}^d_x \to \mathbb{R}^+ \quad \text{representing coagulated or stuck particles.}$$

It of course remains to explain how those quantities evolve in time. For free particles, we assume that each one moves with its velocity that does not change (as long as it remains free). This is a simplification and more realistic models should consider how this velocity may change (influence of chemoattractants, stochastic jumps...). As we focus mainly on the coagulation phenomenon, this assumption is however reasonable.

Free particles may move freely, interact one with another and coagulate. They may also meet already coagulated particles, interact and combine with them. For stuck particles, the situation is simpler. They do not move and may therefore only interact with free particles that occupy the same position in space.

We consequently consider the following system of equations:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' - \beta(v)\rho(t,x) f(t,x,v) \qquad (1.1)$$

$$\frac{\partial \rho}{\partial t} = \int_{\mathbb{R}^{2d}} \alpha(v, v') f(t, x, v') f(t, x, v) \, dv' dv + \rho(t, x) \int_{\mathbb{R}^d_v} \beta(v) f(t, x, v) \, dv \tag{1.2}$$

supplied with initial data $0 \le f^0(x, v) \in L^1(\mathbb{R}^{2d})$ and $0 \le \rho^0(x) \in L^1(\mathbb{R}^d_x)$.

The functions $\alpha(v, v')$ and $\beta(v)$ are collision or coagulation kernels and give the probability that two free particles with velocities v and v' will coagulate for α or one free particle with velocity v will coagulate with a stuck particle for β.

The collision kernels $\alpha(v, v')$, $\beta(v)$ are nonnegative. In most physical situations they behave polynomially; moreover by Galilean invariance, α should essentially depend on the relative velocity of two particles $v - v'$. For these reasons we assume that they satisfy the following domination property: there exists a constant $C > 0$ such that

$$\alpha(v, v') \le C|v - v'|^a, \;\; \beta(v) \le C|v|^a, \quad \text{for some } a \in \mathbb{R}. \tag{1.3}$$

Note nevertheless that some of the results that we shall present here can be generalized to abstract kernels (only integrability conditions and dependence on $v - v'$ assumed), even measure-valued kernels.

In some biological situations, coagulation between two cells touching each other would always occur. This would correspond to $a = +\infty$ and would lead to a sort of sticky particles dynamics. Even in dimension 1, the analysis of such models is quite difficult (see for instance [53]) and especially so for the modified models that one would obtain in this case.

The main result in the chapter is the characterization of the asymptotic behavior depending on a. It is obvious from the equations that the mass associated to the population of free particles may only decrease and the mass associated to the population of coagulated particles may only increase. Hence the main issue as $t \to +\infty$ is whether all free particles finally coagulate or if some of them remain free. We show that this depends only on the strength of the interactions (i.e. the value of a). The analysis is based on precise dispersion estimates for kinetic equations.

With respect to classical kinetic coagulation models, the existence and uniqueness theory is quite simple as a priori estimates are obtained in a standard way. It is nevertheless included for the sake of completeness.

We may summarize the results of the chapter with the following

Theorem 5.1.1 *Assume that the integral kernels are non-negative, satisfy* (1.3) *and* $a + d > 0$. *For any* $0 \le f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $0 \le \rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ *and such that for some* $\eta > 0$ *there holds that*

$$f^0(x, v) \le \frac{C}{1 + |v|^{\max(a,0)+d+\eta}}, \; \textit{for a.e. } (x, v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v, \tag{1.4}$$

there exists a weak solution of the system (1.1)–(1.2) *with initial data* $f(0, x, v) = f^0$ *and* $\rho(0, x) = \rho^0$. *If this weak solution can be approximated strongly in* $L^\infty(0, T, L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v) \times L^1(\mathbb{R}^d_x))$ *by a sequence of smooth solutions then this weak solution is unique. Moreover, there exists a function* $g_\infty(x, v)$ *such that*

$$\left\| f(t, x, v) - g_\infty\left(\frac{x}{t}, t(v - \frac{x}{t})\right) \right\| \to 0 \quad \textit{as } t \to \infty$$

in the norm of $W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$. *Furthermore,*

- *if $a > 1-d$ (or $a > 1$ if $d = 1$) and f^0 and ρ^0 are compactly supported in x, the amount of mass $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ is bounded from below by a positive constant independently of time.*
- *if $-d < a \leq 1-d$, the amount of mass $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ is strictly positive for all times but converges to zero as t goes to infinity.*

These results were the subject of [69]. Existence and uniqueness are dealt with in the second section, where we also explain what do we mean by weak solution. The traveling wave form of the solution is proved in the last section. Section 5.3 investigates the issue of vanishing free particles.

The non-negativity of the kernels and the condition (1.3) will be assumed (with $C = 1$) for the rest of the chapter, with no further mention.

5.2 Existence and Uniqueness

In this section we state our concept of solution and prove existence and uniqueness under certain decay assumptions for the initial data and for the integral kernels.

Definition 5.2.1 *A* weak solution *of the system (1.1)–(1.2) in the time interval* $[0,T]$ *is a pair of nonnegative functions* $f \in L^\infty([0,T],\ L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v))$, $\rho \in L^\infty([0,T],\ L^1(\mathbb{R}^d_x))$ *with initial data* $0 \leq f^0(x,v) \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ *and* $0 \leq \rho^0(x) \in L^1(\mathbb{R}^d_x)$ *and which satisfies the following weak formulation:*

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} \frac{\partial \varphi}{\partial t} f\, dtdxdv - \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} \varphi(0,x,v) f^0(x,v)\, dxdv \\
&-\int_0^T \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} v \cdot \nabla_x \varphi f\, dtdxdv \\
&\quad = -\int_0^T \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} \varphi f \left[\int_{\mathbb{R}_{v'}} \alpha(v,v') f(t,x,v')\, dv' + \beta(v)\rho(t,x) \right] dtdxdv
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^d_x} \frac{\partial \psi}{\partial t} \rho\, dxdt - \int_{\mathbb{R}^d_x} \psi(0,x)\rho(0,x)\, dx \\
&\quad = \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x)\rho(t,x) \int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv\, dtdx \\
&\quad + \int_0^T \int_{\mathbb{R}^d_x} \psi(t,x) \int_{\mathbb{R}^d_v} \int_{\mathbb{R}_{v'}} \alpha(v,v') f(t,x,v) f(t,x,v')\, dv' dvdxdt,
\end{aligned}
$$

for every $\varphi \in \mathcal{D}([0,T[\times \mathbb{R}^d_x \times \mathbb{R}^d_v)$ *and every* $\psi \in \mathcal{D}([0,T[\times \mathbb{R}^d_x)$.

A-priori estimates will show that for initial data in an appropriate class the property $\partial_t f \in L^\infty([0,T], W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v)))$ holds, so that the statement $f(0,x,v) = f^0(x,v)$ makes sense. To give a meaning for $\rho(0,x) = \rho^0(x)$ is easier, because this holds in $L^1(\mathbb{R}^d_x)$.

We need to introduce some extra notation:

Definition 5.2.2 *The* density function *associated with the population f is given by*

$$\rho_f(t,x) = \int_{\mathbb{R}^d_v} f(t,x,v)\ dv.$$

Definition 5.2.3 *The* total mass *of the system is represented by the quantity*

$$M = \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} f(t,x,v)\,dxdv + \int_{\mathbb{R}^d_x} \rho(t,x)\,dx$$

We explain here some conventions that will be used in the chapter. We will use $B(r)$ to refer to a ball centered at the origin with radius r. The space on which this ball is considered will either be clear from the context or indicated by a proper subscript. $B(r)^c$ denotes the complement of such a ball in its corresponding space. We use $|A|$ to represent the Lebesgue measure of a set A.

The following stability result essentially implies the existence result in Theorem 5.1.1 (as the approximation of our system does not pose any problem)

Theorem 5.2.4 *Consider $a > -d$. Assume that $f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $\rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ and that for some $\epsilon > 0$ the following bound is verified:*

$$f^0(x,v) \le \frac{C}{1+|v|^{\max\{a,0\}+d+\epsilon}} \quad a.e.\ (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v. \tag{2.5}$$

Then, any sequence $\{(f_n,\ \rho_n)\}$ of smooth solutions to (1.1)–(1.2) converges weakly in any $L^p([0,T]\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$, $1 \le p \le \infty$, to a weak solution $(f,\ \rho)$ of (1.1)–(1.2).

Remark 5.2.5 The assumptions are quite reasonable from the point of view of applications to physics or biology. They imply that

$$(1+|v|^\epsilon)f^0 \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v) \cap L^1(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x)).$$

The way in which the existence result is stated is not completely sharp. This can be seen tracking carefully the proof. We could also use a more technical proof and sharpen the result even more, lowering the integral regularity which is required for the initial data. Generally speaking the less we ask for f^0, the more we have to demand from ρ^0, and vice versa. However the corresponding assumptions are not easy to state; we prefer to restrict to this non optimal form of the result.

Proof. First we outline the proof for the case $a \ge 0$, then we explain the modifications that are needed for the case $a < 0$.

Note that introducing the characteristics curves for (1.1), which are straight lines indeed, we get a representation of $f(t,x,v)$ as

$$f(t,x,v) = f^0(x-vt,v)\,m(t,x,v),$$

with $m \in [0,\ 1]$ a damping factor. This shows that $f(t)$ is nonnegative if f^0 is, and we also get some a priori estimates as a consequence. These are gathered here.

Lemma 5.2.6 *For any $0 < t < T$ and any solution of (1.1)–(1.2), the following estimates hold:*

1. $f(t,x,v) \le f^0(x-vt,v)$.

2. $\|f(t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le \|f^0\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)}$, $1\le p\le\infty$.

3. $\|\rho_f(t)\|_{L^p(\mathbb{R}^d_x)} \le \|f^0\|_{L^1(\mathbb{R}^d_v,L^p(\mathbb{R}^d_x))}$, $1\le p\le\infty$.

4. $\displaystyle\int_{\mathbb{R}^{2d}} \alpha(v,v')f(t,x,v')f(t,x,v)\,dv'dv \in L^\infty([0,T]\times\mathbb{R}^d_x)$.

5. $\displaystyle\int_{\mathbb{R}^d_v} \beta(v)f(t,x,v)\,dv \in L^\infty([0,T]\times\mathbb{R}^d_x)$.

Proof. Estimate *2* follows from *1*; estimate *3* follows from *1* and Minkowsky's inequality. To prove *4*, we recall that for a given $a>0$ there exists a constant $C=2^{\max\{0,\ a-1\}}>0$ such that $|v-v'|^a \le C(|v|^a+|v'|^a)$. Using this fact,

$$\begin{aligned}
&\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v)f(t,x,v')\,dvdv' \\
&\le 2C\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v|^a f(t,x,v)f(t,x,v')\,dvdv' \\
&\le C\|\rho_f\|_\infty \int_{\mathbb{R}^d_v} |v|^a f^0(x-vt,v)\,dv,
\end{aligned}$$

where we used *3* to ensure the finiteness of $\|\rho_f\|_\infty$. Using then (2.5) we conclude the proof of *4*. The one for *5* is similar. □

Next, we integrate the equation (1.2) and knowing that $f(t)$ is nonnegative we infer that $\rho(t)$ has also this property if it does initially. At this stage it is then meaningful to introduce the total mass of the system M, which is conserved during the evolution for classical solutions and therefore trivially non-increasing in the general case.

The conservation of mass shows that $\rho(t)\in L^1(\mathbb{R}^d_x)$ uniformly in time. If we prove that $\rho(t)$ is bounded in some $L^p(\mathbb{R}^d_x)$ space we can show in the usual way the convergence of all the linear terms involved in the weak formulation. Indeed, we can get an estimate for ρ in $L^\infty([0,T]\times\mathbb{R}^d_x)$, as (1.2) is readily integrated and then estimates *4* and *5* of Lemma 5.2.6 allow to deduce it.

The last point to prove the stability result is to show the convergence of the product terms. We recall here an useful result which can be found in [75]. Here $\mathcal{T}$ denotes the transport operator, the left hand side of (1.1).

Lemma 5.2.7 *Suppose that* $\{g_n\}\subset L^1(]0,T[,L^1_{loc}(\mathbb{R}^d_x\times\mathbb{R}^d_v))$ *is weakly relatively compact, and that* $\{\mathcal{T}g_n\}$ *is weakly relatively compact in* $L^1_{loc}(]0,T[\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$. *Then, if* $\{\psi_n\}$ *is a bounded sequence in* $L^\infty(]0,T[,L^\infty_{loc}(\mathbb{R}^d_x\times\mathbb{R}^d_v))$ *that converges a.e., then* $\int g_n\psi_n\,dv$ *is strongly compact in* $L^1(]0,T[,L^1_{loc}(\mathbb{R}^d_x))$.

We describe here how to deal with one of the product terms. We can estimate the corresponding difference

$$\left|\int_0^T\int_{\mathbb{R}^d_x}\psi(t,x)\int_{\mathbb{R}^d_v\times\mathbb{R}^d_{v'}}\alpha(v,v')\big(f(t,x,v)f(t,x,v')-f_n(t,x,v)f_n(t,x,v')\big)\,dvdv'dxdt\right|$$

using terms like

$$\left|\int_0^T \int_{\mathbb{R}^d_x} \psi(t,x) \int_{(B_v(R)\times B_{v'}(R))^c} \alpha(v,v') f_n(v) f_n(v') \; dvdv'dxdt\right|$$

and

$$\int_0^T \int_{\mathbb{R}^d_x} |\psi(t,x)| \; \sup_{v\in B_v(R)} |f_n(v)|$$

$$\times \int_{B_v(R)} \left| \int_{B_{v'}(R)} \alpha(v,v') f_n(v') \; dv' - \int_{B_{v'}(R)} \alpha(v,v') f(v') \; dv' \right| \; dvdxdt,$$

where for the sake of simplicity we have omitted the dependence on x as this does not cause confusion. The aim is to make these quantities less than any given $\epsilon > 0$. To control terms as the first one, we use a parameter $R > 0$ and estimate as follows:

$$\begin{aligned}
&\int_{(B_v(R)\times B_{v'}(R))^c} \alpha(v,v') f_n(v) f_n(v') \; dvdv' \\
&\le \int_{(B_v(R)\times B_{v'}(R))^c} \alpha(v,v') \frac{(|v|^2+|v'|^2)^{r/2}}{(|v|^2+|v'|^2)^{r/2}} f_n(v) f_n(v') \; dvdv' \\
&\le \frac{1}{R^r} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} \alpha(v,v') \left[\sqrt{|v|^2+|v'|^2}\right]^r f_n(v) f_n(v') \; dvdv' \le \frac{C}{R^r}.
\end{aligned}$$

This works thanks to (2.5), for $r > 0$ suitably small. Then, we choose $R^r = \frac{2}{\epsilon} C\|\psi\|_1$ and we can force this type of terms to be smaller than $\epsilon/2$.

It remains to show that the terms involving velocity averages over a compact set can also be made as small as wanted. To use the averaging results it suffices to show that $\mathcal{T} f_n$ is uniformly in $L^{1+\epsilon}([0,T], L^1_{loc}(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ for some $\epsilon > 0$ and in this way we avoid concentration phenomena —thus Dunford–Pettis theorem applies. To do so, notice that we already know that $f_n, \rho_n \in L^1 \cap L^\infty$. The integral $\int_{\mathbb{R}^d_{v'}} \alpha(v,v') f(t,x,v') \; dv'$ is then bounded a.e. (t,x). Finally, $\beta(v) f(t,x,v)$ belongs to $L^\infty([0,T]\times\mathbb{R}^d_x, L^p(\mathbb{R}^d_v))$ for any $p > 1$, as thanks to (2.5) we get

$$(\beta(v) f_n(t,x,v))^p \le \left(\frac{C|v|^a}{1+|v|^{a+d+\epsilon}}\right)^p \le \left(\frac{C}{1+|v|^{d+\epsilon}}\right)^p.$$

The rest of the product terms can be handled with slight variations of the arguments sketched above.

The case $a < 0$: to proceed we introduce for the remaining of the section the notation

$$q := d/|a|,$$

which comes from the fact that $|\cdot|^{-|a|} \in L^q_w(\mathbb{R}^d)$. Basic facts about weak Lebesgue spaces can be found in [42, 147]. The main differences with the previous case are the following:

- The estimates *4-5* of Lemma 5.2.6 are proved in a different way. For *4* we use the Hardy–Littlewood–Sobolev inequality [216], combined with suitable spatial regularity. This would require

$$f^0 \in L^{\frac{2q}{2q-1}}(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x)), \tag{2.6}$$

which by interpolation is always true as thanks to (2.5) our initial datum f^0 belongs to $L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v) \cap L^1(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x))$. Note that $\frac{2q}{2q-1} = \frac{2d}{2d+a}$. The estimate *5* is dealt away combining Hölder's inequality with a layer-cake-type argument. More precisely:

Lemma 5.2.8 *Let $g \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, with $p > q'$. Define*

$$e(p,q) = \frac{(p')^2}{q - p'(1-p')}.$$

Then, for any $\lambda \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \frac{|g(x)|}{|x-\lambda|^{|a|}}\, dx \le C(p,q) \|g\|_1^{1-e(p,q)} \|g\|_p^{e(p,q)}.$$

The use of this result to obtain the estimate *5* requires

$$f^0 \in (L^1 \cap L^{\frac{d}{d+a}+\delta})(\mathbb{R}^d_v, L^\infty(\mathbb{R}^d_x))$$

for some $\delta > 0$ suitably close to zero. This is again implied by (2.5) and $f^0 \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$. Note that the hypothesis (2.6) is contained in this one and these in turn are implied by the assumptions in Theorem 5.2.4.

- To have the convergence of the product terms the procedure is different. First of all, we use Lemma 5.2.7 or a similar result (the ones in [86, 103, 127] for instance) to prove the convergence for a regularized kernel. Secondly we show that the integral against the difference between the regularized kernel and the non regularized one tends to 0 as the parameter of regularization tends to 0 and this uniformly in n. This is easily implied by the uniform bounds on f_n.

□

5.2.1 Uniqueness

We have uniqueness in the class of weak solutions that can be approximated by classical solutions in $L^\infty(0,T,L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v) \times L^1(\mathbb{R}^d_x))$, thanks to the following result.

Proposition 5.2.9 *Any weak solution of (1.1)–(1.2) which is limit of classical solutions and satisfies the assumptions of Theorem 5.2.4 is unique.*

Proof. We consider two solutions (f_1, ρ_1), (f_2, ρ_2). Since those are the limits of classical solutions, justifying the computations performed below is easy: consider them as classical solutions and simply pass to the limit at the end.

Let us introduce the functions $g = f_1 - f_2$ and $h = \rho_1 - \rho_2$. We will conclude uniqueness with a Grönwall argument applied to the integral of $|g| + |h|$. First we compute an equation for g.

$$\begin{aligned}
\partial_t g + v \cdot \nabla_x g = &- g(t,x,v) \int_{\mathbb{R}^d_{v'}} \alpha(v,v') f_1(t,x,v')\, dv' \\
&- f_2(t,x,v) \int_{\mathbb{R}^d_{v'}} \alpha(v,v') g(t,x,v')\, dv' \\
&- h(t,x)\beta(v) f_2(t,x,v) - \beta(v) g(t,x,v) \rho_1(t,x).
\end{aligned}$$

We use this to obtain an equation for $\int |g|\, dxdv$. To do so, an equation for $\frac{d}{dt}\int \phi_n(g)\, dxdv$ is computed first, being ϕ_n a suitable smooth approximation of the sign function —a.e. convergent and matching ± 1 outside a compact set containing the origin—, and then we pass to the limit on that equation. Thus, we get

$$\begin{aligned}\frac{d}{dt}\int |g(t,x,v)|\, dxdv = &-\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |g(t,x,v)|\alpha(v,v')f_1(t,x,v')\, dv'dvdx \\ &-\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} f_2(t,x,v)\alpha(v,v')g(t,x,v')\mathrm{sign}\,[g(t,x,v)]\, dv'dvdx \\ &-\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \beta(v)\rho_1(t,x)|g(t,x,v)|\, dxdv \\ &-\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} h(t,x)\beta(v)f_2(t,x,v)\mathrm{sign}\,[g(t,x,v)]\, dxdv.\end{aligned}$$

Then we compute an equation for h,

$$\begin{aligned}\partial_t h = &\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f_1(t,x,v')g(t,x,v)\, dvdv' \\ &+\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f_2(t,x,v)g(t,x,v')\, dvdv' \\ &+\left(\int_{\mathbb{R}^d_v}\beta(v)f_2(t,x,v)\, dv\right)h(t,x)+\rho_1(t,x)\left(\int_{\mathbb{R}^d_v}\beta(v)g(t,x,v)\, dv\right),\end{aligned}$$

so that, doing as before, we find

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d_x}|h(t,x)|\, dx = &\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f_1(t,x,v')g(t,x,v)\mathrm{sign}[h(t,x)]\, dxdvdv' \\ &+\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f_2(t,x,v)g(t,x,v')\mathrm{sign}[h(t,x)]\, dxdvdv' \\ &+\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} |h(t,x)|\beta(v)f_2(t,x,v)\, dxdv \\ &+\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} \beta(v)g(t,x,v)\rho_1(t,x)\mathrm{sign}[h(t,x)]\, dxdv.\end{aligned}$$

Adding both we get to

$$\begin{aligned}&\frac{d}{dt}\left[\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}|g(t,x,v)|\, dxdv+\int_{\mathbb{R}^d_x}|h(t,x)|\, dx\right] \\ &\leq 2\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} f_2(t,x,v)\alpha(v,v')|g(t,x,v')|\, dxdvdv' \\ &+2\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}|h(t,x)|\beta(v)f_2(t,x,v)\, dxdv.\end{aligned}$$

In case that $a<0$ we use Lemma 5.2.8 to show that

$$\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}|h(t,x)|\beta(v)f_2(t,x,v)\, dxdv \leq C\int_{\mathbb{R}^d_x}|h(t,x)|\, dx$$

and likewise

$$\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} f_2(t,x,v)\alpha(v,v')|g(t,x,v')|\, dxdvdv' \le C\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_{v'}} |g(t,x,v')|\, dxdv'.$$

Whereas if $a > 0$ we achieve the same inequality for the β-integral using the compact support in velocities. The same can be done for the α-integral if the velocity supports of the solutions under consideration are compact, so that $\alpha(v,v')$ can be majored independently of v', which is the case. □

5.3 Large time behaviour

The aim of this section is to investigate the behavior of the solution for large times. From the very form of the equations of the model we can see that f will lose mass progressively, which in principle will be transferred to the population ρ. The issue that we address here is the following: Does the species f eventually vanish completely, thus transferring all its mass to ρ, or some of this mass is going to be lost to infinity? Under some decay assumptions on the initial data we will show that this is not so for the range $a \in]1-d,+\infty]$, while the total transfer of mass is achieved in infinite time for powers $a \in]-d, 1-d]$.

We use the notation

$$M(t) = \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_v} f(t,x,v)\, dxdv$$

for the mass carried by the species f. We will typically need some compactness assumptions on the initial data. Compactness may be assumed in space or in velocity and we introduce the following set of notations and possible assumptions:

$$\text{supp } \rho^0_f \subset B(R) \text{ for some } R < \infty. \tag{3.7}$$

$$\cup_{x\in\mathbb{R}^d_x} \text{supp}_v\ f^0 \subset B(V) \text{ for } V := \operatorname{ess\,sup}\{|v|/f^0(x,v) > 0\} < \infty. \tag{3.8}$$

$$\text{supp } \rho^0 \subset B(\tilde{R}) \text{ for some } \tilde{R} < \infty. \tag{3.9}$$

5.3.1 The non-vanishing case: bounded velocity supports.

We start by the simplest case with bounded compact support in space and velocity. This will be extended in the next subsection but for the sake of a better understanding we present the main arguments (some dispersive inequalities) in this simplified setting. This section is devoted to the proof of the following statement:

Proposition 5.3.1 *For $a + d > 1$, assume that $\rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$, that $f^0 \in (L^1 \cap L^\infty)(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and verifies the hypotheses (3.7)–(3.9). Then, for each weak solution of (1.1)–(1.2) associated to this class of initial data and such that it can be approximated by smooth solutions in $L^\infty_t(0,\infty,L^1(\mathbb{R}^{2d}) \times L^1(\mathbb{R}^d_x))$, there exists a constant $C = C(f^0,\rho^0) > 0$ such that the total mass associated to f satisfies*

$$M(t) \ge C \quad \forall t \ge 0.$$

The upshot is that a certain amount of particles starting with velocities high enough do not get trapped. The proof of the above statement is done through a series of lemmas.

The following result yields some basic useful information:

Lemma 5.3.2 *Assume that $|supp\ \rho_f^0| < +\infty$. Then, the measure of the set $\{v \in \mathbb{R}^d,\ f(t,x,v) > 0\}$ decreases in time like t^{-d}, pointwise in x. This implies*

$$\rho_f(t,x) \leq \frac{|supp\ \rho_f^0| \|f^0\|_\infty}{t^d}. \tag{3.10}$$

Furthermore, for each pair $z = (v,x,t)$ and $z' = (v',x,t)$ such that $f(z) > 0$ and $f(z') > 0$, we have $|v - v'| \leq \frac{2R}{t}$.

Proof. Indeed, a particle starting from a position x_0 with a velocity v reaches the position x at time t if and only if $x = x_0 + tv$. Thus, we have that

$$v = \frac{x - x_0}{t} \in \frac{x - \text{supp } \rho_f^0}{t},$$

the latter being a set of measure $|\text{supp } \rho_f^0| t^{-d}$. □

The estimate (3.10) shows that all the local mass associated to f will eventually vanish; the question is how much of this is going to be transferred to ρ and how much is going to be lost to infinity. As a common framework to deal with this problem, the assumptions of Proposition 5.3.1 will be implicitly taken for granted in all the statements to follow in this section.

To proceed we introduce particular fractions of mass M_ϵ, which account for the contribution of particles with non-vanishing velocities. These masses are going to be non-vanishing for large times if this is true for short times. We do not take care of the remaining part of the initial mass. Define accordingly

$$M_\epsilon(t) = \int_{\mathbb{R}_x^d} \int_{|v|>\epsilon} f(t,x,v)\, dxdv.$$

It can be readily shown that this function satisfies the following equation:

$$\begin{aligned} \frac{dM_\epsilon}{dt} = &- \int_{\mathbb{R}_x^d} \int_{|v|>\epsilon} \int_{\mathbb{R}_{v'}^d} \alpha(v,v') f(t,x,v) f(t,x,v')\, dxdvdv' \\ &- \int_{\mathbb{R}_x^d} \int_{|v|>\epsilon} \beta(v)\rho(t,x) f(t,x,v)\, dxdv = -I - II. \end{aligned} \tag{3.11}$$

The basic estimate for M_ϵ is the following.

Lemma 5.3.3 *The function M_ϵ satisfies*

$$\frac{dM_\epsilon(t)}{dt} \geq -\frac{C}{t^{a+d}} M_\epsilon(t) - C\|\rho\|_{L^\infty(\Omega)} M_\epsilon(t),$$

with $\Omega = \{x \in \mathbb{R}^d / \exists |v| > \epsilon,\ s.t.\ f(t,x,v) > 0\}$. Furthermore, this function does not vanish in finite time.

Proof. To deal with the integral I in (3.11), note that we only have to estimate it in the following set of velocities: $\{v, v' \in \mathbb{R}^d$ such that $v - v' \in \frac{2}{t}\text{supp } \rho_f^0\}$. We can use that $a + d > 0$ to write

$$\int_{\{v'/v-v'\in\frac{2}{t}\text{supp } \rho_f^0\}} |v-v'|^a \, dv' \leq |\mathbb{S}^{d-1}| \int_0^{\frac{2R}{t}} r^{a+d-1} \, dr = \frac{|\mathbb{S}^{d-1}|}{a+d} \frac{(2R)^{a+d}}{t^{a+d}}$$

and then we get the estimate

$$I \leq \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v') \, dx dv dv'$$

$$\leq \|f_0\|_\infty \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} \left(\int_{\{v'/v-v'\in\frac{2}{t}\text{supp } \rho_f^0\}} |v-v'|^a \, dv' \right) f(t,x,v) \, dv dx$$

$$\leq \frac{\|f^0\|_\infty}{a+d} |\mathbb{S}^{d-1}| \frac{(2R)^{a+d}}{t^{a+d}} M_\epsilon(t).$$

To treat the integral II we notice that the integral with respect to x is actually computed over the set Ω. Then, if $a \geq 0$

$$II \leq V^a \|\rho(t)\|_{L^\infty(\Omega)} M_\epsilon(t)$$

and if $a < 0$,

$$II \leq \epsilon^{-|a|} \|\rho(t)\|_{L^\infty(\Omega)} M_\epsilon(t),$$

which in both cases concludes the proof of the differential inequality.

Finally, the later claim follows from the rough estimates

$$\frac{dM_\epsilon}{dt} \geq -(2V)^a \|\rho_f\|_\infty(t) M_\epsilon(t) - V^a \|\rho\|_\infty(t) M_\epsilon(t)$$

if $a \geq 0$ and

$$\frac{dM_\epsilon}{dt} \geq -C(f^0) M_\epsilon(t) - (\epsilon)^{-|a|} \|\rho\|_\infty(t) M_\epsilon(t)$$

if $a < 0$, where Lemma 5.2.8 has been used. □

In order to control the factor $\|\rho\|_{L^\infty(\Omega)}$, we also need to estimate the terms appearing in the right hand side of (1.2).

Lemma 5.3.4 *The function $\rho(t,x)$ satisfies the following inequality:*

$$\frac{\partial \rho}{\partial t}(t,x) \leq \frac{C}{t^{a+2d}} + \frac{C}{t^{a^*}} \rho(t,x),$$

with $a^* = \min\{d, a+d\}$.

Proof. The fact that $a + d > 0$ allows us to estimate

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v') \, dv' dv$$

$$\leq \|f^0\|_\infty \int_{\mathbb{R}^d_v} \int_{\{v'/v-v'\in\frac{2}{t}\text{supp } \rho_f^0\}} |v-v'|^a \, dv' f(t,x,v) \, dv$$

$$\leq \|f^0\|_\infty |\mathbb{S}^{d-1}| \int_0^{\frac{2R}{t}} r^{a+d-1} \, dr \int_{\mathbb{R}^d_v} f(t,x,v) \, dv$$

$$\leq \frac{|\mathbb{S}^{d-1}|}{a+d} \frac{(2R)^{a+d}}{t^{a+d}} \|f^0\|_\infty \frac{|\text{supp } \rho_f^0| \, \|f_0\|_\infty}{t^d} = \|f^0\|_\infty^2 \frac{|\text{supp } \rho_f^0| (2R)^{a+d}}{a+d} \frac{|\mathbb{S}^{d-1}|}{t^{a+2d}},$$

so that
$$\int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f(t,x,v)f(t,x,v')\ dv'dv \le \frac{C}{t^{a+2d}}. \tag{3.12}$$
Next, if $a \ge 0$ we find
$$\int_{\mathbb{R}^d_v} \beta(v)f(t,x,v)\ dv \le V^a \int_{\mathbb{R}^d_v} f(t,x,v)\ dv \le \frac{C}{t^d},$$
whereas if $a < 0$ we have
$$\begin{aligned}\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv &= \int_{|v|\le 1/t} |v|^a f(t,x,v)\ dv + \int_{|v|>1/t} |v|^a f(t,x,v)\ dv \\ &\le |\mathbb{S}^{d-1}|\|f^0\|_\infty \int_0^{1/t} r^{a+d-1}\ dr + t^{|a|}\int_{\mathbb{R}^d_v} f(t,x,v)\ dv \\ &\le \frac{|\mathbb{S}^{d-1}|\|f^0\|_\infty}{(a+d)t^{a+d}} + \frac{C}{t^{a+d}},\end{aligned}$$
where in both cases we have used (3.10). Summing up,
$$\int_{\mathbb{R}^d_v} \beta(v)f(t,x,v)\ dv \le \frac{C}{t^{a^*}}. \tag{3.13}$$
□

Now we go to the core of our method of proof.

Lemma 5.3.5 *Given $t_0 > 2R/V$, if $V > \epsilon \ge \frac{2R}{t_0}$ the sets $\{(x,v)/|x| \le R + Vt_0\}$ and supp $f(2Vt_0/\epsilon,\cdot,\cdot)\cap\{(x,v)/|v|>\epsilon\}$ are disjoint.*

Proof. Any pair $(x,v)\in\{|x|\le R+Vt_0\}\cap \text{supp } f(2Vt_0/\epsilon,\cdot,\cdot)$ satisfies the relation $2Vt_0|v|/\epsilon - R \le R + Vt_0$, and then $|v| \le \frac{R}{t_0}+\frac{\epsilon}{2}$. □

In the next result we obtain some control over the size of the support of $\rho(t)$. This result is the principal technical difference between the case that we are considering here and the non-compactly supported one.

Lemma 5.3.6 *Whenever $t > \overline{\tau} = \max\{\frac{\tilde{R}-R}{V}, 0\}$, we have that supp $\rho(t) \subset B(R+Vt)$.*

Proof. Integrating (1.2) we deduce that
$$\text{supp } \rho(t) \subset \text{supp } \rho_0 \cup \left(\cup_{\tau\le t} \text{ supp } \rho_f(\tau)\right).$$
□

Integration of the inequality for ρ, given by Lemma 5.3.4, yields the estimate
$$\rho(t,x) \le \rho(t_0,x)\exp\left\{\int_{t_0}^t \frac{C}{\tau^{a^*}}\ d\tau\right\} + \int_{t_0}^t \frac{C}{\tau^{a+2d}}\exp\left\{\int_\tau^t \frac{C\ ds}{s^{a^*}}\right\}\ d\tau.$$
If we consider it in the range $t_0 > \overline{\tau}$ and $|x| > R + Vt_0$ we get rid of the first term. From now on we set
$$t_0 := \epsilon t/(2V) > \overline{\tau}$$

(the range of t is restricted accordingly) and thus

$$\rho(t,x) \leq \int_{\epsilon t/(2V)}^{t} \frac{C}{\tau^{a+2d}} \exp\left\{ \int_{\tau}^{t} \frac{C\, ds}{s^{a^*}} \right\} d\tau,$$

so that

$$\rho(t,x) \leq \frac{C(2^{a+2d-1}-1)}{a+2d-1} \frac{1}{t^{a+2d-1}} \exp\left\{ \frac{C(2^{a^*-1}-1)}{a^*-1} \frac{1}{t^{a^*-1}} \right\}.$$

We shall substitute this estimate into the inequality granted by Lemma 5.3.3, with Lemma 5.3.5 assuring that Ω does not include the region $|x| \leq R + Vt_0$. We are left with

$$\frac{dM_\epsilon}{dt}(t) \geq -\frac{C}{t^{a+d}} M_\epsilon(t) - \frac{C}{t^{a+2d-1}} \exp\left\{ \frac{C}{t^{a^*-1}} \right\} M_\epsilon(t).$$

After integration in time,

$$M_\epsilon(t) \geq M_\epsilon(t_0) \exp\left\{ -\int_{t_0}^{t} \frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}}\, d\tau \right\}.$$

If we show that the above integral is convergent we can perform the limit $t \to \infty$ to obtain that M_ϵ does not vanish. Simply note that for τ big enough

$$\frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}} \leq \frac{C}{\tau^{a+d}}.$$

So that, as $a+d-1>0$,

$$\exp\left\{ -\int_{t_0}^{t} \frac{C}{\tau^{a+d}} + \frac{Ce^{\tau^{1-a^*}}}{\tau^{a+2d-1}}\, d\tau \right\} \geq \exp\left\{ C\left(\frac{1}{t^{a+d-1}} - \frac{1}{t_0^{a+d-1}} \right) \right\}.$$

Meaning that

$$M_\epsilon(\infty) \geq M_\epsilon(t_0) \exp\left\{ -\frac{C}{t_0^{a+d-1}} \right\},$$

or that the total mass, which is larger than M_ϵ, may not vanish.

Finally notice that the restrictions concerning the time for the above arguments to be valid are:

- $t \geq \frac{4RV}{\epsilon^2}$ to assure the applicability of Lemma 5.3.5.
- $t > 2\frac{V\overline{\tau}}{\epsilon}$ to be able to control the growth of the supports of both species in an easy way.

It is always possible to work in this range as no mass may vanish in finite time.

5.3.2 Non-vanishing case: unbounded velocity supports.

Here we extend the result of the previous section in order to allow unbounded velocities. As a consequence compactness in velocity will be replaced by a more precise decay assumption.

Only the case $d > 1$ will be considered for the moment; we defer the special case $d = 1$ to the next subsection.

Proposition 5.3.7 *Assume $a+d>1$ and that for some $\eta>0$ we have*

$$f^0(x,v) \le \frac{C}{1+|v|^{\max(a,0)+d+\eta}} \quad a.e.\ (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v. \tag{3.14}$$

We also assume that the compact support conditions (3.7) and (3.9) hold. Then, for any weak solution of (1.1)–(1.2) given by Theorem 5.2.4, there exists a constant $C = C(f^0, \rho^0) > 0$ such that

$$M(t) \ge C \quad \forall t \ge 0.$$

Note that the assumptions imply that $f^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and that $f^0 \in L^1(\mathbb{R}^d_v,\ L^\infty(\mathbb{R}^d_x))$. The rest of the section is devoted to prove this statement. The assumptions of Proposition 5.3.7 are implicitly taken for granted in all the intermediate lemmas.

It is still easy to check that no mass may vanish in finite time.

Lemma 5.3.8 *The functions M_ϵ do not vanish in finite time.*

Proof. Set for any $k>1$

$$M_{\epsilon,k\epsilon}(t) := \int_{\mathbb{R}^d_x} \int_{k\epsilon>|v|>\epsilon} f(t,x,v)\, dxdv.$$

Choose a number $k>1$ such that $M_{\epsilon,k\epsilon}(0)>0$. Then the same proof as in Lemma 5.3.3 with $V = k\epsilon$ ensures that the function $M_{\epsilon,k\epsilon}(t)$ does not vanish in finite time. Thus, being $M_\epsilon(t) \ge M_{\epsilon,k\epsilon}(t)$ the statement follows. □

Let us turn to the crucial estimates in large times. It first goes along the same lines as for the case with full compact support.

Lemma 5.3.9 *The function M_ϵ satisfies*

$$\frac{d\,M_\epsilon}{dt} \ge -\frac{C}{t^{a+d}}\, M_\epsilon - C\, M_\epsilon(t) \sup_{|x|\ge t\epsilon/2} (1+|x|)^{\max(a,0)}\, \rho(t,x)$$

for $t > 2R/\epsilon$.

Proof. Let us start with the integral II in (3.11). Consider first the case $a \ge 0$. As f^0 is compactly supported in x, whenever $f>0$ then $v \in B(x/t, R/t)$ and the following chain of estimates

$$\begin{aligned}\int_{\mathbb{R}^d_x\times\{|v|>\epsilon\}} \beta(v)\, \rho(t,x)\, f(t,x,v)\, dx\, dv &\le \int_{\mathbb{R}^d_x\times\{|v|>\epsilon\}} |v|^a\, \rho(t,x)\, f(t,x,v)\, dx\, dv \\ &\le \frac{C}{t^a} \int_{\mathbb{R}^d_x\times\{|v|>\epsilon\}} (R+|x|)^a\, \rho(t,x)\, f(t,x,v)\, dx\, dv\end{aligned}$$

holds. Since $|v| \ge \epsilon$ any particle with such speed issuing form a point x_0 occupies at time t a position x that verifies $|x-x_0| \ge \epsilon t$. If $t > 2R/\epsilon$, then $|x| \ge \frac{\epsilon t}{2}$. Thus, we have

$$\int_{\mathbb{R}^d_x\times\{|v|>\epsilon\}} \beta(v)\, \rho(t,x)\, f(t,x,v)\, dx\, dv \le C\, M_\epsilon(t) \sup_{|x|\ge t\epsilon/2} (1+|x|)^a\, \rho(t,x).$$

In the case $a < 0$, one simply has

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\, \rho(t,x)\, f(t,x,v)\, dx\, dv$$
$$\leq \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} |v|^a \,\rho(t,x)\, f(t,x,v)\, dx\, dv$$
$$\leq \epsilon^a \int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \rho(t,x)\, f(t,x,v)\, dx\, dv$$
$$\leq \epsilon^a\, M_\epsilon(t) \sup_{|x|\geq t\epsilon/2} \rho(t,x).$$

Combining both, one gets in every case

$$\int_{\mathbb{R}^d_x \times \{|v|>\epsilon\}} \beta(v)\, \rho(t,x)\, f(t,x,v)\, dx\, dv$$
$$\leq C\, M_\epsilon(t) \sup_{|x|>t\epsilon/2} (1+|x|)^{\max(a,0)}\, \rho(t,x).$$

The control of I follows the line of the case with compact support in velocity. Again if $f(t,x,v) > 0$ and $f(t,x,v') > 0$ then $|x - vt| \leq R$ and $|x - v't| \leq R$ so that $|v - v'| \leq 2R/t$ and

$$I \leq C \int_{\mathbb{R}^d_x} \int_{|v|>\epsilon} f(t,x,v) \int_{v' \in B(v,2R/t)} |v - v'|^a\, dv'\, dv\, dx \leq \frac{C}{t^{d+a}}\, M_\epsilon.$$

□

So now we have to exhibit some decay for the x moment of ρ that appears in Lemma 5.3.9. We start with a technical result that will prove useful in the sequel.

Lemma 5.3.10 *The estimate*

$$\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv \leq \frac{C}{t^d}.$$

is verified for a.e. $|x| \geq \epsilon\, t/2$ *and for* $t > 4R/\epsilon$.

Proof. Note that since f^0 is compactly supported in x, in case that $f(t,x,v) > 0$ then $|x - vt| \leq R$. So that $v \in B(x/t, R/t)$ holds under these circumstances. In particular, if $|x| \geq \epsilon\, t/2$ then $|v| > \epsilon/4$ for t large enough ($t > 4R/\epsilon$ indeed).

That means that in the case $a < 0$ we will have $|v| > \epsilon/2 + |v|/2$ and so $|v|^a < (\epsilon/2 + |v|/2)^a$. Obviously if $a > 0$ then $|v|^a \leq (\epsilon + |v|)^a$. Consequently,

$$\int_{\mathbb{R}^d_v} |v|^a f(t,x,v)\ dv \leq \int_{B(x/t,R/t)} |v|^a f^0(x - vt, v)\ dv$$
$$\leq\ C \sup_{v \in \mathbb{R}^d} (\epsilon + |v|)^a\, f^0(x - vt, v) \int_{B(x/t,R/t)} dv \leq \frac{C}{t^d}.$$

This holds also when $a > 0$, as $|v|^a \leq (\epsilon + |v|)^a$ in this case, being the supremum finite thanks to (3.14). □

Next we bound ρ in terms of the quadratic terms of the equation.

Lemma 5.3.11 *The following inequality*

$$\rho(t,x) \leq C \int_0^t \int_{\mathbb{R}^d_{v'}\times\mathbb{R}^d_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\ dvdv' \tag{3.15}$$

holds for any $|x| > \epsilon\, t/2$ *and* $t > \frac{2R}{\epsilon}$.

Proof. If $|x| > \epsilon\, t/2$ then we know that x is not in the initial support of ρ^0 for $t > \frac{2R}{\epsilon}$ and the integration of (1.2) gives

$$\begin{aligned} \rho(t,x) = &\int_0^t \int_{\mathbb{R}^d_{v'}\times\mathbb{R}^d_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\ dvdv' \\ &\times \exp\left\{ \int_\tau^t \int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dvds \right\}\ d\tau. \end{aligned}$$

A direct application of Lemma 5.3.10 yields the estimate

$$\int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dv \leq \int_{\mathbb{R}^d_v} |v|^a\, f(s,x,v)\ dv \leq \frac{C}{s^d}.$$

So finally we have

$$\int_\tau^t \int_{\mathbb{R}^d_v} \beta(v) f(s,x,v)\ dvds \leq C$$

for $\tau \geq 1$. To control the integration between 0 and τ, one simply uses that the integral $\int_{\mathbb{R}^d_v} |v|^a f(s)\, dv$ is bounded for any value of s. This is due to (3.14) and the fact that $a > -d$. Therefore the lemma is proved. □

Now we are to estimate the integral term in (3.15). The following result does the job.

Lemma 5.3.12 *The estimate*

$$\begin{aligned} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\, dvdv' \leq &\frac{C(R)}{|x|^k t^{a+2d-k}} \\ &\times \|(1+|v|)^{k/2} f^0(x-vt,v)\|^2_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \end{aligned}$$

holds true for any $k \geq 0$.

Proof. Use the bound $f \leq f^0(x-vt,v)$ and the compact spatial support of f^0 to get $|x| \leq |x-vt| + |v|\, t \leq R + |v|\, t$ and hence

$$\begin{aligned} &|x|^k \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\, dvdv' \\ &\leq C \int_{|v-v'|\leq C/t} |v-v'|^a\, (R+|v|\, t)^{k/2}\, f(t,x,v) (R+|v'|\, t)^{k/2}\, f(t,x,v')\, dvdv'. \end{aligned}$$

This is in turn dominated by

$$
\begin{aligned}
&C\,t^k \int_{|v-v'|\le C/t} |v-v'|^a\,(1+|v|^{k/2})\,f(t,x,v)(1+|v'|^{k/2})\,f(t,x,v')\,dvdv' \\
\le& C\,t^k\,\|(1+|v|^{k/2})\,f(t)\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \int_{\mathbb{R}^d_v} (1+|v|^{k/2})\,f(t,x,v)\,dv \\
&\times \int_{|v-v'|\le \frac{C}{t}} |v-v'|^a\,dv'.
\end{aligned}
$$

Using Lemma 5.3.10 we finally arrive to

$$
\begin{aligned}
&|x|^k \int_{\mathbb{R}^d_v}\int_{\mathbb{R}^d_{v'}} |v-v'|^a f(t,x,v) f(t,x,v')\,dvdv' \\
&\le \frac{Ct^k}{t^{a+d}}\,\|(1+|v|^{k/2})\,f(t)\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \int_{\mathbb{R}^d_v} (1+|v|^{k/2})\,f(t,x,v)\,dv \\
&\le \frac{C}{t^{a+2d-k}}\,\|(1+|v|^{k/2})\,f(t)\|^2_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)}
\end{aligned}
$$

□

Combining Lemmas 5.3.11 and 5.3.12, we find that

$$
\rho(t,x) \le C\int_0^1 \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} |v-v'|^a f(\tau,x,v)\,f(\tau,x,v')\,dv\,dv' + \int_1^t \frac{C\,d\tau}{|x|^k\,\tau^{a+2d-k}}
$$

holds for $|x| > \epsilon\, t/2$ and t large enough. The first term is easy to bound: we use again Lemma 5.3.12, with the choice $k = 2d + a$. By means of (3.14) and taking into account that $a > 1 - d$, there exists some $\delta > 0$ in order to have

$$
\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} |v-v'|^a f(\tau,x,v)\,f(\tau,x,v')\,dv\,dv' \\
&\quad\le \frac{C}{|x|^{2d+a}}\,\|(1+|v|)^{d+a/2}\,f^0\|_{L^\infty(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le \frac{C}{|x|^{d-1+\delta}}.
\end{aligned}
$$

When computing the supremum over $|x| > \epsilon\, t/2$ we will have the inequality

$$
(1+|x|)^{\max(a,0)}\rho(t,x) \le \frac{C}{|x|^{k-\max(a,0)}} + \frac{C}{|x|^{d-1}} \le \frac{C}{|t|^{k-\max(a,0)}} + \frac{C}{|t|^{d-1+\delta}}
$$

as long as $a + 2d - k > 1$, for some $\delta > 0$. Inserting this in Lemma 5.3.9 gives

$$
\frac{d\,M_\epsilon}{dt} \ge -\frac{C}{t^{a+d}}\,M_\epsilon - \frac{C}{t^{d-1+\delta} + t^{k-\max(a,0)}}\,M_\epsilon, \tag{3.16}
$$

which, as before, shows that M_ϵ does not vanish for large times. Indeed the only constraint on $k-a$ is $k-a < 2d-1$ and one may therefore always have $k-a > 1$. Since we also have $d-1+\delta > 1$, the second coefficient is always integrable in time. This is enough to conclude, as Lemma 5.3.8 assures that M_ϵ does not vanish in finite time.

The case $d = 1$

The proof in the previous section can cover also the case $d = 1$ when some minor changes are introduced, which we indicate here briefly. We can prove the following result:

Proposition 5.3.13 *Assume that $a > 1$ and (3.14) for $d = 1$, together with the compact support hypotheses (3.7) and (3.9). Then, for any weak solution of (1.1)–(1.2) constructed in Theorem 5.2.4, there exists a constant $C = C(f^0, \rho^0) > 0$ such that*

$$M(t) \geq C \quad \forall t \geq 0.$$

We describe below a brief sketch of the modifications required for the proof given in the previous section to work in the present context.

The proof of Lemma 5.3.9 can be modified to give, in this case $d = 1$, the following result.

Lemma 5.3.14 *The function M_ϵ satisfies*

$$\frac{d\, M_\epsilon}{dt} \geq -\frac{C}{t^{a+1}}\, M_\epsilon - C\, M_\epsilon(t)\, t^{-a} \sup_{|x| \geq \epsilon/2\, t} (R + |x|)^a \, \rho(t,x),$$

for $t > 2R/\epsilon$.

We come back to (5.3.11), which we write as

$$\rho(t,x) \leq C\, t \int_0^1 \int_{\mathbb{R}_{v'} \times \mathbb{R}_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\; dv dv' d\tau$$

$$+ C\, t \int_1^t \int_{\mathbb{R}_{v'} \times \mathbb{R}_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\; dv dv' d\tau$$

for $|x| > \epsilon\, t/2$. When inserted into the inequality of Lemma 5.3.14, we find that we have to compute the supremum of the following quantity:

$$Ct \left(\frac{|x|}{t}\right)^a \int_0^1 \int_{\mathbb{R}_{v'} \times \mathbb{R}_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\; dv dv' d\tau$$

$$+ Ct \left(\frac{|x|}{t}\right)^a \int_1^t \int_{\mathbb{R}_{v'} \times \mathbb{R}_v} \alpha(v,v') f(\tau,x,v) f(\tau,x,v')\; dv dv' d\tau.$$

That is, we are dealing with

$$C \left(\frac{|x|}{t}\right)^a t|x|^{-k_0} \int_0^1 \frac{d\tau}{\tau^{a+2-k_0}} + C \left(\frac{|x|}{t}\right)^a t|x|^{-k_\infty} \int_1^t \frac{d\tau}{\tau^{a+2-k_\infty}}$$

where Lemma 5.3.12 was applied twice. To conclude, we need $k_\infty < a+1$ and $k_0 > a+1$ to assure integrability while $k_\infty, k_0 > \max\{a, 2\}$ for compensating the factors in front and getting an overall decay better than t^{-1}. These conditions are compatible only if $a > 1$. The choices $k_\infty = \max\{a, 2\} + \delta$, $k_0 = a + 1 + \delta$ for $\delta > 0$ suitably close to zero conclude with the proof.

5.3.3 The vanishing case

In this section we study what happens in the complementary regime $-a \in [d-1, d[$. The main result is

Proposition 5.3.15 *Assume $-a \in [d-1, d[$ and that there exists some $k > 0$ such that*

$$\int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} (1 + |x|^k + |v|^k) f^0(x,v) \, dxdv < +\infty.$$

Then, for each weak solution of (1.1)–(1.2) given by Theorem 5.2.4, we have that

$$\lim_{t\to\infty} M(t) = 0.$$

Remark 5.3.16 This result shows that there occurs a total transfer of mass from f to ρ. Precise minimal rates of convergence for $M(t)$ are given in the proof for the case of smooth solutions. But note that to assure that $M(t)$ does not vanish in finite time some extra decay assumptions are needed; for instance (1.4) would do — use the proof of Lemma 5.3.8.

Proof. Suppose first that f^0 satisfies (3.7) and (3.8). Then this implies that the support of $\rho_f(t)$ lies within $B(R+Vt)$. Using Jensen's inequality we get

$$\int_{\mathbb{R}^d_x} \rho_f(t,x)^2 \, dx \geq \frac{\left(\displaystyle\int_{\mathbb{R}^d_x} \rho_f(t,x) \, dx \right)^2}{|\mathbb{S}^{d-1}|(R+Vt)^d}.$$

Recalling the differential inequality (3.11) we get

$$\begin{aligned} \frac{dM(t)}{dt} &\leq -\int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} |v-v'|^{-|a|} f(t,x,v) f(t,x,v') \, dxdvdv' \\ &\leq -\frac{t^{|a|}}{(\operatorname{diam} \operatorname{supp} \rho_f^0)^{|a|}} \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_{v'}} f(t,x,v') f(t,x,v) \, dxdvdv' \\ &= -Ct^{|a|} \int_{\mathbb{R}^d_x} \rho_f(x)^2 \, dx, \end{aligned}$$

where we have used Lemma 5.3.2, being diam the diameter of a set. Combining with the previous, we obtain the estimate

$$\frac{dM(t)}{dt} \leq -C \frac{t^{|a|} M(t)^2}{(R+Vt)^d}.$$

This implies logarithmic decay of $M(t)$ in the case $d = 1 + |a|$, and a power decay at the rate $t^{d-1-|a|}$ if $d < 1 + |a| < d+1$. In both cases the mass finally vanishes.

For the general case, we introduce a parameter V and perform the following decomposition of the initial datum:

$$f^0 = g_V^0 + f_V^0 = g_V^0 + f^0(x,v) \chi_{\{|x| \leq V\}} \chi_{\{|v| \leq V\}}.$$

The evolution of the solution is decomposed accordingly:

$$f(t) = g_V(t) + f_V(t) = g_V(t) + f(t,x,v)\chi_{\{|x|\le V\}}\chi_{\{|v|\le V\}}.$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} g_V^0 \, dxdv &= \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} \frac{|x|^k+|v|^k}{|x|^k+|v|^k} g_V^0 \, dxdv \\ &\le \frac{1}{V^k}\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} (|x|^k+|v|^k) f^0 \, dxdv \le \frac{C}{V^k}.\end{aligned}$$

The mass associated to $g_V(t)$ does not increase: this follows from the fact that the function $g_V(t)$ satisfies the equation

$$\partial_t g_V + v\cdot\nabla_x g_V = -g_V(t,x,v)\int_{\mathbb{R}^d_{v'}} \alpha(v,v')f(t,x,v')\,dv' - \beta(v)\rho(t,x)g_V(t,x,v).$$

This implies that

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} g_V(t,x,v)\,dxdv \le \frac{C}{V^k}.$$

On the other hand, we can repeat with the function $f_V(t)$ what we did before, obtaining

$$M_{f_V}(t) \le \frac{C}{\frac{1}{M(f_V^0)} + \frac{\int_0^t \frac{\tau^{|a|}}{(1+\tau)^d}\,d\tau}{V^{|a|+d}}}.$$

To conclude we optimize in V. In case that $d < |a|+1$ the mass decays at least like t to the power of $\frac{k(d-|a|-1)}{d+|a|+k}$. In the borderline case $d = |a|+1$ the mass decays as least as $\log t$ to the power of $\frac{-k}{|a|+d+k}$. We recover the rates of the compactly supported case if we can allow infinite moments for the initial datum. □

5.4 Self-similar solutions

The aim of this section is to show that the solution f can be approximated by a function of the self-similar variables $y = \frac{x}{t}$, $w = t(v-\frac{x}{t})$ for large times. More precisely:

Proposition 5.4.1 *Assume that $a+d>0$, $f^0 \in (L^1\cap L^\infty)(\mathbb{R}^d_x\times\mathbb{R}^d_v)$ and that the compact support hypothesis (3.7) is verified. If $a>0$ assume also that*

$$f^0(x,v) \le \frac{C}{1+|v|^a} \quad a.e.\ (x,v)\in\mathbb{R}^d_x\times\mathbb{R}^d_v. \tag{4.17}$$

Then, for each weak solution of (1.1)–(1.2), given by Theorem 5.2.4, there exists a function $g_\infty(x,v)$ such that

$$\|f(t,x,v) - g_\infty(x/t, t(v-x/t))\| \to 0, \quad as\ t\to\infty,$$

in the norm of $W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$.

The rest of the section outlines a proof for this statement. To begin with, let us introduce the function g defined by $f(t,x,v) = g(t, \frac{x}{t}, t(v - \frac{x}{t}))$, or, in an equivalent way, $g(t,y,w) = f(t, ty, y + \frac{w}{t})$. We have to prove that g has a limit as $t \to +\infty$. The function g satisfies the following equation:

$$\begin{aligned} \frac{\partial g}{\partial t} + \frac{w}{t^2}\nabla_y g = &- g(t,y,w) \int_{\mathbb{R}^d_{w'}} \alpha\left(\frac{w}{t}, \frac{w'}{t}\right) \frac{g(t,y,w')}{t^d}\, dw' \\ &- \frac{1}{t^2}\bar{\rho}(t,y) g(t,y,w) \beta\left(y + \frac{w}{t}\right), \end{aligned} \tag{4.18}$$

with $\rho(t,x) = \frac{1}{t^2}\bar{\rho}(t, \frac{x}{t})$. Now we can prove that $g(t)$ is a Cauchy sequence in the space $W_{x,v} := W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$, that is, for the norm

$$\|g(t)\|_{W_{x,v}} = \sup_{\Delta} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \varphi(x,v)\, g(t,x,v)\, dx\, dv,$$

with $\Delta = \{\varphi \in \mathcal{D}(\mathbb{R}^d_x \times \mathbb{R}^d_v) / |\varphi| \le 1,\ |\nabla_x \varphi| \le 1\}$.

Lemma 5.4.2 *For $0 < s < t$, the following estimate holds*

$$\|g(t) - g(s)\|_{W_{y,w}} \le |s - t| \left[\frac{1}{ts} + \frac{1}{t^{a+d}} + \frac{1}{t^{a^*}}\right] C(R, d, a, f^0, M),$$

being $a^* = \min\{a + d, d\}$.

Proof. We compute

$$\begin{aligned} &\int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \varphi\left(g(t,x,v) - g(s,x,v)\right) dxdv \\ &\quad = \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi\left(g(t,y,w) - g(s,y,w)\right) dydw \\ &\quad = \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) \left(g(t,y,w) - g(t, y + \frac{w}{s} - \frac{w}{t}, w)\right) dydw \\ &\quad + \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) \left(g(t, y + \frac{w}{s} - \frac{w}{t}, w) - g(s,y,w)\right) dydw = I + II \end{aligned}$$

The first term is handled as follows:

$$\begin{aligned} I &= \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} g(t,y,w) \left(\varphi(y,w) - \varphi(y - \frac{w}{s} + \frac{w}{t}, w)\right) dydw \\ &\le \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \int g(t,y,w) |w| \frac{|s-t|}{|ts|}\, dydw \\ &= \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{|ts|} \int |w| f(t, ty, y + \frac{w}{t})\, dydw. \end{aligned}$$

So, thanks to (3.7) we get

$$\begin{aligned} I &\le \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{ts} \int |tw - y| f(t,y,w)\, dydw \\ &\le \|\nabla_y \varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \frac{|s-t|}{ts} \int_{\mathbb{R}^d_r \times \mathbb{R}^d_w} |r| f^0(r,w)\ drdw \\ &\le CM \frac{|s-t|}{ts}. \end{aligned}$$

To deal with the second term, we introduce the function $\phi(\tau) = g(\tau, y + \frac{w}{s} - \frac{w}{\tau}, w)$. Notice that $\phi(t) = g(t, y + \frac{w}{s} - \frac{w}{t}, w)$ and $\phi(s) = g(s, y, w)$. Evaluating (4.18) at points of the form $(t, y + \frac{w}{s} - \frac{w}{t}, w)$, multiplying by $\varphi(y, w)$ and integrating we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\phi(t)\,dydw \\
&= -\int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\phi(t)\int_{\mathbb{R}^d_{w'}} \frac{\alpha(w/t, w'/t)}{t^d} g(t, y + \frac{w}{s} - \frac{w}{t}, w')\,dydwdw' \\
&\quad + \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\frac{1}{t^2}\phi(t)\overline{\rho}(t, y + \frac{w}{s} - \frac{w}{t})\beta(y + w/s)\,dydw = A + B.
\end{aligned}$$

Then we write

$$\begin{aligned}
II &= \left| \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\phi(t)\,dydw - \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\phi(s)\,dydw \right| \\
&\le |t-s| \sup_{\theta\in[s,t]} \left| \left[\frac{\partial}{\partial t}\int_{\mathbb{R}^d_y\times\mathbb{R}^d_w} \varphi(y,w)\phi(t)\,dydw \right]_{t=\theta} \right|.
\end{aligned}$$

Thus if we bound $|A|$ and $|B|$ we are done. Recalling that $g(t, y, w) = f(t, ty, y + \frac{w}{t})$, we have

$$\begin{aligned}
|A| \le & C\|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w\times\mathbb{R}'^d_w} g(t, y + \frac{w}{s} - \frac{w}{t}, w) \\
& \times g(t, y + \frac{w}{s} - \frac{w}{t}, w') \frac{|w - w'|^a}{t^{d+a}}\,dydwdw' \\
= & \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_y\times\mathbb{R}^d_w\times\mathbb{R}'^d_w} f(t, ty + \frac{tw}{s} - w, y + \frac{w}{s}) \\
& \times f(t, ty + \frac{tw}{s} - w, y + \frac{w'}{s}) \frac{|w - w'|^a}{t^{d+a}}\,dydwdw'.
\end{aligned}$$

To continue we change variables inside the integral by means of $r = ty + \frac{tw}{s} - w$, $z = y + \frac{w}{s}$ and $z' = y + \frac{w'}{s}$. In particular, $|w - w'| = s|z - z'|$. In the case $d = 1$, the Jacobian matrix of the mapping $(y, w, w') \mapsto (ty + \frac{tw}{s} - w, y + \frac{w}{s}, y + \frac{w'}{s})$ is

$$\begin{pmatrix} t & t/s - 1 & 0 \\ 1 & 1/s & 0 \\ 1 & 0 & 1/s \end{pmatrix}$$

In the general case each entry corresponds now to a diagonal block of size d and all the elements equal to the corresponding one-dimensional entry. Then the inverse Jacobian reduces to s^d; to see this, transform the matrix in order to have the second and third blocks of the first column equal to zero. Thanks to (3.7) we can use Lemma 5.3.2 and performing along the same lines of (3.12) we deduce

$$\begin{aligned}
|A| &\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z\times\mathbb{R}^d_{z'}} f(t,r,z)f(t,r,z')\frac{1}{t^{d+a}} s^d |s(z-z')|^a\,drdzdz' \\
&\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y\times\mathbb{R}^d_w)} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z\times\mathbb{R}^d_{z'}} f(t,r,z)f(t,r,z')|z-z'|^a\,drdzdz' \\
&\le \frac{C(R,d,a,f^0)}{t^{a+d}} \int_{\mathbb{R}^d_r\times\mathbb{R}^d_z} f(t,r,z)\,drdz \le \frac{C(R,d,a,f^0)M}{t^{a+d}}.
\end{aligned}$$

On the other hand,

$$B = \int_{\mathbb{R}^d_y \times \mathbb{R}^d_w} \varphi(y,w) f(t, ty + \frac{tw}{s} - w, y + \frac{w}{s}) \rho(t, ty + \frac{tw}{s} - w) \beta\left(y + \frac{w}{s}\right) \, dydw.$$

We change to the new variables $z = y + \frac{w}{s}$, $r = ty + \frac{tw}{s} - w$ inside the integral. The Jacobian of the mapping $(y,w) \mapsto (y + \frac{w}{s}, ty + \frac{tw}{s} - w)$ is 1 (transform to have a zero block in the left lower corner). Then,

$$\begin{aligned} |B| &\le C\|\varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \int_{\mathbb{R}^d_r \times \mathbb{R}^d_z} f(t,r,z)|z|^a \rho(t,r) \, drdz \\ &\le \|\varphi\|_{L^\infty(\mathbb{R}^d_y \times \mathbb{R}^d_w)} \sup_r \left(\int_{\mathbb{R}^d_z} f(t,r,z)|z|^a \, dz \right) \int_{\mathbb{R}^d_r} \rho(t,r) \, dr. \end{aligned}$$

If $a < 0$ we estimate like in (3.13) to get

$$|B| \le \frac{CM}{t^{a^*}}.$$

In the case $a \ge 0$ we claim that

$$\int_{\mathbb{R}^d_z} f(t,r,z)|z|^a \, dz \le \frac{C}{t^a}$$

uniformly in r and then we get exactly the same type of estimate. □

The previous claim relies on the following technical result, which covers a slightly more general situation than needed: we assume compact support in x and the decay condition (4.17).

Lemma 5.4.3 *Whenever the condition*

$$f^0(x,v) \le \frac{C}{1 + |x|^{d+\epsilon} + |v|^k} \quad a.e. \ (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v$$

is fulfilled for some $\epsilon > 0$ *and some* $k > 0$ *the following estimate is verified:*

$$\int_{\mathbb{R}^d_v} |v|^k f(t,x,v) \, dv \le \frac{C}{t^d}.$$

Proof. Just follow the chain of inequalities:

$$\begin{aligned} &\int_{\mathbb{R}^d_v} |v|^k f(t,x,v) \, dv \le \int_{\mathbb{R}^d_v} |v|^k f^0(x - vt, v) \, dv \le \int_{\mathbb{R}^d_v} \sup_{\xi \in \mathbb{R}^d} |\xi|^k f^0(x - vt, \xi) \, dv \\ &\le \frac{1}{t^d} \int_{\mathbb{R}^d_x} \sup_{\xi \in \mathbb{R}^d} |\xi|^k f^0(x,\xi) \, dx \le \frac{1}{t^d} \int_{\mathbb{R}^d_x} \sup_{\xi \in \mathbb{R}^d} \frac{C|\xi|^k}{1 + |\xi|^k + |x|^{d+\epsilon}} \, dx \le \frac{1}{t^d} \int_{\mathbb{R}^d_x} \frac{C \, dx}{1 + |x|^{d+\epsilon}}. \end{aligned}$$

□

As a consequence there exists a function $g(\infty, y, w)$ such that $g(t) \to g(\infty)$ in the norm of $W^{-1,1}(\mathbb{R}^d_y, L^1(\mathbb{R}^d_w))$ and that $\|g(\infty) - g(t)\|_{W_{y,w}} \le \frac{C}{t}$. But note that, being 1 the Jacobian of the mapping $(x,v) \mapsto (x/t, t(v - x/t))$, we have that

$$\|g(\infty, y, w) - g(t, y, w)\|_{W_{y,w}} = \|f(t,x,v) - g(\infty, x/t, t(v - x/t))\|_{W_{x,v}}.$$

Thus we can get from the self-similar variables to the original ones. We have proved that

$$\|f(t,x,v) - g(\infty, x/t, t(v - x/t))\|_{W(x,v)} \to 0 \ \text{ as } t \to \infty.$$

Chapter 6

An evolution model to the transport of morphogenes

The purpose of this chapter is to give further insights into the modeling of the dynamics of morphogenic functions, which is a very important theme of research in developmental Biology. Presently the available models rely on linear diffusive mechanisms that some recent experimental findings call into question. Our proposal is to substitute the diffusive mechanisms of these models by a non-linear, flux-limited diffusion mechanism, which in our opinion gives a more accurate description of the phenomena under consideration. To present our results we will take first a short detour into the realm of flux-limited diffusion equations, that will serve to motivate our perspective about the biological problem and to state clearly what are the mathematical ideas behind these equations at the same time. Meanwhile the biological problem that we are interested in will be explained in detail. Only then will we proceed with the concrete mathematical model for the problem in question. The results of this chapter are contained in [13] and [70].

6.1 Introduction to flux-limited diffusion equations

This story is best understood if we trace it back to the deduction of the heat equation as a model for heat conduction. This "classical" theory of heat conduction arises from the pioneering works of J. Fourier [96]. He postulated the following relation between the heat flux q and the temperature u

$$q(t, x) = -k\nabla u(t, x),$$

which was named Fourier's law afterwards. In combination with the equation that encodes the conservation of energy

$$\partial_t u + \mathrm{div}_x q = 0,$$

J. Fourier obtained the linear parabolic heat equation

$$\partial_t u = k\Delta u.$$

This was later reinterpreted in terms of Brownian motion and originated the theory of linear diffusion equations. One of the most important drawbacks of this model is that of its infinite propagation speed (although, this gap is only present at the continuous level, the Brownian motion of a finite collection of particles does not exhibit this feature). This is easily seen from the representation formula for the solutions of the initial value problem in $\mathbb{R}^N$, namely the convolution in space of the initial datum with the heat kernel

$$u(t, \cdot) = G_t * u_0, \tag{1.1}$$

being

$$G_t(x) = \frac{1}{(4\pi kt)^{\frac{N}{2}}} e^{\frac{-|x|^2}{4kt}}.$$

This causes that any non-negative initial datum that is compactly supported becomes everywhere positive for any later time. This feature of the heat equation and related diffusion (Fokker–Planck) models makes them physically unrealistic. Another way in which these models might not be the best suited in certain circumstances is that they introduce an instantaneous smoothing on the initial configuration; this is again easily deduced form (1.1). In particular, any initial discontinuity like a shock is instantaneously dissolved.

Several attempts have been tried in order to have a mathematical theory that can correct the infinite speed of propagation that comes with Fourier's theory. Here we are interested in a promising approach that was started by P. Rosenau and coworkers [193]. Their idea is to modify the heat equation considering a flux-limited diffusion process (to keep it simple we consider one spatial dimension)

$$\partial_t u = [G(u_x)]_x.$$

The related flux is $q = -G(u_x)$; conditions are imposed on the function G in order that the flux be a monotone function of the gradient which saturates at a finite value, no matter the size of the gradients. In practical terms, using such a device we impose ad hoc the maximum speed of propagation. This is a way out to our problem, as we can impose that the maximum free speed in a given medium be that of sound (or that of light, depending on the context).

Let us follow closely a more concrete proposal given in [194]. To obtain a flux q that saturates as the gradients become unbounded he related u and q through the velocity v, defined by the relation

$$q = uv.$$

For the heat equation we would get

$$v = -k\frac{u_x}{u},$$

so that if $|u_x/u| \uparrow \infty$ then so will v, which is the type of phenomena that we want to correct. Rosenau proposed the alternative relation

$$k\frac{u_x}{u} = \frac{-v}{\sqrt{1 - \frac{v^2}{c^2}}},$$

being c the speed of sound. This forces v to stay in the subsonic regime, and the speed of sound is only approached if $|u_x/u| \uparrow \infty$. We are led to

$$q = uv = \frac{-ku_x}{\sqrt{1 + (\frac{ku_x}{cu})^2}}$$

Bringing this new flux into the conservation of energy equation, we arrive at the following modification of the heat equation:

$$\partial_t u = \left(\frac{k u u_x}{\sqrt{u^2 + \frac{k^2}{c^2} u_x^2}} \right)_x . \tag{1.2}$$

The equation (1.2) has been obtained in a number of different ways. For instance, Y. Brenier derived it using Monge–Kantorovich's mass transport theory [54] and he named it as the relativistic heat equation (although it isn't an actual relativistic generalization of the usual heat equation). Nowadays there are several attempts to try to deduce it from microscopic principles, let us mention those of macroscopic hyperbolic limits [40] and stochastic processes related to mean curvature flow.

As Brenier points out, equation (1.2) is one among several flux limited diffusion equations used in the theory of radiation hydrodynamics [158]. For instance, another slightly different proposal is that of J.R. Wilson (unpublished, see [158]). He suggests that

$$q = -\nu u \frac{u_x}{u + \frac{\nu}{c}|u_x|}, \tag{1.3}$$

being ν a constant representing a kinematic viscosity. The associated diffusion equation is

$$\partial_t u = \nu \left(\frac{u u_x}{u + \frac{\nu}{c}|u_x|} \right)_x ;$$

the reader can see that it formally interpolates between the standard heat equation (when $c \to \infty$) and the diffusion equation in transparent media with constant speed of propagation c

$$\partial_t u = c \left(u \frac{u_x}{|u_x|} \right)_x ,$$

which is obtained in the limit $\nu \to \infty$. This features can also be deduced from the formula for the flux (1.3): in the limit $\nu \to \infty$ it becomes $q = -c\, u \frac{u_x}{|u_x|}$, while in the regime $c \to \infty$ we get $q = -\nu u_x$, the flux associated with the usual heat equation. The relativistic heat equation shares also these features. Analogous formulas have been proposed for the study of mass diffusion in liquids [235] and the study of charge carriers in sub micrometer electronic devices [161]. Other models of nonlinear degenerate parabolic equations with flux saturations have been introduced in [78, 81, 194], for instance.

To summarize, the idea is to modify the flux formula of diffusion theory in such a way that the correct limit behaviors that we want to reproduce and the fact that the flux cannot violate causality are obtained. Let us mention in passing that the same type of behavior could be obtained using the porous media equation [226]; for the applications we have in mind this model is not so well suited, due to the fact that the speed of propagation depends on the particular initial datum rather than on intrinsic properties of the particles.

These models have been used in the study of heat and mass transfer in turbulent fluids [45]; they also appear in the study of radiation transfer related to astrophysical systems [158], to name a few areas. Our motivation comes from the transport of morphogenes in biological systems and we will comment on this in Section 6.2.

A general class of flux limited diffusion equations and the properties of the relativistic heat equation have been studied in a series of papers [18, 19, 20, 21, 22, 23, 24], where the well-posedness of the Cauchy, the Neumann and the Dirichlet problem for the relativistic heat equation and related models is proved. These works constitute the starting point for our research.

6.2 The biological problem: setting and main results

The driving idea behind this chapter is to contribute with some new ideas to the modeling of the dynamics of Sonic Hedgehog morphogenetic function, as a mean to study its consequences on the evolution of some transcription factors and on cellular differentiation in the embryonic neural tube. Since the discovery of the Drosophila Hedgehog (Hh) mutation and gene, Hh signalling has been found to play multiple roles in development, homeostasis and disease (reviewed in [126, 162]).

Let us give some hints about the biological motivation of this problem. In vertebrates the Hh family comprises three proteins which act as secreted, intercellular factors that affect cell fate, differentiation, survival, and proliferation in the developing embryo and in many organs at one time or another. Sonic Hedgehog (Shh) is the most broadly expressed member of the Hh family. Within the central nervous system, the development of the early vertebrate ventral neural tube [130] and of the later dorsal brain [198] depends on Shh signaling. In the early neural tube, it is proposed to act as a morphogen to specify ventral fates.

Shh signaling has also an important role in tumor formation: the deregulation of the Shh pathway leads to the development of various tumors, including those in skin, prostate and brain [196]. Recent findings in developmental genetics afford an alternative way to think of solid cancer that contrasts with the single cell focus derived from molecular and cell biology and their emphasis on the cell cycle and the concepts of gene mutations [196, 199]. In this context, cancer is interpreted as a patterning disease, in which cells are playing out abnormal developmental programs. Thus, in tumorigenesis and in embryogenesis, patterning signals and pathways play critical roles.

Recently, a mathematical model that analyzes the Shh signaling network within the early chick neural tube has been proposed [138, 202], with the main purpose of investigating temporal effects of morphogen transport and intracellular signaling. Let us describe it briefly in this paragraph. The model employs a reaction-diffusion equation to describe spatial transport of Shh, as suggested in [139]. In [202] morphogen transport and Shh signaling pathway in responding cells are represented by a set of differential equations. The idea is to analyze the morphogenetic patterning of the vertebrate embryonic neural tube along the dorsoventral axis. This axis represents a natural privileged direction for the description of Shh propagation. Actually, the system is symmetric with respect to this axis and this justifies the reduction to one dimension. The basic ingredients employed to construct the equations are Fick's second law and the law of mass action. The first is applied to describe Shh diffusion from the floor plate source along the dorso-ventral axis, in one spatial dimension. The law of mass action, valid for single step reactions, states that the rate of a chemical reaction is proportional to the concentration of the participating molecules. It is used to describe the rate of change in the concentrations of the various proteins involved in the Shh signaling pathway. All chemical reactions are considered to be reversible and to have already reached a stationary state during tissue patterning. To summarize, the model consists of a partial differential equation describing Shh diffusion inside the neural tube and of seven ordinary coupled differential equation representing temporal evolution of protein concentrations. All the variables thus depend on both time and space.

Despite the reaction terms that are present in the equation for the evolution of Shh concentration, the predictions obtained by this model show the typical behavior corresponding to a diffusive transport mechanism. These results do not reproduce the experimental observation that Shh is able to exert a long range action in specifying the ventral progenitor cell pattern [55, 107]. Actually it has been shown that this morphogen can signal over a range of at least 15-20 cell diameters. This means that progenitor cells situated at distance up to 300 microns from the morphogen source are exposed to a Shh concentration sufficient to trigger the activation of the transduction pathway and induce phenotype switch. Furthermore, numerous studies have pointed out that not only the quantity of morphogen is relevant to determine cell response, but also the duration of exposure [84, 170], consistent with a transient mixed phenotype in central cells [197].

From the biological point of view, although the existence of morphogens and their action is no longer in doubt, their transport mechanism and the process of formation of their concentration gradient is still under debate [44, 140]. The model presented in [202] is not able to reproduce a fundamental feature of the Shh pathway: the long-range signaling activity of this morphogen, for instance up to 15-20 cell diameters in the chick neural tube [55] or of Hh in the fly imaginal disc [108]. Furthermore, as all the models based on reaction-diffusion equations, it involves unphysical spreading out of the morphogen to all the precursor cell field soon after the secretion.

These facts motivate our approach to the problem trying to overcome the difficulty of an infinite speed of propagation of the signal transmission by proposing a new model with a non-linear diffusion mechanism with finite speed of propagation. In this way, the system behaves more like an hyperbolic system than a linear parabolic model.

Modeling transport mechanism using diffusion equations implies a general problem: the introduction of an unphysical spreading of the morphogen to all the precursor cell field soon after secretion. If the morphogen propagates too fast within the neural tube, along the dorso-ventral axis, its concentration gradient decreases too quickly to be able to induce cell phenotype switch. One of the main reasons why the model presented in [202] encounters difficulties in representing the long range signaling property of Shh resides in the features of the linear diffusion mechanism, in particular the infinite speed of propagation of the Shh signal. As a consequence, the cells in the neural tube receive instantaneously the information and this is a problem since, as we have commented before, not only the amount of morphogen matters, but also the time of exposure. Actually it seems reasonable to overcome this difficulty in this type of model, even when the diffusion is coupled with reaction terms in the equation. We thus propose to modify the usual Fick's second law employed to explicate morphogen propagation pretty much in the same way that Fourier's law is modified to obtain the relativistic heat equation instead of the classical heat equation. This modification would allow to control the spreading of the morphogen along the neural tube without having to introduce accessory mechanisms limiting the diffusion.

This proposal of ours leads us to analyze a mixed initial-boundary value problem associated with a nonlinear flux–limited reaction–diffusion system

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = (\mathbf{a}(u,u_x))_x - f(t-\tau, u(t,x))\, u(t,x) + g(t,u(t,x)), & \text{in} \quad]0,T[\times]0,L[\\
-\mathbf{a}(u(t,0),u_x(t,0)) = \beta > 0 \quad \text{and} \quad u(t,L) = 0 & \text{on} \quad t \in]0,T[, \\
u(0,x) = u_0(x) & \text{in} \quad x \in]0,L[,
\end{cases}
\tag{2.4}
$$

being

$$
\mathbf{a}(z,\xi) := \nu \frac{|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}},
$$

where the boundary conditions must be interpreted in a weak sense —which will be explained later on— and the functions f and g are nonlinear with respect to u and depend on u through a coupling system of ordinary differential equations. Recall that ν represents a kinematic viscosity and c stands for the maximum speed allowed. The parameter τ represents a delay in the process of signaling pathway cell internalization. The analysis of a simplified version of these models is the main purpose of the present chapter. In addition to the biological or physical motivations, the mathematical analysis of this equation poses several difficulties, making even more interesting its study, such as the existence and evolution of fronts as well as the study of its finite speed of propagation, the related lack of regularity and the set-up of an appropriate functional framework to give a meaning to the differential operator and the boundary conditions. To deal with this mathematical problems we need to combine and extend the applicability of different techniques coming from parabolic and hyperbolic contexts such as Crandall-Liggett's theorem, Minty–Browder's technique, the entropy concept of solution and the method of doubling variables due to S. Kruzhkov.

Our main result concerning (2.4) with $f = g = 0$ is the following:

Theorem 6.2.1 *For any initial datum $0 \le u_0 \in L^\infty(]0,L[)$ there exists a unique bounded entropy solution u of (2.4) in $Q_T =]0,T[\times]0,L[$ for every $T > 0$ such that $u(0) = u_0$.*

The rest of the chapter is structured as follows. First we will take a detour to explain with detail the mathematical problems and techniques related to these models in an intuitive way; this will serve as a bridge before we proceed to our concrete problem. Section 6.4 serves to introduce all the tools needed to develop the theory: a suitable integration by parts formula, lower semi-continuity results and a functional calculus, in order to be able to give a sense to the differential operator. In Section 6.5 we discuss the associated elliptic problem: we define what a solution is, and then we prove existence and uniqueness of such a solution. Next, this material is used to define an accretive operator and construct a nonlinear semigroup, which accounts for solving our problem (2.4) in a mild sense; all this is the content of Section 6.6. In Section 6.7 we prove that the mild solution previously constructed can be characterized in more operative terms, as a so-called entropy solution —a concept which is also introduced in that section—. Then we prove a comparison criterion which in particular entails uniqueness of entropy solutions, thus proving Theorem 6.2.1.

6.3 The mathematics behind the heuristics

A broad class of evolution problems of the form

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, \nabla u)$$

have been studied in [16, 17] and in [18, 19, 20, 21, 22, 23], under several restrictions for the operator $\mathbf{a}(u, \nabla u)$ (basically to be the derivative of a convex function with linear growth as $|\nabla u| \to \infty$). The mathematical theory developed to perform this study is presently quite involved. The purpose of this section is to illustrate the heuristics that motivate all the subtleties of the theory; we will have to readapt them later for the study of the model presented in the next section. To this aim it will be instructive to restrict ourselves to a reference problem among the previous class. We will consider the Cauchy problem for a concrete equation —rescaling to one all physical constants—,

$$\frac{\partial u}{\partial t} = \operatorname{div} \left(\frac{|u|\nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) \quad \text{in } [0, T[\times \mathbb{R}^N \tag{3.5}$$

with non-negative initial data; arguments can be suitably adapted to cover domains other than $\mathbb{R}^N$. Although, $\mathbf{a}(u, \nabla u)$ will be a shorthand for the operator enclosed in the divergence still.

The strategy to study these equations falls under the standard paradigm: propose a sequence of approximating problems that you are able to cope with and then try to pass to the limit in some way keeping as much information as you can. For the purpose of this section we will proceed as we had a sequence of smooth solutions to the very equation (3.5) and try to figure out what is to be expected for the limit.

We start wondering about uniqueness. This question poses no problem at a formal level, as we are going to see. Given two solutions u_1, u_2 we subtract both equations, multiply by $\mathrm{sign}_0(\mathrm{u}_1 - \mathrm{u}_2)$ and integrate in the whole $\mathbb{R}^N$, thus obtaining

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} |u_1 - u_2| \, dx &= \int_{\mathbb{R}^N} \mathrm{sign}_0(\mathrm{u}_1 - \mathrm{u}_2) \frac{d}{dt}(u_1 - u_2) \, dx \\ &= \int_{\mathbb{R}^N} \mathrm{sign}_0(\mathrm{u}_1 - \mathrm{u}_2) \operatorname{div} \left[\mathbf{a}(u_1, \nabla u_1) - \mathbf{a}(u_2, \nabla u_2) \right] \, dx. \end{aligned}$$

Integrating by parts we get to

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} |u_1 - u_2| \, dx &= -2 \int_{\mathbb{R}^N} \delta(u_1 - u_2) \nabla (u_1 - u_2) (\mathbf{a}(u_1, \nabla u_1) - \mathbf{a}(u_2, \nabla u_2)) \, dx \\ &= -2 \int_{[u_1 = u_2]} (\nabla u_1 - \nabla u_2)(\mathbf{a}(u_1, \nabla u_1) - \mathbf{a}(u_1, \nabla u_2)) \, dx \end{aligned}$$

and the monotonicity properties of $\mathbf{a}(u, \cdot)$ allows us to conclude that the above quantity has a well defined sign, so that

$$\int_{\mathbb{R}^N} |u_1(t, x) - u_2(t, x)| \, dx \le \int_{\mathbb{R}^N} |u_1(0, x) - u_2(0, x)| \, dx \quad \forall t > 0.$$

This contraction principle yields uniqueness as a particular consequence, but we have "proved" much more than that. It is even possible to substitute the absolute values by the positive part above.

The monotonicity properties of these operators stem from the convexity of the associated Lagrangian. Namely, the class of operators and Lagrangians studied in [16, 17, 18, 19, 20, 21, 22, 23] is such that

$$\mathbf{a}(z,\xi) = \nabla_\xi F(z,\xi), \quad \text{being } \xi \mapsto F(z,\xi) \text{ a convex function.}$$

So that

$$F(z,\xi_1) - F(z,\xi_2) \geq \nabla_\xi F(z,\xi_2)\cdot(\xi_1-\xi_2)$$

and

$$F(z,\xi_2) - F(z,\xi_1) \geq \nabla_\xi F(z,\xi_1)\cdot(\xi_2-\xi_1).$$

Adding both we obtain

$$(\nabla F(z,\xi_1) - \nabla F(z,\xi_2))\cdot(\xi_2-\xi_1) \geq 0$$

or, in an equivalent way,

$$(\nabla u_1 - \nabla u_2)\cdot(\mathbf{a}(z,\nabla u_1) - \mathbf{a}(z,\nabla u_2)) \geq 0.$$

To construct a real uniqueness proof from these computations is a much harder task. Let us mention in passing that there is no clear way to get these computations to work in any other L^p space. For instance in the L^2 case we would get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(u_1-u_2)^2\,dx = -\int_{\mathbb{R}^N}\nabla(u_1-u_2)\,(\mathbf{a}(u_1,\nabla u_1) - \mathbf{a}(u_2,\nabla u_2))\,dx$$

and it's by no means clear what should we do to cope with the right hand side.

6.3.1 About a priori estimates

First of all, let us point out the fact that several explicit solutions for a related model, the diffusion equation in transparent media

$$u_t = \text{div}\left(u\frac{Du}{|Du|}\right),$$

are known [22]. The solutions we are talking about consist in spherically symmetric solutions that are radially evolving fronts expanding at constant speed. This implies that we cannot hope to get regular solutions in general. That is, presumably $u \in BV_x$ (this will be recovered again from the abstract framework) but $u_t \notin L^1$. For the above mentioned solutions u_t is a Radon measure, but this not need to be the case in general. In fact, the parabolicity of the problem is so weak when $|\nabla u| \to \infty$ that the resulting limit equation is hyperbolic and propagates initial singularities. The results in [21] support these ideas.

Let us derive formally some a priori estimates for the solutions of the equation.

- Non-negativity is obtained multiplying the equation by $u^- = -\min\{u,0\}$ and integrating, so that

$$\int_{\mathbb{R}^N} u^-\frac{\partial u}{\partial t}\,dx = -\int_{\mathbb{R}^N}\nabla u^-\frac{|u|\nabla u^-}{\sqrt{u^2+|\nabla u|^2}}\,dx$$

and then

$$\int_{\mathbb{R}^N} (u^-(t))^2\,dx \le \int_{\mathbb{R}^N} (u^-(0))^2\,dx$$

Thus, we are entitled to remove the absolute values in the numerator of $\mathbf{a}(u,\nabla u)$ hereafter.

- Direct integration of the equation shows that $\int_{\mathbb{R}^N} u(t)\,dx$ is preserved during evolution.

- Multiply the equation by u^{p-1}, $p>1$ and integrate:

$$\int_{\mathbb{R}^N} u^{p-1}\frac{\partial u}{\partial t}\,dx = -(p-1)\int_{\mathbb{R}^N} u^{p-2}\nabla u \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\,dx$$

  and so

$$\int_{\mathbb{R}^N} u(t)^p\,dx \le \int_{\mathbb{R}^N} u(0)^p\,dx$$

  By semicontinuity properties, the same holds for $p=\infty$.

- A regularity estimate. Multiply the equation by u and integrate to get

$$\int_{\mathbb{R}^N} u\frac{\partial u}{\partial t}\,dx = -\int_{\mathbb{R}^N} \frac{u(\nabla u)^2}{\sqrt{u^2+|\nabla u|^2}}\,dx \le \int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} u|\nabla u|\,dx \tag{3.6}$$

  (this inequality will occupy us for some time, we defer the discussion to the next paragraph) which we recast as

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^N} |\nabla u(t)^2|\,dxdt \le (T+1)\int_{\mathbb{R}^N} u(0)^2\,dx.$$

  This poorer-than-expected estimate is related to the degeneracy provoked by the presence of the factor u (rather than ∇u naked) in the numerator of $\mathbf{a}(u,\nabla u)$. As no lower bound for the solution is known, this spoils the regularity of the solution close to the zeroth level set: only a BV estimate for u^2 is available in the limit, instead of a BV estimate for u, which could be the thing to be expected from the linear growth of the Lagrangian with respect to the gradient. This motivates the use of truncation functions.

To have a correct derivation of these formal estimates would require to multiply the equation by the very solution —or a suitable truncation— and to be able to give a rigorous meaning to this operation. Even in the more favored case of having that $\operatorname{div}\mathbf{a}(u,Du)$ is a Radon measure this would require to integrate such a measure against BV functions. We can solve this problem arguing that $\operatorname{div}\mathbf{a}(u,Du)$ does indeed define an element of $BV(\mathbb{R}^N)^*$, as $\mathbf{a}(u,Du)$ is bounded. Then Anzellotti's results [26] can be used in order to integrate by parts under such framework. More precisely, his results allow to give a meaning to the pairing $\mathbf{z}\,Du$ of a vector field $\mathbf{z}$ against the derivative of a function u, being $\mathbf{z}\in\{\psi\in L^\infty(\mathbb{R}^N,\mathbb{R}^N)/\operatorname{div}\psi\in L^p(\mathbb{R}^N)\}$ and $u\in(BV\cap L^{p'})(\mathbb{R}^N)$, for $1\le p\le N$ and $N\ge 2$. Time dependence has to be included, so what we will really have is $\operatorname{div}\mathbf{a}(u,Du)\in(L^1(0,T,BV(\mathbb{R}^N)))^*$ and we will be able to use test functions in $L^1(0,T,BV(\mathbb{R}^N))$; Anzellotti's theory has to be extended to the time-dependent case.

Assume then that we are able to integrate by parts, for the sake of obtaining a priori estimates or whatsoever. Being the expected regularity for the solutions that poor, what meaning is attached to $\mathbf{a}(u, Du)$? The main problem here is how to give a sense to Du that works nicely with the nonlinear context where it appears. The essential properties that we will be interested in are really only a few. Basically we want to regard $\mathbf{a}(u, Du)$ as an almost everywhere defined function, that coincides a.e. with the limit of the regular objects $\mathbf{a}(u, \nabla u)$ and that enables us to say that $|\mathbf{a}(u, \nabla u)| \leq C|u|$. This last property entitles us to transfer the integrability properties of the solution u to the limit of smooth objects $\mathbf{a}(u, \nabla u)$ and so to be able to use Anzellotti's results for partial integration. Such a handy meaning for Du can be borrowed from [41]. If $u \in BV(\mathbb{R}^N)$ we use ∇u to give a meaning to $\mathbf{a}(u, Du)$, if u is of bounded variation only after truncation we can use a procedure like the one in [41] to manufacture the analog of the Radon–Nikodym derivative for this class of functions. Anyway, the thing to be remembered is that we do not consider any information coming from the singular parts of the derivative when giving a meaning to $\mathbf{a}(u, Du)$; this loss of information is treated later using entropy inequalities.

Once we have all these tools at our disposal, we can see that a priori estimates will be obtained multiplying the equation by some truncation of the solution $T(u)$; then the crucial abstract fact that enables us to derive some information is the property that $\mathbf{a}(u, \nabla u)\nabla T(u) \geq 0$, as the reader may check in the previous computations. The extension of this fact to the non-smooth setting is one of the central issues, as it is not completely clear how to deal with the singular parts of the derivative of $T(u)$.

6.3.2 Functional calculus I: the mechanism that provides regularity

Now we elaborate on the inequality in (3.6). This is where the properties of the Lagrangian function are crucial in order to get an estimate on ∇u. It comes from the fact that convexity yields the inequality

$$\mathbf{a}(z, \xi)(\eta - \xi) \leq F(z, \eta) - F(z, \xi).$$

Taking $\eta = 0$ gives

$$\mathbf{a}(z, \xi)\xi \geq F(z, \xi) - F(z, 0).$$

As we previously pointed out we can obtain the relation

$$\int_{\mathbb{R}^N} u u_t \, dx = -\int_{\mathbb{R}^N} \mathbf{a}(u, \nabla u)\nabla u \, dx$$

and then the previous inequality would lead to

$$\int_{\mathbb{R}^N} F(u, \nabla u) \, dx + \int_{\mathbb{R}^N} u u_t \, dx \leq \int_{\mathbb{R}^N} F(u, 0) \, dx.$$

Assuming that a priori estimates are good enough to control the difference $\int_{\mathbb{R}^N} F(u, 0) \, dx - \int_{\mathbb{R}^N} u u_t \, dx$ we would obtain the following regularity estimate for our solution u:

$$\int_{\mathbb{R}^N} F(u, \nabla u) \, dx < \infty.$$

No wonder that these computations can be valid only for regular solutions, and it is very likely that we won't be that lucky. Nevertheless it will go through for the approximated problems. So that we are confronted to the question of giving a meaning to the expression $F(u, \nabla u)$ in the presence of non-smooth functions while retaining an inequality like

$$\mathbf{a}(u, \nabla u)\nabla u \geq F(u, \nabla u) - F(u, 0) \tag{3.7}$$

in the extended setting. It turns out that a suitable extension of the object $F(u, \nabla u)$ to our non-smooth setting can be obtained with recourse to the relaxed energy functionals introduced by Dal Masso for functions of linear growth [82]. Then we get a generalization of the inequality (3.7) that works properly in this setting:

$$F(u, DT(u)) \leq \mathbf{a}(u, Du)DT(u) + F(u, 0)\mathcal{L}^N \quad \text{in the sense of measures.}$$

The derivation is just a passage to the limit in the inequality satisfied by the smooth approximations, as one of the crucial advantages of Dal Masso's extension is the fact that the resulting object $F(u, DT(u))$ is lower semicontinuous in $L^1(\mathbb{R}^N)$.

6.3.3 The Minty-Browder method

We will encounter a very delicate problem when passing to the limit of the approximations in the highly nonlinear expression that constitutes the operator $\mathbf{a}(u, \nabla u)$. No doubt that we can ensure that the related sequence of approximating operators has a limit in some sense, but the central issue here is the following: Can we assure that the limit operator retains the $\mathbf{a}(u, \nabla u)$ structure? For a problem like ours this can be solved in an affirmative sense exploiting the monotonicity properties of our differential operator. The related device is known as Minty–Browder's method; we will describe briefly how does it work, in the framework of a Hilbert space.

Assume that we have an equation for q involving a nonlinear operator $\mathbf{a}(q)$ and to study it we construct a sequence of approximating problems in a Hilbert space H with solutions q_n that converge to a solution q of the limit problem. The sequence given by $\mathbf{a}(q_n)$ will have a limit in some sense that we call ξ. The issue is to prove that $\xi = \mathbf{a}(q)$ indeed. The operator $\mathbf{a}(q)$ is assumed to be *monotone*, meaning that

$$(\mathbf{a}(p) - \mathbf{a}(q)) \cdot (p - q) \geq 0 \quad \forall p, q \in H.$$

Thanks to this property we will have an inequality like

$$(\mathbf{a}(q_n) - \mathbf{a}(p)) \cdot (q_n - p) \geq 0 \quad \text{for } p \in H$$

for the approximating problems. Then we have to be able to assure that in the limit we will retain this structure, that is,

$$(\xi - \mathbf{a}(p)) \cdot (q - p) \geq 0 \quad \text{for } p \in H.$$

Choose now $p = q - tv$, being $t \in \mathbb{R}$ and $v \in H$, so that

$$(\xi - \mathbf{a}(q - tv)) \cdot (tv) \geq 0.$$

Divide the previous by t and let $t \to 0$. If the operator $\mathbf{a}(q)$ is continuous on finite dimensional subspaces we will get to

$$(\xi - \mathbf{a}(q)) \cdot v \geq 0.$$

Replacing v by $-v$ we conclude that $\xi = a(q)$, equality holding in the sense provided by the underlying space.

6.3.4 Functional calculus II: Kruzkov's method for uniqueness

At the beginning we sketched the idea that uniqueness could be obtained multiplying the equation by a suitable regularization $T_n(u_1 - u_2)$ of the sign of the difference of two solutions. Now that we have given some hints about the solutions of this equation we can see that this approach is condemned to failure. This time the computation would read

$$\int_{\mathbb{R}^N} T_n(u_1 - u_2) \frac{\partial}{\partial t}(u_1 - u_2)\, dx = \\ - \int_{\mathbb{R}^N} T_n'(u_1 - u_2) \nabla(u_1 - u_2)(\mathbf{a}(u_1, \nabla u_1) - \mathbf{a}(u_2, \nabla u_2))\, dx$$

and some difficulties arise pretty soon. First, we have no guarantee that $u_1 - u_2$ is of bounded variation after truncation, which seems to be the minimal regularity to be able to perform calculus in our setting.

Second, we would like to use the monotonicity to conclude that the right hand side is non-positive, but in the present form it cannot be applied: the first argument in both $\mathbf{a}(\cdot, \cdot)$ terms is not the same yet, despite the fact that $T_n'(u_1 - u_2)$ approaches a Dirac delta. Nevertheless we could expand the expression in its four factors, but two of them do not have a definite sign. So that $\mathbf{a}(u_1, \nabla u_1) DT_n(u_1 - u_2)$ and $\mathbf{a}(u_2, \nabla u_2) DT_n(u_1 - u_2)$ do not have an a priori definite sign.

And, maybe the most important drawback, the fact that we may be confronting a Dirac measure against presumably non-continuos functions does not disappear by the mere fact that we have regularized the sign function.

Kruzkov's method of doubling variables provides a way out to the third point, as the problem of pointwise convergence is relaxed to a problem concerning L^1 convergence. In this way we can regard the method as some short of regularizing device. We are going to see that it solves also the other caveats that we noticed.

The idea is to regard both solutions as functions of different variables, say (t, x) and (s, y) for instance. So that a kind of decoupling is performed and we don't have to rely on issues of the combined regularity of $u_1 - u_2$, for instance. We mean that if $l \in \mathbb{R}$, we retain the desired properties for some carefully chosen truncations $T(u(x) - l)$ (that is, regularity and the fact that $\mathbf{a}(u_1, Du_1) DT(u_1 - l) \geq 0$ —but this is a long story that deserves an explanation, soon we will come to that), which is now a meaningful object, while the parameter l can be used to bring in the other solution. We only identify the variables to return to $T_n(u_1(x) - u_2(x))$ once we have control over all the superfluous terms that arise due to this maneuver.

The set-up of the procedure is as follows: we multiply each equation (with decoupled variables) by a suitable test function $\phi S(u_i) T(u_i),\ i = 1, 2$, we integrate in the chosen variables and use some functional calculus in order to integrate by parts. After that we integrate in the mute variables and then subtract both resulting equations. The test functions that we shall use are made up of three factors. First, ϕ is a smooth function that approximates a Dirac measure, evaluated in the difference of variables, say $(t - s, x - y)$. This allows to give a meaning to the dualities and also to fuse the decoupled variables once it is possible (basically, once we get rid of all the gradients in the computations). The second factor $S(u_i)$ is an approximation to the sing_0 or sing_0^+ function of the difference of both solutions. As variables are decoupled, one of the solutions is regarded only as a truncation parameter as long as we do not fuse the variables, thus allowing the use of regularity and positivity properties. The third factor $T(u_i)$ is required to stay away from the degeneracy of u_i at zero, a feature that $S(u_i)$ cannot always guarantee —precisely where one of the solutions vanishes; its use could be disregarded if we knew that the solutions are of bounded variation [16, 17].

We need to set up a functional calculus that is able to give a meaning to expressions like

$$\mathbf{a}(u_i, Du_i))D(S(u_i)T(u_i)) = S(u_i)\mathbf{a}(u_i, Du_i)DT(u_i) + T(u_i)\mathbf{a}(u_i, Du_i)DS(u_i),$$

that we will encounter while developing the previous program. To do so we observe that defining

$$J_q(r) = \int_0^r q(s)\, ds$$

we have the relations

$$S(u_i)\mathbf{a}(u_i, Du_i)DT(u_i) = \mathbf{a}(u_i, Du_i)DJ_{T'S}(u_i),$$

$$S(u_i)\mathbf{a}(u_i, Du_i)DT(u_i) = \mathbf{a}(u_i, Du_i)DJ_{T'S}(u_i)$$

and in this way written we pass from three to two factors, having also a functional form that falls under the framework of Anzellotti's results.

6.3.5 The entropy inequalities

If we try to continue with that computations on the previous paragraph we encounter new difficulties pretty soon. At a formal level we have that

$$\mathbf{a}(u, Du)DJ_{T'S}(u) = S(u)\mathbf{a}(u, DT(u))DT(u). \tag{3.8}$$

It's almost like what we started with, $S(u)\mathbf{a}(u, Du)DT(u)$, but the fact that the truncation function has also "shifted" *inside* $\mathbf{a}(u, \cdot)$ is crucial, as it provides a sign for the quantity in (3.8). Recall that many formal estimates that we commented on previously rely on the fact that

$$\mathbf{a}(u, \nabla u) \cdot \nabla u \geq 0,$$

or even

$$\mathbf{a}(u, \nabla T(u)) \cdot \nabla(T(u)) \geq 0.$$

Indeed, $\mathbf{a}(u, \nabla T(u)) \cdot \nabla u = \mathbf{a}(u, \nabla T(u)) \cdot \nabla(T(u))$ when it has a sense, as the factor $\nabla T(u)$ restricts the support accordingly.

In the non-smooth setting (3.8) is no longer true, and we are going to show why. Recall that we said that we neglect all the information coming from the singular part of the derivative of u in order to give a meaning to $\mathbf{a}(u, Du)$, and thus this information is not included when we write down the distributional formulation of the equation. In some situations it is useful to try to restore as much information as we can, and this is the role played by the entropy inequalities, that keep track of the behavior of the singular parts of Du. The necessity for this is clearly seen if we continue with the computations on Kruzkov's procedure. Sloppy writing shows that

$$\begin{aligned} &S(u_i)\mathbf{a}(u_i, Du_i)DT(u_i) = S(u_i)\mathbf{a}(u_i, \nabla u_i)(\nabla T(u_i) + D^sT(u_i)) \\ =&S(u_i)\mathbf{a}(u_i, \nabla T(u_i))\nabla T(u_i) + S(u_i)\mathbf{a}(u_i, \nabla u_i)D^sT(u_i) \geq S(u_i)\mathbf{a}(u_i, \nabla u_i)D^sT(u_i) \end{aligned}$$

and we would be very pleased to ensure the non-negativity of that quantity, but it's clear that we cannot "complete the square" with ∇u_i. Good news are that we can prove that for a nice class of solutions to our problem that quantity above is non-negative indeed. This statement and variants of it constitute what we will call "entropy inequalities". They are crucial for the proof of uniqueness (which as a consequence holds by now only for the class of solutions that fulfill them) and for certain details of the development of the existence theory, as regularity of the solutions or identification of the operator in the limit using Minty–Browder's technique.

6.4 Preliminaries

We refer the reader to the Appendices for a brief survey on non-linear semigroup theory, vector integration and functions of bounded variation in dimension 1, which will appear recurrently in the present chapter. All the notation related to these topics is explained in the corresponding Appendix. Contrary to the general picture sketched in Section 6.3, we will take definite advantage of working in dimension 1.

6.4.1 An integration by parts formula

Given $\mathbf{z} \in W^{1,1}(]0,L[)$ and $u \in BV(]0,L[)$, by $\mathbf{z}Du$ we mean the Radon measure in $]0,L[$ defined as

$$\langle \varphi, \mathbf{z}Du\rangle := \int_0^L \varphi \mathbf{z}\, Du \qquad \forall\, \varphi \in C_c(]0,L[).$$

The following integration by parts formula will be used repeatedly in the sequel.

Lemma 6.4.1 *If* $\mathbf{z} \in W^{1,1}(]0,L[)$ *and* $u \in BV(]0,L[)$, *then*

$$\int_0^L \mathbf{z}Du + \int_0^L u(x)\mathbf{z}'(x)\, dx = \mathbf{z}(L)u(L_-) - \mathbf{z}(0)u(0_+).$$

Proof. Note first that if $u \in BV(]0,L[)$ then $u \in L^\infty(]0,L[)$, which gives a sense to the integral $\int_0^L u(x)\mathbf{z}'(x)\, dx$. To proceed we take the approximating sequence $\{u_n\}$ given by Theorem C.1.3 in the Appendix. Let us see that

$$\lim_{n\to+\infty} \int_0^L \mathbf{z}(x)u_n'(x)\, dx = \int_0^L \mathbf{z}Du. \tag{4.9}$$

For any given $\epsilon > 0$ we can find an $n \in \mathbb{N}$ big enough so that we can have

$$|Du|\left(]0,1/n[\bigcup]L-1/n,L[\right) < \epsilon.$$

Let $\varphi \in \mathcal{D}(]0,L[)$ be such that $0 \le \varphi \le 1$ in $]0,L[$ and $\varphi(x) = 1$ for all $x \in]1/n, L-1/n[$. Then,

$$\left|\int_0^L \mathbf{z}(x)u_n'(x)\, dx - \int_0^L \mathbf{z}Du\right| \le \left|\int_0^L \varphi(x)\mathbf{z}(x)u_n'(x)\, dx - \int_0^L \varphi\mathbf{z}\, Du\right| + \int_0^L (1-\varphi(x))|\mathbf{z}(x)u_n'(x)|\, dx + \int_0^L (1-\varphi)|\mathbf{z}Du|.$$

Now, integrating by parts, by the Dominate Convergence Theorem we get

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_0^L \varphi(x)\mathbf{z}(x)u_n'(x)\,dx\\
&= -\lim_{n\to+\infty}\int_0^L \varphi'(x)\mathbf{z}(x)u_n(x)\,dx - \lim_{n\to+\infty}\int_0^L \varphi(x)\mathbf{z}'(x)u_n(x)\,dx\\
&= -\int_0^L \varphi'(x)\mathbf{z}(x)u(x)\,dx - \int_0^L \varphi(x)\mathbf{z}'(x)u(x)\,dx\\
&= -\int_0^L (\varphi\mathbf{z})'(x)u(x)\,dx = \int_0^L \varphi\mathbf{z}\,Du
\end{aligned}
$$

For the last step we used that Du is a Radon measure and $\varphi\mathbf{z} \in C_c(]0,L[))$. On the other hand,

$$
\limsup_{n\to+\infty}\int_0^L (1-\varphi(x))|\mathbf{z}(x)u_n'(x)|\,dx \le \|\mathbf{z}\|_\infty \int_{]0,1/n[\cup]L-1/n,L[} |u_n'(x)|\,dx \le \epsilon\|\mathbf{z}\|_\infty
$$

(see Lemma C.1.6 in the Appendix). Then

$$
\int_0^L (1-\varphi)|\mathbf{z}Du| \le \epsilon\|\mathbf{z}\|_\infty.
$$

Since $\epsilon > 0$ is arbitrary, (4.9) follows. Finally, by (4.9) and integrating by parts we have

$$
\begin{aligned}
\int_0^L \mathbf{z}Du &= \lim_{n\to+\infty}\int_0^L \mathbf{z}(x)u_n'(x)\,dx\\
&= -\lim_{n\to+\infty}\int_0^L \mathbf{z}'(x)u_n(x)\,dx + \mathbf{z}(L)u(L_-) - \mathbf{z}(0)u(0_+)\\
&= -\int_0^L \mathbf{z}'(x)u(x)\,dx + \mathbf{z}(L)u(L_-) - \mathbf{z}(0)u(0_+).
\end{aligned}
$$

□

6.4.2 Properties of the Lagrangian

We define for $z,\xi \in \mathbb{R}$

$$
\mathbf{a}(z,\xi) := \frac{\nu|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}}. \tag{4.10}
$$

We assume $\mathbf{a}(z,0) = 0$ for all $z \in \mathbb{R}$. Then $\mathbf{a}(z,\xi) = \partial_\xi F(z,\xi)$, being the Lagrangian

$$
F(z,\xi) := \frac{c^2}{\nu}|z|\sqrt{z^2 + \frac{\nu^2}{c^2}\xi^2}.
$$

By the convexity of F in the ξ-variable,

$$
\mathbf{a}(z,\xi)(\eta-\xi) \le F(z,\eta) - F(z,\xi) \qquad \text{for all } z,\xi,\eta \in \mathbb{R} \tag{4.11}
$$

In what follows we will use M as a generic constant whose purpose is always to denote a bound on the L^∞-norm of the solutions. Its value may change from line to line, or

even within the same line. A notation like $M(a,b,\ldots)$ means that the constant M depends upon the specified quantities.
Note that we have

$$c|z||\xi| - \frac{c^2}{\nu} z^2 \leq \mathbf{a}(z,\xi)\xi \leq cM|\xi| \quad \text{for all } z,\xi \in \mathbb{R}, \quad |z| \leq M. \tag{4.12}$$

First inequality in (4.12) comes after setting $\eta = 0$ in (4.11). Moreover, using (4.11) again it is easy to see that

$$(\mathbf{a}(z,\xi) - \mathbf{a}(z,\hat{\xi}))\ (\xi - \hat{\xi}) \geq 0 \tag{4.13}$$

for any $(z,\xi),(z,\hat{\xi}) \in \mathbb{R}\times\mathbb{R}$, $|z| \leq M$ and

$$(\mathbf{a}(z,\xi) - \mathbf{a}(\hat{z},\hat{\xi}))\ (\xi - \hat{\xi}) \geq -C|z-\hat{z}|\,|\xi - \hat{\xi}| \tag{4.14}$$

for any $(z,\xi),(\hat{z},\xi) \in \mathbb{R}\times\mathbb{R}$, $|z|,|\hat{z}| \leq M$. Finally, we have

$$\mathbf{a}(z,\xi)\,\eta \leq c|z||\eta| \quad \text{for all } \xi,\eta,z \in \mathbb{R}. \tag{4.15}$$

6.4.3 A functional calculus

We introduce the following notation to ease the way in which our functional calculus will be written: for any function q let $J_q(r)$ denote its primitive, i.e.,

$$J_q(r) = \int_0^r q(s)\,ds.$$

We are going to give a sense to the expressions $F(u,u')$ and $\mathbf{a}(u,u')u'$ for functions of bounded variation. For this we rely in dal Masso's theory of relaxed energy functionals of functionals with linear growth with respect to the gradient [82]. The general setting is this: Assume that $f:\mathbb{R}\times\mathbb{R}\to[0,\infty[$ is a continuous function convex in its second variable such that

$$0 \leq f(z,\xi) \leq C(1+|\xi|) \qquad \forall (z,\xi) \in \mathbb{R}\times\mathbb{R},\ |z| \leq M. \tag{4.16}$$

for some constant $C \geq 0$ which may depend on M. Given $f(z,\xi)$, we define its recession function as

$$f^0(z,\xi) = \lim_{t\to 0^+} t f\left(z, \frac{\xi}{t}\right).$$

We assume that $f^0(z,\xi) = \varphi(z)\psi^0(\xi)$, with φ Lipschitz continuous and ψ^0 homogeneous of degree 1. Then, working as in [18], if for a fixed function $\phi \in C_c(]0,L[)$ we define the operator $\mathcal{R}_{\phi f} : BV(]0,L[) \to \mathbb{R}$ by

$$\mathcal{R}_{\phi f}(u) := \int_0^L \phi(x) f(u(x),u'(x))\,dx + \int_0^L \phi(x)\psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|, \tag{4.17}$$

we have that $\mathcal{R}_{\phi f}$ is lower semi-continuous respect to the L^1-convergence.
For instance, we discuss here for future usage one of the cases we are mostly interested in: define $\theta(z) = c|z|$; note that $F^0(z,\xi) = \theta(z)\psi^0(\xi)$, with $\psi^0(\xi) = |\xi|$. Therefore,

$$\mathcal{R}_{\phi F}(u) := \int_0^L \phi(x) F(u(x),u'(x))\,dx + \frac{c}{2}\int_0^L \phi(x)|D^s(u^2)|$$

is lower semi-continuous in $BV(]0,L[)$ respect to the L^1-convergence. This construction gives a meaning to the formal expression $F(u,u')$.

Next we move on to the task of giving a meaning to $\mathbf{a}(u,Du)Du$. For this we shall consider the function $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by

$$h(z,\xi) := \mathbf{a}(z,\xi)\cdot\xi.$$

Note that

$$h(z,\xi)\geq 0 \quad \forall \xi, z\in\mathbb{R}. \tag{4.18}$$

We will make use of the following property:

$$h^0(z,\xi) = F^0(z,\xi) \quad \forall \xi, z\in\mathbb{R}. \tag{4.19}$$

Under the present circumstances we can give a meaning to $\mathbf{a}(u,u')u'$ as an operator $\mathcal{R}_{\phi h}: BV(]0,L[)\to\mathbb{R}$, regarding u' as the Radon–Nikodym derivative and consequently $\mathbf{a}(u,u')$ as a function. We will come back to the precise meaning of $\mathbf{a}(u,u')$ in the next paragraph.

There is also need to introduce functionals that can take into account the boundary values, as it is known that for the Dirichlet problem (see [23]) the boundary data is not taken pointwise in general. The following result is a particular case of Theorem 2.4 in [23].

Theorem 6.4.2 *Let f be verifying (4.16) and such that its recession function factorizes as $f^0(z,\xi)=\varphi(z)|\xi|$, being φ Lipschitz continuous. Let also $\phi\in C([0,L])^+$ be given. Then, the functional $\mathcal{F}^0_{\phi f}: BV(]0,L[)\longrightarrow\mathbb{R}$ defined by*

$$\mathcal{F}^0_{\phi f}(u) := \mathcal{R}_{\phi f}(u) + \phi(L)\,|J_\varphi(u)(L_-)|$$

is lower semi-continuous with respect to the L^1-convergence.

We will require the following specialized result.

Corollary 6.4.3 *Let $S=S(z)$ be a Lipschitz function. Let also $\phi\in C([0,L])^+$ be given. The functionals $\mathcal{F}^0_{\phi SF}$, $\mathcal{F}^0_{\phi F}$ $: BV(]0,L[)\longrightarrow\mathbb{R}$ are lower semi-continuous with respect to the L^1-convergence.*

6.4.4 Spaces of truncated functions and associated calculus

As already pointed out in Chapter 6.3, the degeneracy of the equation close to zero forces us to use cutoff functions in order to retain some regularity for the solutions. The tools devised to this aim are detailed here.

We need to take into account the following truncation functions. For $a<b$, let

$$T_{a,b}(r) := \max\{\min\{b,r\},a\}.$$

We denote $T_k = T_{-k,k}$. Given $l\in\mathbb{R}$, we shall also consider the truncation functions $T^l_{a,b}(r) := T_{a,b}(r) - l$. We define the following sets of truncations:

$$\mathcal{T}_r := \{T_{a,b} \;:\; 0<a<b\},\quad \mathcal{T}^+ := \{T^l_{a,b} \;:\; 0<a<b,\ l\in\mathbb{R},\ T^l_{a,b}\geq 0\}.$$

Consider the function space

$$TBV^+(]0,L[) := \left\{ u \in L^1(]0,L[)^+ \;:\; T(u) \in BV(]0,L[), \quad \forall\, T \in \mathcal{T}_r \right\};$$

we want to give a sense to the Radon–Nikodym derivative u' of a function u belonging to $TBV^+(]0,L[)$. Using chain's rule for BV-functions (see, for instance, [3]), and with a similar proof to the one given in Lemma 2.1 of [41], we obtain the following result.

Lemma 6.4.4 *For every $u \in TBV^+(]0,L[)$ there exists a unique measurable function $v :]0,L[\to \mathbb{R}$ such that*

$$(T_{a,b}(u))' = v\chi_{[a<u<b]} \qquad \mathcal{L}^1 - \text{a.e.}, \quad \forall\, T_{a,b} \in \mathcal{T}_r. \tag{4.20}$$

Thanks to this result we *define* u' for a function $u \in TBV^+(]0,L[)$ as the unique function v which satisfies (4.20). This notation will be used throughout in the sequel. Notice that with this result at hand the object $\mathbf{a}(u,u')$ has a sense as a function a.e. defined, for any $u \in TBV^+(]0,L[)$.
The notation ∂_x will also be used in the case of functions of several variables (say t and x), for the same purposes, whenever there is some risk of confusion. Note also that this concept of derivative reduces to the case of the Radon–Nikodym derivative for function of bounded variation, or to the ordinary derivative of smooth functions, then this notation coincides with the standard one.

Sometimes it will be handy to consider a more general class of truncation functions. We denote by $\mathcal{P}$ the set of Lipschitz continuous function $p : [0,+\infty[\to \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough. We recall the following result ([14], Lemma 2).

Lemma 6.4.5 *If $u \in TBV^+(]0,L[)$, then $p(u) \in BV(]0,L[)$ for every $p \in \mathcal{P}$ such that $p(r) = 0$ in a neighborhood of $r = 0$. Moreover, with the above notation $[p(u)]' = p'(u)u'$ $\mathcal{L}^1$-a.e.*

The following straightforward consequence will be useful to deal with products of truncation functions, a situation that we will encounter when using Kruzkov's method to prove uniqueness.

Corollary 6.4.6 *Let $S \in \mathcal{P}^+$ and $T = T^a_{a,b}$. Given $u \in TBV^+(]0,L[)$, then*

$$T(u), S(u)T(u), J_{T'S}(u), J_{TS'}(u) \in BV(]0,L[).$$

Moreover,

$$D(S(u)T(u)) = DJ_{T'S}(u) + DJ_{TS'}(u)$$

and hence, if $\mathbf{z} \in W^{1,1}(]0,L[)$,

$$\mathbf{z}D(T(u)S(u)) = \mathbf{z}DJ_{T'S}(u) + \mathbf{z}DJ_{TS'}(u).$$

As regards boundary values, we will define for $u \in TBV^+(]0,L[)$

$$u(0_+) := \lim_{n\to\infty} T_{\frac{1}{n},n}(u)(0_+) \quad \text{and} \quad u(L_-) := \lim_{n\to\infty} T_{\frac{1}{n},n}(u)(L_-).$$

It is easy to see that the above limits exist [23].

Now we are going to extend the definition of the functionals $\mathcal{R}_{\phi f}$ from the BV setting to this TBV setting, allowing to deal with expressions as $F(u, DT(u))$ and

$\mathbf{a}(u,u')DT(u)$. For technical reasons related to Kruzkov's doubling variable procedure, we also need to give a meaning to $F(u,DT(u))S(u)$ and $\mathbf{a}(u,u')DT(u)S(u)$. Consider to this purpose f as in (4.16). For $u \in TBV^+(]0,L[)$, $\phi \in C_c(]0,L[)$ and $T = T_{a,b} - l \in \mathcal{T}^+$ with $l \in \mathbb{R}$, we introduce the functional

$$\mathcal{R}(\phi f, T)(u) := \mathcal{R}_{\phi f}(T_{a,b}(u)) + \int_{[u \leq a]} \phi(x)(f(u(x),0) - f(a,0))\, dx$$
$$- \int_{[u \geq b]} \phi(x)(f(u(x),0) - f(b,0))\, dx.$$

We have that $\mathcal{R}(\phi f, T)(\cdot)$ is lower semicontinuous in $TBV^+(]0,L[)$ with respect to the L^1-convergence.

Given $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and $u \in TBV^+(]0,L[)$, we define the following Radon measures in $]0,L[$,

$$\langle F(u,DT(u)),\phi\rangle := \mathcal{R}(\phi F, T)(u), \quad \langle F_S(u,DT(u)),\phi\rangle := \mathcal{R}(\phi SF, T)(u),$$
$$\langle h(u,DT(u)),\phi\rangle := \mathcal{R}(\phi h, T)(u), \quad \langle h_S(u,DT(u)),\phi\rangle := \mathcal{R}(\phi Sh, T)(u),$$

for $\phi \in C_c(]0,L[)$. Using (4.17) and (4.19), we compute

$$\begin{cases} F(u,DT(u))^s = \frac{c}{2}\left|D^s(T(u))^2\right| = h(u,DT(u))^s, \\ h(u,DT(u))^{ac} = h(u,(T(u))'), \\ F_S(u,DT(u))^s = |D^s J_{S\theta}(T(u))| = h_S(u,DT(u))^s \\ h_S(u,DT(u))^{ac} = S(u)h(u,(T(u))'). \end{cases} \tag{4.21}$$

6.5 The Elliptic Problem

Our way to solve the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u,u_x))_x & \text{in} \quad]0,T[\times]0,L[\\ -\mathbf{a}(u(t,0),u_x(t,0)) = \beta > 0 \text{ and } u(t,L) = 0 & \text{on} \quad t \in]0,T[, \\ u(0,x) = u_0(x) & \text{in} \quad x \in]0,L[\end{cases} \tag{5.22}$$

rests in the following idea: proceed with a discretization of the time derivative, so that

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} \sim (\mathbf{a}(u(t+\Delta t), u(t+\Delta t)_x))_x\,,$$

fix then a time step Δt, use it to construct a grid on $[0,T]$, assign the "value" u_i to the i-th point of the grid according to the rule

$$\frac{u_i - u_{i-1}}{\Delta t} = (\mathbf{a}(u_i,(u_i)_x))_x\,,$$

and finally use the resulting collection of "values" to approximate the actual solution. Semigroup theory gives the abstract foundations to this schematic procedure. So, given $v \in L^1(]0,L[)$, we are interested in the following abstract problem:

$$\begin{cases} -\left(\mathbf{a}(u,u')\right)' = v & \text{in }]0,L[\\ -\mathbf{a}(u,u')|_{x=0} = \beta > 0, & u(L) = 0, \end{cases} \tag{5.23}$$

where $\mathbf{a}$ is given by (4.10). Note that the time step with its precise physical dimensions is not present in this abstract framework. We introduce the following concept of solution for problem (5.23).

Definition 6.5.1 Given $v \in L^1(]0,L[)$, we say that $u \geq 0$ is an *entropy solution* of (5.23) if $u \in TBV^+(]0,L[)$ and $\mathbf{a}(u,u') \in C([0,L])$ both satisfy the following conditions:

$$v = -D\mathbf{a}(u,u') \qquad \text{in } \mathcal{D}'(]0,L[),$$

$$-\mathbf{a}(u,u')(0) = \beta, \qquad \text{and} \qquad \mathbf{a}(u,u')(L) = -cu(L_-). \tag{5.24}$$

$$h(u, DT(u)) \leq \mathbf{a}(u,u')\ DT(u) \quad \text{as measures} \quad \forall\, T \in \mathcal{T}^+ \tag{5.25}$$

$$h_S(u, DT(u)) \leq \mathbf{a}(u,u')\ DJ_{T'S}(u) \quad \text{as measures } \forall\, S \in \mathcal{P}^+,\ T \in \mathcal{T}^+. \tag{5.26}$$

As the absolutely continuos parts coincide thanks to (4.21), the condition (5.25) can be rewritten as $h(u,DT(u))^s \leq [\mathbf{z}\ DT(u)]^s$, and thus it is equivalent to

$$\frac{c}{2}|D^s((T(u))^2)| \leq \mathbf{z}\ D^sT(u) \quad \text{as measures} \quad \forall\, T \in \mathcal{T}^+. \tag{5.27}$$

Also we have that (5.26) can be rewritten as $h_S(u,DT(u))^s \leq [\mathbf{z}\ DJ_{T'S}(u)]^s$, and is equivalent to

$$|D^s(J_{S\theta}(T(u)))| \leq \mathbf{z}\ D^sJ_{T'S}(u) \quad \text{as measures} \quad \forall\, S \in \mathcal{P}^+,\ T \in \mathcal{T}^+. \tag{5.28}$$

A particular consequence —see (4.18)— is that

$$\mathbf{a}(u,u')DT(u),\ \mathbf{a}(u,u')DJ_{T'S}(u) \geq 0. \tag{5.29}$$

Remark 6.5.2 Let us point out that the definition of solution uses some sloppy conventions. The pointwise-defined function $\mathbf{a}(u,u')$ as it stands might not be continuous, but nevertheless it belongs to $W^{1,1}(]0,L[)$. We will denote by $\mathbf{z}$ its continuous representative and we will shift from one to the other without warning. This won't be misleading, as both have the same boundary values in the sense of traces. Meaning that any time we are to deal with pointwise values of $\mathbf{a}(u,u')$ it will be in the sense of traces. As an illustration of this we will prove here a lower bound for the solutions at the left end of the interval.

The first boundary condition reads formally as

$$\nu \frac{u(0)u'(0)}{\sqrt{u(0)^2 + \frac{\nu^2}{c^2}|u'(0)|^2}} = -\beta.$$

Taking absolute values we would obtain that $c\, u(0) \geq \beta$. But as we have stated before plain evaluation at $x = 0$ might be nonsense. To derive this estimate in the right way we have to deal with boundary traces. Since $|\mathbf{a}(u,u')| \le c|u|$ follows from (4.10) , we may take traces on this inequality and thanks to (5.24) we find that

$$u(0_+) \ge \frac{\beta}{c} > 0. \tag{5.30}$$

We introduce now the main result of this section.

Theorem 6.5.3 *For any $0 \le f \in L^\infty(]0,L[)$ there exists a unique entropy solution $u \in TBV^+(]0,L[)$ of the problem*

$$\begin{cases} u - (\mathbf{a}(u,u'))' = f & \text{in }]0,L[\\ -\mathbf{a}(u,u')|_{x=0} = \beta > 0, & u(L) = 0, \end{cases} \tag{5.31}$$

which satisfies $\|u\|_\infty \le M(\beta, c, \nu, \|f\|_\infty)$. Moreover, let $u, \overline{u}$ be two entropy solutions of (5.31) associated to $f, \overline{f} \in L^1(]0,L[)^+$, respectively. Then,

$$\int_0^L (u-\overline{u})^+\,dx \le \int_0^L (f-\overline{f})^+\,dx.$$

The rest of the section is devoted to the proof of this statement.

6.5.1 Existence of entropy solutions

We begin considering a family of approximating problems and showing a-priori estimates for them. Let $0 \le f \in L^\infty(]0,L[)$. For every $n \in \mathbb{N}$, consider

$$\mathbf{a}_n(z,\xi) := \mathbf{a}(z,\xi) + \frac{1}{n}\xi.$$

As a consequence of the results about pseudo-monotone operators in [56] we know that $\forall n \in \mathbb{N}$ there exists a unique $u_n \in W^{1,2}(]0,L[)$ such that $u_n(L) = 0$ and

$$\int_0^L u_n v\,dx + \int_0^L \mathbf{a}(u_n,u_n')v'\,dx + \frac{1}{n}\int_0^L u_n'v'\,dx - \beta v(0) = \int_0^L fv\,dx \tag{5.32}$$

for all $v \in W^{1,2}(]0,L[)$, $v(L) = 0$.

Lemma 6.5.4 *The functions u_n are non-negative $\forall\, n \in \mathbb{N}$.*

Proof. In fact, taking $v = u_n^- = -\min\{u_n(x), 0\}$ in (5.32), we get

$$\int_0^L u_n u_n^-\,dx + \int_0^L \mathbf{a}(u_n,u_n')(u_n^-)'\,dx + \frac{1}{n}\int_0^L u_n'(u_n^-)'\,dx = \beta u_n^-(0) + \int_0^L f u_n^-\,dx \ge 0.$$

Now,

$$\int_0^L u_n'(u_n^-)'\,dx = -\int_{[u_n<0]} ((u_n^-)')^2\,dx \le 0$$

and

$$\int_0^L \mathbf{a}(u_n,u_n')(u_n^-)'\,dx = -\int_{[u_n<0]} \frac{\nu|u_n|((u_n^-)')^2}{\sqrt{u_n^2 + \frac{\nu^2}{c^2}|u_n'|^2}}\,dx \le 0.$$

Hence,

$$0 \le \int_0^L u_n u_n^- \, dx = -\int_0^L (u_n^-)^2 \, dx$$

and Lemma 6.5.4 holds. □

Now we give a bound for the boundary values of the sequence u_n at zero.

Lemma 6.5.5 *The sequence $\{u_n(0)\}$ is bounded. More precisely, given any $\epsilon \in]0,2[$ such that $\frac{c^2}{\nu} + \frac{\epsilon}{2} > 1$, the following bound holds:*

$$0 \le u_n(0) \le \sqrt{\frac{2c}{\nu\epsilon(2-\epsilon)}}\|f\|_2 + \frac{4c\,\beta}{\nu(2-\epsilon)}. \tag{5.33}$$

Proof. Taking $v = u_n$ in (5.32), we get

$$\int_0^L u_n^2 \, dx + \int_0^L \mathbf{a}(u_n, u_n')u_n' \, dx + \frac{1}{n}\int_0^L ((u_n)')^2 \, dx = \beta u_n(0) + \int_0^L f u_n \, dx. \tag{5.34}$$

Then, dropping nonnegative terms and performing Young's inequality with weights,

$$\int_0^L u_n^2 \, dx \le \beta u_n(0) + \int_0^L f u_n \, dx \le \beta u_n(0) + \frac{\epsilon}{2}\int_0^L u_n^2 \, dx + \frac{1}{2\epsilon}\int_0^L f^2 \, dx$$

and we get

$$\int_0^L u_n^2 \, dx \le \frac{1}{\epsilon(2-\epsilon)}\int_0^L f^2 \, dx + \frac{2}{2-\epsilon}\beta\, u_n(0). \tag{5.35}$$

Taking into account (4.12) we have

$$u_n'\mathbf{a}(u_n, u_n') \ge c\; u_n|u_n'| - \frac{c^2}{\nu}u_n^2.$$

Now we can write $u_n|u_n'| = \frac{1}{2}|(u_n^2)'|$ as $u_n^2 \in W^{1,1}(]0,L)$. Then, from (5.34), we obtain

$$\int_0^L \frac{c}{2}|(u_n^2)'| \, dx + \int_0^L \frac{((u_n)')^2}{n} dx \le \int_0^L \left(\frac{c^2}{\nu} - 1\right) u_n^2 \, dx + \int_0^L f u_n \, dx + \beta u_n(0). \tag{5.36}$$

We use also Young's inequality with the same weights as before on the right hand side of (5.36), thus getting

$$\begin{aligned} \frac{c}{2}u_n(0)^2 &= \left|\int_0^L \frac{c}{2}(u_n^2)' \, dx\right| \le \int_0^L \frac{c}{2}|(u_n^2)'| \, dx \\ &\le \left(\frac{c^2}{\nu} - 1 + \frac{\epsilon}{2}\right)\int_0^L u_n^2 \, dx + \frac{1}{2\epsilon}\int_0^L f^2 \, dx + \beta u_n(0) \end{aligned}$$

or

$$u_n(0)^2 \le \left(\frac{2c}{\nu} + \frac{\epsilon - 2}{c}\right)\int_0^L u_n^2 \, dx + \frac{1}{c\,\epsilon}\int_0^L f^2 \, dx + \frac{2\beta}{c}u_n(0).$$

Now we substitute (5.35) in the previous and we arrive to

$$u_n(0)^2 \le \frac{2c}{\nu\epsilon(2-\epsilon)}\int_0^L f^2 \, dx + \frac{4c\,\beta\, u_n(0)}{\nu(2-\epsilon)}.$$

Then, we have

$$u_n^2(0) - \frac{4c\,\beta}{\nu(2-\epsilon)} u_n(0) - \frac{2c}{\nu\epsilon(2-\epsilon)} \int_0^L f^2 \, dx \le 0,$$

from where we deduce that for all $n \in \mathbb{N}$,

$$0 \le u_n(0) \le \frac{1}{2} \left(\frac{4c\,\beta}{\nu(2-\epsilon)} + \sqrt{\left(\frac{4c\;\beta}{\nu(2-\epsilon)} \right)^2 + \frac{8c}{\nu\epsilon(2-\epsilon)} \|f\|_2^2} \right)$$

and (5.33) follows. □

By (5.36) and Lemma 6.5.5, we get

$$\frac{c}{2} \int_0^L |(u_n^2)'| \, dx + \frac{1}{n} \int_0^L ((u_n)')^2 \, dx \le C \qquad \forall\, n \in \mathbb{N}. \tag{5.37}$$

Lemma 6.5.6 *The sequence $\{u_n \ : \ n \in \mathbb{N}\}$ is uniformly bounded in $L^\infty(]0, L[)$.*

Proof. By (5.33), we know that

$$M = \max\left\{ \|f\|_\infty, \max\{u_n(0) \ : \ n \in \mathbb{N}\} \right\} < +\infty.$$

Then, taking as test function in (5.32) $v = (u_n - M)^+$, we get

$$\int_0^L u_n\,(u_n - M)^+ \; dx + \int_0^L \mathbf{a}(u_n, u_n') \left((u_n - M)^+\right)' \; dx$$
$$+\frac{1}{n} \int_0^L u_n' \left((u_n - M)^+\right)' \; dx - \beta\,(u_n(0) - M)^+ = \int_0^L f\,(u_n - M)^+ \; dx.$$

As the fourth term of the left hand side vanishes and the second and third ones are non negative, we arrive to

$$\int_0^L u_n\,(u_n - M)^+ \; dx \le \int_0^L f\,(u_n - M)^+ \; dx.$$

Hence

$$0 \le \int_0^L \left[(u_n - M)^+\right]^2 \; dx \le \int_0^L (f - M)\,(u_n - M)^+ \; dx \le 0,$$

and consequently, $u_n(x) \le M$ for almost all $x \in [0, L]$ and all $n \in \mathbb{N}$. Then, since u_n is non-negative, we get

$$\|u_n\|_\infty \le \max\left\{ \|f\|_\infty, \max\{u_n(0) \ : \ n \in \mathbb{N}\} \right\} < +\infty \tag{5.38}$$

and Lemma 6.5.6 holds. □

Lemma 6.5.7 *The sequence $\{u_n\}$ is uniformly bounded in $TBV^+(]0, L[)$. Furthermore, there exists a function $0 \le u \in TBV^+(]0, L[) \cap L^\infty(]0, L[)$ such that (up to subsequence) $u_n \to u$ a.e. and strongly in $L^1(]0, L[)$.*

Proof. By Lemma 6.5.6, extracting a subsequence if necessary, we may assume that u_n converges weakly in $L^2(]0,L[)$ to some nonnegative function u as $n \to +\infty$. Moreover, by (5.38), we have that $0 \le u \in L^\infty(]0,L[)$. On the other hand, estimates so far show that $u_n^2 \in W^{1,1}(]0,L[)$ uniformly in n. Thus $u^2 \in BV(]0,L[)$ and hence by virtue of the chain rule $u \in BV$ if we stay away from the zeroth level set, as we can compose with a Lipschitz function and the square root function is so out of a neighborhood of zero. We put this in more precise terms below.

Consider $0 < a < b$. By means of the coarea formula and (5.37), we have

$$\int_0^L |(T_{a,b}(u_n))'|\,dx = \int_a^b \left|D\chi_{[u_n \le t]}\right|(]0,L[)\,dt = \int_a^b \left|D\chi_{[u_n^2 \le t^2]}\right|(]0,L[)\,dt$$

$$= \int_{a^2}^{b^2} \left|D\chi_{[u_n^2 \le s]}\right|(]0,L[)\frac{ds}{2\sqrt{s}} \le \frac{1}{2a}\int_0^L |(u_n^2)'|\,dx \le \frac{C}{a}.$$

This estimate entails equicontinuity properties for the sequence $\{u_n\}$, as we are going to show. Let now $h > 0$ be given; we are going to use the previous estimate with $b > M$ and $a = \sqrt{h}$. Consider the set

$$H = \{x \in [0,L]/x+h \in [0,L] \text{ and } u_n(x+h),\ u_n(x) \ge \sqrt{h}\}.$$

Then

$$\int_H |u_n(x+h) - u_n(x)|\,dx \le \frac{C|h|}{\sqrt{h}}$$

while

$$\int_{[\max\{0,-h\},\min\{L-h,L\}]\setminus H} |u_n(x+h) - u_n(x)|\,dx \le 2\sqrt{h}L.$$

We have proved that

$$\int_{\max\{0,-h\}}^{\min\{L-h,L\}} |u(t,x+h) - u(t,x)|\ dx \le |h|^{\frac{1}{2}} \max\{2L, C\}$$

uniformly in n for any $h \in \mathbb{R}$. Together with Lemma 6.5.6 we can invoke the Frechet–Kolmogorov theorem to get that $\{u_n\}$ is strongly compact in $L^1(]0,L[)$, as desired. Using the above estimate on the gradients we obtain that $u \in TBV^+(]0,L[)$. □

Since $|\mathbf{a}(u_n,u_n')| \le c|u_n|$, by Lemma 6.5.6 we get that

$$\mathbf{a}(u_n,u_n') \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly}^* \text{ in } L^\infty(]0,L[). \tag{5.39}$$

By assumption we have that

$$\mathbf{a}(u_n,u_n') = c\,u_n \mathbf{b}(u_n,u_n')$$

with $|\mathbf{b}(u_n,u_n')| \le 1$ (independent of n), $\|u_n\|_\infty \le M$ and $u_n \to u$ a.e. as $n \to \infty$. So, we may assume that

$$\mathbf{b}(u_n,u_n') \rightharpoonup \mathbf{z}_b$$

weakly* in $L^\infty(]0,L[)$ as $n \to \infty$ and

$$\mathbf{z} = c\,u\,\mathbf{z}_b, \qquad \text{with} \quad \|\mathbf{z}_b\|_\infty \le 1. \tag{5.40}$$

On the other hand, by (5.37),

$$\frac{1}{n}u_n' \to 0 \quad \text{in} \;\; L^2(]0,L[). \tag{5.41}$$

Given $\phi \in \mathcal{D}(]0,L[)$, taking $v=\phi$ in (5.32) we obtain

$$\int_0^L u_n\phi\,dx + \int_0^L \mathbf{a}(u_n,u_n')\phi'\,dx + \frac{1}{n}\int_0^L u_n'\phi'\,dx = \int_0^L f\phi\,dx$$

Letting $n\to+\infty$, having in mind (5.39) and (5.41), we obtain

$$\int_0^L (f-u)\phi\;dx = \int_0^L \mathbf{z}\,\phi'\;dx,$$

that is,

$$f-u = -D\mathbf{z}, \qquad \text{in} \;\; \mathcal{D}'(]0,L[) \tag{5.42}$$

and (as $f-u_n \rightharpoonup f-u$ in $L^2(]0,L[)$)

$$(\mathbf{a}_n(u_n,u_n'))' \rightharpoonup D\mathbf{z} \quad \text{weakly} \;\; \text{in} \;\; L^2(]0,L[).$$

Note that by (5.42), we have $\mathbf{z}\in W^{1,p}(]0,L[)$ for $1\le p\le\infty$ and $D\mathbf{z}=\mathbf{z}'$. The next step is to identify the object $\mathbf{z}$.

Lemma 6.5.8 *The functions $\mathbf{z}(x)$ and $\mathbf{a}(u(x),u'(x))$ coincide for a.e. $x\in]0,L[$.*

Proof. We use Minty–Browder's technique. Let $0<a<b$, let $0\le\phi\in C_c^1(]0,L[)$ and let $g\in C^2([0,L])$. When we write $T_{a,b}'$ we shall mean $\chi_{]a,b[}$. By (4.13), we have that

$$\int_0^L \phi[\mathbf{a}(u_n,u_n')-\mathbf{a}(u_n,g')](u_n-g)'T_{a,b}'(u_n)\,dx \ge 0.$$

Note that introducing suitable terms added and subtracted we might write

$$\int_0^L \phi\,\mathbf{a}(u_n,u_n')(u_n-g)'T_{a,b}'(u_n)\,dx = \int_0^L \phi\,\mathbf{a}_n(u_n,u_n')\,(T_{a,b}(u_n)-g)'\;dx$$
$$-\frac{1}{n}\int_0^L \phi\,u_n'\,(T_{a,b}(u_n)-g)'\;dx + \int_0^L \phi\,\mathbf{a}(u_n,u_n')\,g'\left(1-T_{a,b}'(u_n)\right)\,dx,$$

which, using integration by parts and the fact that $\phi\;u_n'[T_{a,b}(u_n)]'$ is nonnegative, is less or equal than

$$-\int_0^L \left[\mathbf{a}_n(u_n,u_n')\right]'\phi\,(T_{a,b}(u_n)-g)\;dx - \int_0^L (T_{a,b}(u_n)-g)\,\mathbf{a}_n(u_n,u_n')\phi'\,dx$$
$$+\frac{1}{n}\int_0^L \phi\,u_n'\,g'\,dx + \int_0^L \phi\,\mathbf{a}(u_n,u_n')\,g'\left(1-T_{a,b}'(u_n)\right)\,dx.$$

Thus, using that $\phi\,(T_{a,b}(u_n)-g)$ converges strongly in $L^2(]0,L[)$ thanks to the dominated convergence,

$$\begin{aligned}&\lim_{n\to+\infty}\int_0^L \phi\,\mathbf{a}(u_n,u_n')(u_n-g)'T'_{a,b}(u_n)\,dx\\ \le&-\int_0^L \mathbf{z}'\,\phi\,(T_{a,b}(u)-g)\;dx-\int_0^L (T_{a,b}(u)-g)\,\mathbf{z}\,\phi'\,dx\\ &+cM\|g'\|_\infty\int_0^L\phi\left(1-T'_{a,b}(u)\right)\,dx\\ =&\langle\mathbf{z}\,D\,(T_{a,b}(u)-g)\,,\phi\rangle+M\|g'\|_\infty\int_0^L\phi\left(1-T'_{a,b}(u)\right)\,dx\end{aligned}$$

On the other hand, let us denote

$$\begin{aligned}J_{\mathbf{a}}(x,r):=&\int_0^r \mathbf{a}(s,g'(x))\,ds,\\ J_{\mathbf{a}'}(x,r):=&\int_0^r \partial_x[\mathbf{a}(s,g'(x))]\,ds=\int_0^r\frac{\partial\mathbf{a}}{\partial\xi}(s,g'(x))g''(x)\,ds\end{aligned}$$

and observe that

$$\begin{aligned}[J_{\mathbf{a}}(x,T_{a,b}(u_n(x)))]'=&\mathbf{a}\left(T_{a,b}(u_n(x)),g'(x)\right)u_n'(x)T'_{a,b}(u_n(x))+J_{\mathbf{a}'}(x,T_{a,b}(u_n(x)))\\ =&\mathbf{a}\left(u_n(x),g'(x)\right)u_n'(x)T'_{a,b}(u_n(x))+J_{\mathbf{a}'}(x,T_{a,b}(u_n(x))).\end{aligned}$$

As the map $z\mapsto\frac{\partial\mathbf{a}}{\partial\xi}(z,\xi)$ is continuous for fixed ξ, the a.e. convergence of u_n implies that

$$J_{\mathbf{a}'}\left(x,T_{a,b}(u_n(x))\right)\to J_{\mathbf{a}'}\left(x,T_{a,b}(u(x))\right)\text{ a.e.}$$

and we also have that

$$J_{\mathbf{a}}\left(x,T_{a,b}(u_n(x))\right)\to J_{\mathbf{a}}\left(x,T_{a,b}(u_n(x))\right)\quad\text{strongly in }L^1(]0,L[).$$

This later claim follows form a dominated convergence, since $|J_{\mathbf{a}}(x,r)|\le cMr$ and another dominated convergence justifies that $J_{\mathbf{a}}\left(x,T_{a,b}(u_n(x))\right)\to J_{\mathbf{a}}\left(x,T_{a,b}(u_n(x))\right)$ almost everywhere. It follows that

$$[J_{\mathbf{a}}(x,T_{a,b}(u_n(x)))]'\rightharpoonup D[J_{\mathbf{a}}(x,T_{a,b}(u(x)))]\quad\text{weakly}^*\text{ as measures,}$$

for an uniform bound in $L^1(]0,L[)$ of the above sequence is obtained on the aid of the coarea formula and the fact that g'' is bounded (see Proposition C.1.5 in the Appendix).

Consequently, we have

$$\begin{aligned}&\lim_{n\to+\infty}\int_0^L\phi\,\mathbf{a}(u_n,g')(u_n-g)'T'_{a,b}(u_n)\,dx\\ =&\lim_{n\to+\infty}\int_0^L\phi\left\{[J_{\mathbf{a}}(x,T_{a,b}(u_n(x)))]'-J_{\mathbf{a}'}(x,T_{a,b}(u_n(x))\right\}\,dx\\ &-\lim_{n\to+\infty}\int_0^L\phi\,\mathbf{a}(u_n,g')g'T'_{a,b}(u_n)\,dx\\ =&\langle D\left[J_{\mathbf{a}}(x,T_{a,b}(u(x)))\right]-J_{\mathbf{a}'}(x,T_{a,b}(u(x))),\phi\rangle-\int_0^L\phi\,\mathbf{a}(u,g')g'T'_{a,b}(u)\,dx,\end{aligned}$$

as $\mathbf{a}(u_n, g') \to \mathbf{a}(u, g')$ almost everywhere. Then we obtain

$$\begin{aligned} 0 \leq & \langle \mathbf{z}\, D\left(T_{a,b}(u) - g\right), \phi\rangle + M\|g'\|_\infty \int_0^L \phi\left(1 - T'_{a,b}(u)\right)\, dx \\ & + \int_0^L \phi\, \mathbf{a}(u, g') g' T'_{a,b}(u)\, dx \\ & - \langle D\left[J_{\mathbf{a}}(x, T_{a,b}(u(x)))\right] - J_{\mathbf{a}'}(x, T_{a,b}(u(x))), \phi\rangle \end{aligned}$$

for all $0 \leq \phi \in C_c^1(]0, L[)$. This means that, as measures,

$$\begin{aligned} 0 \leq & \left\{a(u, g') g' T'_{a,b}(u) + M\|g'\|_\infty \left(1 - T'_{a,b}(u)\right)\right\} \mathcal{L}^1 \\ & + \mathbf{z}\, D\left(T_{a,b}(u) - g\right) - D\left[J_{\mathbf{a}}(x, T_{a,b}(u(x)))\right] + J_{\mathbf{a}'}(x, T_{a,b}(u(x)))\mathcal{L}^1. \end{aligned}$$

Now we compute

$$\begin{aligned} D_x^{ac}[J_{\mathbf{a}}(u_1, u_2)] = & \nabla J_{\mathbf{a}}(u_1, u_2) \frac{(u_1, u_2)}{\partial x} \\ = & \{J_{\mathbf{a}'}(u_1, u_2), \mathbf{a}(u_2, g'(u_1))\} \cdot \{1, (T_{a,b}(u(x)))'\} \\ = & J_{\mathbf{a}'}(x, T_{a,b}(u(x))) + \mathbf{a}(T_{a,b}(u(x)), g'(x))[T_{a,b}(u(x))]' \end{aligned}$$

using the chain rule for BV functions ([3], Theorem 3.96), being $u_1(x) = x$ and $u_2(x) = T_{a,b}(u(x))$. Then we deduce that the absolutely continuous part of

$$-D\left[J_{\mathbf{a}}(x, T_{a,b}(u(x)))\right] + J_{\mathbf{a}'}(x, T_{a,b}(u(x)))\mathcal{L}^1$$

is

$$-\mathbf{a}(u, g')(T_{a,b}(u))'\, \mathcal{L}^1$$

and thus, comparison of the absolutely continuous parts leads to

$$\mathbf{z}\left(T_{a,b}(u) - g\right)' - \mathbf{a}(u, g')(T_{a,b}(u))' + \mathbf{a}(u, g')\, g' T'_{a,b}(u) + cM\|g'\|_\infty \left(1 - T'_{a,b}(u)\right) \geq 0.$$

If $x \in [a < u < b]$, this reduces to

$$\left(\mathbf{z} - \mathbf{a}(u, g')\right)(u - g)' \geq 0,$$

which holds for all $g \in C^2([0, L])$ and all $x \in \Omega \cap [a < u < b]$, where $\mathcal{L}^1(]0, L[\setminus\Omega) = 0$. Being $x \in \Omega \cap [a < u < b]$ fixed and $\xi \in \mathbb{R}$ given, we find a function g as above such that $g'(x) = \xi$. Then

$$(\mathbf{z}(x) - \mathbf{a}(u(x), \xi))\,(u'(x) - \xi) \geq 0, \ \forall \xi \in \mathbb{R}.$$

By an application of Minty–Browder's method in $\mathbb{R}$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) \quad \text{a.e. on}\, [a < u < b].$$

Since this holds for any $0 < a < b$, we obtain equality a.e. on the points of $]0, L[$ such that $u(x) \neq 0$. Now, by our assumptions on $\mathbf{a}$ and (5.40) we deduce that $\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) = 0$ a.e. on $[u = 0]$. We have proved Lemma 6.5.8.

□

From Lemma 6.5.8 and (5.42) it follows that

$$f - u = -\mathbf{a}(u, u')', \quad \text{in}\, \mathcal{D}'(]0, L[)$$

Lemma 6.5.9 *The flux* $-\mathbf{a}(u,u')$ *verifies the Neumann condition at* $x=0$.

Proof. Let $w \in W^{1,1}(]0,L[)$ such that $w(L)=0$ and consider $w_k \in W^{1,2}(]0,L[)$ with $w_k(L)=0$ for all $k \in \mathbb{N}$, $w_k \to w$ pointwise and $w_k' \to w'$ in $L^1(]0,L[)$. Taking in (5.32) w_k as test function and letting $n \to +\infty$, we get

$$\int_0^L u\, w_k\, dx + \int_0^L \mathbf{z}\, w_k'\, dx - \beta w_k(0) = \int_0^L f w_k\, dx.$$

Then, letting $k \to +\infty$ we arrive to

$$\int_0^L u\, w\, dx + \int_0^L \mathbf{z}\, w'\, dx - \beta w(0) = \int_0^L f w\, dx. \tag{5.43}$$

Fixed $w \in BV(]0,L[)$ such that $w(L_-)=0$, let $w_m \in W^{1,1}(]0,L[)$ with $w_m(L)=0$, $w_m(0)=w(0_+)$ and such that $w_m \to w$ in $L^1(]0,L[)$. Taking in (5.43) w_m as test functions and integrating by parts we get

$$\int_0^L (f-u)w_m\, dx = \int_0^L \mathbf{z}\ w_m'\, dx - \beta w(0_+) = -\int_0^L \mathbf{z}' w_m\, dx - w(0_+)(\mathbf{z}(0)+\beta).$$

Letting $m \to +\infty$, the right hand side becomes

$$\int_0^L \mathbf{z}' w\ dx - \mathbf{z}(0)w(0_+) - \beta w(0_+) = \int_0^L w(f-u)\ dx - \mathbf{z}(0)w(0_+) - \beta w(0_+);$$

recall that $\mathbf{z} \in W^{1,p}(]0,L[)$ for $1 \le p \le \infty$, and that $f-u=\mathbf{z}'$ holds in any $L^p(]0,L[)$. On the other hand, this coincides with

$$\lim_m \int_0^L (f-u)w_m\, dx = \int_0^L (f-u)w\ dx,$$

from where we get $0 = -\mathbf{z}(0)w(0_+) - \beta w(0_+)$ and we are done, since $-\mathbf{a}$ and $-\mathbf{z}$ leave the same trace at the origin. $\square$

Lemma 6.5.10 *Let* $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ *and* $\phi \in C^1([0,L])$, $\phi \ge 0$, *with* $\phi(0)=0$. *Then*

$$\begin{aligned}&\int_0^L \phi F(u, DT(u)) + \phi(L)\frac{c}{2}|(T(u))^2(L_-)|\\ &\le \int_0^L \phi\, \mathbf{z} DT(u) + \int_0^L \phi F(u,0)\, dx\\ &-\phi(L)T(u)(L_-) + \phi(L)|J_\theta(T(0))|\end{aligned} \tag{5.44}$$

and

$$\begin{aligned}&\int_0^L \phi F_S(u, DT(u)) + \phi(L)\,|J_{\theta S}(T(u)(L_-))|\\ &\le \int_0^L \phi\, \mathbf{z} DJ_{T'S}(u) + \int_0^L \phi S(u) F(u,0)\, dx\\ &-\phi(L)\mathbf{z}(L)J_{T'S}(u(L_-)) + \phi(L)\,|J_{\theta S}(T(0))|\,.\end{aligned} \tag{5.45}$$

In particular,

$$F(u, DT(u)) \leq \mathbf{z}DT(u) + F(u,0)\mathcal{L}^1 \quad \text{as measures in }]0,L[. \tag{5.46}$$

$$F_S(u, DT(u)) \leq \mathbf{z}DJ_{T'S}(u)) + S(u)F(u,0)\,\mathcal{L}^1 \text{ as measures in }]0,L[. \tag{5.47}$$

Proof. We will only prove (5.45), the proof of (5.44) being similar. Let $0 \leq \phi \in C^1([0,L])$ with $\phi(0) = 0$. Since $\mathcal{F}^0_{\phi SF}$ is lower semicontinuous with respect to the L^1-convergence (Corollary 6.4.3), letting $n \to \infty$ we obtain

$$\begin{aligned}
&\int_0^L \phi F_S(u, DT(u)) + \phi(L)\,|J_{\theta S}(T(u)(L_-))| \\
&\leq \liminf_{n\to\infty} \int_0^L \phi S(u_n)F(u_n, T(u_n)')\ dx + \phi(L)\,|J_{\theta S}(T(0))| \\
&\leq \limsup_{n\to\infty} \int_0^L \phi S(u_n)F(u_n, T(u_n)')\ dx + \phi(L)\,|J_{\theta S}(T(0))|
\end{aligned}$$

By the convexity (4.11) of F and using that $\mathbf{a}(u_n, T(u_n)')T(u_n)' = \mathbf{a}(u_n, u_n')T(u_n)'$, we have

$$\begin{aligned}
&\int_0^L \phi S(u_n)F(u_n, T(u_n)')\,dx \\
&\leq \int_0^L \phi S(u_n)\mathbf{a}(u_n, T(u_n)')T(u_n)'\ dx + \int_0^L \phi S(u_n)F(u_n,0)dx \\
&= \int_0^L \phi\,\mathbf{a}(u_n, u_n')(J_{T'S}(u_n))'\,dx + \int_0^L \phi S(u_n)F(u_n,0)dx.
\end{aligned}$$

Now we take $v = J_{T'S}(u_n)\phi$ as test function in (5.32)) and we obtain

$$\begin{aligned}
&\int_0^L \phi\,\mathbf{a}(u_n, u_n')(J_{T'S}(u_n))'\,dx + \frac{1}{n}\int_0^L \phi\, u_n'(J_{T'S}(u_n))'\,dx \\
&= \int_0^L (f - u_n)J_{T'S}(u_n)\phi\,dx - \int_0^L J_{T'S}(u_n)\mathbf{a}(u_n, u_n')\phi'\,dx - \frac{1}{n}\int_0^L J_{T'S}(u_n)u_n'\phi'\,dx.
\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}
\limsup_n \int_0^L \phi\,\mathbf{a}(u_n, u_n')(J_{T'S}(u_n))'\,dx &\leq \int_0^L \phi(f-u)J_{T'S}(u)\,dx - \int_0^L J_{T'S}(u)\mathbf{z}\,\phi'\,dx \\
&= \int_0^L \phi\,\mathbf{z}DJ_{T'S}(u) - \phi(L)\mathbf{z}(L)J_{T'S}(u(L_-)).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^L \phi F_S(u, DT(u)) + \phi(L)\,|J_{\theta S}(T(u))(L_-)| \leq \int_0^L \phi\,\mathbf{z}DJ_{T'S}(u) \\
&+ \phi(L)\,|J_{\theta S}(T(0))| - \phi(L)\mathbf{z}(L)J_{T'S}(u(L_-)) + \int_0^L \phi S(u)F(u,0)\,dx
\end{aligned}$$

and (5.45) holds. □

Lemma 6.5.11 *The entropy inequalities (5.25) and (5.26) hold.*

Proof. Using (5.46) and the computations in (4.21) it follows that

$$h(u, DT(u))^s \le (\mathbf{z}\ DT(u))^s.$$

Thanks to (4.21) again,

$$(\mathbf{z}DT(u)))^{ac} = h(u, DT(u))^{ac}$$

and (5.25) holds.

By virtue of (5.47) and (4.21) we have

$$\mathbf{z}DJ_{T'S}(u)) \ge \mathbf{z}(J_{T'S}(u))' + (F_S(u, DT(u)))^s = h_S(u, DT(u))$$

and we obtain (5.26). □

Lemma 6.5.12 *The Dirichlet condition* $\mathbf{a}(u, u')(L) = -cu(L_-)$ *holds.*

Proof. Firstly, observe that by (5.40) we have

$$|\mathbf{z}(L)| \le c\, u(L_-).$$

Then, it is enough to prove the lemma in the case $u(L_-) > 0$. In that case, again by (5.40) and having in mind that $\mathbf{z}$ is continuous in $[0, L]$, we have

$$\mathbf{z}(L) = cu(L_-)\xi, \qquad \text{with} \quad |\xi| \le 1. \tag{5.48}$$

Given $T \in \mathcal{T}^+$, we consider $S := T^{m-1} \in \mathcal{P}^+$ for $m > 1$. Taking singular parts over $x = L$ in (5.45) we have

$$|J_{\theta T^{m-1}}(T(u))(L_-)| \le -\mathbf{z}(L)J_{T^{m-1}T'}(u)(L_-) + |J_{\theta T^{m-1}}(T(0))|\,. \tag{5.49}$$

Consider now $T = T_{d,d'}$ with $0 < d \le u(L_-) \le \|u\|_\infty \le d'$. Using (5.48), the inequality (5.49) particularizes to

$$\frac{c}{2}d^{m+1} + \frac{c}{m+1}\left(u^{m+1}(L_-) - d^{m+1}\right) \le \frac{c}{2}d^{m+1} - \frac{c}{m}\xi\ u(L_-)\left(u^m(L_-) - d^m\right)$$

and letting $d \to 0^+$ we have

$$\frac{c}{m+1}u^{m+1}(L_-) \le -\frac{c}{m}u(L_-)\xi u^m(L_-).$$

Then, since $u(L_-) > 0$, we get $\frac{m}{m+1} \le -\xi$ for all $1 < m$. Therefore, since $|\xi| \le 1$, we have $\xi = -1$. Consequently, using (5.48) we finish the proof. □

6.5.2 Proof of uniqueness

Let $u, \overline{u}$ be entropy solutions of (5.31) associated with $f, \overline{f} \in L^1(]0, L[)^+$, respectively. Let ρ_n be a mollifiers in $\mathbb{R}$ with support in $]-L, L[$ and let $\psi \in \mathcal{D}(]0, L[)$. We define

$$\xi_n(x, y) := \rho_n(x - y)\psi\left(\frac{x + y}{2}\right).$$

We will use the notation $T = T^a_{a,b}$, being $b > a > 2\epsilon > 0$. We also need to consider truncation functions of the form

$$S_{\epsilon,l}(r) := T_\epsilon(r-l)^+ = T_{l,l+\epsilon}(r) - l \in \mathcal{T}^+$$

and

$$S^l_\epsilon(r) := T_{l-\epsilon,l}(r) + \epsilon - l = -T_\epsilon(l-r)^+ + \epsilon \in \mathcal{T}^+.$$

We set $u = u(y)$ and $\overline{u} = \overline{u}(x)$. If we denote $\mathbf{z}(y) = \mathbf{a}(u(y), \partial_y u(y))$ and $\overline{\mathbf{z}}(x) = \mathbf{a}(\overline{u}(x), \partial_x \overline{u}(x))$, we have

$$u - \mathbf{z}' = f \quad \text{and} \quad \overline{u} - \overline{\mathbf{z}}' = \overline{f} \quad \text{in} \quad \mathcal{D}'(]0,L[).$$

Then, multiplying the first equation by $T(u(y))S_{\epsilon,\overline{u}(x)}(u(y))\xi_n(x,y)$, the second by $T(\overline{u}(x))S^{u(y)}_\epsilon(\overline{u}(x))\xi_n(x,y)$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_0^L u(y)T(u(y))T_\epsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,dy \\
&+\int_0^L \xi_n(x,y)\mathbf{z}(y)D_y[T(u(y))S_{\epsilon,\overline{u}(x)}(u(y))] \\
&+\int_0^L T(u(y))S_{\epsilon,\overline{u}(x)}(u(y))\mathbf{z}(y)\ \partial_y\xi_n(x,y)\,dy \\
&=\int_0^L f(y)T(u(y))T_\epsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,dy
\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}
&-\int_0^L \overline{u}(x)T(\overline{u}(x))\left(T_\epsilon(u(y)-\overline{u}(x))^+ - \epsilon\right)\xi_n(x,y)\,dx \\
&+\int_0^L \xi_n(x,y)\overline{\mathbf{z}}(x)D_x[T(\overline{u}(x))S^{u(y)}_\epsilon(\overline{u}(x))] \\
&+\int_0^L T(\overline{u}(x))S^{u(y)}_\epsilon(\overline{u}(x))\overline{\mathbf{z}}(x)\ \partial_x\xi_n(x,y)\,dx \\
&=-\int_0^L \overline{f}(x)T(\overline{u}(x))\left(T_\epsilon(u(y)-\overline{u}(x))^+ - \epsilon\right)\xi_n(x,y)\,dx
\end{aligned} \tag{5.51}$$

All the boundary terms have vanished because for n big enough the x-supports of $\rho_n(x-y)$ and $\psi((x+y)/2)$ are disjoint when $y = 0, L$ or vice versa.

We integrate (5.50) in x and (5.51) in y. Then we add both identities to obtain

$$\begin{aligned}
&\int_0^L \int_0^L [u(y)T(u(y)) - \overline{u}(x)T(\overline{u}(x))]T_\epsilon(u(y) - \overline{u}(x))^+ \xi_n(x,y)\, dxdy \\
&+\epsilon \int_0^L \int_0^L (\overline{u}(x) - \overline{f}(x))T(\overline{u}(x))\xi_n(x,y)\, dxdy \\
&+\int_0^L \left(\int_0^L \xi_n(x,y)\mathbf{z}(y)D_y[T(u(y))S_{\epsilon,\overline{u}(x)}(u(y))] \right) dx \\
&+\int_0^L \int_0^L T(u(y))S_{\epsilon,\overline{u}(x)}(u(y))\mathbf{z}(y)\, \partial_y \xi_n(x,y)\, dydx \qquad (5.52)\\
&+\int_0^L \left(\int_0^L \xi_n(x,y)\overline{\mathbf{z}}(x)D_x[T(\overline{u}(x))S_\epsilon^{u(y)}(\overline{u}(x))] \right) dy \\
&+\int_0^L \int_0^L T(\overline{u}(x))S_\epsilon^{u(y)}(\overline{u}(x))\overline{\mathbf{z}}(x)\, \partial_x \xi_n(x,y)\, dxdy \\
&= \int_0^L \int_0^L [f(y)T(u(y)) - \overline{f}(x)T(\overline{u}(x))]T_\epsilon(u(y) - \overline{u}(x))^+ \xi_n(x,y)\, dxdy.
\end{aligned}$$

Let I denote all the terms at the left hand side of the above identity, but the first one. We defer the proof of the following statement to the end of the section.

Lemma 6.5.13 *The following inequality is satisfied*

$$\begin{aligned}
\frac{1}{\epsilon} I \geq & o(\epsilon) - \int_0^L \left(\int_0^L \xi_n(x,y)\overline{\mathbf{z}}(x)D_x T(\overline{u}(x)) \right) dy \\
&+ \frac{1}{\epsilon} \int_0^L \int_0^L T_\epsilon(u(y) - \overline{u}(x))^+ (T(u(y))\mathbf{z}(y) - T(\overline{u}(x))\overline{\mathbf{z}}(x)) \\
&\qquad \times (\partial_x \xi_n(x,y) + \partial_y \xi_n(x,y))\, dxdy,
\end{aligned}$$

where $o(\epsilon)$ denotes an expression such that $o(\epsilon) \to 0$ as $\epsilon \to 0$.

By the above lemma, dividing (5.52) by ϵ and letting $\epsilon \to 0$ we obtain

$$\begin{aligned}
&\int_0^L \int_0^L [u(y)T(u(y)) - \overline{u}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(y) - \overline{u}(x))\xi_n(x,y)\, dxdy \\
&+ \int_0^L \int_0^L \mathrm{sign}_0^+(u(y) - \overline{u}(x))[T(u(y))\mathbf{z}(y) - T(\overline{u}(x))\overline{\mathbf{z}}(x)]\, \psi'\left(\frac{x+y}{2}\right) \rho_n(x-y)\, dxdy \\
\leq & \int_0^L \int_0^L [f(y)T(u(y)) - \overline{f}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(y) - \overline{u}(x))\xi_n(x,y)\, dxdy \\
&+ \int_0^L \left(\int_0^L \xi_n(x,y)\overline{\mathbf{z}}(x)D_x T(\overline{u}(x)) \right) dy.
\end{aligned}$$

Notice that we used the identity

$$\partial_x \xi_n(x,y) + \partial_y \xi_n(x,y) = \rho_n(x-y)\psi'\left(\frac{x+y}{2}\right)$$

when dealing with the second term above. Letting $n \to \infty$, we find

$$
\begin{aligned}
&\int_0^L [u(x)T(u(x)) - \overline{u}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\psi(x)\,dx \\
&\quad + \int_0^L \mathrm{sign}_0^+(u(x) - \overline{u}(x))[T(u(x))\mathbf{z}(x) - T(\overline{u}(x))\overline{\mathbf{z}}(x)]\psi'(x)\,dx \\
&\leq \int_0^L [f(x)T(u(x)) - \overline{f}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\psi(x)\,dx \\
&\quad + \int_0^L \psi(x)\overline{\mathbf{z}}(x)DT(\overline{u}(x)).
\end{aligned}
$$

Taking now a sequence $\psi_m \uparrow \chi_{]0,L[}$, $\psi_m \in \mathcal{D}(]0,L[)$ in the above formula, we have

$$
\begin{aligned}
&\int_0^L [u(x)T(u(x)) - \overline{u}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\,dx \\
&+ \lim_{m\to\infty} \int_0^L \mathrm{sign}_0^+(u(x) - \overline{u}(x))(T(u(x))\mathbf{z}(x) - T(\overline{u}(x))\overline{\mathbf{z}}(x))\psi'_m(x)\,dx \\
&\leq \int_0^L [f(x)T(u(x)) - \overline{f}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\,dx \\
&+ \int_0^L \overline{\mathbf{z}}(x)\,DT(\overline{u}(x)).
\end{aligned}
\tag{5.53}
$$

Now we deal with the second term in the above expression. As $\mathrm{sign}_0^+(u - \overline{u})T(u) \in BV(]0,L[)$ [23], integration by parts yields

$$
\begin{aligned}
&\lim_{m\to\infty} \int_0^L \mathrm{sign}_0^+(u(x) - \overline{u}(x))[T(u(x))\mathbf{z}(x) - T(\overline{u}(x))\overline{\mathbf{z}}(x)]\psi'_m(x)\,dx \\
&= -\lim_{m\to\infty} \int_0^L \psi_m(x)\{\mathbf{z}(x)D[\mathrm{sign}_0^+(u(x) - \overline{u}(x))T(u(x))] \\
&\qquad - \overline{\mathbf{z}}(x)D[\mathrm{sign}_0^+(u(x) - \overline{u}(x))T(\overline{u}(x))]\} \\
&\quad + \lim_{m\to\infty} \int_0^L \psi_m(x)\{\mathrm{sign}_0^+(u(x) - \overline{u}(x))T(\overline{u}(x))\overline{\mathbf{z}}'(x) \\
&\qquad - \mathrm{sign}_0^+(u(x) - \overline{u}(x))T(u(x))\mathbf{z}'(x)\}\,dx
\end{aligned}
$$

and thus

$$
\begin{aligned}
&= -\int_0^L \mathrm{sign}_0^+(u(x) - \overline{u}(x))T(u(x))\mathbf{z}'(x)\,dx - \int_0^L \mathbf{z}(x)D[\mathrm{sign}_0^+(u(x) - \overline{u}(x))T(u(x))] \\
&\quad + \int_0^L \mathrm{sign}_0^+(u(x) - \overline{u}(x))T(\overline{u}(x))\overline{\mathbf{z}}'(x)\,dx + \int_0^L \overline{\mathbf{z}}(x)D[\mathrm{sign}_0^+(u(x) - \overline{u}(x))T(\overline{u}(x))] \\
&= [\mathbf{z}(0)T(u(0_+)) - \overline{\mathbf{z}}(0)T(\overline{u}(0_+))]\,\mathrm{sign}_0^+(u(0_+) - \overline{u}(0_+)) \\
&\quad - [\mathbf{z}(L)T(u(L_-)) - \overline{\mathbf{z}}(L)T(\overline{u}(L_-))]\,\mathrm{sign}_0^+(u(L_-) - \overline{u}(L_-)).
\end{aligned}
$$

Therefore, (5.53) becomes

$$\begin{aligned}
&\int_0^L [u(x)T(u(x)) - \overline{u}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\,dx \\
&+ [\mathbf{z}(0)T(u(0_+)) - \overline{\mathbf{z}}(0)T(\overline{u}(0_+))]\,\mathrm{sign}_0^+(u(0_+) - \overline{u}(0_+)) \\
&- [\mathbf{z}(L)T(u(L_-)) - \overline{\mathbf{z}}(L)T(\overline{u}(L_-))]\,\mathrm{sign}_0^+(u(L_-) - \overline{u}(L_-)) \\
\le &\int_0^L [f(x)T(u(x)) - \overline{f}(x)T(\overline{u}(x))]\mathrm{sign}_0^+(u(x) - \overline{u}(x))\,dx + \int_0^L \overline{\mathbf{z}}(x)\,DT(\overline{u}(x)).
\end{aligned}$$

Dividing by $b > 0$ and letting $a \to 0^+$ and $b \to 0^+$ in this order, we obtain

$$\begin{aligned}
&\int_0^L (u\chi_{[u>0]} - \overline{u}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u - \overline{u})\,dx \\
&+ \left[\mathbf{z}(0)\mathrm{sign}_0^+(u(0_+)) - \overline{\mathbf{z}}(0)\mathrm{sign}_0^+(\overline{u}(0_+))\right]\mathrm{sign}_0^+(u(0_+) - \overline{u}(0_+)) \\
&- \left[\mathbf{z}(L)\mathrm{sign}_0^+(u(L_-)) - \overline{\mathbf{z}}(L)\mathrm{sign}_0^+(\overline{u}(L_-))\right]\mathrm{sign}_0^+(u(L_-) - \overline{u}(L_-)) \\
\le &\int_0^L (f\chi_{[u>0]} - \overline{f}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u - \overline{u})\,dx + \lim_{b\downarrow 0}\frac{1}{b}\left(\lim_{a\downarrow 0}\int_0^L \overline{\mathbf{z}}\,DT(\overline{u})\right).
\end{aligned} \tag{5.54}$$

By virtue of (5.30) and the fact that $\mathbf{z}(0) = \overline{\mathbf{z}}(0) = -\beta \neq 0$, we have that the second term in (5.54) vanishes. On the other hand, since $\mathbf{z}(L) = -cu(L_-)$ and $\overline{\mathbf{z}}(L) = -c\overline{u}(L_-)$, the third line in (5.54) is nonnegative. Consequently,

$$\begin{aligned}
&\int_0^L (u\chi_{[u>0]} - \overline{u}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u - \overline{u})\,dx \\
\le &\int_0^L (f\chi_{[u>0]} - \overline{f}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u - \overline{u})\,dx - \lim_{b\downarrow 0}\frac{1}{b}\left(\lim_{a\downarrow 0}\int_0^L \overline{\mathbf{z}}\,DT(\overline{u})\right).
\end{aligned} \tag{5.55}$$

To continue we rely on the following auxiliary result:

Lemma 6.5.14 *There holds that $f = 0$ a.e. on $[u = 0]$ and $\overline{f} = 0$ a.e. on $[\overline{u} = 0]$.*

Proof. Let $0 \le \phi \in \mathcal{D}(]0, L[)$ be and let $a,\ \epsilon > 0$. We multiply $f - u = -\mathbf{z}'$ by $T^a_{a,a+\epsilon}(u)\phi \in L^\infty(]0, L[)$ and integrate by parts. Having in mind (4.18), we obtain

$$\begin{aligned}
\int_0^L (f - u)T^a_{a,a+\epsilon}(u)\phi\,dx &= \int_0^L \mathbf{z}\,\phi DT^a_{a,a+\epsilon}(u) + \int_0^L \mathbf{z}\,\phi' T^a_{a,a+\epsilon}(u)\,dx \\
&\ge \int_0^L \mathbf{z}\,\phi' T^a_{a,a+\epsilon}(u)\,dx,
\end{aligned}$$

where we also used (5.29). Dividing by ϵ and letting $\epsilon \to 0^+$, we get

$$\int_0^L (f - u)\chi_{[u>a]}\phi\,dx \ge \int_0^L \mathbf{z}\,\phi'\chi_{[u>a]}\,dx.$$

Hence

$$\begin{aligned}
\int_0^L (f - u)\chi_{[u\le a]}\phi\,dx &= \int_0^L (f - u)\phi\,dx - \int_0^L (f - u)\chi_{[u>a]}(x)\phi\,dx \\
&\le \int_0^L (f - u)\phi\,dx - \int_0^L \mathbf{z}\,\phi'\chi_{[u>a]}\,dx = \int_0^L \mathbf{z}\,\phi'\chi_{[u\le a]}\,dx.
\end{aligned}$$

Then, since $\mathbf{z} = 0$ in $[u = 0]$, letting $a \to 0^+$ we have

$$\int_0^L f\chi_{[u=0]}\phi\,dx = \int_0^L (f-u)\chi_{[u=0]}\phi\,dx \le 0,$$

for all $0 \le \phi \in \mathcal{D}(]0,L[)$. It follows that $f\chi_{[u=0]} = 0$ a.e. in $]0,L[$. Similarly, $\overline{f}\chi_{[\overline{u}=0]} = 0$ a.e. in $]0,L[$ and Lemma 6.5.14 holds. □

Thanks to this Lemma we have

$$\begin{aligned}
&\lim_{b\to 0}\frac{1}{b}\left(\lim_{a\to 0}\int_0^L \overline{\mathbf{z}}DT(\overline{u})\right)\\
&= -\lim_{b\to 0}\frac{1}{b}\lim_{a\to 0}\left(\overline{\mathbf{z}}(0)T(\overline{u}(0_+)) - \overline{\mathbf{z}}(L)T(\overline{u}(L_-)) + \int_0^L T(\overline{u})\overline{\mathbf{z}}'\,dx\right)\\
&= -\lim_{b\to 0}\frac{1}{b}\left(\overline{\mathbf{z}}(0)T_{0,b}(\overline{u}(0_+)) - \overline{\mathbf{z}}(L)T_{0,b}(\overline{u}(L_-)) + \int_0^L T_{0,b}(\overline{u})\overline{\mathbf{z}}'\,dx\right)\\
&= -\overline{\mathbf{z}}(0)\mathrm{sign}_0^+(\overline{u}(0_+)) + \overline{\mathbf{z}}(L)\mathrm{sign}_0^+(\overline{u}(L_-)) - \int_0^L \chi_{[\overline{u}>0]}\overline{\mathbf{z}}'\,dx\\
&= -\overline{\mathbf{z}}(0)\mathrm{sign}_0^+(\overline{u}(0_+)) + \overline{\mathbf{z}}(L)\mathrm{sign}_0^+(\overline{u}(L_-)) - \int_0^L \overline{\mathbf{z}}'\,dx\\
&= \overline{\mathbf{z}}(0)\left(1 - \mathrm{sign}_0^+(\overline{u}(0_+))\right) + \overline{\mathbf{z}}(L)\left(\mathrm{sign}_0^+(\overline{u}(L_-)) - 1\right) = 0,
\end{aligned}$$

where we also used (5.30) and (5.24).

Then, it follows from (5.55) that

$$\int_0^L (u\chi_{[u>0]} - \overline{u}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u-\overline{u})\,dx \le \int_0^L (f\chi_{[u>0]} - \overline{f}\chi_{[\overline{u}>0]})\mathrm{sign}_0^+(u-\overline{u})\,dx.$$

Hence, using Lemma 6.5.14 again, we obtain

$$\int_0^L (u-\overline{u})^+\,dx \le \int_0^L (f-\overline{f})\mathrm{sign}_0^+(u-\overline{u})\,dx \le \int_0^L (f-\overline{f})^+\,dx.$$

This concludes the proof of the uniqueness part of Theorem 6.5.3. □

6.5.3 Proof of Lemma 6.5.13

Recall that $u, \mathbf{z}$ are always functions of y and $\overline{u}, \overline{\mathbf{z}}$ are always functions of x. From now on we shall work with more concise expressions. In order to do so, we shall omit the arguments of u, $\mathbf{z}$, $\overline{u}$ and $\overline{\mathbf{z}}$ except in some cases where we find it useful to remind them. We decompose $I = I_1 + \cdots + I_5$ in the obvious way.

Now, since $\overline{u} - \overline{f} = \overline{\mathbf{z}}'$ and $T(\overline{u}(x))\xi_n(x,y)$ belongs to $L^1(]0,L[)$ as a function of x, we have

$$I_1 = \epsilon\int_0^L\int_0^L \overline{\mathbf{z}}'T(\overline{u})\xi_n\,dxdy.$$

Note also that

$$I_5 = -\int_0^L\int_0^L T(\overline{u})T_\epsilon(u-\overline{u})^+\overline{\mathbf{z}}\partial_x\xi_n\,dxdy + \epsilon\int_0^L\int_0^L T(\overline{u})\overline{\mathbf{z}}\partial_x\xi_n\,dxdy.$$

Moreover,

$$\begin{aligned}
&I_3 - \int_0^L \int_0^L T(\overline{u}) T_\epsilon (u-\overline{u})^+ \overline{\mathbf{z}}\, \partial_x \xi_n \, dxdy \\
&= \int_0^L \int_0^L T(u) T_\epsilon (u-\overline{u})^+ \mathbf{z}\, \partial_y \xi_n \; dxdy - \int_0^L \int_0^L T(\overline{u}) T_\epsilon (u-\overline{u})^+ \overline{\mathbf{z}}\, \partial_x \xi_n \; dxdy \\
&= \int_0^L \int_0^L T_\epsilon (u-\overline{u})^+ \left[T(u) \mathbf{z}\, \partial_y \xi_n - T(\overline{u}) \overline{\mathbf{z}}\, \partial_x \xi_n \right] \; dxdy.
\end{aligned}$$

At this point we substitute the above expressions and we also add and subtract the terms $\int\int T_\epsilon(u-\overline{u})^+ T(u) \overline{\mathbf{z}}\, \partial_x \xi_n \, dxdy$ and $-\int\int T_\epsilon(u-\overline{u})^+ T(\overline{u}) \overline{\mathbf{z}}\, \partial_y \xi_n \, dxdy$. All together reads

$$\begin{aligned}
I = \epsilon & \int_0^L \int_0^L \overline{\mathbf{z}}' \, T(\overline{u}) \xi_n \, dxdy + \epsilon \int_0^L \int_0^L T(\overline{u}) \overline{\mathbf{z}}\, \partial_x \xi_n \, dxdy \\
& + \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y [T(u) S_{\epsilon, \overline{u}(x)}(u)] \right) \; dx + \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x [T(\overline{u}) S_\epsilon^{u(y)}(\overline{u})] \right) \; dy \\
& + \int_0^L \int_0^L T_\epsilon (u-\overline{u})^+ [T(u)\mathbf{z} - T(\overline{u})\overline{\mathbf{z}}] \; (\partial_x \xi_n + \partial_y \xi_n) \, dxdy \\
& + \int_0^L \int_0^L T(\overline{u}) T_\epsilon (u-\overline{u})^+ \overline{\mathbf{z}}\, \partial_y \xi_n \, dxdy - \int_0^L \int_0^L T(u) T_\epsilon (u-\overline{u})^+ \mathbf{z}\, \partial_x \xi_n \, dydx
\end{aligned}$$

Integrating by parts the above expression can be written as

$$\begin{aligned}
& - \epsilon \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x T(\overline{u}) \right) dy \\
& + \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y [T(u) S_{\epsilon, \overline{u}(x)}(u)] \right) \; dx + \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x [T(\overline{u}) S_\epsilon^{u(y)}(\overline{u})] \right) \; dy \\
& + \int_0^L \int_0^L T_\epsilon (u-\overline{u})^+ [T(u)\mathbf{z} - T(\overline{u})\overline{\mathbf{z}}] \; (\partial_x \xi_n + \partial_y \xi_n) \, dxdy \\
& - \int_0^L \int_0^L \xi_n \overline{\mathbf{z}}\, T(\overline{u}) D_y [T_\epsilon (u-\overline{u})^+] \, dx\, dy + \int_0^L \int_0^L \xi_n \mathbf{z}\, T(u) D_x [T_\epsilon (u-\overline{u})^+)] \, dydx
\end{aligned}$$

Using the product functional calculus (Corollary 6.4.6) and rearranging a bit we get

$$\begin{aligned}
I = & - \epsilon \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x T(\overline{u}) \right) dy \\
& + \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y [J_{T' S_{\epsilon, \overline{u}(x)}}(u)] \right) dx + \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x [J_{T' S_\epsilon^{u(y)}}(\overline{u})] \right) dy \\
& + \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y [J_{T S'_{\epsilon, \overline{u}(x)}}(u)] \right) dx + \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x [J_{T (S_\epsilon^{u(y)})'}(\overline{u})] \right) dy \\
& - \int_0^L T(\overline{u}) \left(\int_0^L \xi_n \overline{\mathbf{z}} D_y \, T_\epsilon (u-\overline{u})^+ \right) dx + \int_0^L T(u(y)) \left(\int_0^L \xi_n \mathbf{z} D_x T_\epsilon (u-\overline{u})^+ \right) dy \\
& + \int_0^L \int_0^L T_\epsilon (u-\overline{u})^+ [T(u)\mathbf{z} - T(\overline{u})\overline{\mathbf{z}}] \; (\partial_x \xi_n + \partial_y \xi_n) \, dxdy \\
= & I^1 + I^2 + I^3,
\end{aligned}$$

where I^1 denotes the sum of the first three terms, I^2 denotes the sum from the fourth to the seventh terms and I^3 denotes the last term.

First we estimate I^1. Let us consider the second and third terms in I^1. Since

$$h_{S_{\epsilon,\overline{u}(x)}}(u, DT(u)) \leq \mathbf{z} D_y J_{T' S_{\overline{u}(x)}}(u)$$

thanks to (5.26), using (5.29) we have

$$\int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y J_{T' S_{\epsilon,\overline{u}(x)}}(u) \right) dx \geq 0.$$

In the same way,

$$\int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x [J_{T' S_\epsilon^{u(y)}}(\overline{u})] \right) dy \geq 0.$$

Hence,

$$I^1 \geq -\epsilon \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x T(\overline{u}) \right) dy. \tag{5.56}$$

Now let us deal with I^2. We could deal with the first two integrals here in the same way as before, but it will be better not to get rid of them that soon. Let us write

$$I^2 = I^2(ac) + I^2(s)$$

where $I^2(ac)$ contains the absolutely continuous parts of I^2 while $I^2(s)$ contains its singular parts. First we estimate the absolutely continuous part:

Lemma 6.5.15 *There holds that*

$$\frac{1}{\epsilon} I_2^2(ac) \geq o(\epsilon),$$

where $o(\epsilon)$ denotes an expression such that $o(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. Note that

$$\begin{aligned} I^2(ac) = &\int_0^L \int_0^L \xi_n T(u)\, \mathbf{z}\, \partial_y T_\epsilon (u - \overline{u})^+ \, dydx - \int_0^L \int_0^L \xi_n T(\overline{u})\, \overline{\mathbf{z}}\, \partial_y\, T_\epsilon (u - \overline{u})^+ \, dydx \\ &- \int_0^L \int_0^L \xi_n T(\overline{u})\, \overline{\mathbf{z}}\, \partial_x T_\epsilon (u - \overline{u})^+ \, dxdy + \int_0^L \int_0^L \xi_n T(u)\, \mathbf{z}\, \partial_x\, T_\epsilon (u - \overline{u})^+ \, dxdy. \end{aligned}$$

This can be rearranged as

$$\begin{aligned} I^2(ac) &= \int_0^L \int_0^L \xi_n (\mathbf{z} T(u) - \overline{\mathbf{z}} T(\overline{u})) \left(\partial_y T_\epsilon (u - \overline{u})^+ + \partial_x T_\epsilon (u - \overline{u})^+ \right) dxdy \\ &= \int_0^L \int_0^L \xi_n (\mathbf{z} - \overline{\mathbf{z}}) T(u) (\partial_y T_\epsilon (u - \overline{u})^+ + \partial_x T_\epsilon (u - \overline{u})^+) \, dxdy \\ &\quad + \int_0^L \int_0^L \xi_n \overline{\mathbf{z}} (T(u) - T(\overline{u})) (\partial_y T_\epsilon (u - \overline{u})^+ + \partial_x T_\epsilon (u - \overline{u})^+) \, dxdy \\ &=: A^1 + A^2. \end{aligned}$$

Let us estimate A^1. First, observe that

$$\partial_y T_\epsilon(u-\overline{u}(x))^+(y) = \chi_{\{0<u(y)-\overline{u}(x)<\epsilon\}}(x,y)\partial_y u(y) = \chi_{]\overline{u}(x),\overline{u}(x)+\epsilon[}(u(y))\partial_y u(y)$$

and

$$\partial_x T_\epsilon(u(y)-\overline{u})^+(x) = -\chi_{\{0<u(y)-\overline{u}(x)<\epsilon\}}(x,y)\partial_x \overline{u}(x) = -\chi_{]\overline{u}(x),\overline{u}(x)+\epsilon[}(u(y))\partial_x \overline{u}(x).$$

Taking the previous into account

$$\begin{aligned} A^1 &= \int_0^L \int_0^L \xi_n(\mathbf{z}-\overline{\mathbf{z}})T(u)(\partial_y u - \partial_x \overline{u})\chi_{]\overline{u}(x),\overline{u}(x)+\epsilon[}(u)\, dxdy \\ &\geq -Cb \int_0^L \int_0^L \chi_{[u\geq a]}\xi_n \chi_{]\overline{u}(x),\overline{u}(x)+\epsilon[}(u)\, |u-\overline{u}|\, |\partial_y u - \partial_x \overline{u}|\, dxdy \end{aligned}$$

where (4.14) was used.

Now, observe that $0 \leq u(y)-\overline{u}(x) \leq \epsilon$ and $u(y) \geq a$ imply the fact that $\overline{u}(x) \geq a-\epsilon$. Hence,

$$A^1 \geq -Cb\epsilon \int_0^L \int_0^L \chi_{[u\geq a]}\chi_{[\overline{u}\geq a-\epsilon]}\xi_n\, \chi_{[0\leq u-\overline{u}\leq\epsilon]}|\partial_y u - \partial_x \overline{u}|\, dxdy.$$

The present lower bound is not trivial as $\chi_{[\overline{u}\geq a-\epsilon]}\partial_x \overline{u}$ and $\chi_{[u\geq a]}\partial_y u$ both belong to $L^1(]0,L[)$ of its respective variable. To justify this, we use for instance Lemma 6.4.5 to rewrite $\chi_{[u\geq a]}\partial_y u$ as $\left(T_a^{\|u\|_\infty}(u(y))\right)'$, which is the Radon–Nikodym derivative (or density) of $D\left[T_a^{\|u\|_\infty}(u(y))\right]$ with respect to the Lebesgue measure $\mathcal{L}^1$, hence integrable against it. Similarly

$$\begin{aligned} |A^2| &= \left|\int_0^L \int_0^L \xi_n \overline{\mathbf{z}}(T(u)-T(\overline{u}))(\partial_y u - \partial_x \overline{u})\chi_{[\overline{u}(x),\overline{u}(x)+\epsilon]}(u)\, dxdy\right| \\ &\leq cM \int_0^L \int_0^L \xi_n \chi_{[u\geq a]}\chi_{[\overline{u}\geq a]}\chi_{[0\leq u-\overline{u}\leq\epsilon]}\xi_n |u-\overline{u}|\, |\partial_y u - \partial_x \overline{u}|\, dxdy \\ &\leq cM\epsilon \int_0^L \int_0^L \xi_n \chi_{[u\geq a]}\chi_{[\overline{u}\geq a]}\xi_n \chi_{[0\leq u-\overline{u}\leq\epsilon]}\, |\partial_y u - \partial_x \overline{u}|\, dxdy \\ &\leq cM\epsilon \int_0^L \int_0^L \xi_n \chi_{[u\geq a]}\chi_{[\overline{u}\geq a]}\xi_n \chi_{[0\leq u-\overline{u}\leq\epsilon]}(|\partial_y u| + |\partial_x \overline{u}|)\, dxdy. \end{aligned}$$

On recourse to the coarea formula we can estimate for instance

$$\begin{aligned} &\int_0^L \int_0^L \chi_{[u\geq a]}\chi_{[\overline{u}\geq a]}\xi_n \chi_{[0\leq u-\overline{u}\leq\epsilon]}\, |\partial_y u|\, dx\, dy \\ &\leq \|\psi\|_\infty \|\rho_n\|_\infty \int_0^L \chi_{[\overline{u}\geq a]}\left(\int_{\{y\in]0,L[/\overline{u}(x)\leq u(y)\leq\overline{u}(x)+\epsilon\}} |\partial_y u|\, dy\right) dx \\ &\leq \|\psi\|_\infty \|\rho_n\|_\infty \int_0^L \chi_{[\overline{u}\geq a]}\left(\int_{\overline{u}(x)}^{\overline{u}(x)+\epsilon} \mathrm{Per}([u\geq\lambda])\, d\lambda\right) dx \\ &\leq \|\psi\|_\infty \|\rho_n\|_\infty \int_0^L \chi_{[\overline{u}\geq a]} o(\epsilon)\, dx \leq o(\epsilon), \end{aligned}$$

where $o(\epsilon)$ denotes an expression such that $o(\epsilon) \to 0$ as $\epsilon \to 0$ (note that $\lambda \mapsto \mathrm{Per}([u \geq \lambda]) \in L^1_{loc}(]0,+\infty[)$ because $u \in TBV(]0,L[)^+)$. The point here is that, although $\|\rho_n\|_\infty$ is not uniformly bounded in n, the limit with respect to ϵ will be taken in first place. Alternatively, note also that $[0,L]$ and even $[\overline{u} > a]$ are of finite measure; the limit with respect to a goes after the one for ϵ.

The other three terms are dealt away in the same fashion. The condition $a > 2\epsilon$ is crucial to be able to use the coarea formula in all the cases. Thus

$$\frac{1}{\epsilon} A^1 \geq -Co(\epsilon)$$

and

$$\frac{1}{\epsilon}|A^2| \leq o(\epsilon).$$

Hence,

$$\frac{1}{\epsilon} I^2(ac) \geq o(\epsilon). \tag{5.57}$$

□

Next we estimate the singular part:

Lemma 6.5.16 *There holds that*

$$\frac{1}{\epsilon} I^2(s) \geq o(\epsilon),$$

where $o(\epsilon)$ denotes an expression such that $o(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. Let us set $I^2(s) = I^2(1,s) + I^2(2,s)$, where

$$I^2(1,s) := \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_y^s [J_{TS'_{\epsilon,\overline{u}(x)}}(u)] \right) dx \; - \int_0^L \left(\int_0^L \xi_n T(\overline{u})\, \overline{\mathbf{z}} D_y^s\, T_\epsilon(u-\overline{u})^+ \right) dx$$

and

$$I^2(2,s) := \int_0^L \left(\int_0^L \xi_n \overline{\mathbf{z}} D_x^s [J_{T(S_\epsilon^{u(y)})'}(\overline{u})] \right) dy \; + \int_0^L \left(\int_0^L \xi_n T(u)\, \mathbf{z} D_x^s T_\epsilon(u-\overline{u})^+ \right) dy.$$

Note that using (5.28) we have

$$\mathbf{z} D_y^s J_{TS'_{\epsilon,\overline{u}(x)}}(u) \geq \left| D_y^s J_{T\theta}(S_{\epsilon,\overline{u}(x)}(u)) \right| = \left| D_y^s J_{T\theta}(T_\epsilon(u(y) - \overline{u}(x))^+) \right| = \left| D_y^s J_{T\theta}(u_\epsilon) \right| \tag{5.58}$$

where $u_\epsilon(x,y) = T_{\overline{u}(x),\overline{u}(x)+\epsilon}(u(y))$. Also by (4.15) we have

$$\overline{\mathbf{z}}(x) D_y^s T_\epsilon(u - \overline{u}(x))^+ \leq \theta(\overline{u}(x)) |D_y^s u_\epsilon|. \tag{5.59}$$

First we deal with $I^2(1,s)$. Since the integrand of the first term is positive and the support of $T(\overline{u})$ is contained in $[\overline{u} \geq a]$ we can restrict the domains of integration accordingly in order to bound $I^2(1,s)$ from below by means of

$$\int_{[\overline{u} \geq a]} \left(\int_0^L \xi_n \mathbf{z} D_y^s J_{TS'_{\epsilon,\overline{u}(x)}}(u) \right) dx \; - \int_{[\overline{u} \geq a]} \left(\int_0^L \xi_n T(\overline{u})\, \overline{\mathbf{z}} D_y^s\, T_\epsilon(u-\overline{u})^+ \right) dx,$$

which is in turn bounded from below —use (5.58) and (5.59)— by

$$
\begin{aligned}
&\int_{[\overline{u}\geq a]} \left(\int_0^L \xi_n \left| D_y^s J_{T\theta}(u_\epsilon) \right| \right) dx - \int_{[\overline{u}\geq a]} \left(\int_0^L \xi_n T(\overline{u})\theta(\overline{u}) |D_y^s u_\epsilon| \right) dx \\
=& \int_{[\overline{u}\geq a]} \left(\int_0^L \xi_n T(u_\epsilon)\theta(u_\epsilon) |D_y^c u_\epsilon| \right) dx - \int_{[\overline{u}\geq a]} \left(\int_0^L \xi_n T(\overline{u})\theta(\overline{u}) |D_y^c u_\epsilon| \right) dx \\
&+ \int_{[\overline{u}\geq a]} \left(\int_{\mathcal{J}_{u_\epsilon}} \xi_n \frac{1}{(u_\epsilon)^+(y) - (u_\epsilon)^-(y)} \left(\int_{(u_\epsilon)^-(y)}^{(u_\epsilon)^+(y)} T(s)\theta(s)\, ds \right) |D_y^j u_\epsilon| \right) dx \\
&- \int_{[\overline{u}\geq a]} \left(\int_0^L \xi_n T(\overline{u})\theta(\overline{u}) |D_y^j u_\epsilon| \right) dx =: J_1 + J_2.
\end{aligned}
$$

Here J_1 denotes the sum of the first and second terms of the above expression and J_2 the sum of the third and fourth terms. We used that

$$
D^j \left(J_{T\theta}(u_\epsilon) \right) = \frac{J_{T\theta}(u_\epsilon^+) - J_{T\theta}(u_\epsilon^-)}{u_\epsilon^+ - u_\epsilon^-} D^j u_\epsilon
$$

(chain rule for BV functions; recall that $\mathcal{J}_{u_\epsilon}$ denotes the jump set of u_ϵ) and

$$
J_{T\theta}(u_\epsilon^+) - J_{T\theta}(u_\epsilon^-) = \int_0^{u_\epsilon^+} T(s)\theta(s)\, ds - \int_0^{u_\epsilon^-} T(s)\theta(s)\, ds = \int_{u_\epsilon^-}^{u_\epsilon^+} T(s)\theta(s)\, ds.
$$

Note also that

$$
D_y^c(u_\epsilon) = D_y^c[T_{\overline{u}(x),\overline{u}(x)+\epsilon}(u(y))] = T'_{\overline{u}(x),\overline{u}(x)+\epsilon}(\tilde{u}) D_y^c u(y) = \chi_{]\overline{u}(x),\overline{u}(x)+\epsilon[}(\tilde{u}(y)) D_y^c u(y),
$$

being $\tilde{u}$ the good representative.

Now, since T and θ are Lipschitz continuous, we have

$$
\begin{aligned}
|J_1| \leq& \int_0^L \left(\int_{[\overline{u}\geq a]} \xi_n |T(u_\epsilon)\theta(u_\epsilon) - T(\overline{u})\theta(\overline{u})| |D_y^c u_\epsilon| \right) dx \\
\leq& c\|\psi\|_\infty \|\rho_n\|_\infty \int_{[\overline{u}\geq a]} \left(\int_0^L |u_\epsilon - \overline{u}| \chi_{[\overline{u}(x),\overline{u}(x)+\epsilon]}(u) |D_y^c u| \right) dx \\
\leq& \epsilon \|\psi\|_\infty c \|\rho_n\|_\infty \int_{[\overline{u}\geq a]} \left(\int_{\{y\in]0,L[\ :\ \overline{u}(x)<u(y)<\epsilon+\overline{u}(x)\}} |D_y^c u| \right) dx,
\end{aligned}
$$

being c the Lipschitz constant of $T\theta$. Using the coarea formula, we obtain

$$
|J_1| \leq \epsilon M \|\psi\|_\infty \|\rho_n\|_\infty \int_0^L \chi_{[\overline{u}\geq a]} \left(\int_{\overline{u}(x)}^{\overline{u}(x)+\epsilon} \mathrm{Per}(\{u(y) \geq \lambda\})\, d\lambda \right) dx,
$$

which yields

$$
\frac{1}{\epsilon} |J_1| \leq o(\epsilon). \tag{5.60}
$$

In order to deal with $|J_2|$ let us shorten the expressions introducing

$$
J(u_\epsilon, y) = \frac{1}{(u_\epsilon)^+(y) - (u_\epsilon)^-(y)}.
$$

Working in a similar way as before, we bound $|J_2|$ from above by means of

$$\int_{[\overline{u}\geq a]} \left(\int_{\mathcal{J}_{u_\epsilon}} \xi_n J(u_\epsilon, y) \left(\int_{(u_\epsilon)^-(y)}^{(u_\epsilon)^+(y)} |T(s)\theta(s) - T(\overline{u}(x))\theta(\overline{u}(x))| \, ds \right) |D_y^j u_\epsilon| \right) dx$$

which is itself bounded above by

$$\epsilon \|\psi\|_\infty C \|\rho_n\|_\infty \int_0^L \chi_{[\overline{u}\geq a]} \left(\int_{\overline{u}(x)}^{\overline{u}(x)+\epsilon} \mathrm{Per}(\{u(y) \geq \lambda\}) \, d\lambda \right) dx.$$

For this last step we used that if $s \in](u_\epsilon)^-(y), (u_\epsilon)^+(y)[$ then $|s - \overline{u}(x)| \leq \epsilon$. This shows that

$$\frac{1}{\epsilon} J_2 \geq o(\epsilon).$$

Collecting all these facts, we obtain

$$\frac{1}{\epsilon} I^2(1, s) \geq o(\epsilon).$$

In a similar way we prove that

$$\frac{1}{\epsilon} I^2(2, s) \geq o(\epsilon).$$

Hence

$$\frac{1}{\epsilon} I^2(s) \geq o(\epsilon).$$

□

Both lemmas in combination yield

$$\frac{1}{\epsilon} I^2 \geq o(\epsilon).$$

Collecting all the estimates so far, we have the inequality

$$\frac{1}{\epsilon} I \geq o(\epsilon) - \int_0^L \left(\int_0^L \xi_n \mathbf{z} D_x T(\overline{u}) \right) dy + \frac{1}{\epsilon} I^3$$

and the lemma is proved.

6.6 Semigroup solution

Definition 6.6.1 $(u, v) \in \mathcal{B}_\beta$ if and only if $0 \leq u \in TBV^+(]0, L[)$, $v \in L^1(]0, L[)$ and u is the entropy solution of problem (5.23).

From Theorem 6.5.3 it follows that the operator $\mathcal{B}_\beta$ is T-accretive in $L^1(]0, L[)$ and verifies

$$L^\infty(]0, L[)^+ \subset R(I + \lambda \mathcal{B}_\beta) \quad \text{for all} \quad \lambda > 0. \tag{6.61}$$

In order to get an L^∞-estimate of the resolvent we need to find the steady state solution, that is, the function u_β which is the entropy solution of the problem

$$\begin{cases} -\left(\mathbf{a}(u_\beta, u'_\beta)\right)' = 0 & \text{in} \quad]0, L[\\ \\ -\mathbf{a}(u_\beta, u'_\beta)|_{x=0} = \beta > 0 & \text{and} \quad u_\beta(L) = 0. \end{cases} \tag{6.62}$$

Proposition 6.6.2 *There is a non-increasing function $u_\beta \in C^1(]0,L[)$ that is an entropy solution of the stationary problem (6.62). This solution verifies that $u_\beta \geq \frac{\beta}{c}$ and $(u_\beta)'(L_-) = -\infty$. Moreover, there exists a constant $M := M(c,\beta,\nu,L)$ such that*

$$\|u_\beta\|_\infty \leq M.$$

Proof. We seek for regular solutions. Integrating (6.62) over $]0,L[$ we find that $\mathbf{a}(u_\beta, u'_\beta)(L) = -\beta$. Now, if u_β has to fulfill the weak Dirichlet condition $\mathbf{a}(u_\beta, u'_\beta)(L) = -cu_\beta(L_-)$ then we must have $u_\beta(L_-) = \beta/c$. We will follow this prescription hereafter. If u_β is a solution of the problem (6.62), we have

$$-\left(\mathbf{a}(u_\beta, u'_\beta)\right)' = 0 \iff \nu \frac{u_\beta u'_\beta}{\sqrt{u_\beta^2 + \frac{\nu^2}{c^2}(u'_\beta)^2}} = -\beta \quad \text{a.e. } x \in]0,L[.$$

Then we must assume that $u'_\beta < 0$ (note also that the previous relation ensures that $|u'_\beta| \geq \frac{\beta}{\nu}$). In this way u_β is greater than β/c everywhere. Hence we obtain

$$u'_\beta = -\frac{\beta\, u_\beta}{\nu\sqrt{u_\beta^2 - \left(\frac{\beta}{c}\right)^2}}.$$

Thus, we get that u_β satisfies the ordinary differential equation

$$\frac{u'_\beta\sqrt{u_\beta^2 - \left(\frac{\beta}{c}\right)^2}}{u_\beta} = -\frac{\beta}{\nu}.$$

By means of the change of variable $v^2 = u_\beta^2 - \left(\frac{\beta}{c}\right)^2$ we arrive to the ODE

$$-\frac{\beta}{\nu} = \left(1 - \frac{1}{1+\left(\frac{v}{\beta/c}\right)^2}\right) v'.$$

Then,

$$\int_x^L \left(-\frac{\beta}{\nu}\right) dy = \int_x^L v'(y)\, dy - \int_x^L \frac{v'(y)}{1+\left(\frac{v(y)}{\beta/c}\right)^2}\, dy$$
$$= v(L) - v(x) - \frac{\beta}{c}\arctan\left(\frac{v(L)}{\beta/c}\right) + \frac{\beta}{c}\arctan\left(\frac{v(x)}{\beta/c}\right).$$

Hence, we get

$$x = L - \frac{\nu}{\beta}\sqrt{u_\beta(x)^2 - \left(\frac{\beta}{c}\right)^2} + \frac{\nu}{c}\arctan\left[\frac{c}{\beta}\sqrt{u_\beta(x)^2 - \left(\frac{\beta}{c}\right)^2}\right]. \tag{6.63}$$

If $x = u_\beta^{-1}(y)$ then we can write (6.63) as

$$u_\beta^{-1}(y) = L - \frac{\nu}{\beta}\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2} + \frac{\nu}{c}\arctan\left[\frac{c}{\beta}\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2}\right].$$

Thus,

$$\left(u_\beta^{-1}\right)'(y) = \frac{y}{\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2}} \frac{\nu}{\beta}\left(\frac{\beta^2}{c^2y^2} - 1\right).$$

Consequently, since $(u_\beta)(L_-) = \frac{\beta}{c}$, we obtain that

$$(u_\beta)'(L_-) = \lim_{y\downarrow\frac{\beta}{c}} \frac{1}{\left(u_\beta^{-1}\right)'(y)} = \lim_{y\downarrow\frac{\beta}{c}} \frac{\sqrt{y^2 - \left(\frac{\beta}{c}\right)^2}}{y}\left(\frac{\beta}{\nu}\right)\left(\frac{c^2y^2}{\beta^2 - c^2y^2}\right) = -\infty.$$

Finally, since u_β satisfies $-(\mathbf{a}(u_\beta(x), u'_\beta(x)))' = 0$ if $x \in]0, L[$ and satisfies the boundary conditions also, we have that u_β is an entropy solution of the problem (6.62). As an aside, it is easily proved that

$$M \leq \frac{\beta}{c}\sqrt{1 + \left(\frac{cL}{\nu} + \frac{\pi}{2}\right)^2}$$

from the relation

$$0 = L - \frac{\nu}{\beta}\sqrt{u_\beta(0)^2 - (\beta/c)^2} + \frac{\nu}{c}\arctan\left[\frac{c}{\beta}\sqrt{u_\beta(0)^2 - (\beta/c)^2}\right].$$

□

Remark 6.6.3 In fact dimensional analysis shows that

$$u_\beta(x) = \frac{\beta}{c}\sqrt{1 + \psi\left(\frac{L-x}{L}\right)}$$

for some dimensionless function $\psi : [0, L] \to \mathbb{R}^+$ with $\psi(0) = 0$. The first two terms of an asymptotic expansion around $x = L$ are

$$\frac{\beta}{c} + \left(\frac{3(2-\sqrt{2})}{2\sqrt{2}} \frac{\beta}{\nu}\sqrt{\frac{\beta}{c}}\right)(L-x)^{\frac{2}{3}}.$$

The following homogeneity property of the operator $\mathcal{B}_\beta$ will be important to get an L^∞-estimate of the resolvent.

Proposition 6.6.4 *Let $u \in L^1(]0, L[)^+$. Then, for $\mu,\ \lambda,\ \beta > 0$ the following relation holds:*

$$(I + \lambda\mathcal{B}_\beta)^{-1}(\mu u) = \mu\left(I + \lambda\mathcal{B}_{\frac{\beta}{\mu}}\right)^{-1}(u). \tag{6.64}$$

Moreover, let $u \in L^\infty(]0, L[)^+$ and $\lambda > 0$. If for some $\beta_2 > 0$ we have that

$$(I + \lambda\mathcal{B}_{\beta_2})^{-1}(u) \in BV(]0, L[),$$

then for any $\beta_1 \leq \beta_2$ we have the following inequality:

$$(I + \lambda\mathcal{B}_{\beta_1})^{-1}(u) \leq (I + \lambda\mathcal{B}_{\beta_2})^{-1}(u) \quad a.e.\ x \in]0, L[. \tag{6.65}$$

Proof. The very definition of the operator entails the fact that, if $u \in D(\mathcal{B}_{\frac{\beta}{\mu}})$, then $\mu u \in D(\mathcal{B}_\beta)$ and $\mathcal{B}_\beta(\mu u) = \mu \mathcal{B}_{\frac{\beta}{\mu}}(u)$. Then, we have

$$v := (I + \lambda \mathcal{B}_\beta)^{-1}(\mu u) \iff v + \lambda \mathcal{B}_\beta(v) = \mu u \iff \frac{1}{\mu} v + \frac{1}{\mu} \lambda \mathcal{B}_{\mu \frac{\beta}{\mu}}(v) = u$$

$$\iff \frac{1}{\mu} v + \lambda \mathcal{B}_{\frac{\beta}{\mu}}(\frac{v}{\mu}) = u \iff \left(I + \lambda \mathcal{B}_{\frac{\beta}{\mu}}\right)^{-1}(u) = \frac{v}{\mu},$$

from where (6.64) follows.

Finally, let us see that (6.65) holds. Let $u_i := (I + \lambda \mathcal{B}_{\beta_i})^{-1}(u)$, $i = 1, 2$. Then, u_i is an entropy solution of the problem

$$\begin{cases} u_i - \lambda\left(\mathbf{a}(u_i, u_i')\right)' = u & \text{in }]0, L[\\ \\ -\mathbf{a}(u_i, u_i')|_{x=0} = \beta_i > 0 & \text{and } \ u(L) = 0. \end{cases}$$

Consider a sequence p_n of non negative increasing approximations to the sign_0^+ function, each of them vanishing in a neighborhood of the origin. Under the present hypotheses, $p_n(u_1 - u_2) \in BV(]0, L[)$. Therefore, having in mind (4.13), we get

$$\begin{aligned} &\int_0^L (u_1 - u_2) p_n(u_1 - u_2)\, dx \\ &= \int_0^L \lambda \left(\left(\mathbf{a}(u_1, u_1')\right)' - \left(\mathbf{a}(u_2, u_2')\right)' \right) p_n(u_1 - u_2)\, dx \\ &= - \int_0^L \lambda \left(\mathbf{a}(u_1, u_1') - \mathbf{a}(u_2, u_2')\right) D(p_n(u_1 - u_2)) \\ &\quad + \lambda \left[\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\right] p_n(u_1 - u_2)(L_-) \\ &\quad - \lambda \left[\mathbf{a}(u_1, u_1')(0_+) - \mathbf{a}(u_2, u_2')(0_+)\right] p_n(u_1 - u_2)(0_+) \\ &\le \lambda \left[\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\right] p_n(u_1 - u_2)(L_-) + \lambda(\beta_1 - \beta_2) p_n(u_1 - u_2)(0_+) \\ &\le \lambda \left(\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\right) p_n(u_1 - u_2)(L_-). \end{aligned}$$

Then, taking limit as $n \to +\infty$ we get

$$\int_0^L (u_1 - u_2)^+ \, dx \le \lambda \left[\mathbf{a}(u_1, u_1')(L_-) - \mathbf{a}(u_2, u_2')(L_-)\right] \text{sign}_0^+(u_1 - u_2)(L_-) \le 0,$$

since $\mathbf{a}(u_i, u_i')(L_-) = -c\, u_i(L_-)$, $i = 1, 2$. Therefore $u_1 \le u_2$ as desired. □

Proposition 6.6.5 *For $u \in L^\infty(]0, L[)^+$ and $\lambda > 0$, we have the inequality*

$$0 \le (I + \lambda \mathcal{B}_\beta)^{-1}(u) \le \mu u_\beta,$$

being u_β the entropy solution of the stationary problem (6.62) constructed in Proposition 6.6.2 and $\mu = \max\left\{\frac{c\|u\|_\infty}{\beta}, 1\right\}$.

Proof. As a consequence of Proposition 6.6.2 we have that $(u_\beta, 0) \in \mathcal{B}_\beta$, from where it follows that

$$(I + \lambda \mathcal{B}_\beta)^{-1} (u_\beta) = u_\beta. \tag{6.66}$$

On the other hand, since $u_\beta \geq \frac{\beta}{c}$, we have that $0 \leq u \leq \max\{\frac{c\|u\|_\infty}{\beta}, 1\} u_\beta$. Define then $\mu := \max\{\frac{c\|u\|_\infty}{\beta}, 1\}$. Hence, by Proposition 6.6.4 and having in mind (6.66), we get

$$\begin{aligned} 0 \leq & (I + \lambda \mathcal{B}_\beta)^{-1} (u) \leq (I + \lambda \mathcal{B}_\beta)^{-1} (\mu u_\beta) \\ =& \mu \left(I + \lambda \mathcal{B}_{\frac{\beta}{\mu}} \right)^{-1} (u_\beta) \leq \mu (I + \lambda \mathcal{B}_\beta)^{-1} (u_\beta) = \mu u_\beta \end{aligned}$$

as desired. □

Remark 6.6.6 Needless to say, a sharper definition of μ could be given. Namely,

$$\mu = \max \left\{ 1, \inf \left\{ K \geq 0 / \int_0^L (u - K u_\beta)^+ \, dx = 0 \right\} \right\}.$$

We won't need this improvement of the estimate in Proposition 6.6.5.

Next we introduce the main result of this section, which paves the way for the operator $\mathcal{B}_\beta$ to generate an order-preserving semigroup.

Theorem 6.6.7 *The operator $\mathcal{B}_\beta$ is* T*-accretive in $L^1(]0, L[)$ and verifies the range condition*

$$\overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)} = L^1(]0, L[)^+ \subset R(I + \lambda \mathcal{B}_\beta) \quad \textit{for all} \quad \lambda > 0.$$

Proof. Theorem 6.5.3 yields the T-accretivity of the operator $\mathcal{B}_\beta$ and the fulfillment of condition (6.61) also. To prove the density of $D(\mathcal{B}_\beta)$ in $L^1(]0, L[)^+$, we shall prove that $\mathcal{D}(]0, L[)^+ \subseteq \overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)}$. Let $0 \leq v \in \mathcal{D}(]0, L[)$. By (6.61), $v \in R(I + \frac{1}{n}\mathcal{B}_\beta)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D(\mathcal{B}_\beta)$ such that $(u_n, n(v - u_n)) \in \mathcal{B}_\beta$. Since $u_n = (I + \frac{1}{n}\mathcal{B}_\beta)^{-1}(v)$, by Proposition 6.6.5 we get

$$\|u_n\|_\infty \leq M := M(\beta, c, \nu, L, \|v\|_\infty). \tag{6.67}$$

The function u_n verifies the equation

$$n(v - u_n) = -D\mathbf{a}(u_n, u_n') \quad \text{in} \quad \mathcal{D}'(]0, L[).$$

Let $\epsilon > 0$ and consider $S_\epsilon := T_{\epsilon, \|v\|_\infty}$. We multiply the equation above by $v - S_\epsilon(u_n)$ and integrate by parts. What we get is

$$\begin{aligned} \int_0^L (v - S_\epsilon(u_n)) n(v - u_n) \, dx = & \int_0^L \mathbf{a}(u_n, u_n')(Dv - DS_\epsilon(u_n)) \\ & - c u_n(L_-) S_\epsilon(u_n)(L_-) + \beta S_\epsilon(u_n)(0_+). \end{aligned}$$

Note that due to (5.29) we have

$$\int_0^L \mathbf{a}(u_n, u_n') DS_\epsilon(u_n) \geq 0.$$

Taking into account (6.67) we get the estimate

$$\int_0^L (v - S_\epsilon(u_n))(v - u_n)\, dx \leq \frac{1}{n}\left[\int_0^L \mathbf{a}(u_n, u_n')Dv\right] + \frac{1}{n}\beta S_\epsilon(u_n)(0_+) \leq \frac{C}{n}.$$

Letting $\epsilon \to 0^+$, we obtain

$$\int_0^L (v - u_n)^2\, dx \leq \frac{C}{n},$$

and thus $u_n \to v$ in $L^2(]0,L[)$ as $n \to \infty$. Moreover, we have $u_n \to v$ in $L^1(]0,L[)$ as $n \to \infty$. Therefore $v \in \overline{D(\mathcal{B}_\beta)}^{L^1(]0,L[)}$ and the proof of the density of $D(\mathcal{B}_\beta)$ in $L^1(]0,L[)^+$ is complete.

To finish the proof of the theorem we only need to show that the operator $\mathcal{B}_\beta$ is closed in $L^1(]0,L[) \times L^1(]0,L[)$. Given $(u_n, v_n) \in \mathcal{B}_\beta$ such that $u_n \to u$ and $v_n \to v$ in $L^1(]0,L[)$, we need to prove that $(u,v) \in \mathcal{B}_\beta$. Since $(u_n, v_n) \in \mathcal{B}_\beta$, we have that $u_n \in TBV^+(]0,L[)$ and $\mathbf{z}_n := \mathbf{a}(u_n, u_n') \in C([0,L])$ —under the usual conventions— satisfy the following relations:

$$v_n = -D\mathbf{z}_n \qquad \text{in } \mathcal{D}'(]0,L[), \tag{6.68}$$

$$h(u_n, DT(u_n)) \leq \mathbf{z}_n DT(u_n) \quad \text{as measures } \ \forall\, T \in \mathcal{T}^+ \tag{6.69}$$

$$h_S(u_n, DT(u_n)) \leq \mathbf{z}_n DJ_{T'S}(u_n) \quad \text{as measures } \forall\, S \in \mathcal{P}^+\ T \in \mathcal{T}^+,$$

$$-\mathbf{z}_n(0) = \beta \quad \text{and} \quad \mathbf{z}_n(L) = -cu_n(L_-). \tag{6.70}$$

Let $T = T_{a,b} \in \mathcal{T}_r$. Multiplying (6.68) by $T(u_n)$ and applying integration by parts we get

$$\int_0^L v_n T(u_n)\, dx = \int_0^L \mathbf{z}_n DT(u_n) - \mathbf{z}_n(L)T(u_n(L_-)) - \beta T(u_n(0_+)).$$

We deduce that

$$\int_0^L \mathbf{z}_n DT(u_n) \leq b(\beta + \|v\|_1) \leq C. \tag{6.71}$$

Here we used (6.70) to note that the term related to $\mathbf{z}_n(L)$ can be disregarded as it has the right sign. Anyway $\mathbf{z}_n(L)$ is bounded by $\|\mathbf{z}_n\|_\infty$, a quantity which is controlled in terms of $\|\mathbf{z}_n\|_{W^{1,1}}$, being the later uniformly bounded. On the other hand, by (6.69) and having in mind (4.12) and (4.21), we get

$$\int_0^L \mathbf{z}_n DT(u_n) \geq \frac{c}{2}\int_0^L |D([T(u_n)]^2)| - \frac{c^2}{\nu}\int_0^L T(u_n)^2\, dx. \tag{6.72}$$

By (6.71) and (6.72), it follows that

$$\int_0^L |D([T(u_n)]^2)| \leq \frac{2c}{\nu}\int_0^L T(u_n)^2\, dx + \frac{2C}{c} \leq \frac{2cLb^2}{\nu} + \frac{2C}{c} = C. \tag{6.73}$$

Using the coarea formula as in the proof of Theorem 6.5.3 we deduce from (6.73) that

$$\int_0^L |DT(u_n)| \leq \frac{C}{2a} \qquad \forall\, n \in \mathbb{N}.$$

Then, since the total variation is lower semi-continuous in $L^1(]0,L[)$, we have

$$\int_0^L |DT(u)| \leq \liminf_{n\to\infty} \int_0^L |DT(u_n)| \leq \frac{C}{2a}.$$

Hence $T(u) \in BV(]0,L[)$ and consequently $u \in TBV^+(]0,L[)$.

Since $\mathbf{z}_n = c|u_n|\mathbf{b}(u_n,u_n')$ with $|\mathbf{b}(u_n,u_n')| \leq 1$, we have that for all measurable subsets $E \subset]0,L[$ the following inequality holds:

$$\int_E |\mathbf{z}_n|\, dx \leq c\int_E |u_n|\, dx.$$

As $\{u_n\}$ converges strongly, it is equi-integrable. Therefore, by Dunford-Pettis's Theorem, we can assume that

$$\mathbf{z}_n \rightharpoonup \mathbf{z} \quad \text{weakly in } L^1(]0,L[). \tag{6.74}$$

Moreover, since $|\mathbf{b}(u_n,u_n')| \leq 1$ we also can assume that

$$\mathbf{b}(u_n,u_n') \rightharpoonup \mathbf{z}_b \quad \text{weakly}^* \text{ in } L^\infty(]0,L[). \tag{6.75}$$

As $u_n \to u$ in $L^1(]0,L[)$, we obtain from (6.74) and (6.75) that

$$\mathbf{z} = c\, u\, \mathbf{z}_b. \tag{6.76}$$

As $v_n \to v$ in $L^1(]0,L[)$, we easily deduce from (6.74) and (6.68) that

$$v = -D\mathbf{z} \qquad \text{in } \mathcal{D}'(]0,L[). \tag{6.77}$$

Hence by (6.76) and (6.77) it follows that $\mathbf{z} \in W^{1,1}(]0,L[) \subset C([0,L])$.

Lemma 6.6.8 *The functions $\mathbf{z}(x)$ and $\mathbf{a}(u(x),u'(x))$ coincide for a.e. $x \in]0,L[$.*

Proof. We use Minty–Browder's technique. Let $0 < a < b$, let $0 \leq \phi \in C_c^1(]0,L[)$ and let $g \in C^2([0,L])$. We recall that $T'_{a,b}$ shall mean $\chi_{]a,b[}$. By (4.13), we have that

$$\int_0^L \phi[\mathbf{z}_n - \mathbf{a}(u_n,g')]T'_{a,b}(u_n)(u_n-g)'\, dx \geq 0. \tag{6.78}$$

Let us denote

$$J_{\mathbf{a}}(x,r) := \int_0^r \mathbf{a}(s,g'(x))\, ds,$$
$$J_{\mathbf{a}'}(x,r) := \int_0^r \partial_x[\mathbf{a}(s,g'(x))]\, ds = \int_0^r \frac{\partial \mathbf{a}}{\partial \xi}(s,g'(x))g''(x)\, ds$$

and observe that

$$-\mathbf{a}\left(T_{a,b}(u_n(x)),g'(x)\right)[T_{a,b}(u_n(x))]' = -D^{ac}\left[J_{\mathbf{a}}(x,T_{a,b}(u_n(x)))\right] + J_{\mathbf{a}'}(x,T_{a,b}(u_n(x))).$$

This we will substitute into (6.78). Note now that, using (6.69) we can get

$$\int_0^L \phi[\mathbf{z}_n D^s T_{a,b}(u_n) - D^s J_{\mathbf{a}}(x,T_{a,b}(u_n))] \geq \int_0^L \phi[h(u_n,DT_{a,b}(u_n))^s - D^s J_{\mathbf{a}}(x,T_{a,b}(u_n))].$$

This lower bound is non-negative, as we have the following result:

Lemma 6.6.9 *The following inequality*

$$(DJ_{\mathbf{a}}(x, T_{a,b}(u_n)))^s \leq h(u_n, DT_{a,b}(u_n))^s$$

holds true.

Proof. We begin the proof stating two properties that concern the function h^0. Thanks to (4.19) we have that $h^0(z,\xi) = c|z|\,|\xi| = \theta(z)|\xi|$. In connection with (4.15) this leads to

$$\mathbf{a}(z,\xi)\eta \leq h^0(z,\eta) \quad \forall \xi, \eta, z \in \mathbb{R} \tag{6.79}$$

Using the chain rule and Volpert's averaged superposition we compute

$$\begin{aligned}(DJ_{\mathbf{a}}(x, T_{a,b}(u_n)))^s &= \overline{\mathbf{a}}(T_{a,b}(u_n), g'(x))\ D^sT_{a,b}(u_n)\\ &= \overline{\mathbf{a}}(T_{a,b}(u_n), g'(x))\frac{D^sT_{a,b}(u_n)}{|D^sT_{a,b}(u_n)|}\ |D^sT_{a,b}(u_n)|\end{aligned}$$

$$= \int_0^1 \mathbf{a}(\tau(T_{a,b}(u_n))^+ + (1-\tau)(T_{a,b}(u_n))^-, g'(x))\ d\tau \frac{D^sT_{a,b}(u_n)}{|D^sT_{a,b}(u_n)|}\ |D^sT_{a,b}(u_n)|.$$

Notice that by (6.79) the previous is bounded above by

$$\int_0^1 h^0\left(\tau(T_{a,b}(u_n))^+ + (1-\tau)(T_{a,b}(u_n))^-, \frac{D^sT_{a,b}(u_n)}{|D^sT_{a,b}(u_n)|}\right)\ d\tau\ |D^sT_{a,b}(u_n)|.$$

Using the decomposition of h^0 this is rewritten as

$$\int_0^1 \theta(\tau(T_{a,b}(u_n))^+ + (1-\tau)(T_{a,b}(u_n))^-)\ d\tau\ \left|\frac{D^sT_{a,b}(u_n)}{|D^sT_{a,b}(u_n)|}\right| |D^sT_{a,b}(u_n)|.$$

Thanks to the chain rule again, this is transformed into $|D^sJ_\theta(T_{a,b}(u_n))|$, which coincides with $h(u_n, DT_{a,b}(u_n))^s$ and conclusion follows. □

Then we can combine this information with (6.78), obtaining that

$$\begin{aligned}0 \leq &\int_0^L \phi\,[\mathbf{z}_n DT_{a,b}(u_n) - DJ_{\mathbf{a}}(x, T_{a,b}(u_n(x)))]\\ &+ \int_0^L \phi\left[J_{\mathbf{a}'}(x, T_{a,b}(u_n(x))) - \mathbf{z}_n g' T'_{a,b}(u_n)) + g'T'_{a,b}(u_n)\mathbf{a}(u_n, g')\right]\ dx.\end{aligned}$$

Now, since

$$\begin{aligned}&\int_0^L \phi\,\mathbf{z}_n[DT_{a,b}(u_n) - g'T'_{a,b}(u_n)\ dx]\\ &= \int_0^L \phi\,\mathbf{z}_n D[T_{a,b}(u_n) - g] + \int_0^L \phi\,\mathbf{z}_n g'(1 - T'_{a,b}(u_n))\,dx\\ &= -\int_0^L v_n\phi\,(T_{a,b}(u_n) - g)\ dx - \int_0^L \phi'\mathbf{z}_n\,(T_{a,b}(u_n) - g)\ dx\\ &\quad + \int_0^L \phi\,\mathbf{z}_n g'(1 - T'_{a,b}(u_n))\,dx\end{aligned}$$

we get

$$\lim_{n\to+\infty}\int_0^L \phi\,\mathbf{z}_n[DT_{a,b}(u_n)-g'T'_{a,b}(u_n)\,dx]$$
$$\leq \langle \mathbf{z}D(T_{a,b}(u)-g),\phi\rangle+\|g'\|_\infty\int_0^L |\mathbf{z}|\phi\left(1-T'_{a,b}(u)\right)\,dx.$$

On the other hand, the almost everywhere convergence of u_n implies that

$$J_{\mathbf{a}'}\left(x,T_{a,b}(u_n(x))\right)\to J_{\mathbf{a}'}\left(x,T_{a,b}(u(x))\right)\text{ a.e.}$$

and we also have that

$$DJ_{\mathbf{a}}(x,T_{a,b}(u_n(x)))\rightharpoonup DJ_{\mathbf{a}}(x,T_{a,b}(u(x)))\,\text{weakly}^*\text{ as measures.}$$

As a consequence, we get

$$\lim_{n\to+\infty}\int_0^L \phi\left[J_{\mathbf{a}'}(x,T_{a,b}(u_n(x)))\;dx-DJ_{\mathbf{a}}(x,T_{a,b}(u_n(x)))+g'T'_{a,b}(u_n)\mathbf{a}(u_n,g')\;dx\right]$$
$$=\langle J_{\mathbf{a}'}(x,T_{a,b}(u))-DJ_{\mathbf{a}}(x,T_{a,b}(u),\phi\rangle+\int_0^L \phi\,g'T'_{a,b}(u)\mathbf{a}(u,g')\;dx.$$

Consequently we obtain

$$\begin{aligned}0\leq&\langle \mathbf{z}\,D\left(T_{a,b}(u)-g\right),\phi\rangle+\|g'\|_\infty\int_0^L|\mathbf{z}|\phi\left(1-T'_{a,b}(u)\right)\,dx\\&+\int_0^L\phi\,g'T'_{a,b}(u)\mathbf{a}(u,g')\,dx-\langle DJ_{\mathbf{a}}(x,T_{a,b}(u(x)))-J_{\mathbf{a}'}(x,T_{a,b}(u(x))),\phi\rangle\end{aligned}$$

for all $0\leq\phi\in C^1_c(]0,L[)$. This means that, as measures,

$$\begin{aligned}0\leq&\mathbf{z}\,D\left(T_{a,b}(u)-g\right)-DJ_a(x,T_{a,b}(u(x)))+J_{\mathbf{a}'}(x,T_{a,b}(u(x)))\mathcal{L}^1\\&+\left\{\mathbf{a}(u,g')g'T'_{a,b}(u)+|\mathbf{z}|\|g'\|_\infty\left(1-T'_{a,b}(u)\right)\right\}\mathcal{L}^1.\end{aligned}$$

We deduce, as it was done in Lemma 6.5.8, that the absolutely continuous part of

$$-D\left[J_{\mathbf{a}}(x,T_{a,b}(u(x)))\right]+J_{\mathbf{a}'}(x,T_{a,b}(u(x)))$$

is

$$-\mathbf{a}(u,g')(T_{a,b}(u))'\,\mathcal{L}^1.$$

Then we obtain the inequality

$$\mathbf{z}\left(T_{a,b}(u)-g\right)'-\mathbf{a}(u,g')(T_{a,b}(u))'+\mathbf{a}(u,g')\,g'T'_{a,b}(u)+|\mathbf{z}|\|g'\|_\infty\left(1-T'_{a,b}(u)\right)\geq 0$$

after taking absolutely continuous parts in the previous relation. If $x\in[a<u<b]$, this reduces to

$$\left(\mathbf{z}-\mathbf{a}(u,g')\right)(u-g)'\geq 0,$$

which holds for all $g\in C^2([0,L])$ and all $x\in\Omega\cap[a<u<b]$, where $\mathcal{L}^1(]0,L[\setminus\Omega)=0$. Being $x\in\Omega\cap[a<u<b]$ fixed and $\xi\in\mathbb{R}$ given, we find g as above such that $g'(x)=\xi$. Then

$$(\mathbf{z}(x)-\mathbf{a}(u(x),\xi))\,(u'(x)-\xi)\geq 0,\;\;\forall\xi\in\mathbb{R}.$$

By an application of Minty–Browder's method in $\mathbb{R}$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) \quad \text{a.e. on } [a < u < b].$$

Since this holds for any $0 < a < b$, we obtain the identification a.e. on the points of $]0, L[$ such that $u(x) \neq 0$. Now, by our assumptions on $\mathbf{a}$ and (6.76) we deduce that $\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) = 0$ a.e. on $[u = 0]$. The Lemma is proved. □

To finish the proof we only need to show that

$$\frac{c}{2}|D^s(T(u)^2)| \leq \mathbf{z}D^s T(u) \quad \text{as measures} \quad \forall\, T \in \mathcal{T}^+,$$

$$|D^s(J_{S\theta}(T(u))| \leq \mathbf{z}D^s J_{T'S}(u) \quad \text{as measures} \quad \forall\, S,\, T \in \mathcal{T}^+,$$

$$-\mathbf{a}(u, u')(0) = \beta \quad \text{and} \quad \mathbf{a}(u, u')(L) = -cu(L_-).$$

These proofs are similar to those in the previous section. □

Once we get to this point we can use Theorem 6.6.7 to justify that Crandall-Liggett's Theorem B.1.4 applies in our situation. Thus we get that for any $0 \leq u_0 \in L^1(]0, L[)$ there exists a unique mild solution $u \in C([0, T]; L^1(]0, L[))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}_\beta u(t) \ni 0, \qquad u(0) = u_0.$$

Moreover, $u(t) = T_\beta(t)u_0$ for all $t \geq 0$, where $(T_\beta(t))_{t\geq 0}$ is the semigroup in $L^1(]0, L[)^+$ generated by Crandall-Liggett's exponential formula, i.e.,

$$T_\beta(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B}_\beta\right)^{-n} u_0.$$

On the other hand, as the operator $\mathcal{B}_\beta$ is T-accretive we have that the comparison principle also holds for $T_\beta(t)$. Meaning that, if $u_0, \overline{u}_0 \in L^1(]0, L[)^+$, we have the estimate

$$\|(T_\beta(t)u_0 - T_\beta(t)\overline{u}_0)^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1. \tag{6.80}$$

Using Crandall-Liggett's exponential formula and (6.64) we get that for all $u_0 \in L^1(]0, L[)^+$,

$$T_\beta(t)(\mu u_0) = \mu T_{\frac{\beta}{\mu}}(t)(u_0) \quad \text{for all } t > 0. \tag{6.81}$$

Finally, as a consequence of (6.80) and (6.81) we get an L^∞-bound for the evolution semigroup. More precisely, for $u \in L^\infty(]0, L[)^+$ we have —with the notation of Proposition 6.6.5—

$$0 \leq T_\beta(t)(u) \leq \mu u_\beta, \quad \forall\, t \geq 0.$$

6.7 Existence and uniqueness of solutions of the parabolic problem

This section deals with the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u, u_x))_x & \text{in} \quad]0, T[\times]0, L[\\[2ex] -\mathbf{a}(u(t, 0), u_x(t, 0)) = \beta > 0 \text{ and } u(t, L) = 0 & \text{on} \quad t \in]0, T[, \\[2ex] u(0, x) = u_0(x) & \text{in} \quad x \in]0, L[. \end{cases} \tag{7.82}$$

To make precise our notion of solution we need to recall the following definitions given in [15]. We set $Q_T =]0,T[\times]0,L[$ for the spatio-temporal domain.

First we give a meaning to products like $\mathbf{z}DT(u)$. Note that if

$$w \in L^1(0,T;BV(]0,L[)) \cap L^\infty(Q_T)$$

and $\mathbf{z} \in L^1(Q_T)$ is such that there exists an element $\xi \in [L^1(0,T;BV(]0,L[))]^*$ with $D_x\mathbf{z} = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated with $(\mathbf{z},\xi)$, the distribution $\mathbf{z}D_x w$ in Q_T by means of

$$\langle \mathbf{z}D_x w, \varphi\rangle = -\langle \xi, \varphi w\rangle - \int_0^T\int_0^L \mathbf{z}(t,x)w(t,x)\partial_x\varphi(t,x)\,dxdt \tag{7.83}$$

for all $\varphi \in \mathcal{D}(Q_T)$.

Our concept of solution for the problem (7.82) is the following.

Definition 6.7.1 A measurable function $u :]0,T[\times]0,L[\to \mathbb{R}^+$ is an *entropy solution* of (7.82) in $Q_T =]0,T[\times]0,L[$ if

$$\begin{cases} u \in C([0,T];L^1(]0,L[)) \\ u(0,x) = u_0(x), \quad x \in]0,L[. \\ T(u(\cdot)) \in L^1_{loc,w}(0,T,BV(]0,L[)) \text{ for all } T \in \mathcal{T}_r \\ \mathbf{z}(t) := \mathbf{a}(u(t),\partial_x u(t)) \in L^1(Q_T) \end{cases}$$

and the equation is satisfied in the following sense:

(i) the time derivative u_t of u in $\mathcal{D}'(Q_T)$ belongs to $[L^1(0,T;BV(]0,L[))]^*$ and satisfies

$$\int_0^T \langle u_t(t),\psi(t)\rangle\,dt = -\int_0^T\int_0^L u(t,x)\Theta(t,x)\,dxdt \tag{7.84}$$

for all test function $\psi \in L^1(0,T;BV(]0,L[))$ compactly supported in time such that $\psi(t) = \int_0^t \Theta(s)\,ds$ as a Pettis integral and $\Theta \in L^1_w(0,T;BV(]0,L[))\cap L^\infty(Q_T)$.

(ii) The relation $D_x\mathbf{z} = u_t$ holds in $\mathcal{D}'(Q_T)$ and for any $w \in L^1(0,T;BV(]0,L[))$, the distribution $\mathbf{z}D_x w$ defined by (7.83) is a Radon measure in Q_T. The following integration by parts formula

$$\int_{Q_T} \mathbf{z}D_x w \ + \langle u_t, w\rangle = \beta\int_0^T w(t,0_+)\,dt \ - c\int_0^T u(t,L_-)w(t,L_-)\,dt. \tag{7.85}$$

is fulfilled by the above objects, for all $w \in L^1(0,T;BV(]0,L[))$.

(iii) Given any truncations $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and any $\eta \in \mathcal{D}(Q_T)$, the following entropy inequality is satisfied:

$$\begin{aligned} &\int_{Q_T} \eta h_S(u,DT(u))\,dt + \int_{Q_T} \eta h_T(u,DS(u))\,dt \\ &\le \int_{Q_T} J_{TS}(u)\partial_t\eta\ dxdt - \int_{Q_T} \mathbf{a}(u,\partial_x u)\partial_x\eta\ T(u)S(u)\ dxdt. \end{aligned}$$

Definition 6.7.2 We say that u is a *bounded entropy solution* of (7.82) if u is and entropy solution of (7.82) and satisfies that

$$\sup_{0\leq t\leq T} \|u(t)\|_{L^\infty(]0,L[)} < +\infty.$$

In the following result we get a positive lower bound for $u(t,0_+)$ that holds for any entropy solution.

Lemma 6.7.3 *If u is an entropy solution of (7.82) in $Q_T =]0,T[\times]0,L[$, then*

$$u(t,0_+) \geq \frac{\beta}{c} > 0, \qquad \textit{for almost all } t \in]0,T[. \tag{7.86}$$

Proof. For any $n\in\mathbb{N}$, let v_n be the function defined by

$$v_n(x) := \begin{cases} -nx+1, & 0 < x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x < L. \end{cases}$$

Being $0 \leq \phi \in \mathcal{D}(]0,T[)$ fixed and taking w in (7.85) as $w_n(t) := \phi(t)v_n$, we get

$$\int_{Q_T} \mathbf{z} D_x w_n + \langle u_t, w_n\rangle = \beta \int_0^T \phi(t)\,dt. \tag{7.87}$$

By (7.84), we have

$$\langle u_t, w_n\rangle = -\int_0^T \phi'(t)\int_0^L u(t,x)v_n(x)\,dxdt.$$

Using the dominated convergence theorem it follows that

$$\lim_{n\to\infty} \langle u_t, w_n\rangle = 0. \tag{7.88}$$

On the other hand, given $\varphi \in \mathcal{D}(Q_T)$, relations (7.83) and (7.85) in combination yield

$$\langle \mathbf{z} D_x w_n, \varphi\rangle = \int_0^T \phi(t)\int_0^L \mathbf{z}(t,x)\varphi(t,x)v_n'(x)\,dxdt.$$

Hence,

$$\int_{Q_T} \mathbf{z}(t,x)D_x w_n(t,x) = -\int_0^T \phi(t) n \int_0^{\frac{1}{n}} \mathbf{z}(t,x)\,dxdt. \tag{7.89}$$

Now, by (7.87), (7.88) and (7.89), we get

$$\beta\int_0^T \phi(t)\,dt = -\lim_{n\to\infty}\int_0^T \phi(t)\,n\int_0^{\frac{1}{n}} \mathbf{z}(t,x)\,dxdt.$$

Assume now that for some $n\in\mathbb{N}$ we have that $u \in L^\infty(]0,1/n[)$; if this is not the case then $u \geq \beta/c$ in a neighborhood of zero and we are done. Then, since $|\mathbf{z}(t,x)| \leq$

$cu(t,x) \leq cT_{a,\|u\|_\infty}(u(t,x))$ for any $0 < a < \|u\|_\infty$ and almost every $x \in]0,1/n[$, using Fatou's Lemma we obtain that

$$\beta \int_0^T \phi(t)\, dt \leq c \int_0^T \phi(t) \left[\lim_{n\to\infty} n \int_0^{\frac{1}{n}} T_{a,\|u\|_\infty}(u(t,x))\, dx \right] dt$$
$$= c \int_0^T \phi(t) \left(T_{a,\|u\|_\infty}(u(t)) \right)(0_+)\, dt,$$

thanks to the definition of the trace operator. Thus $\beta \leq c \left(T_{a,\|u\|_\infty}(u(t)) \right)(0_+)$ for any $a > 0$, from where (7.86) follows. □

As regards the existence and uniqueness of bounded entropy solutions we have the following result.

Theorem 6.7.4 *For any initial datum $0 \leq u_0 \in L^\infty(]0,L[)$ there exists a unique bounded entropy solution u of (7.82) in $Q_T =]0,T[\times]0,L[$ for every $T > 0$. Moreover, if $u(t)$, $\overline{u}(t)$ are bounded entropy solutions of (7.82) in $Q_T =]0,T[\times]0,L[$ corresponding to initial data u_0, $\overline{u}_0 \in L^\infty(]0,L[)^+$ respectively, then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all} \quad t \geq 0.$$

In particular, we have uniqueness of bounded entropy solutions for (7.82).

The remaining sections of this chapter constitute a proof for this statement. The time regularization procedure introduced in Definition C.2.1 of the Appendix will be required several times. For future usage we point out here the following features of our particular case:

Remark 6.7.5 Let u a bounded entropy solution of (7.82) in Q_T and consider the time regularization given in Definition C.2.1. Given $p \in \mathcal{P}^+$, it is easy to see that

$$|D_x(\phi p(u))^\tau(t)|(]0,L[) \leq \frac{1}{\tau} \int_{t-\tau}^t |D_x(\phi(s)p(u(s)))|(]0,L[)\, ds.$$

Then, by the lower-semi-continuity of the total variation respect to the L^1-convergence, we have

$$|D_x(\phi(t)p(u(t)))|(]0,L[) \leq \liminf_{\tau\to 0} |D_x(\phi p(u))^\tau(t)|(]0,L[)$$
$$\leq \limsup_{\tau\to 0} \frac{1}{\tau} \int_{t-\tau}^t |D_x(\phi(s)p(u(s))|(]0,L[)\, ds.$$

Since the map $t \mapsto |D_x(\phi(t)p(u(t)))|(]0,L[)$ belongs to $L^1_{\text{loc}}([0,T])$, we have that almost all $t \in [0,T]$ is a Lebesgue point of this map. So, for almost all $t \in [0,T]$, we have

$$\frac{1}{\tau} \int_{t-\tau}^t |D_x(\phi(s)p(u(s))|(]0,L[)\, ds \overset{\tau\to 0}{\longrightarrow} |D_x(\phi(t)p(u(t))|(]0,L[),$$

and consequently,

$$|D_x(\phi p(u))^\tau(t)|(]0,L[) \overset{\tau\to 0}{\longrightarrow} |D_x(\phi(t)p(u(t))|(]0,L[) \text{ a.e. } t. \tag{7.90}$$

The following trick will be also useful a number of times.

Lemma 6.7.6 *Let q be a non-decreasing function and $r, \overline{r} \geq 0$. Then, the inequality*

$$J_q(r) - J_q(\overline{r}) \leq q(r)(r - \overline{r})$$

holds. As a consequence, given $\tau \in \mathbb{R}$, given real functions u, ϕ and changing variables we obtain the following inequality

$$\int_{-\infty}^{+\infty} \frac{u(t) - u(t-\tau)}{\tau} q(u(t))\phi(t)\ dt \geq \int_{-\infty}^{+\infty} J_q(u(t)) \frac{\phi(t) - \phi(t+\tau)}{\tau}\ dt$$

whenever it makes sense.

6.7.1 Proof of the comparison principle

Let $b > a > 2\epsilon > 0$. We will use the notation $T = T_{a,b}^a$. We also need to consider truncation functions of the form

$$S_{\epsilon,l}(r) := T_\epsilon(r - l)^+ = T_{l,l+\epsilon}(r) - l \in \mathcal{T}^+$$

and

$$S_\epsilon^l(r) := T_{l-\epsilon,l}(r) + \epsilon - l \in \mathcal{T}^+ = -T_\epsilon(l - r)^+ + \epsilon,$$

where $l \geq 0$. Let us denote

$$J_{T,\epsilon,l}^+(r) = \int_0^r T(s)T_\epsilon(s-l)^+\ ds,$$

$$J_{T,\epsilon,l}^-(r) = -\int_0^r T(s)T_\epsilon(l-s)^+\ ds.$$

Then, $J_{TS_{\epsilon,l}}(r) = J_{T,\epsilon,l}^+(r)$ and $J_{TS_\epsilon^l}(r) = J_{T,\epsilon,l}^-(r) + \epsilon J_T(r)$.

Let u, $\overline{u}$ be two entropy solutions of (7.82) corresponding to the initial conditions $u_0, \overline{u_0} \in \left(L^1(]0,L[)\right)^+$ respectively. Then, if $\mathbf{z}(t) := \mathbf{a}(u(t), \partial_x u(t))$, $\overline{\mathbf{z}}(t) := \mathbf{a}(\overline{u}(t), \partial_x \overline{u}(t))$ and $l_1, l_2 > \epsilon$, we have

$$\begin{aligned} &-\int_0^T \int_0^L J_{T,\epsilon,l_1}^+(u(t))\partial_t\eta(t)\ dxdt \\ &+\int_0^T \int_0^L \eta(t)[h_T(u(t), D_x S_{\epsilon,l_1}(u(t))) + h_{S_{\epsilon,l_1}}(u(t), D_x T(u(t)))]\ dt \\ &+\int_0^T \int_0^L \mathbf{z}(t)\partial_x\eta(t)T(u(t))S_{\epsilon,l_1}(u(t))\ dxdt \leq 0 \end{aligned} \tag{7.91}$$

and

$$\begin{aligned} &-\int_0^T \int_0^L J_{T,\epsilon,l_2}^-(\overline{u}(t))\partial_t\eta(t)\ dxdt - \epsilon\int_0^T \int_0^L J_T(\overline{u}(t))\partial_t\eta(t)\ dxdt \\ &+\int_0^T \int_0^L \eta(t)[h_T(\overline{u}(t), D_x S_\epsilon^{l_2}(\overline{u}(t))) + h_{S_\epsilon^{l_2}}(\overline{u}(t), D_x T(\overline{u}(t)))]\ dt \\ &+\int_0^T \int_0^L \overline{\mathbf{z}}(t)\partial_x\eta(t)T(\overline{u}(t))S_\epsilon^{l_2}(\overline{u}(t))\ dxdt \leq 0 \end{aligned} \tag{7.92}$$

for all $\eta \in C^\infty(Q_T)$ which are non-negative and factorize as $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\rho \in \mathcal{D}(]0,L[)$.

We choose two different pairs of variables (t,x), (s,y) and consider u, $\mathbf{z}$ as functions of (t,x) and $\overline{u}$, $\overline{\mathbf{z}}$ as functions of (s,y). Let $0 \leq \phi \in \mathcal{D}(]0,T[)$ and $\psi \in \mathcal{D}(]0,L[)$; let also ρ_m and $\tilde{\rho}_n$ be sequences of mollifiers in $\mathbb{R}$. Define

$$\eta_{m,n}(t,x,s,y) := \rho_m(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right).$$

Being (s,y) fixed, we substitute $l_1 = \overline{u}(s,y)$ in (8.145) to get

$$\begin{aligned}
&-\int_0^T\int_0^L J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))\partial_t\eta_{m,n}\,dxdt \\
&+\int_0^T\int_0^L \eta_{m,n}[h_T(u(t,x),D_xS_{\epsilon,\overline{u}(s,y)}(u(t,x))) + h_{S_{\epsilon,\overline{u}(s,y)}}(u(t,x),D_xT(u(t,x)))]\,dt \\
&+\int_0^T\int_0^L \mathbf{z}(t,x)\partial_x\eta_{m,n}T(u(t,x))S_{\epsilon,\overline{u}(s,y)}(u(t,x))\,dxdt \leq 0.
\end{aligned} \tag{7.93}$$

Similarly, for (t,x) fixed, if we take $l_2 = u(t,x)$ in (8.146) we get

$$\begin{aligned}
&-\int_0^T\int_0^L J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))\partial_s\eta_{m,n}\,dyds - \epsilon\int_0^T\int_0^L J_T(\overline{u}(s,y))\partial_s\eta_{m,n}\,dyds \\
&+\int_0^T\int_0^L \eta_{m,n}[h_T(\overline{u}(s,y),D_yS_\epsilon^{u(t,x)}(\overline{u}(s,y))) + h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y),D_yT(\overline{u}(s,y)))]\,ds \\
&+\int_0^T\int_0^L \overline{\mathbf{z}}(s,y)\partial_y\eta_{m,n}T(\overline{u}(s,y))S_\epsilon^{u(t,x)}(\overline{u}(s,y))\,dyds \leq 0.
\end{aligned} \tag{7.94}$$

We integrate (8.147) in (s,y) and (8.148) in (t,x). Then we add the two resulting inequalities. What we get is

$$
\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))\partial_t\eta_{m,n}+J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))\partial_s\eta_{m,n}\right)\,dsdtdydx\\
&-\epsilon\int_{Q_T\times Q_T}J_T(\overline{u}(s,y))\partial_s\eta_{m,n}\,dsdtdydx\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(u(t,x),D_xS_{\epsilon,\overline{u}(s,y)}(u(t,x)))\,dsdtdy\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(\overline{u}(s,y),D_yS_\epsilon^{u(t,x)}(\overline{u}(s,y)))\,dsdtdx\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_{S_{\epsilon,\overline{u}(s,y)}}(u(t,x),D_xT(u(t,x)))\,dsdtdy\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\,dsdtdx\\
&+\int_{Q_T\times Q_T}\mathbf{z}(tx)\partial_x\eta_{m,n}T(u(t,x))S_{\epsilon,\overline{u}(s,y)}(u(t,x))\,dsdtdydx\\
&+\int_{Q_T\times Q_T}\overline{\mathbf{z}}(s,y)\partial_y\eta_{m,n}T(\overline{u}(s,y))S_\epsilon^{u(t,x)}(\overline{u}(s,y))\,dsdtdydx\le 0.
\end{aligned}
\tag{7.95}
$$

Since

$$\int_{Q_T\times Q_T}\eta_{m,n}h_{S_{\epsilon,\overline{u}(s,y)}}(u(t,x),D_xT(u(t,x)))\,dsdtdy\ge 0$$

and

$$\int_{Q_T\times Q_T}\eta_{m,n}h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\,dsdtdx\ge 0$$

thanks to (4.18), we might neglect fifth and sixth terms above. We could do the same for the third and fourth terms, but we keep them as they will be helpful later. Next we bring in the terms

$$\int \mathbf{z}(t,x)\partial_y\eta_{m,n}T(u(t,x))S_{\epsilon,\overline{u}(s,y)}(u(t,x))\,dsdtdydx$$

and

$$\int \overline{\mathbf{z}}(s,y)\partial_x\eta_{m,n}T(\overline{u}(s,y))S_\epsilon^{u(t,x)}(\overline{u}(s,y))\,dsdtdydx$$

added and subtracted and we combine them with the seventh and eighth terms in

(7.95). Summing up all computations so far we get

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))\partial_t\eta_{m,n}+J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))\partial_s\eta_{m,n}\right)\,dsdtdydx\\
&-\epsilon\int_{Q_T\times Q_T}J_T(\overline{u}(s,y))\partial_s\eta_{m,n}\,dsdtdydx\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(u(t,x),D_xS_{\epsilon,\overline{u}(s,y)}(u(t,x)))\,dsdtdy\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_T(\overline{u}(s,y),D_yS_\epsilon^{u(t,x)}(\overline{u}(s,y)))\,dsdtdx\\
&-\int_{Q_T\times Q_T}\overline{\mathbf{z}}(s,y)\partial_x\eta_{m,n}T(\overline{u}(s,y))S_\epsilon^{u(t,x)}(\overline{u}(s,y))\,dsdtdydx\\
&-\int_{Q_T\times Q_T}\mathbf{z}(t,x)\partial_y\eta_{m,n}T(u(t,x))S_{\epsilon,\overline{u}(s,y)}(u(t,x))\,dsdtdydx\\
&+\int_{Q_T\times Q_T}T_\epsilon(u(t,x)-\overline{u}(s,y))^+[T(u(t,x))\mathbf{z}(t,x)-T(\overline{u}(s,y))\overline{\mathbf{z}}(s,y)]\\
&\qquad\times(\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\,dsdtdydx\\
&+\epsilon\int_{Q_T\times Q_T}T(\overline{u}(s,y))\overline{\mathbf{z}}(s,y)(\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\,dsdtdydx\le 0.
\end{aligned}\tag{7.96}$$

Recall that u, $\mathbf{z}$ are always functions of (t,x) and $\overline{u}$, $\overline{\mathbf{z}}$ are always functions of (s,y). From now on we shall work with more concise expressions. In order to do so, we shall omit the arguments of u, $\mathbf{z}$, $\overline{u}$ and $\overline{\mathbf{z}}$ except in some cases where we find it useful to remind them. We will also omit the differentials of the integrals.

Let I be the sum of the third up to the sixth terms of the above inequality. Working as in the proof of uniqueness of Theorem 3 in [17], we obtain that $\frac{1}{\epsilon}I\ge\|\phi\|_\infty\|\psi\|_\infty o(\epsilon)$; the techniques to obtain this result are pretty similar to those used in the proof of Lemma 6.5.13. Hence, by (8.149), it follows that

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}}(u)\partial_t\eta_{m,n}+J^-_{T,\epsilon,u}(\overline{u})\partial_s\eta_{m,n}\right)\\
&+\int_{Q_T\times Q_T}T_\epsilon(u-\overline{u})^+[T(u)\mathbf{z}-T(\overline{u})\overline{\mathbf{z}}](\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\\
&+\epsilon\int_{Q_T\times Q_T}T(\overline{u})\overline{\mathbf{z}}(\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\le-\epsilon\,o(\epsilon)+\epsilon\int_{Q_T\times Q_T}J_T(\overline{u})\partial_s\eta_{m,n}.
\end{aligned}$$

Then, dividing by ϵ and letting $\epsilon \to 0$ we get

$$-\int_{Q_T\times Q_T}\Big(J^+_{T,\mathrm{sign},\overline{u}}(u)\partial_t\eta_{m,n}+J^-_{T,\mathrm{sign},u}(\overline{u})\partial_s\eta_{m,n}\Big)$$
$$+\int_{Q_T\times Q_T}\mathrm{sign}_0^+(u-\overline{u})[T(u)\mathbf{z}-T(\overline{u})\overline{\mathbf{z}}](\partial_x\eta_{m,n}+\partial_y\eta_{m,n})$$
$$+\int_{Q_T\times Q_T}T(\overline{u})\overline{\mathbf{z}}(\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\le\int_{Q_T\times Q_T}J_T(\overline{u})\partial_s\eta_{m,n}$$

where

$$J^+_{T,sign,l}(r)=\int_0^r T(s)\mathrm{sign}_0^+(s-l)ds \quad l\in\mathbb{R},\ r\ge 0$$

and

$$J^-_{T,sign,l}(r)=-\int_0^r T(s)\mathrm{sign}_0^+(l-s)ds \quad l\in\mathbb{R},\ r\ge 0.$$

Now, letting $m\to\infty$ we obtain

$$-\int_0^T\int_0^T\int_0^L\Big(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\chi_n+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\chi_n\Big)$$
$$+\int_0^T\int_0^T\int_0^L\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))[T(u(t,x))\mathbf{z}(t,x)-T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x)]\partial_x\chi_n$$
$$+\int_0^T\int_0^T\int_0^L T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x)\partial_x\chi_n\le\int_0^T\int_0^T\int_0^L J_T(\overline{u}(s,x))\partial_s\chi_n$$

where

$$\chi_n(t,s,x):=\tilde{\rho}_n(t-s)\phi(\frac{t+s}{2})\psi(x).$$

We set $\psi=\psi_k\in\mathcal{D}(]0,L[)\uparrow\chi_{]0,L[}$ in the last expression. Taking limit as $k\to+\infty$ we obtain

$$-\int_0^T\int_0^T\int_0^L\Big(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\kappa_n(t,s)\Big)$$
$$+\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L\kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x\psi_k(x)$$
$$-\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L\kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x))\partial_x\psi_k(x)$$
$$+\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L\kappa_n(t,s)T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x)\partial_x\psi_k(x)$$
$$\le\int_0^T\int_0^T\int_0^L J_T(\overline{u}(s,x))\partial_s\kappa_n(t,s),\tag{7.97}$$

where $\kappa_n(t,s):=\tilde{\rho}_n(t-s)\phi(\frac{t+s}{2})$. Rewrite that inequality as

$$\int_0^T\int_0^T\int_0^L J_T(\overline{u}(s,x))\partial_s\kappa_n(t,s)\ge +A-B+C$$
$$-\int_0^T\int_0^T\int_0^L\Big(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\kappa_n(t,s)\Big)$$

being A, $-B$ and C the second, third and fourth terms above.

Lemma 6.7.7 *The following inequalities hold true:*

1. *for the second term above,*

$$A \geq -\beta \int_0^T \int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+) - \overline{u}(s,0_+))T(u(t,0_+))\,dtds$$

$$+c\int_0^T \int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-) - \overline{u}(s,L_-))T(u(t,L_-))\,dtds,$$

2. *for the third term above,*

$$\begin{aligned} -B \geq &\beta \int_0^T \int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+) - \overline{u}(s,0_+))T(\overline{u}(s,0_+))\,dtds \\ &-c\int_0^T \int_0^T \overline{u}(s,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-) - \overline{u}(s,L_-))T(\overline{u}(s,L_-))\,dtds, \end{aligned}$$

3. *for the fourth term above,*

$$C \geq c\int_0^T \int_0^T \overline{u}(s,L_-)\kappa_n(t,s)T(\overline{u}(s,L_-))\,dtds - \beta\int_0^T \int_0^T \kappa_n(t,s)T(\overline{u}(s,0_+))\,dtds.$$

Proof. To prove the first point, denote

$$\begin{aligned} I_k :=& \int_0^T \int_0^T \int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x) - \overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x\psi_k(x) \\ =& \int_0^T \int_0^T \int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x) - \overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x(\psi_k(x)-1). \end{aligned}$$

Set

$$H_n(s,r) := \kappa_n(r,s)\mathrm{sign}_0^+(u(r) - \overline{u}(s))T(u(r)).$$

For $\tau > 0$, we define the function $(\kappa_n(s))^\tau$ as a Dunford integral (see Remark C.2.1)

$$(\kappa_n(s))^\tau(t) := \frac{1}{\tau}\int_t^{t+\tau} H_n(s,r)\,dr.$$

In fact, this defines an s-parametric family of Dunford integrals. Note that

$$r \mapsto \kappa_n(r,s) \in \mathcal{D}(]0,T[)$$

and

$$r \mapsto \mathrm{sign}_0^+(u(r) - \overline{u}(s))T(u(r)) \in L^1_{loc}(0,T,BV(]0,L[)).$$

Then, using (ii) of Definition 6.7.1 with $w = (\kappa_n(s))^\tau(\psi_k - 1)$,

$$
\begin{aligned}
I_k = &\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (\kappa_n(s))^\tau(t)\mathbf{z}(t,x)\partial_x[\psi_k(x)-1]\,dxdtds \\
= &+\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L D_x[(\kappa_n(s))^\tau(t)(\psi_k(x)-1)]\mathbf{z}(t,x) \\
&-\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L D_x(\kappa_n(s))^\tau(t)\mathbf{z}(t,x)(\psi_k(x)-1) \\
= &-\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L [\psi_k(x)-1]\mathbf{z}(t,x)D_x((\kappa_n(s))^\tau(t)))\,dtds \\
&-\lim_{\tau\to 0}\int_0^T \langle u_t, (\kappa_n(s))^\tau(\psi_k(x)-1)\rangle\,ds \\
&+c\lim_{\tau\to 0}\int_0^T\int_0^T u(t,L_-)(\kappa_n(s))^\tau(t)(L_-)\,dtds \\
&-\beta\lim_{\tau\to 0}\int_0^T\int_0^T (\kappa_n(s))^\tau(t)(0_+)\,dtds \\
:= &I_k^1 + I_k^2 + I_k^3 + I_k^4.
\end{aligned}
$$

Notice that

$$I_k^3 = c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(u(t,L_-))\,dtds$$

and

$$I_k^4 = -\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(u(t,0_+))\,dtds.$$

By Remark 6.7.5, we get

$$|D_x((\kappa_n(s))^\tau(t))|(]0,L[) \stackrel{\tau\to 0}{\longrightarrow} |D_x(\kappa_n(t,s)\mathrm{sign}_0^+(u(t)-\overline{u}(s))T(u(t)))|(]0,L[). \tag{7.98}$$

Using (8.151), we obtain the bound

$$|I_k^1| \le c\|u\|_{L^\infty(Q_T)}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))|D_x(\kappa_n(t,s)\mathrm{sign}_0^+(u(t)-\overline{u}(s))T(u(t)))|\;dtds,$$

which implies $\lim_{k\to\infty} I_k^1 = 0$. Let us deal with I_k^2. We have

$$I_k^2 = \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L u(t,x)\frac{H_n(s,t+\tau)-H_n(s,t)}{\tau}(\psi_k(x)-1)\,dxdtds.$$

Define the function

$$q(\tau) := \text{sign}_0^+(\tau - \overline{u}(s,x))T(\tau)$$

Using Lemma 6.7.6 and the fact that $H_n(s,t) = q(u(t))\kappa_n(t,s)$ we get

$$\begin{aligned}
I_k^2 &= \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{u(t,x)-u(t-\tau,x)}{\tau}H_n(s,t)\,dxdtds \\
&\geq \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\kappa_n(t,s)\frac{J_q(u(t,x))-J_q(u(t-\tau,x))}{\tau}\,dxdtds \\
&= \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))J_q(u(t,x))\frac{\kappa_n(t,s)-\kappa_n(t+\tau,s)}{\tau}\,dxdtds \\
&= -\int_0^T\int_0^T\int_0^L (1-\psi_k(x))J_q(u(t,x))\partial_t\kappa_n(t,s)\,dxdtds,
\end{aligned}$$

from where it follows that $\lim_{k\to\infty} I_k^2 \geq 0$. Taking into account the above facts, we get a proof for the first point of the Lemma.

Now we deal with the second point. First we set

$$\begin{aligned}
&-\lim_{k\to\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\text{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x)\partial_x\psi_k(x) \\
&= -\lim_{k\to\infty}\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (\kappa_n(t))^\tau(s)\overline{\mathbf{z}}(s,x)\partial_x(\psi_k(x)-1),
\end{aligned}$$

being

$$(\kappa_n(t))^\tau(s) = \frac{1}{\tau}\int_s^{s+\tau} \text{sign}_0^+(u(t,x)-\overline{u}(r,x))T(\overline{u}(r,x))\kappa_n(t,r)\,dr.$$

To study the integral above we decompose it and use integration by parts as follows:

$$\begin{aligned}
-\int_0^T\int_0^T\int_0^L & (\kappa_n(t))^\tau(s)\overline{\mathbf{z}}(s,x)\partial_x(\psi_k(x)-1) \\
&= \int_0^T\int_0^T\int_0^L [\psi_k(x)-1]\overline{\mathbf{z}}(s,x)D_x((\kappa_n(t))^\tau(s)))\,dsdt \\
&\quad + \int_0^T \langle \overline{u}_s, (\kappa_n(t))^\tau(s)(\psi_k(x)-1)\rangle\,dt \\
&\quad - c\int_0^T\int_0^T \overline{u}(s,L_-)(\kappa_n(t))^\tau(s)(L_-)\,dtds \\
&\quad + \beta\int_0^T\int_0^T (\kappa_n(t))^\tau(s)(0_+)\,dtds.
\end{aligned}$$

All the terms are treated in the same way as before, except the second one. This one can be written down as

$$\begin{aligned}
&\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{\overline{u}(s-\tau,x)-\overline{u}(s,x)}{\tau}\text{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\kappa_n(t,s) \\
&= -\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{\overline{u}(s-\tau,x)-\overline{u}(s,x)}{\tau}\kappa_n(t,s)\left(\overline{u}(s-\tau,x)+\overline{u}(s,x)\right) \\
&\quad + \int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{\overline{u}(s-\tau,x)-\overline{u}(s,x)}{\tau}\kappa_n(t,s) \\
&\qquad \times\left[\text{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))+(\overline{u}(s-\tau,x)+\overline{u}(s,x))\right] := I_1 + I_2.
\end{aligned}$$

We show that the first term will converge to zero after taking the limits in τ and k, as

$$\begin{aligned}-I_1 &= \int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{\overline{u}(s-\tau,x)^2-\overline{u}(s,x)^2}{\tau}\kappa_n(t,s)\\ &= \int_0^T\int_0^T\int_0^L (1-\psi_k(x))\overline{u}(s,x)^2\frac{\kappa_n(t,s+\tau)-\kappa_n(t,s)}{\tau}.\end{aligned}$$

To deal with the second term, notice that the mapping

$$q(r) = \left[\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))+r+\overline{u}(s,x))\right]$$

is non-decreasing in r, so that defining $Q(r)=\int_0^r q(s)\,ds$ we get

$$Q(b)-Q(a)\geq (b-a)q(a)$$

or

$$Q(a)-Q(b)\leq (a-b)q(a).$$

Choose now $a=\overline{u}(s-\tau)$ and $b=\overline{u}(s)$, which leads us to

$$I_2 \geq \int_0^T\int_0^T\int_0^L (1-\psi_k(x))\kappa_n(t,s)\frac{Q(\overline{u}(s-\tau))-Q(\overline{u}(s))}{\tau}.$$

This enables us to show, as it has been done before, that

$$\lim_{k\to\infty}\lim_{\tau\to 0} I_2\geq 0,$$

which concludes the proof of the second point of the Lemma.

The proof of the third point is performed in a similar way. □

From (8.150), by Lemma 6.7.7 we have

$$\begin{aligned}&-\int_0^T\int_0^T\int_0^L \left(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\kappa_n(t,s)\right)\,dtdsdx\\ &+c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(u(t,L_-))\,dtds\\ &-c\int_0^T\int_0^T \overline{u}(s,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(\overline{u}(s,L_-))\,dtds\\ &-\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(u(t,0_+))\,dtds\\ &+\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(\overline{u}(s,0_+))\,dtds\\ &+c\int_0^T\int_0^T \overline{u}(s,L_-)\kappa_n(t,s)T(\overline{u}(s,L_-))\,dtds-\beta\int_0^T\int_0^T \kappa_n(t,s)T(\overline{u}(s,0_+))\,dtds\\ &\leq \int_0^T\int_0^T\int_0^L J_T(\overline{u}(s,x))\partial_s\kappa_n(t,s)\ dtdsdx.\end{aligned}\tag{7.99}$$

By Lemma 6.7.3, we have

$$u(t,0_+) \geq \frac{\beta}{c} > 0, \qquad \overline{u}(s,0_+) \geq \frac{\beta}{c} > 0 \qquad \text{for almost all } t,s>0. \tag{7.100}$$

Now we let $a \to 0$, then we divide by b and finally we let $b \to 0$ in (8.155). We have that $J^+_{T,\text{sign},\overline{u}(s,x)}(u(t,x))$ converges to $(u(t,x)-\overline{u}(s,x))^+$, while $J^-_{T,\text{sign},u(t,x)}(\overline{u}(s,x))$ does to $(u(t,x)-\overline{u}(s,x))^+ - u(t,x)$. Thus, after these processes (8.155) has become

$$\begin{aligned}
&-\int_0^T\int_0^T\int_0^L (u(t,x)-\overline{u}(s,x))^+(\partial_t\kappa_n(t,s)+\partial_s\kappa_n(t,s))\ dtdsdx\\
&+c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))\mathrm{sign}_0^+(u(t,L_-))\,dtds\\
&-c\int_0^T\int_0^T \overline{u}(s,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))\mathrm{sign}_0^+(\overline{u}(s,L_-))\,dtds\\
&-\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))[\mathrm{sign}_0(u(t,0_+)-\mathrm{sign}_0(\overline{u}(s,0_+))]dtds\\
&+c\int_0^T\int_0^T \overline{u}(s,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(\overline{u}(s,L_-))\,dtds-\beta\int_0^T\int_0^T\kappa_n(t,s)\,dtds\\
&\leq \int_0^T\int_0^T\int_0^L \overline{u}(s,x)\partial_s\kappa_n(t,s)\ dtdsdx.
\end{aligned} \tag{7.101}$$

Now we analyze term by term. Having in mind (8.156), the fourth term in (7.101) vanishes. Moreover, the sum of the second and third terms in (7.101) is non-negative (argue case by case). Finally, using that $\overline{u}_s = D_x(\overline{\mathbf{z}})$ in the sense given in (ii) of Definition 6.7.1 with $w = w(s) = \kappa_n(s,t)$ —which does not depend on x and has t as a mute variable —, it follows that

$$\begin{aligned}
\int_0^T\int_0^T\int_0^L \overline{u}(s,x)\partial_s\kappa_n(t,s)\,dxdtds &= -\int_0^T \langle \overline{u}_s, \kappa_n(\cdot,t)\rangle\,dt\\
&= c\int_0^T\int_0^T \overline{u}(s,L_-)\kappa_n(t,s)\,dtds-\beta\int_0^T\int_0^T\kappa_n(t,s)\,dtds.
\end{aligned}$$

Therefore, this contribution vanishes when combined with the fifth and sixth terms in (7.101). All together yields

$$-\int_0^T\int_0^T\int_0^L (u(t,x)-\overline{u}(s,x))^+(\partial_t\kappa_n(t,s)+\partial_s\kappa_n(t,s))\ dtdsdx \leq 0.$$

Letting $n\to\infty$,

$$-\int_0^T\int_0^L (u(t,x)-\overline{u}(t,x))^+\phi'(t)\ dxdt \leq 0.$$

Since this is true for all $0 \leq \phi \in \mathcal{D}(]0,T[)$, we have

$$\frac{d}{dt}\int_0^L (u(t,x)-\overline{u}(t,x))^+\,dx \leq 0.$$

Hence

$$\int_0^L (u(t,x) - \overline{u}(t,x))^+ \, dx \le \int_0^L (u_0(x) - \overline{u}_0(x))^+ \, dx \qquad \text{for all } \; t \ge 0,$$

which finishes the uniqueness part.

6.7.2 Existence of bounded entropy solution

Given $0 \le u_0 \in L^1(]0,L[)$, let $u(t) = T_\beta(t)u_0$, being $(T_\beta(t))_{t\ge 0}$ the semigroup in $L^1(]0,L[)^+$ generated by the accretive operator $\mathcal{B}_\beta$. Then, according to the general theory of nonlinear semigroups, we have that $u(t)$ is a mild solution of the abstract Cauchy problem

$$u'(t) + \mathcal{B}_\beta u(t) \ni 0, \qquad u(0) = u_0.$$

Let us prove that, assuming $0 \le u_0 \in L^\infty(]0,L[)$, then this mild solution u is also a bounded entropy solution of (7.82) in Q_T. We divide the proof in several steps.

Step 1. Approximation with Crandall-Ligget's scheme.

Let $T > 0$, $K \in \mathbb{N}$, $\Delta t = \frac{T}{K}$, $t_n = n\Delta t$, $n = 0, \ldots, K$. We define inductively u^{n+1}, $n = 0, \ldots, K-1$, to be the unique entropy solution of the problem

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \left(\mathbf{a}(u^{n+1}, (u^{n+1})')\right)' = 0 & \text{in } \;]0,L[\\ -\mathbf{a}(u^{n+1}(0), (u^{n+1})'(0)) = \beta > 0 & \text{and } \; u^{n+1}(L_-) = 0 \end{cases} \tag{7.102}$$

in the sense of Definition 6.5.1, where $u^0 = u_0$.

If we set

$$u^K(t) := u^0 \chi_{[0,t_1]}(t) + \sum_{n=1}^{K-1} u^n \chi_{]t_n,t_{n+1}]}(t),$$

we get that u^K converges uniformly to $u \in C([0,T], L^1(]0,L[))$ as $K \to \infty$, thanks to Crandall-Liggett's Theorem.

We also define

$$\xi^K(t) := \sum_{n=0}^{K-1} \frac{u^{n+1} - u^n}{\Delta t} \chi_{]t_n,t_{n+1}]}(t)$$

and

$$\mathbf{z}^K(t) := \mathbf{a}(u^1, (u^1)')\chi_{[0,t_1]}(t) + \sum_{n=1}^{K-1} \mathbf{a}(u^{n+1}, (u^{n+1})')\chi_{]t_n,t_{n+1}]}(t).$$

Since u^{n+1} is the entropy solution of (7.102), we have the relations

$$\xi^K(t) = D_x \mathbf{z}^K(t) \;\; \text{in} \;\; \mathcal{D}'(]0,L[), \;\; \forall t \in]0,T] \tag{7.103}$$

$$\mathbf{z}^K(t)(L) = -c u^K(t+\Delta t)(L_-), \;\; \forall t \in]0, T-\Delta t] \tag{7.104}$$

$$-\mathbf{z}^K(t)(0) = \beta, \quad \forall t \in [0,T]. \tag{7.105}$$

We also have that for all $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and $\forall t \in]0, T-\Delta t]$, the following inequalities are satisfied:

$$h(u^K(t+\Delta t), D_x T(u^K(t+\Delta t)) \le \mathbf{z}^K(t) D_x T(u^K(t+\Delta t)) \quad \text{as measures} \tag{7.106}$$

$$h_S(u^K(t+\Delta t), D_xT(u^K(t+\Delta t)) \leq \mathbf{z}^K(t)D_xJ_{T'S}(u^K(t+\Delta t)) \quad \text{as measures.} \tag{7.107}$$

Note that (7.106) is equivalent to

$$\frac{c}{2}|D_x^s([T(u^K(t+\Delta t))]^2)| \leq \mathbf{z}^K(t)D_x^sT(u^K(t+\Delta t)) \quad \text{as measures.} \tag{7.108}$$

In the same way, (7.107) is equivalent to

$$|D^s(J_{S\theta}(T(u^K(t+\Delta t)))| \leq \mathbf{z}^K(t)D^sJ_{T'S}(u^K(t+\Delta t)) \quad \text{as measures.} \tag{7.109}$$

Note also that using (4.12) and (4.21) we can write

$$\begin{aligned} h(u^{n+1}, D_xT(u^{n+1})) &= \mathbf{a}(u^{n+1},(u^{n+1})')(T(u^{n+1}))'\mathcal{L}^1 + \frac{c}{2}|D_x^s[(T(u^{n+1}))^2]| \\ &\geq \frac{c}{2}|((T(u^{n+1}))^2)'|\mathcal{L}^1 - \frac{c^2}{\nu}(T(u^{n+1}))^2\mathcal{L}^1 + \frac{c}{2}|D_x^s[(T(u^{n+1}))^2]| \\ &= \frac{c}{2}|D[(T(u^{n+1}))^2]| - \frac{c^2}{\nu}(T(u^{n+1}))^2\mathcal{L}^1; \end{aligned}$$

since $\mathbf{a}(u^{n+1},(u^{n+1})')D_xT(u^{n+1}) \geq h(u^{n+1}, D_xT(u^{n+1}))$ as measures in $]0,L[$ we get the following inequality as measures

$$\mathbf{z}^K(t)D_xT(u^K(t+\Delta t)) \geq \frac{c}{2}|D_x([T(u^K(t+\Delta t))]^2)| - \frac{c^2}{\nu}(T(u^K(t+\Delta t)))^2. \tag{7.110}$$

Lemma 6.7.8 *There exists $M := M(\beta, c, \nu, L, \|u_0\|_\infty)$ such that*

$$\|u^K(t)\|_\infty \leq M \qquad \forall\, K \in \mathbb{N} \text{ and } \quad \forall t \in [0,T]. \tag{7.111}$$

Consequently, $\|u(t)\|_\infty \leq M \quad \forall t \in [0,T]$.

Proof. Since

$$(I + \Delta t\mathcal{B}_\beta)^{-1}(u^n) = u^{n+1}, \qquad \text{for } n = 0, \ldots, K-1,$$

setting $\mu := \max\{\frac{c\|u_0\|_\infty}{\beta}, 1\}$ and using Proposition 6.6.5 we get that

$$0 \leq u^1 = (I + \Delta t\mathcal{B}_\beta)^{-1}(u_0) \leq \mu u_\beta.$$

Then, repeating this process, we obtain

$$\begin{aligned} 0 \leq u^{n+1} &= (I + \Delta t\mathcal{B}_\beta)^{-1}(u^n) \leq (I + \Delta t\mathcal{B}_\beta)^{-1}(\mu u_\beta) \\ &= \mu\left(I + \Delta t\mathcal{B}_{\frac{\beta}{\mu}}\right)^{-1}(u_\beta) \leq \mu(I + \Delta t\mathcal{B}_\beta)^{-1}(u_\beta) = \mu u_\beta \end{aligned}$$

and the statement follows.

□

Step 2. Passage to the limit.

Lemma 6.7.8 assures that $\|\mathbf{z}^K(t)\|_\infty \leq C$ for all $K \in \mathbb{N}$ and a.e. $t \in [0,T]$. Then we may assume that

$$\mathbf{z}^K \rightharpoonup \mathbf{z} \in L^\infty(Q_T) \quad \text{weakly}^*. \tag{7.112}$$

Moreover, since u^K converges uniformly to u in $C([0,T],L^1(]0,L[))$ and

$$\mathbf{z}^K(t) = c\,u^K(t+\Delta t)\mathbf{b}(u^K(t+\Delta t),\partial_x u^K(t+\Delta t)) \quad \forall t \in]0,T-\Delta t],$$

with $\|\mathbf{b}(u^K(t+\Delta t),\partial_x u^K(t+\Delta t))\|_\infty \leq 1$, we may also assume that

$$\mathbf{b}(u^K(t+\Delta t),\partial_x u^K(t+\Delta t)) \rightharpoonup \mathbf{z}_\mathbf{b}(t) \in L^\infty(Q_T) \text{ weakly}^*$$

and

$$\mathbf{z}(t) = c\,u(t)\mathbf{z}_\mathbf{b}(t) \qquad \text{for almost all } \ t \in [0,T]. \tag{7.113}$$

Given $w \in BV(]0,L[)$, it follows from (7.103) and (7.111) that for each $t \in]0,T]$,

$$\left|\int_0^L \xi^K(t,x)w(x)\,dx\right| = \left|-\int_0^L \mathbf{z}^K(t)Dw + \mathbf{z}^K(L)w(L_-) + \beta w(0_+)\right|$$

$$\leq C\|w\|_{BV(]0,L[)} + |\mathbf{z}^K(L)w(L_-)| \leq (C + c\mu\|u_\beta\|_\infty)\|w\|_{BV(]0,L[)},$$

Here the continuous injection of $BV(]0,L[)$ into $L^\infty(]0,L[)$ was used. Thus,

$$\|\xi^K(t)\|_{BV(]0,L[)^*} \leq C, \ \forall \ K \in \mathbb{N} \text{ and } t \in]0,T].$$

Consequently, $\{\xi^K\}$ is a bounded sequence in $L^\infty(0,T;BV(]0,L[)^*)$. Now, since $L^\infty(0,T;BV(]0,L[)^*)$ is a vector subspace of the dual space $\left(L^1(0,T;BV(]0,L[))\right)^*$, we can find a subnet ξ^α of ξ^K such that

$$\xi^\alpha \rightharpoonup \xi \in \left(L^1(0,T;BV(]0,L[))\right)^* \quad \text{weakly}^*.$$

Now we are to identify this limit ξ. We recall that not every subnet of a sequence qualifies a a subsequence of the original one, but nevertheless any quantities that converge for $K \to \infty$ converge also along the subnet α.

Lemma 6.7.9 *The limit ξ is the time derivative u_t of u in $\mathcal{D}'(Q_T)$ and (7.84) holds.*

Proof. Let $\psi \in L^1(0,T;BV(]0,L[))$ compactly supported in time and such that there exists $\Theta \in L^1_w(0,T;BV(]0,L[)) \cap L^\infty(Q_T)$ verifying $\psi(t) = \int_0^t \Theta(s)\,ds$, the integral being a Pettis integral, as in (7.84) . Then, for Δt small enough

$$\int_0^T\int_0^L \sum_{n=0}^{K-1} \frac{u^{n+1}-u^n}{\Delta t}\chi_{]t_n,t_{n+1}]}(t)\psi(t)\,dxdt = \int_0^T\int_0^L \frac{u^K(t+\Delta t)-u^K(t)}{\Delta t}\psi(t)\,dxdt$$

$$= -\frac{1}{\Delta t}\int_0^{t_1}\int_0^L u^0\psi(t)\,dxdt + \int_{t_1}^{t_K}\int_0^L u^K(t)\frac{\psi(t-\Delta t)-\psi(t)}{\Delta t}\,dxdt.$$

Taking limits along α we obtain

$$\int_0^T \langle\psi(t),\xi(t)\rangle\,dt = -\int_0^T\int_0^L u(t,x)\Theta(t,x)\,dxdt. \tag{7.114}$$

Substitute now $\psi \in \mathcal{D}(Q_T)$ into (7.114). Taking into account that

$$\psi(t) = \int_0^t \Theta(s)\,ds, \quad \Theta(s) = \frac{\partial\psi}{\partial s}$$

conclusion follows. □

Step 3. Fulfillment of the equation.

We begin with

Lemma 6.7.10 *The relation*

$$u_t = D_x \mathbf{z} \tag{7.115}$$

holds in $\mathcal{D}'(Q_T)$.

Proof. In fact, given $\psi \in \mathcal{D}(Q_T)$, we have that $\psi \in L^1(0,T;BV(]0,L[))$. Then, by (7.114) and (7.112), we get

$$\begin{aligned}\langle u_t, \psi\rangle &= \lim_\alpha \langle \xi^\alpha, \psi\rangle = \lim_\alpha \int_0^T \int_0^L \psi(t)\xi^\alpha(t)\,dxdt = \lim_\alpha \int_0^T \int_0^L \psi(t) D_x \mathbf{z}^\alpha(t)\,dxdt \\ &= -\lim_\alpha \int_0^T \int_0^L \partial_x \psi(t)\mathbf{z}^\alpha(t)\,dxdt = -\int_0^T \int_0^L \partial_x\psi(t)\mathbf{z}\,dxdt = \langle \psi, D_x \mathbf{z}\rangle\end{aligned}$$

and (7.115) holds (the representation of the duality product is done by means of Remark C.2.2). □

Next, we are going to prove that $u_t = D_x\,\mathbf{z}$ holds in the stronger sense given by Definition 6.7.1. To do this, let us first prove:

Lemma 6.7.11 *The distribution* $\mathbf{z}D_x w$ *in* Q_T *defined by (7.83) is a Radon measure in* Q_T *for all* $w \in L^1(0,T;BV(]0,L[))$.

Proof. Let $\varphi \in \mathcal{D}(Q_T)$, then

$$\begin{aligned}\langle \mathbf{z}Dw, \varphi\rangle &= -\langle u_t, \varphi w\rangle - \int_0^T \int_0^L \mathbf{z}(t,x)w(t,x)\partial_x\varphi(t,x)\,dxdt \\ &= -\langle u_t - \xi^\alpha, \varphi w\rangle - \langle \xi^\alpha, \varphi w\rangle - \int_0^T \int_0^L \mathbf{z}(t,x)w(t,x)\partial_x\varphi(t,x)\,dxdt \\ &= -\langle u_t - \xi^\alpha, \varphi w\rangle - \int_0^T \int_0^L D_x\mathbf{z}^\alpha(t)\varphi(t)w(t)\,dxdt \\ &\quad - \int_0^T \int_0^L \mathbf{z}(t,x)w(t,x)\partial_x\varphi(t,x)\,dxdt\end{aligned}$$

and integrating by parts

$$\begin{aligned}\langle \mathbf{z}Dw, \varphi\rangle = &-\langle u_t - \xi^\alpha, \varphi w\rangle + \int_0^T \int_0^L \mathbf{z}^\alpha(t,x)D_x w(t,x)\varphi(t,x)\,dxdt \\ &+ \int_0^T \int_0^L (\mathbf{z}^\alpha(t,x) - \mathbf{z}(t,x))\,w(t,x)\partial_x\varphi(t,x)\,dxdt.\end{aligned}$$

Then, taking limit in α and having in mind (7.112) we obtain

$$\langle \mathbf{z}Dw, \varphi\rangle = \lim_\alpha \int_0^T \int_0^L \mathbf{z}^\alpha(t,x)D_x w(t,x)\varphi(t,x)\,dxdt. \tag{7.116}$$

Therefore, we have

$$|\langle \mathbf{z}Dw, \varphi\rangle| \le \|\varphi\|_\infty cM \int_0^T |D_x w(t)|(]0,L[)\,dt.$$

Hence, $\mathbf{z}Dw$ is a Radon measure in Q_T. □

Lemma 6.7.12 *The relation $u_t = D_x \mathbf{z}$ holds in the sense given by (ii) of Definition 6.7.1.*

Proof. It follows from (7.116) and integrating by parts that

$$\int_{Q_T} \mathbf{z} D_x w = \lim_\alpha \int_0^T \int_0^L \mathbf{z}^\alpha(t) D_x w(t)\, dt = -\lim_\alpha \int_0^T \int_0^L w(t,x) D_x \mathbf{z}^\alpha(t,x)\, dxdt$$

$$+\lim_\alpha \left[\int_0^T \mathbf{z}^\alpha(t,L) w(t,L_-)\, dt - \int_0^T \mathbf{z}^\alpha(0) w(t,0_+)\, dt \right].$$

Using (7.104) this becomes

$$\int_{Q_T} \mathbf{z} D_x w = \lim_\alpha \left[-\langle \xi^\alpha, w\rangle - c\int_0^T u^\alpha(t+\Delta t)(L_-) w(t,L_-)\, dt + \beta \int_0^T w(t,0_+)\, dt \right]$$

$$= -\langle u_t, w\rangle - c\int_0^T u(t)(L_-) w(t,L_-)\, dt + \beta \int_0^T w(t,0_+)\, dt.$$

The convergence of the boundary term followed from the fact that $w(t,L_-) \in L^1(]0,T[)$ together with

$$u^\alpha(t+\Delta t, L_-) \in L^\infty(]0,T[) \quad \text{uniformly on} \quad \alpha,$$

which is due to (7.111) and the definition of trace. So finally (7.85) holds. □

Step 4. Regularity and some auxiliary inequalities.

Let $T = T_{a,b}$ be any cut-off function and let $0 \le \phi \in \mathcal{D}(Q_T)$. We multiply (7.102) by $T(u^{n+1})\phi(t)$, $t \in]t_n, t_{n+1}]$ and integrate in $]t_n, t_{n+1}] \times]0, L[$. Adding from $n = 0$ to $n = K-1$, we have

$$\sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \int_0^L \frac{u^{n+1}-u^n}{\Delta t} \phi T(u^{n+1})\, dxdt + \int_0^T \int_0^L \mathbf{z}^K(t) D_x(\phi T(u^K(t+\Delta t)))\, dt = 0. \tag{7.117}$$

Since ϕ has compact support in time in $]0,T[$, using Lemma 6.7.6 for K large enough we have

$$-\sum_{n=0}^{K-1} \int_{t_n}^{t_{n+1}} \int_0^L \frac{u^{n+1}-u^n}{\Delta t} T(u^{n+1})\phi\, dxdt \le \int_0^T \int_0^L J_T(u^K(t)) \frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\, dxdt.$$

Hence, from (7.117) it follows that

$$\int_0^T \int_0^L \mathbf{z}^K(t) D_x(\phi T(u^K(t+\Delta t)))\, dt \le \int_0^T \int_0^L J_T(u^K(t)) \frac{\phi(t)-\phi(t-\Delta t)}{\Delta t}\, dxdt. \tag{7.118}$$

Assume now that $0 \le \phi \in \mathcal{D}(]0,T[)$. Arguing as before, we have

$$\int_0^L \frac{u^{n+1}-u^n}{\Delta t} \phi(t) T(u^{n+1})\, dx + \int_0^L \mathbf{a}(u^{n+1}, (u^{n+1})') D_x[\phi(t) T(u^{n+1})]$$

$$= \beta \phi(t) T(u^{n+1}(0_+)) - c\,\phi(t) u^{n+1}(L_-) T(u^{n+1}(L_-)).$$

Adding from $n = 0$ to $n = K - 1$ and integrating in time we arrive to

$$\int_0^T \int_0^L \mathbf{z}^K(t)\phi(t) D_x T(u^K(t+\Delta t))\, dt$$

$$\leq \int_0^T \int_0^L J_T(u^K(t)) \frac{\phi(t) - \phi(t-\Delta t)}{\Delta t}\, dxdt + \int_0^T \beta\phi(t) T(u^K(t+\Delta t, 0_+))\, dt. \quad (7.119)$$

As a first application of these computations we can prove the following regularity property for the solution we are constructing.

Lemma 6.7.13 *The function $u(t,x)$ defined as the limit of Crandall-Liggett approximations verifies*

$$T(u(\cdot)) \in L^1_{loc,w}(0,T,BV(]0,L[)) \quad (7.120)$$

for any $T \in \mathcal{T}_r$.

Proof. Given $\epsilon > 0$, we take into (7.119) any test $0 \leq \phi \in \mathcal{D}(]0,T[)$ such that $\phi(t) = 1$ for $t \in]\epsilon, T-\epsilon[$. Having in mind (7.106) and (7.111), we get

$$\int_\epsilon^{T-\epsilon} \int_0^L \mathbf{z}^K(t) D_x T(u^K(t+\Delta t))\, dt$$

$$\leq \int_0^T \int_0^L J_T(u^K(t)) \frac{\phi(t) - \phi(t-\Delta t)}{\Delta t}\, dxdt + \int_0^T \beta T(u^K(t, 0_+))\, dt \leq C.$$

On the other hand, by (7.110)

$$\int_\epsilon^{T-\epsilon} \int_0^L \mathbf{z}^K(t) D_x T(u^K(t+\Delta t))\, dt$$
$$\geq \frac{c}{2} \int_\epsilon^{T-\epsilon} \int_0^L |D_x([T(u^K(t+\Delta t))]^2)|\, dt - \int_\epsilon^{T-\epsilon} \int_0^L \frac{c^2}{\nu} (T(u^K(t+\Delta t)))^2\, dt.$$

Hence

$$\int_\epsilon^{T-\epsilon} \int_0^L |D_x[(T(u^K(t+\Delta t)))^2]|\, dt \leq \frac{2C}{c} + \frac{2cLTb^2}{\nu} = C.$$

Using the coarea formula it follows that

$$\begin{aligned}
\int_0^L |D_x T_{a,b}(u^K(t+\Delta t))|\, dx &= \int_a^b |D_x \chi_{[T_{a,b}(u^K(t+\Delta t,\cdot)) \leq \lambda]}|(]0,L[)\, d\lambda \\
&= \int_a^b |D_x \chi_{[\{T_{a,b}(u^K(t+\Delta t,\cdot))\}^2 \leq \lambda^2]}|(]0,L[)\, d\lambda \\
&= \int_{a^2}^{b^2} |D_x \chi_{[\{T_{a,b}(u^K(t+\Delta t,\cdot))\}^2 \leq s]}|(]0,L[) \frac{ds}{2\sqrt{s}} \\
&\leq \frac{1}{2a} \int_0^L |D_x[T_{a,b}(u^K(t+\Delta t))]^2|)
\end{aligned}$$

and this entails the estimate

$$\int_\epsilon^{T-\epsilon} \int_0^L |D_x T(u^K(t+\Delta t))|\, dt \leq C. \quad (7.121)$$

Moreover, the map $t \mapsto \|T(u^K(t))\|_{BV(]0,L[)}$ is measurable (Lemma 5 of [14]). Then by Fatou's Lemma and (7.121), it follows that

$$\int_{\epsilon}^{T-\epsilon} \liminf_{K\to\infty} \int_0^L |D_x T(u^K(t+\Delta t))| \, dt \leq \liminf_{K\to\infty} \int_{\epsilon}^{T-\epsilon} \int_0^L |D_x T(u^K(t+\Delta t))| \, dt \leq C. \tag{7.122}$$

Now, since the total variation is lower semi-continuous in $L^1(]0,L[)$, we have

$$\int_0^L |D_x T(u(t))| \leq \liminf_{K\to\infty} \int_0^L |D_x T(u^K(t))|.$$

Thus, we deduce that $T(u(t)) \in BV(]0,L[)$ for almost all $t \in]0,T[$ and consequently $u(t) \in TBV^+(]0,L[)$. Then, by (7.122), applying again Lemma 5 of [14], we reach the conclusion of the Lemma. □

Step 5. Identification of the field.

Let us now prove that

$$\mathbf{z}(t) = \mathbf{a}(u(t), \partial_x u(t)) \quad \text{a.e. } t \in]0,T[. \tag{7.123}$$

Let $0 \leq \phi \in \mathcal{D}(Q_T)$ and $g \in C^2([0,L])$. Assume that $\phi = \eta(t)\rho(x)$ with $\eta \in \mathcal{D}(]0,T[)$ and $\rho \in \mathcal{D}(]0,L[)$. Let $0 < a < b$ and $T = T_{a,b}$. Recall that $T'(r)$ means $\chi_{]a,b[}(r)$. Recall also that

$$J_{\mathbf{a}}(x,r) = \int_0^r \mathbf{a}(s, g'(x)) \, ds \quad \text{and} \quad J_{\mathbf{a}'}(x,r) = \int_0^r \partial_x[\mathbf{a}(s, g'(x))] \, ds$$

For simplicity, we will use the following notation

$$D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) := D_x \left[J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) \right] - J_{\mathbf{a}'}(x, T(u^K(t+\Delta t))).$$

Using Volpert's averaged superposition

$$\overline{\mathbf{a}}(T(u^K(t+\Delta t)), g'(x)) = \int_0^1 \mathbf{a}(\tau T(u^K(t+\Delta t))^+ + (1-\tau)T(u^K(t+\Delta t))^-, g'(x)) \, d\tau$$

and the chain rule we can write

$$\begin{aligned} D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) &= \mathbf{a}(T(u^K(t+\Delta t)), g')\partial_x T(u^K(t+\Delta t)) \\ &\quad + \overline{\mathbf{a}}(T(u^K(t+\Delta t)), g') D_x^s T(u^K(t+\Delta t)). \end{aligned}$$

As a consequence we find out that

$$[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^{ac} = \mathbf{a}(u^K(t+\Delta t), g')\partial_x[T(u^K(t+\Delta t))]. \tag{7.124}$$

Using (7.124) we have that

$$\int_0^T \int_0^L \phi\, \mathbf{z}^K(t) D_x[T(u^K(t+\Delta t)) - g]\, dt$$

$$- \int_0^T \int_0^L \phi \left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), g')g'\, dx \right] dt$$

$$= \int_0^T \int_0^L \phi \left[\mathbf{z}^K(t) D_x T(u^K(t+\Delta t))\ dx - \mathbf{z}^K(t) g' + \mathbf{a}(u^K(t+\Delta t), g')g'\, dx \right] dt$$

$$- \int_0^T \int_0^L \phi \left\{ [D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^{ac}\ dx + [D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^s \right\} dt$$

$$= \int_0^T \int_0^L \phi \left(\mathbf{a}(u^K(t+\Delta t), g') - \mathbf{z}^K(t) \right) \left(g' - \partial_x T(u^K(t+\Delta t)) \right) dxdt$$

$$+ \int_0^T \int_0^L \phi \left[\mathbf{z}^K(t) D_x^s T(u^K(t+\Delta t)) - [D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^s \right] dt,$$

which, thanks to (4.13) is bounded from below by

$$\int_0^T \int_0^L \phi \left[\mathbf{z}^K(t) D_x^s T(u^K(t+\Delta t))) - [D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^s \right] dt.$$

Then, using (7.108) we obtain that

$$\int_0^T \int_0^L \phi\, \mathbf{z}^K(t) D_x[T(u^K(t+\Delta t)) - g]\, dt$$

$$- \int_0^T \int_0^L \phi \left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), g')g'\ dx \right] dt$$

$$\geq \int_0^T \int_0^L \phi \left[\frac{c}{2} \left| D_x^s (T(u^K(t+\Delta t))^2) \right| - [D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^s \right] dt.$$

Performing as in the proof of Lemma 6.6.9, we get

$$\int_0^T \int_0^L \phi \left[\frac{c}{2} \left| D_x^s (T(u^K(t+\Delta t))^2) \right| - [D_2 (J_{\mathbf{a}}(x, T(u^K(t+\Delta t)))]^s \right] dt \geq 0.$$

Collecting everything so far we arrive to

$$\begin{aligned} &\int_0^T \int_0^L \phi\, \mathbf{z}^K(t) D_x[T(u^K(t+\Delta t)) - g]\, dt \\ &- \int_0^T \int_0^L \phi \left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), g')g'\ dx \right] dt \geq 0. \end{aligned} \tag{7.125}$$

Now we shall bound from above the first term. This is done by means of Lemma 6.7.6. Use (7.103) for Δt small enough to get

$$\int_0^T \int_0^L \phi(t,x) T(u^K(t+\Delta t)) D_x \mathbf{z}^K(t) dt = \int_0^T \int_0^L \phi(t,x) T(u^K(t+\Delta t)\xi^K(t)\, dxdt$$

$$\geq \int_0^T \int_0^L \frac{\phi(t-\Delta t, x) - \phi(t,x)}{\Delta t} J_T(u^K(t))\, dxdt.$$

Then, integrating by parts, we have

$$\int_0^T \int_0^L \phi\, \mathbf{z}^K(t) D_x(T(u^K(t+\Delta t)) - g)\, dt \leq - \int_0^T \int_0^L \frac{\phi(t-\Delta t) - \phi(t)}{\Delta t} J_T(u^K(t))\, dxdt$$

$$+ \int_0^T \int_0^L \phi g\ \xi^K(t)\, dtdx - \int_0^T \int_0^L \partial_x \phi\, \mathbf{z}^K(t) [T(u^K(t+\Delta t)) - g]\, dxdt.$$

Thanks to this inequality we arrive from (7.125) to

$$\begin{aligned} &- \int_0^T \int_0^L \frac{\phi(t-\Delta t) - \phi(t)}{\Delta t} J_T(u^K(t))\, dtdx + \int_0^T \int_0^L \phi(t) g \xi^K(t)\, dxdt \\ &- \int_0^T \int_0^L \partial_x \phi(t)\ \mathbf{z}^K(t) [T(u^K(t+\Delta t)) - g]\, dxdt \\ &- \int_0^T \int_0^L \phi(t) \left[D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) - \mathbf{a}(u^K(t+\Delta t), g') g' \right]\, dt \geq 0. \end{aligned} \tag{7.126}$$

Taking limit along α in (7.126) and having in mind that

$$D_2 J_{\mathbf{a}}(x, T(u^K(t+\Delta t))) \rightharpoonup D_2 J_{\mathbf{a}}(x, T(u(t))) \quad \text{weakly}^* \text{ as measures}$$

we obtain

$$\begin{aligned} &\int_0^T \int_0^L \partial_t \phi J_T(u(t))\, dxdt + \langle u_t, \phi g \rangle \ - \int_0^T \int_0^L [T(u(t)) - g] \mathbf{z}(t) \partial_x \phi\, dxdt \\ &+ \int_0^T \int_0^L \phi \left[-D_2 J_{\mathbf{a}}(x, T(u(t))) + \mathbf{a}(u(t), g') g'\ dx \right]\ dt \geq 0. \end{aligned} \tag{7.127}$$

By (7.85),

$$\langle u_t, \phi g \rangle = - \int_0^T \int_0^L \mathbf{z}(t) g\, \partial_x \phi\, dxdt - \int_0^T \int_0^L \mathbf{z}(t) g' \phi\, dxdt$$

and we can rearrange (7.127) in the following way

$$\begin{aligned} &\int_0^T \int_0^L \partial_t \phi J_T(u(t))\, dxdt - \int_0^T \int_0^L \mathbf{z}(t) g' \phi\, dxdt - \int_0^T \int_0^L T(u(t)) \mathbf{z}(t) \partial_x \phi\, dxdt \\ &+ \int_0^T \int_0^L \phi \left[-D_2 J_{\mathbf{a}}(x, T(u(t))) + \mathbf{a}(u(t), g') g'\ dx \right]\ dt \geq 0. \end{aligned} \tag{7.128}$$

Our next step will be to use again Lemma 6.7.6 for τ small enough, as follows

$$\int_0^T\int_0^L \partial_t\phi(t,x)J_T(u(t,x))\,dxdt = \lim_{\tau\to 0}\int_0^T\int_0^L \frac{\eta(t-\tau)-\eta(t)}{-\tau}J_T(u(t,x))\rho(x)\,dxdt$$

$$\leq \lim_{\tau\to 0}\int_0^T\int_0^L u(t,x)\rho(x)\frac{d}{dt}(\eta T(u))^\tau(t,x)\,dxdt,$$

where we used the time regularization given in Definition C.2.1. On recourse to (7.84), we have

$$\int_0^T\int_0^L u(t)\rho\frac{d}{dt}(\eta T(u))^\tau(t)\,dxdt = -\langle u_t, \rho(\eta T(u))^\tau(\cdot)\rangle$$

which we recast as

$$-\lim_\alpha\langle \xi^K, \rho(\eta T(u))^\tau(\cdot)\rangle = -\lim_\alpha\int_0^T\left\langle D_x\mathbf{z}^K(t), \rho\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T(u(s))\,ds\right\rangle dt$$

and after integration by parts equals to

$$\begin{aligned}
&\lim_\alpha\int_0^T\int_0^L \mathbf{z}^K(t)D_x\left(\rho\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T(u(s))\,ds\right)dt\\
&=\lim_\alpha\int_0^T\int_0^L \partial_x\rho\,\mathbf{z}^K(t)\frac{1}{\tau}\int_{t-\tau}^t \eta(s)T(u(s))\,dsdxdt + \lim_\alpha\int_0^T\int_0^L \rho\,\mathbf{z}^K(t)D_x[(\eta T(u))^\tau(t)]\,dt\\
&=\int_0^T\frac{1}{\tau}\int_{t-\tau}^t\eta(s)\int_0^L T(u(s))\mathbf{z}(t)\partial_x\rho\,dxdsdt + \lim_\alpha\int_0^T\int_0^L \rho\,\mathbf{z}^K(t)\partial_x[(\eta T(u))^\tau(t)]\,dxdt\\
&\quad+\lim_\alpha\int_0^T\int_0^L \rho\,\mathbf{z}^K(t)D_x^s[(\eta T(u))^\tau(t)]\,dt.
\end{aligned}$$

This is bounded above by

$$\int_0^T\frac{1}{\tau}\int_{t-\tau}^t\eta(s)\int_0^L T(u(s))\mathbf{z}(t)\partial_x\rho\,dxdsdt + \int_0^T\frac{1}{\tau}\int_{t-\tau}^t\eta(s)\int_0^L \rho\,\mathbf{z}(t)\partial_x(T(u(s)))\,dxdsdt$$

$$+\int_0^T\frac{1}{\tau}\int_{t-\tau}^t\eta(s)\int_0^L cM\rho|D_x^s[T(u(s))]|\,dsdt.$$

Taking limits when $\tau\to 0$ in the previous chain of inequalities we arrive to

$$\begin{aligned}
\int_0^T\int_0^L \partial_t\phi(t)J_T(u(t))\,dxdt &\leq \int_0^T\eta(t)\int_0^L T(u(t))\mathbf{z}(t)\partial_x\rho\,dxdt\\
+\int_0^T\eta(t)\int_0^L \rho\,\mathbf{z}(t)\partial_x T(u(t))\,dxdt &+ cM\int_0^T\eta(t)\int_0^L\rho|D_x^s[T(u(t))]|\,dt.
\end{aligned}$$

From (7.128), all gathered together reads

$$\begin{aligned}
0\leq &-\int_0^T\int_0^L\phi(t)\mathbf{z}(t)g'\,dxdt + \int_0^T\eta(t)\int_0^L\rho\,\mathbf{z}(t)\partial_x(T(u(t)))\,dxdt\\
&+cM\int_0^T\eta(t)\int_0^L\rho|D_x^sT(u(t))|\,dt + \int_0^T\int_0^L\phi\left[-D_2J_{\mathbf{a}}(x,T(u(t))) + \mathbf{a}(u(t),g')g'dx\right]dt.
\end{aligned}$$

Using (7.124) this is written as

$$0 \leq cM \int_0^T \eta(t) \int_0^L \rho |D_x^s T(u(t))| \, dt - \int_0^T \int_0^L \phi [D_2 J_{\mathbf{a}}(x, T(u(t))]^s \, dt$$
$$+ \int_0^T \int_0^L [g' - \partial_x(T(u(t)))][\mathbf{a}(u(t), g') - \mathbf{z}(t)]\phi \, dxdt.$$

As measures, this is translated into

$$cM|D_x^s T(u(t))| - [D_2 J_{\mathbf{a}}(x, T(u(t))]^s + [g' - \partial_x(T(u(t)))][\mathbf{a}(u(t), g') - \mathbf{z}(t)]\mathcal{L}^2 \geq 0.$$

Taking the absolutely continuous part and particularizing to points $x \in [a < u(t) < b]$, this reduces to

$$[g' - \partial_x u(t)][\mathbf{a}(u(t), g') - \mathbf{z}(t)] \geq 0,$$

an inequality which holds for all $g \in C^2([0, L])$ and all $(t, x) \in S \cap [a < u < b]$, where $S \subseteq]0, T[\times]0, L[$ is such that $\mathcal{L}^2(]0, T[\times]0, L[\setminus S) = 0$. Being $(t, x) \in S \cap [a < u < b]$ fixed and $\xi \in \mathbb{R}$ given, we can find a function g as above such that $g'(x) = \xi$. Then

$$(\mathbf{z}(t, x) - \mathbf{a}(u(t), \xi)) (\partial_x u(t, x) - \xi) \geq 0, \ \forall \xi \in \mathbb{R}, \quad \forall (t, x) \in S \cap [a < u < b].$$

By an application of Minty–Browder's method in $\mathbb{R}$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{a}(u(t, x), \partial_x u(t, x)) \quad \text{a.e. on} \quad Q_T \cap [a < u < b].$$

Since this holds for any $0 < a < b$, we obtain (7.123) a.e. on the points of Q_T such that $u(t, x) \neq 0$. Now, by our assumptions on $\mathbf{a}$ and (7.113) we deduce that $\mathbf{z}(x) = \mathbf{a}(u(x), u'(x)) = 0$ a.e. on $[u = 0]$. We have finally proved that

$$\mathbf{z}(t) = \mathbf{a}(u(t), \partial_x u(t)) \quad \text{a.e.} \, t \in]0, T[.$$

Step 6. The entropy inequality.

Given $S \in \mathcal{P}^+$, $T \in \mathcal{T}^+$ and $\phi \in \mathcal{D}(Q_T)$, working as in the proof of (7.118) we can get the following inequality

$$\int_0^T \int_0^L \phi \, \mathbf{z}^K(t) D_x[T(u^K(t + \Delta t)) S(u^K(t + \Delta t))] \, dt$$
$$\leq \int_0^T \int_0^L J_{TS}(u^K(t)) \frac{\phi(t) - \phi(t - \Delta t)}{\Delta t} \, dxdt \tag{7.129}$$
$$- \int_0^T \int_0^L \mathbf{z}^K(t) T(u^K(t + \Delta t)) S(u^K(t + \Delta t)) \partial_x \phi \, dxdt.$$

The fact that

$$\left\{ \mathbf{z}^K(t) D_x[T(u^K(t + \Delta t)) S(u^K(t + \Delta t))] \right\}$$

is a bounded sequence in $L^1_{loc}(0, T; \mathcal{M}(]0, L[))$ follows. Using now Lemma 6.4.6 we can write

$$\mathbf{z}^K(t) D_x[T(u^K(t + \Delta t)) S(u^K(t + \Delta t))]$$
$$= \mathbf{z}^K(t) D_x J_{T'S}(u^K(t + \Delta t)) + \mathbf{z}^K(t) D_x J_{S'T}(u^K(t + \Delta t))$$

and we get that the sequences of positive — due to (7.107)— measures

$$\{\mathbf{z}^K(t)D_xJ_{T'S}(u^K(t+\Delta t))\}$$

and

$$\{\mathbf{z}^K(t)D_xJ_{S'T}(u^K(t+\Delta t))\}$$

are both bounded in $L^1_{loc}(0,T;\mathcal{M}(]0,L[))$. This allows us to define, up to subsequence, the objects $\mu_T^S, \mu_S^T \in \mathcal{M}(Q_T)$ by means of

$$\langle\phi,\mu_S^T\rangle = \lim_K \int_0^T\int_0^L \mathbf{z}^K(t)D_xJ_{T'S}(u^K(t+\Delta t))\phi\,dt, \quad \forall\phi\in C_c(Q_T),$$

$$\langle\phi,\mu_T^S\rangle = \lim_K \int_0^T\int_0^L \mathbf{z}^K(t)D_xJ_{S'T}(u^K(t+\Delta t))\phi\,dt, \quad \forall\phi\in C_c(Q_T).$$

Then, passing to the limit in (7.129), we obtain

$$\begin{aligned}\langle\phi,\mu_S^T\rangle + \langle\phi,\mu_T^S\rangle &\le \int_0^T\int_0^L J_{TS}(u(t))\partial_t\phi(t)\,dxdt\\ -\int_0^T\int_0^L &\mathbf{z}(t)T(u(t))S(u(t))\partial_x\phi\,dxdt, \quad \forall\phi\in\mathcal{D}(Q_T),\end{aligned} \tag{7.130}$$

The entropy inequalities, as stated in Definition 6.7.1, (iii), can be obtained for the solution already constructed as a direct consequence of the following result.

Lemma 6.7.14 *For any truncations $S,T\in\mathcal{T}^+$, we have that the inequality*

$$\mu_S^T \ge h_S(u,DT(u))$$

holds in the sense of measures.

Proof. From the entropy inequalities (7.109) we can get, using (4.21), that

$$\mathbf{z}^K(t)D_x^sJ_{T'S}(u^K(t+\Delta t)) \ge F_S(u^K(t+\Delta t),D_x(T(u^K(t+\Delta t))))^s \quad \forall t\in]0,T-\Delta t]. \tag{7.131}$$

We recall also that

$$\partial_xJ_{T'S}(u^K(t+\Delta t)) = S(u^K(t+\Delta t))\partial_xT(u^K(t+\Delta t)). \tag{7.132}$$

Let $0\le\phi\in C_c(Q_T)$. We decompose

$$\int_0^T\int_0^L \phi S(u^K(t+\Delta t))\mathbf{z}^K(t)\partial_xT(u(t))\,dxdt$$

$$=\int_0^T\int_0^L \phi S(u^K(t+\Delta t))\mathbf{a}(u^K(t+\Delta t),\partial_xT(u^K(t+\Delta t)))\partial_xT(u(t))\,dxdt$$

$$+\int_0^T\int_0^L \phi S(u^K(t+\Delta t))[\mathbf{z}^K(t)-\mathbf{a}(u^K(t+\Delta t),\partial_xT(u^K(t+\Delta t)))]\partial_xT(u(t))\;dxdt$$

On the aid of the convexity (4.11) of F we have that

$$\int_0^T\int_0^L \phi S(u^K(t+\Delta t))\mathbf{a}(u^K(t+\Delta t),\partial_x T(u^K(t+\Delta t)))\partial_x T(u(t))\,dxdt$$

$$\leq \int_0^T\int_0^L \phi S(u^K(t+\Delta t))\mathbf{z}^K(t)\partial_x T(u^K(t+\Delta t))\,dxdt$$

$$+\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u(t)))\,dt$$

$$-\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u^K(t+\Delta t)))\,dt.$$

Using (7.132) we rewrite the right hand side as

$$\int_0^T\int_0^L \phi\left[\mathbf{z}^K(t)D_x J_{T'S}(u^K(t+\Delta t))-\mathbf{z}^K(t)D_x^s J_{T'S}(u^K(t+\Delta t))\right]\,dt$$

$$+\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u(t)))\,dt$$

$$-\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u^K(t+\Delta t)))\,dt,$$

which, thanks to (7.131) and (4.21) is in turn bounded above by

$$\int_0^T\int_0^L \phi\,\mathbf{z}^K(t)D_x J_{T'S}(u^K(t+\Delta t))\,dt$$

$$-\int_0^T\int_0^L \phi F_S(u^K(t+\Delta t),D_x T(u^K(t+\Delta t)))^s\,dt$$

$$+\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u(t)))\,dt$$

$$-\int_0^T\int_0^L \phi F_S(u^K(t+\Delta t),D_x(T(u^K(t+\Delta t))))^{ac}\,dxdt.$$

We recast it as

$$\int_0^T\int_0^L \phi\,\mathbf{z}^K(t)D_x J_{T'S}(u^K(t+\Delta t))\,dt$$

$$+\int_0^T\int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t),\partial_x T(u(t)))\,dt$$

$$-\int_0^T\int_0^L \phi F_S(u^K(t+\Delta t),D_x T(u^K(t+\Delta t)))\,dt.$$

Using the previous chain of estimates, we get

$$\int_0^T \int_0^L \phi S(u^K(t+\Delta t))\mathbf{z}^K(t)\partial_x T(u(t))\,dxdt$$

$$\leq \int_0^T \int_0^L \phi\, \mathbf{z}^K(t) D_x J_{T'S}(u^K(t+\Delta t))\,dt$$

$$+ \int_0^T \int_0^L \phi S(u^K(t+\Delta t))F(u^K(t+\Delta t), \partial_x T(u(t)))\,dt$$

$$- \int_0^T \int_0^L \phi F_S(u^K(t+\Delta t), D_x T(u^K(t+\Delta t)))\,dt$$

$$+ \int_0^T \int_0^L \phi S(u^K(t+\Delta t))[\mathbf{z}^K(t) - \mathbf{a}(u^K(t+\Delta t), \partial_x T(u^K(t+\Delta t)))]\partial_x T(u(t))\,dxdt.$$

Note that the last term vanishes in the limit $K \to \infty$. Thanks to the lower-semicontinuity properties,

$$\int_0^T \int_0^L \phi F_S(u(t), D_x T(u(t)))\,dt$$
$$\leq \liminf_{k\to\infty} \int_0^T \int_0^L \phi F_S(u^K(t+\Delta t), D_x T(u^K(t+\Delta t)))\,dt.$$

Thus passing to the limit as $K \to \infty$ in the previous inequality yields

$$\int_0^T \int_0^L \phi S(u(t))\mathbf{z}(t)\partial_x T(u(t))\,dxdt$$

$$\leq \langle \mu_S^T, \phi \rangle + \int_0^T \int_0^L \phi S(u(t))F(u(t), \partial_x T(u(t)))\,dt$$

$$- \int_0^T \int_0^L \phi F_S(u(t), D_x T(u(t)))\,dt$$

$$= \langle \mu_S^T, \phi \rangle - \int_0^T \int_0^L \phi F_S(u(t), D_x T(u(t)))^s\,dt$$

where we used the definition of μ_S^T. Put it in another way,

$$\langle h_S(u, D_x T(u))^{ac}, \phi \rangle \leq \langle \mu_S^T, \phi \rangle - \langle h_S(u, D_x T(u))^s, \phi \rangle,$$

and hence the result. □

6.8 Some qualitative features

Entropy solutions verify the following balance of mass law.

Lemma 6.8.1 *The entropy solution u of (7.82) with initial datum $u_0 \in L^\infty(]0,L[)^+$ satisfies*

$$\int_0^L u(t,x)\,dx = \int_0^L u_0(x)\,dx + \beta t - c\int_0^t u(s,L_-)\,ds \quad \textit{for all } t \geq 0. \tag{8.133}$$

Proof. Let $t > 0$ be fixed. By the definition of entropy solution, we know that u satisfies the relation

$$\int_{Q_t} \mathbf{z} D_x w + \langle u_s, w\rangle = \beta\int_0^t w(s,0_+)\,ds - c\int_0^t u(s,L_-)w(s,L_-)\,ds \tag{8.134}$$

for all $w \in L^1(0,t;BV(]0,L[))$. Taking $w(s) = \chi_{]0,L[}$ in (8.134), we get

$$\langle u_s, \chi_{]0,L[}\rangle = \beta t - c\int_0^t u(s,L_-)\,ds. \tag{8.135}$$

Recall that u is the limit of Crandall–Liggett's approximating scheme; we stress the fact that (see step 2 of the previous section)

$$\langle u_s, w\rangle = \lim_\alpha \langle \xi^K, w\rangle \quad \text{for all } w \in L^1(0,t;BV(]0,L[)). \tag{8.136}$$

Taking again $w(s) = \chi_{]0,L[}$ in (8.136), we obtain that

$$\begin{aligned}
\langle u_s, \chi_{]0,L[}\rangle &= \lim_\alpha \langle \xi^K, \chi_{]0,L[}\rangle = \lim_\alpha \int_0^t \langle \xi^K, \chi_{]0,L[}\rangle_{BV^*-BV}\ d\tau \\
&= \lim_\alpha \int_0^t \frac{1}{\Delta s}\sum_{n=0}^{K-1}\int_0^L [u^{n+1}(x) - u^n(x)]\chi_{]s_n,s_{n+1}]}(\tau)\,dx d\tau \\
&= \lim_\alpha \sum_{n=0}^{K-1}\left\{\int_0^L u^{n+1}(x)\ dx - \int_0^L u^n(x)\ dx\right\} \\
&= \lim_\alpha \int_0^L u^K(x)\ dx - \int_0^L u^0(x)\ dx = \int_0^L u(t,x)\,dx - \int_0^L u^0(x)\ dx.
\end{aligned} \tag{8.137}$$

Finally, from (8.135) and (8.137), we get (8.133) and we finished the proof. □

Let u_β the stationary solution given in Proposition 6.6.2. Particularization of (8.133) to this solution yields

$$\int_0^L u_\beta(x)\,dx = \int_0^L u_\beta(x)\,dx + \beta t - c\int_0^t u_\beta(L_-)\,ds \quad \text{for all } t \geq 0.$$

Hence

$$u_\beta(L_-) = \frac{\beta}{c}. \tag{8.138}$$

Then, since u_β is a non-increasing function with $u_\beta \geq \frac{\beta}{c}$, and obviously is non-constant, we have

$$u_\beta(0_+) > \frac{\beta}{c}. \tag{8.139}$$

We shall denote by U_0 the entropy solution of (7.82) with initial datum $u_0 = 0$, that is,

$$U_0(t) = T_\beta(t)(0) \quad \text{for all } t \geq 0.$$

By comparison we have

$$U_0(t) \leq T_\beta(t)(u_\beta) = u_\beta \quad \text{for all } t \geq 0. \tag{8.140}$$

As a way to study more precise upper estimates we introduce the following concept of super-solution. To do so we require the space $\mathcal{T}^-$, standing for non-positive truncations of the form $T^l_{a,b}$.

Definition 6.8.2 Given $0 \leq u_0 \in L^\infty(]0,L[)$, we say that a measurable function $u :]0,T[\times]0,L[\to \mathbb{R}$ is an *entropy super-solution* of the problem (7.82) in $Q_T =]0,T[\times]0,L[$ if $u \in C([0,T];L^1(]0,L[)) \cap L^\infty(Q_T)$, $u(0) \geq u_0$, $T_{a,b}(u(\cdot)) \in L^1_{loc,w}(0,T,BV(]0,L[))$ for all $0 < a < b$, and $\mathbf{z}(t) := \mathbf{a}(u(t), \partial_x u(t)) \in L^1(Q_T)$ is such that:

(i) The following inequalities hold for almost every $t \in [0,T]$

$$\mathbf{z}(t, 0_+) \leq -\beta \quad \text{and} \quad \mathbf{z}(t, L_-) \geq -cu(t, L_-). \tag{8.141}$$

(ii) The following inequality is satisfied:

$$\begin{aligned} &\int_{Q_T} h_S(u, DT(u))\phi + \int_{Q_T} h_T(u, DS(u))\phi \\ \leq\ &\int_{Q_T} J_{TS}(u)\partial_t\phi - \int_0^T \int_0^L \mathbf{a}(u(t), \nabla u(t)) \cdot \partial_x\phi T(u(t))S(u(t))dxdt, \end{aligned} \tag{8.142}$$

for any $\phi \in \mathcal{D}((0,T)\times]0,L[)$, $\phi \geq 0$, and any $T \in \mathcal{T}^+$, $S \in \mathcal{T}^-$.

Note that taking $T(r) = 1$ and $S(r) = -1$, for all $r \in \mathbb{R}$, from (8.142), we get

$$\frac{\partial u}{\partial t} \geq \mathbf{a}(u(\cdot), \nabla u(\cdot))_x \quad \text{in } \mathcal{D}'(Q_T). \tag{8.143}$$

We can not use these truncation functions directly, instead we can use $T = T_{\frac{1}{n},\frac{2}{n}} + 1$ and $S = T_{\frac{1}{n},\frac{2}{n}} - 1$, and so obtain (8.143) by a limit process.

Working as in the proof of Theorem 2 in [21] we have the following comparison principle between entropy super-solutions and entropy solutions.

Theorem 6.8.3 *Assume that u is an entropy solution of (7.82) corresponding to initial datum $u_0 \in (L^\infty(]0,L[))^+$, and $\overline{u}$ is an entropy super-solution of (7.82) corresponding to initial datum $\overline{u}_0 \in (L^\infty(]0,L[))^+$ such that $\overline{u}(t) \in BV(]0,L[)$ for almost all $0 < t < T$. Then*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{for all } \ t \geq 0. \tag{8.144}$$

Proof. Let $b > a > 2\epsilon > 0$, $T(r) := T_{a,b}(r) - a$. We need to consider truncation functions of the form $S_{\epsilon,l}(r) := T_\epsilon(r-l)^+ = T_{l,l+\epsilon}(r) - l \in \mathcal{T}^+$, and $S^l_\epsilon(r) := T_\epsilon(r-l)^- = -T_\epsilon(l-r)^+ = T_{l-\epsilon,l}(r) - l$, if $l > \epsilon$. Observe that $S^l_\epsilon \in \mathcal{T}^-$. Let us denote

$$J^+_{T,\epsilon,l}(r) = \int_0^r T(s) T_\epsilon(s-l)^+ \, ds,$$

$$J^-_{T,\epsilon,l}(r) = \int_0^r T(s) T_\epsilon(s-l)^- \, ds = -\int_0^r T(s) T_\epsilon(l-s)^+ \, ds.$$

Since u is an entropy solution of (7.82) and $\overline{u}$ is an entropy super-solution of (7.82), if $\mathbf{z}(t) := \mathbf{a}(u(t), \partial_x u(t))$, $\overline{\mathbf{z}}(t) := \mathbf{a}(\overline{u}(t), \partial_x \overline{u}(t))$ and $l_1, l_2 > \epsilon$, we have

$$\begin{aligned} &-\int_0^T \int_0^L J^+_{T,\epsilon,l_1}(u(t)) \partial_t \eta(t) \, dxdt \\ &+\int_0^T \int_0^L \eta(t) [h_T(u(t), D_x S_{\epsilon,l_1}(u(t))) + h_{S_{\epsilon,l_1}}(u(t), D_x T(u(t)))] \, dt \\ &+\int_0^T \int_0^L \mathbf{z}(t) \partial_x \eta(t) \; T(u(t)) S_{\epsilon,l_1}(u(t)) \, dxdt \le 0, \end{aligned} \tag{8.145}$$

and

$$\begin{aligned} &-\int_0^T \int_0^L J^-_{T,\epsilon,l_2}(\overline{u}(t)) \partial_t \eta \, dxdt \\ &+\int_0^T \int_0^L \eta(t) [h_T(\overline{u}(t), D_x S^{l_2}_\epsilon(\overline{u}(t))) + h_{S^{l_2}_\epsilon}(\overline{u}(t), D_x T(\overline{u}(t)))] \, dt \\ &+\int_0^T \int_0^L \overline{\mathbf{z}}(t) \partial_x \eta(t) \; T(\overline{u}(t)) S^{l_2}_\epsilon(\overline{u}(t)) \, dxdt \le 0, \end{aligned} \tag{8.146}$$

for all $\eta \in C^\infty(Q_T)$, with $\eta \ge 0$, $\eta(t,x) = \phi(t)\rho(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\rho \in \mathcal{D}(]0,L[)$.

We choose two different pairs of variables (t,x), (s,y) and consider u, $\mathbf{z}$ as functions in (t,x), $\overline{u}$, $\overline{\mathbf{z}}$ in (s,y). Let $0 \le \phi \in \mathcal{D}(]0,T[)$, $\psi \in \mathcal{D}(]0,L[)$, ρ_m and $\tilde{\rho}_n$ sequences of mollifier in $\mathbb{R}$. Define

$$\eta_{m,n}(t,x,s,y) := \rho_m(x-y) \tilde{\rho}_n(t-s) \phi\left(\frac{t+s}{2}\right) \psi\left(\frac{x+y}{2}\right).$$

For (s,y) fixed, if we take in (8.145) $l_1 = \overline{u}(s,y)$, we get

$$\begin{aligned} &-\int_0^T \int_0^L J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x)) \partial_t \eta_{m,n} \, dxdt \\ &+\int_0^T \int_0^L \eta_{m,n} [h_T(u(t,x), D_x S_{\epsilon,\overline{u}(s,y)}(u(t,x))) + h_{S_{\epsilon,\overline{u}(s,y)}}(u(t,x), D_x T(u(t,x)))] \, dt \\ &+\int_0^T \int_0^L \mathbf{z}(t,x) \partial_x \eta_{m,n} \; T(u(t,x)) \; S_{\epsilon,\overline{u}(s,y)}(u(t,x)) \, dxdt \le 0. \end{aligned} \tag{8.147}$$

Similarly, for (t,x) fixed, if we take in (8.146) $l_2 = u(t,x)$ we get

$$
\begin{aligned}
&-\int_0^T \int_0^L J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))\partial_s \eta_{m,n}\ dyds \\
&+\int_0^T \int_0^L \eta_{m,n}[h_T(\overline{u}(s,y), D_y S_\epsilon^{u(t,x)}(\overline{u}(s,y))) + h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y), D_y T(\overline{u}(s,y)))]\, ds \\
&+\int_0^T \int_0^L \overline{\mathbf{z}}(s,y)\partial_y \eta_{m,n}\ T(\overline{u}(s,y))\ S_\epsilon^{u(t,x)}(\overline{u}(s,y))\, dyds \leq 0.
\end{aligned}
\tag{8.148}
$$

Observe that since $a > 2\epsilon$, if $\overline{u}(s,y) \leq \epsilon$ or $u(t,x) \leq \epsilon$ the integrals in (8.147) and (8.148) are zero. We integrate (8.147) in (s,y), (8.148) in (t,x), and add the two inequalities. Since

$$
\int_{Q_T \times Q_T} \eta_{m,n} h_{S_{\epsilon,\overline{u}(s,y)}}(u(t,x), D_x T(u(t,x)))\ dsdtdy \geq 0
$$

we get

$$
\begin{aligned}
&-\int_{Q_T \times Q_T} \left(J^+_{T,\epsilon,\overline{u}(s,y)}(u(t,x))\partial_t \eta_{m,n} + J^-_{T,\epsilon,u(t,x)}(\overline{u}(s,y))\partial_s \eta_{m,n} \right)\ dsdtdydx \\
&+\int_{Q_T \times Q_T} \eta_{m,n} h_T(u(t,x), D_x S_{\epsilon,\overline{u}(s,y)}(u(t,x)))\ dsdtdy \\
&+\int_{Q_T \times Q_T} \eta_{m,n} h_T(\overline{u}(s,y), D_y S_\epsilon^{u(t,x)}(\overline{u}(s,y)))\ dsdtdx \\
&-\int_{Q_T \times Q_T} \overline{\mathbf{z}}(s,y)\partial_x \eta_{m,n} T(\overline{u}(s,y)) S_\epsilon^{u(t,x)}(\overline{u}(s,y))\ dsdtdydx \\
&-\int_{Q_T \times Q_T} \mathbf{z}(t,x)\partial_y \eta_{m,n} T(u(t,x)) S_{\epsilon,\overline{u}(s,y)}(u(t,x))\ dsdtdydx \\
&+\int_{Q_T \times Q_T} \eta_{m,n} h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y), D_y T(\overline{u}(s,y)))\ dsdtdx \\
&+\int_{Q_T \times Q_T} T_\epsilon^+(u(t,x) - \overline{u}(s,y))[T(u(t,x))\mathbf{z}(t,x) - T(\overline{u}(s,y))\overline{\mathbf{z}}(s,y)] \\
&\times(\partial_x \eta_{m,n} + \partial_y \eta_{m,n})\ dsdtdydx \leq 0.
\end{aligned}
\tag{8.149}
$$

Let I_2 be the sum of the second up to the fifth terms of the above inequality. From now on, since u, $\mathbf{z}$ are always functions of (t,x), and $\overline{u}$, $\overline{\mathbf{z}}$ are always functions of (s,y), to make our expression shorter, we shall omit the arguments except when they appear as sub-index and in some additional cases where we find it useful to remind them. We also omit the differentials of the integrals.

Working as in the proof of uniqueness of Theorem 3 in [17], we obtain that $\frac{1}{\epsilon}I_2 \geq \|\phi\|_\infty\|\psi\|_\infty o(\epsilon)$. Hence, by (8.149), it follows that

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\epsilon,\overline{u}}(u)\partial_t\eta_{m,n}+J^-_{T,\epsilon,u}(\overline{u})\partial_s\eta_{m,n}\right)\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h_{S_\epsilon^{u(t,x)}}(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\\
&+\int_{Q_T\times Q_T}T^+_\epsilon(u-\overline{u})[T(u)\mathbf{z}-T(\overline{u})\overline{\mathbf{z}}](\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\leq \epsilon o(\epsilon).
\end{aligned}$$

Then, dividing by ϵ and letting $\epsilon\to 0$ we get

$$\begin{aligned}
&-\int_{Q_T\times Q_T}\left(J^+_{T,\mathrm{sign},\overline{u}}(u)\partial_t\eta_{m,n}+J^-_{T,\mathrm{sign},u}(\overline{u})\partial_s\eta_{m,n}\right)\\
&+\int_{Q_T\times Q_T}\eta_{m,n}h(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\\
&+\int_{Q_T\times Q_T}\mathrm{sign}_0^+(u-\overline{u})[T(u)\mathbf{z}-T(\overline{u})\overline{\mathbf{z}}](\partial_x\eta_{m,n}+\partial_y\eta_{m,n})\leq 0.
\end{aligned}$$

where

$$J^+_{T,sign,l}(r)=\int_0^r T(r')sign_0^+(r'-l)dr' \quad l\in\mathbb{R},\ r\geq 0$$

and

$$J^-_{T,sign,l}(r)=\int_0^r T(r')sign_0^-(r'-l)dr' \quad l\in\mathbb{R},\ r\geq 0.$$

Now, letting $m\to\infty$, we obtain

$$\begin{aligned}
&-\int_0^T\int_0^T\int_0^L\left(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\chi_n+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\chi_n\right)\\
&+\int_{Q_T\times Q_T}\chi_n h(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\\
&+\int_0^T\int_0^T\int_0^L\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))[T(u(t,x))\mathbf{z}(t,x)-T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x)]\partial_x\chi_n\leq 0
\end{aligned}$$

where $\chi_n(t,s,x):=\tilde{\rho}_n(t-s)\phi(\frac{t+s}{2})\psi(x)$. We set $\psi=\psi_k\in\mathcal{D}(]0,L[)\uparrow\chi_{]0,L[}$ in the last expression and we take limit as $k\to+\infty$. Then we have

$$\begin{aligned}
&-\int_0^T\int_0^T\int_0^L\left(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\kappa_n(t,s)\right)\\
&+\int_{Q_T\times Q_T}\kappa_n h(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\\
&+\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L\kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x\psi_k(x)\\
&-\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L\kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x))\partial_x\psi_k(x)\leq 0,
\end{aligned} \tag{8.150}$$

where $\kappa_n(t,s) := \tilde{\rho}_n(t-s)\phi(\frac{t+s}{2})$. Let us study the second and third terms of the above expression. Let

$$\begin{aligned}I_k :=& \int_0^T\int_0^T\int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x\psi_k(x)\\ =& \int_0^T\int_0^T\int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(u(t,x))\mathbf{z}(t,x)\partial_x(\psi_k(x)-1).\end{aligned}$$

Let $H_n(s,r) := \kappa_n(r,s)\mathrm{sign}_0^+(u(r)-\overline{u}(s))T(u(r))$. For $\tau > 0$, we define the function $(\kappa_n(s))^\tau$, as the Dunford integral (see Remark 6.7.5)

$$(\kappa_n(s))^\tau(t) := \frac{1}{\tau}\int_t^{t+\tau} H_n(s,r)\ dr.$$

Then,

$$\begin{aligned}I_k =& \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (\kappa_n(s))^\tau(t)\mathbf{z}(t,x)\partial_x[\psi_k(x)-1]\,dxdtds\\ =& -\lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L [\psi_k(x)-1]\mathbf{z}(t,x)D_x((\kappa_n(s))^\tau(t)))\,dsdt\\ &-\lim_{\tau\to 0}\int_0^T \langle u_t, (\kappa_n(s))^\tau(\psi_k(x)-1)\rangle\,ds\\ &+c\lim_{\tau\to 0}\int_0^T\int_0^T u(t,L_-)(\kappa_n(s))^\tau(t)(L_-)\,dtds\\ &-\beta\lim_{\tau\to 0}\int_0^T\int_0^T (\kappa_n(s))^\tau(t)(0_+)\,dtds\\ :=& I_k^1+I_k^2+I_k^3+I_k^4.\end{aligned}$$

Notice that

$$I_k^3 = c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(u(t,L_-))\,dtds$$

and

$$I_k^4 = -\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(u(t,0_+))\,dtds.$$

By Remark 6.7.5, we get

$$|D_x((\kappa_n(s))^\tau(t))|(]0,L[) \stackrel{\tau\to 0}{\longrightarrow} |D_x(\kappa_n(t,s)\mathrm{sign}_0^+(u(t)-\overline{u}(s))T(u(t)))|(]0,L[). \quad (8.151)$$

Using (8.151), we get

$$|I_k^1| \le c\|u\|_{L^\infty(Q_T)}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))|D_x(\kappa_n(t,s)\mathrm{sign}_0^+(u(t)-\overline{u}(s))T(u(t)))|\ dtds,$$

which implies $\lim_{k\to\infty} I_k^1 = 0$. Let us deal with I_k^2. We have

$$I_k^2 = \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L u(t,x)\frac{H_n(s,t+\tau)-H_n(s,t)}{\tau}(\psi_k(x)-1)\,dxdtds.$$

Let

$$q(\tau) := \mathrm{sign}_0^+(\tau - \overline{u}(s,x))T(\tau), \quad Q(r) := \int_0^r q(\tau)\,d\tau.$$

Since q is non-decreasing, $Q(r) - Q(\overline{r}) \le q(r)(r-\overline{r})$. Then, changing variables, since $H_n(s,t) = q(u(t))\kappa_n(t,s)$

$$\begin{aligned} I_k^2 &= \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\frac{u(t,x)-u(t-\tau,x)}{\tau}H_n(s,t)\,dxdtds \\ &\ge \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))\kappa_n(t,s)\frac{Q(u(t,x))-Q(u(t-\tau,x))}{\tau}\,dxdtds \\ &= \lim_{\tau\to 0}\int_0^T\int_0^T\int_0^L (1-\psi_k(x))Q(u(t,x))\frac{\kappa_n(t,s)-\kappa_n(t+\tau,s)}{\tau}\,dxdtds \\ &= -\int_0^T\int_0^T\int_0^L (1-\psi_k(x))Q(u(t,x))\partial_t\kappa_n(t,s)\,dxdtds, \end{aligned} \tag{8.152}$$

from where it follows that $\lim_{k\to\infty} I_k^2 \ge 0$. Taking into account the above facts, we get

$$\begin{aligned} \lim_{k\to\infty} I_k &\ge -\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(u(t,0_+))\,dtds \\ &+c\int_0^T\int_0^T u(t,L_-)\kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(u(t,L_-))\,dtds. \end{aligned} \tag{8.153}$$

To deal with the third term in (8.150) we introduce

$$H_n(t,s) = \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))$$

and

$$(\kappa_n(t))^\tau(s) = \frac{1}{\tau}\int_s^{s+\tau} H_n(t,r)\,dr.$$

Then, integrating by parts and owing to (8.143) we get to

$$\begin{aligned} &-\int_0^T\int_0^T\int_0^L (\kappa_n(t))^\tau(s)\overline{\mathbf{z}}(s,x)\partial_x(\psi_k(x)-1)\,dxdtds \\ &\qquad\ge \int_0^T\int_0^T\int_0^L D_x(\kappa_n(t))^\tau(s)\overline{\mathbf{z}}(s,x)(\psi_k(x)-1)\,dxdtds \\ &\qquad+\int_0^T\int_0^T\int_0^L \overline{u}_s(\kappa_n(t))^\tau(s)(\psi_k(x)-1)\,dxdtds \\ &\qquad+\int_0^T\int_0^T (\kappa_n(t))^\tau(s)(L_-)\overline{\mathbf{z}}(s,L_-)\,dtds \\ &\qquad-\int_0^T\int_0^T (\kappa_n(t))^\tau(s)(0_+)\overline{\mathbf{z}}(s,0_+)\,dtds. \end{aligned}$$

The sum of the first and the second term can be shown to be non-negative after the passage to the limit in τ and k and rearranging conveniently in between. Thus,

$$
\begin{aligned}
-\lim_{k\to+\infty}&\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x))\partial_x\psi_k(x)\\
&\geq \int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(\overline{u}(s,L_-))\overline{\mathbf{z}}(s,L_-)\,dtds\\
&-\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(\overline{u}(s,0_+))\overline{\mathbf{z}}(s,0_+)\,dtds
\end{aligned}
$$

or, invoking the boundary conditions,

$$
\begin{aligned}
&-\lim_{k\to+\infty}\int_0^T\int_0^T\int_0^L \kappa_n(t,s)\mathrm{sign}_0^+(u(t,x)-\overline{u}(s,x))T(\overline{u}(s,x))\overline{\mathbf{z}}(s,x))\partial_x\psi_k(x)\\
&\geq -c\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))T(\overline{u}(s,L_-))\overline{u}(s,L_-)\,dtds\\
&+\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))T(\overline{u}(s,0_+))\,dtds
\end{aligned}
\tag{8.154}
$$

From (8.150), by (8.153) and (8.154), we have

$$
\begin{aligned}
&-\int_0^T\int_0^T\int_0^L \Big(J^+_{T,\mathrm{sign},\overline{u}(s,x)}(u(t,x))\partial_t\kappa_n(t,s)+J^-_{T,\mathrm{sign},u(t,x)}(\overline{u}(s,x))\partial_s\kappa_n(t,s)\Big)\,dtdsdx\\
&+\int_{Q_T\times Q_T}\kappa_n h(\overline{u}(s,y),D_yT(\overline{u}(s,y)))\,dtdsdx\\
&+c\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-))\\
&\qquad\times\{u(t,L_-)T(u(t,L_-))-\overline{u}(s,L_-)T(\overline{u}(s,L_-))\}\,dtds\\
&+\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))\,\{T(\overline{u}(s,0_+))-T(u(t,0_+))\}\,dtds\leq 0.
\end{aligned}
\tag{8.155}
$$

Notice now that, as $h(\mathbf{z},\xi)\leq C|\mathbf{z}||\xi|$, we are enable to write

$$
\int_0^L h(\overline{u}(s,y),D_yT_{0,b}(\overline{u}(s,y)))\leq Cb\int_0^b Per([\overline{u}\geq\lambda])\,d\lambda,
$$

thanks to the coarea formula. Since the mapping $\lambda\mapsto Per([\overline{u}\geq\lambda])$ is integrable, we deduce that

$$
\lim_{b\to0^+}\frac{1}{b}\int_{Q_T\times Q_T}\kappa_n h(\overline{u}(s,y),D_yT_{0,b}(\overline{u}(s,y)))\,dtdsdx=0.
$$

Then, letting $a \to 0$, dividing by b and letting $b \to 0$ in (8.155), we obtain,

$$
\begin{aligned}
&-\int_0^T\int_0^T\int_0^L (u(t,x)-\overline{u}(s,x))^+(\partial_t\kappa_n(t,s)+\partial_s\kappa_n(t,s))\ dtdsdx \\
&+c\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-)) \\
&\qquad\times\left\{-u(t,L_-)\mathrm{sign}_0^+(u(t,L_-)-\overline{u}(s,L_-)\mathrm{sign}_0^+(\overline{u}(s,L_-))\right\}\ dtds \\
&+\beta\int_0^T\int_0^T \kappa_n(t,s)\mathrm{sign}_0^+(u(t,0_+)-\overline{u}(s,0_+))\left\{\mathrm{sign}_0(\overline{u}(s,0_+)-\mathrm{sign}_0(u(t,0_+)\right\}\ dtds \le 0.
\end{aligned}
$$

By Lemma 6.7.3 and the boundary conditions, we have

$$
u(t,0_+)\ge\frac{\beta}{c}>0,\qquad \overline{u}(s,0_+)>0\qquad \text{for almost every } t,s>0 \tag{8.156}
$$

and the third term above vanishes. Therefore,

$$
-\int_0^T\int_0^T\int_0^L (u(t,x)-\overline{u}(s,x))^+(\partial_t\kappa_n(t,s)+\partial_s\kappa_n(t,s))\ dtdsdx\le 0.
$$

Letting $n\to\infty$,

$$
-\int_0^T\int_0^L (u(t,x)-\overline{u}(t,x))^+\phi'(t)\ dxdt\le 0.
$$

Since this is true for all $0\le\phi\in\mathcal{D}(]0,T[)$, we have

$$
\frac{d}{dt}\int_0^L (u(t,x)-\overline{u}(t,x))^+\,dx\le 0.
$$

Hence

$$
\int_0^L (u(t,x)-\overline{u}(t,x))^+\,dx\le\int_0^L (u_0(x)-\overline{u}_0(x))^+\,dx\qquad\text{for all}\ \ t\ge 0,
$$

which finishes the proof. □

Proposition 6.8.4 *There are values $0\le a<L$, $c_2>0$ and $\mu>1$ such that*

$$
u(t,x)=\left(\mu\frac{\beta}{c}+c_2t-\frac{\beta}{\nu}\frac{\mu}{\sqrt{\mu^2-1}}x\right)\chi_{]0,a+ct[}(x)
$$

is a super-solution of the problem (7.82) in the time interval $[0,(L-a)/c]$ and with initial datum

$$
u_0(x)=\left(\mu\frac{\beta}{c}-\frac{\beta}{\nu}\frac{\mu}{\sqrt{\mu^2-1}}x\right)\chi_{]0,a[}(x).
$$

Proof. To ease notation we write $u(t,x)=\varphi(t,x)\chi_{]0,a+ct[}(x)$ and set $\gamma(t):=\varphi(t,a+ct)$. We compute

$$
\partial_t u=c_2\chi_{]0,a+ct[}(x)+c\gamma(t)\delta_{x=a+ct},
$$

$$
D_xu(t)=-\frac{\beta}{\nu}\frac{\mu}{\sqrt{\mu^2-1}}\chi_{]0,a+ct[}(x)-\gamma(t)\delta_{x=a+ct},
$$

$$\mathbf{a}(u(t), u(t)_x) = \frac{-\frac{\beta\mu}{\sqrt{\mu^2-1}}\varphi(t,x)}{\sqrt{\varphi(t,x)^2 + \frac{\beta^2}{c^2}\frac{\mu^2}{\mu^2-1}}}\chi_{]0,a+ct[}(x).$$

For the sake of a lighter notation we decompose

$$\mathbf{a}(u(t), u(t)_x) = \psi(t,x)\chi_{]0,a+ct[}(x).$$

The super-solution boundary condition (8.141) now reads

$$\mathbf{a}(u(t), u(t)_x)(x=0) = \frac{-\frac{\beta\mu}{\sqrt{\mu^2-1}}\left(\mu\frac{\beta}{c} + c_2 t\right)}{\sqrt{\left(\mu\frac{\beta}{c} + c_2 t\right)^2 + \frac{\beta^2}{c^2}\frac{\mu^2}{\mu^2-1}}} \leq -\beta$$

and its readily seen to be satisfied without placing any restriction on the parameters.

Next we study the fulfillment of the entropy condition (8.142). Let $T \in \mathcal{T}^+$ and $S \in \mathcal{T}^-$. As

$$J_{TS}(u(t)) = J_{TS}(\varphi(t))\chi_{]0,a+ct[}$$

we can compute

$$\frac{\partial}{\partial t}J_{TS}(u(t)) = cJ_{TS}(\gamma(t))\delta_{x=a+ct} + c_2[TS(\varphi(t))]\chi_{]0,a+ct[}$$

and so

$$\int_0^T\int_0^L J_{TS}(u(t,x))\partial_t\phi(t,x)\,dxdt = -\int_0^T\int_0^L \phi(t)\frac{\partial}{\partial t}J_{TS}(u(t,x))\,dxdt$$

$$= -c\int_0^T \phi(t,a+ct)J_{TS}(\gamma(t))\,dt - c_2\int_0^T\int_0^{a+ct}[TS(\varphi(t,x))]\phi(t,x)\,dxdt.$$

We also have that

$$\begin{aligned}
&-\int_0^T\int_0^L \mathbf{a}(u(t),u(t)_x)\partial_x\phi(t)T(u(t))S(u(t))\,dxdt \\
&= -\int_0^T\int_0^{a+ct} \psi(t,x)T(\varphi(t,x))S(\varphi(t,x))\partial_x\phi\,dxdt \\
&= \int_0^T\int_0^{a+ct} \partial_x\left[TS(\varphi(t,x))\psi(t,x)\right]\phi(t)\,dxdt \\
&\quad -\int_0^T \phi(t,a+ct)[TS](\gamma(t))\psi(t,a+ct)\,dt. \\
&= \int_0^T\int_0^{a+ct} \partial_x\left[TS(u(t))\right]\psi(t,x)\phi(t)\,dxdt \\
&\quad +\int_0^T\int_0^{a+ct} \phi(t)[TS(u(t))]\partial_x\psi(t,x)\,dxdt \\
&\quad -\int_0^T \phi(t,a+ct)[TS](\gamma(t))\psi(t,a+ct)\,dt.
\end{aligned}$$

On the other hand,

$$h_S(u(t), DT(u(t)))^{ac} = \partial_x T(u(t))S(u(t))\psi(t,x)\chi_{]0,a+ct[}(x)$$

and

$$h_T(u(t), DS(u(t)))^{ac} = \partial_x S(u(t))T(u(t))\psi(t,x)\chi_{]0,a+ct[}(x).$$

We compute now the corresponding singular parts. Using the chain rule several times and the fact that the derivative has no Cantor part, as it was done in [21], we obtain

$$h_T(u(t), DS(u(t)))^s = |D^s J_{\theta T}(S(u(t)))| = |D^j J_{\theta TS'}(u(t))| = J_{\theta TS'}(\gamma(t))\delta_{x=a+ct}.$$

Similarly

$$h_S(u(t), DT(u(t)))^s = J_{\theta T'S}(\gamma(t))\delta_{x=a+ct}.$$

Note that, integrating by parts,

$$J_{\theta(TS)'}(\gamma(t)) = -cJ_{TS}(\gamma(t)) + [\theta TS](\gamma(t)),$$

and consequently

$$h_T(u(t), DS(u(t)))^s + h_S(u(t), DT(u(t)))^s = [-cJ_{TS}(\gamma(t)) + [\theta TS](\gamma(t))]\,\delta_{x=a+ct}.$$

Thus

$$\begin{aligned}&\int_{Q_T} h_S(u(t), DT(u(t)))\phi + \int_{Q_T} h_T(u(t), DS(u(t)))\phi(t)\\ &= \int_0^T\int_0^L \phi(t,x)\psi(t,x)\partial_x[T(u(t))S(u(t))]\chi_{]0,a+ct[}(x)\,dxdt\\ &\quad+\int_0^T \phi(t,a+ct)\,[-cJ_{TS}(\gamma(t)) + [\theta TS](\gamma(t))]\,dt.\end{aligned}$$

Altogether, the fulfillment of the entropy condition (8.142) reduces to check that

$$\begin{aligned}\int_0^T \phi(t,a+ct)[\theta TS](\gamma(t))\,dt &\le -c_2\int_0^T\int_0^{a+ct}[TS(\varphi(t,x))]\phi(t,x)\,dxdt\\ +\int_0^T\int_0^L \phi[TS(u(t))]\partial_x\psi\chi_{]0,a+ct[}(x)\,dxdt &- \int_0^T \phi(t,a+ct)[TS](\gamma(t))\psi(t,a+ct)\,dt.\end{aligned}$$

Now we use that $0 > \psi(t,a+ct) \ge -c\gamma(t)$ to reduce the situation further to

$$c_2\int_0^T\int_0^{a+ct}[TS(\varphi(t,x))]\phi(t,x)\,dxdt \le \int_0^T\int_0^L \phi(t,x)[TS(u(t,x))]\partial_x\psi\chi_{]0,a+ct[}(x)\,dxdt.$$

This condition would be fulfilled if we were able to show that $u_t \le \mathbf{a}(u,u_x)_x$ almost everywhere in $]0,a+ct[$. That is, we are to show that

$$c_2 \le \partial_x\psi(t,x) \quad \text{a.e. in} \quad]0,a+ct[.$$

A bit of calculus shows that

$$\partial_x\psi(t,x) = \frac{\frac{\beta^4\mu^4}{c^2(\mu^2-1)^2}}{\left(\varphi(t,x)^2 + \frac{\beta^2}{c^2}\frac{\mu^2}{\mu^2-1}\right)^{3/2}}.$$

This function attains its minimum value precisely where $\varphi(t,x)$ has its maximum value. For the region that interests us, this maximum value is

$$\varphi\left(\frac{L-a}{c},0\right)=\mu\frac{\beta}{c}+\frac{c_2}{c}(L-a).$$

Then, the final condition over the parameters reads

$$c_2\leq\frac{\frac{\beta^4\mu^4}{c^2(\mu^2-1)^2}}{\left(\frac{\beta^2}{c^2}\left(\mu^2+\frac{\mu^2}{\mu^2-1}\right)+\left(\frac{c_2}{c}(L-a)\right)^2+2\frac{\mu\beta c_2}{c^2}(L-a)\right)^{3/2}}.$$

Therefore, for $\mu>1$ sufficiently close to one we have that $u(t,x)$ is a super-solution of (7.82). □

The previous result will allow to demonstrate an upper bound for the speed of propagation associated with our model. Next we introduce the concept of sub-solution, which will enable us to derive lower bounds for this speed of propagation.

Definition 6.8.5 Given $0\leq u_0\in L^\infty(]0,L[)$, we say that a measurable function $u:]0,T[\times]0,L[\to\mathbb{R}$ is an *entropy sub-solution* of the problem (7.82) in $Q_T=]0,T[\times]0,L[$ if $u\in C([0,T];L^1(]0,L[))\cap L^\infty(Q_T)$, $u(0)\leq u_0$, $T_{a,b}(u(\cdot))\in L^1_{loc,w}(0,T,BV(]0,L[))$ for all $0<a<b$, and $\mathbf{z}(t):=\mathbf{a}(u(t),\partial_x u(t))\in L^1(Q_T)$, such that:

(i) The following inequalities hold for almost every $t\in[0,T]$

$$\mathbf{z}(t,0_+)\geq-\beta,\quad \mathbf{z}(t,L_-)\leq -cu(t,L_-).$$

(ii) The following inequality is satisfied:

$$\begin{aligned}&\int_{Q_T}h_S(u,DT(u))\phi+\int_{Q_T}h_T(u,DS(u))\phi \\ \geq\ &\int_{Q_T}J_{TS}(u)\partial_t\phi-\int_0^T\int_0^L\mathbf{a}(u(t),\nabla u(t))\cdot\partial_x\phi T(u(t))S(u(t))dxdt,\end{aligned}\tag{8.157}$$

for any $\phi\in\mathcal{D}((0,T)\times]0,L[)$, $\phi\geq0$, and any $T\in\mathcal{T}^+$, $S\in\mathcal{T}^-$.

Note that taking $T(r)=1$ and $S(r)=-1$, for all $r\in\mathbb{R}$, from (8.157), we get

$$\frac{\partial u}{\partial t}\leq\mathbf{a}(u(\cdot),\nabla u(\cdot))_x\quad\text{in}\quad\mathcal{D}'(Q_T).\tag{8.158}$$

We can not use these truncation functions directly, instead we can use $T=T_{\frac{1}{n},\frac{2}{n}}+1$ and $S=T_{\frac{1}{n},\frac{2}{n}}-1$, and so obtain (8.158) by a limit process.

With a similar proof of one of Theorem 6.8.3, we obtain the following result.

Theorem 6.8.6 *If $u(t)$ is a bounded entropy solution of (7.82) in $Q_T=]0,T[\times]0,L[$ corresponding to initial data u_0 and $\overline{u}(t)$ is an entropy sub-solution corresponding to initial data $\overline{u}_0\in L^\infty(]0,L[)^+$ such that*

$$\overline{u}(t,0_+)\geq\frac{\beta}{c}>0\qquad\textit{for almost every } t>0.$$

Then,

$$\|(\overline{u}(t)-u(t))^+\|_1\leq\|(\overline{u}_0-u_0)^+\|_1\qquad\text{for all}\quad t\geq0.$$

Proposition 6.8.7 *The function*

$$u(t,x) = \frac{\beta}{c^2 t}(ct - x)\chi_{]0,ct[}(x).$$

is a sub-solution with zero initial datum, as long as $t \le L/c$.

Proof. Notice that $u(t,x)$ represents the family of straight lines joining the points $(0, \frac{\beta}{c})$ and $(ct, 0)$. As these are smooth in almost every point we can stick to classical computations when trying to check the requirements for $u(t,x)$ to be a sub-solution. We compute over $]0, ct[$

$$u_t = \frac{\beta x}{c^2 t^2},$$

$$\partial_x u = -\frac{\beta}{c^2 t},$$

$$\mathbf{a}(u, u_x) = \frac{-\frac{\beta\nu}{c^2 t}(1 - \frac{x}{ct})}{\sqrt{\frac{\nu^2}{c^4 t^2} + (1 - \frac{x}{ct})^2}} = \frac{\frac{\beta\nu}{c^2 t}(x - ct)}{\sqrt{\frac{\nu^2}{c^2} + (x - ct)^2}}$$

and

$$\mathbf{a}(u, u_x)_x = \frac{\beta\nu^3}{c^4 t}\left(\frac{\nu^2}{c^2} + (x - ct)^2\right)^{-\frac{3}{2}}.$$

Then we can check that $u_t \le \mathbf{a}(u, u_x)_x$ holds on $]0, ct[$ and therefore on the whole spatial interval. These computations are valid for $t \le \frac{L}{c}$, the limit configuration being the segment $\frac{\beta}{cL}(L - x)$. As regards the boundary conditions, we notice that the one pertaining $x = L$ is automatically fulfilled for $t < \frac{L}{c}$, while

$$\mathbf{a}(u, u_x)(0_+) = \frac{-\frac{\beta\nu}{c^2 t}}{\sqrt{\frac{\nu^2}{c^4 t^2} + 1}} > -\beta.$$

□

Once we have all this machinery at our disposal we can prove easily that the speed of the signal is precisely c:

Theorem 6.8.8 *We have*

$$supp(U_0(t)) =]0, ct[\quad for \;\; 0 < t \le \frac{L}{c}.$$

Proof. This is an straightforward consequence of Propositions 6.8.7 and 6.8.4 and the corresponding comparison results (Theorem 6.8.6 and 6.8.3). □

Chapter 7

Perspectives for future work

The aim of this final part of the Thesis is to address several issues linked with the previous work in a natural way. These constitute possible continuations and extensions of the material contained in this memory. In some of the following problems the advisors of this Thesis are involved.

1. In the first part of the memory, self-gravitating systems were described using an approach that was free of short range interactions. These approaches might be too idealized in certain circumstances, as we might not be able to control a certain number of important contributions to the dynamics. For instance, effects related to the presence of different species whose material properties and ways of interacting with their environment are not completely understood. The influence of close encounters could be also important under several circumstances.

   In taking into account these uncontrollable features in the non-relativistic framework, the Vlasov–Poisson system appears complemented with Fokker-Planck terms (white noise in velocities), which yields the following kinetic equation

   $$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = L(f),$$

   where $L(f) = \mathrm{div}_v\,(\beta v f) + \sigma \Delta_v f$ ($\beta \geq 0$ and $\sigma > 0$ are constants related to the collisions between particles). Performing the coupling with Poisson's equation we obtain what is known as the Vlasov–Poisson–Fokker–Planck system [71, 72]. The overall feature in the case without friction ($\beta = 0$) is that, no matter the smallness of the diffusion coefficient σ, the dynamics associated with the Vlasov–Poisson system is now completely destroyed as all solutions go to zero! This fact might not be what we were willing to obtain when we committed ourselves to incorporate a small source of stochasticity to our system.

   Our proposal is that other sources of stochasticity might be brought in that do not alter the overall dynamics in such a radical way and are maybe able to respect some of the interesting structures that were present in the original Vlasov–Poisson system. We suggest then the use of the following model

   $$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = FL(f) \\ \Delta_x \phi = 4\pi\rho, \end{cases}$$

where $FL(f)$ stands for a nonlinear flux–limited term similar to that studied here. This work is already in progress in collaboration with V. Caselles. This is not the first time that this type of diffusions appear in connection with the field of astrophysics; the interested reader can consult [145] and Chapter 7 in [158].

The technical difficulties accompanying the study of the Cauchy problem for the suggested model come from two main sources. One is to deduce uniqueness. We know that the conditions ensuring uniqueness for the Vlasov–Poisson system (we typically refer to [152], as we may not have classical solutions) and for the relativistic heat equation are quite different and probably a combination of both has to be imposed. The second is to pass to the limit in the nonlinear expression furnished by the flux limiter. For this we need some compactness in the spatial variable that is likely to be obtained using Kruzkov's doubling variables method. To succeed here we must have a (spatially) continuous force field. We cannot assume that we will get classical solutions, so that this has to be justified using bounds on velocity moments. Thus we would have to mimic the proof in [152], taking care of the extra term and being able to give a meaning to the representation formula for the density, which uses the characteristics of the Vlasov equation. This is not completely straightforward; at least the energy functional can be shown to be non-increasing and then the kinetic energy can be shown to be finite for all times. To conclude, note that when we consider both Vlasov–Poisson and the relativistic heat equation separately, none of them gives nice regularizing properties, thus we might not get smooth solutions to our model. The Vlasov–Poisson equation does not regularize at all the distribution function f, although it is true that it provides a regularization mechanism at the macroscopic level (averaging lemmas furnish regularity for ρ, see [86, 103, 106, 127] for instance). Regarding the relativistic heat equation, we would get some regularization of our data in the sense that we will get some integrability for $\nabla_v f$ out of nothing, but at the same time the equation may spoil all other types of smoothness we would start with. To summarize, for the solutions of the complete model we only expect some integrability properties for $\nabla_v f$ and some regularity for ρ. Once one succeeds in proving that the system is well-posed, more interesting questions arise related to the qualitative behavior of the solutions and the compatibility with the special solutions associated with the Vlasov–Poisson system.

2. We also think that flux-limited diffusions have something to say about the mathematical modeling of chemotactic processes. The mathematical study of Chemotaxis started with the work of Patlak [169] and was driven by the papers of Keller and Segel, where they introduced a model to study the aggregation of Dictyostelium Discoideum due to an attractive chemical substance [131] and made some further comments and studies [132, 133]. We refer to [134] for a review about the first years of research on the Keller–Segel model.

   The original Keller–Segel model consists of an advection-diffusion system constituted by two coupled parabolic equations:

$$\begin{cases} \partial_t n = \mathrm{div}_x(D_n \nabla_x n - \chi n \nabla_x S) + H(n,S), \\ \\ \partial_t S = D_S \Delta S + K(n,S), \end{cases} \tag{0.1}$$

where $n = n(t, x)$ is the cell density at position x and time t, $S = S(t, x)$ being the density of the chemo-attractant. The positive definite terms D_S and D_n are the diffusivities of the chemo-attractant and of the cells, respectively, and $\chi \geq 0$ is the chemotactic sensitivity.

Let us briefly comment on the main aspects of model (0.1) in order to well understand its derivation from a microscopic approach and how to improve or incorporate some new fundamentals of the chemotaxis:

- Assuming that the chemical population undergoes a linear diffusion process seems reasonable in a preliminary approach. But in general the substance S does not only diffuse in the substrate, as it can also be produced by bacteria themselves. Recent results suggest that the chemical attractant acts by local diffusion but also has the possibility to jump over more long distances, like what happens for a Lévy process. That might be a research line.

- It is not completely clear how the term $\mathrm{div}_x(\chi n \nabla_x S)$ induces per se the optimal movement of the cells towards the pathway determined by the chemoattractant. In our opinion this term could be modified in order that the flux density of particles be optimized along the trajectory induced by the chemoattractant, i.e. minimizing the functional

$$\int \chi n \, dS = \int \chi n \sqrt{1 + |\nabla_x S|^2}$$

with respect to S, dS being the measure of the curve defined by S. This provides an alternative term in the corresponding Euler-Lagrange equation of type

$$\mathrm{div}_x \left(\chi n \frac{\nabla_x S}{\sqrt{1 + |\nabla_x S|^2}} \right).$$

Of course, this term coincides with $\mathrm{div}_x(\chi n \nabla_x S)$ when $|\nabla_x S|$ is very small. But in the case $|\nabla_x S| \sim 0$ it will be necessary to compare this scale with the rest of the scales of the problem in order to be correct.

- It does not seem realistic to think that the cells or bacteria are moving by simple (linear Fokker-Planck) diffusion, $\mathrm{div}_x(D_n \nabla_x n)$, by assuming that D_n is constant. There are some possibilities to modify this approach based on incorporating real phenomena related with cell or bacteria motion (cilium activation or elasticity properties of the membrane, among others) that could be represented by a nonlinear limited flux that would allow for richer and more realistic dynamics: finite speed of propagation c, preservation of fronts along the evolution or formation of biological patterns. This corresponds with an optimal mass transportation approach as it has been motivated previously and would be represented by terms of the type

$$\mathrm{div}_x \left(D_n n \frac{\nabla_x n}{\sqrt{n^2 + \frac{D_n^2}{c^2} |\nabla_x n|^2}} \right).$$

Considering these facts we propose to modify the Keller–Segel model by accounting for flux limitation mechanisms and imposing optimal transport of the population n following the chemical signal S, as follows

$$\begin{cases} \partial_t n = \mathrm{div}_x \left(D_n \dfrac{n \nabla_x n}{\sqrt{n^2 + \frac{D_n^2}{c^2} |\nabla_x n|^2}} - n\chi \dfrac{\nabla_x S}{\sqrt{1 + |\nabla_x S|^2}} \right) + H_2(n, S), \\ \partial_t S = \mathrm{div}_x (D_S \cdot \nabla_x S) + H_1(n, S); \end{cases}$$

where the functions $H_i(n, S)$, $i = 1, 2$, describe extra interactions between both populations. Let us point out that this particular model can be derived from first principles using Kinetic Theory for Active Particles [40]. We plan to perform the mathematical study of this model in collaboration with J.M. Mazón.

3. As regards the results in Chapter 6, we want to conclude the study of qualitative features of the solutions. For instance, it is likely that once a suitable concept of sub-solutions is found we will be able to ensure that the speed of propagating fronts is precisely c. The study of the convergence to the steady state solution constitutes another interesting open problem. We would also like to study the existence and uniqueness of solutions to the complete models (with source terms and coupling), the technical problem here is that non-linear semigroup theory does not work nicely when time-dependent source terms that do also depend on the solution itself are incorporated into the equations.

4. Concerning relativistic kinetic models, there are a number of questions that would be interesting to address. The study of dispersive behavior for the Einstein–Vlasov system is still a wide source of open problems. Namely, only results for small initial data [185, 191] and shells of outgoing matter [10] are available; here dispersion is understood in the sense that certain norms of the metric quantities decay with time. It would be desirable to generalize these results to a broader class of initial data. This problem is related to a couple of central questions. First, we have to understand what the precise meaning of dispersion is. This may depend on the system of coordinates, but just getting results in Schwarzschild coordinates would constitute an important goal. Moreover, there may be several meaningful ways in which a solution to the Einstein–Vlasov system disperses. Secondly, it is likely that if a solution shows such dispersive behavior then we shall have that such solution is global, at least in the set of coordinates used to describe its dispersive behavior. And we must not forget that we are not able to deal with the related issues for the simpler Nordström–Vlasov system, not even to give a physically meaningful (i.e Lorentz invariant) definition of dispersive behavior. We would like to find functionals that are able to generalize in a meaningful way the second spatial moment of solutions to the Vlasov–Poisson system or similar useful macroscopic quantities. These problems should constitute a good starting point before we confront them for the Einstein–Vlasov system.

   Another challenging problem is that of proving (or disproving) the stability of some static solutions to the spherically symmetric Einstein–Vlasov system; it should be kept in mind that under this formulation of the Einstein–Vlasov system we will be able to test stability only against spherically symmetric perturbations. Our intention is to study these problems in collaboration with S. Calogero.

5. In the dynamics of multiphase fluids described by a fluid-kinetic approach, particles are also affected by their interaction with the surrounding fluid. This could include and favor the possibility of fragmentation or coagulation for particles or droplets, which modifies the density or the velocity of the particles and also of the fluid. In the context of fragmentation this problem has been recently analyzed in the Navier–Stokes–Boltzmann coupling, see [128]. In this memory we have studied the coagulation properties of a mixture of two species under a kinetic formulation. It could be of great interest to extend this study to the interaction with a fluid.

6. The foundations of flux-limited models and related diffusion mechanisms remain at a phenomenological level, although some new results have been achieved by using hyperbolic limits of a kinetic system in [40]. However, a rigorous derivation of them from first principles is still lacking. We think that this gap could be filled by using some ideas from stochastic processes and diffusion by mean curvature flows; this is by now a work in progress.

Appendix A

Vector integration

This material can be found in [85]. Through the following let X be a Banach space and $\|\cdot\|$ its norm.

Definition A.1.1 *Let (Ω, Σ, μ) be a finite measure space and let $f : \Omega \to X$.*

1. *The function f is called* simple *if there exist $x_1, x_2, \ldots x_n \in X$ and $E_1, E_2, \ldots E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}(w) = 1$ if $w \in E_i$ and $\chi_{E_i}(w) = 0$ if $w \notin E_i$.*

2. *The function f is said to be* strongly measurable *if there exists a sequence of simple functions $\{f_n\}$ for which $\lim_n \|f_n - f\| = 0$ μ-almost everywhere.*

3. *The function f is said to be* weakly (or scalarly) measurable *if for each $x^* \in X^*$ the numerical function $x^* f$ is μ-measurable.*

If X is separable then both notions of measurability coincide.

Definition A.1.2 *A function $f : \Omega \to X^*$ is said to be* weakly* measurable *if the scalar functions $x^{**} f$ are μ-measurable for each $x^{**} \in X^{**}$ belonging to the image of X under the natural imbedding into X^{**}.*

Definition A.1.3 *Let (Ω, Σ, μ) be a finite measure space. A strongly measurable function is called* Bochner integrable *if there exists a sequence of simple functions $\{f_n\}$ such that*

$$\lim_n \int_\Omega \|f_n - f\| \, d\mu = 0.$$

If that's the case, we define for each $E \in \Sigma$ its Bochner integral *as*

$$\int_E f \, d\mu := \lim_n \int_E f_n \, d\mu,$$

where $\int_E f_n \, d\mu$ is defined as an element of X in the obvious way.

Definition A.1.4 *Let (Ω, Σ, μ) be a finite measure space. A weakly measurable function f is called* Dunford integrable *if $x^* f \in L^1(\Omega, \mu)$ for all $x^* \in X^*$. The* Dunford integral *of f over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ such that*

$$x_E^{**}(x^*) = \int_E x^* f \, d\mu \quad \forall x^* \in X^*.$$

Definition A.1.5 *Let* (Ω, Σ, μ) *be a finite measure space. A Dunford integrable function* f *such that* $x_E^{**} \in X$ *for each* $E \in \Sigma$ *is said to be* Pettis integrable*, being* $x_E^{**} \in X$ *its* Pettis integral *over* $E \in \Sigma$*.*

The Dunford and Pettis integrals coincide when X is reflexive. This may not be the case if X is not reflexive.

Appendix B

Nonlinear Semigroup theory

Here X will be a real Banach space with norm $\| \|$. A mapping A from X to the set 2^X of parts of X will be called an operator in X. Given $x \in X$ the value of A at x will be denoted by Ax. Let $D(A) := \{x \in X \ : \ Ax \neq \emptyset\}$ stand for the (effective) domain of A and $R(A) := \cup\{Ax \ : \ x \in D(A)\}$ for its range. The subset $\{(x, y) \in X \times X \ : \ y \in Ax\}$ is called the graph of A. Sometimes we identify A with its graph. The closure of the operator A is defined to be the closure of its graph in $X \times X$.

Our aim is to study evolution problems of this kind:

$$\begin{cases} u'(t) + Au(t) \ni 0 \quad \text{on } \ t \in]0, T[\\ \\ u(0) = x. \end{cases} \tag{1.1}$$

A mild solution of this problem is a continuous function $u \in C([0, T], X)$ which is the uniform limit of solutions to time-discretized problems given by the implicit Euler scheme of the form

$$\frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}} + Av(t_i) \ni 0.$$

This is stated in more precise terms below:

Definition B.1.1 *Let $\epsilon > 0$.*

1. *An* ϵ-discretization *of $u' + Au \ni 0$ consists on a system of difference relations*

$$\frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni 0$$

determined by a partition $0 \leq t_0 < t_1 < \cdots < t_N \leq T$ with $t_i - t_{i-1} \leq \epsilon$, $i = 1, \ldots, N$, $t_0 \leq \epsilon$ and $T - t_N \leq \epsilon$. This will be denoted by $D_A(t_0, \ldots, t_N)$.

2. *A* solution of the discretization *$D_A(t_0, \ldots, t_N)$ is a piecewise constant function $v : [0, T] \to X$ whose values $v(0) = v_0$, $v(t) = v_i$ for $t \in]t_{i-1}, t_i]$, $i = 1, \ldots, N$ satisfy*

$$\frac{v_i - v_{i-1}}{t_i - t_{i-1}} + Av_i \ni 0, \quad i = 1, \ldots, N.$$

3. *A* mild solution *of $u' + Au \ni 0$ in $[0, T]$ is a continuous function $u \in C([0, T]; X)$ such that, for each $\epsilon > 0$ there is an ϵ-discretization $D_A(t_0, \ldots, t_N)$ having a solution v which satisfies*

$$\|u(t) - v(t)\| \leq \epsilon \quad \textit{for } t_0 \leq t \leq t_N.$$

Note that to solve uniquely the discretized equations we need that the inverse of the operator $I + \lambda A$ be a single-valued operator, being I the identity operator and $\lambda \in \mathbb{R}$. This motivates the following definition.

Definition B.1.2 *An operator A in X is* accretive *if*

$$\|x - \hat{x}\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\|$$

whenever $\lambda > 0$ and $(x, y),\ (\hat{x}, \hat{y}) \in A$.

To be able to talk in a meaningful way about the inverse of the operator $(I + \lambda A)$ we need to state clearly that

$$(I + \lambda A)x = x + \lambda(Ax)$$

and

$$(I + \lambda A)^{-1}x = \{y \in X\ :\ x \in (I + \lambda A)y\}.$$

Note that Definition B.1.2 encodes the non-expansivity of the map $(I + \lambda A)^{-1}$. This property of the operator implies uniqueness of solutions for the discretized equations. In case A is accretive, we denote $J_\lambda^A = (I + \lambda A)^{-1}$ and we call J_λ^A the resolvent of A.

Next we introduce the notion of evolution semigroup.

Definition B.1.3 *Let D be a subset of X. A family of mappings $S(t) : D \to D,\ (t \geq 0)$ satisfying*

$$S(t + s)x = S(t)S(s)x \quad \textit{for all } t, s \geq 0,\ x \in D$$

and

$$\lim_{t \to 0} S(t)x = x \quad \textit{for } x \in D$$

is called a strongly continuous semigroup *on D.*

One may now associate to every operator A in X a strongly continuous semigroup $(S^A(t))_{t \geq 0}$ by the following definitions:

$$D(S^A) := \{x \in X\ :\ \exists!\, \text{mild solution } u_x \text{ of } u' + Au \ni 0\, \text{on }]0, +\infty[\, \text{with } u_x(0) = x\}.$$

For $t \geq 0$ and $x \in D(S^A)$ we set $S^A(t) := u_x(t)$. We also denote $S^A(t)$ by e^{-tA} and we call $(e^{-tA})_{t \geq 0}$ the semigroup generated by $-A$.

Theorem B.1.4 (Crandall–Liggett) *If A is accretive and satisfies the range condition $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$ then $-A$ generates a semigroup of contractions $(e^{-tA})_{t \geq 0}$ on $\overline{D(A)}$. Moreover, for $x_0 \in \overline{D(A)}$ and $0 \leq t \leq \infty$,*

$$\lim_{\lambda \downarrow 0,\ k\lambda \to t} (J_\lambda)^k x_0 = e^{-tA} x_0$$

holds uniformly for compact subintervals of $[0, \infty[$.

We deduce

$$e^{-tA}x = \lim_{n \to \infty} \left(I + \frac{t}{n} A \right)^{-n} x \quad \text{for } x \in \overline{D(A)}.$$

Our next set of definitions are aimed to discuss the interplay between a certain ordering on X and the action of the semigroup. The natural setting for this is the following:

Definition B.1.5 *A* Banach lattice *is a Banach space X together with a mapping*

$$(x, y) \in X \times X \mapsto x \vee y \in X$$

which is continuous and satisfies the the following properties:

$$x \vee x = x, \quad x \vee y = y \vee x,$$

$$\lambda(x \vee y) = (\lambda x) \vee (\lambda y),$$

$$(x \vee y) \vee z = x \vee (y \vee z), \quad (x \vee y) + z = (x + z) \vee (y + z),$$

for $\lambda \geq 0$ and $x, y, z \in X$. This mapping defines an order "$\leq$" on X according to

$$x \leq y \text{ if and only if } x \vee y = y$$

for which $x \vee y = \max\{x, y\}$.

Definition B.1.6 *Let X be a Banach Lattice and $S : D(S) \subset X \to X$.*

1. *S is* order-preserving *if $Sx \leq S\hat{x}$ whenever $x, \hat{x} \in D(S)$ and $x \leq \hat{x}$.*
2. *S is a* T-contraction *if*

$$\|(Sx - S\hat{x})^+\| \leq \|(x - \hat{x})^+\| \quad \text{for } x,\ \hat{x} \in D(S).$$

3. *An operator A in X is* T-accretive *if*

$$\|(x - \hat{x})^+\| \leq \|(x - \hat{x} + \lambda(y - \hat{y}))^+\| \quad \text{for } (x, y),\ (\hat{x}, \hat{y}) \in A \text{ and } \lambda > 0.$$

All this material was borrowed from [15, 43], where wider expositions on the subject can be found.

Appendix C

Functions of bounded variation in one dimension

For bounded variation functions the reader may consult [3, 15, 93]. Let $]0,L[\subset \mathbb{R}$ an interval, we say that a function $u \in L^1(]0,L[)$ is of bounded variation if its distributional derivative Du is a Radon measure on $]0,L[$ with bounded total variation $|Du|(]0,L[) < +\infty$. We denote by $BV(]0,L[)$ the space of all function of bounded variation in $]0,L[$. It is well know that given $u \in BV(]0,L[)$ there exists a function $\tilde{u}$ in the equivalence class of u, called a good representative of u with the following properties. If $\mathcal{J}_u$ is the set of atoms of Du, i.e., $x \in \mathcal{J}_u$ if and only if $Du(\{x\}) \neq 0$, then $\tilde{u}$ is continuous in $]0,L[\backslash \mathcal{J}_u$ and has a jump discontinuity at any point of $\mathcal{J}_u$:

$$\tilde{u}(x)^- := \lim_{y\uparrow x} \tilde{u}(y) = Du(]0,x[), \qquad \tilde{u}(x)^+ := \lim_{y\downarrow x} \tilde{u}(y) = Du(]0,x]) \quad \forall\, x \in \mathcal{J}_u.$$

Consequently,

$$\tilde{u}(x)^+ - \tilde{u}(x)^- = Du(\{x\}) \quad \forall\, x \in \mathcal{J}_u.$$

The set $\mathcal{J}_u$ is denumerable. Moreover, $\tilde{u}$ is differentiable at $\mathcal{L}^1$ a.e. point of $]0,L[$, and the derivative $\tilde{u}'$ is the density of Du with respect to $\mathcal{L}^1$. For $u \in BV(]0,L[)$, the measure Du decomposes into its absolutely continuous and singular parts $Du = D^{ac}u + D^s u$. Then $D^{ac}u = \tilde{u}'\,\mathcal{L}^1$. Obviously, if $u \in BV(]0,L[)$ then $u \in W^{1,1}(]0,L[)$ if and only if $D^s u \equiv 0$; in this case we have $Du = \tilde{u}'\,\mathcal{L}^1$. When we deal with pointwise valued BV functions we always shall use the good representative. Hence, in the case $u \in W^{1,1}(]0,L[)$, we shall assume that $u \in C([0,L])$.

Hereafter we will use the following notation, which might not be the standard one: Given $u \in BV(]0,L[)$, we set

$$\int_0^L Du := Du(]0,L[).$$

C.1 Some remarkable results

Next we state without proof a series of results that will be used often in Chapter 6.

Theorem C.1.1 (Lower semicontinuity of the total variation) *Let $\{u_n\} \subset BV(]0,L[)$ and $u_n \to u$ in $L^1(]0,L[)$. Then*

$$|Du|(]0,L[) \leq \liminf_{n\to\infty} |Du_n|(]0,L[).$$

Theorem C.1.2 (Traces) *There exist a bounded linear mapping*

$$Tr : BV(]0, L[) \to \mathbb{R}^2$$

such that

$$\int_0^L u\varphi' \, dx = -\int_0^L \varphi \, Du + Tr(u) \cdot (-\varphi(0), \varphi(L))$$

for all $u \in BV(]0, L[)$ *and* $\varphi \in C^1(]0, L[)$. *Moreover,*

$$Tr(u) = \lim_{r \to 0^+} \left(\frac{1}{r} \int_0^r u \, dx, \; \frac{1}{r} \int_{L-r}^L u \, dx \right).$$

We shall use the notation $Tr(u) = (u(0_+), u(L_-))$, which is coherent with the customary interpretation of the trace as "boundary values".

Theorem C.1.3 (Approximation by smooth functions) *Let* $u \in BV(]0, L[)$. *Then, there exists a sequence* $u_n \in C^\infty(]0, L[) \cap BV(]0, L[)$ *such that*

1. $\|u_n\|_\infty \le \|u\|_\infty$ *for all* $n \in \mathbb{N}$
2. $u_n \to u$ *in* $L^1(]0, L[)$
3. $\int_0^L |u_n'(x)| \, dx \to \int_0^L |Du|$
4. $u_n(L) = u(L_-), \quad u_n(0) = u(0_+)$ *for all* $n \in \mathbb{N}$

Definition C.1.4 *Let* $u, \; u_n \in BV(]0, L[)$. *We say that* $\{u_n\}$ *converges* weakly* *to* u *in* $BV(]0, L[)$ *if* $\{u_n\}$ *converges to* u *in* $L^1(]0, L[)$ *and* $\{Du_n\}$ *converges weakly* to* Du, *meaning that*

$$\lim_{n \to \infty} \int_0^L \phi Du_n = \int_0^L \phi Du \quad \forall \phi \in C_c(]0, L[).$$

Proposition C.1.5 *Let* $\{u_n\} \subset BV(]0, L[)$. *Then* $\{u_n\}$ *converges weakly* to* u *in* $BV(]0, L[)$ *if and only if* $\{u_n\}$ *is bounded in* $BV(]0, L[)$ *and converges to* u *in* $L^1(]0, L[)$.

Lemma C.1.6 *Let* μ_n *be a sequence of measures on* $]0, L[$ *that is weakly* convergent to a measure* μ. *Then*

$$\mu(K) \ge \limsup_{n \to \infty} \mu_n(K)$$

for any compact $K \subset]0, L[$.

Theorem C.1.7 (Coarea formula) *Let* $u \in BV(]0, L[)$. *Then*

$$\chi_{\{x \in]0,L[\; : \; u(x) > t\}} \in BV(]0, L[) \quad \textit{for a.e. } t \in \mathbb{R}.$$

and

$$|Du|(]0, L[) = \int_{-\infty}^{\infty} \mathrm{Per}(\{x \in]0, L[\; : \; u(x) > t\}) \, dt$$

being $\mathrm{Per}(\{x \in]0, L[\; : \; u(x) > t\}) = D\chi_{\{x \in]0,L[\; : \; u(x) > t\}}(]0, L[)$. *Note that this reduces to a counting measure.*

Theorem C.1.8 (Embeddings) *The embedding $BV(]0,L[) \to L^\infty(]0,L[)$ is continuous and the embeddings $BV(]0,L[) \to L^p(]0,L[)$ are compact for all $1 \leq p < \infty$.*

Theorem C.1.9 (Radon–Nikodym) *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\lambda : \mathcal{A} \to \mathbb{R}$ be a real measure which is absolutely continuous with respect to μ. Then there exists a unique $f \in L^1(\mu)$ such that*

$$\lambda(E) = \int_E f \, d\mu \quad \forall E \in \mathcal{A}.$$

We say that f is the Radon–Nikodym derivative of λ with respect to μ.

Proposition C.1.10 *Let $u \in BV(]0,L[)$. Then Du is absolutely continuous with respect to the total variation $|Du|$, hence it has a Radon–Nikodym derivative with respect to $|Du|$, that we will denote by $\frac{Du}{|Du|}$.*

Theorem C.1.11 (Chain rule) *Let $u \in BV(]0,L[)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then, $v = f \circ u$ belongs to $BV(]0,L[)$. Moreover,*

$$Dv = f'(u)u' \, \mathcal{L} + \frac{f(u^+) - f(u^-)}{u^+ - u^-} D^j u + f'(\tilde{u}) D^c u.$$

The formula for the derivative of the composition can be written in a compact way using Volpert's averaged superposition. Let

$$\overline{f_u}(x) = \int_0^1 f'(\tau u^+(x) + (1-\tau)u^-(x)) \, d\tau,$$

being $(u^+(x), u^-(x)) = (u(x)^+, u(x)^-)$ if $x \in \mathcal{J}_u$ and $(u^+(x), u^-(x)) = (\tilde{u}(x), \tilde{u}(x))$ if $x \in]0,L[\setminus \mathcal{J}_u$. Then

$$Dv = \overline{f_u} \, Du.$$

A vectorial version of the chain rule can be found in [3], Theorems 3.96 and 3.101.

C.2 Interplay with vector integration

For the rest of the paragraph we shall inquire about certain aspects of Bochner and related spaces having $BV(]0,L[)$ as target space. We stress the fact that $BV(]0,L[)$ is not separable and thus the notions of strong and weak measurability do not coincide.

It is well known (see for instance [213]) that the dual space $\left[L^1(0,T; BV(]0,L[))\right]^*$ is isometric to the space

$$L^\infty(0,T; BV(]0,L[)^*, BV(]0,L[))$$

of all weakly* measurable functions $f : [0,T] \to BV(]0,L[)^*$ such that the supremum of the set $\{|\langle w, f \rangle| \ : \ \|w\|_{BV(]0,L[)} \leq 1\}$ in the vector lattice of measurable real functions belongs to $L^\infty([0,T])$. Moreover, the duality pairing is

$$\langle w, f \rangle = \int_0^T \langle w(t), f(t) \rangle \, dt,$$

for $w \in L^1(0,T;BV(]0,L[))$ and $f \in L^\infty(0,T;BV(]0,L[)^*,BV(]0,L[))$.

By $L^1_w(0,T,BV(]0,L[))$ (resp. $L^1_{loc,w}(0,T,BV(]0,L[)))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(]0,L[)$ such that $\int_0^T \|w(t)\|\,dt < \infty$ (resp. $t \in [0,T] \to \|w(t)\|$ is in $L^1_{loc}(]0,T[)$). Observe that, since $BV(]0,L[)$ has a separable predual (see [3]), it follows easily that the map $t \in [0,T] \to \|w(t)\|$ is measurable.

The techniques of vector integration allow to set up a procedure to obtain a (time) regularization of functions with values in some non-separable Banach space. We detail it for the case that interests us in the following

Definition C.2.1 *Given $\phi \in \mathcal{D}(]0,T[)$ and $w \in L^1_{loc}(0,T;BV(]0,L[))$, we define $(\phi w)^\tau$ as the Dunford integral*

$$(\phi w)^\tau(t) := \frac{1}{\tau}\int_{t-\tau}^{t} \phi(s)w(s)\ ds \in BV(]0,L[)^{**},$$

that is

$$\langle(\phi w)^\tau(t),\eta\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \langle\phi(s)w(s),\eta\rangle\ ds \quad \forall\,\eta \in BV(]0,L[)^*.$$

In [14] it is shown that $(\phi w)^\tau \in C([0,T];BV(]0,L[))$, so that the integral defining $(\phi w)^\tau$ is in fact a Pettis integral. Note that being $\eta \in BV(]0,L[)^*$ fixed the map $t \mapsto \langle\phi(t)w(t),\eta\rangle$ belongs to $L^1_{loc}([0,T])$, hence almost any $t \in [0,T]$ is a Lebesgue point for it. Moreover, $(\phi w)^\tau$ admits a weak derivative in $L^1_w(0,T,BV(0,L)) \cap L^\infty(Q_T)$, see [14], which values

$$\frac{\phi(t)w(t) - \phi(t-\tau)w(t-\tau)}{\tau}.$$

Remark C.2.2 Under certain circumstances the BV^*-BV pairing can be represented by means of the standard integral of the product. For instance, given $u \in L^\infty(]0,L[)$ we can define the functional $T_u : BV(]0,L[) \to \mathbb{R}$ by means of $\langle T_u,v\rangle := \int_0^L u\,v$. Trivially $|\langle T_u,v\rangle| \le \|u\|_\infty\|v\|_1 \le \|u\|_\infty\|v\|_{BV}$ and then $T_u \in BV(]0,L[)^*$.

Let us remark that a complete description of BV^* cannot be achieved without the use of the axiom of choice.

Appendix D

Miscellaneous material

D.1 The linear transport equation

It is the simplest kinetic equation that we can think of, and represents the free streaming of a bunch of particles. It reads:

$$\frac{\partial}{\partial t} f(t,x,v) + v\nabla_x f(t,x,v) = 0.$$

We complement it with an initial datum $f(0,x,v) = f^0(x,v) : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}^+$. Its solution is explicitly given by the formula

$$f(t,x,v) = f^0(x-vt,v).$$

It follows immediately that all the norms $\|f(t)\|_{L^p(\mathbb{R}^6)}$ are preserved during evolution. It is also easy to see that the energy

$$H = \frac{1}{2}\int_{\mathbb{R}^6} v^2 f(t,x,v)\,dxdv$$

remains constant. We also notice that the system is Galilean invariant and satisfies

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^6} x^2 f(t,x,v)\,dxdv = 4H.$$

This implies that solutions are statistically dispersive with a t^2 rate.

We have several decay estimates for the solutions of this problem. We are interested in the following:

1. $|\rho(t,x)| \le \frac{1}{t^3}\|f^0\|_{L^1(\mathbb{R}^3_x, L^\infty(\mathbb{R}^3_v))}$

2. $\|\rho(t)\|_{L^{5/3}(\mathbb{R}^3_x)} \le \frac{C}{t^{6/5}}\|f^0\|^{2/5}\left(\int_{\mathbb{R}^6} x^2 f^0(x,v)\,dxdv\right)^{3/5}.$

Part of this material is borrowed from [173], where more information can be found.

D.2 Performing matchings with the Schwarzschild solution

Any compactly supported solution of the spherically symmetric Einstein–Vlasov system is continuously matched with an exterior Schwarzschild solution. The family of solutions introduced by K. Schwarzschild is the general solution representing vacuum in the framework of General Relativity under spherical symmetry; as a particular case this family contains Minkowsky's solution. This matching processes is analogous to the trivial prolongation by zero to the whole space of a compactly supported solution to the Vlasov–Poisson; in the relativistic setting this is no longer trivial.

The Schwarzschild solution of parameter H is given by the metric

$$-\left(1-\frac{2H}{r}\right)\,dt^2+\left(1-\frac{2H}{r}\right)^{-1}\,dr^2+r^2\,dw^2.$$

Coming back to the EV system, we use the definition of local mass

$$m(t,r)=4\pi\int_0^r s^2h(t,s)\,ds=\frac{r}{2}(1-e^{-2\lambda})$$

to guess on one hand that

$$e^{2\lambda}=\left(1-\frac{2m(t,r)}{r}\right)^{-1}$$

(thus we obtained one of the metric coefficients) and on the other hand that, being $R(t)$ the radius of our configuration,

$$m(t,R(t))=H.$$

So that for $r\geq R(t)$ we get

$$e^{2\lambda}=\left(1-\frac{2H}{r}\right)^{-1}.$$

For sure there holds that $r>2H$ in this case, since for $r\geq R(t)$ we have the condition $2m(t,r)\leq r$. To match the other metric coefficient with the corresponding one in Schwarzschild's metric, just write

$$\mu(r)+\lambda(r)=-\int_r^\infty 4\pi s(h+p^{\mathrm{rad}})e^{2\lambda}\,ds,$$

so that given $r\geq R(t)$ we know that $\mu+\lambda=0$ and accordingly $e^{\mu(r)+\lambda(r)}=1$. Thus,

$$e^{2\mu}=1-\frac{2H}{r}\quad\text{for } r\geq R(t).$$

Appendix E

Resumen y resultados

[1] La presente memoria de Tesis se estructura en torno a varios modelos matemáticos basados en ecuaciones diferenciales en derivadas parciales (EDPs) de tipo cinético que aparecen asociadas a varios fenómenos de los campos de la Astrofísica y la Biología.

¿Cúal es el significado de una EDP cinética? Los modelos cinéticos proporcionan descripciones estadísticas de sistemas compuestos por un gran número de partículas que interactúan entre sí. El objetivo es incorporar al nivel de descripción dado por la EDP de transporte las propiedades microscópicas de interacción entre partículas (las cuales han de ser deducidas a partir de primeros principios). Aquí aparece una interesante controversia entre modelos construídos a partir de leyes de interacción básicas y modelos fenomenológicos (que usualmente describen cantidades macrocópicas). En este sentido, las descripciones cinéticas proporcionan una escala intermedia entre la descripción microscópica y la macroscópica. El rango de aplicación de estos modelos es muy amplio, y lo suficientemente flexible como para admitir que muy diferentes entidades jueguen el rol de las partículas acerca de las que versan los modelos cinéticos: desde los sistemas más grandes que imaginar podamos (galaxias y el propio Universo) hasta objetos realmente pequeños para los cuáles los efectos cuánticos pueden ser importantes, como puede ser el caso de sistemas compuestos por partículas atómicas. Más aún, el concepto de partícula puede incluir individuos con la capacidad de tomar decisiones: vehículos en una carretera o células en contextos biológicos, por ejemplo. Por otra parte, la versatilidad de estos modelos permite incluir toda una serie de interacciones de distinta naturaleza en el sistema de EDPs, como pueden ser interacciones de largo alcance en el caso de potenciales gravitatorios o electrostáticos, interacciones de corto alcance, como procesos de agregación o coagulación de varias partículas, o incluso efectos difusivos. Otro aspecto a destacar es la posibilidad de obtener otros modelos interesantes llevando a cabo límites macroscópicos a partir de la EDP cinética, ya sean de tipo parabólico difusivo (régimen de campo bajo) o de tipo hiperbólico (régimen de campo alto), que incorporen las propiedades de los niveles microscópicos. Los efectos de cualquiera de estas elecciones son cruciales para las propiedades cualitativas del transporte mediante flujos (parabólicos o hiperbólicos) de estructuras geométricas, frentes, patrones, etc. En este contexto la pregunta natural que surge es: ¿qué tipo de descripción se ajusta mejor al estudio de una realidad física concreta? Ésta es una de las ideas conductoras de la presente memoria.

[1] The present chapter constitutes an almost literal translation to spanish of Chapter 1: Introduction and results.

Uno de los principales aspectos conceptuales relacionados con esta Tesis es la discusión sobre cómo modelar pequeñas fluctuaciones, que podrían ser originadas a partir de interacciones no incluidas inicialmente en el modelo matemático, de forma que la dinámica del sistema (patrones, frentes, configuraciones especiales, singularidades, etc) no sea destruída (desvanecimiento). Los límites macroscópicos de tipo parabólico e hiperbólico son las dos formas —aparentemente dispares— de tratar este problema que encontramos en la literatura. Estos límites van asociados a los conceptos de difusión y dispersión respectivamente. La Tesis también trata con una situación intermedia: las correcciones de flujo limitado, que inducen un comportamiento más cercano a las ecuaciones hiperbólicas que a las parabólicas (Fokker–Planck), aunque las soluciones de los sistemas que incluyen este tipo de términos heredan características de ambos campos, como son la velocidad finita de propagación del soporte o los efectos regularizantes en el interior del soporte.

La presente memoria consta de dos partes bien diferenciadas. En la primera analizaremos varios conceptos de dispersión en el marco de la dinámica de materia tipo Vlasov clásica o relativista, dando ejemplos concretos y condiciones implicando dispersión, coagulación, aparición de configuraciones espaciales estacionarias o "breathing modes". El cometido de esta parte es conectar toda una serie de conceptos y resulta por ello bastante menos técnica que la segunda, dedicada al análisis de un sistema con limitación de flujo, motivado por el estudio del transporte de morfógenos. El lector puede escoger el punto de partida que mejor le parezca sin temor a perderse, pues desde el punto de vista matemático y descriptivo ambas partes son bastante independientes. El plan de la Tesis se basa en la dirección que va desde los niveles de descripción (matemática) microscópico al macroscópico.

Describamos brevemente los contenidos de la mísma. En la primera parte de esta memoria utilizaremos descripciones cinéticas que sólo incluyen interacciones de largo alcance (salvo en el último capítulo de la mísma, en el que incluiremos otro tipo de interacciones). Veremos que este escenario permite gran variedad de posibilidades y en consecuencia una dinámica muy rica. Dependiendo de ciertos parámetros macroscópicos de la condiciones iniciales podemos observar la aparición de configuraciones espaciales (o patrones), "breathing modes", dinámicas dispersivas e incluso fenómenos más complejos. Daremos ejemplos explícitos de todo ello en cuanto tratemos con sistemas concretos. Estas estructuras pueden ser eventualmente robustas (como es el caso de las configuraciones estacionarias estables o el de los solitones) y no por ello carecer de dinámica interna. Estas características no deben ser subestimadas, en cuanto que hay un número importante de ejemplos en el mundo real que coinciden con este tipo de descripción. Por ejemplo la dinámica tumoral, la dinámica galáctica o los halos de materia oscura, por nombrar algunos en los que estaremos interesados en esta memoria.

Es por tanto importante remarcar que tenemos la posibilidad de modelar un amplio conjunto de fenómenos utilizando ecuaciones hiperbólicas que pueden describir comportamiento dispersivo en algunos regímenes. Incluso podemos aumentar nuestras posibilidades si permitimos otro tipo de interacciones. Como ejemplo de ello trataremos en el último capítulo de la primera parte el caso de sistemas hiperbólicos que incluyen también mecanismos de coagulación (formación de agregados).

Una cuestión muy importante es la relativa al uso de términos difusivos en nuestras ecuaciones. Estos términos están relacionados con algún tipo de aleatoriedad, ya sea porque un modelo explícito no es conocido o porque haya demasiados factores a tener en cuenta que podrían convertir al modelo en algo demasiado complejo. Lo importante aquí es que si sabemos que esta variables incontrolables no tienen una influencia muy grande sobre los fenómenos en consideración, querremos que esto se refleje en nuestras ecuaciones.

No cualquier forma de aleatoriedad servirá para un problema dado; hacer una elección correcta de este tipo de términos se convierte en una cuestión decisiva. Sabemos que los mecanismos de difusión estándar (lineal, relacionados con movimiento browniano) suavizarán absolutamente todo, y este mecanismo de uniformización podría destruir en mucho casos la mayoría de la estructura en la que pudiéramos estar interesados para el tipo de problemas que antes hemos mencionado. Este es precisamente el caso si a las ecuaciones cinéticas que se usan para describir la evolución de un sistema gravitatorio afectado por interacciones de largo alcance les añadimos términos de tipo Fokker–Planck. Aquí aparece una idea recurrente: la confrontación entre dispersión y difusión, idea que está en el fondo de muchas de las cuestiones que se van a tratar en esta memoria. El comportamiento dispersivo resulta compatible con otras estructuras complejas que las descripciones cinéticas pueden asumir: respeta las invarianzas (leyes de conservación) del sistema y puede coexistir con la aparición de configuraciones estables, por ejemplo. Este no tiene por qué ser el caso de los mecanismos de difusión estándar (lineal), que no preservan cantidades físicamente relevantes (más bien las disipan) y tienden a destruir toda la dinámica, sin importar lo pequeña que pudiera ser su contribución. De hecho, son las colas de las distribuciones Gaussianas las únicas estructuras que sobreviven a la dinámica, inclusive siendo inicialmente muy pequeñas.

Dado que necesitaremos que nuestros modelos sean capaces de preservar estructuras macroscópicas (que de hecho son físicamente observables), la búsqueda de mecanismos de difusión alternativos que sean capaces de ello está más que justificada. Tales difusiones han de ser necesariamente no lineales. Muchos de estos mecanismos han sido investigados en la literatura física y matemática; estaremos interesados en todos aquellos que cumplan además la propiedad adicional de que la velocidad de propagación sea finita (cual es el caso de las ecuaciones de transporte cinéticas). Las ecuaciones de los medios porosos proporcionan este tipo de descripciones, aunque la velocidad de propagación no es una característica intrínseca de las leyes que gobiernan los fenómenos observables, sino que depende de la configuración inicial. Un enfoque capaz de cumplir con todas las condiciones buscadas es el de los mecanismos de limitación de flujo, que será objeto de estudio en la segunda parte de esta memoria. Estos modelos permiten estructuras robustas, como son frentes que se propagan, e introducen nuevos fenómenos como ondas viajeras singulares; veremos algunas de ellas utilizando modelos macroscópicos. De hecho, los resultados contenidos en esta Tesis junto con los de [200] demuestran que la aplicación de este tipo de argumentos al problema del transporte de morfógenos implica la eliminación de la difusión (que para este problema no tiene base física) e induce la preservación de estructuras dinámicas como pueden ser la propagación de frentes o las respuestas biológicas ante ellos, hechos en perfecta concordancia con los resultados experimentales. Por otra parte, hay varios intentos recientes para intentar deducir los términos de limitación de flujo a partir de primeros principios microscópicos; podemos mencionar aquí los límites hiperbólicos de sistemas cinéticos para un sistema de quimiotaxis con flujo limitado [40] o la difusión mediante procesos estocásticos relacionados con flujos por curvatura media. Podemos observar las diferencias cualitativas entre una ecuación de difusión lineal y una ecuación de flujo limitado que permite propagación de frentes en la Figura 1.1 de la introducción en inglés.

A modo de resumen, tendremos por un lado ecuaciones cinéticas para estudiar comportamiento dispersivo y estados estacionarios en Astrofísica, y también utilizaremos estas ecuaciones para estudiar el comportamiento a tiempo largo de la dinámica de poblaciones biológicas; el carácter de estas ecuaciones es principalmente hiperbólico. Por otro lado tenemos las ecuaciones de flujo limitado, que exhiben un mezcla de comportamientos hiperbólico y parabólico, y que serán utilizadas para modelar el transporte de morfógenos en el embrión. Estas ecuaciones pueden jugar un papel en Astrofísica, como alternativa a las ecuaciones de Fokker–Planck.

A continuación detallamos los problemas que vamos a estudiar, los modelos matemáticos que utilizamos para ello y los resultados que hemos obtenido.

E.2 Sistemas gravitatorios

En esta memoria de Tesis estudiaremos algunos modelos cinéticos que sirven para describir sistemas gravitatorios. Trabajaremos por ello siempre en tres dimensiones espaciales. Nos centramos principalmente en el comportamiento a tiempo largo de sus soluciones y en ciertas propiedades de sus soluciones estacionarias. Estos temas tienen aplicaciones en el campo de la dinámica galáctica, y de hecho presentamos una aplicación directa de varias de estas ideas a uno de los temas actuales más candentes de la Astrofísica: el modelado de halos de materia oscura.

Vamos a introducir en este párrafo varios modelos matemáticos ampliamente usados para describir sistemas gravitatorios. El tipo de objetos en los que estamos pensando son galaxias, grandes agrupaciones de galaxias e incluso halos de materia oscura. Estos sistemas comparten la característica de estar compuestos por un gran número de entidades individuales, o partículas (digamos las estrellas de una galaxia, las propias galaxias vistas dentro de una agrupación galáctica, etc), que evolucionan bajo interacciones gravitatorias. Estas características también están presentes en otros sistemas físicos relevantes, como son los gases o los plasmas por ejemplo; la única diferencia es que las leyes de interacción son de naturaleza distinta. Veremos que la teoría cinética proporciona un marco común en el cual todos estos sistemas se pueden tratar en pie de igualdad.

Es razonable suponer que las galaxias actuales se encuentran aproximadamente en un estado estacionario, queriendo esto decir que varían de forma tan lenta que esto no resulta apreciable en nuestra escala temporal y por tanto los podemos considerar como objetos estáticos (los físicos se refieren a estas configuraciones con el nombre de estados o equilibrios metaestables). De forma que una representación mediante modelos estáticos parece conformar un acercamiento coherente al estudio de estos objetos. Esto nos insta a encontrar y analizar soluciones estacionarias de nuestros modelos (cinéticos). Es también interesante estudiar en qué forma evolucionan las soluciones dinámicas hacia semejantes estados, y estudiar si una configuración dada puede realmente vagar en torno a o evolucionar hacia tales estados, o exhibir por contra un comportamiento muy diferente. Una vez que hayamos presentado los modelos nos centraremos en el comportamiento a tiempo largo de sus soluciones y en ciertas propiedades de sus soluciones estáticas.

Una primera aproximación inocente sería modelar sistemas gravitatorios como colecciones de masas puntuales evolucionando bajo interacciones gravitatorias. Esto es, un problema de N cuerpos. Es bien conocido el hecho de que este modelo no se puede resolver de forma analítica, pero tenemos una dificultad más importante, de naturaleza más práctica que teórica. Usualmente el número de partículas que conforma estos sistemas es enorme, de forma que no hay manera de ni tampoco interés en registrar toda esta abrumadora cantidad de información.

Este primer intento se abandona en pos de modelos más sofisticados y pretensiones algo menos ambiciosas: la descripción del comportamiento dinámico a grosso modo puede ser suficiente para la mayoría de los propósitos. Los modelos continuos se ajustan mejor que los discretos al cometido de llevar a cabo tales predicciones teóricas; en particular los más ampliamente utilizados en este campo son los modelos cinéticos.

De hecho, el propósito de la teoría cinética en su formulación general no es otro que la descripción de gases a una escala intermedia entre la microscópica y la hidrodinámica. Éstos cubren un rango importante de aplicaciones, puesto que los gases son, hablando con cierta libertad, sistemas con un número muy grande de partículas que son descritos a un nivel estadístico. Para este tipo de problemas una descripción de la posición y la velocidad de cada partícula es irrelevante, pero la descripción del propio sistema no se puede reducir al cálculo de una velocidad promedio en una posición e instante de tiempo dados (que sería el caso con modelos de tipo fluido). Queremos poder incorporar la posibilidad de tener más de una velocidad en cada punto y por tanto la descripción se ha de llevar a cabo al nivel del espacio de fases.

De forma que pretendemos una descripción estadística de nuestro sistema, en términos de lo que se conoce como función de distribución, que depende del tiempo t y de las coordenadas del espacio de fases (x, v) —para el caso de modelos relativistas es mejor usar (x, p), siendo p el momento. El valor de esta función en un punto dado corresponde al número probable de partículas que encontraremos en un volumen infinitesimal en torno a ese punto en el espacio de fases. Conviene retener que toda la información macroscópica está codificada en este objeto matemático, por complicado que pudiera ser. Sólo es necesario saber cómo extraerla, lo cuál explicaremos para cada uno de los modelos que vamos a considerar.

Como ya hemos dicho, hay varias descripciones posibles de los sistemas gravitatorios, las cuáles dependen obviamente del tipo de efectos que queramos considerar y de los que prefiramos ignorar. Pero todas ellas comparten una serie de principios comunes. Lo primero de todo, el objeto central es la función de distribución $f(t, x, v)$, que describe la evolución del conjunto de partículas en sentido estadístico. De forma que dos exigencias básicas son que esta función sea no negativa y localmente integrable en espacio de fases, para que la interpretación física que hemos mencionado anteriormente tenga sentido. Un segundo punto crucial es dar una ley para la evolución de la función de distribución. Aquí es donde aparecen las diferencias entre los distintos modelos, pero en cualquier caso todas las ecuaciones que codifican las leyes para la evolución de f se obtienen a partir del mismo principio: la llamada ecuación de Vlasov. Esta ecuación establece que la derivada material (o total) de f es igual a la tasa de cambio a lo largo de las trayectorias de las partículas en el espacio de fases. Denotaremos a esta tasa de cambio mediante $C(f)$. Tales cambios están asociados a interacciones de corto alcance. Aquí se pueden incluir colisiones entre partículas —en un sentido amplio— o fenómenos de coagulación entre ellas, que dan lugar típicamente a términos bilineales; también podemos considerar efectos de fragmentación, que se pueden incluir usando términos de tipo lineal.

Calculemos ahora la derivada material de f. Bajo el dominio de la mecánica clásica las leyes de Newton establecen que para las trayectorias de las partículas la derivada de la posición es la velocidad y la derivada de la velocidad es la fuerza que se ejerce sobre la partícula, digamos F —todo esto ha de ser modificado adecuadamente para considerar modelos relativistas. Entonces la derivada material de f se puede escribir como

$$\frac{Df}{Dt} = \frac{df}{dt} + v \cdot \nabla_x f + F \cdot \nabla_v f$$

y de este modo la ecuación de Vlasov resulta

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = C(f).$$

Dependiendo del tipo de interacciones de largo alcance F y del tipo de segundos miembros $C(f)$ obtendremos diferentes modelos; consideraremos algunos de ellos sucesivamente.

E.3 El caso clásico

Estudiaremos en primer lugar el sistema clásico de Vlasov–Poisson, que proporciona una descripción estadística de un conglomerado de muchas partículas que evolucionan en ausencia de colisiones y de acuerdo a las leyes de Newton de la gravitación en el potencial gravitatorio autogenerado $\phi(t,x)$. La función de distribución $f(t,x,v)$ del conjunto y su densidad asociada

$$\rho(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\, dv$$

satisfacen (con $G = 1$)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ \Delta_x \phi = 4\pi\rho. \end{cases}$$

La dinámica de las soluciones de este sistema se estudia en el Capítulo 2, concentrándonos especialmente en su comportamiento dispersivo. Este estudio se lleva a cabo mediante un análisis detallado de las posibles formas en las que una solución puede exhibir comportamiento dispersivo. Nos concentramos en dispersión fuerte (en el sentido de las normas), en lo que llamamos dispersión total o parcial —dos nociones que cuantifican la cantidad de masa que una solución pierde por el infinito— y en la dispersión estadística (crecimiento ilimitado de la varianza de la función de densidad). Encontramos que la aparición de cualquiera de ellas está profundamente vinculada a los valores de ciertos parámetros macroscópicos de los sistemas a estudiar: su masa M, su momento lineal Q y su energía H, definidas por

$$M = \int_{\mathbb{R}^6} f\, dvdx, \quad Q = \int_{\mathbb{R}^6} v\, f\, dvdx,$$

y

$$H = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f\, dvdx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla_x \phi|^2 dx = E_{\rm kin} - E_{\rm pot},$$

siendo todas ellas cantidades conservadas. Daremos ejemplos de todos ellos a lo largo del capítulo 2, a partir de ciertas construcciones de anillos de materia que se desplazan hacia el exterior y modificaciones de éstos. La relación entre todas estas nociones de dispersión también se ha clarificado. Tenemos el siguiente resultado:

Proposición 1 *Sea f una solución regular del sistema de Vlasov–Poisson. Equivalen:*

1. *f es dispersiva en sentido fuerte.*
2. *f es totalmente dispersiva.*
3. *La energía potencial se desvanece para $t \to \infty$.*

Además, si cualquiera de las anteriores posibilidades ocurre, entonces f cumple la siguiente desigualdad

$$H \geq \frac{Q^2}{2M} \, .$$

Finalmente, si f es total o parcialmente dispersiva entonces es también estadísticamente dispersiva.

A continuación estudiamos las ratios de dispersión más rápidas permitidas para la dispersión fuerte, extendiendo y recuperando con una prueba distinta los resultados en [87]. El nuevo resultado (Proposition 2.3.3) asegura que

$$\|\rho(t)\|_p \geq C(1+t)^{-\frac{3(p-1)}{p}} \quad \text{para } t \gg 1, \quad p \in]1, \infty].$$

También completamos los resultados dados en [87] acerca de la ocurrencia de dispersión estadística en el régimen $H > \frac{Q^2}{2M}$, analizando el caso límite $H = \frac{Q^2}{2M}$ que aquellos resultados no cubrían. Nuestra conclusión es que también se tiene dispersión estadística en este caso, genéricamente con una ratio al menos lineal en tiempo (Proposition 2.3.8). Damos ejemplos de soluciones estadísticamente dispersivas tales que $H < \frac{Q^2}{2M}$; puesto que en este régimen existen soluciones no dispersivas (por ejemplo estados estacionarios), la dinámica bajo la condición $H < \frac{Q^2}{2M}$ aparenta ser mucho más complicada y su comprensión necesitará bastante trabajo en el futuro. Como ejemplo de esto presentamos soluciones que aún siendo dispersivas permanecen en la región de estabilidad de estados estacionarios estables.

Posteriormente comprobamos todas las herramientas introducidas con las soluciones de Kurth [136], que constituyen el ejemplo más conocido de soluciones dinámicas del sistema de Vlasov–Poisson, ya que se pueden describir de forma casi explícita. Finalmente tratamos también otras dos clases de soluciones que exhiben comportamiento dinámico interesante: soluciones periódicas en tiempo (para las que deducimos la relación $H < -\frac{Q^2}{2M}$ entre sus parámetros macroscópicos —Proposition 2.3.12) y soluciones virializadas. Para esta segunda clase generalizamos el teorema virial acerca de sistemas de N cuerpos [177] al continuo en el Lemma 2.3.13, conectando la identidad virial con una cierta condición de crecimiento del sistema (a saber, que su varianza espacial crezca en tiempo estríctamente más despacio que t^2).

E.4 Generalizaciones relativistas

Cuando los efectos relativistas pasan a ser importantes el sistema de Vlasov–Poisson deja de ser una descripción válida y otros modelos han de ser usados. Hoy día se acepta que la generalización correcta del sistema de Vlasov–Poisson es el sistema de Einstein–Vlasov, en el cuál la ley de Poisson se sustituye por el acoplamiento con las ecuaciones de Einstein de la Relatividad General. El sistema que resulta es bastante complejo y actualmente todavía no demasiado bien comprendido. En consecuencia una estrategia común para su análisis es reducirlo a situaciones con simetría, o bien tratar con otras generalizaciones relativistas más sencillas.

El sistema de Einstein–Vlasov en simetría esférica se escribe en coordenadas de Schwarzschild (tomando unidades tales que G=c=1) como

$$\partial_t f + e^{\mu-\lambda}\frac{v}{\sqrt{1+|v|^2}}\cdot\nabla_x f - \left(\lambda_t\frac{x\cdot v}{r} + e^{\mu-\lambda}\mu_r\sqrt{1+|v|^2}\right)\frac{x}{r}\cdot\nabla_v f = 0,$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 h\,,$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p^{\mathrm{rad}}\,,$$

siendo

$$p^{\mathrm{rad}}(t,r) = \int_{\mathbb{R}^3}\left(\frac{x\cdot v}{r}\right)^2 f\frac{dv}{\sqrt{1+|v|^2}}$$

la presión radial,

$$h(t,r) = \int_{\mathbb{R}^3}\sqrt{1+|v|^2}f\,dv$$

la densidad de energía y λ, μ los potenciales métricos —en el sentido de que la métrica queda completamente determinada a partir de estas dos funciones—. Estudiamos esta formulación en el Capítulo 3.

Los parámetros macroscópicos relevantes para una solución del sistema de Einstein–Vlasov en simetría esférica son la ADM masa (o energía) H y la masa total en reposo M, definidas por

$$H = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sqrt{1+|v|^2}\,f\,dvdx, \quad M = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\lambda} f\,dvdx.$$

Estas cantidades permanecen constantes para las soluciones regulares. Otra cantidad que resulta ser importante es el redshift central, definido mediante $Z_c := e^{-\mu(0)} - 1$. Corresponde al redshift de un fotón emitido desde el centro de la galaxia, y no necesariamente ha de permanecer constante durante la evolución de las soluciones.

Demostramos una identidad virial muy general para soluciones de este sistema (Lemma 3.4.1) que posteriormente particularizamos al caso de estados estacionarios y de esta forma obtenemos una identidad que relaciona algunos de sus parámetros macroscópicos. El resultado es el siguiente:

Proposición 2 *Sea f una solución estática y con soporte compacto del sistema de Einstein–Vlasov en simetría esférica con ADM masa (o energía) H, masa en reposo M y redshift central Z_c. Entonces se tiene la siguiente desigualdad:*

$$Z_c \geq \left|\frac{M}{H} - 1\right|.$$

Estudiamos también dos clases particulares de estados estacionarios. Para los estados estacionarios de tipo Jeans (soluciones estáticas dependientes de cantidades conservadas a lo largo de geodésicas, ver Capítulo 3 para una definición precisa) con radio R demostramos que

$$e^{\mu(0)} \leq \min\left\{1, \frac{M}{H}\right\} \sqrt{1-\frac{2H}{R}},$$

mientras que para configuraciones de tipo anular con radio interno R_1 probamos que

$$R_1 \leq \frac{18H}{\ln\left(\left|\frac{M}{H}-1\right|+1\right)}.$$

Otra generalización relativista del sistema de Vlasov–Poisson que vamos a considerar en esta memoria es el sistema de Norsdtröm–Vlasov, el cuál constituye un modelo no físico pero que aún así incorpora algunas características interesantes de la Teoría de la Relatividad General (por medio de una teoría escalar de la gravitación) y es más manejable que el sistema de Einstein–Vlasov. Por tanto constituye un buen banco de pruebas. Consideraremos este sistema en la siguiente formulación

$$\partial_t f + \frac{p}{\sqrt{e^{2\phi}+|p|^2}} \cdot \nabla_x f - \nabla_x\left(\sqrt{e^{2\phi}+|p|^2}\right) \cdot \nabla_p f = 0,$$
$$\partial_t^2 \phi - \Delta_x \phi = -e^{2\phi} \int_{\mathbb{R}^3} f \, \frac{dp}{\sqrt{e^{2\phi}+|p|^2}} \, .$$

La función ϕ determina la métrica del espaciotiempo subyacente y por tanto puede ser considerada como una suerte de potencial. La energía local y el momento de una solución (f, ϕ) se definen respectivamente como (i=1,2,3)

$$h(t,x) = \int_{\mathbb{R}^3} \sqrt{e^{2\phi}+|p|^2}\, f\, dp + \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}|\nabla_x \phi|^2,$$
$$q_i(t,x) = \int_{\mathbb{R}^3} p_i f\, dp - \partial_t \phi\, \partial_i \phi\,,$$

donde ∂_i denota la derivada parcial a lo largo de x^i. La energía total y el momento total

$$H = \int_{\mathbb{R}^3} h(t,x)\, dx, \quad Q = \int_{\mathbb{R}^3} q(t,x)\, dx$$

son cantidades conservadas a lo largo de la evolución. Además, las soluciones del sistema de Nordström–Vlasov satisfacen también la conservación de la masa total en reposo

$$M = \int_{\mathbb{R}^3} \rho(t,x)\, dx = \int_{\mathbb{R}^6} f(t,x,v)\, dx dp.$$

Estudiamos varios aspectos de este sistema en los Capítulos 2 y 3. En el primero de ellos demostramos una estimación de dispersión en términos de la energía local h:

Proposición 3 *Sea (f,ϕ) una solución del sistema de Norsdtröm–Vlasov con masa M, energía H y momento Q. Supóngase que*

$$H^2 - HM - |Q|^2 > 0.$$

Entonces existe un instante de tiempo t_0 y constantes positivas $0 < C_1 < C_2$ tales que la varianza espacial $\Delta_x(t)$ de la función unitaria de densidad de energía

$$\Delta_x(t) = \int_{\mathbb{R}^3} |x - \bar{h}(t)|^2 \frac{h(t,x)}{H}\, dx, \quad \textit{donde} \quad \bar{h}(t) = \int_{\mathbb{R}^3} x \frac{h(t,x)}{H}\, dx,$$

verifica

$$C_1 t^2 \leq \Delta_x(t) \leq C_2 t^2 \qquad \forall t > t_0\,.$$

En el Capítulo 3 establecemos una identidad (Lemma 3.2.1) que toda solución dinámica satisface y a continuación la restringimos a estados estacionarios. Obtenemos que la energía de los estados estacionarios regulares está acotada por su masa (Theorem 3.2.2), una propiedad que está en claro paralelismo con el hecho de que las soluciones estacionarias del sistema de Vlasov–Poisson verifiquen que $H < 0$. De hecho, este resultado motivó el desarrollo de la Proposición 2.

E.5 Estudio de los halos de materia oscura

El primer bloque conceptual de esta memoria (prácticamente toda la primera parte), dedicado al modelado de sistema gravitatorios, concluye con una aplicación del sistema de Vlasov–Poisson al modelado matemático de halos de materia oscura, en el Capítulo 4. Éstos consisten en estructuras esféricas que envuelven a cada galaxia, constituidas a partir de algún tipo de materia exótica indetectable mediante mediciones directas. Sólo constatamos su presencia a partir de sus efectos gravitatorios, siendo estos realmente fuertes, pues se cree que estas estructuras contribuyen con nueve décimas partes a la masa total de la configuración resultante. Aunque este paradigma tiene varios detractores es una tendencia comúnmente aceptada en Astrofísica en la actualidad, y se han dedicado muchos esfuerzos a la elaboración de modelos para los perfiles de densidad de estos objetos. Éstos perfiles suelen pertenecer a una de las siguientes categorías: modelos fenomenológicos (ajustes a datos) y simulaciones numéricas. El perfil de densidad de Navarro-Frenk-White [164] es el más popular de entre los originados mediante simulaciones, mientras que el perfil Isotermo [109] y el perfil de Burkert [58] constituyen una buena representación de los modelos fenomenológicos. Ninguno de estos modelos tiene radio finito, lo cuál no es físicamente razonable. Aún más controvertido es el hecho de que los modelos generados mediante simulaciones numéricas predicen un valor infinito de la densidad en el centro de la configuración.

Nuestra propuesta consiste en generar perfiles de densidad para los halos de materia oscura utilizando una familia triparamétrica de soluciones del sistema de Vlasov–Poisson: los polítropos isótropos. Estos modelos vienen corroborados por toda una serie de fundamentos teóricos (desde la misma ecuación consolidada que los origina, pasando por sus propiedades de reescalado, hasta toda una serie de teorías termodinámicas que tratan de ámbitos mucho más generales que el nuestro — pero todavía en debate en la comunidad física) y son relativamente fáciles de manejar. Comparamos estos modelos politrópicos con aquellos que antes comentamos utilizando un criterio de mínimos cuadrados. Una vez que se optimizan los parámetros de ajuste se obtienen resultados bastante buenos: los errores son del orden de un 3%. Además, el origen analítico de nuestros modelos nos permite efectuar un desarrollo en torno al origen y sugerir la fórmula (6.4) para los perfiles de densidad de materia oscura en el centro, que subsana la divergencia de las simulaciones numéricas en la región central.

E.6 Comportamiento asintótico de un modelo de coagulación

La primera parte de esta memoria concluye con el Capítulo 5, donde estudiaremos un modelo cinético de coagulación que describe dos especies de partículas (típicamente moléculas o células). Las partículas pueden adoptar dos estados distintos: un estado "libre" en el cuál simplemente se mueven con una velocidad dada o un estado de agregación en el cuál las partículas ya no se mueven más. Los fenómenos de coagulación, agregación o adhesión de partículas a un grupo, ya sean éstas células, lípidos, gotas, proteínas, etc, son de fundamental importancia en procesos biológicos y biotecnológicos, ver por ejemplo [1, 92, 100, 217]. Ésta es la motivación principal para el modelo que estudiamos.

La distribución de partículas libres será representada mediante f y la función de densidad de las partículas agregadas mediante ρ. Como modelo matemático para la anterior situación en dimensión d consideramos el siguiente sistema de ecuaciones:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' - \beta(v)\rho(t,x) f(t,x,v) \tag{6.1}$$

$$\frac{\partial \rho}{\partial t} = \int_{\mathbb{R}^{2d}} \alpha(v,v') f(t,x,v') f(t,x,v)\, dv' dv + \rho(t,x) \int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv \tag{6.2}$$

completado con datos iniciales $0 \le f^0(x,v) \in L^1(\mathbb{R}^{2d})$ y $0 \le \rho^0(x) \in L^1(\mathbb{R}^d_x)$.

Las funciones $\alpha(v,v')$ y $\beta(v)$ representan núcleos de colisión o coagulación, y proporcionan, en el caso de α, la probabilidad de que dos partículas libres con velocidades v y v' coagulen, y en el caso de β, la probabilidad de que una partícula libre con velocidad v coagule con una partícula ya detenida. Estos núcleos de colisión seran no negativos y cumplirán una propiedad de dominación, motivada por consideraciones físicas (esto se explica con detalle en el Capítulo 5): ha de existir una constante $C > 0$ tal que

$$\alpha(v,v') \le C|v - v'|^a, \ \beta(v) \le C|v|^a, \quad \text{para algún } a \in \mathbb{R}. \tag{6.3}$$

Nos concentramos principalmente en el estudio del comportamiento asintótico de las soluciones de este modelo. A partir de las propias ecuaciones es fácil observar que la masa correspondiente a las partículas libres sólo puede disminuir y que la masa asociada a las partículas coaguladas aumentará. Por tanto la principal cuestión cuando $t \to +\infty$ es si todas las partículas libres terminarán por coagular o si algunas de ellas permanecerán eternamente libres. Demostramos que esta alternativa depende únicamente de la fuerza de las interacciones, codificada en el valor de a. Nuestro análisis se basa en estimaciones de dispersión para ecuaciones cinéticas. Mostramos también que para tiempos suficientemente grandes la distribución de partículas libres exhibe comportamiento autosemejante. Nuestros resultados quedan resumidos en el siguiente:

Teorema 1 *Supóngase que los núcleos* α, β *son no negativos, que verifican la condición* (6.3) *y la relación* $a + d > 0$. *Entonces, para cualquier configuración inicial* $0 \le f^0 \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $0 \le \rho^0 \in L^1 \cap L^\infty(\mathbb{R}^d_x)$ *tal que para algún* $\eta > 0$ *se verifique*

$$f^0(x,v) \le \frac{C}{1 + |v|^{\max\{a,0\}+d+\eta}}, \ \textit{para casi todo } (x,v) \in \mathbb{R}^d_x \times \mathbb{R}^d_v,$$

existe una solución débil del sistema (6.1)–(6.2) *con datos iniciales* $f(0,x,v) = f^0$ *y* $\rho(0,x) = \rho^0$. *Si esta solución débil se puede aproximar fuertemente en* $L^\infty(0,T,L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v) \times L^1(\mathbb{R}^d_x))$ *por una sucesión de soluciones regulares entonces esta solución débil es única. Además, existe una función* $g_\infty(x,v)$ *tal que*

$$\left\| f(t,x,v) - g_\infty\left(\frac{x}{t}, t(v - \frac{x}{t})\right)\right\| \to 0 \quad para\ t \to \infty$$

en la norma de $W^{-1,1}(\mathbb{R}^d_x, L^1(\mathbb{R}^d_v))$. *Finalmente,*

- *si* $a > 1 - d$ *(o si* $a > 1$ *en el caso de que* $d = 1$*) y tanto* f^0 *como* ρ^0 *son compáctamente soportadas en la variable* x*, la cantidad de masa* $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ *está acotada inferiormente por una constante positiva que es independiente del tiempo.*

- *si* $-d < a \leq 1 - d$*, la cantidad de masa* $\int_{\mathbb{R}^{2d}} f(t,x,v)\, dxdv$ *es estrictamente positiva para todo instante de tiempo, pero converge a cero cuando* t *se hace infinito.*

E.7 Un modelo de flujo limitado para el transporte de morfógenos

La última parte de esta memoria está dedicada al estudio del transporte de morfógenos en sistemas biológicos. Es éste un problema clásico, que data de los trabajos de Turing [225], Meinhardt [101, 157], Wolpert [234] o Lander [139], que plantearon la cuestión como uno de los principales problemas a la hora de comprender el transporte de proteínas mediante "signaling pathways": subyace la cuestión de si los gradientes morfogénicos son generados mediante difusión o no.

Nos concentramos en un problema algo más concreto: el estudio de la dinámica de la función morfogénica "Sonic Hedgehog" (Shh), que juega un papel muy importante en la evolución de algunos factores de trasncripción y en el proceso de diferenciación celular en el tubo neural del embrión. Estos procesos son de capital importancia para la biología del desarrollo. Por ejemplo, dentro del sistema nervioso central el desarrollo del tubo neural en vertebrados y más adelante del cerebro [198] dependen del proceso de señalización del Sonic Hedgehog. Este proceso de señalización tiene también un papel importante en la formación de tumores cancerígenos: los fallos en la regulación del "Shh pathway" provocan el desarrollo de varios tipos de tumores, incluyendo aquellos en la piel, la próstata y el cerebro [196].

Actualmente existen modelos matemáticos para estudiar este problema [202], pero desde nuestro punto de vista su utilización de mecanismos de difusión no es realista en este contexto. Como remedio a esta situación proponemos la supresión del mecanismo de difusión y en su lugar la introducción de un mecanismo de limitación de flujo; todo ello está explicado en detalle en el Capítulo 6.

Nuestro objetivo en esta segunda parte de la memoria es el análisis de un problema mixto asociado con un sistema de reacción-difusión no lineal con limitación de flujo para la concentración de Shh, $u(t,x)$, dado por

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u,u_x))_x - f(t-\tau, u(t,x))\, u(t,x) + g(t, u(t,x)), & \text{en} \quad]0,T[\times]0,L[\\ -\mathbf{a}(u(t,0), u_x(t,0)) = \beta > 0 \ \text{ y } \ u(t,L) = 0 & \text{en} \quad t \in]0,T[\,, \\ u(0,x) = u_0(x) & \text{en} \quad x \in]0,L[\,. \end{cases}$$

donde

$$\mathbf{a}(z,\xi) := \nu \frac{|z|\xi}{\sqrt{z^2 + \frac{\nu^2}{c^2}|\xi|^2}}\,,$$

siendo f la concentración de receptores transmembranales en las células, representando g la concentración del complejo que liga el morfógeno al receptor y estando incluida la dependencia con respecto a u mediante el acoplamiento con un sistema de siete ecuaciones diferenciales ordinarias. Estas siete ecuaciones modelan las tasas de cambio de las concentraciones de las proteínas involucradas en el "signaling pathway" provenientes de la cascada bioquímica interior a las células, véase [200]. El significado de las constantes físicas c, ν, τ se explica en el Capítulo 6.

Trataremos de hacer avanzar el conocimiento acerca de este problema biológico, y para ello estudiaremos como primer paso un modelo simplificado sin términos fuente. Este estudio se lleva a cabo en el Capítulo 6 utilizando teoría de semigrupos no lineal. El modelo simplificado en consideración resulta

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u,u_x))_x & \text{en} \quad]0,T[\times]0,L[\\ -\mathbf{a}(u(t,0), u_x(t,0)) = \beta > 0 \text{ y } u(t,L) = 0 & \text{en} \quad t \in]0,T[\,, \\ u(0,x) = u_0(x) & \text{en} \quad x \in]0,L[\,. \end{cases} \tag{7.4}$$

Rápidamente nos damos cuenta de que (y esto es algo que también se ha comprobado mediante simulaciones numéricas) la condición de contorno tipo Dirichlet ha de ser relajada a una condición de borde del tipo de la del problema del obstáculo. Esto se refleja en el hecho de que frentes que se propagan dinámicamente alcanzarán eventualmente el borde y persistirán tras ello (ver Figura 1.1 de la introducción en inglés). La formulación precisa de estas cuestiones se puede encontrar en el Capítulo 6.

La utilización de la teoría de semigrupos nos proporcionará soluciones de tipo "mild" para nuestro problema, pero nuestro propósito será caracterizar estas soluciones en términos más operativos. Para nosotros la teoría de semigrupos será la herramienta que nos proporcione un adecuado esquema aproximante a los problemas parabólicos. De manera que analizaremos los problemas elípticos asociados, construiremos el semigrupo correspondiente y lo utilizaremos para generar una sucesión de soluciones aproximadas, para las cuales se puede demostrar convergencia a una solución razonable del problema parabólico. También somos capaces de demostrar unicidad para esta clase de soluciones "razonables" (las llamamos "bounded entropic") de la ecuación parabólica. De hecho somos capaces de probar un resultado bastante más fuerte, pues las soluciones de la ecuación verifican una propiedad de contracción. Todo ello se obtiene mediante el uso de la técnica del desdoblamiento de variables de Kruzkov, convenientemente adaptada. Los resultados quedan recogidos en el siguiente enunciado.

Teorema 2 *Para cualquier dato inicial* $0 \leq u_0 \in L^\infty(]0,L[)$ *existe una única solución* u *del problema (7.4) in* $Q_T =]0,T[\times]0,L[$ *de tipo "bounded entropic", para cualquier* $T > 0$. *Además, si* $u(t)$, $\overline{u}(t)$ *son soluciones de (7.4) en* $Q_T =]0,T[\times]0,L[$ *de tipo "bounded entropic" asociadas a datos iniciales* u_0, $\overline{u}_0 \in L^\infty(]0,L[)^+$ *respectivamente, entonces*

$$\|(u(t) - \overline{u}(t))^+\|_1 \leq \|(u_0 - \overline{u}_0)^+\|_1 \qquad \text{para todo} \quad t \geq 0.$$

En particular, tenemos unicidad en la clase de soluciones "bounded entropic" del problema (7.4).

Durante este estudio hemos determinado también un perfil estacionario (Proposition 6.6.2) hacia el cuál todas las soluciones de este modelo parecen converger. También analizamos la velocidad de propagación de los frentes, demostrando que la velocidad de propagación de la señal entrante es precisamente c.

Bibliography

[1] M. Aizerman, T.A. Bak: Convergence to equilibrium in a system of reacting polymers. *Comm. Math. Phys.* **65**, 203–320 (1979)

[2] D.J. Aldous: Deterministic and stochastic models for coalescence (aggregation, coagulation): a review of the mean–field theory for probabilists. *Bernouilli* **5**, 3–48 (1999)

[3] L. Ambrosio, N. Fusco, D. Pallara: *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs (2000)

[4] A.A. Amsden, P.J. O'Rourke: The T.A.B. method for numerical calculation of spray droplet breakup. Presented at the *International Fuels and Lubricants Meeting and Exposition*, Toronto, Ontario, 2 Nov. 1987

[5] H. Andréasson: The Einstein–Vlasov System/Kinetic Theory. *Living Rev. Relativity* **8** (2005)

[6] H. Andréasson: Sharp bounds on 2m/r of general spherically symmetric static objects. *J. Diff. Eqns.* **245**, 2243–2266 (2008)

[7] H. Andréasson: Sharp bounds on the critical stability radius for relativistic charged spheres. *Commun. Math. Phys.* **288**, 715–730 (2009)

[8] H. Andréasson: On global existence for the spherically symmetric Einstein–Vlasov system in Schwarzschild coordinates. *Indiana Univ. Math. J.* **56**, 523–552 (2007)

[9] H. Andréasson, S. Calogero, G. Rein: Global classical solutions to the spherically symmetric Nordström–Vlasov system. *Math. Proc. Cambridge. Phil. Soc.* **138**, 533–539 (2005)

[10] H. Andréasson, M. Kunze, G. Rein: Global existence for the spherically symmetric Einstein–Vlasov system with outgoing matter. *Commun. Part. Diff. Eqns.* **33**, 656–668 (2008)

[11] H. Andréasson, G. Rein: A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein–Vlasov system. *Class. Quantum Grav.* **23**, 3659–3677 (2006)

[12] H. Andréasson, G. Rein: On the steady states of the spherically symmetric Einstein–Vlasov system. *Class. Quantum Grav.* **24**, 1809–1832 (2007)

[13] F. Andreu, J. Calvo, J.M. Mazón, J. Soler: On a nonlinear flux-limited equation arising in the transport of morphogenes. Submitted

[14] F. Andreu, V. Caselles, J.M. Mazón: Existence and uniqueness of solution for a parabolic quasilinear problem for linear growth functionals with L^1 data. *Math. Ann.* **322**, 139–206 (2002)

[15] F. Andreu, V. Caselles, J.M. Mazón: *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*. Progress in Mathematics **223**, Birkhauser (2004)

[16] F. Andreu, V. Caselles, J.M. Mazón: A Strongly Degenerate Quasilinear Equation: the Elliptic Case. *Ann. Scuola Normale Superiore di Pisa CL. Sci.* **V (3)**, 555—587 (2004)

[17] F. Andreu, V. Caselles, J.M. Mazón: A Strongly Degenerate Quasilinear Equation: the Parabolic Case. *Arch. Rat. Mech. Anal.* **176**, 415–453 (2005)

[18] F. Andreu, V. Caselles, J.M. Mazón: A Strongly Degenerate Quasilinear Elliptic Equation. *Nonlinear Analysis; Theory, Methods and Applications* **61**, 637–669 (2005)

[19] F. Andreu, V. Caselles, J.M. Mazón: The Cauchy Problem for a Strongly Degenerate Quasilinear Equation. *J. Europ. Math. Soc.* **7**, 361–393 (2005)

[20] F. Andreu, V. Caselles, J.M. Mazón: Some regularity results on the 'relativistic' heat equation. *J. Diff. Equations* **245**, 3639–3663 (2008)

[21] F. Andreu, V. Caselles, J.M. Mazón, S. Moll: Finite Propagation Speed for Limited Flux Diffusion Equations. *Arch. Rat. Mech. Anal.* **182**, 269–297 (2006)

[22] F. Andreu, V. Caselles, J.M. Mazón, S. Moll: A Diffusion Equation in Transparent Media. *J. Evol. Equat.* **7**, 113–143 (2007)

[23] F. Andreu, V. Caselles, J.M. Mazón, S. Moll: The Dirichlet problem associated to the relativistic heat equation. To appear in *Math. Ann.*

[24] F. Andreu, V. Caselles, J.M. Mazón, J. Soler, M. Verbeni: Radially Symmetric Solutions of a Tempered Diffusion Equation. The porous medium-flux-limited case. Preprint

[25] V.A. Antonov: Solution of the problem of stability of a stellar system with the Emden density law and spherical velocity distribution. *J. Leningrad Univ. Ser. Mekh. Astro.* **7**, 135–146 (1962)

[26] G. Anzellotti: Pairings Between Measures and Bounded Functions and Compensated Compactness. *Ann. di Matematica Pura et Appl.* **IV (135)**, 293–318 (1983)

[27] E.R. Arriola, J. Soler: A variational approach to the Schrdinger–Poisson system: asymptotic behaviour, breathers and stability. *Journal of Statistical Physics.* **103**, 1069–1105 (2001)

[28] A.A. Arsen'ev: Global existence of a weak solution of Vlasov's system of equations. *U.S.S.R. Comp. Math. and Math. Phys.* **15**, 131–143 (1975)

[29] V. Avila-Reese, P. Coli, S. Gottlöber, C. Firmani, C. Maulbetsch: The dependence on enviroment of cold dark matter halo properties. *Astrophys. J.* **634**, 51–69 (2005)

[30] C. Baranger: Modeling of oscillations, breakup and collisions for droplets: the establishment of kernels for the T.A.B. model. *Math. Models Methods Appl. Sci.* **14 (5)**, 775–794 (2004)

[31] C. Bardos, P. Degond: Global existence for the Vlasov–Poisson equation in 3 space variables with small initial data. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **2**, 101–118 (1985)

[32] J. Batt: Global symmetric solutions of the initial value problem in stellar dynamics. *J. Diff. Eqns.* **25**, 342–364 (1977)

[33] J. Batt: Asymptotic properties of spherically symmetric self-gravitating mass systems for $t \to \infty$. *Transport Theory and Stat. Phys.* **16**, 763–768 (1987)

[34] J. Batt, H. Beresticky, P. Degond, B. Perthame: Some Families of Solutions of the Vlasov–Poisson system. *Arch. Rat. Mech. Anal.* **104**, 79–103 (1988)

[35] J. Batt, W. Faltenbacher, E. Horst: Stationary spherically symmetric models in stellar dynamics. *Arch. Rat. Mech. Anal.* **93**, 159–183 (1986)

[36] E. Battaner, E. Florido: The Rotation Curve of Spiral Galaxies and its Cosmological Implications. *Fund. Cosmic Phys.* **21**, 1–154 (2000)

[37] E. Battaner, E. Florido: Are rotation curves in NGC 6946 and the Milky Way magnetically supported? *Astron. Nachrichten* **328**, 92–98 (2007)

[38] E. Battaner, J.L. Garrido, M. Membrado, E. Florido: Magnetic fields as an alternative explanation for the rotation curves of spiral galaxies. *Nature* **360**, 652–653 (1992)

[39] K.G. Begeman, A.H. Broeils, R.H. Sanders: Extended rotation curves of spiral galaxies - Dark haloes and modified dynamics. *Monthly Notices Royal Astronomical Society* **249**, 523–537 (1991)

[40] N. Bellomo, A. Bellouquid, J. Nieto, J. Soler: Multiscale derivation of biological tissue models and flux-limited chemotaxis from binary mixtures of multicellular growing systems. To appear in *Math. Models Methods Appl. Sci.*

[41] Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J.L. Vazquez: An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations. *Ann. Scuola Normale Superiore di Pisa* **IV (22)**, 241–273 (1995)

[42] P. Benilan, H. Brezis, M.G. Crandall: A semilinear equation in $L^1(\mathbb{R}^N)$. *Ann. Scuola Normale Superiore di Pisa CL. Sci.* **IV (2)** 523–555 (1975)

[43] Ph. Bénilan, M.G. Crandall, A. Pazy: *Evolution Equations Governed by Accretive Operators*. Book in preparation

[44] S. Bergmann, Z. Tamari, E. Schejter, B.Z. Shilo, N. Barkai: Re-examining the Stability of the Bicoid Morphogen Gradient. *Cell* **132**, 15–17 (2008)

[45] M. Berstch, R. Dal Passo: Hyperbolic phenomena in a strongly degenerate parabolic equation. *Arch. Rat. Mech. Anal.* **117**, 349–387 (1992)

[46] P. Bett, V. Eke, C.S. Frenk, A. Jenkins, J. Helly, J. Navarro: The spin and shape of dark matter haloes in the Millennium simulation of a Λ cold dark matter universe. *Monthly Notices Royal Astronomical Society* **376**, 215–232 (2007)

[47] J.J. Binney, N.W. Evans: Cuspy dark matter haloes and the galaxy. *Monthly Notices Royal Astronomical Society* **327**, L27–L31 (2001)

[48] J. Binney, S. Tremaine: *Galactic dynamics*. Princeton University Press, Princeton (1987)

[49] S. Blais-Oullette, P. Amram, C. Carignan: Accurate Determination of the Mass Distribution in Spiral Galaxies. II. Testing the Shape of Dark Halos. *The Astronomical Journal* **121**, 1952–1964 (2001)

[50] W.J.G. de Block, A. Bosma: A constant-density halo around DDO 47. *Astron. & Astrophys.* **385**, 816–846 (2002)

[51] W.J.G. de Block, A. Bosma, S. Mc Gaugh: Simulating observations of dark matter dominated galaxies: towards the optimal halo profile. *Monthly Notices Royal Astronomical Society* **340**, 657–678 (2003)

[52] W.J.G. de Block, S.S. Mc Gaugh, V.C. Rubin, A. Bosma: Mass density profiles of low surface brightness galaxies. *Astrophys. J. (Lett.)* **552**, L23–L26 (2001)

[53] F. Bouchut, F. James: Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness. *Comm. Partial Differential Equations* **24**, 2173–2189 (1999)

[54] Y. Brenier: Extended Monge–Kantorovich Theory. In *Optimal Transportation and Applications*, Lectures given at the C.I.M.E. Summer School help in Martina Franca, L.A. Caffarelli and S. Salsa (eds.). Lecture Notes in Math. **1813**, Springer-Verlag pp. 91–122 (2003)

[55] J. Briscoe, Y. Chen, T.M. Jessell, G. Struhl: A hedgehog-insensitive form of patched provides evidence for direct long-range morphogen activity of sonic hedgehog in the neural tube. *Mol. Cell* **7**, 1279–1291 (2001)

[56] F. Browder: Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains. *Proc. Nat. Acad. Sci. USA* **74**, 2659–2611 (1977)

[57] J.S. Bullock, T.S. Kollatt, Y. Sigad, R.S. Somerville, A.V. Kravtsov, A.A. Klypin, J.R. Primack, A. Dekel: Profiles of dark haloes: evolution, scatter and environment. *Monthly Notices Royal Astronomical Society* **321**, 559–575 (2001)

[58] A. Burkert: The structure of dark matter halos in dwarf galaxies. *Astrophys. J.* **447**, L25–L28 (1995)

[59] S. Calogero: Spherically symmetric steady states of galactic dynamics in scalar gravity. *Class. Quantum. Grav.* **20**, 1729–1741 (2003)

[60] S. Calogero: Global classical solutions to the 3D Nordström–Vlasov system. *Commun. Math. Phys.* **266**, 343–353 (2006)

[61] S. Calogero, J. Calvo, O. Sánchez, J. Soler: Dispersive behavior in galactic dynamics. *Discr. Cont. Dyn. Syst. Series B* **14**, 1–16 (2010)

[62] S. Calogero, J. Calvo, O. Sánchez, J. Soler: Virial inequalities for steady states in relativistic galactic dynamics. To appear in *Nonlinearity*

[63] S. Calogero, H. Lee: The non-relativistic limit of the Nordström–Vlasov system. *Commun. Math. Sci.* **2**, 19–34 (2004)

[64] S. Calogero, G. Rein: On classical solutions of the Nordström–Vlasov system. *Commun. Part. Diff. Eqns.* **28**, 1863–1885 (2003)

[65] S. Calogero, G. Rein: Global weak solutions to the Nordström–Vlasov system. *J. Diff. Eqns.* **204**, 323–338 (2004)

[66] S. Calogero, O. Sánchez, J. Soler: Asymptotic behavior and orbital stability of galactic dynamics in relativistic scalar gravity. *Arch. Rat. Mech. Anal.* **194**, 743–773 (2009)

[67] J. Calvo: On dispersion rates for solutions to the Vlasov–Poisson system. To appear in *Transport Theory and Statistical Physics*

[68] J. Calvo, E. Florido, O. Sánchez, E. Battaner, J. Soler, B. Ruíz-Granados: On an unified theory of cold dark matter halos based on collisionless Boltzmann-Poisson polytropes. *Physica A, Statistical Mechanics and its Applications* **388**, 2321–2330 (2009)

[69] J. Calvo, P.E. Jabin: Large time asymptotics for a modified coagulation model. Submitted.

[70] J. Calvo, J.M. Mazón, J. Soler, M. Verbeni: Some qualitative properties of the solutions of a nonlinear flux-limited equation arising in the transport of morphogenes. In preparation

[71] A. Carpio: Long-time Behaviour for Solutions of the Vlasov–Poisson–Fokker–Planck Equation. *Math. Meth. Appl. Sci.* **21**, 985–1014 (1998)

[72] J.A. Carrillo, J. Soler, J.L. Vazquez: Asymptotic behaviour and selfsimilarity for the three dimensional Vlasov–Poisson–Fokker–Planck system. *Journal of Functional Analysis* **141**, 99–132 (1996)

[73] F. Castella: Propagation of space moments in the Vlasov–Poisson Equation and further results. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **16**, 503–533 (1999)

[74] A. Castillo-Morales, J. Jiménez-Vicente, E. Mediavilla, E. Battaner: Non-circular motion evidence in the circumnuclear region of M100 (NGC 4321). *Monthly Notices Royal Astronomical Society* **380**, 489–498 (2007)

[75] C. Cercignani, R. Illner, M. Pulvirenti: *The Mathematical Theory of Dilute Gases.* Applied Mathemetical Sciences **106**, Springer-Verlag

[76] P.H. Chavanis: Gravitational instability of isothermal and polytropic spheres. *Astron. & Astrophys.* **401**, 15–42 (2003)

[77] P.H. Chavanis, C. Sire: Anomalous diffusion and collapse of self-gravitating Langevin particles in D dimensions. *Phys. Rev. E* **69**, 016116 (2004)

[78] A. Chertock, A. Kurganov, P. Rosenau: Formation of discontinuities in flux-saturated degenerate parabolic equations. *Nonlinearity* **16**, 1875–1898 (2003)

[79] Y. Choquet-Bruhat: Problème de Cauchy pour le système intégro différentiel d'Einstein-Liouville. *Ann. Inst. Fourier* **21**, 181–201 (1971)

[80] F. Combes, P. Boissé, A. Mazure, A. Blanchard: *Galaxies and Cosmology.* Springer 2004

[81] J.-F. Coulombel, F. Golse, T. Goudon: Diffusion Approximation and Entropy-based Moment Closure for Kinetic Equations. *Asymptotic Analysis* **45**, 1–39 (2005)

[82] G. Dal Masso: Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals. *Manuscripta Math.* **30**, 387–416 (1980)

[83] H.T. Davis: *Introduction to nonlinear differential and integral equations.* Dover publications, Inc., New York (1962)

[84] E. Dessaud, L.L. Yang, K. Hill, B. Cox, F. Ulloa, A. Ribeiro, A. Mynett, B.G. Novitch, J. Briscoe: Interpretation of the sonic hedgehog morphogen gradient by a temporal adaptation mechanism. *Nature* **450**, 717–720 (2007)

[85] J. Diestel, J.J. Uhl Jr.: *Vector Measures.* Math. Surveys **15**, Amer. Math. Soc., Providence (1977)

[86] R. DiPerna, P.L. Lions, Y. Meyers: L^p regularity of velocity averages. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **8**, 271–287 (1991)

[87] J. Dolbeault, O. Sanchez, J. Soler: Asymptotic behavior for the Vlasov–Poisson system in the stellar dynamics case. *Arch. Rat. Mech. Anal.* **171**, 301–327 (2004)

[88] R.L. Drake: A general mathematical survey of the coagulation equation. In *Topics in Current Aerosol Research (Part 2)*, volume 3 of International Reviews in Aerosol Physics and Chemistry, G. M. Hidy and J. R. Brock (eds.). Pergamon Press, Oxford, UK, pp. 203-376 (1972)

[89] G. Dufour, M. Massot, Ph. Villedieu: Etude d'un modèle de fragmentation secondaire pour les brouillards de gouttelettes. *C. R. Math. Acad. Sci. Paris* **336** **(5)**, 447–452 (2003)

[90] J. Ehlers: *Survey of general relativity theory.* In Israel W (ed.) Relativity, Astrophysics and Cosmology. Dordrecht, Reidel (1973)

[91] M. Escobedo, S. Mischler: Dust and self-similarity for the Smoluchowski coagulation equation. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **23** 331–362 (2006)

[92] O.R. Etingin, R.L. Silverstein, H.M. Friedman, D.P. Hajjar: Viral activation of the coagulation cascade: molecular interactions at the surface of infected endothelial cells. *Cell* **61** 657–662, (1990)

[93] L.C. Evans, R.F. Gariepy: *Measure Theory and Fine Properties of Functions.* CRC Press (1999)

[94] C. Féron, J. Hjorth: Simulated dark-matter halos as a test of nonextensive statistical mechanics. *Phys. Rev. E* **77**, 022106 (2008)

[95] M. Fjällborg, J. M. Heinzle, C. Uggla: Self-gravitating stationary spherically symmetric systems in relativistic galactic dynamics. *Math. Proc. Cambridge Phil. Soc.* **143**, 731–752 (2007)

[96] J. Fourier: *Théorie Analithique de la Chaleur* (1822)

[97] C.S. Frenk, C.M. Baugh, S. Cole, C. Lacey: Numerical and Analytical Modelling of Galaxy Formation and Evolution. In *Dark and visible matter in galaxies*, M. Persic and P. Salucci (eds). ASP Conference Series **117**, pp 335–347 (1997)

[98] S. Friedrich: Global Small Solutions of the Vlasov-Nordström System. Preprint

[99] I. Gasser, P.-E. Jabin, B. Perthame: Regularity and propagation of moments in some nonlinear Vlasov systems. *Proc. Roy. Soc. Edinburgh Sect. A* **130**, 1259–1273 (2000)

[100] U. Gasser, E. Weeks, A. Schoeld, P.N. Pursey, D.A. Weitz: Real-space imaging of nucleation and growth in colloidal crystallization. *Science* **292**, 258–262 (2001)

[101] A. Gierer, H. Meinhardt: A theory of biological pattern formation. *Kybernetik* **12**, 30–39 (1972)

[102] R. Glassey, W. Strauss: Remarks on collisionless plasmas. *Contemporary Mathematics* **28**, 269–279 (1984)

[103] F. Golse, P.L. Lions, B. Perthame, R. Sentis: Regularity of the Moments of the Solution of a Transport Equation. *Journal of Functional Analysis* **76**, 110–125 (1988)

[104] F. Golse, F. Poupaud: Limite fluide des équations de Boltzmann des semiconducteurs pour une statistique de Fermi-Dirac. *J. on Asympt. Analysis* **6**, 135–160 (1992)

[105] F. Golse, L. Saint-Raymond: The Navier-Stokes Limit of the Boltzmann Equation for Bounded Collision Kernels. *Invent. Math.* **155**, 81–161 (2004)

[106] F. Golse, L. Saint-Raymond: Velocity Averaging in L^1 for the Transport Equation. *C.R. Acad. Sci.* **334**, 557–562 (2002)

[107] A. Gritli-Linde, P. Lewis, A.P. McMahon, A. Linde: The whereabouts of a morphogen: direct evidence for short- and graded long-range activity of hedgehog signaling peptides. *Dev. Biol.* **236**, 364–386 (2001)

[108] I. Guerrero, C. Chiang: A conserved mechanism of Hedgehog gradient formation by lipid modifications. *Trends Cell Biol.* **17**, 1–5 (2007)

[109] J. Gunn, J.R. Gott: On the infall of matter into clusters of galaxies and some effects on their evolution. *Astrophys. J.* **176**, 1 (1972)

[110] Y. Guo, G. Rein: Isotropic steady states in galactic dynamics revised. *Commun. Math. Phys.* **209**, 607–629 (2001).

[111] Y. Guo, G. Rein: Stable models of elliptical galaxies. *Monthly Notices Royal Astronomical Society* **344**, 1296–1306 (2003)

[112] E. Hayashi: The inner structure of ΛCDM haloes - II. Halo mass profiles and low surface brightness galaxy rotation curves. *Monthly Notices Royal Astronomical Society* **355**, 794–812 (2004)

[113] A. Helmi: Velocity Trends in the Debris of Sagittarius and the Shape of the Dark Matter Halo of Our Galaxy. *Astrophys. J. (Lett.)* **610**, L97 (2004)

[114] M. Hénon: Numerical experiments on the stability of spherical stellar systems. *Astron. & Astrophys.* **24**, 229–238 (1973)

[115] G. Hinshaw *et al.* (20 co-authors): Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Data Processing, Sky Maps, and Basic Results. *Astrophys. J. Suppl.* **180**, 225–245 (2009)

[116] M. Hoeft, J.P. Mücket, S. Gotlöber: Velocity dispersion profiles in dark matter halos. *Astrophys. J.* **602**, 162–169 (2004)

[117] H. Hoekstra, H.K.C. Yee, M.D. Gladders: Properties of galaxy dark matter halos from weak lensing. *Astrophys. J.* **606**, 67–77 (2004)

[118] E. Horst: On the asymptotic growth of the solutions of the Vlasov–Poisson system. *Math. Meth. Appl. Sci.* **16**, 75–85 (1993)

[119] E. Horst: On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I. *Math. Meth. Appl. Sci.* **3**, 229–248 (1981)

[120] E. Horst: On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation II. *Math. Meth. Appl. Sci.* **4**, 19–32 (1982)

[121] E. Horst: Global strong solutions of Vlasov's equation — necessary and sufficient conditions for their existence. *Partial Differential Equations, Banach Center Publications* **19**, 143–153 (1987)

[122] E. Horst: Symmetric plasmas and their decay. *Comm. Math. Phys.* **126**, 613–633 (1990)

[123] E Horst, R Hunze: Weak solutions of the initial value problem for the unmodified non-linear Vlasov equation. *Math. Meth. Appl. Sci.* **6**, 262–279 (1984)

[124] H.J. Hwang, A.D. Rendall, J.L. Velázquez: Optimal gradient estimates and asymptotic behavior for the Vlasov–Poisson system with small initial data. Preprint: math.AP/0606389

[125] R. Illner, G. Rein: Time decay of the solutions of the Vlasov–Poisson system in the plasma physical case. *Math. Meth. Appl. Sci.* **19**, 1409–1413 (1996)

[126] P.W. Ingham, A.P. McMahon: Hedgehog signaling in animal development: paradigms and principles. *Genes Dev.* **15**, 3059–3087 (2001)

[127] P.E. Jabin: Averaging Lemmas and Dispersion Estimates for kinetic equations. To appear in *Riv. Mat. Univ. Parma*

[128] P.E. Jabin, J. Soler: A kinetic description of particle fragmentation. *Math. Mod. Meth. Appl. Sci.* **16, 6**, 933–948 (2006)

[129] P.E. Jabin, J. Soler: A coupled Boltzmann & Navier–Stokes fragmentation model induced by a fluid-particle-spring interaction. To appear in *SIAM Multiscale Modeling and Simulation*

[130] T.M. Jessell: Neuronal specification in the spinal cord: inductive signals and transcriptional codes. *Nat. Rev. Genet.* **1**, 20–29 (2000)

[131] E.F. Keller, L.A. Segel: Initiation of slide mold aggregation viewed as an instability. *J. Theor. Biol.* **26**, 399–415 (1970)

[132] E.F. Keller, L.A. Segel: Model for chemotaxis. *J. Theor. Biol.* **30**, 225–234 (1971)

[133] E.F. Keller, L.A. Segel: Traveling Bands of chemotactic Bacteria: A Theoretical Analysis. *J. Theor. Biol.* **30**, 235–248 (1971)

[134] E.F. Keller: Assessing the Keller-Segel model: how has it fared? In *Biological growth and spread*, Proc. Conf., Heidelberg, 1979. Springer Berlin pp. 379–387 (1980)

[135] T. Kronbreger, M.P. Leubner, E. van Kampen: Dark matter density profiles: A comparison of nonextensive theory with N-body simulations. *Astron. & Astrophys.* (to appear) (2007)

[136] R. Kurth: A global particular solution to the initial value problem of the stellar dynamics. *Quart. Appl. Math.* **36**, 325–329 (1978)

[137] R. Kurth: Das Anfangswertproblem der Stellardynamik. *Z. Astrophys.* **30**, 213–229 (1952)

[138] K. Lai, M.J. Robertson, D.V. Schaffer: The Sonic hedgehog signaling system as a bistable genetic switch. *Biophys. J.* **86**, 2748–2757 (2004)

[139] A.D. Lander: Do morphogen gradients arise by diffusion? *Dev. Cell* **2**, 785–96 (2002)

[140] A.D. Lander: Morpheus unbound: reimagining the morphogen gradient. *Cell* **128**, 245–256 (2007)

[141] J.C. Lasheras, C. Eastwood, C. Martínez-Bazán, J.L. Montañés: A review of statistical models for the break-up of an immiscible fluid immersed into a fully developed turbulent flow. *Int. J. Multiphase Flow* **28**, 247–278 (2002)

[142] P. Laurencot, S. Mischler: From the discrete to the continuous coagulation- fragmentation equations. *Proc. Roy. Soc. Edinburgh Sect. A* **132**, 1219–1248 (2002)

[143] P. Laurencot, S. Mischler: On coalescence equations and related models. In *Topics in Current Aerosol Research (Part 2), Modelling and Simulation in Science, Engineering and Technology*, P. Degond, L. Pareschi, and G. Russo (eds.). Birkhauser pp. 321–342 (2004)

[144] P. Laurencot, S. Mischler: The continuous coagulation–fragmentation equations with diffusion. *Arch. Rational Mech. Anal.* **162**, 45–99 (2002)

[145] C.D. Levermore, G.C. Pomraning: A Flux-Limited Diffusion Theory. *Astrophys. J.* **248**, 321–334 (1981)

[146] P. Lévy: *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris (1954)

[147] E. H. Lieb, M. Loss: *Analysis.* Providence: American Math. Soc. (1996)

[148] E.M. Lifshitz, L.P. Pitaevskii: *Physical Kinetics.* Pergamon Press, New York (1981)

[149] M. Limousin, J.P. Kneib, S. Bardeau, P. Natarajan , O. Czoske, I. Smail, H. Ebeling, G.P. Smith: Truncation of galaxy dark matter halos in high density enviroments. *Astron. & Astrophys.* **461**, 881–891 (2007)

[150] D. Lynden-Bell: Statistical mechanics of violent relaxation in stellar systems. *Monthly Notices Royal Astronomical Society* **136**, 101–121 (1967)

[151] P.L. Lions: The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **1**, 109–145 (1984)

[152] P.L. Lions, B. Perthame: Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system. *Invent. Math.* **105**, 415–430 (1991)

[153] G. Loeper: Uniqueness of the solution to the Vlasov–Poisson system with bounded density. *Journal de Mathematiques Pures et Appliquees* **86**, 68–79 (2006)

[154] D. Marchesini, E. Donghia, G. Chincarini, C. Firman, P. Conconi, E. Molinari, A. Zacchei: Hα Rotation Curves: The Soft Core Question. *Astrophys. J.* **575**, 801–814 (2002)

[155] T. Matos, D. Nuñez, A. Sussman: A general relativistic approach to the Navarro-Frenk-White galactic halos. *Class. Quantum Grav.* **21**, 5275–5293 (2004)

[156] S. McGaugh, V.C. Rubin, W.J.G. de Block: FREE High-Resolution Rotation Curves of Low Surface Brightness Galaxies. I. Data. *The Astronomical Journal* **122**, 2381–2395 (2001)

[157] H. Meinhardt: Space-dependent cell determination under the control of a morphogen gradient. *J. theor. Biol.* **74**, 307–321 (1978)

[158] D. Mihalas, B. Mihalas: *Foundations of radiation hydrodynamics.* Oxford University Press (1984)

[159] B. Moore, F. Governato, T. Quinn, J. Stadel, G. Lake: Resolving the structure of cold dark matter halos. *Astrophys. J.* **499**, L5–L8 (1998)

[160] B. Moore, T. Quinn, F. Governato, J. Stadel, G. Lake: Cold collapse and the core catastrophe. *Monthly Notices Royal Astronomical Society* **310**, 1147–1152 (1999)

[161] E. Morel: Diffusion-limit asymptotics of the transport equation, the P1/3 equations, and two flux-limited diffusion theories. *J. Quantitative Spectr. & Radiative Transfer* **65**, 769–778 (2000)

[162] J.L. Mullor, P. Sánchez, A. Ruiz i Altaba: Pathways and consequences: hedgehog signaling in human disease. *Trends Cell Biol.* **12**, 562–569 (2002)

[163] A. Munier, M. Feix, E. Fijalkow, J.R. Burgan, J. Gutierrez: Time Dependent Solution for a Self-gravitating Star System. *Astron. & Astrophys.* **78**, 65–74 (1979)

[164] J.F. Navarro, C.S. Frenk, S.D.M. White: The structure of cold dark matter halos. *Astrophys. J.* **462**, 563–575 (1996)

[165] J.F. Navarro, C.S. Frenk, S.D.M. White: A universal density profile from hierarchical clustering. *Astrophys. J.* **490**, 493–508 (1997)

[166] J.F. Navarro, M. Steimetz: Dark halo and disk galaxy scaling laws in hierarchical universes. *Astrophys. J.* **538**, 477–488 (2000)

[167] J. Nieto, F. Poupaud, J. Soler: High field limit for Vlasov-Poisson-Fokker-Planck equations. *Arch. Rat. Mech. Anal.* **158**, 29–59 (2001)

[168] G. Nordström: Zur Theorie der Gravitation vom Standpunkt des Relativitätprinzips. *Ann. Phys., Lpz* **42**, 533 (1913)

[169] C.S. Patlak: Random walk with persistant and external bias. *Bull. Math. Biophy.* **15**, 311–338 (1953)

[170] H.C. Park, J. Shin, B. Appel: Spatial and temporal regulation of ventral spinal cord precursor specification by Hedgehog signaling. *Development* **131**, 5959–5969 (2004)

[171] I. Pérez, R. Fox, K. Freeman: Gas flow and dark matter in the inner parts of early-type barred galaxies. *Astron. & Astrophys.* **424**, 799–815 (2004)

[172] B. Perthame: Time decay, propagation of low moments and dispersive effects for kinetic equations. *Commun. Part. Diff. Eqns.* **21**, 659–686 (1996)

[173] B. Perthame: Mathematical tools for kinetic equations. *Bulletin (new series) of the American Mathematical Society* **41 (2)**, 205–244 (2004)

[174] K. Pfaffelmoser: Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data. *J. Diff. Eqns.* **95**, 281–303 (1992)

[175] A.R. Plastino, A. Plastino: Stellar polytropes and Tsallis' entropy. *Physics Letters A* **174**, 384–386 (1993)

[176] A.R. Plastino: S_q entropy and self-gravitating systems. *Europhysics News* **36**, 207–209 (2005)

[177] H. Pollard: A sharp form of the virial theorem. *Bull. Amer. Math. Soc.* **LXX**, 703–705 (1964)

[178] F. Poupaud: Mathematical Theory of Kinetic Equations For Transport Modelling in Semiconductors. In *Series on Advances in Mathematics for Applied Sciences, Kinetic Theory and Computing*, B. Perthame edt, World Scientific **22**, pp 141–168 (1994)

[179] F. Poupaud, J. Soler: Parabolic Limit and Stability of the Vlasov-Poisson-Fokker-Planck System. *Math. Models Methods Appl. Sci.* **10**, 1027–1045 (2000)

[180] G. Rein: *Collisionless Kinetic Equations from Astrophysics—The Vlasov–Poisson System.* Handbook of Differential Equations, Evolutionary Equations. Vol. 3. Eds. C.M. Dafermos and E. Feireisl, Elsevier (2007)

[181] G. Rein: *The Vlasov-Einstein system with surface symmetry.* Habilitationsschrift, München (1995)

[182] G. Rein: Static solutions of the spherically symmetric Vlasov- Einstein system. *Math. Proc. Cambridge Phil. Soc.* **115**, 559–570 (1994)

[183] G. Rein: Static shells for the Vlasov–Poisson and Vlasov-Einstein systems. *Indiana Univ. Math. J.* **48**, 335–346 (1999)

[184] G. Rein: Stationary and static stellar dynamic models with axial symmetry. *Nonlinear Analysis; Theory, Methods and Applications* **41**, 313–344 (2000)

[185] G. Rein, A.D. Rendall: Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. *Commun. Math. Phys.* **150**, 561–583 (1992). Erratum: *Commun. Math. Phys.* **176**, 475–478 (1996)

[186] G. Rein, A.D. Rendall: The Newtonian limit of the spherically symmetric Vlasov-Einstein system. *Commun. Math. Phys.* **150**, 585–591 (1992)

[187] G. Rein, A. D. Rendall: Smooth static solutions of the spherically symmetric Vlasov-Einstein system. *Ann. Inst. Henri Poincaré, Physique Théorique* **59**, 383–397 (1993)

[188] G. Rein, A.D. Rendall: Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. *Math. Proc. Cambridge Phil. Soc.* **128**, 363–380 (2000)

[189] G. Rein, A.D. Rendall, J. Schaeffer: A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system. *Commun. Math. Phys.* **168**, 467–478 (1995)

[190] G. Rein, A.D. Rendall, J. Schaeffer: Critical collapse of collisionless matter – a numerical investigation. *Phys. Rev. D* **58**, 044007 (1998)

[191] A.D. Rendall: An introduction to the EinsteinVlasov system. In *Mathematics of Gravitation, Part I: Lorentzian Geometry and Einstein Equations*, Proceedings of the Workshop Mathematical Aspects of Theories of Gravitation, Chrúsciel, P.T., (ed.). Polish Academy of Sciences, Institute of Mathematics, Warsaw, Poland, vol. 41(I) of Banach Center Publications, pp. 35–68 (1997)

[192] R. Robert: Unicité de la solucion faible à support compact de l'équation de Vlasov–Poisson. *C. R. Acad. Sci. Paris* **324**, Série I, 873–877 (1997)

[193] P. Rosenau, P.S. Hagan, R.L. Northcutt, D.S. Cohen: Delayed Diffusion due to Flux Limitation. *Phys. Letters A* **142**, 26–30 (1989)

[194] P. Rosenau: Tempered Diffusion: A Transport Process with Propagating Front and Inertial Delay. *Phys. Rev. A* **46**, 7371–7374 (1992)

[195] W. Rudin: *Principles of Mathematical Analysis.* McGrawHill, U.S.A. (1976)

[196] A. Ruiz i Altaba: How the Hedgehog outfoxed the crab: interference with Hedgehog-Gli signaling as anti-cancer therapy? In A. Ruiz i Altaba, Hedgehog-Gli signaling in human disease, Springer Science + Business Media, New York, USA, 2006.

[197] A. Ruiz i Altaba: Coexpression of HNF-3 beta and Isl-1/2 and mixed distribution of ventral cell types in the early neural tube. *Int. J. Dev. Biol.* **40**, 1081–1088 (1996)

[198] A. Ruiz i Altaba, V. Palma, N. Dahmane: Hedgehog-Gli signaling and the growth of the brain. *Nat. Rev. Neurosci.* **3**, 24–33 (2002)

[199] A. Ruiz i Altaba, B. Stecca, P. Sánchez: Hedgehog-Gli signaling in brain tumors: stem cells and paradevelopmental programs in cancer. *Cancer Lett.* **204**, 145–157 (2004)

[200] A. Ruiz i Altaba, O. Sánchez, J. Soler, M. Verbeni *et al.*: Transport of morphogens and nonlinear flux–limited diffusion. Preprint

[201] G. Russo, P. Smereka: Kinetic theory for bubbly flow I and II. *SIAM J. Appl. Math.* **56** 327–371 (1996)

[202] K. Saha, D.V. Schaffer: Signal dynamics in Sonic hedgehog tissue patterning. *Development* **133**, 889–900 (2006)

[203] P. Salucci: The constant-density region of the dark haloes of spiral galaxies. *Monthly Notices Royal Astronomical Society* **320**, L1–L5 (2001)

[204] P. Salucci, A. Borriello: The dark matter distribution in disk galaxies. *Monthly Notices Royal Astronomical Society* **323**, 285–292 (2001)

[205] P. Salucci, M. Persic: Dark Halos around Galaxies. In *Dark and visible matter in galaxies*, M. Persic and P. Salucci (eds.). ASP Conference Series 117, pp 1–27 (1997)

[206] O. Sánchez, J. Soler: Orbital stability for polytropic galaxies. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **23**, 781–802 (2006)

[207] W.C. Saslaw: *Gravitational physics of stellar and galactic systems.* Cambridge Monographs on Mathematical Physics (1985)

[208] J. Schaeffer: Global existence for the Poisson-Vlasov System with Nearly Symmetric Data. *J. Diff. Eqns.* **69**, 111–148 (1987)

[209] J. Schaeffer: Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. *Commun. Part. Diff. Eqns.* **16**, 1313–1335 (1991)

[210] J. Schaeffer: A class of counterexamples to Jeans's theorem for the Einstein–Vlasov system. *Commun. Math. Phys.* **204**, 313–327 (1999)

[211] R.W. Schmidt, S.W. Allen: The dark matter haloes of massive, relaxed galaxy clusters observed with Chandra. *Monthly Notices Royal Astronomical Society*, **379**,209–221 (2007)

[212] A. Schulze: Existence of axially symmetric solutions to the Vlasov–Poisson system depending on Jacobi's integral. *Commun. Math. Sci.* **6**, 711–727 (2008)

[213] L. Schwartz: Fonctions mesurables et *-scalairement mesurables, mesures banachiques majorées, martingales banachiques, et propiété de Radon-Nikodým. Sém. Maurey-Schawartz, 19674-75, Ecole Polytech., Centre de Math.

[214] S.L. Shapiro, S.A. Teukolsky: Relativistic stellar dynamics on the computer, I & II. *Astrophys. J.* **298**, 34–79 (1985)

[215] Y. Sofue, V. Rubin: Rotation curves of spiral galaxies. *Ann. Rev. Astron. & Astrophys.* **39**, 137–174 (2001)

[216] E.M. Stein: *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press **30**, New Jersey (1970)

[217] D. Stern, P. Nawroth, D. Handley, W. Kisiel: An endothelial cell-dependent pathway of coagulation. *Proc. Nac. Acad. Sci. USA* **82 (8)** 2523–2527 (1985)

[218] W. Strauss: *Non-linear wave equations.* Providence: American Math. Soc. (1992)

[219] R.A. Swaters, B.F. Madore, F.C. van den Bosch, M. Balcells: The central mass distribution in dwarf and low surface brightness galaxies. *Astrophys. J.* **583**, 732–751 (2003)

[220] A. Taruya, M. Sakagami: Gravothermal catastrophe and Tsallis' generalized entropy of self-gravitating systems. *Physica A* **307**, 185–206 (2002)

[221] A. Taruya, M. Sakagami: Long-term evolution of stellar self-gravitating systems away from thermal equilibrium: connections with nonextensive statistics. *Phys. Rev. Lett.* **90**, 181101-1 – 181101-4 (2003)

[222] S. Tremaine, M. Hénon, D. Lynden-Bell: H-functions and mixing in violent relaxation. *Monthly Notices Royal Astronomical Society* **219**, 285–297 (1986)

[223] C. Tsallis: Nonextensive statistics: theoretical, experimental and computational evindences and connections. *Brazilian Journal of Physics* **29**, 1–35 (1999)

[224] C. Tsallis: Possible generalization of Boltzmann-Gibbs statistics *J. Stat. Phys.* **52**, 479–487 (1988)

[225] A.M. Turing: The chemical basis of morphogenesis. *Philos. Trans. R. Soc. Lond. Ser. B Biol. Sci.* **237**, 37–72 (1952)

[226] J.L. Vázquez: Perspectives in nonlinear diffusion: between analysis, physics and geometry. In *Proceedings of the International Congress of Mathematics*, Madrid, Spain, (2006)

[227] L. Velázquez, F. Guzmán: Alternative model of the Antonov problem. *Phys. Rev. E* **68**, 066116 (2003)

[228] R.M. Wald: *General Relativity.* The University of Chicago Press, Chicago (1984)

[229] S.D.M. White, D. Zaritsky: Models for Galaxy Halos in an Open Universe. *Astrophys. J.* **394**, 1–6 (1992)

[230] F.A. Williams. *Combustion Theory.* Benjamin/Cummings, Menlo Park, CA (1985)

[231] G. Wolansky: Static solutions of the Vlasov-Einstein system. *Arch. Rat. Mech. Anal.* **156**, 205–230 (2001)

[232] G. Wolansky: On nonlinear stability of polytropic galaxies. *Ann. Inst. Henri Poincaré, Analyse non linéaire* **16**, 15–48 (1999)

[233] S. Wollman: Global in time solutions to the three-dimensional Vlasov–Poisson System. *J. Math. Anal. Appl.* **176**, 76–91 (1993)

[234] L. Wolpert: Positional information and the spatial pattern of cellular differentiation. *J. Theor. Biol.* **25**, 1–47 (1969)

[235] M. Zakari, D. Jou: Nonequilibrium Lagrange Multipliers and Heat-Flux Saturation. *J. Non-Equilib. Thermodyn.* **20**, 342–349 (1995)

[236] D. Zaritsky, S.D.M. White: The massive halos of spiral galaxies. *Astrophys. J.* **435**, 599–610 (1994)

[237] J. Zavala, D. Nuñez, R.A. Sussman, L.G. Cabral-Rosetti, T. Matos: Stellar polytropes and Navarro-Frenk-White halo models: comparison with observations. *J. Cosm. Astroparticles Phys.* **06**, 008–029 (2006)

[238] Ya.B. ZelĎovich, I.D. Novikov: *Relativistic Astrophysics, Vol. 1.* Chicago University Press, Chicago (1971)

[239] Ya.B. ZelĎovich, M.A. Podurets: The evolution of a system of gravitationally interacting point masses. *Soviet Astronomy-AJ*, **9** 742–749 (1965). Translated from *Astronomicheskii Zhurnal* **42**, 963–973 (1965)

[240] P. Zhidkov: On global solutions for the Vlasov–Poisson system. *Electronic Journal of Differential Equations* **58**, 1–11 (2004)