Cohomologies of monoids and the classification of monoidal groupoids

TESIS DOCTORAL

Por

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Programa de doctorado en Física y Matemáticas (FisyMat)

Universidad de Granada

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3 de marzo de 2016

Editor: Universidad de Granada. Tesis Doctorales
Autora: María Calvo Cervera
ISBN: 978-84-9163-012-8
URI: http://hdl.handle.net/10481/44554

A Lourdes

Contents

Prólogo

Esta memoria de tesis doctoral es presentada por Dña. María Calvo Cervera para optar al título de Doctora en Matemáticas por la Universidad de Granada, dentro del programa oficial de Doctorado en Física y Matemáticas (FisyMat). Se realiza por tanto de acuerdo con las normas que regulan las enseñanzas oficiales de Doctorado y del Título de Doctor en la Universidad de Granada, aprobadas por Consejo de Gobierno de la Universidad en su sesión de 2 de Mayo de 2012, donde se especifica que *"la tesis doctoral consistirá en un trabajo original de investigación elaborado por el candidato en cualquier campo del conocimiento que se enmarcará en alguna de las líneas del programa de doctorado en el que está matriculado. Para garantizar, con anterioridad a su presentación formal, la calidad del trabajo desarrollado se aportará, al menos, una publicación aceptada o publicada en un medio de impacto en el ámbito de conocimiento de la tesis doctoral firmada por el doctorando, que incluya parte de los resultados de la tesis. La tesis podrá ser desarrollada y, en su caso, defendida, en los idiomas habituales para la comunicación científica en su campo de conocimiento. Si la redacción de la tesis se realiza en otro idioma, deberá incluir un resumen en español."*.

La presente memoria ha sido redactada en base a cinco artículos de investigación, tres de los cuales fueron publicados entre los años 2013-2015 [10, 16, 15] y los otros dos [12, 13] actualmente están pendientes de publicación. Dichos artículos se han seleccionado teniendo en cuenta sobre todo su coherencia temática, pero también su extensión en orden a que la tesis tenga un tamaño razonable. Todas estos trabajos están sometidos o han aparecido en revistas de relevancia internacional, incluidas todas ellas en el Journal of Citations Reports e incluidas en las bases de datos MathSciNet (American Mathematical Society) y Zentralblatt für Mathematik (European Mathematical Society).

Para optar a la mención internacional en el título de doctor, la mayor parte de la memoria está escrita en inglés, idioma que actualmente es de mayoritario uso en la comunicación científica en el ámbito de las matemáticas, respetando así el idioma en que los artículos de investigación recopilados han sido o serán publicados. Al redactarse en una lengua no oficial, sin embargo, incluimos un resumen también en español.

Los resultados novedosos presentados en la memoria han sido obtenidos a lo largo de los últimos años bajo la supervisión del Dr. Antonio Martínez Cegarra en el Departamento de Álgebra de la Universidad de Granada. En este tiempo, la doctoranda ha sido alumna del Programa Oficial de Doctorado en Física y Matemáticas (FisyMat); desde Marzo de 2013 ha disfrutado de una Beca de Formación de Profesorado Universitario (FPU12/0112), financiada por el Ministerio de Educación, Cultura y Deportes español, y ha realizado sus investigaciones en el marco del Grupo de Investigación FQM-168, financiado por la Junta de Andalucía, y del Proyecto de Investigación MTM2011-22554, financiado por la Dirección General de Investigación del Gobierno de España. Durante los meses de Mayo, Junio y Julio de 2013, la doctoranda realizó una estancia de investigación en el École Polytechnique Fédérale de Lausanne, Suiza, y durante los meses de Septiembre, Octubre, Noviembre y Diciembre de 2014, realizó otra estancia en la Queen Mary University of London (Reino Unido).

Declaración de la doctoranda

María Calvo Cervera,

CERTIFICA:

Que la tesis titulada *Cohomologies of monoids and the classification of monoidal groupoids* , presentada para optar al Grado de Doctor en Matemáticas, ha sido realizada por ella misma, bajo la supervisión del Dr. Antonio Martínez Cegarra, en el Departamento de Álgebra de la Universidad de Granada.

Granada, 3 de marzo de 2016

María Calvo Cervera

Declaración del director

Antonio Martínez Cegarra, doctor en Matemáticas y catedrático de Álgebra de la Universidad de Granada

CERTIFICA:

Que la tesis titulada *Cohomologies of monoids and the classification of monoidal groupoids*, presentada por María Calvo Cervera para optar al Grado de Doctor en Matemáticas, ha sido realizada bajo su supervisión, en el Departamento de Álgebra de la Universidad de Granada.

Granada, 3 de marzo de 2016

Antonio Martínez Cegarra

Sobre derechos de autor

La doctoranda Dña. María Calvo Cervera y el director de la tesis D. Antonio Martínez Cegarra, garantizan que, hasta donde su conocimiento alcanza, en la realización de la presente tesis doctoral se han respetado los derechos de otros autores a ser citados, cuando se han utilizado sus resultados o publicaciones.

Granada, 3 de marzo de 2016

Doctoranda Director de la tesis

María Calvo Cervera Antonio Martínez Cegarra

Sobre la Mención Internacional

Con el fin de obtener la Mención Internacional en el Título de Doctor, se han cumplido en lo que atañe a esta tesis y a su defensa los siguientes requisitos:

1. Esta Memoria ha sido escrita en inglés, con un resumen y conclusiones en español.
2. Esta tesis ha sido evaluada por dos investigadores externos pertenecientes a centros no españoles.
3. Uno de los miembros del tribunal proviene de una universidad no española.
4. La defensa de la tesis se realiza en inglés.
5. La doctoranda ha realizado dos estancias de tres meses cada una en centros de investigación no españoles, uno en la École Polytechnique Fédérale de Lausanne (Suiza) y otro en la Queen Mary University of London, Reino Unido.

Granada, 3 de marzo de 2016

Doctoranda | Director de la tesis

María Calvo Cervera | Antonio Martínez Cegarra

Agradecimientos

Son muchas las personas que a lo largo de estos cuatro años me han ayudado, de una u otra manera, a que esta tesis haya salido adelante con éxito. Hablo de discusiones delante de una pizarra, pero también de cafés, comidas, charlas, sonrisas, excursiones por la montaña... esos momentos que sirven para recargar pilas después de horas delante de un papel, y que producen, al volver al trabajo, ese momento en el que de repente todo cuadra, y que es una de las razones por la que amamos resolver problemas.

Una tesis son muchas horas, la mayoría de las cuales en solitario, es cierto, pero son aquellas pasadas en compañía las que marcan la diferencia. Dentro de unos años apenas recordaré cómo eran esas demostraciones que tanto tiempo empleé en obtener, en escribir, en reescribir... lo que sí recordaré será lo que cada una de las personas que me han acompañado estos años me han enseñado, una forma de trabajar, de pensar, de enfrentarse a los problemas. Quisiera destacar aquí alguna de esas personas, no sólo por lo que han contribuido al inicio, desarrollo o fin de esta tesis, sino por lo que en el proceso he aprendido de ellos.

A Antonio, mi director, he de agradecerle su paciencia, su insistencia, su ánimo, cuando me atascaba y afirmaba rotundamente que aquello no salía. Por supuesto que al final él tenía razón. Pero, sobre todo, lo que he aprendido de él es a tener paciencia y no contentarme hasta tener un trabajo bien hecho.

A Benjamín, mi eterno compañero de batallas matemáticas, burocráticas, informáticas, docentes, personales... tu inmensa tranquilidad, no importa qué tengas entre manos, es algo de admirar. Gracias por responder siempre a mis preguntas, del tipo que sean y por los medios que sean.

Al Departamento de Álgebra, y en especial a mi grupo de investigación, por haberme acogido y haber hecho tan agradable el trabajo de estos años.

Thanks to the people from École Polytechnique Fédérale de Lausanne and Queen Mary University of London, for giving me the opportunity to work with them and live some good experiences there.

A mis padres y mi hermano, sin los que hoy no estaría escribiendo estas líneas. Por haberme apoyado durante estos años en aquello que he querido emprender, por enseñarme a que en esta vida hay que trabajar para conseguir lo que uno quiere y, sobre todo, por mostrarme que nunca hay que dejar de aprender.

A Paco, porque siempre ha estado ahí, escuchando mis problemas, sin entender una palabra algunas veces, otras indicándome el camino que yo no quería ver. Porque me da la paz y la tranquilidad que necesito.

Y por último, la persona sin la cual todos estos años, desde que llegué a Granada, hubiesen sido bastante diferentes. Con Lourdes aprendí a amar las matemáticas, a disfrutar resolviendo problemas. Ha estado ahí siempre desde el primer día que empecé la carrera, primero ella sola y luego con Canela. Gracias por esos paseos en los que nos desahogábamos del mundo. Pero sobre todo, gracias por enseñarme una forma de ver la vida que ahora es también un poco mía:

Aprende a enseñar, enseñando aprenderás.

Abstract

Monoidal categories have been studied and used extensively in the literature. Monoidal small groupoids, as found in particular in algebra and algebraic topology, are important as mathematical objects in their own right. Most of the work in this thesis is motivated by the structural analysis of several kinds of these monoidal groupoids, whose final aim is to state and prove precise cohomological classification theorems for them. Some of these results are established by means of known cohomology theories for monoids, but others need suitable new ones. Therefore, the memory also contributes to the study of monoids under an homological point of view.

This thesis is divided into five chapters, that contained the results obtained, and a conclusion chapter in Spanish. All chapters can be read quite independently, although most of the terminology and some technical arguments are shared between them. Apart from a few minor notational changes that have been made to unify our presentation, and that the full bibliography has been collected at the end of the thesis, Chapter 1 has appeared as [10] in the journal Semigroup Forum (2013), Chapter 3 as [16] in Semigroup Forum (2015), Chapter 4 as [15] in Mathematics (2015), while Chapters 2 and 5 correspond to the papers [12] and [13], which are pending of publication.

In Chapter 1 we analyze the structure of arbitrary monoidal groupoids, that is, small categories $\mathcal{M}$ in which all arrows are invertible, enriched with a monoidal structure by a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an unit object I, and corresponding coherent associativity and unit constraints $\boldsymbol{a}_{x,y,z} : (x \otimes y) \otimes z \cong x \otimes (y \otimes z)$, $\boldsymbol{l}_x : \mathrm{I} \otimes x \cong x$, and $\boldsymbol{r}_x : x \otimes \mathrm{I} \cong x$. Strongly inspired by Schreier's analysis of group extensions [67] and its extension to fibrations of categories by Grothendieck [45] (but also by works of Sinh [69], Breen [8], *et al.*), we develop a 3-dimensional Schreier-Grothendieck factor set theory for monoidal groupoids. More precisely, we classify monoidal groupoids by, what we call, *Schreier systems for monoidal groupoids*, or *non-abelian* 3*-cocycles on monoids*. That is, systems of data consisting of a monoid M, a family of (not necessarily commutative) groups $\mathcal{A}(a)$, parameterized by the elements $a \in M$ of the monoid, a family of group homomorphisms $a_* : \mathcal{A}(b) \to \mathcal{A}(ab)$ and $a^* : \mathcal{A}(b) \to \mathcal{A}(ba)$ between these groups, and a list of elements $\lambda_{a,b,c} \in \mathcal{A}(abc)$, satisfying various requirements.

When we focus in the special case of *monoidal abelian groupoids* then our classification results are stated in a more enjoyable and precise way by means of Leech cohomology theory of monoids [53]. In fact, any monoidal abelian groupoid is clas-

sified by a Schreier system in which now every group $\mathcal{A}(a)$ is abelian, so that the data for a Schreier system turn just to be a three cocycle for a Leech cohomology group of the monoid M. Although these results are mainly of algebraic interest, we would like to stress their potential interest in homotopy theory since there are natural isomorphisms between Leech cohomology groups of a monoid M and certain Gabriel-Zisman's cohomology groups [37] of the classifying space $\overline{W}M$ of the monoid.

In Chapter 2 we deal with the computability of Leech (co)homology groups of finite cyclic monoids, whose structure and classification was first stated by Frobenius [34]. Although (co)homology groups of any finite cyclic group have been well-known since they were computed in 1949 by Eilenberg [27], this is not the case for finite cyclic monoids. Indeed, to our knowledge, the Leech cohomology groups of a cyclic monoid have been computed only for the infinite case (i.e., for the monoid of the additive monoid $\mathbb{N}$ of natural numbers) and up to dimension 2 for the finite case by Leech in [53]. Then, because higher cohomology groups arise with interest for us (mainly due to our interpretation of the 3rd cohomology groups in Chapter 1), we dedicate this chapter to compute all the (co)homology groups of any finite cyclic monoid.

In Chapter 3 we change to work with commutative monoids. The category of commutative monoids is tripleable (monadic) over the category of sets [58], and so it is natural to specialize Barr-Beck cotriple cohomology [2] to define a cohomology theory for commutative monoids. This was done in the 1990s by Grillet [40, 41, 42, 43]. In this chapter, our goal is to interpret these 3rd cohomology group in terms of *strictly symmetric* (or *strictly commutative*) monoidal abelian groupoids [25, 56, 66], that is, monoidal abelian groupoids, but now endowed with coherent and natural isomorphisms $\boldsymbol{c}_{x,y} : x \otimes y \cong y \otimes x$, satisfying the symmetry and strictness conditions $\boldsymbol{c}_{y,x}\, \boldsymbol{c}_{x,y} = id_{x\otimes y}$ and $\boldsymbol{c}_{x,x} = id_{x\otimes x}$. The monoid M of connected components of such a monoidal groupoid becomes commutative and our main result here is that the complete invariant for the classification of any strictly symmetric monoidal abelian groupoid is provided by a Grillet 3-cohomology class of M. This classification result generalizes the well-known one for strictly commutative Picard categories by Deligne [25], Fröhlich and Wall [36], and Sinh [69].

So far, we have dealt with Leech cohomology theory for arbitrary monoids and Grillet cohomology theory for commutative monoids. For a commutative monoid, these two cohomology theories differ beyond dimension 2. Indeed, one easily argues that Leech cohomology groups do not take properly account of the commutativity of the monoid, in contrast to what happens with Grillet ones. To some extent, however, Grillet's symmetric cohomology theory at degrees greater than 2 seem to be a little too 'strict' (for example, symmetric 3-cohomology groups of a group are always zero). Therefore, in Chapters 4 and 5, we present new approaches for cohomology theories of commutative monoids, mainly motivated for the problem of classifying both braided and symmetric monoidal abelian groupoids.

In Chapter 4 we define and study a new cohomology theory, consisting of what we call *commutative cohomology groups of a commutative monoid*. We came to them inspired in the (second level) *cohomology groups of abelian groups* by Eilenberg and

Mac Lane [31, 55], and their definition is based on the cohomology theory of simplicial sets by Gabriel and Zisman [37]. To compute these cohomology groups up to dimension 3, we conclude here with a manageable truncated at dimension 4 cochain complex, which we call the complex of *commutative cochains*. By means of these commutative cocycles we establish interpretation results for the commutative cohomology groups at dimensions up to 3. In particular, we prove that equivalence classes of *braided monoidal abelian groupoids*, that is, monoidal abelian groupoids with coherent and natural isomorphisms $\boldsymbol{c}_{x,y} : x \otimes y \cong y \otimes x$ but without the symmetry ($\boldsymbol{c}_{y,x}\, \boldsymbol{c}_{x,y} = id$) and strictness ($\boldsymbol{c}_{x,x} = id$) requirements [50], are classified by means of commutative 3-cohomology classes of commutative monoids.

Finally, in Chapter 5, we introduce and study, for any integer $r \geq 1$, a *rth level cohomology theory* for commutative monoids. The rth level cohomology groups provide a generalization to commutative monoids of Eilenberg-Mac Lane's rth level cohomology groups for abelian groups [31, 55], which, recall, compute the cohomology of the spaces $K(G, r)$. Furthermore, this theory recover, at its first level, Leech cohomology on commutative monoids, and, at its second level, the commutative cohomology theory treated in the previous Chapter 4. Regarding the third level cohomology groups, we find, among them, the invariants for classifying *symmetric monoidal abelian groupoids*, that is, monoidal abelian groupoids with coherent and natural isomorphisms $\boldsymbol{c}_{x,y} : x \otimes y \cong y \otimes x$ satisfying the symmetry condition ($\boldsymbol{c}_{y,x}\, \boldsymbol{c}_{x,y} = id$) but not the strictness one ($\boldsymbol{c}_{x,x} = id$), and so we complete the lists of invariants for equivalence classes of monoidal abelian groupoids. A relevant part of the chapter is dedicated to give explicit computations of these higher level cohomology groups for cyclic monoids.

Chapter 1

Structure and classification of monoidal groupoids

This chapter deals with *monoidal groupoids* $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$, that is, categories $\mathcal{M}$ in which all arrows are invertible, enriched with a monoidal structure by a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an unit object I, and corresponding coherent associativity and unit constraints $\boldsymbol{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $\boldsymbol{l}_X : \mathrm{I} \otimes X \to X$, and $\boldsymbol{r}_X : X \otimes \mathrm{I} \to X$. Our main objective is to state and prove precise classification theorems for monoidal groupoids and their homomorphisms. In this classification, two monoidal groupoids, say $\mathcal{M}$ and $\mathcal{M}'$, are *equivalent* whenever they are connected by a monoidal equivalence $(F, \varphi) : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$, and two monoidal functors $(F, \varphi), (F', \varphi') : \mathcal{M} \to \mathcal{M}'$ which are related by a monoidal natural isomorphism, $\delta : (F, \varphi) \overset{\sim}{\Rightarrow} (F', \varphi')$, are considered the same.

The particular case of *categorical groups* is well known since it was dealt with by Sinh in 1975. Recall that a categorical group [50] (also called a *Gr-category* [8, 69] and a *weak* 2*-group* [1]) is a monoidal groupoid $\mathcal{M}$ in which every object X is invertible, in the sense that there is another object X^* and an isomorphism $X \otimes X^* \to \mathrm{I}$. In [69], she proved that, for any group G, any G-module A, and any Eilenberg-Mac Lane cohomology class $c \in H^3(G, A)$, there exists a categorical group $\mathcal{M}$, unique up to monoidal equivalence, such that G is the group of isomorphism classes of objects of $\mathcal{M}$, $A = \mathrm{Aut}_{\mathcal{M}}(\mathrm{I})$ is the (abelian) group of automorphisms in $\mathcal{M}$ of the unit object, and the G-action and the cohomology class c are canonically deduced from the functoriality of the tensor and the naturality and coherence of the constraints of $\mathcal{M}$. This fact was historically relevant since it pointed out the utility of categorical groups in homotopy theory: as $H^3(G, A) = H^3(BG, A)$ is the 3th cohomology group of the classifying space BG of the group G with local coefficients in A, for any triplet of data (G, A, c) as above, there exists a path-connected CW-complex X, unique up to homotopy equivalence, such that $\pi_i X = 0$ if $i \neq 1, 2$, $\pi_1 X = G$, $\pi_2 X = A$ as G-module, and $c \in H^3(G, A)$ is the unique non-trivial Postnikov invariant of X. Therefore, categorical groups arise as algebraic homotopy 2-types of path-connected spaces. Indeed, strict categorical groups -i.e. categorical groups in which all the structure constraints are identities- are the same as *crossed modules*, whose use in homotopy theory goes back to Whitehead (1949) (see [9] for the history).

However, many illustrative examples such as the category $\mathcal{A}z_R$ of central separable algebras over a commutative ring R, or the fundamental groupoid πX of a Stasheff A_4-space X (of any topological monoid, for instance), show the ubiquity of monoidal groupoids in several branches of mathematics, and therefore the interest to study these categorical structures in their own right. But the situation with monoidal groupoids is more difficult than with categorical groups. Let us stress the main two differences between both situations. On the one hand, *the induced structure by the tensor product on the set of connected components of a monoidal groupoid is that of a monoid*, rather than a group, as it happens in the categorical group case. On the other hand, if $\mathcal{M}$ is a categorical group, then the isotropy groups $\mathrm{Aut}_{\mathcal{M}}(X)$, $X \in \mathrm{Ob}\mathcal{M}$, are all abelian and all isomorphic to $\mathrm{Aut}_{\mathcal{M}}(\mathrm{I})$, while *a monoidal groupoid may have some isotropy groups that are not isomorphic to* $\mathrm{Aut}_{\mathcal{M}}(\mathrm{I})$*, as well as some noncommutative isotropy groups.* Think of the simple example $\mathfrak{Fin}$ of finite sets and bijective functions between them, whose monoidal structure is given by disjoint union construction: Its monoid of isomorphism classes of objects is $\mathbb{N}$, the additive monoid of natural numbers, and its isotropy groups are the symmetric groups $\mathfrak{S}_n$.

Strongly inspired by Schreier's analysis of group extensions [67] and its extension to fibrations of categories by Grothendieck [45] (but also by works of Sinh [69], Breen [8], *et al.*), the structure of the monoidal groupoids is analyzed in this chapter, where we develop a 3-dimensional Schreier-Grothendieck factor set theory for monoidal groupoids, which indeed involves a 2-dimensional one for the monoidal functors between monoidal groupoids, and even a 1-dimensional one for the monoidal transformations between them. More precisely, our general conclusions on this issue concerning to monoidal groupoids can be summed up by saying that we give explicit quasi-inverses biequivalences

$$\mathbf{MonGpd} \underset{\Sigma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\approx}}}} \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd},$$

between the 2-category of monoidal groupoids and the 2-category of what we call *Schreier systems for monoidal groupoids*, or *non-abelian* 3*-cocycles on monoids*. That is, systems of data

$$(M, \mathcal{A}, \Theta, \lambda)$$

consisting of a monoid M, a family of (non-necessarily commutative) groups $\mathcal{A} = (\mathcal{A}(a))_{a\in M}$ parameterized by the elements of the monoid, a family of group homomorphisms

$$\Theta = (\mathcal{A}(b) \xrightarrow{a_*} \mathcal{A}(ab) \xleftarrow{b^*} \mathcal{A}(a))_{a,b\in M},$$

and a normalized map

$$\lambda : M \times M \times M \longrightarrow \bigcup_{a\in M} \mathcal{A}(a) \ \mid\ \lambda_{a,b,c} \in \mathcal{A}(abc),$$

satisfying various requirements. In the 2-category $\mathbf{Z^3_{\text{n-ab}}Mnd}$ every equivalence is actually an isomorphism, so that our classification results are effective.

When we focus in the special case of *monoidal abelian groupoids*, that is, monoidal groupoids $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ whose isotropy groups $\mathrm{Aut}_{\mathcal{M}}(X)$, $X \in \mathrm{Ob}\mathcal{M}$, are all abelian, then our classification results are stated in a more enjoyable and precise way by means of Leech cohomology theory of monoids [53]. The biequivalences above restrict to quasi-inverses biequivalences

$$\mathbf{MonAbGpd} \underset{\Sigma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\approx}}}} \mathbf{Z^3Mnd},$$

between $\mathbf{MonAbGpd}$, the full 2-subcategory of monoidal abelian groupoids, and $\mathbf{Z^3Mnd}$, the full 2-subcategory given by those Schreier systems $(M, \mathcal{A}, \Theta, \lambda)$ in which every group $\mathcal{A}(a)$ of $\mathcal{A}$ is abelian. But, the data $\mathcal{A}$ and Θ that occur in any such a Schreier system constitute just a coefficient system, denoted now only by $\mathcal{A}$, for Leech cohomology groups $H^n_{\mathrm{L}}(M, \mathcal{A})$ of the monoid M, and $\lambda \in Z^3_{\mathrm{L}}(M, \mathcal{A})$ is a normalized 3-cocycle. From this observation, we achieve the classification both of the monoidal abelian groupoids and of the monoidal functors between them, by means of the cohomology groups $H^3_{\mathrm{L}}(M, \mathcal{A})$ and $H^2_{\mathrm{L}}(M, \mathcal{A})$. Although these results are mainly of algebraic interest, we would like to stress their potential interest in homotopy theory since, as we will observe in the Chapter 4, there are natural isomorphisms $H^n_{\mathrm{L}}(M, \mathcal{A}) \cong H^n(\overline{W}M, \mathcal{A})$, between Leech cohomology groups of a monoid M and Gabriel-Zisman's cohomology groups of the classifying space $\overline{W}M$ of the monoid with twisted coefficients in $\mathcal{A}$ [37, Appendix II].

The plan of the chapter, briefly, is as follows. After this introduction, there are four sections. Section 1.1 comprises some notations and basic results concerning monoidal groupoids and the 2-category that they form, as well as a list of some striking examples of them. The main Section 1.2 includes our 'Schreier-Grothendieck theory' for monoidal groupoids. This is a quite long and technical section, but crucial to our conclusions, where we describe the 2-category $\mathbf{Z^3_{\text{n-ab}}Mnd}$ of non-abelian 3-cocycles on monoids, and we show in detail how this 2-category is biequivalent to the 2-category $\mathbf{MonGpd}$ of monoidal groupoids. Section 1.3 focuses in the special case of monoidal abelian groupoids. In a first subsection we briefly review some aspects concerning Leech cohomology of monoids $H^n_{\mathrm{L}}(M, \mathcal{A})$. In the second subsection we include our main classification results concerning monoidal abelian groupoids in terms of Leech cohomology groups. And, finally, a third subsection is devoted to revisit the 2-category of categorical groups, in order to show how the results here obtained imply the already known for them.

1.1 Preliminaires: The 2-category of monoidal groupoids

In this section we fix notations and terminology, as well as we review some necessary aspects and results from the background of monoidal categories that will be used in what follows.

A *monoidal category* $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ consists of a category $\mathcal{M}$, a functor

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}, \quad (X, Y) \mapsto X \otimes Y,$$

(the *tensor product*) a distinguished object $\mathrm{I} \in \mathcal{M}$ (the *unit object*), and natural isomorphisms

$$\boldsymbol{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z), \quad \boldsymbol{l}_X : \mathrm{I} \otimes X \xrightarrow{\simeq} X, \quad \boldsymbol{r}_X : X \otimes \mathrm{I} \xrightarrow{\simeq} X,$$

(called the *associativity, left unit, and right unit constraints*, respectively), such that, for all objects X, Y, Z, T of $\mathcal{M}$, the diagrams below (called the *associativity pentagon* and the *triangle for the unit*) commute.

$$\begin{tikzcd} ((X \otimes Y) \otimes Z) \otimes T \arrow[r, "\boldsymbol{a}"] \arrow[d, "\boldsymbol{a}\otimes 1"'] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\boldsymbol{a}"] & X \otimes (Y \otimes (Z \otimes T)) \\ (X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\boldsymbol{a}"] & & X \otimes ((Y \otimes Z) \otimes T) \arrow[u, "1\otimes\boldsymbol{a}"'] \end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd} (X \otimes \mathrm{I}) \otimes Y \arrow[rr, "\boldsymbol{a}"] \arrow[dr, "\boldsymbol{r}\otimes 1"'] & & X \otimes (\mathrm{I} \otimes Y) \arrow[dl, "1\otimes\boldsymbol{l}"] \\ & X \otimes Y & \end{tikzcd} \tag{1.2}$$

Observe that usually we write the structure constraints without label of objects, since their source and target make it clear what constraint isomorphism it is. A monoidal category is called *strictly unitary* when the unit constraints $\boldsymbol{l}_X$, $\boldsymbol{r}_X$ are identity arrows, while it is called *strit* if $\boldsymbol{a}_{X,Y,Z}$ is also the identity.

In any monoidal category $\boldsymbol{r}_{\mathrm{I}} = \boldsymbol{l}_{\mathrm{I}}$ and for any objects X, Y the triangles below commute [50, Proposition 1.1].

$$\begin{tikzcd} (X \otimes Y) \otimes \mathrm{I} \arrow[rr, "\boldsymbol{a}"] \arrow[dr, "\boldsymbol{r}"'] & & X \otimes (Y \otimes \mathrm{I}) \arrow[dl, "1\otimes\boldsymbol{r}"] \\ & X \otimes Y & \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{I} \otimes X) \otimes Y \arrow[rr, "\boldsymbol{a}"] \arrow[dr, "\boldsymbol{l}\otimes 1"'] & & \mathrm{I} \otimes (X \otimes Y) \arrow[dl, "\boldsymbol{l}"] \\ & X \otimes Y & \end{tikzcd} \tag{1.3}$$

If $\mathcal{M}, \mathcal{M}'$ are monoidal categories, then a *monoidal functor*

$$F = (F, \varphi) : \mathcal{M} \to \mathcal{M}' \tag{1.4}$$

consists of a functor $F : \mathcal{M} \to \mathcal{M}'$, a family of natural isomorphisms

$$\varphi_{X,Y} : FX \otimes' FY \xrightarrow{\simeq} F(X \otimes Y),$$

and an isomorphism $\varphi_0 : \mathrm{I}' \xrightarrow{\cong} F\mathrm{I}$, such that the following diagrams commute:

$$\begin{tikzcd}
(FX \otimes' FY) \otimes' FZ \arrow[r, "\varphi\otimes 1"] \arrow[d, "a'"'] & F(X \otimes Y) \otimes' FZ \arrow[r, "\varphi"] & F((X \otimes Y) \otimes Z) \arrow[d, "Fa"] \\
FX \otimes' (FY \otimes' FZ) \arrow[r, "1\otimes'\varphi"] & FX \otimes' F(Y \otimes Z) \arrow[r, "\varphi"] & F(X \otimes (Y \otimes Z))
\end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd}
FX \otimes' \mathrm{I}' \arrow[r, "1\otimes'\varphi_0"] \arrow[d, "r'"'] & FX \otimes' F\mathrm{I} \arrow[d, "\varphi"] \\
FX & F(X \otimes \mathrm{I}), \arrow[l, "Fr"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I}' \otimes' FX \arrow[r, "\varphi_0\otimes 1"] \arrow[d, "l'"'] & F\mathrm{I} \otimes' FX \arrow[d, "\varphi"] \\
FX & F(\mathrm{I} \otimes X), \arrow[l, "Fl"']
\end{tikzcd} \tag{1.6}$$

When $F\mathrm{I} = \mathrm{I}'$ and $\varphi_0 = 1_{\mathrm{I}'}$, the identity, then the monoidal functor F is qualified as *strictly unitary*. When each of the isomorphisms $\varphi_{X,Y}$, φ_0 is an identity, the monoidal functor is called *strict*.

The composition of monoidal functors $\mathcal{M} \xrightarrow{F} \mathcal{M}' \xrightarrow{F'} \mathcal{M}''$ will be denoted by juxtaposition, that is, $F'F : \mathcal{M} \to \mathcal{M}''$. Recall that its structure constraints are obtained from those of F and F' respectively, by the compositions

$$F'FX \otimes'' F'FY \xrightarrow{\varphi'} F'(FX \otimes' FY) \xrightarrow{F'\varphi} F'F(X \otimes Y)\ ,$$
$$\mathrm{I}'' \xrightarrow{\varphi'_0} F'\mathrm{I}' \xrightarrow{F'\varphi_0} F'F\,\mathrm{I}.$$

The composition of monoidal functors is associative and unitary, so that the category **MonCat** of monoidal categories is defined. Actually, this is the underlying category of a 2-category, also denoted by **MonCat**, whose 2-arrows are the *morphisms* of monoidal functors or *monoidal natural transformations*. If $F, F' : \mathcal{M} \to \mathcal{M}'$ are monoidal functors, then a morphism between them

$$\delta : F \Rightarrow F' \tag{1.7}$$

is a natural transformation on the underlying functors such that, for all objects X, Y of $\mathcal{M}$, the following coherence diagrams commute:

$$\begin{tikzcd}
FX \otimes' FY \arrow[r, "\varphi"] \arrow[d, "\delta_X\otimes'\delta_Y"'] & F(X \otimes Y) \arrow[d, "\delta_{X\otimes Y}"] \\
F'X \otimes' F'Y \arrow[r, "\varphi'"] & F'(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{I}' \arrow[dl, "\varphi_0"'] \arrow[dr, "\varphi'_0"] & \\
F\mathrm{I} \arrow[rr, "\delta_{\mathrm{I}}"] & & F'\mathrm{I}
\end{tikzcd} \tag{1.8}$$

In this 2-category, the "vertical composition" of 2-cells, denoted by

$$\begin{tikzcd}
\mathcal{M} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\delta"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta'"] & \mathcal{M}'
\end{tikzcd}
\ \overset{\circ}{\mapsto}\
\begin{tikzcd}
\mathcal{M} \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "F''"'] \arrow[r, phantom, "\Downarrow\delta'\circ\delta"] & \mathcal{M}',
\end{tikzcd}$$

is given by the ordinary vertical composition of natural transformations, that is, the component of $\delta' \circ \delta$ at any object X of $\mathcal{M}$ is given by the composition in $\mathcal{M}'$

$$(\delta' \circ \delta)_X = \delta'_X \circ \delta_X : FX \xrightarrow{\delta_X} F'X \xrightarrow{\delta'_X} F''X. \tag{1.9}$$

Similarly, the "horizontal composition"

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{M}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\delta'"] & \mathcal{M}'' \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F'F"] \arrow[r, bend right, "G'G"'] \arrow[r, phantom, "\Downarrow\delta'\delta"] & \mathcal{M}'' \end{tikzcd},$$

is given by the usual horizontal composition of natural transformations:

$$\delta'\delta = G'\delta \circ \delta' F = \delta' G \circ F'\delta : F'F \Rightarrow G'G. \tag{1.10}$$

The following known lemma will be useful in the sequel (cf. [20, Lemma 1.1], for example). Let

$$\mathbf{MonCat}_{\mathrm{u}} \subseteq \mathbf{MonCat}$$

denote the 2-subcategory of the 2-category of monoidal categories which is full on 0-cells and 2-cells, but whose 1-cells are the strictly unitary monoidal functors.

Lemma 1.1 *The inclusion* $\mathbf{MonCat}_{\mathrm{u}} \overset{\approx}{\hookrightarrow} \mathbf{MonCat}$ *is a biequivalence.*

Proof: For any monoidal categories $\mathcal{M}$ and $\mathcal{M}'$, a quasi-inverse to the inclusion functor $i : \mathbf{MonCat}_{\mathrm{u}}(\mathcal{M}, \mathcal{M}') \hookrightarrow \mathbf{MonCat}(\mathcal{M}, \mathcal{M}')$,

$$(\ )^{\mathrm{u}} : \mathbf{MonCat}(\mathcal{M}, \mathcal{M}') \to \mathbf{MonCat}_{\mathrm{u}}(\mathcal{M}, \mathcal{M}'), \tag{1.11}$$

which should be called the *normalization functor*, works as follows: For any given monoidal functor $F = (F, \varphi) : \mathcal{M} \to \mathcal{M}'$, let $\Psi_F = (\psi_X)_{X \in \mathrm{Ob}\mathcal{M}}$ be the family of isomorphisms in $\mathcal{M}'$

$$\psi_X = \begin{cases} 1_{FX} : FX \to FX & \text{if} \quad X \neq \mathrm{I} \\ \varphi_0^{-1} : F\mathrm{I} \to \mathrm{I}' & \text{if} \quad X = \mathrm{I}. \end{cases}$$

Then, F can be deformed to a new monoidal functor, $F^{\mathrm{u}} = (F^{\mathrm{u}}, \varphi^{\mathrm{u}}) : \mathcal{M} \to \mathcal{M}'$, in a unique way such that $\Psi_F : F \overset{\sim}{\Rightarrow} F^{\mathrm{u}}$ becomes an isomorphism. Namely,

$$F^{\mathrm{u}}X = \begin{cases} FX & \text{if} \quad X \neq \mathrm{I} \\ \mathrm{I}' & \text{if} \quad X = \mathrm{I}, \end{cases} \qquad F^{\mathrm{u}}(X \xrightarrow{f} Y) = \ (F^{\mathrm{u}}X \xrightarrow{\psi_Y \circ Ff \circ \psi_X^{-1}} F^{\mathrm{u}}Y),$$

$$\varphi^{\mathrm{u}}_{X,Y} = \psi_{X \otimes Y} \circ \varphi_{X,Y} \circ (\psi_X \otimes \psi_Y)^{-1}, \quad \varphi^{\mathrm{u}}_0 = \psi_{\mathrm{I}} \circ \varphi_0 = 1_{\mathrm{I}'}.$$

Furthermore, any morphism $\delta : F \Rightarrow G$ gives rise to the morphism

$$\delta^{\mathrm{u}} = \Psi_G^{-1} \circ \delta \circ \Psi_F : F^{\mathrm{u}} \Rightarrow G^{\mathrm{u}},$$

which is explicitly given by

$$\delta^{\mathrm{u}}_X = \begin{cases} \delta_X : FX \to GX & \text{if} \quad X \neq \mathrm{I} \\ \varphi_0 \circ \delta_{\mathrm{I}} \circ \varphi_0^{-1} = 1_{\mathrm{I}'} : \mathrm{I}' \to \mathrm{I}' & \text{if} \quad X = \mathrm{I}. \end{cases}$$

These mappings $F \mapsto F^{\mathrm{u}}$, $\delta \mapsto \delta^{\mathrm{u}}$, describe the normalization functor (1.11).

Since, by construction, $(\ )^{\mathrm{u}}\, i = id$, the identity functor, and we have the natural isomorphism $\Psi : id \overset{\sim}{\Rightarrow} i\,(\ )^{\mathrm{u}}$, $F \mapsto \Psi_F$, both functors i and $(\ )^{\mathrm{u}}$ are mutually quasi-inverse. □

A monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is called a *monoidal equivalence* when there exists a monoidal functor $F' : \mathcal{M}' \to \mathcal{M}$ and isomorphisms of monoidal functors $1_{\mathcal{M}} \overset{\sim}{\Rightarrow} F'F$, $FF' \overset{\sim}{\Rightarrow} 1_{\mathcal{M}'}$. Two monoidal categories are *equivalent* if they are connected by a monoidal equivalence. By Saavedra [66, I, Proposition 4.4.2], we have the following useful result:

Proposition 1.1 *A monoidal functor $(F, \varphi) : \mathcal{M} \to \mathcal{M}'$ is a monoidal equivalence if and only if the underlying functor $F : \mathcal{M} \to \mathcal{M}'$ is an equivalence of categories; that is, if and only if the functor F is full, faithful and each object of $\mathcal{M}'$ is isomorphic to an object of the form FX for some $X \in \mathcal{M}$.*

In this chapter, we are going to work with the full 2-subcategory of **MonCat** given by the monoidal groupoids, that is, of monoidal categories whose morphisms are all invertible, hereafter denoted by

MonGpd.

This 2-category of monoidal groupoids contains as a full 2-subcategory the well known 2-category of categorical groups, denoted by

CatGp,

whose objects, recall, are those monoidal groupoids in every object is invertible. The inclusion **CatGp** $\hookrightarrow$ **MonGpd** has a right biadjoint 2-functor

$$\mathcal{P}ic : \mathbf{MonGpd} \to \mathbf{CatGp}$$

that assigns to each monoidal groupoid $\mathcal{M}$ its *Picard categorical group* [66, 2.5.1],

$$\mathcal{P}ic(\mathcal{M}) \subseteq \mathcal{M},$$

which is defined as the monoidal full subgroupoid of $\mathcal{M}$ given by the invertible objects.

1.1.1 Examples

To help motivate the reader we shall show some classic and striking instances of monoidal groupoids. The most basic example of a monoidal groupoid is perhaps the defined by the category $\mathfrak{Fin}$ of finite sets and bijective functions between them, whose monoidal structure is given by means of the disjoint union construction, which arises in the study of categories of representations of the symmetric groups $\mathfrak{S}_n$ (see Joyal [48]). Indeed, $\mathfrak{Fin}$ is equivalent to the strict monoidal groupoid $\mathfrak{G}$ defined as the disjoint union of the symmetric groups $\mathfrak{S}_n$, $n \in \mathbb{N}$. More precisely, $\mathfrak{G}$ has objects the natural numbers $n \in \mathbb{N}$ and the hom-sets are given by

$$\mathfrak{G}(m,n) = \begin{cases} \mathfrak{G}_n & \text{if } m = n \\ \emptyset & \text{if } m \neq n\,. \end{cases}$$

Composition is multiplication in the symmetric groups, and the tensor product is given by the obvious map $\mathfrak{G}_m \times \mathfrak{G}_n \to \mathfrak{G}_{m+n}$.

Ring theory is a good source of many interesting monoidal groupoids. For example, following Fröhlich and Wall [35], let R be any given commutative ring. Then, the monoidal category of R-modules, $\mathcal{M}od_R$, whose monoidal structure is given by the usual tensor product of R-modules, $(M,N) \mapsto M \otimes_R N$, contains as an interesting monoidal subcategory the so-called *monoidal groupoid of R-progenerators*, usually denoted by

$$\mathcal{G}en_R,$$

whose objects are the faithful, finitely generated projective R-modules, and whose morphisms are the module isomorphisms between them. The invertible objects in $\mathcal{G}en_R$ are the invertible R-modules, i.e. rank 1 projectives. Hence,

$$\mathcal{P}ic(\mathcal{G}en_R) = \mathcal{P}ic_R,$$

is the monoidal groupoid known as the *Picard categorical group of R*. Similarly, the monoidal category of associative R-algebras with identity, $\mathcal{A}lg_R$, whose monoidal structure is given by the ordinary tensor product of R-algebras, $(A,B) \mapsto A \otimes_R B$, contains a striking instance of a monoidal groupoid: the so-called *monoidal groupoid of Azumaya R-algebras*, denoted by

$$\mathcal{A}z_R,$$

whose objects are the central separable R-algebras and whose morphisms are the R-algebra isomorphisms. Forgetting algebra structure and taking the endomorphism ring define, respectively, two remarkable monoidal functors: $\mathrm{Lin}_R : \mathcal{A}z_R \to \mathcal{G}en_R$ and $\mathrm{End}_R : \mathcal{G}en_R \to \mathcal{A}z_R$. The *Morita monoidal groupoid of R-algebras*,

$$\mathcal{M}\mathcal{A}lg_R,$$

is defined to have objects R-algebras, and a morphism $A \to B$ is an isomorphism class of a Morita equivalence $\mathcal{M}od_A \xrightarrow{\sim} \mathcal{M}od_B$ (or, equivalently, an isomorphism class of an invertible (left) $A \otimes_R B^{\mathrm{op}}$-module). An object A of this monoidal groupoid is invertible if and only if there is another object B such that $A \otimes_R B$ is Morita equivalent to R. It follows that A must be an Azumaya R-algebra. Conversely, if A is Azumaya, the isomorphism $A \otimes_R A^{\mathrm{op}} \cong \mathrm{End}_R(\mathrm{Lin}_R(A))$ shows that, since $\mathrm{End}_R(\mathrm{Lin}_R(A))$ is Morita equivalent to R, A^{op} provides a quasi-inverse of A. Hence,

$$\mathcal{P}ic(\mathcal{M}\mathcal{A}lg_R) = \mathcal{B}r_R$$

is the *Brauer categorical group of* R, whose objects are the same as those of $\mathcal{A}z_R$, i.e. the Azumaya R-algebras, but whose morphisms are here iso-classes of Morita equivalences between them.

Every monoidal groupoid arises from an elemental categorical construction: If $\mathcal{B}$ is any bicategory [71], then the *monoidal groupoid of endomorphisms of an object* $b \in \mathcal{B}$, denoted by

$$\mathcal{E}nd(b),$$

has objects the 1-cells $f : b \to b$ in $\mathcal{B}$ and morphisms the invertible 2-cells $f \overset{\sim}{\Rightarrow} f'$ between them. The monoidal structure on $\mathcal{E}nd(b)$ is given by the horizontal composition of cells in the bicategory. The *categorical group of autoequivalences* of b is

$$\mathcal{A}ut(b) = \mathcal{P}ic(\mathcal{E}nd(b)),$$

that is, the monoidal full subgroupoid of equivalences $b \overset{\sim}{\to} b$ in the bicategory. If, for example, we take $\mathcal{B} = \mathbf{Cat}$, the 2-category of categories, and $\mathbb{C}$ is any category, then the monoidal groupoid

$$\mathcal{E}nd(\mathbb{C})$$

has objects the functors $F : \mathbb{C} \to \mathbb{C}$ and the morphisms are the natural equivalences $F \overset{\sim}{\Rightarrow} G$. The composition in $\mathcal{E}nd(\mathbb{C})$ is given by the usual vertical composition of natural transformations, while the composition of the functors and the horizontal composition of the natural transformations define its (strict) monoidal structure. These monoidal groupoids of endofunctors are relevant in several frameworks, since a pseudo-action of a monoidal category $\mathcal{M}$ on a category $\mathbb{C}$ is the same thing as a monoidal functor $\mathcal{M} \to \mathcal{E}nd(\mathbb{C})$. For instance, a Deligne action [24] of a monoid M on a category $\mathbb{C}$, is just a monoidal functor $M \to \mathcal{E}nd(\mathbb{C})$, from the discrete monoidal category that M defines to the monoidal groupoid of endofunctors of $\mathbb{C}$.

The *Picard categorical group of a category* $\mathbb{C}$ is

$$\mathcal{P}ic(\mathbb{C}) = \mathcal{A}ut(\mathbb{C}),$$

that is, the monoidal full subgroupoid of $\mathcal{E}nd(\mathbb{C})$ given by the autoequivalences $\mathbb{C} \overset{\sim}{\to} \mathbb{C}$. If, for example, A is any ring and we take $\mathbb{C} = {}_A\mathcal{M}od_A$, the category of A-bimodules, then, by Morita's theory, there is a monoidal equivalence

$$\mathcal{A}ut({}_A\mathcal{M}od_A) \simeq \mathcal{P}ic_A,$$

where $\mathcal{P}ic_A$ is the Picard categorical group of the ring, that is, the categorical group of invertible A-bimodules with isomorphisms, whose monoidal structure is given by the usual monoidal product of A-bimodules $(M, N) \mapsto M \otimes_A N$. The case where $\mathbb{C} = G$, a group regarded as a category with only one object, is also well-known: the monoidal groupoid

$$\mathcal{E}nd(G)$$

can be described as having objects the group of endomorphisms $f : G \to G$ and morphisms $u : f \Rightarrow g$ those elements $u \in G$ such that $f = C_u f'$, where $C_u : G \to G$ is the inner automorphism $C_u(v) = uvu^{-1}$ given by conjugation with u. Composition of morphisms is multiplication in G, and the (strict) monoidal structure is defined by

$$(f \overset{u}{\to} f') \otimes (g \overset{v}{\to} g') = \ (fg \xrightarrow{uf'(v)} f'g').$$

The corresponding Picard categorical group of invertible objects

$$\mathcal{A}ut(G),$$

is the *categorical group of automorphisms of* G. It is the internal groupoid in the category of groups whose group of objects is $\mathrm{Aut}(G)$, the group of automorphisms of G, and whose group of arrows is the holomorph group $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$. Thus, $\mathcal{A}ut(G)$ is precisely the categorical group corresponding to the universal crossed module $G \overset{C}{\to} \mathrm{Aut}(G)$ by the well-known Verdier equivalence between the category of Whitehead crossed modules and the category of strict categorical groups, see [9] for the history.

Algebraic Topology is also a natural setting where monoidal groupoids appear with recognized interest: Recall that the fundamental groupoid πX, of a space X, is the category having X as set of objects, and whose morphisms $[\omega] : x \to y$ $(x, y \in X)$ are relative end points homotopy classes of paths $\omega : [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. Composition in πX is induced by the usual concatenation of paths and constant paths provide the identities. Any continuous map $f : X \to Y$ induces a functor $f_* : \pi X \to \pi Y$ given by

$$f_*\big(x \xrightarrow{[\omega]} y\big) \ = \ \big(f(x) \xrightarrow{[f\omega]} f(y)\big),$$

so that the fundamental groupoid construction, $X \mapsto \pi X$, is a functor from the category of topological spaces to the category of groupoids. If $f, g : X \to Y$ are two maps, then a homotopy $\alpha : f \Rightarrow g$, $\alpha : [0,1] \to Y^X$, induces a natural isomorphism $\alpha_* : f_* \Rightarrow g_*$ defined, for any point $x \in X$, by

$$\alpha_*(x) = [\alpha(-)(x)] : f(x) \to g(x).$$

Moreover, it is easy to see that if two homotopies $\alpha, \beta : f \Rightarrow g$ are related by a relative end maps homotopy, $\alpha \Rrightarrow \beta$, then both induce the same natural isomorphism, that is, if $[\alpha] = [\beta]$ in the track groupoid πY^X, then $\alpha_* = \beta_* : f_* \Rightarrow g_*$.

Suppose now that $X = (X, m, e, \alpha, \lambda, \rho)$ is any given homotopy coherent associative H-space, i.e. a Stasheff A_4-space [70] (any topological monoid, for instance).

This means that we have a topological space X, which is endowed with a continuous multiplication map $m : X \times X \to X$, a point $e \in X$, and homotopies

$$\begin{tikzcd}
X\times X\times X \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "\overset{\alpha}{\Rrightarrow}"] & X\times X \arrow[d, "m"] \\
X\times X \arrow[r, "m"'] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "e\times 1"] \arrow[d, "1\times e"'] \arrow[dr, "1"] & X\times X \arrow[d, "m"] \arrow[dl, phantom, "\overset{\lambda}{\Lleftarrow}"] \\
X\times X \arrow[r, "m"'] \arrow[ur, phantom, "\overset{\rho}{\Rrightarrow}"] & X,
\end{tikzcd}$$

which are homotopy coherent, in the sense that there are homotopies as below.

$$\begin{tikzcd}[column sep=small]
 & X^4 \arrow[rr, "m\times1\times1"] \arrow[dl, "1\times1\times m"'] \arrow[dr, "1\times m\times 1"] & & X^3 \arrow[dr, "m\times1"] \arrow[dl, phantom, "\Downarrow\alpha\times1"] & & & & X^4 \arrow[rr, "m\times1\times1"] \arrow[dl, "1\times1\times m"'] \arrow[dr, phantom, "="] & & X^3 \arrow[dl, "1\times m"'] \arrow[dr, "m\times1"] & \\
X^3 \arrow[dr, "1\times m"'] \arrow[rr, phantom, "\overset{1\times\alpha}{\Lleftarrow}"] & & X^3 \arrow[rr, "m\times1"] \arrow[dl, "1\times m"] & & X^2 \arrow[dl, "m"] & \Rrightarrow & X^3 \arrow[rr, "m\times1"] \arrow[dr, "1\times m"'] & & X^2 \arrow[dr, "m"] \arrow[rr, phantom, "\overset{\alpha}{\Lleftarrow}"] & & X^2 \arrow[dl, "m"] \\
 & X^2 \arrow[rr, "m"'] \arrow[ur, phantom, "\Downarrow\alpha" near end] & & X & & & & X^2 \arrow[rr, "m"'] \arrow[ur, phantom, "\Downarrow\alpha"] & & X &
\end{tikzcd}$$

$$\begin{tikzcd}
X^2 \arrow[rr, "1"] \arrow[d, "1\times e\times 1"'] \arrow[dr, phantom, "\Uparrow 1\times\lambda"] & & X^2 \arrow[d, "m"] \\
X^3 \arrow[r, "m\times1"'] \arrow[urr, "1\times m"] & X^2 \arrow[r, "m"'] \arrow[ur, phantom, "\Uparrow\alpha"] & X
\end{tikzcd}
\quad\Rrightarrow\quad
\begin{tikzcd}
X^2 \arrow[rr, "1"] \arrow[d, "1\times e\times 1"'] \arrow[dr, phantom, "\Uparrow \rho\times1"] & & X^2 \arrow[d, "m"] \\
X^3 \arrow[r, "m\times1"'] \arrow[urr, "m\times 1"] & X^2 \arrow[r, "m"'] \arrow[ur, phantom, "="] & X
\end{tikzcd}$$

Since the functor $X \mapsto \pi X$ preserves products, the multiplication map $m : X\times X \to X$ induces a tensor product

$$m_* : \pi X \times \pi X \cong \pi(X \times X) \longrightarrow \pi X,$$

and the homotopies α, λ, and ρ, induce corresponding associativity, left unit, and right unit constraints (which satisfy the pentagon and triangle axioms (1.1), (1.2) thanks to the existence of the homotopies $\Rrightarrow$ above), we have thus defined *the fundamental monoidal groupoid of the H-space*

$$\pi X = (\pi X, m_*, e, \alpha_*, \lambda_*, \rho_*).$$

Let us stress that πX is a categorical group whenever X is group-like (for instance $X \simeq \Omega(Y, y_0)$, any loop space).

1.2 Schreier-Grothendieck theory for monoidal groupoids

The Schreier extension theorem [67] gives a cohomological classification of extensions of (non-abelian) groups, $1 \to A \to E \to G \to 1$, in terms of equivalence classes of the so-called *Schreier systems for group extensions* or *non-abelian 2-cocycles on groups.* That is, by means of systems of data

$$(G, A, \Theta, \lambda), \tag{1.12}$$

consisting of groups G and A, a family of automorphisms $\Theta = (A \xrightarrow{a_*} A)_{a\in G}$, and a family of elements $\lambda = (\lambda_{a,b} \in A)_{a,b\in G}$, satisfying:

$$\lambda_{a,b} \circ (ab)_*(f) \circ \lambda_{a,b}^{-1} = a_*(b_*(f)), \qquad 1_*(f) = f,$$
$$a_*(\lambda_{b,c}) \circ \lambda_{a,bc} = \lambda_{a,b} \circ \lambda_{ab,c}, \qquad \lambda_{a,1} = 1 = \lambda_{1,a},$$

where f is any element of the group A. Any such Schreier system gives rise to a group extension

$$1 \to A \to \Sigma(G, A, \Theta, \lambda) \to G \to 1, \tag{1.13}$$

where $\Sigma(G, A, \Theta, \lambda)$ is the group defined by considering on the set $A \times G$ the product $(f, a) \circ (g, b) = (f \circ a_*(g) \circ \lambda_{a,b}, ab)$, and any group extension can be obtained in this way up to isomorphism. Actually, the construction of the group extension (1.13), from each Schreier system (1.12), defines the function on objects of an equivalence of categories between the category of Schreier systems for group extensions, whose morphisms

$$(p, q, \varphi) : (G, A, \Theta, \lambda) \to (G', A', \Theta', \lambda')$$

are triplets consisting of homomorphisms $p : G \to G'$, $q : A \to A'$, and a family of elements $\varphi = (\varphi_a \in A')_{a\in G}$, satisfying:

$$\varphi_a \circ p(a)_*(q(f)) \circ \varphi_a^{-1} = q(a_*(f)),$$
$$q(\lambda_{a,b}) \circ \varphi_{ab} = \varphi_a \circ p(a)_*(\varphi_b) \circ \lambda'_{p(a),p(b)},$$

and the category of extensions of groups, whose morphisms are commutative diagrams

$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] \arrow[d, "q"'] & E \arrow[r] \arrow[d, "\phi"'] & H \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & G' \arrow[r] & E' \arrow[r] & H' \arrow[r] & 1.
\end{tikzcd}$$

Several generalizations to monoid extensions of Schreier theory are known in the literature: Rédey [64], Leech [53, 54], Inassaridze [47], etc. To classify fibrations between categories, Grothendieck [45] raised to a categorical level Schreier's theorem by means of the theory of pseudo-functors, and higher analogues problems were studied, among others, by Sinh in [69], where she performed the categorical group classification; Breen [8], who treated with non-abelian 3-cocycles of groups for the classification of extensions of groups by categorical groups; Carrasco and Cegarra in [17], where they carried out the classification of central extensions of categorical groups; Ulbrich [72], who classified extensions of Picard categories; Cegarra and Garzón in [21], where a classification of torsors over a category under a categorical group is done; or Cegarra and Khmaladze [22, 23], where it is performed the classification both of braided and symmetric graded categorical groups, later on extended to the fibred cases by Calvo, Cegarra and Quang in [11]. We are inspired in all these works to make below a corresponding analysis of monoidal groupoids, whence we achieve a 3-dimensional Schreier-Grothendieck factor set theory for the classification of monoidal groupoids, which indeed involves a 2-dimensional one for monoidal functors between monoidal groupoids, and even a 1-dimensional one for the monoidal transformations between them.

1.2.1 Schreier systems for monoidal groupoids

Keeping the Schreier-Grothendieck theory in mind, we introduce below 3-dimensional *Schreier systems for monoidal groupoids*, or *non-abelian* 3*-cocycles on monoids*, which will be showed as appropriate minimal systems of "descent datum" to build a survey of all monoidal groupoids up to monoidal equivalences.

Definition 1.1 *A* Schreier system (for a monoidal groupoid) $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ *consists of the following data:*

- *a monoid* M*,*
- *a family of groups* $\mathcal{A} = (\mathcal{A}(a))_{a\in M}$*,*
- *a family of homomorphisms* $\Theta = (\mathcal{A}(b) \xrightarrow{a_*} \mathcal{A}(ab) \xleftarrow{b^*} \mathcal{A}(a))_{a,b\in M}$*,*
- *a family of elements* $\lambda = (\lambda_{a,b,c} \in \mathcal{A}(abc))_{a,b,c\in M}$*.*

These data must satisfy the following seven conditions:

- *For any* $a, b, c \in M$*,* $h \in \mathcal{A}(a)$*,* $g \in \mathcal{A}(b)$*, and* $f \in \mathcal{A}(c)$*,*

$$\lambda_{a,b,c} \circ (ab)_*(f) \circ \lambda^{-1}_{a,b,c} = a_*(b_*(f)), \tag{1.14}$$

$$\lambda_{a,b,c} \circ c^*(a_*(g)) \circ \lambda^{-1}_{a,b,c} = a_*(c^*(g)), \tag{1.15}$$

$$\lambda_{a,b,c} \circ c^*(b^*(h)) \circ \lambda^{-1}_{a,b,c} = (bc)^*(h). \tag{1.16}$$

- *For any* $a, b, c, d \in M$*,*

$$a_*(\lambda_{b,c,d}) \circ \lambda_{a,bc,d} \circ d^*(\lambda_{a,b,c}) = \lambda_{a,b,cd} \circ \lambda_{ab,c,d}. \tag{1.17}$$

- *For any* $a, b \in M$*,* $g \in \mathcal{A}(a)$*, and* $f \in \mathcal{A}(b)$*,*

$$a_*(f) \circ b^*(g) = b^*(g) \circ a_*(f). \tag{1.18}$$

- *For any* $a \in M$ *and* $f \in \mathcal{A}(a)$*,*

$$e_*(f) = f = e^*(f), \tag{1.19}$$

where $e \in M$ *is the unit.*

- *For any* $a, b \in M$*,*

$$\lambda_{e,a,b} = \lambda_{a,e,b} = \lambda_{a,b,e} = 1. \tag{1.20}$$

Example 1.1 A Schreier system as above with $\lambda = 1$ (i.e., such that $\lambda_{a,b,c} = 1$ for all $a, b, c \in M$) is the same thing as a pair of data $(M, (\mathcal{A}, \Theta))$ consisting of a monoid M together with an internal group object $(\mathcal{A}, \Theta) \in \mathbf{Gp}(\mathbf{Mnd}\downarrow_M)$, in the comma category of monoids over M. We refer to Wells [74, Theorem 6] for details, but briefly let us say that, for that identification, one regards $\mathcal{A}$ as the monoid obtained as the disjoint union of the groups $\mathcal{A}(a)$, $a \in M$, with multiplication given by $(f, a)(g, b) = (a_*(f) \circ b^*(g), ab)$. This multiplication is associative thanks to (1.14), (1.15), and (1.16), and it is unitary, with $(1, e)$ its unit, owing to (1.19). The monoid homomorphism $\bigcup_{a\in M} \mathcal{A}(a) \to M$ is the obvious projection $(f, a) \mapsto a$, and the internal group operation is defined by the map $\bigcup_{a\in M} \mathcal{A}(a) \times_M \bigcup_{a\in M} \mathcal{A}(a) \to \bigcup_{a\in M} \mathcal{A}(a)$, $((f, a), (g, a)) \mapsto (f \circ g, a)$, which is plainly recognized to be a monoid homomorphism thanks to the centralizing condition (1.18).

Surjective monoid homomorphisms $E \to M$ endowed with a principal homogenous internal $(\mathcal{A}, \Theta)$-action in $\mathbf{Mon}\downarrow_M$ (i.e., internal $(\mathcal{A}, \Theta)$-torsors) are classified by means of Leech *non-abelian 2-cocycles of M with coefficients in $\mathcal{A}$*. That is, by families $\lambda = (\lambda_{a,b})$ of elements $\lambda_{a,b} \in \mathcal{A}(ab)$, one for each $a, b \in M$, such that

$$a_*(\lambda_{b,c}) \circ \lambda_{a,bc} = c^*(\lambda_{a,b}) \circ \lambda_{ab,c}, \quad \lambda_{e,a} = 1 = \lambda_{a,e},$$

for all $a, b, c \in M$; see Leech [53, Section 3] and Wells [74, Theorems 1 and 7].

Remark 1.1 Regarding any group as a groupoid with exactly one object, it was observed by Grothendieck [45] that a non-abelian 2-cocycle (G, A, Θ, λ) for a group extension of a group G by a group A, as in (1.12), can be identified as a normal pseudo-functor on G that associates the group A to the unique object of A. Similarly, as one identifies any monoid with the monoidal discrete category it defines, then a Schreier system $(M, \mathcal{A}, \Theta, \lambda)$ for a monoidal groupoid, as in Definition 1.1, can be viewed as a group valuated normal monoidal pseudo-functor on M, in the sense of Carrasco-Cegarra [17, Definition 1.6], that associates the group $\mathcal{A}(a)$ to each object $a \in M$.

Next we explain how Schreier systems, as in Definition 1.1, come characteristically associated to monoidal groupoids.

1.2.2 Schreier systems associated to monoidal groupoids

From now on, in this chapter we denote the tensor product by juxtaposition, that is, $XY = X \otimes Y$.

For any monoidal groupoid $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ let

$$M(\mathcal{M}) = \mathrm{Ob}\mathcal{M}/_{\cong} \tag{1.21}$$

be the monoid of isomorphism classes $a = [X]$ of objects $X \in \mathcal{M}$ where multiplication is induced by the tensor product, that is, $[X][Y] = [XY]$ and whose unit is $e = [\mathrm{I}]$.

The construction $\mathcal{M} \mapsto M(\mathcal{M})$ turns the category of monoidal groupoids into a fibred category over the category of monoids. To determine its fiber over a monoid, we shall proceed as Schreier did for extensions of a group:

We start by choosing a cleavage for $\mathcal{M}$ over $M(\mathcal{M})$, that is, for each $a \in M(\mathcal{M})$, let us choose an object $X_a \in a$, and for any other $X \in a$, we fix a morphism $\Gamma = \Gamma_X : X \to X_a$. In particular, we take

$$X_e = \mathrm{I}, \quad \Gamma_{\mathrm{I}X_a} = \boldsymbol{l}_{X_a} : \mathrm{I}X_a \to X_a, \quad \Gamma_{X_a\mathrm{I}} = \boldsymbol{r}_{X_a} : X_a\mathrm{I} \to X_a. \tag{1.22}$$

Then, we have the following family of isotropy groups of the groupoid $\mathcal{M}$ parameterized by the elements of $M(\mathcal{M})$:

$$\mathcal{A}(\mathcal{M}) = \big(\mathrm{Aut}_{\mathcal{M}}(X_a)\big)_{a\in M(\mathcal{M})}. \tag{1.23}$$

We also have the family of group homomorphisms

$$\Theta(\mathcal{M}) = \big(\mathrm{Aut}_{\mathcal{M}}(X_b) \xrightarrow{a_*} \mathrm{Aut}_{\mathcal{M}}(X_{ab}) \xleftarrow{b^*} \mathrm{Aut}_{\mathcal{M}}(X_a)\big)_{a,b\in M(\mathcal{M})}, \tag{1.24}$$

which, for any $a, b \in M(\mathcal{M})$, carry automorphisms of $\mathcal{M}$, say $f : X_b \to X_b$ and $g : X_a \to X_a$, to the automorphisms $a_*(f) : X_{ab} \to X_{ab}$ and $b^*(g) : X_{ab} \to X_{ab}$, respectively determined by the commutativity of the squares below.

$$\begin{tikzcd} X_aX_b \arrow[r, "1f"] \arrow[d, "\Gamma"'] & X_aX_b \arrow[d, "\Gamma"] \\ X_{ab} \arrow[r, dotted, "a_*(f)"] & X_{ab} \end{tikzcd} \qquad \begin{tikzcd} X_aX_b \arrow[r, "g1"] \arrow[d, "\Gamma"'] & X_aX_b \arrow[d, "\Gamma"] \\ X_{ab} \arrow[r, dotted, "b^*(g)"] & X_{ab} \end{tikzcd} \tag{1.25}$$

Furthermore, for any three elements $a, b, c \in M(\mathcal{M})$, there is a unique

$$\lambda_{a,b,c} \in \mathrm{Aut}_{\mathcal{M}}(X_{abc})$$

making commutative the diagram

$$\begin{tikzcd} (X_aX_b)X_c \arrow[r, "\Gamma 1"] \arrow[d, "\boldsymbol{a}"'] & X_{ab}X_c \arrow[r, "\Gamma"] & X_{abc} \arrow[d, dotted, "\lambda_{a,b,c}"] \\ X_a(X_bX_c) \arrow[r, "1\Gamma"] & X_aX_{bc} \arrow[r, "\Gamma"] & X_{abc}. \end{tikzcd} \tag{1.26}$$

Then, letting

$$\lambda(\mathcal{M}) = \big(\lambda_{a,b,c} \in \mathrm{Aut}_{\mathcal{M}}(X_{abc})\big)_{a,b,c\in M(\mathcal{M})}, \tag{1.27}$$

we have:

Proposition 1.2 *For any monoidal groupoid $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$, the associated quadruplet*

$$\Delta(\mathcal{M}) = \big(M(\mathcal{M}), \mathcal{A}(\mathcal{M}), \Theta(\mathcal{M}), \lambda(\mathcal{M})\big), \tag{1.28}$$

given by (1.21), (1.23), (1.24), *and* (1.27), *is a Schreier system.*

Proof: In all the diagrams below, those inner regions labelled with (A) commute by the naturality of the associativity constraint, those labelled with (B) are commutative because $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a functor, and the other commute by the references therein.

For any $a, b, c \in M(\mathcal{M})$, $h \in \mathrm{Aut}_{\mathcal{M}}(X_a)$, $g \in \mathrm{Aut}_{\mathcal{M}}(X_b)$, and $f \in \mathrm{Aut}_{\mathcal{M}}(X_c)$, the conditions in (1.14), (1.15), and (1.16), follow, respectively, from the commutativity of the outside regions in the following three diagrams in $\mathcal{M}$:

$$\begin{tikzcd}
X_{abc} \arrow[d, "(ab)_*(f)"'] \arrow[rrrrr, bend left=20, "\lambda_{a,b,c}", "(1.26)"'] \arrow[dr, phantom, "(1.25)"] & X_{ab}X_c \arrow[l, "\Gamma"'] \arrow[d, "1f"] \arrow[dr, phantom, "(B)"] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] \arrow[d, "(11)f"] \arrow[dr, phantom, "(A)"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] \arrow[d, "1(1f)"] \arrow[dr, phantom, "(1.25)"] & X_aX_{bc} \arrow[r, "\Gamma"] \arrow[d, "1b_*(f)"] \arrow[dr, phantom, "(1.25)"] & X_{abc} \arrow[d, "a_*b_*(f)"] \\
X_{abc} \arrow[rrrrr, bend right=20, "\lambda_{a,b,c}"', "(1.26)"] & X_{ab}X_c \arrow[l, "\Gamma"'] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] & X_aX_{bc} \arrow[r, "\Gamma"] & X_{abc}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{abc} \arrow[d, "c^*a_*(g)"'] \arrow[rrrrr, bend left=20, "\lambda_{a,b,c}", "(1.26)"'] \arrow[dr, phantom, "(1.25)"] & X_{ab}X_c \arrow[l, "\Gamma"'] \arrow[d, "a_*(g)1"] \arrow[dr, phantom, "(1.25)"] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] \arrow[d, "(1g)1"] \arrow[dr, phantom, "(A)"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] \arrow[d, "1(g1)"] \arrow[dr, phantom, "(1.25)"] & X_aX_{bc} \arrow[r, "\Gamma"] \arrow[d, "1c^*(g)"] \arrow[dr, phantom, "(1.25)"] & X_{abc} \arrow[d, "a_*c^*(g)"] \\
X_{abc} \arrow[rrrrr, bend right=20, "\lambda_{a,b,c}"', "(1.26)"] & X_{ab}X_c \arrow[l, "\Gamma"'] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] & X_aX_{bc} \arrow[r, "\Gamma"] & X_{abc}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{abc} \arrow[d, "c^*b^*(h)"'] \arrow[rrrrr, bend left=20, "\lambda_{a,b,c}", "(1.26)"'] \arrow[dr, phantom, "(1.25)"] & X_{ab}X_c \arrow[l, "\Gamma"'] \arrow[d, "b^*(h)1"] \arrow[dr, phantom, "(1.25)"] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] \arrow[d, "(h1)1"] \arrow[dr, phantom, "(A)"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] \arrow[d, "h(11)"] \arrow[dr, phantom, "(B)"] & X_aX_{bc} \arrow[r, "\Gamma"] \arrow[d, "h1"] \arrow[dr, phantom, "(1.25)"] & X_{abc} \arrow[d, "(bc)^*(h)"] \\
X_{abc} \arrow[rrrrr, bend right=20, "\lambda_{a,b,c}"', "(1.26)"] & X_{ab}X_c \arrow[l, "\Gamma"'] & (X_aX_b)X_c \arrow[l, "\Gamma 1"'] \arrow[r, "\boldsymbol{a}"] & X_a(X_bX_c) \arrow[r, "1\Gamma"] & X_aX_{bc} \arrow[r, "\Gamma"] & X_{abc}
\end{tikzcd}$$

Since, for any $a, b \in M(\mathcal{M})$, $g \in \mathrm{Aut}_{\mathcal{M}}(X_a)$, and $f \in \mathrm{Aut}_{\mathcal{M}}(X_b)$, we have the commutative diagram

$$\begin{tikzcd}
X_{ab} \arrow[d, "b^*(g)"'] \arrow[rrr, bend left, "a_*(f)", "(1.25)"'] \arrow[r, phantom, "(1.25)" below, shift right=6] & X_aX_b \arrow[l, "\Gamma"'] \arrow[r, "1f"] \arrow[d, "g1"'] \arrow[r, phantom, "(B)" below, shift right=6] & X_aX_b \arrow[r, "\Gamma"] \arrow[d, "g1"] \arrow[r, phantom, "(1.25)" below, shift right=6] & X_{ab} \arrow[d, "b^*(g)"] \\
X_{ab} \arrow[rrr, bend right, "a_*(f)"', "(1.25)"] & X_aX_b \arrow[l, "\Gamma"'] \arrow[r, "1f"'] & X_aX_b \arrow[r, "\Gamma"] & X_{ab},
\end{tikzcd}$$

it follows that the homomorphisms a_* and b^* in (1.24) are centralizing, that is, condition in (1.18) holds. Moreover, when $a = e$ or $b = e$, the naturality of the unit constraints gives the commutativity of the squares

$$\begin{tikzcd}
\mathrm{I}X_b \arrow[r, "1f"] \arrow[d, "\Gamma=\boldsymbol{l}"'] & \mathrm{I}X_b \arrow[d, "\Gamma=\boldsymbol{l}"] \\
X_b \arrow[r, "f"] & X_b,
\end{tikzcd}
\qquad
\begin{tikzcd}
X_a\mathrm{I} \arrow[r, "g1"] \arrow[d, "\Gamma=\boldsymbol{r}"'] & X_a\mathrm{I} \arrow[d, "\Gamma=\boldsymbol{r}"] \\
X_a \arrow[r, "g"] & X_a,
\end{tikzcd}$$

whence $e_*(f) = f$ and $e^*(g) = g$. That is, the normalization conditions in (1.19) hold.

Furthermore, the 3-cocycle condition (1.17), for any $a, b, c, d \in M(\mathcal{M})$, follows from the commutativity of the following diagram

$$\begin{tikzcd}
X_{abcd} \arrow[rrr, "\lambda_{ab,c,d}"] \arrow[dddddddd, bend right=40, "d^*(\lambda_{a,b,c})"', "(1.25)"] & & & X_{abcd} \arrow[rrr, "\lambda_{a,b,cd}"] & & & X_{abcd} \\
X_{abc}X_d \arrow[u, "\Gamma"'] \arrow[dddddd, bend right=20, "\lambda_{a,b,c}1"'] & \arrow[r, phantom, "(1.26)"] & & X_{ab}X_{cd} \arrow[u, "\Gamma"] & \arrow[r, phantom, "(1.26)"] & & X_aX_{bcd} \arrow[u, "\Gamma"'] \\
(X_{ab}X_c)X_d \arrow[u, "\Gamma 1"'] \arrow[rr, "\boldsymbol{a}"] & & X_{ab}(X_cX_d) \arrow[ur, "1\Gamma"] & (B) & (X_aX_b)X_{cd} \arrow[ul, "\Gamma 1"'] \arrow[rr, "\boldsymbol{a}"] & & X_a(X_bX_{cd}) \arrow[u, "1\Gamma"'] \\
& (A) & & (X_aX_b)(X_cX_d) \arrow[ul, "\Gamma(11)"'] \arrow[ur, "(11)\Gamma"] \arrow[ddrr, "\boldsymbol{a}"] & & (A) & \\
& & & & & & \\
\arrow[r, phantom, "(1.26)"] & ((X_aX_b)X_c)X_d \arrow[uuul, "(\Gamma 1)1"] \arrow[uurr, "\boldsymbol{a}"] \arrow[d, "\boldsymbol{a}1"] & & (1.1) & & X_a(X_b(X_cX_d)) \arrow[uuur, "1(1\Gamma)"'] & \arrow[l, phantom, "(1.26)"] \\
& (X_a(X_bX_c))X_d \arrow[rrrr, "\boldsymbol{a}"] \arrow[dl, "(1\Gamma)1"'] & & (A) & & X_a((X_bX_c)X_d)) \arrow[u, "1\boldsymbol{a}"'] \arrow[dr, "1(\Gamma 1)"] & \\
(X_aX_{bc})X_d \arrow[rrrrrr, "\boldsymbol{a}"] \arrow[d, "\Gamma 1"] & & & (1.26) & & & X_a(X_{bc}X_d) \arrow[d, "1\Gamma"'] \\
X_{abc}X_d \arrow[d, "\Gamma"] & & & & & & X_aX_{bcd} \arrow[d, "\Gamma"'] \arrow[uuuuuu, bend right=20, "1\lambda_{b,c,d}"'] \\
X_{abcd} \arrow[rrrrrr, "\lambda_{a,bc,d}"] & & & & & & X_{abcd}. \arrow[uuuuuuuuu, bend right=40, "a_*(\lambda_{b,c,d})"', "(1.25)"]
\end{tikzcd}$$

Finally, recalling the selections (1.22), it is plain to see that the normalization conditions in (1.20) are direct consequence of the coherence triangles in (1.1) and (1.3). This completes the proof. □

The Schreier system in (1.28), associated to a monoidal groupoid, depends on the selection of the cleavage made for its construction. However, as we shall prove, different choices produce *equivalent* Schreier systems.

We next explain how each Schreier system gives rise, by the so-called Grothendieck construction (cf. [17, 1.3]), to a monoidal groupoid.

1.2.3 The monoidal groupoid defined by a Schreier system

Let $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ any given Schreier system. Then, a monoidal groupoid

$$\Sigma(\mathcal{S}) = (\Sigma(\mathcal{S}), \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}) \tag{1.29}$$

is defined as follows: an object of $\Sigma(\mathcal{S})$ is an element $a \in M$. If $a \neq b$ are different elements of the monoid M, then there is no morphisms in $\Sigma(\mathcal{S})$ between them, whereas if $a = b$, then a morphism $f : a \to a$ is an element f of the group $\mathcal{A}(a)$, that is

$$\Sigma(\mathcal{S})(a, b) = \begin{cases} \emptyset & \text{if} \quad a \neq b, \\ \mathcal{A}(a) & \text{if} \quad a = b. \end{cases}$$

The composition of morphisms is given by the group operation of $\mathcal{A}(a)$, that is,

$$(a \xrightarrow{f} a) \circ (a \xrightarrow{f'} a) = (a \xrightarrow{f \circ f'} a).$$

The tensor product $\otimes : \Sigma(\mathcal{S}) \times \Sigma(\mathcal{S}) \to \Sigma(\mathcal{S})$ is defined by

$$(a \xrightarrow{g} a) \otimes (b \xrightarrow{f} b) = \ (ab \xrightarrow{a_*(f) \circ b^*(g)} ab)\ ,$$

which is a functor thanks to the centralizing condition (1.18). In effect, we have

$$(a \xrightarrow{1} a) \otimes (b \xrightarrow{1} b) = \ (ab \xrightarrow{a_*(1) \circ b^*(1)} ab)\ = ab \xrightarrow{1} ab.$$

and, for any $g, g' : a \to a$ and $f, f' : b \to b$, we have

$$\begin{aligned} (g \circ g') \otimes (f \circ f') \ &= \ a_*(f \circ f') \circ b^*(g \circ g') \ = \ a_*(f) \circ a_*(f') \circ b^*(g) \circ b^*(g') \\ &\overset{(1.18)}{=} \ a_*(f) \circ b^*(g) \circ a_*(f') \circ b^*(g') \ = \ (g \otimes f) \circ (g' \otimes f'). \end{aligned}$$

The associativity isomorphisms are

$$\lambda_{a,b,c} : (ab)c \to a(bc).$$

These are natural thanks to conditions (1.14), (1.15), and (1.16). In effect, for any $h : a \to a$, $g : b \to b$, and $f : c \to c$,

$$\begin{aligned}
\lambda_{a,b,c} \circ ((h \otimes g) \otimes f) &= \lambda_{a,b,c} \circ ((a_*(g) \circ b^*(h)) \otimes f) \\
&= \lambda_{a,b,c} \circ (ab)_*(f) \circ c^*(a_*(g) \circ b^*(h)) \\
&= \lambda_{a,b,c} \circ (ab)_*(f) \circ c^*(a_*(g)) \circ c^*(b^*(h)) \\
&\overset{(1.14)}{=} a_*(b_*(f)) \circ \lambda_{a,b,c} \circ c^*(a_*(g)) \circ c^*(b^*(h)) \\
&\overset{(1.15)}{=} a_*(b_*(f)) \circ a_*(c^*(g)) \circ \lambda_{a,b,c} \circ c^*(b^*(h)) \\
&\overset{(1.16)}{=} a_*(b_*(f)) \circ a_*(c^*(g)) \circ (bc)^*(h) \circ \lambda_{a,b,c} \\
&= a_*(b_*(f) \circ c^*(g)) \circ (bc)^*(h) \circ \lambda_{a,b,c} \\
&= (h \otimes (b_*(f) \circ c^*(g))) \circ \lambda_{a,b,c} \\
&= (h \otimes (g \otimes f)) \circ \lambda_{a,b,c}.
\end{aligned}$$

The pentagon coherence condition in (1.1) just says that, for any $a, b, c, d \in M$, the diagram

$$\begin{tikzcd}
((ab)c)d \arrow[r, "\lambda_{ab,c,d}"] \arrow[d, "d^*(\lambda_{a,b,c})"'] & (ab)(cd) \arrow[r, "\lambda_{a,b,cd}"] & a(b(cd)) \\
(a(bc))d \arrow[rr, "\lambda_{a,bc,d}"] & & a((bc)d) \arrow[u, "a_*(\lambda_{b,c,d})"']
\end{tikzcd}$$

must be commutative, what holds because of the 3-cocycle condition (1.17).

The unit object is $\mathrm{I} = e$, the unit element of the monoid M, and the unit constraints are both identities, that is, for any $a \in M$,

$$\boldsymbol{l}_a = 1 = \boldsymbol{r}_a : a \to a.$$

These are natural due to the equalities in (1.19). In effect, for any $f : a \to a$, we have

$$\begin{aligned}
\boldsymbol{l}_a \circ (1 \otimes f) = 1 \otimes f = e_*(f) \circ a^*(1) &\overset{(1.19)}{=} f \circ 1 = f = f \circ \boldsymbol{l}_a, \\
\boldsymbol{r}_a \circ (f \otimes 1) = f \otimes 1 = a_*(1) \circ e^*(f) &\overset{(1.19)}{=} 1 \circ f = f = f \circ \boldsymbol{r}_a.
\end{aligned}$$

The coherence triangle for the unit in (1.2) commutes owing to the normalization condition $\lambda_{a,e,b} = 1$ in (1.20). □

As we will show, both constructions $\mathcal{S} \mapsto \Sigma(\mathcal{S})$, as above, and $\mathcal{M} \mapsto \Delta(\mathcal{M})$, as in (1.28), are convenient to express the strong relationship between Schreier systems and monoidal groupoids. Previously, we need the notions of *morphisms* between Schreier systems and their *deformations*, that we establish below.

1.2.4 The 2-category of Schreier systems

The Schreier systems introduced in 1.1, or non-abelian 3-cocycles of monoids, are the objects of a 2-category in which all 2-cells are invertible, denoted by

$$\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd},$$

whose cells and their compositions are defined as follows:

Morphisms of Schreier systems

If $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$, $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$ are two Schreier systems, then a *morphism* $\wp = (p, q, \varphi) : \mathcal{S} \to \mathcal{S}'$ consists of the following data:

- a monoid homomorphism $p : M \to M'$,
- a family of group homomorphisms $q = \big(\mathcal{A}(a) \xrightarrow{q_a} \mathcal{A}'(p(a))\big)_{a\in M}$,
- a family of elements $\varphi = \big(\varphi_{a,b} \in \mathcal{A}'(p(ab))\big)_{a,b\in M}$,

satisfying the following three conditions:

- For any $a, b \in M$, $g \in \mathcal{A}(a)$, and $f \in \mathcal{A}(b)$,

$$\begin{aligned}\varphi_{a,b} \circ p(a)_*(q_b(f)) \circ \varphi_{a,b}^{-1} &= q_{ab}(a_*(f)),\\ \varphi_{a,b} \circ p(b)^*(q_a(g)) \circ \varphi_{a,b}^{-1} &= q_{ab}(b^*(g)).\end{aligned} \tag{1.30}$$

- For any $a, b, c \in M$,

$$q_{abc}(\lambda_{a,b,c}) \circ \varphi_{ab,c} \circ p(c)^*(\varphi_{a,b}) = \varphi_{a,bc} \circ p(a)_*(\varphi_{b,c}) \circ \lambda'_{p(a),p(b),p(c)}. \tag{1.31}$$

- $$\varphi_{e,e} = 1. \tag{1.32}$$

Observe that, taking $b = c = e$ in the above equality (1.31), we deduce that, for any $a \in M$, $\varphi_{a,e} \circ \varphi_{a,e} = \varphi_{a,e} \circ p(a)_*(\varphi_{e,e}) = \varphi_{a,e}$ in the group $\mathcal{A}'(p(a))$, whence $\varphi_{a,e} = 1$. Similarly, $\varphi_{e,a} = 1$.

Deformations

Let $\wp = (p, q, \varphi) : \mathcal{S} \to \mathcal{S}'$ and $\bar{\wp} = (\bar{p}, \bar{q}, \bar{\varphi}) : \mathcal{S} \to \mathcal{S}'$ be morphisms between Schreier systems $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ and $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$.

If $p \neq \bar{p}$ are different homomorphisms, then there is no deformation in $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$ between $\wp$ and $\bar{\wp}$.

If $p = \bar{p}$, then a *deformation*

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "{\wp=(p,q,\varphi)}"] \arrow[r, bend right, "{\bar{\wp}=(p,\bar{q},\bar{\varphi})}"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{S}' \end{tikzcd}$$

is a family of elements $\delta = \big(\delta_a \in \mathcal{A}'(p(a))\big)_{a\in M}$ satisfying the following two conditions:

- For any $a \in M$ and $f \in \mathcal{A}(a)$,

$$\delta_a^{-1} \circ \bar{q}_a(f) \circ \delta_a = q_a(f). \tag{1.33}$$

- For any $a, b \in M$,

$$\delta_{ab} \circ \varphi_{a,b} = \bar{\varphi}_{a,b} \circ p(a)_*(\delta_b) \circ p(b)^*(\delta_a). \tag{1.34}$$

Observe that, taking $a = b = e$ in the above equality (1.34), we deduce that $\delta_e = \delta_e \circ \delta_e$ in the group $\mathcal{A}'(e')$ (where e' is the unit in M'), whence $\delta_e = 1$.

Vertical composition of deformations

For any Schreier systems $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ and $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$, the vertical composition in the 2-category $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$ of deformations

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left=50, "\wp=(p,q,\varphi)"] \arrow[r, "(p,\bar{q},\bar{\varphi})" description] \arrow[r, bend right=50, "\bar{\bar{\wp}}=(p,\bar{\bar{q}},\bar{\bar{\varphi}})"'] \arrow[r, phantom, shift left=3, "\Downarrow\delta"] \arrow[r, phantom, shift right=3, "\Downarrow\bar{\delta}"] & \mathcal{S}' \end{tikzcd} \tag{1.35}$$

is the deformation $\bar{\delta} \circ \delta : \wp \Rightarrow \bar{\bar{\wp}}$ obtained by pointwise multiplication, that is,

$$\bar{\delta} \circ \delta = \big(\bar{\delta}_a \circ \delta_a \in \mathcal{A}'(p(a))\big)_{a\in M}. \tag{1.36}$$

The identity deformation on each morphism $\wp : \mathcal{S} \to \mathcal{S}'$ is

$$1_{\wp} = \big(1 \in \mathcal{A}'(p(a))\big)_{a\in M} : \wp \Rightarrow \wp.$$

Thus, every deformation $\delta : \wp \Rightarrow \wp'$ becomes invertible (with $\delta^{-1} = (\delta_a^{-1} \in \mathcal{A}'(p(a)))_{a\in M}$) and therefore, in this 2-category of Schreier systems, the hom-categories $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}(\mathcal{S}, \mathcal{S}')$ are groupoids.

Horizontal composition of morphisms

For any $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$, $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$, and $\mathcal{S}'' = (M'', \mathcal{A}'', \Theta'', \lambda'')$ Schreier systems, the horizontal composition of two morphisms

$$\mathcal{S} \xrightarrow{\wp=(p,q,\varphi)} \mathcal{S}' \xrightarrow{\wp'=(p',q',\varphi')} \mathcal{S}''$$

is the morphism

$$\wp'\wp = (p'p, q'q, \varphi\varphi') : \mathcal{S} \to \mathcal{S}'', \tag{1.37}$$

where $p'p : M \to M''$ is the composite of p and p', and

$$
\begin{aligned}
q'q &= \big(q'_{p(a)}q_a : \mathcal{A}(a) \to \mathcal{A}''(p'p\,(a))\big)_{a\in M},\\
\varphi\varphi' &= \big(q'_{p(ab)}(\varphi_{a,b}) \circ \varphi'_{p(a),p(b)} \in \mathcal{A}''(p'p(ab))\big)_{a,b\in M}.
\end{aligned}
$$

The identity morphism on a Schreier system $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ is

$$1_{\mathcal{S}} = (1_M, 1_{\mathcal{A}}, 1) : \mathcal{S} \to \mathcal{S}, \tag{1.38}$$

where 1_M is the identity map on M, $1_{\mathcal{A}} = \big(1_{\mathcal{A}(a)}\big)_{a\in M}$, and $1 = \big(1 \in \mathcal{A}(ab)\big)_{a,b\in M}$.

Horizontal composition of deformations

The horizontal composition of deformations

$$
\begin{tikzcd}
\mathcal{S} \arrow[r, bend left, "\wp=(p,q\,,\varphi)"] \arrow[r, bend right, "\bar{\wp}=(p,\bar{q},\bar{\varphi})"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{S}' \arrow[r, bend left, "\wp'=(p',q',\varphi')"] \arrow[r, bend right, "\bar{\wp}'=(p',\bar{q}',\bar{\varphi}')"'] \arrow[r, phantom, "\Downarrow\delta'"] & \mathcal{S}'
\end{tikzcd} \tag{1.39}
$$

is the deformation $\delta'\delta : \wp'\wp \Rightarrow \bar{\wp}'\bar{\wp}$ defined by

$$\delta'\delta = \big(\delta'_{p(a)} \circ q'_{p(a)}(\delta_a) \in \mathcal{A}''(p'p(a))\big)_{a\in M}. \tag{1.40}$$

For later use, we prove here the lemma below.

Lemma 1.2 *Let* $(p, q\,, \varphi) : (M, \mathcal{A}, \Theta, \lambda) \to (M', \mathcal{A}', \Theta', \lambda')$ *be any Schreier system morphisms. Then, the following statements are equivalent:*

(i) $(p, q\,, \varphi)$ *is an isomorphism.*

(ii) $(p, q\,, \varphi)$ *is an equivalence.*

(iii) The homomorphisms $p : M \to M'$ *and* $q_a : \mathcal{A}(a) \to \mathcal{A}'(p(a))$*,* $a \in M$*, are all isomorphisms.*

Proof: *(i)* $\Rightarrow$ *(ii)* is obvious.

(ii) $\Rightarrow$ *(iii)*. First observe that, for any Schreier system $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$, a morphism $\wp : \mathcal{S} \to \mathcal{S}$ with a deformation $\delta : \wp \Rightarrow 1_{\mathcal{S}}$ is necessarily of the form $\wp = (1_M, q(\delta), \varphi(\delta))$, for some family $\delta = (\delta_a \in \mathcal{A}(a))_{a\in M}$, with $\delta_e = 1$, where $q(\delta) = (q(\delta)_a : \mathcal{A}(a) \to \mathcal{A}(a))_{a\in M}$ consists of the inner automorphisms given by $q(\delta)_a(f) = \delta_a^{-1} \circ f \circ \delta_a$, and $\varphi(\delta) = (\varphi(\delta)_{a,b} \in \mathcal{A}(ab))_{a,b\in M}$ consists of the elements obtained by the formula $\varphi(\delta)_{a,b} = \delta_{ab}^{-1} \circ a_*(\delta_b) \circ b^*(\delta_a)$.

Then, the existence of a morphism $(p', q', \varphi') : \mathcal{S}' \to \mathcal{S}$, where $\mathcal{S}$ is as above and $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$, with deformations $\delta : (p', q', \varphi')(p, q, \varphi) \Rightarrow 1_{\mathcal{S}}$ and $\delta' : (p, q, \varphi)(p', q', \varphi') \Rightarrow 1_{\mathcal{S}'}$, implies that $p'p = 1_M$, $pp' = 1_{M'}$, so p is an isomorphism, and also that

$$q'_{p(a)}q_a = q(\delta)_a, \quad q_a q'_{p(a)} = q(\delta')_{p(a)},$$

for all $a \in M$. Hence q_a and $q'_{p(a)}$ are both isomorphisms since $q(\delta)_a$ and $q(\delta')_{p(a)}$ are automorphisms.

$(iii) \Rightarrow (i)$. The inverse $(p, q, \varphi)^{-1} = (p', q', \varphi')$ is given by taking

$$p' = p^{-1}, \quad q' = \big(q^{-1}_{p'(a')}\big)_{a' \in M'}, \quad \varphi' = \big(q'_{a'b'}(\varphi^{-1}_{p'(a'),p'(b')})\big)_{a',b' \in M'}.$$

□

1.2.5 The classifying biequivalence

The following theorem, where it is stated that the 2-categories of monoidal groupoids and Schreier systems are biequivalent is the main result of this section.

Theorem 1.1 (Classification of monoidal groupoids) *The assignment given by the monoidal groupoid construction* (1.29)*, $\mathcal{S} \mapsto \Sigma(\mathcal{S})$, is the function on objects of a 2-functor*

$$\Sigma : \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd} \xrightarrow{\approx} \mathbf{MonGpd}, \tag{1.41}$$

which establishes a biequivalence between the 2-category of Schreier systems and the 2-category of monoidal groupoids. More precisely (cf. [71, p. 570]), for any two Schreier systems $\mathcal{S}$ and $\mathcal{S}'$, the functor

$$\Sigma : \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}(\mathcal{S}, \mathcal{S}') \xrightarrow{\simeq} \mathbf{MonGpd}(\Sigma(\mathcal{S}), \Sigma(\mathcal{S}')) \tag{1.42}$$

is an equivalence of groupoids, and for any monoidal groupoid $\mathcal{M}$, there is a monoidal equivalence

$$J_{\mathcal{M}} : \Sigma(\Delta(\mathcal{M})) \xrightarrow{\simeq} \mathcal{M}, \tag{1.43}$$

where $\Delta(\mathcal{M})$ is the Schreier system (1.28) *associated to $\mathcal{M}$.*

Proof: We have already described Σ on objects of the 2-category $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$, its effect on morphisms and deformations is as follows:

Σ on morphisms

Let $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$, $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$ be Schreier systems. Then, the 2-functor Σ carries any morphism $\wp = (p, q, \varphi) : \mathcal{S} \to \mathcal{S}'$ to the strictly unitary monoidal functor $\Sigma(\wp) : \Sigma(\mathcal{S}) \to \Sigma(\mathcal{S}')$ given by

$$\big(a \xrightarrow{f} a\big) \mapsto \big(p(a) \xrightarrow{q_a(f)} p(a)\big), \tag{1.44}$$

and whose structure isomorphisms are

$$\varphi_{a,b} : p(a)p(b) \to p(ab), \tag{1.45}$$

which are well defined since $p(a)p(b) = p(ab)$ and $\varphi_{a,b} \in \mathcal{A}'(p(ab))$, for any $a, b \in M$.

Since the maps $q_a : \mathcal{A}(a) \to \mathcal{A}'(p(a))$ are homomorphisms, it follows that $\Sigma(\wp)$ is a functor. Furthermore, the isomorphisms (1.45) are natural since, for any morphisms $f : b \to b$ and $g : a \to a$ in $\Sigma(\mathcal{S})$, the squares in $\Sigma(\mathcal{S}')$

$$\begin{tikzcd} p(a)p(b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "p(a)_*(q_b(f))"'] & p(ab) \arrow[d, "q_{ab}(a_*(f))"] \\ p(a)p(b) \arrow[r, "\varphi_{a,b}"] & p(ab) \end{tikzcd} \qquad \begin{tikzcd} p(a)p(b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "p(b)^*(q_a(g))"'] & p(ab) \arrow[d, "q_{ab}(b^*(f))"] \\ p(a)p(b) \arrow[r, "\varphi_{a,b}"] & p(ab) \end{tikzcd} \tag{1.46}$$

commute owing to condition (1.30). The coherence condition (1.5) for $\Sigma(\wp)$ just says that the diagrams

$$\begin{tikzcd} (p(a)p(b))p(c) \arrow[r, "p(c)^*(\varphi_{a,b})"] \arrow[d, "\lambda'_{p(a),p(b),p(c)}"'] & p(ab)p(c) \arrow[r, "\varphi_{ab,c}"] & p((ab)c) \arrow[d, "q_{abc}(\lambda_{a,b,c})"] \\ p(a)(p(b)p(c)) \arrow[r, "p(a)_*(\varphi_{b,c})"] & p(a)p(bc) \arrow[r, "\varphi_{a,bc}"] & p(a(bc)) \end{tikzcd} \tag{1.47}$$

must commute, what follows from (1.31). Finally, conditions (1.6) are both consequence of the normality condition (1.32) of φ, that is, of the equalities $\varphi_{a,e} = 1 = \varphi_{e,a}$.

For $\wp' = (p', q', \varphi') : \mathcal{S}' \to \mathcal{S}''$ another Schreier system morphism, the composite monoidal functor $\Sigma(\wp')\Sigma(\wp) : \mathcal{S} \to \mathcal{S}''$ is given by

$$\Sigma(\wp')\Sigma(\wp)\big(a \xrightarrow{f} a\big) = \Sigma(\wp')\big(p(a) \xrightarrow{q_a(f)} p(a)\big) = \big(\, p'p(a) \xrightarrow{q'_{p(a)}(q_a(f))} p'p(a)\ \big),$$

together with the structure isomorphisms obtained by composing in $\Sigma(\mathcal{S}'')$

$$p'p(a)p'p(b) \xrightarrow{\varphi'_{p(a),p(b)}} p'(p(a)p(b)) \xrightarrow{q'_{p(ab)}(\varphi_{a,b})} p'p(ab)\ .$$

Hence, taking into account the definition of the composition $\wp'\wp$ in (1.37) and the definition of Σ, simple comparison gives that $\Sigma(\wp')\Sigma(\wp) = \Sigma(\wp'\wp)$. Moreover, it is straightforward to see that Σ carries identity morphisms on Schreier systems $1_{\mathcal{S}} = (1_M, 1_{\mathcal{A}}, 1)$, see (1.38), to identity monoidal functors, that is, $\Sigma(1_{\mathcal{S}}) = 1_{\Sigma(\mathcal{S})}$ for any Schreier system $\mathcal{S}$. Therefore, $\Sigma : \mathbf{Z^3_{\text{n-ab}}Mnd} \to \mathbf{MonGpd}$ is indeed a functor.

Σ on deformations

Given Schreier systems $\mathcal{S}$ and $\mathcal{S}'$ as above, any deformation

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "\wp=(p,q,\varphi)"] \arrow[r, bend right, "\bar{\wp}=(p,\bar{q},\bar{\varphi})"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{S}' \end{tikzcd}$$

is mapped by the 2-functor Σ to the isomorphism of monoidal functors

$$\begin{tikzcd} \Sigma(\mathcal{S}) \arrow[r, bend left, "\Sigma(\wp)"] \arrow[r, bend right, "\Sigma(\bar{\wp})"'] \arrow[r, phantom, "\Downarrow\Sigma(\delta)"] & \Sigma(\mathcal{S}') \end{tikzcd}$$

just defined by the family of isomorphisms in $\Sigma(\mathcal{S}')$

$$\Sigma(\delta)_a = \delta_a : p(a) \to p(a), \quad a \in M, \tag{1.48}$$

which are natural thanks to condition (1.33). Moreover, so defined, $\Sigma(\delta) : \Sigma(\wp) \Rightarrow \Sigma(\bar{\wp})$ is monoidal, that is, conditions (1.8) hold, owing to (1.34) and the equality $\delta_e = 1 \in \mathcal{A}'(e')$.

For any two vertically composable deformations $\delta : \wp \Rightarrow \bar{\wp}$ and $\bar{\delta} : \bar{\wp} \Rightarrow \bar{\bar{\wp}}$, as in (1.35), the equality $\Sigma(\bar{\delta} \circ \delta) = \Sigma(\bar{\delta}) \circ \Sigma(\delta)$ is easily verified from (1.36) and (1.9), as well as the equality $\Sigma(1_\wp) = 1_{\Sigma(\wp)}$, for any morphism $\wp : \mathcal{S} \to \mathcal{S}'$. Hence (1.42) is a functor.

Furthermore, for

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "\wp=(p,q\,,\varphi)"] \arrow[r, bend right, "\bar{\wp}=(p,\bar{q},\bar{\varphi})"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{S}' \arrow[r, bend left, "\wp'=(p',q',\varphi')"] \arrow[r, bend right, "\bar{\wp}'=(p',\bar{q}',\bar{\varphi}')"'] \arrow[r, phantom, "\Downarrow\delta'"] & \mathcal{S}'' \end{tikzcd}$$

any two horizontally composable deformations as in (1.39), we have the equality $\Sigma(\delta'\delta) = \Sigma(\delta')\,\Sigma(\delta)$, since, for any $a \in M$,

$$\begin{aligned} (\Sigma(\delta')\,\Sigma(\delta))_a &\overset{(1.10)}{=} \Sigma(\delta')_{\Sigma(\bar{\wp})(a)} \circ \Sigma(\wp')(\Sigma(\delta)_a) \\ &\overset{(1.44),(1.48)}{=} \delta'_{p(a)} \circ q'_{p(a)}(\delta_a) \overset{(1.40)}{=} (\delta'\delta)_a \overset{(1.48)}{=} \Sigma(\delta'\delta)_a. \end{aligned}$$

The above confirms that (1.41), $\Sigma : \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd} \to \mathbf{MonGpd}$, is actually a 2-functor.

(1.42) **is full and faithful**

For any two Schreier systems $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ and $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$, the functor $\Sigma : \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}(\mathcal{S}, \mathcal{S}') \to \mathbf{MonGpd}(\Sigma(\mathcal{S}), \Sigma(\mathcal{S}'))$ is plainly recognized to be faithful, by (1.48). To prove that it is full, let $\delta : \Sigma(\wp) \Rightarrow \Sigma(\bar{\wp})$ be any isomorphism of monoidal functors, where $\wp = (p, q\,, \varphi), \bar{\wp} = (\bar{p}, \bar{q}, \bar{\varphi}) : \mathcal{S} \to \mathcal{S}'$ are morphisms of Schreier systems. Then, for any $a \in M$, it must be $p(a) = \bar{p}(a)$, since $\delta_a : p(a) \to \bar{p}(a)$ is an isomorphism in the skeletal category $\Sigma(\mathcal{S}')$, and moreover $\delta_a \in \mathcal{A}'(p(a))$. Any element $f \in \mathcal{A}(a)$ defines a morphism $f : a \to a$ in $\Sigma(\mathcal{S})$, and the naturality of δ implies the commutativity of the square in $\Sigma(\mathcal{S}')$

$$\begin{tikzcd} p(a) \arrow[r, "q_a(f)"] \arrow[d, "\delta_a"'] & p(a) \arrow[d, "\delta_a"] \\ p(a) \arrow[r, "\bar{q}_a(f)"] & p(a), \end{tikzcd}$$

whence $\delta_a^{-1} \circ \bar{q}_a(f) \circ \delta_a = q_a(f)$. That is, condition (1.33) for the family $\delta = \big(\delta_a \in \mathcal{A}'(p(a))\big)_{a\in M}$ being a deformation of Schreier system morphisms from $\wp$ to $\bar{\wp}$, holds. Furthermore, for any $a,b \in M$, the coherence condition (1.8) for $\delta : \Sigma(\wp) \Rightarrow \Sigma(\bar{\wp})$ gives the commutativity of

$$\begin{tikzcd} p(a)\,p(b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "p(a)_*(\delta_b)\circ p(b)^*(\delta_a)"'] & p(ab) \arrow[d, "\delta_{ab}"] \\ p(a)\,p(b) \arrow[r, "\bar{\varphi}_{a,b}"] & p(ab), \end{tikzcd}$$

whence condition (1.34) follows. Therefore, $\delta = \big(\delta_a\big)_{a\in M} : \wp \Rightarrow \bar{\wp}$ is actually a deformation in $\mathbf{Z^3_{\text{n-ab}}Mnd}$, and clearly $\Sigma(\delta) = \delta$.

(1.42) **is essentially surjective**

Suppose $F = (F, \varphi) : \Sigma(\mathcal{S}) \to \Sigma(\mathcal{S}')$ is any given monoidal functor, where $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ and $\mathcal{S}' = (M', \mathcal{A}', \Theta', \lambda')$ are Schreier systems. By Lemma 1.1, there is no loss of generality in assuming that F is strictly unitary, that is, $\varphi_0 : e' \to F(e)$ is the identity isomorphism.

If we denote by $p : M \to M'$ the map given by the action of the monoidal functor F on objects, that is, $p(a) = F(a)$ for any $a \in M$, then the action of the functor F on morphisms can be written, for any $a \in M$ and $f \in \mathcal{A}(a)$, in the form

$$F\big(a \xrightarrow{f} a\big) = \big(p(a) \xrightarrow{q_a(f)} p(a)\big)$$

for a map $q_a : \mathcal{A}(a) \to \mathcal{A}'(p(a))$, which is indeed a group homomorphism since F is a functor. Let $q = \big(q_a : \mathcal{A}(a) \to \mathcal{A}'(p(a))\big)_{a\in M}$ denote the family of these group homomorphisms. Since we have the structure isomorphism $\varphi_{a,b} : p(a)\,p(b) \to p(ab)$ and $\varphi_0 : e' \to p(e)$, it must be $p(a)p(b) = p(ab)$ and $p(e) = e'$. Therefore, p is an homomorphism of monoids.

The so obtained triplet $\wp = (p, q, \varphi)$, where $\varphi = \big(\varphi_{a,b} \in \mathcal{A}'(p(ab))\big)_{a,b\in M}$, is actually a morphism of Schreier systems $\wp : \mathcal{S} \to \mathcal{S}'$ and, by construction, $\Sigma(\wp) = F$. In effect, the naturality of the isomorphisms $\varphi_{a,b} : p(a)\,p(b) \to p(ab)$ gives the commutativity of the squares (1.46), whence condition (1.30) holds. Moreover, condition (1.31) follows from the coherence condition (1.5) which, in this case, just says that the diagrams (1.47) are commutative. The normalization condition (1.32), $\varphi_{e,e} = 1$, is consequence of the coherence squares (1.6), since F is assumed to be strictly unitary, that is, since $\varphi_0 = 1$.

The monoidal equivalence (1.43)

We keep the notations used in Subsection 1.2.2 to define the Schreier system $\Delta(\mathcal{M})$. The mapping

$$(a \xrightarrow{f} a) \;\mapsto\; (X_a \xrightarrow{f} X_a)$$

is easily recognized as an equivalence of categories $J_{\mathcal{M}} : \Sigma(\Delta(\mathcal{M})) \xrightarrow{\sim} \mathcal{M}$, which, by Proposition 1.1, defines a strictly unitary monoidal equivalence when it is endowed with the family of isomorphisms $\varphi_{a,b} = \Gamma_{X_aX_b} : X_aX_b \to X_{ab}$, $a, b \in M(\mathcal{M})$. Note that their required naturality holds, since for any $a, b \in M(\mathcal{M})$, $g \in \mathrm{Aut}_{\mathcal{M}}(X_a)$, and $f \in \mathrm{Aut}_{\mathcal{M}}(X_b)$, we have the the commutative diagram

$$\begin{tikzcd}
X_aX_b \arrow[r, "g1"] \arrow[d, "\Gamma"'] \arrow[rr, bend left, "gf"] \arrow[r, phantom, shift left=3ex, "(B)"] & X_aX_b \arrow[r, "1f"] \arrow[d, "\Gamma"] \arrow[dl, phantom, "(1.25)"] & X_aX_b \arrow[d, "\Gamma"] \arrow[dl, phantom, "(1.25)"] \\
X_{ab} \arrow[r, "b^*(g)"'] \arrow[rr, bend right, "a_*(f)\circ b^*(g)"'] & X_{ab} \arrow[r, "a_*(f)"'] \arrow[d, phantom, shift left=1ex, "="] & X_{ab}
\end{tikzcd}$$

where the commutativity of the region labelled (B) is consequence of the fact that $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a functor. The needed coherence conditions (1.5) and (1.6) follow from the commutativity of diagrams (1.26) and the choices of the morphisms Γ's made in (1.22), respectively. $\square$

1.2.6 The Schreier system construction biequivalence

The above stated biequivalence between the 2-category of monoidal groupoids and the 2-category of Schreier systems (1.41), $\Sigma : \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd} \xrightarrow{\approx} \mathbf{MonGpd}$, is injective on objects, morphisms, and deformations. Moreover, for any Schreier system $\mathcal{S}$, the equality $\Delta\Sigma(\mathcal{S}) = \mathcal{S}$ holds. Hence, the assignment $\mathcal{M} \mapsto \Delta(\mathcal{M})$, given by the Schreier system construction (1.28), is the function on objects of a biequivalence, quasi-inverse of Σ,

$$\Delta : \mathbf{MonGpd} \xrightarrow{\approx} \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}, \tag{1.49}$$

uniquely determined up to pseudo-natural equivalence by $\Delta\Sigma = 1_{\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}}$ and the existence of a pseudo-natural equivalence $J : \Sigma\Delta \overset{\approx}{\Rightarrow} 1_{\mathbf{MonGpd}}$, whose component at any monoidal groupoid $\mathcal{M}$ is the monoidal equivalence (1.43), $J_{\mathcal{M}} : \Sigma\Delta(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$. For completeness, we shall next show how the pseudo-functor Δ and the pseudo-equivalence J work.

Δ on monoidal functors

Suppose $F : \mathcal{M} \to \mathcal{M}'$ is any given monoidal functor between monoidal groupoids $\mathcal{M}$ and $\mathcal{M}'$. Let $F^{\mathrm{u}} : \mathcal{M} \to \mathcal{M}'$ be the strictly unitary monoidal funtor associated to F by the normalization functor (1.11), and let $(\Gamma_X : X \to X_a)_{a \in M(\mathcal{M})}$ and $(\Gamma'_{X'} : X' \to X'_{a'})_{a' \in M(\mathcal{M}')}$ be the cleavages used for constructing the Schreier systems $\Delta(\mathcal{M})$ and $\Delta(\mathcal{M}')$, respectively. Then,

$$\Delta(F) = (p(F), q(F)\,, \varphi(F)) : \Delta(\mathcal{M}) \to \Delta(\mathcal{M}')$$

is the morphism of Schreier systems where:

- $p = p(F) : M(\mathcal{M}) \to M(\mathcal{M}')$ is the homomorphism of monoids defined by

$$p(a) = [FX_a] = [F^{\mathrm{u}}X_a], \quad a \in M(\mathcal{M}).$$

- $q = q(F) = \big(\mathrm{Aut}_{\mathcal{M}}(X_a) \xrightarrow{q_a} \mathrm{Aut}_{\mathcal{M}'}(X'_{p(a)})\big)_{a\in M(\mathcal{M})}$, is the family of group homomorphisms which carry an automorphism $f : X_a \to X_a$, for any $a \in M(\mathcal{M})$, to the unique automorphism $q_a(f) : X'_{p(a)} \to X'_{p(a)}$ in $\mathcal{M}'$ making the square below commutative.

$$\begin{tikzcd} F^{\mathrm{u}}X_a \arrow[r, "F^{\mathrm{u}}f"] \arrow[d, "\Gamma'"'] & F^{\mathrm{u}}X_a \arrow[d, "\Gamma'"] \\ X'_{p(a)} \arrow[r, dotted, "q_a(f)"] & X'_{p(a)} \end{tikzcd}$$

- $\varphi = \varphi(F) = \big(\varphi_{a,b} \in \mathrm{Aut}_{\mathcal{M}'}(X'_{p(ab)})\big)_{a,b\in M(\mathcal{M})}$, is the family of automorphisms in $\mathcal{M}'$ determined by the commutativity of the diagrams

$$\begin{tikzcd} F^{\mathrm{u}}X_a\, F^{\mathrm{u}}X_b \arrow[r, "\varphi^{\mathrm{u}}"] \arrow[d, "\Gamma'\Gamma'"'] & F^{\mathrm{u}}(X_aX_b) \arrow[r, "F^{\mathrm{u}}\Gamma"] & F^{\mathrm{u}}X_{ab} \arrow[d, "\Gamma'"] \\ X'_{p(a)}X'_{p(b)} \arrow[r, "\Gamma'"] & X'_{p(a)p(b)} \arrow[r, dotted, "\varphi_{a,b}"] & X'_{p(ab)}. \end{tikzcd}$$

J on monoidal functors

The component of the pseudo-natural equivalence $J : \Sigma\Delta \overset{\sim}{\Rightarrow} 1$ at any monoidal functor $F : \mathcal{M} \to \mathcal{M}'$, is the isomorphism

$$\begin{tikzcd} \Sigma\Delta(\mathcal{M}) \arrow[r, "\Sigma\Delta(F)"] \arrow[d, "J_{\mathcal{M}}"'] \arrow[dr, phantom, "\overset{\sim}{\Rightarrow}"] & \Sigma\Delta(\mathcal{M}') \arrow[d, "J_{\mathcal{M}'}"] \\ \mathcal{M} \arrow[r, "F"'] & \mathcal{M}' \end{tikzcd}$$

defined by the isomorphisms of the cleavage in $\mathcal{M}'$, $\Gamma' : FX_a \overset{\sim}{\to} X'_{p(a)}$, $a \in M(\mathcal{M})$.

Δ on morphisms of monoidal functors

Let $F, \bar{F} : \mathcal{M} \to \mathcal{M}'$ be monoidal functors as above, and suppose $\delta : F \Rightarrow \bar{F}$ is any morphism between them. Then,

$$\begin{tikzcd} \Delta(\mathcal{M}) \arrow[r, bend left, "\Delta(F)"] \arrow[r, bend right, "\Delta(\bar{F})"'] \arrow[r, phantom, "\Downarrow\Delta(\delta)"] & \Delta(\mathcal{M}') \end{tikzcd}$$

is the deformation $\Delta(\delta) = \big(\Delta(\delta)_a \in \mathrm{Aut}_{\mathcal{M}'}(X'_{p(a)})\big)_{a\in M(\mathcal{M})}$, consisting of the automorphisms in $\mathcal{M}'$ determined by the commutativity of the diagrams below.

$$\begin{tikzcd} F^{\mathrm{u}}X_a \arrow[r, "\Gamma'"] \arrow[d, "\delta^{\mathrm{u}}"'] & X'_{p(a)} \arrow[d, dotted, "\Delta(\delta)_a"] \\ \bar{F}^{\mathrm{u}}X_a \arrow[r, "\Gamma'"] & X'_{p(a)} \end{tikzcd}$$

Since $\Delta : \mathbf{MonGpd} \xrightarrow{\approx} \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$ is a biequivalence and, by Lemma 1.2 every equivalence in $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$ is actually an isomorphism, we have the following theorem as a corollary:

Theorem 1.2 (i) *For any Schreier system* $(M, \mathcal{A}, \Theta, \lambda)$*, there is a monoidal groupoid* $\mathcal{M}$ *with an isomorphism* $\Delta(\mathcal{M}) \cong (M, \mathcal{A}, \Theta, \lambda)$*.*

(ii) *Two monoidal groupoids* $\mathcal{M}$ *and* $\mathcal{M}'$ *are equivalent if and only if their associated Schreier systems* $\Delta(\mathcal{M})$ *and* $\Delta(\mathcal{M}')$ *are isomorphic.*

1.3 Classification of monoidal abelian groupoids

This section focuses in the special case of *monoidal abelian groupoids*, that is, monoidal groupoids $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ whose isotropy groups $\mathrm{Aut}_{\mathcal{M}}(X)$, $X \in \mathrm{Ob}\mathcal{M}$, are all abelian (cf. [6, Definition 2.11.3 and Example 2.11.4], where the notion of abelian groupoid is discussed under a categorical point of view). To start, we shall observe that some of the isotropy groups of any monoidal groupoid are always abelian:

Proposition 1.3 (i) *If* $\mathcal{S} = (M, \mathcal{A}, \Theta, \lambda)$ *is any Schreier system, then, for any invertible element* $a \in M$*, the group* $\mathcal{A}(a)$ *is abelian.*

(ii) *If* $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ *is any monoidal groupoid, then, for any invertible object* $X \in \mathcal{M}$*, the group* $\mathrm{Aut}_{\mathcal{M}}(X)$ *is abelian.*

Proof: (i) The group $\mathcal{A}(e)$ is abelian due to conditions (1.18) and (1.19): for any $f, g \in \mathcal{A}(e)$,

$$f \circ g = e_*(f) \circ e^*(g) = e^*(g) \circ e_*(f) = g \circ f.$$

For any invertible element $a \in M$, the homomorphism $a_* : \mathcal{A}(e) \to \mathcal{A}(a)$ is actually an isomorphism, with inverse $(a^{-1})_* : \mathcal{A}(a) \to \mathcal{A}(e)$, since, by (1.14), (1.20), and (1.19), we have

$$a_*(a^{-1})_* = (aa^{-1})_* = e_* = 1_{\mathcal{A}(e)}, \quad (a^{-1})_* a_* = (a^{-1}a)_* = e_* = 1_{\mathcal{A}(a)}.$$

Hence, $\mathcal{A}(a)$ is abelian as $\mathcal{A}(e)$ is.

(ii) Let $\Delta(\mathcal{M})$ be the associated Schreier system to the monoidal groupoid $\mathcal{M}$, as in (1.28). If $X \in \mathcal{M}$ is any invertible object, then $a = [X] \in M(\mathcal{M})$ is an invertible element of the associated monoid (1.21), whence, by part (i), the group $\mathrm{Aut}_{\mathcal{M}}(X_a)$ is abelian. Since the isomorphism $\Gamma : X \to X_a$ induces a group isomorphism $\mathrm{Aut}_{\mathcal{M}}(X) \cong \mathrm{Aut}_{\mathcal{M}}(X_a)$, the result follows. $\square$

Thus, for example, every categorical group is a monoidal abelian groupoid. The classification of categorical groups was given by Sinh in [69], by means of Eilenberg-Mac Lane group cohomology groups $H^3(G, A)$, and our aim here is to give a similar solution to the more general problem of classifying monoidal abelian groupoids, now by means of monoid cohomology groups $H^3(M, \mathcal{A})$. To this end, we shall briefly review below some basic aspects concerning the cohomology theory of monoids that we are going to use, which is a generalization of Eilenberg-Mac Lane's cohomology of groups due to Leech [53].

In what follows, we will use additive notation for abelian groupoids. Thus, the identity morphism of an object X of an abelian groupoid $\mathcal{M}$ will be denoted by 0_X; if $f : X \to Y$, $g : Y \to Z$ are morphisms, their composite is written by $g + f : X \to Z$, while the inverse of f is $-f : Y \to X$.

1.3.1 Leech cohomology of monoids

Let **Ab** denote the category of abelian groups, which will be written additively.

If $\mathbb{C}$ is a small category, then the category of (left) $\mathbb{C}$-modules has objects the functors $\mathcal{A} : \mathbb{C} \to \mathbf{Ab}$ from $\mathbb{C}$ into abelian groups, with morphisms the natural transformations. This is an abelian category with enough injectives and projectives, and the abelian groups

$$H^n(\mathbb{C}, \mathcal{A}) = \mathrm{Ext}^n_{\mathbb{C}}(\mathbb{Z}, \mathcal{A}), \tag{1.50}$$

where $\mathbb{Z} : \mathbb{C} \to \mathbf{Ab}$ is the constant functor with value $\mathbb{Z}$, are the cohomology groups of the category $\mathbb{C}$ with coefficients in the $\mathbb{C}$-module $\mathcal{A}$, studied by Roos [65] and Watts [73], among other authors. Cohomology theory of small categories is itself a basis for other cohomology theories, in particular for Leech cohomology theory of monoids, which is defined as follows:

A monoid M gives rise to a category $\mathbb{D}M$ with object set M and arrow set $M \times M \times M$, with $(a, b, c) : b \to abc$. Composition is given by

$$(a', abc, c')(a, b, c) = (a'a, b, cc'),$$

and the identity morphism of any object a is $1_a = (e, a, e)$, where e is the unit element of M. This construction $M \mapsto \mathbb{D}M$ defines a functor $\mathbb{D} : \mathbf{Mnd} \to \mathbf{Cat}$, which maps a monoid homomorphism $p : M \to M'$ to the functor $\mathbb{D}p : \mathbb{D}M \to \mathbb{D}M'$ given by $(\mathbb{D}p)(a, b, c) = (p(a), p(b), p(c))$.

If we say that a $\mathcal{A} : \mathbb{D}M$-module, $\mathbb{D}M \to \mathbf{Ab}$, carries the element $a \in M$ to the abelian group $\mathcal{A}(a)$ and carries the morphism (a, b, c) to the group homomorphism $a_*c^* : \mathcal{A}(b) \to \mathcal{A}(abc)$, then we see that such a $\mathbb{D}M$-module, is a system of data consisting of two families of abelian groups and homomorphisms, respectively,

$$(\mathcal{A}(a))_{a\in M}, \quad (\mathcal{A}(b) \xrightarrow{a_*} \mathcal{A}(ab) \xleftarrow{b^*} \mathcal{A}(a))_{a,b\in M}$$

such that, for any $a, b, c \in M$,

$$\begin{aligned}
(ab)_* &= a_* b_* : \mathcal{A}(c) \to \mathcal{A}(abc),\\
c^* a_* &= a_* c^* : \mathcal{A}(b) \to \mathcal{A}(abc),\\
c^* b^* &= (bc)^* : \mathcal{A}(a) \to \mathcal{A}(abc),
\end{aligned}$$

and, for any $a \in M$, $e_* = 1_{\mathcal{A}(a)} = e^* : \mathcal{A}(a) \to \mathcal{A}(a)$. Since $\mathcal{A}(a)$, $a \in M$, are abelian groups, we will now use additive notation.

Leech cohomology groups $H^n_{\mathrm{L}}(M, \mathcal{A})$ [53], of a monoid M with coefficients in a $\mathbb{D}M$-module $\mathcal{A}$, are defined to be those of its associated category $\mathbb{D}M$, that is,

$$H^n_{\mathrm{L}}(M, \mathcal{A}) = H^n(\mathbb{D}M, \mathcal{A}).$$

For computing these cohomology groups there is a cochain complex, called the *standard normalized cochain complex* of M with coefficients in $\mathcal{A}$,

$$C^\bullet_{\mathrm{L}}(M, \mathcal{A}), \tag{1.51}$$

which is defined in degree $n > 0$ by

$$C^n_{\mathrm{L}}(M, \mathcal{A}) = \left\{ f \in \prod_{(a_1,\dots,a_n)\in M^n} \mathcal{A}(a_1 \cdots a_n) \mid f(a_1, \dots, a_n) = 0 \text{ whenever some } a_i = e \right\}$$

and $C^0_{\mathrm{L}}(M, \mathcal{A}) = \mathcal{A}(e)$. The coboundary operator

$$\partial^n : C^n_{\mathrm{L}}(M, \mathcal{A}) \to C^{n+1}_{\mathrm{L}}(M, \mathcal{A})$$

is given, for $n = 0$, by $(\partial^0 f)(a) = a_*(f) - a^*(f)$, while, for $n > 0$,

$$\begin{aligned}
(\partial^n f)(a_1, \dots, a_{n+1}) = &(a_1)_* f(a_2, \dots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) +\\
&(-1)^{n+1} (a_{n+1})^* f(a_1, \dots, a_n).
\end{aligned}$$

By [53, Chapter II, 2.12], we have

$$H^n_{\mathrm{L}}(M, \mathcal{A}) = H^n\big(C^\bullet_{\mathrm{L}}(M, \mathcal{A})\big).$$

It will be useful for our purposes to describe the natural properties of the Leech cohomology on the category obtained by the Grothendieck construction on the functor that associates to any monoid M the category of $\mathbb{D}M$-modules and, to any homomorphism $p : M \to M'$, the functor p^* that carries any $\mathbb{D}M'$-module, say $\mathcal{A}'$, to the $\mathbb{D}M$-module $p^*\mathcal{A}'$ determined by

$$(\mathcal{A}'(p(a)))_{a\in M}, \quad \big(\mathcal{A}'(p(b)) \xrightarrow{p(a)_*} \mathcal{A}'(p(ab)) \xleftarrow{p(b)^*} \mathcal{A}'(p(a))\big)_{a,b\in M}.$$

The resulting category by the Grothendieck construction, which may be heuristically viewed as the category obtained by tying the categories of $\mathbb{D}M$-modules together in some natural fashion, is denoted by

$$\mathcal{M}od_{\mathbb{D}}.$$

It has objects pairs $(M,\mathcal{A})$, where M is a monoid and $\mathcal{A}$ is a $\mathbb{D}M$-module. Morphisms are pairs

$$(p,q):(M,\mathcal{A})\to(M',\mathcal{A}')$$

consisting of a monoid homomorphism $p:M\to M'$ together with a morphism of $\mathbb{D}M$-modules $q:\mathcal{A}\to p^*\mathcal{A}'$, that is, a family of group homomorphisms

$$q=\ \big(\mathcal{A}(a)\xrightarrow{q_a}\mathcal{A}(p(a))'\big)_{a\in M}\ ,$$

satisfying, for any $a,b\in M$,

$$q_{ab}\,a_* = p(a)_*\,q_b : \mathcal{A}(b)\to\mathcal{A}'(p(ab)), \tag{1.52}$$

$$q_{ab}\,b^* = p(b)^*\,q_a : \mathcal{A}(a)\to\mathcal{A}'(p(ab)). \tag{1.53}$$

Composition is defined by $(p',q')(p,q)=(p'p,q'q)$, where $q'q=(q'_{p(a)}q_a)_{a\in M}$.

Any morphism $(p,q):(M,\mathcal{A})\to(M',\mathcal{A}')$ as above yields homomorphisms

$$H^n_{\mathrm{L}}(M,\mathcal{A})\xrightarrow{q_*}H^n_{\mathrm{L}}(M,p^*\mathcal{A}')\xleftarrow{p^*}H^n_{\mathrm{L}}(M',\mathcal{A}')$$

induced by the morphisms of cochain complexes

$$C^\bullet_{\mathrm{L}}(M,\mathcal{A})\xrightarrow{q_*}C^\bullet_{\mathrm{L}}(M,p^*\mathcal{A}')\xleftarrow{p^*}C^\bullet_{\mathrm{L}}(M',\mathcal{A}'),$$

which are given on cochains by

$$(q_*f)(a_1,\dots,a_n)=q_{a_1\cdots a_n}(f(a_1,\dots,a_n)),\quad (p^*f')(a_1,\dots,a_n)=f'(p(a_1),\dots,p(a_n)).$$

1.3.2 The classification theorems

The biequivalence in Theorem 1.1 restricts to a biequivalence between the full 2-subcategory of the 2-category of monoidal groupoids given by the monoidal abelian groupoids, denoted by

MonAbGpd,

and the full 2-subcategory of the 2-category of Schreier systems given by those Schreier systems $(M,\mathcal{A},\Theta,\lambda)$ in which every group $\mathcal{A}(a)$, $a\in M$, is abelian. Hereafter, this latter 2-category will be called the 2-*category of Leech* 3-*cocycles of monoids*, and denoted by

$\mathbf{Z^3Mnd}$,

since its cells have the following cohomological interpretation:

0-cells. According to Definition 1.1, a Schreier system $\mathcal{S}$ in $\mathbf{Z^3Mnd}$ is just a triplet $\mathcal{S} = (M, \mathcal{A}, h)$ consisting of a monoid M, a $\mathbb{D}M$-module $\mathcal{A}$, and a 3-cocycle $h \in Z^3_{\mathrm{L}}(M, \mathcal{A})$.

1-cells. If $\mathcal{S} = (M, \mathcal{A}, h)$ and $\mathcal{S}' = (M', \mathcal{A}', h')$ are in $\mathbf{Z^3Mnd}$, then a morphism of Schreier systems (see Subsection 1.2.4), $\wp = (p, q\,, g) : \mathcal{S} \to \mathcal{S}'$, is the same thing as a morphism $(p, q) : (M, \mathcal{A}) \to (M', \mathcal{A}')$ in $\mathcal{M}od_{\mathbb{D}}$, together with a 2-cochain $g \in C^2_{\mathrm{L}}(M, p^*\mathcal{A}')$ such that $q_*h = p^*h' + \partial^2 g$.

2-cells. If $\wp = (p, q, g) : \mathcal{S} \to \mathcal{S}'$ and $\bar{\wp} = (\bar{p}, \bar{q}, \bar{g}) : \mathcal{S} \to \mathcal{S}'$ are morphisms in $\mathbf{Z^3Mnd}$, then (see Subsection 1.2.4) there is no deformation between them unless $p = \bar{p}$ and $q = \bar{q}$. In such a case, a deformation $f : \wp \Rightarrow \bar{\wp}$ consists of a 1-cochain $f \in C^1_{\mathrm{L}}(M, p^*\mathcal{A}')$, such that $g = \bar{g} + \partial^1 f$.

Hence, our first result here comes as a direct consequence of Theorem 1.1:

Theorem 1.3 *The quasi-inverse biequivalences* (1.41) *and* (1.49) *restrict to corresponding quasi-inverse biequivalences*

$$\mathbf{MonAbGpd} \underset{\Sigma}{\overset{\Delta}{\underset{\approx}{\rightleftarrows}}} \mathbf{Z^3Mnd}. \tag{1.54}$$

Closely related to the category $\mathbf{Z^3Mnd}$ is the *category of Leech* 3*-cohomology classes of monoids*, denoted by

$$\mathbf{H^3Mnd}, \tag{1.55}$$

which plays a fundamental role to state our classification theorem below. Its objects are triplets $(M, \mathcal{A}, c)$, where M is a monoid, $\mathcal{A}$ is a $\mathbb{D}M$-module, and $c \in H^3_{\mathrm{L}}(M, \mathcal{A})$ is 3-cohomology class of M with coefficients in $\mathcal{A}$. An arrow

$$(p, q) : (M, \mathcal{A}, c) \to (M', \mathcal{A}', c')$$

is a morphism $(p, q) : (M, \mathcal{A}) \to (M', \mathcal{A}')$ in $\mathcal{M}od_{\mathbb{D}}$, such that

$$p^*(c') = q_*(c) \in H^3_{\mathrm{L}}(M, p^*\mathcal{A}').$$

Observe that a morphism (p, q) is an isomorphism in $\mathbf{H^3Mnd}$ if and only if $p : M \to M'$ is an isomorphism of monoids and $q : \mathcal{A} \to p^*\mathcal{A}'$ is an isomorphism of $\mathbb{D}M$-modules.

We have the *cohomology class functor*

$$\begin{gathered} \mathrm{cl} : \mathbf{Z^3Mnd} \to \mathbf{H^3Mnd}, \\ (M, \mathcal{A}, h) \mapsto (M, \mathcal{A}, [h]) \\ (p, q\,, g) \mapsto (p, q) \end{gathered} \tag{1.56}$$

where $[h] \in H^3_{\mathrm{L}}(M, \mathcal{A})$ denotes the cohomology class of $h \in Z^3_{\mathrm{L}}(M, \mathcal{A})$. This functor clearly carries two isomorphic morphisms of $\mathbf{Z^3Mnd}$ to the same morphism in

$\mathbf{H^3Mnd}$, whence composition with the pseudo-functor Δ above gives a functor

$$\mathrm{Cl} = \mathrm{cl}\,\Delta : \mathbf{MonAbGpd} \to \mathbf{H^3Mnd}, \tag{1.57}$$

that we call *the classifying functor*, because of the theorem below.

Theorem 1.4 (Classification of monoidal abelian groupoids) *(i) For M any monoid , $\mathcal{A}$ any $\mathbb{D}M$-module, and $c \in H^3_{\mathrm{L}}(M, \mathcal{A})$ any cohomology class, there is a monoidal abelian groupoid $\mathcal{M}$ with an isomorphism $\mathrm{Cl}(\mathcal{M}) \cong (M, \mathcal{A}, c)$.*

(ii) A monoidal functor between monoidal abelian groupoids $F : \mathcal{M} \to \mathcal{M}'$ is an equivalence if and only if $\mathrm{Cl}(F) : \mathrm{Cl}(\mathcal{M}) \to \mathrm{Cl}(\mathcal{M}')$ is an isomorphism.

(iii) For any isomorphism $(p, q) : \mathrm{Cl}(\mathcal{M}) \xrightarrow{\cong} \mathrm{Cl}(\mathcal{M}')$, there exists a monoidal equivalence $F : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ such that $\mathrm{Cl}(F) = (p, q)$.

(iv) If $\mathrm{Cl}(\mathcal{M}) = (M, \mathcal{A}, c)$ and $\mathrm{Cl}(\mathcal{M}') = (M', \mathcal{A}', c')$, then, for any morphism $(p, q) : \mathrm{Cl}(\mathcal{M}) \to \mathrm{Cl}(\mathcal{M}')$ in $\mathbf{H^3Mnd}$, there is a (non-natural) bijection

$$\big\{[F] : \mathcal{M} \to \mathcal{M}' \mid \mathrm{Cl}(F) = (p, q)\big\} \cong H^2_{\mathrm{L}}(M, p^*\mathcal{A}')$$

between the set of isomorphism classes of those monoidal functors $F : \mathcal{M} \to \mathcal{M}'$ which are carried by the classifying funtor to (p, q) and the elements of the second cohomology group of M with coefficients in the $\mathbb{D}M$-module $p^*\mathcal{A}'$.

Proof: (i) Given any object $(M, \mathcal{A}, c) \in \mathbf{H^3Mnd}$, let us choose any 3-cocycle $h \in Z^3(M, \mathcal{A})$ such that $[h] = c$. Then, letting $\mathcal{M} = \Sigma(M, \mathcal{A}, h)$, we have

$$\mathrm{Cl}(\mathcal{M}) = \mathrm{cl}(\Delta\Sigma(M, \mathcal{A}, h)) = \mathrm{cl}(M, \mathcal{A}, h) = (M, \mathcal{A}, c).$$

(ii) Since the pseudo-functor $\Delta : \mathbf{MonAbGpd} \to \mathbf{Z^3Mnd}$ is a biequivalence, it suffices to prove that a morphism in $\mathbf{Z^3Mnd}$, say $(p, q, g) : (M, \mathcal{A}, h) \to (M', \mathcal{A}', h')$, is an equivalence if and only if the induced $(p, q) : (M, \mathcal{A}, [h]) \to (M', \mathcal{A}', [h'])$ is an isomorphism in $\mathbf{H^3Mnd}$, that is, if and only if $p : M \to M'$ is an isomorphism of monoids and $q : \mathcal{A} \to p^*\mathcal{A}'$ is an isomorphism of $\mathbb{D}M$-modules. Hence, the result follows from Lemma 1.2.

(iv) Let $\mathcal{M}$, $\mathcal{M}'$ be monoidal abelian groupoids. Suppose $\Delta(\mathcal{M}) = (M, \mathcal{A}, h)$ and $\Delta(\mathcal{M}') = (M', \mathcal{A}', h')$, so that $\mathrm{Cl}(\mathcal{M}) = (M, \mathcal{A}, [h])$ and $\mathrm{Cl}(\mathcal{M}') = (M', \mathcal{A}', [h'])$, and let $(p, q) : \mathrm{Cl}(\mathcal{M}) \to \mathrm{Cl}(\mathcal{M}')$ be any given morphism in $\mathbf{H^3Mnd}$. The equivalence between the hom-groupoids

$$\mathbf{MonAbGpd}(\mathcal{M}, \mathcal{M}') \overset{\Delta}{\simeq} \mathbf{Z^3Mnd}(\Delta(\mathcal{M}), \Delta(\mathcal{M}')),$$

induces a bijection, $[F] \mapsto [\Delta(F)]$,

$$\big\{[F] : \mathcal{M} \to \mathcal{M}' \mid \mathrm{Cl}(F) = (p, q)\big\} \cong \big\{[p, q, g] : (M, \mathcal{A}, h) \to (M', \mathcal{A}', h')\big\}$$

between the set of iso-classes $[F]$ of those monoidal functors $F : \mathcal{M} \to \mathcal{M}'$ with $\mathrm{Cl}(F) = (p, q)$, and the set of iso-classes $[(p, q), g]$ of morphisms of the form

$$(p, q, g) : (M, \mathcal{A}, h) \to (M', \mathcal{A}', h')$$

in the 2-category of Leech 3-cocycles. Since $p^*\mathrm{cl}(h') = q_*\mathrm{cl}(h)$, both 3-cocycles p^*h' and q_*h represent the same class in the cohomology group $H^3_{\mathrm{L}}(M, p^*\mathcal{A}')$. Therefore, it must exist a 2-cochain $g_0 \in C^2_{\mathrm{L}}(M, p^*\mathcal{A}')$ such that $q_*h = p^*h' + \partial^2 g_0$. Hence, $(p, q, g_0) : (M, \mathcal{A}, h) \to (M', \mathcal{A}', h')$ is a morphism in $\mathbf{Z^3Mnd}$. Furthermore, observe that any other morphism in $\mathbf{Z^3Mnd}$ realizing the same morphism (p, q) of $\mathbf{H^3Mnd}$ is necessarily written in the form $(p, q, g_0 \circ g)$ for some $g \in Z^2_{\mathrm{L}}(M, p^*\mathcal{A}')$ and, moreover, both morphisms (p, q, g_0) and $(p, q, g_0 \circ g)$ are isomorphic if and only if $g = \partial^1 f$ for some $f \in C^1_{\mathrm{L}}(M, p^*\mathcal{A}')$. That is, there is a bijection

$$H^2_{\mathrm{L}}(M, p^*\mathcal{A}') \cong \big\{[p, q, g] : (M, \mathcal{A}, h) \to (M', \mathcal{A}', h')\big\}$$

given by $[g] \mapsto [p, q, g_0 \circ g]$.

(iii) Let $(p, q) : \mathrm{Cl}(\mathcal{M}) \xrightarrow{\cong} \mathrm{Cl}(\mathcal{M}')$ any given isomorphism in $\mathbf{H^3Mnd}$. By the already proven part (iv), there exists a monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ such that $\mathrm{Cl}(F) = (p, q)$, which, by part (ii) is an equivalence. □

The functor $\mathbf{MonAbGpd} \to \mathcal{M}od_{\mathbb{D}}$, $\mathcal{M} \mapsto (M(\mathcal{M}), \mathcal{A}(\mathcal{M}))$, obtained by composing the classifying functor (1.57) with the forgetful functor $\mathbf{H^3Mnd} \to \mathcal{M}od_{\mathbb{D}}$, $(M, \mathcal{A}, c) \mapsto (M, \mathcal{A})$, turns the 2-category of monoidal abelian groupoids into a fibred 2-category over the category $\mathcal{M}od_{\mathbb{D}}$. It follows from the above results that, for any fixed monoid M and $\mathbb{D}M$-module $\mathcal{A}$, the mappings $[h] \mapsto [\Sigma(M, \mathcal{A}, h)]$ and $\mathcal{M} \mapsto [h(\mathcal{M})]$ describe mutually inverse bijections between the set $H^3_{\mathrm{L}}(M, \mathcal{A})$ and the set of equivalence classes of monoidal groupoids in the fibre 2-category over $(M, \mathcal{A})$. However, this latter set is conceptually a little too rigid, since the strict requirements $M(\mathcal{M}) = M$ and $\mathcal{A}(\mathcal{M}) = \mathcal{A}$, for a monoidal abelian groupoid $\mathcal{M}$, are not very natural. We shall show below how to relax them.

Definition 1.2 *For any given monoid M and any $\mathbb{D}M$-module $\mathcal{A}$, we say that a monoidal abelian groupoid $\mathcal{M}$ is* of type $(M, \mathcal{A})$ *if there are given*

- *a monoid isomorphism $i : M \cong M(\mathcal{M})$,*
- *a family of group isomorphisms $\mathbf{j} = \big(j_X : \mathcal{A}(a) \cong \mathrm{Aut}_{\mathcal{M}}(X)\big)_{a \in M,\, X \in i(a)}$,*

such that,

- *If $X, Y \in i(a)$ then, for any morphism $h : X \to Y$ in $\mathcal{M}$ and any $g \in \mathcal{A}(a)$,*

$$j_Y(g) = h + j_X(g) - h.$$

- *If $X \in i(a)$ and $Y \in i(b)$, then, for any $f \in \mathcal{A}(b)$ and $g \in \mathcal{A}(a)$,*

$$j_{XY}(a_*(f)) = 1_X\, j_Y(f), \quad j_{XY}(b^*(g)) = j_X(g)\, 1_Y.$$

If $(p, q) : (M, \mathcal{A}) \to (M', \mathcal{A}')$ is any morphism in the category $\mathcal{M}od_{\mathbb{D}}$, and $\mathcal{M}$ and $\mathcal{M}'$ are monoidal abelian groupoids of respective types $(M, \mathcal{A})$ and $(M', \mathcal{A}')$, then a monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is said to be of type (p, q) *whenever*

- *If $X \in i(a)$, then $FX \in i'(p(a))$, and, for any $g \in \mathcal{A}(a)$, $j'_{FX} q_a(g) = F(j_X(g))$.*

Two monoidal abelian groupoids of the same type $(M, \mathcal{A})$, $(\mathcal{M}, i, \mathbf{j})$ and $(\mathcal{M}', i', \mathbf{j}')$, are defined to be equivalent *if there exists a monoidal equivalence $F : \mathcal{M} \to \mathcal{M}'$ of type $(1, 1)$, that is, whenever*

- *If $X \in i(a)$, then $FX \in i'(a)$, and, for any $g \in \mathcal{A}(a)$, $j'_{FX}(g) = F(j_X(g))$.*

If we denote by

$$\mathbf{MonAbGpd}(M, \mathcal{A})$$

the set of equivalence classes $[\mathcal{M}, i, \mathbf{j}]$ of those monoidal abelian groupoids $(\mathcal{M}, i, \mathbf{j})$ of type $(M, \mathcal{A})$, then we are ready to summarize our results on the classification of monoidal abelian groupoids and their homomorphisms in a bit more classical terms:

Theorem 1.5 (i) *For any monoidal abelian groupoid $\mathcal{M}$, there exists a monoid M and a $\mathbb{D}M$-module $\mathcal{A}$ such that $\mathcal{M}$ is of type $(M, \mathcal{A})$.*

(ii) *For any monoid M and any $\mathbb{D}M$-module $\mathcal{A}$, there is a natural bijection*

$$\mathbf{MonAbGpd}(M, \mathcal{A}) \cong H^3_{\mathrm{L}}(M, \mathcal{A}) \tag{1.58}$$

given by

$$[\mathcal{M}, i, \mathbf{j}] \;\mapsto\; c(\mathcal{M}) = \mathbf{j}_*^{-1} i^*([h(\mathcal{M})]),$$

where $h(\mathcal{M})$ is the 3-cocycle obtained as in (1.27), and

$$H^3_{\mathrm{L}}(M(\mathcal{M}), \mathcal{A}(\mathcal{M})) \xrightarrow{i^*} H^3_{\mathrm{L}}(M, i^*\mathcal{A}(\mathcal{M}))) \xrightarrow{\mathbf{j}_*^{-1}} H^3_{\mathrm{L}}(M, \mathcal{A})$$

the induced isomorphisms on cohomology groups by the isomorphism

$$(i, \mathbf{j}) : (M, \mathcal{A}) \cong (M(\mathcal{M}), \mathcal{A}(\mathcal{M}))$$

in the category $\mathcal{M}od_{\mathbb{D}}$. In the other direction, the bijection is induced by the mapping that carries a 3-cocycle $h \in Z^3_{\mathrm{L}}(M, \mathcal{A})$ to the monoidal abelian groupoid $\Sigma(M, \mathcal{A}, h)$, given by the construction (1.29).

(iii) *If $\mathcal{M}$ is of type $(M, \mathcal{A})$ and $\mathcal{M}'$ is of type $(M', \mathcal{A}')$, then for every monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ there exists a morphism $(p, q) : (M, \mathcal{A}) \to (M', \mathcal{A}')$ in the category $\mathcal{M}od_{\mathbb{D}}$, such that F is of type (p, q).*

(iv) *If $\mathcal{M}$ is of type $(M, \mathcal{A})$, $\mathcal{M}'$ is of type $(M', \mathcal{A}')$, and $(p, q) : (M, \mathcal{A}) \to (M', \mathcal{A}')$ is any morphism in the category $\mathcal{M}od_{\mathbb{D}}$, then there exists $F : \mathcal{M} \to \mathcal{M}'$ a monoidal functor of type (p, q) if and only if*

$$p^*(c(\mathcal{M}')) = q_*(c(\mathcal{M})) \in H^3_{\mathrm{L}}(M, p^*\mathcal{A}').$$

In such a case, isomorphism classes of monoidal functors $F : \mathcal{M} \to \mathcal{M}'$ of type (p, q) are in bijection with the elements of the group

$$H^2_{\mathrm{L}}(M, p^*\mathcal{A}').$$

Proof: All the statements here are direct consequence of those in Theorem 1.4, after two quite obvious observations, namely: (1) A monoidal abelian groupoid $\mathcal{M}$ is of type $(M,\mathcal{A})$ if and only if there is given an isomorphism $(i,\mathbf{j}):(M,\mathcal{A})\cong(M(\mathcal{M}),\mathcal{A}(\mathcal{M}))$ in the category $\mathcal{M}od_{\mathbb{D}}$. (2) if $(p,q):(M,\mathcal{A})\to(M',\mathcal{A}')$ is a morphism in the category $\mathcal{M}od_{\mathbb{D}}$, and $\mathcal{M}$ and $\mathcal{M}'$ any monoidal groupoids of respective types $(M,\mathcal{A})$ and $(M',\mathcal{A}')$, then a monoidal functor $F:\mathcal{M}\to\mathcal{M}'$ is of type (p,q) if and only if the square below in the category $\mathcal{M}od_{\mathbb{D}}$ commutes.

$$\begin{tikzcd}
(M,\mathcal{A}) \arrow[r, "{(i,\mathbf{j})}"] \arrow[d, "{(p,q)}"'] & (M(\mathcal{M}),\mathcal{A}(\mathcal{M})) \arrow[d, "{(p(F),q(F))}"] \\
(M',\mathcal{A}') \arrow[r, "{(i',\mathbf{j}')}"] & (M(\mathcal{M}'),(\mathcal{A}(\mathcal{M}')).
\end{tikzcd}$$

$\square$

Remark 1.2 The category of monoids is tripleable over the category of sets. In [74, Theorem 8], Wells identified the category $\mathbf{Ab}(\mathbf{Mnd}\downarrow_M)$ of abelian group objects in the comma category of monoids over a monoid M with the category of $\mathbb{D}M$-modules (see Example 1.1), and he proved that with a dimension shift both Barr-Beck cotriple cohomology theory [2, 5] and Leech cohomology theory of monoids are the same. Hence, for any monoid M and any $\mathbb{D}M$-module $\mathcal{A}$, Duskin [26] and Gleen [38] general interpretation theorem for cotriple cohomology classes shows that equivalence classes of 2-*torsors* over M under $\mathcal{A}$ are in bijection with elements of the cohomology group $H^3_{\mathrm{L}}(M,\mathcal{A})$.

A very similar result follows from the general result by Pirashvili [60, 61] and Baues-Dreckmann [3] about the classification of track categories. From this result, the elements of $H^3_{\mathrm{L}}(M,\mathcal{A})$ are in bijection with equivalence classes of *linear track extensions* of (the category) M by the $\mathbb{D}M$-module (*natural system on* M in their terminology) $\mathcal{A}$.

Indeed, the three terms '2-torsor over M under $\mathcal{A}$', 'linear track extension of M by $\mathcal{A}$', and '*strict* monoidal abelian groupoid of type $(M,\mathcal{A})$', are plainly recognized to be synonymous: simply take into account that an internal groupoid in the category of monoids is the same thing as a strict monoidal groupoid, together with Lemmas 2.2 and 2.3 in [18] (or [19, Theorem 3.3]).

However, we must stress that while it is relatively harmless to consider monoidal abelian groupoids as 'strict', since by Mac Lane Coherence Theorem for Monoidal Categories [56, 50] every monoidal abelian groupoid is equivalent to a strict one, we understand it is not so when dealing with their homomorphisms, since *not every monoidal functor is isomorphic to a strict one.* Indeed, it is possible to find two strict monoidal abelian groupoids, say $\mathcal{M}$ and $\mathcal{M}'$, which are related by a monoidal equivalence between them but, however, there is no strict equivalence either from $\mathcal{M}$ to $\mathcal{M}'$ nor from $\mathcal{M}'$ to $\mathcal{M}$. For this reason, if for establishing the bijection (1.58) we want to use only strict monoidal abelian groupoids and strict equivalences between them, as we need to do for applying Duskin or Pirashvili classification results, then we must define two strict monoidal abelian groupoids $\mathcal{M}$ and $\mathcal{M}'$ to be *equivalent* if there is a zig-zag chain of strict equivalences as $\mathcal{M} \leftarrow \mathcal{M}_1 \rightarrow \cdots \leftarrow \mathcal{M}_n \rightarrow \mathcal{M}'$. Although two strict monoidal abelian groupoids in the same equivalence class can always be linked by one intervening pair of strict equivalences, this phenomenon, we think, obscures unnecessarily the conclusions. Moreover, the facts stated in Theorem 1.5(iv) clearly fail for strict monoidal functors.

1.3.3 Classification of categorical groups revisited

As we recalled before, a categorical group is a monoidal groupoid $\mathcal{M}$ in which every object is invertible or, equivalently, such that its associated monoid of connected components $M(\mathcal{M})$ is a group. By Proposition 1.3, every categorical group is abelian, so that

$$\mathbf{CatGp} \subseteq \mathbf{MonAbGpd}$$

is the full 2-subcategory of the 2-category of monoidal abelian groupoids given by the categorical groups. We shall denote by

$$\mathbf{Z^3Mnd}|_{\mathbf{Gp}} \subseteq \mathbf{Z^3Mnd}$$

the full 2-subcategory of the 2-category of Leech 3-cocycles of monoids whose objects are those $\mathcal{S} = (G, \mathcal{A}, h)$ in $\mathbf{Z^3Mnd}$ where G is a group. Then, the biequivalences (1.54) in Theorem 1.3 restrict to corresponding quasi-inverse biequivalences

$$\mathbf{CatGp} \underset{\Sigma}{\overset{\Delta}{\rightleftarrows}} \mathbf{Z^3Mnd}|_{\mathbf{Gp}}. \tag{1.59}$$

But now we shall note that this latter 2-category $\mathbf{Z^3Mnd}|_{\mathbf{Gp}}$ is essentially the same as its full 2-subcategory, called the 2-*category of Eilenberg-Mac Lane* 3-*cocycles of groups* [20] and denoted by

$$\mathbf{Z^3Gp} \subseteq \mathbf{Z^3Mnd}|_{\mathbf{Gp}},$$

which is defined by those $\mathcal{S} = (G, \mathcal{A}, h)$ as above, but in which the family of groups $\mathcal{A}$ is constant, that is, where $\mathcal{A}(a) = \mathcal{A}(e)$ for all $a \in G$, and all automorphisms $a^* : \mathcal{A}(e) \to \mathcal{A}(e)$, $a \in G$, are identities. Observe that such a $\mathcal{S}$ is then described simply as a triple $\mathcal{S} = (G, A, h)$, where G is a group, A $(= \mathcal{A}(e))$ is a G-module with left action $(a, f) \mapsto {}^a f = a_*(f)$, and $h \in Z^3(G, A)$ is an ordinary normalized 3-cocycle of the group G with coefficients in the G-module A.

A morphism $(p, q, g) : (G, A, h) \to (G', A', h')$ in $\mathbf{Z^3Gp}$ then consists of a group homomorphism $p : G \to G'$, a homomorphism of G-modules

$$q : A \to p^* A',$$

and a normalized 2-cochain $g \in C^2(G, p^*A')$ such that $q_*(h) = p^*(h') + \partial^2 g$. If

$$(p,q,g),(\bar{p},\bar{q},\bar{g}) : (G,A,h) \to (G',A',h')$$

are two morphisms in $\mathbf{Z^3Gp}$, then there is no deformation between them unless $p = \bar{p}$ and $q = \bar{q}$, and, in such a case, a deformation $\delta : (p,q,g) \Rightarrow (p,q,\bar{g})$ consists of a 1-cochain $f \in C^1(G, p^*A')$, such that $g = \bar{g} + \partial^1 f$.

We have a 2-functor

$$(\ )_e : \mathbf{Z^3Mnd}|_{\mathbf{Gp}} \to \mathbf{Z^3Gp}$$

that is given on objects by

$$(G,\mathcal{A},h) \mapsto (G,\mathcal{A}(e),\widehat{h}), \tag{1.60}$$

where the action in $\mathcal{A}(e)$ is defined on each $f \in \mathcal{A}(e)$, by means of the isomorphisms $\mathcal{A}(e) \overset{a_*}{\to} \mathcal{A}(a) \overset{a^*}{\leftarrow} \mathcal{A}(e)$, $a \in G$, of $\mathcal{A}$, by the equations

$$a^*({}^a f) = a_*,$$

while the component at any $(a,b,c) \in G \times G \times G$ of the 3-cocycle $\widehat{h} \in Z^3(G,\mathcal{A}(e))$ is defined, by means of the isomorphism $\mathcal{A}(e) \xrightarrow{(abc)^*} \mathcal{A}(abc)$, by

$$(abc)^*(\widehat{h}(a,b,c)) = h(a,b,c).$$

A morphism $(p,q,g) : (G,\mathcal{A},h) \to (G',\mathcal{A}',h')$ in $\mathbf{Z^3Mnd}|_{\mathbf{Gp}}$ is mapped by the 2-functor $(\ )_e$ to the morphism

$$(p,q_e,\widehat{g}) : (G,\mathcal{A}(e),\widehat{h}) \to (G',\mathcal{A}'(e),\widehat{h}'), \tag{1.61}$$

where $\widehat{g} \in C^2(G, p^*\mathcal{A}'(e))$ is the 2-cochain whose component at any pair $(a,b) \in G \times G$ is determined by the isomorphism $p(ab)^* : \mathcal{A}'(e) \to \mathcal{A}'(p(ab))$ such that

$$p(ab)^*(\widehat{g}(a,b)) = g(a,b),$$

whereas a deformation $f : (p,q,g) \Rightarrow (p,q,g')$ in $\mathbf{Z^3Mnd}|_{\mathbf{Gp}}$ is carried to the the deformation in $\mathbf{Z^3Gp}$

$$\widehat{f} : (p,q_e,\widehat{g}) \Rightarrow (p,q_e,\widehat{g'}),$$

where $\widehat{f} \in C^1(G, p^*\mathcal{A}'(e))$ is the 1-cochain defined by the isomorphisms $p(a)^* : \mathcal{A}'(e) \to \mathcal{A}'(p(a))$, $a \in G$, such that

$$p(a)^*(\widehat{f}(a)) = f(a).$$

All the needed verifications to prove that so defined $(\ )_e$ is actually a 2-functor are quite straightforward. For example, we see that $\widehat{h}$ in (1.60) is a 3-cocycle and that the homomorphism $q_e : \mathcal{A}(e) \to p^*\mathcal{A}'(e)$ in (1.61) is of G-modules, as follows:

$$
\begin{aligned}
&(abcd)^*({}^a\widehat{h}(b,c,d)+\widehat{h}(a,bc,d)+\widehat{h}(a,b,c))\\
&\qquad=(bcd)^*a^*({}^a\widehat{h}(b,c,d)+(abcd)^*(\widehat{h}(a,bc,d)+d^*(abc)^*(\widehat{h}(a,b,c))\\
&\qquad=(bcd)^*a_*(\widehat{h}(b,c,d))+h(a,bc,d)+d^*(h(a,b,c))\\
&\qquad=a_*(h(b,c,d))+h(a,bc,d)d^*(h(a,b,c))=h(a,b,cd)+h(ab,c,d)\\
&\qquad=(abcd)^*(\widehat{h}(a,b,cd)+\widehat{h}(ab,c,d)),
\end{aligned}
$$

whence ${}^a\widehat{h}(b,c,d)+\widehat{h}(a,bc,d)+\widehat{h}(a,b,c)=\widehat{h}(a,b,cd)+\widehat{h}(ab,c,d)$.

$$p(a)^*({}^{p(a)}q_e(f))=p(a)_*(q_e(f))\overset{(1.52)}{=}q_aa_*(f)=q_a(a^*({}^af))\overset{(1.53)}{=}p(a)^*(q_e({}^af)),$$

whence $q_e({}^af)={}^{p(a)}q_e(f)$.

Proposition 1.4 *The 2-functors inclusion in and $(\ )_e$ are mutually quasi-inverse biequivalences*

$$\mathbf{Z^3Gp}\ \underset{in}{\overset{(\ )_e}{\underset{\longrightarrow}{\overset{\longleftarrow}{\approx}}}}\ \mathbf{Z^3Mnd}|_{\mathbf{Gp}}.$$

Proof: We have $(\ )_e\,in=1$, while the pseudo-equivalence $in\,(\ )_e\simeq 1$ is given, at any object $(G,\mathcal{A},h)$, by the isomorphism

$$(1_G,q,1):(G,\mathcal{A}(e),\widehat{h})\overset{\sim}{\Rightarrow}(G,\mathcal{A},h),$$

where $q=(\mathcal{A}(e)\xrightarrow{a^*}\mathcal{A}(a))_{a\in G}$. □

Hence, by composing the biequivalences above with those in (1.59), we get the following (already known, see [20, Theorem 3.3]) cohomological description of the 2-category of categorical groups:

Theorem 1.6 *The 2-functors $\Delta_e=(\ )_e\Delta$ and $\Sigma_e=\Sigma\,in$,*

$$\mathbf{CatGp}\ \underset{\Sigma_e}{\overset{\Delta_e}{\underset{\longleftarrow}{\overset{\longrightarrow}{\approx}}}}\ \mathbf{Z^3Gp} \tag{1.62}$$

are quasi-inverse biequivalences.

Let us now denote by $\mathbf{H^3Gp}\subseteq\mathbf{H^3Mnd}$ the full subcategory of the category of Leech 3-cohomology classes of monoids (1.55), given by the Eilenberg-Mac Lane 3-cohomology classes of groups. An object in $\mathbf{H^3Gp}$ is then a triple (G,A,c), where G is a group, A is a G-module, and $c\in H^3(G,A)$. An arrow $(p,q):(G,A,c)\to(G',A',c')$ in $\mathbf{H^3Gp}$ consists of a group homomorphism $p:G\to G'$ and an homomorphism of G-modules $q:A\to p^*A'$ such that $p^*(c')=q_*(c)\in H^3(G,p^*A')$.

We have the *cohomology class functor*

$$\mathrm{cl}: \mathbf{Z^3Gp} \to \mathbf{H^3Gp}.$$
$$(G, A, h) \mapsto (G, A, [h])$$
$$(p, q, g) \mapsto (p, q)$$

This functor carries isomorphic morphisms of $\mathbf{Z^3Gp}$ to the same morphism in $\mathbf{H^3Gp}$; it is surjective on objects; it reflects isomorphisms: if $(p, q, g) : (G, A, h) \to (G', A', h')$ is any morphism in $\mathbf{Z^3Gp}$ such that the maps p and q are invertible, then the morphism of $\mathbf{Z^3Gp}$

$$(p^{-1}, q^{-1}, p^{*-1}q_*^{-1}(-g)) : (G', A', h') \to (G, A, h)$$

is an inverse of (p, q, g); and it is full: if $(p, q) : \mathrm{cl}(G, A, h) \to \mathrm{cl}(G', A', h')$ is any morphism in $\mathbf{H^3Gp}$, then $p^*[h']$ and $q_*[h]$ both represent the same class in $H^3(G, p^*A')$, so there is $g \in C^2(G, p^*A')$ such that $q_*(h) = p^*(h') + \partial^2 g$. Then, $(p, q, g) : (G, A, h) \to (G', A', h')$ is a morphism in $\mathbf{Z^3Gp}$ with $\mathrm{cl}(p, q, g) = (p, q)$. Observe that any other realization of (p, q) is of the form $(p, q, g \circ g')$ with $g' \in Z^2(G, p^*A')$ and, moreover, that there is a deformation $(p, q, g) \Rightarrow (p, q, g \circ g')$ if and only if $g' = \partial^1 f$ for some $f \in C^1(G, p^*A')$.

Hence, the *classifying functor*

$$\mathrm{Cl} = \mathrm{cl}\,\Delta_e : \mathbf{CatGp} \to \mathbf{H^3Gp}$$

has the following properties:

Theorem 1.7 ([69] Classification of categorical groups) (*i*) *For any group G, any G-module A, and any cohomology class $c \in H^3(G, A)$, there is a categorical group $\mathcal{M}$ with an isomorphism $\mathrm{Cl}(\mathcal{M}) \cong (G, A, c)$.*

(*ii*) *A monoidal functor between categorical groups $F : \mathcal{M} \to \mathcal{M}'$ is an equivalence if and only if the induced $\mathrm{Cl}(F) : \mathrm{Cl}(\mathcal{M}) \to \mathrm{Cl}(\mathcal{M}')$ is an isomorphism.*

(*iii*) *If $\mathcal{M}$ and $\mathcal{M}'$ are categorical groups, then, for $(p, q) : \mathrm{Cl}(\mathcal{M}) \xrightarrow{\cong} \mathrm{Cl}(\mathcal{M}')$ any isomorphism, there is a monoidal equivalence $F : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ such that $\mathrm{Cl}(F) = (p, q)$.*

(*iv*) *If $\mathrm{Cl}(\mathcal{M}) = (G, A, c)$ and $\mathrm{Cl}(\mathcal{M}') = (G', A', c')$, then, for any morphism $(p, q) : \mathrm{Cl}(\mathcal{M}) \to \mathrm{Cl}(\mathcal{M}')$ in $\mathbf{H^3Gp}$, there is a (non-natural) bijection*

$$\big\{[F] : \mathcal{M} \to \mathcal{M}' \mid \mathrm{Cl}(F) = (p, q)\big\} \cong H^2(G, p^*A'),$$

between the set of isomorphism classes of those monoidal functors $F : \mathcal{M} \to \mathcal{M}'$ which are carried by the classifying funtor to (p, q) and the second cohomology group of G with coefficients in the G-module p^*A'.

Chapter 2

Computability of the (co)homology of cyclic monoids

Recall from 1.3.1 that Leech cohomology groups of a monoid M are defined to be those of its category $\mathbb{D}M$, that is, if $\mathcal{A} : \mathbb{D}M \to \mathbf{Ab}$ is any $\mathbb{D}M$-module (called in what follows left $\mathbb{D}M$-module), then

$$H^n_{\mathrm{L}}(M, \mathcal{A}) = \mathrm{Ext}^n_{\mathbb{D}M}(\mathbb{Z}, \mathcal{A}) = R^n\mathrm{Hom}_{\mathbb{D}M}(\mathbb{Z}, -)(\mathcal{A}) = R^n\mathrm{Hom}_{\mathbb{D}M}(-, \mathcal{A})(\mathbb{Z}), \quad (2.1)$$

where, for any two left $\mathbb{D}M$-modules $\mathcal{A}$ and $\mathcal{A}'$, $\mathrm{Hom}_{\mathbb{D}M}(\mathcal{A}, \mathcal{A}')$ denotes the abelian group of morphisms of $\mathbb{D}M$-modules between them, and $\mathbb{Z} : \mathbb{D}M \to \mathbf{Ab}$ is the constant functor defined by the abelian group of integers $\mathbb{Z}$. Similarly, for $\mathcal{B} : \mathbb{D}M^{op} \to \mathbf{Ab}$ any right $\mathbb{D}M$-module, the homology groups of M with coefficients in $\mathcal{B}$ [51, Definition 2.1] are defined by

$$H_n^{\mathrm{L}}(M, \mathcal{B}) = \mathrm{Tor}_n^{\mathbb{D}M}(\mathcal{B}, \mathbb{Z}) = L_n(- \otimes_{\mathbb{D}M} \mathbb{Z})(\mathcal{B}) = L_n(\mathcal{B} \otimes_{\mathbb{D}M} -)(\mathbb{Z}), \quad (2.2)$$

where, for any left $\mathbb{D}M$-module $\mathcal{A}$, the tensor product $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{A}$ is the abelian group defined as the coend of the bifunctor $\mathbb{D}M^{op} \times \mathbb{D}M \to \mathbf{Ab}$ which carries each pair $(x, y) \in M \times M$ to the tensor product abelian group $\mathcal{B}(x) \otimes \mathcal{A}(y)$.

It is remarkable that, when coefficients are taken in ordinary M-modules (regarded as constant on objects $\mathbb{D}M$-modules), Leech (co)homology groups agree with those by Eilenberg and Mac Lane [57, Chapter X, 5], see 2.1.6 below for some details. In particular, Eilenberg-Mac Lane (co)homology groups of groups are instances of Leech (co)homology groups of monoids.

This chapter deals with the (co)homology of finite cyclic monoids $C_{m,q}$, whose structure and classification by means of the index m and the period q was first stated by Frobenius [34]. As we noted in the abstract, the (co)homology groups of any finite cyclic group $C_q = C_{0,q}$ were computed by Eilenberg [27, Section 11], while for finite cyclic monoids of index $m \geq 1$, the cohomology groups $H^n_{\mathrm{L}}(C_{m,q}, \mathcal{A})$ of a cyclic monoid of index $m \geq 1$ have been computed only for $n \leq 2$ by Leech in [53, Chapter II, 7.20, 7.21]. However, because higher cohomology groups are interesting (as we have shown in the previous chapter, for instance), the aim of this chapter is to compute all the (co)homology groups of any finite cyclic monoid $C_{m,q}$.

Briefly, the contents of the chapter are as follows. In Section 2.1, while fixing notation and terminology, we review some basic constructions concerning the (co)homology of monoids. Section 2.2 is mainly dedicated to studying the *trace maps* associated to any $\mathbb{D}C_{m,q}$-module, which are a key tool in our deliberations. Section 2.3 is devoted to the construction of a specific free resolution of the trivial $\mathbb{D}C_{m,q}$-module $\mathbb{Z}$, which allows us to determine, in the final Section 2.4, the groups $H^n_{\mathrm{L}}(C_{m,q}, \mathcal{A})$ and $H_n^{\mathrm{L}}(C_{m,q}, \mathcal{B})$. The (co)homology of $C_{m,q}$ is proven to be periodic with a period of $2q/\gcd(m,q)$.

2.1 Notations and preliminaries

2.1.1 Left $\mathbb{D}M$-modules

A left $\mathbb{D}M$-module (called simply $\mathbb{D}M$-module in 1.3.1) is an abelian group valued functor on the category $\mathbb{D}M$.

For instance, let $\mathbb{Z} : \mathbb{D}M \to \mathbf{Ab}$ be the $\mathbb{D}M$-module that associates to each element $x \in M$ the free abelian group on the generator (x), $\mathbb{Z}(x)$, and to each $x, y \in M$ the isomorphisms of abelian groups

$$\mathbb{Z}(y) \xrightarrow{x_*} \mathbb{Z}(xy) \xleftarrow{y^*} \mathbb{Z}(x)$$

given on generators by $x_*(y) = (xy) = y^*(x)$. This is isomorphic to the $\mathbb{D}M$-module defined by the constant functor on $\mathbb{D}M$ which associates the abelian group $\mathbb{Z}$ to any $x \in M$.

For two left $\mathbb{D}M$-modules $\mathcal{A}$ and $\mathcal{A}'$, a morphism between them (i.e., a natural transformation) $f : \mathcal{A} \to \mathcal{A}'$ consists of homomorphisms $f_x : \mathcal{A}(x) \to \mathcal{A}'(x)$, such that, for any $x, y \in M$, the squares below commute.

$$\begin{tikzcd}
\mathcal{A}(y) \arrow[r, "x_*"] \arrow[d, "f_y"'] & \mathcal{A}(xy) \arrow[d, "f_{xy}"] & \mathcal{A}(x) \arrow[l, "y^*"'] \arrow[d, "f_x"] \\
\mathcal{A}'(x) \arrow[r, "x_*"] & \mathcal{A}'(xy) & \mathcal{A}'(x) \arrow[l, "y^*"']
\end{tikzcd}$$

The category of left $\mathbb{D}M$-modules, denoted by $\mathbb{D}M$-Mod, is an abelian category with enough projective and injective objects. We refer to [53, Chapter I, 1] for details, but recall that the set of morphisms between two $\mathbb{D}M$-modules $\mathcal{A}$ and $\mathcal{A}'$, denoted by $\mathrm{Hom}_{\mathbb{D}M}(\mathcal{A}, \mathcal{A}')$, is an abelian group by pointwise addition, that is, if $f, g : \mathcal{A} \to \mathcal{A}'$ are morphisms, then $f + g : \mathcal{A} \to \mathcal{A}'$ is defined by setting $(f+g)_x = f_x + g_x$, for each $x \in M$. The zero $\mathbb{D}M$-module is the constant functor $0 : \mathbb{D}M \to \mathbf{Ab}$ defined by the trivial abelian group 0, and a sequence of $\mathbb{D}M$-modules $\mathcal{A} \to \mathcal{A}' \to \mathcal{A}''$ is exact if and only if the induced sequences of abelian groups $\mathcal{A}(x) \to \mathcal{A}'(x) \to \mathcal{A}''(x)$ are exact, for all $x \in M$.

2.1.2 Free left $\mathbb{D}M$-modules

Let $\mathbf{Set}\!\downarrow_M$ be the comma category of sets over the underlying set of M, that is, the category whose objects $S = (S, \pi)$ are sets S endowed with a map $\pi : S \to M$ and whose morphisms are maps $\varphi : S \to T$ such that $\pi\varphi = \pi$. There is a *forgetful functor* $\mathcal{U} : \mathbb{D}M\text{-Mod} \to \mathbf{Set}\!\downarrow_M$, which carries any $\mathbb{D}M$-module $\mathcal{A}$ to the disjoint union set

$$\mathcal{U}\mathcal{A} = \coprod_{x\in M} \mathcal{A}(x) = \{(x,a) \mid x \in M,\, a \in \mathcal{A}(x)\},$$

endowed with the projection map $\pi : \mathcal{U}\mathcal{A} \to M$, $\pi(x,a) = x$. A morphism $f : \mathcal{A} \to \mathcal{A}'$ is sent to the map $\mathcal{U}f : \mathcal{U}\mathcal{A} \to \mathcal{U}\mathcal{A}'$ given by $\mathcal{U}f(x,a) = (x, f_x(a))$. There is also a *free left $\mathbb{D}M$-module* functor $\mathcal{F} : \mathbf{Set}\!\downarrow_M \to \mathbb{D}M\text{-Mod}$, which is defined as follows: If $S = (S,\pi)$ is any set over M, then $\mathcal{F}S$ is the $\mathbb{D}M$-module such that, for each $x \in M$,

$$\mathcal{F}S(x) = \mathbb{Z}\{(u,s,v) \in M \times S \times M \mid u\pi(s)v = x\}$$

is the free abelian group with generators all triplets (u,s,v), where $u,v \in M$ and $s \in S$, such that $u\pi(s)v = x$. We usually write (e,s,e) simply by s (where $e \in M$ is the unit), so that each element of $s \in S$ is regarded as an element $s \in \mathcal{F}S(\pi s)$. For any $x,y \in M$, the homomorphisms

$$\mathcal{F}S(y) \xrightarrow{x_*} \mathcal{F}S(xy) \xleftarrow{y^*} \mathcal{F}S(x)$$

are defined on generators by $x_*(u,s,v) = (xu,s,v)$ and $y^*(u,s,v) = (x,s,vy)$.

If $\varphi : S \to T$ is any map of sets over M, the induced morphism $\mathcal{F}\varphi : \mathcal{F}S \to \mathcal{F}S'$ is given, at each $x \in M$, by the homomorphism such that $(\mathcal{F}\varphi)_x(u,s,v) = (u,\varphi(s),v)$.

Proposition 2.1 *The functor $\mathcal{F}$ is left adjoint to the functor $\mathcal{U}$. Thus, for $S = (S,\pi)$ any set over M and any left $\mathbb{D}M$-module $\mathcal{A}$, there is a natural isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}S, \mathcal{A}) \;\cong\; \prod_{s\in S} \mathcal{A}(\pi s).$$

Proof: At any set S over M, the unit of the adjunction is the map

$$\epsilon : S \to \mathcal{U}\mathcal{F}S = \{(x,a) \mid x \in M,\, a \in \mathcal{F}S(x)\}, \qquad s \mapsto (\pi s, s).$$

If $\mathcal{A}$ is a $\mathbb{D}M$-module and $\varphi : S \to \mathcal{U}\mathcal{A}$ is any map over M, then the unique morphism of $\mathbb{D}M$-modules $f : \mathcal{F}S \to \mathcal{A}$ such that $(\mathcal{U}f)\,\epsilon = \varphi$ is determined by the equations

$$f_x(u,s,v) = u_* v^* \varphi(s),$$

for any $x \in M$ and $(u,s,v) \in M \times S \times M$ with $u\pi(s)v = x$. Since giving a map over M, $\varphi : S \to \mathcal{U}\mathcal{A}$, is the same thing as giving a list $(\varphi(s))_{s\in S} \in \prod_{s\in S} \mathcal{A}(\pi s)$, the isomorphism $\mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}S, \mathcal{A}) \cong \prod_{s\in S} \mathcal{A}(\pi s)$ follows. $\square$

2.1.3 Right $\mathbb{D}M$-modules

The category of *right* $\mathbb{D}M$*-modules* is defined to be the category of functors $\mathcal{B}$: $\mathbb{D}M^{op} \to \mathbf{Ab}$. A right $\mathbb{D}M$-module $\mathcal{B}$ provides us with abelian groups $\mathcal{B}(x)$, $x \in M$, and homomorphisms

$$\mathcal{B}(y) \xleftarrow{x^*} \mathcal{B}(xy) \xrightarrow{y_*} \mathcal{B}(x),$$

for each $x, y \in M$, such that the equations below hold.

$$\begin{array}{ll} y^*x^* = (xy)^* : \mathcal{B}(xyz) \to \mathcal{B}(z), & x_*y_* = (xy)_* : \mathcal{B}(zxy) \to \mathcal{B}(z), \\ e_* = e^* = id_{\mathcal{B}(x)} : \mathcal{B}(x) \to \mathcal{B}(x), & x^*y_* = y_*x^* : \mathcal{B}(xzy) \to \mathcal{B}(z). \end{array}$$

2.1.4 Tensor product of $\mathbb{D}M$-modules

If $\mathcal{B}$ is a right $\mathbb{D}M$-module and $\mathcal{A}$ is any left $\mathbb{D}M$-module, their tensor product $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{A} = \int^{\mathbb{D}M} \mathcal{B} \otimes \mathcal{A}$ is the abelian group coend [58, Chapter IX, 6] of the functor $\mathcal{B} \otimes \mathcal{A} : \mathbb{D}M^{op} \times \mathbb{D}M \to \mathbf{Ab}$ defined by $(\mathcal{B} \otimes \mathcal{A})(x, y) = \mathcal{B}(x) \otimes \mathcal{A}(y)$. That is, $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{A}$ is the abelian group generated by elements of the form $b \otimes a$, where $b \in \mathcal{B}(x)$ and $a \in \mathcal{A}(x)$, $x \in M$, subject to the relations

$$\begin{array}{lll} (b + b') \otimes a = b \otimes a + b' \otimes a, & \text{for} & b, b' \in \mathcal{B}(x),\ a \in \mathcal{A}(x), x \in M, \\ b \otimes (a + a') = b \otimes a + b \otimes a', & \text{for} & b \in \mathcal{B}(x),\ a, a' \in \mathcal{A}(x), x \in M, \\ y_*b \otimes a = b \otimes y^*a, & \text{for} & b \in \mathcal{B}(xy),\ a \in \mathcal{A}(x),\ x, y \in M, \\ y^*b \otimes a = b \otimes y_*a, & \text{for} & b \in \mathcal{B}(yx),\ a \in \mathcal{A}(x),\ x, y \in M. \end{array}$$

Proposition 2.2 *For $S = (S, \pi)$ any set over M and $\mathcal{B}$ any right $\mathbb{D}M$-module, there is a natural isomorphism of abelian groups*

$$\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{F}S \ \cong \ \bigoplus_{s \in S} \mathcal{B}(\pi s).$$

Proof: As an abelian group, $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{F}S$ is generated by the elements

$$b \otimes (u, s, v) = b \otimes u_*v^*(e, s, e) = b \otimes u_*v^*s = u^*v_*b \otimes s,$$

with $u, v \in M$, $s \in S$, and $b \in \mathcal{B}(u\pi(s)v)$. The claimed isomorphism carries such a generator $b \otimes (u, s, v)$ to the element $u^*v_*b \in \mathcal{B}(\pi s)$. Its inverse map carries any element $b \in \mathcal{B}(\pi s)$ to the generator $b \otimes s$ of $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{F}S$. $\square$

2.1.5 Computing the (co)homology of a monoid.

From Proposition 2.1, it easily follows that every free left $\mathbb{D}M$-module is projective. Then, if

$$\mathcal{F}_\bullet \xrightarrow{\epsilon} \mathbb{Z} : \qquad \cdots \to \mathcal{F}_2 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is any free resolution of $\mathbb{Z}$ in the category of left $\mathbb{D}M$-modules, then the cohomology groups of M with coefficients in a left $\mathbb{D}M$-module $\mathcal{A}$, defined in (2.1), can be computed by means of the induced cochain complex of abelian groups

$$\mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}_\bullet,\mathcal{A}) : 0 \to \mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}_0,\mathcal{A}) \xrightarrow{\partial^*} \mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}_1,\mathcal{A}) \xrightarrow{\partial^*} \mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}_2,\mathcal{A}) \to \cdots$$

by $H^n_{\mathrm{L}}(M,\mathcal{A}) = H^n_{\mathrm{L}}\big(\mathrm{Hom}_{\mathbb{D}M}(\mathcal{F}_\bullet,\mathcal{A})\big)$, and the homology groups of M with coefficients in a right $\mathbb{D}M$-module $\mathcal{B}$, defined in (2.2), by means of the induced chain complex

$$\mathcal{B}\otimes_{\mathbb{D}M}\mathcal{F}_\bullet : \ \cdots \to \mathcal{B}\otimes_{\mathbb{D}M}\mathcal{F}_2 \xrightarrow{id\otimes\partial} \mathcal{B}\otimes_{\mathbb{D}M}\mathcal{F}_1 \xrightarrow{id\otimes\partial} \mathcal{B}\otimes_{\mathbb{D}M}\mathcal{F}_0 \to 0$$

as $H^n_{\mathrm{L}}(M,\mathcal{B}) = H^{\mathrm{L}}_n\big(\mathcal{B}\otimes_{\mathbb{D}M}\mathcal{F}_\bullet\big)$.

2.1.6 Eilenberg-Mac Lane (co)homology

There is a full exact embedding from the category of ordinary left M-modules into the category of left $\mathbb{D}M$-modules. This carries any left M-module A, with M-action $(x,a)\mapsto xa$, to the left $\mathbb{D}M$-module, also denoted by A, defined by $A(x)=A$, for all $x\in M$, together with the homomorphisms $x_*, x^* : A\to A$ given by $x_*a = xa$ and $x^*a = a$ [53, Chapter III, Lemma 1.9]. Similarly, there is a full exact embedding from the category of ordinary right M-modules into the category of right $\mathbb{D}M$-modules, which carries any right M-module B to the right $\mathbb{D}M$-module, also denoted by B, defined by $B(x)=B$, for all $x\in M$, together the homomorphisms $x_*, x^* : B\to B$ given by $x^*b = bx$ and $x_*b = b$.

When, for A any left M-module and B any right M-module, one applies the functors $\mathrm{Hom}_{\mathbb{D}M}(-,A)$ and $B\otimes_{\mathbb{D}M} -$ to the standard free resolution of the left $\mathbb{D}M$-module $\mathbb{Z}$ in [53, Chaper II, 2.2], then one obtains a cochain complex isomorphic to $\mathrm{Hom}_{\mathbb{Z}M}(\mathrm{B}(M),A)$ and a chain complex isomorphic to $B\otimes_{\mathbb{Z}M}\mathrm{B}(M)$, respectively. Here, $\mathbb{Z}M$ is the monoid ring and $\mathrm{B}(M)$ the bar resolution of $\mathbb{Z}$ as a left M-module. It follows that

$$H^n_{\mathrm{L}}(M,A) = \mathrm{Ext}^n_{\mathbb{Z}M}(\mathbb{Z},A), \qquad H^{\mathrm{L}}_n(M,B) = \mathrm{Tor}^{\mathbb{Z}M}_n(B,\mathbb{Z}).$$

That is, the Leech (co)homology groups $H^n_{\mathrm{L}}(M,A)$ and $H^{\mathrm{L}}_n(M,B)$ agree with those by Eilenberg and Mac Lane [57, Chapter X, 5] (see the proof of [53, Chapter III, Corollary 1.15] for more details). In particular, Eilenberg-Mac Lane (co)homology groups of groups are instances of Leech (co)homology groups of monoids [1].

[1] If G is a group, regarded as a category with only one object, then G and $\mathbb{D}G$ become equivalent categories due to the functor $F : G\to\mathbb{D}G$ given by $F(x) = (x,e,x^{-1}) : e\to e$. Consequently, the categories of G-modules and of $\mathbb{D}G$-modules are equivalent. This gives an alternative and easier proof that, for groups, both the Leech and the Eilenberg-Mac Lane (co)homology theories are equivalent.

2.2 Cyclic monoids and trace maps

The structure of finite cyclic monoids was first stated by Frobenius [34]. Briefly, let us recall that, if $\sim$ is any non-equality congruence on the additive monoid of natural numbers, $\mathbb{N} = \{0, 1, \dots\}$, then the least $m \geq 0$ such that $m \sim x$ for some $x \neq m$ is called the *index* of the congruence, and the least $q \geq 1$ such that $m \sim m + q$ is called its *period*. Hence,

$$x \sim y \text{ if and only if either } x = y < m, \text{ or } x, y \geq m \text{ and } x \equiv y \mod q.$$

The quotient $\mathbb{N}/\!\sim$ is called the *cyclic monoid of index m and period q*, and is denoted here by $C_{m,q}$. As $\mathbb{N}$ is a free monoid on the generator 1, every finite cyclic monoid is isomorphic to a proper quotient of $\mathbb{N}$ and, therefore, to a monoid $C_{m,q}$ for some $m \geq 0$ and $q \geq 1$.

Since every element of $C_{m,q}$ can be written uniquely in the form $[x]$ with $0 \leq x < m + q$, the underlying set of this monoid can be described as the set

$$C_{m,q} = \{0, 1, \dots, m, m + 1, \dots, m + q - 1\}.$$

Hereafter, we use this description. In these terms, the projection map $\wp : \mathbb{N} \to C_{m,q}$ is given by

$$\wp(x) = \begin{cases} x & \text{if } x < m + q \\ x - kq & \text{if } m + kq \leq x < m + (k + 1)q, \end{cases}$$

and the addition in $C_{m,q}$, which is denoted by the symbol $\oplus$ to avoid confusion with the addition $+$ of $\mathbb{N}$, is given by

$$x \oplus y = \wp(x + y).$$

Furthermore, we use the notation $r \cdot x$, for any $r \in \mathbb{N}$ and $x \in C_{m,q}$, to denote the element of $C_{m,q}$ defined recursively by

$$0 \cdot x = 0, \quad (r + 1) \cdot x = (r \cdot x) \oplus x. \tag{2.3}$$

In other words, $r \cdot x = x \overset{(r\text{-times})}{\oplus \cdots \oplus} x = \wp(x \overset{(r\text{-times})}{+ \cdots +} x) = \wp(rx)$. For instance, $2 \cdot 8 = 7$ in $C_{2,9}$.

From now on, $C = C_{m,q}$ denotes the finite cyclic monoid of index m and period q. We assume that $m + q \geq 2$, so that $C_{m,q}$ is not the zero monoid.

The following two families of homomorphisms are crucial for our deliberations.

Definition 2.1 *Let $\mathcal{A}$ be a left $\mathbb{D}C$-module. For each $x \in C$, $x \geq 1$, the* 'trace map'

$$\mathrm{T} : \mathcal{A}(x) \longrightarrow \mathcal{A}(m \oplus (x - 1)) \tag{2.4}$$

is the homomorphism defined by

$$\mathrm{T}(a) = \sum_{t=0}^{m+q-1} t^*(m+q-t-1)_* a - \sum_{s=0}^{m-1} s^*(m-s-1)_* a.$$

Also, for each $x \in C$, let

$$\mathrm{S} : \mathcal{A}(x) \to \mathcal{A}(x \oplus 1) \tag{2.5}$$

be the homomorphism defined by $\mathrm{S}(a) = 1_* a - 1^* a$.

The following subgroups will be used later.

$$\begin{aligned} &\mathcal{A}^{\mathrm{T}}(x) = \{a \in \mathcal{A}(x) \mid \mathrm{T}(a) = 0\}, \quad \mathcal{A}_{\mathrm{T}}(x) = \{\mathrm{T}(a) \mid a \in \mathcal{A}(x)\}, \\ &\mathcal{A}^{\mathrm{S}}(x) = \{a \in \mathcal{A}(x) \mid \mathrm{S}(a) = 0\}, \quad \mathcal{A}_{S}(x) = \{\mathrm{S}(a) \mid a \in \mathcal{A}(x)\}. \end{aligned} \tag{2.6}$$

Lemma 2.1 *For any left $\mathbb{D}C$-module $\mathcal{A}$, the squares below commute.*

$$\begin{tikzcd} \mathcal{A}(x) \arrow[r, "1_*"] \arrow[d, "1^*"'] & \mathcal{A}(x \oplus 1) \arrow[d, "\mathrm{T}"] \\ \mathcal{A}(x \oplus 1) \arrow[r, "\mathrm{T}"] & \mathcal{A}(m \oplus x) \end{tikzcd} \quad \begin{tikzcd} \mathcal{A}(x) \arrow[r, "\mathrm{T}"] \arrow[d, "1^*"'] & \mathcal{A}(m \oplus (x-1)) \arrow[d, "1^*"] \\ \mathcal{A}(x \oplus 1) \arrow[r, "\mathrm{T}"] & \mathcal{A}(m \oplus x) \end{tikzcd} \quad \begin{tikzcd} \mathcal{A}(x) \arrow[r, "\mathrm{T}"] \arrow[d, "1_*"'] & \mathcal{A}(m \oplus (x-1)) \arrow[d, "1_*"] \\ \mathcal{A}(x \oplus 1) \arrow[r, "\mathrm{T}"] & \mathcal{A}(m \oplus x) \end{tikzcd}$$

Proof: To prove that $\mathrm{T}\,1_* = \mathrm{T}\,1^*$, let $a \in \mathcal{A}(x)$. On the one hand,

$$\begin{aligned} \mathrm{T}(1_* a) &= \sum_{t=0}^{m+q-1} t^*(m+q-t-1)_* 1_* a - \sum_{s=0}^{m-1} s^*(m-s-1)_* 1_* a \\ &= \sum_{t=0}^{m+q-1} t^*\big((m+q-t-1) \oplus 1\big)_* a - \sum_{s=0}^{m-1} s^*((m-s-1) \oplus 1)_* a \\ &= m_* a + \sum_{t=1}^{m+q-1} t^*(m+q-t)_* a - m_* a - \sum_{s=1}^{m-1} s^*(m-s)_* a \\ &= \sum_{t=1}^{m+q-1} t^*(m+q-t)_* a - \sum_{s=1}^{m-1} s^*(m-s)_* a, \end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\mathrm{T}(1^*a) &= \sum_{t=0}^{m+q-1} 1^*t^*(m+q-t-1)_*a - \sum_{s=0}^{m-1} 1^*s^*(m-s-1)_*a \\
&= \sum_{t=0}^{m+q-1} (1\oplus t)^*(m+q-t-1)_*a - \sum_{s=0}^{m-1}(1\oplus s)^*(m-s-1)_*a \\
&= \sum_{t=0}^{m+q-2} (1+t)^*(m+q-t-1)_*a + m^*a - \sum_{s=0}^{m-2}(1+s)^*(m-s-1)_*a - m^*a \\
&= \sum_{t=0}^{m+q-2} (1+t)^*(m+q-t-1)_*a - \sum_{s=0}^{m-2}(1+s)^*(m-s-1)_*a,
\end{aligned}$$

whence, by comparison, the result follows.

The other two equalities, $1^*\,\mathrm{T} = \mathrm{T}\,1^*$ and $1_*\,\mathrm{T} = \mathrm{T}\,1_*$, follow easily from the commutativity of the monoid C. □

Lemma 2.2 *For any left $\mathbb{D}C$-module $\mathcal{A}$, the sequences*

$$\mathcal{A}(x) \xrightarrow{\mathrm{S}} \mathcal{A}(x\oplus 1) \xrightarrow{\mathrm{T}} \mathcal{A}(m\oplus x), \quad \mathcal{A}(x) \xrightarrow{\mathrm{T}} \mathcal{A}(m\oplus(x-1)) \xrightarrow{\mathrm{S}} \mathcal{A}(m\oplus x),$$

are semiexact, that is, $\mathrm{T}\,\mathrm{S}=0$ *and* $\mathrm{S}\,\mathrm{T}=0$.

Proof: It is a direct consequence of Lemma 2.1 since

$$\mathrm{TS} = \mathrm{T}\,1_* - \mathrm{T}\,1^* = 0, \quad \mathrm{ST} = 1_*\,\mathrm{T} - 1^*\,\mathrm{T} = \mathrm{T}\,1_* - \mathrm{T}\,1^* = 0.$$

□

2.3 A resolution of $\mathbb{Z}$ by free $\mathbb{D}C$-modules

It is possible to calculate the (co)homology of cyclic monoids efficiently by a clever choice of resolution. We construct here a specific free resolution of the trivial $\mathbb{D}C$-module $\mathbb{Z}$,

$$\mathcal{F}_\bullet \xrightarrow{\epsilon} \mathbb{Z}: \qquad \cdots \to \mathcal{F}_2 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{2.7}$$

as follows.

For each integer $r \geq 0$, choose symbols $\mathbf{v}_r$ and $\mathbf{w}_r$. Then, recalling the notation (2.3),

- *$\mathcal{F}_{2r}$ is the free $\mathbb{D}C$-module on the unitary set over C, $\{\mathbf{v}_r\} \xrightarrow{\pi} C$, where $\pi\mathbf{v}_r = r\cdot m$.*

- $\mathcal{F}_{2r+1}$ *is the free* $\mathbb{D}C$*-module on the unitary set over* C, $\{\mathbf{w}_r\} \xrightarrow{\pi} C$, *where* $\pi\mathbf{w}_r = r \cdot m \oplus 1$.

- *The augmentation* $\epsilon : \mathcal{F}_0 \to \mathbb{Z}$ *is the morphism of* $\mathbb{D}C$*-modules determined by*

$$\epsilon_0(\mathbf{v}_0) = (0) \in \mathbb{Z}(0).$$

- *For each* $r \geq 0$, *the differential* $\partial : \mathcal{F}_{2r+2} \to \mathcal{F}_{2r+1}$ *is the morphism of* $\mathbb{D}C$*-modules determined by*

$$\partial_{\pi\mathbf{v}_{r+1}}(\mathbf{v}_{r+1}) = \mathrm{T}(\mathbf{w}_r),$$

where $\mathrm{T} : \mathcal{F}_{2r+1}(r \cdot m \oplus 1) \to \mathcal{F}_{2r+1}((r+1) \cdot m)$ *is the trace map* (2.4).

- *For each* $r \geq 0$, *the differential* $\partial : \mathcal{F}_{2r+1} \to \mathcal{F}_{2r}$ *is the morphism of* $\mathbb{D}C$*-modules determined by*

$$\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r) = \mathrm{S}(\mathbf{v}_r),$$

where $\mathrm{S} : \mathcal{F}_{2r}(r \cdot m) \to \mathcal{F}_{2r}(r \cdot m \oplus 1)$ *is the homomorphism* (2.5).

Proposition 2.3 $\mathcal{F}_\bullet \xrightarrow{\epsilon} \mathbb{Z}$, *defined as above, is an augmented complex of* $\mathbb{D}C$*-modules.*

Proof: The sequence $\mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \xrightarrow{\epsilon} \mathbb{Z}$ is semiexact, that is, $\epsilon\partial = 0$, since

$$\epsilon_1\partial_1(\mathbf{w}_0) = \epsilon_1(1_*\mathbf{v}_0 - 1^*\mathbf{v}_0) = 1_*\epsilon_0(\mathbf{v}_0) - 1^*\epsilon_0(\mathbf{v}_0) = 1_*(0) - 1^*(0) = (1) - (1) = 0.$$

For any $r \geq 1$, the sequence $\mathcal{F}_{2r+1} \xrightarrow{\partial} \mathcal{F}_{2r} \xrightarrow{\partial} \mathcal{F}_{2r-1}$ is semiexact, since

$$\begin{aligned} \partial_{\pi\mathbf{w}_r}\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r) &= \partial_{\pi\mathbf{w}_r}(1_*\mathbf{v}_r - 1^*\mathbf{v}_r) = 1_*\partial_{\pi\mathbf{v}_r}(\mathbf{v}_r) - 1^*\partial_{\pi\mathbf{v}_r}(\mathbf{v}_r) \\ &= 1_*\mathrm{T}(\mathbf{v}_r) - 1^*\mathrm{T}(\mathbf{v}_r) = \mathrm{ST}(\mathbf{v}_r) = 0. \end{aligned}$$

Finally, for any $r \geq 0$, the sequence $\mathcal{F}_{2r+2} \xrightarrow{\partial} \mathcal{F}_{2r+1} \xrightarrow{\partial} \mathcal{F}_{2r}$ is also semiexact, since

$$\begin{aligned} &\partial_{\pi\mathbf{v}_{r+1}}\partial_{\pi\mathbf{v}_{r+1}}(\mathbf{v}_{r+1}) = \partial_{\pi\mathbf{v}_{r+1}}\mathrm{T}(\mathbf{w}_r) = \\ &\quad = \partial_{\pi\mathbf{v}_{r+1}}\Big(\sum_{t=0}^{m+q-1} t^*(m+q-t-1)_*\mathbf{w}_r\Big) - \partial_{\pi\mathbf{v}_{r+1}}\Big(\sum_{s=0}^{m-1} s^*(m-s-1)_*\mathbf{w}_r\Big) \\ &\quad = \sum_{t=0}^{m+q-1} t^*(m+q-t-1)_*\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r) - \sum_{s=0}^{m-1} s^*(m-s-1)_*\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r) \\ &\quad = \sum_{t=0}^{m+q-1} t^*(m+q-t-1)_*\mathrm{S}(\mathbf{w}_r) - \sum_{s=0}^{m-1} s^*(m-s-1)_*\mathrm{S}(\mathbf{w}_r) = \mathrm{TS}(\mathbf{w}_r) = 0. \end{aligned}$$

$\square$

We are now ready to establish the main result of this section.

Theorem 2.1 *$\mathcal{F}_\bullet \stackrel{\epsilon}{\to} \mathbb{Z}$, defined as above, is a free resolution of the $\mathbb{D}C$-module $\mathbb{Z}$.*

Proof: We only have to prove its exactness or, equivalently, that, for any fixed $x \in C$, the augmented complex of abelian groups

$$\mathcal{F}_\bullet(x) \stackrel{\epsilon_x}{\to} \mathbb{Z}(x): \qquad \cdots \to \mathcal{F}_2(x) \stackrel{\partial_x}{\to} \mathcal{F}_1(x) \stackrel{\partial_x}{\to} \mathcal{F}_0(x) \stackrel{\epsilon_x}{\to} \mathbb{Z}(x) \to 0, \tag{2.8}$$

is exact. To do so, we are going to show that it has a contracting homotopy. That is, there are homomorphisms $\phi : \mathbb{Z}(x) \to \mathcal{F}_0(x)$ and $\Phi : \mathcal{F}_n(x) \to \mathcal{F}_{n+1}(x)$ for $n \geq 0$, such that $\epsilon_x \phi = id_{\mathbb{Z}(x)}$, $\phi\,\epsilon_x + \partial_x \Phi = id_{\mathcal{F}_0(x)}$, and for $n \geq 1$, $\Phi\partial_x + \partial_x\Phi = id_{\mathcal{F}_n(x)}$.

These homomorphisms ϕ and Φ are defined on the generators and extended linearly. Recall that $\mathbb{Z}(x)$ is the free abelian group on the generator (x) and, for each $r \geq 0$, $\mathcal{F}_{2r}(x)$ is the free abelian group on the set

$$\{(u, \mathbf{v}_r, v) \mid u, v \in C \text{ with } u \oplus r\cdot m \oplus v = x\},$$

and $\mathcal{F}_{2r+1}(x)$ is the free abelian group on the set

$$\{(u, \mathbf{w}_r, v) \mid u, v \in C \text{ with } u \oplus r\cdot m \oplus v \oplus 1 = x\}.$$

Then, we define

- $\phi : \mathbb{Z}(x) \to \mathcal{F}_0(x)$ to be the homomorphism determined by

$$\phi(x) = (0, \mathbf{v}_0, x).$$

and, for $r \geq 0$,

- $\Phi : \mathcal{F}_{2r+1}(x) \to \mathcal{F}_{2r+2}(x)$ to be the homomorphism determined by

$$\Phi(u, \mathbf{w}_r, v) = \begin{cases} 0 & \text{if} \quad u < m+q-1 \\ (0, \mathbf{v}_{r+1}, v) & \text{if} \quad u = m+q-1 \end{cases}$$

- $\Phi : \mathcal{F}_{2r}(x) \to \mathcal{F}_{2r+1}(x)$ to be the homomorphism determined by

$$\Phi(u, \mathbf{v}_r, v) = \sum_{t=0}^{u-1} (t, \mathbf{w}_r, v \oplus (u-t-1)).$$

So defined, we prove that these homomorphisms establish a contracting homotopy on the augmented chain complex (2.8) as follows.

$\underline{\epsilon_x\phi = id_{\mathbb{Z}(x)}}$, since

$$\epsilon_x\phi(x) = \epsilon_x(0, \mathbf{v}_0, x) = \epsilon_x(x^*\mathbf{v}_0) = x^*\epsilon_0(\mathbf{v}_0) = x^*(0) = (x).$$

$\underline{\partial_x\Phi + \phi\,\epsilon_x = id_{\mathcal{F}_0(x)}}$, since, for any $u, v \in C$ with $u \oplus v = x$,

$$\begin{aligned}
\partial_x\Phi(u,\mathbf{v}_0,v) &= \partial_x\Big(\sum_{t=0}^{u-1}(t,\mathbf{w}_0,v\oplus(u-t-1))\Big) = \partial_x\Big(\sum_{t=0}^{u-1}t_*(v\oplus(u-t-1))^*\mathbf{w}_0\Big)\\
&= \sum_{t=0}^{u-1}t_*(v\oplus(u-t-1))^*\partial_1(\mathbf{w}_0) = \sum_{t=0}^{u-1}t_*(v\oplus(u-t-1))^*(1_*\mathbf{v}_0 - 1^*\mathbf{v}_0)\\
&= \sum_{t=0}^{u-1}(t+1)_*\wp(u+v-t-1)^*\mathbf{v}_0 - \sum_{t=0}^{u-1}t_*\wp(u+v-t)^*\mathbf{v}_0\\
&= u_*v^*\mathbf{v}_0 - \wp(u+v)^*\mathbf{v}_0 = u_*v^*\mathbf{v}_0 - x^*\mathbf{v}_0 = (u,\mathbf{v}_0,v) - (0,\mathbf{v}_0,x),
\end{aligned}$$

$$\begin{aligned}
\phi\,\epsilon_x(u,\mathbf{v}_0,v) &= \phi\,\epsilon_x(u_*v^*\mathbf{v}_0) = \phi(u_*v^*\epsilon_0(\mathbf{v}_0)) = \phi(u_*v^*(0)) = \phi(u\oplus v) = \phi(x)\\
&= (0,\mathbf{v}_0,x),
\end{aligned}$$

and therefore $(\partial_x\Phi + \phi\,\epsilon_x)(u,\mathbf{v}_0,v) = (u,\mathbf{v}_0,v)$, for any generator $(u,\mathbf{v}_0,v)$ of $\mathcal{F}_0(x)$.

$\underline{\partial_x\Phi + \Phi\partial_x = id_{\mathcal{F}_{2r+1}}(x)}$, since for any generator $(u,\mathbf{w}_r,v)$ of $\mathcal{F}_{2r+1}(x)$ with $u < m+q-1$,

$$\begin{aligned}
(\partial_x\Phi + \Phi\partial_x)(u,\mathbf{w}_r,v) &= \Phi\partial_x(u,\mathbf{w}_r,v) = \Phi\partial_x(u_*v^*\mathbf{w}_r) = \Phi\big(u_*v^*\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r)\big)\\
&= \Phi\big(u_*v^*(1_*\mathbf{v}_r - 1^*\mathbf{v}_r)\big) = \Phi\big((u+1)_*v^*\mathbf{v}_r - u_*(v\oplus 1)^*\mathbf{v}_r)\big)\\
&= \Phi(u+1,\mathbf{v}_r,v) - \Phi(u,\mathbf{v}_r,\wp(v+1))\\
&= \sum_{t=0}^{u}(t,\mathbf{w}_r,\wp(u+v-t)) - \sum_{t=0}^{u-1}(t,\mathbf{w}_r,\wp(u+v-t))\\
&= (u,\mathbf{w}_r,\wp(v)) = (u,\mathbf{w}_r,v),
\end{aligned}$$

while for generators $(m+q-1,\mathbf{w}_r,v)$ of $\mathcal{F}_{2r+1}(x)$, we have

$$\begin{aligned}
&\partial_x\Phi(m+q-1,\mathbf{w}_r,v) = \partial_x(0,\mathbf{v}_{r+1},v) = \partial_x(v^*\mathbf{v}_{r+1}) = v^*\partial_{\pi\mathbf{v}_{r+1}}(\mathbf{v}_{r+1})\\
&= \sum_{t=0}^{m+q-1} v^*t_*(m+q-t-1)^*\mathbf{w}_r - \sum_{t=0}^{m-1} v^*t_*(m-t-1)^*\mathbf{w}_r\\
&= \sum_{t=0}^{m+q-1} (t,\mathbf{w}_r,\wp(v+m+q-t-1)) - \sum_{t=0}^{m-1}(t,\mathbf{w}_r,\wp(v+m-t-1))\\
&= \sum_{t=0}^{m+q-1} (t,\mathbf{w}_r,\wp(v+m+q-t-1)) - \Phi(m,\mathbf{v}_r,v),
\end{aligned}$$

$$\begin{aligned}
\Phi\partial_x(m+q-1,\mathbf{w}_r,v) &= \Phi\big((m+q-1)_* v^* \partial_{\pi\mathbf{w}_r}(\mathbf{w}_r)\big)\\
&= \Phi\big((m+q-1)_* v^*(1_*\mathbf{v}_r - 1^*\mathbf{v}_r)\big)\\
&= \Phi(m_* v^* \mathbf{v}_r) - \Phi\big((m+q-1)_*(1\oplus v)^*\mathbf{v}_r)\big)\\
&= \Phi(m,\mathbf{v}_r,v) - \Phi(m+q-1,\mathbf{v}_r,\wp(1+v))\\
&= \Phi(m,\mathbf{v}_r,v) - \sum_{t=0}^{m+q-2}(t,\mathbf{w}_r,\wp(v+m+q-t-1)),
\end{aligned}$$

whence $(\partial_x\Phi+\Phi\partial_x)(m+q-1,\mathbf{w}_r,v) = (m+q-1,\mathbf{w}_r,\wp(v)) = (m+q-1,\mathbf{w}_r,v)$.

And, finally, we prove that $\underline{\partial_x\Phi+\Phi\partial_x = id_{\mathcal{F}_{2r}}(x)}$. To do so, let $(u,\mathbf{v}_r,v)$ be any fixed generator of $\mathcal{F}_{2r}(x)$. Then, on the one hand,

$$\begin{aligned}
\partial_x\Phi(u,\mathbf{v}_r,v) &= \partial_x\Big(\sum_{t=0}^{u-1}\big(t,\mathbf{w}_r,(v\oplus(u-t-1))\big)\Big) = \partial_x\Big(\sum_{t=0}^{u-1} t_*(v\oplus(u-t-1))^*\mathbf{w}_r\Big)\\
&= \sum_{t=0}^{u-1} t_*(v\oplus(u-t-1))^*\partial_{\pi\mathbf{w}_r}(\mathbf{w}_r) = \sum_{t=0}^{u-1} t_*(v\oplus(u-t-1))^*(1_*\mathbf{v}_r - 1^*\mathbf{v}_r)\\
&= \sum_{t=0}^{u-1}(t+1)_*\wp(u+v-t-1)^*\mathbf{v}_r - \sum_{t=0}^{u-1} t_*\wp(u+v-t)^*\mathbf{v}_r\\
&= u_*v^*\mathbf{v}_r - \wp(u+v)^*\mathbf{v}_r = u_*v^*\mathbf{v}_r - (u\oplus v)^*\mathbf{v}_r = (u,\mathbf{v}_r,v) - (0,\mathbf{v}_r,u\oplus v),
\end{aligned}$$

while, on the other hand, we have

$$\begin{aligned}
&\Phi\partial_x(u,\mathbf{v}_r,v) = \Phi\big(u_*v^*\partial_{\pi\mathbf{v}_r}(\mathbf{v}_r)\big)\\
&= \Phi\Big(u_*v^*\sum_{t=0}^{m+q-1} t_*(m+q-t-1)^*\mathbf{w}_{r-1} - u_*v^*\sum_{t=0}^{m-1} t_*(m-t-1)^*\mathbf{w}_{r-1}\Big)\\
&= \sum_{t=0}^{m+q-1}\Phi\big(u\oplus t,\mathbf{w}_{r-1},v\oplus(m+q-t-1)\big) - \sum_{t=0}^{m-1}\Phi\big(u\oplus t,\mathbf{w}_{r-1},v\oplus(m-t-1)\big).
\end{aligned}$$

Now, if $l\geq 0$ is integer such that $lq<u\leq(l+1)q$, then it is easy to see that the various t, with $0\leq t\leq m+q-1$ (resp. $0\leq t\leq m-1$), such that $u\oplus t = m+q-1$, that is, $\wp(u+t) = u+q-1$, are just those of the form $t = m+(k+1)q-1-u$ for $0\leq k\leq l$ (resp. $0\leq k\leq l-1$). Hence,

$$\begin{aligned}
\Phi\partial_x(u,\mathbf{v}_r,v) &= \sum_{k=0}^{l}(0,\mathbf{v}_r,v\oplus(u-kq)) - \sum_{k=0}^{l-1}(0,\mathbf{v}_r,v\oplus(u-(k+1)q))\\
&= (0,\mathbf{v}_r,v\oplus u),
\end{aligned}$$

and thus we get $(\partial_x\Phi+\Phi\partial_x)(u,\mathbf{v}_r,v) = (u,\mathbf{v}_r,v)$. This makes complete the proof. □

2.4 The (co)homology groups of C

By Theorem 2.1, the (co)homology groups of the cyclic monoid C of index m and period q can be computed by means of the complex $\mathcal{F}_\bullet$ in (2.7) as

$$H^n_{\mathrm{L}}(C,\mathcal{A}) = H^n_{\mathrm{L}}\mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_\bullet,\mathcal{A}),$$
$$H_n^{\mathrm{L}}(C,\mathcal{B}) = H_n^{\mathrm{L}}(\mathcal{B}\otimes_{\mathbb{D}C}\mathcal{F}_\bullet),$$

for $\mathcal{A}$ any left $\mathbb{D}C$-module and $\mathcal{B}$ any right $\mathbb{D}C$-module.

Now, for each $r\geq 0$, the $\mathbb{D}C$-module $\mathcal{F}_{2r}$ is free on the unitary set $\{\mathbf{v}_r\}$ with $\pi\mathbf{v}_r = r\cdot m$, while $\mathcal{F}_{2r+1}$ is free on the unitary set $\{\mathbf{w}_r\}$ with $\pi\mathbf{w}_r = r\cdot m\oplus 1$. Then, by Proposition 2.1, there are natural isomorphisms

$$\mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_{2r},\mathcal{A})\cong\mathcal{A}(r\cdot m),\quad \mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_{2r+1},\mathcal{A})\cong\mathcal{A}(r\cdot m\oplus 1),$$

respectively given by $f\mapsto f_{\pi\mathbf{v}_r}(\mathbf{v}_r)$ and $g\mapsto g_{\pi\mathbf{w}_r}(\mathbf{w}_r)$, which make the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_{2r},\mathcal{A}) \arrow[r, "\partial^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_{2r+1},\mathcal{A}) \arrow[r, "\partial^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{D}C}(\mathcal{F}_{2r+2},\mathcal{A}) \arrow[d, "\cong"'] \\
\mathcal{A}(r\cdot m) \arrow[r, "\mathrm{S}"] & \mathcal{A}(r\cdot m\oplus 1) \arrow[r, "\mathrm{T}"] & \mathcal{A}((r+1)\cdot m),
\end{tikzcd}$$

commutative, where S and T are the homomorphisms (2.5) and (2.4) in Definition 2.1. Therefore, recalling the notations in (2.6), we obtain:

Theorem 2.2 *Let $\mathcal{A}$ be any left $\mathbb{D}C$-module. Then,*

$$H^0_{\mathrm{L}}(C,\mathcal{A})\cong\mathcal{A}^{\mathrm{S}}(0),$$

and, for any $r\geq 0$,

$$H^{2r+1}_{\mathrm{L}}(C,\mathcal{A})\cong\frac{\mathcal{A}^{\mathrm{T}}(r\cdot m\oplus 1)}{\mathcal{A}_{\mathrm{S}}(r\cdot m)},\qquad H^{2r+2}_{\mathrm{L}}(C,\mathcal{A})\cong\frac{\mathcal{A}^{\mathrm{S}}((r+1)\cdot m)}{\mathcal{A}_{\mathrm{T}}(r\cdot m\oplus 1)}. \tag{2.9}$$

For instance, let us consider the $\mathbb{D}C$-module $\mathbb{Z}$ for coefficients. In this case, for any $x\in C$, $x\geq 1$, the trace map $\mathrm{T}:\mathbb{Z}(x)\to\mathbb{Z}(m\oplus(x-1))$ is the homomorphism of multiplication by q, since

$$\begin{aligned}
\mathrm{T}(x) &= \sum_{i=0}^{m+q-1} t^*(m+q-t-1)_*(x) - \sum_{i=0}^{m-1} t^*(m-t-1)_*(x)\\
&= \sum_{i=0}^{m+q-1}(m\oplus(x-1)) - \sum_{i=0}^{m-1}(m\oplus(x-1)) = q(m\oplus(x-1)),
\end{aligned}$$

while, for all x, $\mathrm{S}:\mathbb{Z}(x)\to\mathbb{Z}(x\oplus 1)$ is the zero homomorphism, since

$$\mathrm{S}(x) = 1_*(x) - 1^*(x) = (1\oplus x)-(x\oplus 1) = 0.$$

Therefore, $H^0_{\mathrm{L}}(C,\mathbb{Z}) \cong \mathbb{Z}$ and, for any $r \geq 0$,

$$H^{2r+1}_{\mathrm{L}}(C,\mathbb{Z}) \cong 0, \quad H^{2r+2}_{\mathrm{L}}(C,\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}.$$

We should note that the isomorphisms (2.9) in the particular case when $r = 0$, that is,

$$H^1_{\mathrm{L}}(C,\mathcal{A}) \cong \frac{\mathcal{A}^{\mathrm{T}}(1)}{\mathcal{A}_{\mathrm{S}}(0)}, \qquad H^2_{\mathrm{L}}(C,\mathcal{A}) \cong \frac{\mathcal{A}^{\mathrm{S}}(m)}{\mathcal{A}_{\mathrm{T}}(1)},$$

were proven by Leech in [53, Chapter II, 7.20, 7.21].

As for homology, if $\mathcal{B}$ is any right $\mathbb{D}C$-module, by Proposition 2.2, there are natural isomorphisms

$$\mathcal{B} \otimes_{\mathbb{D}C} \mathcal{F}_{2r} \cong \mathcal{B}(r \cdot m), \quad \mathcal{B} \otimes_{\mathbb{D}C} \mathcal{F}_{2r+1} \cong \mathcal{B}(r \cdot m \oplus 1),$$

respectively given on generators by $a' \otimes (u, \mathbf{v}_r, v) \mapsto u^* v_* a'$ and $a' \otimes (u, \mathbf{w}_r, v) \mapsto u^* v_* a'$, which make the diagram

$$\begin{tikzcd}
\mathcal{B} \otimes_{\mathbb{D}C} \mathcal{F}_{2r+2} \arrow[r, "id\otimes\partial"] \arrow[d, "\cong"'] & \mathcal{B} \otimes_{\mathbb{D}C} \mathcal{F}_{2r+1} \arrow[r, "id\otimes\partial"] \arrow[d, "\cong"'] & \mathcal{B} \otimes_{\mathbb{D}C} \mathcal{F}_{2r} \arrow[d, "\cong"'] \\
\mathcal{B}((r+1)\cdot m) \arrow[r, "\mathrm{T}"] & \mathcal{B}(r \cdot m \oplus 1) \arrow[r, "\mathrm{S}"] & \mathcal{B}(r \cdot m),
\end{tikzcd}$$

commutative, where, for each $x \in C$, $x \geq 1$, the homomorphism $\mathrm{T} : \mathcal{B}(m \oplus (x-1)) \longrightarrow \mathcal{B}(x)$ is the *'trace map'*, defined by

$$\mathrm{T}(b) = \sum_{t=0}^{m+q-1} t^*(m+q-t-1)_* b - \sum_{t=0}^{m-1} t^*(m-t-1)_* b,$$

and, for any $x \in C$, $\mathrm{S} : \mathcal{B}(x \oplus 1) \to \mathcal{B}(x)$ is the homomorphism defined by $\mathrm{S}(b) = 1_* b - 1^* b$.

Then, introducing the subgroups (parallel to those in (2.6))

$$\begin{aligned}
&\mathcal{B}^{\mathrm{T}}(x) = \{b \in \mathcal{B}(m \oplus (x-1)) \mid \mathrm{T}(b) = 0\}, && \mathcal{B}_{\mathrm{T}}(x) = \{\mathrm{T}(b) \mid b \in \mathcal{B}(m \oplus (x-1)\}, \\
&\mathcal{B}^{\mathrm{S}}(x) = \{b \in \mathcal{B}(x \oplus 1) \mid \mathrm{S}(b) = 0\}, && \mathcal{B}_{\mathrm{S}}(x) = \{\mathrm{S}(b) \mid b \in \mathcal{B}(x \oplus 1)\},
\end{aligned}$$

we have the following.

Theorem 2.3 *Let $\mathcal{B}$ be any right $\mathbb{D}C$-module. Then,*

$$H_0^{\mathrm{L}}(C,\mathcal{B}) \cong \frac{\mathcal{B}(0)}{\mathcal{B}_{\mathrm{S}}(0)},$$

and, for any $r \geq 0$,

$$H^{\mathrm{L}}_{2r+1}(C,\mathcal{B}) \cong \frac{\mathcal{B}^{\mathrm{S}}(r \cdot m)}{\mathcal{B}_{\mathrm{T}}(r \cdot m \oplus 1)}, \qquad H^{\mathrm{L}}_{2r+2}(C,\mathcal{B}) \cong \frac{\mathcal{B}^{\mathrm{T}}(r \cdot m \oplus 1)}{\mathcal{B}_{\mathrm{S}}((r+1)\cdot m)}. \tag{2.10}$$

Thus, for example,

$$H_1^{\mathrm{L}}(C,\mathcal{B}) \cong \frac{\mathcal{B}^{\mathrm{S}}(0)}{\mathcal{B}_{\mathrm{T}}(1)}, \qquad H_2^{\mathrm{L}}(C,\mathcal{B}) \cong \frac{\mathcal{B}^{\mathrm{T}}(1)}{\mathcal{B}_{\mathrm{S}}(m)}.$$

It is well-known that the (co)homology of a finite cyclic group $C_q = C_{0,q}$ is periodic with a period of 2. Indeed, when $m = 0$, isomorphisms (2.9) and (2.10) state that, for any left $\mathbb{D}C_q$-module $\mathcal{A}$ and right $\mathbb{D}C_q$-module $\mathcal{B}$, and any integer $r \geq 0$, there are isomorphisms

$$H^{2r+1}_{\mathrm{L}}(C_q,\mathcal{A}) \cong \mathcal{A}^{\mathrm{T}}(1)/\mathcal{A}_{\mathrm{S}}(0), \quad H^{2r+2}_{\mathrm{L}}(C_q,\mathcal{A}) \cong \mathcal{A}^{\mathrm{S}}(0)/\mathcal{A}_{\mathrm{T}}(1),$$
$$H_{2r+1}^{\mathrm{L}}(C_q,\mathcal{B}) \cong \mathcal{B}^{\mathrm{S}}(0)/\mathcal{B}_{\mathrm{T}}(1), \quad H_{2r+2}^{\mathrm{L}}(C_q,\mathcal{B}) \cong \mathcal{B}^{\mathrm{T}}(1)/\mathcal{B}_{\mathrm{S}}(0),$$

whence the periodicity of the (co)homology of C_q follows trivially. The following proposition states that, from dimension 3 onwards, the (co)homology of any finite cyclic monoid C is periodic with a period of $2q/(m,q)$, where (m,q) denotes the greatest common divisor of the index and the period. More precisely,

Proposition 2.4 *Let $p, n \geq 3$ be integers such that $p \equiv n \bmod 2q/(m,q)$. Then, for any left $\mathbb{D}C$-module $\mathcal{A}$ and right $\mathbb{D}C$-module $\mathcal{B}$, there are isomorphisms*

$$H^p_{\mathrm{L}}(C,\mathcal{A}) \cong H^n_{\mathrm{L}}(C,\mathcal{A}), \quad H_p^{\mathrm{L}}(C,\mathcal{B}) \cong H_n^{\mathrm{L}}(C,\mathcal{B}). \tag{2.11}$$

If $m = 1$, then there are also isomorphisms

$$H^n_{\mathrm{L}}(C_{1,q},\mathcal{A}) \cong H^2_{\mathrm{L}}(C_{1,q},\mathcal{A}), \quad H_n^{\mathrm{L}}(C_{1,q},\mathcal{B}) \cong H_2^{\mathrm{L}}(C_{1,q},\mathcal{B}),$$

for any $n \geq 2$ such that $n \equiv 2 \bmod 2q$.

Proof: Let $p, n \geq 3$ be integers such that $p \equiv n \bmod 2q/(m,q)$. Then, $p \equiv n \bmod 2$ and we can write $p = 2r+1$ and $n = 2s+1$ or $p = 2r+2$ and $n = 2s+2$ for some integers $r, s \geq 1$ satisfying $r \equiv s \bmod q/(m,q)$ or, equivalently, satisfying that $rm \equiv sm \bmod q$. Hence, $r \cdot m = s \cdot m$, $r \cdot m \oplus 1 = s \cdot m \oplus 1$, and $(r \cdot m) \oplus m = (s \cdot m) \oplus m$, whence the isomorphisms in (2.11) follow from those in (2.9) and (2.10).

Suppose now that the cyclic monoid is of index one[2], and let $r \geq 0$ be such that $r \equiv 0 \bmod q$. Then $r \oplus 1 = 1$, and therefore

$$H^{2r+2}_{\mathrm{L}}(C_{1,q},\mathcal{A}) \cong \frac{\mathcal{A}^{\mathrm{S}}(r\oplus 1)}{\mathcal{A}_{\mathrm{T}}(r\oplus 1)} = \frac{\mathcal{A}^{\mathrm{S}}(1)}{\mathcal{A}_{\mathrm{T}}(1)} \cong H^2_{\mathrm{L}}(C_{1,q},\mathcal{A}),$$

$$H_{2r+2}^{\mathrm{L}}(C_{1,q},\mathcal{B}) \cong \frac{\mathcal{B}^{\mathrm{T}}(r\oplus 1)}{\mathcal{B}_{\mathrm{S}}(r\oplus 1)} = \frac{\mathcal{B}^{\mathrm{T}}(1)}{\mathcal{B}_{\mathrm{S}}(1)} \cong H_2^{\mathrm{L}}(C_{1,q},\mathcal{B}).$$

□

Our results in Theorems 2.2 and 2.3 specify in a simpler form for (co)homology with coefficients in C-modules (see 2.1.6).

[2] A cyclic monoid of index $m = 1$ and period q is the same thing that a cyclic group of order q with a identity adjoined.

Corollary 2.1 *(i) Let A be any left C-module. Then,*

$$H^0_{\mathrm{L}}(C,A) \cong A^{\mathrm{S}},$$

and, for any $r \geq 0$,

$$H^{2r+1}_{\mathrm{L}}(C,A) \cong A^{\mathrm{T}}/A_{\mathrm{S}}, \qquad H^{2r+2}_{\mathrm{L}}(C,A) \cong A^{\mathrm{S}}/A_{\mathrm{T}},$$

where $\mathrm{S},\mathrm{T} : A \to A$ *are the homomorphisms given by*

$$\mathrm{S}(a) = 1_*a - a, \quad \mathrm{T}(a) = m_* \sum_{t=0}^{q-1} t_*a,$$

$A^{\mathrm{T}} = \mathrm{KerT}$, $A_{\mathrm{T}} = \mathrm{ImT}$, $A^{\mathrm{S}} = \mathrm{KerS}$, *and* $A_S = \mathrm{ImS}$.

(ii) Let B be any right C-module. Then,

$$H_0^{\mathrm{L}}(C,B) \cong B/B_{\mathrm{S}},$$

and, for any $r \geq 0$,

$$H^{\mathrm{L}}_{2r+1}(C,B) \cong B^{\mathrm{S}}/B_{\mathrm{T}}, \qquad H^{\mathrm{L}}_{2r+2}(C,B) \cong B^{\mathrm{T}}/B_{\mathrm{S}},$$

where $\mathrm{S},\mathrm{T} : B \to B$ *are the homomorphisms given by*

$$\mathrm{S}(b) = b - 1^*b, \quad \mathrm{T}(b) = m^* \sum_{t=0}^{q-1} t^*b,$$

$B^{\mathrm{T}} = \mathrm{KerT}$, $B_{\mathrm{T}} = \mathrm{ImT}$, $B^{\mathrm{S}} = \mathrm{KerS}$, *and* $B_S = \mathrm{ImS}$.

The isomorphism $H^2_{\mathrm{L}}(C,A) \cong A^{\mathrm{S}}/A_{\mathrm{T}}$ is already known, see [44, Proposition 4.1] for a recent proof. As an immediate consequence of the above corollary, we see that the Eilenberg-Mac Lane (co)homology of any C is periodic with a period of 2, that is,

Corollary 2.2 *Let A be a left C-module and let B be a right C-module. For any $r \geq 0$, there are natural isomorphisms*

$$H^{2r+1}_{\mathrm{L}}(C,A) \cong H^1_{\mathrm{L}}(C,A), \quad H^{2r+2}_{\mathrm{L}}(C,A) \cong H^2_{\mathrm{L}}(C,A),$$
$$H^{\mathrm{L}}_{2r+1}(C,B) \cong H^{\mathrm{L}}_1(C,B), \quad H^{\mathrm{L}}_{2r+2}(C,B) \cong H^{\mathrm{L}}_2(C,B).$$

If A is any abelian group, regarded as a left or right C-module on which the monoid acts trivially, then, for any $a \in A$, $\mathrm{S}(a) = a - a = 0$, that is, $\mathrm{S} = 0 : A \to A$ is the zero homomorphism, while $\mathrm{T}(a) = \sum_{i=0}^{q-1} a = q\,a$, that is, the trace map $\mathrm{T} = q : A \to A$ is multiplication by q. Therefore,

$$H^0_{\mathrm{L}}(C,A) \cong A \cong H_0^{\mathrm{L}}(C,A),$$

and, for all $r \geq 0$,

$$\begin{array}{rcccl} H^{2r+1}_{\mathrm{L}}(C,A) & \cong & \mathrm{Ker}(q:A\to A) & \cong & H^{\mathrm{L}}_{2r+2}(C,A),\\ H^{2r+2}_{\mathrm{L}}(C,A) & \cong & \mathrm{Coker}(q:A\to A) & \cong & H^{\mathrm{L}}_{2r+1}(C,A). \end{array}$$

Observe that the (co)homology groups of the finite cyclic monoid C with coefficients in the abelian group do not depend on the index m. Indeed, they agree with those of the cyclic group C_q. Actually, this fact is not surprising because it is well-known that the (co)homology groups of any commutative monoid with trivial coefficients coincide with those of its group reflection (i.e., its image under the left adjoint of the forgetful functor from groups to monoids) [33, Proposition 4.4], and the group reflection of C is just C_q.

To conclude, we particularize to the case when the coefficients are *symmetric* $\mathbb{D}C$-modules. Recall that, if M is any *commutative* monoid, a left $\mathbb{D}M$-module $\mathcal{A}$ is called symmetric if, for any $x,y\in M$, $y_*=y^*:\mathcal{A}(x)\to\mathcal{A}(xy)$. Symmetric $\mathbb{D}M$-modules are equivalent to abelian group objects in the comma category of commutative monoids over M [43, Chap. XXII, 2], and therefore they are the coefficients for the cotriple cohomology theory [2] of commutative monoids (see Chapter 3). See also the recent approach to the (co)homology of commutative monoids by Kurdiani and Pirashvili in [52]. Symmetric right $\mathbb{D}M$-modules are defined similarly, and Theorems 2.2 and 2.3 give the following.

Corollary 2.3 (i) *Let $\mathcal{A}$ be any symmetric left $\mathbb{D}C$-module. Then,*

$$H^0_{\mathrm{L}}(C,\mathcal{A})\cong\mathcal{A}(0),$$

and, for any $r\geq 0$,

$$H^{2r+1}_{\mathrm{L}}(C,\mathcal{A})\cong\mathcal{A}^{\mathrm{T}}(r\cdot m\oplus 1),\qquad H^{2r+2}_{\mathrm{L}}(C,A)\cong\frac{\mathcal{A}((r+1)\cdot m)}{\mathcal{A}_{\mathrm{T}}(r\cdot m\oplus 1)},$$

where, for any $x\in C$, $x\geq 1$, $\mathrm{T}:\mathcal{A}(x)\to\mathcal{A}(m\oplus(x-1))$ is the trace map given by

$$\mathrm{T}(a)=(m+q)\big((m+q-1)_*a\big)-m\big((m-1)_*a\big),$$

$A^{\mathrm{T}}(x)=\mathrm{KerT}$, *and* $A_{\mathrm{T}}(x)=\mathrm{ImT}$.

(ii) *Let $\mathcal{B}$ be any symmetric right $\mathbb{D}C$-module. Then,*

$$H_0^{\mathrm{L}}(C,\mathcal{B})\cong\mathcal{B}(0),$$

and, for any $r\geq 0$,

$$H^{\mathrm{L}}_{2r+1}(C,\mathcal{B})\cong\frac{\mathcal{B}(r\cdot m)}{\mathcal{B}_{\mathrm{T}}(r\cdot m\oplus 1)},\qquad H^{2r+2}_{\mathrm{L}}(C,\mathcal{B})\cong\mathcal{B}^{\mathrm{T}}(r\cdot m\oplus 1),$$

where, for any $x\in C$, $x\geq 1$, $\mathrm{T}:\mathcal{B}(m\oplus(x-1))\to\mathcal{B}(x)$ is the trace map given by

$$\mathrm{T}(b)=(m+q)\big((m+q-1)_*b\big)-m\big((m-1)_*b\big),$$

$B^{\mathrm{T}}(x)=\mathrm{KerT}$, *and* $B_{\mathrm{T}}(x)=\mathrm{ImT}$.

Chapter 3

On the third cohomology group of commutative monoids

The category of commutative monoids is tripleable (monadic) over the category of sets [58], and so it is natural to specialize Barr-Beck cotriple cohomology [2] to define a cohomology theory for commutative monoids. This was done in the 1990s by Grillet, to whose papers [40, 41, 42] and book [43] we refer the readers interested in cohomology theory for commutative monoids. Although in 3.1.1 we review the basic facts about the resulting Grillet's cohomology, let us briefly recall here that, for each commutative monoid M, its cohomology groups in this theory, $H^n_{\mathrm{G}}(M,\mathcal{A})$, take coefficients in $\mathbb{H}M$-modules, that is, abelian group valued functors $\mathcal{A}$ on the category $\mathbb{H}M$. This category $\mathbb{H}M$ has as objects the elements of M and as morphisms pairs $(a,b) : a \to ab$, $a, b \in M$. Since these cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ can be computed, at less in low dimensions, by means of *symmetric cochains*, they are usually referred as the *symmetric cohomology groups* of the commutative monoid M.

Recall that for an arbitrary monoid M, that is, non necessarily commutative, and $\mathcal{A}$ any $\mathbb{D}M$-module there are defined Leech cohomology groups $H^n_{\mathrm{L}}(M,\mathcal{A})$ (see 1.3.1). When the monoid M is commutative, and $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$ is any $\mathbb{H}M$-module, then both cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ and $H^n_{\mathrm{L}}(M,\mathcal{A})$ are defined, where the coefficients for the Leech cohomology are here obtained by composing $\mathcal{A}$ with canonical functor $\mathbb{D}M \to \mathbb{H}M$, $(a,b,c) \mapsto (b,ac)$. The $\mathbb{D}M$-modules so obtained are precisely the symmetric $\mathbb{D}M$-modules introduced at the end of the previous chapter. Although in dimension one we have that $H^1_{\mathrm{G}}(M,\mathcal{A}) = H^1_{\mathrm{L}}(M,\mathcal{A})$, in higher dimensions the cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ and $H^n_{\mathrm{L}}(M,\mathcal{A})$ are, however, different. Indeed, one easily argues that Leech cohomology groups do not take properly account of the commutativity of the monoid, in contrast to what happens with Grillet ones. Thus, for example, while $H^2_{\mathrm{L}}(M,\mathcal{A})$ classifies *all* group coextensions of M by $\mathcal{A}$ [53, 2.4.9], [74, Theorem 2], the symmetric 2-cohomology group $H^2_{\mathrm{G}}(M,\mathcal{A})$ classifies *commutative* group coextensions of M by $\mathcal{A}$ [43, Chapter V.4].

In Section 1.3, we gave a natural interpretation for Leech 3-cohomology classes in terms of *monoidal abelian groupoids.* More concretely, in Theorem 1.4, it was stated that monoidal equivalence classes of monoidal abelian groupoids are in one-to-one correspondence with isomorphism classes of triples $(M,\mathcal{A},k)$, consisting of a (non necessarily commutative) monoid M, a $\mathbb{D}M$-module $\mathcal{A}$, and a Leech 3-cohomology class $k\in H^3_{\mathrm{L}}(M,\mathcal{A})$.

In this chapter, our goal is to state and prove a similar interpretation for Grillet symmetric 3-cohomology classes, now in terms of *strictly symmetric* (or *strictly commutative*) monoidal abelian groupoids [25, 56, 66], that is, monoidal abelian groupoids, but now endowed with coherent and natural isomorphisms $\boldsymbol{c}_{x,y}: x\otimes y\cong y\otimes x$, satisfying the symmetry and strictness conditions $\boldsymbol{c}_{y,x}\,\boldsymbol{c}_{x,y}=id_{x\otimes y}$ and $\boldsymbol{c}_{x,x}=id_{x\otimes x}$. Our result here can be summarized as follows (see Theorem 3.1):

- *Each symmetric 3-cocycle $h\in Z^3_{\mathrm{G}}(M,\mathcal{A})$, of a commutative monoid M with coefficients in an $\mathbb{H}M$-module $\mathcal{A}$, gives rise to a strictly symmetric monoidal abelian groupoid*
$$\Sigma(M,\mathcal{A},h).$$
- *For any strictly symmetric monoidal abelian groupoid $\mathcal{M}$, there exist a commutative monoid M, an $\mathbb{H}M$-module, a symmetric 3-cocycle $h\in Z^3_{\mathrm{G}}(M,\mathcal{A})$, and a symmetric monoidal equivalence*
$$\Sigma(M,\mathcal{A},h)\simeq\mathcal{M}.$$
- *For any two symmetric 3-cocycles $h\in Z^3_{\mathrm{G}}(M,\mathcal{A})$ and $h'\in Z^3_{\mathrm{G}}(M',\mathcal{A}')$, there is a symmetric monoidal equivalence*
$$\Sigma(M,\mathcal{A},h)\simeq\Sigma(M',\mathcal{A}',h')$$
if and and only if there exist an isomorphism of monoids $i: M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong i^*\mathcal{A}'$, such that the equality of cohomology classes below holds.
$$[h]=\psi_*^{-1}i^*[h']\in H^3_{\mathrm{G}}(M,\mathcal{A})$$

Thus, triples $(M,\mathcal{A},k)$, with M a commutative monoid, $\mathcal{A}$ an $\mathbb{H}M$-module, and $k\in H^3_{\mathrm{G}}(M,\mathcal{A})$ a symmetric 3-cohomology class, provide complete invariants for the classification of strictly symmetric monoidal abelian groupoids, where two of them connected by a symmetric monoidal equivalence are considered the same.

Our result particularizes to strictly commutative Picard categories by giving, as a corollary, Deligne's well-known classification for them [25], also proved independently by Fröhlich and Wall in [36] and by Sinh in [68, 69]. Indeed, in the very special case where $M=G$ is an abelian group, any abelian group valued functor on $\mathbb{H}G$ is naturally equivalent to the constant functor given by an abelian group A, and the symmetric 3-cohomology group $H^3_{\mathrm{G}}(G,A)$ vanishes, whence Deligne's result follows: *Strictly commutative Picard categories are classified by pairs (G,A) of abelian groups.*

The organization of the chapter is simple. After this introduction, it contains two sections. The first is dedicated to stating a minimum of necessary concepts and terminology, by reviewing some facts concerning Grillet cohomology of commutative monoids (Subsection 3.1.1) and symmetric monoidal groupoids (Subsection 3.1.2). The second section comprises our classification theorem for strictly symmetric monoidal abelian groupoids by means of symmetric 3-cohomology classes.

3.1 Preliminaries

The aim of this section is to review some necessary aspects and results about cohomology of commutative monoids and symmetric monoidal categories that will be used throughout the chapter. For the cohomology theory of commutative monoids we mainly refer the reader to Grillet [43, Chapters V, XII, XIII, and XIV], and for symmetric monoidal (= tensor) categories to Mac Lane [56, 58] and Saavedra [66].

3.1.1 Grillet cohomology of commutative monoids: Symmetric cocycles

Like most of cohomology theories in Algebra, the cohomology of commutative monoids is a particular instance of the cotriple cohomology by Barr and Beck [2]. Briefly, let us recall that the category of commutative monoids is tripleable over the category of sets and, for any given commutative monoid M, the resulting cotriple $(\mathbb{G},\varepsilon,\delta)$ in the comma category $\mathbf{CMnd}{\downarrow_M}$, of commutative monoids over M, is as follows. For each commutative monoid $X \overset{p}{\to} M$ over M,

$$\mathbb{G}(X \overset{p}{\to} M) = \mathbb{N}[X] \overset{\overline{p}}{\to} M,$$

where $\mathbb{N}[X]$ is the free commutative monoid on the underlying set X, and $\overline{p}$ is the homomorphism such that $\overline{p}[x] = p(x)$ for any $x \in X$. The counit $\delta : \mathbb{G} \to id$ sends $X \to M$ to the homomorphism in the comma category $\delta : \mathbb{N}[X] \to X$ such that $\delta[x] = x$, and the comultiplication $\varepsilon : \mathbb{G} \to \mathbb{G}^2$ carries each $X \to M$ to the homomorphism $\mathbb{N}[X] \to \mathbb{N}[\mathbb{N}[X]]$ such that $\varepsilon[x] = [[x]]$, for $x \in X$. This cotriple produces a simplicial object $\mathbb{G}_\bullet$ in the category of endofunctors on $\mathbf{CMnd}{\downarrow_M}$, which is defined by $\mathbb{G}_n = \mathbb{G}^{n+1}$, with face and degeneracy operators $d_i = \mathbb{G}^{n-i}\delta\mathbb{G}^i : \mathbb{G}_n \to \mathbb{G}_{n-1}$ and $s_i = \mathbb{G}^{n-i}\varepsilon\mathbb{G}^i : \mathbb{G}_n \to \mathbb{G}_{n+1}$, $0 \le i \le n$. Then, for any abelian group object $\mathbf{A}$ in $\mathbf{CMnd}{\downarrow_M}$, one obtains a cosimplicial abelian group $\mathrm{Hom}(\mathbb{G}_\bullet(1_M),\mathbf{A})$, whose associated cochain complex obtained by taking alternating sums of the coface operators $\big(\partial^n = \sum\limits_{i=0}^{n+1}(-1)^i d_i^*\big)$

$$0 \to \mathrm{Hom}(\mathbb{G}(1_M),\mathbf{A}) \overset{\partial^0}{\to} \mathrm{Hom}(\mathbb{G}^2(1_M),\mathbf{A}) \overset{\partial^1}{\to} \mathrm{Hom}(\mathbb{G}^3(1_M),\mathbf{A}) \overset{\partial^2}{\to} \cdots$$

provides the *cotriple cohomology groups of the commutative monoid M with coefficients in* $\mathbf{A}$ by

$$H^n_{\mathbb{G}}(M,\mathbf{A}) = H^n(\mathrm{Hom}(\mathbb{G}_\bullet(1_M),\mathbf{A})).$$

In [40], Grillet observes that, for any given commutative monoid M, the category of abelian group objects in $\mathbf{CMnd}\downarrow_M$, is equivalent to the category of $\mathbb{H}M$-modules, that is, abelian group valued functors

$$\mathcal{A} : \mathbb{H}M \to \mathbf{Ab},$$

where $\mathbb{H}M$ is the category with object set M and arrow set $M\times M$, where $(a,b) : a \to ab$. Composition is given by $(ab,c)(a,b) = (a,bc)$, and the identity of an object a is (a,e), with e the unit in M. An $\mathbb{H}M$-module, thus consists of abelian groups $\mathcal{A}(a)$, $a\in M$, and homomorphisms $b_* : \mathcal{A}(a) \to \mathcal{A}(ab)$, $a,b\in M$, such that, for any $a,b,c\in M$, $b_*c_* = (bc)_* : \mathcal{A}(a)\to\mathcal{A}(abc)$ and, for any $a\in M$, $e_* = id_{\mathcal{A}(a)}$. We refers to [43, Chap. XXII, 2] for details but, briefly, let us say that the abelian group object defined by an $\mathbb{H}M$-module $\mathcal{A}$ can be written as

$$E(M,\mathcal{A}) \to M,$$

where the *crossed product* commutative monoid $E(M,\mathcal{A})$ is the set $\bigcup_{a\in M}\mathcal{A}(a)\times\{a\}$ of all ordered pairs (u_a,a) with $a\in M$ and $u_a\in\mathcal{A}(a)$, with multiplication given by

$$(u_a,a)(u_b,b) = (a_*u_b + b_*u_a, ab).$$

The monoid homomorphism $E(M,\mathcal{A}) \to M$ is the obvious projection $(u_a,a)\mapsto a$, and the internal group operation

$$E(M,\mathcal{A})\times_M E(M,\mathcal{A}) \xrightarrow{+} E(M,\mathcal{A})$$

is defined by $(u_a,a)+(v_a,a) = (u_a+v_a,a)$.

Furthermore, in [40, 41, 42], Grillet shows an algebraically more lucid description of the low dimensional cohomology groups

$$H^n_{\mathrm{G}}(M,\mathcal{A}) := H^{n-1}_{\mathbb{G}}(M, E(M,\mathcal{A})\to M)$$

by means of a specific manageable complex (see also the recent work [52])

$$C_{\mathrm{G}}(M,\mathcal{A}):\ 0\to C^1_{\mathrm{G}}(M,\mathcal{A}) \xrightarrow{\partial} C^2_{\mathrm{G}}(M,\mathcal{A}) \xrightarrow{\partial} C^3_{\mathrm{G}}(M,\mathcal{A}) \xrightarrow{\partial} C^4_{\mathrm{G}}(M,\mathcal{A}), \qquad (3.1)$$

called the complex of (normalized on $e\in M$) *symmetric cochains* on M with values in $\mathcal{A}$, which is defined as follows (below, $\bigcup_{a\in M}\mathcal{A}(a)$ is the disjoint union set of the groups $\mathcal{A}(a)$):

• A *symmetric* 1*-cochain* is a function $f : M\to\bigcup_{a\in M}\mathcal{A}(a)$, satisfying $f(a)\in\mathcal{A}(a)$ and $f(e)=0$.

• A *symmetric* 2*-cochain* is a function $g : M^2 \to \bigcup_{a\in M} \mathcal{A}(a)$, verifying that $g(a,b) \in \mathcal{A}(ab)$,

$$g(a,b) = g(b,a),$$

and $g(a,e) = 0$.

• A *symmetric* 3*-cochain* is a function $h : M^3 \to \bigcup_{a\in M} \mathcal{A}(a)$, verifying that $h(a,b,c) \in \mathcal{A}(abc)$,

$$h(c,b,a) + h(a,b,c) = 0, \quad h(a,b,c) + h(b,c,a) + h(c,a,b) = 0 \tag{3.2}$$

and $h(a,b,e) = 0$.

• A *symmetric* 4*-cochain* is a function $t : M^4 \to \bigcup_{a\in M} \mathcal{A}(a)$, verifying that $t(a,b,c,d) \in \mathcal{A}(abcd)$,

$$\begin{array}{l} t(a,b,b,a) = 0, \quad t(d,c,b,a) + t(a,b,c,d) = 0, \\ t(a,b,c,d) - t(b,c,d,a) + t(c,d,a,b) - t(d,a,b,c) = 0, \\ t(a,b,c,d) - t(b,a,c,d) + t(b,c,a,d) - t(b,c,d,a) = 0, \end{array}$$

and $t(a,b,c,e) = 0$.

These symmetric n-cochains constitute, under pointwise addition, the abelian groups $C^n_{\mathrm{G}}(M,\mathcal{A})$ in (3.1), $1 \leq n \leq 4$. The coboundary homomorphisms are defined by

• $(\partial^1 f)(a,b) = -a_* f(b) + f(ab) - b_* f(a)$,

• $(\partial^2 g)(a,b,c) = -a_* g(b,c) + g(ab,c) - g(a,bc) + c_* g(a,b)$,

• $(\partial^3 h)(a,b,c,d) =$

$$- a_* h(b,c,d) + h(ab,c,d) - h(a,bc,d) + h(a,b,cd) - d_* h(a,b,c). \tag{3.3}$$

The following lemma will be useful here and in the following chapters.

Lemma 3.1 *Let $\mathcal{A}$ be an $\mathbb{H}M$-module, where M is any commutative monoid, and let $h : M^3 \to \bigcup_{a\in M} \mathcal{A}(a)$ be a function with $h(a,b,c) \in \mathcal{A}(abc)$. Then h satisfies the symmetry conditions*

$$h(a,b,c) + h(c,b,a) = 0, \quad h(a,b,c) + h(b,c,a) + h(c,a,b) = 0, \tag{3.4}$$

if and only if it satisfies either (3.5) *or* (3.6) *below.*

$$h(a,b,c) - h(b,a,c) + h(b,c,a) = 0 \tag{3.5}$$

$$h(a,b,c) - h(a,c,b) + h(c,a,b) = 0 \tag{3.6}$$

Proof: The implications (3.4) $\Rightarrow$ (3.5) and (3.4) $\Rightarrow$ (3.6) are easily seen. To see that (3.5) $\Rightarrow$ (3.4), observe that, making the permutation $(a,b,c) \mapsto (c,b,a)$, equation (3.5) is written as $h(b,c,a) = h(c,b,a) + h(b,a,c)$. If we carry this to (3.5), we obtain

$$h(a,b,c) - h(b,a,c) + h(c,b,a) + h(b,a,c) = h(a,b,c) + h(c,b,a) = 0,$$

that is, the first condition in (3.4) holds. But then, we get also the second one simply by replacing the term $h(b,a,c)$ with $-h(c,a,b)$ in (3.5). The proof that (3.6) $\Rightarrow$ (3.4) is parallel. $\square$

The groups

$$Z^n_{\mathrm{G}}(M,\mathcal{A}) = \mathrm{Ker}\big(\partial^n : C^n_{\mathrm{G}}(M,\mathcal{A}) \to C^{n+1}_{\mathrm{G}}(M,\mathcal{A})\big),$$
$$B^n_{\mathrm{G}}(M,\mathcal{A}) = \mathrm{Im}\big(\partial^{n-1} : C^{n-1}_{\mathrm{G}}(M,\mathcal{A}) \to C^n_{\mathrm{G}}(M,\mathcal{A})\big),$$

are respectively called the groups of *symmetric n-cocycles* and *symmetric n-coboundaries* on M with values in $\mathcal{A}$. By [42, Theorems 1.3 and 2.12], there are natural isomorphisms

$$H^n_{\mathrm{G}}(M,\mathcal{A}) \cong Z^n_{\mathrm{G}}(M,\mathcal{A})/B^n_{\mathrm{G}}(M,\mathcal{A}) \tag{3.7}$$

for $n = 1,2,3$.

The elements of $H^1_{\mathrm{G}}(M,\mathcal{A}) = Z^1_{\mathrm{G}}(M,\mathcal{A})$ are *derivations* of M in $\mathcal{A}$, that is, functions $f : M \to \bigcup_{a\in M}\mathcal{A}(a)$ with $f(a) \in \mathcal{A}(a)$, such that $f(ab) = a_*f(b) + b_*f(a)$.

The elements of $H^2_{\mathrm{G}}(M,\mathcal{A})$ have a natural interpretation in terms of *commutative group coextension* of the commutative monoid M by the $\mathbb{H}M$-module $\mathcal{A}$. This is the classification result by Grillet in [43, Chapter V.4], whose proof is a good illustration of the one we give of our result in this chapter. We shall not present Grillet's proof here but, briefly, let us recall that in the correspondence between symmetric 2-cohomology classes and isomorphism classes of commutative group coextensions, each symmetric 2-cocycle $g \in Z^2_{\mathrm{G}}(M,\mathcal{A})$ is carried to the coextension

$$E(M,\mathcal{A},g) \to M,$$

where the *twisted crossed product* commutative monoid is the set $\bigcup_{a\in M}\mathcal{A}(a)\times\{a\}$ of all pairs (u_a,a) with $a \in M$ and $u_a \in \mathcal{A}(a)$, with multiplication defined by

$$(u_a,a)(u_b,b) = (a_*u_b + b_*u_a + g(a,b), ab).$$

This multiplication is unitary ($(0,e)$ is the unit) since g is normalized, that is, $g(a,e) = 0 = g(e,a)$; and it is associative and commutative due to g being a symmetric 2-cocycle, that is, because of the equalities $a_*g(b,c) + g(a,bc) = g(ab,c) + c_*g(a,b)$ and $g(a,b) = g(b,a)$. The homomorphism $E(M,\mathcal{A},g) \to M$ is the projection $(u_a,a) \mapsto a$, and, for each $a \in M$, the simply transitive group action of the group $\mathcal{A}(a)$ on the fiber set over a is given by

$$u_a \cdot (v_a,a) = (u_a + v_a, a).$$

3.1.2 Strictly symmetric monoidal abelian groupoids

Recall from Section 1.3 that a groupoid $\mathcal{M}$ is termed abelian if its isotropy (or vertex) groups $\mathrm{Aut}_{\mathcal{M}}(x)$, $x \in \mathrm{Ob}\mathcal{M}$, are all abelian. As there, we use additive notation for the composition in these abelian groupoids.

Example 3.1 Any abelian group A can be regarded as an abelian groupoid $\mathcal{M}$ with only one object, say a, and $\mathrm{Aut}_{\mathcal{M}}(a) = A$. For many purposes it is convenient to distinguish A from the one-object groupoid $\mathcal{M}$; the notation $(K(A,1),a)$ for $\mathcal{M}$ is not bad (its nerve or classifying space [39, I, Example 1.4] is precisely the pointed Eilenberg-Mac Lane minimal complex $K(A,1)$ with base-vertex a), and we shall use it below.

A groupoid in which there is no morphisms between different objects is called *totally disconnected*. It is easily seen that any totally disconnected abelian groupoid is a disjoint union of abelian groups, or, more precisely, of the form

$$\bigcup_{a\in M} (K(\mathcal{A}(a),1),a),$$

for some family of abelian groups $(\mathcal{A}(a))_{a\in M}$.

A *strictly symmetric* (or *strictly commutative*) *monoidal abelian groupoid*

$$\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$$

is a monoidal abelian groupoid $(\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ (see 1.1 for details) endowed with natural isomorphisms $\boldsymbol{c}_{x,y} : x \otimes y \to y \otimes x$ (called the *symmetry constraint*) such that the following three conditions are satisfied.

- $(0_y \otimes \boldsymbol{c}_{x,z}) + \boldsymbol{a}_{y,x,z} + (\boldsymbol{c}_{x,y} \otimes 0_z) = \boldsymbol{a}_{y,z,x} + \boldsymbol{c}_{x,y\otimes z} + \boldsymbol{a}_{x,y,z}$, (3.8)

$$\begin{tikzcd}
 & (y\otimes x)\otimes z \arrow[r, "\boldsymbol{a}"] & y\otimes(x\otimes z) \arrow[dr, "0\otimes\boldsymbol{c}"] & \\
(x\otimes y)\otimes z \arrow[ur, "\boldsymbol{c}\otimes 0"] \arrow[dr, "\boldsymbol{a}"'] & & & y\otimes(z\otimes x) \\
 & x\otimes(y\otimes z) \arrow[r, "\boldsymbol{c}"] & (y\otimes z)\otimes x \arrow[ur, "\boldsymbol{a}"'] &
\end{tikzcd}$$

- $\boldsymbol{c}_{y,x} + \boldsymbol{c}_{x,y} = 0_{x\otimes y}$, (3.9)

$$\begin{tikzcd}
x\otimes y \arrow[rr, "\boldsymbol{c}"] \arrow[dr, equal] & & y\otimes x \arrow[dl, "\boldsymbol{c}"] \\
 & x\otimes y &
\end{tikzcd}$$

- $\boldsymbol{c}_{x,x} = 0_{x\otimes x} : x\otimes x \to x \otimes x.$ (3.10)

These axioms guarantee the coherence of the constraints in the following sense (see Mac Lane [56, Theorem 5.1] and Fröhlich and Wall [36, Theorem 5.2]).

Fact 3.1 (Coherence Theorem) *Let $\mathcal{M}$ be a strictly symmetric monoidal abelian groupoid. Then, commutativity holds in every diagram in $\mathcal{M}$ with vertices iterated instances of the functors $x \mapsto x$, the identity, $* \mapsto \mathrm{I}$, which selects the unit object, $(x,y) \mapsto x \otimes y$, $(x,y) \mapsto y \otimes x$, and $x \mapsto x \otimes x$, the diagonal functor, and whose edges are expanded instances of $\boldsymbol{a}$, $-\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{l}$, $-\boldsymbol{l}$, $\boldsymbol{r}$, and $-\boldsymbol{r}$.*

Below there is a convenient way to express this coherence in practice (see Deligne [25, 1.4.1] and Fröhlich and Wall [36, Theorem (5.3)]). Recall that, for any set M, the free commutative monoid $\mathbb{N}[M]$ consists of commutative words in M, which are unordered sequences $[a_1, \dots, a_n]$ of elements of M; unordered means that for any permutation σ, $[a_{\sigma 1}, \dots, a_{\sigma n}] = [a_1, \dots, a_n]$. Multiplication in $\mathbb{N}[M]$ is given by concatenation:

$$[a_1, \dots, a_n][b_1, \dots, b_m] = [a_1, \dots, a_n, b_1, \dots, b_m],$$

and the unit is $e = [\ \ ]$, the empty word.

Lemma 3.2 *Let $(x_a)_{a \in M}$ be any family of objects of a strictly symmetric monoidal abelian groupoid $\mathcal{M}$. If $\mathbb{N}[M]$ is the free commutative monoid generated by the index set M, then, there exists a map $F : \mathbb{N}[M] \to \mathrm{Ob}\mathcal{M}$ with $F[a] = x_a$, $a \in M$, and isomorphisms $\varphi_{f,g} : Ff \otimes Fg \cong F(fg)$, $f, g \in \mathbb{N}[M]$, and $\varphi_0 : \mathrm{I} \to Fe$, satisfying the equations below.*

- $\varphi_{fg,h} + (\varphi_{f,g} \otimes 0_{Fh}) = \varphi_{f,gh} + (0_{Ff} \otimes \varphi_{g,h}) + \boldsymbol{a}_{Ff,Fg,Fh}$,

$$\begin{tikzcd}
(Ff \otimes Fg) \otimes Fh \arrow[r, "\varphi\otimes 0"] \arrow[d, "\boldsymbol{a}"'] & F(fg) \otimes Fh \arrow[r, "\varphi"] & F(fgh) \arrow[d, equal] \\
Ff \otimes (Fg \otimes Fh) \arrow[r, "0\otimes\varphi"] & Ff \otimes F(gh) \arrow[r, "\varphi"] & F(fgh)
\end{tikzcd}$$

- $\varphi_{g,f} + \boldsymbol{c}_{Ff,Fg} = \varphi_{f,g}$.

$$\begin{tikzcd}
Ff \otimes Fg \arrow[r, "\boldsymbol{c}"] \arrow[d, "\varphi"'] & Fg \otimes Ff \arrow[d, "\varphi"] \\
F(fg) \arrow[r, equal] & F(gf)
\end{tikzcd}$$

- $\varphi_{f,e} + (0_{Ff} \otimes \varphi_0) = \boldsymbol{r}_{Ff}, \quad \varphi_{e,f} + (\varphi_0 \otimes 0_{Ff}) = \boldsymbol{l}_{Ff}$

$$\begin{tikzcd}
Ff \otimes \mathrm{I} \arrow[r, "0\otimes\varphi_0"] \arrow[d, "\boldsymbol{r}"'] & Ff \otimes Fe \arrow[d, "\varphi"] \\
Ff \arrow[r, equal] & Ff
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I} \otimes Ff \arrow[r, "\varphi_0\otimes 0"] \arrow[d, "\boldsymbol{l}"'] & Fe \otimes Ff \arrow[d, "\varphi"] \\
Ff \arrow[r, equal] & Ff
\end{tikzcd}$$

Proof: Let us choose a total order for the index set M, so that any $f \in \mathbb{N}[M]$ can be uniquely expressed as a sequence in increasing order

$$f = [a_1, \dots, a_n], \quad a_1 \leq \dots \leq a_n.$$

Then, we define $F : \mathbb{N}[M] \to \mathrm{Ob}\mathcal{M}$ by putting $Fe = \mathrm{I}$, $F[a] = x_a$, and, recursively,

$$F[a_1, \dots, a_n] = F[a_1, \dots, a_{n-1}] \otimes x_{a_n}$$

for $n > 1$. We have the identity isomorphism $\varphi_0 = 0_{\mathrm{I}} : \mathrm{I} \to Fe$ and, for any $f, g \in \mathbb{N}[M]$, it is clear that there is an isomorphism

$$\varphi_{f,g} : Ff \otimes Fg \cong F(fg)$$

coming from instances of $\boldsymbol{a}$, $-\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{l}$ and $\boldsymbol{r}$. It follows from the Coherence Theorem above that these isomorphisms $\varphi_{f,g}$ so obtained satisfy all the requirements in the lemma. □

If $\mathcal{M}$, $\mathcal{M}'$ are strictly symmetric monoidal abelian groupoids, then a *symmetric monoidal functor* $F = (F, \varphi, \varphi_0) : \mathcal{M} \to \mathcal{M}'$ is a monoidal functor (1.4) verifying

- $\varphi_{y,x} + \boldsymbol{c}'_{Fx,Fy} = F\boldsymbol{c}_{x,y} + \varphi_{x,y}$. (3.11)

$$\begin{tikzcd} Fx \otimes Fy \arrow[r, "\boldsymbol{c}'"] \arrow[d, "\varphi"'] & Fy \otimes Fx \arrow[d, "\varphi"] \\ F(x \otimes y) \arrow[r, "F\boldsymbol{c}"] & F(y \otimes x) \end{tikzcd}$$

Suppose $F' : \mathcal{M} \to \mathcal{M}'$ is another symmetric monoidal functor. Then, a *symmetric isomorphism* $\theta : F \Rightarrow F'$ is a monoidal isomorphism (1.7).

With compositions given in a natural way, strictly symmetric monoidal abelian groupoids, symmetric monoidal functors, and symmetric isomorphisms form a 2-category. A symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is called a *symmetric monoidal equivalence* if it is an equivalence in this 2-category, that is, when there exist a symmetric monoidal functor $F' : \mathcal{M}' \to \mathcal{M}$ and symmetric isomorphisms $\theta : F'F \cong id_{\mathcal{M}}$ and $\theta' : FF' \cong id_{\mathcal{M}'}$.

Our goal is to show a classification for strictly symmetric monoidal abelian groupoids, where two of them that are connected by a symmetric monoidal equivalence are considered the same. To do that, we will use the fact below by Saavedra [66, I, 4.4.5], where it is shown how to transport the symmetric monoidal structure on an abelian groupoid along an equivalence on its underlying groupoid. Recall that a functor between (not necessarily abelian) groupoids $F : \mathcal{M} \to \mathcal{M}'$ is an equivalence (of categories) if and only if the induced map on the sets of iso-classes of objects

$$\mathrm{Ob}\mathcal{M}/_{\cong} \to \mathrm{Ob}\mathcal{M}'/_{\cong}, \quad [x] \mapsto [Fx], \tag{3.12}$$

is a bijection, and the induced homomorphisms on the automorphism groups

$$\mathrm{Aut}_{\mathcal{M}}(x) \to \mathrm{Aut}_{\mathcal{M}'}(Fx), \quad u \mapsto Fu \tag{3.13}$$

are all isomorphisms [46, Chapter 6, Corollary 2].

Fact 3.2 (Transport of Structure) *Let $F : \mathcal{M} \to \mathcal{M}'$ be an equivalence between abelian groupoids, so that there is a functor $F' : \mathcal{M}' \to \mathcal{M}$ with natural equivalences $\theta : id_{\mathcal{M}} \cong F'F$ and $\theta' : FF' \cong id_{\mathcal{M}'}$ satisfying*

$$\theta' F + F\theta = id_F, \quad F'\theta' + \theta F' = id_{F'}.$$

(i) Any strictly symmetric monoidal structure on $\mathcal{M}$ can be transported to one on $\mathcal{M}'$ such that the functors F and F' underlie symmetric monoidal functors, and the natural equivalences θ and θ' turn to be symmetric isomorphisms.

(ii) If both $\mathcal{M}$ and $\mathcal{M}'$ have a strictly symmetric monoidal structure, then any symmetric monoidal structure on F can be transported to one on F' such that θ and θ' become symmetric isomorphisms. Hence, a symmetric monoidal functor is a symmetric monoidal equivalence if and only if the underlying functor is an equivalence.

Concerning Fact 3.2(i), let us point out that for any strictly symmetric monoidal structure $(\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ on the abelian groupoid $\mathcal{M}$, the structure transported onto the abelian groupoid $\mathcal{M}'$, that is $(\mathcal{M}', \otimes, \mathrm{I}', \boldsymbol{a}', \boldsymbol{l}', \boldsymbol{r}', \boldsymbol{c}')$, by means of (F, F', θ, θ') is such that the monoidal product $\otimes$ is the dotted functor in the commutative square

$$\begin{tikzcd}
\mathcal{M}' \times \mathcal{M}' \arrow[r, dotted, "\otimes"] \arrow[d, "F' \times F'"'] & \mathcal{M}' \\
\mathcal{M} \times \mathcal{M} \arrow[r, "\otimes"] & \mathcal{M}, \arrow[u, "F"']
\end{tikzcd}$$

and the unit object is $F\mathrm{I}$. The functors F and F' are endowed with the isomorphisms

$$\varphi_{x,y} = -F(\theta_x \otimes \theta_y) : Fx \otimes Fy \to F(x \otimes y), \quad \varphi_0 = 0_{FI} : F\mathrm{I} \to F\mathrm{I}, \tag{3.14}$$

$$\varphi'_{x',y'} = \theta_{F'x' \otimes F'y'} : F'x' \otimes F'y' \to F'(x' \otimes y'), \quad \varphi'_0 = \theta_{\mathrm{I}} : \mathrm{I} \to F'\mathrm{I}',$$

and then the constraints $\boldsymbol{a}', \boldsymbol{l}', \boldsymbol{r}'$ and the symmetry $\boldsymbol{c}'$ are given by those isomorphisms uniquely determined by the equations (1.5), (1.6), and (3.11), respectively.

Concerning Fact 3.2(ii), let us recall that if both $\mathcal{M}$ and $\mathcal{M}'$ are strictly symmetric monoidal abelian groupoids, then for any symmetric monoidal structure (F, φ, φ_0) on F, the structure $(F', \varphi', \varphi'_0)$ transported on the functor F' is such that the isomorphisms

$$\varphi'_{x',y'} : F'x' \otimes F'y' \to F'(x' \otimes y'), \quad \varphi'_0 : \mathrm{I} \to F'\mathrm{I}',$$

are the uniquely determined by the dotted arrows making commutative the diagrams below.

$$\begin{tikzcd}
FF'x' \otimes FF'y' \arrow[r, "\theta'_{x'} \otimes \theta'_{y'}"] \arrow[d, "\varphi"'] & x' \otimes y' \arrow[d, "-\theta'_{x' \otimes y'}"] \\
F(F'x' \otimes F'y') \arrow[r, dotted, "F\varphi'"] & FF'(x' \otimes y')
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F\mathrm{I} \arrow[d, dotted, "F\varphi'_0"] \\
\mathrm{I}' \arrow[ur, "\varphi_0"] \arrow[r, "-\theta'_{\mathrm{I}'}"'] & FF'\mathrm{I}'
\end{tikzcd}$$

3.2 The classification theorem

The framework of our discussion below comes suggested by the known classification theorems for strictly commutative Picard categories given in [25], [36] and [68], for categorical groups and Picard categories in [69], for braided categorical groups in [50], for graded categorical groups, braided graded categorical groups, graded Picard categories and strictly commutative graded Picard categories in [20, 22, 23], for braided fibred categorical groups, fibred Picard categories and strictly commutative fibred Picard categories in [11], and for monoidal groupoids in chapter 1.

Let M be a commutative monoid and let $\mathcal{A}$ be an $\mathbb{H}M$-module. Each symmetric 3-cocycle $h \in Z^3_{\mathrm{G}}(M,\mathcal{A})$ gives rise to a strictly symmetric monoidal abelian groupoid

$$\Sigma(M,\mathcal{A},h) \tag{3.15}$$

which should be thought of a sort of 2-*dimensional twisted crossed product of* M *by* $\mathcal{A}$, and it is built as follows: Its underlying groupoid is the totally disconnected groupoid

$$\textstyle\bigcup_{a\in M}(K(\mathcal{A}(a),1),a), \tag{3.16}$$

where, recall from Example 3.1, each $(K(\mathcal{A}(a),1),a)$ denotes the groupoid having a as its unique object and $\mathcal{A}(a)$ as the automorphism group of a. Thus, an object of $\Sigma(M,\mathcal{A},h)$ is an element $a \in M$; if $a \neq b$ are different elements of the monoid M, then there is no morphisms in $\Sigma(M,\mathcal{A},h)$ between them, whereas its isotropy group at any $a \in M$ is $\mathcal{A}(a)$.

The tensor functor

$$\otimes : \Sigma(M,\mathcal{A},h) \times \Sigma(M,\mathcal{A},h) \to \Sigma(M,\mathcal{A},h)$$

is given on objects by multiplication in M, so $a \otimes b = ab$, and on morphisms by the group homomorphisms

$$\otimes : \mathcal{A}(a) \times \mathcal{A}(b) \to \mathcal{A}(ab), \quad u_a \otimes u_b = b_*u_a + a_*u_b. \tag{3.17}$$

The unit object is $\mathrm{I} = e$, the unit element of the monoid M, and the structure constraints and the symmetry isomorphisms are

$$\begin{aligned}
\boldsymbol{a}_{a,b,c} &= h(a,b,c) : (ab)c \to a(bc),\\
\boldsymbol{c}_{a,b} &= 0_{ab} : ab \to ba,\\
\boldsymbol{r}_a &= 0_a : ae \to a,\\
\boldsymbol{l}_a &= 0_a : ea \to a,
\end{aligned}$$

which are easily seen to be natural since $\mathcal{A}$ is an abelian group valued functor. The coherence condition (1.1) holds thanks to the cocycle condition $\partial^3 h = 0$ in (3.3), while (3.8) easily follows from the cochain equations in (3.2). The normalization condition

$h(a,e,b)=0$, easily deduced from being $h(a,b,e)=0$, implies the coherence condition (1.2), and those in (3.9) and (3.10) are obviously verified.

In next Theorem 3.1, we observe how any strictly symmetric monoidal abelian groupoid is symmetric monoidal equivalent to such a 2-dimensional crossed product. Previously, we combine the transport process in Fact 3.2 with the generalization of Brandt's Theorem [7], which asserts that every groupoid is equivalent as a category to a totally disconnected groupoid [46, Chapter 6, Theorem 2], to obtain the following.

Lemma 3.3 *Any strictly symmetric monoidal abelian groupoid is symmetric monoidal equivalent to one which is totally disconnected and whose symmetry and unit constraints are all identities.*

Proof: Let $\mathcal{M}=(\mathcal{M},\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r},\boldsymbol{c})$ be any given strictly symmetric monoidal abelian groupoid.

Let $M=\mathrm{Ob}\mathcal{M}/_{\cong}$ be the set of isomorphism classes $[x]$ of objects of $\mathcal{M}$, and let us choose, for each $a\in M$, a representative object $x_a\in a$, with $x_{[\mathrm{I}]}=\mathrm{I}$.

In a first step, let us assume that all the symmetry constraints are identities, that is, $x\otimes y=y\otimes x$ and $\boldsymbol{c}_{x,y}=0_{x\otimes y}$, for any objects x,y of $\mathcal{M}$, and also that $\mathrm{I}\otimes\mathrm{I}=\mathrm{I}$, and $\boldsymbol{l}_\mathrm{I}=\boldsymbol{r}_\mathrm{I}=0_\mathrm{I}$, the identity of the unit object. Then, let us form the totally disconnected abelian groupoid

$$\mathcal{M}'=\bigcup\nolimits_{a\in M}(K(\mathcal{A}(a),1),a),$$

whose set of objects is M, and whose isotropy group at any object $a\in M$ is $\mathcal{A}(a)=\mathrm{Aut}_\mathcal{M}(x_a)$.

This groupoid $\mathcal{M}'$ is equivalent to the underlying groupoid $\mathcal{M}$. To give a particular equivalence $F:\mathcal{M}\to\mathcal{M}'$, let us choose, for each $a\in M$ and each $x\in a$, an isomorphism $\theta_x: x\cong x_a$ in $\mathcal{M}$. In particular, for every $a\in M$, we take $\theta_{\mathrm{I}\otimes x_a}=\boldsymbol{l}_{x_a}$ and $\theta_{x_a\otimes\mathrm{I}}=\boldsymbol{r}_{x_a}$. Note that this selection implies that $\theta_\mathrm{I}=\theta_{\mathrm{I}\otimes\mathrm{I}}=\boldsymbol{r}_\mathrm{I}=\boldsymbol{l}_\mathrm{I}=0_\mathrm{I}$. Then, let $F:\mathcal{M}\to\mathcal{M}'$ be the functor which acts on objects by $Fx=[x]$, and on morphisms $u:x\to y$ by $Fu=\theta_y+u-\theta_x$. We have also the more obvious functor $F':\mathcal{M}'\to\mathcal{M}$, which is defined on objects by $F'a=x_a$, and on morphisms $u:a\to a$ by $F'u=u$. We have the natural isomorphisms $\theta: id_\mathcal{M}\cong F'F$, and $\theta': FF'\cong id_{\mathcal{M}'}$, where $\theta'_a=-\theta_{x_a}$, which clearly satisfy the equalities $\theta'F+F\theta=id_F$ and $F'\theta'+\theta F'=id_{F'}$.

Therefore, according to Fact 3.2, we can transport the given symmetric monoidal structure of $\mathcal{M}$ to a corresponding one on $\mathcal{M}'$ by means of (F,F',θ,θ'), so that we get a totally disconnected strictly symmetric monoidal abelian groupoid $\mathcal{M}'=(\mathcal{M}',\otimes,\mathrm{I}',\boldsymbol{a}',\boldsymbol{r}',\boldsymbol{c}')$, and a symmetric monoidal equivalence $F=(F,\varphi,\varphi_0):\mathcal{M}\to\mathcal{M}'$. Now, a quick analysis of the structure on $\mathcal{M}'$ points out that its unit object is $F\mathrm{I}=[\mathrm{I}]$ and that, for any object $a\in\mathrm{Ob}\mathcal{M}'=M$,

$$\begin{aligned}\boldsymbol{r}'_a &\overset{(1.6)}{=} F(\boldsymbol{r}_{x_a})+\varphi_{x_a,\mathrm{I}}+(0_a\otimes\varphi_0) \overset{(3.14)}{=} F(\boldsymbol{r}_{x_a})+\varphi_{x_a,\mathrm{I}}+(0_a\otimes 0_{[\mathrm{I}]}) = F(\boldsymbol{r}_{x_a})+\varphi_{x_a,\mathrm{I}}\\ &\overset{(3.14)}{=} \theta_{x_a}+\boldsymbol{r}_{x_a}-\theta_{x_a\otimes\mathrm{I}}+\theta_{x_a\otimes\mathrm{I}}-(\theta_{x_a}\otimes 0_\mathrm{I})-\theta_{x_a\otimes\mathrm{I}}\\ &= \theta_{x_a}+\boldsymbol{r}_{x_a}-(\theta_{x_a}\otimes 0_\mathrm{I})-\theta_{x_a\otimes\mathrm{I}} \overset{(\text{naturality of } \boldsymbol{r})}{=} \boldsymbol{r}_{x_a}-\theta_{x_a\otimes\mathrm{I}}=0_{x_a}=0_a,\end{aligned}$$

similarly, $\boldsymbol{l}'_a = 0_a$, while, for any $a, b \in M$,

$$\begin{aligned}
\boldsymbol{c}'_{a,b} &\overset{(3.11)}{=} -\varphi_{x_b,x_a} + F(\boldsymbol{c}_{x_a,x_b}) + \varphi_{x_a,x_b} \overset{(\boldsymbol{c}=0)}{=} -\varphi_{x_b,x_a} + \varphi_{x_a,x_b} \\
&\overset{(3.14)}{=} \theta_{x_b\otimes x_a} - \theta_{x_b} \otimes \theta_{x_a} - \theta_{x_b\otimes x_a} + \theta_{x_a\otimes x_b} + \theta_{x_a} \otimes \theta_{x_b} - \theta_{x_a\otimes x_b} \\
&\text{(since } x_a \otimes x_b = x_b \otimes x_a\text{)} \\
&= \theta_{x_b\otimes x_a} - \theta_{x_b} \otimes \theta_{x_a} + \theta_{x_a} \otimes \theta_{x_b} - \theta_{x_a\otimes x_b} \\
&\text{(since } \theta_{x_a} \otimes \theta_{x_b} = \theta_{x_b} \otimes \theta_{x_a}\text{, by the naturality of } \boldsymbol{c}_{x_a,x_b} = 0_{x_a\otimes x_b}\text{)} \\
&= \theta_{x_b\otimes x_a} - \theta_{x_a\otimes x_b} = 0_{x_a\otimes x_b} = 0_{ab}.
\end{aligned}$$

Thus, $\mathcal{M}$ is symmetric monoidal equivalent to $\mathcal{M}'$, which is a totally disconnected strictly symmetric abelian groupoid whose unit and symmetry constraints are all identities.

Hence, it suffices to prove now that the given strictly symmetric monoidal abelian groupoid $\mathcal{M}$ is symmetric monoidal equivalent to another one whose symmetry constraints are all identities and whose unit constraint at the unit object is also the identity. Even more, following Deligne [25], we can prove that there is a symmetric monoidal abelian groupoid $\mathcal{N} = (\mathcal{N}, \bar{\otimes})$ whose constraints are all trivial (i.e., $\boldsymbol{a} = 0$, $\boldsymbol{c} = 0$, $\boldsymbol{l} = 0$, and $\boldsymbol{r} = 0$) with a symmetric monoidal equivalence $\mathcal{N} \simeq \mathcal{M}$:

Let $\mathbb{N}[M]$ be the free commutative monoid generated by M, which we shall regard as a strictly symmetric monoidal discrete groupoid (i.e., with only identities as morphisms). It follows from Lemma 3.2 that there is a symmetric monoidal functor

$$F = (F, \varphi, \varphi_0) : \mathbb{N}[M] \to \mathcal{M}$$

such that $F[a] = x_a$, for any $a \in M$. Then, we define $\mathcal{N}$ to be the abelian groupoid whose set of objects is $\mathbb{N}[M]$, and whose hom-sets are defined by

$$\mathrm{Hom}_{\mathcal{N}}(f, g) = \mathrm{Hom}_{\mathcal{M}}(Ff, Fg).$$

Composition in $\mathcal{N}$ is given by that in $\mathcal{M}$, so that we have a full, faithful, and essentially surjective functor (i.e., an equivalence)

$$F : \mathcal{N} \to \mathcal{M}, \quad (f \xrightarrow{u} g) \mapsto (Ff \xrightarrow{u} Fg).$$

The monoidal functor $\bar{\otimes} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is defined by multiplication in $\mathbb{N}[M]$ on objects, and on morphisms by

$$(f \xrightarrow{u} g) \otimes (f' \xrightarrow{u'} g') = (ff' \xrightarrow{u\bar{\otimes}u'} gg'),$$

where $u\bar{\otimes}u'$ is the dotted morphism in the commutative square in $\mathcal{M}$

$$\begin{tikzcd}
Ff \otimes Ff' \arrow[r, "u\otimes u'"] \arrow[d, "\varphi_{f,f'}"'] & Fg \otimes Fg' \arrow[d, "\varphi_{g,g'}"] \\
F(ff') \arrow[r, dotted, "u\bar{\otimes}u'"] & F(gg').
\end{tikzcd} \tag{3.18}$$

So defined $\mathcal{N} = (\mathcal{N}, \bar{\otimes})$ is a strictly symmetric monoidal abelian groupoid with all the constraints being identities. To prove this claim, the following equalities on morphisms in $\mathcal{N}$ should be verified

$$u\bar{\otimes}u' = u'\bar{\otimes}u, \quad 0_e\bar{\otimes}u = u, \quad u\bar{\otimes}0_e = u, \quad (u\bar{\otimes}u')\bar{\otimes}u'' = u\bar{\otimes}(u'\bar{\otimes}u''). \tag{3.19}$$

But these follow from the naturality of the structure constraints of $\mathcal{M}$, $\boldsymbol{c}$, $\boldsymbol{l}$, $\boldsymbol{r}$, and $\boldsymbol{a}$, respectively. For example, given any $u \in \mathrm{Hom}_{\mathcal{N}}(f,g)$, we have the diagram

$$\begin{tikzcd}
Ff \otimes \mathrm{I} \arrow[r, "0_{Ff}\otimes\varphi_0"] \arrow[d, "u\otimes 0_{\mathrm{I}}"'] \arrow[rr, bend left, "\boldsymbol{r}_{Ff}"] \arrow[dr, phantom, "(A)"] & Ff \otimes Fe \arrow[r, "\varphi_{f,e}"] \arrow[d, "u\otimes 0_{Fe}"] \arrow[u, phantom, "(C)"] \arrow[dr, phantom, "(B)"] & Ff \arrow[d, "u"] \\
Fg \otimes \mathrm{I} \arrow[r, "0_{Fg}\otimes\varphi_0"] \arrow[rr, bend right, "\boldsymbol{r}_{Fg}"'] & Fg \otimes Fe \arrow[r, "\varphi_{g,e}"] \arrow[d, phantom, "(C)"] & Fg
\end{tikzcd}$$

where the outside region commutes by naturality of $\boldsymbol{r}$, those labelled with (C) commute because $(F, \varphi, \varphi_0) : \mathbb{N}[M] \to \mathcal{M}$ is a symmetric monoidal functor, and the square (A) commutes due to $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ being a functor. It follows that the square (B) is also commutative and then that $u\bar{\otimes}0_e = u$. The other three equations in (3.19) are proved similarly, and we leave them to the reader.

Owing to the commutativity of the squares (3.18), the isomorphisms $\varphi_{f,f'}$ are natural on morphisms of $\mathcal{N}$ and, therefore, $F = (F, \varphi, \varphi_0) : \mathcal{N} \to \mathcal{M}$ is actually a symmetric monoidal functor, whence, by Fact 3.2 (*ii*), a symmetric monoidal equivalence. □

We are now ready to prove the main result in this chapter, namely, the *classification of strictly symmetric monoidal abelian groupoids.*

Theorem 3.1 (*i*) *For any strictly symmetric monoidal abelian groupoid $\mathcal{M}$, there is a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A}$, a symmetric 3-cocycle $h \in Z^3_{\mathrm{G}}(M, \mathcal{A})$, and a symmetric monoidal equivalence*

$$\Sigma(M, \mathcal{A}, h) \simeq \mathcal{M}.$$

(*ii*) *For any two commutative 3-cocycles $h \in Z^3_{\mathrm{G}}(M, \mathcal{A})$ and $h' \in Z^3_{\mathrm{G}}(M', \mathcal{A}')$, there is a symmetric monoidal equivalence*

$$\Sigma(M, \mathcal{A}, h) \simeq \Sigma(M', \mathcal{A}', h')$$

if and only if there exist an isomorphism of monoids $i : M \cong M'$ and a natural isomorphism $\psi : \mathcal{A} \cong i^*\mathcal{A}'$, such that the equality of cohomology classes below holds.

$$[h] = \psi_*^{-1} i^*[h'] \in H^3_{\mathrm{G}}(M, \mathcal{A})$$

Proof: (i) By Lemma 3.3, we can suppose that $\mathcal{M}$ is totally disconnected and that all its symmetry and unit constraints are identities. In assuming that hypothesis, let us write the underlying groupoid as $\mathcal{M} = \bigcup_{a\in M}(K(\mathcal{A}(a),1),a)$, where $M = \mathrm{Ob}\mathcal{M}$ and, for each $a \in M$, $\mathcal{A}(a) = \mathrm{Aut}_{\mathcal{M}}(a)$. Then, a system of data $(M,\mathcal{A},h)$, such that $\Sigma(M,\mathcal{A},h) = \mathcal{M}$ as symmetric monoidal abelian groupoids, is defined as follows:

• *The monoid M.* The function on objects of the tensor functor $\otimes : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ determines a multiplication on M, simply by putting $ab = a\otimes b$, for any $a, b \in M$. If we write $e \in M$ for the the unit object of $\mathcal{M}$, then this multiplication on M is unitary, since the unit is strict. Furthermore, it is associative and commutative since, being $\mathcal{M}$ totally disconnected, the existence of the associativity and symmetry constraints $(ab)c \to a(bc)$ and $ab \to ba$ forces the equalities $(ab)c = a(bc)$ and $ab = ba$. Thus, M becomes a commutative monoid.

• *The $\mathbb{H}M$-module $\mathcal{A}$.* The group homomorphisms $\otimes : \mathcal{A}(a)\times\mathcal{A}(b) \to \mathcal{A}(ab)$ have an associative, commutative, and unitary behaviour, in the sense that the equalities

$$(u_a \otimes u_b)\otimes u_c = u_a \otimes (u_b \otimes u_c),\;\; u_a\otimes u_b = u_b \otimes u_a,\;\; 0_e \otimes u_a = u_a, \tag{3.20}$$

hold. These follow from the abelianess of the groups of automorphisms in $\mathcal{M}$, since the diagrams below commute due to the naturality of the structure constraints.

$$\begin{tikzcd}
(ab)c \arrow[r, "\boldsymbol{a}_{a,b,c}"] \arrow[d, "(u_a\otimes u_b)\otimes u_c"'] & a(bc) \arrow[d, "u_a\otimes(u_b\otimes u_c)"] \\
(ab)c \arrow[r, "\boldsymbol{a}_{a,b,c}"] & a(bc)
\end{tikzcd}
\qquad
\begin{tikzcd}
ab \arrow[r, "0_{ab}"] \arrow[d, "u_a\otimes u_b"'] & ba \arrow[d, "u_b\otimes u_a"] \\
ab \arrow[r, "0_{ab}"] & ba
\end{tikzcd}
\qquad
\begin{tikzcd}
ae = a \arrow[r, "0_a"] \arrow[d, "u_a\otimes 0_e"'] & a \arrow[d, "u_a"] \\
ae = a \arrow[r, "0_a"] & a
\end{tikzcd}$$

Then, if write $b_* : \mathcal{A}(a) \to \mathcal{A}(ab)$ for the homomorphism such that

$$b_*u_a := 0_b \otimes u_a = u_a \otimes 0_b,$$

the equalities

$$\begin{aligned}
&(bc)_*(u_a) = 0_{bc}\otimes u_a = (0_b\otimes 0_c)\otimes u_a \overset{(3.20)}{=} 0_b\otimes(0_c\otimes u_a) = b_*(c_*u_a),\\
&e_*u_a = 0_e\otimes u_a \overset{(3.20)}{=} u_a,
\end{aligned} \tag{3.21}$$

show that the assignments $a \mapsto \mathcal{A}(a)$, $(a,b) \mapsto b_* : \mathcal{A}(a) \to \mathcal{A}(ab)$, define an $\mathbb{H}M$-module. Observe that this $\mathbb{H}M$-module determines the monoidal product $\otimes$ of $\mathcal{M}$, since

$$\begin{aligned}
u_a\otimes u_b &= (u_a+0_a)\otimes(0_b+u_b) = (u_a\otimes 0_b)+(0_a\otimes u_b) \overset{(3.20)}{=} (0_b\otimes u_a)+(0_a\otimes u_b)\\
&= b_*u_a + a_*u_b.
\end{aligned}$$

• *The symmetric 3-cocycle $h \in Z^3_{\mathrm{G}}(M,\mathcal{A})$.* The associativity constraints of $\mathcal{M}$ can be written in the form $\boldsymbol{a}_{a,b,c} = h(a,b,c)$, for some list $\big(h(a,b,c)\in\mathcal{A}(abc)\big)_{a,b,c\in M}$. Since the symmetry constraints are all identities, for any $(a,b,c)\in M^3$, equation (3.8) gives

$$h(a,b,c) - h(b,a,c) + h(b,c,a) = 0, \tag{3.22}$$

which is (3.4), and thus, by Lemma 3.1 we get symmetric cochain conditions in (3.2). Now, from (1.3) it follows that $h(a,b,e)=0$. Hence $h\in C^3_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 3-cochain. Finally, the coherence condition in (1.1) gives the equations

$$a_*h(b,c,d)+h(a,bc,d)+d_*h(a,b,c)=h(a,b,cd)+h(ab,c,d),$$

which means that $\partial^3 h=0$ in (3.3), so that $h\in Z^3_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 3-cocycle.

Since an easy comparison shows that $\mathcal{M}=\Sigma(M,\mathcal{A},h)$, the proof of this part is complete.

(ii) We first assume that there exist an isomorphism of monoids $i:M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong i^*\mathcal{A}'$, such that $\psi_*[h]=i^*[h']\in H^3_{\mathrm{G}}(M,i^*\mathcal{A}')$. This means that there is a symmetric 2-cochain $g\in C^2_{\mathrm{G}}(M,i^*\mathcal{A}')$ such that the equalities below hold.

$$\psi_{abc}h(a,b,c)=h'(ia,ib,ic)+(ia)_*g(b,c)-g(ab,c)+g(a,bc)-(ic)_*g(a,b) \tag{3.23}$$

Then, we have a symmetric monoidal isomorphism

$$\Sigma(i,\psi,g)=(F,\varphi,\varphi_0):\Sigma(M,\mathcal{A},h)\to\Sigma(M',\mathcal{A}',h'), \tag{3.24}$$

whose underlying functor acts by

$$F(a\xrightarrow{u}a)=(ia\xrightarrow{\psi_a u_a}ia),$$

and whose structure isomorphisms are given by

$$\begin{aligned}&\varphi_{a,b}=g(a,b):(ia)\,(ib)\to i(ab),\\ &\varphi_0=0_{e'}:e'\to ie=e'.\end{aligned}$$

In effect, so defined, it is easy to see that F is an isomorphism between the underlying groupoids. Verifying the naturality of the isomorphisms $\varphi_{a,b}$, that is, the commutativity of the squares

$$\begin{tikzcd}(ia)(ib) \arrow[r, "\varphi_{a,b}"] \arrow[d, "(ia)_*\psi_b(u_b)+(ib)_*\psi_a(u_a)"'] & i(ab) \arrow[d, "\psi_{ab}(a_*u_b+b_*u_a)"] \\ (ia)(ib) \arrow[r, "\varphi_{a,b}"] & i(ab),\end{tikzcd} \tag{3.25}$$

for $u_a\in\mathcal{A}(a)$, $u_b\in\mathcal{A}(b)$, is equivalent (since the groups $\mathcal{A}'(i(ab))$ are abelian) to verify the equalities

$$\psi_{ab}(a_*u_b+b_*u_a)=(ia)_*\psi_b(u_b)+(ib)_*\psi_a(u_a), \tag{3.26}$$

which hold since the naturality of $\psi:\mathcal{A}\cong i^*\mathcal{A}'$ just says that

$$\psi_{ab}(a_*u_b)=(ia)_*\psi_b(u_b). \tag{3.27}$$

The coherence condition (1.5) is verified as follows

$$\begin{aligned}
&\varphi_{a,b\otimes c} + (0_{Fa}\otimes\varphi_{b,c}) + \boldsymbol{a}'_{Fa,Fb,Fc} = \varphi_{a,bc} + (ia)_*\varphi_{b,c} + h'(ia,ib,ic) \qquad (3.28)\\
&\quad = g(a,bc) + (ia)_*g(b,c) + h'(ia,ib,ic) \overset{(3.23)}{=} \psi_{abc}h(a,b,c) + g(ab,c) + (ic)_*g(a,b)\\
&\quad = \psi_{abc}h(a,b,c) + \varphi_{ab,c} + i(c)_*\varphi_{a,b} = F(\boldsymbol{a}_{a,b,c}) + \varphi_{a\otimes b,c} + (\varphi_{a,b}\otimes 0_{Fc}),
\end{aligned}$$

whilst the conditions in (1.6) and (3.11) trivially follow from the symmetric cochain conditions $g(a,e)=0_{ia}$ and $g(a,b)=g(b,a)$, respectively.

Conversely, suppose that

$$F=(F,\varphi,\varphi_0):\Sigma(M,\mathcal{A},h)\to\Sigma(M',\mathcal{A}',h')$$

is any symmetric monoidal equivalence. By a similar result than Lemma 1.1 (see [23, Lemma 3.1]), there is no loss of generality in assuming that F is strictly unitary in the sense that $\varphi_0=0_{e'}:e'\to e'=Fe$.

As the underlying functor establishes an equivalence between the underlying groupoids,

$$F:\bigcup_{a\in M}(K(\mathcal{A}(a),1),a)\simeq\bigcup_{a'\in M'}(K(\mathcal{A}'(a'),1),a'),$$

and these are totally disconnected, it is necessarily an isomorphism. Let us write $i:M\cong M'$ for the bijection describing the action of F on objects; that is, such that $ia=Fa$, for each $a\in M$. Then, i is actually an isomorphism of monoids, since the existence of the structure isomorphisms $\varphi_{a,b}:(ia)(ib)\to i(ab)$ forces the equality $(ia)(ib)=i(ab)$.

Let us write $\psi_a:\mathcal{A}(a)\cong\mathcal{A}'(ia)$ for the isomorphism giving the action of F on automorphisms $u_a:a\to a$, that is, such that $\psi_a u_a=Fu_a$, for each $u_a\in\mathcal{A}(a)$, and $a\in M$. The naturality of the automorphisms $\varphi_{a,b}$ tell us that the equalities (3.26) hold (see diagram (3.25)). These, for the case when $u_a=0_a$, give the equalities in (3.27), which amounts to saying that $\psi:\mathcal{A}\cong i^*\mathcal{A}'$ is a homomorphism of $\mathbb{H}M$-modules.

Writing now $g(a,b)=\varphi_{a,b}$, for each $a,b\in M$, the equations $g(a,e)=0_{ia}$ and $g(a,b)=g(b,a)$ hold just due to the coherence equations (1.6) and (3.11), and thus we have a symmetric 2-cochain $g=\big(g(a,b)\in\mathcal{A}'(i(ab))\big)_{a,b\in M}$, which satisfies the equations (3.23) owing to the coherence equations (1.5), as we can see just by retracting our steps in (3.28). This means that $\psi_*(h)=i^*(h')+\partial^2 g$ and, therefore, we have that $\psi_*[h]=i^*[h']\in H^3_{\mathrm{G}}(M,i^*\mathcal{A}')$, whence $[h]=\psi_*^{-1}i^*[h']\in H^3_{\mathrm{G}}(M,\mathcal{A})$, as required. □

Remark 3.1 Let

Symmetric 3-cocycles

denote the *category of symmetric* 3*-cocycles of commutative monoids.* That is, the category whose objects are triplets $(M,\mathcal{A},h)$ with M a commutative monoid, $\mathcal{A}$ an $\mathbb{H}M$-module, and $h\in Z^3_{\mathrm{G}}(M,\mathcal{A})$ a symmetric 3-cocycle, and whose arrows

$$(i,\psi,[g]):(M,\mathcal{A},h)\to(M',\mathcal{A}',h')$$

are triples consisting of a monoid homomorphism $i : M \to M'$, a natural transformation $\psi : \mathcal{A} \to i^*\mathcal{A}'$, and the equivalence class $[g]$ of a symmetric 2-cochain $g \in C^2_{\mathrm{G}}(M, i^*\mathcal{A}')$ such that $\psi_*(h) = i^*(h') + \partial^2 g$ (i.e., equation (3.23) holds). Two such cochains $g, g' \in C^2_{\mathrm{G}}(M, i^*\mathcal{A}')$ are equivalent if there is a symmetric 1-cochain $f \in C^1_{\mathrm{G}}(M, i^*\mathcal{A}')$ such that $g = g' + \partial^1 f$. Composition in this category of 3-cocycles is defined in a natural way: The composite of $(i, \psi, [g])$ with $(i', \psi', [g']) : (M', \mathcal{A}', h') \to (M'', \mathcal{A}'', h'')$ is the arrow

$$(i'i, \psi' i\ \psi, [(\psi' i)_*(g) + i^*(g')]) : (M, \mathcal{A}, h) \to (M'', \mathcal{A}'', h''),$$

where $i'i : M \to M''$ is the composite homomorphism of i' and i, $\psi' i\ \psi : \mathcal{A} \to (i'i)^*\mathcal{A}''$ is the natural transformation such that $(\psi' i\ \psi)_a = \psi'_{ia}\psi_a$, the composite homomorphism of $\psi'_{ia} : \mathcal{A}'(ia) \to \mathcal{A}''(i'ia)$ with $\psi_a : \mathcal{A}(a) \to \mathcal{A}'(ia)$, for each $a \in M$, and $(\psi' i)_*(g) + i^*(g') \in C^2_{\mathrm{G}}(M, (i'i)^*\mathcal{A}'')$ is the symmetric 2-cochain given by

$$((\psi' i)_*(g) + i^*(g'))(a, b) = \psi'_{i(ab)} g(a, b) + g'(ia, ib).$$

The identity arrow of any object $(M, \mathcal{A}, h)$ is the triple $(1_M, 1_{\mathcal{A}}, [0])$.

With a slight adaptation of the arguments in the proof of part (ii), Theorem 3.1 can be formulated as an equivalence of categories

Symmetric 3-cocycles $\simeq$ Strictly symmetric monoidal abelian groupoids

between the category of symmetric 3-cocycles and the category of strictly symmetric monoidal abelian groupoids, $\mathcal{M}$, with iso-classes, $[F] : \mathcal{M} \to \mathcal{M}'$, of symmetric monoidal functors, $F : \mathcal{M} \to \mathcal{M}'$, as arrows. The equivalence of categories is given by the constructions (3.15) on objects and (3.24) on morphisms, that is,

$$\big((M, \mathcal{A}, h) \xrightarrow{(i,\psi,[g])} (M', \mathcal{A}', h')\big) \ \mapsto\ \big(\Sigma(M, \mathcal{A}, h) \xrightarrow{[\Sigma(i,\psi,g)]} \Sigma(M', \mathcal{A}', h')\big).$$

A *strictly commutative Picard category* [25, Definition 1.4.2] is a strictly symmetric monoidal abelian groupoid $\mathcal{P} = (\mathcal{P}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ in which, for any object x, there is an object x^* with an arrow $x \otimes x^* \to \mathrm{I}$. Actually, the hypothesis of being abelian is superfluous here since a monoidal groupoid in which every object has a quasi-inverse is always abelian Proposition 1.3 (ii). Next, we obtain Deligne's classification result for these Picard categories as a corollary of Theorem 3.1 and the lemma below, which is a consequence of Lemma 3.1 and a result by Mac Lane [55, Theorem 4].

Lemma 3.4 *Let G be any abelian group. For any abelian group A, regarded as a constant $\mathbb{H}G$-module, the symmetric 3-cohomology group of G with coefficients in A is zero, that is, $H^3_{\mathrm{G}}(G, A) = 0$.*

For any abelian groups G and A, let $\Sigma(G, A, 0)$ be the strictly symmetric monoidal abelian groupoid built as in (3.15), for the constant functor $A : \mathbb{H}G \to \mathbf{Ab}$ and the zero 3-cocycle $0 : G^3 \to A$. Since G is a group, $\Sigma(G, A, 0)$ is actually a strictly commutative Picard category. Then, we have

Corollary 3.1 (Deligne [25], Fröhlich-Wall [36], Sinh [68]) (i) *For any strictly commutative Picard category $\mathcal{P}$, there exist abelian groups G and A and a symmetric monoidal equivalence*

$$\Sigma(G, A, 0) \simeq \mathcal{P}.$$

(ii) *For any abelian groups G, G', A and A', there is a symmetric monoidal equivalence*

$$\Sigma(G, A, 0) \simeq \Sigma(G', A', 0)$$

if and and only if there are isomorphisms $G \cong G'$ and $A \cong A'$.

Proof: (i) Let $\mathcal{P}$ be a strictly commutative Picard category. By Theorem 3.1, there are a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A}$, a 3-cocycle $h \in Z^3_{\mathrm{G}}(M, \mathcal{A})$, and a symmetric monoidal equivalence $\Sigma(M, \mathcal{A}, h) \simeq \mathcal{P}$.

Then, $\Sigma(M, \mathcal{A}, h)$ is a strictly commutative Picard category as $\mathcal{P}$ is and, therefore, for any $a \in M$, it must exist another $a^* \in M$ with a morphism $a \otimes a^* = aa^* \to \mathrm{I} = e$ in $\Sigma(M, \mathcal{A}, h)$. Since the groupoid $\Sigma(M, \mathcal{A}, h)$ is totally disconnected, it must be $aa^* = e$ in M, which means that $a^* = a^{-1}$ is an inverse of a in M. Therefore, $M = G$ is actually an abelian group.

Let $A = \mathcal{A}(e)$ be the abelian group attached by $\mathcal{A}$ at the unit of G. Then, a natural isomorphism $\phi : A \cong \mathcal{A}$ is defined such that, for any $a \in G$, $\phi_a = a_* : A = \mathcal{A}(e) \to \mathcal{A}(a)$. Therefore, Theorem 3.1$(ii)$ and Lemma 3.4 give the existence of a symmetric monoidal equivalence

$$\Sigma(G, \mathcal{A}, h) \simeq \Sigma(G, A, 0),$$

whence a symmetric monoidal equivalence $\Sigma(G, A, 0) \simeq \mathcal{P}$ follows.

(ii) This follows directly form Theorem 3.1(ii). □

Remark 3.2 As in Remark 3.1, the classification result above can be formulated in terms of an equivalence between the category of strictly commutative Picard categories, with iso-classes of symmetric monoidal functors as morphisms, and the category of pairs (G, A) of abelian groups, with morphisms

$$(i, \psi, k) : (G, A) \to (G', A')$$

triplets consisting of two group homomorphisms $i : G \to G'$, $\psi : A \to A'$, and a cohomology class $k \in H^2_{\mathrm{G}}(G, A') = \mathrm{Ext}_{\mathbb{Z}}(G, A')$, where composition is given by

$$(i', \psi', k')(i, \psi, k) = (i'i, \psi'\psi, \psi'_*(k) + i^*(k')).$$

Chapter 4

A cohomology theory for commutative monoids

Leech cohomology groups of monoids are useful for the classification of interesting monoidal structures (see 1.3, for instance). Nevertheless, when dealing with commutative monoids, Grillet symmetric cohomology groups (see 3.1.1) keep track of their commutativity, unlike Leech cohomology groups. To some extent, however, Grillet symmetric cohomology theory at degrees greater than 2 seem to be a little too 'strict' (for example, when $M = G$ is any abelian group, its symmetric 3-cohomology groups $H^3_{\mathrm{G}}(G, \mathcal{A})$ are all zero, see Lemma 3.4). Therefore, in this chapter, we present a different approach for a cohomology theory of commutative monoids, which is inspired in the (second level) *cohomology of abelian groups* by Eilenberg and Mac Lane [31, 55] and based on the cohomology theory of simplicial sets by Gabriel and Zisman [37, Appendix II].

In the same manner that every monoid M, regarded as a constant simplicial monoid, has associated a *classifying* simplicial set $\overline{W}M$ [31] satisfying that, for any $\mathbb{D}M$-module $\mathcal{A}$, $H^n_{\mathrm{L}}(M, \mathcal{A}) = H^n(\overline{W}M, \mathcal{A})$ (see Section 4.1), when the monoid M is commutative it also has associated an iterated classifying simplicial set $\overline{W}(\overline{W}M)$. The Gabriel-Zisman's cohomology groups of this simplicial set are used to define the *commutative cohomology groups* of M, denoted $H^n_{\mathrm{c}}(M, \mathcal{A})$, by

$$H^n_{\mathrm{c}}(M, \mathcal{A}) = H^{n+1}(\overline{W}^2 M, \mathcal{A}),$$

where $\mathcal{A}$ is an $\mathbb{H}M$-module (see 3.1.1). For instance, when $M = G$ is an abelian group, as the simplicial set $\overline{W}^2 G$ is an Eilenberg-Mac Lane's minimal complex $K(G, 2)$, for any abelian group A (regarded as a constant coefficient system on G), the commutative cohomology groups $H^n_{\mathrm{c}}(G, A)$ are precisely the Eilenberg-Mac Lane cohomology groups of the abelian group G with coefficients in A [31, 55] (also denoted by $H^n_{\mathrm{ab}}(G, A)$ in [22, 50]).

In this chapter, we are mainly interested in the cohomology groups $H^n_{\mathrm{c}}(M, \mathcal{A})$ for $n \leq 3$ for any commutative monoid with coefficients in an $\mathbb{H}M$-module. Hence, in

Section 4.2 most of our work is dedicated to showing how these commutative cohomology groups can be defined 'concretely' by manageable and computable *commutative cocycles*, such as Grillet did for the cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ by using *symmetric cocycles*. Thus, for any $\mathbb{H}M$-module $\mathcal{A}$ on a commutative monoid M, we exhibit a 4-truncated complex of *commutative cochains* $C^\bullet_{\mathrm{c}}(M,\mathcal{A})$, such that

$$H^n_{\mathrm{c}}(M,\mathcal{A}) \cong H^n C^\bullet_{\mathrm{c}}(M,\mathcal{A}), \quad n \leq 3,$$

whose construction is based on the construction of the reduced complexes $A(G,2)$ by Eilenberg and Mac Lane [31] to compute the (co)homology groups of the spaces $K(G,2)$. Furthermore, the existence of a monomorphism $C^\bullet_{\mathrm{G}}(M,\mathcal{A}) \hookrightarrow C^\bullet_{\mathrm{c}}(M,\mathcal{A})$, where the first is Grillet's 4-truncated complex of symmetric cochains in (3.1), easily allows one to state the relationships among the symmetric, commutative, and Leech low dimensional cohomology groups of commutative monoids (see Theorem 4.2):

$$\begin{aligned}
H^1_{\mathrm{G}}(M,\mathcal{A}) &\cong H^1_{\mathrm{c}}(M,\mathcal{A}) \cong H^1_{\mathrm{L}}(M,\mathcal{A}),\\
H^2_{\mathrm{G}}(M,\mathcal{A}) &\cong H^2_{\mathrm{c}}(M,\mathcal{A}) \hookrightarrow H^2_{\mathrm{L}}(M,\mathcal{A}),\\
H^3_{\mathrm{G}}(M,\mathcal{A}) &\hookrightarrow H^3_{\mathrm{c}}(M,\mathcal{A}) \to H^3_{\mathrm{L}}(M,\mathcal{A}),
\end{aligned}$$

where, in general, the inclusions $H^2_{\mathrm{c}}(M,\mathcal{A}) \hookrightarrow H^2_{\mathrm{L}}(M,\mathcal{A})$ and $H^3_{\mathrm{G}}(M,\mathcal{A}) \hookrightarrow H^3_{\mathrm{c}}(M,\mathcal{A})$ are strict, whereas the homomorphism $H^3_{\mathrm{c}}(M,\mathcal{A}) \to H^3_{\mathrm{L}}(M,\mathcal{A})$ is neither injective nor surjective.

For $n = 1, 2$, because of the the isomorphisms $H^n_{\mathrm{G}}(M,\mathcal{A}) \cong H^n_{\mathrm{c}}(M,\mathcal{A})$, there is nothing new to say about how to interpret these latter: Elements of $H^1_{\mathrm{c}}(M,\mathcal{A})$ are derivations, and elements of $H^2_{\mathrm{c}}(M,\mathcal{A})$ are iso-classes of (abelian-group) commutative monoid coextensions.

Then, in Section 4.3 of the chapter, we focus our attention on the commutative cohomology groups $H^3_{\mathrm{c}}(M,\mathcal{A})$, to whose elements we give a natural interpretation in terms of equivalence classes of *braided monoidal abelian groupoids* $(\mathcal{M},\otimes,\boldsymbol{c})$, that is, monoidal abelian groupoids $(\mathcal{M},\otimes)$ endowed with coherent and natural isomorphisms (the *braidings*) $\boldsymbol{c}_{x,y} : x\otimes y \cong y \otimes x$ [50], defined as for strictly symmetric abelian monoids (subsection 3.1.2) but now not necessarily satisfying the symmetry condition $\boldsymbol{c}_{x,y}\circ \boldsymbol{c}_{y,x} = id_{x\otimes y}$ nor the strictness condition $\boldsymbol{c}_{x,x} = id_{x\otimes x}$. The result, which was in fact our main motivation to seek the cohomology theory we present, can be summarized as follows (see Theorem 4.3 for details): Stating that any two triplets $(M,\mathcal{A},k)$ and $(M',\mathcal{A}',k')$, where $k \in H^3_{\mathrm{c}}(M,\mathcal{A})$ and $k' \in H^3_{\mathrm{c}}(M',\mathcal{A}')$, are isomorphic whenever there are isomorphisms $i : M \cong M'$ and $\psi : \mathcal{A} \cong i^*\mathcal{A}'$, such that $\psi_*^{-1} i^* k' = k$, then

> "*There is a one-to-one correspondence between equivalence classes of braided monoidal abelian groupoids* $(\mathcal{M},\otimes,\boldsymbol{c})$ *and classes of triplets* $(M,\mathcal{A},k)$*, with* $k \in H^3_{\mathrm{c}}(M,\mathcal{A})$*.*"

This classification theorem, which extends that given by Joyal and Street in [50, Section 3] for braided categorical groups, leads to bijections

$$H^3_{\mathrm{c}}(M,\mathcal{A}) \cong \mathrm{Ext}^2_{\mathrm{c}}(M,\mathcal{A})$$

expressing a natural interpretation of commutative 3-cohomology classes as equivalence classes of certain commutative 2-dimensional coextensions of M by $\mathcal{A}$.

4.1 Preliminaries on cohomology of simplicial sets

Cohomology theory of small categories (defined in (1.50)) is in itself a basis for other cohomology theories, in particular for the cohomology theory of simplicial sets with twisted coefficients defined by Gabriel and Zisman in [37, Appendix II]. Briefly, recall that the simplicial category, Δ, consists of the finite ordered sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, with weakly order-preserving maps between them, and that the category of simplicial sets is the category of functors $X : \Delta^{op} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of sets, with morphisms the natural transformations. The category Δ is generated by the injections $d^i : [n-1] \to [n]$ (cofaces), which omit the ith element, and the surjections $s^i : [n+1] \to [n]$ (codegeneracies), which repeat the ith element, $0 \leq i \leq n$, subjet to the well-known cosimplicial identities: $d^j d^i = d^i d^{j-1}$ if $i < j$, etc. (see [58]). Hence, in order to define a simplicial set it suffices to give the sets of its n-simplices $X_n = X([n])$ together with maps

$$\begin{aligned} d_i &= (d^i)^* : X_n \to X_{n-1}, \quad 0 \leq i \leq n \quad \text{(the face maps)},\\ s_i &= (s^i)^* : X_n \to X_{n+1}, \quad 0 \leq i \leq n \quad \text{(the degeneracy maps)}, \end{aligned} \tag{4.1}$$

satisfying the well-known basic simplicial identities: $d_i d_j = d_{j-1} d_i$ if $i < j$, etc. The *category of simplices* of a simplicial set X, Δ/X, has as objects pairs (x, n) with $x \in X_n$, and a morphism $(\alpha, x) : (\alpha^* x, m) \to (x, n)$ consists of a map $\alpha : [m] \to [n]$ in Δ together with a simplex $x \in X_n$. A *coefficient system on* X is a functor $\mathcal{A} : \Delta/X \to$ $\mathbf{Ab}$, and the *cohomology groups of the simplicial set* X *with coefficients in* $\mathcal{A}$ are, by definition,

$$H^n(X, \mathcal{A}) = H^n(\Delta/X, \mathcal{A}).$$

We point out below two useful facts. The first of them is an easy consequence of being the maps d^i, s^j and the cosimplicial identities a set of generators and relations for Δ, and the second one is dual of Theorem 4.2 in [37, Appendix II] and takes into account the Normalization Theorem.

Fact 4.1 *Let X be a simplicial set. In order to define a functor $\pi : \Delta/X \to \mathbb{C}$ it suffices to give objects $\pi x \in \mathbb{C}$, $x \in X_n$, $n \geq 0$, together with morphisms*

$$\pi d_i x \xrightarrow{\pi(d^i,x)} \pi x \xleftarrow{\pi(s^i,x)} \pi s_i x, \quad x \in X_n,\ 0 \leq i \leq n,$$

satisfying the equations:

$$\begin{cases} \pi(d^j,x)\pi(d^i,d_jx) = \pi(d^i,x)\pi(d^{j-1},d_ix) : \pi d_id_jx \to \pi x, & i<j, \\ \pi(s^j,x)\pi(d^i,s_jx) = \pi(d^i,x)\pi(s^{j-1},d_ix) : \pi d_is_jx \to \pi x, & i<j, \\ \pi(s^i,x)\pi(d^i,s_ix) = id_{\pi x} = \pi(s^i,x)\pi(d^{i+1},s_ix) : \pi d_is_ix \to \pi x, & \\ \pi(s^j,x)\pi(d^i,s_jx) = \pi(d^{i-1},x)\pi(s^j,d_{i-1}x) : \pi d_is_jx \to \pi x, & i>j+1, \\ \pi(s^j,x)\pi(s^i,s_jx) = \pi(s^i,x)\pi(s^{j+1},s_ix) : \pi s_is_jx \to \pi x, & i \leq j. \end{cases}$$

If $\mathcal{A} : \Delta/X \to \mathbf{Ab}$ is any coefficient system on a simplicial set X, then, for any simplex $x \in X_n$, we denote by $\mathcal{A}(x)$ the abelian group $\mathcal{A}(x)$, and by $(\alpha,x)_* : \mathcal{A}(\alpha^*x) \to \mathcal{A}(x)$ the homomorphism $\mathcal{A}(\alpha,x)$ associated to any morphism (α,x) in Δ/X.

Fact 4.2 *Let $\mathcal{A} : \Delta/X \to \mathbf{Ab}$ be a coefficient system on a simplicial set X. A n-cochain of X with coefficients in $\mathcal{A}$ is a map $\lambda : X_n \to \bigcup_{x\in X_n}\mathcal{A}(x)$ such that $\lambda(x) \in \mathcal{A}(x)$ for each $x \in X_n$. Thus, $\prod_{x\in X_n}\mathcal{A}(x)$ is the abelian group of such n-cochains. As $n \geq 0$ varies, these define a cosimplicial abelian group $\Delta \to \mathbf{Ab}$, $[n] \mapsto \prod_{x\in X_n}\mathcal{A}(x)$, whose cosimplicial operators*

$$\prod_{x\in X_{n-1}}\mathcal{A}(x) \xrightarrow{d^i_*} \prod_{x\in X_n}\mathcal{A}(x) \xleftarrow{s^i_*} \prod_{x\in X_{n+1}}\mathcal{A}(x),$$

$0 \leq i \leq n$, are respectively given by the formulas

$$d^i_*(\lambda)(x) = (d^i,x)_*(\lambda(d_ix)), \quad s^i_*(\lambda)(x) = (s^i,x)_*(\lambda(s_ix)).$$

Then, if

$$C^\bullet(X,\mathcal{A}) : \ \ 0 \to C^0(X,\mathcal{A}) \to \cdots \to C^n(X,\mathcal{A}) \xrightarrow{\partial} C^{n+1}(X,\mathcal{A}) \to \cdots$$

denotes its associated normalized cochain complex, where

$$C^n(X,\mathcal{A}) = \bigcap_{i=0}^{n-1} \ker(s^i_* : \textstyle\prod_{x\in X_n}\mathcal{A}(x) \to \prod_{x\in X_{n-1}}\mathcal{A}(x)),$$

is the abelian group of normalized n-cochains, with coboundary $\partial = \sum(-1)^i d^i_*$, there is a natural isomorphism

$$H^n(X,\mathcal{A}) \cong H^n(C^\bullet(X,\mathcal{A})).$$

Many cohomology theories for algebraic systems find fundament in the cohomology of simplicial sets. In particular, Leech cohomology theory for monoids (see subsection 1.3.1), as we explain below. Previously, recall that a *simplicial monoid* is a contravariant functor from the simplicial category to the category of monoids, $X : \Delta^{op} \to \mathbf{Mnd}$.

Thus, each X_n is a monoid and the face and degeneracy operators (4.1) are homomorphisms. Every simplicial monoid X has associated a *classifying* simplicial set

$$\overline{W}X : \Delta^{op} \to \mathbf{Set}, \tag{4.2}$$

which is defined as follows (this is WX in [31]): $(\overline{W}X)_0 = \{e\}$, the unitary set, and

$$(\overline{W}X)_{n+1} = X_n \times X_{n-1} \times \cdots \times X_0.$$

Write the elements of $(\overline{W}X)_{n+1}$ in the form $(x_n, \ldots, x_0)$. The face and degeneracy maps are defined by $s_0(e) = (e)$, by $d_i(x_0) = e$, $i = 0, 1$, and for $n > 0$ by

$$\begin{cases} d_0(x_n, \ldots, x_0) = (x_{n-1}, \ldots, x_0), & \\ d_{i+1}(x_n, \ldots, x_0) = (d_i x_n, \ldots, d_1 x_{n-i+1}, d_0 x_{n-i} \cdot x_{n-i-1}, x_{n-i-2}, \ldots, x_0), & i < n, \\ d_{n+1}(x_n, \ldots, x_0) = (d_n x_n, \ldots, d_1 x_1), & \\ s_0(x_n, \ldots, x_0) = (e, x_n, \ldots, x_0), & \\ s_{i+1}(x_n, \ldots, x_0) = (s_i x_n, \ldots, s_0 x_{n-i}, e, x_{n-i-1}, \ldots, x_0), & i < n, \\ s_{n+1}(x_n, \ldots, x_0) = (s_n x_n, \ldots, s_0 x_0, e), & \end{cases}$$

where e is the unit in the corresponding monoid.

For example, given any monoid M, let $M : \Delta^{op} \to \mathbf{Mnd}$ denote the constant M simplicial monoid, that is, the simplicial monoid given by $M_n = M$, $n \geq 0$, and by letting each d_i and s_i on M_n be the identity map on M. Then the $\overline{W}$-construction on it produces the so-called classifying simplicial set of the monoid

$$\overline{W}M : \Delta^{op} \to \mathbf{Set}, \quad [n] \mapsto M^n, \tag{4.3}$$

whose face and degeneracy maps are given by the familiar formulas

$$\begin{aligned} d_i(a_1, \ldots, a_n) &= \begin{cases} (a_2, \ldots, a_n) & i = 0, \\ (a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n) & 0 < i < n, \\ (a_1, \ldots, a_{n-1}) & i = n, \end{cases} \\ s_i(a_1, \ldots, a_n) &= (a_1, \ldots, a_{i-1}, e, a_i, \ldots, a_n) \qquad\qquad 0 \leq i \leq n. \end{aligned}$$

where $e \in M$ the unit.

Recall now the category $\mathbb{D}M$ from 1.3.1. Then, there is a functor $\pi : \Delta/\overline{W}M \to \mathbb{D}M$ such that $\pi(a_1, \ldots, a_n) = a_1 \cdots a_n$, and

$$\begin{aligned} \pi(d^i, (a_1, \ldots, a_n)) &= \begin{cases} (a_1, a_2 \cdots a_n, e) : a_2 \cdots a_n \to a_1 \cdots a_n, & i = 0, \\ id : a_1 \cdots a_n \to a_1 \cdots a_n, & 0 < i < n, \\ (e, a_1 \cdots a_{n-1}, a_n) : a_1 \cdots a_{n-1} \to a_1 \cdots a_n, & i = n, \end{cases} \\ \pi(s^i, (a_1, \ldots, a_n)) &= id : a_1 \cdots a_n \to a_1 \cdots a_n, \qquad\qquad 0 \leq i \leq n. \end{aligned}$$

Then, by composition with π, any $\mathbb{D}M$-module $\mathcal{A}$ defines a coefficient system on $\overline{W}M$, also denoted by $\mathcal{A} : \Delta/\overline{W}M \to \mathbf{Ab}$, and therefore the cohomology groups

$H^n(\overline{W}M, \mathcal{A})$ are defined. By Fact 4.2, these cohomology groups can be computed from the cochain complex $C^\bullet(\overline{W}M, \mathcal{A})$, which is given in degree $n > 0$ by

$$C^n(\overline{W}M, \mathcal{A}) = \Big\{\lambda \in \prod_{(a_1,\dots,a_n)\in M^n} \mathcal{A}(a_1\cdots a_n) \mid \lambda(a_1,\dots,a_n) = 0 \text{ if some } a_i = e\Big\}$$

and $C^0(\overline{W}M, \mathcal{A}) = \mathcal{A}(e)$. The coboundary $\partial^n : C^n(\overline{W}M, \mathcal{A}) \to C^{n+1}(\overline{W}M, \mathcal{A})$ is given, for $n = 0$, by $(\partial^0\lambda)(a) = a_*\lambda - a^*\lambda$, while, for $n > 0$,

$$\begin{aligned}(\partial^n\lambda)(a_1,\dots,a_{n+1}) = (a_1)_*\lambda(a_2,\dots,a_n) + \sum_{i=1}^{n}(-1)^i\lambda(a_1,\dots,a_ia_{i+1},\dots,a_{n+1})\\ + (-1)^{n+1}(a_{n+1})^*\lambda(a_1,\dots,a_n).\end{aligned}$$

Observe now that this cochain complex $C^\bullet(\overline{W}M, \mathcal{A})$ is precisely the one defined in (1.51), and thus it follows that there are natural isomorphisms

$$H^n_{\mathrm{L}}(M, \mathcal{A}) \cong H^n(\overline{W}M, \mathcal{A}).$$

4.2 A cohomology theory for commutative monoids

Let us return now to the case where M is a *commutative* monoid, whose unit is e. Under this hypothesis, the simplicial set $\overline{W}M$ in (4.3) is again a simplicial monoid, with the product monoid structure on each M^n. We can then perform the $\overline{W}$-construction (4.2) on it, which gives the simplicial set (actually, a commutative simplicial monoid)

$$\overline{W}^2M : \Delta^{op} \to \mathbf{Set},$$

whose set of n-simplices is

$$(\overline{W}^2M)_n = \begin{cases} \{e\} & i = 0, 1,\\ M^{n-1} \times M^{n-2} \times \cdots \times M & i \geq 2.\end{cases}$$

Writing an $n+1$-simplex x of $\overline{W}^2M$ in the form

$$x = \big(x^k_j\big)_{1\le j\le k\le n} = ((x^n_1,\dots,x^n_n),\dots,(x^2_1,x^2_2),x^1_1), \tag{4.4}$$

where each $(x^k_1,\dots,x^k_k) \in M^k$ is a k-simplex of $\overline{W}M$, its faces and degeneracies are respectively defined by $d_i(x) = \big(y^m_l\big)$ and $s_i(x) = \big(z^v_u\big)$, where

$$y^m_l = \begin{cases} x^m_l & m<n-i,\\ x^{m+1}_{l+1}x^m_l & m=n-i,\\ x^{m+1}_l & m>n-i, l<m-n+i,\\ x^{m+1}_l x^{m+1}_{l+1} & m>n-i, l=m-n+i,\\ x^{m+1}_{l+1} & m>n-i, l>m-n+i,\end{cases} \qquad z^v_u = \begin{cases} x^v_u & v\le n-i,\\ e & v=n-i+1,\\ x^{v-1}_u & v>n-i+1, u<v-n+i-1,\\ e & v>n-i+1, u=v-n+i-1,\\ x^{v-1}_{u-1} & v>n-i+1, u>v-n+i-1.\end{cases}$$

Recall now, from 3.1.1, that abelian group valued functors on the category $\mathbb{H}M$ provide the coefficients for Grillet cohomology groups of a commutative monoid M. There is a functor $\pi : \Delta/\overline{W}^2 M \to \mathbb{H}M$ which, taking into account Fact 4.1, is determined by $\pi x = \prod x_j^k$, for each $n+1$-simplex $x = \big(x_j^k\big)_{1\leq j\leq k\leq n}$ of $\overline{W}^2 M$ as in (4.4), where the product $\prod x_j^k$ is in the monoid M over all $0 \leq j \leq k \leq n$, together with the homomorphisms

$$\pi(d^i, x) = \begin{cases} (\pi d_0 x, x_1^n x_2^n \cdots x_n^n) : \pi d_0 x \to \pi x, & i = 0, \\ (\pi d_i x, x_1^{n+1-i}) : \pi d_i x \to \pi x, & 0 < i \leq n, \\ (\pi d_{n+1} x, x_n^n x_{n-1}^{n-1} \cdots x_1^1) : \pi d_{n+1} x \to \pi x, & i = n+1, \end{cases}$$
$$\pi(s^i, x) = id : \pi s_i x = \pi x \to \pi x, \qquad 0 \leq i \leq n.$$

Therefore, by composition with π, any $\mathbb{H}M$-module gives rise to a coefficient system on the simplicial set $\overline{W}^2 M$, equally denoted by

$$\mathcal{A} : \Delta/\overline{W}^2 M \to \mathbf{Ab},$$

whence the cohomology groups of $\overline{W}^2 M$ with coefficients in $\mathcal{A}$ are defined. Note that these cohomology groups are trivial at dimensions 0 and 1. Then, making a dimensional shift, we state the following definition.

Definition 4.1 *Let M be a commutative monoid. For any $\mathbb{H}M$-module $\mathcal{A}$, the* commutative cohomology groups *of M with coefficients in $\mathcal{A}$, denoted $H_c^n(M, \mathcal{A})$, are defined by*

$$H_c^n(M, \mathcal{A}) = H^{n+1}(\overline{W}^2 M, \mathcal{A}), \quad n \geq 1.$$

Example 4.1 Let $M = G$ be an abelian group. Then, the simplicial set $\overline{W}^2 G$ is an Eilenberg-Mac Lane's minimal complex $K(G, 2)$ [31, Theorem 17.4], [59, Theorem 23.2]. For any abelian group A, regarded as a constant functor $A : \mathbb{H}G \to \mathbf{Ab}$, the commutative cohomology groups $H_c^n(G, A) = H^{n+1}(K(G, 2), A)$ define the *second level* or *abelian Eilenberg-Mac Lane cohomology theory of the abelian group* G [28, 29, 30, 31, 55] (these are denoted also by $H_{\mathrm{ab}}^n(G, A)$ in [50, 22] and by $H_1^n(G, A)$ in [11]). Although these cohomology groups arise from algebraic topology, they come with algebraic interest. Briefly, recall that there are natural isomorphisms [32, (26.1), (26.3), (26.4)]

$$H_c^1(G, A) \cong \mathrm{Hom}(G, A), \quad H_c^2(G, A) \cong \mathrm{Ext}(G, A), \quad H_c^3(G, A) \cong \mathrm{Quad}(G, A),$$

where $\mathrm{Hom}(G, A)$ is the group of homomorphisms from G to A, $\mathrm{Ext}(G, A)$ is the group of abelian group extensions of G by A, and $\mathrm{Quad}(G, A)$ is the abelian group of quadratic maps from G to A, that is, functions $q : G \to A$ such that $f(x, y) = q(x+y) - q(x) - q(y)$ is a bilinear function of $x, y \in G$. A precise classification theorem for *braided categorical groups* [50, Definition 3.1] in terms of cohomology classes $k \in H_c^3(G, A)$

was proved by Joyal and Street in [50, Theorem 3.3] (see Corollary 4.1 for an approach here to that issue).

Let us stress that, among the $\mathrm{Ext}^{\mathrm{n}}$ groups in the category of abelian groups, only $\mathrm{Ext}^0(G,A)\cong H^1_{\mathrm{c}}(G,A)$ and $\mathrm{Ext}^1(G,A)\cong H^2_{\mathrm{c}}(G,A)$ are relevant since all groups $\mathrm{Ext}^n(G,A)$ vanish for $n\geq 2$. However, for example, it holds that $H^3_{\mathrm{c}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})\cong \mathbb{Z}/2\mathbb{Z}\neq 0$.

In this chapter we are only interested in the cohomology groups $H^n_{\mathrm{c}}(M,\mathcal{A})$ for $n\leq 3$. Both for theoretical and computational interests, it is appropriate to have a more manageable cochain complex than $C^\bullet(\overline{W}^2M,\mathcal{A})$ to compute the lower commutative cohomology groups $H^n_{\mathrm{c}}(M,\mathcal{A})$, such as Grillet did for computing the cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ by means of symmetric cochains (see Subsection 3.1.1). We shall exhibit below such a (truncated) complex, denoted by

$$C^\bullet_{\mathrm{c}}(M,\mathcal{A}):\ 0\to C^1_{\mathrm{c}}(M,\mathcal{A})\xrightarrow{\partial^1} C^2_{\mathrm{c}}(M,\mathcal{A})\xrightarrow{\partial^2} C^3_{\mathrm{c}}(M,\mathcal{A})\xrightarrow{\partial^3} C^4_{\mathrm{c}}(M,\mathcal{A}), \tag{4.5}$$

and referred to as the complex of (normalized) *commutative cochains* on M with values in $\mathcal{A}$. The construction of this complex is heavily inspired on that given by Eilenberg and Mac Lane of the complexes $A(G,2)$ [31] for computing the (co)homology groups of the spaces $K(G,2)$, and it is as follows:

- A *commutative* 1*-cochain* $f\in C^1_{\mathrm{c}}(M,\mathcal{A})$ is a function $f:M\to\bigcup_{a\in M}\mathcal{A}(a)$ with $f(a)\in\mathcal{A}(a)$, such that $f(e)=0$.

- A *commutative* 2*-cochain* $g\in C^2_{\mathrm{c}}(M,\mathcal{A})$ is a function $g:M^2\to\bigcup_{a\in M}\mathcal{A}(a)$ with $g(a,b)\in\mathcal{A}(ab)$, such that $g(a,b)=0$ if a or b are equal to e.

- A *commutative* 3*-cochain* $(h,\mu)\in C^2_{\mathrm{c}}(M,\mathcal{A})$ is a pair of functions

$$h:M^3\to\textstyle\bigcup_{a\in M}\mathcal{A}(a),\quad \mu:M^2\to\bigcup_{a\in M}\mathcal{A}(a)$$

with $h(a,b,c)\in\mathcal{A}(abc)$ and $\mu(a,b)\in\mathcal{A}(ab)$, such that $h(a,b,c)=0$ whenever some of a,b, or c are equal to e, and $\mu(a,b)=0$ if a or b are equal to e.

- A *commutative* 4*-cochain* $(t,\gamma,\delta)\in C^2_{\mathrm{c}}(M,\mathcal{A})$ is a triplet of functions

$$t:M^4\to\textstyle\bigcup_{a\in M}\mathcal{A}(a),\quad \gamma,\delta:M^3\to\bigcup_{a\in M}\mathcal{A}(a)$$

with $t(a,b,c,d)\in\mathcal{A}(abcd)$ and $\gamma(a,b,c),\delta(a,b,c)\in\mathcal{A}(abc)$, such that $t(a,b,c,d)=0$ whenever some of a,b,c, or d are equal to e, and $\gamma(a,b,c)=0=\delta(a,b,c)$ if some of a,b, or c are equal to e.

Under pointwise addition, these commutative n-cochains form the abelian groups $C^n_{\mathrm{c}}(M,\mathcal{A})$ in (4.5), $1\leq n\leq 4$. The coboundary homomorphisms are defined by

$\partial^1 f=g$, where

$$g(a,b)=-a_*f(b)+f(ab)-b_*f(a),$$

$\partial^2 g = (h, \mu)$, where

$$\begin{aligned} h(a,b,c) &= -a_*g(b,c) + g(ab,c) - g(a,bc) + c_*g(a,b), \\ \mu(a,b) &= g(a,b) - g(b,a), \end{aligned}$$

$\partial^3(h,\mu) = (t, \gamma, \delta)$, where

$$\begin{aligned} t(a,b,c,d) &= -a_*h(b,c,d) + h(ab,c,d) - h(a,bc,d) + h(a,b,cd) - d_*h(a,b,c), \\ \gamma(a,b,c) &= -b_*\mu(a,c) + \mu(a,bc) - c_*\mu(a,b) + h(a,b,c) - h(b,a,c) + h(b,c,a), \\ \delta(a,b,c) &= -a_*\mu(b,c) + \mu(ab,c) - b_*\mu(a,c) - h(a,b,c) + h(a,c,b) - h(c,a,b). \end{aligned}$$

A quite straightforward verification shows that (4.5) is actually a truncated cochain complex, that is, the equalities $\partial^2\partial^1 = 0$ and $\partial^3\partial^2 = 0$ hold.

A basic result here is the following, whose proof is quite long and technical and we give it in Subsection 4.2.1 so as not to obstruct the natural flow of the chapter.

Theorem 4.1 *Let M be any commutative monoid and let $\mathcal{A}$ be an $\mathbb{H}M$-module. For each $n \leq 3$, there is a natural isomorphism*

$$H^n_{\mathrm{c}}(M,\mathcal{A}) \cong H^n\big(C^{\bullet}_{\mathrm{c}}(M,\mathcal{A})\big). \tag{4.6}$$

From this theorem, for $n \leq 3$, we have isomorphisms

$$H^n_{\mathrm{c}}(M,\mathcal{A}) \cong Z^n_{\mathrm{c}}(M,\mathcal{A})/B^n_{\mathrm{c}}(M,\mathcal{A}) \tag{4.7}$$

where

$$\begin{aligned} Z^n_{\mathrm{c}}(M,\mathcal{A}) &= \mathrm{Ker}\big(\partial^n : C^n_{\mathrm{c}}(M,\mathcal{A}) \to C^{n+1}_{\mathrm{c}}(M,\mathcal{A})\big), \\ B^n_{\mathrm{c}}(M,\mathcal{A}) &= \mathrm{Im}\big(\partial^{n-1} : C^{n-1}_{\mathrm{c}}(M,\mathcal{A}) \to C^n_{\mathrm{c}}(M,\mathcal{A})\big), \end{aligned}$$

are referred as the groups of *commutative n-cocycles* and *commutative n-coboundaries* on M with values in $\mathcal{A}$, respectively.

After Theorem 4.1 and the isomorphisms in (3.7), Grillet symmetric cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$ and the commutative ones $H^n_{\mathrm{c}}(M,\mathcal{A})$ are closely related, for $n \leq 3$ through the natural injective cochain map

$$\begin{tikzcd} 0 \arrow[r] & C^1_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^1"] \arrow[d, equal, "i_1=id"'] & C^2_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^2"] \arrow[d, hook, "i_2"'] & C^3_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^3"] \arrow[d, hook, "i_3"'] & C^4_{\mathrm{G}}(M,\mathcal{A}) \arrow[d, hook, "i_4"'] \\ 0 \arrow[r] & C^1_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^1"] & C^2_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^2"] & C^3_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^3"] & C^4_{\mathrm{c}}(M,\mathcal{A}), \end{tikzcd}$$

which is the identity map, $i_1(f) = f$, on 1-cochains, the inclusion map, $i_2(g) = g$, on 2-cochains, and on 3- and 4-cochains is defined by the simple formulas $i_3(h) = (h, 0)$ and $i_4(t) = (t, 0, 0)$, respectively. The only non-trivial verification here concerns the equality $\partial^3 i_3 = i_4\partial^3$, that is, $\partial^3(h,0) = (\partial^3 h,0,0)$, for any $h \in C^3_{\mathrm{G}}(M,\mathcal{A})$, but it easily follows from Lemma 3.1.

From now on, we shall regard the complex of symmetric cochains as a subcomplex of the complex of commutative cochains, via the above injective cochain map. Thus,

$$C^\bullet_{\mathrm{G}}(M,\mathcal{A}) \subseteq C^\bullet_{\mathrm{c}}(M,\mathcal{A}). \tag{4.8}$$

Theorem 4.2 *For any commutative monoid M, and any $\mathbb{H}M$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^1_{\mathrm{G}}(M,\mathcal{A}) \cong H^1_{\mathrm{c}}(M,\mathcal{A}), \tag{4.9}$$

$$H^2_{\mathrm{G}}(M,\mathcal{A}) \cong H^2_{\mathrm{c}}(M,\mathcal{A}), \tag{4.10}$$

and a natural monomorphism

$$H^3_{\mathrm{G}}(M,\mathcal{A}) \hookrightarrow H^3_{\mathrm{c}}(M,\mathcal{A}). \tag{4.11}$$

Proof: The equalities $Z^1_{\mathrm{G}}(M,\mathcal{A}) = Z^1_{\mathrm{c}}(M,\mathcal{A})$ and $B^2_{\mathrm{G}}(M,\mathcal{A}) = B^2_{\mathrm{c}}(M,\mathcal{A})$ are clear. Further $Z^2_{\mathrm{G}}(M,\mathcal{A}) = Z^2_{\mathrm{c}}(M,\mathcal{A})$, since the cocycle condition on a commutative 2-cochain g implies the symmetry condition $g(a,b) = g(b,a)$. Hence, the isomorphisms (4.9) and (4.10) follow from those in (3.7) and (4.7), for $n=1$ and $n=2$ respectively.

The homomorphism in (4.11) is the composite of

$$H^3_{\mathrm{G}}(M,\mathcal{A}) \overset{(3.7)}{\cong} H^3 C^\bullet_{\mathrm{G}}(M,\mathcal{A}) \xrightarrow{(4.8)} H^3 C^\bullet_{\mathrm{c}}(M,\mathcal{A}) \overset{(4.7)}{\cong} H^3_{\mathrm{c}}(M,\mathcal{A}),$$

so it suffices to prove that the induced by (4.8) on the third cohomology groups is injective. To do so, suppose $h \in C^3_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 3-cochain such that $i_3(h) = (h,0) \in B^2_{\mathrm{c}}(M,\mathcal{A})$ is a commutative 3-coboundary, that is, $(h,0) = \partial^2 g$ for some $g \in C^2_{\mathrm{c}}(M,\mathcal{A})$. This means that the equalities

$$h(a,b,c) = -a_*g(b,c) + g(ab,c) - g(a,bc) + c_*g(a,b), \quad 0 = g(a,b) - g(b,a),$$

hold, whence $g \in C^2_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 2-cochain, and $h = \partial^2 g \in B^2_{\mathrm{G}}(M,\mathcal{A})$, is actually a symmetric 2-coboundary. It follows that the inclusion map $i_3 : Z^3_{\mathrm{G}}(M,\mathcal{A}) \hookrightarrow Z^3_{\mathrm{c}}(M,\mathcal{A})$ induces a injective map in cohomology $H^3C^\bullet_{\mathrm{G}}(M,\mathcal{A})\big) \hookrightarrow H^3C^\bullet_{\mathrm{c}}(M,\mathcal{A})$, as required. $\square$

Remark 4.1 The inclusion $H^3_{\mathrm{G}}(M,\mathcal{A}) \subseteq H^3_{\mathrm{c}}(M,\mathcal{A})$ is, in general, strict. Let G be any abelian group, and let $A : \mathbb{H}G \to \mathbf{Ab}$ be the constant $\mathbb{H}G$-module defined by any other abelian group A, as in Example 4.1. Then, by Lemma 3.4 we have that $H^3_{\mathrm{G}}(G,A) = 0$. However, for instance, it holds that $H^3_{\mathrm{c}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \neq 0$.

From Theorem 4.2 and the interpretations given in Subsection 3.1.1 of Grillet cohomology groups we know that $H^1_{\mathrm{c}}(M,\mathcal{A})$ classifies derivations of M in $\mathcal{A}$ while $H^2_{\mathrm{c}}(M,\mathcal{A})$ classifies commutative group coextensions of M by $\mathcal{A}$.

4.2.1 Proof of Theorem 4.1.

We start by specifying the relevant truncation of the cochain complex $C^\bullet(\overline{W}^2M, \mathcal{A})$ that, by Fact 4.2, yields cocycles and coboundaries on the commutative monoid M at dimensions ≤ 3. To do so, we need to pay attention to the 6-dimensional truncated part of $\overline{W}^2M$:

$$\overline{W}^2M : \cdots \quad M^{10} \rightrightarrows M^6 \rightrightarrows M^3 \rightrightarrows M \rightrightarrows e \rightrightarrows e$$

whose face and degeneracy operators given by

$$d_i(b_1,b_2,a_1) = \begin{cases} a_1 & i=0, \\ b_2a_1 & i=1, \\ b_1b_2 & i=2, \\ b_1 & i=3; \end{cases} \quad d_i(c_1,c_2,c_3,b_1,b_2,a_1) = \begin{cases} (b_1,b_2,a_1) & i=0, \\ (c_2b_1,c_3b_2,a_1) & i=1, \\ (c_1c_2,c_3,b_2a_1) & i=2, \\ (c_1,c_2c_3,b_1b_2) & i=3, \\ (c_1,c_2,b_1) & i=4; \end{cases}$$

$$d_i(d_1,d_2,d_3,d_4,c_1,c_2,c_3,b_1,b_2,a_1) = \begin{cases} (c_1,c_2,c_3,b_1,b_2,a_1) & i=0, \\ (d_2c_1,d_3c_2,d_4c_3,b_1,b_2,a_1) & i=1, \\ (d_1d_2,d_3,d_4,c_2b_1,c_3b_2,a_1) & i=2, \\ (d_1,d_2d_3,d_4,c_1c_2,c_3,b_2a_1) & i=3, \\ (d_1,d_2,d_3d_4,c_1,c_2c_3,b_1b_2) & i=4, \\ (d_1,d_2,d_3,c_1,c_2,b_1) & i=5; \end{cases}$$

$$s_i(a_1) = \begin{cases} (e,e,a_1) & i=0, \\ (e,a_1,e) & i=1, \\ (a_1,e,e) & i=2; \end{cases} \quad s_i(b_1,b_2,a_1) = \begin{cases} (e,e,e,b_1,b_2,a_1) & i=0, \\ (e,b_1,b_2,e,e,a_1) & i=1, \\ (b_1,e,b_2,e,a_1,e) & i=2, \\ (b_1,b_2,e,a_1,e,e) & i=3; \end{cases}$$

$$s_i(c_1,c_2,c_3,b_1,b_2,a_1) = \begin{cases} (e,e,e,e,c_1,c_2,c_3,b_1,b_2,a_1) & i=0, \\ (e,c_1,c_2,c_3,e,e,e,b_1,b_2,a_1) & i=1, \\ (c_1,e,c_2,c_3,e,b_1,b_2,e,e,a_1) & i=2, \\ (c_1,c_2,e,c_3,b_1,e,b_2,e,a_1,e) & i=3, \\ (c_1,c_2,c_3,e,b_1,b_2,1,a_1,e,e) & i=4. \end{cases}$$

Hence, (with a dimensional shift) the cochain complex $C^\bullet(\overline{W}^2M, \mathcal{A})$ for low degrees is

$$0 \to C^1(\overline{W}^2M,\mathcal{A}) \xrightarrow{\partial} C^2(\overline{W}^2M,\mathcal{A}) \xrightarrow{\partial} C^3(\overline{W}^2M,\mathcal{A}) \xrightarrow{\partial} C^4(\overline{W}^2M,\mathcal{A}), \tag{4.12}$$

where:

A 1-*cochain* $\lambda \in C^1(\overline{W}^2 M, \mathcal{A})$ is a function $\lambda : M \to \bigcup_{a\in M} \mathcal{A}(a)$ with $\lambda(a) \in \mathcal{A}(a)$, such that $\lambda(e) = 0$.

A 2-*cochain* $\lambda \in C^2(\overline{W}^2 M, \mathcal{A})$ is a function

$$\lambda : M^2 \times M \to \textstyle\bigcup_{a\in M} \mathcal{A}(a),$$

with $\lambda(b_1, b_2, a_1) \in \mathcal{A}(b_1b_2a_1)$, such that $\lambda(e, e, a_1) = 0 = \lambda(e, a_1, e) = \lambda(a_1, e, e)$.

A 3-*cochain* $\lambda \in C^3(\overline{W}^2 M, \mathcal{A})$ is a function

$$\lambda : M^3 \times M^2 \times M \to \textstyle\bigcup_{a\in M} \mathcal{A}(a),$$

with $\lambda(c_1, c_2, c_3, b_1, b_2, a_1) \in \mathcal{A}(c_1c_2c_3b_1b_2a_1)$, such that

$$\begin{aligned} 0 &= \lambda(e, e, e, b_1, b_2, a_1) = \lambda(e, b_1, b_2, e, e, a_1) \\ &= \lambda(b_1, e, b_2, e, a_1, e) = \lambda(b_1, b_2, e, a_1, e, e). \end{aligned}$$

A 4-*cochain* $\lambda \in C^4(\overline{W}^2 M, \mathcal{A})$ is a function

$$\lambda : M^4 \times M^3 \times M^2 \times M \to \textstyle\bigcup_{a\in M} \mathcal{A}(a),$$

such that $\lambda(d_1, d_2, d_3, d_4, c_1, c_2, c_3, b_1, b_2, a_1) \in \mathcal{A}(d_1d_2d_3d_4c_1c_2c_3b_1b_2a_1)$, and

$$\begin{aligned} 0 = {} & \lambda(e, e, e, e, c_1, c_2, c_3, b_1, b_2, a_1) = \lambda(e, c_1, c_2, c_3, e, e, e, b_1, b_2, a_1) \\ = {} & \lambda(c_1, e, c_2, c_3, e, b_1, b_2, e, e, a_1) = \lambda(c_1, c_2, e, c_3, b_1, e, b_2, e, a_1, e) \\ = {} & \lambda(c_1, c_2, c_3, e, b_1, b_2, e, a_1, e, e). \end{aligned}$$

The coboundary homomorphisms are given by

$$(\partial^1\lambda)(b_1, b_2, a_1) = (b_1b_2)_*\lambda(a_1) - (b_1)_*\lambda(b_2a_1) + (a_1)_*\lambda(b_1b_2) - (b_2a_1)_*\lambda(b_1),$$

$$\begin{aligned} (\partial^2\lambda)(c_1, c_2, c_3, b_1, b_2, a_1) = {} & (c_1c_2c_3)_*\lambda(b_1, b_2, a_1) - (c_1)_*\lambda(c_2b_1, c_3b_2, a_1) \\ & + (b_1)_*\lambda(c_1c_2, c_3, b_2a_1) - (a_1)_*\lambda(c_1, c_2c_3, b_1b_2) \\ & + (c_3b_2a_1)_*\lambda(c_1, c_2, b_1), \end{aligned}$$

$$\begin{aligned} & (\partial^3\lambda)(d_1, d_2, d_3, d_4, c_1, c_2, c_3, b_1, b_2, a_1) = \\ & \qquad (d_1d_2d_3d_4)_*\lambda(c_1, c_2, c_3, b_1, b_2, a_1) - (d_1)_*\lambda(d_2c_1, d_3c_2, d_4c_3, b_1, b_2, a_1) \\ & \qquad + (c_1)_*\lambda(d_1d_2, d_3, d_4, c_2b_1, c_3b_2, a_1) - (b_1)_*\lambda(d_1, d_2d_3, d_4, c_1c_2, c_3, b_2a_1) \\ & \qquad + (a_1)_*\lambda(d_1, d_2, d_3d_4, c_1, c_2c_3, b_1b_2) - (d_4c_3b_2a_1)_*\lambda(d_1, d_2, d_3, c_1, c_2, b_1). \end{aligned}$$

Then, the claimed isomorphisms (4.6) follows from the existence of the following diagram of abelian group homomorphisms

$$
\begin{tikzcd}
0 \arrow[r] & C^1(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^1"] \arrow[d, "\phi_1"'] & C^2(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^2"] \arrow[d, "\phi_2"'] & C^3(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^3"] \arrow[d, "\phi_3"] \arrow[ddl, "\Gamma_2"'] & C^4(\overline{W}^2M,\mathcal{A}) \arrow[d, "\phi_4"] \arrow[ddl, "\Gamma_3"'] \\
0 \arrow[r] & C^1_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^1"] \arrow[d, "\psi_1"'] & C^2_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^2"] \arrow[d, "\psi_2"'] & C^3_{\mathrm{c}}(M,\mathcal{A}) \arrow[r, "\partial^3"] \arrow[d, "\psi_3"'] & C^4_{\mathrm{c}}(M,\mathcal{A}) \arrow[d, "\psi_4"] \\
0 \arrow[r] & C^1(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^1"] & C^2(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^2"] & C^3(\overline{W}^2M,\mathcal{A}) \arrow[r, "\partial^3"] & C^4(\overline{W}^2M,\mathcal{A})
\end{tikzcd}
$$

which satisfy the equalities: $\partial^n\phi_n = \phi_{n+1}\partial^n$ and $\partial^n\psi_n = \psi_{n+1}\partial^n$, for $1 \le n \le 3$; $\phi_n\psi_n = id$, for $0 \le n \le 4$; $\psi_1\phi_1 = id$; $\psi_2\phi_2 = \Gamma_2\partial^2 + id$; and $\psi_3\phi_3 = \Gamma_3\partial^3 + \partial^2\Gamma_2 + id$.

These homomorphisms are defined as follows:

- $\phi_1 = \psi_1 = id$;
- $\phi_2(\lambda) = g$, where $g(a,b) = \lambda(a,b,e)$;
- $\psi_2(g) = \lambda$, where $\lambda(b_1,b_2,a_1) = (a_1)_*g(b_1,b_2) - (b_1)_*g(b_2,a_1)$;
- $\Gamma_2(\lambda) = \lambda'$, where $\lambda'(b_1,b_2,a_1) = \lambda(b_1,b_2,e,e,a_1,e) - \lambda(b_1b_2,e,e,e,e,a_1)$;
- $\phi_3(\lambda) = (h,\mu)$, where

$$
\begin{aligned}
&h(a,b,c) = \lambda(a,b,c,e,e,e),\\
&\mu(a,b) = \lambda(a,e,e,e,e,b) - \lambda(e,a,e,e,b,e) + \lambda(e,e,a,b,e,e);
\end{aligned}
$$

- $\psi_3(h,\mu) = \lambda$, where

$$
\begin{aligned}
\lambda(c_1,c_2,c_3,b_1,b_2,a_1) = {} & (b_1b_2a_1)_*h(c_1,c_2,c_3) + (c_1c_2b_1)_*h(c_3,b_2,a_1)\\
& - (c_1c_2a_1)_*h(c_3,b_1,b_2) + (c_1c_2a_1)_*h(b_1,c_3,b_2)\\
& - (c_1a_1)_*h(c_2,b_1,c_3b_2) + (c_1a_1)_*h(c_2,c_3,b_1b_2)\\
& + (c_1c_2b_2a_1)_*\mu(c_3,b_1);
\end{aligned}
$$

- $\Gamma_3(\lambda) = \lambda'$, where

$$
\begin{aligned}
\lambda'(c_1,c_2,c_3,b_1,b_2,a_1) = {} & -\lambda(c_1c_2,e,e,c_3,e,e,e,b_1,b_2,a_1) + \lambda(c_1,c_2,e,c_3,e,b_1,b_2,e,e,a_1)\\
& - (a_1)_*\lambda(c_1,c_2,c_3,e,e,e,b_1b_2,e,e,e) + (a_1)_*\lambda(c_1c_2,e,c_3,e,e,e,e,e,b_1b_2,e)\\
& - (a_1)_*\lambda(c_1c_2,c_3,e,e,e,e,e,e,e,b_1b_2) + (b_1)_*\lambda(c_1c_2,c_3,e,e,e,e,e,e,e,b_2a_1)\\
& - (b_1)_*\lambda(c_1c_2,e,c_3,e,e,e,e,e,b_2a_1,e) + (c_1)_*\lambda(c_2,b_1,c_3b_2,e,e,e,e,e,a_1,e)\\
& - (c_1)_*\lambda(c_2,c_3b_1b_2,e,e,e,e,e,e,e,a_1) + (c_1c_2)_*\lambda(e,c_3,e,e,b_1b_2,e,e,e,e,a_1)\\
& - (c_1c_2)_*\lambda(e,e,c_3,e,b_1,b_2,e,e,a_1,e) + (c_1c_2b_1)_*\lambda(c_3,e,e,e,e,e,e,b_2,a_1,e)\\
& - (c_1c_2b_1)_*\lambda(e,c_3,e,e,e,b_2,a_1,e,e,e) + (c_1c_2a_1)_*\lambda(e,c_3,e,e,e,b_1,b_2,e,e,e)\\
& - (c_1c_2a_1)_*\lambda(c_3,e,e,e,e,e,e,b_1,b_2,e) - (c_1c_2a_1)_*\lambda(e,e,c_3,e,b_1,e,b_2,e,e,e);
\end{aligned}
$$

- $\phi_4(\lambda) = (t, \gamma, \delta)$, where

$$
\begin{aligned}
t(a,b,c,d) &= \lambda(a,b,c,d,e,e,e,e,e,e),\\
\gamma(a,b,c) &= \lambda(a,e,e,e,e,e,b,c,e) - \lambda(e,a,e,e,e,b,c,e,e,e)\\
&\quad + \lambda(e,e,a,e,b,e,c,e,e,e) - \lambda(e,e,e,a,b,c,e,e,e,e),\\
\delta(a,b,c) &= \lambda(a,b,e,e,e,e,e,e,e,c) - \lambda(a,e,b,e,e,e,e,e,c,e)\\
&\quad + \lambda(a,e,e,b,e,e,e,c,e,e) + \lambda(e,a,b,e,e,e,c,e,e,e)\\
&\quad - \lambda(e,a,e,b,e,c,e,e,e,e) + \lambda(e,e,a,b,c,e,e,e,e,e);
\end{aligned}
$$

- $\psi_4(t, \gamma, \delta) = \lambda$, where

$$
\begin{aligned}
\lambda(d_1,d_2,d_3,d_4,c_1,c_2,c_3,b_1,b_2,a_1) &= (c_1c_2c_3b_1b_2a_1)_*t(d_1,d_2,d_3,d_4)\\
&- (d_1d_2d_3c_1a_1)_*t(c_2,d_4c_3,b_1,b_2) - (d_1d_2d_3c_1b_2a_1)_*t(d_4,c_3,c_2,b_1)\\
&+ (d_1d_2d_3c_1a_1)_*t(d_4,c_2c_3,b_1,b_2) + (d_1d_2d_3c_1a_1)_*t(c_2,b_1,d_4c_3,b_2)\\
&- (d_1d_2d_3c_1a_1)_*t(d_4,c_2b_1,c_3,b_2) + (d_1d_2d_3c_1a_1)_*t(c_2b_1,d_4,c_3,b_2)\\
&+ (d_1d_2c_1a_1)_*t(d_3,c_2,b_1,d_4c_3b_2) - (d_1d_2c_1a_1)_*t(d_3,c_2,d_4c_3,b_1b_2)\\
&- (d_1b_1b_2a_1)_*t(d_2,c_1,d_3c_2,d_4c_3) + (d_1d_2c_1a_1)_*t(d_3,d_4,c_2c_3,b_1b_2)\\
&- (d_1d_2b_1b_2a_1)_*t(d_3,d_4,c_1,c_2c_3) - (d_1b_1b_2a_1)_*t(d_2,d_3,d_4,c_1c_2c_3)\\
&+ (d_1b_1b_2a_1)_*t(d_2,d_3,c_1c_2,d_4c_3) - (d_1d_2b_1b_2a_1)_*t(c_1,d_3,d_4,c_2c_3)\\
&- (d_1d_2b_1b_2a_1)_*t(d_3,c_1,c_2,d_4c_3) + (d_1d_2b_1b_2a_1)_*t(c_1,d_3,c_2,d_4c_3)\\
&+ (d_1d_2b_1b_2a_1)_*t(d_3,c_1,d_4,c_2c_3) - (d_1d_2d_3b_1b_2a_1)_*t(d_4,c_1,c_2,c_3)\\
&+ (d_1d_2d_3b_1b_2a_1)_*t(c_1,d_4,c_2,c_3) - (d_1d_2d_3b_1b_2a_1)_*t(c_1,c_2,d_4,c_3)\\
&+ (d_1d_2d_3d_4c_1a_1)_*t(c_2,c_3,b_1,b_2) - (d_1d_2d_3d_4c_1a_1)_*t(c_2,b_1,c_3,b_2)\\
&- (d_1d_2d_3c_1c_2b_1)_*t(d_4,c_3,b_2,a_1) + (d_1d_2c_2c_3b_1b_2a_1)_*\delta(d_3,d_4,c_1)\\
&- (d_1d_2d_3c_1b_2a_1)_*\delta(d_4,c_3,c_2b_1) + (d_1d_2d_3c_1b_1b_2a_1)_*\delta(d_4,c_3,c_2)\\
&- (d_1d_2d_3c_3b_1b_2a_1)_*\gamma(d_4,c_1,c_2) - (d_1d_2d_3d_4c_1b_2a_1)_*\gamma(c_3,c_2,b_1)\\
&+ (d_1d_2d_3c_1b_2a_1)_*\gamma(d_4c_3,c_2,b_1).
\end{aligned}
$$

A quite tedious, but totally straightforward, verification shows that these homomorphisms ϕ_n, ψ_n, and Γ_n satisfy the claimed properties implying that the truncated cochain complexes $C^\bullet_c(M,\mathcal{A})$ in (4.5) and $C^\bullet(\overline{W}^2M,\mathcal{A})$ in (4.12) are homologoical isomorphic.

4.3 Classifying braided monoidal abelian groupoids by 3-cohomology classes

This section is dedicated to showing a precise cohomological classification of braided monoidal abelian groupoids. The case of monoidal abelian groupoids was studied in Section 1.3, where their classification was solved by means of Leech 3-cohomology classes of monoids. Strictly symmetric monoidal abelian groupoids have been classified in Section 3.2, in this case by Grillet 3-cohomology classes of commutative monoids. Here, we show how every braided monoidal abelian groupoid invariably has a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A} : \mathbb{H}M \to \mathbf{Ab}$, and a commutative 3-cohomology class $k \in H^3_{\mathrm{c}}(M, \mathcal{A})$ associated with it. Furthermore, the triplet $(M, \mathcal{A}, k)$ thus obtained is an appropriate system of 'descent data' to rebuild the braided abelian groupoid up to braided equivalence.

Recall from section 1.3 that a groupoid $\mathcal{M}$ is termed abelian if its isotropy (or vertex) groups $\mathrm{Aut}_{\mathcal{M}}(x)$, $x \in \mathrm{Ob}\mathcal{M}$, are all abelian and that we use additive notation for the composition. Moreover, in Example 3.1, we introduce the notation $K(A, 1)$ for one-object groupoids with group of automorphisms A. It was also pointed out that any abelian totally disconnected groupoid is of the form $\bigcup_{a \in M} K(\mathcal{A}(a), 1)$, for some family of abelian groups $(\mathcal{A}(a))_{a \in M}$.

Braided monoidal categories have been studied extensively in the literature and we refer to Saavedra [66], and Joyal and Street [50] for the background. We intend to work with *braided monoidal abelian groupoids* (or *braided monoidal abelian groupoids*)

$$\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c}), \tag{4.13}$$

which consist of a monoidal abelian groupoid $(\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ and natural isomorphisms $\boldsymbol{c}_{x,y} : x \otimes y \to y \otimes x$ (the *braidings*), such that the two coherence conditions below hold.

$$(0_y \otimes \boldsymbol{c}_{x,z}) + \boldsymbol{a}_{y,x,z} + (\boldsymbol{c}_{x,y} \otimes 0_z) = \boldsymbol{a}_{y,z,x} + \boldsymbol{c}_{x,y\otimes z} + \boldsymbol{a}_{x,y,z}, \tag{4.14}$$

$$\begin{tikzcd}
 & (y \otimes x) \otimes z \arrow[r, "\boldsymbol{a}"] & y \otimes (x \otimes z) \arrow[dr, "0\otimes \boldsymbol{c}"] & \\
(x \otimes y) \otimes z \arrow[ur, "\boldsymbol{c}\otimes 0"] \arrow[dr, "\boldsymbol{a}"'] & & & y \otimes (z \otimes x) \\
 & x \otimes (y \otimes z) \arrow[r, "\boldsymbol{c}"] & (y \otimes z) \otimes x \arrow[ur, "\boldsymbol{a}"'] &
\end{tikzcd}$$

$$(\boldsymbol{c}_{x,z} \otimes 0_y) - \boldsymbol{a}_{x,z,y} + (0_x \otimes \boldsymbol{c}_{y,z}) = -\boldsymbol{a}_{z,x,y} + \boldsymbol{c}_{x\otimes y,z} - \boldsymbol{a}_{x,y,z}. \tag{4.15}$$

$$\begin{tikzcd}
 & x \otimes (z \otimes y) \arrow[r, "-\boldsymbol{a}"] & (x \otimes z) \otimes y \arrow[dr, "\boldsymbol{c}\otimes 0"] & \\
x \otimes (y \otimes z) \arrow[ur, "0\otimes \boldsymbol{c}"] \arrow[dr, "-\boldsymbol{a}"'] & & & (z \otimes x) \otimes y \\
 & (x \otimes y) \otimes z \arrow[r, "\boldsymbol{c}"] & z \otimes (x \otimes y) \arrow[ur, "-\boldsymbol{a}"'] &
\end{tikzcd}$$

For further use, we recall that in any braided monoidal abelian groupoid $\mathcal{M}$ the equalities below are satisfied (see [50, Propositions 2.1]).

$$\boldsymbol{l}_x + \boldsymbol{c}_{x,\mathrm{I}} = \boldsymbol{r}_x, \quad \boldsymbol{r}_x + \boldsymbol{c}_{\mathrm{I},x} = \boldsymbol{l}_x. \tag{4.16}$$

Example 4.2 Every commutative 3-cocycle $(h,\mu)\in Z^3_{\mathrm{c}}(M,\mathcal{A})$, gives rise to a braided monoidal abelian groupoid

$$\Sigma(M,\mathcal{A},(h,\mu)) = (\Sigma(M,\mathcal{A},(h,\mu)),\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r},\boldsymbol{c}), \tag{4.17}$$

that should be thought of as a 2-*dimensional twisted crossed product of M by $\mathcal{A}$*, and it is built as follows: Its underlying groupoid is the totally disconnected groupoid $\bigcup_{a\in M} K(\mathcal{A}(a),1)$ in (3.16).

The tensor product

$$\otimes : \Sigma(M,\mathcal{A},(h,\mu))\times\Sigma(M,\mathcal{A},(h,\mu))\to\Sigma(M,\mathcal{A},(h,\mu))$$

is given as in (3.17). The unit object is $\mathrm{I}=e$, the unit of the monoid M, and the structure constraints and the braiding isomorphisms are

$$\begin{gathered}\boldsymbol{a}_{a,b,c} = h(a,b,c) : (ab)c\to a(bc),\\ \boldsymbol{c}_{a,b} = \mu(a,b) : ab\to ba,\\ \boldsymbol{l}_a = 0_a : ea = a\to a,\ \ \boldsymbol{r}_a = 0_a : ae = a\to a,\end{gathered}$$

which are easily seen to be natural since the groups $\mathcal{A}(a)$, $a\in M$ are abelian. The coherence conditions (1.1), (4.14), and (4.15) follow from the 3-cocycle condition $\partial^3(h,\mu)=(0,0,0)$, while the coherence condition (1.2) holds due to the normalization condition $h(a,e,b)=0$.

Example 4.3 A braided monoidal abelian groupoid is called *strict* if all its structure constraints $\boldsymbol{a}_{x,y,z}$, $\boldsymbol{l}_x$, and $\boldsymbol{r}_x$ are identities. Seeing a monoid as a category with only one object, it is easy to identify a braided strict monoidal abelian groupoid with an *abelian track monoid*, in the sense of Baues-Jibladze [4] and Pirashvili [62], endowed with a braided structure. Porter [63] and Joyal-Street [49, Section 3, Example 4 (a preliminary manuscript of [50])] show a natural way to produce braided strict monoidal abelian groupoids from crossed modules in the category of monoids **Mnd**. We recall that construction in this example.

A *crossed module* in the category **Mnd** is a triplet (G,M,∂) consisting of a monoid M, a group G endowed with a M-action by a monoid homomorphism $M\to\mathrm{End}(G)$, written $(a,g)\mapsto {}^a g$, and an homomorphism $\partial: G\to M$ satisfying

$$\partial({}^a g)\,a = a\,\partial g,\ \ {}^{\partial g}g'\,g = g\,g'.$$

Roughly speaking, these two conditions say that the action of M on G behaves like an abstract conjugation. Note that when the monoid M is a group, we have the ordinary notion of crossed module by Whitehead [75]. Observe that, if $\partial g = e$, then $g\,g' = g'\,g$ for all $g'\in G$; that is, the subgroup $\{g\mid \partial g=e\}$ is contained in the center of G and, therefore, it is abelian. The crossed module is termed *abelian* whenever, for any $a\in M$, the subgroup $\{g\mid \partial g\,a = a\}\subseteq G$ is abelian. If, for example, the group G

is abelian, or the monoid M is cancellative (a group, for instance), then the crossed module is abelian.

A *bracket operation* for a crossed module (G, M, ∂) is a function $\{\ ,\ \} : M \times M \to G$ satisfying

$$\partial\{a,b\}\, b\, a = ab, \quad \{e,b\} = 1 = \{a,e\}, \quad \{\partial g, a\}\, {}^{a}g = g, \quad \{a, \partial g\}\, g = {}^{a}g,$$
$$\{ab, c\} = {}^{a}\{b,c\}\, \{a,c\}, \quad \{a, bc\} = \{a,b\}\, {}^{b}\{a,c\}.$$

where $1 \in G$ is the unit. This operation should be thought as an abstract commutator.

Each abelian crossed module with a bracket operator yields a braided strict monoidal abelian groupoid $\mathcal{M} = \mathcal{M}(G, M, \partial, \{\ ,\ \})$ as follows. Its objects are the elements of the monoid M, and a morphism $g : a \to b$ in $\mathcal{M}$ is an element $g \in G$ with $a = \partial g\, b$. The composition of two morphisms $a \overset{g}{\to} b \overset{h}{\to} c$ is given by multiplication in G, $a \overset{gh}{\to} c$. The tensor product is

$$(a \overset{g}{\to} b) \otimes (c \overset{h}{\to} d) = (ac \xrightarrow{g\, {}^{b}h} bd),$$

and the braiding is provided by the bracket operator via the formula

$$\boldsymbol{c}_{a,b} = \{a,b\} : ab \to ba.$$

In the very special case where M and G are commutative, the action of M on G is trivial, and ∂ is the trivial homomorphism (i.e., ${}^{a}g = g$ and $\partial g = e$, for all $a \in M$, $g \in G$), then a bracket operator $\{\ ,\ \} : M \times M \to G$ amounts to a bilinear map, that is, a function satisfying

$$\{e,b\} = 1 = \{a,e\}, \quad \{ab,c\} = \{a,c\}\, \{b,c\}, \quad \{a,bc\} = \{a,b\}\, \{a,c\}.$$

Thus, for example, when $M = \mathbb{N}$ is the additive monoid of non-negative integers and $G = \mathbb{Z}$ is the abelian group of integers, a bracket $\mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is given by $\{p,q\} = pq$. Also, if G is any multiplicative abelian group, then any $g \in G$ defines a bracket $\mathbb{N} \times \mathbb{N} \to G$ by $\{p,q\} = g^{pq}$.

Suppose $\mathcal{M}$, $\mathcal{M}'$ are braided monoidal abelian groupoids. A *braided monoidal functor*

$$F = (F, \varphi, \varphi_0) : \mathcal{M} \to \mathcal{M}' \tag{4.18}$$

consists of a monoidal functor (1.4) satisfying :

$$\varphi_{y,x} + \boldsymbol{c}_{Fx,Fy} = F\boldsymbol{c}_{x,y} + \varphi_{x,y}. \tag{4.19}$$

$$\begin{array}{ccc} Fx \otimes Fy & \xrightarrow{\boldsymbol{c}} & Fy \otimes Fx \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ F(x \otimes y) & \xrightarrow{F\boldsymbol{c}} & F(y \otimes x) \end{array}$$

If $F' : \mathcal{M} \to \mathcal{M}'$ is another braided monoidal functor, then, a *monoidal isomorphism* $\delta : F \Rightarrow F'$ is a monoidal natural transformation (1.7).

Example 4.4 Let $(h,\mu),(h',\mu') \in Z^3_{\mathrm{c}}(M,\mathcal{A})$ be commutative 3-cocycles of a commutative monoid. Then, any commutative cochain $g \in C^2_{\mathrm{c}}(M,A)$ such that $(h,\mu) = (h',\mu') + \partial^2 g$ induces a braided monoidal isomorphism

$$F(g) = (id, g, 0_e) : \Sigma(M,\mathcal{A},(h,\mu)) \cong \Sigma(M,\mathcal{A},(h',\mu')) \tag{4.20}$$

which is the identity functor on the underlying groupoids, and whose structure isomorphisms are given by $\varphi_{a,b} = g(a,b) : ab \to ab$ and $\varphi_0 = 0_e : e \to e$, respectively. Since the groups $\mathcal{A}(ab)$ are abelian, these isomorphisms $\varphi_{a,b}$ are natural. The coherence conditions (1.5) and (4.19) follows from the equality $(h,\mu) = (h',\mu') + \partial^2 g$, whilst the conditions in (1.6) trivially hold because of the normalization conditions $g(a,e) = 0_a = g(e,a)$.

If $f \in C^1_{\mathrm{c}}(M,\mathcal{A})$ is any commutative 1-cochain, and $g' = g + \partial^1 f \in C^2_{\mathrm{c}}(M,\mathcal{A})$, then an isomorphism of braided monoidal functors $\delta(f) : F(g) \Rightarrow F(g')$ is defined by putting $\delta(f)_a = f(a) : a \to a$, for each $a \in M$. So defined, δ is natural because of the abelian structure of the groups $\mathcal{A}(a)$; the first condition in (1.8) holds owing to the equality $g' = g + \partial^1 f$, and the second one thanks to the normalization condition $f(e) = 0_e$ of f.

With compositions defined in a natural way, braided monoidal abelian groupoids, braided monoidal functors, and monoidal isomorphisms form a 2-category [37, Chapter V, Section 1]. A braided monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is called a *braided monoidal equivalence* if it is an equivalence in this 2-category of braided monoidal abelian groupoids. From [66, I, Proposition 4.4.2] it follows that a braided monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is a braided monoidal equivalence if and only if the underlying functor is an equivalence of groupoids, that is, if and only if and only if the induced map on the sets of iso-classes of objects (3.12) is a bijection, and the induced homomorphisms on the automorphism groups (3.13) are all isomorphisms.

Remark 4.2 From the Coherence Theorem for monoidal categories [50, Corollary 1.4, Example 2.4], it follows that every braided monoidal abelian groupoid is braided monoidal equivalent to a braided strict one, that is, to one in which all the structure constraints $\boldsymbol{a}_{x,y,z}$, $\boldsymbol{l}_x$, and $\boldsymbol{r}_x$ are identities (see Example 4.3). This suggests that it is relatively harmless to consider braided monoidal abelian groupoids as strict. However, it is not so harmless when dealing with their homomorphisms since not every braided monoidal functor is isomorphic to a strict one (i.e., one as in (4.18) in which the structure isomorphisms $\varphi_{x,y}$ and φ_0 are all identities). Indeed, it is possible to find two braided strict monoidal abelian groupoids, say $\mathcal{M}$ and $\mathcal{M}'$, that are related by a braided monoidal equivalence between them but there is no strict monoidal equivalence either from $\mathcal{M}$ to $\mathcal{M}'$ nor from $\mathcal{M}'$ to $\mathcal{M}$.

Our goal is to state a classification for braided monoidal abelian groupoids, where two of them connected by a braided monoidal equivalence are considered the same. The main result in this section is the following:

Theorem 4.3 (Classification of Braided Monoidal Abelian Groupoids)
(i) *For any braided monoidal abelian groupoid $\mathcal{M}$, there exist a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A}$, a commutative* 3*-cocycle $(h, \mu) \in Z^3_{\mathrm{c}}(M, \mathcal{A})$, and a braided monoidal equivalence*

$$\Sigma(M, \mathcal{A}, (h, \mu)) \simeq \mathcal{M}.$$

(ii) *For any two commutative* 3*-cocycles $(h, \mu) \in Z^3_{\mathrm{c}}(M, \mathcal{A})$, $(h', \mu') \in Z^3_{\mathrm{c}}(M', \mathcal{A}')$, there is a braided monoidal equivalence*

$$\Sigma(M, \mathcal{A}, (h, \mu)) \simeq \Sigma(M', \mathcal{A}', (h', \mu'))$$

if and and only if there exist an isomorphism of monoids $i : M \cong M'$ and a natural isomorphism $\psi : \mathcal{A} \cong i^*\mathcal{A}'$, such that the equality of cohomology classes below holds.

$$[h, \mu] = \psi_*^{-1} i^*[h', \mu'] \in H^3_{\mathrm{c}}(M, \mathcal{A})$$

Proof: (i) Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ be any given braided monoidal abelian groupoid.

In a first step, we assume that $\mathcal{M}$ is totally disconnected and *strictly unitary*, in the sense that its unit constraints $\boldsymbol{l}_x$ and $\boldsymbol{r}_x$ are all identities. Then, a system of data $(M, \mathcal{A}, (h, \mu))$, such that $\Sigma(M, \mathcal{A}, (h, \mu)) = \mathcal{M}$ as braided abelian groupoids, is defined as follows:

• *The monoid M.* Let $M = \mathrm{Ob}\mathcal{M}$ be the set of objects of $\mathcal{M}$. As in the proof of Theorem 3.1 (i), the tensor functor determines a multiplication on M which is associative, commutative and unitary.

• *The $\mathbb{H}M$-module $\mathcal{A}$.* For each $a \in M = \mathrm{Ob}\mathcal{M}$, let $\mathcal{A}(a) = \mathrm{Aut}_{\mathcal{M}}(a)$ be the vertex group of the underlying groupoid at a. The group homomorphisms $\otimes : \mathcal{A}(a) \times \mathcal{A}(b) \to \mathcal{A}(ab)$ verifies the equalities (3.20), since the diagrams below commute due to the naturality of the structure constraints and the braiding.

$$\begin{tikzcd}
(ab)c \arrow[r, "\boldsymbol{a}_{a,b,c}"] \arrow[d, "(u_a\otimes u_b)\otimes u_c"'] & a(bc) \arrow[d, "u_a\otimes(u_b\otimes u_c)"] \\
(ab)c \arrow[r, "\boldsymbol{a}_{a,b,c}"] & a(bc)
\end{tikzcd}
\qquad
\begin{tikzcd}
ab \arrow[r, "\boldsymbol{c}_{a,b}"] \arrow[d, "u_a\otimes u_b"'] & ba \arrow[d, "u_b\otimes u_a"] \\
ab \arrow[r, "\boldsymbol{c}_{a,b}"] & ba
\end{tikzcd}
\qquad
\begin{tikzcd}
ea = a \arrow[r, "0_a"] \arrow[d, "0_e\otimes u_a"'] & a \arrow[d, "u_a"] \\
ea = a \arrow[r, "0_a"] & a
\end{tikzcd}$$

Then, writing as in the proof of Theorem 3.1 (i), $b_* : \mathcal{A}(a) \to \mathcal{A}(ab)$ for the homomorphism such that $b_*u_a := 0_b \otimes u_a = u_a \otimes 0_b$, the equalities (3.21) hold and so the assignments $a \mapsto \mathcal{A}(a)$, $(a, b) \mapsto b_* : \mathcal{A}(a) \to \mathcal{A}(ab)$, define an abelian group valued functor on $\mathbb{H}M$, as required.

• *The* 3-*cocycle* $(h,\mu)\in Z^3_{\mathrm{c}}(M,\mathcal{A})$. The associativity constraint and the braiding of $\mathcal{M}$ are necessarily written in the form $\boldsymbol{a}_{a,b,c}=h(a,b,c)$ and $\boldsymbol{c}_{a,b}=\mu(a,b)$, for some given lists $\big(h(a,b,c)\in\mathcal{A}(abc)\big)_{a,b,c\in M}$ and $\big(\mu(a,b)\in\mathcal{A}(ab)\big)_{a,b\in M}$. Since $\mathcal{M}$ is strictly unitary, equations in (1.2) and (1.3) give the normalization conditions $h(a,e,b)=0=h(e,a,b)=h(a,b,e)$ for h, while equations in (4.16) imply the normalization conditions $\mu(a,e)=0=\mu(e,a)$ for μ. Thus, $(h,\mu)\in C^3_{\mathrm{c}}(M,\mathcal{A})$ is a commutative 3-cochain, which is actually a 3-cocycle since the coherence conditions (1.1), (4.14), and (4.15) are now written as

$$\begin{aligned}&h(a,b,cd)+h(ab,c,d)=a_*h(b,c,d)+h(a,bc,d)+d_*h(a,b,c)\\&b_*\mu(a,c)+h(b,a,c)+c_*\mu(a,b)=h(b,c,a)+\mu(a,bc)+h(a,b,c),\\&b_*\mu(a,c)-h(a,c,b)+a_*\mu(b,c)=-h(c,a,b)+\mu(ab,c)-h(a,b,c),\end{aligned}$$

which amount to the cocycle condition $\partial^3(h,\mu)=(0,0,0)$.

Since an easy comparison (see Example 4.2) shows that $\mathcal{M}=\Sigma(M,\mathcal{A},(h,\mu))$, the proof of this part is complete, under the hypothesis of being $\mathcal{M}$ totally disconnected and strictly unitary.

It remains to prove that the braided monoidal abelian groupoid $\mathcal{M}$ is braided monoidal equivalent to another one $\mathcal{M}'$ that is totally disconnected and strictly unitary. To do that, we proceed as in the proof of Lemma 3.3. We begin by assuming that $\mathcal{M}$ is strictly unitary (see Remark 4.2). Then we combine the transport process by Saavedra [66, I, 4.4.5] and Joyal-Street [50, Example 2.4], which shows how to transport the braided monoidal structure on a monoidal abelian groupoid along an equivalence on its underlying groupoid, with the generalized Brandt's theorem, which asserts that every groupoid is equivalent (as a category) to a totally disconnected groupoid [46, Chapter 6, Theorem 2]. We leave the details to the reader since they are very similar to those in Lemma 3.3.

(*ii*) We follow the same lines than the ones used in the proof of Theorem 3.1 (*ii*).

Suppose there is an isomorphism of monoids $i:M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong i^*\mathcal{A}'$, such that $\psi_*[h,\mu]=i^*[h',\mu']\in H^3_{\mathrm{c}}(M,i^*\mathcal{A}')$. Then there is a commutative 2-cochain $g\in C^2_{\mathrm{c}}(M,i^*\mathcal{A}')$ verifying

$$\psi_{abc}h(a,b,c)=h'(ia,ib,ic)+(ia)_*g(b,c)-g(ab,c)+g(a,bc)-(ic)_*g(a,b), \tag{4.21}$$

$$\psi_{ab}\mu(a,b)=\mu'(ia,ib)-g(a,b)+g(b,a). \tag{4.22}$$

and so we have a braided isomorphism

$$\Sigma(i,\psi,g)=(F,\varphi,\varphi_0):\Sigma(M,\mathcal{A},(h,\mu))\to\Sigma(M',\mathcal{A}',(h',\mu')) \tag{4.23}$$

which is defined as follows. The underlying functor acts by $F(u_a:a\to a)=(\psi_a(u_a):ia\to ia)$. The structure isomorphisms of F are given by $\varphi_{a,b}=g(a,b):(ia)\,(ib)\to i(ab)$, and $\varphi_0=0_{e'}:e'\to ie=e'$. So defined, it is plain to see that F is an isomorphism between the underlying groupoids. The isomorphisms $\varphi_{a,b}$ are natural if the equalities as in (3.26) hold, but these are a consequence of the naturality of ψ.

The coherence condition (1.5) can be verified using (4.21) as in (3.28), while condition (4.19) as follows

$$\varphi_{b,a} + \boldsymbol{c}_{Fa,Fb} = g(b,a) + \mu'(ia,ib) \overset{(4.22)}{=} \psi_{ab}(\mu(a,b)) + g(a,b) = F(\boldsymbol{c}_{a,b}) + \varphi_{a,b}. \quad (4.24)$$

Finally, the conditions in (1.6) trivially follow from the equalities $g(a,e) = 0_{ia} = g(e,a)$.

Now, take

$$F = (F, \varphi, \varphi_0) : \Sigma(M, \mathcal{A}, (h,\mu)) \to \Sigma(M', \mathcal{A}', (h',\mu'))$$

any braided equivalence, which, by [22, Lemma 18], we assume is strictly unitary in the sense that $\varphi_0 = 0_{e'} : e' \to e' = Fe$. Since the underlying functor is an equivalence between totally disconnected groupoids, it has to be an isomorphism.

As in Theorem 3.1 (*ii*), we write $i : M \cong M'$ for the bijection describing the action of F on objects, and $\psi_a : \mathcal{A}(a) \cong \mathcal{A}'(ia)$ for the isomorphism giving the action of F on automorphisms. Then i is an isomorphism of monoids and $\psi : \mathcal{A} \cong i^*\mathcal{A}'$ a natural isomorphism between the $\mathbb{H}M$-modules.

Finally, if we denote $g(a,b) = \varphi_{a,b}$, for each $a, b \in M$, the equations $g(a,e) = 0_{ia} = g(e,a)$ hold due to the coherence equations (1.6), and thus we have a commutative 2-cochain

$$g(F) = \big(g(a,b) \in \mathcal{A}'(i(ab))\big)_{a,b\in M}, \quad (4.25)$$

which satisfies the equations (4.21) and (4.22) due to the coherence equations (1.5) and (4.19), respectively. Hence, $\psi_*(h,\mu) = i^*(h',\mu') - \partial^2 g$ and, therefore, we have that $[h,\mu] = \psi_*^{-1} i^*[h',\mu'] \in H^3_{\mathrm{c}}(M, \mathcal{A})$. □

A *braided categorical group* [50, Section 3] is a braided monoidal abelian groupoid $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ in which, for any object x, there is an object x^* with an arrow $x \otimes x^* \to \mathrm{I}$. By Proposition 1.3 (*ii*) we know that the hypothesis of being abelian is not needed here, since every monoidal groupoid in which every object has a quasi-inverse is always abelian. The cohomological classification of these braided categorical groups was stated and proved by Joyal and Street [50, Theorem 3.3] by means of Eilenberg-Mac Lane's commutative cohomology groups $H^3_{\mathrm{c}}(G, A)$, of abelian groups G with coefficients in abelian groups A (see Example 4.1). Next, we obtain Joyal-Street's classification result as a corollary of Theorem 4.3.

Corollary 4.1 (*i*) *For any abelian groups G and A, and any 3-cocycle $(h,\mu) \in Z^3_{\mathrm{c}}(G, A)$, the braided abelian groupoid $\Sigma(G, A, (h,\mu))$ is a braided categorical group.*

(*ii*) *For any braided categorical group $\mathcal{M}$, there exist abelian groups G and A, a 3-cocycle $(h,\mu) \in Z^3_{\mathrm{c}}(G, A)$, and a braided monoidal equivalence*

$$\Sigma(G, A, (h,\mu)) \simeq \mathcal{M}.$$

(*iii*) *For two commutative 3-cocycles $(h,\mu) \in Z^3_{\mathrm{c}}(G, A)$ and $(h',\mu') \in Z^3_{\mathrm{c}}(G', A')$, where G, G', A and A' are abelian groups, there is a braided monoidal equivalence*

$$\Sigma(G, A, (h, \mu)) \simeq \Sigma(G', A', (h', \mu'))$$

if and only if there exist isomorphism of groups $i : G \cong G'$ and $\psi : A \cong A'$, such that the equality of cohomology classes below holds.

$$[h, \mu] = \psi_*^{-1} i^* [h', \mu'] \in H^3_{\mathrm{c}}(G, A)$$

Proof: (i) Recall from Example 4.1 that we are here regarding A as the constant $\mathbb{H}G$-module it defines. Since G is a group, for any object a of $\Sigma(G, A, (h, \mu))$ (i.e., any element $a \in G$) we have $a \otimes a^{-1} = aa^{-1} = e = \mathrm{I}$. Thus, $\Sigma(G, A, (h, \mu))$ is actually a braided categorical group.

(ii) Let $\mathcal{M}$ a braided categorical group. By Theorem 4.3(i), there are a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A}$, a commutative 3-cocycle $(h, \mu) \in Z^3_{\mathrm{c}}(M, \mathcal{A})$, and a braided monoidal equivalence $\Sigma(M, \mathcal{A}, (h, \mu)) \simeq \mathcal{M}$. Then, $\Sigma(M, \mathcal{A}, (h, \mu))$ is a braided categorical group as $\mathcal{M}$ is and, for any $a \in M$, it must exist another $a^* \in M$ with a morphism $a \otimes a^* = aa^* \to \mathrm{I} = e$ in $\Sigma(M, \mathcal{A}, (h, \mu))$; this implies that $aa^* = e$ in M, since the groupoid is totally disconnected, whence $a^* = a^{-1}$ is an inverse of a in M. Therefore, $M = G$ is actually an abelian group.

Let $\mathcal{A}(e)$ be the abelian group attached by $\mathcal{A}$ at the unit of G. Then, a natural isomorphism $\phi : \mathcal{A} \cong \mathcal{A}(e)$ is defined such that, for any $a \in G$, $\phi_a = a_*^{-1} : \mathcal{A}(a) \to \mathcal{A}(e)$. Therefore, if we take $(h', \mu') = \phi_*(h, \mu) \in Z^3_{\mathrm{c}}(G, \mathcal{A}(e))$, Theorem 4.3$(ii)$ gives the existence of a braided equivalence $\Sigma(G, A, (h, \mu)) \simeq \Sigma(G, \mathcal{A}(e), (h', \mu'))$, whence $\Sigma(G, \mathcal{A}(e), (h', \mu'))$ and the given $\mathcal{M}$ are braided monoidal equivalent.

(iii) This follows directly form Theorem 4.3(ii). □

The classification result in Theorem 4.3 involves an interpretation of the elements of $H^3_{\mathrm{c}}(M, \mathcal{A})$ in terms of certain 2-dimensional coextensions of M by $\mathcal{A}$, such as the elements of $H^2_{\mathrm{c}}(M, \mathcal{A})$ are interpreted as commutative monoid coextensions. To state this fact, in next definition we regard any commutative monoid M as a braided abelian discrete monoidal groupoid (i.e., whose only morphisms are the identities), on which the tensor product is multiplication in M. Thus, if $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ is any braided monoidal abelian groupoid, a braided monoidal functor $p : \mathcal{M} \to M$ is the same thing as a map $p : \mathrm{Ob}\mathcal{M} \to M$ satisfying $p(x) = p(y)$ whenever $\mathrm{Hom}_{\mathcal{M}}(x, y) \neq \emptyset$, $p(x \otimes y) = p(x)\, p(y)$, and $p(\mathrm{I}) = e$.

Definition 4.2 *Let M be a commutative monoid, and let $\mathcal{A}$ be an $\mathbb{H}M$-module. A* braided 2-coextension *of M by $\mathcal{A}$ is a surjective braided monoidal functor $p : \mathcal{M} \twoheadrightarrow M$, where $\mathcal{M}$ is a braided monoidal abelian groupoid, such that, for any $a \in M$, it is given an (associative and unitary) action of the groupoid $K(\mathcal{A}(a), 1)$ on the fibre groupoid $p^{-1}(a)$ by means of a functor*

$$K(\mathcal{A}(a), 1) \times p^{-1}(a) \to p^{-1}(a), \quad (u, x \xrightarrow{f} y) \mapsto (x \xrightarrow{u \cdot f} y)$$

which is simply-transitive, in the sense that the induced functor

$$K(\mathcal{A}(a),1)\times p^{-1}(a) \to p^{-1}(a)\times p^{-1}(a), \quad (u,f)\mapsto (u\cdot f, f),$$

is an equivalence, and satisfies

$$(u\cdot f)\otimes(v\cdot g) = (a_*v + b_*u)\cdot(f\otimes g), \tag{4.26}$$

for every $a,b \in M$, $u\in\mathcal{A}(a)$, $v\in\mathcal{A}(b)$, $f: x\to y \in p^{-1}(a)$, *and* $g: z\to t\in p^{-1}(b)$.

Let us point out that if $p(x)=p(y)$, for some $x,y\in \mathrm{Ob}\mathcal{M}$, then $\mathrm{Hom}_{\mathcal{M}}(x,y)\neq\emptyset$ since the functor $K(\mathcal{A}(a),1)\times p^{-1}(a)\to p^{-1}(a)\times p^{-1}(a)$, for $a=p(x)$, is essentially surjective. Furthermore, the functoriality of the action means that if f,f' are composible arrows in $p^{-1}(a)$ then, for any $u,u'\in\mathcal{A}(a)$, we have

$$(u+u')\cdot(f+f') = u\cdot f + u'\cdot f'.$$

In particular,

$$f + u\cdot f' = u\cdot(f+f') = u\cdot f + f'. \tag{4.27}$$

Remark 4.3 These braided 2-coextensions can be seen as a sort of (braided, non-strict) *linear track extensions* in the sense of Baues, Dreckmann, and Jibladze [3, 4]. Briefly, note that to give a commutative 2-coextension $p:\mathcal{M}\twoheadrightarrow M$, as above, is equivalent to giving a surjective braided monoidal functor $p:\mathcal{M}\twoheadrightarrow M$ satisfying

$$p(x)=p(y) \ \text{ if and only if } \ \mathrm{Hom}_{\mathcal{M}}(x,y)\neq\emptyset,$$

together with a family of isomorphisms of groups $\big(\psi_x : \mathcal{A}(p(x))\cong \mathrm{Aut}_{\mathcal{M}}(x)\big)_{x\in\mathrm{Ob}\mathcal{M}}$ satisfying

$$\psi_y u = f + \psi_x u - f, \quad f\in\mathrm{Hom}_{\mathcal{M}}(x,y),$$
$$\psi_x u\otimes\psi_y v = \psi_{x\otimes y}((px)_*v + (py)_*u), \quad x,y\in\mathrm{Ob}\mathcal{M}.$$

The family of isomorphisms $(\psi_x)_{x\in\mathrm{Ob}\mathcal{M}}$ and the action of $\mathcal{A}$ on $\mathcal{M}$ are related each other by the equations $u\cdot f = f+\psi_x(u)$, for any $x\in\mathrm{Ob}\mathcal{M}$, $u\in\mathcal{A}(p(x))$, and $f\in\mathrm{Hom}_{\mathcal{M}}(x,y)$.

Let $\mathrm{Ext}^2_{\mathrm{c}}(M,\mathcal{A})$ denote the set of equivalence classes of such braided 2-coextensions of M by $\mathcal{A}$, where two of them, say $p:\mathcal{M}\twoheadrightarrow M$ and $p':\mathcal{M}'\twoheadrightarrow M$, are *equivalent* whenever there is a braided monoidal equivalence $F:\mathcal{M}\to\mathcal{M}'$ such that $p'F=p$ and $F(u\cdot f)=u\cdot F(f)$, for any morphism $f:x\to y$ in $\mathcal{M}$ and $u\in\mathcal{A}(p(x))$. Then, we have

Theorem 4.4 (Classification of braided 2-coextensions) *For any commutative monoid* M, *and any* $\mathbb{H}M$*-module* $\mathcal{A}$, *there is a natural bijection*

$$H^3_{\mathrm{c}}(M,\mathcal{A})\cong \mathrm{Ext}^2_{\mathrm{c}}(M,\mathcal{A}).$$

Proof: This is a consequence of Theorem 4.3 with only a slight adaptation of the arguments used for its proof. For any 3-cocycle $(h,\mu)\in Z^3_{\mathrm{c}}(M,\mathcal{A})$, the braided monoidal abelian groupoid $\Sigma(M,\mathcal{A},(h,\mu))$ in (4.17) comes with a natural structure of braided 2-coextension of M by $\mathcal{A}$, in which the surjective braided functor

$$\pi:\Sigma(M,\mathcal{A},(h,\mu))\twoheadrightarrow M$$

is given by the identity map on objects, $\pi(a)=a$. The fibre groupoid over any $a\in M$ is just $\pi^{-1}(a)=K(\mathcal{A}(a),1)$, and the action functor

$$K(\mathcal{A}(a),1)\times\pi^{-1}(a)\to\pi^{-1}(a)$$

is given by addition in $\mathcal{A}(a)$, that is, $u\cdot v=u+v$. If $(h',\mu')\in Z^3_{\mathrm{c}}(M,\mathcal{A})$ in any other 3-cocycle such that $(h,\mu)=(h',\mu')-\partial^2 g$, for some 2-cochain $g\in C^2_{\mathrm{c}}(M,\mathcal{A})$, then the associated braided monoidal isomorphism in (4.20),

$$F(g):\Sigma(M,\mathcal{A},(h,\mu))\to\Sigma(M,\mathcal{A},(h',\mu')),$$

is easily recognized as an equivalence between the braided coextensions

$$\Sigma(M,\mathcal{A},(h,\mu))\twoheadrightarrow M$$

and

$$\Sigma(M,\mathcal{A},(h',\mu'))\twoheadrightarrow M.$$

Thus, we have a well-defined map

$$H^3_{\mathrm{c}}(M,\mathcal{A})\to\mathrm{Ext}^2_{\mathrm{c}}(M,\mathcal{A}),\quad [h,\mu]\mapsto[\Sigma(M,\mathcal{A},(h,\mu))\overset{\pi}{\twoheadrightarrow}M].$$

To see that it is injective, suppose $(h,\mu),(h',\mu')\in Z^3_{\mathrm{c}}(M,\mathcal{A})$, such that the associated braided 2-coextension are made equivalent by a braided monoidal functor, say

$$F:\Sigma(M,\mathcal{A},(h,\mu))\to\Sigma(M,\mathcal{A},(h',\mu')),$$

which can be assumed to be strictly unitary [22, Lemma 18]. Then, the 2-cochain $g(F)\in C^2_{\mathrm{c}}(M,\mathcal{A})$ built in (4.25) satisfies that $(h,\mu)=(h',\mu')-\partial^2 g$, whence $[h,\mu]=[h',\mu']\in H^3_{\mathrm{c}}(M,\mathcal{A})$.

Finally, to prove that the map is surjective, let $p:\mathcal{M}\twoheadrightarrow M$ be any given braided 2-coextension of M by $\mathcal{A}$. By Theorem 4.3(i) and Lemma 4.1 below, we can assume that $\mathcal{M}=\Sigma(M',\mathcal{A}',(h',\mu'))$, for some commutative monoid M', an $\mathbb{H}M'$-module $\mathcal{A}'$, and a 3-cocycle $(h',\mu')\in Z^3_{\mathrm{c}}(M',\mathcal{A}')$. Then, a monoid isomorphism $i:M\cong M'$ and a natural isomorphism $\psi:\mathcal{A}\cong i^*\mathcal{A}'$ come determined by the equations $p(ia)=a$ and $\psi_a(u)=u\cdot 0_{ia}$, for any $a\in M$ and $u\in\mathcal{A}(a)$. Furthermore, taking $(h,\mu)=\psi_*^{-1}i^*(h',\mu')\in Z^3_{\mathrm{c}}(M,\mathcal{A})$, the braided monoidal isomorphism in (4.23) for the 2-cochain $g=0$,

$$F(0):\Sigma(M,\mathcal{A},(h,\mu))\cong\Sigma(M',\mathcal{A}',(h',\mu')),$$

is then easily seen as an equivalence between $\pi:\Sigma(M,\mathcal{A},(h,\mu))\twoheadrightarrow M$ and $p:\mathcal{M}\twoheadrightarrow M$. □

Lemma 4.1 *Let $p' : \mathcal{M}' \twoheadrightarrow M$ be a braided 2-coextension of M by $\mathcal{A}$, and suppose that $\mathcal{M}$ is any braided monoidal abelian groupoid which is braided monoidal equivalent to $\mathcal{M}'$. Then $\mathcal{M}$ can be endowed with a braided 2-coextension structure of M by $\mathcal{A}$, say $p : \mathcal{M} \twoheadrightarrow M$, such that $p : \mathcal{M} \twoheadrightarrow M$ and $p' : \mathcal{M}' \twoheadrightarrow M$ are equivalent braided 2-coextensions.*

Proof: Let $F = (F, \varphi) : \mathcal{M} \to \mathcal{M}'$ be a braided monoidal equivalence. Then, a braided 2-coextension structure of $\mathcal{M}$ is given as follows: Let

$$p = p'F : \mathcal{M} \to M$$

be the braided monoidal functor composite of p' and F. This is clearly surjective, since p' is and F is essentially surjective. For every $a \in M$, let $K(\mathcal{A}(a), 1) \times p^{-1}(a) \to p^{-1}(a)$ be the action defined by $(u, x \xrightarrow{f} y) \mapsto (x \xrightarrow{u \cdot f} y)$, where $u \cdot f$ is unique arrow in $\mathcal{M}$ such that

$$F(u \cdot f) = u \cdot Ff. \tag{4.28}$$

This is a simply-transitive well-defined action since F is a full, faithful, and essentially surjective functor. In order to check equation (4.26), we have

$$\begin{array}{lll} F((u \cdot f) \otimes (v \cdot g)) + \varphi_{x\otimes z} & = \varphi_{y\otimes t} + F(u \cdot f) \otimes F(v \cdot g) & (\text{nat. of } \varphi) \\ & = \varphi_{y\otimes t} + (u \cdot Ff) \otimes (v \cdot Fg) & (4.28) \\ & = \varphi_{y\otimes t} + (a_* u + b_* v) \cdot (Ff \otimes Fg) & ((4.26) \text{ for } \mathcal{M}') \\ & = (a_* v + b_* v) \cdot F(f \otimes g) + \varphi_{x\otimes z} & (\text{nat. of } \varphi, (4.27)) \\ & = F((a_* v + b_* u) \cdot (f \otimes g)) + \varphi_{x\otimes z} & (4.28) \end{array}$$

and the result follows since F is faithful and $\varphi_{x\otimes z}$ is an isomorphism. Thus we have defined the braided 2-coextension $\mathcal{M} \twoheadrightarrow M$, which is clearly equivalent to the original one by means of F. $\square$

Chapter 5

Higher cohomologies of commutative monoids

In [57, Chapter X, Section 12], Mac Lane explains how to define, for each integer $r \geq 0$, the rth level cohomology groups of a (skew) commutative DGA-algebra (differential graded augmented algebra) over a commutative ring K, say D: Take the commutative DGA-algebra $\mathbf{B}^r(D)$, obtained by iterating r times the reduced bar construction on D, and then, for any K-module A, define

$$H^n(D, r; A) = H^n\big(\mathrm{Hom}_K(\mathbf{B}^r(D), A\big), \quad n = 0, 1, \dots,$$

where $\mathrm{Hom}_K(\mathbf{B}^r(D), A)$ is the cochain complex obtained by applying the functor $\mathrm{Hom}_K(-, A)$ to the underlying chain complex of K-modules $\mathbf{B}^r(D)$.

This process may be applied, for example, when $D = \mathbb{Z}G$ is the group ring of an abelian group G, regarded as a trivially graded DGA-ring, augmented by $\alpha : \mathbb{Z}G \to \mathbb{Z}$ with $\alpha(x) = 1$ for all $x \in G$. Thus, the Eilenberg-Mac Lane rth level cohomology groups of the abelian group G with coefficients in an abelian group A are defined by

$$H^n(G, r; A) = H^n(\mathbb{Z}G, r; A). \tag{5.1}$$

In particular, the first level cohomology groups $H^n(G, 1; A) = H^n(G, A)$ are the ordinary cohomology groups of G with coefficients in the trivial G-module A [57, Chapter IV, Corollary 5.2]. These rth level cohomology groups of abelian groups were studied primarily with interest in Algebraic Topology. For instance, they have a topological interpretation in terms of the Eilenberg-Mac Lane spaces $K(G, r)$, owing to the isomorphisms $H^n(G, r; A) \cong H^n\big(K(G, r), A\big)$ [31, Theorem 20.3]. However, they early found application in solving purely algebraic problems. For example, we could recall that central group extensions of G by A are classified by cohomology classes in $H^2(G, 1; A)$, while abelian group extensions of G by A are classified by cohomology classes in $H^3(G, 2; A)$ [32, Section 26, (26.2), (26.3)]; or that second level cohomology classes in $H^4(G, 2; A)$ classify braided categorical groups [50, Theorem 3.3], while third level cohomology classes in $H^5(G,3;A)$ classify Picard categories [69, II, Proposition 5].

Here, we introduce a generalization of Eilenberg-Mac Lane's theory for abelian groups to commutative monoids. The obtained rth level cohomology groups of a commutative monoid M, denoted by

$$H^n(M,r;\mathcal{A}),$$

enjoy many desirable properties, whose study this chapter and a companion paper [14] are mainly dedicated to. In our development, the role of coefficients is now played by $\mathbb{H}M$-modules, which, recall, are abelian group objects in the comma category of commutative monoids over M (see 3.1.1).

For any given commutative monoid M, the category of chain complexes of $\mathbb{H}M$-modules is an abelian category. In Section 5.1, we show that it is also a symmetric monoidal category, with a distributive tensor product $\mathcal{A}\otimes_{\mathbb{H}M}\mathcal{B}$, and whose unit object is $\mathbb{Z}$, the concentrated in degree zero complex defined by the constant $\mathbb{H}M$-module given by the abelian group $\mathbb{Z}$ of integers. Hence, commutative *DGA-algebras over* $\mathbb{H}M$ arise as internal commutative monoids $\mathcal{A}$ in the symmetric monoidal category of complexes of $\mathbb{H}M$-modules, endowed with a morphism of internal monoids $\mathcal{A}\to\mathbb{Z}$.

Quite similarly as for ordinary commutative DGA-algebras over a commutative ring, a reduced bar construction $\mathcal{A}\mapsto\mathbf{B}(\mathcal{A})$ works on these DGA-algebras over $\mathbb{H}M$. Thus, $\mathbf{B}(\mathcal{A})$ is obtained from $\mathcal{A}$ by first totalizing the double complex of $\mathbb{H}M$-modules

$$\textstyle\bigoplus_{p\geq 0}\mathcal{A}/\mathbb{Z}\otimes_{\mathbb{H}M}\overset{(p\text{ factors })}{\cdots}\otimes_{\mathbb{H}M}\mathcal{A}/\mathbb{Z}\,,$$

and then enriching the (suitably graded) totalized complex of $\mathbb{H}M$-modules with a multiplicative structure by a shuffle product. We do this in Section 5.2, where we also define, for any $\mathbb{H}M$-module $\mathcal{B}$, the rth level cohomology groups of $\mathcal{A}$ with coefficients in $\mathcal{B}$ by

$$H^n(\mathcal{A},r;\mathcal{B})=H^n\big(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}),\mathcal{B})\big),\quad n=0,1,\dots\,.$$

Next, in Section 5.3 we briefly study free $\mathbb{H}M$-modules. These arise as a left adjoint construction to a forgetful functor from the category of $\mathbb{H}M$-modules to the comma category of sets over the underlying set of M. In particular, in Section 5.4 we introduce the free $\mathbb{H}M$-module on the identity map $id_M:M\to M$, denoted by $\mathcal{Z}M$. This becomes a (trivially graded) commutative DGA-algebra over $\mathbb{H}M$ and then, for each integer positive r, we define the rth level cohomology groups of a commutative monoid M with coefficients in an $\mathbb{H}M$-module $\mathcal{A}$ by

$$H^n(M,r;\mathcal{A})=H^n(\mathcal{Z}M,r;\mathcal{A}).\tag{5.2}$$

When $M=G$ is an abelian group, for any integer $r\geq 0$, $\mathbf{B}^r(\mathcal{Z}G)$ is isomorphic to the constant DGA-algebra over $\mathbb{H}G$ defined by the Eilenberg-Mac Lane DGA-ring $\mathbf{B}^r(\mathbb{Z}G)$ ($=A_N(G,r)$ in [31, Section 14]). Hence, for any abelian group A, viewed as a constant $\mathbb{H}G$-module, the cohomology groups $H^n(G,r;A)$ defined as in (5.2) are naturally isomorphic to those by Eilenberg and Mac Lane in (5.1), which, recall, compute the cohomology groups of the spaces $K(G,r)$ as $H^n(G,r;A)\cong H^n\big(K(G,r),A\big)$. In the companion paper [14] we show that, for any commutative monoid M, there are isomorphisms

$$H^n(M,r;\mathcal{A})\cong H^n(\overline{W}^rM,\mathcal{A}),$$

where $H^n(\overline{W}^rM,\mathcal{A})$, $n\geq 0$, are Gabriel-Zisman cohomology groups [37, Appendix II] of the underlying simplicial set of the simplicial monoid $\overline{W}^rM$, obtained by iterating the $\overline{W}$ construction on the constant simplicial monoid defined by M.

An analysis of the complex $\mathbf{B}(\mathcal{Z}M)$, for M any commutative monoid, leads us in Proposition 5.4 to identify the cohomology groups $H^n(M,1;\mathcal{A})$ with the standard cohomology groups $H^n_{\mathrm{L}}(M,\mathcal{A})$ by Leech [53] (see Subsection 1.3.1). Recall that Leech cohomology groups of a (not necessarily commutative) monoid M take coefficients in $\mathbb{D}M$-modules. When the monoid M is commutative, there is a natural functor $\mathbb{D}M\to\mathbb{H}M$ which is the identity on objects and carries a morphism (x,y,z) of $\mathbb{D}M$ to the morphism (y,xz) of $\mathbb{H}M$. Via this functor, every $\mathbb{H}M$-module $\mathcal{A}$ is regarded as a (symmetric) $\mathbb{D}M$-module and we prove that, for any commutative monoid M and $\mathbb{H}M$-module $\mathcal{A}$, there are natural isomorphisms

$$H^n(M,1;\mathcal{A})\cong H^n_{\mathrm{L}}(M,\mathcal{A}),\quad n=0,1,\dots.$$

For any $r\geq 2$, we show explicit descriptions of the complexes $\mathbf{B}^r(\mathcal{Z}M)$ truncated at dimensions $\leq r+3$, which are useful both for theoretical and computational interests concerning the cohomology groups $H^n(M,r;\mathcal{A})$ for $n\leq r+2$. Some conclusions here summarize as follows:

- $H^0(M,r;\mathcal{A})\cong H^0(M,1;\mathcal{A})\cong H^0_{\mathrm{L}}(M,\mathcal{A})\cong\mathcal{A}(e)$,

where $\mathcal{A}(e)$ is the abelian group attached by $\mathcal{A}$ at the identity e of the monoid.

- $H^n(M,r;\mathcal{A})=0$, for $0<n<r$,
- $H^r(M,r;\mathcal{A})\cong H^1(M,1;\mathcal{A})\cong H^1_{\mathrm{L}}(M,\mathcal{A})\cong H^1_{\mathrm{G}}(M,\mathcal{A})$,
- $H^{r+1}(M,r;\mathcal{A})\cong H^3(M,2;\mathcal{A})\cong H^2_{\mathrm{G}}(M,\mathcal{A})$.

where $H^n_{\mathrm{G}}(M,\mathcal{A})$ denotes the n-cohomology group by Grillet (see 3.1.1).

- $H^4(M,2;\mathcal{A})\cong H^3_{\mathrm{c}}(M,\mathcal{A})$,

where $H^3_{\mathrm{c}}(M,\mathcal{A})$ is the commutative 3-cohomology group defined in Chapter 4.

- $H^{r+2}(M,r;\mathcal{A})\cong H^5(M,3;\mathcal{A})$, for $r\geq 3$.
- There are natural inclusions $H^3_{\mathrm{G}}(M,\mathcal{A})\subseteq H^5(M,3;\mathcal{A})\subseteq H^3_{\mathrm{c}}(M,\mathcal{A})$.

Most of these cohomology groups above have known algebraic interpretations. For example, elements of $H^1(M,1;\mathcal{A}) = H^1_{\mathrm{L}}(M,\mathcal{A})$ are *derivations* [53, Chapter II, 2.7]. Cohomology classes in $H^2(M,1;\mathcal{A}) = H^2_{\mathrm{L}}(M,\mathcal{A})$ are isomorphism classes of *group coextensions* [53, Chapter V, Section 2] (or [74, Theorem 2]), while elements of $H^3(M,2;\mathcal{A}) = H^2_{\mathrm{G}}(M,\mathcal{A})$ classify *abelian group coextensions* [43, Chapter V, Section 4]. Cohomology classes in $H^3(M,1;\mathcal{A}) = H^3_{\mathrm{L}}(M,\mathcal{A})$ are equivalence classes of *monoidal abelian groupoids* (Theorem 1.4), elements of $H^4(M,2;\mathcal{A}) = H^3_{\mathrm{c}}(M,\mathcal{A})$ are equivalence classes of *braided monoidal abelian groupoids* (Theorem 4.3), and elements of $H^3_{\mathrm{G}}(M,\mathcal{A})$ are equivalence classes of *strictly commutative monoidal abelian groupoids* (Theorem 3.1). Thus, among them, only the cohomology groups $H^5(M,3;\mathcal{A})$ are pending of interpretation, and we solve this in Section 5.5. Here we give a natural interpretation of the cohomology classes in $H^5(M,3;\mathcal{A})$ in terms of equivalence classes of *symmetric monoidal abelian groupoids*, that is, braided monoidal abelian groupoids $\mathcal{M}$, whose braiding constraint $\boldsymbol{c}_{x,y}: x\otimes y \cong y\otimes x$ satisfy the symmetry condition $\boldsymbol{c}_{y,x}\boldsymbol{c}_{x,y} = id_{x\otimes y}$. The classification of symmetric monoidal abelian groupoids we give extends that, above refereed, by Sinh in [69, II, Proposition 5] for Picard categories.

In last Section 5.6, we compute the cohomology groups $H^n(M,r;\mathcal{A})$, for $n\leq r+2$, when M is any cyclic monoid.

5.1 Commutative differential graded algebras over $\mathbb{H}M$

Throughout this chapter M denotes, as in the preceding Chapters 3 and 4, a *commutative monoid* whose unit is e.

In [43, Chapter XII, Section 2] Grillet observes that the category of abelian group objects in the slice category of commutative monoids over M, $\mathbf{CMon}\!\downarrow_M$, is equivalent to the category of abelian group valued functors on the small category $\mathbb{H}M$ (see Subsection 3.1.1 for more details). This category of functors from $\mathbb{H}M$ into the category of abelian groups will be denoted by

$$\mathbb{H}M\text{-Mod}$$

and called the category of $\mathbb{H}M$*-modules.* For instance, let

$$\mathbb{Z}:\mathbb{H}M\to\mathbf{Ab},\quad x\mapsto\mathbb{Z}(x), \tag{5.3}$$

be the $\mathbb{H}M$-module which associates to each element $x\in M$ the free abelian group on the generator (x), and to each pair (x,y) the isomorphism of abelian groups $y_*:\mathbb{Z}(x)\to\mathbb{Z}(xy)$ given on the generator by $y_*(x)=(xy)$. This is isomorphic to the $\mathbb{H}M$-module defined by the constant functor on $\mathbb{H}M$ which associates the abelian group of integers $\mathbb{Z}$ to any $x\in M$.

For two $\mathbb{H}M$-modules $\mathcal{A}$ and $\mathcal{B}$, a morphism between them (i.e., a natural transformation) $f:\mathcal{A}\to\mathcal{B}$ consists of homomorphisms $f_x:\mathcal{A}(x)\to\mathcal{B}(x)$, such that, for any $x, y \in M$, the square below commutes.

$$\begin{tikzcd} \mathcal{A}(x) \arrow[r, "f_x"] \arrow[d, "y_*"'] & \mathcal{B}(x) \arrow[d, "y_*"] \\ \mathcal{A}(xy) \arrow[r, "f_{xy}"] & \mathcal{B}(xy) \end{tikzcd}$$

The category of $\mathbb{H}M$-modules is abelian and we refer to [57, Chapter IX, Section 3] for details. Recall that the set of morphisms between two $\mathbb{H}M$-modules $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathrm{Hom}_{\mathbb{H}M}(\mathcal{A},\mathcal{B})$, is an abelian group by objectwise addition, that is, if $f, g : \mathcal{A} \to \mathcal{B}$ are morphisms, then $f+g : \mathcal{A} \to \mathcal{B}$ is defined by setting $(f+g)_x = f_x+g_x$, for each $x \in M$. The zero $\mathbb{H}M$-module is the constant functor $0 : \mathbb{H}M \to \mathbf{Ab}$ defined by the trivial abelian group 0, and the direct sum of two $\mathbb{H}M$-modules $\mathcal{A}$ and $\mathcal{B}$ is given by taking direct sum at each object, that is, $(\mathcal{A}\oplus\mathcal{B})(x) = \mathcal{A}(x)\oplus\mathcal{B}(x)$. Similarly, all limits and colimits (in particular, kernels, images, cokernels, etc.) in the category $\mathbb{H}M$-Mod are pointwise constructed.

Remark 5.1 Every abelian group A defines a *constant* $\mathbb{H}M$-module, equally denoted by A, such that $A(x) = A$ and $y_* = id_A : A(x) \to A(xy)$, for any $x, y \in M$. In this way, the category of abelian groups becomes a full subcategory of the category of $\mathbb{H}M$-modules.

When $M = G$ is an abelian group, then this inclusion $\mathbf{Ab} \hookrightarrow \mathbb{H}G$-Mod is actually an equivalence of categories. In the other direction, we have the functor associating to each $\mathbb{H}G$-module $\mathcal{A}$ the abelian group $\mathcal{A}(e)$, and there is natural isomorphism of $\mathbb{H}G$-modules $\mathcal{A} \cong \mathcal{A}(e)$ whose component at each $x \in G$ is the isomorphism of abelian groups $x_*^{-1} : \mathcal{A}(x) \to \mathcal{A}(e)$.

5.1.1 Tensor product of $\mathbb{H}M$-modules

For any two $\mathbb{H}M$-modules $\mathcal{A}$, $\mathcal{B}$, their *tensor product*, denoted by $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$, is the $\mathbb{H}M$-module defined as follows: It attaches to any $x \in M$ the abelian group defined by the coequalizer sequence of homomorphisms

$$\bigoplus_{uvw=x} \mathbb{Z}(u) \otimes \mathcal{A}(v) \otimes \mathcal{B}(w) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \bigoplus_{zt=x} \mathcal{A}(z) \otimes \mathcal{B}(t) \twoheadrightarrow (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x),$$

where, for any two abelian groups A and B, $A \otimes B$ denotes their tensor product as $\mathbb{Z}$-modules, the direct sum on the left is taken over all triples $(u, v, w) \in M^3$ such that $uvw = x$, the direct sum on the middle is over all pairs $(z, t) \in M^2$ with $zt = x$, and the homomorphisms ϕ and ψ are defined by

$$\phi\big((u) \otimes a_v \otimes b_w\big) = u_* a_v \otimes b_w \in \mathcal{A}(uv) \otimes \mathcal{B}(w),$$
$$\psi\big((u) \otimes a_v \otimes b_w\big) = a_v \otimes u_* b_w \in \mathcal{A}(v) \otimes \mathcal{B}(uw),$$

for all $u, v, w \in M$ with $uvw = x$, $a_v \in \mathcal{A}(v)$, and $b_w \in \mathcal{B}(w)$. For any pair $(x, y) \in M^2$, the homomorphism

$$y_* : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x) \to (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(xy)$$

is given on generators by

$$y_*\big(a_z \otimes b_t\big) = y_* a_z \otimes b_t = a_z \otimes y_* b_t, \qquad \big(a_z \in \mathcal{A}(z),\, b_t \in \mathcal{B}(t), zt = x\big).$$

If $f : \mathcal{A} \to \mathcal{A}'$ and $g : \mathcal{B} \to \mathcal{B}'$ are morphisms of $\mathbb{H}M$-modules, then there is an induced one $f \otimes g : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} \to \mathcal{A}' \otimes_{\mathbb{H}M} \mathcal{B}'$ such that, for each $x \in M$, the homomorphism

$$(f \otimes g)_x : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})(x) \to (\mathcal{A}' \otimes_{\mathbb{H}M} \mathcal{B}')(x)$$

is given on generators by

$$(f \otimes g)_x\big(a_z \otimes b_t\big) = f_z a_z \otimes g_t b_t, \qquad \big(a_z \in \mathcal{A}(z),\, b_t \in \mathcal{B}(t), zt = x\big).$$

Thus, we have a distributive tensor functor

$$- \otimes_{\mathbb{H}M} - : \mathbb{H}M\text{-Mod} \times \mathbb{H}M\text{-Mod} \to \mathbb{H}M\text{-Mod}.$$

Further, there are canonical isomorphisms of $\mathbb{H}M$-modules

$$\begin{aligned} &\boldsymbol{l}_{\mathcal{A}} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathcal{A} \cong \mathcal{A}, \quad \boldsymbol{c}_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} \cong \mathcal{B} \otimes_{\mathbb{H}M} \mathcal{A}, \\ &\boldsymbol{a}_{\mathcal{A},\mathcal{B},\mathcal{C}} : \mathcal{A} \otimes_{\mathbb{H}M} (\mathcal{B} \otimes_{\mathbb{H}M} \mathcal{C}) \cong (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) \otimes_{\mathbb{H}M} \mathcal{C}, \end{aligned}$$

respectively defined by the formulas

$$\begin{aligned} &\boldsymbol{l}_{zt}((z) \otimes a_t) = z_* a_t, \quad \boldsymbol{c}_{zt}(a_z \otimes b_t) = b_t \otimes a_z, \\ &\boldsymbol{a}_{yzt}(a_y \otimes (b_z \otimes c_t)) = (a_y \otimes b_z) \otimes c_t, \end{aligned}$$

which are easily proven to be natural and coherent in the sense of [56, Theorem 5.1]. Therefore, $\mathbb{H}M$-Mod is a symmetric monoidal category. We will usually treat the constraints above as identities, so we think of $\mathbb{H}M$-Mod as a symmetric strict monoidal category.

5.1.2 Tensor product of complexes of $\mathbb{H}M$-modules

The (positive) complexes of $\mathbb{H}M$-modules

$$\mathcal{A} = \cdots \to \mathcal{A}_2 \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0$$

and the morphisms between them also form an abelian symmetric monoidal category, where the tensor product $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$ of two complexes of $\mathbb{H}M$-modules $\mathcal{A}$ and $\mathcal{B}$ is the graded $\mathbb{H}M$-module whose component of degree n is

$$(\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_n = \bigoplus_{p+q=n} \mathcal{A}_p \otimes_{\mathbb{H}M} \mathcal{B}_q,$$

and whose differential $\partial^{\otimes}$, at any $x \in M$,

$$\partial_x^{\otimes} : (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_n(x) \to (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_{n-1}(x),$$

is defined on generators by the Leibniz formula

$$\partial_x^{\otimes}(a_z \otimes b_t) = \partial_z a_z \otimes b_t + (-1)^p\ a_z \otimes \partial_t b_t.$$

for all $z, t \in M$ such that $zt = x$, $a_z \in \mathcal{A}_p(z)$, $b_t \in \mathcal{B}_q(t)$, and $p, q \geq 0$ such that $p + q = n$.

In this monoidal category, the unit object is $\mathbb{Z}$, defined in (5.3), regarded as a complex concentrated in degree zero. The structure constraints

$$\begin{array}{l} \boldsymbol{l}_{\mathcal{A}} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathcal{A} \cong \mathcal{A}, \quad \boldsymbol{c}_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B} \cong \mathcal{B} \otimes_{\mathbb{H}M} \mathcal{A}, \\ \boldsymbol{a}_{\mathcal{A},\mathcal{B},\mathcal{C}} : \mathcal{A} \otimes_{\mathbb{H}M} (\mathcal{B} \otimes_{\mathbb{H}M} \mathcal{C}) \cong (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) \otimes_{\mathbb{H}M} \mathcal{C}, \end{array} \tag{5.4}$$

are respectively defined by the formulas

$$\begin{array}{l} \boldsymbol{l}_{xy}((x) \otimes a_y) = x_* a_y, \\ \boldsymbol{c}_{xy}\big(a_x \otimes b_y\big) = (-1)^{pq}\ b_y \otimes a_x, \\ \boldsymbol{a}_{xyz}\big(a_x \otimes (b_y \otimes c_z)\big) = (a_x \otimes b_y) \otimes c_z, \end{array}$$

for any $x, y, z \in M$, $a_x \in \mathcal{A}_p(x)$, $b_y \in \mathcal{B}_q(y)$, and $c_z \in \mathcal{C}_r(z)$. As for $\mathbb{H}M$-modules, we will treat these constraints as identities.

5.1.3 Commutative differential graded algebras over $\mathbb{H}M$

A commutative *differential graded algebra* (DG-algebra) $\mathcal{A}$ over $\mathbb{H}M$ is defined to be a commutative monoid in the symmetric monoidal category of complexes of $\mathbb{H}M$-modules, see [58, Chapter VII, Section 3]. Hence, $\mathcal{A}$ is a complex of $\mathbb{H}M$-modules equipped with a *multiplication morphism* of complexes $\circ : \mathcal{A} \otimes_{\mathbb{H}M} \mathcal{A} \to \mathcal{A}$ satisfying the associativity $\circ(\circ \otimes id) = \circ(id \otimes \circ)$ and the commutativity $\circ\, \boldsymbol{c} = \circ$, and a *unit morphism* of complexes $\iota : \mathbb{Z} \to \mathcal{A}$ satisfying $\circ(\iota \otimes id_{\mathcal{A}}) = \boldsymbol{l}_{\mathcal{A}}$. We write

$$1 = \iota_e(e) \in \mathcal{A}_0(e)$$

and, for any $x, y \in M$, $a_x \in \mathcal{A}_p(x)$, and $a_y \in \mathcal{A}_q(y)$,

$$a_x \circ a_y = \circ_{xy}(a_x \otimes a_y) \in \mathcal{A}_{p+q}(xy),$$

so that the algebra structure on the complex $\mathcal{A}$ gives us multiplication homomorphisms of abelian groups

$$\mathcal{A}_p(x) \otimes \mathcal{A}_q(y) \to \mathcal{A}_{p+q}(xy), \quad a_x \otimes a_y \mapsto a_x \circ a_y,$$

and a *unit* $1 \in \mathcal{A}_0(e)$, satisfying

$$x_* a_y \circ a_z = x_*(a_y \circ a_z) = a_y \circ x_* a_z, \tag{5.5}$$

$$a_x \circ a_y = (-1)^{pq}\, a_y \circ a_x, \tag{5.6}$$

$$1 \circ a_x = a_x = a_x \circ 1, \tag{5.7}$$

$$a_x \circ (a_y \circ a_z) = (a_x \circ a_y) \circ a_z, \tag{5.8}$$

$$\partial_{xy}(a_x \circ a_y) = \partial_x a_x \circ a_y + (-1)^p\, a_x \circ \partial_y a_y, \tag{5.9}$$

for all $x, y, z \in M$, $a_x \in \mathcal{A}_p(x)$, $a_y \in \mathcal{A}_q(y)$, and $a_z \in \mathcal{A}_r(z)$.

In these terms, a morphism $f : \mathcal{A} \to \mathcal{B}$ of commutative DG-algebras over $\mathbb{H}M$ is a morphism of complexes of $\mathbb{H}M$-modules such that $f_{xy}(a_x \circ a_y) = f_x a_x \circ f_y a_y$, and $f_e(1) = 1$.

The category of commutative DG-algebras over $\mathbb{H}M$ is symmetric monoidal. The tensor product of two of them $\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}$ is given by their tensor product as complexes of $\mathbb{H}M$-modules endowed with multiplication such that, for $u, v, x, y \in M$, $a_u \in \mathcal{A}_p(u)$, etc.,

$$(a_u \otimes b_x) \circ (a_y \otimes b_z) = (a_u \circ a_y) \otimes (b_x \circ b_z)$$

and with unit $1 \otimes 1 \in (\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B})_0(e)$. Observe that the canonical isomorphisms in (5.4) are actually of DG-algebras whenever the data $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ therein are DG-algebras over $\mathbb{H}M$.

Commutative DG-algebras over $\mathbb{H}M$ which are concentrated in degree zero are the same as commutative monoids in the symmetric monoidal category of $\mathbb{H}M$-modules, and they are simply called *algebras over* $\mathbb{H}M$ or $\mathbb{H}M$*-algebras*. For example, $\mathbb{Z}$ is an $\mathbb{H}M$-algebra with multiplication the unit constraint $\boldsymbol{l} : \mathbb{Z} \otimes_{\mathbb{H}M} \mathbb{Z} \cong \mathbb{Z}$ and unit the identity $id : \mathbb{Z} \to \mathbb{Z}$. In other words, $\mathbb{Z}$ is an $\mathbb{H}M$-algebra whose unit is $(e) \in \mathbb{Z}(e)$ and whose multiplication homomorphisms $\mathbb{Z}(x) \otimes \mathbb{Z}(y) \to \mathbb{Z}(xy)$ are given by $(mx) \circ (ny) = mn(xy)$, where mn is multiplication of m and n in the ring $\mathbb{Z}$.

The augmented case is relevant. A commutative *differential graded augmented algebra* (DGA-algebra) $\mathcal{A}$ over $\mathbb{H}M$ is a commutative DG-algebra over $\mathbb{H}M$ as above equipped with a homomorphism of commutative DG-algebras $\epsilon : \mathcal{A} \to \mathbb{Z}$ (the *augmentation*). Such an augmentation is entirely determined by its component of degree 0, which is a morphism of $\mathbb{H}M$-algebras $\epsilon : \mathcal{A}_0 \to \mathbb{Z}$ such that $\epsilon\,\partial = 0$. Morphisms of commutative DGA-algebras over $\mathbb{H}M$ are those of commutative DG-algebras which are compatible with the augmentations (i.e., $\epsilon f = \epsilon$).

Remark 5.2 When $M = G$ is a group, the equivalence between the category of abelian groups and the category of $\mathbb{H}G$-modules, described in Remark 5.1, is symmetric monoidal and, therefore, produces an equivalence between the category of commutative DGA-rings and the category of commutative DGA-algebras over $\mathbb{H}G$. Thus every commutative DGA-ring A defines a *constant* commutative DGA-algebra over $\mathbb{H}G$, equally denoted by A, and each commutative DGA-algebra over $\mathbb{H}G$, $\mathcal{A}$, gives rise to the DGA-ring $\mathcal{A}(e)$, which comes with a natural isomorphism of DGA-algebras $\mathcal{A} \cong \mathcal{A}(e)$ whose component at each $x \in G$ is the isomorphism of augmented chain complexes $x_*^{-1} : \mathcal{A}(x) \to \mathcal{A}(e)$.

5.2 The Bar construction on commutative DGA-algebras over $\mathbb{H}M$

Let $\mathcal{A}$ be any given commutative DGA-algebra over $\mathbb{H}M$. As we explain below, $\mathcal{A}$ determines a new commutative DGA-algebra over $\mathbb{H}M$, denoted by $\mathbf{B}(\mathcal{A})$ and called the *bar construction* on $\mathcal{A}$.

Previously to describe $\mathbf{B}(\mathcal{A})$, let us introduce complexes of $\mathbb{H}M$-modules $\bar{\mathcal{A}}$, $\mathrm{S}\bar{\mathcal{A}}$, and $\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}}$ for each integer $p \geq 0$, and a double complex of $\mathbb{H}M$-modules $\mathrm{T}^\bullet\mathrm{S}\mathcal{A}$, as follows:

The *reduced complex* $\bar{\mathcal{A}} = \cdots \to \bar{\mathcal{A}}_2 \overset{\partial}{\to} \bar{\mathcal{A}}_1 \overset{\partial}{\to} \bar{\mathcal{A}}_0$ is defined to be the cokernel of the unit morphism $\iota : \mathbb{Z} \to \mathcal{A}$. That is, $\bar{\mathcal{A}} = \cdots \to \mathcal{A}_2 \overset{\partial}{\to} \mathcal{A}_1 \overset{\partial}{\to} \mathcal{A}_0/\iota\mathbb{Z}$. Note that ι embeds $\mathbb{Z}$ as a direct summand of the underlying complex $\mathcal{A}$, since, being $\epsilon : \mathcal{A} \to \mathbb{Z}$ the augmentation, $\epsilon\iota = id_{\mathbb{Z}}$. We will use below the following notation: For any $x \in M$ and each chain a_x of the chain complex $\mathcal{A}(x)$, $\tilde{\epsilon}(a_x)$ is the integer which express $\epsilon_x(a_x)$ as a multiple of the generator (x) of the abelian group $\mathbb{Z}(x)$, that is, such that

$$\epsilon_x(a_x) = \tilde{\epsilon}(a_x)(x). \tag{5.10}$$

The complex $\mathrm{S}\bar{\mathcal{A}}$ is the *suspension* of $\bar{\mathcal{A}}$, that is, the complex of $\mathbb{H}M$-modules defined by $(\mathrm{S}\bar{\mathcal{A}})_{p+2} = \mathcal{A}_{p+1}$, $(\mathrm{S}\bar{\mathcal{A}})_1 = \mathcal{A}_0/\iota\mathbb{Z}$, $(\mathrm{S}\bar{\mathcal{A}})_0 = 0$, and differential $-\partial$. The *suspension map* is then the morphism of complexes $\mathrm{S} : \bar{\mathcal{A}} \to \mathrm{S}\bar{\mathcal{A}}$, of degree 1, defined by

$$\mathrm{S}_p = id_{\bar{\mathcal{A}}_p} : \bar{\mathcal{A}}_p \to (\mathrm{S}\bar{\mathcal{A}})_{p+1} = \bar{\mathcal{A}}_p.$$

Note that the sign in the differential of $\mathrm{S}\bar{\mathcal{A}}$ is taken so that the equality $\partial\mathrm{S} = -\mathrm{S}\partial$ holds.

For each $p \geq 1$, let $\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}}$ be the complex of $\mathbb{H}M$-modules defined by the iterated tensor product

$$\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}} = \mathrm{S}\bar{\mathcal{A}} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} \mathrm{S}\bar{\mathcal{A}} \qquad (p \text{ factors}).$$

Thus, for any integer $n \geq 0$ and $x \in M$, the abelian group $(\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}})_n(x)$ is generated by elements $\mathrm{S}\bar{a}_{x_1} \otimes \cdots \otimes \mathrm{S}\bar{a}_{x_p}$, that we write as

$$[a_{x_1} \,|\, \cdots \,|\, a_{x_p}], \tag{5.11}$$

where the $x_i \in M$ are elements of the monoid such that $x_1 \cdots x_p = x$, and the $a_{x_i} \in \mathcal{A}_{r_i}(x_i)$ are chains of the complexes of abelian groups $\mathcal{A}(x_i)$ whose degrees satisfy that $p + r_1 + \cdots + r_p = n$. On such a generator (5.11), the differential $\partial^{\otimes}$ of $\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}}$ at x,

$$\partial_x^{\otimes} : (\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}})_n(x) \to (\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}})_{n-1}(x),$$

acts by

$$\partial_x^{\otimes}[a_{x_1}|\cdots|a_{x_p}] = -\sum_{i=1}^{p}(-1)^{e_{i-1}}[a_{x_1}|\cdots|a_{x_{i-1}}|\partial_{x_i}a_{x_i}|a_{x_{i+1}}|\cdots|a_{x_p}],$$

where the exponents e_i of the signs are $e_0 = 0$ and, for $i \geq 1$,

$$e_i = i + r_1 + \cdots + r_i,$$

and $\partial_{x_i} : \mathcal{A}_{r_i}(x_i) \to \mathcal{A}_{r_i-1}(x_i)$ is the differential of $\mathcal{A}$ at x_i. Remark that the elements (5.11) are normalized, in the sense that $[a_{x_1} \,|\, \cdots \,|\, a_{x_p}] = 0$ whenever some $a_{x_i} = x_{i*}1 \in \mathcal{A}_0(x_i)$.

For $p = 0$, we take $\mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}$ to be $\mathbb{Z}$, but where we write $[\ ]$ for the unit $(e) \in \mathbb{Z}(e)$. Thus, $\mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}$ is the concentrated in degree 0 complex of $\mathbb{H}M$-modules such that, for any $x \in M$, $\mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}(x)$ is the free abelian group on the element $x_*[\ ]$ ($= [\ ]$ if $x = e$), and, for each $x, y \in M$, $y_* : \mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}(x) \to \mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}(xy)$ is determined by $y_*x_*[\ ] = (yx)_*[\ ]$.

The double complex of $\mathbb{H}M$-modules

$$\mathrm{T}^{\bullet}\mathrm{S}\mathcal{A} = \cdots \to \mathrm{T}^2\mathrm{S}\bar{\mathcal{A}} \xrightarrow{\partial^\circ} \mathrm{T}^1\mathrm{S}\bar{\mathcal{A}} \xrightarrow{\partial^\circ} \mathrm{T}^0\mathrm{S}\bar{\mathcal{A}}$$

is then constructed, thanks to the multiplication $\circ$ in $\mathcal{A}$, by the morphisms of complexes of $\mathbb{H}M$-modules $\partial^\circ : \mathrm{T}^p\mathrm{S}\bar{\mathcal{A}} \to \mathrm{T}^{p-1}\mathrm{S}\bar{\mathcal{A}}$, which are of degree -1 (so that $\partial^\circ\partial^{\otimes} = -\partial^{\otimes}\partial^\circ$) and defined, at any $x \in M$, by the homomorphisms

$$\partial_x^\circ : (\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}})_n(x) \to (\mathrm{T}^{p-1}\mathrm{S}\bar{\mathcal{A}})_{n-1}(x)$$

given on generators as in (5.11) by

$$\begin{aligned}\partial_x^\circ[a_{x_1}|\cdots|a_{x_p}] = {}& \tilde{\epsilon}_{x_1}(a_{x_1})\ x_{1*}[a_{x_2}|\cdots|a_{x_p}] \\ &+ \sum_{i=1}^{p-1}(-1)^{e_i}[a_{x_1}|\cdots|a_{x_{i-1}}|a_{x_i} \circ a_{x_{i+1}}|a_{x_{i-1}}|\cdots|a_{x_p}] \\ &+ (-1)^{e_p}\ \tilde{\epsilon}_{x_p}(a_{x_p})\, x_{p\,*}[a_{x_1}|\cdots|a_{x_{p-1}}]\end{aligned}$$

(recall the notation $\tilde{\epsilon}$ from (5.10), and note that the first (resp. last) summand in the above formula is zero whenever the degree r_1 of a_{x_1} in the chain complex $\mathcal{A}(x_1)$ (resp. r_p of a_{x_p}) is higher than zero).

All in all, we are now ready to present the bar construction $\mathbf{B}(\mathcal{A})$. As a graded $\mathbb{H}M$-module

$$\mathbf{B}(\mathcal{A}) = \cdots \to \mathbf{B}(\mathcal{A})_2 \xrightarrow{\partial} \mathbf{B}(\mathcal{A})_1 \xrightarrow{\partial} \mathbf{B}(\mathcal{A})_0$$

is defined by the $\mathbb{H}M$-modules

$$\mathbf{B}(\mathcal{A})_n = \bigoplus\nolimits_{p\geq 0}(\mathrm{T}^p\mathrm{S}\bar{\mathcal{A}})_n\ .$$

Notice that $\partial^{\otimes}\mathbf{B}(\mathcal{A})_n \subseteq \mathbf{B}(\mathcal{A})_{n-1}$, $\partial^\circ\mathbf{B}(\mathcal{A})_n \subseteq \mathbf{B}(\mathcal{A})_{n-1}$, and that $(\partial^{\otimes} + \partial^\circ)^2 = 0$. Thus, $\mathbf{B}(\mathcal{A})$ becomes a complex of $\mathbb{H}M$-modules with differential

$$\partial = \partial^{\otimes} + \partial^\circ : \mathbf{B}(\mathcal{A})_n \to \mathbf{B}(\mathcal{A})_{n-1}.$$

Proposition 5.1 $\mathbf{B}(\mathcal{A})$ *is a commutative DGA-algebra over* $\mathbb{H}M$*, with multiplication*

$$\circ : \mathbf{B}(\mathcal{A}) \otimes_{\mathbb{H}M} \mathbf{B}(\mathcal{A}) \to \mathbf{B}(\mathcal{A})$$

defined, for integers $m, n \geq 0$ *and* $x, y \in M$*, by the homomorphisms of abelian groups*

$$\circ : \mathbf{B}(\mathcal{A})_m(x) \otimes \mathbf{B}(\mathcal{A})_n(y) \to \mathbf{B}(\mathcal{A})_{m+n}(xy)$$

given by the shuffle products

$$[a_{x_1}|\cdots|a_{x_p}] \circ [a_{x_{p+1}}|\cdots|a_{x_{p+q}}] = \sum_\sigma (-1)^{e(\sigma)} \left[a_{x_{\sigma^{-1}(1)}}|\cdots|a_{x_{\sigma^{-1}(p+q)}}\right]$$

for any $x_i \in M$ *and* $a_{x_i} \in \mathcal{A}_{r_i}(x_i)$, $i = 1, \ldots, p+q$, *such that* $x_1 \cdots x_p = x$, $x_{p+1} \cdots x_{p+q} = y$, $p + \sum_{i=1}^p r_i = m$, *and* $q + \sum_{j=1}^q r_{p+j} = n$, *where the sum is taken over all* (p,q)*-shuffles* σ *and, for each* σ, *the exponent of the sign is* $e(\sigma) = \sum(1+r_i)(1+r_{p+j})$ *summed over all pairs* $(i, p+j)$ *such that* $\sigma(i) > \sigma(p+j)$.

The unit is $[\ ] \in \mathbf{B}(\mathcal{A})_0(e)$, *that is, the unit morphism* $\iota : \mathbb{Z} \to \mathbf{B}(\mathcal{A})$ *is the isomorphism of* $\mathbb{H}M$*-modules* $\iota : \mathbb{Z} \cong \mathbf{B}(\mathcal{A})_0$ *given by* $\iota_x(x) = x_*[\ ]$, *for any* $x \in M$, *and the augmentation* $\epsilon : \mathbf{B}(\mathcal{A}) \to \mathbb{Z}$ *is defined by the isomorphism of* $\mathbb{H}M$*-modules* $\epsilon = \iota^{-1} : \mathbf{B}(\mathcal{A})_0 \cong \mathbb{Z}$ *such that* $\epsilon_x(x_*[\ ]) = (x)$, *for any* $x \in M$.

Proof: We give an indirect proof, by using that the category of $\mathbb{H}M$-modules is closely related to the category $\mathbb{Z}M$-Mod, of ordinary modules over the monoid ring $\mathbb{Z}M$.

There is an exact faithful functor $\Gamma : \mathbb{H}M\text{-Mod} \to \mathbb{Z}M\text{-Mod}$, which carries any $\mathbb{H}M$-module $\mathcal{A}$ to the $\mathbb{Z}M$-module defined by the abelian group $\Gamma\mathcal{A} = \bigoplus_{x\in M} \mathcal{A}(x)$, with M-action of an element $y \in M$ on an element $a_x \in \mathcal{A}(x)$ given by $y\, a_x = y_* a_x \in \mathcal{A}(xy)$. This functor Γ is left adjoint to the functor which associates to any $\mathbb{Z}M$-module A the constant on objects $\mathbb{H}M$-module defined by the underlying abelian group A, with $y_* : A \to A$, for any $y \in M$, the homomorphism of multiplication by y [52].

It is plain to see that Γ is a symmetric strict monoidal functor, that is, $\Gamma\mathbb{Z} = \mathbb{Z}M$, for any $\mathbb{H}M$-modules $\mathcal{A}$ and $\mathcal{B}$, $\Gamma(\mathcal{A} \otimes_{\mathbb{H}M} \mathcal{B}) = \Gamma\mathcal{A} \otimes_{\mathbb{Z}M} \Gamma\mathcal{B}$, and it carries the associativity, unit, and commutativity constraints of the monoidal category of $\mathbb{H}M$-modules to the corresponding ones of the category of $\mathbb{Z}M$-modules. Then, the same properties hold for the induced functor Γ from the symmetric monoidal category of complexes of $\mathbb{H}M$-modules to the the symmetric monoidal category of complexes of $\mathbb{Z}M$-modules. It follows that Γ transforms commutative monoids in the category of complexes of $\mathbb{H}M$-modules (i.e. commutative DG-algebras over $\mathbb{H}M$) to commutative monoids in the category of $\mathbb{Z}M$-modules (i.e., commutative DG-algebras over $\mathbb{Z}M$), and therefore Γ also transform commutative DGA-algebras over $\mathbb{H}M$ to commutative DGA-algebras over the monoid ring $\mathbb{Z}M$.

Now, given $\mathcal{A}$, a commutative DGA-algebra over $\mathbb{H}M$, let $\mathbf{B}(\Gamma\mathcal{A})$ be the commutative DGA-algebra over $\mathbb{Z}M$ obtained by applying the ordinary Eilenberg-Mac Lane bar construction on $\Gamma\mathcal{A}$ [57, Chapter X, Theorem 12.1]. A direct comparison shows that $\mathbf{B}(\Gamma\mathcal{A}) = \Gamma\mathbf{B}(\mathcal{A})$ as complexes of $\mathbb{Z}M$-modules, and also that its multiplication, unit, and augmentation are, respectively, just the morphisms

$$\mathbf{B}(\Gamma\mathcal{A}) \otimes_{\mathbb{Z}M} \mathbf{B}(\Gamma\mathcal{A}) = \Gamma\big(\mathbf{B}(\mathcal{A}) \otimes_{\mathbb{H}M} \mathbf{B}(\mathcal{A})\big) \xrightarrow{\Gamma\circ} \Gamma\mathbf{B}(\mathcal{A}) = \mathbf{B}(\Gamma\mathcal{A})\ ,$$

$$\mathbb{Z}M = \Gamma\mathbb{Z} \xrightarrow{\Gamma\iota} \Gamma\mathbf{B}(\mathcal{A}) = \mathbf{B}(\Gamma\mathcal{A}), \qquad \mathbf{B}(\Gamma\mathcal{A}) = \Gamma\mathbf{B}(\mathcal{A}) \xrightarrow{\Gamma\epsilon} \Gamma\mathbb{Z} = \mathbb{Z}M.$$

Then, as $\mathbf{B}(\Gamma\mathcal{A})$ is actually a commutative DGA-algebra over $\mathbb{Z}M$, it follows that the equalities

$$\Gamma\big(\circ(\circ\otimes id_{\mathbf{B}(\mathcal{A})})\big) = \Gamma\big(\circ(id_{\mathbf{B}(\mathcal{A})}\otimes\circ)\big), \quad \Gamma(\circ\,\boldsymbol{c}_{\mathbf{B}(\mathcal{A}),\mathbf{B}(\mathcal{A})}) = \Gamma\circ,$$

$$\Gamma(\circ(\iota\otimes id_{\mathbf{B}(\mathcal{A})})) = \Gamma\boldsymbol{l}_{\mathbf{B}(\mathcal{A})}, \quad \Gamma(\circ(\epsilon\otimes\epsilon)) = \Gamma(\epsilon\circ), \quad \Gamma(\epsilon\iota) = \Gamma id_{\mathbb{Z}}.$$

hold. Therefore, the result, that is, that $\mathbf{B}(\mathcal{A})$ is a commutative DGA-algebra over $\mathbb{H}M$, follows since the functor Γ is faithful. □

Remark 5.3 Observe, as in [31, Section 7], that the shuffle product $\circ$ on $\mathbf{B}(\mathcal{A})$ can also be expressed by the recursive formula below, where $\alpha = [a_{x_1}|\cdots|a_{x_p}] \in \mathbf{B}(\mathcal{A})_r(x)$, $\beta = [b_{y_1}|\cdots|b_{y_q}] \in \mathbf{B}(\mathcal{A})_s(y)$, $a_z \in \mathcal{A}_m(z)$ and $b_t \in \mathcal{A}_n(t)$.

$$[\alpha \mid a_z]\circ[\beta \mid b_t] = [[\alpha \mid a_z]\circ\beta \mid b_t] + (-1)^{r(n+s+1)}[\alpha\circ[\beta \mid b_t] \mid a_z] \tag{5.12}$$

Let us stress the *suspension* morphism of complexes of $\mathbb{H}M$-modules, of degree 1 (hence satisfying $\partial\,\mathrm{S} = -\mathrm{S}\,\partial$),

$$\mathrm{S}:\mathcal{A}\to\mathbf{B}(\mathcal{A}), \tag{5.13}$$

which is defined, at any $x\in M$, by $\mathrm{S}_x a_x = [a_x] \in \mathbf{B}(\mathcal{A})(x)$, for any chain a_x of $\mathcal{A}(x)$.

Such as Mac Lane does in [57, Chapter X, Section 12] for ordinary commutative DGA-algebras over a commutative ring, the cohomology of a commutative DGA-algebra over $\mathbb{H}M$ can be defined in "stages" or "levels". If $\mathcal{A}$ is any commutative DGA-algebra over $\mathbb{H}M$, then $\mathbf{B}(\mathcal{A})$ is again a commutative DGA-algebra over $\mathbb{H}M$, so an iteration is possible to form $\mathbf{B}^r(\mathcal{A})$ for each integer $r\geq 1$. Hence, we define the *rth level cohomology groups of $\mathcal{A}$ with coefficients in an $\mathbb{H}M$-module $\mathcal{B}$*, denoted by $H^n(\mathcal{A},r;\mathcal{B})$, as

$$H^n(\mathcal{A},r;\mathcal{B}) = H^n\big(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}),\mathcal{B})\big), \quad n=0,1,\dots\,,$$

where $\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}),\mathcal{B})$ is the cochain complex obtained by applying the functor $\mathrm{Hom}_{\mathbb{H}M}(-,\mathcal{B})$ to the underlying chain complex of $\mathbb{H}M$-modules $\mathbf{B}^r(\mathcal{A})$.

Remark 5.4 When the bar construction above is applied on the constant DGA-algebra over $\mathbb{H}M$ defined by a commutative DGA-ring A, the result is just the constant DGA-algebra over $\mathbb{H}M$ defined by the commutative DGA-ring obtained by applying on A the Eilenberg-Mac Lane reduced bar construction. Hence, the notation $\mathbf{B}(A)$ is not confusing.

If $\mathcal{A}$ and $\mathcal{B}$ are commutative DGA-algebras over $\mathbb{H}M$, then we say that two morphisms of DGA-algebras $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ form a *contraction* whenever $fg = id_{\mathcal{B}}$, and there exists an homotopy of morphisms of complexes $\Phi : gf \Rightarrow id_{\mathcal{A}}$ satisfying the conditions

$$\Phi g = 0,\ f\Phi = 0,\ \Phi\Phi = 0. \tag{5.14}$$

Paralleling the proof by Eilenberg and Mac Lane of [31, Theorem 12.1], one proves the following:

Lemma 5.1 *If $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ form a contraction of commutative DGA-algebras over $\mathbb{H}M$, then the induced $\mathbf{B}(f) : \mathbf{B}(\mathcal{A}) \to \mathbf{B}(\mathcal{B})$ and $\mathbf{B}(g) : \mathbf{B}(\mathcal{B}) \to \mathbf{B}(\mathcal{A})$ also form a contraction.*

5.3 Free $\mathbb{H}M$-modules

Let $\mathbf{Set}{\downarrow_M}$ be the comma category of sets over the underlying set of M; that is, the category whose objects $S = (S, \pi)$ are sets S endowed with a map $\pi : S \to M$, and whose morphisms are maps $\varphi : S \to T$ such that $\pi\varphi = \pi$. There is a *forgetful functor*

$$\mathcal{U} : \mathbb{H}M\text{-Mod} \to \mathbf{Set}{\downarrow_M},$$

which carries any $\mathbb{H}M$-module $\mathcal{A}$ to the disjoint union set

$$\mathcal{U}\mathcal{A} = \bigcup_{x\in M} \mathcal{A}(x) = \{(x, a_x) \mid x \in M,\ a_x \in \mathcal{A}(x)\},$$

endowed with the projection map $\pi : \mathcal{U}\mathcal{A} \to M$, $\pi(x, a_x) = x$. A morphism $f : \mathcal{A} \to \mathcal{B}$ is sent to the map $\mathcal{U}f : \mathcal{U}\mathcal{A} \to \mathcal{U}\mathcal{B}$ given by $\mathcal{U}f(x, a_x) = (x, f_x a_x)$. There is also a *free $\mathbb{H}M$-module* functor

$$\mathcal{Z} : \mathbf{Set}{\downarrow_M} \to \mathbb{H}M\text{-Mod}, \tag{5.15}$$

which is defined as follows: If S is any set over M, then $\mathcal{Z}S$ is the $\mathbb{H}M$-module such that, for each $x \in M$,

$$\mathcal{Z}S(x) = \mathbb{Z}\{(u, s) \in M \times S \mid u\,\pi(s) = x\}$$

is the free abelian group with generators all pairs (u, s), where $u \in M$ and $s \in S$, such that $u\,\pi(s) = x$. We usually write (e, s) simply by s; so that each element of $s \in S$ is regarded as an element $s \in \mathcal{Z}S(\pi s)$. For any $x, y \in M$, the homomorphism

$$y_* : \mathcal{Z}S(x) \to \mathcal{Z}S(xy)$$

is defined on generators by $y_*(u, s) = (uy, s)$. If $\varphi : S \to T$ is any map of sets over M, the induced morphism $\mathcal{Z}\varphi : \mathcal{Z}S \to \mathcal{Z}T$ is given, at each $x \in M$, by the homomorphism $(\mathcal{Z}\varphi)_x : \mathcal{Z}S(x) \to \mathcal{Z}T(x)$ defined on generators by $(\mathcal{Z}\varphi)_x(u, s) = (u, \varphi s)$.

Proposition 5.2 *The functor $\mathcal{Z}$ is left adjoint to the functor $\mathcal{U}$. Thus, for S any set over M, to each $\mathbb{H}M$-module $\mathcal{A}$ and each list of elements $a_s \in \mathcal{A}(\pi s)$, one for each $s \in S$, there is an unique morphism of $\mathbb{H}M$-modules $f : \mathcal{Z}S \to \mathcal{A}$ with $f_{\pi s}(s) = a_s$ for every $s \in S$.*

Proof: At any set S over M, the unit of the adjunction is the map

$$\nu : S \to \mathcal{U}\mathcal{Z}S = \{(x, a_x) \mid x \in M,\ a_x \in \mathcal{Z}S(x)\}, \qquad s \mapsto (\pi s, s).$$

If $\mathcal{A}$ is an $\mathbb{H}M$-module and $\varphi : S \to \mathcal{U}\mathcal{A}$ is any map over M, then, the unique morphism of $\mathbb{H}M$-modules $f : \mathcal{Z}S \to \mathcal{A}$ such that $(\mathcal{U}f)\,\nu = \varphi$ is determined by the equations $f_x(u, s) = u_*\varphi(s)$, for any $x \in M$ and $(u, s) \in M \times S$ with $u\,\pi(s) = x$. □

The category $\mathbf{Set}\downarrow_M$ has a symmetric monoidal structure, where the tensor product of two sets over M, say S and T, is the cartesian product set of $S \times T$ with $\pi(s, t) = \pi(s)\pi(t)$. The unit object is provided by the unitary set $\{e\}$ with $\pi(e) = e \in M$, and the associativity, unit, and commutativity constraints are the obvious ones. Hereafter, the category $\mathbf{Set}\downarrow_M$ will be considered with this monoidal structure[1].

Proposition 5.3 *The free $\mathbb{H}M$-module functor* (5.15) *is symmetric monoidal, that is, there are natural and coherent isomorphisms of $\mathbb{H}M$-modules*

$$\mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T, \quad \mathcal{Z}\{e\} \cong \mathbb{Z},$$

for S and T any sets over M.

Proof: For S, T any given sets over M, the isomorphism $f : \mathcal{Z}(S \times T) \cong \mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T$ is the morphism of $\mathbb{H}M$-modules such that, for any $(s, t) \in S \times T$, $f_{\pi(s,t)}(s, t) = s \otimes t$. Observe that, for any $x \in M$, the abelian group $\mathcal{Z}(S \times T)(x)$ is free with generators the elements $(u, s, t) = u_*(s, t)$, with $u \in M$, $s \in S$, and $t \in T$, such that $u\,\pi(s)\,\pi(t) = x$, while $(\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$ is the abelian group generated by the elements $(u, s) \otimes (v, t) = u_*s \otimes v_*t$, with $u, v \in M$, $s \in S$, and $t \in T$, such that $u\,\pi(s)\,v\,\pi(t) = x$, with the relations $u_*s \otimes v_*t = (uv)_*(s \otimes t)$. Then, the homomorphism $f_x : \mathcal{Z}(S \times T)(x) \to (\mathcal{Z}S \otimes_{\mathbb{H}M} \mathcal{Z}T)(x)$, which acts on elements of the basis by $f_x(u_*(s, t)) = u_*(s \otimes t)$, is clearly an isomorphism of abelian groups.

The isomorphism $f : \mathcal{Z}\{e\} \cong \mathbb{Z}$ is the morphism of $\mathbb{H}M$-modules such that $f_e(e) = e$. Observe that, for any $x \in M$, the isomorphism f_x is the composite

$$\mathcal{Z}\{e\}(x) = \mathbb{Z}\{(u, e) \mid ue = x\} = \mathcal{Z}\{(x, e)\} \cong \mathbb{Z}(x).$$

It is straightforward to see that the isomorphisms f above are natural and coherent, so that $\mathcal{Z}$ is actually a symmetric monoidal functor. □

[1] The category $\mathbf{Set}\downarrow_M$ has a different monoidal structure where the tensor product is given by the fibre-product $S \times_M T$ with $\pi(s, t) = \pi(s) = \pi(t)$.

Corollary 5.1 *For S and T any two sets over M, the tensor product $\mathbb{H}M$-module $\mathcal{Z}S\otimes_{\mathbb{H}M}\mathcal{Z}T$ is free on the set of elements $s\otimes t$, $s\in S$, $t\in T$, with $\pi(s\otimes t)=\pi(s)\pi(t)$.*

Since the functor $\mathcal{Z}$ is symmetric monoidal, it transports commutative monoids in $\mathbf{Set}\downarrow_M$ to commutative monoids in $\mathbb{H}M$-Mod, that is, to algebras over $\mathbb{H}M$. As a commutative monoid in the symmetric monoidal category $\mathbf{Set}\downarrow_M$ is merely a commutative monoid over M, that is, a commutative monoid S endowed with a homomorphism $\pi: S\to M$, the corollary below follows.

Corollary 5.2 *If S is a commutative monoid over M, then the free $\mathbb{H}M$-module $\mathcal{Z}S$ is an algebra over $\mathbb{H}M$. The multiplication morphism $\circ:\mathcal{Z}S\otimes_{\mathbb{H}M}\mathcal{Z}S\to\mathcal{Z}S$ is the composite*

$$\mathcal{Z}S\otimes_{\mathbb{H}M}\mathcal{Z}S\cong\mathcal{Z}(S\times S)\xrightarrow{\mathcal{Z}\mathrm{m}}\mathcal{Z}S,$$

where $\mathrm{m}: S\times S\to S$ is the homomorphism of multiplication in S, $\mathrm{m}(s,s')=ss'$, and the unit morphism $\iota:\mathbb{Z}\to\mathcal{Z}S$ is the composite $\mathbb{Z}\cong\mathcal{Z}\{e\}\xrightarrow{\mathcal{Z}\mathrm{i}}\mathcal{Z}S$, where $\mathrm{i}:\{e\}\to S$ is the trivial homomorphism mapping the unit of M to the unit of S.

5.4 The cohomology groups $H^n(M,r;\mathcal{A})$

Let us consider the commutative monoid M over itself with $\pi=id_M: M\to M$. Then, by Corollary 5.2, the free $\mathbb{H}M$-module $\mathcal{Z}M$ is an algebra over $\mathbb{H}M$. Explicitly, this is described as follows: For each $x\in M$,

$$\mathcal{Z}M(x)=\mathbb{Z}\{(u,v)\mid uv=x\}$$

is the free abelian group with generators all pairs $(u,v)\in M\times M$ such that $uv=x$. For any $x,y\in M$, the homomorphism $y_*:\mathcal{Z}M(x)\to\mathcal{Z}M(xy)$ is given on generators by $y_*(u,v)=(yu,v)$, and the homomorphism of multiplication

$$\circ:\mathcal{Z}M(x)\otimes\mathcal{Z}M(y)\to\mathcal{Z}M(xy)$$

is defined on generators by $(u,v)\otimes(w,t)\mapsto(u,v)\circ(w,t)=(uw,vt)$, for any u,v,w,t in M such that $uv=x$ and $wt=y$. The unit is $(e,e)\in\mathcal{Z}M(e)$. We see each element $x\in M$ as an element of $\mathcal{Z}M(x)$ by means of the identification $x=(e,x)$, so that that any generator (u,v) of $\mathcal{Z}M(x)$ can be write as u_*v.

By Proposition 5.2, if $\mathcal{A}$ is any $\mathbb{H}M$-module, for any list of elements $a_x\in\mathcal{A}(x)$, one for each $x\in M$, there is an unique morphism of $\mathbb{H}M$-modules $f:\mathcal{Z}M\to\mathcal{A}$ such that each homomorphism $f_x:\mathcal{Z}M(x)\to\mathcal{A}(x)$ verifies that $f_x(x)=a_x$ (explicitly, f_x acts on generators by $f_x(u,v)=u_*a_v$). Furthermore, it is plain to see that, if $\mathcal{A}$ is an algebra over $\mathbb{H}M$, then f is a morphism of algebras if and only if $a_e=1$ and $a_x\circ a_y=a_{xy}$ for all $x,y\in M$.

Hereafter, we regard $\mathcal{Z}M$ as a commutative DGA-algebra over $\mathbb{H}M$ with the trivial grading, that is, with $(\mathcal{Z}M)_n = 0$ for $n > 0$ and $(\mathcal{Z}M)_0 = \mathcal{Z}M$, and with augmentation the morphism of $\mathbb{H}M$-algebras

$$\epsilon : \mathcal{Z}M \to \mathbb{Z},$$

such that, for any $x \in M$, $\epsilon_x(x) = (x) \in \mathbb{Z}(x)$. Then, we define, for each integer $r \geq 1$, *the rth level cohomology groups of the commutative monoid M with coefficients in an $\mathbb{H}M$-module $\mathcal{A}$* by

$$H^n(M, r; \mathcal{A}) = H^n(\mathcal{Z}M, r; \mathcal{A}), \quad n = 0, 1, \dots , \tag{5.16}$$

or, in other words,

$$H^n(M, r; \mathcal{A}) = H^n\big(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{Z}M), \mathcal{A})\big),$$

where $\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{Z}M), \mathcal{A})$ is the cochain complex obtained by applying the abelian group valued functor $\mathrm{Hom}_{\mathbb{H}M}(-, \mathcal{A})$ to the neglected chain complex of $\mathbb{H}M$-modules $\mathbf{B}^r(\mathcal{Z}M)$.

Remark 5.5 When $M = G$ is an abelian group, $\mathcal{Z}G$ is isomorphic to the constant DGA-algebra over $\mathbb{H}G$ defined by the commutative DGA-ring $\mathcal{Z}G(e)$ (see Remark 5.2), which is itself isomorphic to the trivially graded DGA-ring defined by the group ring $\mathbb{Z}G$ with augmentation the ring homomorphism $\alpha : \mathbb{Z}G \to \mathbb{Z}$ such that $\alpha(x) = 1$ for any $x \in G$. To see this, observe that $\mathcal{Z}G(e)$ is the commutative ring whose underlying abelian group is freely generated by the elements of the form (x^{-1}, x), $x \in G$, with multiplication such that $(x^{-1}, x) \circ (y^{-1}, y) = ((xy)^{-1}, xy)$, and unit $(e, e) = e$. The map $(x^{-1}, x) \mapsto x$ clearly determines a ring isomorphism between $\mathcal{Z}G(e)$ and the group ring $\mathbb{Z}G$, which is compatible with the corresponding augmentations.

Hence, for any integer $r \geq 1$, $\mathbf{B}^r(\mathcal{Z}G) \cong \mathbf{B}^r(\mathbb{Z}G)$ (see Remark 5.4)[2], and therefore for any abelian group A, regarded as a constant $\mathbb{H}G$-module, there are natural isomorphisms

$$\mathrm{Hom}_{\mathbb{H}G}(\mathbf{B}^r(\mathcal{Z}G), A) \cong \mathrm{Hom}_{\mathbb{H}G}(\mathbf{B}^r(\mathbb{Z}G), A) \cong \mathrm{Hom}(\mathbf{B}^r(\mathbb{Z}G), A)$$

showing that the rth level cohomology groups $H^n(G, r; A)$ in (5.16) agree with those by Eilenberg and Mac Lane in [31], which compute the cohomology of the spaces $K(G, r)$ by means of natural isomorphisms $H^n(K(G, r), A) \cong H^n(G, r; A)$.

From now on, this section is dedicated to show explicit cochain descriptions for some of these cohomology groups, starting with those of first level

$$H^n(M, 1; \mathcal{A}) = H^n\big(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}(\mathcal{Z}M), \mathcal{A})\big).$$

[2] The commutative DGA-rings $\mathbf{B}^r(\mathbb{Z}G)$ are denoted by $A_N(G, r)$ in [31]

Let us analyze the underlying complex $\mathbf{B}(\mathcal{Z}M)$. For any integer $n \geq 1$,

$$\mathbf{B}(\mathcal{Z}M)_n = \overline{\mathcal{Z}M} \otimes_{\mathbb{H}M} \overset{(n \text{ factors})}{\cdots} \otimes_{\mathbb{H}M} \overline{\mathcal{Z}M},$$

where $\overline{\mathcal{Z}M} = \mathcal{Z}M/\iota\mathbb{Z} = \mathcal{Z}M/\mathcal{Z}\{e\} \cong \mathcal{Z}M^*$ is a free $\mathbb{H}M$-module on $M^* = M \setminus \{e\}$ with $\pi : M^* \to M$ the inclusion map. Then, by construction and Proposition 5.3, we have that

- *The $\mathbb{H}M$-module $\mathbf{B}(\mathcal{Z}M)_0$ is free on the unitary set $\{[\ ]\}$ with $\pi[\ ] = e$ and, for any $n \geq 1$, $\mathbf{B}(\mathcal{Z}M)_n$ is a free $\mathbb{H}M$-module generated by the set over M consisting of n-tuples of elements of M*

$$\alpha_n = [x_1|\cdots|x_n], \quad \textit{with} \;\; \pi\alpha_n = x_1\cdots x_n,$$

which we call generic n-cells *of* $\mathbf{B}(\mathcal{Z}M)$, *with the relations $\alpha_n = 0$ whenever some $x_i = e$.*

- *The differential $\partial : \mathbf{B}(\mathcal{Z}M)_n \to \mathbf{B}(\mathcal{Z}M)_{n-1}$ is the morphism of $\mathbb{H}M$-modules such that, for each $x \in M$ and any generic n-cell $[x_1|\cdots|x_n]$ with $x_1\cdots x_n = x$,*

$$\partial_x[x_1|\cdots|x_n] = x_{1*}[x_2|\cdots|x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$$
$$+ (-1)^n x_{n*}[x_1|\cdots|x_{n-1}].$$

Hence, Proposition 5.2 gives the following.

Theorem 5.1 *For any $\mathbb{H}M$-module $\mathcal{A}$, the cohomology groups $H^n(M,1;\mathcal{A})$ can be computed as the cohomology groups of the* cochain complex of normalized 1st level cochains of M with values in $\mathcal{A}$,

$$C(M,1;\mathcal{A}) :\ 0 \to C^0(M,1;\mathcal{A}) \xrightarrow{\partial^0} C^1(M,1;\mathcal{A}) \xrightarrow{\partial^1} C^2(M,1;\mathcal{A}) \xrightarrow{\partial^2} \cdots, \quad (5.17)$$

where

- *$C^0(M,1;\mathcal{A}) = \mathcal{A}(e)$, and for $n \geq 1$, $C^n(M,1;\mathcal{A})$ is the abelian group, under pointwise addition, of functions*

$$f : M^n \to \textstyle\bigcup_{x\in M}\mathcal{A}(x)$$

such that $f(x_1,\ldots,x_n) \in \mathcal{A}(x_1\cdots x_n)$ and $f(x_1,\ldots,x_n) = 0$ whenever some $x_i = e$,

- *$\partial^0 = 0$, and for $n \geq 1$, the coboundary $\partial^n : C^n(M,1;\mathcal{A}) \to C^{n+1}(M,1;\mathcal{A})$ is given by*

$$(\partial^n f)(x_1,\cdots,x_{n+1}) = x_{1*}f(x_2,\cdots,x_{n+1}) + \sum_{i=1}^{n}(-1)^i f(x_1,\cdots,x_ix_{i+1},\cdots,x_{n+1})$$
$$+ (-1)^{n+1}x_{n+1*}f(x_1,\cdots,x_n).$$

Let us now recall that Leech cohomology groups [53] (see 1.3.1) of a (not necessarily commutative) monoid M, $H^n_{\mathrm{L}}(M,\mathcal{A})$, take coefficients in $\mathbb{D}M$-modules. When the monoid M is commutative, as it is in our case, there is a full functor $\mathbb{D}M \to \mathbb{H}M$, which is the identity on objects and carries a morphism $(x,y,z) : y \to xyz$ of $\mathbb{D}M$ to the morphism $(y,xz) : y \to xyz$ of $\mathbb{H}M$. Composition with this functor induces a full embedding of $\mathbb{H}M$-Mod into $\mathbb{D}M$-mod, whose image consists of the symmetric $\mathbb{D}M$-modules (see the end of Section 2.4).

As a direct inspection shows that, for any $\mathbb{H}M$-module $\mathcal{A}$, the cochain complex $C(M,1;\mathcal{A})$ in (5.17) coincides with the cochain complex in (1.51) next theorem follows.

Proposition 5.4 *For any $\mathbb{H}M$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^n(M,1;\mathcal{A}) \cong H^n_{\mathrm{L}}(M,\mathcal{A}), \quad n = 0,1,\ldots .$$

We now analyze the complex of $\mathbb{H}M$-modules $\mathbf{B}^r(\mathcal{Z}M)$ for $r \geq 2$ any integer. By construction,

- $\mathbf{B}^r(\mathcal{Z}M)_0$ *is the free $\mathbb{H}M$-module on the unitary set consisting of the 0-tuple*
$$[\ ], \quad \textit{with } \pi[\ ] = e,$$
which we call the generic 0*-cell of* $\mathbf{B}^r(\mathcal{Z}M)$,

and, for $n \geq 1$,

$$\mathbf{B}^r(\mathcal{Z}M)_n = \bigoplus_{p+\sum n_i = n} \overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_1} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} \overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_p}.$$

Since $\overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_0 = 0$ while, for $n_i \geq 1$, $\overline{\mathbf{B}^{r-1}(\mathcal{Z}M)}_{n_i} = \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_i}$, it follows by induction on r that

- $\mathbf{B}^r(\mathcal{Z}M)_n = 0$ *for* $0 < n < r$,

and that, for any $r \leq n$,

$$\mathbf{B}^r(\mathcal{Z}M)_n = \bigoplus_{\substack{n_1,\ldots,n_p \geq r-1 \\ p+\sum n_i = n}} \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_1} \otimes_{\mathbb{H}M} \cdots \otimes_{\mathbb{H}M} \mathbf{B}^{r-1}(\mathcal{Z}M)_{n_p}.$$

Then, if we denote by $|_r$ the symbol $|$ used for the tensor product in the construction of $\mathbf{B}^r(\mathcal{Z}M)$ from $\mathbf{B}^{r-1}(\mathcal{Z}M)$, by Proposition 5.3 and induction, we see that

- $\mathbf{B}^r(\mathcal{Z}M)_n$, *for $r \leq n$, is a free $\mathbb{H}M$-module generated by the set over M consisting of all p-tuples, which we call* generic n-cells *of* $\mathbf{B}^r(\mathcal{Z}M)$,
$$\alpha_n = [\alpha_{n_1}|_r\alpha_{n_2}|_r \cdots |_r\alpha_{n_p}], \qquad \textit{with } \ \pi\alpha_n = \pi\alpha_{n_1}\cdots\pi\alpha_{n_p},$$
of generic n_i-cells of $\mathbf{B}^{r-1}(\mathcal{Z}M)$, such that $n_i \geq r-1$ and $p + \sum n_i = n$, with the relations $\alpha_n = 0$ whenever some $\alpha_{n_i} = 0$.

Let us stress that a generic n-cell α_n of any $\mathbf{B}^r(\mathcal{Z}M)$ is actually a generator of the abelian group $\mathbf{B}^r(\mathcal{Z}M)_n(\pi\alpha_n)$. Indeed, for each $x \in M$, $\mathbf{B}^r(\mathcal{Z}M)_n(x)$ is the free abelian group generated by the elements $u_*\alpha_n$ with u an element of M and the α_n any non-zero generic n-cell of $\mathbf{B}^r(\mathcal{Z}M)$ such that $u\,\pi\alpha_n = x$. Arbitrary elements of the groups $\mathbf{B}^r(\mathcal{Z}M)_n(x)$, are referred as *$n$-chains of* $\mathbf{B}^r(\mathcal{Z}M)$.

For any $r \geq 1$, the multiplication $\circ_r$ of $\mathbf{B}^r(\mathcal{Z}M)$ is given by the morphism of $\mathbb{H}M$-modules

$$\circ_r : \mathbf{B}^r(\mathcal{Z}M)_n \otimes_{\mathbb{H}M} \mathbf{B}^r(\mathcal{Z}M)_m \to \mathbf{B}^r(\mathcal{Z}M)_{n+m}$$

which, according to Proposition 5.2, are determined on generic cells by the shuffle product

$$[\alpha_{n_1}|_r \cdots |_r \alpha_{n_p}] \circ_r [\alpha_{n_{p+1}}|_r \cdots |_r \alpha_{n_{p+q}}] = \sum_\sigma (-1)^{e(\sigma)} [\alpha_{n_{\sigma^{-1}(1)}}|_r \cdots |_r \alpha_{n_{\sigma^{-1}(p+q)}}],$$

where the sum is taken over all (p,q)-shuffles σ and $e(\sigma) = \sum (1+n_i)(1+n_{p+j})$ summed over all pairs $(i, p+j)$ such that $\sigma(i) > \sigma(p+j)$. In particular, for $r = 1$,

$$[x_1|\cdots|x_n] \circ_1 [x_{n+1}|\cdots|x_{n+m}] = \sum_\sigma (-1)^{e(\sigma)} [x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(n+m)}], \tag{5.18}$$

where the sum is taken over all (n,m)-shuffles σ and $e(\sigma)$ is the sign of the shuffle.

Then, for $r \geq 2$,

- *the boundary $\partial : \mathbf{B}^r(\mathcal{Z}M)_n \to \mathbf{B}^r(\mathcal{Z}M)_{n-1}$ is the morphism of $\mathbb{H}M$-modules recursively defined, on any generic n-cell $\alpha_n = [\alpha_{n_1}|_r \cdots |_r \alpha_{n_p}]$ of $\mathbf{B}^r(\mathcal{Z}M)$ with $\pi\alpha_n = x$ and $\pi\alpha_{n_i} = x_i$, by*

$$\begin{aligned}\partial_x \alpha_n = & -\sum_{i=1}^{p} (-1)^{e_{i-1}} [\alpha_{n_1}|_r \cdots |_r \alpha_{n_{i-1}}|_r \partial_{x_i}\alpha_{n_i}|_r \alpha_{n_{i+1}}|_r \cdots |_r \alpha_{n_p}] \\ & + \sum_{i=1}^{p-1} (-1)^{e_i} [\alpha_{n_1}|_r \cdots |_r \alpha_{n_{i-1}}|_r \alpha_{n_i} \circ_{r-1} \alpha_{n_{i+1}}|_r \alpha_{n_{i+2}}|_r \cdots |_r \alpha_{n_p}],\end{aligned}$$

where the exponents e_i of the signs are $e_i = i + \sum n_i$.

In the above formula, the term $\partial_{x_i}\alpha_{n_i}$, which refers to the differential of α_{n_i} in $\mathbf{B}^{r-1}(\mathcal{Z}M)$, or $\alpha_{n_i} \circ_{r-1} \alpha_{n_{i+1}}$, is not in general a generic cell of $\mathbf{B}^{r-1}(\mathcal{Z}M)$ but a chain; the term is to be expanded by linearity.

Recall now that we have the embedding suspensions (5.13), $\mathrm{S} : \mathbf{B}^{r-1}(\mathcal{Z}M) \hookrightarrow \mathbf{B}^r(\mathcal{Z}M)$, through which we identify any generic $(n-1)$-cell α_{n-1} of $\mathbf{B}^{r-1}(\mathcal{Z}M)$ with the generic n-cell $\mathrm{S}\alpha_{n-1} = [\alpha_{n-1}]$ of $\mathbf{B}^r(\mathcal{Z}M)$. Hence, by induction, one proves that any generic n-cell of any $\mathbf{B}^r(\mathcal{Z}M)$ can be uniquely written in the form

$$\alpha_n = [x_1|_{k_1} x_2|_{k_2} \cdots |_{k_{m-1}} x_m]$$

with $x_i \in M$, $1 \leq m$, $1 \leq k_i \leq r$, and $r + \sum_{i=1}^{m-1} k_i = n$. So written, we have $\pi\alpha_n = x_1 \cdots x_m$, and $\alpha_n = 0$ if $x_i = e$ for some i. Observe that if some $k_i = r$, then $n \geq 2r$. Indeed, the generic n-cells of lowest n appearing in $\mathbf{B}^r(\mathcal{Z}M)$ but not in $\mathbf{B}^{r-1}(\mathcal{Z}M)$ are those generic $2r$-cells of the form $[x_1|_r x_2]$. Thus, via the suspension morphism, $\mathbf{B}^{r-1}(\mathcal{Z}M)_{n-1}$ is identified with $\mathbf{B}^r(\mathcal{Z}M)_n$ for $r \leq n < 2r$, while $\mathbf{B}^{r-1}(\mathcal{Z}M)_{n-1} \subsetneq \mathbf{B}^r(\mathcal{Z}M)_n$ for $n \geq 2r$. In particular, we have the commutative diagram of suspensions

$$\begin{tikzcd}
\mathbf{B}(\mathcal{Z}M)_4 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathbf{B}(\mathcal{Z}M)_3 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathbf{B}(\mathcal{Z}M)_2 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}(\mathcal{Z}M)_1 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}(\mathcal{Z}M)_0 \arrow[d] \\
\mathbf{B}^2(\mathcal{Z}M)_5 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathbf{B}^2(\mathcal{Z}M)_4 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}^2(\mathcal{Z}M)_3 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}^2(\mathcal{Z}M)_2 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & 0 \\
\mathbf{B}^3(\mathcal{Z}M)_6 \arrow[r] \arrow[d, equal, "\mathrm{S}^{r-3}"'] & \mathbf{B}^3(\mathcal{Z}M)_5 \arrow[r] \arrow[d, equal, "\mathrm{S}^{r-3}"'] & \mathbf{B}^3(\mathcal{Z}M)_4 \arrow[r] \arrow[d, equal, "\mathrm{S}^{r-3}"'] & \mathbf{B}^3(\mathcal{Z}M)_3 \arrow[r] \arrow[d, equal, "\mathrm{S}^{r-3}"'] & 0 \\
\mathbf{B}^r(\mathcal{Z}M)_{r+3} \arrow[r] & \mathbf{B}^r(\mathcal{Z}M)_{r+2} \arrow[r] & \mathbf{B}^r(\mathcal{Z}M)_{r+1} \arrow[r] & \mathbf{B}^r(\mathcal{Z}M)_r \arrow[r] & 0
\end{tikzcd}$$

where in the bottom row is $r \geq 3$, and

- $\mathbf{B}^2(\mathcal{Z}M)_4$ *is the free* $\mathbb{H}M$*-module on the set of suspensions of the non-zero generic* 3*-cells* $[x_1|x_2|x_3]$ *of* $\mathbf{B}(\mathcal{Z}M)$ *together the non-zero generic* 4*-cells*

$$[x_1\|x_2],$$

with $\pi[x_1\|x_2] = x_1x_2$*, and whose differential is (*$x = x_1x_2$*)*

$$\partial_x[x_1\|x_2] = [x_1|x_2] - [x_2|x_1].$$

- $\mathbf{B}^2(\mathcal{Z}M)_5$ *is the free* $\mathbb{H}M$*-module on the set of suspensions of the non-zero generic* 4*-cells* $[x_1|x_2|x_3|x_4]$ *of* $\mathbf{B}(\mathcal{Z}M)$ *together the non-zero generic* 5*-cells*

$$[x_1\|x_2|x_3],\ [x_1|x_2\|x_3],$$

with $\pi[x_1\|x_2|x_3] = x_1x_2x_3 = \pi[x_1|x_2\|x_3]$*, and whose differential is (*$x = x_1x_2x_3$*)*

$$\begin{aligned}\partial_x[x_1\|x_2|x_3] = &- x_{2*}[x_1\|x_3] + [x_1\|x_2x_3] - x_{3*}[x_1\|x_2] \\ &+ [x_1|x_2|x_3] - [x_2|x_1|x_3] + [x_2|x_3|x_1],\end{aligned}$$

$$\begin{aligned}\partial_x[x_1|x_2\|x_3] = &- x_{1*}[x_2\|x_3] + [x_1x_2\|x_3] - x_{2*}[x_1\|x_3] \\ &- [x_1|x_2|x_3] + [x_1|x_3|x_2] - [x_3|x_1|x_2].\end{aligned}$$

- *$\mathbf{B}^3(\mathcal{Z}M)_6$ is the free $\mathbb{H}M$-module on the set of double suspensions of the non-zero generic 4-cells $[x_1|x_2|x_3|x_4]$ of $\mathbf{B}(\mathcal{Z}M)$, together with the suspensions of the non-zero generic 5-cells $[x_1\|x_2|x_3]$ and $[x_1|x_2\|x_3]$ of $\mathbf{B}^2(\mathcal{Z}M)$, and the non-zero generic 6-cells*

$$[x_1\|\|x_2],$$

with $\pi[x_1\|\|x_2]=x_1x_2$, whose differential is ($x=x_1x_2$)

$$\partial_x[x_1\|\|x_2]=-[x_1\|x_2]-[x_2\|x_1].$$

Therefore, from Proposition 5.2, we get the following.

Theorem 5.2 *For any $\mathbb{H}M$-module $\mathcal{A}$, the cohomology groups $H^n(M,r;\mathcal{A})$, for $n\leq r+2$, are isomorphic to the cohomology groups of the truncated* cochain complexes of normalized rth level cochains of M with values in $\mathcal{A}$, $C(M,r;\mathcal{A})$,

$$\begin{tikzcd} C(M,r;\mathcal{A}): & 0 \arrow[r] & C^0(M,r;\mathcal{A}) \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & C^r(M,r;\mathcal{A}) \arrow[dllll] \\ & & C^{r+1}(M,r;\mathcal{A}) \arrow[r] & C^{r+2}(M,r;\mathcal{A}) \arrow[r] & C^{r+3}(M,r;\mathcal{A}) \end{tikzcd} \tag{5.19}$$

where $C^0(M,r;\mathcal{A})=\mathcal{A}(e)$, and the remaining non-trivial parts occur in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^1(M,1;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^2(M,1;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^3(M,1;\mathcal{A}) \arrow[r] & C^4(M,1;\mathcal{A}) \\ 0 \arrow[r] & C^2(M,2;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^3(M,2;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^4(M,2;\mathcal{A}) \arrow[r] \arrow[u, "S^*"'] \arrow[d, equal] & C^5(M,2;\mathcal{A}) \arrow[u, "S^*"'] \\ 0 \arrow[r] & C^3(M,3;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^4(M,3;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^5(M,3;\mathcal{A}) \arrow[r] \arrow[d, equal] & C^6(M,3;\mathcal{A}) \arrow[u, "S^*"'] \arrow[d, equal] \\ 0 \arrow[r] & C^r(M,r;\mathcal{A}) \arrow[r] & C^{r+1}(M,r;\mathcal{A}) \arrow[r] & C^{r+2}(M,r;\mathcal{A}) \arrow[r] & C^{r+3}(M,r;\mathcal{A}) \end{tikzcd} \tag{5.20}$$

where in the bottom row is $r\geq 3$, and

- *$C^4(M,2;\mathcal{A})$ is the abelian group, under pointwise addition, of pairs of functions (g,μ), where*

$$g:M^3\to\bigcup_{x\in M}\mathcal{A}(x) \qquad \mu:M^2\to\bigcup_{x\in M}\mathcal{A}(x),$$

with $g(x,y,z)\in\mathcal{A}(xyz)$ and $\mu(x,y)\in\mathcal{A}(xy)$, which are normalized in the sense that they take the value 0 whenever some of their arguments are equal to the unit e of the monoid.

• *The coboundary* $\partial : C^3(M,2;\mathcal{A}) = C^2(M,1;\mathcal{A}) \to C^4(M,2;\mathcal{A})$ *acts on a normalized* 2*-cochain* f *of* M *in* $\mathcal{A}$ *by* $\partial f = (g,\mu)$*, where*

$$\begin{aligned} g(x,y,z) &= -x_* f(y,z) + f(xy,z) - f(x,yz) + z_* f(xy), \\ \mu(x,y) &= f(x,y) - f(y,x). \end{aligned}$$

• $C^5(M,2;\mathcal{A})$ *is the abelian group of triplets* (h,γ,δ) *consisting of normalized functions*

$$h : M^4 \to \bigcup_{x\in M} \mathcal{A}(x) \qquad \gamma,\delta : M^3 \to \bigcup_{x\in M} \mathcal{A}(x),$$

with $h(x,y,z,t) \in \mathcal{A}(xyzt)$ *and* $\gamma(x,y,z), \delta(x,y,z) \in \mathcal{A}(xyz)$.

• *The coboundary* $\partial : C^4(M,2;\mathcal{A}) \to C^5(M,2;\mathcal{A})$ *acts on a* 2*nd level* 4*-cochain* (g,μ) *by* $\partial(g,\mu) = (h,\gamma,\delta)$*, where*

$$\begin{aligned} h(x,y,z,t) &= -x_* g(y,z,t) + g(xy,z,t) - g(x,yz,t) + g(x,y,zt) - t_* g(x,y,z), \\ \gamma(x,y,z) &= -y_* \mu(x,z) + \mu(x,yz) - z_* \mu(x,y) + g(x,y,z) - g(y,x,z) + g(y,z,x), \\ \delta(x,y,z) &= -x_* \mu(y,z) + \mu(xy,z) - y_* \mu(x,z) - g(x,y,z) + g(x,z,y) - g(z,x,y). \end{aligned}$$

• $C^6(M,3;\mathcal{A})$ *is the abelian group of quadruples* (h,γ,δ,ξ) *consisting of normalized functions*

$$h : M^4 \to \bigcup_{x\in M} \mathcal{A}(x), \quad \gamma,\delta : M^3 \to \bigcup_{x\in M} \mathcal{A}(x), \quad \xi : M^2 \to \bigcup_{x\in M} \mathcal{A}(x),$$

with $h(x,y,z,t) \in \mathcal{A}(xyzt)$, $\gamma(x,y,z), \delta(x,y,z) \in \mathcal{A}(xyz)$*, and* $\xi(x,y) \in \mathcal{A}(xy)$.

• *The coboundary* $\partial : C^5(M,3;\mathcal{A}) = C^4(M,2;\mathcal{A}) \to C^6(M,3;\mathcal{A})$ *acts on a* 3*rd-level* 5*-cochain by* $\partial(g,\mu) = (h,\gamma,\delta,\xi)$*, where*

$$\begin{aligned} h(x,y,z,t) &= x_* g(y,z,t) - g(xy,z,t) + g(x,yz,t) - g(x,y,zt) + t_* g(x,y,z), \\ \gamma(x,y,z) &= y_* \mu(x,z) - \mu(x,yz) + z_* \mu(x,y) - g(x,y,z) + g(y,x,z) - g(y,z,x), \\ \delta(x,y,z) &= x_* \mu(y,z) - \mu(xy,z) + y_* \mu(x,z) + g(x,y,z) - g(x,z,y) + g(z,x,y) \\ \xi(x,y) &= -\mu(x,y) - \mu(y,x). \end{aligned}$$

The following corollaries follow directly from the form of the cochain complex (5.19) and the commutativity of the diagram (5.20).

Corollary 5.3 *For any* $r \geq 1$, $H^0(M,r;\mathcal{A}) \cong \mathcal{A}(e)$.

Corollary 5.4 *For any* $0 < n < r$, $H^n(M,r;\mathcal{A}) = 0$.

Corollary 5.5 *For any* $r \geq 2$, $H^r(M,r;\mathcal{A}) \cong H^1(M,1;\mathcal{A})$.

Corollary 5.6 *For any* $r \geq 2$, $H^{r+1}(M,r;\mathcal{A}) \cong H^3(M,2;\mathcal{A})$*, and there is a natural monomorphism* $H^3(M,2;\mathcal{A}) \hookrightarrow H^2(M,1;\mathcal{A})$.

Corollary 5.7 *For any $r \geq 3$, $H^{r+2}(M,r;\mathcal{A}) \cong H^5(M,3;\mathcal{A})$, and there is a natural monomorphism $H^5(M,3;A) \hookrightarrow H^4(M,2;\mathcal{A})$.*

Let us now recall that Grillet cohomology groups $H^n_{\mathrm{G}}(M,\mathcal{A})$, for $1 \leq n \leq 3$, can be computed as the cohomology groups of the truncated cochain complex $C_{\mathrm{G}}(M,\mathcal{A})$ in (3.1). There is natural injective cochain map

$$\begin{tikzcd}
0 \arrow[r] & C^1_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^1_{\mathrm{G}}"] \arrow[d, equal, "i_1=id"'] & C^2_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^2_{\mathrm{G}}"] \arrow[d, hook, "i_2"'] & C^3_{\mathrm{G}}(M,\mathcal{A}) \arrow[r, "\partial^3_{\mathrm{G}}"] \arrow[d, hook, "i_3"'] & C^4_{\mathrm{G}}(M,\mathcal{A}) \arrow[d, hook, "i_4"'] \\
0 \arrow[r] & C^3(M,3;\mathcal{A}) \arrow[r, "\partial^3"] & C^4(M,3;\mathcal{A}) \arrow[r, "\partial^4"] & C^5(M,3;\mathcal{A}) \arrow[r, "\partial^5"] & C^6(M,3;\mathcal{A}),
\end{tikzcd} \tag{5.21}$$

which is the identity map, $i_1(f) = f$, on symmetric 1-cochains, the map $i_2(g) = -g$ on symmetric 2-cochains, and on symmetric 3- and 4-cochains is defined by the simple formulas $i_3(h) = (h,0)$ and $i_4(t) = (-t,0,0,0)$, respectively. The only non-trivial verification here concerns the equality $\partial^5 i_3 = i_4 \partial^3$, that is, $\partial^5(h,0) = (-\partial^3 h,0,0,0)$, for any $h \in C^3_{\mathrm{G}}(M,\mathcal{A})$, but it easily follows from Lemma 3.1.

Proposition 5.5 *For any $\mathbb{H}M$-module $\mathcal{A}$, the injective cochain map* (5.21) *induces natural isomorphisms*

$$H^1_{\mathrm{G}}(M,\mathcal{A}) \cong H^1(M,1;\mathcal{A}), \qquad H^2_{\mathrm{G}}(M,\mathcal{A}) \cong H^3(M,2;\mathcal{A}),$$

and a natural monomorphism

$$H^3_{\mathrm{G}}(M,\mathcal{A}) \hookrightarrow H^5(M,3;\mathcal{A}).$$

Proof: From diagram (5.21), it follows directly that $\ker \partial^1_{\mathrm{G}} = \ker \partial^3$ and $i_2 \mathrm{Im}\, \partial^1_{\mathrm{G}} = \mathrm{Im}\, \partial^3$. Further, $i_2 \ker \partial^2_{\mathrm{G}} = \ker \partial^4$, since the condition $\partial^4 f = 0$ on a cochain $f \in C^4(M,3;\mathcal{A}) = C^2(M,1;\mathcal{A})$ implies the symmetry condition $f(x,y) = f(y,x)$. Then,

$$H^1_{\mathrm{G}}(M,\mathcal{A}) = \ker \partial^1_{\mathrm{G}} = \ker \partial^3 \cong H^3(M,3;\mathcal{A}) \cong H^1(M,1;\mathcal{A}),$$

and

$$H^2_{\mathrm{G}}(M,\mathcal{A}) = \frac{\ker \partial^2_{\mathrm{G}}}{\mathrm{Im}\, \partial^1_{\mathrm{G}}} \cong \frac{i_2 \ker \partial^2_{\mathrm{G}}}{i_2 \mathrm{Im}\, \partial^1_{\mathrm{G}}} = \frac{\ker \partial^4}{\mathrm{Im}\, \partial^3} \cong H^4(M,3;\mathcal{A}) \cong H^3(M,2;\mathcal{A}).$$

To prove that the induced homomorphism $H^3_{\mathrm{G}}(M,\mathcal{A}) \to H^5(M,3;\mathcal{A})$ is injective, suppose $h \in C^3_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 3-cochain such that $i_3 h = \partial^4 g$ for some $g \in C^4(M,3;\mathcal{A}) = C^2(M,1;\mathcal{A})$. This means that the equalities

$$h(x,y,z) = x_* g(y,z) - g(xy,z) + g(x,yz) - z_* g(x,y), \quad 0 = g(x,y) - g(y,x),$$

hold. Then, $g \in C^2_{\mathrm{G}}(M,\mathcal{A})$ is a symmetric 2-cochain, and $h = -\partial^2 g$ is actually a symmetric 2-coboundary. It follows that the injective map $i_3 : \ker \partial^3_{\mathrm{G}} \hookrightarrow \ker \partial^5$ induces a injective map in cohomology

$$H^3 C_{\mathrm{G}}(M,\mathcal{A})\big) \hookrightarrow H^5 C(M,3;\mathcal{A}),$$

as required. □

To complete the list of relationships between the cohomology groups $H^n(M,r;\mathcal{A})$ with those already known in the literature, let us note that a direct comparison of the cochain complex (5.19) with the cochain complex in (4.5), which computes the lower *commutative cohomology groups* $H^n_{\mathrm{c}}(M,\mathcal{A})$, gives the following.

Proposition 5.6 *For any $\mathbb{H}M$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^1(M,1;\mathcal{A}) \cong H^1_{\mathrm{c}}(M,\mathcal{A}), \quad H^3(M,2;\mathcal{A}) \cong H^2_{\mathrm{c}}(M,\mathcal{A}), \quad H^4(M,2;\mathcal{A}) \cong H^3_{\mathrm{c}}(M,\mathcal{A}).$$

5.5 Cohomological classification of symmetric monoidal abelian groupoids

This section is dedicated to showing a precise classification for symmetric monoidal abelian groupoids, by means of the 3rd level cohomology groups of commutative monoids $H^5(M,3;\mathcal{A})$.

Symmetric monoidal categories have been studied extensively in the literature and we refer to Mac Lane [56] and Saavedra [66] for the background. Recall from section 1.3 that an abelian groupoid $\mathcal{M}$ is a groupoid whose isotropy groups $\mathrm{Aut}_{\mathcal{M}}(x)$, $x \in \mathrm{Ob}\mathcal{M}$, are all abelian and that composition is written additively.

A *symmetric monoidal abelian groupoid*

$$\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$$

consists of a braided monoidal abelian groupoid whose braiding, called now symmetry, verifies

$$\boldsymbol{c}_{y,x} + \boldsymbol{c}_{x,y} = 0_{x\otimes y}. \tag{5.22}$$

Remark that in a symmetric monoidal abelian groupoid if coherence condition (4.14) hold then (4.15) is also verified.

Example 5.1 For any 3rd level 5-cocycle (g,μ) in $Z^5(M,3;\mathcal{A})$, it can be defined a symmetric monoidal abelian groupoid

$$\Sigma(M,\mathcal{A},(g,\mu)) = (\Sigma(M,\mathcal{A},(g,\mu)), \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c}),$$

that should be thought of as a 2-*dimensional twisted crossed product of M by $\mathcal{A}$*, and it is defined as follows: Its underlying groupoid is the totally disconnected groupoid in (3.16), that is, its set of objects is M and its isotropy group at any object $x \in M$ is $\mathcal{A}(x)$.

The tensor product

$$\otimes : \Sigma(M, \mathcal{A}, (g, \mu)) \times \Sigma(M, \mathcal{A}, (g, \mu)) \to \Sigma(M, \mathcal{A}, (g, \mu))$$

is defined as in (3.17). The unit object is $\mathrm{I} = e$, the unit of the monoid M, and the structure constraints are

$$\begin{aligned}
\boldsymbol{a}_{x,y,z} &= g(x, y, z) : (xy)z \to x(yz), \\
\boldsymbol{c}_{x,y} &= \mu(x, y) : xy \to yx, \\
\boldsymbol{l}_x &= 0_x : ex = x \to x \\
\boldsymbol{r}_x &= 0_x : xe = x \to x,
\end{aligned}$$

which are easily seen to be natural since $\mathcal{A}$ is an abelian group valued functor. The coherence conditions (1.1), (4.14), and (5.22) follow from the 5-cocycle condition $\partial^5(h, \mu) = (0, 0, 0, 0)$, while the coherence condition (1.2) comes from the normalization condition $h(x, e, y) = 0$.

If $\mathcal{M}$, $\mathcal{M}'$ are symmetric monoidal abelian groupoids, then a *symmetric monoidal functor* $F = (F, \varphi, \varphi_0) : \mathcal{M} \to \mathcal{M}'$ is a braided functor (4.18), while a *symmetric monoidal isomorphism* $\delta : F \Rightarrow F'$, where $F' : \mathcal{M} \to \mathcal{M}'$ is another symmetric monoidal functor, is just a monoidal isomorphism (1.7). Therefore, symmetric monoidal abelian groupoids, symmetric monoidal functors, and symmetric monoidal isomorphisms form a 2-category [37, Chaper V, Section 1]. A symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ is called a *symmetric monoidal equivalence* if it is an equivalence in this 2-category.

Our goal is to show a classification for symmetric monoidal abelian groupoids, where two symmetric monoidal abelian groupoids connected by a symmetric monoidal equivalence are considered the same, as stated in the theorem below.

Theorem 5.3 (Classification of Symmetric Monoidal Abelian Groupoids)

(*i*) *For any symmetric monoidal abelian groupoid $\mathcal{M}$, there exist a commutative monoid M, an $\mathbb{H}M$-module $\mathcal{A}$, a* 3*rd level* 5*-cocycle $(g, \mu) \in Z^5(M, 3; \mathcal{A})$, and a symmetric monoidal equivalence*

$$\Sigma(M, \mathcal{A}, (g, \mu)) \simeq \mathcal{M}.$$

(*ii*) *For two* 3*rd level* 5*-cocycles $(g, \mu) \in Z^5(M, 3; \mathcal{A})$ and $(g', \mu') \in Z^5(M', 3; \mathcal{A}')$, there is a symmetric monoidal equivalence*

$$\Sigma(M, \mathcal{A}, (g, \mu)) \simeq \Sigma(M', \mathcal{A}', (g', \mu'))$$

if and and only if there exist an isomorphism of monoids $i : M \cong M'$ and a natural isomorphism $\psi : \mathcal{A} \cong i^*\mathcal{A}'$, such that the equality of cohomology classes below holds.

$$[g, \mu] = \psi_*^{-1} i^* [g', \mu'] \in H^5(M, 3; \mathcal{A})$$

Proof: We proceed as in the proof of Theorems 3.1 and 4.3.

(*i*) Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}, \boldsymbol{c})$ be any given symmetric monoidal abelian groupoid. By the coherence theorem [56], we can assume that $\mathcal{M}$ is strictly unitary, that is, that both unit constraints $\boldsymbol{l}$ and $\boldsymbol{r}$ are identities. Then, we observe that $\mathcal{M}$ is symmetric monoidal equivalent to another one that is totally disconnected. Indeed, by the generalized Brandt's theorem [46, Chapter 6, Theorem 2], there is a totally disconnected groupoid, say $\mathcal{M}'$, with an equivalence of groupoids $\mathcal{M} \to \mathcal{M}'$. Hence, by Saavedra [66, I, 4.4], we can transport the symmetric monoidal structure along this equivalence so that $\mathcal{M}'$ becomes a strictly unitary symmetric monoidal abelian groupoid and the equivalence a symmetric monoidal one.

Hence, we assume that $\mathcal{M}$ is totally disconnected and strictly unitary. Then, a triplet $(M, \mathcal{A}, (g, \mu))$, such that $\Sigma(M, \mathcal{A}, (g, \mu)) = \mathcal{M}$ as symmetric monoidal abelian groupoids, can be defined as follows:

- *The monoid M and the $\mathbb{H}M$-module $\mathcal{A}$.* We define them in the same way than in the proof of Theorem 4.3 (*i*).
- *The 3rd level 5-cocycle $(g, \mu) \in Z^5(M, 3; \mathcal{A})$.* We write the associativity constraint and the symmetry of $\mathcal{M}$ as $\boldsymbol{a}_{x,y,z} = g(x, y, z)$ and $\boldsymbol{c}_{x,y} = \mu(x, y)$, for some given lists $\big(g(x, y, z) \in \mathcal{A}(xyz)\big)_{x,y,z\in M}$ and $\big(\mu(x, y) \in \mathcal{A}(xy)\big)_{x,y\in M}$. Since $\mathcal{M}$ is strictly unitary, equations in (1.2) and (1.3) implies the normalization conditions $g(x, e, y) = 0 = g(e, x, y) = g(x, y, e)$ for g, while the normalization conditions $\mu(x, e) = 0 = \mu(e, x)$ for μ follows from (4.16). Thus, $(g, \mu) \in C^5(M, 3; \mathcal{A})$ is a 3rd level 5-cochain. By the coherence conditions (1.1), (4.14), (4.15) and (5.22) we have that

$$
\begin{aligned}
&g(x, y, zt) + g(xy, z, y) = x_*g(y, z, y) + g(x, yz, y) + t_*g(x, y, z)\\
&y_*\mu(x, z) + g(y, x, z) + z_*\mu(x, y) = g(y, z, x) + \mu(x, yz) + g(x, y, z),\\
&y_*\mu(x, z) - g(x, z, y) + x_*\mu(y, z) = -g(z, x, y) + \mu(xy, z) - g(x, y, z),\\
&\mu(x, y) + \mu(y, x) = 0,
\end{aligned}
$$

Hence, we obtain the required cocycle condition $\partial^3(g, \mu) = (0, 0, 0)$. From a direct comparison we have $\mathcal{M} = \Sigma(M, \mathcal{A}, (g, \mu))$ as symmetric monoidal abelian groupoids, and so the proof of this part is complete.

(*ii*) Let $i : M \cong M'$ be an isomorphism and $\psi : \mathcal{A} \cong i^*\mathcal{A}'$ a natural isomorphism such that $\psi_*[g, \mu] = i^*[g', \mu'] \in H^5(M, 3; i^*\mathcal{A}')$. This implies that there is a 3rd level 4-cochain $f \in C^4(M, 3; i^*\mathcal{A}') = C^2(M, 1; i^*\mathcal{A}')$ such that

$$\psi_{xyz}g(x,y,z) = g'(ix,iy,iz) + (ix)_*f(y,z) - f(xy,z) + f(x,yz) - (iz)_*f(x,y), \tag{5.23}$$

$$\psi_{xy}\mu(x, y) = \mu'(ix, iy) - f(x, y) + f(y, x). \tag{5.24}$$

Then, there is a symmetric monoidal isomorphism

$$\Sigma(i, \psi, f) = (F, \varphi, \varphi_0) : \Sigma(M, \mathcal{A}, (g, \mu)) \to \Sigma(M', \mathcal{A}', (g', \mu')).$$

whose action on objects and arrows is $F(a_x : x \to x) = (\psi_x a_x : ix \to ix)$, and so it is an isomorphism between the underlying groupoids. The constraints of F are given by $\varphi_{x,y} = f(x,y) : (ix)\,(iy) \to i(xy)$, which are natural by the naturality of ψ, and $\varphi_0 = 0_{e'} : e' \to ie = e'$.

The coherence conditions (1.5) and (4.19) are obtained as a consequence of equations (5.23) and (5.24), respectively, whereas the conditions in (1.6) trivially follow from the normalization conditions $f(x,e) = 0_{ix} = f(e,x)$.

Conversely, suppose we have

$$F = (F, \varphi, \varphi_0) : \Sigma(M, \mathcal{A}, (g,\mu)) \to \Sigma(M', \mathcal{A}', (g',\mu'))$$

a symmetric monoidal equivalence. By [22, Lemma 18], there is no loss of generality in assuming that F is strictly unitary in the sense that $\varphi_0 = 0_{e'} : e' \to e' = Fe$. Since the underlying groupoids are totally discnnected, F is an isomorphism.

We now have an isomorphism of monoids $i : M \cong M'$ (the bijection established by F between the object sets) and a natural isomorphism $\psi : \mathcal{A} \cong i^*\mathcal{A}'$ (the isomorphism on the automorphism groups). Finally, if we write $f(x,y) = \varphi_{x,y}$, for each $x, y \in M$, we have a 3rd level 4-cochain $f(F) = \big(f(x,y) \in \mathcal{A}'(i(xy))\big)_{x,y\in M}$, since the equations $f(x,e) = 0_{ix} = f(e,x)$ hold due to (1.6). Equations (5.23) and (5.24) follow from to the coherence equations (1.5) and (4.19). This means that $\psi_*(g,\mu) = i^*(g',\mu') + \partial^4 f$ and, therefore, we have that $\psi_*[g,\mu] = i^*[g',\mu'] \in H^5(M,3;i^*\mathcal{A}')$, whence $[g,\mu] = \psi_*^{-1} i^*[g',\mu'] \in H^5(M,3;\mathcal{A})$. $\square$

5.6 Cohomology of cyclic monoids

In this section we compute the cohomology groups $H^n(C,r;\mathcal{A})$, for $n \le r+2$, when C is any cyclic monoid. The method we employ follows similar lines to the one used by Eilenberg and Mac Lane in [32, Section 14 and Section 15], for computing higher level cohomology of cyclic groups, though the generalization to monoids is highly nontrivial.

5.6.1 Cohomology of finite cyclic monoids

The structure of finite cyclic monoids was recalled in Section 2.2. From now on, $C = C_{m,q}$ denotes the finite cyclic monoid of index m and period q. We assume that $m + q \ge 2$, so that C is not the zero monoid.

We remember now the notation $k \cdot m = \wp(km)$ from (2.3), and introduce, to any pair $x, y \in C$, the useful integer

$$s(x,y) = \frac{(x+y) - (x \oplus y)}{q},$$

which satisfies $s(x,y) \ge 1$ if $x + y \ge m + q$, whereas $s(x,y) = 0$ if $x + y < m + q$. It follows directly from the associativity in C that the cocycle property below holds.

$$s(y,z) + s(x, y \oplus z) = s(x \oplus y, z) + s(x,y). \tag{5.25}$$

We begin by constructing a specific commutative DGA-algebra over $\mathbb{H}C$, denoted by

$$\mathcal{R} = \mathcal{R}(C),$$

which is homologically equivalent to $\mathbf{B}(\mathcal{Z}C)$ but algebraically simpler and more lucid. For each integer $k = 0, 1, \ldots$, let us choose unitary sets over C, $\{\mathbf{v}_k\}$ and $\{\mathbf{w}_k\}$, with

$$\pi \mathbf{v}_k = k \cdot m, \quad \pi \mathbf{w}_k = k \cdot m \oplus 1, \tag{5.26}$$

and define

$$\begin{cases} \mathcal{R}_{2k} & = & \text{the free } \mathbb{H}C\text{-module on } \{\mathbf{v}_k\}, \\ \mathcal{R}_{2k+1} & = & \text{the free } \mathbb{H}C\text{-module on } \{\mathbf{w}_k\}. \end{cases} \tag{5.27}$$

The augmentation $\alpha : \mathcal{R}_0 \to \mathbb{Z}$, the differential $\partial : \mathcal{R}_n \to \mathcal{R}_{n-1}$, and the multiplication $\circ : \mathcal{R} \otimes_{\mathbb{H}C} \mathcal{R} \to \mathcal{R}$ are determined by the equations

$$\alpha \mathbf{v}_0 = 1, \ \partial \mathbf{v}_{k+1} = (m+q)\big((m+q-1)_* \mathbf{w}_k\big) - m\big((m-1)_* \mathbf{w}_k\big), \ \partial \mathbf{w}_k = 0, \tag{5.28}$$

$$\mathbf{v}_k \circ \mathbf{v}_l = \tbinom{k+l}{k} \mathbf{v}_{k+l}, \quad \mathbf{w}_k \circ \mathbf{w}_l = 0, \quad \mathbf{v}_k \circ \mathbf{w}_l = \tbinom{k+l}{k} \mathbf{w}_{k+l} = \mathbf{w}_l \circ \mathbf{v}_k, \tag{5.29}$$

and the unit is $\mathbf{v}_0$.

Proposition 5.7 *$\mathcal{R}$, defined as above, is a commutative DGA-algebra over $\mathbb{H}C$.*

Proof: By Proposition 5.2, the mapping in (5.28), $\mathbf{v}_{k+1} \mapsto \partial \mathbf{v}_{k+1}$, determines a morphism of $\mathbb{H}C$-modules $\partial : \mathcal{R}_{2k+2} \to \mathcal{R}_{2k+1}$ since

$$(m+q-1) \oplus \pi \mathbf{w}_k \overset{(5.26)}{=} (m+q-1) \oplus k \cdot m \oplus 1 = m \oplus k \cdot m = \pi \mathbf{v}_{k+1},$$
$$(m-1) \oplus \pi \mathbf{w}_k \overset{(5.26)}{=} (m-1) \oplus k \cdot m \oplus 1 = \wp(m+km) = \pi \mathbf{v}_{k+1},$$

and therefore $\partial \mathbf{v}_{k+1} \in \mathcal{R}_{2k+1}(\pi \mathbf{v}_{k+1})$. Similarly, by Proposition 5.2, we see that the formulas in (5.29) determine a multiplication morphism of $\mathbb{H}C$-modules since $k \cdot m \oplus l \cdot m = (k+l) \cdot m$ and $k \cdot m \oplus l \cdot m \oplus 1) = (k+l) \cdot m \oplus 1$. Associativity condition (5.8) follows from the equality on combinatorial numbers

$$\binom{k+l+t}{k} + \binom{l+t}{t} = \frac{(k+l+t)!}{k!\,l!\,t!} = \binom{k+l+t}{k+l} + \binom{k+l}{l},$$

while condition (5.9) holds thanks to the equality

$$\binom{k+l-1}{k-1} + \binom{k+l-1}{k} = \binom{k+l}{k},$$

and the remaining conditions in (5.5)-(5.7) are quite obviously verified. $\square$

In next proposition we shall define a morphism $f : \mathbf{B}(\mathcal{Z}C) \to \mathcal{R}$. Previously, observe that the graded $\mathbb{H}C$-module $\{\mathcal{R}_n\}$ admits another structure of commutative graded algebra over $\mathbb{H}C$ (although it does not respect the differential structure), whose multiplication is determined by the simpler formulas

$$\mathbf{v}_k \bullet \mathbf{v}_l = \mathbf{v}_{k+l}, \quad \mathbf{w}_k \bullet \mathbf{w}_l = 0, \quad \mathbf{v}_k \bullet \mathbf{w}_l = \mathbf{w}_{k+l} = \mathbf{w}_l \bullet \mathbf{v}_k.$$

Proposition 5.8 *A morphism $f : \mathbf{B}(\mathcal{Z}C) \to \mathcal{R}$, of DGA-algebras over $\mathbb{H}C$, may be defined by the recursive formulas*

$$\begin{cases} f[\,] & = \mathbf{v}_0, \\ f[x] & = x((x-1)_*\mathbf{w}_0), \\ f[x \mid y] & = \begin{cases} 0 & \text{if } \ x+y < m+q, \\ ((x \oplus y)\text{-}m)_*\Big(\sum\limits_{i=0}^{s(x,y)\text{-}1}(iq)_*\mathbf{v}_1\Big) & \text{if } \ x+y \geq m+q, \end{cases} \\ f[x \mid y \mid \sigma] & = f[x \mid y] \bullet f[\sigma], \end{cases} \tag{5.30}$$

where $\sigma = [z|\cdots]$ is any cell of dimension 1 or greater.

Proof: This is divided into four parts. Note first that, from the inequalities

$$m + s(x,y)q \leq (x \oplus y) + s(x,y)q = x + y < 2m + 2q - 1,$$

it follows that $s(x,y)q < m + 2q - 1$. Therefore, for any $0 \leq i < s(x,y)$, we have $iq = \wp(iq) \in C$ and the formula above for $f[x \mid y]$ is well defined.

Part 1. We prove in this step that the assignment in (5.30) extends to a morphism of complexes of $\mathbb{H}C$-modules. This follows from Proposition 5.2, since one verifies recursively that

$$f[x_1 \mid \cdots \mid x_n] \in \mathcal{R}_n(x_1 \oplus \cdots \oplus x_n)$$

as follows: The case when $n = 0$ is obvious. When $n = 1$, it holds since $\mathbf{w}_0 \in \mathcal{R}_1$ and $(x_1 - 1) \oplus \pi\mathbf{w}_0 = (x_1 - 1) \oplus 1 = x_1$, and for $n = 2$ since $\mathbf{v}_1 \in \mathcal{R}_2$ and

$$((x_1 \oplus x_2)\text{-}m) \oplus \pi\mathbf{v}_1 = ((x_1 \oplus x_2)\text{-}m) \oplus m = x_1 \oplus x_2.$$

Then, for $n \geq 3$, induction gives $f[x_1|\cdots|x_n] = f[x_1|x_2] \bullet f[x_3|\cdots|x_n] \in \mathcal{R}_2(x_1 \oplus x_2) \bullet \mathcal{R}_{n-2}(x_3 \oplus \cdots \oplus x_n) \subseteq \mathcal{R}_n(x_1 \oplus \cdots \oplus x_n)$.

Part 2. We prove now that $\partial f = f\partial$.

For a 1-cell $[x]$ of $\mathbf{B}(\mathcal{Z}C)$, we have $\partial f[x] = x((x-1)_*\partial\mathbf{w}_0) \overset{(5.28)}{=} 0 = f\partial[x]$.

For a 2-cell $[x \mid y]$, we have

$$f\partial[x \mid y] = x_* f[y] - f[x \oplus y] + y_* f[x].$$

To compare with $\partial f[x \mid y]$, we shall distinguish three cases:

- *Case $x + y < m + q$.* In this case $\partial f[x \mid y] = 0$, and also

$$f\partial[x \mid y] = y((x+y-1)_*\mathbf{w}_0) - (x+y)((x+y-1)_*\mathbf{w}_0) + x((x+y-1)_*\mathbf{w}_0) = 0.$$

-*Case* $x+y \geq m+q$ *and* $x \oplus y = m$. Here, $(x-1) \oplus y = m+q-1 = x \oplus (y-1)$. Then,

$$\begin{aligned}
\partial f[x \mid y] &= \sum_{i=0}^{s(x,y)\text{-}1} (m+q)((iq \oplus (m+q-1))_*\mathbf{w}_0)\text{-}m((iq \oplus (m-1))_*\mathbf{w}_0) \\
&= (m+q)((m+q-1)_*\mathbf{w}_0)\text{-}m((m-1)_*\mathbf{w}_0) \\
&\quad + \sum_{i=1}^{s(x,y)\text{-}1} (m+q)((m+q-1)_*\mathbf{w}_0)\text{-}m((m+q-1))_*\mathbf{w}_0) \\
&= (m+q)((m+q-1)_*\mathbf{w}_0)\text{-}m((m-1)_*\mathbf{w}_0) \\
&\quad + (s(x,y)-1)q((m+q-1)_*\mathbf{w}_0) \\
&= (m+s(x,y)q)((m+q-1)_*\mathbf{w}_0)\text{-}m((m-1)_*\mathbf{w}_0) \\
&= (x+y)((m+q-1)_*\mathbf{w}_0)\text{-}m((m-1)_*\mathbf{w}_0) = f\partial[x \mid y].
\end{aligned}$$

-*Case* $x+y \geq m+q$ *and* $x \oplus y > m$. In this case, $(x-1) \oplus y = (x \oplus y)-1 = x \oplus (y-1)$, whence

$$\begin{aligned}
&\partial f[x \mid y] = \sum_{i=0}^{s(x,y)\text{-}1} ((x \oplus y)\text{-}m) \oplus iq)_*\partial\mathbf{v}_1 \\
&= \sum_{i=0}^{s(x,y)\text{-}1} (m+q)\big(((x \oplus y)\text{-}m) \oplus ((iq \oplus (m+q-1))_*\mathbf{w}_0\big) \qquad (5.31) \\
&- \sum_{i=0}^{s(x,y)\text{-}1} m\big(((x \oplus y)\text{-}m) \oplus (iq \oplus (m-1))_*\mathbf{w}_0\big) = \sum_{i=0}^{s(x,y)\text{-}1} (m+q)((x \oplus y)-1)_*\mathbf{w}_0) \\
&- \sum_{i=0}^{s(x,y)\text{-}1} m((x \oplus y)-1))_*\mathbf{w}_0) = qs(x,y)((x \oplus y)-1)_*\mathbf{w}_0) \\
&= (y-(x \oplus y)+x)((x \oplus y)-1)_*\mathbf{w}_0) = f\partial[x \mid y].
\end{aligned}$$

For a 3-cell $[x \mid y \mid z]$, we have to prove that $f\partial[x \mid y \mid z] = 0$ or, equivalently, that

$$x_*f[y \mid z] + f[x \mid y \oplus z] = z_*f[x \mid y] + f[x \oplus y \mid z]. \qquad (5.32)$$

Since $x+(y \oplus z) = x \oplus y \oplus z + s(x, y \oplus z)q$, it follows that

$$x \oplus ((y \oplus z)\text{-}m) = ((x \oplus y \oplus z)\text{-}m) \oplus \wp(s(x, y \oplus z)q),$$

whenever $y \oplus z \geq m$. Then, we can write

$$x_*f[y \mid z] = \begin{cases} 0, & \text{if } s(y,z)=0,\\ (x \oplus ((y \oplus z)\text{-}m))_*\Big(\sum\limits_{i=0}^{s(y,z)\text{-}1} (i \cdot q)_* \mathbf{v}_1\Big), & \text{if } s(y,z) \geq 1, \end{cases}$$
$$= \begin{cases} 0, & \text{if } s(y,z)=0,\\ ((x \oplus y \oplus z)\text{-}m)_*\Big(\sum\limits_{i=0}^{s(y,z)\text{-}1} \wp\big(s(x, y\oplus z)q + iq\big)_* \mathbf{v}_1\Big), & \text{if } s(y,z) \geq 1. \end{cases}$$

As

$$f[x \mid y \oplus z] = \begin{cases} 0, & \text{if } s(x, y \oplus z)=0,\\ ((x \oplus y \oplus z)\text{-}m)_*\Big(\sum\limits_{i=0}^{s(x,y\oplus z)\text{-}1} (i \cdot q)_* \mathbf{v}_1\Big), & \text{if } s(x, y\oplus z) \geq 1, \end{cases}$$

one concludes the formula

$$x_*f[y \mid z] + f[x \mid y \oplus z] = \begin{cases} 0, & \text{if } s(y,z)=0=s(x,y\oplus z),\\ ((x \oplus y \oplus z)\text{-}m)_*\Big(\sum\limits_{i=0}^{s(y,z)+s(x,y\oplus z)\text{-}1} (i \cdot q)_* \mathbf{v}_1\Big), & \text{otherwise.} \end{cases}$$

Similarly, one sees that

$$z_*f[x \mid y] + f[x \oplus y \mid z] = \begin{cases} 0, & \text{if } s(x,y)=0=s(x \oplus y, z),\\ ((x \oplus y \oplus z)\text{-}m)_*\Big(\sum\limits_{i=0}^{s(x,y)+s(x\oplus y,z)\text{-}1} (i \cdot q)_* \mathbf{v}_1\Big), & \text{otherwise,} \end{cases}$$

and the equality in (5.32) follows by comparison using (5.25).

Finally, for a cell $[x \mid y \mid z \mid t \mid \cdots] = [x \mid y \mid z \mid \tau]$ of dimension higher than 3 we use the formulas

$$\partial[a \mid x \mid b] = [\partial[a \mid x] \mid b] + [a \mid \partial[x \mid b]], \tag{5.33}$$

which holds for any even chain a and any other chain b of $\mathbf{B}(\mathcal{Z}C)$, and

$$\partial(c \bullet d) = c \bullet \partial d, \tag{5.34}$$

which holds for any chains $c, d \in \mathcal{R}$. Thus, as we know that $f\partial[x \mid y \mid z] = 0$, induction gives

$$\begin{aligned} f\partial[x \mid y \mid z \mid \tau] &\overset{(5.33)}{=} f[\partial[x \mid y \mid z] \mid \tau] + f[x \mid y \mid \partial[z \mid \tau]] \\ &= f\partial[x \mid y \mid z] \bullet f[\tau] + f[x \mid y] \bullet f\partial[z \mid \tau] = f[x \mid y] \bullet \partial f[z \mid \tau] \\ &\overset{(5.34)}{=} \partial(f[x \mid y] \bullet f[z \mid \tau]) = \partial f[x \mid y \mid z \mid \tau] \end{aligned}$$

Part 3. Here we show that f preserves products. It is enough to prove that $f(\sigma \circ \tau) = f(\sigma) \circ f(\tau)$ for cells $\sigma = [x_1 \mid \cdots \mid x_n]$ and $\tau = [y_1 \mid \cdots \mid y_{n'}]$ of $\mathbf{B}(\mathcal{Z}C)$.

As in [32, page 99], a term $T = \pm[t_1 \mid \cdots \mid t_{n+n'}]$ in the shuffle product (5.18) of σ and τ is called *mixed* whenever there exists an index i such that t_{2i-1} is an x of σ and t_{2i} an y of τ, or vice versa. Choose the first index i for each mixed T, and let T' be the term obtained from T by interchanging t_{2i-1} with t_{2i}. Since $f[x, y]$ is symmetric,

$$\begin{aligned} f(T) &= f[t_1 \mid t_2] \bullet \cdots \bullet f[t_{2i-1} \mid t_{2i}] \bullet f[t_{2i+1} \mid \cdots] \\ &= f[t_1 \mid t_2] \bullet \cdots \bullet f[t_{2i} \mid t_{2i-1}] \bullet f[t_{2i+1} \mid \cdots] = f(T'). \end{aligned}$$

Since T and T' have opposite signs, the results cancel and $f(\sigma \circ \tau) = \sum f(T)$, with summation taken only over the unmixed terms, and where the sign of each term due the shuffle is always plus. If $n = 2r + 1$ and $n' = 2r' + 1$ are both odd, there are no unmixed terms, so $f(\sigma \circ \tau) = 0$ in agreement with the fact that $f(\sigma) \circ f(\tau) = 0$ (since $\mathbf{w}_k \circ \mathbf{w}_l = 0$). If $n = 2r$ and $n' = 2r'$ are both even, the unmixed terms T are obtained by taking all shuffles of the r pairs $(x_1, x_2), ..., (x_{2r-1}, x_{2r})$ through the pairs $(y_1, y_2), ..., (y_{2r'-1}, y_{2r'})$. For any such a shuffle

$$f(T) = f[x_1 \mid x_2] \bullet \cdots \bullet f[x_{2r-1} \mid x_{2r}] \bullet f[y_1 \mid y_2] \bullet \cdots \bullet f[y_{2r'-1}, y_{2r'}] = f(\sigma) \bullet f(\tau)$$

and the number of such shuffles is $\binom{r+r'}{r}$, hence

$$f(\sigma \circ \tau) = \binom{r + r'}{r} f(\sigma) \bullet f(\tau) = f(\sigma) \circ f(\tau),$$

as desired. For $n = 2r$ and $n' = 2r' + 1$, the unmixed terms T are as above but with the last argument $y_{2r'+1}$ always at the end. Hence, for each of them

$$f(T) = f[x_1 \mid x_2] \bullet \cdots \bullet f[x_{2r-1} \mid x_{2r}] \bullet f[y_1 \mid y_2] \bullet \cdots \bullet f[y_{2r'-1}, y_{2r'}] \bullet f[y_{2r'+1}] = f(\sigma) \bullet f(\tau),$$

and therefore $f(\sigma \circ \tau) = \binom{r+r'}{r} f(\sigma) \bullet f(\tau) = f(\sigma) \circ f(\tau)$. The remaining case $n = 2r+1$ and $n' = 2r'$ is treated similarly. $\square$

Proposition 5.9 *A morphism $g : \mathcal{R} \to \mathbf{B}(\mathcal{Z}C)$, of DGA-algebras over $\mathbb{H}C$, may be defined by the recursive formulas*

$$\begin{cases} g\mathbf{v}_0 &= [\ ], \\ g\mathbf{w}_k &= [g\mathbf{v}_k \mid 1], \\ g\mathbf{v}_{k+1} &= \sum\limits_{t<m+q} (m + q - t - 1)_* [g\mathbf{w}_k \mid t] - \sum\limits_{s<m} (m - s - 1)_* [g\mathbf{w}_k \mid s]. \end{cases} \tag{5.35}$$

Proof: Part 1. We show here that the assignment in (5.35) extends to a morphism of complexes of $\mathbb{H}C$-modules. By Proposition 5.2, we have to verify that $g\mathbf{v}_k \in \mathbf{B}(\mathcal{Z}C)_{2k}(k \cdot m)$ and $g\mathbf{w}_k \in \mathbf{B}(\mathcal{Z}C)_{2k+1}(k \cdot m \oplus 1)$. Clearly $g\mathbf{v}_0 = [\ ] \in \mathbf{B}(\mathcal{Z}C)_0(0)$. Assume that $g\mathbf{v}_k \in \mathbf{B}(\mathcal{Z}C)_{2k}(k \cdot m)$. Then, we have

$$g\mathbf{w}_k = [g\mathbf{v}_k \mid 1] \in \mathbf{B}(\mathcal{Z}C)_{2k+1}(k \cdot m \oplus 1),$$

as required. Moreover, for any $t < m+q$ and $s < m$,

$$(m+q-t-1)_*[g\mathbf{w}_k \mid t], (m-s-1)_*[g\mathbf{w}_k \mid s] \in \mathbf{B}(\mathcal{Z}C)_{2k+2}((k+1)\cdot m),$$

since

$$(m+q-t-1) \oplus k\cdot m \oplus 1 \oplus t = (k+1)\cdot m = (m-s-1) \oplus k \cdot m \oplus 1 \oplus s.$$

Whence $g\mathbf{v}_{k+1} \in \mathbf{B}(\mathcal{Z}C)_{2k+2}((k+1)\cdot m)$.

Part 2. Here we shall prove, as an auxiliary result, that

$$g\mathbf{v}_k \circ [1] = g\mathbf{w}_k, \quad g\mathbf{w}_k \circ [1] = 0, \tag{5.36}$$

where $\circ = \circ_1$ is the shuffle product (5.18) of $\mathbf{B}(\mathcal{Z}C)$. Clearly $g\mathbf{v}_0 \circ [1] = [\;] \circ [1] = [1] = [g\mathbf{v}_0 \mid 1] = g\mathbf{w}_0$. Assuming the result for $g\mathbf{v}_k$, we have

$$g\mathbf{w}_k \circ [1] = g\mathbf{v}_k \circ [1] \circ [1] = g\mathbf{v}_k \circ ([1 \mid 1] - [1 \mid 1]) = 0,$$

from where, in addition, it follows that, for any $t \in C$,

$$[g\mathbf{w}_k \mid t] \circ [1] = [g\mathbf{w}_k \mid t \mid 1] - [g\mathbf{w}_k \circ [1] \mid t] = [g\mathbf{w}_k \mid t \mid 1],$$

whence

$$\begin{aligned} g\mathbf{v}_{k+1} \circ [1] &= \sum_{t<m+q} (m+q-t-1)_*[g\mathbf{w}_k \mid t \mid 1] - \sum_{s<m} (m-s-1)_*[g\mathbf{w}_k \mid s \mid 1] \\ &= [g\mathbf{v}_{k+1} \mid 1] = g\mathbf{w}_{k+1}. \end{aligned}$$

Part 3. We now prove recursively that $\partial g = g\partial$.

For argument $\mathbf{w}_0$ is immediate: $\partial g\mathbf{w}_0 = \partial[1] = 0$. For argument $\mathbf{v}_{k+1}$, first observe that $\partial g\mathbf{w}_k = 0$ gives, for ant $t \in C$,

$$\begin{aligned} \partial[g\mathbf{w}_k \mid t] = \partial[g\mathbf{v}_k \mid 1 \mid t] &\overset{(5.33)}{=} [\partial[g\mathbf{v}_k \mid 1] \mid t] + [g\mathbf{v}_k \mid \partial[1 \mid t]] \\ &= [\partial g\mathbf{w}_k \mid t] + [g\mathbf{v}_k \mid \partial[1 \mid t]] = [g\mathbf{v}_k \mid \partial[1 \mid t]] \\ &= 1_*[g\mathbf{v}_k \mid t] - [g\mathbf{v}_k \mid 1 \oplus t] + t_*[g\mathbf{v}_k \mid 1] \\ &= 1_*[g\mathbf{v}_k \mid t] - [g\mathbf{v}_k \mid 1 \oplus t] + t_* g\mathbf{w}_k. \end{aligned}$$

Then,

$$\begin{aligned} \partial g\mathbf{v}_{k+1} &= \sum_{t<m+q} (m+q-t-1)_*\partial[g\mathbf{w}_k \mid t] - \sum_{t<m} (m-t-1)_*\partial[g\mathbf{w}_k \mid t] \\ &= \sum_{t<m+q-1} (m+q-t)_*[g\mathbf{v}_k \mid t] - (m+q-t-1)_*[g\mathbf{v}_k \mid 1+t] \\ &\quad + (m+q-1)_* g\mathbf{w}_k + 1_*[g\mathbf{v}_k \mid m+q-1] - [g\mathbf{v}_k \mid m] + (m+q-1)_* g\mathbf{w}_k \\ &\quad - \sum_{t<m} (m-t)_*[g\mathbf{v}_k \mid t] - (m-t-1)_*[g\mathbf{v}_k \mid 1+t] + (m-1)_* g\mathbf{w}_k \\ &= -1_*[g\mathbf{v}_k \mid m+q-1] + (m+q-1)\big((m+q-1)_* g\mathbf{w}_k\big) \\ &\quad + 1_*[g\mathbf{v}_k \mid m+q-1] - [g\mathbf{v}_k \mid m] + (m+q-1)_* g\mathbf{w}_k + [g\mathbf{v}_k \mid m] \\ &\quad - m(m-1)_* g\mathbf{w}_k = (m+q)\big((m+q-1)_* g\mathbf{w}_k\big) - m\big((m-1)_* g\mathbf{w}_k\big) = g\partial\mathbf{v}_{k+1}. \end{aligned}$$

And for argument $\mathbf{w}_{k+1}$,

$$\begin{aligned}\partial g\mathbf{w}_{k+1} &\overset{(5.36),(5.7)}{=} \partial g\mathbf{v}_{k+1} \circ [1] \\ &= \Big((m+q)\big((m+q-1)_* g\mathbf{w}_k\big) - m\big((m-1)_*(g\mathbf{w}_k)\big)\Big) \circ [1] \overset{(5.36)}{=} 0.\end{aligned}$$

Part 4. Here we show that g preserves products by proving that $g(a \circ b) = ga \circ gb$ for $a, b \in \{\mathbf{v}_k, \mathbf{w}_l\}$. For the case when $a = \mathbf{w}_k$ and $b = \mathbf{w}_l$, we have

$$g\mathbf{w}_k \circ g\mathbf{w}_l \overset{(5.36)}{=} g\mathbf{v}_k \circ [1] \circ g\mathbf{w}_l \overset{(5.36)}{=} 0 = g(\mathbf{w}_k \circ \mathbf{w}_l).$$

To prove the remaining cases, first observe that if $g\mathbf{v}_k \circ g\mathbf{v}_l = g(\mathbf{v}_k \circ \mathbf{v}_l)$ for some k and l, then

$$\begin{aligned}g\mathbf{w}_k \circ g\mathbf{v}_l = g\mathbf{v}_k \circ [1] \circ g\mathbf{v}_l = g\mathbf{v}_k \circ g\mathbf{v}_l \circ [1] = g(\mathbf{v}_k \circ \mathbf{v}_l) \circ [1] = \\ = \binom{k+l}{k} g\mathbf{v}_{k+l} \circ [1] = \binom{k+l}{k} g\mathbf{w}_{k+l} = g(\mathbf{w}_k \circ \mathbf{v}_l).\end{aligned}$$

Next, we show that $g\mathbf{v}_k \circ g\mathbf{v}_l = g(\mathbf{v}_k \circ \mathbf{v}_l)$ by induction. The case when $k = 0$ or $l = 0$ is immediate, since $g\mathbf{v}_0 = [\ ]$. Now, using that, for any $t, s \in C$,

$$[g\mathbf{w}_k \mid t] \circ [g\mathbf{w}_l \mid s] \overset{(5.12)}{=} [[g\mathbf{w}_k \mid t] \circ g\mathbf{w}_l \mid s] + [g\mathbf{w}_k \circ [g\mathbf{w}_l \mid s], t],$$

we have

$$\begin{aligned}g\mathbf{v}_{k+1} \circ g\mathbf{v}_{l+1} = &\sum_{s<m+q} (m+q-s-1)_* \Big[\sum_{t<m+q} (m+q-t-1)_* [g\mathbf{w}_k \mid t] \circ g\mathbf{w}_l \mid s\Big] \\ &- \sum_{s<m+q} (m+q-s-1)_* \Big[\sum_{t<m} (m-t-1)_* [g\mathbf{w}_k \mid t] \circ g\mathbf{w}_l \mid s\Big] \\ &+ \sum_{t<m+q} (m+q-t-1)_* \Big[g\mathbf{w}_k \circ \sum_{s<m+q} (m+q-s-1)_* [g\mathbf{w}_l \mid s] \mid t\Big] \\ &- \sum_{t<m+q} (m+q-t-1)_* \Big[g\mathbf{w}_k \circ \sum_{s<m} (m-s-1)_* [g\mathbf{w}_l \mid s] \mid t\Big] \\ &- \sum_{t<m} (m-t-1)_* \Big[g\mathbf{w}_k \circ \sum_{s<m+q} (m+q-s-1)_* [g\mathbf{w}_l \mid s] \mid t\Big] \\ &+ \sum_{t<m} (m-t-1)_* \Big[g\mathbf{w}_k \circ \sum_{s<m} (m-s-1)_* [g\mathbf{w}_l \mid s] \mid t\Big] \\ &- \sum_{s<m} (m-s-1)_* \Big[\sum_{t<m+q} (m+q-t-1)_* [g\mathbf{w}_k \mid t] \circ g\mathbf{w}_l \mid s\Big] \\ &+ \sum_{s<m} (m-s-1)_* \Big[\sum_{t<m} (m-t-1)_* [g\mathbf{w}_k \mid t] \circ g\mathbf{w}_l \mid s\Big],\end{aligned}$$

and then, by induction,

$$\begin{aligned}
g\mathbf{v}_{k+1}\circ g\mathbf{v}_{l+1} = &\sum_{s<m+q}(m+q-s-1)_*[g\mathbf{v}_{k+1}\circ g\mathbf{w}_l \mid s] \\
&+\sum_{t<m+q}(m+q-t-1)_*[g\mathbf{w}_k\circ g\mathbf{w}_{l+1}\mid t] \\
&-\sum_{s<m}(m-s-1)_*[g\mathbf{v}_{k+1}\circ g\mathbf{w}_l\mid s]-\sum_{t<m}(m-t-1)_*[g\mathbf{w}_k\circ g\mathbf{w}_{l+1}\mid t] \\
&= \binom{k+l+1}{k+1}\Big(\sum_{s<m+q}(m+q-s-1)_*[g\mathbf{w}_{k+l+1}\mid s] \\
&-\sum_{s<m}(m-s-1)_*[g\mathbf{w}_{k+l+1}\mid s]\Big) \qquad (5.37)\\
&+\binom{k+l+1}{k}\Big(\sum_{t<m+q}(m+q-t-1)_*[g\mathbf{w}_{k+l+1}\mid t] \\
&-\sum_{t<m}(m-t-1)_*[g\mathbf{w}_{k+l+1}\mid t]\Big) \qquad (5.38)\\
&= \binom{k+l+1}{k+1}g\mathbf{v}_{k+l+2}+\binom{k+l+1}{k}g\mathbf{v}_{k+l+2}=\binom{k+l+2}{k+1}g\mathbf{v}_{k+l+2} \\
&= g(\mathbf{v}_{k+1}\circ\mathbf{v}_{l+1}).
\end{aligned}$$

□

Now, we are ready to establish the following key result.

Theorem 5.4 *The morphisms $f:\mathbf{B}(\mathcal{Z}C)\to\mathcal{R}$ and $g:\mathcal{R}\to\mathbf{B}(\mathcal{Z}C)$, as defined above, form a contraction.*

Proof: Part 1. We start by showing that the composite fg is the identity. Clearly $fg\mathbf{v}_0=f[\,]=\mathbf{v}_0$. Then, induction gives

$$\begin{aligned}
fg\mathbf{w}_k &\overset{(5.36)}{=} f(g\mathbf{v}_k\circ[1]) = fg\mathbf{v}_k\circ f[1]=\mathbf{v}_k\circ\mathbf{w}_0=\mathbf{w}_k,\\
fg\mathbf{v}_{k+1} &= \sum_{t<m+q}(m+q-t-1)_*f[g\mathbf{v}_k|1\mid t]-\sum_{s<m}(m-s-1)_*f[g\mathbf{v}_k|1|s]\\
&=\sum_{t<m+q}(m+q-t-1)_*(f[g\mathbf{v}_k]\bullet f[1\mid t])-\sum_{s<m}(m-s-1)_*(f[g\mathbf{v}_k]\bullet f[1|s])\\
&=\mathbf{v}_k\bullet f[1|m+q-1]=\mathbf{v}_k\bullet\mathbf{v}_1=\mathbf{v}_{k+1}.
\end{aligned}$$

Part 2. Here, we describe the composite gf. Clearly $gf[\,]=[\,]$ and $gf[x]=x((x-1)_*[1])$. For those 2-cells $[x\mid y]$ such that $x+y<m+q$ we have $gf[x\mid y]=0$, and, as we prove below, the effect of gf on the 2-cells $[x\mid y]$ with $x+y\geq m+q$ is described

by the formula

$$
\begin{aligned}
gf[x \mid y] = & \sum_{t=x+y-m-q}^{m+q-1} (x+y-t-1)_*[1 \mid t] + \sum_{t=0}^{r-1}(m+r-t-1)_*[1 \mid t] \qquad (5.39)\\
& - \sum_{t=0}^{m-1}(m+r-t-1))_*[1 \mid t] + \sum_{i=1}^{s(x,y)-1} \sum_{t=(i-1)q+r}^{iq+r-1} (m+iq+r-t-1)_*[1 \mid t] \\
& + \sum_{i=1}^{s(x,y)-1} \sum_{t=m}^{m+q-1} (m+iq+r-t-1)_*[1 \mid t],
\end{aligned}
$$

where we write $x+y = m+s(x,y)q+r$ with $0 \leq r < q$ (so that $x \oplus y = m+r$). Concerning the two last terms, note that $(s(x,y)-1)q+r < m+q$ whenever $s(x,y) \geq 2$, since $m+s(x,y)q+r = x+y < 2m+2q$.

In effect, by definition of f and g, we have

$$
gf[x \mid y] = \sum_{i=0}^{s(x,y)-1} \Big(\sum_{t=0}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 \mid t] - \sum_{t=0}^{m-1} \wp(m+iq+r-t-1)_*[1 \mid t]\Big).
$$

Then, since for any $i \geq 1$ and $t < r$ is $\wp\big(m+(i+1)q+r-t-1\big) = \wp(m+iq+r-t-1)$, we see that

$$
\begin{aligned}
gf[x \mid y] = & \sum_{t=r}^{m+q-1} (m+q+r-t-1)_*[1 \mid t] + \sum_{t=0}^{r-1}(m+r-t-1)_*[1 \mid t] \\
& - \sum_{t=0}^{m-1}(m+r-t-1)_*[1 \mid t] \\
& + \sum_{i=1}^{s(x,y)-1} \Big(\sum_{t=r}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 \mid t] \\
& - \sum_{t=r}^{m-1} \wp(m+iq+r-t-1)_*[1 \mid t]\Big) \\
= & \sum_{t=0}^{r-1}(m+r-t-1)_*[1 \mid t] - \sum_{t=0}^{m-1}(m+r-t-1)_*[1 \mid t] \\
& + \sum_{i=0}^{s(x,y)-1} \sum_{t=r}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 \mid t] \\
& - \sum_{i=1}^{s(x,y)-1} \sum_{t=r}^{m-1} \wp(m+iq+r-t-1)_*[1 \mid t],
\end{aligned}
$$

from where (5.39) follows thanks to the equalities

$$\sum_{t=r}^{m+q-1} \wp(m+(i+1)q+r-t-1)_*[1 \mid t] =$$
$$\sum_{l=0}^{i-1} \sum_{t=lq+r}^{(l+1)q+r-1} (m+(l+1)q+r-t-1)_*[1 \mid t] + \sum_{t=iq+r}^{m+q-1} (m+(i+1)q+r-t-1)_*[1 \mid t],$$
$$\sum_{t=r}^{m-1} \wp(m+iq+r-t-1)_*[1 \mid t] = \sum_{l=1}^{i-1} \sum_{t=(l-1)q+r}^{lq+r-1} (m+lq+r-t-1)_*[1 \mid t]$$
$$+ \sum_{t=(i-1)q+r}^{m-1} (m+iq+r-t-1)_*[1 \mid t].$$

Finally, to complete the description of the composite gf, for generic cells $[x \mid y \mid \sigma]$ of dimensions greater than 2 we have the formula

$$gf[x \mid y \mid \sigma] = [gf[x,y] \mid gf[\sigma]]. \tag{5.40}$$

In effect, as $gf[x \mid y \mid \sigma] = g(f[x,y] \bullet f[\sigma])$, by linearity, it suffices to observe that, for any $k \geq 1$,

$$g(\mathbf{v}_1 \bullet \mathbf{w}_k) = [g\mathbf{v}_1 \mid g\mathbf{w}_k], \quad g(\mathbf{v}_1 \bullet \mathbf{v}_k) = [g\mathbf{v}_1 \mid g\mathbf{v}_k],$$

or, equivalently, that $g\mathbf{w}_{k+1} = [g\mathbf{v}_1 \mid g\mathbf{w}_k]$ and $g\mathbf{v}_{k+1} = [g\mathbf{v}_1 \mid g\mathbf{v}_k]$. But these last equations are immediate for $k=1$, and for higher k by a straightforward induction.
Part 3. We establish here a homotopy Φ from gf to the identity, which is determined by the recursive formulas

$$\begin{cases} \Phi[\ ] = 0, \\ \Phi[x] = \sum\limits_{t<x} (x-t-1)_*[1 \mid t], \\ \Phi[x \mid y \mid \sigma] = [\Phi[x] \mid y \mid \sigma] + [gf[x \mid y] \mid \Phi[\sigma]]. \end{cases} \tag{5.41}$$

Since, for any $t < x$ in C, $(x-t-1) \oplus 1 \oplus t = x$, we see that $\pi\Phi[x] = x$ and then, by recursion, that $\pi\Phi[x \mid y \mid \sigma] = x \oplus y \oplus \pi[\sigma]$. Hence, by Proposition 5.2, the formulas above determine an endomorphism of the complex of $\mathbb{H}C$-modules $\mathbf{B}(\mathcal{Z}C)$, which is of differential degree $+1$.

Next, we prove that $\Phi : gf \Rightarrow id$ is actually a homotopy:

<u>For a 1-cell</u> $[x]$ is $\Phi\partial[x] = 0$, and

$$\begin{aligned} \partial\Phi[x] &= \sum_{t<x} (x-t)_*[t] - (x-t-1)_*[1+t] + (x-1)_*[1] = -[x] + x((x-1)_*[1]) \\ &= -[x] + gf[x], \end{aligned}$$

as required.

For a 2−cell $[x \mid y]$ we have

$$
\begin{aligned}
(\partial\Phi+\Phi\partial)[x \mid y] &= \sum_{t<x}(x-t-1)_*(1_*[t \mid y]-[1+t,y]+[1 \mid t\oplus y]-y_*[1 \mid t]) \\
&+\sum_{t<y}(x\oplus(y-t-1))_*[1 \mid t]-\sum_{t<x\oplus y}((x\oplus y)-t-1)_*[1 \mid t] \\
&+\sum_{t<x}((x-t-1)\oplus y)_*[1 \mid t]=\sum_{t<x}(x-t)_*[t \mid y]-(x-t-1)_*[1+t \mid y] \\
&+\sum_{t<x}(x-t-1)_*[1 \mid t\oplus y]-\sum_{t<x}((x-t-1)\oplus y)_*[1 \mid t] \\
&+\sum_{t<y}(x\oplus(y-t-1))_*[1 \mid t]-\sum_{t<x\oplus y}((x\oplus y)-t-1)_*[1 \mid t] \\
&+\sum_{t<x}((x-t-1)\oplus y)_*[1 \mid t]=-[x,y]+\sum_{t<x}(x-t-1)_*[1 \mid t\oplus y] \\
&+\sum_{t<y}(x\oplus(y-t-1))_*[1 \mid t]-\sum_{t<x\oplus y}((x\oplus y)-t-1)_*[1 \mid t].
\end{aligned}
\tag{5.42}
$$

If $s(x,y)=0$ then, for any $t<x$, $t\oplus y=t+y$ and $x\oplus(y-t-1)=x+y-t-1=(x\oplus y)-t-1$. Therefore

$$
\sum_{t<x}(x-t-1)_*[1 \mid t+y]+\sum_{t<y}(x+y-t-1)_*[1 \mid t]-\sum_{t<x+y}(x+y-t-1)_*[1 \mid t]=0,
$$

and, since $gf[x \mid y]=0$, it follows that $(\partial\Phi+\Phi\partial)[x \mid y]=-[x \mid y]+gf[x \mid y]$, as required.

If $s(x,y)>0$, the composite $gf[x \mid y]$ has been computed in (5.39) and, writing as there $x+y=m+s(x,y)q+r$ with $0\le r<q$, we have

$$
\begin{aligned}
\sum_{t<x}(x-t-1)_*[1 \mid t\oplus y] &= \sum_{l=1}^{s(x,y)-1}\sum_{\substack{t<x\\ m+lq\le t+y<m+(l+1)q}}(x-t-1)_*[1 \mid t+y-lq] \\
&+\sum_{\substack{t<x\\ t+y<m+q}}(x-t-1)_*[1 \mid t+y]+\sum_{\substack{t<x\\ m+s(x,y)q\le t+y}}(x-t-1)_*[1 \mid t+y-s(x,y)q].
\end{aligned}
$$

Now, making the changes $u=t+y-lq$, $u=t+y$, and $u=t+y-s(x,y)q$ in the respective terms, and then renaming the u again by t, we obtain

$$
\begin{aligned}
\sum_{t<x}(x-t-1)_*[1 \mid t\oplus y] &= \sum_{i=1}^{s(x,y)-1}\sum_{t=m}^{m+q-1}(m+iq+r-t-1)_*[1 \mid t] \\
&+\sum_{t=y}^{m+q-1}(x+y-t-1)_*[1 \mid t]+\sum_{t=m}^{m+r-1}(m+r-t-1)_*[1 \mid t].
\end{aligned}
$$

Similarly, we have

$$\sum_{t<y}(x\oplus(y-t-1))_*[1\mid t]=\sum_{i=1}^{s(x,y)-1}\sum_{t=(i-1)q+r}^{iq+r-1}(m+iq+r-t-1)_*[1\mid t]$$
$$+\sum_{t=x+y-m-q}^{y-1}(x+y-t-1)_*[1\mid t]+\sum_{t=0}^{r-1}(m+r-t-1)_*[1\mid t],$$

and

$$\sum_{t<x\oplus y}((x\oplus y)-t-1)_*[1\mid t]=\sum_{t=0}^{m+r-1}(m+r-t-1)_*[1\mid t].$$

Hence, a direct comparison with (5.39) gives that $(\partial\Phi+\Phi\partial)[x\mid y]=-[x\mid y]+gf[x\mid y]$, as required.

Finally, we prove that $(\partial\Phi+\Phi\partial)(\tau)=-\tau+gf(\tau)$ if τ is of dimension 3 or greater. To do so, previously observe that, for any generic cell γ of $\mathbf{B}(\mathcal{Z}C)$, we have

$$\partial[gf[x\mid y]\mid\Phi(\gamma)]=[gf[x\mid y]\mid\partial\Phi(\gamma)].\tag{5.43}$$

To prove it, by linearity, it suffices to check that $\partial[g\mathbf{v}_1\mid 1\mid\beta]=[g\mathbf{v}_1\mid\partial[1\mid\beta]]$, for any generic cell β:

$$\partial[g\mathbf{v}_1\mid 1\mid\beta]\overset{(5.33)}{=}[\partial[g\mathbf{v}_1\mid 1]\mid\beta]+[g\mathbf{v}_1\mid\partial[1\mid\beta]]$$
$$\overset{(5.35)}{=}[\partial g\mathbf{w}_1\mid\beta]+[g\mathbf{v}_1\mid\partial[1\mid\beta]]=[g\partial\mathbf{w}_1\mid\beta]+[g\mathbf{v}_1\mid\partial[1\mid\beta]]=[g\mathbf{v}_1\mid\partial[1\mid\beta]].$$

Now, according to the definition in (5.41), on chains c of $\mathbf{B}(\mathcal{Z}C)$ of dimensions 2 or greater, we can write $\Phi(c)=\Phi_1(c)+\Phi_2(c)$, where Φ_1 and Φ_2 are the morphisms of $\mathbb{H}C$-modules given on generic cells by $\Phi_1[x\mid y\mid\sigma]=[\Phi[x]\mid y\mid\sigma]$ and $\Phi_2[x\mid y\mid\sigma]=[gf[x\mid y]\mid\Phi(\sigma)]$. Then, for the generic cell $\tau=[x\mid y\mid z\mid\rho]$, as

$$\partial\tau=[\partial[x\mid y]\mid z\mid\rho]-[x\mid\partial[y\mid z\mid\rho]]=[\partial[x\mid y\mid z]\mid\rho]+[x\mid y\mid\partial[z\mid\rho]],$$

we have

$$\begin{aligned}\Phi\partial(\tau)&=\Phi_1[\partial[x\mid y]\mid z\mid\rho]-\Phi_1[x\mid\partial[y\mid z\mid\rho]]+\Phi_2[\partial[x\mid y\mid z]\mid\rho]\\&\quad+\Phi_2[x\mid y\mid\partial[z\mid\rho]]\\&=[\Phi\partial[x\mid y]\mid z\mid\rho]-[\Phi[x]\mid\partial[y\mid z\mid\rho]]\\&\quad+[gf\partial[x\mid y\mid z]\mid\Phi[\rho]]+[gf[x\mid y]\mid\Phi\partial[z\mid\rho]]\\&=[\Phi\partial[x\mid y]\mid z\mid\rho]-[\Phi[x]\mid\partial[y\mid z\mid\rho]]+[gf[x\mid y]\mid\Phi\partial[z\mid\rho]],\end{aligned}$$

since $f\partial[x\mid y\mid z]=0$ by (5.32). Furthermore, by using (5.33) and (5.43), we have

$$\begin{aligned}\partial\Phi(\tau)&=\partial[\Phi[x]\mid y\mid z\mid\rho]+\partial[gf[x\mid y]\mid\Phi[z\mid\rho]]\\&=[\partial\Phi[x\mid y]\mid z\mid\rho]+[\Phi[x]\mid\partial[y\mid z\mid\rho]]+[gf[x\mid y]\mid\partial\Phi[z\mid\rho]],\end{aligned}$$

whence, by the already proven above and induction on the dimension of ρ, we get

$$\begin{aligned}
(\partial\Phi + \Phi\partial)(\tau) &= [\partial\Phi[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid \partial\Phi[z \mid \rho]] + [\Phi\partial[x \mid y] \mid z \mid \rho] \\
&\quad + [gf[x \mid y] \mid \Phi\partial[z \mid \rho]] \\
&= [(\partial\Phi + \Phi\partial)[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid (\partial\Phi + \Phi\partial)[z \mid \rho]] \\
&= [-[x \mid y] + gf[x \mid y] \mid z \mid \rho] + [gf[x \mid y] \mid -[z \mid \rho] + gf[z \mid \rho]] \\
&= -[x \mid y \mid z \mid \rho] + [gf[x \mid y] \mid gf[z \mid \rho]] \overset{(5.40)}{=} -\tau + gf(\tau),
\end{aligned}$$

as required.

This completes the proof of Theorem 5.4, since the conditions in (5.14) are easily verified. □

If $\mathcal{A}$ is any $\mathbb{H}C$-module, by Proposition 5.4, the first level cohomology groups $H^n(C,1;\mathcal{A})$ are precisely Leech cohomology groups $H^n_{\mathrm{L}}(C,\mathcal{A})$. Hence, by Theorem 5.4, these can be computed as $H^n_{\mathrm{L}}(C,\mathcal{A}) = H^n(\mathrm{Hom}_{\mathbb{H}C}(\mathcal{R},\mathcal{A}))$. Since, by Proposition 5.2, there are natural isomorphisms

$$\mathrm{Hom}_{\mathbb{H}C}(\mathcal{R}_{2k},\mathcal{A}) \cong \mathcal{A}(k\cdot m), \quad \mathrm{Hom}_{\mathbb{H}C}(\mathcal{R}_{2k+1},\mathcal{A}) \cong \mathcal{A}(k\cdot m \oplus 1).$$

we obtain the following already known result.

Proposition 5.10 (Corollary 2.3 in Chapter 2) *Let $C = C_{m,q}$ be the cyclic monoid of index m and period q. Then, for any $\mathbb{H}C$-module $\mathcal{A}$ and any integer $k \geq 0$, there is a natural exact sequence of abelian groups*

$$0 \to H^{2k+1}_{\mathrm{L}}(C,\mathcal{A}) \longrightarrow \mathcal{A}(k\cdot m \oplus 1) \xrightarrow{\partial} \mathcal{A}(km \oplus m) \longrightarrow H^{2k+2}_{\mathrm{L}}(C,\mathcal{A}) \to 0,$$

where ∂ is given by $\partial(a) = (m+q)\big((m+q-1)_* a\big) - m\big((m-1)_* a\big)$.

We consider now the rth level cohomology groups of $C = C_{m,q}$ with $r \geq 2$. By Theorem 5.4 and an iterated use of Lemma 5.1 we conclude that the complexes of $\mathbb{H}C$-modules $\mathbf{B}^r(\mathcal{Z}C)$ and $\mathbf{B}^{r-1}(\mathcal{R})$ are homotopy equivalent. Therefore, for any $\mathbb{H}C$-module $\mathcal{A}$, there are natural isomorphisms

$$H^n(C,r,\mathcal{A}) \cong H^n\big(\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}^{r-1}(\mathcal{R}),\mathcal{A})\big).$$

An analysis of the complexes $\mathbf{B}^{r-1}(\mathcal{R})$ tell us that $\mathbf{B}^{r-1}(\mathcal{R})_n = 0$ for $0 < n < r$, and that we have the diagram of suspensions

$$\begin{tikzcd}
\mathcal{R}_4 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathcal{R}_3 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathcal{R}_2 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathcal{R}_1 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & 0 \\
\mathbf{B}(\mathcal{R})_5 \arrow[r] \arrow[d, hook, "\mathrm{S}"'] & \mathbf{B}(\mathcal{R})_4 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}(\mathcal{R})_3 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & \mathbf{B}(\mathcal{R})_2 \arrow[r] \arrow[d, equal, "\mathrm{S}"'] & 0 \\
\mathbf{B}^2(\mathcal{R})_6 \arrow[r] & \mathbf{B}^2(\mathcal{R})_5 \arrow[r] & \mathbf{B}^2(\mathcal{R})_4 \arrow[r] & \mathbf{B}^2(\mathcal{R})_3 \arrow[r] & 0
\end{tikzcd}$$

where

• $\mathbf{B}(\mathcal{R})_4$ *is the free* $\mathbb{H}C$*-module on the binary set consisting of the suspension of the* 3*-cell* $\mathbf{w}_1$ *of* $\mathcal{R}$ *and the* 4*-cell*

$$[\mathbf{w}_0|\mathbf{w}_0]$$

with $\pi[\mathbf{w}_0 \mid \mathbf{w}_0] = \wp(2)$*, whose differential is* $\partial([\mathbf{w}_0 \mid \mathbf{w}_0]) = \mathbf{w}_0 \circ \mathbf{w}_0 = 0$*,*

• $\mathbf{B}(\mathcal{R})_5$ *is the free* $\mathbb{H}C$*-module on the set consisting of the suspension of the* 4*-cell* $\mathbf{v}_2$ *of* $\mathcal{R}$ *together the* 5*-cells*

$$[\mathbf{w}_0 \mid \mathbf{v}_1],\ [\mathbf{v}_1 \mid \mathbf{w}_0]$$

with $\pi[\mathbf{w}_0 \mid \mathbf{v}_1] = m \oplus 1 = \pi[\mathbf{v}_1|\mathbf{w}_0]$*, and whose differential is*

$$\partial[\mathbf{w}_0 \mid \mathbf{v}_1] = \mathbf{w}_1 - (m+q)\big((m+q-1)_*[\mathbf{w}_0 \mid \mathbf{w}_0]\big) + m\big((m-1)_*[\mathbf{w}_0 \mid \mathbf{w}_0]\big),$$
$$\partial[\mathbf{v}_1 \mid \mathbf{w}_0] = -\mathbf{w}_1 - (m+q)\big((m+q-1)_*[\mathbf{w}_0 \mid \mathbf{w}_0]\big) + m\big((m-1)_*[\mathbf{w}_0 \mid \mathbf{w}_0]\big).$$

• $\mathbf{B}^2(\mathcal{R})_6$ *is the free* $\mathbb{H}C$*-module on the set consisting of the double suspension of the* 4*-cell* $\mathbf{v}_2$ *of* $\mathcal{R}$*, the suspension of the* 5*-cells* $[\mathbf{w}_0 \mid \mathbf{v}_1]$ *and the* $[\mathbf{v}_1 \mid \mathbf{w}_0]$ *of* $\mathbf{B}(\mathcal{R})_5$*, and the* 6*-cell*

$$[\mathbf{w}_0 \parallel \mathbf{w}_0]$$

with $\pi[\mathbf{w}_0 \parallel \mathbf{w}_0] = \wp(2)$*, whose differential is*

$$\partial[\mathbf{w}_0 \parallel \mathbf{w}_0] = 0.$$

Then, by Proposition 5.2, there are natural isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_2, \mathcal{A}) \cong \mathcal{A}(1),\\
&\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_3, \mathcal{A}) \cong \mathcal{A}(m),\\
&\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_4, \mathcal{A}) \cong \mathcal{A}(m\oplus 1) \times \mathcal{A}(\wp(2)),\\
&\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R})_5, \mathcal{A}) \cong \mathcal{A}(2\cdot m) \times \mathcal{A}(m\oplus 1) \times \mathcal{A}(m\oplus 1),\\
&\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}^2(\mathcal{R})_6, \mathcal{A}) \cong \mathcal{A}(2\cdot m) \times \mathcal{A}(m\oplus 1) \times \mathcal{A}(m\oplus 1) \times \mathcal{A}(\wp(2)).
\end{aligned}$$

In these terms the truncated complex $\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}(\mathcal{R}), \mathcal{A})$ is written as

$$0 \to \mathcal{A}(1) \xrightarrow{\partial^1} \mathcal{A}(m) \xrightarrow{\partial^2} \mathcal{A}(m\oplus 1) \times \mathcal{A}(\wp(2)) \xrightarrow{\partial^3} \mathcal{A}(2\cdot m) \times \mathcal{A}(m\oplus 1) \times \mathcal{A}(m\oplus 1), \quad (5.44)$$

where the coboundaries are given by

$$\partial^1(a) = -(m+q)\big((m+q-1)_* a\big) + m\big((m-1)_* a\big),$$

$\partial^2 = 0$ is the morphism zero, and

$$\begin{aligned}\partial^3(a,b) = \Big(&-(m+q)\big((m+q-1)_*a\big) + m\big((m-1)_*a\big),\\ &a-(m+q)\big((m+q-1)_*b\big) + m\big((m-1)_*b\big),\\ &-a-(m+q)\big((m+q-1)_*b\big) + m\big((m-1)_*b\big)\Big),\end{aligned}$$

while the truncated complex $\mathrm{Hom}_{\mathbb{H}C}(\mathbf{B}^2(\mathcal{R}),\mathcal{A})$ is written as

$$\begin{aligned}0\to\mathcal{A}(1) &\overset{\partial^1}{\to} \mathcal{A}(m) \overset{\partial^2}{\to} \mathcal{A}(m\oplus 1)\times\mathcal{A}(\wp(2))\\ &\overset{\partial^3}{\to} \mathcal{A}(2\cdot m)\times\mathcal{A}(m\oplus 1)\times\mathcal{A}(m\oplus 1)\times\mathcal{A}(\wp(2)),\end{aligned} \tag{5.45}$$

where ∂^1 and ∂^2 are the same as above whereas ∂^3 acts by

$$\begin{aligned}\partial^3(a,b) = \Big(&(m+q)\big((m+q-1)_*a\big) - m\big((m-1)_*a\big),\\ &-a+(m+q)\big((m+q-1)_*b\big) - m\big((m-1)_*b\big),\\ &a+(m+q)\big((m+q-1)_*b\big) - m\big((m-1)_*b\big),\ 0\Big).\end{aligned}$$

Then, as an immediate consequence of (5.44) and (5.45), we have

Theorem 5.5 *Let $C = C_{m,q}$ be the cyclic monoid of index m and period q. Then, for any $\mathbb{H}C$-module $\mathcal{A}$, there is a natural exact sequence of abelian groups*

$$0\to H^2(C,2;\mathcal{A}) \longrightarrow \mathcal{A}(1) \overset{\partial}{\longrightarrow} \mathcal{A}(m) \longrightarrow H^3(C,2;\mathcal{A})\to 0$$

where $\partial(a) = (m+q)\big((m+q-1)_*a\big) - m\big((m-1)_*a\big)$, and natural isomorphisms

$$\begin{aligned}H^4(C,2,\mathcal{A}) &\cong H^5(C,3;\mathcal{A})\\ &\cong \left\{ b\in\mathcal{A}(\wp(2)) \;\middle|\; \begin{array}{l}(m+q)^2\wp(2m+q-2)_*b = m^2\wp(2m-2)_*b,\\ 2(m+q)(m+q-1)_*b = 2m(m-1)_*b,\end{array} \right\}.\end{aligned}$$

Note that in the case when the cyclic monoid is of index $m = 1$, the above description of $H^4(C,2;\mathcal{A})$ adopts the simpler form

$$H^4(C,2;\mathcal{A}) \cong \left\{ b\in\mathcal{A}(\wp(2)) \;\middle|\; \begin{array}{l}(q+1)^2q_*b = b,\\ 2(q+1)q_*b = 2b,\end{array} \right\},$$

while when $m \geq 2$,

$$H^4(C,2;\mathcal{A}) \cong \left\{ b\in\mathcal{A}(\wp(2)) \;\middle|\; \begin{array}{l}(2mq+q^2)\wp(2m-2)_*b = b,\\ 2(m+q)(m+q-1)_*b = 2m(m-1)_*b,\end{array} \right\}.$$

Corollary 5.8 *For any finite cyclic monoid C, any integer $r \geq 1$, and any $\mathbb{H}C$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^{r+1}(C, r; \mathcal{A}) \cong H^2_{\mathrm{L}}(C, \mathcal{A}) \cong H^2_{\mathrm{G}}(C, \mathcal{A}).$$

Proof: A direct comparison of the exact sequence in Theorem 5.5 with the sequence in Proposition 5.10, for the case when $k = 0$, gives $H^3(C, 2; \mathcal{A}) \cong H^2_{\mathrm{L}}(C, \mathcal{A})$. Then, the result follows since $H^3(C, 2; \mathcal{A}) \cong H^2_{\mathrm{G}}(C, \mathcal{A})$ by Proposition 5.5, and $H^{r+1}(C, r; \mathcal{A}) \cong H^3(C, 2; \mathcal{A})$ by Corollary 5.6. $\square$

Corollary 5.9 *For any finite cyclic monoid C, any integer $r \geq 2$, and any $\mathbb{H}C$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^{r+2}(C, r; \mathcal{A}) \cong H^3_{\mathrm{c}}(C, \mathcal{A}).$$

Proof: By Corollary 5.7, $H^{r+2}(C, r; \mathcal{A}) \cong H^5(C, 3; \mathcal{A})$, for any $r \geq 3$. Since, by Theorem 5.5, $H^5(C, 3; \mathcal{A}) \cong H^4(C, 2, \mathcal{A})$, the result follows by Proposition 5.6. $\square$

For instance, if A is any abelian group viewed as a constant $\mathbb{H}C$-module, then $H^4(C, 2; A)$ is isomorphic to the subgroup of A consisting of those elements b such that

$$\left|\begin{array}{l}(m+q)^2b = m^2b,\\ 2qb = 0,\end{array}\right. \Leftrightarrow \left|\begin{array}{l}(2mq+q^2)b = 0,\\ 2qb = 0,\end{array}\right. \Leftrightarrow \left|\begin{array}{l}q^2b = 0,\\ 2qb = 0,\end{array}\right. \Leftrightarrow (2q, q^2)\, b = 0,$$

where $(2q, q^2) = q(2, q)$ is the greatest common divisor of 2 and q. This leads to the following isomorphism, which is analogous to the proven by Eilenberg- Mac Lane for the third abelian cohomology group of the cyclic group C_q with coefficients in A [32, Section 21].

Corollary 5.10 *For any finite cyclic monoid C, any integer $r \geq 2$, and any abelian group A, there is a natural isomorphism*

$$H^{r+2}(C, r; A) \cong \mathrm{Hom}_{\mathbf{Ab}}\big(\mathbb{Z}/(2q, q^2)\mathbb{Z}, A\big).$$

5.6.2 Cohomology of the infinite cyclic monoid

In this subsection we focus on the additive monoid of natural numbers $C_\infty = \mathbb{N}$. As before, we start by introducing a commutative DGA-algebra over $\mathbb{H}C_\infty$, $\mathcal{R}$, simpler than $\mathbf{B}(\mathcal{Z}C_\infty)$.

For each integer $k = 0, 1, \ldots$, let us choose unitary sets over C_∞, $\{\mathbf{w}_0\}$ and $\{\mathbf{v}_k\}$, with $\pi\mathbf{w}_0 = 1$ and $\pi\mathbf{v}_k = k$. Then,

$$\begin{cases} \mathcal{R}_0 &= \text{the free } \mathbb{H}C_\infty\text{-module on } \{\mathbf{v}_0\},\\ \mathcal{R}_1 &= \text{the free } \mathbb{H}C_\infty\text{-module on } \{\mathbf{w}_0\},\\ \mathcal{R}_n &= 0,\ n \geq 2\end{cases}$$

The differential $\partial = 0$ is zero. The augmentation is the canonical isomorphism $\mathcal{R}_0 \cong \mathbb{Z}$, and the multiplication on $\mathcal{R}$ is by determined by the rules $\mathbf{v}_0 \circ \mathbf{v}_0 = \mathbf{v}_0$, $\mathbf{v}_0 \circ \mathbf{w}_0 = \mathbf{w}_0$ and $\mathbf{w}_0 \circ \mathbf{w}_0 = 0$.

Theorem 5.6 *There are DGA-algebra morphisms $f : \mathbf{B}(\mathcal{Z}C_\infty) \to \mathcal{R}$ and $g : \mathcal{R} \to \mathbf{B}(\mathcal{Z}C_\infty)$, determined by the formulas*

$$\begin{cases} f[\,] &= \mathbf{v}_0, \\ f[x] &= x((x-1)_*\mathbf{w}_0) \end{cases}$$

$$\begin{cases} g\mathbf{v}_0 &= [\,], \\ g\mathbf{w}_0 &= [1], \end{cases}$$

which form a contraction.

Proof: It is plain to see that the above assignments extends to well defined morphisms of DGA algebras over $\mathbb{H}C_\infty$. Indeed, for f, we have $f[\,] = \mathbf{v}_0 \in \mathcal{R}_0(0)$, and

$$(x-1) \oplus \pi\mathbf{w}_0 = (x-1) \oplus 1 = x,$$

which implies $f[x] \in \mathcal{R}_1(x)$. Moreover $\partial f = 0 = f\partial$, and so f is a morphism of complexes of $\mathbb{H}C_\infty$-modules. We now check that f preserves products, indeed,

$$\begin{aligned} &f([\,] \circ [\,]) = f[\,] = \mathbf{v}_0 = \mathbf{v}_0 \circ \mathbf{v}_0 = f[\,] \circ f[\,], \\ &f([\,] \circ [x]) = f[x] = x(x-1)_*\mathbf{w}_0 = \mathbf{v}_0 \circ x(x-1)_*\mathbf{w}_0 = f[\,] \circ f[x], \\ &f([x] \circ [y]) = f[x \mid y] - f[y \mid x] = 0 = f[x] \circ f[y]. \end{aligned}$$

On the other hand, $g\mathbf{v}_0 \in \mathcal{R}_0(0)$, $g\mathbf{w}_0 \in \mathcal{R}_1(1)$ and $\partial g = 0 = g\partial$, so g is a morphism of complexes of $\mathbb{H}C$-modules. To show that g preserves products it is also straightforward, as we see below.

$$\begin{aligned} &g(\mathbf{v}_0 \circ \mathbf{v}_0) = g\mathbf{v}_0 = [\,] = [\,] \circ [\,] = g\mathbf{v}_0 \circ g\mathbf{v}_0, \\ &g(\mathbf{v}_0 \circ \mathbf{w}_0) = g\mathbf{w}_0 = [1] = [\,] \circ [1] = g\mathbf{v}_0 \circ g\mathbf{w}_0, \\ &g(\mathbf{w}_0 \circ \mathbf{w}_0) = 0 = [1] \circ [1] = g\mathbf{w}_0 \circ g\mathbf{w}_0. \end{aligned}$$

To prove that they form a contraction, we begin by showing that the composite fg is the identity:

$$fg\mathbf{v}_0 = f[\,] = \mathbf{v}_0, \quad fg\mathbf{w}_0 = f[1] = 0_*\mathbf{w}_0 = \mathbf{w}_0.$$

We describe now the homotopy $\Phi : gf \Rightarrow id$ by the formula

$$\begin{cases} \Phi[\,] = 0, \\ \Phi[x] = \sum\limits_{0 \le t < x} (x-t-1)_*[1 \mid t], \\ \Phi[x \mid \sigma] = [\Phi[x] \mid \sigma], \end{cases}$$

with σ any cell of dimension greater than 1. It is plain to see that, so defined, Φ is a morphism of $\mathbb{H}C_\infty$-module since, for any $t < x$,

$$(x - t - 1) \oplus 1 \oplus t = x,$$

and thus $\pi\Phi[x] = x$ and $\pi\Phi[x \mid \sigma] = \pi[\Phi[x] \mid \sigma] = x \oplus \pi[\sigma] = \pi[x \mid \sigma]$. We show now that $\partial\Phi + \Phi\partial = gf - id$.

<u>For a 1-cell</u> $[x]$ is $\Phi\partial[x] = 0$, and

$$\begin{aligned}\partial\Phi[x] &= \sum_{t<x}(x-t)_*[t] - (x-t-1)_*[1+t] + (x-1)_*[1] = -[x] + x((x-1)_*[1]) \\ &= -[x] + gf[x],\end{aligned}$$

as required.

<u>For a 2−cell</u> $[x \mid y]$ we have

$$\begin{aligned}(\partial\Phi + \Phi\partial)[x \mid y] &= \sum_{t<x}(x-t-1)_*(1_*[t \mid y] - [1+t, y] + [1 \mid t+y] - y_*[1 \mid t]) \\ &+ \sum_{t<y}(x+y-t-1))_*[1 \mid t] - \sum_{t<x\oplus y}(x+y-t-1)_*[1 \mid t] \\ &+ \sum_{t<x}((x+y-t-1)_*[1 \mid t] = \sum_{t<x}(x-t)_*[t \mid y] - (x-t-1)_*[1+t \mid y] \\ &+ \sum_{t<x}(x-t-1)_*[1 \mid t+y] - \sum_{t<x}(x+y-t-1)_*[1 \mid t] \\ &+ \sum_{t<y}(x+y-t-1)_*[1 \mid t] - \sum_{t<x+y}(x+y-t-1)_*[1 \mid t] \\ &+ \sum_{t<x}(x+y-t-1)_*[1 \mid t] = -[x, y] + \sum_{t<x}(x-t-1)_*[1 \mid t+y] \\ &+ \sum_{t<y}(x+y-t-1)_*[1 \mid t] - \sum_{t<x+y}(x+y-t-1)_*[1 \mid t] = -[x \mid y].\end{aligned}$$

Since $gf[x \mid y] = 0$, it follows that $(\partial\Phi + \Phi\partial)[x \mid y] = -[x \mid y] + gf[x \mid y]$, as required.

Finally, for <u>τ a cell of dimension 3 or greater</u>, we prove that $(\partial\Phi + \Phi\partial)(\tau) = -\tau + gf(\tau)$. Recall that for a cell $\tau = [x \mid y \mid \sigma]$ we have the formula

$$\partial[x \mid y \mid \sigma] = [\partial[x \mid y] \mid \sigma] - [x \mid \partial[y \mid \sigma]],$$

and thus,

$$\Phi\partial[x \mid y \mid \sigma] = [\Phi\partial[x \mid y] \mid \sigma] - [\Phi[x] \mid \partial[y \mid \sigma]].$$

On the other hand, from (5.33), we obtain

$$\partial\Phi[x \mid y \mid \sigma] = \partial[\Phi[x] \mid y \mid \sigma] = [\partial[\Phi[x] \mid y] \mid \sigma] + [\Phi[x] \mid \partial[y \mid \sigma]].$$

Hence, by the already proven above we have

$$\Phi\partial[x \mid y \mid \sigma] + \partial\Phi[x \mid y \mid \sigma] = [\Phi\partial[x \mid y] \mid \sigma] + [\partial\Phi[x \mid y] \mid \sigma] = -[x \mid y \mid \sigma],$$

and, since $gf[x \mid y \mid \sigma] = 0$, we conclude that Φ is indeed an homotopy between the identity and the composite gf. The remaining conditions (5.14), in order to form f, g a contraction, are straightforward to check. $\square$

By Proposition 5.4, there are isomorphisms $H^n(C_\infty, 1; \mathcal{A}) \cong H^n_{\mathrm{L}}(C_\infty, \mathcal{A})$, for any $\mathbb{H}C_\infty$-module $\mathcal{A}$. Then, as a consequence of Theorem 5.6, we recover the computation by Leech of the cohomology groups of the monoid C_∞ [53, Theorem 6.8].

Proposition 5.11 *For any $\mathbb{H}C_\infty$-module $\mathcal{A}$, there are natural isomorphisms*

$$H^0_{\mathrm{L}}(C_\infty, \mathcal{A}) \cong \mathcal{A}(0),\ \ H^1_{\mathrm{L}}(C_\infty, \mathcal{A}) \cong \mathcal{A}(1),$$

and for every $n \geq 2$, $H^n_{\mathrm{L}}(C_\infty, \mathcal{A}) = 0$.

We now pay attention to the second level cohomology groups of C_∞. By Theorem 5.6 and Lemma 5.1, $H^n(C_\infty, 2; \mathcal{A}) \cong H^n\big(\mathrm{Hom}_{\mathbb{H}C_\infty}(\mathbf{B}(\mathcal{R}), \mathcal{A})\big)$. An analysis of $\mathbf{B}(\mathcal{R})$ tell us that

$$\begin{cases} \mathbf{B}(\mathcal{R})_{2k} & = & \text{the free } \mathbb{H}C_\infty\text{-module on } \{\mathbf{v}_k\}, \\ \mathbf{B}(\mathcal{R})_{2k+1} & = & 0, \end{cases}$$

where, recall, $\pi\mathbf{v}_k = k$; the augmentation is the canonical isomorphism $\mathbf{B}(\mathcal{R})_0 \cong \mathbb{Z}$ and the product is given by

$$\mathbf{v}_k \circ \mathbf{v}_l = \binom{k+l}{k}\mathbf{v}_{k+l}.$$

Hence,

Proposition 5.12 *For any $\mathbb{H}C_\infty$-module $\mathcal{A}$, and any integer $k \geq 0$,*

$$H^{2k}(C_\infty, 2; \mathcal{A}) \cong \mathcal{A}(k), \quad H^{2k+1}(C_\infty, 2; \mathcal{A}) = 0.$$

From Corollary 5.6, it follows that

Corollary 5.11 *For any $\mathbb{H}C_\infty$-module $\mathcal{A}$, and any integer $r \geq 2$,*

$$H^{r+1}(C_\infty, r; \mathcal{A}) = 0.$$

We finish by specifying the 3rd level 5-cohomology group of C_∞.

Proposition 5.13 *For any $\mathbb{H}C_\infty$-module $\mathcal{A}$, and any integer $r \geq 3$, there is a natural isomorphism*

$$H^{r+2}(C_\infty, r; \mathcal{A}) \cong \{a \in \mathcal{A}(2) \mid 2a = 0\}.$$

Proof: By Corollary 5.7, $H^{r+2}(C_\infty, r; \mathcal{A}) \cong H^5(C_\infty, 3; \mathcal{A})$. An analysis of $\mathbf{B}^2(\mathcal{R})$ tell us that $\mathbf{B}^2(\mathcal{R})_4 = \mathbf{B}(\mathcal{R})_3 = 0$, $\mathbf{B}^2(\mathcal{R})_5 = \mathbf{B}(\mathcal{R})_4$ is the free $\mathbb{H}C_\infty$-module on $\{\mathbf{v}_2\}$, where $\pi\mathbf{v}_2 = 2$, $\mathbf{B}^2(\mathcal{R})_6$ is the free $\mathbb{H}C_\infty$-module on $\{[\mathbf{v}_1 \parallel \mathbf{v}_1]\}$, with $\pi[\mathbf{v}_1 \parallel \mathbf{v}_1] = 2$, and the differential is

$$\partial[\mathbf{v}_1 \parallel \mathbf{v}_1] = -2\mathbf{v}_2.$$

Whence, for any $\mathbb{H}C_\infty$-module $\mathcal{A}$, $H^5(C_\infty, 3; \mathcal{A}) \cong \{a \in \mathcal{A}(2) \mid 2a = 0\}$. $\square$

Resumen

Las categorías monoidales surgen en distintas ramas de las matemáticas y han sido, por tanto, ampliamente estudiadas en la literatura. Los grupoides monoidales pequeños, que aparecen por ejemplo en álgebra y en topología algebraica, son objetos matemáticos importantes en sí mismos. La mayor parte del trabajo presentado en esta tesis está motivado por el análisis de distintos tipos de grupoides monoidales, y su objetivo último es probar teoremas de clasificación cohomológica para ellos. Algunos de estos resultados han sido establecidos usando teorías de cohomología ya conocidas y estudiadas, mientras que otros han necesitado el desarrollo de nueva teorías. Por lo tanto, esta memoria contribuye también al estudio de monoides bajo un punto de vista homológico.

La tesis se encuentra divida en cinco capítulos. Estos capítulos pueden leerse de forma bastante independiente, aunque comparten gran parte de terminología y argumentos técnicos. Exceptuando algunos cambios de notación realizados para unificar la presentación, y que la bibliografía se encuentra recopilada al final, el Capítulo 1 ha sido publicado como [10] en la revista Semigroup Forum (2013), el Capítulo 3 como [16] en Semigroup Forum (2015), el Capítulo 4 como [15] en Mathematics (2015), mientras que los Capítulos 2 y 5 corresponden a los artículos [12] y [13], que se encuentran actualmente pendientes de publicación.

En el Capítulo 1 analizamos la estructura de grupoides monoidales arbitrarios $(\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$, es decir, categorías pequeñas $\mathcal{M}$ cuyos morfismos son todos invertibles y enriquecidas con un producto tensor

$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}, \qquad (X, Y) \mapsto X \otimes Y,$$

un objeto unidad $\mathrm{I} \in \mathcal{M}$ y los isomorfismos naturales

$$\boldsymbol{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z), \quad \boldsymbol{l}_X : \mathrm{I} \otimes X \xrightarrow{\simeq} X, \quad \boldsymbol{r}_X : X \otimes \mathrm{I} \xrightarrow{\simeq} X,$$

de asociatividad y unidad izquierda y derecha, respectivamente. Estos isomorfismos han de verificar, para cualesquiera objetos X, Y, Z, T de $\mathcal{M}$, los siguientes diagramas (llamados usualmente el pentágono de asociatividad y el triángulo de la unidad):

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes T \arrow[r, "\boldsymbol{a}"] \arrow[d, "\boldsymbol{a}\otimes 1"'] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\boldsymbol{a}"] & X \otimes (Y \otimes (Z \otimes T)) \\
(X \otimes (Y \otimes Z))T \arrow[rr, "\boldsymbol{a}"] & & X \otimes ((Y \otimes Z) \otimes T) \arrow[u, "1\otimes \boldsymbol{a}"']
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] (X \otimes \mathrm{I}) \otimes Y \arrow[rr, "\boldsymbol{a}"] \arrow[dr, "\boldsymbol{r}\otimes 1"'] & & X \otimes (\mathrm{I} \otimes Y) \arrow[dl, "1\otimes \boldsymbol{l}"] \\ & X \otimes Y & \end{tikzcd}$$

Inspirados por trabajos de Schreier [67], Grothendieck [45], Sinh [69] y Breen [8], entre otros, desarrollamos una teoría de Schreier-Grothendieck 3-dimensional para grupoides monoidales. Concretamente, nuestras conclusiones al respecto se resumen en la existencia de biequivalencias

$$\mathbf{MonGpd} \underset{\Sigma}{\overset{\Delta}{\rightleftarrows}} \mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd},$$

entre la 2-categoría de grupoides monoidales y la 2-categoría de lo que hemos denominado *sistemas de Schreier para grupoides monoidales* o *3-cociclos no abelianos para monoides*. Se trata de sistemas de datos

$$(M, \mathcal{A}, \Theta, \lambda)$$

consistentes en un monoide M, una familia de grupos $\mathcal{A} = (\mathcal{A}(a))_{a\in M}$ (no necesariamente abelianos) indizados por elementos del monoide, una familia de homomorfismos de grupos

$$\Theta = (\mathcal{A}(b) \xrightarrow{a_*} \mathcal{A}(ab) \xleftarrow{b^*} \mathcal{A}(a))_{a,b\in M},$$

y una aplicación normalizada

$$\lambda : M \times M \times M \longrightarrow \bigcup_{a\in M} \mathcal{A}(a) \ \mid\ \lambda_{a,b,c} \in \mathcal{A}(abc),$$

satisfaciendo una serie de axiomas. En esta categoría $\mathbf{Z}^{\mathbf{3}}_{\text{n-ab}}\mathbf{Mnd}$ toda equivalencia es un isomorfismo y por tanto nuestros resultados de clasificación son efectivos.

Una vez alcanzada esta clasificación nos centramos en el caso de grupoides monoidales abelianos, es decir, grupoides monoidales $\mathcal{M} = (\mathcal{M}, \otimes, \mathrm{I}, \boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r})$ cuyos grupos de isotropía $\mathrm{Aut}_{\mathcal{M}}(X)$, $X \in \mathrm{Ob}\mathcal{M}$, son todos abelianos. Aquí los resultados pueden ser expresados de una forma más precisa por medio de la teoría de cohomología de Leech [53]. La cohomología de Leech para un monoide M toma coeficientes en funtores desde la categoría $\mathbb{D}M$ a grupos abelianos, llamados usualmente $\mathbb{D}M$-módulos. Esta categoría $\mathbb{D}M$ tiene por objetos el conjunto de elementos de M y por morfismos el conjunto $M \times M \times M$, con $(a, b, c) : b \to abc$. Así, los grupos de cohomología de M con coeficientes en $\mathcal{A} : \mathbb{D}M \to \mathbf{Ab}$, denotados por $H^n_{\mathrm{L}}(M, \mathcal{A})$, están definidos como los grupos de cohomología de la categoría $\mathbb{D}M$.

Las biequivalencias arriba especificadas se restringen en el caso de grupoides monoidales abelianos a biequivalencias

$$\mathbf{MonAbGpd} \underset{\Sigma}{\overset{\Delta}{\rightleftarrows}} \mathbf{Z}^{\mathbf{3}}\mathbf{Mnd},$$

entre **MonAbGpd**, la 2-subcategoría plena de grupoides monoidales abelianos, y $\mathbf{Z^3Mnd}$, la 2-subcategoría plena dada por aquellos sistemas de Schreier $(M, \mathcal{A}, \Theta, \lambda)$ cuyos grupos $\mathcal{A}(a)$ de $\mathcal{A}$ son todos abelianos. Además, en este caso, el par de datos $\mathcal{A}$ y Θ constituyen un sistema de coeficientes para la cohomología de Leech del monoide M, es decir, un $\mathbb{D}M$-módulo $\mathcal{A}$, y $\lambda \in Z^3_{\mathrm{L}}(M, \mathcal{A})$ es un 3-cociclo normalizado. Gracias a esta observación obtenemos la clasificación de los grupoides monoidales abelianos y de los funtores monoidales entre ellos a través de los grupos de cohomología $H^3_{\mathrm{L}}(M, \mathcal{A})$ y $H^2_{\mathrm{L}}(M, \mathcal{A})$. Estos resultados generalizan los ya obtenidos por Sinh [69] para la clasificación de grupos categóricos.

Aunque estos resultados son de interés principalmente algebraico, nos gustaría indicar su potencial uso en teoría de homotopía ya que, como probamos en el Capítulo 4, existen isomorfismos naturales $H^n_{\mathrm{L}}(M, \mathcal{A}) \cong H^n(\overline{W}M, \mathcal{A})$, entre los grupos de cohomología de Leech de un monoide M y los grupos de cohomología de Gabriel-Zisman [37, Appendix II] del espacio clasificante $\overline{W}M$ de un monoide con coeficientes en $\mathcal{A}$.

En el Capítulo 2 trabajamos con los grupos de homología y de cohomología de Leech. Recordemos que los grupos de cohomología de Leech para un monoide M son los de su categoría $\mathbb{D}M$, es decir, si $\mathcal{A} : \mathbb{D}M \to \mathbf{Ab}$ es un $\mathbb{D}M$-módulo (llamados en este capítulo $\mathbb{D}M$-módulos izquierda), entonces

$$H^n_{\mathrm{L}}(M, \mathcal{A}) = \mathrm{Ext}^n_{\mathbb{D}M}(\mathbb{Z}, \mathcal{A}) = R^n\mathrm{Hom}_{\mathbb{D}M}(\mathbb{Z}, -)(\mathcal{A}) = R^n\mathrm{Hom}_{\mathbb{D}M}(-, \mathcal{A})(\mathbb{Z}),$$

donde, para cualesquiera dos $\mathbb{D}M$-módulos $\mathcal{A}$ y $\mathcal{A}'$, $\mathrm{Hom}_{\mathbb{D}M}(\mathcal{A}, \mathcal{A}')$ denota el grupo abeliano de morfismos de $\mathbb{D}M$-módulos entre ellos, y $\mathbb{Z} : \mathbb{D}M \to \mathbf{Ab}$ es el funtor constantemente el grupo de enteros $\mathbb{Z}$. Análogamente, para $\mathcal{B} : \mathbb{D}M^{op} \to \mathbf{Ab}$ un $\mathbb{D}M$-módulo derecha, los grupos de homología de M con coeficientes en $\mathcal{B}$ [51, Definición 2.1] están definidos como

$$H^{\mathrm{L}}_n(M, \mathcal{B}) = \mathrm{Tor}^{\mathbb{D}M}_n(\mathcal{B}, \mathbb{Z}) = L_n(- \otimes_{\mathbb{D}M} \mathbb{Z})(\mathcal{B}) = L_n(\mathcal{B} \otimes_{\mathbb{D}M} -)(\mathbb{Z}),$$

donde, para cualquier $\mathbb{D}M$-módulo izquierda $\mathcal{A}$, el producto tensor $\mathcal{B} \otimes_{\mathbb{D}M} \mathcal{A}$ es el grupo abeliano definido como el coend del bifuntor $\mathbb{D}M^{op} \times \mathbb{D}M \to \mathbf{Ab}$ que lleva cualquier par $(x, y) \in M \times M$ al producto tensor de grupos abelianos $\mathcal{B}(x) \otimes \mathcal{A}(y)$.

En este capítulo calculamos los grupos de (co)homología de Leech de monoides cíclicos finitos $C_{m,q}$, cuya estructura y clasificación por medio del índice m y el período q fue establecido por primera vez por Frobenius [34]. Aunque los grupos de (co)homología de cualquier grupo cíclico finito son bien conocidos desde que fueron calculados en 1949 por Eilenberg [27], no es así para los monoides cíclicos finitos. De hecho, hasta donde sabemos, los grupos de cohomología de Leech de monoides cíclicos han sido únicamente calculados para el caso infinito (es decir, para el monoide aditivo $\mathbb{N}$ de números naturales), y hasta dimensión 2 para el caso finito por Leech en [53]. Por tanto, puesto que los grupos de cohomología superiores son interesantes para nosotros (principalmente debido a nuestra interpretación del tercer grupo de cohomología en el Capítulo 1), dedicamos este capítulo a calcular todos los grupos de cohomología de cualquier monoide cíclico finito.

En el Capítulo 3 pasamos a trabajar con monoides conmutativos. La categoría de monoides conmutativos es tripleable sobre la categoría de conjuntos [58], y por tanto es natural usar la cohomología del cotriple de Barr-Beck [2] para definir una teoría de cohomología para monoides conmutativos. Esto fue realizado en los años 90 por Grillet [40, 41, 42, 43]. Recordemos que, para cualquier monoide conmutativo M, sus grupos de cohomología en esta teoría, $H^n_{\mathrm{G}}(M,\mathcal{A})$, toman coeficientes en grupos abelianos objetos en la categoría coma de monoides conmutativos sobre M. Dichos grupos abelianos objetos resultan ser $\mathbb{H}M$-módulos, es decir, funtores con valores en la categoría de grupos abelianos desde la categoría $\mathbb{H}M$, categoría que tiene por objetos los elementos del monoide M y por morfismos pares $(a,b): a \to ab$. Como los grupos de cohomología $H^n_{\mathrm{G}}(M,\mathcal{A})$ pueden ser calculados, al menos en dimensiones bajas, por medio de *cocadenas simétricas*, estos grupos son normalmente denominados *grupos de cohomología simétricos* del monoide conmutativo M.

Gracias a estas cocadenas simétricas, en este capítulo interpretamos el tercer grupo de cohomología de Grillet en términos de grupoides monoidales estrictamente conmutativos, es decir, grupoides monoidales abelianos dotados de isomorfismos naturales y coherentes $\boldsymbol{c}_{x,y} : x \otimes y \cong y \otimes x$, satisfaciendo las condiciones $\boldsymbol{c}_{y,x}\,\boldsymbol{c}_{x,y} = id_{x\otimes y}$ y $\boldsymbol{c}_{x,x} = id_{x\otimes x}$. Concretamente, nuestro resultado de clasificación establece que las ternas $(M,\mathcal{A},k)$, donde M es un monoide conmutativo, $\mathcal{A}$ es un $\mathbb{H}M$-módulo, y $k \in H^3_{\mathrm{G}}(M,\mathcal{A})$ es un clase de 3-cohomología simétrica, son los invariantes para la clasificación de grupoides monoidales abelianos estrictamente conmutativos. Esta clasificación generaliza la ya conocida para categorías de Picard estrictamente conmutativas obtenida por Deligne [25], Fröhlich y Wall [36], y Sinh [69].

Hasta ahora hemos trabajado con la teoría de cohomología de Leech para monoides arbitrarios y con la de Grillet para monoides conmutativos. Para un monoide conmutativo M ambos grupos de cohomología $H^n_{\mathrm{G}}(M,\mathcal{A})$ y $H^n_{\mathrm{L}}(M,\mathcal{A})$ están definidos, donde los coeficientes para la cohomología de Leech aquí son obtenidos componiendo $\mathcal{A}: \mathbb{H}M \to \mathbf{Ab}$ con el funtor canónico $\mathbb{D}M \to \mathbb{H}M$, $(a,b,c) \mapsto (b,ac)$. Aunque en dimensión uno ambos grupos de cohomología coinciden, en dimensiones superiores difieren. De hecho, se puede argumentar fácilmente que los grupos de cohomología de Leech no tienen en cuenta la conmutatividad del monoide, al contrario de lo que ocurre con la de Grillet. Por ejemplo, mientras que $H^2_{\mathrm{L}}(M,\mathcal{A})$ clasifica *todas* las coextensiones de grupos de M por $\mathcal{A}$ [53, 2.4.9], [74, Theorem 2], el segundo grupo de cohomología simétrico $H^2_{\mathrm{G}}(M,\mathcal{A})$ clasifica las coextensiones de grupos *conmutativas* [43, Chapter V.4].

Sin embargo, los grupos de cohomología de Grillet parecen ser un poco “estrictos” en dimensiones mayores que dos (por ejemplo, el tercer grupo de cohomología es cero para el caso de un grupo). De ahí que, en los Capítulos restantes 4 y 5, presentemos nuevas aproximaciones a la cohomología de monoides conmutativos, principalmente motivados por el problema de clasificar tanto grupoides monoidales abelianos trenzados como simétricos.

En el Capítulo 4 definimos y estudiamos una nueva teoría de cohomología, consistente en lo que hemos denominado *grupos de cohomología conmutativos para un* *monoide conmutativo.* Para definirla nos hemos inspirado en los grupos de cohomología (segundo nivel) para grupos abelianos de Eilenberg-Mac Lane [31, 55] y está basada en la teoría de cohomología de Gabiel-Zisman [37, Appendix II] para conjuntos simpliciales. Un ejemplo de la cohomología de Gabriel-Zisman es precisamente la cohomología de Leech. En efecto, si vemos un monoide M como un monoide simplicial podemos asociarle un conjunto simplicial clasificante $\overline{W}M$ [31] y para cada $\mathbb{D}M$-módulo $\mathcal{A}$ obtener los grupos de cohomología de Gabriel-Zisman $H^n(\overline{W}M, \mathcal{A})$, que resultan ser los grupos de cohomología de Leech, es decir, $H^n(\overline{W}M, \mathcal{A}) \cong H^n_{\mathrm{L}}(M, \mathcal{A})$. Cuando el monoide es conmutativo entonces $\overline{W}M$ es de nuevo un monoide simplicial y podemos iterar esta construcción obteniendo un nuevo conjunto clasificante $\overline{W}(\overline{W}M)$. Los grupos de cohomología de este conjunto simplicial son usados para definir los *grupos de cohomología conmutativos* de M, denotados por $H^n_{\mathrm{c}}(M, \mathcal{A})$, como

$$H^n_{\mathrm{c}}(M, \mathcal{A}) = H^{n+1}(\overline{W}^2 M, \mathcal{A}),$$

donde $\mathcal{A}$ es un $\mathbb{H}M$-módulo. Por ejemplo, si $M = G$ es un grupo abeliano, como el conjunto simplicial $\overline{W}^2 G$ es un complejo minimal de Eilenberg-Mac Lane $K(G, 2)$, para cualquier grupo abeliano A (visto como un $\mathbb{H}G$-módulo constante) los grupos de cohomología conmutativos $H^n_{\mathrm{c}}(G, A)$ son precisamente los grupos de cohomología de Eilenberg-Mac Lane para el grupo abeliano G con coeficientes en A [31, 55].

Para calcular estos grupos de cohomología hasta dimensión 3 definimos un complejo de cocadenas, truncado en dimensión 4, más manejable que el original y que denominamos complejos de *cocadenas conmutativas.* Gracias a estas cocadenas podemos interpretar estos grupos hasta dimensión 3. En particular, probamos que los grupoides monoidales abelianos trenzados [50], es decir, grupoides monoidales abelianos dotados de isomorfismos naturales y coherentes $\boldsymbol{c}_{x,y} : x \otimes y \cong y \otimes x$ (sin necesidad de satisfacer las condiciones $\boldsymbol{c}_{y,x}\, \boldsymbol{c}_{x,y} = id$ ni $\boldsymbol{c}_{x,x} = id$), son clasificados mediante ternas $(M, \mathcal{A}, k)$ donde M es un monoide conmutativo, $\mathcal{A}$ un $\mathbb{H}M$-módulo y $k \in H^3_{\mathrm{c}}(M, \mathcal{A})$. Este resultado generaliza el dado por Joyal-Street [50] para grupos categóricos trenzados.

Finalmente, en el Capítulo 5, introducimos y estudiamos, para cualquier entero $r \geq 1$, una *teoría de cohomología de r nivel* para monoides. Esta teoría de cohomología de r nivel es una generalización de la teoría de Eilenberg-Mac Lane para grupos abelianos a monoides conmutativos. Los grupos de cohomología de r nivel de un monoide conmutativo M, denotados por

$$H^n(M, r; \mathcal{A}),$$

tienen muchas buenas propiedades, a cuyo estudio este capítulo y un artículo compañero [14] están principalmente dedicados. En nuestro desarrollo, el papel de los coeficientes es jugado ahora por los $\mathbb{H}M$-módulos, que, recordemos, son grupos abelianos objetos en la categoría coma de monoides conmutativos sobre M.

Para cualquier monoide conmutativo M, la categoría de complejos de cadenas de $\mathbb{H}M$-módulos es una categoría abeliana. Ésta categoría abeliana es de hecho una categoría monoidal simétrica con un producto tensor $\mathcal{A}\otimes_{\mathbb{H}M}\mathcal{B}$, donde $\mathcal{A}$ y $\mathcal{B}$ son $\mathbb{H}M$-módulos. Por tanto, podemos definir DGA-álgebras conmutativas sobre $\mathbb{H}M$ como monoides conmutativos internos en la categoría monoidal simétrica de complejos de $\mathbb{H}M$-módulos, enriquecidos con un morfismo de monoides internos $\mathcal{A}\to\mathbb{Z}$ ($\mathbb{Z}$ es el $\mathbb{H}M$-módulo constantemente $\mathbb{Z}$).

De forma similar al caso de DGA-álgebras conmutativas ordinarias sobre un anillo conmutativo, una construcción bar reducida $\mathcal{A}\mapsto\mathbf{B}(\mathcal{A})$ puede ser definida para estas DGA-álgebras sobre $\mathbb{H}M$. En efecto, $\mathbf{B}(\mathcal{A})$ se obtiene a partir de $\mathcal{A}$ totalizando en primer lugar el complejo doble de $\mathbb{H}M$-módulos

$$\bigoplus\nolimits_{p\geq 0}\mathcal{A}/\mathbb{Z}\otimes_{\mathbb{H}M}\overset{(p\text{ factores })}{\cdots}\otimes_{\mathbb{H}M}\mathcal{A}/\mathbb{Z}\,,$$

y después enriqueciendo el (adecuadamente graduado) complejo totalizado de $\mathbb{H}M$-módulos con una estructura multiplicativa por un producto shuffle. Una vez definida esta construcción introducimos los grupos de cohomología de r nivel de $\mathcal{A}$ con coeficientes en un $\mathbb{H}M$-módulo $\mathcal{B}$ como

$$H^n(\mathcal{A},r;\mathcal{B})=H^n\big(\mathrm{Hom}_{\mathbb{H}M}(\mathbf{B}^r(\mathcal{A}),\mathcal{B})\big),\quad n=0,1,\ldots\,.$$

Llegados a este punto definimos lo que son $\mathbb{H}M$-módulos libres, que surgen como una construcción adjunta izquierda al funtor olvido desde la categoría de $\mathbb{H}M$-módulos a la categoría coma de conjuntos sobre el conjunto M. En particular, introducimos el $\mathbb{H}M$-módulo libre sobre la aplicación identidad $id_M: M\to M$, denotado por $\mathcal{Z}M$. Dicho $\mathbb{H}M$-módulo resulta ser además una DGA-álgebra conmutativa sobre $\mathbb{H}M$ (con graduación trivial) y, por tanto, para cada entero positivo r podemos definir los grupos de cohomología de r nivel de un monoide conmutativo M con coeficientes en un $\mathbb{H}M$-módulo $\mathcal{A}$ como

$$H^n(M,r;\mathcal{A})=H^n(\mathcal{Z}M,r;\mathcal{A}).\tag{5.46}$$

Esta teoría de cohomología recupera, en su primer nivel, la cohomología de Leech para monoides conmutativos, y en su segundo nivel la teoría de cohomología conmutativa introducida en el capítulo anterior. En cuanto a los grupos de cohomología de tercer nivel, encontramos entre ellos los invariantes para clasificar *grupoides monoidales abelianos simétricos*, es decir, grupoides monoidales abelianos dotados de isomorfismos naturales y coherentes $\boldsymbol{c}_{x,y}: x\otimes y\cong y\otimes x$ satisfaciendo la condición $\boldsymbol{c}_{y,x}\,\boldsymbol{c}_{x,y}=id$ pero no $\boldsymbol{c}_{x,x}=id$. Es decir, los grupoides monoidales abelianos simétricos son clasificados mediante ternas $(M,\mathcal{A},k)$, donde M es un monoide conmutativo, $\mathcal{A}$ un $\mathbb{H}M$-módulo y $k\in H^5(M,3;\mathcal{A})$. De esta forma completamos la lista de invariantes para las clases de equivalencia de grupoides monoidales abelianos. Este resultado generaliza el obtenido por Sinh [69, II, Proposición 5] para categorías de Picard.

Para terminar, dedicamos la última parte del capítulo a calcular grupos de cohomología de primer, segundo y tercer nivel para monoides cíclicos.

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