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One-sided limits of a function at a point in a drug metabolism context as explained by non-compulsory secondary students

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We present the results of an exploratory and descriptive study performed with Spanish students in Non-Compulsory Secondary Education focusing on how they explain the meaning of both one-sided limits within a temporal phenomenon (drug metabolism) given by a graphical non-authentic model. We organised the given explanations according to the following options: Only calculation; interpretation in a neighbourhood (locally); meaning of the value (pointwise); direction of approaching in time with regard to right-sided limit. We also highlight a particular attention to other elements of the model apart from limiting notions and some difficulty to give sense to right-sided limit, possibly because the direction of approximation is contrary to the natural progression of time.

Keywords: Partial modelling activity, one-sided limits, non-authentic drug metabolism model, spontaneous extension of a model, graphics.

PROBLEM

Mathematical modelling has been integrated in international programmes of students' assessment (e.g., PISA) and it has been a fundamental part of the mathematics education curricula for students aged 6–15 years old for several years (OECD, 2013). In some countries, such as Spain, mathematics teaching in Non-Compulsory Secondary Education (16–17 years old) is intended as a preparation for tertiary studies. However it seems not to keep this trend but rather emphasizes abstract concepts and procedures from advanced mathematical activity (Crouch & Haines, 2004). Successful teachers' training programmes in design and assessment of modelling classroom proposals (Ortiz, Rico, & Castro, 2006) make possible to transfer such proposals to pre-university education.

Mathematical modelling is a field of mathematics education widely explored (Fischbein, 1987; De Lange, 1987; Niss, Blum, & Galbraith, 2007; Swetz, 1991).

This study aimed to explore how students perform a pre-modelled task related to the concept of limit of a function at a point. Concretely, we choose a task contextualised in how a human body removes a drug in two days period, focusing on both one-sided limits of the amount of drug at the moment at which a new dose of drug is introduced.

Since we had previously characterised a misconception about the limit of a function related to the continuity (Fernández-Plaza, Rico, & Ruiz-Hidalgo, 2013a, 2013b), we considered for this study a function with a discontinuity at the limit point.

The outcomes of this study provide a better understanding of students' learning of the concept of finite limit of a function at a point, specifically jump discontinuities where there is an instantaneous change to the function at a point in the domain. We examined students' learning of advanced mathematical content, but also their understanding of continuity as a tool to model real phenomena.

The specific aim we propose for this study is:

"To describe student meanings of one-sided limits of a function at a point as they explore a given graphical model describing a temporal phenomena, and the possible influence of variable time on left-sided and right-sided limits interpretations."

THEORETICAL FRAMEWORK

We provide a brief description of what we understand by a model and how to model. We then establish a relevant distinction between modelling and application, with particular reference to the concept of limit of a function at a point.

Notion of model, applications and modelling procedure

We consider a mathematical model as a mathematical structure that approaches or describes certain relationships within a phenomenon in order to explore, understand, explain and eventually control it (Swetz, 1989). As Fischbein (1987, p. 21) notes, not only physical facts can be modelled but also concepts can be associated with a model and properties of the abstract concept may be better understood from the corresponding model.

Niss and colleagues (2007, pp. 10–11) stress a significant distinction between an application and a modelization. Modelling focuses on finding mathematical knowledge from a certain part of the real world, for example, the cycloid is the model related to the motion of a point in a wheel as it rolls along a straight line without slippage. In contrast, application focuses on the opposite direction. Given a model the problem is finding what parts of the real world are susceptible of being modelled by such a model. For example, the inverted cycloid provides a solution to the Brachistochrone and Tautochrone Problems.

The modelling ability as considered by PISA 2015 draft framework (OECD, 2013) (called *mathematising*) involves capabilities such as:

> To structure the field or situation to be modelled,
>
> To make assumptions
>
> To translate the reality into a mathematical structure
>
> To work on the mathematical model to obtain findings
>
> To reinterpret these findings in terms of the real situation, and
>
> To establish limited or generalized conditions to validate or modify the model.

Greefrath and Riess (2013) summarise the modelling procedure into five steps, Understanding of the problem; approach selection; performing; explanation of results; checking results, calculations and approach. They developed and implemented a solution plan (it consists of these five steps with questions and clarifying points) with 6th grade students. In spite of some students engaged appropriately with this aid, other of them had some difficulties.

We pay special attention to the last two aforementioned capabilities: Interpreting, criticizing and modifying a given model. To sum up, modelling is the process to find a mathematical structure which approaches relationships within a phenomenon, which consists of an understanding of the problem, approach selection (assumptions), performing (translation of the reality to the mathematical structure and work on it to obtain findings), explanation of results (to reinterpret the findings in terms of the real situation) and checking results (to establish limited or generalized conditions to validate or modify the model).

Applications and Modelling related to the concept of limit of a function at a point

Classical problems, which were modelled using the concept of the limit of a function at a point, dealt with movement of an object and variations in magnitudes with respect to time.

According to the distinction between application and modelling, other phenomena may involve relationships between variables and time and the limit concept could be applied. The basic question to which the concept of limit of a function at a point tries to give an answer is the following:

> Given a flow of amount of a magnitude along an interval of time, $y = f(t)$ and an instant $t = t0$, obtain the best approximation L of the amount of magnitude near $t = t0$, providing that $t = t0$ is an accumulation point of the interval of time. If it does not exist, explore the reasons why not. L is the best approximation of $f(t)$ near $t = t0$, if for any K approximation, there exists an instant tK, such that $|t - t0| < |t0 - tK|$ implies $|f(t) - L| < |f(t) - K|$.

This question leads us to four different models related to the concept of finite limit of a function at a point depending on the properties of the function:

Instantaneous invariance model, related to the existence of limit. If the image of the point is the same as the limit, we referred to a *continuous model*, otherwise, it is a *hole model*.

The non-existence of the limit leads to the other models:

Jump or instantaneous change, when there are both one-sided best approximations, but they are different, so there is not a best approximation of the function at any neighbourhood centered at $t = t_0$.

Asymptotic change model, when one or both of the one-sided best approximations do not exist but there is a tendency to plus or minus infinity.

Oscillating change model, when none of one-sided best approximations exist, either finite or infinite.

For this study we are going to consider students' work on a jump model, because the asymptotic one involves infinity and oscillating one is not usually taught at their educational level.

METHOD

This is an exploratory and descriptive study based on a survey method. We designed and implemented a questionnaire including open-ended and closed-ended questions. This paper is focused on the following one:

> A patient is given a 0.05 mg injection of a drug daily, and each day 40% of the drug in the body is eliminated. The following graph (Figure 1) corresponds to the function $y = f(t)$ that relates time to the amount of the drug in the body during the first two days of treatment. Interpret $\lim_{t\to 1^-} f(t)$ (from now on *Lim_Left*) and $\lim_{t\to 1^+} f(t)$ (*Lim_Right*)

The sample was composed of 36 Spanish students in the first year of Non-Compulsory Secondary Education (grade 11th), 16–17 years of age, who were taking Mathematics for the Science and Technology track. The students were chosen deliberately based on their availability.

The survey was administered to the sample described above in the middle of the academic year 2010/2011 during a regular session of the math class (1 hour) counting on the collaboration of the teacher. Subjects had received prior instruction on the concept of limit by their teacher.

The analysis of students' answers (interpretations of left-sided limit and right-sided limit) is based on a *content analysis methodology*. Firstly, characterization of students' interpretation of left and right-sided limits. Secondly, detection of different approaches used to the interpretations of one-sided limits. Finally, detection of spontaneous attempts related to a further analysis of the model.

According to limit models framework, the task describes a phenomenon with a jump instantaneous change. However, the real phenomenon is continuous. According to Andresen (2007, p. 2044) is a non-authentic model. We discarded the time taken to inject the drug which is very small in comparison with the unit of the variable t (days), otherwise the function would seem to have a vertical line at $t = 1$.

It is important to note that the task does not consider the whole modelling procedure, because a model is given beforehand and students only have to interpret the value of the one-sided limits according to the provided model. However, our results show that some students spontaneously focused on other aspects of the model and tried to develop or modify it. So we argue that in part they were doing modelling activity and therefore bringing into play the two last capabilities according to PISA 2015 framework.

RESULTS

We describe the interpretations provided by the students of left-sided and right-sided limits, also different

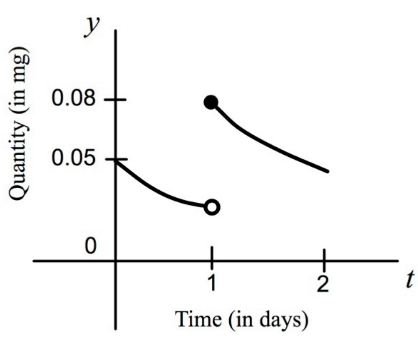

Figure 1: Graphical model of the situation described on the text

approaches to conceive them, as well as the modelling actions of students apart from the requirements of the task.

Students' interpretations of left-sided limit from the model

The different interpretations of the left-sided limit can be organised into these main categories, developed from answers, considering two dimensions: Information included from the graph and Contextualization. Answers in "Other Category" related to each dimension are incomplete or vague.

Dimension 1. *How much information from the graph is included*

Only the value of the left-sided limit. Some answers only provide the specific value of the left-sided limit (Example 1).

Example 1. " = 60% of 0.05 which is 0.03"

Local / pointwise interpretation. Some answers describe the behaviour of the function around the point t=1 (locally) (Example 2), particularly using specific terms such as, "to approach," "to tend," or "to get closer and closer," or focusing on the possibility of the limit to be reached, while other ones focus exclusively on the "left side" of the point t=1 making explicit the meaning of the left-sided limit value (pointwise) (Example 3).

Example 2. "The limit as t tends to 1 from the left represents how the amount of drug is decreasing along the day and at the end of the day increases by 0.05 mg due to the injection"

Example 3. "To know the remaining amount of drug in the body after the first dose"

Dimension 2. *Contextualization*

Contextualized. Students interpret the meaning of left-sided limit in terms of the real situation. The clearest interpretations used infinitesimal expressions such as "just before" or "before" (Example 4), or "at the moment drug has been removed" (Example 5).

Example 4. "To know amount of drug that patient has before 40% is removed"

Example 5. "As time goes on, body is removing drug until the moment in which a 40% has already been removed"

Decontextualized. Students interpret the meaning of left-sided limit in a purely mathematical context, not in the real situation (Example 1).

Table 1 shows the frequencies of each interpretation category. These categories are mutually exclusive.

Dimension 1	Frequencies (N=36)
Only value	2
Local	19
Pointwise	10
No answer/Other	5

Table 1: Frequencies of interpretations of Lim_left related to Dimension 1

Table 2 shows the frequencies of contextualized and decontextualized interpretations of left-sided limit.

Dimension 2	Frequencies (N=36)
Contextualized	29
Decontextualized	4
No answer/Other	3

Table 2: Frequencies of interpretations of Lim_left related to Dimension 2

Students' interpretations of right-sided limit from the model

The different interpretations of the right-sided limit can be organised into the aforementioned categories, but there is a new singular category:

To relate the tendency of t to 1 from the right to stepping back in time. Some students become aware that tending to 1 from the right implies "counting" backwards in time (Example 4). However, we note the contrary fact in the example 5.

Example 4. "As we get close to 1 from the right, we see that the drug has just been injected and the patient has not eliminated any amount of drug"

Example 5. "If we take limit from the right, the approximate amount of drug will tend to 0 mg"

Table 3 shows the frequencies of each interpretation category. These categories also are mutually exclusive:

Dimension 1	Frequencies (N=36)
Only value	1
Local / Tendency as stepping back in time	7
Pointwise	11
Tendency as stepping forward in time	13
No answer/Other	4

Table 3: Frequencies of interpretations of Lim_Right related to Dimension 1

An example of contextualized interpretation of right-sided limit is Example 6.

> Example 6. "As t approaches 1 from the right we see drug has just been injected and the patient has not removed any amount of drug"

Table 4 shows the frequencies of contextualized and decontextualized interpretations of right-sided limit.

Dimension 2	Frequencies (N=36)
Contextualized	28
Decontextualized	6
No answer/Other	2

Table 4: Frequencies of interpretations of Lim_Right related to Dimension 2

Different approaches of left-sided and right-sided limit interpretations

Even though students clearly identified the value of the left-sided limit, we consider that there are two different approaches when the right-sided limit is interpreted: (a) *formal*, according to the formal notion of right-sided limit, and (b) *contextual*, according to

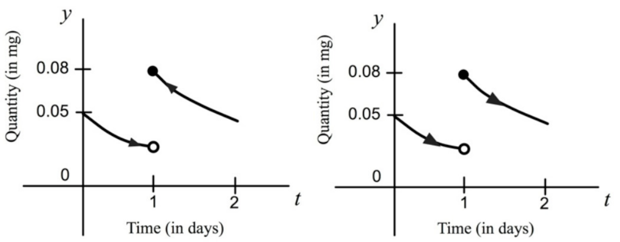

Figure 2: Formal (left) and contextual (right) direction of approximation

the natural development of the phenomenon along the time, as expressed in Figure 2 and Example 5.

Table 5 shows the frequencies of formal and contextual approaches related to students' pairs of interpretations (N= number of formal left-sided limit direction of approximation).

Approach	Frequencies (N=30)
Formal	19
Contextual	11

Table 5: Frequencies of formal or contextual pairs of interpretations

Unexpected students' attempts of further analysis of the model

Students spontaneously focused on other aspects of the provided model (8 out of 36), such as:

To set an absolute start-end of the day. 2 out 8 students arbitrarily do not consider the day as a measure of time between two instants of time, but like a day in the calendar (from 00:00 a.m. today to 00:00 a.m. next day) (Example 8).

To consider that the total amount eliminated during a day is constant for every day. 2 out of 8 students do not consider that the 40% is taken out of the current amount of drug in the body, but of 0.05 mg dose, so the eliminated amount of drug is constant just like the length of the jump (Example 9 and Figure 3).

To extend the model to other days. 7 out of 8 students generalised the model to next days (inductive reasoning, Example 9). Some of them (3 out of 7) consider as well that the velocity of elimination is increasing day to day, because the amount of drug to remove in the same interval of time (1 day) is higher (Example 10), but in fact the phenomenon reaches a stationary behaviour around 0.05 mg of drug eliminated per day.

Discussion on the arbitrary setting of the hole in the graph. Only one student discussed the arbitrary place of the hole in the graph, i.e., the image of the function at t =1, because the limit is independent of what happens at the point t =1. He stressed the finite jump (Example 11).

> Example 8. " [Lim_Left] For example, let us suppose that the injection is administered at 0:00 a.m. As the time goes the amount of drug in the

body is decreasing. 24 hours later the amount has reached 0.03 mg…"

Example 9. "[Lim_Left] The patient eliminates 0.02 mg per day. [Figure 3 provided]"

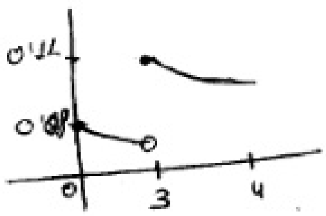

Figure 3: Inductive reasoning from the model (Example 9)

Example 10. "[Lim_Right] After 24 hours on the next day amount has increased by 0.05 from the remaining 0.03, that is 0.08. So, each day 1, 2, 3… there would be a jump and step by step the patient will have a higher amount of drug in his body"

Example 11. "[Lim_Right]…it has been set the hole at the end of the first day and the image at the beginning of the second one, but it could be done anyway. There is a finite jump because the one-sided limits are different"

Finally, given the similarity between right-sided limit at $t = 2$ (0.048 mg) and the amount of drug at $t = 0$ (0.05 mg), Example 12 could be interpreted as establishing by the student the equality between both values graphically rather than by calculation.

Example 12. "We observe that the patient takes the drug and so the amount in the body increases, along the day, the amount of drug is reducing until the same level when the first dose was administered."

DISCUSSION AND CONCLUSIONS

According to the aim we draw the following conclusions:

The pointwise interpretations of both one-sided limits have been slightly frequent (10 and 11 out of 36), so we can tell that some students have a routine procedural conception of both one-sided limits. On the other hand, the local interpretations are more frequent for the limit from the left (19 out of 36) than are those from the right (7 out of 36), possibly due to the natural progression of time that produced some conflicts of interpretations as is shown by the answers (11 out of 36). Such a kind of conflicts was reported by other studies, such as Blázquez (2000). For further research, new examples with independent variable different from time could be chosen.

It is important to mention the spontaneous references to other aspects of the model (8 out of 36), such as the invariance of the daily-eliminated amount, the gradual increment of the amount of drug in the body in the future as well as the velocity of elimination. Only one student suggested that the image of $t = 1$ could be either 0.03 or 0.08. These actions are characteristic of modelling activity (OECD, 2013).

Surprisingly, no student discussed about the necessity of taking into account the time employed to inject the drug in order to check the continuity of the real phenomenon. For further research, no model would be provided in order to elicit students' own proposals and specific classroom proposals could be planned according to PISA recommendations.

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