Precision Physics in High Energy Colliders: New Techniques of Calculating Radiative Corrections

Benjamin John Page

Universidad de Granada

Departamento de Física Teórica y del Cosmos
University of Granada
Spain
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Director/es de la Tesis

Doctorando

Fdo.: Roberto Pittau

Fdo.: Ben Page

Abstract

In this thesis we make two investigations in the context of precision physics, firstly in the multi-leg and secondly in the multi-loop frontier. In the first part, we approach the problem of making automated NLO phenomenological studies of gluon fusion driven Higgs production. Here we calculate the R_2 vertices necessary to use the effective theory of Higgs/gluon interactions at one loop in MADGRAPH5_AMC@NLO for both a scalar and pseudoscalar coupling. This has allowed for a study of the CP properties of the Higgs/top-quark coupling where it is found that correlations between the di-jet azimuthal angles in Higgs + 2 jets provide valuable CP information. In the second part we work in the framework of the Four Dimensional Regularization (FDR) approach, where we develop two-loop translation rules between FDR and dimensional regularization in massless QCD in the form of a coupling constant shift. To achieve this we set up a framework for comparison between the two strategies and develop an algorithm for the automation of the FDR defining expansion at two loops. During the investigation we find that a naive global prescription in the scheme breaks locality and universality in correlation functions with external fermions and so we introduce a "sub-prescription" to deal with this. In this way we solve a problem analogous to the breakdown of unitarity at two loops in the four dimensional helicity method and provide explicit evidence of the consistency of FDR at two loops. Furthermore we gain insight into the relation between the FDR approach and a canonical counterterm picture, witnessing a direct cancellation between FDR sub-vacua and counterterms.

Física de Precisión en los Colisionadores de Alta Energía: Nuevas Técnicas de Cálculo de Correcciones Radiativas

Benjamin John Page

Universidad de Granada

Departamento de Física Teórica y del Cosmos
Universidad de Granada
España
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Abstracto

En esta tesis, llevo a cabo dos investigaciones en el contexto de física de precisión, primero en la frontera "multi-leg" y segundo en la frontera "multi-loop". En la primera parte, considero el problema de poder hacer estudios fenomenológicos automatizados NLO de producción del boson de Higgs dominada por la fusion de gluones. Aquí calculo los vértices necesarios para utilizar la teoría efectiva de interacciones del boson de Higgs y gluones para cálculos de 1-loop en MADGRAPH5_AMC@NLO para casos de acoplamiento escalar y pseudoescalar. Esto ha permitido el estudio del las propiedades de CP del acoplamiento del quark top y boson de Higgs, donde se encuentra que las correlaciones entre los ángulos azimutales en el sistema de di-jet en la producción de Higgs + 2 jets proporcionan información valiosa.

En la segunda parte, trabajo con el método de "Four Dimensional Regularization" (FDR) donde desarrollo reglas de transición de dos-lazos entre FDR y regularización dimensional en QCD sin masas en la forma de un cambio de constante de acoplamiento. Para lograr esto, construyo un método para comparar los dos estrategias y desarrollo un algoritmo para la automatización del "FDR defining expansion" a 2-loops. Durante la investigación encuentro que una "global prescription" ingenua en el esquema rompe la localidad y la universalidad en funciones de correlación con fermiones externos y por eso introduzco una "sub-prescription". De esta manera, soluciono un problema que es análago al experimentado en el método de "Four Dimensional Helicity" donde se rompe la unitaridad y proporciono pruebas explícitas de la consistencía de FDR a 2-loops. Además mi análisis ayuda a entender la relación entre el método FDR y un punto de visto canónico de contratérminos ya que muestra una cancelación directa entre contratérminos y "FDR sub-vacua".

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I would like to thank all those who have supported me through my studies. To Roberto - thank you for answering many stupid questions. To Mum, Dad and Laura, thanks for always being there to talk to. To Maria, I'm not sure I could have done it without you.

In loving memory of David Ronald Page.

Contents

Chapter 1

Introduction

Modern particle physics is the culmination of a long quest for an improved understanding of the nature of the universe in which we live. In searching for insight into the mechanisms which drive and the components which make up our world we have been compelled to investigate things on an increasingly small scale. The most adept tool we have for this search is the high energy particle collider, currently well represented by the Large Hadron Collider (LHC) at CERN in Geneva, Switzerland. Many years of investigation have led to the conclusion that the small scale structure of our universe is governed by the rules of quantum field theory (QFT) and that all collider data can be explained, within experimental limits, by the Standard Model (SM). Yet, when we move outside the world of colliders we find a number of phenomena whose explanation remains elusive. The hierarchy problem, dark matter and dark energy, the quantum nature of gravity, neutrino masses and the strong CP problem are just some of the indications that our understanding of the universe is incomplete.

Nevertheless, in spite of this evidence of the incompleteness of our picture, all current collider data is consistent with the theoretical predictions given by the SM. As we push the energy and luminosity frontiers at the LHC it is expected that new physics will emerge from small, but statistically significant discrepancies between the theoretical predictions of the SM and experimental measurement. It is then the work of the model builder to try to solve the problems of our incomplete picture whilst remaining within the tight constraints of SM data. If there truly are new physics signals hiding in the experimental statistics then it is important that we have very precise SM predictions with which to compare them. In this thesis we are going to introduce the need for such "precision physics" and make two dif-

ferent investigations into problems experienced in such work. Firstly, in this chapter we outline the necessity of such work, the relevant physics, and the typical technology that one employs. Chapter 2 is our first investigation into the practical problems of making predictions, where we perform necessary calculations to make automated "loop corrections" in an effective theory of Higgs/gluon interactions. In Chapter 3 we lay out the FDR method of calculation within perturbative QFT, and in chapter 4 we perform work to be able to make translations between the DR$/\overline{MS}$ scheme and FDR. We present our conclusions in chapter 5.

1.1   Precision Physics at the LHC

The Large Hadron Collider (LHC) is the largest, highest energy particle collider currently available. The LHC is a proton collider, set to begin its second run at a centre of mass energy of 13 TeV. These proton collisions often result in the creation of many particles not in the initial state and the precise measurement of the properties of these outgoing particles allows one to elucidate the underlying dynamics. Accordingly, the LHC is a machine reaching historically unprecedented levels of experimental precision and in order to undertake new physics searches one must be able to match the experimental precision on the theory side. In practice, this proves a more tricky proposition than perhaps naively thought as the current state of QFT is that it is not possible to compute observables 100% precisely, and so one must use an approximation. Perturbative Quantum Field Theory (pQFT) provides such an approximation which is well understood in a way which is possible to systematically improve. It prescribes that all observables can be given as perturbative series in the relevant coupling constants such that one can truncate the series in order to reach a certain precision. The orders of approximation are commonly referred to as leading order (LO), next-to-leading order (NLO), next-to-next-to-leading order (NNLO) etc. Over a number of decades this has been shown to be a valid and useful strategy in situations of weak coupling, where the perturbative series parameter is small.

Of the fundamental forces, the one that presides at the LHC is Quantum Chromodynamics (QCD), as the machine is a proton collider. Naively this could cause problems as at low energies QCD is a strongly coupled theory. However, as QCD is asymptotically free, at LHC energies a large amount of the physics can still be described in the framework of pQFT. Nevertheless, at these energies the strong coupling is of $O(0.1)$ and so the task of making

theoretical predictions is notoriously difficult as to reach an appropriate level of approximation, one must move beyond leading order in the perturbative series. This is made increasingly tough at the LHC as there are many jets, and so it is necessary to be able to describe processes with a large number of outgoing particles.

For this reason, the technical difficulty of this task precludes the knowledge of any desired observable to arbitrary precision and so the community is forced to focus its efforts in improving perturbative approximations. Currently, the field of Higgs physics makes strong demands of precision physics. In measuring Higgs observables one must have strong control over the production cross-section which is predominantly driven by gluon fusion. Ever since the calculation of the larger than expected NLO corrections it has been known that the perturbative expansion converges slowly in this channel. For this reason a great deal of effort has been put into improving the theoretical precision, recently culminating in the calculation of the inclusive cross section to N^3LO [1].

When one considers $pp \rightarrow WZ$ it is also found that there is some degree of tension between theory and data, in this case all measured cross sections are bigger than the theory predictions. Similarly for W^+W^- production there was recently tension as combined CMS and ATLAS results placed the cross section at 3.5σ beyond the theory predictions [2, 3, 4]. For this reason, studies of weak boson production have been performed including NNLO results for both ZZ and W^+W^- [5, 6].

Also in recent history, top physics has been a driving force for precision calculation. Historically, the Tevatron top forward backward asymmetry has been an outstanding experimental deviation from SM predictions, but recently this has been weakened through the calculation of NNLO corrections [7]. In general, the large $t\bar{t}$ cross section makes it of great importance to have precise predictions [8].

Additionally, the understanding of the non-perturbative QCD phenomena at the LHC is greatly improved through improved NNLO techniques. Jets at hadron colliders are ubiquitous and important to study due to their sensitivity to the strong coupling, parton distribution functions (PDFs) and physics beyond the standard model (BSM). The high experimental precision available from the LHC again begets the need for NNLO studies [9]. What's more, the study of PDFs themselves is required at the LHC due to their large contribution to the overall theoretical uncertainties, and so efforts must be undertaken to extract them from NNLO predictions [10, 11].

1.2 Quantum Chromodynamics at Colliders

At the LHC as one collides protons we are immediately brought into the world of Quantum Chromodynamics. QCD, as it is more commonly known, is the currently accepted theory for describing the strong interactions which govern the proton, a bound state of the model. The strength of its coupling make it firmly different to weakly coupled theories at low energies, but it displays an important property known as asymptotic freedom which allows it to be described in the same framework as weakly coupled theories in colliders. Whilst the coupling strength at modern collider energies is small enough to be handled perturbatively, it is large enough that the slow convergence of the series causes programs of precision physics to be dominated by QCD calculations. For this reason, the discussion of this thesis focuses around QCD.

QCD is the special case of Yang-Mills theory [12] with $N_f = 6$ fermions connected by an $SU(N_c = 3)$ gauge symmetry. The covariantly gauge fixed Lagrangian of the model is

$$
\begin{aligned}
\mathcal{L}_{\text{QCD}} = \overline{\psi}_j^f \left(i\slashed{D}_{jk} - m_f \right) \psi_k^f &- \frac{1}{4} G^{\mu\nu a} G_{\mu\nu a} + \partial_\mu \bar{c}^a \partial^\mu c^a \\
&+ g f^{abc} \left(\partial^\mu \bar{c}^a \right) G_\mu^b c^c - \frac{1}{2\xi} \left(\partial_\mu G^{\mu a} \right).
\end{aligned}
\tag{1.2.1}
$$

The index f runs over our N_f quark flavours each with different masses. The quarks themselves transform in the fundamental representation of $SU(N_c)$, whereas the gluon field $G^{\mu a}$ transforms in the adjoint representation and so the gauge covariant derivative is given by

$$
D_{jk}^\mu = \partial^\mu - ig G^{\mu a} t_{jk}^a,
\tag{1.2.2}
$$

and the field strength tensor is non abelian, such that

$$
G_{\mu\nu a} = \partial_\mu G_{\nu a} - \partial_\nu G_{\mu a} - g f_{abc} G_{\mu b} G_{\nu c}.
\tag{1.2.3}
$$

The matrices t^a are the generators of $SU(N_c)$ and the coefficients f_{abc} are the completely antisymmetric "structure factors" of the group. The fields c^a are Fadeev-Popov ghosts and are necessary to perform the gauge fixing covariantly. In contrast to QED, where the ghosts decouple, the non-abelian nature of QCD renders them a necessity. In the Standard Model all of the quarks have different, independent masses. However it is often useful to consider the limit $m_f = 0$ for calculational purposes. The Feynman rules for this model can be found in appendix A.1.

At low energies, QCD is strongly coupled i.e. $g >> 1$. For this reason, if one wishes to describe the low energy phenomena of QCD the perturbative toolbox is not applicable and currently the most promising results are found in lattice calculations [13], however at collider energies all is not lost. It is well known that when one renormalizes a quantum field theory the effective value of the coupling changes with the characteristic energy scale that is being probed. In QCD this running of the coupling means that the coupling decreases with increasing energy, such that as the energy scale increases arbitrarily the coupling flows to 0. This is known as asymptotic freedom, and at the LHC it is the case that the coupling is weak enough to be worked perturbatively as for example at the Z mass [14],

$$\alpha_S(m_Z) = 0.1190 \pm 0.0012 \text{ (68\% C.L.)}. \tag{1.2.4}$$

1.3   Regularization

In QCD, as in many quantum field theories, when one attempts to move beyond leading order in the perturbative expansion it is discovered that the naively constructed theory is inconsistent and calculations give infinite results. The canonical solution to this problem is to define a "regulated" theory through which one can recover the original theory by taking an appropriate limit. One is then able to perform all necessary calculation in the regulated theory, with the hopes of recovering physical results by taking this limit. Over the years many different methods of regularization have been invented such as cut off regularization, Pauli-Villars [15], dimensional regularization [16] and more recently differential renormalization [17, 18, 19, 20], implicit regularization [21, 22] and (to be discussed in this thesis) FDR [23]. Though incomplete, this list is quite long as it has proven difficult to construct regularizations that don't obscure properties of the original, unregulated theory.

Gauge theories such as QCD have historically proven especially difficult to regulate as many regulators will break the gauge invariance of the theory. At the level of amplitude calculations, the gauge invariance of the theory manifests in the form of Ward-Takahashi identities [24]. Schematically these can be thought of as relationships between correlation functions with differing numbers of legs. For gauge theories these are a necessary requirement for the unitarity of the theory, and so if the regulated theory is not gauge invariant it is inconsistent. An important condition needed for the proof of the Ward identities in gauge theories is that the loop integrals are defined in

such a way that two integrals related by a shift of the integration momentum are identical, this is known as "shift invariance".

Cut-off regularization is conceptually quite simple, especially when considered in the context of effective field theory, but is unsuitable for use in any physical theory because it breaks both shift and gauge invariance. Pauli-Villars regularization was originally used in the context of QED, where it maintains gauge invariance, but was unsuitable for more general, non-abelian gauge theories. Here, dimensional regularization was the first method which respects gauge and shift invariance, and for this reason it is heavily used in modern precision physics.

In practice, when making predictions for colliders, the most common tool is dimensional regularization. Principally this is because, when working in gauge theories, it has been the simplest available tool which has respected all symmetries of the necessary theories. It also has the practical advantage of regulating both IR and UV infinities with the same parameter, thereby reducing the complexity in calculations. Since its invention, a number of sister techniques have been developed in order to solve particular difficulties presented in the standard formulation. For example, in supersymmetric theories it is required that the number of degrees of freedom of fermionic and bosonic states coincide, however, when one moves away from 4 dimensions, this no longer holds. In 1979, the method of dimensional reduction [25] was invented to solve this issue by changing the number of dimensions through compactification thereby preserving the relation between fermionic and bosonic degrees of freedom. More recently the four dimensional helicity method, FDH [26], was developed in order to facilitate the use of on-shell helicity methods within unitarity based loop calculations.

1.4   Theory Meets Experiment: Cross Sections

When working within particle physics, wishing to connect theory to data, the standard tool is the cross section (or sometimes the related branching ratio). It is a way to express the likelihood of interaction between particles within the context of scattering experiments. For asymptotic incoming and outgoing states the general form for a $2 \rightarrow n$ cross section where the centre of mass energy is much greater than the incoming masses is given by

$$\sigma(s) = \frac{1}{2s} \int \left[\Pi_{i=1}^{n} \frac{d^3 q_i}{(2\pi)^3 2E_i} \right] \left[(2\pi)^4 \delta^4 \left(\Sigma_i k_i^{\mu} - (p_1 + p_2)^{\mu} \right) \right] |\mathcal{M}|^2. \quad (1.4.1)$$
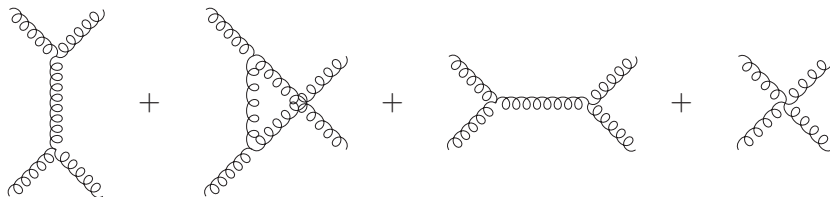
Here, p_1 and p_2 are the momenta of the incoming particles, $s = (p_1 + p_2)^2$ and k_i are the momenta of the n outgoing particles. The first factor in

the integral gives the density of states for the final state particles and the second enforces momentum conservation between the incoming and outgoing particles. The final factor $|\mathcal{M}|^2$ is the squared matrix element for the process. At the LHC, the incoming states are protons, which are strongly bound states of QCD and so one cannot naively use (1.4.1). Fortunately, at high enough energies, the asymptotic freedom of QCD results in the validity of the so-called *factorization formula*,

$$\frac{d\sigma}{dX} = \sum_{j,k} \int_{\hat{X}} f_j(x_1) f_k(x_2) \frac{d\hat{\sigma}_{j,k}}{d\hat{X}} F(\hat{X} \to X). \qquad (1.4.2)$$

This expresses that we can calculate the cross section from the collision of two bound states (e.g. protons) by convolving the partonic cross sections ($\hat{\sigma}_{j,k}$) with the parton densities (f_j) and a function F which contains the hadronization physics[1]. The partonic cross sections can again be calculated perturbatively, but the PDFs and hadronization effects are of a non-perturbative origin and must be determined from data as no practical tools exist to compute them from first principles. In this thesis our focus lies in techniques for the computation of these partonic cross sections.

When computing a cross section, phase space integration is usually handled numerically and most of the difficulty lies in the evaluation of the matrix element. Typically this calculation can only be handled perturbatively, usually through the Feynman diagram expansion. The leading order (LO) approximation is given in terms of tree level Feynman diagrams, which can be converted into mathematical expressions through the use of Feynman rules. Consider, for instance, the process $gg \to gg$, given at tree level by the diagrammatic expression
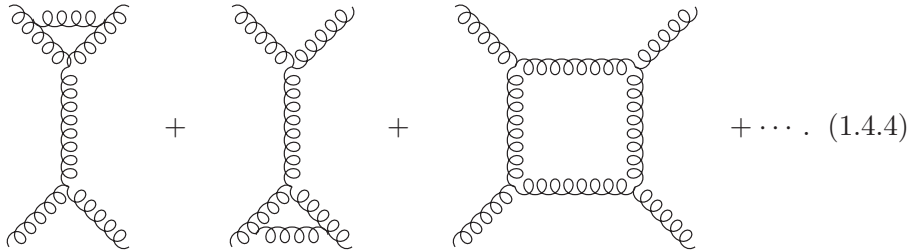

$$(1.4.3)$$

This is purely gluonic at tree level, and very important at the LHC due to the high gluon flux. Naively, the next order of approximation is given by

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[1]For final states which do not involve hadrons, F is 1.

loop diagrams, i.e.

$$+ \cdots . \quad (1.4.4)$$
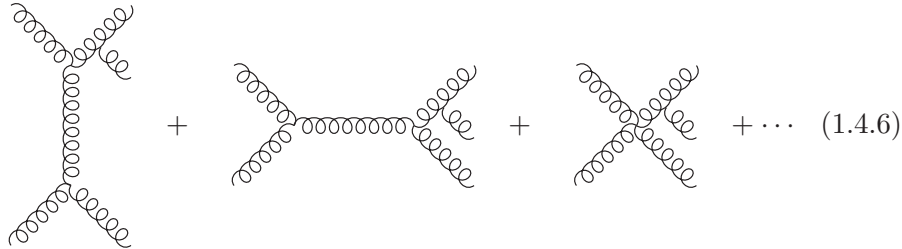
Here we have not listed the full expression as it contains 92 terms. In a loop diagram, the unconstrained degree of freedom is integrated over. For a covariant method of gauge fixing, e.g. via Faddeev-Popov ghosts, all of these integrals are given in terms of rational functions of tensor structure and local propagators, for example

$$\int \mathrm{d}^4 q \frac{q^\mu}{q^2 (q + k_1 + k_2)^2} \quad \text{or} \quad \int \mathrm{d}^4 q \frac{1}{q^2 (q - k_1)^2 (q + k_2)^2}. \quad (1.4.5)$$

Looking at the unregulated integrals in equation (1.4.5) we can see how the need for regularization manifests at the cross section level - the two presented integrals are not well defined as they are *divergent*. As it turns out, these divergences are the result of a naive treatment failing to correctly incorporate important physical effects. The first integral contains a UV divergence, as the integral does not go to zero fast enough for large q^2 and so the theory needs to be renormalized. The second integral, for on-shell kinematics displays an IR divergence. The origin of this divergence is that when there are massless fields in the theory, a $2 \rightarrow 2$ process is indistinguishable from a $2 \rightarrow 3$ process with a sufficiently soft/collinear outgoing massless particle [27, 28]. That is, beyond tree level, the definition of $2 \rightarrow 2$ is not a priori well defined, and a good definition is "sufficiently inclusive" - this is known as IR safety. In practice these infinities are removed in two separate ways. Firstly we must add contributions from the cross section of the $2 \rightarrow 3$ where one cannot resolve the extra parton. This is known as the real contribution[2] and requires the calculation of diagrams

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[2]In contrast, the one-loop diagrams of equation (1.4.4) are known as the virtual contribution.

such as

$$+\cdots \quad (1.4.6)$$

where again we have not listed the full expression as it has 25 terms. Secondly, to remove the collinear poles due to initial state radiation, one absorbs them into a renormalization of the PDFs. In order to be able to perform this in any practical way, one employs a regularization procedure as described in section 1.3. When coupled with an appropriate method of renormalization, all IR safe observables are well defined and can be calculated by virtue of equation (1.4.2).

In practice, this discussion needs to be greatly extended to become workable. When making computations with a large number of legs, even at tree level, the number of Feynman diagrams which one is required to draw grows exponentially. Similarly, when we move to to higher loop orders, these calculations involve a very large number of terms. This is a natural consequence of the perturbative expansion, but it is exacerbated in gauge theories such as QCD due to self interactions and the ghost fields. Furthermore, the Feynman rules for each interaction are complicated and lead to large analytic expressions for any given diagram, made even more complicated by the QCD colour structure. Overall it is a hard problem to even generate the required amplitudes. For this reason, it is more often the case that the generation of the amplitudes is performed using more sophisticated techniques than Feynman diagrams. At tree level recursion relations such as BCFW [29, 30], Dyson-Schwinger and Berends-Giele [31] provide valuable tools for the generation of amplitudes. Beyond tree level, techniques such as Open Loops [32] attempt to generalize the recursion relations to NLO calculations.

The generation of the amplitudes aside, it is typically of great difficulty to perform the integrals found in higher order calculations. In the context of loop integrals, one finds a very large number of difficult Feynman integrals to evaluate. The standard strategy to compute amplitudes composed of loop integrals is to reduce them to a smaller set of "master integrals" which one then directly computes. For multi-leg or multi-loop processes this reduction is often computationally complex but in the last few years at NLO the practical possibilities have been greatly improved by the development

of numerically efficient methods such as OPP [33, 34] and unitarity inspired techniques [35, 36, 37, 38, 39]. At higher orders, the situation is increasingly more complex and the whilst there exist general techniques, such as generalizations of OPP/unitarity together with integration by parts relations in various forms of the Laporta algorithm [40], they are not yet efficient enough. The other key component is the evaluation of the master integrals. For one loop integrals the basis of master integrals is known and all relevant integrals have been calculated. However, at higher orders the situation is not so simple - integral bases for all final state multiplicities have not been found, and many relevant Feynman integrals prove difficult to compute.

The other technical difficulties lie with the real radiation. The picture previously presented of the infra-red divergences cancelling between real and virtual is practically difficult to realize in a numerical way as the divergences arise differently. In the real contribution, the divergence occurs when integrating over a certain region of phase space, whereas in the virtual part the divergences arise in the evaluation of the matrix element. To make these two cancel, one commonly adds a part to the real, and subtracts the same part from the virtual in such a way that the divergences manifestly cancel separately in each contribution. Consider the situation for an m body cross section at NLO

$$\sigma^{\text{NLO}} = \int_m d\sigma^B + \int_m \left(d\sigma^V + \int_1 d\sigma^A \right) + \int_{m+1} \left(d\sigma^R - d\sigma^A \right). \quad (1.4.7)$$

Here σ^{NLO} is the NLO cross section, $\int_n$ is the n body phase space integral, $d\sigma^V$ is the differential virtual cross section, $d\sigma^R$ is the differential real cross section. The crucial part is $d\sigma^A$, an approximation to the singular behaviour of $d\sigma^R$. This is subtracted from the real part directly, and then integrated over the phase-space of the unresolved parton and added to the virtual part. The main difficulty is the construction of an approximation that can be easily analytically integrated in order to cancel the IR poles in $d\sigma^V$.

A number of strategies to write down these IR counterterms exist. Those commonly implemented in NLO generators are Catani-Seymour subtraction [41] and FKS [42]. Beyond NLO the construction of the counterterms proves more intricate and whilst there are a number of available procedures such as Antenna subtraction [43], it is not currently clear which strategy is most successful. Nevertheless, with a given strategy it is possible to perform such phase space integration numerically.

Chapter 2

Automating NLO Corrections to gg $\rightarrow$ H $+ n$ Jets

2.1 Gluon Gluon Fusion and Effective Theories

As discussed in section 1.1 it is important to identify physics processes which require higher theoretical precision in order to allow experiment to make discriminating claims about deviations from the SM. Since the discovery of a SM-like Higgs boson it has been of importance to investigate the properties of the Higgs boson at the LHC in order to elucidate the nature of Electroweak symmetry breaking. Efforts in this direction require very precise determination of its production.

At the LHC, the main process driving Higgs production is that of gluon-gluon fusion (the first diagram in figure 2.1). At first glance this is not to be expected - gluon gluon fusion is a loop induced process and so naively one would expect it to be small. However, a variety of factors conspire to make this the dominant production channel. Firstly, relative to other production mechanisms the process is enhanced by the top Yukawa coupling which is of $O(1)$ due to the top being so massive. Secondly, this production mechanism directly produces a Higgs without requiring the creation of any other particles (as is the case in $gg \rightarrow Ht\bar{t}$), and so has a comparitively low energy cost. Finally, the gluon flux is high at the LHC - i.e. the gluon PDF dominates at the LHC.

Recently, a lot of effort has been put into pushing the loop frontier for this production channel and the fully inclusive production process has been

calculated at N^3LO [1] [1]. In tandem, it is also of importance to advance the limits of multi-leg calculations. This has been our motivation for the work presented in this chapter on automating the NLO corrections to $gg \rightarrow H + n$ jets.

For such an important process, the fact that it is loop induced makes the act of obtaining a quantitive prediction more difficult. In order to calculate even the leading order one must undergo a loop calculation. A common technique to overcome this difficulty is that of effective field theory (EFT). This tool allows us to discard irrelevant degrees of freedom when performing calculations and reduce the complexity of our calculations. In the case of gluon fusion, it is of benefit to employ an EFT which has a direct coupling between the Higgs and gluon fields.

To understand this, let us roughly illustrate how we arrive at this EFT from the SM. In the SM the interactions between the Higgs boson and gluons are mediated by a fermion loop giving diagrams such as those in figure 2.1. Importantly, the coupling of the Higgs to a fermion is proportional to its mass and so the largest contribution to a Higgs/gluon transition amplitude is due to diagrams with a top loop, as it is by far the most massive quark at $\sim 173\text{GeV}$ [44]. Schematically, as long as the higgs mass meets the constraint $m_h < 2m_t$, where m_t is the top mass, then it is fair to consider an expansion where the top mass is larger than the other scales in the problem, i.e. taking $m_t \rightarrow \infty$. If we calculate the amplitudes of Higgs to n gluons in the limit of a large top mass, we see that only the three diagrams in figure 2.1 have a non-zero contribution.
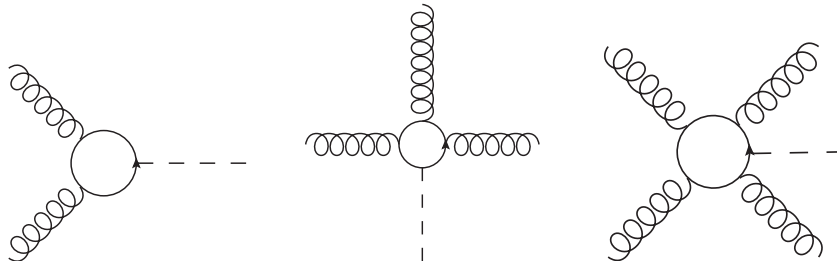

Figure 2.1: The contributing 1 loop diagrams linking an incoming Higgs to outgoing gluons in the infinite top mass limit.

Let us sketch an argument to understand this. As the top mass is the

[1]Indeed, this is the first process relevant to hadron colliders to have been calculated at N^3LO.

only scale in the integrals, the diagrams must behave as $\sim m_t^{d+1}/M_w$, where d is the mass dimension of the diagram and there is an extra m_t/M_w factor coming from the $t\bar{t}H$ Yukawa coupling. This implies that if the mass dimension of the diagram is less than -1, the diagram will vanish in the limit $m_t \to \infty$. The mass dimension of the diagram with two external gluons is $+2$, and with every additional gluon leg we have another fermion propagator in the calculation which reduces the dimension by 1. We can therefore discard diagrams with five external legs or more.

This suggests the properties of the EFT found by integrating out the top quark it is an EFT with a direct Higgs/gluon field coupling. The full EFT that describes these interactions is given by the following Lagrangian [45, 46]

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}g_H H G^a_{\mu\nu}G^{a\mu\nu}, \tag{2.1.1}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, H is the Higgs field and g_H is the Wilson coefficient, known to N^4LO [47] but here given to $O(\alpha_S^2)$

$$g_H = \frac{\alpha_S}{3\pi v}\left(1 + \frac{11}{4}\frac{\alpha_S}{\pi}\right) + O(\alpha_S^3). \tag{2.1.2}$$

The Feynman rules for this model correspond to a Hgg, $Hggg$ and $Hgggg$ vertex which we give in appendix A.2. Within the correct kinematic region we are now able to use this effective field theory to use tree level technology to calculate the leading order effects. What's more, we can now perform calculations that would previously require two loop technique using only one-loop methods.

This technology readily allows us to study Higgs processes through improved control over the perturbative corrections to its production. Further to this, we can also probe the nature of the Higgs/top coupling with a similar method. Whilst the Standard Model predicts that the Higgs boson is a scalar field, other models predict that it may also couple to the top quark in a pseudo-scalar like way. We can understand this in the same framework through a pseudo-scalar effective Lagrangian [48]

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}g_A A G^a_{\mu\nu}\tilde{G}^{a\mu\nu}, \tag{2.1.3}$$

where the dual field strength tensor $\tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma}$, A is the pseudoscalar field and g_A is the effective coupling, given exactly to all orders in α_S by [49]

$$g_A = -\frac{\alpha_S}{2\pi v}. \tag{2.1.4}$$

The Feynman rules for this Lagrangian are also given in appendix A.2.

With this in hand we can state the aim of this chapter of the thesis. We wish to undertake all necessary calculations to be able to perform multi-leg computations at NLO in HEFT complete with the pseudo-scalar in an automated fashion using programs such as MADGRAPH5_AMC@NLO [50] or HELAC-NLO [51]. In the following sections we will discuss the limiting factor - virtual corrections, and how they are mastered with the OPP method. Beyond this we will present our work on computing the last required component to be able to perform automated computations with the OPP method - effective R_2 Feynman rules.

2.2 The Problem of Virtual Corrections

Over the last 20 years, many strides have been made towards the ultimate goal of being able to automatically perform NLO^2 computations in order to boost the available precision of QCD dominated observables for hadron colliders such as the LHC. As discussed in section 1.4 there are two principal components of this style of calculation, each of comparable complexity - the virtual and real emission parts. Frameworks for the automation of real emission were developed quite early on [42, 52, 53, 41, 54, 51], but only recently has the automation of the virtual part received the necessary boost to be able to automatically treat high multiplicity processes at NLO.

The principal issue with the virtual part is its complexity. In any observable relevant to a hadronic collider, one must deal with high final state multiplicities. Firstly, this means that we will be faced with a large number of Feynman diagrams. Secondly, in the language of computing virtual corrections, this implies that each diagram will have a large number of legs and we must be able to deal with this. Multi-leg calculations within the framework of QCD are of great complexity, principally due to the Feynman rules. For any given diagram with a large number of legs, these lead to a very complicated numerator structure in the loop integrals. In principle, the ideas necessary for computing the loop integrals have been well established for a number of years - one exploits the Lorentz covariance of the integrals [55], together with the fact that one can span the momentum basis with external loop momenta [56] to reduce tensorial loop integrals to a small set of so-called "master integrals"[3]. At one loop this basis of master integrals is simply four scalar integrals - the box, triangle, bubble and tadpole as given

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[2]Including also LO loop induced processes.

[3]This process is explored in more depth in section 4.3.1.
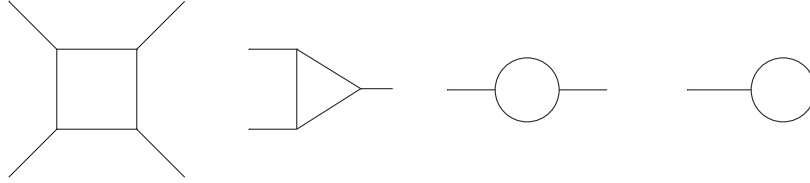
Figure 2.2: The four master integrals in one loop calculations - box, triangle, bubble and tadpole graphs.

in figure 2.2. However, in practice, the original analytical techniques are not well suited for multi-leg processes as the complexity rises. For example, even the rank-4 box creates a large amount of algebra when computed in Passarino-Veltman reduction.

This led to the search for methods which could better leverage current computing power. Implicitly this meant solutions that did not perform this manipulation analytically, but numerically. Around 2007, two methods arose to solve this problem conclusively - unitarity inspired methods and the OPP method. Both approaches take advantage of analytical properties of the loop integrals in order to perform a reduction to the scalar basis. The principal difference between the two techniques is that unitarity methods work by manipulating the integrals, whereas the OPP method works at the integrand level. The work presented in this chapter of the thesis is a study of a component that must be computed separately in order to apply the OPP method, within the context of physics programs such as MadGraph5_aMC@NLO or HELAC-NLO, to the study of Higgs $+n$ jets processes at the LHC.

2.3   The OPP Method

The Ossola, Papadopoulos, Pittau (OPP) method is a technique for the rapid reduction of one-loop tensorial integrals to a linear combination of the one-loop master integrals[4]. In this section we will introduce the OPP method in order to motivate the specific calculation we undertake in this chapter.

The insight of the OPP method is that it is possible to perform reduction of one-loop amplitudes directly at the integrand level. In a nutshell, one can rewrite the numerator of a dimensionally regulated integrand in terms

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[4]See figure 2.2.

of a linear combination of products of denominators *plus* a piece which is "spurious" - vanishes upon integration. Let us illustrate what this means with a generic DR (sub-)amplitude. We begin by placing everything over a common denominator, writing the integrand as

$$A(\bar{q}, n) = \frac{N(\bar{q}, n)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}, \quad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2, \qquad (2.3.1)$$

where all barred objects are explicitly n-dimensional, $\bar{q}$ is the loop momenta and p_i are linear combinations of external momenta. The OPP reduction makes use of four dimensional algebra, and so one must separate out the $(n-4)$ dimensional parts of the numerator, and recover them later. We write this splitting as

$$N(\bar{q}, n) = N(q, 4) + \tilde{N}(q, \tilde{q}, \epsilon), \qquad (2.3.2)$$

where q and $\tilde{q}$ are the 4 and $(n-4)$ dimensional parts of $\bar{q}$ respectively. In general we will use a similar notation throughout this discussion, representing quantities in n-dimensions with bars, $(n-4)$ dimensions with tildes and leaving unmarked the objects living in 4-dimensions. The $\tilde{N}$ part gives a necessary contribution called R_2 which we leave to section 2.4.1 to discuss.

The OPP method tells us that the 4-dimensional numerator of the integrand $N(q, 4)$, can be written as

$$
\begin{aligned}
N(q, 4) = &\sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3} D_i \\
&+ \sum_{i_0 < i_1 < i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2} D_i \\
&+ \sum_{i_0 < i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0} D_i \\
&+ \tilde{P}(q) \prod_i D_i.
\end{aligned}
\qquad (2.3.3)
$$

To understand this equation, consider placing it into the expression for $A(\bar{q}, n)$. If we temporarily ignore the difference between barred and unbarred denominators we see that the terms without tildes, a, b, c and d in

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equation (2.3.3), correspond to tadpole, bubble, triangle and box integrals respectively. Their tilded counterparts, $\tilde{a}, \tilde{b}, \tilde{c}$ and $\tilde{d}$, still depend on the loop momenta q, however, the dependence is such that these terms are spurious and vanish after integration. Furthermore, this dependence is universal and was determined once and for all in the original OPP paper [33].

The difference to this picture that is caused by the numerator being in terms of 4 dimensional D_i and the denominator in terms of the n dimensional $\bar{D}_i$ can be understood by employing the following identity

$$\frac{1}{\bar{D}_i} = \frac{1}{D_i}\left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right). \tag{2.3.4}$$

This tells us that the naive cancellation between numerator and denominator occurs, but we also create extra terms, known as the R_1 contribution, which we will discuss in section 2.4.1.

All of the information in the amplitude is then encoded in the coefficients a, b, c and d, and OPP gives a method to extract these coefficients. Notably, these coefficients correspond to *all* of the "cut constructible" information. That is, information which is recoverable through its analytic structure because it contains logarithms or other functions with branch cuts. After some work, one finds that the parts R_1 and R_2, collectively known as R, are polynomials in scales of the problem and are known as "rational parts". The method of extracting these coefficients which OPP employs is to take advantage of the simplifications of the system of equations which occur when one chooses a value of q for which some denominators are 0. This simplifies the determination to a triangular system of equations from which one can easily obtain the coefficients. An automated implementation of the method is available in the package CutTools [34]. Together with an efficient method for generating the numerator function $N(q, 4)$, such as that of OpenLoops or HELAC-NLO, this provides an extremely rapid way of performing one-loop virtual calculations.

In the following sections we will discuss rational parts within the context of the OPP method and the difficulties these cause for the automation of one-loop calculations.

2.4   Rational Parts

In the discussion of section 2.3 we note how the OPP method allows rapid access to the cut constructible part of an amplitude, but fails to account for the so-called "rational parts". It should be noted that these contributions

arise in any attempt to perform integral reduction and are not unique to OPP. Before we examine how one recovers these rational parts in the OPP method, let us give a clear demonstration of what they are and why they are present.

In any given virtual amplitude calculation which is complicated enough to require the use of integral reduction techniques, one necessarily finds contributions to the expressions which, when taking the $n \to 4$ limit, cannot be expressed as a numerical constant pre-multiplying a master integral. To demonstrate this, let us take a specific example and perform the classic technique of Passarino-Veltman reduction [55]. Over the course of this short calculation we will find a constant term due to a cancellation of the form ϵ/ϵ, where $\epsilon = (n-4)$. Consider the following tensor integral

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{\bar{q}^\mu \bar{q}^\nu}{\bar{q}^2 (\bar{q}+p)^2} = A \bar{g}^{\bar{\mu}\bar{\nu}} + B p^\mu p^\nu. \qquad (2.4.1)$$

Here the barring is to make explicit the objects that live in n dimensions. The next step is to construct two equations to determine the coefficients A and B by contracting with $g^{\bar{\mu}\bar{\nu}}$ and p^μ. Crucially, as the numerator $\bar{q}$ are n dimensional objects, when we perform contraction with $g^{\bar{\mu}\bar{\nu}}$ we must recognise that this is an n dimensional metric such that $g^{\bar{\mu}\bar{\nu}} g_{\bar{\mu}\bar{\nu}} = n$. The two equations which we construct are then

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{\bar{q}^2}{\bar{q}^2 (\bar{q}+p)^2} = nA + p^2 B$$
$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{(q \cdot p) \bar{q}^\mu}{\bar{q}^2 (\bar{q}+p)^2} = A p^\mu + B p^2 p^\mu. \qquad (2.4.2)$$

Now we perform the standard method of reconstructing the numerators in terms of the denominators and repeating the process for any remaining tensor integrals[5]. This allows us to write a matrix equation for the coefficients A and B,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{(n-1)p^2} \begin{pmatrix} p^2 & -p^2 \\ -1 & n \end{pmatrix} \begin{pmatrix} 0 \\ I_1 \end{pmatrix}, \qquad (2.4.3)$$

where

$$I_1 = \frac{p^2}{4} \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\bar{D}_0 \bar{D}_1}. \qquad (2.4.4)$$

The rational part turns up when taking the limit $n \to 4$. The integral I_1 contains a pole in ϵ. In calculating A and B we have multiplied I_1 by a

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[5]This is covered in more detail in section 4.3.1.

function of $n = 4 + \epsilon$, and so when we take the limit $\epsilon \to 0$ these leave a finite contribution of the form ϵ/ϵ. This finite contribution is proportional to p^2 - a rational part.

2.4.1 Rational Parts in OPP

Now that we understand the origin of rational parts when computing one-loop amplitudes, we shall discuss how they arise in the OPP method. In this context as we shall see it turns out that in OPP there is a natural splitting of the rational terms into two categories - R_1 and R_2. As it turns out, it is possible to automatically calculate R_1 within the OPP framework, however R_2 requries special, theory dependent attention.

Type 1 - R_1

R_1 can be thought of as the consequence of cancelling a numerator computed in 4 dimensions with a denominator that exists in n-dimensions. In order to explain this, let us take a simple example integrand

$$A = \frac{D_0 D_1}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3}. \tag{2.4.5}$$

Here we can imagine that an OPP treatment has reconstructed the numerator, in 4-dimensions, as $N(q) = D_0 D_1$. To proceed, we apply equation (2.3.4) leading to the naive (i.e. ignoring bars) result, along with a number of integrands with $\tilde{q}^2$ in the numerator

$$A = \frac{1}{\bar{D}_2 \bar{D}_3} - \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_2 \bar{D}_3} - \frac{\tilde{q}^2}{\bar{D}_1 \bar{D}_2 \bar{D}_3} - \frac{\tilde{q}^4}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3}. \tag{2.4.6}$$

The integrands with $\tilde{q}^2$ in the numerator are R_1 parts.

In principle this method can be used to extract any R_1 part, however it is not practical for calculating R_1 on a large scale - we have only presented it as an effort to make its origin clearer. The computation of R_1 in a practical sense is easily automated because one can recover these terms by shifting the mass of all propagators by $\tilde{q}^2$. In this way, one can find the coefficients of all relevant R_1 integrals in a method very similar to that of OPP [57]. As this is a solved problem, we move onto the topic of relevance for this chapter - R_2.

Type 2 - R_2

R_2 is the second category of rational part encountered in the OPP method. At present, no automatic methods exist for its calculation that are inequivalent to simply computing the component by hand or computing the full theory for $n \neq 4$. As we shall see, the R_2 component can be fully recovered by calculating a small set of tree-level Feynman rules. This takes place in a theory dependent way, and so in order to be able to use the OPP method with a new theory, one must undertake this calculation. This chapter describes our calculation of these rules for HEFT complete with a pseudoscalar as described in section 2.1.

In order to be able to perform this calculation, let us describe how to extract the R_2 part in some detail. We begin by remembering that the OPP method is based upon a treatment of the numerator using 4-dimensional algebra. However, the numerator function in dimensional regularization exists in n dimensions and so this treatment misses the part of the numerator which lives in the $(n - 4)$ dimensional space. Schematically we can see the origin of the R_2 contribution by writing the integrand as

$$A(\bar{q}, n) = \frac{N(\bar{q}, n)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}} = \frac{N(q, 4)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}} + \frac{\tilde{N}(q, \tilde{q}, \epsilon)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}. \quad (2.4.7)$$

Here we take the integrand of a one-loop amplitude defined in n dimensions with n-dimensional loop momenta $\bar{q}$. The original numerator function, $N(\bar{q}, n)$, is defined explicitly in terms of these two objects. This is because, in DR, the gauge fields live in n dimensions and so when constructing the expressions corresponding to Feynman diagrams one must compute any fully contracted metrics as $\bar{g}^{\bar{\mu}\bar{\nu}} \bar{g}_{\bar{\mu}\bar{\nu}} = n$ [6]. The OPP treatment effectively separates the numerator on the LHS into a purely 4-dimensional part, which can be captured by the numerical treatment, and an $\epsilon = (n - 4)$ dimensional part which cannot be. Note that the 4 dimensional part of the RHS, $N(q, 4)$ is the exact same function N, but evaluated with $\bar{q} = q$, $n = 4$. Explicitly then, the R_2 component of $A(\bar{q}, n)$ is given by the second term of the RHS of equation (2.4.7), i.e.

$$R_2 \left[A(\bar{q}, n) \right] = \frac{\tilde{N}(q, \tilde{q}, \epsilon)}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}. \quad (2.4.8)$$

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[6] The FDH flavour of dimensional regularization allows us to take this n as 4, so as we shall see the $(n - 4)$ component is independently gauge invariant. However, it turns out that this is a quirk of one loop calculations as this treatment is inconsistent at higher loop orders [58].

In order to be able to fully understand how to calculate this R_2 component, we need a precise definition of how to separate $N(\bar{q}, n)$ into it's 4 and ϵ dimensional parts.

This separation is achieved, at a practical level, by considering which objects live in 4 dimensions, and which live in n dimensions. We work in the 't Hooft-Veltman (TV) scheme of dimensional regularization. In this scheme, all *external* objects are considered to live in 4-dimensions, whereas *internal* objects live in n dimensions. We can understand this in a simple manner by realizing that this amounts to constructing a virtual amplitude in n dimensions and then demoting everything with an uncontracted Lorentz index to 4 dimensions. For example

$$\int \frac{\mathrm{d}^n \bar{q}}{\mu_R^\epsilon} \frac{\bar{q}^2 \bar{q}^{\bar{\mu}} \bar{q}^{\bar{\nu}}}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \rightarrow \int \frac{\mathrm{d}^n \bar{q}}{\mu_R^\epsilon} \frac{\bar{q}^2 q^\mu q^\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2}, \qquad (2.4.9)$$

where we have denoted objects living in n dimensions with a bar, and objects living in 4 dimensions without. We can see this to be valid as these uncontracted indices indicate that we are to contract with a four dimensional object, be it a momentum or possibly an external wave function. This is clear because a vector which lives in $(n-4)$ dimensions will always be orthogonal to one living in 4. More explicitly for any 4-dimensional vector v it holds that

$$\bar{q} \cdot v = q \cdot v, \quad v^\mu \bar{g}_{\bar{\mu}\bar{\nu}} = v_\nu, \quad \bar{\gamma}^{\bar{\mu}} v_\mu = \gamma^\mu v_\mu. \qquad (2.4.10)$$

Therefore, as the external indices are always to be contracted with a 4-dimensional object, we never need to treat the $(n-4)$ dimensional part, so the objects with uncontracted indices can be considered in 4 dimensions. It is important to realize that any occurrence of a squared loop momentum, when constructed from Feynman rules, always exists in n dimensions, that is it is a $\bar{q}^2$. Furthermore, any internally contracted gamma matrices and metrics live in n dimensions, such that

$$\bar{\gamma}^{\bar{\mu}} \bar{\gamma}_{\bar{\mu}} = \bar{g}^{\bar{\mu}\bar{\nu}} \bar{g}_{\bar{\mu}\bar{\nu}} = n. \qquad (2.4.11)$$

With a clear understanding of how to construct the amplitudes in n dimensions we can now discuss how to separate out the R_2 part. At a conceptual level, what we wish to do is separate the $(n-4)$ dimensional part from the 4 dimensional part. That is, we would take our amplitude and, in the numerator, apply the identities[7]

$$\bar{q} = q + \tilde{q}, \quad \bar{\gamma}_{\bar{\mu}} = \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}, \quad \bar{g}_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} + \tilde{g}_{\tilde{\mu}\tilde{\nu}}. \qquad (2.4.12)$$

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[7]The $(n-4)$ dimensional gamma matrices freely anti-commute with the 4 dimensional ones.

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Here we represent objects living in n dimensions with a bar, 4 dimensions with no barring and $(n-4)$ dimensions with a tilde. The R_2 part is now any integral which has a numerator that contains a part that lives in $(n-4)$ dimensions. It should be noted that, in order to be able to perform the integration, we also rewrite the integral measure as

$$\mathrm{d}^n \bar{q} = \mathrm{d}^4 q \, \mathrm{d}^\epsilon \tilde{q}. \qquad (2.4.13)$$

There are other ways extracting R_2 which are equivalent to a splitting of objects into 4 dimensional and $(n-4)$ dimensional parts at the *beginning* of the calculation. Indeed, to ease the calculation in section 2.5, we perform this separation differently.

A priori it could be the case that these R_2 parts are a purely analytical piece which must always be calculated independently for any number of legs. Fortunately, one can show that the entire R_2 contribution can be recovered by using effective R_2 Feynman rules. The calculation needs to be performed once and for all for a given theory and this approach has indeed been used for a number of theories, including QCD [59] and the Electroweak model [60]. The reason for this simplicity is that R_2 is an effect of UV origin, i.e. it vanishes when the loop which one is computing is UV finite. At one-loop it is quite simple to show that with an increasing number of legs, the amplitudes under consideration become more and more UV finite. This can be understood by simple power counting arguments[8]. The degree of divergence is related to the mass dimension of the integrals, such that once the mass dimension of the integrals is less than 0 they converge. In a renormalizable theory one can show that when a loop has more than four legs the mass dimension of all integrals is indeed less than 0, and therefore there is no R_2 for more than four legs. In the case of HEFT, which we will consider in the next section, the logic is similar and we find that there are no R_2 contributions for more than five legs. This argumentation implies that the R_2 contribution can be fully captured by a series of effective Feynman rules with (for HEFT) three to five external legs.

2.5 R_2 for HEFT + Pseudoscalar

As we have discussed in section 2.4.1, if we wish to use the OPP method for HEFT with a pseudo scalar we must calculate the R_2 Feynman rules for the theory. Once this calculation is performed, HEFT will be open to

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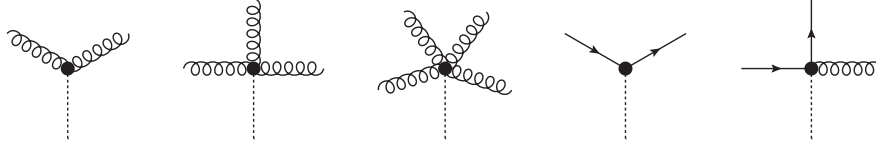[8]These are explained in detail in section 3.2.4.

Figure 2.3: Feynman diagrams representing the relevant processes to which we calculate the R_2 part of the NLO QCD corrections. As we work in the theory of HEFT + pseudoscalar we compute the R_2 component with both the dotted line as the Higgs and as the pseudoscalar.

NLO studies in frameworks such as MADGRAPH5_AMC@NLO or HELAC-NLO. These R_2 vertices that we compute come from the NLO corrections to a number of processes. Specifically, we will need to compute R_2 for Higgs/Pseudoscalar to gluons and to quarks. The relevant processes for which we will need to compute the R_2 component of the one loop QCD corrections are given in figure 2.3.

In order to calculate the R_2 parts of the NLO corrections to the diagrams of figure 2.3, we will need to develop some technology, which we do in the following sections. Firstly we will need to be able to construct the relevant Feynman diagrams to our calculation. For the specific calculation of the R_2 part of the QCD corrections to $H \to gg$ we list the diagrams in figure 2.4 [9]. In general, drawing all of the Feynman diagrams is non-trivial as we increase
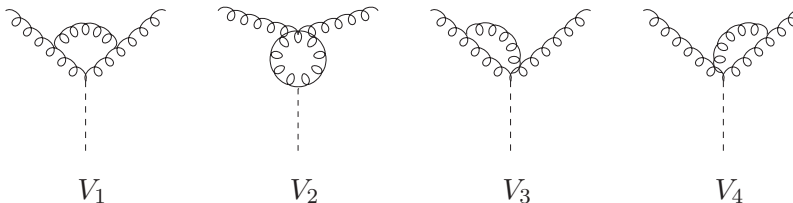

Figure 2.4: The four diagrams that contribute to the 1-loop corrections to the Higgs-Gluon-Gluon vertex in the Higgs effective field theory. V_3 and V_4 belong to the same topology.

the number of legs and so we employ the program QGRAF [61] to generate

[9]We omit the diagram which is zero in dimensional regularization due to scalelessness.

them. The next step is to convert the diagrams into analytic, dimensionally regulated expressions. To do this we need the Feynman rules for both HEFT and QCD which are given in appendices A.2 and A.1 respectively, which we implemented in a FORM [62] code.

In the following sections we shall discuss the methodology which we employed to calculate the R_2 component of the one loop QCD corrections to the processes in figure 2.3. In summary we:

1. Perform the colour algebra.

2. Perform gamma algebra to put integrals into a regular form.

3. Perform Feynman parameterization.

4. Extract the R_2 contribution from the now simple integrals.

5. Perform the loop momentum integration.

6. Integrate all Feynman parameters.

2.5.1   A Naive Approach to Colour Algebra

The first challenge that we face is to be able to thoroughly understand the colour structure of our problem. What's more, all of the discussion here will be useful later on in chapter 4. The colour structure, built as dictated by the Feynman diagrams, can often be reduced to much simpler structures. In this way, the complexity of our calculation decreases as parts which previously did not communicate are brought together.

In our calculations we are working with the $SU(3)$ colour algebra, however it is often more illuminating to work considering a more general $SU(N_c)$ algebra. One benefit is that during the calculation one often finds contributions which are independent due to their structure in N_c, which can aid in cross checks. In this section, our aim is to reduce the strings of colour structure that we find to a minimal basis. For example, one would expect the following style of relationship

$$c^{m_1 n_1 n_2} c^{m_2 n_1 n_2} = A \cdot \delta^{m_1 m_2}. \tag{2.5.1}$$

That is, this combination of colour factors should reduce to a form proportional to the Kronecker delta, as it is the only structure available within the algebra. In general, and especially at two-loop in chapter 4, we will meet long strings of colour factors with repeated, summed over indices which we will wish to reduce to a simpler form.

To see how colour structure arises in our diagrams we look to the feynman rules in appendix A. We see that within the bosonic Feynman rules, the colour structure always arises as structure factors c^{abc}. Therefore, in our calculation of the R_2 Feynman rules with external gluons, we only ever need consider strings of structure factors. However, as we also discuss fermions lines, we notice that the generators of the $SU(N_c)$ algebra, t^a, also enter our calculation and so we need to be able to deal with these as well. These matrices are in the fundamental representation, and so are N_c by N_c, traceless matrices. Here we present a number of properties of these matrices which will prove crucial in our task of simplifying the colour structure of our diagrams

$$[t^a, t^b] = i c^{abc} t_c. \tag{2.5.2}$$

$$t^a_{ij} t^a_{kl} = \frac{1}{2} \left[\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right], \tag{2.5.3}$$

$$\text{Tr}(t^a t^b) = \frac{\delta^{ab}}{2}. \tag{2.5.4}$$

Note that for a given generator t^a_{ij}, the indices i and j are in the fundamental representation and the index a is in the adjoint representation. The anticommutation relation, equation (2.5.2), provides the link between the structure factor and the generator matrices. Equation (2.5.3) describes the completeness relation obeyed by the generators. Finally we have equation (2.5.4), which describes the normalization of the generators[10]. As it turns out, we can combine the commutation relation and equation (2.5.4) to write the structure factor c^{abc} in terms of the generating matrices, i.e.

$$c^{abc} = -2i Tr([t^a, t^b] t^c). \tag{2.5.5}$$

With this information in hand, we are now able to state our naive approach to colour algebra. In general, finding good methods for manipulating the colour structure is a topic under study, however during our calculations there is no need for such advanced treatments. For this reason we make no suggestion that the following methods will scale in any reasonable way. To reduce colour factors we:

1. Use equation (2.5.5) to write the colour factors as traces of generators.

2. Use the completeness relation, equation (2.5.3), to remove any pairs of generators with the same adjoint indices.

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[10]Often in the literature this normalization is parametrised with a coefficient T_f, so in our scheme, $T_f = 1/2$.

3. Use the relations for the trace of a product of generators (equation (2.5.4)) and a single (traceless) generator to further simplify the result.

During this process, one will often meet dirac deltas with repeatedly summed over fundamental indices. As these represent the dimensionality of the matrices they are given by N_c, i.e. $\delta_{ii} = N_c$. It is important to remember to write the traces in component form in order to fully exploit the generator completeness relation, i.e.

$$\text{Tr}(A) = A_{ii}, \tag{2.5.6}$$

where A is a matrix in colour space.

A few examples of results easily found with this naive application of colour algebra are

$$c^{m_1 n_1 n_2} c^{m_2 n_1 n_2} = N_c \cdot \delta^{m_1 m_2}, \tag{2.5.7}$$

$$c^{m_1 n_1 n_2} c^{m_2 n_2 n_3} c^{m_3 n_3 n_1} = \frac{N_c}{2} \cdot c^{m_1 m_2 m_3}. \tag{2.5.8}$$

2.5.2 Gamma Algebra

The next step which we must undertake is to peform any necessary gamma algebra. In the context of the vertices with external gluons this section is inert, but for external fermions the discussion of the following sections would be incorrect without taking into account the gamma matrices. As stated in section 2.4.1 we may take multiple approaches to extract the R_2 component of our amplitudes. The simplistic approach which we take in section 2.5.4 is only possible because all n dimensional objects are easily collected. In the context of the one loop corrections with fermion lines this is not the case because the gamma matrices stop this. Specifically we find strings of gamma matrices separating n dimensional objects

$$\left(\cdots \bar{\gamma}_{\bar{\mu}} \gamma^{\nu_1} \cdots \gamma^{\nu_n} \bar{\gamma}^{\bar{\mu}} \cdots \right) \quad \text{or} \quad \left(\cdots \bar{\slashed{q}} \gamma^{\nu_1} \cdots \gamma^{\nu_n} \bar{\slashed{q}} \cdots \right). \tag{2.5.9}$$

In order to separate the n dimensional objects explicitly we can use the anti-commutation relation,

$$\left\{ \bar{\gamma}^{\bar{\mu}}, \bar{\gamma}^{\bar{\nu}} \right\} = 2 \bar{g}^{\bar{\mu}\bar{\nu}}, \tag{2.5.10}$$

to bring the two $\bar{\gamma}^{\bar{\mu}}$ and $\bar{\slashed{q}}$ together. This then allows us to use the identities

$$\bar{\gamma}^{\bar{\mu}} \bar{\gamma}_{\bar{\mu}} = n, \tag{2.5.11}$$

$$\bar{\slashed{q}} \bar{\slashed{q}} = \bar{q}^2. \tag{2.5.12}$$

Once this has been performed it is simple to access all relevant n dimensional objects for the computation of the R_2 part. It is sufficient to separate out only the $\bar{q}^2$ terms as, strings containing only a single $\bar{\slashed{q}}$ do not make a contribution to the R_2 part.

2.5.3 Feynman Parameterization

At this point we perform a Feynman parameterization of our integrals. It is not typical to perform the Feynman parameterization early on in a loop computation as one introduces a large number of difficult Feynman parameter integrals. However, due to the nature of our calculation, in the following sections we see that the Feynman parameter integrals which we perform are quite simple.

We perform the parameterization as it greatly aids in the momentum integration which we perform in the following section. Feynman integrals, even scalar ones, are not trivial to perform because of the momentum scales that we find in the denominator. In practice it is best to employ a method which exploits as much symmetry as possible to perform the integration. The angular dependence that one finds in the integrand often complicates this. We can see that the origin of this dependence is the dot product found in the denominator

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 = \bar{q}^2 + p_i^2 + 2q \cdot p_i - m_i^2. \tag{2.5.13}$$

Feynman parameterization solves this problem by combining all of the denominators in a linear fashion, at the expense of introducing a new integration variable for each denominator. The result is a denominator structure which is a perfect square and so a shift in the integration momentum removes all angular dependence of the integral. Let us consider this process with a general, potentially tensorial Feynman integral with numerator $f(\bar{q})$, we perform Feynman parameterization as

$$I = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{f(\bar{q})}{([\bar{q} + p_0]^2 - m_0^2) \cdots ([\bar{q} + p_{r-1}]^2 - m_{r-1}^2)} \tag{2.5.14}$$

$$= (r - 1)! \int [dx] \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{f(\bar{q})}{\left(\sum_{i=0}^{r-1} x_i([\bar{q} + p_i]^2 - m_i^2) \right)^r}, \tag{2.5.15}$$

where, without loss of generality, we use n dimensional $\bar{q}$. Here $[dx]$ means that we integrate over the space of Feynman parameters. That is, there are r Feynman parameters each integrated from 0 to 1 under the constraint that

they all sum to one. More explicitly we can write this as

$$\int [dx] = \int_0^1 \mathrm{d}x_0 \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{r-1}\, \delta \left(1 - \sum_{i=0}^{i=r-1} x_i \right). \qquad (2.5.16)$$

Expanding out the denominator and using the delta function constraint to set the prefactor of $\bar{q}^2$ to 1 we can see that our expression is equal to

$$(r-1)! \int [dx] \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{f(\bar{q})}{\left(\bar{q}^2 + \sum_i \left\{ 2x_i p_i \cdot q + x_i [p_i^2 - m_i^2] \right\} \right)^r}. \qquad (2.5.17)$$

Now it is apparent that we can complete the square, suggesting we shift the denominator in order to remove the dot products, $\bar{q} \to \bar{q} - \Pi$ such that

$$I = (r-1)! \int [dx] \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{f(\bar{q} - \Pi)}{(\bar{q}^2 - \chi^2)^r}, \qquad (2.5.18)$$

$$\chi = \sum_i x_i (m_i^2 - p_i^2) + \sum_{i,j} x_i x_j\, p_i \cdot p_j, \qquad (2.5.19)$$

$$\Pi = \sum x_i p_i. \qquad (2.5.20)$$

2.5.4 One Loop Momentum Integration

Now that we are in a Feynman parameterized form, all of our integrals only have mass scales. To reduce our integrals to a simple common form we exploit the Lorentz covariance and replace

$$q^{\mu_1} \cdots q^{\mu_{2r}} \to \frac{2}{(2r+2)!!} q^{2r} g^{\mu_1 \ldots \mu_{2r}}, \qquad (2.5.21)$$

where $g^{\mu_1 \ldots \mu_{2r}}$ is the totally symmetric metric tensor with $2r$ indices. The prefactor here is given by the reciprocal of $g^{\mu_1 \ldots \mu_{2r}} g_{\mu_1 \ldots m u_{2r}}$. This is because, as we are in the 't Hooft Veltman scheme, the indices live in 4 dimensions. In this way we avoid introducing any n dependent pre-factors. In general, our expression is in terms of integrals of the form

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{(q^2)^i (\bar{q}^2)^j}{(\bar{q}^2 - \chi^2)^k}. \qquad (2.5.22)$$

At this point, we can now quite easily separate the R_2 parts of our amplitude. To do this we simply have to make the separations

$$\bar{q}^2 = q^2 + \tilde{q}^2 \quad \text{and} \quad n = 4 + \epsilon. \qquad (2.5.23)$$

Any integral which contains an object that lives in ϵ dimensions in the numerator is part of R_2. The integrals which do not we discard as they contribute to $N(q, 4)$. We must also throw away any terms with a prefactor of ϵ^r, where $r > 1$, as these vanish (even when hitting a UV pole) in the limit $\epsilon \to 0$. Next we discard any integrals which do not have a $\tilde{q}^2$ or ϵ in the numerator, as these are contributions to the $N(q, 4)$ part that OPP can handle. Our expression for the R_2 part of our (sub-amplitude) is now given in terms of integrals of the form

$$J_{i,j,r} = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{(q^2)^i (\tilde{q}^2)^j}{(\bar{q}^2 - \chi^2)^r}, \qquad (2.5.24)$$

where they possibly have a prefactor of ϵ. In fact, as we will soon prove, the integrals with a prefactor of ϵ vanish if $j \neq 0$. In the following steps we will evaluate this integral, throwing away terms of $O(\epsilon)$.

We start by remembering that we can write the n dimensional measure as $\mathrm{d}^n \bar{q} = \mathrm{d}^4 q \, \mathrm{d}^\epsilon \tilde{q}$. Importantly we must perform the ϵ dimensional integration first, or the integral is not well defined. We begin by Wick rotating to Euclidean space. The Wick rotation in n dimensions, in our metric convention, is given by $\bar{q}^2 \to -\bar{q}^2$. Therefore, to be consistent we must Wick rotate the 4 and ϵ dimensional parts accordingly, i.e. $q^2 \to -q^2$ and $\tilde{q}^2 \to -\tilde{q}^2$. We can now perform the integration over $\tilde{q}$

$$J_{i,j,r} = (-1)^{i+j+r} \frac{i\pi^{\frac{\epsilon}{2}} \Gamma(j + \frac{\epsilon}{2}) \Gamma(r - j - \frac{\epsilon}{2})}{\mu_R^\epsilon \Gamma(\frac{\epsilon}{2}) \Gamma(r)} \int d^4 q \frac{q^{2i}}{(q^2 + \chi)^{r-j-\frac{\epsilon}{2}}}. \qquad (2.5.25)$$

For $j > 0$ we can simplify the ratio of gamma functions involving j by using the gamma function recursion relation and expanding to $O(\epsilon)$, such that

$$\frac{\Gamma(j + \frac{\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})} = (j - 1)! \frac{\epsilon}{2} + O(\epsilon^2). \qquad (2.5.26)$$

For the $j = 0$ case this ratio is clearly 1 to all orders in epsilon. However, the $j = 0$ case only ever arises as an R_2 contribution when the entire integral is premultiplied by an ϵ. In this way, we can return to the practical $j = 0$ case by dropping the factor of $(j - 1)!/2$.

Now we can perform the integral over the 4-dimensional q, giving

$$(-1)^{i+j+r} \frac{i\pi^{2+\frac{\epsilon}{2}} (j - 1)! \frac{\epsilon}{2} \Gamma(i + 2) \Gamma(r - j - i - 2 - \frac{\epsilon}{2})}{\mu_R^\epsilon \Gamma(r)} \chi^{i+j+2-r+\frac{\epsilon}{2}}. \qquad (2.5.27)$$

Computing this integral introduces a Γ-function that potentially has a pole as $\epsilon \to 0$. In order to compute the $O(1)$ part of this expression, as there is

a prefactor of ϵ we need to know the pole part of this Γ-function. Writing $d/2 = j + i + 2 - k$, we find the pole part to be

$$\text{p.p.} \left\{ \Gamma \left(-\frac{d+\epsilon}{2} \right) \right\} = \begin{cases} 0, & \text{if } d \text{ is negative} \\ \frac{(-1)^{d/2}}{(d/2)!} \text{p.p.} \left\{ \Gamma(-\frac{\epsilon}{2}) \right\} & \text{if } d \text{ is positive.} \end{cases} \tag{2.5.28}$$

This implies that for $d < 0$ the integral is $O(\epsilon)$. Note that d corresponds to the mass dimension of the integral as $\epsilon \to 0$. Therefore, the R_2 integrals only contribes as $\epsilon \to 0$ if the integrals are UV divergent.

Expanding now to $O(1)$ we can write down a simple formula for our general integral, for $j > 0$

$$J_{i,j,r} = -i\pi^2 \frac{(j-1)!(i+1)!}{(r-1)!(j+i+2-r)!} \chi^{i+j+2-r}. \tag{2.5.29}$$

As previously mentioned we can now use this to recover the $j = 0$ case when it is multiplied by ϵ

$$\epsilon J_{i,0,r} = -2i\pi^2 \frac{(i+1)!}{(r-1)!(i+2-r)!} \chi^{i+2-r}. \tag{2.5.30}$$

Furthermore, note that any term with a $\tilde{q}$ *and* a factor of ϵ will be $O(\epsilon)$ as it is given by $\epsilon J_{i,j,r}$.

Having computed all momentum integrals, our expression is now given in terms of various powers of Π^μ and χ. Expanding these in terms of scales and feynman parameters can easily be achieved using computer algebra, and so we are left with integrating the Feynman parameters that arise.

2.5.5 Feynman Parameter Monomials

We can observe from the discussion of section 2.5.4 that our expression is given in terms of only *positive* powers of χ and Π. Therefore, when we use equations (2.5.20) and (2.5.19) to expand these functions in terms of the momenta and masses we will always end up with polynomials in the Feynman parameters. The remaining, quite tractable task is to integrate monomials of these parameters over the Feynman parameter space. Here we will deal with n Feynman parameters, where in our practical application $n = r - 1$. We can see that for any monomial of Feynman parameters, integrating over x_0 simply implements the delta function constraint. That is, defining A_n as

$$A_n = 1 - (x_1 + \cdots + x_n), \tag{2.5.31}$$

then we make the substitution $x_0 = A_n$. The result is still a polynomial in the Feynman parameters $x_1, \ldots, x_n$, so we have to be able to perform integrals of the form

$$X(\{j\}) = \int dx_1 x_1^{j_1} \int dx_2 x_2^{j_2} \cdots \int dx_n x_n^{j_n}, \qquad (2.5.32)$$

where $\{j\}$ is the set of powers associated with each Feynman parameter and the integral over x_i is from 0 to A_{i-1}. To perform this integral, let us first observe that

$$\int dx_n x_n^i A_n^j = A_{n-1}^{i+j+1} \sum_{m=0}^{j} \binom{j}{m} \frac{(-1)^m}{i+m+1}, \qquad (2.5.33)$$

where we have pulled x_n out of A_n using the binomial theorem and integrated over x_n. Whilst this might seem peculiar, if we consider the case $i = j_n$ and $j = 0$ we have just solved the innermost integral in such a way that the next integral we have to perform looks again like the LHS of equation (2.5.33). In this way, we can use this formula to recurse through all of the integrals.

We can further simplify the RHS of (2.5.33) by casting it as a simple integral

$$\sum_{m=0}^{j} \binom{j}{m} \frac{(-1)^m}{i+m+1} = (-1)^i \int_{-1}^{0} dx\, x^i (1+x)^j \equiv F_{i,j}. \qquad (2.5.34)$$

We can perform this integral by using IBP to recurse to $F_{0,i+j}$ which is simple to evaluate, i.e.

$$F_{i,j} = (-1)^i \int_{-1}^{0} dx\, x^i (1+x)^j = -(-1)^i \frac{i}{j+1} F_{i-1,j+1} = \frac{F_{0,i+j}}{\binom{i+j}{i}}$$
$$= \frac{1}{(j+i+1)\binom{i+j}{i}}. \qquad (2.5.35)$$

To find a simple form for $X(\{j\})$ we first use equation (2.5.33) to perform all n Feynman parameter integrals. Next we write it in the simplified form of equation (2.5.35) at which point we notice that it is in the form of a telescoping product

$$X(\{j\}) = \prod_{i=0}^{n-1} \left[\left(i+1+\sum_{m=0}^{i} j_{n-m} \right) \binom{i+\sum_{m=0}^{i} j_{n-m}}{j_{n-i}} \right]^{-1}$$
$$= \frac{\prod_{i=1}^{n} j_i!}{(n+\sum_{i=1}^{n} j_i)!}. \qquad (2.5.36)$$

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In equation (2.5.36) we have now solved all arising full Feynman parameter integrals in such a way that we only need to replace them with a number. At this point we have performed all integrals over auxilliary parameters, be they momenta or Feynman parameters and so we now have the expressions for the R_2 components simply as linear combinations of tensor/gamma matrix structure.

2.6 Checks

For all of the vertices in figure 2.3 we performed two different calculations of the R_2 component. The first is as described in this chapter. The second was performed by separating the R_2 contributing integrals before Feynman parameterization which were then independently calculated. The two calculations gave identical results, which we present in appendix A.3.

A further check can be performed by considering the difference between FDH/DRed ($\lambda_{HV} = 0$) and DR ($\lambda_{HV} = 1$). Noticably, the Feynman rules with external gluons are devoid of λ_{HV}. We can see that this should be the case by considering the following argument. Firstly, we note that at NLO the difference between FDH and DR can be understood as a shift in the coupling constant α_S, such that

$$(\alpha_S)_{\text{FDH}} = (\alpha_S)_{\text{DR}} \left[1 + c \cdot (\alpha_S)_{\text{DR}}\right], \qquad (2.6.1)$$

for some constant c. This shift in coupling constant is relevant to physical combinations, for example we could consider the inclusive cross section in one scheme, and convert it to the other by using equation (2.6.1). However, within HEFT, α_S enters for the first time at one-loop level. Therefore, employing equation (2.6.1) leads to no change at $O(A \cdot \alpha_S)$. That is, physical combinations are the same between DR and FDH. For this reason the R_2 diagrams with external gluons do not depend on λ_{HV}. Considering the $Hq\bar{q}$ vertex, we see that it vanishes on-shell, so its λ_{HV} dependence is of no concern. Moreover, we are required that the physical combination
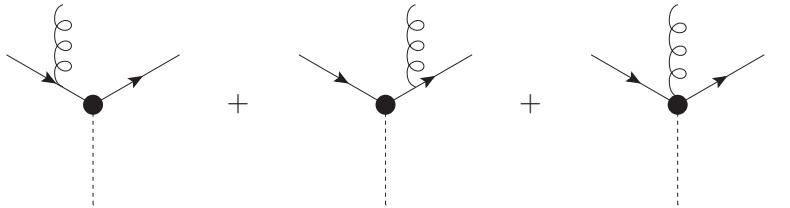
does not depend on λ_{HV}, which we have verified.

2.7 HEFT Phenomenology

The R_2 vertices which we present in appendix A.3 were published in [63] and [64], where they were used to perform studies of Higgs phenomenology. As stated at the beginning of this chapter, it is currently of immense importance to study Higgs properties and one fundamental question is the CP nature of the Higgs/top-quark coupling, i.e. whether the Higgs couples to the top quark as a scalar or pseudo-scalar. At the LHC, to be able to study the CP properties of the Higgs/top-quark interaction one must focus on Higgs production as the decay modes involving top quarks cannot be effectively studied. Further to this, one must design CP sensitive observables to be able to extract these properties from the production rates. In inclusive Higgs production one needs to consider a final state with at least two jets to be able to construct such observables. This is because information on the CP properties of the interaction is encoded in the jet correlations. The aim of the work of [64] was to make studies of these CP sensitive observables to NLO precision in QCD and was performed in the context of the MADGRAPH5_AMC@NLO framework. This implies the ability to perform NLO corrections to $t\bar{t}H$ and gluon fusion. The first requires the standard QCD R_2 counterterms known for a long time [65], but as gluon fusion is a loop induced process, to perform NLO studies in a one-loop generator we must work in the effective field theory framework of this chapter and so we require its corresponding R_2 counterterms. In the following we will discuss the results obtained in [64] made possible by the calculations of this thesis - the gluon fusion-like production channels.

The framework employed was to consider Higgs + jet(s) production whilst parameterizing the CP properties of the Higgs using a linear combination of the scalar and pseudoscalar effective field theories. Specifically we use

$$\mathcal{L} = -\frac{1}{4}\Big[c_\alpha g_H\, G_{\mu\nu}^a G^{a,\mu\nu} + s_\alpha g_A\, G_{\mu\nu}^a \widetilde{G}^{a,\mu\nu}\Big] X_0\,, \qquad (2.7.1)$$

as we are only considering NLO QCD corrections. Here, s_α and c_α are the sine and cosine of a mixing angle α and X_0 is a general scalar boson, identified with the SM Higgs in the limit $\alpha = 0$. This framework allows us to easily investigate three different characteristic scenarios as presented in table 2.1 - pure scalar (0^+), pure pseudoscalar (0^-) and (maximally) mixed ($0^\pm$).

With this physics aim, studies were carried out in MADGRAPH5_AMC@NLO, simulating events for the LHC at a center of mass energy of $\sqrt{s} = 8$ and 13 TeV with the mass of the X_0 resonance at 125 GeV. Parton density functions

scenario	parameter choice
0^+(SM)	$c_\alpha = 1$
0^-	$c_\alpha = 0$
$0^\pm$	$c_\alpha = 1/\sqrt{2}$

Table 2.1: Studied scenarios for gluon fusion.

were evaluated in the NNPDF parameterization [66]. The sources of uncertainty within this simulation are from missing higher order corrections, PDF uncertainty and uncertainty on the measurement of α_S. The PDF uncertainty in NLO predictions is computed along with the uncertainty in $\alpha_S(m_Z)$ according to [14] with $\alpha_S(m_Z)$ taken to be distributed as a gaussian around

$$\alpha_S^{(NLO)}(m_Z) = 0.1190 \pm 0.0012 \ (68\% \ \text{C.L.}),\tag{2.7.2}$$

according to [67, 68]. For leading order this is not possible, so the uncertainty comes solely from the PDF and a fixed $\alpha_S^{(LO)}(m_Z) = 0.130$ is taken [69, 70]. The uncertainty from the missing higher order corrections is estimated by scale variation - specifically μ_R and μ_F are varied independently by a factor of 2 around a central scale μ_0. This central value is given by $\mu_0 = H_T/2$, where H_T is the sum of the transverse masses of the particles in the final state. Parton shower and hadronisation was achieved with HERWIG6 [71] and the employed jet algorithm was anti-k_T [72] as implemented in FASTJET [73]. Finally, the imposed acceptance cuts were

$$p_T(j) > 30 \ \text{GeV}, \qquad |\eta(j)| < 4.5.\tag{2.7.3}$$

We begin by stating the results for inclusive cross sections for Higgs plus jet production for gluon fusion. We present the leading order, next to leading order (NLO) and NLO with parton shower for Higgs + jet and Higgs + 2 jets in tables 2.2 and 2.3 respectively. Here the acceptance cuts of equation (2.7.3) were imposed. The number in brackets is the integration error, the first uncertainty comes from scale variation and the second from the combined PDF+α_S uncertainty.

Let us make a few observations. Firstly, the ratio $\sigma_{\text{NLO}}/\sigma_{\text{NLO+PS}}$ is ~ 0.85 as the extra radiation generated by the parton shower spreads the energy of the unshowered partons such that there are more events which fail to pass the acceptance cuts. Effectively the reconstructed jets after the parton shower are different to those from the fixed order prediction. Secondly, the mixed scalar/pseudoscalar scenario, $0^\pm$ is equal to the average

scenario		σ_{LO} (pb)	σ_{NLO} (pb)	$\sigma_{\mathrm{NLO+PS}}$ (pb)
LHC8	0^+	$4.002(4)\ ^{+46.8}_{-29.6}\pm3.3\%$	$5.484(7)\ ^{+17.0}_{-16.8}\pm1.2\%$	$4.618\ ^{+21.8}_{-18.8}\pm1.2\%$
	0^-	$9.009(9)\ ^{+46.8}_{-29.6}\pm3.3\%$	$12.34(2)\ ^{+17.1}_{-16.8}\pm1.2\%$	$10.38\ ^{+21.7}_{-18.8}\pm1.2\%$
	$0^\pm$	$6.511(6)\ ^{+46.8}_{-29.6}\pm3.3\%$	$8.860(14)\ ^{+16.9}_{-16.8}\pm1.2\%$	$7.474\ ^{+21.7}_{-18.8}\pm1.2\%$
LHC13	0^+	$10.67(1)\ ^{+41.7}_{-27.5}\pm2.6\%$	$14.09(2)\ ^{+16.2}_{-14.9}\pm1.1\%$	$12.08\ ^{+19.8}_{-16.7}\pm1.0\%$
	0^-	$24.01(2)\ ^{+41.7}_{-27.5}\pm2.6\%$	$31.67(6)\ ^{+16.2}_{-14.9}\pm1.1\%$	$27.14\ ^{+20.3}_{-16.4}\pm1.0\%$
	$0^\pm$	$17.36(2)\ ^{+41.7}_{-27.5}\pm2.6\%$	$22.83(3)\ ^{+16.2}_{-14.9}\pm1.1\%$	$19.59\ ^{+19.5}_{-16.6}\pm1.0\%$

Table 2.2: LO and NLO cross sections for $X_0 + 1$ jet at the 8- and 13-TeV LHC, for the three scenarios defined in table 2.1. The K factor, σ_{NLO}/σ_{LO} is ~ 1.35. The integration error in the last digit(s) (in parentheses), and the fractional scale variation (left) and PDF($+\alpha_s$) (right) uncertainties are also reported. In addition to fixed-order results, the PS-matched NLO cross sections and are shown displaying an R factor, $\sigma_{NLO+PS}/\sigma_{NLO}$ of ~ 0.85.

of the pure scalar and pure pseudoscalar scenarios. This implies that there are no interference effects.

Now that we see that the CP properties of the Higgs/top-quark interaction can affect the inclusive cross section, we move to the discussion of distributions. Here we wish to discuss jet-jet correlations as they are known tools to determine Higgs CP properties. We begin with figures 2.5 and 2.6. Here we present shape comparisons for both the rapidity, η, and transverse momentum, p_T, distributions of the scalar boson X_0 and the leading jet respectively. We first apply only the acceptance cuts and then we cut on the invariant mass of the jet system, discarding events with $m(j_1, j_2) < 250\mathrm{GeV}$ and $m(j_1, j_2) < 500\mathrm{GeV}$. This cut is more typically associated with vector boson fusion, because it leads to enhancement of the t-channel contributions. This is because, kinematically, strongly forward and backward jets make up the majority of the high $m(j_1, j_2)$ events which this cut selects for. In this case it leads to the enhancement of contributions which are more sensitive to the CP properties of X_0. In making this cut we can see that proportion of events with hard X_0 p_T increases, and the resonance is produced increasingly centrally. In parallel the leading jet is produced more and more forward/backward as expected due to the enhancement of the t-channel contributions. However, in these two figures we see that these observables do not provide any discriminatory power between the three scenarios, though

scenario		σ_{LO} (pb)	σ_{NLO} (pb)	$\sigma_{\mathrm{NLO+PS}}$ (pb)
LHC8	0^+	$1.351(1)\ ^{+67.1}_{-36.8}\pm4.3\%$	$1.702(6)\ ^{+19.7}_{-20.8}\pm1.7\%$	$1.276\ ^{+29.4}_{-23.9}\pm1.7\%$
	0^-	$2.951(3)\ ^{+67.2}_{-36.8}\pm4.4\%$	$3.660(15)\ ^{+19.1}_{-20.6}\pm1.7\%$	$2.755\ ^{+29.8}_{-24.1}\pm1.8\%$
	$0^\pm$	$2.142(2)\ ^{+67.1}_{-36.8}\pm4.4\%$	$2.687(10)\ ^{+19.6}_{-20.8}\pm1.7\%$	$2.022\ ^{+29.7}_{-24.1}\pm1.8\%$
LHC13	0^+	$4.265(4)\ ^{+61.5}_{-34.9}\pm3.3\%$	$5.092(23)\ ^{+15.4}_{-17.9}\pm1.2\%$	$4.025\ ^{+23.9}_{-21.3}\pm1.2\%$
	0^-	$9.304(9)\ ^{+61.6}_{-34.9}\pm3.4\%$	$11.29(4)\ ^{+16.0}_{-18.2}\pm1.2\%$	$8.701\ ^{+24.6}_{-21.6}\pm1.3\%$
	$0^\pm$	$6.775(6)\ ^{+61.5}_{-34.9}\pm3.3\%$	$8.055(35)\ ^{+15.8}_{-18.2}\pm1.2\%$	$6.414\ ^{+24.4}_{-21.5}\pm1.2\%$

Table 2.3: Same as table 2.2, but for $X_0 + 2$ jets. Here the K factor, σ_{NLO}/σ_{LO}, is ~ 1.22 and the R factor, $\sigma_{NLO+PS}/\sigma_{NLO}$ is ~ 0.77

this is expected. Now moving to figure 2.7 we see distributions for the rapidity difference of the jets, $\Delta\eta$, and the difference in azimuthal angle, $\Delta\phi$. In general it is expected that these will be sensitive observables for the CP nature of the Higgs/top-quark coupling, however the rapidity difference offers little discriminatory power. On the other hand, the difference in azimuthal angle does exhibit different shapes in each scenarios, with the difference becoming more pronounced under the cuts on $m(j_1, j_2)$. Most interestingly is that reordering the jets in pseudorapdity, as opposed to the usual p_T jet ordering, strengthens this difference.

In conclusion, it is confirmed through these results that the difference in azimuthal angle between the two jets in in Higgs + 2 jets production is a sensitive probe of the CP nature of the Higgs/top-quark coupling.
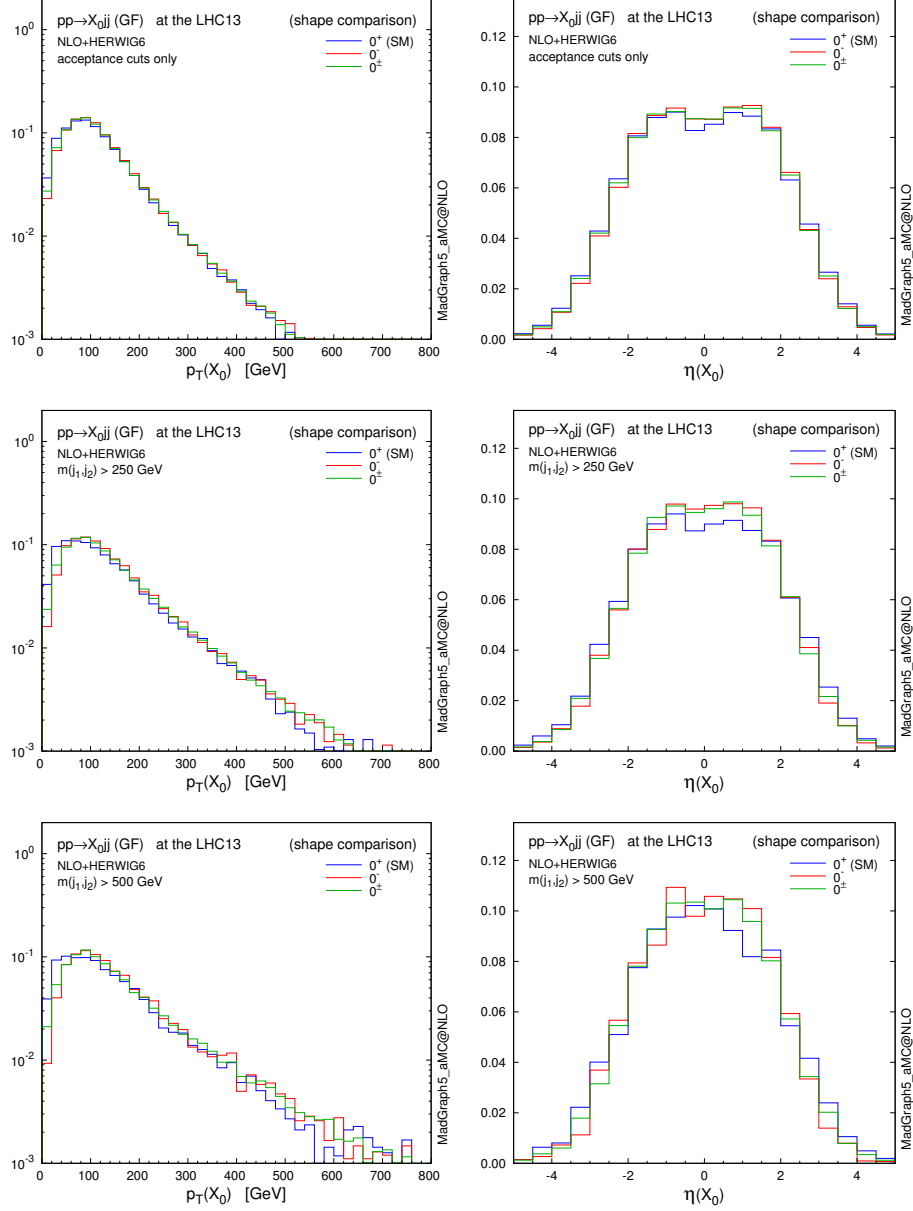
Figure 2.5: Normalized distributions (shape comparison) in p_T and η of the resonance X_0, with the acceptance cuts for jets (top), plus $m(j_1, j_2) > 250$ GeV (centre) and 500 GeV (bottom). The three spin-0 hypotheses are defined in table 2.1.

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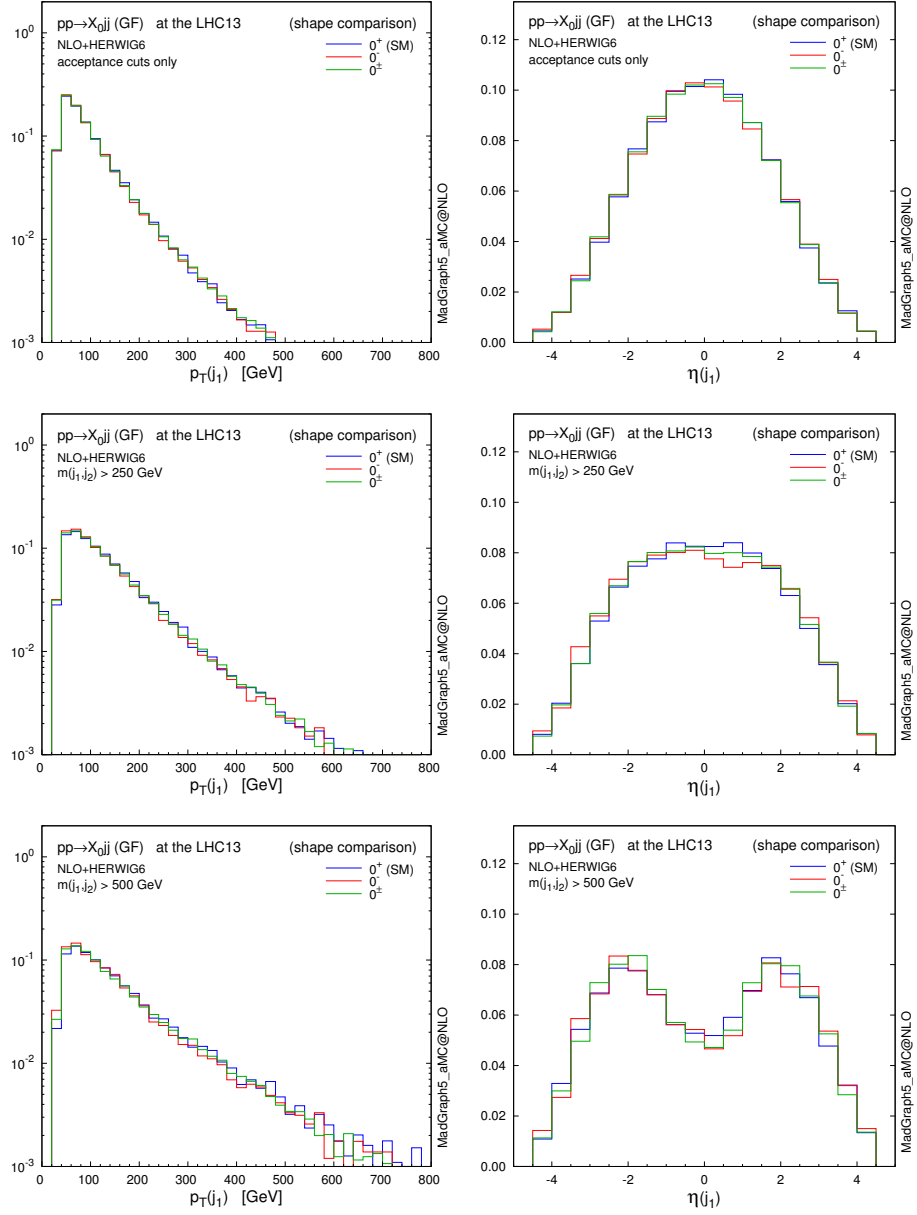
Figure 2.6: Same as fig. 2.5, but for the leading jet.
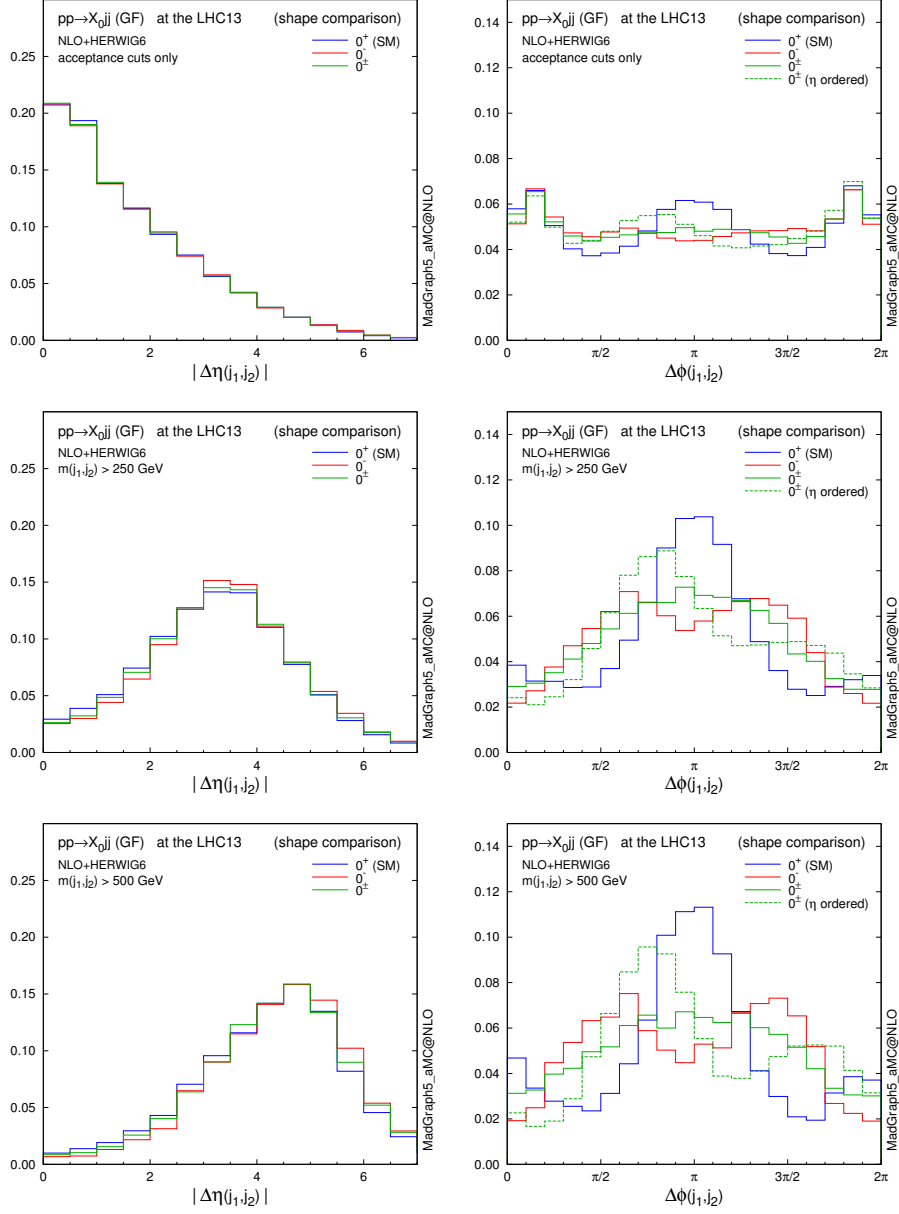
Figure 2.7: Same as fig. 2.5, but for $\Delta\eta$ and $\Delta\phi$ distributions between the two tagging jets. For $\Delta\phi$, the distribution with the additional η jet ordering is also shown by a dashed line for the $0^{\pm}$ case.

Chapter 3

The FDR Approach to QFT

Calculations in perturbative quantum field theory are notoriously difficult. Working in a theory such as QCD and trying to make precise predictions for processes at hadron colliders such as the LHC is a daunting prospect. Processes that require high precision beget higher loop integrals that are technically demanding and currently cannot be performed in any general way. Processes that have large numbers of legs exhibit a great deal of complexity through large algebraic expressions, which the community has only recently begun to be able to handle.

An interesting question to ask in all of this complication is how much we are limited by our current tools. One of the immediate technical difficulties that one finds in perturbative calculations is handling the intermediate infinitees that arise in the calculations. The standard method of doing this is to use dimensional regularization (DR)[16] alongside a suitable renormalization procedure. We analytically continue our integrals to n dimensions and then in order to obtain finite results as $n \to 4$ we renormalize the theory, introducing counterterms or subtractions. All of this is a great deal of intermediate work. What's more, a regularization scheme defined in a continuous number of dimensions is not immediately amenable to numerical approaches.

FDR or "Four/Finite Dimensional Regularization/Renormalization" [23] is a new technique for performing calculations in Quantum Field Theory aiming to provide a solution to these problems by defining a purely 4-dimensional integral. It provides all of the important properties of a "good" regulator such as preserving gauge and shift invariance [23, 74] of the integral whilst also providing new features such as remaining in a fixed number of dimensions and not needing counter-terms.

When attempting to automate one loop calculations the complexity becomes a limiting factor. Standard techniques for dealing with this complexity manipulate the n-dimensional calculation in 4-dimensions as much as possible. Amplitude reduction techniques at one loop, such as OPP and generalized unitarity are based (in their most frequent uses) on performing the difficult 4-dimensional part first and then afterwards recovering extra terms[1] resulting from the n-dimensional nature of the regularization. In the context of generalized unitarity, to help this endeavour, a new flavour of dimensional regularization, FDH [26, 75], was introduced to aid this effort even further. FDR, then, is a natural conclusion of keeping this work in four dimensions.

In principle the reason for using methods based upon dimensional continuation has been one of necessity. DR was the first method available that preserved both the gauge invariance and unitarity of the calculation and no other method had succeeded beyond one loop. For this reason we have only seen success in different flavours of dimensional continuation. FDR, despite not being a dimensional method, has been possible because of inspiration from many years of experience with DR. We imagine that we *could* define a 4-dimensional multi loop integration,

$$\int [\mathrm{d}^4 q_1] \cdots [\mathrm{d}^4 q_l] J(q_1, \cdots, q_l). \tag{3.0.1}$$

In order to be a good definition of integration which respects gauge invariance, we must satisfy a number of properties. It must

- Be coincident with normal integration for UV finite integrals.

- Be independent of any UV cutoff, but dependent on the renormalization scale μ_R.

- Be invariant under the shift of any integration variable:

$$\begin{aligned}
&\int [\mathrm{d}^4 q_1] \cdots [\mathrm{d}^4 q_l] J(q_1, \cdots, q_l) \\
&= \int [\mathrm{d}^4 q_1] \cdots [\mathrm{d}^4 q_l] J(q_1 + p_1, \cdots, q_l + p_l).
\end{aligned} \tag{3.0.2}$$

---

[1]Historical note - trying to solve this difficulty was one of the original motivations for FDR.
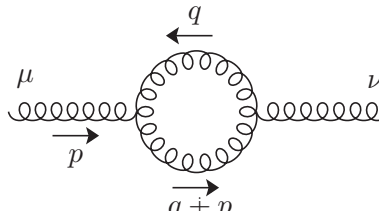
- Maintain simplifications between numerators and denominators:

$$\int [\mathrm{d}^4 q_1] \cdots [\mathrm{d}^4 q_l] \frac{\overline{q}_i^2 - M_i^2}{(\overline{q}_i^2 - M_i^2)^m \cdots}$$
$$= \int [\mathrm{d}^4 q_1] \cdots [\mathrm{d}^4 q_l] \frac{1}{(\overline{q}_i^2 - M_i^2)^{m-1} \cdots}. \tag{3.0.3}$$

The FDR integral *is* such a definition. In the following chapter we shall discuss how FDR realizes these properties all in a manner that is UV finite by construction, and in 4-dimensions.

3.1 A Practical Exposition

In order to start to understand the FDR approach and introduce a number of the concepts required, we will begin with a practical example. This way, one can see where all of the elements come in, step by step. Then, in the following sections, we shall explain each idea in much greater detail. As our example, we shall consider the gluonic contribution to the 1-loop gluon propagator corrections

$$= \frac{g^{\mu\nu}(q^2 + p \cdot q + 5/2p^2) - p^\mu p^\nu + 5q^\mu q^\nu + \frac{5}{2}q^\mu p^\nu + \frac{5}{2}p^\mu q^\nu}{q^2(q+p)^2}. \tag{3.1.1}$$

Here we have suppressed the regulation in (3.1.1) in order to now emphasise what it means to regulate in FDR. When interpreted in the FDR sense we write the regulated form as

$$\int [\mathrm{d}^4 q] \frac{g^{\mu\nu}(\overline{q}^2 + p \cdot q + 5/2p^2) - p^\mu p^\nu + 5q^\mu q^\nu + \frac{5}{2}q^\mu p^\nu + \frac{5}{2}p^\mu q^\nu}{\overline{q}^2(\overline{q}+p)^2}, \tag{3.1.2}$$

where for notational convenience we write

$$(\overline{q}+p)^2 \equiv \overline{q}^2 + p^2 + 2q \cdot p \quad \text{and} \quad \overline{q}^2 = q^2 - \mu^2. \tag{3.1.3}$$

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The FDR treatment demands that here we introduce a **fictitious mass** to each propagator through $q^2 \to \bar{q}^2 = q^2 - \mu^2$. Crucially, for this process to be consistent and to keep gauge invariance this must also be performed in the numerator. To be clear, the q^2 in both the numerator and denominator have been promoted to $\bar{q}^2$. This is known as the **global prescription**. To proceed, one now uses the standard techniques of reconstructing numerators in terms of denominators. This gives us

$$\int [\mathrm{d}^4 q] \frac{1}{\bar{q}^2 (\bar{q} + p)^2} \left(2 g^{\mu\nu} p^2 - p^\mu p^\nu + 5 q^\mu q^\nu + \frac{5}{2} q^\mu p^\nu + \frac{5}{2} p^\mu q^\nu \right) \qquad (3.1.4)$$

We note that the global prescription results in the $\bar{q}^2$ exactly cancelling with the denominator. What is more, dot products cancel without creating any μ^2 like terms because

$$q \cdot p = \frac{1}{2} \left((\bar{q} + p)^2 - \bar{q}^2 - p^2 \right). \qquad (3.1.5)$$

Finally, we have been able to set a number of integrals to zero. This is for two different reasons. Firstly, UV divergent scaleless integrals are zero in FDR, i.e.

$$\int [\mathrm{d}^4 q] \frac{1}{\bar{q}^{2\alpha}} = 0, \quad (\alpha < 2). \qquad (3.1.6)$$

Secondly, FDR integrals are shift invariant, in our case allowing us to write

$$\int [\mathrm{d}^4 q] \frac{1}{(\bar{q} + p)^2} = \int [\mathrm{d}^4 q] \frac{1}{\bar{q}^2} = 0. \qquad (3.1.7)$$

At this point we are left with tensor integrals that have loose indices and so we need to use Passarino-Veltman reduction (PV) in FDR. For the rank 1 integral there are no surprises and the direct analogue to the DR calculation follows, i.e.

$$\int [\mathrm{d}^4 q] \frac{q^\mu}{\bar{q}^2 (\bar{q} + p)^2} = -\frac{p^\mu}{2} \int [\mathrm{d}^4 q] \frac{1}{\bar{q}^2 (\bar{q} + p)^2}. \qquad (3.1.8)$$

The rank 2 PV reduction, however, does not proceed in direct coincidence with the DR calculation. Our starting point is the analogous

$$\int [\mathrm{d}^4 q] \frac{q^\mu q^\nu}{\bar{q}^2 (\bar{q} + p)^2} = A g^{\mu\nu} + B p^\mu p^\nu. \qquad (3.1.9)$$

In the FDR calculation, contracting with the metric tensor leads to

$$\int [\mathrm{d}^4 q] \frac{q^2}{\bar{q}^2 (\bar{q} + p)^2} = \int [\mathrm{d}^4 q] \frac{\mu^2}{\bar{q}^2 (\bar{q} + p)^2} = 4A + p^2 B. \qquad (3.1.10)$$
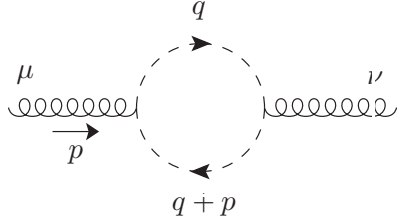
Note that $q^\mu q^\nu g_{\mu\nu} = q^2 \neq \overline{q}^2$. This means that, unlike that which one finds in the DR calculation, the integrals of (3.1.10) are non-zero. We end up with an integral with μ^2 in the numerator. These integrals are known as **extra integrals** and are crucial for maintaining gauge invariance. What's more, in FDR $g^{\mu\nu}g_{\mu\nu} = 4$. From here, the logic of PV is entirely standard and leads us to the result of

$$\int [\mathrm{d}^4 q] \frac{q^\mu q^\nu}{\overline{q}^2 (\overline{q}+p)^2}$$
$$= \int [\mathrm{d}^4 q] \frac{1}{\overline{q}^2 (\overline{q}+p)^2} \left[p^2 g^{\mu\nu} \left(-\frac{1}{12} + \frac{1}{3}\frac{\mu^2}{p^2} \right) + p^\mu p^\nu \left(\frac{1}{3} - \frac{1}{3}\frac{\mu^2}{p^2} \right) \right].$$
$$(3.1.11)$$

Inserting these results into the expression for our gluonic diagram, (3.1.4), we find the result

$$\int [\mathrm{d}^4 q] \frac{1}{\overline{q}^2 (\overline{q}+p)^2} \left[p^2 g^{\mu\nu} \left(\frac{19}{12} + \frac{5}{3}\frac{\mu^2}{p^2} \right) - p^\mu p^\nu \left(\frac{11}{6} + \frac{5}{3}\frac{\mu^2}{p^2} \right) \right]. \quad (3.1.12)$$

To demonstrate the gauge invariance of the method we must calculate a gauge invariant set of diagrams, so we also add in the ghost contribution

$$= \int [\mathrm{d}^4 q] \frac{1}{\overline{q}^2 (\overline{q}+p)^2} \left[p^2 g^{\mu\nu} \left(\frac{1}{12} - \frac{1}{3}\frac{\mu^2}{p^2} \right) + p^\mu p^\nu \left(\frac{1}{6} + \frac{1}{3}\frac{\mu^2}{p^2} \right) \right], \quad (3.1.13)$$

where the momentum routing follows the fermion lines. The sum is the manifestly gauge invariant result

$$\int [\mathrm{d}^4 q] \frac{p^2 g^{\mu\nu} - p^\mu p^\nu}{\overline{q}^2 (\overline{q}+p)^2} \left[\frac{5}{3} + \frac{4}{3}\frac{\mu^2}{p^2} \right]. \quad (3.1.14)$$

We are now left with the final task of evaluating this scalar integral in FDR. The FDR integral can be defined in two different, but equivalent ways. Firstly we can define it through the subtraction of the **FDR vacuum**, found through the **FDR defining expansion**. Alternatively we can perform the defining expansion directly, throwing away scaleless integrals, until all

integrals are manifestly UV finite and so can be performed in 4 dimensions. For the moment, we shall simply state that the above integral can be defined through the relation

$$
\int [\mathrm{d}^4 q] \frac{1}{\bar{q}^2 (\bar{q}+p)^2} \equiv \lim_{\mu^2} \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \left(\frac{1}{\bar{q}^2 (\bar{q}+p)^2} - \frac{1}{\bar{q}^4} \right)
$$
$$
= -i\pi^2 \left(\log \frac{p^2}{\mu_R^2} - 2 \right)
$$

(3.1.15)

Similarly, the extra integral can be defined through (throwing away trivially zero terms in DR)

$$
\int [\mathrm{d}^4 q] \frac{\mu^2}{\bar{q}^2 (\bar{q}+p)^2} \equiv \lim_{\mu^2} \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \left(\frac{\mu^2}{\bar{q}^2 (\bar{q}+p)^2} - \frac{4(q \cdot p)^2 \mu^2}{\bar{q}^8} \right)
$$
$$
= -\frac{i\pi^2}{3} p^2.
$$

(3.1.16)

Here we have used the notation $\lim_{\mu^2}$ to indicate that we should remove the fictitious mass taking the limit $\mu^2 \to 0$ apart from where we find a fictitious IR divergence and here we trade $\mu^2 \to \mu_R^2$. This is how the renormalization scale enters in FDR.

3.2 The Main Ingredients

3.2.1 Global Prescription

After writing down the expression corresponding to a Feynman diagram taking $g^{\mu\nu} g_{\mu\nu} = 4$, the first step in an FDR calculation is to introduce the fictitious mass μ^2. One of the defining features of FDR is that it maintains the gauge invariance of the original theory. It does this by maintaining the cancellation structure of the unregulated Feynman integrals. However, introducing the fictitious mass in the denominator alone would break gauge invariance - so we must consistently introduce the mass into the numerator. More technically speaking, we must introduce the fictitious mass such that cancellations are maintained in any equivalent form.

The process amounts to "barring" all squared momenta that come from Feynman rules. To begin, let us consider the style of integral that could appear in a purely bosonic calculation. Here this process is quite simple, and we can also easily see why it is important. Consider the unregulated integrand

$$
\frac{q^2 - 4q^\mu q^\nu}{(q^2 - m^2)^3}.
$$

(3.2.1)

Here it is understood that the q^2 has come from Feynman rules. To regulate this in FDR one must consistently perform the deformation of introducing the fictitious mass, therefore the regulated form is given by

$$\int [d^4q] \frac{\overline{q}^2 - 4q^\mu q^\nu}{(\overline{q}^2 - m^2)^3}. \tag{3.2.2}$$

When one performs Passarino-Veltman reduction on the above integral, the result is non-zero,

$$\int [d^4q] \frac{-\mu^2}{(\overline{q}^2 - m^2)^3}. \tag{3.2.3}$$

This is an "extra integral", a constant at one loop that is required to maintain gauge invariance (see section 3.2.3).

If we now move to a theory with fermions it is naively less clear how to promote $q^2 \to \overline{q}^2$ as the numerator is written in terms of a string of gamma matrices. The guiding principle here is that we should have all representations of the numerator resulting in the same global prescription. Consider a diagram with an internal fermion loop. Here the gamma matrices are traced over, and once the trace is computed we find ourselves in a situation with q^2 once again. In this traced form it is now clear how to "bar" the q^2. If we wish to do this without performing the trace then the following algorithm can be employed to give the same results[2]. For a given trace where all internal indices have been summed over, find the first loop momenta $\not{q}$ and promote it to $\not{q} \pm \mu$ where the sign is arbitrary. The next $\not{q}$ is then promoted with a sign that is alternated depending on the number of enclosed gamma matrices. Explicitly we can write this as

$$(\cdots \not{q} \gamma^{\alpha_1} \cdots \gamma^{\alpha_n} \not{q} \cdots) = (\cdots (\not{q} \pm \mu) \gamma^{\alpha_1} \cdots \gamma^{\alpha_n} (\not{q} \mp (-1)^n \mu) \cdots). \tag{3.2.4}$$

Where the intervening gamma matrices could (but need not) be contracted with appropriate tensors. The next $\not{q}$ is promoted according to the number of γ between it and the preceding $\not{q}$ and we continue with all $\not{q}$. The above algorithm also can be shown to work for strings of gamma matrices which are *not* traced over. This is because one can write a string of gamma matrices as a trace.

When moving to two loops the prescription is more complicated because of the variety of possible products between loop momenta which need to maintain their cancellation structure. As it is possible to write diagrams

---

[2]This only applies in a situation where there is a single fermion line, if there are multiple that are connected the situation is more complicated, see [74].

in terms of local propagators depending on loop momenta as q_1, q_2 and $q_{12} = q_1 + q_2$ then we have three different squares which we need to promote - q_1^2, q_2^2 and q_{12}^2. There also exist three scalar products between these loop momenta, but these are all linearly dependent on the squares, for example

$$q_1 \cdot q_2 = \frac{1}{2} \left(q_{12}^2 - q_1^2 - q_2^2 \right). \tag{3.2.5}$$

In a purely bosonic calculation it suffices to write the (sub-)amplitude in terms of these squares and promote them. Here we point out a confusing case. Consider the factorizable integrand

$$\frac{q_1 \cdot q_2}{\left(q_1^2 - M^2 \right) \left(q_2^2 - M^2 \right)}. \tag{3.2.6}$$

Here we do not rewrite $q_1 \cdot q_2$ in terms of squares and then promote because there is no cancellation possible. From the perspective of rewriting in terms of squares we would see that the q_{12}^2 term does not cancel.

A subtlety in promoting $q_i^2 \to \overline{q}_i^2 = q_i^2 - \mu^2$ at two loop is related to maintaining the algebraic consistency of the global prescription. As explained more completely in section 3.2.3, whilst there is only one μ^2 we must keep a record of the origin of the μ^2 in order to make a consistent defining expansion. At a practical level[3] this means that (notationally) our promotion in the numerator occurs as $q_i^2 \to \overline{q}_i^2 = q_i^2 - \mu_i^2$.

In order to correctly regulate fermions at two loops we must once again work algorithmically. To make this discussion concrete, consider the following trace in the numerator

$$\Gamma^{\mu\nu} = \text{Tr} \left(\slashed{q}_1 \gamma^\mu \slashed{q}_1 \slashed{q}_2 \gamma^\nu \slashed{q}_2. \right) \tag{3.2.7}$$

After global prescription we find

$$\overline{\Gamma}^{\mu\nu} = \Gamma^{\mu\nu} + \mu_1^2 \, \text{Tr} \left(\gamma^\mu \slashed{q}_2 \gamma^\nu \slashed{q}_2 \right) + \mu_2^2 \, \text{Tr} \left(\slashed{q}_1 \gamma^\mu \slashed{q}_1 \gamma^\nu \right) \\ + \mu_1^2 \mu_2^2 \, \text{Tr} \left(\gamma^\mu \gamma^\nu \right) - 16 \tilde{\mu}_{12}^2 q_1^\mu q_2^\nu, \tag{3.2.8}$$

where $\tilde{\mu}_{12}^2 = \frac{1}{2} \left(\mu_{12}^2 - \mu_1^2 - \mu_2^2 \right)$. This is obtained by first replacing, one after the other, $q_1 \to \overline{q}_1$ and $q_2 \to \overline{q}_2$. Then we simultaneously apply these replacements and subtract any double counting. The resulting $\mu_1 \mu_2$ term is then realized as μ_{12}^2. When making these replacements one should adhere to the rule of equation (3.2.4) to determine the signs of the μ_i.

---

[3]Even more practically speaking, without external fermions this is not a distinction one has to care about when performing the global prescription. Without external fermions one can simply perform the cancellation without extra μ_i^2 terms appearing.

Whilst this completes the definition of the global prescription, this method is not necessarily the easiest to implement. At a practical level, as packages, such as FORM[62], are available to easily perform the traces over fermion lines, the simplest form of the algorithm is given by

1. Trace all internal fermion loops.

2. Rewrite all loop momenta scalar products as squares and reconstruct these squares in terms of denominators.

3. Perform the FDR μ^2 deformation, now only in the denominator[4].

If we also have external fermions as we cannot trace we have the extra step of performing the global prescription along this line. This can be simply performed as

4. Anti-commute $\not{q}_i$ such that all possible applications of $\not{q}_i\not{q}_i = q_i^2$ are made

5. At two loops, make the substitution $\not{q}_1\not{q}_2 \to \not{q}_1\not{q}_2 - \tilde{\mu}_{12}^2$.

It is possible to perform the anti-commutation of step 4 before global prescription as it results in simply another equivalent representation. It is sufficient to perform step 5 as by performing the anti-commutation the *only* remaining string of gamma matrices involving loop momenta are either single $\not{q}_i$ or $\not{q}_1\not{q}_2$. This is the approach used in section 4.5.

3.2.2 The FDR Defining Expansion and Renormalization

Whilst all manipulations in FDR are valid in 4 dimensions and so standard strategies of reduction to master integrals are still valid within the technique, eventually one arrives at the point of needing to calculate these integrals. At this point one must reach for the definition of the FDR integral. The FDR integral is defined by using partial fractions/stepwise Taylor expansion identities to remove the large momentum configurations. At one loop the operative identity is

$$\frac{1}{(\overline{q}+p)^2 - M^2} = \frac{1}{\overline{q}^2}\left[1 + \frac{M^2 - 2q \cdot p}{(\overline{q}+p)^2 - M^2}\right] \qquad (3.2.9)$$

As one can see, the first term on the RHS depends on no scales, and the second term on the right hand side is more UV convergent - this is no

---

[4]For no external fermions, there are no longer any places to perform it in the numerator.

accident. To understand what we are doing here, consider expanding a propagator in the large $\bar{q}^2$ limit (taking $p = 0$ for simplicity)

$$
\begin{aligned}
\frac{1}{\bar{q}^2 - M^2} &= \frac{1}{\bar{q}^2} \frac{1}{1 - \frac{M^2}{\bar{q}^2}} \\
&= \frac{1}{\bar{q}^2} \left(1 + \frac{M^2}{\bar{q}^2} + \left(\frac{M^2}{\bar{q}^2} \right)^2 + \cdots \right)
\end{aligned}
\tag{3.2.10}
$$

Obviously within our integral this is not exact as $\bar{q}^2$ is not always large. However, we can see that each term, by power counting, is progressively more and more UV convergent. Note that as this is a geometric series, we can factor out an appropriate power of $M^2/\bar{q}^2$ at any point allowing us to resum

$$
= \frac{1}{\bar{q}^2} \left(1 + \frac{M^2}{\bar{q}^2} \left(1 + \left(\frac{M^2}{\bar{q}^2} \right) + \cdots \right) \right)
\tag{3.2.11}
$$

$$
= \frac{1}{\bar{q}^2} \left(1 + \frac{M^2}{\bar{q}^2 - M^2} \right).
\tag{3.2.12}
$$

We now end up with the form of equation (3.2.9). From this argumentation we can see what equation (3.2.9) is doing - in effect we are separating out UV divergent pieces. What's more, *repeated* application of (3.2.9) will have the effect of resumming further down the series. The FDR integral is *defined* through this operation - we expand an integrand into manifestly UV divergent and convergent parts and then throw away the UV divergence. Crucially we can achieve this without introducing any IR divergences that are worse than logarithmic. The simplest example is

$$
\begin{aligned}
I &= \int [\mathrm{d}^4 q] \frac{1}{(\bar{q}^2 - M^2)} \\
&= \int [\mathrm{d}^4 q] \left(\frac{1}{\bar{q}^2} + \frac{M^2}{\bar{q}^2 (\bar{q}^2 - M^2)} \right) \\
&= \int [\mathrm{d}^4 q] \left(\frac{1}{\bar{q}^2} + \frac{M^2}{\bar{q}^4} + \frac{M^4}{\bar{q}^4 (\bar{q}^2 - M^2)} \right) \\
&= \int [\mathrm{d}^4 q] \frac{M^4}{\bar{q}^4 (\bar{q}^2 - M^2)}
\end{aligned}
\tag{3.2.13}
$$

In the final line we have used the fact that scaleless UV divergent integrals in FDR are zero.

Notice that in performing this expansion we have introduced a fictitious, logarithmic, IR divergence. This is a general statement in FDR, by removing the UV divergences we introduce fictitious IR ones. What is the origin of this divergence? Consider the logarithmic scaleless integral which we just set to zero. Our aim is to remove all UV divergences in a gauge invariant way. Working at the integrand level, the defining expansion manifestly maintains gauge invariance, but there is a problem with discarding logarithmically UV divergent integrals. Let us investigate by parameterizing this via a cut off

$$\int_\Lambda \frac{1}{\bar{q}^4} = \int_0^\Lambda \frac{1}{(q^2 - \mu^2)^2}. \qquad (3.2.14)$$

Note that this integral contains more than just a UV divergence, it contains the IR divergence as $\mu \to 0$. Our aim, however, is to only throw away the UV divergences. To do this, without introducing any IR divergences we only want to throw away the high frequency components of this integral. Separating the high momentum region from the low momentum region requires a separation scale, which represents an arbitrariness in how we remove our divergences. This is where the renormalization scale, μ_R, enters. Performing this split we find

$$\int_{\mu_R}^\Lambda \frac{1}{(q^2 - \mu^2)^2} + \int_0^{\mu_R} \frac{1}{(q^2 - \mu^2)^2}. \qquad (3.2.15)$$

Our intention is to only throw away the first term. Once we have arranged our calculation to be IR convergent (through making our calculation "sufficiently inclusive") we can exploit this. The fictitious divergences we create correspond exactly to the IR divergence of (3.2.15). Therefore, reintroducing the IR divergence of the (3.2.15) is equivalent to trading μ for μ_R in the fictitious divergence. This mechanism can be shown to apply at all orders in perturbation theory.

We did not just throw away a logarithmic integral, however. We also threw away a quadratically divergent vacuum integral. How do we interpret this? Similarly to the logarithmic case, we are free to throw away the UV behaviour. However, here we lack the previous problem in the IR. In the IR the integral behaves as μ^2 and so falls to 0 in the limit $\mu^2 \to 0$. We therefore see a general strategy - we can completely remove UV divergences and only consider the finite part by taking $\mu^2 \to 0$ and trading $\mu \to \mu_R$ where we find a fictitious IR divergence. In FDR, polynomial divergences decouple completely and UV divergences leave behind a dependence on the renormalization scale.

One may wonder how the discarding of the vacuum relates to a canonical picture of counterterms and order by order renormalization. Naturally, one must be able to view the FDR procedure at any given order of perturbation theory from the perspective of counterterms, even though their explicit calculation is unnecessary. We understand that this is the case as the method renders amplitudes finite in a gauge invariant and unitary way. However, we shall see in chapter 4 that for this to be true the discussion of the global prescription we made in section 3.2.1 is incomplete without a further "sub-prescription" which we present in section 4.4.

3.2.3 Extra Integrals

"Extra integrals" in FDR are integrals with a μ^2 in the numerator. These terms are necessary to maintain gauge invariance and are defined through the need to be algebraically consistent. Consider the extra integral

$$g^{\mu\nu} \int [\mathrm{d}^4 q] \frac{-\mu^2}{(\overline{q}^2 - M^2)^3} = \int [\mathrm{d}^4 q] \left(\frac{g^{\mu\nu}}{(\overline{q}^2 - M^2)^2} + \frac{g^{\mu\nu} M^2 - 4q^\mu q^\nu}{(\overline{q}^2 - M^2)^3} \right).$$
(3.2.16)

We can see that though naively one would drop these extra integrals as $\mu^2 \to 0$, this is inconsistent with the right hand side of (3.2.16). Here, the implication is that these integrals must be expanded *as if* the μ^2 were a q^2. This is made more intuitive by writing the above as

$$g^{\mu\nu} \int [\mathrm{d}^4 q] \frac{\mu^2}{(\overline{q}^2 - M^2)^3} = - \int [\mathrm{d}^4 q] \frac{\overline{q}^2 - 4q^\mu q^\nu}{(\overline{q}^2 - M^2)^3}.$$
(3.2.17)

Typically one will want to calculate these through the following trick of converting them into vacuum integrals, resulting in a simple calculation. Consider the following integral as an example of the mechanism

$$\int [\mathrm{d}^4 q] \frac{\mu^2}{(\overline{q}^2 - M^2)^3} = \lim_{\mu^2 \to 0} \mu^2 \int d^n q \left(\frac{1}{(\overline{q}^2 - M^2)^3} - V_6 \left[\frac{1}{(\overline{q}^2 - M^2)^3} \right] \right),$$
(3.2.18)

where we use the notation V_6 to note that once the μ^2 is factorized out, we find the vacuum part of the integrand *as if* it were in 6 dimensions.[5] The first term on the RHS is manifestly zero as there is no IR divergence as $\mu^2 \to 0$. The second term is to be expanded performing the expansion as if

---

[5]This is reminiscent to the idea of representing $\tilde{q}^2$ integrals as "dimensionally shifted" integrals in DR.

the μ^2 was a q^2. Computing this we find:

$$\int [\mathrm{d}^4 q] \frac{\mu^2}{(\bar{q}^2 - M^2)^3} = -\int d^4 q \frac{\mu^2}{\bar{q}^6}, \qquad (3.2.19)$$

where we have directly taken $n \to 4$ as the integral is UV finite. This is now a rather simple integral to perform. At any loop order one can see that a similar treatment is always available. The vacuum structure of an n loop integral, due to the defining expansion, is always in terms of n loop vacuum bubbles and arbitrary integrals in up to $(n-1)$ loops. This indicates that the calculation of the extra integrals is always a degree easier than computing the other integrals. For this reason they are often regarded as trivial. At one loop we can see that *all* extra integrals are given by constants. Further, at one loop one can see a relationship between the extra integrals in FDR and the ϵ-dimensional integrals in DR, for example

$$\int d^n q \frac{\tilde{q}^2}{(q^2 - M^2)^3} = -\int [\mathrm{d}^4 q] \frac{\mu^2}{(q^2 - M^2)^3}. \qquad (3.2.20)$$

This holds true at one loop for any naive swapping of $\mu^2 \to -\tilde{q}^2$.

When extending to two loop the same logic as in the one loop case applies. However, for reasons very similar to that demonstrated in equation (3.2.16) the μ^2 at two loop must now be labelled to indicate to which loop momenta it corresponds. That is the label tells us if we should expand as if it were a q_1, q_2 or q_{12} (as these expansions are not the same). For example, this means that:

$$\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\mu_1^2}{\overline{D}_1^3 \overline{D}_2 \overline{D}_{12}} \neq \int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\mu_2^2}{\overline{D}_1^3 \overline{D}_2 \overline{D}_{12}}$$
$$\neq \int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\mu_{12}^2}{\overline{D}_1^3 \overline{D}_2 \overline{D}_{12}}, \qquad (3.2.21)$$

where $\overline{D}_i = \bar{q}_i^2 - M_i^2$, as the differing expansions give different results for each μ_i^2. It should be noted that when calculating the extra integrals at two loop using this trick to relate them to the "negative vacuum", as in equation (3.2.18), we find that term by term the integrals are not UV convergent - only the resultant expression is.

3.2.4 Power Counting at 2-Loop

In order to be able to perform an FDR defining expansion at two loops, one needs to have a solid understanding of when the integrals are convergent and

when they are divergent. The principal tool that we have for this is power counting. At one loop this situation is quite simple. Consider the integrand

$$J = \frac{q^\mu q^\nu}{(q^2 - m^2)^2}. \tag{3.2.22}$$

To understand whether or not this integral is divergent we consider what happens in the large loop momentum limit. One can see that this can be performed by simply counting the "power" of q. If this power is sufficiently large then the integral diverges, and below a dimensionality dependent threshold it converges. In this example the numerator contributes $+2$ and the denominator contributes -4. Interpreting this as a 4-dimensional Feynman integral then the measure of phase space also contributes $+4$. This integral therefore has a degree of divergence of $+2$, commonly referred to as "quadratically divergent". At one loop we calculate our discriminant α through

$$\alpha = (\text{Power of } q \text{ in denominator}) - (\text{Power of } q \text{ in numerator}). \tag{3.2.23}$$

In 4 dimensions our integral is convergent if and only if $\alpha > 4$.

When moving to 2-loop this logic becomes more complicated. There is more than a single way for the large momentum behaviour to cause the integral to diverge. It is enough[6] to consider the two loop vacuum bubble with equal internal masses M

$$\frac{1}{(q_1^2 - M^2)^{\alpha_1/2}(q_2^2 - M^2)^{\alpha_2/2}(q_{12}^2 - M^2)^{\alpha_{12}/2}} \tag{3.2.24}$$

Here our two loop momenta are q_1 and q_2 with their sum cast as $q_{12} = q_1 + q_2$. The operative question is for what values of $\{\alpha_i\}$ is this integral divergent? First we consider all loop momenta to be large. Assuming that we are working in a 4-dimensional theory, then parameterizing with a cutoff Λ, the large momentum behaviour is

$$\Lambda^{8-(\alpha_1+\alpha_2+\alpha_{12})}. \tag{3.2.25}$$

If this exponent is non-negative, then the integral is said to be "globally divergent". We also have to consider the case where one loop momentum is larger than another. This can happen in three ways, but can be exemplified by considering q_2 fixed and q_1 large. In this situation, q_{12} effectively becomes q_1 and we find that the large momentum behaviour of this sub-integral is

$$\Lambda^{4-(\alpha_1+\alpha_{12})}. \tag{3.2.26}$$

---

[6]Tensorial cases follow from calculating α_i by also subtracting the powers of appropriate q_i in the numerator.

If this exponent is non-negative then the integral is said to be "sub-divergent". In general we can say that our integral is (UV) convergent if the following conditions are met

$$\alpha_1 + \alpha_2 + \alpha_{12} > 8 \qquad (3.2.27)$$
$$\alpha_1 + \alpha_2 > 4 \qquad (3.2.28)$$
$$\alpha_1 + \alpha_{12} > 4 \qquad (3.2.29)$$
$$\alpha_2 + \alpha_{12} > 4. \qquad (3.2.30)$$

There is a second style of power counting to consider, the IR power counting. This helps us understand the style of IR divergence that an integral is displaying. An IR divergence can occur in all three of the loop momenta - q_1, q_2 or q_{12}. Consider the two loop vacuum bubble with some massless propagators

$$\frac{1}{(q_1^2)^{\beta_1/2} D_1^{\gamma_1/2} (q_2^2)^{\beta_2/2} D_2^{\gamma_2/2} (q_{12}^2)^{\beta_{12}/2} D_{12}^{\gamma_{12}/2}}, \qquad (3.2.31)$$

where $D_i = q_i^2 - M^2$. We can see that UV power counting parameters $\alpha_i = \beta_i + \gamma_i$, but this view gives us an explicit handle on massless propagators. Here we wish to consider the divergence behaviour as $q_i \to 0$, an IR divergence. If we parameterize the behaviour with a vanishing mass μ we can see that the propagators protected by masses do not contribute and it behaves as

$$\lim_{\mu \to 0} \mu^{4-\beta_i}. \qquad (3.2.32)$$

This is only divergent if $\beta_i \geq 4$. In FDR it is not necessary to create an IR divergence, through the vacuum expansion, that is "worse" than logarithmic. That is, we should never find an integral with $\beta > 4$.

There is a final style of two loop power counting that proves useful in the algorithm for the defining expansion given in section 3.2.5. We often wish to consider if the integral over q_i is convergent ignoring the other factors. That is, we could consider if the integral over q_1 is convergent ignoring *both* q_2 and q_{12}. This is given by the simple 1 loop power counting of $\alpha_i > 4$.

3.2.5 Two Loop Defining Expansion Algorithm

The FDR defining expansion is the crux of the FDR technique. Its existence allows for a finite, shift invariant regularization of Feynman integrals. Naively one should only need to apply the defining expansion to the master integrals of any given process. Instead of automating this, therefore,

one could simply perform this process by hand and provide a dictionary of two loop integrals calculated in FDR. However the extra integrals make this somewhat more difficult as the set of "master" extra integrals is quite large. Nevertheless, as indicated in section 3.2.3, they are quite simple to calculate in terms of vacuum integrals. Therefore, in any practical calculation one will need an automated way of performing the FDR defining expansion in order to calculate the extra integrals. What's more, in section 4.5 it will prove invaluable as we can immediately find the 2-loop vacuum and so all work is reduced to manipulating 2-loop vacuum bubble integrals and arbitrary one-loop integrals, thereby avoiding a large amount of 2-loop work. In this section we will present a systematic implementation of the FDR defining expansion at both one and two loops. These have been implemented in a FORM code which is used extensively throughout the work of this chapter.

We will begin with the algorithm for calculating the FDR vacuum of a given 1-loop integral. This is relevant also for two loop calculations as one calculates factorizable two loop integrals by separately calculating each factor, which is defined through the one loop vacuum subtraction. The structure of the algorithm is inherently recursive. Roughly speaking, one judiciously applies the stepwise Taylor expansion identities to an expression term by term until each term is either a vacuum or a UV finite integral. The truly important part of the algorithm, therefore is how to make this judicious expansion - what operation should one make on an arbitary term? The rest of the algorithm is simply ensuring that we make these operations termwise and stop when we have the vacuum. In order to focus on the important part, let us factor away the logic of the algorithm that makes it termwise and terminating by defining two functions - V and V_s. The first function V, when applied to any linear combination of Feynman integrals such as a (sub-)amplitude or even a single integral will give its vacuum part. The function V_s however will only make a single step towards this goal, applying whichever expansion identity is necessary only once. Letting our expression be a sum over Feynman integrals I_i, and denoting any FDR vacuum integral as I_{vac}, any UV divergent integral which is not a vacuum integral as I_{inf} and any UV finite integral integral be as I_{fin} we can write our algorithm as

$$V\left[\sum_i I_i\right] = \sum_i V[I_i], \tag{3.2.33}$$

$$V\left[I_{\text{inf}}\right] = V[V_s[I_{\text{inf}}]], \tag{3.2.34}$$

$$V\left[I_{\text{vac}}\right] = I_{\text{vac}}, \tag{3.2.35}$$

$$V\left[I_{\text{fin}}\right] = 0. \tag{3.2.36}$$

Equation (3.2.33) tells us we should apply the algorithm term by term. That is, for any sum of integrals we should apply the function to each integral. In the language of functional programming we are "mapping" the vacuum function over our expression. Equation (3.2.34) tells us to apply the stepwise identities and perform vacuum extraction on the output. Equations (3.2.35) and (3.2.36) tell our function to stop processing a given integral if it is either a vacuum term or finite[7] (throwing away the finite terms)[8]. In this way the function will terminate when all remaining terms are vacuum integrals. As in standard functional style, the definitions are applied top to bottom, what's more the definitions of V_s must be applied in the order in which they are presented. The key is now to define V_s. In the one loop case this is simple - one applies the FDR fundamental identity [9]

$$\frac{1}{(\overline{q}+p)^2 - M^2} = \frac{1}{\overline{q}^2}\left[1 + \frac{M^2 - 2q\cdot p}{(\overline{q}+p)^2 - M^2}\right] \qquad (3.2.9)$$

To clarify the distinction between V and V_s as well as the action of V_s, consider the following equations

$$V\left[\frac{1}{(\overline{q}^2 - M^2)^2}\right] = \frac{1}{\overline{q}^4}, \qquad (3.2.37)$$

$$V_s\left[\frac{1}{(\overline{q}^2 - M^2)^2}\right] = \frac{1}{\overline{q}^2(\overline{q}^2 - M^2)} + \frac{M^2}{\overline{q}^2(\overline{q}^2 - M^2)^2}. \qquad (3.2.38)$$

As you can see, the one loop implementation is remarkably simple. This is because there is only one possible course of action and the resulting integrals are *always* more UV convergent or more "vacuum like". What is more, as the algorithm never uses the FDR fundamental identity on finite integrals it is impossible to create a fictitious IR divergence which is "worse" than logarithmic. At two loop this is not the case.

To begin constructing a two loop algorithm, we must introduce a number of identities that help us extract the sub-vacua. These identities effectively perform a stepwise Taylor expansion for the case $q_i > q_j$. The numerator

---

[7]See section 3.2.4 for the required definitions of UV finite and infinite.

[8]The generalisation of the algorithm to only give the finite part or simply split into finite and vacuum are achieved by setting the appropriate parts to zero.

[9]There is some ambiguity in which propagator to apply this to, but (importantly) it doesn't actually matter. At the level of an efficient implementation, however, it is wise to first choose the momentumless propagator.

identities are trivial, but worth stating for completeness

$$q_1 = q_{12} - q_2 \tag{3.2.39}$$

$$q_2 = q_{12} - q_1 \tag{3.2.40}$$

$$q_{12} = q_1 + q_2 \tag{3.2.41}$$

In the context of the defining expansion, it is important to view these as (simple) Taylor expansions when a given loop momentum is large. From this perspective, there should be 6 identities, but 3 are redundant as addition is commutative. The denominator Taylor expansions are more complicated and are given by

$$
\begin{aligned}
\frac{1}{\bar{q}_1^2} &= \frac{1}{\bar{q}_2^2}\left(1 - \frac{q_{12}^2 - 2q_2 \cdot q_{12}}{\bar{q}_1^2}\right), \\
\frac{1}{\bar{q}_1^2} &= \frac{1}{\bar{q}_{12}^2}\left(1 + \frac{q_2^2 + 2q_1 \cdot q_2}{\bar{q}_1^2}\right), \\
\frac{1}{\bar{q}_2^2} &= \frac{1}{\bar{q}_1^2}\left(1 - \frac{q_{12}^2 - 2q_1 \cdot q_{12}}{\bar{q}_2^2}\right), \\
\frac{1}{\bar{q}_2^2} &= \frac{1}{\bar{q}_{12}^2}\left(1 + \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_2^2}\right), \\
\frac{1}{\bar{q}_{12}^2} &= \frac{1}{\bar{q}_2^2}\left(1 - \frac{q_1^2 + 2q_1 \cdot q_2}{\bar{q}_{12}^2}\right), \\
\frac{1}{\bar{q}_{12}^2} &= \frac{1}{\bar{q}_1^2}\left(1 - \frac{q_2^2 + 2q_1 \cdot q_2}{\bar{q}_{12}^2}\right).
\end{aligned}
\tag{3.2.42}
$$

These can be derived through the application of (3.2.9) to expand each loop momenta when another is large. To understand, consider the q_{12} identities. It is quite natural to think of expanding q_{12} around the point where either q_1 or q_2 is big. The remaining four are analogous as shift invariance makes the naming of the integration variables q_1, q_2 or q_{12} arbitrary. So, for example, expanding q_1 around the point where q_2 is large involves the use of q_{12}. Naively speaking, one may expect it wise to give these identities raised to an arbitrary power. However, we do not write them in this way in order to be compact, it is important algorithmically to only perform one step and then reassess the UV convergence of the integrals. With the correct tools at hand, we can now describe the algorithm. Inherently there will be a lot of symmetry in the different loop momenta as we engage in the following discussion - so it is best to keep this in mind.

The first step is to attempt to apply the FDR fundamental identity in order to make a single sub-integral finite, without introducing IR divergences

worse than logarithmic. That is, once the sub-integral is finite we move onto the next step. (Once a sub-integral is finite we wish to arrange the other factor to be a 1-loop vacuum.) Letting f_i be the application of the fundamental identity in loop momenta i, we have the following definition for all of the loop momenta, $\{q_i\}$

$$V_s[I] = f_i[I], \quad \text{where } I \text{ has an infinite sub-integral } i. \tag{3.2.43}$$

Note that this rule is enough to generate the global vacuum. The rule generates at least two terms, the first has exactly the same UV convergence as the original I, and, only thinking of the first term, repeated application of this rule to it will move through all of the loop momenta q_i until the resulting first term is a global vacuum. The sub-integral finiteness condition effectively stops us from applying the fundamental identity when we would create an IR divergence worse than $\log(\mu^2)$. For example, the correct thing to do here

$$\frac{1}{\overline{q}_1^4 \overline{D}_1 \overline{D}_2 \overline{D}_{12}}, \tag{3.2.44}$$

where $\overline{D}_i = \overline{q}_i^2 - M^2$, is to apply f_2 or f_{12}.

The next step deals with an "extreme" case which can occur - a divergent integral with two finite sub-integrals. Consider

$$\frac{(q_1 \cdot p)(q_2 \cdot p)^3 (q_{12} \cdot p)}{\overline{q}_1^4 \overline{D}_1 \overline{D}_2 \overline{q}_{12}^4 \overline{D}_{12}} \tag{3.2.45}$$

This integral has finite sub-integrals if we consider the sub-integral in q_1 or in q_{12}. It is divergent for fixed q_1 and large q_2, i.e. $\alpha_2 = 1$. We cannot proceed with denominator expansions as we will always reach an unnacceptable IR divergence. This is rectified by making an appropriate numerator Taylor expansion, (here $q_2 = q_{12} - q_1$). Writing the n_i to mean "expand numerator momentum i", we can generally fix this problem by performing

$$V_s[I] = n_i[I], \quad \text{where } I \text{ has two finite sub-integrals, not in } q_i. \tag{3.2.46}$$

The next, and more obvious task given that we have a convergent sub integral, is to try and arrange the other factor to be a one loop divergence. The first step is to remove all scales

$$V_s[I] = f_{i \neq j}[I] \quad \text{where } I \text{ has a finite sub-integral in } q_j. \tag{3.2.47}$$

This is the step that could move us into the situation of having two finite sub-integrals, whilst still being UV divergent. From now, if our integral is

still divergent then it is in quite a simple form - all scales must be in the convergent sub-integral. We can write a schematic version of the general form as

$$f(q_1)\frac{q_2^a q_{12}^b}{q_2^{2c}\bar{q}_{12}^{2d}},$$ (3.2.48)

where a and b represent powers of the loop momenta. The remaining task is then to expand in such a way as to write the divergence of this factor as a one loop vacuum. One "problem" is that if at any point we split $q_2 = q_{12}-q_1$ (or similarly for q_{12}) then it is quite possible for this to render infinite our finite sub-integral in q_1. However, this possibility is handled by the next application of V_s which attempts all of the previous steps first. The choice of whether to expand in q_2 or q_{12} here is arbitrary, so we are free to chose. However, our choice must be made in in such a way that a few steps down the line, we do not make the opposite choice. That is, any good algorithm cannot make an expansion considering q_2 large and then a few terms later decide that q_{12} should be large - one would find an infinite loop. In an effort to reduce the number of terms, our methodology is to expand assuming the most common numerator momentum to be large. In equation (3.2.48) this means that if $b < a$, then we expand q_{12}, otherwise we expand q_2. For the case $b = a$ we make an arbitrary choice based upon lexicographic ranking of the loop momenta. With this criteria, one first applies the numerator identities as much as possible, without creating IR divergences, and when this is no longer possible, one applies the appropriate denominator identity. This can help the IR convergence to permit more numerator expansions. We write this as

$$V_s[I] = n_i[I],$$ where i is the least common numerator momentum not in the finite sub-integral and application of n_i will not produce an IR divergence. (3.2.49)

$$V_s[I] = f_{i/j}[I],$$ where i and j are the momenta not in the finite sub-integral, and i is the least common. (3.2.50)

Here application of $f_{i/j}$ uses the appropriate identity from equation (3.2.42) for expanding q_i in terms of large q_j. This completes the algorithm definition - application will exactly extract the FDR vacuum part. Moreover, the description we have given here can be applied without any re-naming of the loop momenta, and so one can always check that the final integrand and the initial integrand agree numerically.

Chapter 4

QCD Renormalization in FDR

Having introduced the FDR approach in the previous chapter we now wish to discuss the investigation we undertook within said framework. In order to make predictions in a model with a given regularization scheme, one must have control over the running of the appropriate parameters. As of yet it has not been possible to undertake two loop studies within FDR which require this control of the two loop renormalization, and we wish to rectify this.

Our aim, therefore, in this chapter is to compute the relation between $\mathrm{DR}/\overline{MS}$ and FDR within QCD caused by the different treatments of the UV divergences [76]. In principle this allows us to quickly and universally establish the difference between results computed in $\mathrm{DR}/\overline{MS}$ and FDR once the differences in infra-red divergent structure have been taken into account. We should note that understanding of the infra-red structure of FDR regulated amplitudes is equally important, for example, one needs an IR translation in order to be able to map to PDFs. Both translations are needed and we simply choose to start with the UV structure.

In this chapter we will begin by reviewing the canonical method of renormalization with counterterms, and then we will move on to setting up and performing the calculation of the effective FDR renormalization constants in DR. Schematically this is achieved by computing in DR the vacuum part of the integrand which is thrown away by the FDR defining expansion.

4.1 QCD and Canonical Renormalization

4.1.1 The Renormalized QCD Lagrangian

In section 1.2 we introduced the Lagrangian of QCD and although this is what one finds when trying to write down a picture consistent with gauge invariance, after gauge fixing, this necessarily breaks down when one attempts to use it to calculate higher order corrections. Canonically[77], in order to extract physical results from the naively infinite calculations one must have a proper procedure to both regulate and renormalize the theory. For our discussion of canonical renormalization we shall consider our calculations to be regulated in dimensional regularization. Renormalizing our theory involves taking the original quantities in the theory and rewriting them in terms of their renormalized counterparts. In QCD this involves the following renormalizations

$$
\begin{aligned}
G_{\mu a}^B &\to Z_G \cdot G_{\mu a}^R, \\
g^B &\to Z_g \cdot g^R, \\
\psi^B &\to Z_\psi \cdot \psi^R, \\
c_a^B &\to Z_c \cdot c_a^R, \\
\xi^B &\to Z_\xi \cdot \xi^R.
\end{aligned}
\tag{4.1.1}
$$

Our renormalized Lagrangian is now markedly less simple. Terms that previously could be factored together (for example the gauge covariant derivative and the field strength tensor) no longer display this structure in the renormalized theory

$$
\begin{aligned}
\mathcal{L}_{\text{QCD}} = {}& iZ_\psi^2 \cdot \overline{\psi}^{Rf} \gamma_\mu \partial^\mu \psi^{Rf} \\
& + Z_G^2 \cdot \frac{1}{2} \left(\partial^\nu G_{\mu a}^R \partial^\mu G_{\nu a}^R - \left[1 - \frac{1}{\xi^R} \right] \left(\partial^\mu G_{\mu a}^R \right)^2 \right) \\
& + Z_\psi^2 Z_G \cdot g^R \overline{\psi}_j^{Rf} t_{jk}^a \gamma^\mu G_\mu^{Ra} \psi_k^{Rf} \\
& + Z_g Z_G^3 \cdot g^R \left(\partial^\mu G^{R\nu a} \right) G_{\mu b}^R G_{\nu c}^R f^{abc} \\
& - Z_g Z_G^4 \cdot \frac{1}{4} g^R G_{\mu a}^R G_{\nu a}^R G^{R\mu b} G^{R\nu c} f^{abe} f^{cde} \\
& + Z_c^2 \cdot \partial_\mu \overline{c}^{Ra} \partial^\mu c^{Ra} \\
& + Z_g Z_c Z_G \cdot g^R f^{abc} \left(\partial^\mu \overline{c}^{Ra} \right) G_\mu^{Rb} c^{Rc}
\end{aligned}
\tag{4.1.2}
$$

This process of renormalization is then performed in such a way that our Green's functions are finite. In the case of the gauge fixing renormalization,

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in order to keep the renormalized propagator finite we are constrained to have $Z_\xi = Z_G^{-2}$ [78][1]. Note that whilst the structure is complicated, the number of operators that we have is much larger than the degrees of freedom which we have. That is, the renormalization constants for different operators are related. These relations are known as Slavnov-Taylor identities[79]. The structure of the renormalization is protected by the gauge invariance of the bare theory.

4.1.2 Perturbative Renormalization and Counterterms

In order to make a perturbative treatment of this renormalized theory we are forced to rewrite equation (4.1.2). We can only perform a perturbative expansion around the free theory, and we don't a priori know the values of the renormalization constants - we must determine these perturbatively. This forces us to write our Lagrangian in two parts. The first very similar to the bare Lagrangian, but now in terms of renormalized fields, and the second is known as the counterterm Lagrangian.

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^R + \mathcal{L}_{\text{Counterterms}}, \tag{4.1.3}$$

$$\begin{aligned}
\mathcal{L}_{\text{QCD}}^R = &\, \overline{\psi}_j^{Rf} i\gamma_\mu D_{jk}^\mu \psi_k^{Rf} - \frac{1}{4} G^{\mu\nu a R} G_{\mu\nu a}^R \\
&+ \partial_\mu \bar{c}^{aR} \partial^\mu c^{aR} + g^R f^{abc} \left(\partial^\mu \bar{c}^{aR} \right) G_\mu^{bR} c^c \\
&- \frac{1}{2\xi_R} \left(\partial_\mu G^{\mu a R} \right),
\end{aligned} \tag{4.1.4}$$

$$\begin{aligned}
\mathcal{L}_{\text{Counterterms}} = &\, i \left(Z_\psi^2 - 1 \right) \cdot \overline{\psi}^{Rf} \gamma_\mu \partial^\mu \psi^{Rf} \\
&+ \left(Z_G^2 - 1 \right) \cdot \frac{1}{2} \left(\partial_\nu G^{\mu a} \partial_\mu G^{\nu a} - \left(1 - \frac{1}{\xi_R} \right) (\partial_\mu G^{\mu a})^2 \right) \\
&+ \left(Z_\psi^2 Z_G - 1 \right) \cdot g^R \overline{\psi}_j^{Rf} t_{jk}^a \gamma^\mu G_\mu^{Ra} \psi_k^{Rf} \\
&+ \left(Z_g Z_G^3 - 1 \right) \cdot g^R \left(\partial^\mu G^{\nu a R} \right) G_{\mu b}^R G_{\nu c}^R f^{abc} \\
&- \left(Z_g Z_G^4 - 1 \right) \cdot \frac{1}{4} g^R G_{\mu a}^R G_{\nu a}^R G^{R\mu b} G^{R\nu c} f^{abe} f^{cde} \\
&+ \left(Z_c^2 - 1 \right) \cdot \partial_\mu \bar{c}^{aR} \partial^\mu c^{aR} \\
&+ \left(Z_g Z_c^2 Z_G - 1 \right) \cdot g^R f^{abc} \left(\partial^\mu \bar{c}^{aR} \right) G_\mu^{bR} c^{cR}.
\end{aligned} \tag{4.1.5}$$

The Feynman rules for this renormalized theory are the same as the set in the bare theory given in appendix A.1 (with the replacement $g \to g_R$), but

---

[1]This is only for a minimal scheme. More generally we can set $Z_\xi = (c+1)Z_G^{-2}$ for finite c, but the only result that this has is to change the counterterm to a form that is both dependent on ξ and non-transverse.

we also have a new set of counterterm Feynman rules which can be found in appendix A.4.

The Z_i are then defined in such a way that we remove all UV divergences. For a given theory, it is not trivial that one necessarily *can* remove the divergences without introducing new terms in the bare Lagrangian and the property of a theory which allows this is called **renormalizability**. In order to realize this within our standard perturbative framework one performs calculations order by order, defining the Z_i such that the counterterm diagrams cancel the divergences. That is, we expand the renormalization constants in our theory as

$$Z^i = 1 + \alpha_S^R Z_1^i + \left(\alpha_S^R\right)^2 Z_2^i + \cdots .$$

(4.1.6)

In order to represent this expansion in the diagrammatic expressions, we will represent the order of the renormalization that enters by an appropriate number of crosses in the counter-term insertion.

At one loop this appears as a simple subtraction. For example, consider the renormalized gluon propagator

(4.1.7)

As stated in appendix A.4 the exact form of the counterterm is

$$(Z_{GG} - 1) \left(p^\mu p^\nu - g^{\mu\nu} p^2\right) .$$

(4.1.8)

This is an interesting, and quite constraining consequence of renormalizability. For equation (4.1.7) to be finite, the pole structure of the remainder of the expression must be cancelled by the counterterm. A priori simply imposing Lorentz covariance tells us that we have these two tensor structures. Renormalizability gives us a relation between the coefficients of the pole parts of each structure[2]. This gives us quite a strong check on our result.

This extra diagram that has turned up can be thought of as a tree-level like contribution (this is more easily seen in vertex renormalizations). When one moves to two and higher loop orders, one finds that there are contributions of the form of all lower orders. As an example, let us consider

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[2]In case of the propagator this is also true for the finite part, however this follows from the extra constraint of gauge invariance. In higher point correlation functions gauge invariance is not so constraining but renormalizability still implies a tree level structure for the pole part.

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the counterterms that contribute to the two-loop renormalization of the gluon propagator
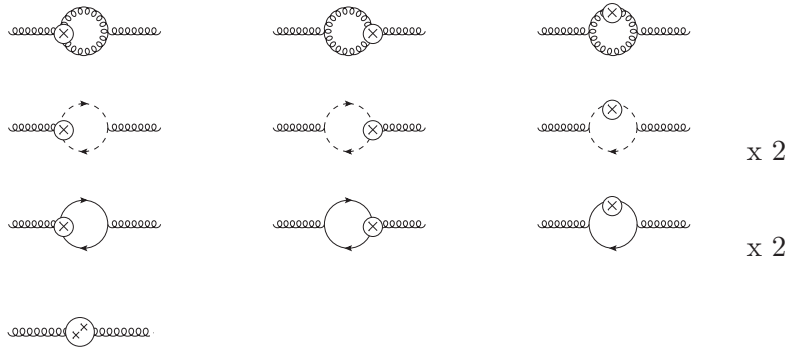
As we can see, here we not only get extra diagrams similar to the tree level (the double crossed counterterm) but also a plethora of diagrams similar to the one loop. These play an important role in our calculation. When computing a bare two-loop diagram, one finds a more complicated divergence structure than in the one-loop case. In the one loop case, the poles of any given integral were always "local" - that is, they were proportional to a polynomial in the relevant scales, e.g. p^2/ϵ. When moving to two loop the bare amplitudes contain both double and single UV poles. The double UV poles are still local, but the single UV poles can be "non-local" which means that they can depend on a logarithm of a relevant scale, e.g. $\log(p^2)/\epsilon$. Absorbing logarithms into the renormalization constants Z_i is not possible because the theory would become non-local under radiative corrections. However, the loop counterterms in equation (4.1.2) give us hope. They contain insertions of the 1-loop renormalization, and so themselves contain terms of the form $\log(p^2)/\epsilon$. If the theory is suitably renormalizable (as QCD is) these loop diagrams then give us the exact contribution required to make the resulting contribution to the second order CT completely local.

Conversely, imagine that for some reason these arguments about locality did not concern you. Do we still need these loop counterterms? Could we not simply define a subtraction that removes all poles, regardless of their locality? The answer, however, is no. These loop counterterms also provide a route to maintaining gauge invariance and unitarity in the finite part.

Consider the simple example of one particle *reducible* N_f corrections to the gluonic propagator in QCD at two-loop. This is a gauge invariant set of diagrams and so it is fair to consider on its own. As this is factorizable the

correct approach is to calculate the one loop bare N_f contribution to $O(\epsilon)$

$$\text{(diagram)} \quad = i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{1}{\epsilon}\Pi_{-1} + \Pi_0 + \epsilon\Pi_1\right) \quad \text{(4.1.9)}$$

and then we "square" it

$$\text{(diagram)} \tag{4.1.10}$$

$$= i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{1}{\epsilon^2}\Pi_{-1}^2 + \frac{1}{\epsilon}2\Pi_{-1}\Pi_0 + \Pi_0^2 + \Pi_{-1}\Pi_0 + O(\epsilon)\right)$$

In order to calculate the renormalized result, we can make use of the factorizability again and realize that the 1PR set of diagrams is the square of the renormalized one loop. The renormalized one loop in $\overline{MS}$ subtracts the pole to give

$$\text{(diagram)} \quad + \quad \text{(diagram)} \tag{4.1.11}$$

$$= i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\Pi_0 + O(\epsilon)\right)$$

Therefore the renormalized two loop result is simply given by

$$= i\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\Pi_0^2 + O(\epsilon) \tag{4.1.12}$$

From this it is quite clear that simply subtracting poles is not enough - the counterterms also remove finite constants necessary for maintaining unitarity. It is necessary to remove these extra constants in this case because, DR does not recognize that there are some ϵ/ϵ terms which are unnecessary.

4.2 FDR vs Renormalization

4.2.1 FDR as a Canonical Renormalization Scheme

In order to find the relationship between an FDR style subtraction and a standard counterterm picture we must find a way to define both of them

in a single calculation. The way we achieve this is by considering the value of the FDR amplitude as the definition of the FDR "scheme". To find the renormalization constants in this scheme we start with an amplitude renormalized with unspecified constants and then *define* them to coincide with the FDR amplitude.

To be able to do this we will need to be able to relate a bare dimensionally regulated amplitude and an FDR renormalized amplitude. Naively, as the FDR integral can be written as the difference of two DR integrals we expect some connection. However, upon closer inspection we notice a difficulty because the two methods treat infra-red divergences differently. Our aim is to understand the renormalization procedure so IR information is extraneous. However, disentangling the two can be tricky as in dimensional regularization one can often find cancellation between IR and UV poles. For example consider the one loop integral

$$\int d^n q \frac{1}{q^4} \tag{4.2.1}$$

This is zero because the poles of two different origins cancel. In order to avoid disentangling these two structures we can work in off-shell kinematics. The amplitudes are only IR divergent when on-shell and so by working off shell we have nothing to disentangle. What's more this offers us a powerful relation which will lead us to the FDR integral. Consider the following identity

$$\int d^n q \lim_{\mu^2 \to 0} \frac{1}{(q^2 - m^2 - \mu^2)} = \lim_{\mu^2 \to 0} \int d^n q \frac{1}{(q^2 - m^2 - \mu^2)}. \tag{4.2.2}$$

This is an example of a general principle - if one adds a fictitious mass to an IR convergent integral then the result is the same if it is removed before or after the act of integration.

With these ideas in hand, let us now move to our task - constructing an FDR amplitude from a bare, dimensionally regulated (sub-)amplitude, J. An important consideration here is the different state counting relationships between DR and FDR. In general $g^{\mu\nu} g_{\mu\nu} = n_\gamma$ where in DR $n_\gamma = n$ and in FDR $n_\gamma = 4$. Naively this could cause problems, so we shall keep this in mind in the following treatment. Our first step in making this look more "FDR-like" is to perform the global prescription $J \to \bar{J}$

$$\int d^n q_1 d^n q_2 J(q_1, q_2) = \lim_{\mu^2 \to 0} \int d^n q_1 d^n q_2 \bar{J}(q_1, q_2). \tag{4.2.3}$$

As we avoid the problem of IR divergences by calculating off shell we can take the μ^2 limit after integration. To be clear - the symbol $\bar{J}$ represents introducing the ficitious mass μ^2 in the same way which one would perform the FDR global prescription[3]. We now perform the FDR defining expansion on this expression, separating finite from vacuum at the integrand level to give

$$\lim_{\mu^2 \to 0} \int d^n q_1 d^n q_2 \left[J_F(q_1, q_2) + J_V(q_1, q_2) \right]. \tag{4.2.4}$$

Here we can return to the "problem" of different state countings in FDR vs DR. In the finite part J_F there are no $1/\epsilon$ poles by construction, and so the state counting parameter n_γ can be set to four here. Crucially this should not be performed in J_V as we would lose finite terms.

The final step that we must take is to split the above into two integrals. However, we can no longer do this naively as the two terms are not individually IR convergent. This is fixed by the FDR prescription[4] of only taking the $lim_{\mu^2 \to 0}$ in places where μ^2 appears polynomially and taking the limit $\mu^2 \to \mu_R^2$ in places where it appears logarithmically. This successfully takes care of this fictitious IR divergence, such that we end up with

$$\left(\lim_{\mu^2} \int_\epsilon J_F \right) + \left(\lim_{\mu^2} \int_\epsilon J_V \right) = \left(\int [d^4 q_1][d^4 q_2] J \right) + \left(\lim_{\mu^2} \int_\epsilon V[J]|_{n_\gamma = n} \right). \tag{4.2.5}$$

Here the first term is recognized as the FDR integral. The second term cannot be directly identified with the dimensionally regulated FDR vacuum as the algebra required to arrive at this point has taken the state counting parameter $n_\gamma = n$. However, in performing this calculation the technology is equivalent at a practical level to calculating the FDR vacuum. This completes our relating of an off shell, dimensionally regulated, bare amplitude to an off shell FDR amplitude.

At this point we need to discuss renormalizing the bare amplitude in order to give the FDR result. As explained in section 4.1.2, renormalizing the theory introduces higher order interactions which we must include. At one loop, this manifests itself as a constant, which we call Z_1, premultiplying the tree level structure[5], which we call M_0. Writing the bare one loop correlation function as M_1 we find that the renormalized correlation function

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[3]Described in section 3.2.1.

[4]See section 3.2.2.

[5]In the case of two point correlation functions tree-level is a misnomer and we really mean the inverse propagator.

M_1^R is given by

$$M_1^R = M_1 + Z_1 \cdot M_0. \tag{4.2.6}$$

Our task is then to define Z_1 such that M_1^R is the FDR renormalized result. From the opposite perspective, this implies that we can determine the renormalization constant corresponding to the FDR scheme from knowledge of the FDR result, i.e.

$$(Z_1)_{FDR} \cdot M_0 = (M_1)_{FDR} - M_1. \tag{4.2.7}$$

Note that it is non-trivial that the FDR method is equivalent to subtracting a gauge invariant term from the DR regulated bare correlation function M_1, and this sets strong constraints on the FDR method. Our previous discussion of relating the bare correlation function simplifies this determination greatly as using equation (4.2.5) on M_1 we can see that this simplifies to

$$(Z_1)_{FDR} \cdot M_0 = - \left. V[M_1] \right|_{n_\gamma = n}, \tag{4.2.8}$$

where V implies the V term from equation (4.2.5). That is, we can determine the renormalization constant of this FDR "scheme" simply by knowing the structure of the vacuum extraction. It is worth pointing out that, unlike $\overline{MS}$, $(Z_1)_{FDR}$ contains a universal finite part as well as poles.

Promoting this discussion to 2 loop level is made more complicated because of the introduction of loop counterterms. These are diagrams which contain insertions of the one loop Green's function renormalization. In order to have a consistent renormalization scheme description of our regularization then these renormalization insertions *must* be the ones that we have calculated at one loop.

For our FDR scheme, this means that we must (unlike in an $\overline{MS}$ scheme) include constant parts in the counterterm insertions which come from the factors of n that hit poles in the vacuum. Understanding this allows us to write down the two loop analogue of equation (4.2.7)

$$(Z_2)_{FDR} \cdot M_0 = (M_2)_{FDR} - M_2 - \left. (\text{Loop CTs}) \right|_{FDR}, \tag{4.2.9}$$

where Z_2 is the two loop contribution to the renormalization of the correlation function, M_2 is the bare dimensionally regulated two loop correlation function and $(M_2)_{FDR}$ is the two loop FDR regulated correlation function. From this equation we can determine the implied renormalization constant in DR when one uses the FDR method. Once again this simplifies greatly using equation (4.2.5)

$$(Z_2)_{FDR} \cdot M_0 = - \left. V[M_2] \right|_{n_\gamma = n} - \left. (\text{Loop CTs}) \right|_{FDR}. \tag{4.2.10}$$

Similar to the one loop case, we are able to determine $(Z_2)_{FDR}$ only with knowledge of the FDR vacuum. What's more, in order for the FDR method to be a good definition of quantum field theory, we must find dramatic cancellations on the RHS of (4.2.10) in order for it to match the form of the LHS.

4.2.2 Renormalizing Massless QCD in FDR

Here we begin the exposition of the calculation we performed in the framework described in section 4.2.1. It should be noted that these calculations are theory dependent and so if one wished to develop translation rules between $\overline{MS}$ and FDR for a theory other than QCD, the calculation would need to be repeated. In order to establish the UV translation between FDR and DR/$\overline{MS}$ at a given loop order it is sufficient (given universality) to calculate all of the renormalization constants in QCD as given in (4.1.1). As finite integrals in FDR are coincident with unregulated integrals we find the same condition as in $\overline{MS}$: $Z_\xi = Z_G^{-2}$, leaving us with 4 renormalization constants to calculate. Our methodology was to extract them by renormalizing the correlation functions given in figure 4.1. It is enough to only use 4 correlation functions, however, in order to check the consistency of our calculations we have extracted the coupling constant renormalization, Z_{α_S}[6] from all three point vertices in the theory. This explicitly verifies both the universality of the renormalizations and the treatments of bosonic and fermionic sectors.

Let us briefly describe the first few common steps of the calculation at both one and two loop. Just as in our work on integrand reduction at one loop in chapter 2, here the complexity obliges us to employ computer algebra systems. Once again we use QGRAF to generate the diagrams, implementing our QCD model by hand. The momentum routing conventions that we have used for the three point correlation funtions are all momenta k_i incoming, such that

$$k_1 + k_2 + k_3 = 0. \qquad (4.2.11)$$

Next, to generate the expressions corresponding to any given diagram we use FORM, together with an in-house implementation in of the Feynman rules from appendices A.1 and A.4. We should note that, in order to greatly simplify the calculational load, we are working in the Feynman gauge, $\xi_R = 1$.[7] Also note that as we are using the method described in section 4.2.1

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[6]From here onwards, we shall discuss the α_S instead of g_s, where $Z_{\alpha_S} = Z_{g_S}^2$.

[7]The degree of simplification here can be huge. At two loop, a diagram which is a
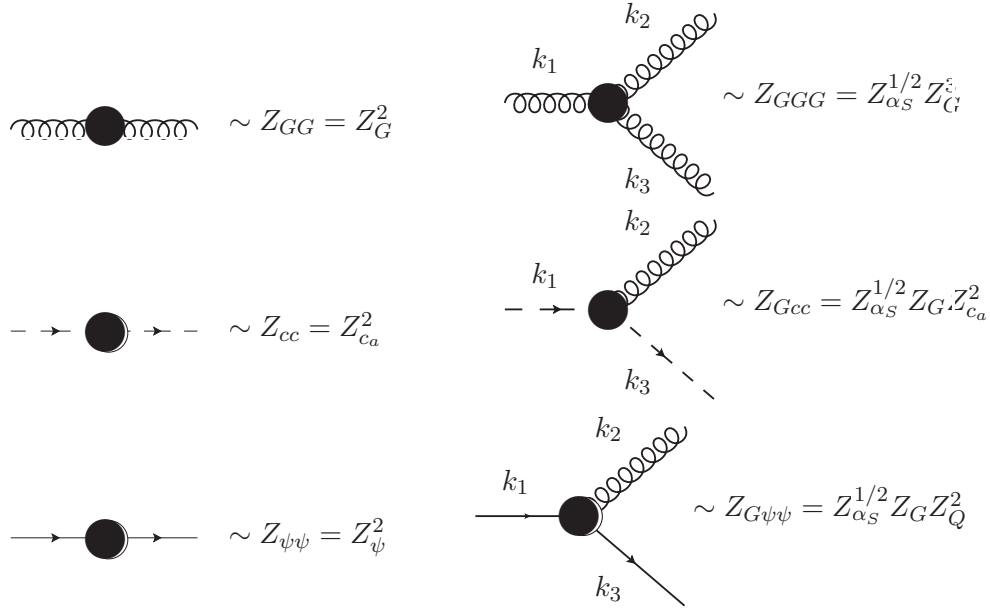
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Figure 4.1: The collection of correlation functions which we will renormalize in order to discern the renormalization constants of QCD in FDR. We also list their relation to QCD's renormalization constants given the Slavnov-Taylor identities. All momenta are incoming.

we are constructing the off-shell dimensionally regulated integral and so we take $g^{\mu\nu} g_{\mu\nu} = n$. Next, with the algebraic expressions corresponding to the diagrams in hand, we find it wise to perform the colour algebra first in order to collect similar integrals.[8] The procedure that we have used is that described in section 2.5.1. At this point we continue the procedure of section 4.2.1 and perform the global prescription and vacuum expansion as described in sections 3.2.1 and 3.2.5 respectively.

All of the work to be described in the following sections was implemented using a mixture of Mathematica and FORM in order to strike a balance between the user friendliness of Mathematica and the speed of FORM.

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single term in the Feynman gauge could be hundreds of terms in the general R_ξ gauge.

[8]This is essentially because we are only considering low-point correlation functions. The most complex colour structure is in the 4 point vertex, but this only occurs at one loop and so is easily manageable.

4.2.3   A One Loop Warmup

We obtain the one loop renormalization constants of QCD in FDR using equation (4.2.8) to calculate them in terms of the one loop vacuum. We remind the reader that we are constructing FDR from the DR integral, and so need to take $g^{\mu\nu}g_{\mu\nu} = n_\gamma = n$, as well as use off shell kinematics, though we choose no special point. The resulting expression is a series of vacuum integrals regulated in dimensional regularization. Examples from the renormalization of the gluon propagator include

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{q^\mu q^\nu}{\bar{q}^6}, \quad \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{q^\mu q^\nu (q \cdot p)^2}{\bar{q}^8}, \quad \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{q^\mu q^\nu (q \cdot p)^4}{\bar{q}^{10}}. \qquad (4.2.12)$$

Importantly they are always of a form where the integral only depends on μ^2. It is interesting to note that we do not have any FDR extra integrals in our vacuum. We can see that this is always the case as after performing the global prescription we can perform cancellations between numerator and denominator without introducing μ^2 terms, we then take the vacuum at this point. At this stage we drop all integrals which represent polynomial divergences due to the FDR prescription of removing the fictitious μ^2, leaving us with only logarithmically divergent integrals to be evaluated at $\mu = \mu_R$ [9]. This expression can be greatly simplified using consistency conditions for 1-loop regularizations[21]. These are relations between tensor integrals of a single mass scale, such of those in equation and (4.2.12), and a scalar integral. In section 4.3.2 we will derive these relations in full generality using integration by parts relations, but at this point the relevant equation is

$$\int d^n q \frac{q^{\alpha_1} \cdots q^{\alpha_{2r}}}{(q^2 - \mu^2)^{r+2}} = 2 * \frac{1}{(2r+2)!!} g^{\alpha_1 \cdots \alpha_{2r}} \int d^n q \frac{1}{(q^2 - \mu^2)^2}. \qquad (4.2.13)$$

Here the $g^{\alpha_1 \cdots \alpha_{2r}}$ is the totally symmetric metric tensor with $2r$ indices, and $m!!$ denotes the double factorial of m. This identity allows us to reduce our vacuum expressions to scalars without introducing any dependence on n such that it is entirely proportional to the fundamental logarithmic divergence in four dimensions -

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\bar{q}^4}\bigg|_{\mu=\mu_R} = -i\pi^2 \left(\frac{2}{\epsilon} + L\right) + O(\epsilon), \qquad (4.2.14)$$

where $L = \gamma_e + \log(\pi)$ (γ_e is the Euler-Mascheroni constant) and $n = 4 + \epsilon$. In this way we can see that the constant part we subtract is entirely due to n_γ hitting a $1/\epsilon$ pole.

The full process to calculate these renormalization constants is then

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[9]See the end of section 3.2.2.

1. Construct the appropriate DR one loop amplitude, keeping $n_\gamma = n$.

2. Perform the global prescription.

3. Perform the FDR defining expansion to find the vacuum.

4. Drop polynomial integrals and use consistency relations to relate to the one loop scalar vacuum.

5. Use equation (4.2.14) to evaluate the scalar vacuum at $\mu = \mu_R$.

6. Use equation (4.2.7) to find the renormalization constant for the correlation function.

The results of this process we give in appendix B.1. The finite part in the renormalization constant gives rise to a coupling constant shift with respect to $\mathrm{DR}/\overline{MS}$ which we describe in section 4.5.

Something further can be divined from the process we have just explored. Starting with a different perspective we can see that the fact that this is always achievable amounts to a proof that, at one loop, the treatment of UV infinities in FDR is equivalent to that of DRed and FDH. More precisely stated - the coupling constant shift elucidated in section 4.5 is the same as in DRed and FDH. Intuitively this is because our calculations tell us that the finite part in our renormalization constant is only due to FDR's state counting $n_\gamma = 4$, which is also the of the finite difference between DRed/FDH and DR.

4.3 Two Loop Renormalization with External Gauge States

At this point, we wish to use the discussion presented in section 4.2.1 to calculate the effective FDR renormalization constants. We present this work in two parts - firstly the correlation functions with external ghosts and gluons which we present here, and secondly the correlation functions with external quarks which we present in section 4.4. The reason for this is that the quark sector of QCD presents extra conceptual difficulties which need to be stressed, however all of the technology which we present in this section is still needed.

Let us begin to discuss the effective FDR renormalization constants in QCD at two loop. The procedure is is similar to the one loop procedure described in section 4.2.3. In principle, however, with the addition of the

loop counterterms there is more to consider and in practice the complexity of the two loop vacuum is much higher. The task is to perform the calculation prescribed by equation (4.2.10). At two-loop to reduce the complexity of the calculation in the three point functions we take advantage of the off-shell nature by using a symmetric off-shell point

$$k_1^2 = k_2^2 = k_3^2 = M^2, \tag{4.3.1}$$

where k_i is the momenta flowing into leg i. This point is euclidean in nature, and has often been used for defining momentum subtraction schemes[80]. For a massless theory this eases our calculation by reducing it to a single scale problem. Note that momentum conservation is still respected and so all scalar products are defined in terms of M^2, for example

$$k_1 \cdot k_2 = -\frac{1}{2}M^2. \tag{4.3.2}$$

In principle there are two distinct parts to be calculated - the dimensionally regulated FDR vacuum (taking $g^{\mu\nu}g_{\mu\nu} = n_\gamma = n$) and the DR loop counterterms. The construction of the relevant expressions has been discussed previously and so here we shall discuss our general strategy for their calculation before delving into the specifics. We perform the work by reducing the integrals to a non over complete basis of integrals and then integrating. The exact nature of the basis differs between the global vacua, sub vacua and loop counterterms. Note that in the case of the sub vacua and loop counterterms a well chosen basis will make it easier to divine any relationship.

In practice the style of work required for the global vacua and the sub-vacua is very different. Inherently the sub-vacua are given by factorizable one loop integrals *with scales* and so a complete one loop toolbox will be necessary. When calculating the global vacuum our endeavour is simpler as we only need to be able to handle two loop vacuum bubbles. For this reason, an explanation of our work is most naturally achieved by splitting the vacuum work into global and sub-vacua.

To begin discussing this work we must first take a detour to discuss some common technologies. Specifically we will need to understand integral reduction techniques and integration by parts identities. These are two technologies which allow us to take a collection of integrals and reduce them to linear combinations of a smaller set of integrals. Within our work these technologies will allow us to greatly simplify expressions and write them in terms of a small number of scalar integrals. We will start with integral reduction in section 4.3.1 and then discuss integration by parts identities in section 4.3.2.

4.3.1 Integral Reduction

Integral reduction is the process of taking an expression defined in terms of a large number of tensor integrals and exploiting their properties to rewrite them in terms of a smaller basis. At one loop these techniques are enough to cast scattering amplitudes in terms of 4 styles of scalar integrals - the tadpole, bubble, triangle and box. Until recently, in NLO calculations, techniques to achieve this style of process were the bottleneck in reaching calculations with large numbers of legs. However, with the recent invention of the OPP method and unitarity methods these can be performed in a fast and automated way. Nevertheless in our work we will not need such calculational power and so we discuss the original formulation of these techniques, invented by Passarino and Veltman [55].

The powerful idea of PV reduction is to exploit the Lorentz covariance of the integrals to rewrite tensor integrals in terms of other integrals whose numerator is given in terms of reducible scalar products - scalar products that can be written in terms of denominators. In our work we will need to apply these relations to integrals with both momentum and mass scales at both one and two loop. To begin we will start with simple examples without momentum scales. These will be necessary when considering vacuum integrals where the only scale is the FDR fictitious mass μ^2. As we are discussing integral reduction within the context of our calculation, all of our integrals are dimensionally regulated. Consider the integral

$$I^\mu = \int d^n q \frac{q^\mu}{(q^2 - m^2)^2}. \tag{4.3.3}$$

The integrand of (4.3.3) is a Lorentz covariant function, and our regulator (DR) does not break Lorentz covariance. Therefore, the final answer must be Lorentz covariant. However, after integration we have no rank 1 Lorentz covariant tensors that we can use to build the answer. Therefore, by Lorentz covariance, we can say that $I^\mu = 0$. Indeed, this will follow for any integral without momentum scales of odd rank - there are no odd rank tensors. If we move to even ranks, e.g. rank 2

$$I^{\mu\nu} = \int d^n q \frac{q^\mu q^\nu}{(q^2 - m^2)^3}, \tag{4.3.4}$$

then we have a Lorentz covariant object available - the metric tensor. We

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therefore observe that

$$I^{\mu\nu} = I_2 \cdot g^{\mu\nu} \tag{4.3.5}$$

$$I_2 = \frac{1}{n}\left[\int d^n q \frac{1}{(q^2 - m^2)^3} + \int d^n q \frac{m^2}{(q^2 - m^2)^3}\right]. \tag{4.3.6}$$

Where we have solved for I_2 by contracting with the metric and cancelling the resulting q^2 in the numerator[10]. Note the introduction of the dimensionality of our integral, n. This arises because when contracting with the metric we compute in n-dimensions $g^{\mu\nu}g_{\mu\nu} = n$.

When we move to higher (even) ranks we have a number of metric tensors to take into account. Consider the rank 4 example

$$I^{\mu\nu\rho\sigma} = \int d^n q \frac{q^\mu q^\nu q^\rho q^\sigma}{(q^2 - m^2)^4}, \tag{4.3.7}$$

$$I^{\mu\nu\rho\sigma} = I_4 \left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right). \tag{4.3.8}$$

In (4.3.8) we only have one coefficient because $I^{\mu\nu\rho\sigma}$ is manifestly symmetric in all of its indices. When we move to two loop, however, it is not necessarily the case and so we would need to maintain independent coefficients.

In our work, because of the sub-vacuum and the loop counterterms we also require the use of PV reduction in the case where we have momentum scales. Let us explain the differences with the simple, concrete 1-loop example of the rank one bubble with no internal masses

$$B^\mu = \int d^n q \frac{q^\mu}{q^2(q + k_1)^2} \tag{4.3.9}$$

To begin, we observe that this integral depends on a single momentum scale k_1. The integral can only depend on k_1 and the only Lorentz covariant object we have is k_1^μ, so our answer must be proportional. That is

$$B^\mu = A \cdot k_1^\mu, \tag{4.3.10}$$

$$A = \frac{1}{k_1^2} \int d^n q \frac{q \cdot k_1}{q^2(q + k_1)^2}. \tag{4.3.11}$$

However, we can reconstruct this scalar product by noting the identity

$$q \cdot k_i = \frac{1}{2}[(q + k_i)^2 - q^2 - k_i^2]. \tag{4.3.12}$$

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[10] We will work to simplify this expression further in section 4.3.2.

Now, shifting the integral as well as setting scaleless terms to zero in the appropriate places, we find the scalarized result

$$B^\mu = -\frac{1}{2}k_1^\mu \int d^n q \frac{1}{q^2(q+k_1)^2} \qquad (4.3.13)$$

In the case of integrals with momentum scales, the automation is highly non trivial. As we have a large number of these in our work we shall now discuss an algorithm to perform this task.

Automating Integral Reduction at One Loop

The technique of PV reduction is conceptually quite pleasing, but in practice quite technically demanding. In our QCD driven calculation, the nature of the FDR vacuum makes it infeasible to perform this by hand. For any given integral, the high rank will lead to a long calculation, but what is yet more problematic is the fact that there are a large number of these integrals which all need to be individually reduced. Here then, we shall discuss the necessary techniques for reducing tensor integrals to a scalar basis in our work. The tensor integrals that we will be considering are of two and three point functions with possibly repeated propagators of arbitrary rank, for example

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{(q \cdot k_1)^2 (q \cdot k_2)}{\overline{q}^2 (\overline{q} + k_1)^2} \quad \text{or} \quad \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{q^\mu q^\nu (q \cdot k_3)^2}{\overline{q}^4 (\overline{q} + k_1)^2 (\overline{q} - k_2)^2}.$$

In this work, we have one large advantage - the symmetric off shell point as given in equation (4.3.1). This leads to a dramatic reduction in complexity in the three point correlation functions. First of all when performing PV reduction one often introduces inverse Gram determinant factors that can make algebraic simplification of the result strongly difficult. In our case, as we have only one scale - M^2 - one can always write these as a numeric factor premultiplying a particular power of our scale. Finding cancellations of the gram determinant in our final result is therefore entirely unnecessary. What is more, the symmetric point reduces the size of our scalar basis as a number of integrals are the same, for example

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\overline{q}^2 (\overline{q} + k_1)^2} = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\overline{q}^2 (\overline{q} + k_2)^2} = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\overline{q}^2 (\overline{q} + k_3)^2}. \quad (4.3.14)$$

Let us now discuss the ideas that we need to construct the algorithm. In the following we will consider integrals including the FDR fictitious mass

μ^2, that is all denominators will be barred. One can infer the cases without μ^2 by setting it to zero. The first step we must take is to reconstruct all reducible numerators in terms of propagators. This means that we make use of equation (4.3.12), which we state here including the FDR fictitious mass

$$q \cdot k_i = \frac{1}{2}[(\overline{q} + k_i)^2 - \overline{q}^2 - k_i^2].\qquad(4.3.12)$$

We make this substitution in any situation where the propagator-like terms on the RHS cancel completely, thereby decreasing the rank of the integral. This process is commonly referred to as "numerator reconstruction".

We are now left in a situation where, at the **integrand** level we only have irreducible loop momenta in the numerator[11]. Examples of this include

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{q^\mu}{\overline{q}^2 (\overline{q} + k_1)^2}, \quad \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{(q \cdot k_2)}{\overline{q}^2 (\overline{q} + k_1)^2}.\qquad(4.3.15)$$

In the first case we have a loose tensor index which is not a scalar product and so cannot be reconstructed. In the second case we are not able to reconstruct the scalar product in terms of the available denominators. However, note that as we can factorize the scalar product and take k_2 outside of the integral, solving the first problem will also solve the second. From now on we implicitly assume that we always do this, allowing us to discuss as if we only have integrals with loose tensor indices.

The first step in performing a PV reduction is to find the form the integrated expression must take as dictated by Lorentz covariance. This means writing a linear combination, with coefficients to be determined, of all of the tensors of appropriate rank, which can be built with the momentum scales we have available and are compatible with any symmetries of the integral. In our work it is not necessary to be able to do this for arbitrary rank or points and so we will not describe a general algorithm to be able to do this, however, let us elucidate what the this means with a few examples. Consider the rank 2 two point function in k_1 and its related equation

$$B^{\mu\nu} = \int d^n q \frac{q^\mu q^\nu}{\overline{q}^2 (\overline{q} + k_1)^2} = b_0 g^{\mu\nu} + b_1 k_1^\mu k_1^\nu.\qquad(4.3.16)$$

The RHS of this equation contains the two tensors $g^{\mu\nu}$ and $k_1^\mu k_1^\nu$, the two tensors we can build of rank 2 using k_1. We also have to respect a symmetry

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[11]Technically this is not true and it is this subtle observation that allows for OPP style reductions. For now we shall ignore this.

in swapping the tensor indices, $\mu \leftrightarrow \nu$. Here, enforcing this has no extra effect, but consider the case of the rank two 3 point integral

$$C^{\mu\nu} = \int d^n q \frac{q^\mu q^\nu}{\overline{q}^2(\overline{q}+k_1)^2(\overline{q}-k_2)^2}$$
$$= c_0 g^{\mu\nu} + c_1 \left(k_1^\mu k_1^\nu + k_2^\mu k_2^\nu\right) + c_2 \left(k_1^\mu k_2^\nu + k_2^\mu k_1^\nu\right). \tag{4.3.17}$$

Here we have less coefficients than tensors because the RHS must respect (as the integral does) the swaps $\mu \leftrightarrow \nu$ and $k_1 \leftrightarrow -k_2$.

We will continue using $C^{\mu\nu}$ as an example of a general strategy, but bear in mind that we will need to be able to treat tensors of higher rank and integrals with doubled propagators. In order to solve for the c_i we construct equations by contracting with relevant tensors. The first is the metric tensor $g^{\mu\nu}$. This leads to

$$C^\mu_\mu = \int d^n q \frac{q^2}{\overline{q}^2(\overline{q}+k_1)^2(\overline{q}-k_2)^2} = c_0 n + (2c_1 - c_2)M^2.$$
$$= \int \frac{d^n q}{\mu_R^\epsilon} \left(\frac{1}{(\overline{q}+k_1)^2(\overline{q}-k_2)^2} + \frac{\mu^2}{\overline{q}^2(\overline{q}+k_1)^2(\overline{q}-k_2)^2} \right) \tag{4.3.18}$$
$$= \int \frac{d^n q}{\mu_R^\epsilon} \frac{1}{(\overline{q}+k_1)^2(\overline{q}-k_2)^2}$$

Here, as we are in dimensional regularization $g^{\mu\nu}g_{\mu\nu} = n$. We reconstruct the square, and are able to set the second term to 0 as it vanishes[12] in the limit $\mu^2 \to 0$. Here the integrals with which we are left are scalars. Implicitly, if we can find three such equations we can solve for the c_i in terms of scalar integrals. Note that if we had chosen an integral with higher rank than 2 as our example then we would still have loose vectors, we will discuss how to deal with this in due course. Our next step is to contract with the vector k_1^μ

$$C^{\mu\nu}k_{1\mu} = \int \frac{d^n q}{\mu_R^\epsilon} \frac{(q \cdot k_1)q^\nu}{\overline{q}^2(\overline{q}+k_1)^2(\overline{q}-k_2)^2}$$
$$= \left(c_0 + \left(c_1 - \frac{c_2}{2} \right) M^2 \right) k_1^\nu + \left(-\frac{c_1}{2} + c_2 \right) M^2 k_2^\nu$$
$$= \frac{1}{2} \int \frac{d^n q}{\mu_R^\epsilon} \left(\frac{q^\nu}{\overline{q}^2(\overline{q}-k_2)^2} - \frac{q^\nu}{(\overline{q}+k_1)^2(\overline{q}-k_2)^2} - \frac{(p^2)q^\nu}{\overline{q}^2(\overline{q}+k_1)^2(\overline{q}-k_2)^2} \right),$$
$$\tag{4.3.19}$$

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[12]For this style of term to give a non-zero contribution the integral would have to contribute as $1/\mu^2$.

where again we have reconstructed the scalar product. The choice of k_1 vs k_2 here is arbitrary as the integral is symmetric in the two. We could have chosen to contract with one of the other two tensors in equation (4.3.17), but on the LHS of equation (4.3.19) we would only be able to reconstruct one scalar product and the other adds unnecessary complication.

At this point, in order to find our next two equations for the c_i we need to find the coefficients of k_1^ν and k_2^ν. However, unlike in equation (4.3.18), we do not have scalar integrals and so this is not manifest. In order to deal with this, we need to repeat all previous steps in this process, so the algorithm must recurse. Before we do this, however, we should note that in a few integrals, for reasons of simplicity, it is necessary to make a shift in the integration momentum so that at least one of the denominators is momentumless. In doing this we should make use of the symmetric kinematic point to rewrite all *scalar* integrals in terms of a single one *à la* equation (4.3.14).

A special comment should be made when performing integral reduction in the context of removing the fictitious mass according to the FDR prescription. When one reconstructs the products, it is possible that one of the integrals with which we end up has only the fictitious mass as its scale. In accordance with the FDR prescription we will discard all those which diverge polynomially.

In summary, our algorithm for integral reduction is as follows:

1. Reconstruct numerators then shifting integrals to have one momentumless propagator. Then apply any symmetric kinematic point identities, dropping polynomially divergent vacuum integrals.

2. Write down the equation prescribed by Lorentz covariance, respecting symmetries in indices and momenta.

3. Construct up to two equations, first by contracting with a momenta and the second (if possible) by contracting with the metric tensor.

4. Repeat the reduction process on each equation if they are not in terms of scalar integrals.[13]

5. Extract the independent equations using the linear independence of tensors and solve them.

Using this approach we see why we always reach scalars - we only ever create scalar products in the numerator which can be reconstructed.

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[13]This does imply that in one recursion the step of writing down the Lorentz covariance equations is inert, but this is to keep the explanation simple.

4.3.2 Integration By Parts

Another tool that we will require to reach a non-redundant basis of integrals is integration by parts (IBP) identities [81]. Once we have reduced our integrals to a scalar basis according to the techniques of section 4.3.1 we will find that we have scalar integrals which look very similar, but have different numbers of propagators. We can often then use the technique of integration by parts to show that they are related. To give an example, we will soon show that there is a relation between

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{(q^2 - m^2)^3} \quad \text{and} \quad \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{(q^2 - m^2)^2}.$$

Integration by parts identities are relations between integrals that arise from the observation that the integration of a total derivative is zero. This can be seen as a consequence of the shift invariance of the dimenionsally regulated integral. To demonstrate this concretely let us write down the definition of shift invariance, taking the time to taylor expand the shifted function

$$\int d^n q f(q) = \int d^n q f(q + p) \tag{4.3.20}$$

$$= \int d^n q f(q) + p^\mu \int d^n q \frac{\partial}{\partial q^\mu} f(q) + \cdots \tag{4.3.21}$$

The first term on either side of this integral is the same, so the infinite sum on the right hand side must be equal to zero. One can demonstrate that any term in this sum is identically zero by differentiating with respect to an appropriate number of p, and then setting $p^\mu = 0$ (as this identity is true for any value of p^μ). The first term in this series is exactly enough to allow us to write down the integration by parts identity. Clearly, then, any definition of an integral which is shift invariant will permit integration by parts - including FDR [82].

To see how we can use these identities, let us find the relation we previously alluded to. We start by taking the integral of a judicious total derivative and setting it to zero

$$0 = \int d^n q \frac{\partial}{\partial q^\mu} \left(\frac{q^\mu}{(q^2 - m^2)^2} \right) \tag{4.3.22}$$

If we now take the time to expand the derivative we gain a relation between a set of integrals. One important thing to remember when performing this

calculation is that we are working in n dimensions, so that

$$\frac{\partial q^\mu}{\partial q^\mu} = n. \tag{4.3.23}$$

Knowing this we can take equation (4.3.22) and rearrange to give

$$\int d^n q \frac{m^2}{(q^2 - m^2)^3} = \frac{n-4}{4} \int d^n q \frac{1}{(q^2 - m^2)^2}. \tag{4.3.24}$$

It is fairly easy to see that choosing an arbitrary power of the propagator within the original derivative will get us a more general version of this relationship

$$(n - 2\alpha) \int d^n q \frac{1}{(q^2 - m^2)^\alpha} = 2\alpha \int d^n q \frac{m^2}{(q^2 - m^2)^{\alpha+1}} \tag{4.3.25}$$

This style of relation is not limited to one style of propagator or even one loop[14]. In common practice these identities are most often used to reduce a large set of irreducible two loop integrals down to a set of master integrals. In standard one loop computations their applicability is limited because the derivative always creates doubled propagators, which are topologically forbidden. However, within our calculation of the two loop vacuum part we find these relations invaluable. Within both the subvacuum and loop counterterms one often ends up with one loop integrals with doubled propagators. In the following sections there is one set of relations that turn up multiple times and so we shall discuss them here. Further use of these techniques will be necessary, but there are a number of details better understood in context.

When computing the one loop vacuum in 4D, without tensor reduction, we find a basis of integrals of the form

$$\int d^n q \frac{q^{\alpha_1} \cdots q^{\alpha_{2r}}}{(q^2 - \mu^2)^{r+2}}. \tag{4.3.26}$$

Application of PV reduction in this case creates a large number of integrals with μ^2 in the numerator. We can then apply, in general, equation (4.3.25)

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[14]A similar style of identity is available in FDR, though they are structurally quite different as FDR's manipulations take place in 4-dimensions and we have the fictitious mass μ^2 to take into account.

with $m^2 = \mu^2$. The explicit proof of this requires an amount of algebra involving combinatorics, but in the end we find that

$$\int d^n q \frac{q^{\alpha_1} \cdots q^{\alpha_{2r}}}{(q^2 - \mu^2)^{r+2}} = \frac{2}{(2r+2)!!} g^{\alpha_1 \cdots \alpha_{2r}} \int d^n q \frac{1}{(q^2 - \mu^2)^2}. \qquad (4.2.13)$$

Though this form is not very transparent, evaluating for specific values of r gives very intuitive answers. The prefactor is exactly what you would expect doing the reduction, throwing away all of the mass terms and setting n to 4 - the mass dimension of the integral. It is of no surprise, therefore that similar identities hold for analogous integrals of different mass dimension.

4.3.3   Reducing The Global Vacuum

Now that we have undergone a technological interlude, we must return to the task of calculating the renormalization constants in DR which give us the FDR scheme. As previously noted, we will break the calculation down into its constituent parts. We begin with the general method for calculating the global vacuum of either a two or three point correlation function.

The global vacuum is defined through the FDR defining expansion as the parts of the FDR vacuum which are not sub-vacua. It is given by a series of two loop integrals which depend *only* on the FDR fictitious mass μ^2. These integrals come in two different flavours - factorizable and "pure" two loop integrals. Factorizable vacuum integrals are simply a product of two one loop vacuum integrals, depending only on μ^2. Some examples of this are

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\overline{q}_1^4} \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_2^4}, \quad \text{or} \quad \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{q_1^\mu q_1^\nu}{\overline{q}_1^6} \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_2^4}. \qquad (4.3.27)$$

We note that in this discussion we are not limited to scalar integrals, but we will also have to be able to deal with tensor integrals. Pure two loop global vacuum integrals are similar, depending only on the fictitious mass μ^2, but they also have propagators in q_{12}. Some representative examples of pure vacua are

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_1^4 \overline{q}_2^2 \overline{q}_{12}^2},$$

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_1^\mu q_1^\nu}{\overline{q}_1^4 \overline{q}_2^4 \overline{q}_{12}^2}, \qquad (4.3.28)$$

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_1^\mu q_1^\nu (q_1 \cdot p)(q_2 \cdot p)}{\overline{q}_1^4 \overline{q}_2^4 \overline{q}_{12}^4}.$$

Our aim now is to reduce this expression to a non-redundant basis of scalar, logarithmically divergent integrals. The first step of calculation is to throw away the polynomial divergences that vanish in the $\mu^2 \to 0$ limit. We are left with logarithmically divergent integrals and to be able to calculate these, we wish to reduce them to scalar integrals through PV reduction. Firstly, the factorizable integrals can be handled using the one loop consistency relations. Next, for the pure integrals we can remember the discussion of section 4.3.1. It is always possible to reduce to scalars as in the intermediate steps we can reconstruct products of the form $q_i \cdot q_j$ in terms of denominators unless, in an intermediate step, we produce an integral with $q_1 \cdot q_2$ in the numerator which factorises. However, here, we can again apply the one-loop consistency relations to each factor. We should note that there is a subtlety in the PV reduction of pure two loop vacuum integrals that is not present in one loop vacuum integrals. Specifically, at two loop, we have no guarantee that there is a symmetry in the exchange of tensor indices (often it is there, but not manifest). Let us write down the Lorentz covariance equation for a representative integral

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_1^\mu q_1^\nu q_2^\rho q_2^\sigma}{\overline{q}_1^8 \overline{q}_2^2 \overline{q}_{12}^2} = a_1 \cdot g^{\mu\nu} g^{\rho\sigma} + a_2 \cdot g^{\mu\rho} g^{\nu\sigma} + a_3 \cdot g^{\mu\sigma} g^{\nu\rho}. \quad (4.3.29)$$

As it is not clear that we can exchange the indices in the numerator, we must leave open the possibility that tensors on the RHS have different coefficients. Nevertheless, we can always reduce to scalars.

Once our expression is in terms of scalars we have a redundant basis of integrals. For example

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_2^\mu q_2^\nu}{\overline{q}_1^4 \overline{q}_2^4 \overline{q}_{12}^2} = \frac{1}{n} g^{\mu\nu} \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \left(\frac{1}{\overline{q}_1^4 \overline{q}_2^2 \overline{q}_{12}^2} + \frac{\mu^2}{\overline{q}_1^4 \overline{q}_2^4 \overline{q}_{12}^2} \right). \quad (4.3.30)$$

In performing the PV reduction we meet a large number of scalar integrals with μ^2 in the numerator and we can show that they are dependent through IBP relations. In fact, we can actually relate them to two integrals, the logarithmic factorizable integral and the pure logarithmic two loop integral

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_1^4} \frac{1}{\overline{q}_2^4}, \quad (4.3.31)$$

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_1^4 \overline{q}_2^2 \overline{q}_{12}^2}. \quad (4.3.32)$$

Unlike in the case of the one-loop consistency relations we have not used the IBP relations to find a simple formula to relate any given tensor integral to

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these fundamental integrals. If it exists, it is currently not worth the effort due to the simple way in which we can use the IBP relations. The following relation allows us to relate any integral with μ^2 in the numerator to other integrals of higher mass dimension without this μ^2: [15]

$$\mu^2 V_{abc} = \hat{O}\, V_{abc}, \tag{4.3.33}$$

$$\hat{O} = \frac{n - 3(a - 1)}{3(a - 1)} \mathbf{1}_- + 2c \left(\mathbf{1}_- \mathbf{2}_- \mathbf{12}_+ - \mathbf{1}_-^2 \mathbf{12}_+ \right) \tag{4.3.34}$$
$$+ (a - 1)\left(\mathbf{12}_- - \mathbf{2}_- \right)$$

$$V_{abc} = \int \frac{d^n q_1}{\mu_R^\epsilon} \frac{d^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_1^{2a} \overline{q}_2^{2b} \overline{q}_{12}^{2c}} \tag{4.3.35}$$

Here $\hat{O}$ is an operator and we have used the compact notation that $\boldsymbol{i}_\pm$ is an operator which increases or decreases the count of an denominator i within V_{abc}. We can see that appropriate use of this can always bring us back to the scalar logarithmic integrals. Firstly, note that the integrals on the RHS are of higher mass dimension than on the LHS, or alternatively every application of the identity always decreases $a + b + c$. This implies that the integral leads back towards cases of logarithmic mass dimension - our two fundamental integrals. Next, note that the only situation which could move away from the V_{abc} structure is if the $\mathbf{1}_-^2$ operator hits an integral where $a = 1$ as this would put a $\overline{q}_1^2$ in the numerator. However, the naming of q_1, q_2 and q_{12} is arbitrary and so we can always rename to avoid this and if we could not, then we have already stepped past logarithmic.

Employing this process has allowed us to write our global vacuum in terms of the logarithmic factorizable and pure integrals. As in the case of the sub vacuum this has been allowed at the cost of introducing a prefactor which depends on n. A priori we do not have reason to expect a specific form for this prefactor but our experience allows us to write, in every tested case, the following expression for a global vacuum

$$\int d^n q_1 d^n q_2 \left(\left[\alpha + (n_\gamma - 4)\alpha' + \frac{\gamma + (n_\gamma - 4)\gamma'}{n} \right] \frac{1}{\overline{q}_1^4 \overline{q}_2^2 \overline{q}_{12}^2} \right.$$
$$\left. + \left[\beta + (n_\gamma - 4)\beta' - \frac{\gamma + (n_\gamma - 4)\gamma'}{n} \right] \frac{1}{\overline{q}_1^4 \overline{q}_2^4} \right). \tag{4.3.36}$$

Here, in order to show the simple structure, we explicitly label the n_γ. This form is quite simple and perhaps it suggests that there exists some currently

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[15]The derivation of this relation is quite technical and offers no further clarification, so we leave this for appendix C.2.

not understood structure. That is, intermediate steps produce greatly more complicated structure in n than a simple $1/n$ factor, but when cast into this form it disappears. Further, the coefficients of the $1/n$ structures are inexplicably linked between the two integrals. We have no explanation at this time.

We now move onto the calculation of the sub vacuum, to complete the calculation of the FDR vacuum in n dimensions.

4.3.4   Reducing The Sub-Vacuum

The calculation of the sub-vacuum that we perform amounts to the application of standard one-loop techniques to reach a simple form. In the following we shall discuss the sub-vacuum of two or three point correlation functions in full generality. In order to do this we shall suppress all structure that is not explicitly part of the integrals, i.e. factors of gamma matrices, metrics and other tensors. Note that due to our kinematic point we can discuss two and three point functions at the same time - in the case of the two point correlation functions the style of integral which produces three point one loop integrals simply never occur. We shall begin by noting that, after vacuum extraction, all dimensionally regulated sub vacuum terms can be written as

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \, f^{\alpha_1 \cdots \alpha_m}(q_1) \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \, v_{\alpha_1 \cdots \alpha_m}(q_2). \tag{4.3.37}$$

That is, they can be factorised into two one loop integrals of functions f and v. Here f is a function that when integrated in n-dimensions exists as $n \to 4$. The function v is a series of one loop vacuum integrals. We write the tensor indices to highlight that the numerator structure is contracted between the two integrals, but the numerator may also contain products with other momenta (e.g. $q_1 \cdot k_i$), contractions with gamma matrices or even external Lorentz indices. Note that these indices are all n-dimensional, so that if we have a factors of gamma matrices, these are to be treated n dimensionally, such that after integral reduction if we find $g_{\mu\nu}\gamma^\mu\gamma^\nu$ then this must be set to n. To clarify our notation in equation (4.3.37) we list some examples

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{q_1^\alpha q_1^\beta (q_1 \cdot k_1)}{\overline{q}_1^6 (\overline{q}_1 + k_1)^2} \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_{2\alpha} q_{2\beta}}{\overline{q}^6},$$

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{q_1^\alpha q_1^\beta \slashed{q}_1}{\overline{q}_1^4 (\overline{q}_1 + k_1)^2 (\overline{q}_1 - k_2)^2} \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{q_{2\alpha} q_{2\beta}}{\overline{q}^6}. \tag{4.3.38}$$

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Note that it is likely that all sub vacuum terms only appear in this form after an appropriate renaming of q_1 and q_2. Further, note that even though the integral in q_1 exists in the $n \to 4$ limit, as it is multiplied by an *infinite* integral in q_2 we cannot set n to 4 as we need to keep track of any ϵ/ϵ terms.

The first step is to perform the exact reduction of the q_2 integral using the one loop consistency relations, equation (4.2.13), whilst also dropping any integrals in q_2 which go as μ^r, $r > 0$. We are safe to do this as the integral in q_1 can by at worse logarithmic, i.e. μ^0 behaviour. This is because, by construction, the finite factor of the sub-vacuum cannot have worse than a logarithmic IR divergence in μ^2. From this point onwards the factor in q_2 is given by the fundamental logarithmic divergence in four dimensions. That is, our general form is now

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \, f'(q_1) \int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q_2}^4}, \qquad (4.3.39)$$

where f' is now different to the original f because of the steps just taken. It will possibly involve factors of q_1^2 in the numerator.

From here, all work takes place in the factor which is an integral in q_1. Our aim is to reduce this factor into a minimal basis. To illustrate our approach let us consider the following integral

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{(k_1^2 + 2q_1 \cdot k_1) q^\mu q^\nu}{\overline{q}_1^6 (\overline{q} + k_1)^2}. \qquad (4.3.40)$$

As required, this is finite as $n \to 4$. The first step is to reconstruct the product, leading to

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{q^\mu q^\nu}{\overline{q}_1^6} - \frac{q^\mu q^\nu}{\overline{q}_1^4 (\overline{q} + k_1)^2} \right). \qquad (4.3.41)$$

This is the difference of two UV infinite functions, however, the poles cancel between the two such that the difference remains finite. As we shall see, this is a general feature and dictates the ultimate form of sub vacuum expression once reduced. The two integrals here are representative of the challenges found in the complete calculation. First we deal with the integrals that depend only on μ^2. If they are of logarithmic nature then we relate them to the fundamental logarithmic divergence using the one-loop consistency relations. If they are polynomial in nature we simply discard as they vanish as $\mu^2 \to 0$. For the second style of integral we must now use Passarino Veltman techniques as discussed in section 4.3.1.

Here we can see that the result will involve an integral with a doubled propagator therefore we must employ the IBP relations as discussed in section 4.3.2. In an arbitrary sub-vacua, because of our off-shell kinematic point there are only two possible integrals with doubled propagators which we will need to relate to their cousins with single propagators. These two are

$$I_1 = \int d^n q \frac{1}{\overline{q}^4 (\overline{q} + k_1)^2}, \tag{4.3.42}$$

$$I_2 = \int d^n q \frac{1}{\overline{q}^4 (\overline{q} - k_1)^2 (\overline{q} + k_2)^2}. \tag{4.3.43}$$

To find their expressions in terms of scalar integrals with single propagators we must employ the technique of integration by parts. The derivation of these relations is somewhat technical and not very illuminating so we leave this to appendix C.2. We find that

$$I_1 = \frac{1}{M^2} \int \frac{d^n q}{\mu_R^\epsilon} \left(\frac{1}{\overline{q}^4} + \frac{3 - n}{\overline{q}^2 (\overline{q} + k_1)^2} \right), \tag{4.3.44}$$

$$I_2 = \frac{1}{M^4} \int \frac{d^n q}{\mu_R^\epsilon} \left(\frac{M^2}{2} \frac{4 - n}{\overline{q}^2 (\overline{q} - k_1)^2 (\overline{q} + k_2)^2} + \frac{1}{\overline{q}^4} + \frac{3 - n}{\overline{q}^2 (\overline{q} + k_1)^2} \right). \tag{4.3.45}$$

Note that our original expressions for I_1 and I_2 are UV finite, i.e. there are no poles in ϵ. However, there do contain IR divergences as μ^2 goes to zero. We can see that this is still respected in by our new form. As we take $n \to 4$ in these expressions there is no UV pole as the $1/\epsilon$ in each integral cancels. What's more the $\log \mu^2$ due to the double propagators is now being held in the fundamental logarithmic vacuum.

At this point we are ready to write down the general form which our sub-vacuum must take

$$\int \frac{d^n q_1}{\mu_R^\epsilon} \frac{d^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_2^4} \left[\frac{A(n)}{\overline{q}_1^4} - \frac{A'(n)}{\overline{q}_1^2 (\overline{q}_1 + k_1)^2} + M^2 \frac{B(n)}{\overline{q}_1^2 (\overline{q}_1 - k_1)^2 (\overline{q}_1 + k_2)^2} \right]. \tag{4.3.46}$$

We have been successful in reducing our expression to scalar integrals and in doing so we have had to allow their coefficients to depend upon n. Note that we label the coefficients of the fundamental logarithmic vacuum and the two point function $A(n)$ and $A'(n)$ respectively. This is to signify that they coincide at $n = 4$, i.e. $A(4) = A'(4)$, such that the integral over q_1 has no UV pole. We recall that this is because one factor of the sub-vacuum is UV finite, but as we have written it in terms of UV infinite integrals their coefficients must conspire to cancel the pole. Finally, we point out that

there are so few integrals because many are related through the symmetric off-shell kinematic point.

At this point, we are able to make two judicious steps which will help us make cancellations with the loop counterterms calculated in section 4.3.5. The first is to note that the A' and B terms in equation (4.3.46) are IR convergent, and so we can take the limit $\mu^2 \to 0$ within the integral here. Further, as the q_1 integral is finite as $\epsilon \to 0$ we do not need to keep the $O(\epsilon)$ terms in the q_2 integral. We write this as a $\epsilon \to 0$ next to the q_2 integral (though we do keep the poles)

$$V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left[\frac{A(n)}{\overline{q}_1^4} - \frac{A'(n)}{q_1^2(q_1+k_1)^2} + M^2 \frac{B(n)}{q_1^2(q_1-k_1)^2(q_1+k_2)^2} \right], \quad (4.3.47)$$

$$\text{where} \quad V_0 = \left[\int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\overline{q}_2^4} \Big|_{\epsilon \to 0} \right]. \quad (4.3.48)$$

We remind the reader that this is the form for both the two point and three point functions, however $B(n) = 0$ in the two point functions as these integrals do not appear by construction.

4.3.5 Reducing The Loop Counterterms

The loop counterterms were introduced in section 4.1.2 as a natural consequence of removing the infinities through renormalization of the bare parameters in the Lagrangian. This creates new, higher order interactions which must be included in calculations of radiative corrections. The first step, therefore, is to renormalize the theory at one-loop, just as we did in section 4.2.3. The necessary renormalizations are given in appendix B.1. For the following discussion, to ease making contact with the FDR vacuum, we will rewrite the renormalization constants in terms of the one-loop fundamental logarithmic divergence. This essentially amounts to not having expanded the logarithmic divergence in terms of ϵ in the first place, i.e. we rewrite

$$- i\pi^2 \left(\frac{2}{\epsilon} + L + c \right) \to (1 + c[n-4]) \, V_0, \quad (4.3.49)$$

where we take $n = 4 + \epsilon$ and $L = \gamma_e + \log(\pi)$ and V_0 is defined in equation (4.3.48). It is important to remember that the counterterm insertions *must* contain any constant part that we absorb at the one loop level (c in equation 4.3.49) otherwise we are not consistently performing the renormalization.

To give an example of the kind of diagram which we are computing here, consider the diagrams given in figure 4.2 as examples of contributing

to the loop counterterms in the renormalization of gluon two point, gluon three point and gluon/ghost/anti-ghost functions respectively. Note the
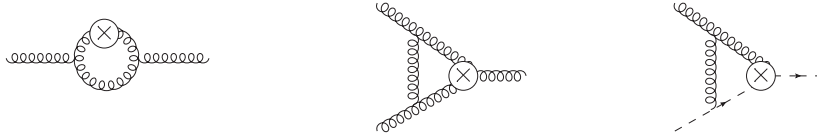

Figure 4.2: Example loop counterterm diagrams required for calculating the FDR scheme renormalization constants.

diagram given as example of a gluon two point function loop counterterm. This contains a counterterm insertion along a propagator and so has two propagators with the same momentum flowing through them - a topology not found in NLO calculations. For this reason, similar to the sub-vacuum calculation we will have one loop integrals with doubled propagators.

With these diagrams in hand, we face a very similar treatment to that of section 4.3.4, using one loop technology. There is, however, one principal difference - these integrals lack the FDR fictitious mass μ^2. In the discussion of section 4.2.1 introduce the fictitious mass only in the bare integral, in order to extract the FDR integral. This might suggest that we could have differences in IR behaviour, but this was the benefit of performing the calculation off-shell - we are completely IR convergent and all divergences are of UV origin.

The work to reduce to scalar integrals that we perform is only slightly different now that we lack a μ^2. Let us again choose a representative integral which allows us to explore the process and highlight the subtleties

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{q_1^\mu q_1^\nu}{q_1^4 (q_1 + k_1)^2}. \tag{4.3.50}$$

This integral is purely UV divergent, as required, and the first step is to perform integral reduction. We observe that the difference between this case and the case where there are IR divergences is minor - we can find the reduction by taking the reduction with μ^2 and taking $\mu^2 = 0$ inside the integral. Note that this is not taking a limit, it is simply an algebraic trick. The only place where this results in a different calculation to the $\mu^2 \neq 0$ case is if we find an integral whose only scale is μ^2. Without the support of this scale, these integrals are now scaleless and so we can set them to zero, for example

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{q_1^4} = 0. \tag{4.3.51}$$

If this integral were in terms of $\bar{q}$ we would not be able to set it to 0 as it diverges logarithmically in μ^2. In equation (4.3.51), we remember that this integral is zero because it contains both an IR divergence and a UV divergence which cancel. In the presence of (even a vanishing) mass μ^2, the two cannot cancel because they are regulated in different ways.

In performing the PV reduction we often end up with scalar integrals with doubled propagators. In the case of equation (4.3.50) we find the scalar integral

$$\bar{I}_1 = \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{q_1^4 (q_1 + k_1)^2}. \tag{4.3.52}$$

Similar to the treatment of the subvacua in section 4.3.4, in considering the loop counterterms for any of our correlation functions the only other example of an integral with a doubled propagator that we meet is

$$\bar{I}_2 = \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{q_1^4 (q_1 - k_1)^2 (q_1 + k_2)^2}. \tag{4.3.53}$$

Before we move to rewrite them using IBP relations, let us note something that naively may be of concern - these two integrals are IR divergent, i.e. they contain a $1/\epsilon$ pole of IR origin. After a short pause we realize that this is no problem - the off-shell kinematic point assures us that the correlation function is IR convergent. These intermediate IR poles are, in fact, representations of the original UV pole. This is made more transparent by using IBP to remove the doubled propagator. Once again we can use the relations with μ^2 to find the relevant equations by taking $\mu^2 = 0$ inside the integral. This tells us that

$$\bar{I}_1 = = \frac{1}{M^2} \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{3-n}{q_1^2 (q_1 + k_1)^2} \right), \tag{4.3.54}$$

$$\bar{I}_2 = = \frac{1}{M^4} \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{M^2}{2} \frac{4-n}{q_1^2 (q_1 - k_1)^2 (q_1 + k_2)^2} + \frac{3-n}{q_1^2 (q_1 + k_1)^2} \right). \tag{4.3.55}$$

Once again, we see that the $1/\epsilon$ pole manifests in a UV divergent integral.

Finally we can see that we can reduce the loop counterterms down to scalars and due to the benefits of the off-shell kinematic point we can write them as

$$V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left[\frac{a'(n)}{q_1^2 (q_1 + p)^2} - M^2 \frac{b(n)}{q_1^2 (q_1 - k_1)^2 (q_1 + k_2)^2} \right], \tag{4.3.56}$$

Note that here we have named our coefficients in judicious correspondence with the names that we gave in our discussion of sub vacuum in section

4.3.4. In this way, we are also aided by the rewriting of the $1/\epsilon$ pole from the counterterm insertion in terms of the fundamental logarithmic vacuum. Finally, note that, as in the case of the sub-vacuum, this is a description valid for both two and three point correlation functions, but in the case of two point correlation functions $b(n) = 0$ by construction.

4.3.6 FDR Renormalized Correlation Functions

At this point we have all of the components required to compute the renormalization constants of our chosen correlation functions with external gauge states in QCD in FDR at two loop. Let us see how they fit together, as there are a few interesting comments that we can make.

We recall the equation defining the second order correction to the renormalization constant for any correlation function

$$(Z_2)_{FDR} \cdot M_0 = -\left. V[M_2] \right|_{n_\gamma = n} - \left. (\text{Loop CTs}) \right|_{FDR}. \qquad (4.2.10)$$

Here, we remind the reader that Z_2 is the second order correction to the renormalization constant for our given correlation function, M_0 is the tree level form of the correlation function[16] and $\left. V[M_2] \right|_{n_\gamma = n}$ represents the FDR vacuum of the 2-loop corrections to the correlation function calculated with $n_\gamma = n$. For the following discussion we split the vacuum term into the global vacuum (GV) and sub-vacuum (SV)

$$(Z_2)_{FDR} \cdot M_0 = -\left. (\text{GV}[M_2] + \text{SV}[M_2]) \right|_{n_\gamma = n} - \left. (\text{Loop CTs}) \right|_{FDR}. \qquad (4.3.57)$$

As previously discussed, this form sets some strong constraints for the FDR vacuum which we must now find as we put the components together. Firstly, the LHS factorises the tree level, so as radiative corrections in general allow for more complex structures there must be some dramatic cancellation in the RHS. Secondly, we want the form of Z_2 to be local - containing no logs of any scales. In order for this to happen, the logs in the sub vacuum and the loop counterterms must cancel.

If we remember the forms given in sections 4.3.4 and 4.3.5 on the sub vacua and loop counterterms respectively then we can write down the equation

$$\text{SV} + \text{Loop CTs} =$$
$$V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left[\frac{A(n)}{\overline{q}_1^4} + \frac{a'(n) - A'(n)}{q_1^2 (q_1 + k_1)^2} + M^2 \frac{B(n) - b(n)}{q_1^2 (q_1 - k_1)^2 (q_1 + k_2)^2} \right]. \qquad (4.3.58)$$

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[16] See footnote 5 on page 69.

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We see, therefore, that in order to drop the logarithms we have strong constraints on the coefficient functions, specifically both $[A'(n) - a'(n)]$ and $[B(n) - b(n)]$ should be $O(\epsilon)$. However, upon performing the calculation, we have found a much stronger condition: $A'(n) = a'(n)$ and $B(n) = b(n)$, such that

$$\text{SV} + \text{Loop CTs} = V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{A(n)}{\overline{q}_1^4}. \qquad (4.3.59)$$

We do not have a thorough explanation of this phenomena to hand at this time. It is interesting to note that we can quite easily repeat the discussion of this chapter using the four dimensional helicity scheme, FDH. The principal difference is that when constructing the FDR integral from the FDH, there is no longer any difference in the state counting parameter n_γ. That is, in FDH $n_\gamma = 4$ and so we can achieve the FDH version of this calculation by setting $n_\gamma = 4$ in our construction of the vacuum [17]. This has the effect of changing the coefficient functions in the numerators of equation (4.3.58). Here we *also* find the exact cancellation of integrals with momentum scales, leaving again the form of equation (4.3.59).

Having noticed exactly how the sub-vacuum and loop counterterms cancel we see that only the part depending on μ^2 remains and so contributes to the determination of the renormalization constant. This leads us to conjecture that one could calculate the renormalization constants of the FDR scheme without calculating the loop counterterms. In our calculation we see that we are able to extract this term directly, without performing the reduction of the entire sub vacuum by observing that it represents the IR divergence, as $\mu^2 \to 0$ of the sub vacuum. In our case we have explicitly performed this by extracting the IR divergence using the identity

$$\frac{1}{(\overline{q} + k_i)^2} = \frac{1}{M^2} \left(1 - \frac{2(q \cdot k_i) + \overline{q}^2}{(\overline{q} + k_i)^2} \right) \qquad (4.3.60)$$

This works in a similar way to the FDR vacuum extraction - the second and third terms are less IR divergent than the first. Consider as an example

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{q^\mu q^\nu}{\overline{q}^6 (\overline{q} + k_1)^2} = \frac{1}{M^2} \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{q^\mu q^\nu}{\overline{q}^6} - \frac{q^\mu q^\nu [2(q \cdot k_i) + \overline{q}^2]}{\overline{q}^6 (\overline{q} + k_1)^2} \right)$$
$$(4.3.61)$$

The first term is exactly a one-loop vacuum integral which, after application of the one-loop consistency relations, contributes to the $A(n)$ part of the

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sub vacuum. The rest of the terms are all IR convergent, and so we can take the $\mu^2 \rightarrow 0$ limit inside the integral, implying that they will reduce to A' style terms which then match with counterterms. Intuitively this observation of how to extract the renormalization constants makes sense - the renormalization scale dependence of the finite part is given by the dependence on μ^2, which behaves as a fictitious IR divergence in μ^2. We could therefore find the renormalization constants of our given correlation function directly from the vacuum by extracting its IR divergent part.

Before moving on to the relations that we need to finally compute the renormalization constants in the FDR scheme, we remark that when one brings together the sub vacuum and the global vacuum we notice a correspondence in their pole structure. To be able to express this, let us write a Laurent series in $\epsilon = n - 4$ form for both the global vacuum and sub vacuum of our correlation functions

$$\text{GV} = \frac{G_{-2}}{\epsilon^2} + \frac{G_{-1}}{\epsilon} + G_0, \tag{4.3.62}$$

$$\text{SV} = \left(\frac{S^a_{-2}}{\epsilon^2} + \frac{S^a_{-1}}{\epsilon} + S^a_0 \right) + \left(\frac{S^b_{-2}}{\epsilon^2} + \frac{S^b_{-1}}{\epsilon} + S^b_0 \right). \tag{4.3.63}$$

Note that equation (4.3.63) is not simply the Laurent series in epsilon of the sub vacuum, but the sub vacuum written as the sum of two different Laurent series - S^a and S^b. We point out that as the sub-vacuum has only a $1/\epsilon$ pole then clearly $S^a_{-2} = -S^b_{-2}$. In order to explain the origin of these two series, let us recall the general form of the reduced sub-vacuum from section 4.3.4

$$V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left[\frac{A(n)}{\bar{q}_1^4} - \frac{A'(n)}{q_1^2 (q_1 + k_1)^2} + M^2 \frac{B(n)}{q_1^2 (q_1 - k_1)^2 (q_1 + k_2)^2} \right]. \tag{4.3.47}$$

Here we have achieved a separation into the part which depends only on μ^2 and a part which, in practice, cancels with the counterterms. From this perspective, the series S^a is the μ^2 dependent part, the $A(n)$ term and the series S^b corresponds to the other part, the A' and B terms. Understanding this, we can state that in our experience, the relation we then find by explicit calculation for all of our correlation functions is

$$S_{a-2} = -S_{b-2} = -2G_{-2}. \tag{4.3.64}$$

Naively one may expect this to be due to a consistency relation that is not known to us which constrains the amplitude. However, we can take any two loop integral and reduce it to the form of equation (4.3.47). Experimentation

therefore leads us to conjecture that equation (4.3.64) is true for any two loop integral, not just correlation functions.

We are now ready to compute the renormalization constant for our given correlation function. We leave discussion of the extraction of the renormalization of α_S and the various fields to section 4.5. The previous discussion allows us to compute this by integrating the global vacuum and the $A(n)$ term from the sub vacuum. That is, we can write a general form of the renormalization constant as

$$Z_i = \left(\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{G_1(n)}{\bar{q}_1^4 \bar{q}_2^4} + \frac{G_2(n)}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} \right) + V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{A(n)}{\bar{q}_1^4}, \qquad (4.3.65)$$

where all integrals are evaluated at the point $\mu = \mu_R$. Here, the G_i terms represent the contribution from the global vacuum, where we have written the n dependence of the coefficients from equation (4.3.36) as the $G_i(n)$. The A term is the result of the sub vacuum/loop counterterm cancellation. The integrals required to compute equation (4.3.65) are presented in appendix C.1 evaluated at the point $\mu = \mu_R$. We now comment on the form of the renormalization constants in the FDR scheme. In general we can write the renormalization constants in the form

$$Z_i = \frac{c_{-2}}{\epsilon^2} + \frac{c_{-1}}{\epsilon} + c_{00} + c_{01} f_{11}. \qquad (4.3.66)$$

where the c_i are coefficients that depend on the correlation function we renormalize. At one-loop the FDR scheme is equivalent to FDH and DRED, but at two loop this is not the case - the renormalization constants differ by a finite amount[18]. What is noticeably distinct from an MS inspired scheme is that this finite difference involves the coefficient f_{11}, a numerical constant given in terms of the polygamma function[19]. This is a pure two loop renormalization effect, and so did not turn up in the work of Pittau and Donati[74]. Another further point that we can observe is that, despite the integrals in equation (4.3.65) all containing terms proportional to π^6, these cancel in the final result. Notably this is an effect of the choice of 1-loop renormalization. If we had chosen to subtract V and not V_0 we would find a π^6 contribution[20].

This completes the discussion of the renormalization of correlation functions with external ghosts and gluons in QCD. Now we must consider correlation functions with external quarks and the extra difficulties which arise.

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[18]Note that their $1/\epsilon$ structure is the same because this is controlled by the 1-loop renormalization, see section 4.5.

[19]See appendix C.1.

[20]This would have no effect on the one-loop amplitude as there is no difference between V and V_0 as $\epsilon \to 0$.

4.4 Two Loop Renormalization with External Quarks

In the discussion of section 4.3 we restricted ourselves to the renormalization of correlation functions with external gluons and ghosts. We have made this separation as when we move to extend this work to the case of external fermions, we find that there is no longer the clean cancellation between the sub-vacuum and loop counterterms that we observe in section 4.3.6. A priori this doesn't have to be disastrous - previously we were surprised by how clean the cancellation was. We do not require this neat cancellation between the loop counterterms and sub vacuum, it is only required that the cancellation leaves us with a renormalization constant which is local and universal. However, we find that in a naive treatment this turns out to not be the case. Therefore it seems that our calculation appears to have catastrophic results - a naive global prescription performs a non-local subtraction at two loop when we have external fermions.

In this section we discuss the origin of this sickness and in turn its requisite remedy. In order to cure the patient we realize that there are "extra" extra integrals required in the case of external fermions which restore this sub-vacuum/loop counterterm cancellation.

4.4.1 Inconsistency in Iterated Integration

At two loops, the naive global prescription fails because it breaks a key property required for computing loop integrals - in a two loop diagram, one should be able to compute a sub-diagram and insert the integrated form into the full diagram and get the same answer. We can see that this fails by considering the expected cancellation in the N_f contribution to the renormalization of the fermion propagator -

$$(\text{S.V.})\left[\ \text{}\ \right] + (N_f)\left[\ \text{}\ \right] = 0, \qquad (4.4.1)$$

where we use the notation $(N_f)[x]$ to indicate that we discuss only the parts proportional to N_f and with the notation $(S.V.)[x]$ we consider the parts of the sub vacuum which we expect to cancel the counterterms[21]. When we calculate the loop counterterm diagram, we find that it is exactly zero in DR because of the form of the insertion. That is, the counterterm insertion is *purely transverse*, and for this reason the diagram evaluates to zero, regardless of the renormalization. When we calculate the sub vacuum

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[21]See section 4.3.6 for a more precise definition.

of the two loop diagram we find that it does not evaluate to zero. This is quite striking as it was only the form of the counterterm which caused the diagram to give zero. Naively speaking, as the fermion loop is transverse at one loop, it should manifest as a transverse sub-diagram.

For this reason, let us consider the sub diagram more closely. We recall that in section 3.1 we performed a very similar calculation - the flavourless corrections to the gluon propagator. The sub diagram which we wish to insert here will take a very similar form which we parameterize as

$$
\begin{array}{c}
q_2 \\
{}^{\mu}\!\!\quad\quad\quad\quad{}^{\nu} \\
\overrightarrow{q_1}\quad\quad q_{12}
\end{array}
\quad = \int [\mathrm{d}^4 q_2] \frac{q_1^2 g^{\mu\nu} - q_1^\mu q_1^\nu}{\overline{q}_2^2 \overline{q}_{12}^2} \left[A + B \frac{\mu_2^2}{q_1{}^2} \right]. \quad (4.4.2)
$$

Inserting this into equation (4.4.1) in place of the fermion loop yields a startling result - indeed the S.V. term is zero as we had expected. This is of great concern - we have performed the integration in two ways which should be equivalent and found different answers.

To begin to understand the origin of this problem, let us consider a specific integral which arises:

$$
\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\gamma_\mu \gamma_\nu q_2^\mu q_2^\nu}{\overline{q}_1^4 (\overline{q}_1 + p)^2 \overline{q}_2^2 \overline{q}_{12}^2} = \int [\mathrm{d}^4 q_1] \frac{\gamma_\mu \gamma_\nu}{\overline{q}_1^4 (\overline{q}_1 + p)^2} \int [\mathrm{d}^4 q_2] \frac{q_2^\mu q_2^\nu}{\overline{q}_2^2 \overline{q}_{12}^2},
$$
$$(4.4.3)$$

where we have still not performed any global prescription. Notably, once we have performed the global prescription this integral is identically zero. We write the RHS in a factorized form to highlight that the integral over q_2 is exactly that of the fermion loop sub diagram. We will only be consistent with the one loop result if we treat this sub-diagram in the same way as the one loop diagram. We should therefore be able to perform integral reduction on the q_2 integral. Recalling this integral reduction from equation (3.1.11) we find the result
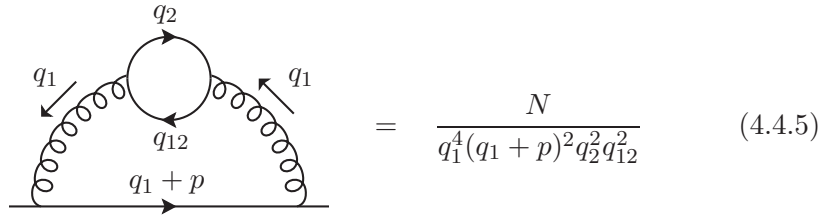
$$
\int [\mathrm{d}^4 q_1] \frac{1}{\overline{q}_1^4 (\overline{q}_1 + p)^2} \int [\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{\overline{q}_2^2 \overline{q}_{12}^2}. \quad (4.4.4)
$$

Here, we mark the $\hat{\mu}_2^2$ with a hat because it is acting only on the sub integral, that is we wish to perform the defining expansion in the sub integral only. This will be further explained in section 4.4.2. Importantly, this means that as we are making efforts to be consistent with the one loop result, the integral no longer evaluates to its naive value of zero. In the coming section we will consider this in more detail and provide a general strategy that restores the cancellation between sub vacua and loop counterterms.

4.4.2 The "Sub-prescription"

As we have discussed, we can see that the problem which the global prescription experiences is an incorrect treatment of the sub diagram. In order to fix this, we here introduce the idea of the "sub-prescription". This prescription allows to identify where we should introduce further extra integrals in order to implement the correct promotion of the sub diagram in the context of the larger diagram.
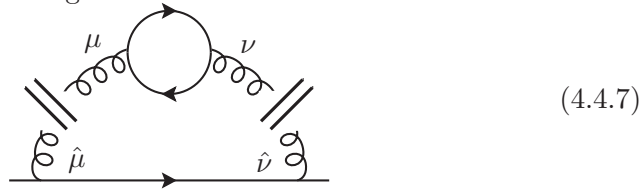
The process which we wish to undergo is to subtract the incorrect promotion of $q_2^2 \to \overline{q}_2^2$ in the sub diagram and then add back the correct promotion. In this section we will discuss them in the context of a standard FDR calculation and leave the discussion of how this is incorporated into the calculation of the effective FDR renormalization constants to section 4.4.3. Let us begin by writing the full *unregulated integrand* of our example diagram, where we have performed the trace over the fermion loop.

$$
\begin{array}{cc}
\text{(diagram)} & = \quad \dfrac{N}{q_1^4(q_1+p)^2 q_2^2 q_{12}^2} \qquad (4.4.5)
\end{array}
$$

$$
N = 4\gamma_\mu \left(\slashed{q}_1 + \slashed{p} \right) \gamma_\nu \left\{ -g^{\mu\nu} q_2 \cdot q_{12} + q_{12}^\mu q_2^\nu + q_2^\mu q_{12}^\nu \right\}, \qquad (4.4.6)
$$

where, for clarity, we have only kept the parts of the expression relevant to the integration. We wish to identify all promotions of the sub graph integration momentum q_2. The first step is to disconnect the divergent sub graph from the rest of the diagram:

$$
\text{(diagram)} \qquad (4.4.7)
$$

We denote the separation of the parts "external" to the sub diagram by placing hats on the lorentz indices (here this is just a notational convenience to help perform our strategy). We can identify the numerator terms which cause sub divergencies in q_2 as

$$
\begin{aligned}
N_2^{\text{div}} &= 4\gamma_{\hat{\mu}} \left(\slashed{q}_1 + \slashed{p} \right) \gamma_{\hat{\nu}} \left\{ -g^{\mu\nu} q_2^2 + 2q_2^\mu q_2^\nu \right\} \\
&= -4\gamma^{\hat{\nu}} \left(\slashed{q}_1 + \slashed{p} \right) \gamma_{\hat{\nu}} q_2^2 + 8\slashed{q}_2 \left(\slashed{q}_1 + \slashed{p} \right) \slashed{q}_2
\end{aligned} \qquad (4.4.8)
$$

Algebraically the hats do not make any difference other than to denote the fact that they are of an origin which is external to the sub diagram, so all standard identities apply, for example

$$\gamma_{\hat{\alpha}}\gamma^{\hat{\alpha}} = \gamma_{\alpha}\gamma^{\hat{\alpha}} = 4 \qquad \gamma_{\hat{\alpha}}q_2^{\alpha} = \slashed{q}_2. \tag{4.4.9}$$

We can extract the promotion which the global prescription performs by performing the anti-commutation to bring all q_2^2 terms out, giving an extra integral of the form

$$\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{4(4-4)(\slashed{q}_1 + \slashed{p})\hat{\mu}_2^2}{\overline{q}_1^4(\overline{q}_1 + p)^2 \overline{q}_2^2 \overline{q}_{12}^2} = 0, \tag{4.4.10}$$

where this cancels due to 4-dimensional gamma algebra. To perform the correct sub-promotion we look at the sub diagram and perform the promotion which it suggests, i.e. we bar only the q_2 internal to the sub graph. In this case we only bar the term with q_2^2, which we can see as there are no hats. In general, any $\hat{q}_2$ are not to be promoted. We find the correct promotion of the sub diagram gives rise to an extra integral of the form

$$\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{-8(\slashed{q}_1 + \slashed{p})\hat{\mu}_2^2}{\overline{q}_1^4(\overline{q}_1 + p)^2 \overline{q}_2^2 \overline{q}_{12}^2}. \tag{4.4.11}$$

To perform the sub-prescription we must then subtract the incorrect global promotion and reintroduce the sub promotion, i.e.

$$\text{sub-prescription} = -(\text{global promotion}) + (\text{sub promotion})$$
$$= \int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{-8(\slashed{q}_1 + \slashed{p})\hat{\mu}_2^2}{\overline{q}_1^4(\overline{q}_1 + p)^2 \overline{q}_2^2 \overline{q}_{12}^2}. \tag{4.4.12}$$

We then calculate this integral by first performing the FDR integral over q_2. To perform this illustratively let us begin by using a Feynman parameterization:

$$I = -8 \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\overline{q}_1^4(\overline{q}_1 + p)^2} \int [\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{\overline{q}_2^2 \overline{q}_{12}^2}$$
$$= -8 \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\overline{q}_1^4(\overline{q}_1 + p)^2} \int_0^1 dx \int [\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{(\overline{q}_2^2 - [-\overline{q}_1^2 \, x(1-x)])^2} \tag{4.4.13}$$

Notice that the q_1 in the extra integral is explicitly barred. Performing this parameterization allows us to easily calculate this integral by recalling the

discussion of extra integrals in section 3.2.3. We replace the integral by minus the vacuum content, making the integral trivial to perform, giving

$$
\begin{aligned}
I &= 8i\pi^2 \left(\int_0^1 dx\, x(1-x) \right) \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\bar{q}_1^2 (\bar{q}_1 + p)^2} \\
&= \frac{2i\pi^2}{3} \slashed{p} \int [\mathrm{d}^4 q_1] \frac{1}{\bar{q}_1^2 (\bar{q}_1 + p)^2}
\end{aligned}
\tag{4.4.14}
$$

In this specific example, this extra contribution is exactly what is required to again cleanly cancel the counterterm in equation (4.4.1). To extend this treatment to the case of multiple divergent sub graphs, one only need to repeat this process for each divergent sub-graph. For further examples, see appendix D.

4.4.3 Renormalizing the Quark Sector

Now that we have the sub-prescription, we can link back to our discussion of the renormalization of the correlation functions with external quarks. We can see how to incorporate the sub-prescription into the calculation of the renormalization constants for the correlation functions by returning to the relation of the FDR amplitude to the renormalized DR amplitude. Firstly we recall where we performed the global prescription, equation (4.2.3)

$$
\int d^n q_1 d^n q_2 J(q_1, q_2) = \lim_{\mu^2 \to 0} \int d^n q_1 d^n q_2 \bar{J}(q_1, q_2). \tag{4.2.3}
$$

It is clear that here we must also perform the sub-prescription in the case of fermions. This has an effect when we decompose the DR bare amplitude into the FDR vacuum and integral. We recall equation (4.2.5),

$$
\left(\lim_{\mu^2} \int_\epsilon J_F \right) + \left(\lim_{\mu^2} \int_\epsilon J_V \right) = \left(\int [d^4 q_1][d^4 q_2] J \right) + \left(\lim_{\mu^2} \int_\epsilon V[J]|_{n_\gamma = n} \right). \tag{4.2.5}
$$

Here, both the vacuum and the FDR integral now contain contributions from the sub-prescription. In practice this means that we can follow the discussion of section 4.3, calculating the global vacuum, sub vacuum and counterterms and then add in the terms from the sub-prescription. We remind the reader that this means that in performing the calculation of the FDR vacuum one must take care to use the DR space time dimensionality n when constructing the amplitude.

A further consideration to make is how to fit the new extra integrals into a sub vacuum like framework. If we wish them to engage in the cancellation

at the integral level, they must be set into the same basis. Naively, when we perform the integration in the divergent extra sub-integral, this leaves us with a constant which does not fit into our basis. The solution is to first re-express the sub integral using the extra integral vacuum trick and then write the other factor as the difference of the DR regulated integral and vacuum. Then we can use the integration by parts identities to relate the extra integral factor to V_0 and finally, if necessary, perform integral reduction on the other factor. As an example, consider
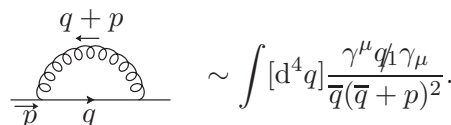
$$
\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{\overline{q}_1^2(\overline{q}_1+p)^2 \overline{q}_2^4 \overline{q}_{12}^2}
$$
$$
= -\int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{\mu^2}{\overline{q}_2^6} \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{1}{\overline{q}_1^2(\overline{q}_1+p)^2} - \frac{1}{\overline{q}_1^4} \right) \qquad (4.4.15)
$$
$$
= \frac{(4-n)}{4} V_0 \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \left(\frac{1}{\overline{q}_1^2(\overline{q}_1+p)^2} - \frac{1}{\overline{q}_1^4} \right)
$$

With this understanding in hand we are able to present the renormalization constants of the quark sector of massless QCD in appendix B.1. As this has been the first calculation involving the sub-prescription, let us comment on a few points of interest. Principally there are a number of diagrams for which the extra extra integrals immediately vanish because the act of removing the "bad prescription" and replacing it with the "good prescription" which we describe in section 4.4.2 is inert. That is, the part removed is equal to the part reintroduced, or both actions do nothing. Consider the following two loop sub vacuum/counterterm cancellation,


$$
= 0 \qquad (4.4.16)
$$

Here we find that the cancellation occurs immediately without the need of performing the sub-prescription. We can understand this by realizing that the sub diagram is simplistic enough to not require global prescription at one loop. I.e. we have no need for it in the diagram

$$
\sim \int [\mathrm{d}^4 q] \frac{\gamma^\mu q\!\!\!/_1 \gamma_\mu}{\overline{q}(\overline{q}+p)^2}. \qquad (4.4.17)
$$

Similarly, in the renormalization of the quark/anti-quark/gluon correlation function, the diagrams with a quark propagator insertion do not require sub-prescription to match their counterterms.

There is a further case where the diagram/counterterm cancellation happens without sub-prescription. This is when the bosonic part of the divergent sub diagram does not reconnect with the rest of the graph. Consider the following two diagrams which have the same divergent sub graph, the triangular part with the gluon three point vertex,
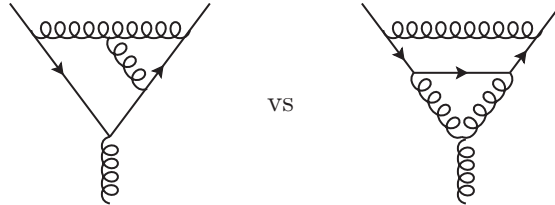
The first requires a sub-prescription, whilst the second does not. We can understand this because in the second, the part of the sub diagram which is incorrectly identified by the global prescription in the first, remains external and so is treated correctly as it is untreated by the global prescription.

4.4.4 The Sub-prescription and FDR vs FDH

In section 4.3 we observed that as well as renormalizing à la FDR starting from a DR viewpoint we could also work in the framework of the four dimensional helicity scheme (FDH). Here, the crucial difference is that in DR one constructs the amplitudes taking the dimensionality of the fields, n_γ to be n, but in FDH the amplitudes are constructed with $n_\gamma = 4$. It is then an interesting question to see what happens if we move the renormalization of correlation functions with external fermions to an FDH based approach. We find that in FDH we observe cancellation between sub vacuum and counterterms *without* the introduction of the sub-prescription. That is, for external fermions FDH suggests that we *do not* make the sub-prescription, whilst DR suggests that we *do* perform it.

This seemingly inconsistent result begins to make sense when considered in the context of the recent literature on FDH. In 2012, Kilgore [58] showed that at two loop FDH is not unitary in the presence of external fermions. More specifically it was shown that the standard renormalization procedure does not give results consistent with the literature at two loops and even fails to make the result finite at three loops. However, FDH has been shown to give completely consistent results in purely gluonic QCD.

It is interesting to consider that our work here demonstrates that without the sub-prescription FDR would fall into the same trap as FDH. In some ways this is not strongly surprising as without the sub-prescription

they both employ the same 4-dimensional algebra and preserve numerator/denominator cancellation to enforce gauge invariance - hence why the sub vacua and counterterms match when working in an FDH based framework. Since the discovery of the non-unitarity of FDH a number of attempts to resolve it have been made - [83, 84]. It would be interesting to investigate a sub-prescription inspired approach to solving the issues experienced by FDH. Here we stress that similar to FDH, a naive application of the global prescription in FDR yields physically consistent results when applied to purely gluonic QCD. That is, in the calculations of section 4.3 we have not included the sub-prescription and still found consistently universal results with the quark sector. Therefore, it would be of immediate interest to understand why purely gluonic QCD does not require the sub-prescription and to see if this has an analogy in the context of FDH.

4.5 Parameter Renormalization, Universality and Shifts

In the previous sections we have calculated the renormalization constants in the FDR scheme for all of the correlation functions given in figure 4.1. As stated in figure 4.1, these renormalization constants are related to the renormalization constants of the bare parameters of the theory through the Slavnov-Taylor identities. We can solve for the renormaliation constants of the parameters giving us

$$Z_G = Z_{GG}^{1/2}, \tag{4.5.1}$$

$$Z_{c_a} = Z_{cc}^{1/2}, \tag{4.5.2}$$

$$Z_\psi = Z_{\psi\psi}^{1/2}, \tag{4.5.3}$$

$$Z_{\alpha_S} = \frac{Z_{GGG}^2}{Z_{GG}^3} = \frac{Z_{Gcc}^2}{Z_{GG}Z_{cc}^2} = \frac{Z_{G\psi\psi}^2}{Z_{GG}Z_{\psi\psi}^2}. \tag{4.5.4}$$

The last expression shows us that for a consistent renormalization method the renormalization constants we extract from each three point vertex are related, i.e. a good method of renormalization treats all correlation functions in a universal way. It is therefore a strong test of FDR's approach to verify the universality of the renormalization constants. The renormalization constants for all of the parameters are presented in appendix B.1, and we find that computing Z_{α_S} in all possible ways gives the same result. In this way, we verify the universality of FDR's approach to renormalization.

Note that the renormalization constants presented in appendix B.1 are expanded as a series in $\epsilon = n - 4$. Over the course of this work, we have been surprised at the benefits gained by keeping the calculation at the unintegrated level. Specifically, if we were to have expanded the global and sub-vacua in ϵ, we would not have witnessed such a dramatically clean cancellation. An interesting question, therefore, is whether or not we find that the universality of the renormalization constants is present *before* expanding around $n = 4$, and thereby true to all orders in ϵ. From the perspective of $\mathrm{DR}/\overline{MS}$ we have no reason to expect this - we have multiple representations of the same $1/\epsilon$ pole, and no reason to expect that they would not mix when comparing the different calculations of the renormalization constants. To discuss this, let us consider the form of Z_{α_S} as extracted from vertex j. We can write this as

$$
\begin{aligned}
Z_{\alpha_S}|_j = \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} & \left(\frac{d_1^j}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2} + \frac{d_2^j}{\bar{q}_1^4 \bar{q}_2^4} \right) \\
& + d_3^j \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4} \left[\int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\bar{q}_2^4} \Big|_{\epsilon \to 0} \right] \\
& + d_4^j \left[\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4} \Big|_{\epsilon \to 0} \right] \left[\int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\bar{q}_2^4} \Big|_{\epsilon \to 0} \right] .
\end{aligned}
\tag{4.5.5}
$$

Here all coefficients d_i^j are functions of n, that a priori could depend on the vertex we choose to calculate Z_{α_S}. Upon investigation, using the two vertices $j = GGG$ and $j = Gcc$ we find that indeed this coefficients do depend on the vertex used - that is universality is *not* manifest. However, we observe two things. Firstly, as the two expressions coincide when integrated and as $n \to 4$, then the difference must be $O(\epsilon)$. Secondly, the first coefficient, d_1^j, is the same between the two vertices so there is *some* universality between the two vertices. When we compute this difference we find that it can be written

$$
Z_{\alpha_S}|_{GGG} - Z_{\alpha_S}|_{Gcc} \sim \left(\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4} - \left[\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4} \Big|_{\epsilon \to 0} \right] \right)^2 ,
\tag{4.5.6}
$$

that is, the difference is proportional to a quantity which vanishes as $n \to 4$. The form of this leads us to an interesting suggestion. The second term, V_0 is only present in our calculation because we renormalize the one loop by subtracting the $1/\epsilon$ pole and a universal constant. However, at one loop it would make no difference as $n \to 4$ to subtract exactly V, as the difference is vanishing. In doing this all V_0 in our calculations would be replaced by V

and so the difference in equation (4.5.6) would vanish, such that we indeed would have universality at all orders in ϵ.

Now we move to the principal relation of physical interest - the rule for understanding the UV effects of switching schemes from DR/$\overline{MS}$ to FDR. We parameterize this as a shift in the coupling constant. The two schemes effectively differ by the different constants that they subtract at one and two loop level. As this is universal it amounts to a shift in the coupling constant, which we write up to NNLO as

$$\frac{\alpha_{\overline{MS}}}{4\pi} = \frac{\alpha_{FDR}}{4\pi}\left[1 + c_1 \cdot \frac{\alpha_{FDR}}{4\pi} + c_2 \cdot \left(\frac{\alpha_{FDR}}{4\pi}\right)^2\right]. \tag{4.5.7}$$

To find c_1 and c_2 we realize that we can relate the two coupling constants through their original connection to the bare coupling,

$$\alpha_0 = Z_{\overline{MS}} \cdot \alpha_{\overline{MS}} = Z_{FDR} \cdot \alpha_{FDR}. \tag{4.5.8}$$

Writing it in this form, it is clear that the constants c_i find their origin the ratio of the two renormalization constants. Let us write out the renormalization constants to NNLO,

$$Z_{\overline{MS}} = 1 + \alpha_{\overline{MS}}\left[\frac{A}{\epsilon}\right] + \alpha_{\overline{MS}}^2\left[\frac{C}{\epsilon^2} + \frac{D}{\epsilon}\right] \tag{4.5.9}$$

$$Z_{FDR} = 1 + \alpha_{FDR}\left[\frac{A}{\epsilon} + B\right] + \alpha_{FDR}^2\left[\frac{C}{\epsilon^2} + \frac{D'}{\epsilon} + E\right]. \tag{4.5.10}$$

For the purpose of clarity, we have left out the appropriate factors of $(e^{\gamma_e}\pi)^\epsilon$ which define the difference between MS and $\overline{MS}$ (and are also present in the FDR scheme). Therefore the $\overline{MS}$ renormalization constants are given strictly in terms of poles, but the FDR scheme also contains constant parts. A number of the coefficients are the same between the two renormalization constants, but notably the single pole of the NNLO correction is different between the two. Now we can insert the expressions for the two Zs into equation (4.5.8), and so we find the following relations,

$$c_1 = D, \tag{4.5.11}$$
$$c_2 = E, \tag{4.5.12}$$
$$D' = D + 2AB. \tag{4.5.13}$$

The third equation is a consistency constraint which we need to satisfy. It is essentially a statement that the difference between the $1/\epsilon$ poles in the

second order corrections to the renormalization constants is controlled by the constant difference at one loop.

In order to be able to find values for c_1 and c_2, we need the explicit value of the renormalization constant in $\overline{MS}$. Further, if we extend this calculation to all correlation functions we have independent determinations of the QCD renormalization constants which we can compare to literature in order to ascertain the soundness of our computational machinery. To be able to calculate the renormalization constants in $\overline{MS}$ we need to be able to extract the pole part of our correlation functions. Remembering our discussion in section 4.2.1 we realize that we can do this by taking the pole part of the FDR vacuum as

$$
\begin{aligned}
\text{p.p.}\left[\int_\epsilon J\right] &= \text{p.p.}\left[\left(\lim_{\mu^2}\int_\epsilon J_F\right) + \left(\lim_{\mu^2}\int_\epsilon V[J]|_{n_\gamma=n}\right)\right] \\
&= \text{p.p.}\left[\left(\lim_{\mu^2}\int_\epsilon V[J]|_{n_\gamma=n}\right)\right],
\end{aligned}
\tag{4.5.14}
$$

where we use the subscript ϵ to denote DR at the appropriate loop order. Equation (4.5.14) then tells us that the pole part of a dimensionally regulated integrand J is given by the pole part of the FDR vacuum of J as we can split J into the FDR vacuum and the FDR finite part which, by construction, has no poles in ϵ. We can then calculate the renormalization constants in $\overline{MS}$ up to two loop using

$$
(Z_1)_{\overline{MS}} \cdot M_0 = \text{p.p.}\left(-\left.V[M_1]\right|_{n_\gamma=n}\right),
\tag{4.5.15}
$$

$$
(Z_2)_{\overline{MS}} \cdot M_0 = \text{p.p.}\left(-\left.V[M_2]\right|_{n_\gamma=n} - \left.(\text{Loop CTs})\right|_{\overline{MS}}\right).
\tag{4.5.16}
$$

Here Z_1 and Z_2 are respectively the one and two loop renormalization constants of the given correlation function and M_0 is the tree level form[22] of the correlation function. We remind the reader that the loop counterterms in equation (4.5.16) are constructed using the $\overline{MS}$ 1-loop counterterm insertions which, in comparison to FDR, have no finite part. A technical subtlety here is that, unlike in the computations for the FDR scheme, the loop counterterms and sub-vacua do not cancel exactly. In general we find that they differ by terms of $O(1/\epsilon)$. In principle this is of no concern as we are only required to find that the non-local terms drop, i.e. there are no terms of the form $\log(M^2)/\epsilon$, however in practice it is worth being aware of. We have computed these renormalization constants and give them in appendix B.2.

---

[22]See footnote 5 on page 69.

Further, we have compared them to literature [85, 86, 87] and found them to be in agreement, supporting the soundness of our computational machinery.

At this point we have everything in hand to state the coupling constant shift between DR/$\overline{MS}$ and FDR. Firstly, let us note that we verify that the FDR scheme satisfies equation (4.5.13). Next we present the values of c_i in the coupling constant shift:

$$c_1 = \frac{-N_c}{3},$$
$$c_2 = N_c^2 \left[-\frac{85}{18} - 8f_{11} \right] + \frac{N_f}{6} \left[N_c \left(-6 + 4f_{11} \right) + T_2 \left(9 + 8f_{11} \right) \right], \tag{4.5.17}$$
$$\text{where} \quad T_2 = \frac{N_c^2 - 1}{2N_c}. \tag{4.5.18}$$

As previously stated, the FDR scheme is equivalent to FDH/DRed at NLO in the UV as to this order the shift in the coupling is the same [75]. However, the schemes differ at NNLO in the UV.

Chapter 5

Conclusions and Perspectives

In this thesis we have made two different attacks on the problem of loop computations in precision physics. The first was an investigation in the context of the multi-leg frontier. Here we worked in the framework of the OPP method, which allows automatic integral reduction for one loop virtual amplitudes. In order to be able to use the OPP method with HEFT and its pseudoscalar analogue we have computed the required effective R_2 Feynman rules for both Lagrangians. In the latter publication this allowed for the first non-standard model one loop studies in the MADGRAPH5_AMC@NLO framework, a study of the CP properties of the top quark/Higgs interaction. Here it was shown that di-jet correlations in Higgs + 2 jets can provide a valuable tool for discerning the CP properties of this coupling. In the time since this R_2 investigation was undertaken, a package for the automatic computation of both R_2 and UV counterterms in renormalizable quantum field theories has been built, NLOCT [88]. In this way, within the context of frameworks such as MADGRAPH5_AMC@NLO or HELAC-NLO there is currently no need for further computations of R_2 vertices in renormalizable theories to be performed manually. In the future, one may still need to undergo such tasks by hand in non-renormalizable theories, but there are also plans for the extension of NLOCT to non-renormalizable theories.

The second attack which we have made in this thesis has been on the loop frontier in massless QCD. We have undertaken an investigation within the framework of FDR, a four dimensional approach for regularizing quantum field theories. Our chief goal has been to understand the renormalization of α_S in this scheme and to develop transition rules between FDR and DR/$\overline{MS}$.

To achieve this, we have developed a framework to identify the effective renormalization constants implied by the FDR approach and applied them

to DR amplitudes in a canonical counterterm picture. In this endeavour, we have been greatly aided by our automation of the FDR defining expansion at both one and two loops. Over the course of this calculation we have been able to prove the identity of the one-loop UV treatments performed by FDR and DRed/FDH. We have also seen exactly how the FDR vacuum fits into a canonical picture of order by order renormalization through the observation of how the FDR vacuum cleanly matches the loop counterterms. What's more, we have identified the need, when studying processes with external fermions, of the sub-prescription within the FDR treatment. This has been necessary to preserve the locality and universality of the FDR approach.

In summary we have calculated the renormalization constants of massless QCD in FDR through the lens of dimensional regularization. The universality of the renormalization constants which we have found amount to an explicit confirmation of the existence of a finite approach for computing predictions in gauge theories concretely within 4-dimensions. The computation of the renormalization constants has allowed us to understand the relationship between the renormalization of α_S in FDR and DR. For any QCD calculation in DR, free of IR divergence effects, this allows one to elucidate the FDR result.

There are still many challenges in the road ahead for FDR. A simple next step is to extend the calculation presented here to the case of massive QCD and it would not prove difficult to repeat the work for other theories such as the electro-weak model. In order to be able to approach complete two loop calculations within the context of FDR one also needs an FDR prescription for the double real phase space. Additionally, for processes with incoming quarks one would need a prescription to map the IR treatments of FDR and CDR in order to match to PDFs which are all given in terms of conventional dimensional regularization. Nevertheless, now with a full understanding of renormalization in FDR under control, it is now possible to be able to undertake investigation into purely numerical approaches for the virtual amplitudes.

Another potentially fruitful ground for further investigation is the sub-prescription presented in this thesis. Firstly it needs to be better understood in contexts where it does not appear, such as calculations within purely gluonic QCD. What's more, it would be very intriguing to understand the analogy between the respective problems suffered by FDR with a naive global prescription and FDH at two loop. Specifically we have observed that the breakdown of a naive global prescription, like the failure of unitarity suffered in FDH, occurs only in the case of external fermions, and not with external gluons and ghosts. It would therefore be interesting to investigate this du-

ality between FDR and FDH and to see if the sub-prescription proposed in
this thesis could be extended to solve the problem of FDH.

Acknowledgements

Appendix A

Feynman Rules

Here we present the complete collection of Feynman rules relevant to the calculations performed in this thesis. Firstly in appendix A.1 we present the Feynman rules for bare QCD. Next in appendix A.2 we present the Feynman rules for the HEFT, followed in appendix A.3 by the R_2 vertices for HEFT + pseudoscalar. Finally we present in appendix A.4 the counterterm vertices in renormalized massless QCD.

A.1   Bare Massless QCD

$$s_1 \quad \xrightarrow{q} \quad s_2 \qquad = i\delta^{s_1 s_2} \frac{1}{\not{q}}$$

$$\mu \qquad = -ig\gamma^\mu t^a_{s_1 s_2}$$

Figure A.1: The Feynman rules for the quark sector of massless QCD. All momenta are incoming.

$$\mu, m \xrightarrow{q} \nu, n \quad = -i\delta^{mn}\frac{1}{q^2}\left(g^{\mu\nu} + (\xi - 1)\frac{q^\mu q^\nu}{q^2}\right)$$

$$m \xrightarrow{q} n \quad = i\delta^{mn}\frac{1}{q^2}$$

$$
\begin{aligned}
p_2, \mu_2, m_2 \\
p_1, \mu_1, m_1 \quad &= g\, f^{m_1 m_2 m_3}\left[\; g^{\mu_1 \mu_2}\,(p_1 - p_2)^{\mu_3}\right. \\
&\qquad\qquad\quad +g^{\mu_2 \mu_3}\,(p_2 - p_3)^{\mu_1} \\
&\qquad\qquad\quad \left. +g^{\mu_3 \mu_1}\,(p_3 - p_1)^{\mu_2}\right] \\
p_3, \mu_3, m_3 &
\end{aligned}
$$

$$
\begin{aligned}
\mu_2, m_2 \qquad \mu_3, m_3 \\
\\
\mu_1, m_1 \qquad \mu_4, m_4
\end{aligned}
\quad
\begin{aligned}
= -i\,g^2\Big[\; & f^{m_1 m_2 b} f^{m_3 m_4 b}\,(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) \\
+ & f^{m_1 m_3 b} f^{m_2 m_4 b}\,(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) \\
+ & f^{m_1 m_4 b} f^{m_2 m_3 b}\,(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4})\Big]
\end{aligned}
$$

$$
\begin{aligned}
p_2, \mu_2, m_2 \\
p_1, \mu_1, m_1 \quad &= g f^{m_1 m_2 m_3} p^{\mu_3} \\
p_3, \mu_3, m_3 &
\end{aligned}
$$

Figure A.2: The Feynman rules for the gauge sector of massless QCD. All momenta are incoming. ξ is the gauge parameter which is set to 1 in the Feynman gauge.

A.2 Higgs Effective Field Theory with Pseudoscalar

p_1, μ_1, m_1 p_2, μ_2, m_2

H

$$= -i\,g_H\,\delta^{m_1 m_2}\left(p_2^{\mu_1} p_1^{\mu_2} - g^{\mu_1 \mu_2}\,p_1 \cdot p_2\right)$$

p_2, μ_2, m_2

p_1, μ_1, m_1 p_3, μ_3, m_3

H

$$= -g_H\,g\,f^{m_1 m_2 m_3}\Bigg[\quad g^{\mu_1 \mu_2}\,(p_1 - p_2)^{\mu_3}$$
$$+g^{\mu_2 \mu_3}\,(p_2 - p_3)^{\mu_1}$$
$$+g^{\mu_3 \mu_1}\,(p_3 - p_1)^{\mu_2}\Bigg]$$

μ_2, m_2 μ_3, m_3

μ_1, m_1 μ_4, m_4

H

$$= -i\,g_H\,g^2\,F^{\mu_1 \mu_2 \mu_3 \mu_4}_{m_1 m_2 m_3 m_4}$$

Figure A.3:   The tree level Feynman rules for the HEFT Lagrangian in equation (2.1.1). Momenta are incoming, f^{abc} is the $SU(3)$ structure factor. The dashed line represents the Higgs field. The tensor F is given by (A.2.1).

$$F^{\mu_1 \mu_2 \mu_3 \mu_4}_{m_1 m_2 m_3 m_4} = \Bigg[\quad f^{m_1 m_2 b} f^{m_3 m_4 b}\left(g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}\right)$$
$$+f^{m_1 m_3 b} f^{m_2 m_4 b}\left(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}\right) \qquad (A.2.1)$$
$$+f^{m_1 m_4 b} f^{m_2 m_3 b}\left(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4}\right)\Bigg]$$

$$p_1, \mu_1, m_1 \qquad p_2, \mu_2, m_2$$

$$= -i\, g_A\, \delta^{m_1 m_2}\, \epsilon^{\mu_1 \mu_2 \rho \sigma}\, p_{1\rho}\, p_{2\sigma}$$

$$A$$

$$p_2, \mu_2, m_2$$

$$= -g_A\, g\, f^{m_1 m_2 m_3}\, \epsilon^{\mu_1 \mu_2 \mu_3 \rho}\, (p_1 + p_2 + p_3)_\rho$$

$$p_1, \mu_1, m_1 \qquad p_3, \mu_3, m_3$$

$$A$$

Figure A.4: The tree level Feynman rules for the pseudoscalar Lagrangian in equation (2.1.3). All momenta are incoming, where f^{abc} is the $SU(3)$ structure factor. The dashed line represents the pseudoscalar field. Note that in the limit $p_A \to 0$ the amplitudes vanish.

A.3　HEFT+Psuedoscalar R_2

$$p_1, \mu_1, m_1 \qquad p_2, \mu_2, m_2 \qquad = \frac{ig_H g^2 N_c}{384\pi^2} \delta^{m_1 m_2} T^{\mu_1 \mu_2}$$

H

$$p_2, \mu_2, m_2$$

$$= \frac{g_H g^3 15 N_c}{128\pi^2} f^{m_1 m_2 m_3} \left[\quad g^{\mu_1 \mu_2} (p_1 - p_2)^{\mu_3} \right.$$

$$+ g^{\mu_2 \mu_3} (p_2 - p_3)^{\mu_1}$$

$$p_1, \mu_1, m_1 \qquad p_3, \mu_3, m_3 \qquad \qquad \left. + g^{\mu_3 \mu_1} (p_3 - p_1)^{\mu_2} \right]$$

H

$$\mu_2, m_2 \qquad \mu_3, m_3$$

$$= \frac{ig_H g^4}{128\pi^2} X^{\mu_1 \mu_2 \mu_3 \mu_4}_{m_1 m_2 m_3 m_4}$$

$$\mu_1, m_1 \qquad \mu_4, m_4$$

H

Figure A.5:   The R$_2$ vertices generated by the effective Lagrangian in (2.1.1). All momenta are incoming, N_c is the number of colors. The dashed line represents the Higgs field. The tensors T and X are defined in equations (A.3.1) and (A.3.2) respectively.

$$= \frac{ig_H g^2}{32\pi^2}\left(\frac{N_c^2-1}{2N_c}\right)\delta^{j_1 j_2}\lambda_{\text{HV}}(\not{p}_1 - \not{p}_2)$$

$$= \frac{ig_H g^3}{64\pi^2}\gamma_\mu t^m_{j_2 j_1}\left[\frac{2\lambda_{\text{HV}}+1}{N_c} - (2\lambda_{\text{HV}}+3)N_c\right]$$

Figure A.6:   The R_2 vertices generated by the effective Lagrangian in (2.1.1). Momenta are incoming, N_c is the number of colors, $\lambda_{\text{HV}} = 1$ ($\lambda_{\text{HV}} = 0$) in DR (DRed), quarks are massless and the dashed line is the Higgs.

$$= \ i\, g_A\, \frac{g_s^2 N_c}{96\pi^2}\, \delta^{m_1 m_2}\, \epsilon^{\mu_1 \mu_2 \rho\sigma}\, p_{1\rho}\, p_{2\sigma}$$

$$= \ g_A\, \frac{g_s^3 N_c}{64\pi^2}\, f^{m_1 m_2 m_3}\, \epsilon^{\mu_1 \mu_2 \mu_3 \rho}\, (p_1 + p_2 + p_3)_\rho$$

Figure A.7: The R_2 vertices generated by the effective Lagrangian for the pseudoscalar interactions, equation (2.1.3). All momenta are incoming, N_c is the number of colors. The dashed line represents the pseudoscalar field.

$$T^{\mu_1\mu_2} = \Big[\quad p_1^{\mu_1} p_2^{\mu_2} + 89\, p_1^{\mu_2} p_2^{\mu_1} + 14\, (p_1^{\mu_1} p_1^{\mu_2} + p_2^{\mu_1} p_2^{\mu_2})$$
$$- \left[17\, (p_1^2 + p_2^2) + 93\, (p_1 \cdot p_2) \right] g^{\mu_1\mu_2} \Big], \tag{A.3.1}$$

$$X^{\mu_1\mu_2\mu_3\mu_4}_{m_1 m_2 m_3 m_4} = \Big[\quad C^{m_1 m_2 m_3 m_4} D^{\mu_1\mu_2\mu_3\mu_4} + C^{m_1 m_2 m_4 m_3} D^{\mu_1\mu_2\mu_4\mu_3}$$
$$+ C^{m_1 m_3 m_2 m_4} D^{\mu_1\mu_3\mu_2\mu_4} \Big], \tag{A.3.2}$$

$$C^{m_1 m_2 m_3 m_4} = f^{m_1 bc} f^{m_2 cd} f^{m_3 de} f^{m_4 eb}, \tag{A.3.3}$$

$$D^{\mu_1\mu_2\mu_3\mu_4} = 21 g^{\mu_1\mu_2} g^{\mu_3\mu_4} - 41 g^{\mu_1\mu_3} g^{\mu_2\mu_4} + 21 g^{\mu_1\mu_4} g^{\mu_2\mu_3}. \tag{A.3.4}$$

A.4 Renormalized Massless QCD

$$= i\,(Z_{\psi\psi} - 1)\, \delta^{s_1 s_2} \slashed{q}$$

$$= -i\,(Z_{G\psi\psi} - 1)\, g \gamma^\mu t^a_{s_1 s_2}$$

Figure A.8: Counterterm vertices for the quark sector of massless QCD.

$$\mu, m \quad q \quad \nu, n \qquad = -i(Z_{GG} - 1)\delta^{mn}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$m \quad q \quad n \qquad = i(Z_{cc} - 1)\delta^{mn} q^2$$

$$p_2, \mu_2, m_2$$

$$= (Z_{GGG} - 1)g\, f^{m_1 m_2 m_3} \left[\begin{array}{l} g^{\mu_1\mu_2}\,(p_1 - p_2)^{\mu_3} \\ +g^{\mu_2\mu_3}\,(p_2 - p_3)^{\mu_1} \\ +g^{\mu_3\mu_1}\,(p_3 - p_1)^{\mu_2} \end{array} \right]$$

$$p_1, \mu_1, m_1$$

$$p_3, \mu_3, m_3$$

$$\mu_2, m_2 \qquad \mu_3, m_3$$

$$= -i\,(Z_{GGGG} - 1)\,g^2\, F^{\mu_1\mu_2\mu_3\mu_4}_{m_1 m_2 m_3 m_4}$$

$$\mu_1, m_1 \qquad \mu_4, m_4$$

$$p_2, \mu_2, m_2$$

$$= (Z_{Gcc} - 1)\,g f^{m_1 m_2 m_3} p^{\mu_3}$$

$$p_1, \mu_1, m_1$$

$$p_3, \mu_3, m_3$$

Figure A.9: Counterterm vertices for the gauge sector of massless QCD.

Appendix B

Renormalization Constants

Here we list the renormalization constants for both the $\overline{MS}$ and FDR scheme. We remind the reader that by calling FDR a scheme, we have regularized in DR and defined the scheme by matching to the result as regularized in FDR.

B.1 FDR At Two Loop

Here we list the correlation function renormalization constants calculated in section 4.2.3. As the FDR scheme is equivalent at one-loop to DRed and FDH, one can truncate at $O(\alpha_S)$ to find the one-loop renormalization constants for those two schemes.

$$
\begin{aligned}
Z_{GG} = 1 + & \frac{\alpha_S}{4\pi} \left[\frac{1}{\epsilon} \left(-\frac{10}{3} N_c + \frac{4}{3} N_f \right) + \frac{1}{3} N_c \right] \\
+ & \left(\frac{\alpha_S}{4\pi} \right)^2 \left[\frac{1}{\epsilon^2} \left(-\frac{25}{3} N_c^2 + \frac{10}{3} N_c N_f \right) + \right. \\
& \frac{1}{\epsilon} \left(-\frac{65}{12} N_c^2 + N_f \left(\frac{5}{2} N_c + 2 T_2 \right) \right) + \\
& \frac{39}{16} N_c^2 + \frac{7}{2} N_c^2 f_{11} + \\
& \left. N_f \left(\frac{7}{8} N_c - \frac{3}{2} T_2 + f_{11} \left(-\frac{4}{3} T_2 - \frac{1}{3} N_c \right) \right) \right]
\end{aligned}
\tag{B.1.1}
$$

120

$$Z_{GGG} = 1 + \frac{\alpha_S}{4\pi} \left[\frac{1}{\epsilon} \left(-\frac{4}{3} (N_c + -N_f) \right) + \frac{1}{3} N_c \right]$$
$$+ \left(\frac{\alpha_S}{4\pi} \right)^2 \left[\frac{1}{\epsilon^2} \left(-\frac{13}{2} N_c^2 + 5 N_c N_f \right) + \right.$$
$$\frac{1}{\epsilon} \left(-\frac{59}{24} N_c^2 + N_f \left(\frac{25}{12} N_c + 2T_2 \right) \right) + \qquad \text{(B.1.2)}$$
$$\frac{119}{96} N_c^2 + \frac{5}{4} N_c^2 f_{11} +$$
$$\left. N_f \left(\frac{13}{16} N_c - \frac{3}{2} T_2 + f_{11} \left(-\frac{4}{3} T_2 - \frac{1}{6} N_c \right) \right) \right]$$

$$Z_{cc} = 1 + \frac{\alpha_S}{4\pi} \frac{1}{\epsilon} (-N_c)$$
$$+ \left(\frac{\alpha_S}{4\pi} \right)^2 \left[\frac{1}{\epsilon^2} \left(-4N_c^2 + N_c N_f \right) + \frac{1}{\epsilon} \left(-\frac{37}{24} N_c^2 + \frac{5}{12} N_c N_f \right) + \right.$$
$$\left. \frac{79}{96} N_c^2 + \frac{3}{2} N_c^2 f_{11} + N_f \left(\frac{1}{16} N_c + -\frac{1}{6} N_c f_{11} \right) \right]$$
$$\text{(B.1.3)}$$

$$Z_{Gcc} = 1 + \frac{\alpha_S}{4\pi} \frac{1}{\epsilon} (N_c)$$
$$+ \left(\frac{\alpha_S}{4\pi} \right)^2 \left[\frac{1}{\epsilon^2} \left(\frac{5}{2} N_c^2 \right) + \frac{1}{\epsilon} \left(\frac{3}{4} N_c^2 \right) + \qquad \text{(B.1.4)} \right.$$
$$\left. -\frac{3}{8} N_c^2 + -\frac{3}{4} N_c^2 f_{11} \right]$$

$$Z_{\Psi\Psi} = 1 + \frac{\alpha_S}{4\pi} \left[\frac{1}{\epsilon} (2T_2) + T_2 \right]$$
$$+ \left(\frac{\alpha_S}{4\pi} \right)^2 \left[\frac{1}{\epsilon^2} \left(-2 + 2N_c^2 + 2T_2^2 \right) + \right.$$
$$\frac{1}{\epsilon} \left(-\frac{17}{4} + \frac{17}{4} N_c^2 + \frac{1}{2} T_2^2 - N_f T_2 \right) +$$
$$-\frac{7}{16} + \frac{7}{16} N_c^2 - \frac{5}{8} T_2^2 + f_{11} \left(2 - \frac{1}{3} T_2^2 - 2N_c^2 \right) +$$
$$\left. N_f \left(\frac{1}{4} T_2 + \frac{4}{3} T_2 f_{11} \right) \right]$$
$$\text{(B.1.5)}$$

$$Z_{G\Psi\Psi} = 1 + \frac{\alpha_S}{4\pi}\left[\frac{1}{\epsilon}\left(2\left(N_c + T_2\right)\right) + T_2\right]$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-4 + \frac{25}{2}N_c^2 + 2T_2^2 - N_cN_f\right) + \right.$$
$$\frac{1}{\epsilon}\left(-\frac{21}{4} + \frac{181}{24}N_c^2 + \frac{1}{2}T_2^2 + N_f\left(-T_2 - \frac{5}{12}N_c\right)\right) + $$
$$-\frac{7}{16} - \frac{5}{8}T_2^2 - \frac{73}{96}N_c^2 + f_{11}\left(2 - \frac{1}{3}T_2^2 - \frac{17}{4}N_c^2\right) + $$
$$\left. N_f\left(\frac{1}{4}T_2 - \frac{1}{16}N_c + f_{11}\left(\frac{1}{6}N_c + \frac{4}{3}T_2\right)\right)\right]$$

(B.1.6)

Further, using the Slavnov-Taylor identities we can relate these to the renormalization constants of α_S and the various fields:

$$Z_{\alpha_S} = 1 + \frac{\alpha_S}{4\pi}\left[\frac{1}{\epsilon}\left(\frac{22}{3}N_c + -\frac{4}{3}N_f\right) + -\frac{1}{3}N_c\right]$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{484}{9}N_c^2 + \frac{16}{9}N_f^2 + -\frac{176}{9}N_cN_f\right) + \right.$$
$$\frac{1}{\epsilon}\left(\frac{58}{9}N_c^2 + N_f\left(-2T_2 - \frac{22}{9}N_c\right)\right) + \qquad \text{(B.1.7)}$$
$$-\frac{85}{18}N_c^2 + -8N_c^2f_{11} + $$
$$\left. N_f\left(\frac{3}{2}T_2 - N_c + f_{11}\left(\frac{2}{3}N_c + \frac{4}{3}T_2\right)\right)\right]$$
$$Z_G = 1 + \frac{\alpha_S}{4\pi}\left[\frac{1}{\epsilon}\left(-\frac{5}{3}N_c + \frac{2}{3}N_f\right) + \frac{1}{6}N_c\right]$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-\frac{2}{9}N_f^2 - \frac{50}{9}N_c^2 + \frac{25}{9}N_cN_f\right) + \right.$$
$$\frac{1}{\epsilon}\left(-\frac{175}{72}N_c^2 + N_f\left(\frac{41}{36}N_c + T_2\right)\right) + \qquad \text{(B.1.8)}$$
$$\frac{347}{288}N_c^2 + \frac{7}{4}N_c^2f_{11} + $$
$$\left. N_f\left(\frac{7}{16}N_c - \frac{3}{4}T_2 + f_{11}\left(-\frac{2}{3}T_2 - \frac{1}{6}N_c\right)\right)\right]$$

$$Z_c = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(-\frac{1}{2}N_c\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2 \left[\frac{1}{\epsilon^2}\left(-\frac{17}{8}N_c^2 + \frac{1}{2}N_cN_f\right) + \frac{1}{\epsilon}\left(-\frac{37}{48}N_c^2 + \frac{5}{24}N_cN_f\right) + \right.$$
$$\left. \frac{79}{192}N_c^2 + \frac{3}{4}N_c^2 f_{11} + N_f\left(\frac{1}{32}N_c + -\frac{1}{12}N_c f_{11}\right)\right]$$

(B.1.9)

$$Z_\Psi = 1 + \frac{\alpha_S}{4\pi}\left[\frac{1}{\epsilon}(T_2) + \frac{1}{2}T_2\right]$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2 \left[\frac{1}{\epsilon^2}\left(-1 + N_c^2 + \frac{1}{2}T_2^2\right) + \right.$$
$$\frac{1}{\epsilon}\left(-\frac{17}{8} + \frac{17}{8}N_c^2 - \frac{1}{4}T_2^2 + -\frac{1}{2}N_f T_2\right) +$$
$$-\frac{7}{32} + \frac{7}{32}N_c^2 - \frac{7}{16}T_2^2 + f_{11}\left(1 - \frac{1}{6}T_2^2 - N_c^2\right) +$$
$$\left. N_f\left(\frac{1}{8}T_2 + \frac{2}{3}T_2 f_{11}\right)\right]$$

(B.1.10)

B.2 $\overline{MS}$ At Two Loop

Here we list the $\overline{MS}$ renormalization constants calculated in section 4.5. They coincide with Mihaila [85], Egorian [86] and Muta [87]. Here we take α_S to be $\alpha_S|_{\overline{MS}}$.

$$Z_{GG} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(-\frac{10}{3}N_c + \frac{4}{3}N_f\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2 \left[\frac{1}{\epsilon^2}\left(-\frac{25}{3}N_c^2 + \frac{10}{3}N_cN_f\right) + \right.$$
$$\left. \frac{1}{\epsilon}\left(-\frac{23}{4}N_c^2 + N_f\left(\frac{5}{2}N_c + 2T_2\right)\right)\right]$$

(B.2.1)

$$\text{(B.2.2)}$$

$$Z_{GGG} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(-\frac{4}{3}N_c + \frac{4}{3}N_f\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-\frac{13}{2}N_c^2 + 5N_cN_f\right) + \right.$$
$$\left. \frac{1}{\epsilon}\left(-\frac{71}{24}N_c^2 + N_f\left(\frac{25}{12}N_c + 2T_2\right)\right)\right]$$
$$\text{(B.2.3)}$$

$$Z_{cc} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}(-N_c)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-4N_c^2 + N_cN_f\right) + \frac{1}{\epsilon}\left(-\frac{49}{24}N_c^2 + \frac{5}{12}N_cN_f\right)\right]$$
$$\text{(B.2.4)}$$

$$Z_{Gcc} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}(N_c)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{5}{2}N_c^2\right) + \frac{1}{\epsilon}\left(\frac{3}{4}N_c^2\right)\right]$$
$$\text{(B.2.5)}$$

$$Z_{\Psi\Psi} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}(2T_2)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-2 + 2N_c^2 + 2T_2^2\right) + \right.$$
$$\left. \frac{1}{\epsilon}\left(-\frac{17}{4} + \frac{17}{4}N_c^2 - \frac{3}{2}T_2^2 - N_fT_2\right)\right]$$
$$\text{(B.2.6)}$$

$$Z_{G\Psi\Psi} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}(2N_c + 2T_2)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-4 + \frac{25}{2}N_c^2 + 2T_2^2 - N_cN_f\right) + \right.$$
$$\left. \frac{1}{\epsilon}\left(-\frac{17}{4} + \frac{169}{24}N_c^2 - \frac{3}{2}T_2^2 + N_f\left(-T_2 - \frac{5}{12}N_c\right)\right)\right]$$
$$\text{(B.2.7)}$$

Further, using the Slavnov-Taylor identities we can relate these to the

renormalization constants of α_S and the various fields:

$$Z_{\alpha_S} = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(\frac{22}{3}N_c + -\frac{4}{3}N_f\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{484}{9}N_c^2 + \frac{16}{9}N_f^2 + -\frac{176}{9}N_cN_f\right) + \right. \qquad \text{(B.2.8)}$$
$$\left. \frac{1}{\epsilon}\left(\frac{34}{3}N_c^2 + N_f\left(-2T_2 - \frac{10}{3}N_c\right)\right)\right]$$

$$Z_G = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(-\frac{5}{3}N_c + \frac{2}{3}N_f\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-\frac{2}{9}N_f^2 - \frac{50}{9}N_c^2 + \frac{25}{9}N_cN_f\right) + \right. \qquad \text{(B.2.9)}$$
$$\left. \frac{1}{\epsilon}\left(-\frac{23}{8}N_c^2 + N_f\left(\frac{5}{4}N_c + T_2\right)\right)\right]$$

$$Z_c = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(-\frac{1}{2}N_c\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-\frac{17}{8}N_c^2 + \frac{1}{2}N_cN_f\right) + \frac{1}{\epsilon}\left(-\frac{49}{48}N_c^2 + \frac{5}{24}N_cN_f\right)\right]$$
$$\text{(B.2.10)}$$

$$Z_\Psi = 1 + \frac{\alpha_S}{4\pi}\frac{1}{\epsilon}\left(T_2\right)$$
$$+ \left(\frac{\alpha_S}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-1 + N_c^2 + \frac{1}{2}T_2^2\right) + \right. \qquad \text{(B.2.11)}$$
$$\left. \frac{1}{\epsilon}\left(-\frac{17}{8} + \frac{17}{8}N_c^2 - \frac{3}{4}T_2^2 + -\frac{1}{2}N_fT_2\right)\right]$$

Appendix C

Useful Identities

C.1 Vacuum Integrals

Here we present the small set of integrals that were required in our methodology in order to calculate the renormalization constants of FDR. We begin with the one loop fundamental logarithmic vacuum:

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{1}{\bar{q}^4}\bigg|_{\mu=\mu_R} = -i\pi^2 \left(e^{\gamma_e}\pi\right)^\epsilon \left(\frac{2}{\epsilon} + \epsilon\frac{\pi^2}{24}\right) + O(\epsilon^2), \qquad \text{(C.1.1)}$$

where $n = 4 + \epsilon$, as it shall be for all of the following integrals. This is both necessary for calculating the one loop renormalization but also for calculating the integrals required in the two loop renormalization where it appears both squared, and premultiplied by V_0. For this reason we have given it to $O(\epsilon)$, allowing us to calculate:

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4} \frac{1}{\bar{q}_2^4}\bigg|_{\mu=\mu_R} = -\pi^4 \left(e^{\gamma_e}\pi\right)^{2\epsilon} \left(\frac{4}{\epsilon^2} + \frac{\pi^2}{6}\right) + O(\epsilon)$$

$$\text{(C.1.2)}$$

$$\left[\int \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\bar{q}_2^4}\bigg|_{\epsilon\to0}\right] \int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4}\bigg|_{\mu=\mu_R} = -\pi^4 \left(e^{\gamma_e}\pi\right)^{2\epsilon} \left(\frac{4}{\epsilon^2} + \frac{\pi^2}{12}\right) + O(\epsilon)$$

$$\text{(C.1.3)}$$

The pure two loop logarithmic vacuum is given by [90]:

$$\int \frac{\mathrm{d}^n q_1}{\mu_R^\epsilon} \frac{\mathrm{d}^n q_2}{\mu_R^\epsilon} \frac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2}\bigg|_{\mu=\mu_R} = -\pi^4 \left(e^{\gamma_e}\pi\right)^{2\epsilon} \left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{2} + \frac{\pi^2}{12} + f_{11}\right) + O(\epsilon).$$

$$\text{(C.1.4)}$$

Here, f_{11} is a numerical value that can be calculated in terms of dilogarithms or the polygamma function:

$$f_{11} = \frac{i}{\sqrt{3}}\left(\mathrm{Li}_2(e^{i\frac{\pi}{3}}) - \mathrm{Li}_2(e^{-i\frac{\pi}{3}})\right) \tag{C.1.5}$$

$$= \frac{1}{36}\left[-\psi_1\left(\frac{1}{6}\right) - \psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{2}{3}\right) + \psi_1\left(\frac{5}{6}\right)\right]. \tag{C.1.6}$$

$$= -1.17195361\ldots \tag{C.1.7}$$

C.2   IBP Identities

Here we derive the few integration by parts identities that have been necessary to calculate the FDR scheme renormalization constants.

Firstly, we begin by expanding this derivative in

$$0 = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{\partial}{\partial q^\mu}\left(\frac{q^\mu}{\bar{q}^2(\bar{q}+k_1)^2}\right). \tag{C.2.1}$$

Then it is a simple matter of reconstructing denominators and rearranging to find the identity:

$$\int d^n q \frac{1}{\bar{q}^4(\bar{q}+k_1)^2} = \frac{1}{M^2}\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon}\left(\frac{1}{\bar{q}^4} + \frac{3-n}{\bar{q}^2(\bar{q}+k_1)^2}\right). \tag{C.2.2}$$

To derive the relation for the three point integral with a doubled propagator we need to do more work. We begin by computing the derivative.

$$0 = k_1^\mu \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{\partial}{\partial q^\mu}\left(\frac{1}{\bar{q}^2(\bar{q}-k_1)^2(\bar{q}+k_2)^2}\right), \tag{C.2.3}$$

$$0 = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{\partial}{\partial q^\mu}\left(\frac{q^\mu}{\bar{q}^2(\bar{q}-k_1)^2(\bar{q}+k_2)^2}\right). \tag{C.2.4}$$

The first derivative gives us relations which look similar to shift invariance without the doubled propagator:

$$\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon}\left(\frac{1}{\bar{q}^4(\bar{q}-k_1)^2(\bar{q}+k_2)^2}\right) = \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon}\left(\frac{1}{\bar{q}^4(\bar{q}+k_1)^2(\bar{q}-k_3)^2}\right)$$

$$= \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon}\left(\frac{1}{\bar{q}^4(\bar{q}+k_2)^2(\bar{q}-k_3)^2}\right), \tag{C.2.5}$$

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This then proves useful when we compute the second derivative. Together with equation (C.2.2) we are able to write the equation:

$$
\int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \frac{M^2}{\overline{q}^4 (\overline{q} - k_1)^2 (\overline{q} + k_2)^2}
$$
$$
= \frac{1}{M^2} \int \frac{\mathrm{d}^n q}{\mu_R^\epsilon} \left(\frac{4-n}{2} \frac{M^2}{\overline{q}^2 (\overline{q} - k_1)^2 (\overline{q} + k_2)^2} + \frac{1}{\overline{q}^4} + \frac{3-n}{\overline{q}^2 (\overline{q} + k_1)^2} \right) \tag{C.2.6}
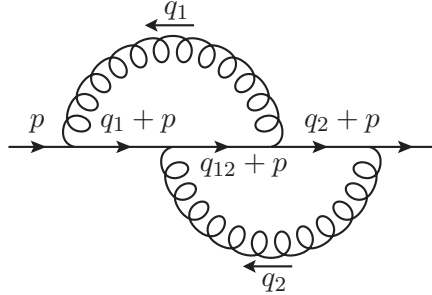$$

In order to derive the recursion relation used to simplify two loop global vacua, we need a global vacuum with an arbitrary number of propagators. Judiciously we compute the folloing total derivative, which can be easily rearranged to give equation (4.3.33):

$$
0 = \int d^n q_1 d^n q_2 \frac{\partial}{\partial q_1^\mu} \frac{q_1^\mu - q_2^\mu}{\overline{q}_1^{\alpha_1} \overline{q}_2^{\alpha_2} \overline{q}_{12}^{\alpha_{12}}} \tag{C.2.7}
$$

Appendix D

Sub-prescription Example

Here we will discuss further examples of the sub-prescription in order to aid the understanding of the reader for future FDR calculations. Consider the following contribution to the two loop corrections to the fermionic propagator in FDR:

$$= \int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2] \frac{\gamma_\beta(\slashed{q}_2 + \slashed{p})\gamma_\alpha(\slashed{q}_{12} + \slashed{p})\gamma^\beta(\slashed{q}_1 + \slashed{p})\gamma^\alpha}{\overline{q}_1^2 \overline{q}_2^2 \overline{D}_1 \overline{D}_2 \overline{D}_{12}} \qquad (\text{D.0.1})$$

$$\text{where} \quad \overline{D}_i = (\overline{q}_i + p)^2$$

Here the fermion momenta follows the fermion line. We wish to discuss how to extract the terms resulting from the sub-prescription, so as in the main text we consider the un-promoted numerator in order to find these terms. In this diagram we have two sub divergences, one for fixed q_1 and another fixed q_2. The terms from the sub-prescription of each can be extracted considering the sub-divergences independently. First we shall consider q_1 fixed. Let us disconnect the divergent sub diagram and write the numerator with its appropriate hatting:
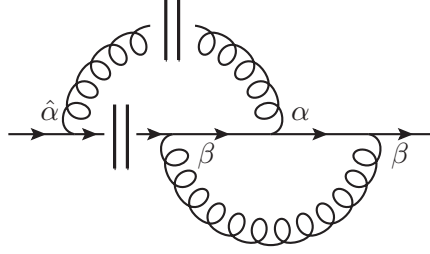
$$N = \gamma_\beta(\slashed{q}_2 + \slashed{p})\gamma_\alpha(\slashed{q}_{12} + \slashed{p})\gamma^\beta(\slashed{q}_1 + \slashed{p})\gamma^{\hat{\alpha}} \tag{D.0.2}$$

Here we see that the numerator terms which give logarithmic divergences in q_2 are:

$$N_{\text{div}}^{(2)} = \gamma_\beta \slashed{q}_2 \gamma_\alpha \slashed{q}_2 \gamma^\beta(\slashed{q}_1 + \slashed{p})\gamma^\alpha \tag{D.0.3}$$

Making the sub promotion here we find:

$$\begin{aligned}
N_{\text{div}}^{(2)} \to \hat{N}_{\text{div}}^{(2)} &= \gamma_\beta(\slashed{q}_2 + \hat{\mu}_2)\gamma_\alpha(\slashed{q}_2 + \hat{\mu}_2)\gamma^\beta(\slashed{q}_1 + \slashed{p})\gamma^{\hat{\alpha}} \\
&= N_{\text{div}}^{(2)} + 4\hat{\mu}_2^2(\slashed{q}_1 + \slashed{p})
\end{aligned} \tag{D.0.4}$$

Next we need to find the global promotion to remove. In this case this can be performed by anticommuting until all q_2 meet, and then we ignore hats and perform the promotion

$$q_2^2 \to q_2^2 - \hat{\mu}_2^2. \tag{D.0.5}$$

Here, due to the 4-dimensional gamma algebra, we find this evaluates to zero. We find the resulting contribution by subtracting the (zero) global promotion and adding in the sub promotion, leading to an extra integral of the form:

$$I = 4\int [\mathrm{d}^4 q_1][\mathrm{d}^4 q_2]\frac{\hat{\mu}_2^2(\slashed{q}_1 + \slashed{p})}{\overline{q}_1^2 \overline{q}_2^2 \overline{D}_1 \overline{D}_2 \overline{D}_{12}}. \tag{D.0.6}$$

We calculate the extra integral by first performing the integral over q_2, a simple calculation because the sub integral is a logarithmically divergent

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extra integral.

$$
\begin{aligned}
I &= 4 \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\overline{q}_1^2 \overline{D}_1} \int [\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{\overline{q}_2^2 \overline{D}_2 \overline{D}_{12}} \\
&= -4 \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\overline{q}_1^2 \overline{D}_1} \int [\mathrm{d}^4 q_2] \frac{\hat{\mu}_2^2}{\overline{q}_2^6} \\
&= 2i\pi^2 \int [\mathrm{d}^4 q_1] \frac{\slashed{q}_1 + \slashed{p}}{\overline{q}_1^2 \overline{D}_1} \\
&= i\pi^2 \int [\mathrm{d}^4 q_1] \frac{\slashed{p}}{\overline{q}_1^2 \overline{D}_1}
\end{aligned}
\tag{D.0.7}
$$

Where we have employed the vacuum trick on the sub integral in the second line. When we move to perform the sub-prescription in the second sub divergence, i.e. fixed q_2, we make a similar treatment and find that the sub-prescription gives an identical contribution due to the symmetry of the diagram.

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