MEANINGS OF THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT A POINT: BACKGROUND AND RESULTS

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In this paper we present a description of previous work carried out by the authors on the general issue of designing and implementing a didactical planning for Spanish students from non-compulsory secondary education, 16-17 years old. The current research has as its aim to describe the meanings that students associate to specific terms from the language, such as, "to approach," "to tend," "to reach," "to exceed," and "to converge." Prior to the study, we reviewed the mathematical use of these terms and we contrast this with the colloquial use of the terms. From the semi-structured interviews used to gather information, we provide an analysis of the written data. It is important to highlight that students have contributed with a variety of meanings, in addition to those from the previous review.

INTRODUCTION

Since the academic year 2009/2010 we have been interested in investigating some problems related to the teaching and learning of the concept of the limit of a function at a point. This concept is important because it is necessary for the learning of the derivative and integral concepts and is more complex than the concept of limit for sequences. Furthermore, it is one of the key concepts that mark off the transition towards the advanced mathematical thinking. By exploring several textbooks we observed a large number of routine tasks about calculating the limit following an intuitive definition based on the idea of approximation. So we carried out an exploratory study about the intuitive meanings that students have about the concept of a finite limit of a function at a point. The students are given tasks using different representations such as verbal, graphic and symbolic representations (Fernández-Plaza, 2011). Some of the results have been presented before both at national and international conferences (Fernández-Plaza, Ruiz-Hidalgo, & Rico, 2011, 2012a, 2012b, 2012c).

Recently, we have gathered information by means of interviews in order to contrast our interpretation of the written records from students and to deepen the personal conceptions that students associate with the following terms from calculus: "limit," "to approach," "to tend," "to converge," "to reach" and "to exceed". The colloquial meaning of these terms has been shown to influence the understanding and this has been reported in several studies (Cornu, 1991; Monaghan, 1991). In (Fernández-Plaza, 2011), the effective use of these terms and other synonyms has been explored, but not the specific meaning implemented by students. By effective use of a term, we mean that students in fact use this term, and not a synonym. For example, for the specific term "to approach", a student may use "to get close" or "to approximate," among others. This does not count as effective use but it is related to "to approach." Below, we describe the main achieved results so far.

MAIN ACHIEVED RESULTS

We summarize the most important results we have found out until the present moment. Firstly, we observed a persistence of misconceptions related to the limit as a non exceedable and unreachable value. This result is consistent with those from Cornu (1991) and Monaghan (1991). Here, we go deeper into the topic in the sense that, some students suggested a link between exceedability and reachability. We consider that this kind of misconceptions could arise from an overgeneralization of the particular case of monotone convergence.

Secondly, we discriminate between *process conceptions*, *object conceptions* and *dual conceptions* of the concept of limit. As *process conceptions*, we understand conceptions where the limit is closely related to a procedure about how to find it. With an *object conception,* the student is able to identify properties of the limit without depending on the process involved. Intermediate conceptions between these two are called *dual conceptions.* Thus when students were requested to discuss about the statement "The limit describes how a function f(x) moves when x moves to certain point," the most of arguments could be classified as one of these three options depending on whether students interpreted the limit as "how" (process conceptions) or "where" (object and dual conceptions) a function moves.

Thirdly, we found conflicts with the arbitrary accuracy of approximation to the limit. Expressions such as "limit can be approximated *as much as you wish*" are taken to mean that some students think that accuracy is bounded in the practical process. We suggest that the ambiguity of the underlined expression could have made students do a crucial distinction between the potential infinite character of the process and its implementation in practice.

Finally, we pointed out the conflicts with the exact or indefinite character of the limit value.

Some subjects considered a limit as an exact number whereas others considered that the limit is an "approximate" number. We suggest, according to Sierpinska (1987), that the latter subjects do not know the exact value of the limit, but only approximations to it, that is to say, the limit is undetermined. The progressive improvement in the interpretation of these results gave rise to talking about *structural aspects*, such as object/process duality of the concept, exact/approximate character of the limit, potential infinite/finite character of the limiting process, reachability and exceedability of the limit. These structural aspects were used to characterize and establish connections between different conceptions about the concept of limit (Fernández-Plaza et al., 2012a). At the same time we tried to characterize the terminology used by students in their answers. We selected the terms "to approach," "to tend," "to reach," "to exceed" and "to converge" among other reasons because they are used in the technical language and they describe different aspects of the concept of limit. Moreover, the influence of their colloquial meanings and everyday use on students' understanding have been reported in the literature. This problem leads us to three questions:

- Which are the different meanings and uses that these terms have in Spanish language?
- What is the terminology that students effectively use to explain their answers?
- What are the explicit definitions and meanings that students associate to these specific terms?

The treatment of the two first questions can be consulted in (Fernández-Plaza, 2011; Fernández-Plaza et al., 2011, 2012b). In the following section we are going to focus on giving answers to the third question.

DESCRIPTION OF THE CURRENT STUDY

We propose to describe how students explicitly define some specific terms from calculus in contrast with the colloquial and technical meanings of these terms. The chosen terms are "to approach," "to tend," "to reach," "to exceed" and "to converge."

Theoretical framework and prior research

We position this study in the research agenda of *Advanced Mathematical Thinking*, from the international group on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006, pp. 147-172). There is no agreement to establish the transition from elementary to advanced mathematical thinking.

The educational stage analyzed assumes a period of transition in which students use elementary techniques to tackle mathematical contents whose development historically, epistemologically, and didactically has an advanced status.

Rico (2012) developed the notion of meaning of a school mathematical concept, based on reference, sign and sense. We analyze the systems of representation, formal aspects or references of the concept, and the phenomena that provides its meaning.

Three components constitute the basis of the meaning of a school mathematical concept:

- *Systems of representation* (sign), defined by a set of signs, graphics and rules, to express and highlight aspects of the abstract concept and to establish relationships with other concepts.
- The *conceptual structure* (reference) that comprises concepts and properties, the derived arguments and propositions and their truth criteria.
- *Phenomenology* (sense) that includes those phenomena (contexts, situations or problems), which are at the centre of the concept and provide sense to it. (Rico, 2012, pp. 52-53)

The mathematical language related to the concept of the limit of a function at a point includes the terms "to approach," "to tend," "to reach," "to exceed" and "to converge." We chose these terms, among other reasons because each of them refers in part to properties and modes of usage associated with the concept of limit, that is to say, the phenomena involved (see Fernández-Plaza, 2011, pp. 14-21).

Conceptual analysis of specific terms

By a conceptual analysis we understand the procedure that leads to establishing the mathematical use of the terms and we want to contrast this use with the colloquial use or use in other disciplines.

We describe the chosen terms below and we also include the colloquial meaning of the term "limit." Monaghan (1991, p. 23) notes that to a mathematician tends to, approaches, converges, and limit are interchangeable. In Spanish "*aproximar*" has two different meanings; the first one expressed by "to approach," (dynamic) and the second one expressed by "to approximate" (static) (Fernández-Plaza, 2011, p. 16).

The sentence "to tend toward a value" means "to approach gradually but never reach the value" (Real Academia Española [RAE], 2001) and expresses a very specific form of approach. Blázquez, Gatica and Ortega (2009) argue that a sequence of numbers approaches a number as a limit if the difference between the terms of the sequence and the limit decreases gradually, but they also argue that a sequence "tends toward a limit" if any arbitrary approximation to the limit can be improved by the terms of the sequence.

A study by Monaghan (1991) concludes that many students do not distinguish between "to tend" and "to approach" in a mathematical context. In a formal sense, to tend toward or to approach a limit is said of a sequence (e.g. the sequence 0.9, 0.99, 0.999… tends toward 1, but also that sequence approaches number 2) according to the definition of limit, but "to approximate" a limit is to give any of the terms of a convergent sequence ("0.999 approximates 1 with an error less than 0.01"). We justify the different distinction between these terms only for Non-University High Education because both of them are applied to the same object (a sequence).

The expression "f(x) tends toward L, when x tends toward a" may cause cognitive conflicts, as Tall and Vinner (1981) note, because x never equals a, so students may consider that f(x) never equals L.

"To reach" means colloquially "to arrive at" or "to come to touch" (RAE, 2001). We interpret the mathematical meaning of "reach" to be that a function reaches the limit if the limit value is the image of the x-point at which the limit is studied (continuity); by extension, the limit can be the image of any other x-value in the domain.

We see that "to exceed" means colloquially "to be above an upper level" (RAE, 2001), excluding the meaning "to be below a lower level". We will say that the limit of a function may be exceeded if we can construct two successive monotone sequences of images that converge to the limit, one ascending and the other descending, for appropriate sequences of values of x that converge at the point at which the limit is studied. The reachability or exceedability of the finite limit of a function can be easily interpreted as global or local concepts, but there is no logical implication of the two concepts.

The term “to converge” means colloquially “to come together from different directions”. In mathematics, this term is equivalent to “to tend” and normally is applied to the limit of sequences and series and is not so often used in connection with the limit of a function at a point. We expected that students could invent a definition for this term in this new mathematical context.

Furthermore, the term “limit” has colloquial meanings that interfere with students’ conceptions of this term, such as ideas of ending, boundary, and what cannot be exceeded (RAE, 2001). The term’s scientific-technical use is related in some disciplines to a subject matter or extreme state in which the behaviour of specific systems changes abruptly (Real Academia de las Ciencias [RAC], 1990).

Prior Research

Monaghan (1991) studied the influence of language on the ideas that students have about the terms mentioned above, when the terms were used in connection with different graphs of functions and examples that school students verbally explained. We underline as a limitation of the approach adopted in this case, that the key terms that the students were asked to use were defined a priori, instead of enabling students to use their own words freely and spontaneously and to infer the appropriate nuances a posteriori.

In previous CERME proceedings there has been published papers related to the learning of the concept of limit of a function. The most relevant one in relation to this study is by Juter (2007) who investigated, among other aspects, how students interpreted the reachability of the limit in a problem solving context.

Method

A semi-structured interview was conducted in an ordinary classroom. The protocol of implementation was the prior request to the students to write their answers on the sheet provided, and the discussion of the answers was audio recorded. The subjects were organised into nine groups with 3-5 in each, in order to facilitate the interaction between the subjects and the researcher.

We focus on the following common question:

Describe how you understand the following terms: “to approach”, “to tend”, “to reach”, “to exceed”, “to converge” in the context of a finite limit of a function at a point.

In order to help the students to better express their conceptions during discussion, we showed them some graphics of functions so that some other characteristics of meaning of these terms could emerge, especially with the terms “to reach” and “to exceed”.

33 subjects out of a total of 36 from the previous study (Fernández-Plaza, 2011) were selected. They were chosen deliberately, according to their previous answers and based on their availability. The subjects attended the second year of non-compulsory secondary school study (17-18 years of age) and they were all studying mathematics. They had received the instruction on the concept of limit according to the current curriculum.

Preliminary results and discussion

We are going to show some preliminary results from the analysis of the written records. Table 1 shows a classification of meanings and frequencies of the selected specific terms. According to these categories, we classify the definitions provided by the students by the codes (Ai, Bj, Ck, Dl, Em) (Note that each student produced at most five definitions). Only 2 out of 33 of the individual productions (the set of their five definitions) have the same code, therefore the differences are relevant (any two students could define 1, 2, 3, 4 or 5 specific terms in a different way).

We observe that 23 out of 33 individual productions establish some distinctions between "to approach" and "to tend." The most relevant differences of meaning between these terms are as follows:

- The possibility of not to reach or exceed the limit. In particular, some students used expressions such as "to approach more and more" to point out a potential infinite character of the process "to tend."
- The technical usage and the subjective view of the term "to tend."

Specific terms	Meanings	Frequencies
To approach	A1. To get as close as possible	14
	A1.1. Not to reach the limit	10
	A1.2. Not to reach and not to exceed the limit	2
	A1.3. To reach but not to exceed the limit	2
	A2. To establish the closest value to the limit	4
	A3. No answer	1
To tend	B1. To approach	5
	B1.1. Not to reach the limit	7
	B1.2. To approach more and more	4
	B2. Technical usage	8
	B3. Subjective	2
	B4. Other	5
	B5. No answer	2

To reach	C1. To arrive at or to touch the limit	27
	C1.1. Not to exceed the limit	3
	C2. To know the exact value of the limit	2
	C3. To know the value of f(x) for a given x	1
To exceed	D1. To surpass the limit of the function f(x)	19
	D2. To surpass the x-value.	7
	D3. To reach the limit and continue (To pass through the limit or x-value)	4
	D4. No answer	3
To converge	E1. The function is above the limit all the time	2
	E2. The function is below the limit all the time	2
	E3. To tend	1
	E4. To reach	1
	E5. The right and left-hand limits are the same.	3
	E6. The function takes the same value than the limit	1
	E7. Two functions or straight lines intersect at one point	9
	E8. Other	4
	E9. No answer	10

Table 1: Classification of meanings and frequencies about the selected specific terms

Below we exemplify answers from some categories in order to clarify their denomination. The other categories are denoted by a "representative" definition, for example, category A1 includes those definitions that express the idea or use the expression "to get as close as possible", so we do not consider it necessary to exemplify all of them:

- **Category B2: Technical usage.** An example of an answer is "*This term is used to indicate the value that x takes in a limit*". It does not state anything about the specific meaning of the action "to tend," that is, it is only a technical word; an agreement. Another answer is "*to tend to a number is to use the closest number to it, for example, if x→1 from the left, we use 0.9. From the right, we use 1.1.*" The term is used to describe a personal rule to calculate a limit.
- **Category B3: Subjective.** Examples of answers are: "*To approach to that number without being aware of it (without intending to obtain it)*" and "*To approach it as much as we want*" indicate a subjective aspect of the definition of the term "to tend."

From Table 1 we discuss the global results:

Most subjects (14 out of 33) consider "to approach" as *to get as close as possible*. Although it is relevant that 10 out of 33 subjects also consider that the function cannot reach the limit. Only 2 out of 33 admitted in addition that the function could in fact reach the limit but never exceed it. In general, "to approach" is considered as an intuitive and incomplete process.

However, the term "to tend" has some particular characteristics different from "to approach", such as a subjective view of its definition (2 out 33) ("*To approach it as much as we want*") or a technical usage (8 out 33) ("*This term is used to indicate the value that x takes in a limit.*"), that is it is an agreement in mathematics.

Regarding "to reach", most subjects (27 out of 33) consider it simply as *to arrive at or to get to touch the limit*. Only 3 out of 33 considered that the limit must not be exceeded. On the other hand, only two subjects considered that the limit is reachable if we can calculate the exact value, while only one subject stated that "to reach" is to know the value of f(x) for a given x, so there could be a possible identification between the limit and the image.

"To exceed" is basically *to surpass the limit or the x-value given* (19 and 7 out of 33), although some subjects (4 out of 33) gave more complete answers, in the sense that a limit or a given x-value are exceeded *if the function reaches them and continues, both above and below them*.

At the beginning, students recognised not to know the term "to converge" in the context of a finite limit of a function at a point. In fact, 9 out of 33 did not answer this question, so the researcher had to encourage them to write whatever they could imagine about any other situation and to invent a definition. Only one subject defined this term as "to tend", and the most frequent meaning (6 out of 33) was *two functions intersect at one point*, and 5 out of 33 define "to converge" as *the right and left-hand limits are the same*, a definition that could be considered suitable in this context. On the other hand, several subjects described situations where the *function is all the time above or below the limit*, that is to say, an asymptotic behaviour of the function, for example, $f(x) = 1/x$ converges to 0 when x tends to $\pm\infty$.

Preliminary conclusions

According to the discussion of the results from Table 1 and the aim proposed at the beginning; to describe how students define explicitly some specific terms from calculus in contrast with a previous conceptual analysis of these terms, we draw the following conclusions about their achievement.

Students interpret the meaning of the selected terms in many different ways, most of them extracted from everyday situations, so we agree with Monaghan (1991) and Cornu (1991) that conflicts between colloquial and formal language are still occurring.

The review of the use of specific terms has predicted partially the meanings that students were going to provide, above all the colloquial meanings. The technical use of the term “to tend” had been conjectured by Fernández-Plaza (2011, p. 36). The observed difficulty some students had to distinguish between “to approach” and “to tend” is consistent with Monaghan (1991) and Blázquez, Gatica and Ortega (2009). All the new meanings of these terms should contribute to enrich this review in order to increase its explicative power.

At the beginning, the term “to converge” had been considered unknown by students in the context of finite limit of a function at a point, but they were able to invent a possible definition for the new context.

It is relevant that exceedability and reachability of the limit are especially connected to the students’ conceptions of the terms “to approach” and “to tend” according to Fernández-Plaza (2011, p. 40).

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