Planar dielectric waveguides in rotation are optical fibers: comparison with the classical model

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Abstract: A novel and simpler method to calculate the main parameters in fiber optics is presented. This method is based in a planar dielectric waveguide in rotation and, as an example, it is applied to calculate the turning points and the inner caustic in an optical fiber with a parabolic refractive index. It is shown that the solution found using this method agrees with the standard (and more complex) method, whose solutions for these points are also summarized in this paper.

OCIS codes: (060.2310) Fiber optics; (060.2270) Fiber Characterization

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1. Introduction

According to wave or physical optics, the electromagnetic waves radiated by an optical source and propagated in an optical fiber can be represented by a train of spherical, planar, etc. wavefronts with the source at the center. A wavefront is defined as the locus of all points in the wave train which exhibit the same phase. Far from source, wavefronts tend to be in a plane. When the wavelength of light is much smaller than the object to be illuminated, the light wave can be represented by a ray, which is drawn perpendicular to the phase front and parallel to the Poynting vector, which indicates the flow of energy. Thus, large-scale optical effects such as reflection and refraction can be analyzed by a simple geometric process called ray tracing.

In the ray theory, to study the transmission of light within an optical fiber, it is essential to take into account the relation between the refractive indexes of the core and the cladding. Geometric ray analysis is introduced as an alternative to the ondulatory model. In many cases, both models are equivalent [1-2].

There are two conditions for establishing guided waves in an optical fiber, one of them is that the waves must be totally reflected twice in the interface core-cladding and the other that

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the total phase shift alter two consecutive reflections must be an integer multiple of 2π. To study internal reflection, we must consider the principles of Geometrical Optics –i.e. the Snell law in simple cases or eikonal equation in more complex situations (equations of Geometrical Optics). Reflection and transmission of Hermites-Gaussian beams on curved interface can be studied by means of complex ray analysis of reflection and transmission [3] without using the ondulatory model, which is more difficult.

A model of cladded multimode fiber taper using geometric and wave theory was provided to study its transmission property and evanescent wave absorption. With this model, it is possible to know the transmission property and evanescent wave absorption of cladded multimode fiber tapers [4].

An idea about an equivalent refractive index for the ray tracing calculation of hollow core waveguides is proposed. A virtual, complex refractive index is uniquely found by minimizing the difference between reflectivity of a virtual monolayer material and a metal substrate coated with a dielectric film. By using this technique, it is possible to calculate transmission losses of a delivery system consisting of a hollow fiber and a tapered hollow waveguide. [5]

As we can see above, the geometrical optics theory can be used to obtain the same result and as a substitution of the ondulatory theory in some cases.

In this paper we present an alternative model of geometrical optics in fiber optics easier than the current one. Essentially, an optical fiber is considered to be a planar waveguide in rotation. This alternative model gives the same results but the mathematics involved are much simpler than those required by conventional explanations, since the large number of equations arising from the cylindrical geometry is reduced.

The innovative aspect of the present paper is the new perspective given to optical fibers, which can be simulated from a planar dielectric waveguide. Good agreement between classical model and our modern model results have been achieved, and, in fact, we have arrived at the same classical results, but using a completely new and much simpler method–a method radically different from the classical one.

2. Classical model

Let there be an arbitrary ray entering one fiber whose refractive index is constant so far. If we define two perpendicular planes (H-H and H-V) whose intersection line is the axis of the fiber then the ray is completely defined by means of two parameters: the angle θ_z (between the ray itself and the horizontal plane H-H) and the angle θ_ϕ (found in the core cross-section between the tangent to the interface and the projection of the ray path, as shown in Figure 1).

If the refractive index of the fiber is constant, these parameters remain constant [6] but, besides the trivial case of rays with $\theta_\phi = \pi/2$ (*meridional rays*), an arbitrary guided ray (*skew rays* if $\theta_\phi \neq \pi/2$) will follow a complex path due to the cylindrical geometry of the fiber (see Fig. 1). Under these conditions the mathematical treatment is relatively simple but the situation in fibers whose refractive index depends on the distance to the axis of the fiber (graded-index profile fibers) is rather different. Due to the continuous refractions, the angles θ_Z and θ_ϕ are no longer constant and they change after each refraction according to the laws of the Geometrical Optics. This makes us use the eikonal equation, whose treatment is complex [6-8], and the parameters derived from its integration prove rather obscure, loosing a big part of their actual physical meaning.

Since a typical graded-index profile fiber consists of N concentric layers with refractive indexes $n_1, n_2, ...n_N$ that decrease from the axis to the outer layers, a ray propagating within such fiber will suffer consecutive refractions until it meets a layer whose refractive index together with the angle of incidence of the ray cause total reflection and make it return [Fig. 1(a) and (b)]. The points where this reflection takes place are called *turning points* and the most external surface they originate is called the *external caustic* of the fiber for one given

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ray. Due to the geometry of the fiber, a skew ray will never intersect the axis of the fiber, so that there is an inner surface (*internal caustic*) between the axis and the path of the ray that it will never intersect. The points of this surface are called *caustic points*.

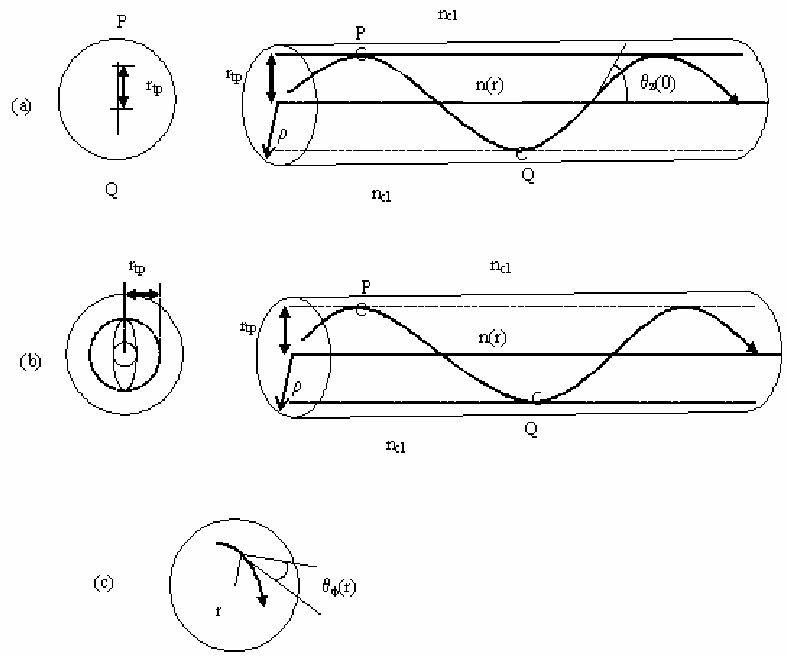

Fig. 1. Trajectory of the rays within the core of a gradual-index fiber with the position of the turning points; (a) shows the trajectory of a meridional ray and (b) an oblique one. In (c), the angle (r) is represented and the azimuthal direction; θ_z and θ_ϕ are projections of the path on a plane perpendicular to the axis of the fiber.

Then, the path of each ray is confined between two surfaces within the fiber: the internal and the external caustic, respectively, formed by the turning points. The knowledge of the position of such points is extremely important in graded-index profile fibers since it informs us about the amount of power that could be guided.

In our model, specifically the turning points and the caustics, whose calculation is rather complex using the eikonal equation arises from a simple physical consideration.

Let there be a graded-index profile fiber with radius ρ and an undefined number of concentric layers, each one with thickness dx and refractive index n_i. Let there be a meridional ray entering the fiber with an angle of incidence θ_{Z1}. If this ray is guided, it will be refracted as it passes from one layer to another with successive angles θ_{Z2} … θ_{Z3} until it reaches the turning point and it is reflected towards the inner layers (see Fig. 2).

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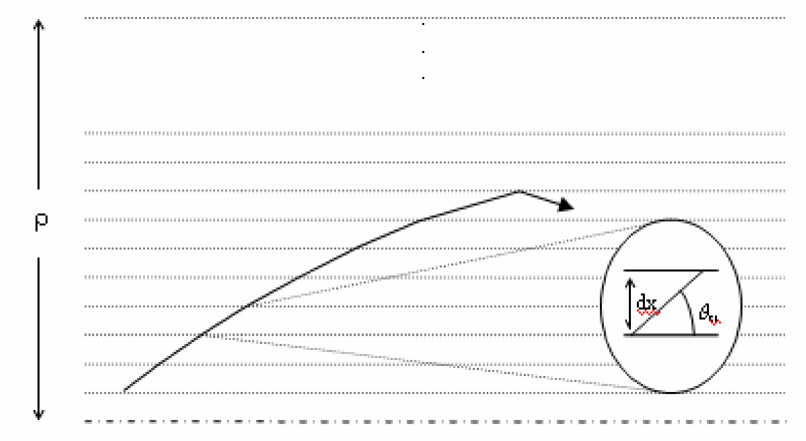

Fig. 2. Path of a meridional ray moving along a graded-index profile fiber. Detail of one layer magnified.

The distance covered by the ray within each layer is

$$dL_i = \frac{dx}{\sin\theta_{Z\,i}} \sin\theta_{\phi\,i} \tag{1}$$

and the time spent in such layer by this same ray is given by

$$dt_i = \frac{dL_i}{v_i} = \frac{n_i dx}{c \sin\theta_{Z\,i}} \sin\theta_{\phi\,i} \tag{2}$$

3. Modern model

The basic idea is that a planar dielectric waveguide in rotation behaves as an optical fiber if the rotation velocity is appropriate; in this way, there is no distinction between the types of rays that are propagated by the fiber. We offer a view that an optical fiber is not a static element through which light is propagated, but rather a system comprising rays and a hypothetical planar waveguide that contains them and that has a different rotation velocity for each. The model consists of a planar dielectric waveguide that spins with variable angular velocity ω around its optical axis. It gives a fuller vision of the physical fundamentals of fiber optics at the same time as demonstrating that the different models can lead to the same result. Skew and meridional rays will be indistinct in the fiber.

Starting from this principle, our model of dielectric waveguide in rotation is equivalent to any fiber optics of similar characteristics (same refractive indexes for core and cladding and semi thickness equivalent to the radius of the fiber).

3.1. Meridional rays in graded-index profile fibers

Let there be a meridional ray going through a planar dielectric waveguide of symmetrical and square section with a graded-index (GRIN) profile (the step-index profile case is studied in [6-8]). The ray will be guided along the fiber as long as total reflection takes place every two opposite interfaces. Nevertheless, if the waveguide rotated with a given angular velocity ω (the waveguide, not the ray), we could make the ray be reflected always on the same face.

Let there be a square planar dielectric waveguide circumscribing our fiber optics with a semi-thickness equal to the radius of the fiber. Let us assume that this planar dielectric

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waveguide can rotate around the axis of the fiber in such a way that the plane of incidence of the guided ray always intersects the same straight line on the waveguide, regardless of where the ray is. We could say that the waveguide "follows" the ray as it moves along the fiber [6]. To satisfy the above conditions, the dielectric waveguide will rotate with an angular velocity ω_i while the ray passes through the layer "*i*". This velocity is given by:

$$\omega_i \to \omega(x) = \frac{d\theta_\phi(x)}{dt} \tag{3}$$

where it is considered a continuous change of the refractive index, which means that the numbers of layers trends to infinity $(N \to \infty)$ and, thus, $\theta_{Zi} \to \theta_Z(x)$, $\theta_{\phi i} \to \theta_\phi(x)$ and $\omega_i \to \omega(x)$. The consideration $N \to \infty$ is not at all restrictive—on the contrary, we can consider it without losing physical or mathematical generality. It is the only way to manage the number of layers of an optical fiber with this profile.

If we introduce dx in both the numerator and denominator, we get:

$$\omega(x) = \frac{d\theta_\phi(x)}{dx}\frac{dx}{dt} = \frac{d\theta_\phi(x)}{dx}\frac{c\sin\theta_Z(x)}{n(x)\sin\theta_\phi(x)} = \begin{cases} 0 & if \quad x > dx \\ \dfrac{\pi c \sin\theta_z(x)}{n(x)2dx} & if \quad x < dx \end{cases} \tag{4}$$

Note that here we have considered

$$\frac{d\theta_\phi(x)}{dx} = \frac{\pi}{2dx} \tag{5}$$

because, for meridional rays, the angle θ_ϕ changes from $\pi/2$ to $-\pi/2$ when it cuts the axis of the fiber. Here, obviously, $2dx$ is the thickness of the central layer.

The meaning of Eq. (4) is the following: while the ray is traveling out of the central layer: our hypothetical waveguide remains at rest but when it enters this layer, the waveguide rotates with an angular speed that allows the ray to be reflected on the same face of the square planar dielectric waveguide when it meets the core-cladding interface. The orientation of this rotation (clockwise or counter clockwise) is free since the results of this model clearly do not depend on it.

In summary, according to our model, a planar dielectric waveguide that rotates "following the ray" through a GRIN fiber would be at rest $(\omega = 0)$ except when the ray enters the central layer of the waveguide. In this situation the waveguide rotates with angular velocity

$$\omega = \frac{\pi c \sin\theta_Z(x)}{n(x)2dx} \tag{6}$$

In step-index profile fibers $(dx = \rho)$, the previous value of ω matches the results obtained in [6].

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3.2. Skew rays in graded-index profile fibers

Let us consider one section of the optical fiber perpendicular to its axis. Since we are working with an optical fibers whose refractive index continuously changes from the inner to the outer layers, the thickness of each layer will trend to 0 $(dx \to 0)$ and thus the length of the projection of one skew ray on this plane can be approached to $dL_i = \cos\theta_\phi(x)dx$ in the layer "*i*" (see Figs. 1 and 3).

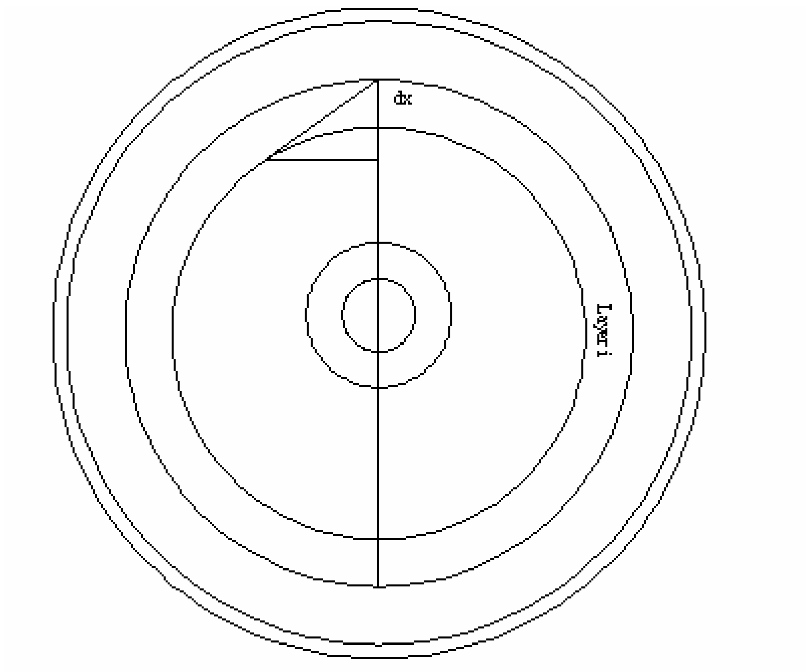

Fig. 3. Section of the fiber on a plane perpendicular to the axis. Projection of the path traveled within each layer over a section perpendicular to the axis.

If the ray traveling in the optical fiber is considered, then the length of the path of a ray within the layer "*i*" will be (see Figs. 1 and 3):

$$dL_i = \frac{\sin\theta_{\phi i}dx}{\sin\theta_{Zi}} \tag{7}$$

Thus the time that the ray spends in each layer is given by:

$$dt_i = \frac{dL_i}{v_i} = \frac{n_i \sin\theta_{\phi i}dx}{c\sin\theta_{Zi}} \tag{8}$$

In the case of meridional rays, we studied the movement of the rays in each layer "*i*", but here this procedure does not make much sense, given the existence of the $N \to \infty$ layer. Therefore, we will study the movement at each point "*x*", and therefore the skew rays $n_i \to n(x)$.

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Once more, we get

$$\omega(x) = \frac{d\theta_\phi(x)}{dx}\frac{dx}{dt} = \frac{d\theta_\phi(x)}{dx}\frac{c\sin\theta_Z(x)}{n(x)\sin\theta_\phi(x)} \tag{9}$$

Since the refractive index decreases from the axis to the outer parts, the angular velocity ω must logically increase as the ray goes to these outer parts. After total internal reflection the ray starts to travel towards the inner layers of the fiber, and thus ω must decrease until it again meets layers with a lower value of the refraction index.

This means that it is possible to find the points where total internal reflection is produced (turning points) and the nearest points from the path of the ray to the axis of the fiber (caustic points) by solving the equation

$$\alpha = \frac{d\omega(x)}{dt} = 0 \tag{10}$$

where α is the angular acceleration of the square planar dielectric waveguide we are considering.

Developing the above equation, we get

$$\alpha = \frac{d\omega(x)}{dt} = \frac{d\omega(x)}{dx}\frac{dx}{dt} = \frac{c\sin\theta_Z(x)}{n(x)\sin\theta_\phi(x)}\frac{d\omega(x)}{dx} = 0 \Rightarrow \frac{d\omega(x)}{dx} = 0 \tag{11}$$

And developing again, we get

$$\begin{gathered}\frac{d\omega(x)}{dx} = \frac{c\sin\theta_Z(x)}{n(x)\sin\theta_\phi(x)}\frac{d^2\theta_\phi(x)}{dx^2} + \\ + c\frac{d\theta_\phi(x)}{dx}\left[\frac{n(x)\sin\theta_\phi(x)\cos\theta_Z(x)\dfrac{d\theta_Z(x)}{dx} - \sin\theta_Z(x)\left[\sin\theta_\phi(x)\dfrac{dn(x)}{dx} + n(x)\cos\theta_\phi(x)\dfrac{d\theta_\phi(x)}{dx}\right]}{n^2(x)\sin^2\theta_\phi(x)}\right] = 0\end{gathered} \tag{12}$$

Consequently,

$$\frac{d^2\theta_\phi(x)}{dx^2} + \frac{d\theta_\phi(x)}{dx}\left[\cot g\theta_Z(x)\frac{d\theta_Z(x)}{dx} - \cot g\theta_\phi(x)\frac{d\theta_\phi(x)}{dx} - \frac{1}{n(x)}\frac{dn(x)}{dx}\right] = 0 \tag{13}$$

Due to their physical meaning, there are two terms in the above equation that vanish at the turning points; $d^2\theta_\phi(x)/dx^2$ represents the rate of change of the angular variation of $\theta_\phi(x)$ as the ray goes through the different layers. As this rate will change from increasing to decreasing at the turning points (from decreasing to increasing at the caustic points —that is, the function goes from decreasing to increasing and, mathematically, it is a continuous and derivable function. This term vanishes at such points. Furthermore, this approximation makes sense in GRIN fibers where there are no strong variations between contiguous layers. As the

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ray goes through the different layers $d\theta_z(x)/dx$ is the change of $\theta_Z(x)$. Due to total reflection, this angle will have a minimum at the turning points and a maximum at the internal caustic ones and, thus, it is 0 at such points.

After these considerations, the final equation is:

$$\cot g\theta_\phi(x)\frac{d\theta_\phi(x)}{dx}+\frac{1}{n(x)}\frac{dn(x)}{dx}=0 \tag{14}$$

and, integrating, we get

$$n(x)\sin\theta_\phi(x)=1 \tag{15}$$

3.3. Example: turning and caustic points in a fiber with parabolic profile

Let there be an index profile like

$$n^2(x)=n_{nu}^2\left[1-2\Delta\left(\frac{x}{\rho}\right)^2\right] \tag{16}$$

If we introduce this index profile into Eq. (15), we get

$$n^2(x)\sin^2\theta_\phi(x)=1\Rightarrow\left(\frac{x}{\rho}\right)^2=\frac{1}{2\Delta}\left(1-\frac{1}{n_{nu}^2\sin^2\theta_\phi(x)}\right) \tag{17}$$

and we finally get the value of x:

$$x=\frac{\rho}{n_{nu}\sqrt{2\Delta}}\sqrt{n_{nu}^2-\frac{1}{\sin^2\theta_\phi(x)}} \tag{18}$$

Apparently, we have only one solution, but so that our model functions, we should have two solutions, as the turning points correspond to the internal and external caustic points. Nevertheless, we will realize that this solution makes sense under certain conditions. Let us compare this expression with the exact solutions for the caustic and turning points in the classical model:

$$r_{ci}=\frac{\rho}{2n_{nu}\sqrt{\Delta}}\sqrt{\left(n_{nu}^2-\overline{\beta}^2\right)-\sqrt{\left(n_{nu}^2-\overline{\beta^2}\right)^2-8\Delta\bar{l}^2n_{nu}^2}} \tag{19}$$

$$r_{pl}=\frac{\rho}{2n_{nu}\sqrt{\Delta}}\sqrt{\left(n_{nu}^2-\overline{\beta}^2\right)+\sqrt{\left(n_{nu}^2-\overline{\beta^2}\right)^2-8\Delta\bar{l}^2n_{nu}^2}} \tag{20}$$

We can simplify the equations above: let us suppose that ρ is very big and the numerical aperture (and therefore Δ) is very small. If we take it into account in the expression of the invariant $\bar{l} = r^2/\rho\, n(r)(d\phi/ds) = (r/\rho)n(r) sen\theta_z(r)\cos\theta_\phi(r)$, then $8\Delta\bar{l}^2 n_{nu}^2 \to 0$. In this case,

$$r_{pl} = \frac{\rho}{2n_{nu}\sqrt{\Delta}}\sqrt{n_{nu}^2 - \overline{\beta}^2 + n_{nu}^2 - \overline{\beta}} = \frac{\rho}{n_{nu}\sqrt{2\Delta}}\sqrt{n_{nu}^2 - \frac{1}{2}\left(\overline{\beta}^2 - \overline{\beta}\right)} \tag{21}$$

$$r_{ci} = \frac{\rho}{2n_{nu}\sqrt{\Delta}}\sqrt{n_{nu}^2 - \overline{\beta}^2 - n_{nu}^2 + \overline{\beta^2}} = 0 \tag{22}$$

The solution above for the inner caustic is logical since for big radii of the fiber, the value of the radius of the inner caustic can be negligible. In the same way, the value of the turning point will be the same as the one we found with our approximation as long as two conditions hold:

$$\text{First: } n_{nu} > \frac{1}{\left|\sin\ \theta_\phi(x)\right|} \tag{23}$$

$$\text{Second: } \frac{1}{\sin^2\theta_\phi(x)} = \frac{1}{2}\left(\overline{\beta}^2 - \overline{\beta}\right) \tag{24}$$

These two conditions demand values of $\cos\theta_Z(x)$ and $\sin\theta_\phi(x)$ very close to 1 (especially this latter one). It means that $\theta_Z(x)$ must be small and $\theta_\phi(x)$ must be near to $\pi/2$. In fact we are assuming fibers with big radii or small numeric aperture, which fits with the request of small values of $\theta_Z(x)$.

Under these conditions, we have reached results completely similar to these arising from the classical model. It might seem just an approach but, since only rays within a cone subtended by small angles $\theta_Z(x)$ are propagated through the fiber, we find that the results reproduced by our model are quite real.

In the same way as our model is capable of giving the same results as the classical model of fiber-optic rays, it is also capable of calculating the radiation losses due to the bending of fibers, coupling between fibres, and other photonic devices; furthermore, with our model, we can design polarization-maintaining optical fibers.

The meridional rays that are propagated by a bended fiber are defined by the axis of the curve and the centre of curvature of the fiber and are identical to those propagated in a bent planar waveguide. In specific, they are propagated with fixed half-periods. The other rays are skewed, because of the asymmetry introduced by the bend, and each ray propagates along a trajectory with a varying half-period.

In the case of the polarization-maintaining optical fibers, there are fibers that have a circular core and an elliptical cladding [8, 10]. In these cases, our model is applicable to these types of fibers.

In the case of the coupling between optical fibers and other photonic devices, with our model, we can also calculate the radiation losses due to coupling between fibers and other photonic devices. To calculate the losses by misalignment (lateral or axial, longitudinal or angular), it is sufficient to know the diameters of the fibers and the surface areas of the cores overlapping between fibers or with devices, the numerical aperture of the fibers or of the

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devices, the angular differences between the axes of the fibers or between the fibers and the devices or the differences of the refractive-index profiles between fibers and devices respectively and all these data can be calculated with our new model.

4. Conclusions

This new model can be used as an alternative to the one that generally appears in books and scientific papers or as a second point of view. Thus it is proved that, starting from two different formalisms in Physics (Optics) we can reach similar results. In this case, we prove again a well known fact: the laws of Geometric Optics are invariant under rotation.

The models explaining the transmission of radiation in optical fibers are complex due to the occurrence of several kinds of rays (meridional, skew …), that arise from the cylindrical geometry of the fiber. We have reported a new and simpler mathematical method based on the rotation of a square planar dielectric waveguide that requires only the study of one parameter (ω) instead of three (3D). Our new model was applied to determine two typical and essential parameters (r_{tp} and r_{ic}) of classical optical fibers. The same results were found between those solutions and the solutions reached with our model.

According to these results, we realize that it is not necessary to develop a specific mathematical treatment for skew rays because they arise directly when considering a planar dielectric waveguide in rotation.

As inferred from Eq. (15), if we work with an optical fiber having the same geometric characteristics but a different refractive index, the study of the radiation transmission requires the introduction of this index into many equations. In our model the guide is the same and only the angular velocity varies.

As we explained at the end of the previous section, our model is capable of accurately accounting for design issues related to optical fibers such as polarization-maintaining fibres, radiation losses due to fiber bending or coupling between fibers and other photonic devices.

In the same way, if we wish to work with another optical fiber whose diameter is the only parameter which changes, this study would just require modification of its angular velocity, ω . This signifies a great simplification, for example, for simulation programs.