Electronic Transactions on Numerical Analysis.
Volume 19, pp. 18-28, 2005.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

ASYMPTOTICS OF POLYNOMIAL SOLUTIONS OF A CLASS OF GENERALIZED LAMÉ DIFFERENTIAL EQUATIONS*

A. MARTÍNEZ-FINKELSHTEIN†, P. MARTÍNEZ-GONZÁLEZ‡, AND R. ORIVE§

Abstract. In this paper we study the asymptotic behavior of sequences of Heine-Stieltjes and Van Vleck polynomials for a class of generalized Lamé differential equations connected with certain equilibrium problems on the unit circle.

Key words. Heine-Stieltjes polynomials, Van Vleck polynomials, zeros, asymptotics.

AMS subject classifications. 33C45.

1. Introduction.

We consider a class of generalized Lamé differential equations

$$A(z)y''(z) + B(z)y'(z) + C(z)y(z) = 0\,, \tag{1.1}$$

where A is a polynomial of degree $p+1$, $p \in \mathbb{N}$, and $\deg B \le p$. If there exists a polynomial C for which (1.1) admits a polynomial solution $y = \varphi$, then C is called a Van Vleck polynomial, and φ is known as a Heine-Stieltjes polynomial. In the simplest case, $p = 1$, (1.1) is the usual hypergeometric equation (and Heine-Stieltjes polynomials are just hypergeometric polynomials), and for $p = 2$ we obtain the so-called Heun equation.

In his well known work, Stieltjes [8] considered the particular setting when A has only simple and real zeros $a_0, \dots, a_p$, and for $j = 1, \dots, p$ the residue r_j of the rational function B/A at a_j is positive. In this situation Stieltjes showed that there exist exactly $\sigma(N) = \binom{N+p-1}{N}$ possible Van Vleck polynomials C for which (1.1) admits a unique (up to normalization) Heine-Stieltjes polynomial $y = \varphi$ of degree N. (In a previous paper, Heine [2] proved that in general $\sigma(N)$ is an upper bound for the number of possible polynomials C). Stieltjes showed that the collection of admissible pairs of polynomials (C, φ) is given by all possible ways the N zeros of φ can be distributed in the open intervals defined by the $p+1$ zeros of A. The location of the zeros is determined by the equilibrium position of unit charges moving under the action of charges of weight r_j fixed at the zeros a_j of A.

There are not so many results on the Heine-Stieltjes polynomials when some of the restrictions imposed by Stieltjes are dropped. In fact, even the existence of these polynomials is an open problem. In a recent paper Grinshpan [1] extended Stieltjes' construction to a class of generalized Lamé equations with coefficients symmetric with respect to the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, so that the zeros of the Heine-Stieltjes polynomials lie on $\mathbb{T}$. More precisely, let in (1.1) $p = 2M$, $M \in \mathbb{N}$, and A be of the form

$$A(z) = z \prod_{j=1}^{M} (z - z_j)(z\overline{z_j} - 1)\,. \tag{1.2}$$

Using ideas from [1] it can be proved that if the residues of B/A satisfy certain conditions (see below), then for each $N \in \mathbb{N}$ there exists a Van Vleck polynomial C for which (1.1) admits a unique, up to normalization, Heine-Stieltjes polynomial of degree N with zeros lying on $\mathbb{T}$.

*Received November 30, 2002. Accepted for publication September 7, 2003. Communicated by J. Arvesú.
†University of Almería and Instituto Carlos I de Física Teórica y Computacional, Granada University, Spain.
‡University of Almería, Spain.
§University of La Laguna, Canary Islands, Spain. Corresponding author, e-mail: rorive@ull.es.

ETNA
Kent State University
etna@mcs.kent.edu

In the classical situation described by Stieltjes, the asymptotic behavior of the Van Vleck and Heine-Stieltjes polynomials, based on the electrostatic interpretation of their zeros, has been described recently in [4] in terms of the equilibrium measures in presence of an external field; see also the bibliography therein for some additional historical background. In this paper we perform the asymptotic analysis of the configuration described in [1] in the spirit of [4].

Let Λ be an infinite subset of $\mathbb{N}$; for each $N \in \Lambda$ define $M = M(N) \in \mathbb{N}$ and consider a 1-parametric family of generalized Lamé differential equations of the type (1.1),

$$A_N(z)y''(z) + B_N(z)y'(z) + C_N(z)y(z) = 0\,, \quad N \in \Lambda\,, \tag{1.3}$$

where A_N are of the form (1.2), and C_N are Van Vleck polynomials. In other words, for each $N \in \Lambda$ there exists a polynomial solution $y = \varphi_N$ of (1.3) of degree N.

We study the asymptotic behavior of the sequences $\{\varphi_N\}$ and $\{C_N\}$ as $N \to \infty$, $N \in \Lambda$, under the following additional assumptions:

$$S(N) := -\frac{1}{2} \sum_{\substack{A_N(\xi)=0,\\ \xi\neq 0}} \operatorname{res} \frac{B_N}{A_N}(\xi) > 0, \quad N \in \Lambda, \qquad \text{and} \qquad \lim_{N\in\Lambda} \frac{S(N)}{N} = \lambda \in [0, \infty) \tag{1.4}$$

(implicitly, we assume the existence of this limit).

The outline of the paper is as follows. In Sect. 2 we state the problem in detail; the minimum energy problem on the unit circle associated with the Lamé equation above is briefly described. The asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials is analyzed in Sect 3. Finally, in Sect 4, our results are illustrated with some simple examples.

2. An equilibrium problem on the unit circle. We are given subsequences $N \in \Lambda$, $M = M(N) \in \mathbb{N}$, and the corresponding Lamé differential equations (1.3), with the following symmetry conditions satisfied for each $N \in \Lambda$:

$$A_N(z) = zQ_N(z)\,, \qquad B_N(z) = Q_N(z) - zR_N(z)\,,$$

where

- Q_N is a polynomial of degree $2M$ with simple zeros and such that $Q_N^*(z) = z^{2M}\overline{Q_N(1/\overline{z})} = Q_N(z)$; in particular, $Q_N(0) \neq 0$ and $Q_N(z) \neq 0$ for $|z| = 1$.
- R_N is a polynomial of degree $2M - 1$ such that if $Q_N(\xi) = Q_N(\xi^*) = 0$, with $\xi^* = 1/\overline{\xi}$, we have

$$\operatorname*{res}_{z=\xi} \frac{R_N}{Q_N}(z) = \operatorname*{res}_{z=\xi^*} \frac{R_N}{Q_N}(z) > 0\,.$$

Under these conditions, we know from [1] that there exists a polynomial C_N of degree $\leq 2M - 1$ such that (1.3) admits a polynomial solution $y = \varphi_N$ of degree N. Its zeros belong to $\mathbb{T}$ and define the unique equilibrium configuration in the external field generated by positive charges fixed at the zeros of Q_N.

Let us denote by z_{Nk}, $k = 1, \ldots, M$, the zeros of Q_N that lie in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (recall that we assume also that $|z_{Nk}| > 0$), and by $z_{Nk}^* = 1/\overline{z_{Nk}}$ their symmetric counterparts in $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$.

At each z_{Nk} we place a negative charge of value $-w_{Nk}$, where

$$w_{Nk} = \operatorname*{res}_{z=z_{Nk}} \frac{R_N}{Q_N}(z) > 0\,, \quad k = 1, \ldots, M\,, \tag{2.1}$$

along with N movable unit positive charges λ_{Nk}, $k = 1, \ldots, N$ on $\mathbb{T}$. The problem consists in minimizing the total (logarithmic) energy of the system. This setting extends the one studied in [1], where $M = N$ and $w_{Nk} = 1$ $(k = 1, \ldots, M)$.

The total energy of the system described is given by

$$E\left(\lambda_{N1}, \ldots, \lambda_{NN}\right) = -\sum_{k<l} \log\left|\lambda_{Nk} - \lambda_{Nl}\right| + \sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl} \log\left|\lambda_{Nk} - z_{Nl}\right|$$

and it is easy to verify that the equilibrium configuration is attained at points $\lambda_{Nk} \in \mathbb{T}$, $k = 1, \ldots, N$, if and only if the resultant electrostatic force at each of them is normal to $\mathbb{T}$, i.e.

$$\operatorname{Im}\left\{\lambda_{Nk}\left[\sum_{l\neq k} \frac{1}{\lambda_{Nk} - \lambda_{Nl}} - \sum_{l=1}^{M} \frac{w_{Nl}}{\lambda_{Nk} - z_{Nl}}\right]\right\} = 0, \quad k = 1, \ldots, N\,.$$

As it was shown in [1], for an equilibrium configuration $\{\lambda_{N1} \ldots, \lambda_{NN}\}$ there exists a polynomial C_N such that

$$\varphi_N(z) = \prod_{j=1}^{N} (z - \lambda_{Nj})$$

is a polynomial solution of (1.3). Moreover, given the polynomial C_N, the polynomial solution of (1.3) is unique (up to a constant factor). Indeed, let us write (1.3) in the following "self-adjoint" form:

$$\left(\frac{z}{P_N(z)} y'\right)' + \frac{C_N}{P_N(z)\, Q_N(z)} y = 0\,, \tag{2.2}$$

where $P_N(z) = \prod_{k=1}^{M} (z - z_{Nk})^{w_{Nk}} (z - z_{Nk}^*)^{w_{Nk}}$ is a function satisfying that $P_N'/P_N = R_N/Q_N$. Thus, if u and v are two polynomial solutions of (2.2), it yields

$$\left(\frac{z}{P_N(z)} u'\right)' v - \left(\frac{z}{P_N(z)} v'\right)' u = 0\,,$$

which implies

$$\left(\frac{z}{P_N(z)} \left(u'v - uv'\right)\right)' = 0\,,$$

and, hence, we obtain that $z\left(u'v - uv'\right) = \mathrm{const} \cdot P_N$. Since $P_N(0) \neq 0$, we conclude that u and v are linearly dependent. Moreover, this argument shows that any other solution of the differential equation cannot be regular at the origin.

3. Asymptotic behavior of Heine-Stieltjes and Van Vleck polynomials. We start by recalling some facts from the logarithmic potential theory (for basic definitions, see e.g. [5, Section I.1] or [7, Appendix]).

If μ is a finite and compactly supported Borel measure on the complex plane $\mathbb{C}$, we denote by $\operatorname{supp}(\mu)$ its support, by

$$V(\mu; z) = \int \log \frac{1}{|z - t|}\, d\mu(t)$$

ETNA
Kent State University
etna@mcs.kent.edu

its logarithmic potential, and by

$$I(\mu) = \iint \log \frac{1}{|z-t|}\, d\mu(t)\, d\mu(z)$$

its logarithmic energy (in fact, doubled).

A function $w : \mathbb{T} \to \mathbb{R}_+$ is an *admissible weight* on $\mathbb{T}$ if w is upper-semicontinuous and the set $\{z \in \mathbb{T} : w(z) > 0\}$ has positive logarithmic capacity . The (admissible) *external field* ψ on $\mathbb{T}$ is defined by

$$w(z) = e^{-\psi(z)}, \quad z \in \mathbb{T},$$

and the weighted energy $I_\psi(\mu)$ of a Borel measure μ on $\mathbb{T}$, by

$$I_\psi(\mu) = I(\mu) + 2\int \psi\, d\mu\,.$$

We now consider the problem of minimization of the weighted energy $I_\psi(\mu)$ in the class $\mathcal{M}$ of positive Borel unit measures on $\mathbb{T}$. The following lemma is a direct consequence of well-known results (see [5, Theorem I.1.3]):

LEMMA 3.1. *If ψ is an admissible weight then there exists a unique $\mu = \mu(\psi) \in \mathcal{M}$* (the equilibrium measure) *such that*

$$I_\psi(\mu) \le I_\psi(\nu)\,, \quad \text{for every } \nu \in \mathcal{M}\,.$$

Moreover, μ is characterized by the following property:

$$\min_{z\in\mathbb{T}} \Big(V(\mu; z) + \psi(z) \Big) = V(\mu; z) + \psi(z)\,, \quad z \in \operatorname{supp}(\mu)\,.$$

In particular, if $\psi = 0$, we have that μ is the normalized Lebesgue measure on $\mathbb{T}$ (called also *Robin distribution* on $\mathbb{T}$).

Recall also that given a unit Borel measure ν on $\mathbb{C} \setminus \mathbb{T}$, the balayage measure $\widehat{\nu}$ of ν onto $\mathbb{T}$ is the unit measure supported on $\mathbb{T}$ such that

$$V(\nu; z) - V(\widehat{\nu}; z) \equiv \text{const}\,, \quad z \in \mathbb{T}\,,$$

(cf. [5, II.4]).

Back to our problem, for each N and $M = M(N)$, we know that given Q_N and R_N as above, there exists a polynomial C_N of degree $\le 2M-1$ for which the differential equation (1.3) admits a unique (up to normalization) polynomial solution $y = \varphi_N$ of degree N, whose zeros are in the equilibrium position described above. We have assumed also the existence of the limit in (1.4).

If as above z_{Nk} denote the M zeros of Q_N which lie in $\mathbb{D}$, w_{Nk} are given by (2.1), and $S = S(N) = w_{N1} + \cdots + w_{NM}$, the expression

$$\nu_N = \frac{1}{S} \sum_{k=1}^{M} w_{Nk} \delta_{z_{Nk}}$$

defines a unit positive measure with support in $\mathbb{D}$. By a weak compactness argument, there exists an infinite subsequence of Λ (that for simplicity we will denote by Λ again) such that ν_N is convergent along Λ in the weak-* topology to a unit measure ν, supported in $\overline{\mathbb{D}}$:

$$\nu_N \longrightarrow \nu\,, \quad N \in \Lambda\,. \tag{3.1}$$

We can construct the symmetric measure ν^*, supported in $\overline{\mathbb{E}}$, by taking $d\nu^*(t) = d\nu\left(1/\overline{t}\right)$.

In the same way, to each Heine-Stieltjes polynomial φ_N with the zeros $\lambda_{Nk} \in \mathbb{T}$, $k = 1, \dots, N$, we can associate the unit counting measure

$$\mu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda_{Nk}} .$$

THEOREM 3.2. *In the conditions above,*

(i) $\mu_N \longrightarrow \mu$, $N \in \Lambda$, *where* $\mu = \mu(\psi)$ *is the equilibrium measure of* $\mathbb{T}$ *in the external field* $\psi = -\lambda V(\nu; \cdot)$.
Furthermore, for the monic Heine-Stieltjes polynomials $\{\varphi_N\}_{N\in\Lambda}$ *it holds locally uniformly in* $\mathbb{C} \setminus \mathbb{T}$ *that*

$$\lim_{N\in\Lambda} |\varphi_N(z)|^{1/N} = \exp\left(-V(\mu; z)\right) . \tag{3.2}$$

(ii) Assume additionally that there exists a compact set $\Delta \subset \mathbb{D} \setminus \{0\}$ *such that*

$$z_{Nk} \in \Delta \quad \text{for } k = 1, \dots, N, \text{ and for all } N \in \Lambda \text{ sufficiently large.} \tag{3.3}$$

Let $\widehat{\nu}$ *denote the balayage of* ν *from* $\mathbb{D}$ *onto* $\mathbb{T}$, *and* $\omega_{\mathbb{T}}$ *stands for the normalized Lebesgue measure (Robin distribution) of* $\mathbb{T}$. *If* $\lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ *is a positive measure on* $\mathbb{T}$ *(for instance, if* $\lambda \in [0,1]$*), then*

$$\mu_N \longrightarrow \lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}\,, \quad N \in \Lambda\,, \tag{3.4}$$

and for the monic Heine-Stieltjes polynomials $\{\varphi_N\}_{N\in\Lambda}$ *it holds*

$$\lim_{N\in\Lambda} |\varphi_N(z)|^{1/N} = \begin{cases} \exp\left(-\lambda\left(V(\nu^*; z) - C\right)\right), & z \in \mathbb{D}, \\ |z|^{1-\lambda} \exp\left(-\lambda V(\nu; z)\right), & z \in \mathbb{E}, \end{cases} \tag{3.5}$$

locally uniformly in $\mathbb{C} \setminus \mathbb{T}$, *where* $C = \int \log|t|\, d\nu(t)$.

Proof. By [1], for each $N \in \Lambda$ the zeros λ_{Nk}, $k = 1, \dots, N$, of the Heine-Stieltjes polynomial φ_N, endowed with a $+1$ charge each, are in the equilibrium position in the external field created by M *negative* charges, of weight $-w_{Nk}$, placed at z_{Nk}, $k = 1, \dots, M$, where w_{Nk} are defined in (2.1). This yields the fact that $\mu_N \to \mu$, $N \in \Lambda$ by standard arguments (see e.g. [4]).

Indeed, the doubled total (discrete) energy of the configuration described above is

$$\begin{aligned} K_N = 2E\left(\lambda_{N1}, \dots, \lambda_{NN}\right) &= -\sum_{k\neq l} \log|\lambda_{Nk} - \lambda_{Nl}| + 2\sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl} \log|\lambda_{Nk} - z_{Nl}| \,, \\ &= N^2 I\left(\mu_N\right) + 2S \sum_{k=1}^{N} \psi_N\left(\lambda_{Nk}\right) \,, \end{aligned}$$

where

$$\psi_N(t) = \frac{1}{S} \sum_{l=1}^{M} w_{Nl} \log|t - z_{Nl}| = -V(\nu_N; t)\,.$$

ETNA
Kent State University
etna@mcs.kent.edu

Now, if $\{r_{Nk}\}_{k=1}^{N}$ is any collection of N pairwise distinct points in $\mathbb{T}$, by the equilibrium conditions,

$$K_N \leq -\sum_{k\neq l} \log |r_{Nk} - r_{Nl}| + 2\sum_{k=1}^{N}\sum_{l=1}^{M} w_{Nl} \log |r_{Nk} - z_{Nl}| \ ;$$

integrating with respect to $d\mu\,(r_{N1}) \ldots d\mu\,(r_{NN})$, we obtain that

$$K_N \leq N\,(N-1)\,I\,(\mu) + 2S\sum_{k=1}^{N}\int \psi_N\,(r_{Nk})\,d\mu\,(r_{Nk}) \ ,$$

which yields

$$K_N \leq N\,(N-1)\,I\,(\mu) + 2NS\int \psi_N d\mu\,. \tag{3.6}$$

On the other hand, if we take $\varepsilon > 0$ and set $K_\varepsilon\,(x,y) = \min\{-\log|x-y|\,, -\log\varepsilon\}$, it is easy to see that

$$N^2 \iint K_\varepsilon\,(x,y) d\mu_N\,(x)\,d\mu_N\,(y) + 2NS\int \psi_N d\mu_N$$
$$= N^2 I\,(\mu_N) + 2NS\int \psi_N d\mu_N - N\log\varepsilon = K_N - N\log\varepsilon\,.$$

The sequence $\{\mu_N\}$ is weakly compact; thus, eventually passing to a subsequence, we may assume that $\mu_N \to \sigma$, $N \in \Lambda$, where σ is a unit measure supported on $\mathbb{T}$. Then dividing (3.6) through by N^2 and taking limits when $N \to \infty$, $N \in \Lambda$, we obtain that

$$\iint K_\varepsilon\,(x,y)\,d\sigma\,(x)\,d\sigma\,(y) + 2\lambda\int \psi\,d\sigma \leq I\,(\mu) + 2\lambda\int \psi\,d\mu\,, \qquad \psi\,(s) = -V(\nu; s)\,.$$

Since $\varepsilon > 0$ is arbitrary, it yields

$$I_{\lambda\psi}\,(\sigma) = I\,(\sigma) + 2\lambda\int \psi\,d\sigma \leq I\,(\mu) + 2\lambda\int \psi\,d\mu = I_{\lambda\psi}\,(\mu) \ .$$

The uniqueness of the equilibrium measure in an external field implies that $\sigma = \mu$.

Formula (3.2) is a direct consequence of the identity

$$\log|\varphi_N(z)| = -N\,V(\mu_N; z)\,, \quad z \in \mathbb{C}\setminus\mathbb{T}\,,$$

valid for monic polynomials φ_N, and the fact that all the zeros of φ_N belong to $\mathbb{T}$.

Now, assume condition (3.3) satisfied; then $\operatorname{supp}(\nu)\cap\mathbb{T} = \emptyset$, $0 \notin \operatorname{supp}\nu$, and we can construct the balayage measure $\widehat{\nu}$ of ν onto $\mathbb{T}$.

Formula (3.4) is a well-known fact; see e. g. [5, Lemma IV.4.4]. It is sufficient to observe that $\mu = \lambda\widehat{\nu} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a unit measure and

$$V(\mu; z) + \psi(z) = \lambda V(\widehat{\nu}; z) + (1-\lambda)\,V(\omega_{\mathbb{T}}; z) - \lambda V(\nu; z) \equiv \text{const}\,, \quad z \in \mathbb{T}\,.$$

By Lemma 3.1, the expression in the right hand side of (3.4) is the equilibrium measure in the external field ψ, and the statement follows from *(i)*.

Having (3.4) we can be more specific in (3.2). Indeed, let

$$g_{\mathbb{D}}(z;t) = \log\left|\frac{1-\overline{t}z}{z-t}\right|$$

denote the Green function of $\mathbb{D}$; the Green potential of ν is given by

$$G(\mathbb{D},\nu;z) = \int g_{\mathbb{D}}(z;t)\,d\nu(t) = \int \log\left|\frac{1-\overline{t}z}{z-t}\right| d\nu(t) = V(\nu;z) - V(\nu^*;z) + C,$$

with $C = \int \log|t|\,d\nu(t)$, and ν^* the measure symmetric to ν with respect to $\mathbb{T}$.

By [5, Theorem II.5.1],

$$V(\widehat{\nu};z) - V(\nu;z) = \begin{cases} -G(\mathbb{D},\nu;z), & z\in\mathbb{D}, \\ 0, & z\in\mathbb{E}. \end{cases}$$

Thus,

$$V(\widehat{\nu};z) = \begin{cases} V(\nu^*;z) - C, & z\in\mathbb{D}, \\ V(\nu;z), & z\in\mathbb{E}. \end{cases}$$

Taking into account that

$$V(\omega_{\mathbb{T}};z) = \begin{cases} 0, & z\in\mathbb{D}, \\ -\log|z|, & z\in\mathbb{E}, \end{cases}$$

we obtain that

$$V(\mu;z) = \lambda V(\widehat{\nu};z) + (1-\lambda)V(\omega_{\mathbb{T}};z) = \begin{cases} \lambda\left(V(\nu^*;z) - C\right), & z\in\mathbb{D}, \\ \lambda V(\nu;(z) - (1-\lambda)\log|z|, & z\in\mathbb{E}, \end{cases}$$

and along with (3.2) this settles the proof. □

REMARKS: For the case where $\lambda > 1$ and $\lambda\widehat{\nu} + (1-\lambda)\omega_{\mathbb{T}}$ is not a positive measure on the whole $\mathbb{T}$, in general it can be said only that the support of μ is contained in the subset of $\mathbb{T}$ where $\lambda\widehat{\nu} + (1-\lambda)\omega_{\mathbb{T}}$ is positive. However, we shall see in the last section that in certain situations more information is available.

Once we know the asymptotic behavior of the Heine-Stieltjes polynomials φ_N, we can obtain the relative asymptotics for the Van Vleck polynomials C_N. For that purpose we rewrite the differential equation for φ_N in terms of the logarithmic derivative of its solution, reducing (1.3) to a Riccati equation (see e.g. [3, I.4.9], [6] or [9, Section 86]).

For each $N\in\Lambda$ let

$$h_N(z) = \frac{1}{N}\frac{\varphi_N'(z)}{\varphi_N(z)} = \int \frac{d\mu_N(x)}{z-x}$$

and

$$k_N(z) = \frac{1}{2S}\frac{R_N(z)}{Q_N(z)} = \frac{1}{2S}\frac{P_N'(z)}{P_N(z)} = \frac{1}{2}\int \frac{d\nu_N(x) + d\nu_N^*(x)}{z-x}.$$

By Theorem 3.2,

$$\lim_{N\in\Lambda} h_N(z) = h(z) = \int \frac{d\mu(x)}{z-x}, \tag{3.7}$$

uniformly in compact subsets of $\mathbb{C}\backslash\mathbb{T}$. Analogously, by (3.3) there exists a compact set $K = \Delta \cup \Delta^* \subset \mathbb{C}\backslash (\mathbb{T} \cup \{0\})$ such that

$$\lim_{N\in\Lambda} k_N(z) = k(z) = \frac{1}{2}\int \frac{d\nu(x) + d\nu^*(x)}{z-x}, \tag{3.8}$$

locally uniformly in $\mathbb{C}\backslash K$.

With these notations, (1.3) can be rewritten as

$$\frac{z}{2S}\left(Nh_N^2(z) + h_N'(z)\right) + \left(\frac{1}{2S} - zk_N(z)\right)h_N(z) + \frac{C_N(z)}{2NSQ_N(z)} = 0\,.$$

Since with assumption (3.3) $\{h_N'\}_{N\in\Lambda}$ is also uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{T}$, taking limits as $N \in \Lambda$ and using (1.4) yields that the limit

$$f(z) = \lim_{N\in\Lambda} \frac{C_N(z)}{N^2 Q_N(z)}$$

exists and satisfies the equation

$$\frac{z}{2\lambda}h^2(z) - zh(z)k(z) + \frac{1}{2\lambda}f(z) = 0.$$

Thus, we have established the following

THEOREM 3.3. *Under assumption (3.3) the Van Vleck polynomials $\{C_N\}_{N\in\Lambda}$ satisfy the following relative asymptotics*

$$\lim_{N\in\Lambda} \frac{C_N(z)}{N^2 Q_N(z)} = zh(z)\left(2\lambda k(z) - h(z)\right), \tag{3.9}$$

where the convergence is uniform in compact subsets of $\mathbb{C}\backslash\{\mathbb{T}\cup K\}$, and where $h(z)$ and $k(z)$ have been defined in (3.7) and (3.8), respectively.

In particular, the Hurwitz theorem can be used together with (3.9) to get information on the accumulation points of the zeros of Van Vleck polynomials C_N.

4. Examples. In this section, some simple examples illustrating the results above will be considered.

EXAMPLE 1. Let $0 < r < 1$ and denote by $\mathbb{U} = \{z \in \mathbb{C} : |z| = r\}$ the circle of radius r centered at the origin. Assume that the zeros of A_N contained in $\mathbb{D}$ are distributed asymptotically uniformly on U. Then obviously (3.1) holds for $\nu = \omega_{\mathbb{U}}$, the equilibrium measure of $\mathbb{U}$. Moreover, condition (3.3) is satisfied.

We have

$$V(\nu; z) = \begin{cases} -\log r, & |z| \le r, \\ -\log|z|, & |z| > r, \end{cases}$$

and for $\lambda \in [0, \infty)$, we have:

$$V(\omega_{\mathbb{T}}; z) = \lambda V(\nu; z), \quad z \in \mathbb{T}\,.$$

Hence, in this case $\mu = \omega_{\mathbb{T}}$, and by Theorem 3.2 for the Heine-Stieltjes polynomials we have

$$\lim_{N\to\infty} |\varphi_N(z)|^{1/N} = \begin{cases} 1, & z \in \mathbb{D}, \\ |z|, & z \in \mathbb{E}, \end{cases}$$

uniformly on compact subsets of the respective domains.

Furthermore, taking into account (3.7) and (3.8),

$$h(z)=\int\frac{d\mu(x)}{z-x}=\begin{cases}0, & z\in\mathbb{D},\\ 1/z, & z\in\mathbb{E},\end{cases}\quad\text{and}\quad k(z)=\begin{cases}1/z, & |z|>1/r,\\ 1/(2z), & r<|z|<1/r,\\ 0, & |z|<r.\end{cases}$$

Consequently,

$$\lim_N\frac{C_N(z)}{N^2Q_N(z)}=\begin{cases}0, & z\in\mathbb{D}\setminus\mathbb{U},\\ (\lambda-1)/z, & 1<|z|<1/r,\\ (2\lambda-1)/z, & |z|>1/r,\end{cases}\tag{4.1}$$

uniformly on compact subsets of $\mathbb{C}\setminus(\mathbb{T}\cup\mathbb{U}\cup\mathbb{U}^*)$, where $U^*=\{z\in\mathbb{C}:\ |z|=1/r\}$.

Using Hurwitz theorem we conclude for instance that if $\lambda\notin\{1/2,1\}$, the zeros of Van Vleck polynomials can accumulate only in $\overline{\mathbb{D}}$ or on the circle U^*. Moreover, if $\lambda=1$, according to (4.1) they might accumulate also in the gap between $\mathbb{D}$ and U^*. In [1] it was considered the case with $S=M=N$ and $Q_N(z)=\left(z^N-r^N\right)\left(1-r^Nz^N\right)$, so that $\varphi_N(z)=z^N-1$ and $C_N(z)=-N^2r^Nz^{N-1}\left(z^N+1\right)$. In this situation the limit points of the zeros of Van Vleck polynomials still lie in $\overline{\mathbb{D}}$. It would be interesting to find an example when these zeros leave the unit disc.

EXAMPLE 2. Assume now that all the zeros of A_N inside $\mathbb{D}$ cluster asymptotically on a single point a, $|a|<1$, so that $\nu=\delta_a$.

Let us consider first the case $\lambda\in[0,1]$, where we know that $\mu=\lambda\widehat{\delta_a}+(1-\lambda)\,\omega_{\mathbb{T}}$. By (3.5), the monic Heine-Stieltjes polynomials have the following n-th root asymptotics:

$$\lim_N|\varphi_N(z)|^{1/N}=\begin{cases}|1-\overline{a}z|^{\lambda}, & z\in\mathbb{D}\setminus\{a\},\\ |z|^{1-\lambda}|z-a|^{\lambda}, & z\in\mathbb{E},\end{cases}$$

uniformly on compact subsets of the respective domains. Furthermore, by (3.7) and (3.8),

$$h(z)=\begin{cases}\lambda/(z-a^*), & z\in\mathbb{D},\\ \lambda/(z-a)+(1-\lambda)/z, & z\in\mathbb{E},\end{cases}$$

and

$$k(z)=\frac{1}{2}\left(\frac{1}{z-a}+\frac{1}{z-a^*}\right),\quad z\neq a,a^*,$$

and

$$2\lambda k(z)-h(z)=\begin{cases}\lambda/(z-a), & z\in\mathbb{D}\setminus\{a\},\\ \lambda/(z-a^*)-(1-\lambda)/z, & z\in\mathbb{E}\setminus\{a^*\},\end{cases}$$

where $a^*=\dfrac{1}{\overline{a}}$. Therefore,

$$\lim_{N\to\infty}\frac{C_N(z)}{N^2Q_N(z)}=\begin{cases}\dfrac{\lambda^2z}{(z-a)(z-a^*)}, & z\in\mathbb{D}\setminus\{a\},\\ z\left(\dfrac{\lambda}{z-a}+\dfrac{1-\lambda}{z}\right)\left(\dfrac{\lambda}{z-a^*}-\dfrac{1-\lambda}{z}\right), & z\in\mathbb{E}\setminus\{a^*\},\end{cases}$$

uniformly on compact subsets of the respective domains.

Using the Hurwitz theorem we see that if $\lambda = 1/2$ or $\frac{1+|a|}{1+2|a|} \leq \lambda \leq 1$, the zeros of Van Vleck polynomials can accumulate only either on $\mathbb{T}$ or at the points 0, a and $1/\overline{a}$. On the other hand, if $0 < \lambda < \frac{1+|a|}{1+2|a|}$, $\lambda \neq 1/2$, an additional accumulation point might appear at $\xi = \frac{\lambda-1}{2\lambda-1}\,\frac{1}{\overline{a}}$.

Assume now $\lambda > 1$; we have seen that (3.4) is still valid for such values of λ for which $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a positive measure on $\mathbb{T}$.

We know (cf. [5, Sect. II.4]) that

$$d\widehat{\delta_a}\,(t) = \frac{1}{2\pi}\,\frac{1-|a|^2}{|t-a|^2}\,d\theta, \quad t = e^{i\theta}\,.$$

Therefore, $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is a positive measure on the whole $\mathbb{T}$ if and only if:

$$|t-a|^2 \leq \frac{\lambda}{\lambda-1}\left(1-|a|^2\right), \quad t \in \mathbb{T};$$

in other words, $\mathbb{T}$ must be contained in the disc centered at a with radius $\left[\left(1-|a|^2\right)\lambda/(\lambda-1)\right]^{1/2}$. This condition is satisfied if

$$\lambda \leq \frac{1+|a|}{2\,|a|}\,,$$

and asymptotic formulas above are valid.

On the other hand, if $\lambda > \frac{1+|a|}{2|a|}$, it is easy to see that $\lambda\widehat{\delta_a} + (1-\lambda)\,\omega_{\mathbb{T}}$ is positive on

$$S_\lambda = \left\{ t \in \mathbb{T} : \ \mathrm{Re}\,(\overline{a}t) \geq \frac{(2\lambda-1)\,|a|^2 - 1}{2\,(\lambda-1)} \right\}.$$

In the general case we can assure only that $\mathrm{supp}(\mu) \subset S_\lambda$, but in our situation we may assert additionally that $\mathrm{supp}\,\mu$ is an arc contained in S_λ. Indeed, for $z \in \mathbb{T}$,

$$V^\mu\,(z) = \mathrm{const} + \lambda V^\nu\,(z) = \mathrm{const} - \lambda \log|z-a|\ .$$

Applying the maximum principle for potentials [5, corollary II.3.3] we conclude that the support is a connected subset of $\mathbb{T}$ (that is, an arc) contained in S_λ. Obviously, this arc is symmetric with respect $z_0 = a/|a|$. When $\lambda \to \infty$, the arc S_λ tends to the limit arc $S_\infty = \left\{t \in \mathbb{T}/\mathrm{Re}\,(\overline{a}t) \geq |a|^2\right\}$.

The considerations above can be extended to the case of an atomic measure, that is, when

$$\nu = \sum_{i=1}^{p} \alpha_i \delta_{a_i}, \quad \text{with} \quad |a_i| < 1,\ \alpha_i > 0,\ i = 1,\dots,p \quad \text{and} \quad \sum_{i=1}^{p} \alpha_i = 1\,.$$

In this case, $\mathrm{supp}(\mu) = \mathbb{T}$ if and only if for all $t \in \mathbb{T}$,

$$\sum_{i=1}^{p} \alpha_i \frac{1-|a_i|^2}{|t-a_i|^2} \geq \frac{\lambda-1}{\lambda}\,.$$

Otherwise, $\mathrm{supp}(\mu)$ will consist of the union of $\leq p$ arcs.

Acknowledgements. The research of A.M.F. and P.M.G. was partially supported by the Ministry of Science and Technology (MCYT) of Spain through the grant BFM2001-3878-C02-02, and by Junta de Andalucía through Grupo de Investigación FQM 0229. A.M.F. acknowledges also the support of the European Research Network on Constructive Complex Approximation (NeCCA), INTAS 03-51-6637, and of NATO Collaborative Linkage Grant “Orthogonal Polynomials: Theory, Applications and Generalizations,” ref. PST.CLG.979738.

The research of R.O. was partially supported by grants from Spanish MCYT (Research Project BFM2001-3411) and Gobierno Autónomo de Canarias (Research Project PI2002/136).

REFERENCES

[1] A. GRINSHPAN, *A minimum energy problem and Dirichlet spaces*, Proc. Amer. Math. Soc., 130 (2002), No. 2, pp. 453-460.
[2] E. HEINE, *Handbuch der kugelfunctionen*, vol. II, G. Reimer, Berlin, 2nd. edition, 1878.
[3] E. KAMKE, *Differentialgleichungen. Lösungsmethoden und Lösungen. I. Gewöhnliche Differentialgleichungen*, Teubner, Stuttgart, 1977.
[4] A. MARTÍNEZ-FINKELSHTEIN AND E. B. SAFF, *Asymptotic properties of Heine-Stieltjes and Van Vleck polynomials*, J. Approx. Theory, 118 (2002), No. 1, pp. 131-151.
[5] E. B. SAFF AND V. TOTIK, *Logarithmic potentials with external fields*, vol. 316 of Grundlehren der Matematischen Wissenschaften, Springer-Verlag, Berlin, 1997.
[6] E. B. SAFF, J. L. ULLMAN AND R. S. VARGA, *Incomplete polynomials: an electrostatic approach*, in Approximation Theory III, Academic Press, New York, 1980, pp. 769-782.
[7] H. STAHL AND V. TOTIK, *General Orthogonal Polynomials*, vol. 43 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.
[8] T. J. STIELTJES, *Sur certain polynomes que verifient une equation differentielle lineaire du second ordre et sur la teorie des fonctions de Lamé*, Acta Math., 6 (1885), pp. 321-326.
[9] D. ZWILLINGER, *Handbook of Differential Equations*, Academic Press, San Diego, CA, 3rd. ed., 1998.