NORM-ATTAINING COMPACT OPERATORS

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Abstract. We show examples of compact linear operators between Banach spaces which cannot be approximated by norm attaining operators. This is the negative answer to an open question posed in the 1970's. Actually, any strictly convex Banach space failing the approximation property serves as the range space. On the other hand, there are examples in which the domain space has Schauder basis. It now makes sense to discuss sufficient conditions on the domain or the range space to ensure that every compact linear operator between them can be approximated by norm attaining operators. We get several basic results in this line and mention some open problems.

1. Introduction

Motivated by the classical Bishop-Phelps theorem of 1961 [6] stating the density of norm-attaining functionals on every Banach space, the study of the density of norm-attaining operators started with J. Lindenstrauss' 1963 paper [20], where the author showed that the Bishop-Phelps theorem is not longer true for operators and gave some partial positive results. We recall that an operator T between two Banach spaces X and Y is said to *attain its norm* whenever there is $x \in X$ with $\|x\| = 1$ such that $\|T\| = \|T(x)\|$ (i.e. the supremum defining the operator norm is actually a maximum). An intensive research about this topic has been developed by, among others, J. Bourgain in the 1970's, J. Partington and W. Schachermayer in the 1980's, and M. Acosta, W. Gowers and R. Payá in the 1990's. We will give a short account on the subject at the beginning of section 3. The expository paper [3] can be used for reference and background.

All known examples of operators which cannot be approximated by norm-attaining ones are non-compact, so the question of whether every linear compact operator between Banach spaces can be approximated by norm-attaining operators seems to be open. It was explicitly asked by J. Diestel and J. Uhl in the 1976 paper [9] (as Problem 4 in page 6) and in their monograph on vector measures [10, p. 217], and also in the 1979 paper by J. Johnson and J. Wolfe [19] (as Question 2 in page 17). More recently, the question also appeared in the 2006 expository paper by M. Acosta [3, p. 16].

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The main aim of this paper is to answer the question in the negative by providing two Banach spaces X and Y and a compact linear operator from X into Y which cannot be approximated by norm-attaining operators. This comes from extending an idea of Lindenstrauss for c_0 to its closed subspaces and applying it to Enflo's counterexample to the approximation problem. Moreover, thanks to an example of W. Johnson and G. Schechtman, the space X can be taken with Schauder basis. It is also possible to get an example where $X = Y$. We also show that for every strictly convex Banach space Y without the approximation property, there exists a Banach space X such that $K(X, Y)$ is not contained in the closure of the set of norm-attaining operators. All of these is the content of section 2.

Once we know that the density of norm-attaining operators in the space of compact operators may fail, it makes sense to study conditions assuring such a density. To do so and imitating what Lindenstrauss did in 1963, we introduce in section 3 two properties that we will call A^k and B^k. We present some new examples and results, and discuss open problems.

Let us finish the introduction with the needed notation. Given two (real or complex) Banach spaces X and Y, we write $L(X, Y)$ for the Banach space of all bounded linear operators from X into Y, endowed with the operator norm. By $K(X, Y)$ and $F(X, Y)$ we denote the subspaces of $L(X, Y)$ of compact operators and finite-rank operators, respectively. We write B_X to denote the closed unit ball of X. The set of all norm-attaining operators from X into Y is denoted by $NA(X, Y)$.

2. Main results

Let us start with the promised counterexample.

Fact 2.1. *There exist compact linear operators between Banach spaces which cannot be approximated by norm-attaining operators.*

The idea for the proof of the above fact comes from extending (the proof of) [20, Proposition 4] to closed subspaces of c_0 and then apply it to Enflo's counterexample to the approximation problem. We state the first ingredient for further use. Recall that a Banach space Y is said to be *strictly convex* if the unit sphere of Y fails to contain non-trivial segments, equivalently, if for every $y \in Y$ with $\|y\| = 1$ and $z \in Y$, $\|y \pm z\| \leqslant 1$ implies $z = 0$.

Lemma 2.2. *Let X be a closed subspace of c_0 and let Y be a strictly convex Banach space. Then, $NA(X, Y) \subseteq F(X, Y)$.*

Proof. Fix $T \in NA(X, Y)$ and $x_0 \in B_X$ such that $\|T(x_0)\| = \|T\| = 1$. As $x_0 \in c_0$, there is $N \in \mathbb{N}$ such that $|x_0(n)| < 1/2$ for every $n \geqslant N$. Now, consider the subspace Z of X given by

$$Z := \{x \in X \,:\, x(i) = 0 \text{ for } 1 \leqslant i \leqslant N\}$$

and observe that for every $z \in Z$ with $\|z\| \leqslant 1/2$, we have

$$\|x_0 \pm z\| \leqslant 1.$$

Therefore,

$$\|T(x_0) \pm T(z)\| \leqslant 1$$

and, being Y strictly convex and $\|T(x_0)\| = 1$, it follows that $T(z) = 0$. Therefore, T vanished on a finite-codimensional space. □

Prior to give the proof of the fact, we have to recall the concept of (Grothendieck) approximation property. We refer to [21] for background. A Banach space X has the *approximation property* if for every compact set K and every $\varepsilon > 0$, there is $R \in F(X, X)$ such that $\|x - R(x)\| < \varepsilon$ for all $x \in K$. It was shown by P. Enflo in 1973 that there are Banach spaces failing the approximation property showing, actually, that there are closed subspaces of c_0 without the approximation property.

Proof of Fact 2.1. Let X be a closed subspace of c_0 failing the approximation property (Enflo's example works, see [21, Theorem 2.d.6]). Then, X^* also fails the approximation property so there is a Banach space Y and a compact operator $T : X \longrightarrow Y$ which cannot be approximated by finite-rank operators (see [21, Theorem 1.3.5]). As we may clearly suppose that Y is separable (considering the closure of $T(X)$) and the approximation property if of isomorphic nature, we may and do suppose that Y is strictly convex (recall that every separable Banach space admits a strictly convex equivalent renorming by an old result of M. Kadec, see [8, §II.2]). Now, Lemma 2.2 shows that T cannot be approximated by norm-attaining operators. □

Next, we would like to present two ways to obtain examples as in Fact 2.1. First, with respect to domain spaces, we observe that the above proof works for arbitrary closed subspaces of c_0 whose dual fails the approximation property.

Theorem 2.3. *For every closed subspace X of c_0 such that X^* fails the approximation property, there exist a Banach space Y and a compact linear operator from X into Y which cannot be approximated by norm-attaining operators.*

Using the result due to W. Johnson and G. Schechtman [18, Corollary JS, p. 127] that there is a closed subspace of c_0 with Schauder basis whose dual fails the approximation property, we may state the following corollary.

Corollary 2.4. *There exist a Banach space X with Schauder basis, a Banach space Y and a compact linear operator T between X and Y which cannot be approximated by norm-attaining operators.*

Dealing with range spaces, the idea of Fact 2.1 can be also squeezed to show that for every strictly convex Banach space Y without the approximation property, an example of the same kind can be constructed. We will use the following characterization of the approximation property, known to A. Grothendieck (see "Proposition" 37 in p. 170 of [14]), which follows easily from the compact factorization of every compact operator through a closed subspace of c_0. A proof of the lemma can be found in [17, Theorem 18.3.2].

Lemma 2.5 (Grothendieck)**.** *A Banach space Y has the approximation property if and only if $F(X, Y)$ is dense in $K(X, Y)$ for every closed subspace X of c_0.*

We are now able to present the promised result.

Theorem 2.6. *Let Y be a strictly convex Banach space without the approximation property. Then, there exist a Banach space X and a compact linear operator from X into Y which cannot be approximated by norm-attaining operators.*

Proof. By Lemma 2.5, there is a closed subspace X of c_0 such that $F(X, Y)$ is not dense in $K(X, Y)$. But Lemma 2.2 implies that $NA(X, Y) \subset F(X, Y)$, so there are compact operators from X into Y which cannot be approximated by norm-attaining operators. □

Compare the result above with the one by M. Acosta [1] of 1999, stating that there is a Banach space X such that for every infinite-dimensional strictly convex Banach space Y, there exists a (non-compact) operator $T \in L(X,Y)$ which cannot be approximated by norm-attaining operators. The case when $Y = \ell_p$ was previously done by W. Gowers [13] in 1990.

Next, we would like to give a result for subspaces of complex $L_1(\mu)$ spaces. We first need to recall the notion of complex strict convexity. A complex Banach space Y is said to be *complex strictly convex* if for every $y \in Y$ with $\|y\| = 1$ and $z \in Y$, the condition $\|y + \theta z\| \leqslant 1$ for every $\theta \in \mathbb{C}$ with $|\theta| = 1$ implies $z = 0$. Clearly, strictly convex spaces are complex strictly convex, but the converse is false, as $L_1(\mu)$ spaces are complex strictly convex, see [16, Proposition 3.2.3]. By an obvious adaption of the proof of Lemma 2.2, we get that, in the complex case, *if X is a closed subspace of c_0 and Y is a complex strictly convex space, then $NA(X,Y) \subseteq F(X,Y)$*. Therefore, the following result follows with the same proof than Theorem 2.6.

Proposition 2.7. *Let μ be a measure and let Y be a closed subspace of the complex space $L_1(\mu)$ without the approximation property. Then, there exist a Banach space X and a compact linear operator from X into Y which cannot be approximated by norm-attaining operators.*

We do not know whether this result is also true in the real case. It is known that there is a Banach space X such that for every measure μ such that $L_1(\mu)$ is infinite dimensional, there is a (non-compact) operator T from X into $L_1(\mu)$ which cannot be approximated by norm-attaining operators (M. Acosta, [2]).

We finish the section providing an example in which the domain and the range space coincides.

Example 2.8. There exist a Banach space Z and a compact operator from Z into Z which cannot be approximated by norm-attaining operators.

Proof. Let X and Y be Banach spaces and fix $T_0 \in K(X,Y)$ with $\|T_0\| = 1$ and $0 < \varepsilon < 1/2$. Write $Z = X \oplus_\infty Y$ (i.e. $\|(x,y)\| = \max\{\|x\|, \|y\|\}$ for $(x,y) \in X \times Y$) and define $S_0 \in K(Z,Z)$ by $S_0(x,y) = (0, T_0(x))$ for every $(x,y) \in X \oplus_\infty Y$, which clearly satisfies $\|S_0\| = 1$. We claim that if there is an operator $S \in NA(Z,Z)$ such that $\|S_0 - S\| < \varepsilon$, then there is $T \in NA(X,Y)$ such that $\|T_0 - T\| < \varepsilon$. Indeed, take $(x_0, y_0) \in B_Z = B_X \times B_Y$ such that $\|S(x_0,y_0)\| = \|S\|$ and write $P_1 : Z \longrightarrow X$ and $P_2 : Z \longrightarrow Y$ for the natural projections. Now, observe that

$$\|P_1 S\| = \|P_1 S - P_1 S_0\| \leqslant \|S - S_0\| < \varepsilon < 1/2$$

so, as $\|S\| \geqslant 1 - \varepsilon > 1/2$, we get that

$$\|P_2 S(x_0,y_0)\| = \|P_2 S\| = \|S\|.$$

Next, take $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = 1$ and define the operator $T \in L(X,Y)$ by

$$T(x) = P_2 S\big(x, x_0^*(x) y_0\big) \qquad \big(x \in X\big).$$

Then, $\|T\| \leqslant \|P_2 S\|$ and $\|T(x_0)\| = \|P_2 S(x_0,y_0)\| = \|P_2 S\|$, so $T \in NA(X,Y)$. On the other hand, for $x \in B_X$,

$$\|T_0(x) - T(x)\| = \big\|P_2 S_0(x, x_0^*(x) y_0) \ - \ P_2 S(x, x_0^*(x) y_0)\big\| \leqslant \|P_2 S_0 - P_2 S\| \leqslant \|S_0 - S\| < \varepsilon,$$

as claimed.

Now, if we take X, Y, and $T_0 \in K(X,Y)$ which cannot be approximated by norm-attaining operators, then $Z = X \oplus_\infty Y$ and $S_0 \in K(Z,Z)$ defined as above, give the desired example. □

3. The properties A^k and B^k

Once we know that the density of norm-attaining operators in the space of compact operators may fail, it makes sense to study the question of when this density holds or not. As this question is too general, and imitating what Lindenstrauss did in 1963, we will introduce two properties, which we will call A^k and B^k.

Let us first present a short account on Lindenstrauss' properties. We refer the reader to the expository paper [3] for a detailed account and also for references. In the cited seminal paper of 1963, J. Lindenstrauss introduced two properties to study norm-attaining operators: a Banach space X (resp. Y) has property A (resp. B) if $NA(X,Y)$ is dense in $L(X,Y)$ for every Banach space Y (resp. every Banach space X). It is shown that c_0 does not have property A since $NA(c_0,Y)$ is not dense in $L(c_0,Y)$ for every strictly convex renorming Y of c_0. Examples of spaces having property A (including reflexive spaces and ℓ_1) and of spaces having property B (including c_0, ℓ_∞ and every finite-dimensional space whose unit ball is a polyhedron) are also shown in this paper. There are many extensions of Lindenstrauss results from which we will comment only a representative sample. With respect to property A, J. Bourgain showed in 1977 that every Banach space with the Radon-Nikodým property (RNP in short) have property A and that, conversely, if a Banach space X has property A in every equivalent norm, then it has the RNP (this direction needs a refinement due to R. Huff, 1980). W. Schachermayer (1983) and B. Godun and S. Tronyanski (1993) showed that "almost" every Banach space can be equivalently renormed to have property A, and J. Uhl (1976) showed that $L_1[0,1]$ does not have property A. With respect to property B, J. Partington proved that every Banach space can be renormed to have property B (1982) and W. Schachermayer showed that $C[0,1]$ fails the property (1983). W. Gowers showed in 1990 that ℓ_p does not have property B for $1<p<\infty$, a result extended by M. Acosta (1999) to all infinite-dimensional strictly convex spaces and to ℓ_1. With respect to pairs of classical Banach spaces not covered by the results above, J. Johnson and J. Wolfe (1979) proved that, in the real case, $NA(C(K),C(S))$ is dense in $L(C(K),C(S))$ for all compact spaces K and S, and C. Finet and R. Payá (1998) showed the same result for the pair $(L_1[0,1],L_\infty[0,1])$.

Our two new properties are the following.

Definition 3.1.

(a) A Banach space X is said to have *property A^k* if $K(X,Y)\cap NA(X,Y)$ is dense in $K(X,Y)$ for every Banach space Y.
(b) A Banach space Y is said to have *property B^k* if $K(X,Y)\cap NA(X,Y)$ is dense in $K(X,Y)$ for every Banach space X.

Even though we do not know whether, in general, property A implies property A^k or property B implies property B^k, all known sufficient conditions for Lindenstrauss properties A and B also implies, respectively, properties A^k and B^k. This is so because the way of establishing the density of norm-attaining operators is by proving that for every operator there can be approximated by compact perturbations of it attaining the norm. This produces the following list of examples. For the definitions, background and concrete references, we refer the reader again to the survey paper [3].

Examples 3.1.

(a) (Bourgain) RNP implies property A^k.
(b) (Schachermayer) Property α implies property A^k.

(c) (Godun–Troyanski) Therefore, Banach spaces admitting long biorthogonal systems (in particular, separable spaces) can be equivalently renormed to have property A^k.
(d) (Choi–Song) Property quasi-α implies property A^k.
(e) (Lindenstrauss) Property β implies property B^k.
(f) (Partington) Therefore, every Banach space can be renormed to have property B^k.
(g) (Acosta–Aguirre–Payá) Property quasi-β implies property B^k.

Concerning the study of norm-attaining compact operators, J. Diestel and J. Uhl (1976) [9] showed that norm-attaining finite-rank operators from $L_1(\mu)$ into any Banach space are dense in the space of all compact operators. This study was continued by J. Johnson and J. Wolfe [19] (1979), who proved that $NA(X,Y) \cap F(X,Y)$ is dense in $K(X,Y)$ whenever X is a $C(K)$ space or Y is an L_1-space or a predual of an L_1-space. The results in [19] are proved in the real case. All the proofs, but the one for $Y = L_1(\mu)$, easily extends to the complex case. Also, the proof for $C(K)$ extends to $C_0(L)$.

Let us enunciate these examples.

Examples 3.2.

(a) (Diestel–Uhl) For every measure μ, $L_1(\mu)$ has property A^k.
(b) (Johnson-Wolfe) $C_0(L)$ has property A^k for every locally compact Hausdorff topological space L.
(c) (Johnson-Wolfe) For every measure μ, every predual of $L_1(\mu)$ have property B^k. In particular, $C(K)$ has property B^k for every compact Hausdorff topological space K.
(d) (Johnson-Wolfe) For every measure μ, the *real* space $L_1(\mu)$ has property B^k.

Next, we write the result of section 2 as (negative) results on properties A^k and B^k.

Examples 3.3.

(a) Property A^k fails in every closed subspace of c_0 whose dual lacks the approximation property.
(b) In particular, there is a Banach space with Schauder basis failing property A^k.
(c) Strictly convex Banach spaces without the approximation property fails property B^k.
(d) Closed subspaces of complex $L_1(\mu)$ spaces failing the approximation property do not have property B^k.

We would like now to discuss some open question and some partial positive results. The main open question in the subject is the following.

Question 3.4. Does every finite-dimensional Banach space have Lindenstrauss property B?

With respect to domain spaces, we have the following open question.

Question 3.5. Does every Banach space whose dual has the approximation property have property A^k?

Observe that a positive answer to Question 3.4 would give a positive answer to the above one. On the other hand, it would be interesting to characterize those Banach spaces having property A^k in every equivalent norm. The approximation property of the dual cannot provide such a characterization, as all reflexive spaces have property A^k.

Let us also comment that the results in [19] about property A^k holds by proving a stronger version of the approximation property of the dual. The argument can be abstracted using the concept of π-property given by J. Lindenstrauss in 1964 (see [7, §5] for background). A Banach space X is said to have the *metric π-property* if there is a net of finite-rank contractive projections $\{P_\alpha\}$ on X converging to the identity in the strong operator topology. We will need the following version of the property: X^* has the metric π-property with w^*-continuous projections, that is, there exists a net of finite-rank contractive projections $\{P_\alpha\}$ on X such that $\{P_\alpha^*\}$ converges to the identity of X^* in the strong operator topology. We include the proof of the following result (which is omitted in [19]) for completeness.

Proposition 3.6. *Let X be a Banach space. Suppose there is a net (P_α) of finite-rank contractive projections on X such that for every $x^* \in X^*$, $(P_\alpha^* x^*) \longrightarrow x^*$ in norm. Then X has property A^k.*

Proof. Let Y be a Banach space and consider $T \in K(X,Y)$. For every α, the operator TP_α attains its norm since $TP_\alpha(B_X) = T(B_{P_\alpha(X)})$ (here we use that P_α is a norm-one projection) and $B_{P_\alpha(X)}$ is compact. We claim that $(TP_\alpha) \longrightarrow T$ in the operator norm, finishing the proof. Indeed, given $\varepsilon > 0$, as T^* is compact, we may find an $\varepsilon/3$-net $x_1^*, \ldots, x_n^* \in X^*$ for $T^*(B_{Y^*})$ and we may find α_0 such that $\|P_\alpha^*(x_i^*) - x_i^*\| < \varepsilon/3$ for $i = 1, \ldots, n$ and every $\alpha \geqslant \alpha_0$. Now, given $y^* \in B_{Y^*}$, we take $i \in \{1, \ldots, n\}$ such that $\|T^*(y^*) - x_i^*\| < \varepsilon/3$ and observe that

$$\|P_\alpha^* T^*(y^*) - T^*(y^*)\| \leqslant \|P_\alpha^* T^*(y^*) - P_\alpha^*(x_i^*)\| + \|P_\alpha^*(x_i^*) - x_i^*\| + \|x_i^* - T^*(y^*)\| < \varepsilon.$$

In other words, $\|TP_\alpha - T\| = \|P_\alpha^* T^* - T^*\| \leqslant \varepsilon$ for every $\alpha \geqslant \alpha_0$. □

It is shown in [19, Proposition 3.2] that every $C(K)$ space satisfies the condition of the above proposition. Actually, the proof also works for $C_0(L)$ spaces. On the other hand, it is easy to show that every $L_1(\mu)$-space satisfies such a condition when μ is finite, giving an alternative proof for the result in [9] stating that $L_1(\mu)$ has property A^k for every measure μ (for non-finite measures, we may just use Proposition 3.15 below).

New examples of spaces with property A^k can be deduced from Proposition 3.6. The first set is the family of preduals of ℓ_1.

Corollary 3.7. *Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1. Then X has property A^k.*

Proof. Let $(x_n^*)_{n\in\mathbb{N}}$ be a Schauder basis of X^* isometrically equivalent to the usual ℓ_1-basis and for every $n \in \mathbb{N}$, let Y_n the linear span of $\{x_1^*, \ldots, x_n^*\}$. In the proof of [12, Corollary 4.1], a sequence of w^*-continuous contractive projections $Q_n : X^* \longrightarrow X^*$ with $Q_n(X^*) = Y_n$ is constructed. The w^*-continuity of Q_n provides then a sequence of finite-rank contractive projections on X satisfying the hypothesis of Proposition 3.6. Let us note that the results in [12] are given in the real case, but the proofs work in the complex case as well. □

We do not know whether the corollary above extends to isometric preduals of arbitrary $L_1(\mu)$ spaces.

Proposition 3.6 applied to spaces with a shrinking monotone Schauder basis. Recall that a Schauder basis of a Banach space X is said to be *shrinking* if its sequence of coordinate functionals is a Schauder basis of X^*.

Corollary 3.8. *Every Banach space with a shrinking monotone Schauder basis has property A^k.*

It is well-known that an unconditional Schauder basis of a Banach space is shrinking if the space does not contain ℓ_1 (see [5, Theorem 3.3.1] for instance), so the following particular case appears.

Corollary 3.9. *Let X be a Banach space with unconditional monotone Schauder basis which does not contain ℓ_1. Then X has property A^k.*

For the class of M-embedded spaces, this last result can be improved removing the unconditionality condition on the basis, by using the 1988 result of G. Godefroy and P. Saphar that Schauder bases in M-embedded spaces with basis constant less than 2 are shrinking (see [15, Corollary III.3.10], for instant). We recall that a Banach space X is said to be *M-embedded* if $X^\perp$ is the kernel of an L_1-projection in X^* (i.e. $X^* = X^\perp \oplus Z$ for some Z and $\|x^\perp + z\| = \|x^\perp\| + \|z\|$ for every $x^\perp \in X^\perp$ and $z \in Z$). We refer the reader to [15] for background.

Corollary 3.10. *Every M-embedded space with monote Schauder basis has property A^k.*

As c_0 is an M-embedded space [15, Examples III.1.4] and M-embeddedness passes to closed subspaces [15, Theorem III.1.6], we get the following interesting particular case

Corollary 3.11. *Every closed subspace of c_0 with monotone Schauder basis has property A^k.*

Compare this result with the example of a closed subspace of c_0 with Schauder basis failing property A^k (Corollary 2.4). It is an interesting question whether Corollary 3.11 extends to every closed subspace of c_0 with the metric approximation property.

We next discuss on range spaces. We first observe that Question 3.4 is equivalent to whether every Banach space with the approximation property have property B^k. Also, it would be interesting to characterize those Banach spaces having property B^k in every equivalent renorming. Here, Theorem 2.6 gives a necessary condition in the separable case (actually, when the space admits a strictly convex equivalent norm).

Corollary 3.12. *Let Y be a separable Banach space satisfying property B^k in every equivalent norm. Then Y has the approximation property.*

On the other hand, property B^k holds for each Banach space with the approximation property satifying that all its finite-dimensional subspaces have Lindenstrauss property B. This is the case of the so-called polyhedral spaces. We recall that a real Banach space is said to be *polyhedral* if the unit balls of all of its finite-dimensional subspaces are polyhedra (i.e. the convex hull of finitely many points). A typical example of polyhedral space is c_0 and hence, so are its closed subspaces. We refer to [11] for background on polyhedral spaces.

Proposition 3.13. *If Y is a polyhedral Banach space with the approximation property, then Y has property B^k. In particular, every closed subspace of the real space c_0 with the approximation property has property B^k.*

Proof. Indeed, let X be a Banach space and take $T \in K(X, Y)$. Since Y has the approximation property, T can be approximated by finite-rank operators. As Y is polyhedral, its finite-dimensional subspaces have polyhedral balls, so they have property B^k by a result of Lindenstrauss [20, Proposition 3]. This shows that every finite-rank operator from X into Y can be approximated by norm-attaining operators, finishing the proof. □

To deal with the complex case, we observe that polyhedrality is equivalent to the fact that the norm of each finite-dimensional subspace can be calculated as the maximum of the absolute value of finitely many functionals. With this idea, and the fact that this is what was used by Lindenstrauss to get property B, the proof above can be extended to the complex case. It is easy to see that closed subspaces of c_0 satisfy this condition.

Proposition 3.14. *Let X be a complex Banach space such that for every finite-dimensional subspace, the norm of the subspace can be calculated as the maximum of the modulus of finitely many functionals. If besides X has the approximation property, then X has property B^k. In particular, every closed subspace of the complex space c_0 with the approximation property has property B^k.*

We may replace polyhedrality in the results above by an stronger form of the approximation property, as it was done in [19, Lemma 3.4]: *a Banach space Y has property B^k provided that there is a net of projections $\{Q_\beta\}$ in Y with $\sup_\beta \|Q_\beta\| < \infty$, converging to Id_Y in the strong operator topology, and such that $Q_\beta(Y)$ has property B^k for every β.* This was used in [19] to show that (real or complex) isometric preduals of $L_1(\mu)$ spaces have property B^k (by a classical result of A. Lazar and J. Lindenstrauss, the projections Q_β can be chosen to have $\|Q_\beta\| = 1$ and $Q_\beta(Y) \equiv \ell_\infty^{n_\beta}$) and also to real $L_1(\mu)$ spaces (here, we may take $Q_\beta(Y) \equiv \ell_1^{n_\beta}$, so the result is only know to be valid in the real case). We do not know of other situations in which the idea above can be applied.

To finish the section, we present a result on the stability of properties A^k and B^k by, respectively, ℓ_1-sums and c_0- or ℓ_∞-sums. The proof is just an adaptation of the corresponding ones for Lindenstrauss properties A and B given in [4, Proposition 3] and [22, Lemma 2], taking into account that when one starts with compact operators, the resulting norm-attaining operators are also compact.

Proposition 3.15. *Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be non-empty families of Banach spaces. Let X denotes the ℓ_1-sum of the family $\{X_i\}$ and let Y denotes the c_0- or ℓ_∞-sum of the family $\{Y_j\}$. Then*

(a) *X has property A^k if and only if X_i does for every $i \in I$.*
(b) *Y has property B^k if and only if Y_j does for every $j \in J$.*

It would be interesting to study other stability results of this kind for some vector-valued function spaces.

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