Tesis Doctoral

Universidad de Granada

[OPERADORES QUE PRESERVAN ORTOGONALIDAD Y HOMOMORFISMOS TERNARIOS]

PROGRAMA OFICIAL DE DOCTORADO EN FÍSICA Y MATEMÁTICAS

FisyMat

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO

Memoria presentada por el

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[ORTOGONALITY PRESERVING OPERATORS AND TERNARY HOMOMORPHISMS]

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DEPARTAMENT OF MATHEMATICAL ANALYSIS

A dissertation submitted by **Jorge José Garcés Pérez,** in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics.

Supervisor: Dr. Antonio M. Peralta

El doctorando D. Jorge José Garcés Pérez y el director de la tesis D. Antonio Miguel Peralta Pereira, Garantizamos, al firmar esta tesis doctoral, que el trabajo ha sido realizado por el doctorando bajo la dirección de los directores de la tesis y hasta donde nuestro conocimiento alcanza, en la realización del trabajo, se han respetado los derechos de otros autores a ser citados, cuando se han utilizado sus resultados o publicaciones.

Granada, 10 de Mayo de 2013

Director de la Tesis　　　　Doctorando

Fdo.: Antonio M. Peralta Pereira　　Fdo.:Jorge José Garcés Pérez

"Verba volant, scripta manent."

— Caio Titus, Roman orator.

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Prólogo

De acuerdo con las normas reguladoras de las enseñanzas oficiales de Doctorado y del Título de Doctor por la Universidad de Granada, aprobadas por Consejo de Gobierno de la Universidad de Granada en su sesión del 2 de Mayo del 2012, la tesis doctoral "puede consistir en el reagrupamiento en una memoria de trabajos de investigación publicados por el doctorando en medios científicos relevantes en su ámbito de conocimiento".

Los artículos elegidos para la memoria deben haber sido publicados o aceptados para su publicación en fecha posterior a la obtención del título de grado y de master universitario. La presente memoria ha sido realizada como compilación de 9 artículos. Todas las publicaciones incluidas en esta memoria han aparecido en revistas de relevancia internacional en el ámbito del Análisis Matemático, referenciadas en la última relación publicada por el Journal of Citations Reports e incluidas en las bases de datos MathSciNet (American Mathematical Society) y Zentralblatt für Mathematik (European Mathematical Society).

Esta memoria ha sido presentada por D. Jorge José Garcés Pérez para optar al título de Doctor en Matemáticas por la Universidad de Granada dentro del programa oficial de doctorado en Física y Matemáticas (FisyMat). Para poder optar a la mención internacional en el título de doctor, la mayor parte de esta memoria está escrita en inglés, idioma que actualmente es el mayoritario para la comunicación científica en el ámbito de las matemáticas. Al redactarse la tesis en una lengua no oficial, incluimos en el primer capítulo un amplio resumen en español. Los capítulos posteriores (escritos en inglés) incluyen (aunque no separadamente) una introducción, los objetivos propuestos, un resumen de los resultados y conclusiones obtenidas, así como la bibliografía utilizada.

Dado el gran número de conceptos y resultados previos que se han de introducir, en lugar de presentarlos todos en un único capítulo, hemos decidido incluir cada uno de ellos justo en el momento en que sea necesario.

Los resultados presentados en esta memoria han sido obtenidos a lo largo de los últimos cinco años bajo la supervisión del Dr. Antonio M. Peralta Pereira en el Departamento de Análisis Matemático de la Universidad de Granada. En este tiempo el doctorado ha sido alumno del Master y del Programa Oficial de Doctorado en Física y Matemáticas (FisyMat); desde Septiembre de 2009 ha disfrutado de una beca de investigación asociada al Proyecto de Excelencia “Aproximación algebraico-analítica de los sistemas no-asociativos y sus aplicaciones FQM-3737”, financiado por la Junta de Andalucía. Entre Septiembre y Diciembre de 2012 el doctorando realizó una estancia de investigación en el Departamento de Matemáticas de la Universidad de Reading (Reino Unido).

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Jorge José Garcés Pérez

Capítulo 1

Introducción

El Capítulo 2 de esta memoria está dedicado a introducir las estructuras algebraico-topológicas en las que llevamos a cabo nuestro trabajo: las C*-álgebras, las JB*-álgebras y los JB*-triples. En dicho Capítulo damos las nociones, resultados y referencias básicas de la teoría de C*-álgebras, JB*-álgebras y JB*-triples.

En el Capítulo 3 hacemos un recorrido histórico por los resultados que han motivado nuestra investigación, desde la década de 1930 hasta nuestros días. El objetivo no es otro que motivar el interés de los problemas que han sido objeto de estudio en esta tesis.

Uno de los ejes principales de esta memoria es el concepto de *ortogonalidad*, y más concretamente, el estudio de los *operadores que preservan ortogonalidad*. Cuatro de los capítulos de esta memoria y muchos de los problemas abiertos presentados en el último capítulo están dedicados al estudio de problemas relacionados con las aplicaciones lineales que preservan ortogonalidad.

El libro "*Théorie des opérations linéaires*" [17], de S. Banach, marca el inicio del Análisis Funcional. En él se define por primera vez el concepto de espacio de Banach y se prueban algunos de los teoremas fundamentales del Análisis Funcional. Por lo que a nuestro trabajo concierne, destacamos los resultados sobre isometrías lineales sobreyectivas entre varios espacios de Banach clásicos, como son los espacios de funciones continuas ($C(K)$-espacios) y los espacios $L^p([0,1])$.

Existe cierto consenso en situar en estos trabajos de Banach (más tarde generalizados por M. Stone en [176]) sobre isometrías lineales sobreyectivas entre espacios de funciones continuas, el origen del estudio de los operadores que preservan ortogonalidad. Si bien es cierto que la propiedad de preservar ortogonalidad, no fue directamente considerada por Banach ni por Stone, la forma de las isometrías lineales sobreyectivas (esto es, un operador de composición con peso) proporciona el primer ejemplo de operador que preserva ortogonalidad. La forma más general del Teorema de Banach (que enunciamos a continuación) es conocida en la actualidad como Teorema de Banach-Stone.

Teorema 1.1.1 [Banach-Stone] *Sean K_1, K_2 espacios compactos y de Hausdorff y sea $T : C(K_1) \to C(K_2)$ una isometría lineal sobreyectiva. Entonces existen una función continua h en $C(K_2)$, con $|h(s)| = 1$, para todo $s \in K_2$, y un homeomorfismo $\varphi : K_2 \to K_1$ tales que*

$$T(f)(s) = h(s)f(\varphi(s)),$$

para cualesquiera $f \in C(K_1), s \in K_2$. $\square$

Como ya hemos mencionado, Banach también considera las isometrías sobreyectivas entre espacios $L^p([0,1])$. Curiosamente, en este caso Banach sí observa que estas aplicaciones son *separadoras*. Citando al propio Banach:

“Etant donée une rotation $y = U(x)$ de $(L^{(p)})$, oú $1 \leq p \neq 2$, autour de 0, si on a pour un couple $x_1(t), x_2(t)$ des fonctions appartenat á $(L^{(p)})$

$$x_1(t)x_2(t) = 0, \textit{persque partout dans } [0,1],$$

alors pour le couple $y_1(t), y_2(t)$, oú $y_1(t) = U(x_1)$ et $y_2 = U(x_2)$, on a également

$$y_1(t)y_2(t) = 0, \textit{persque partout dans } [0,1].''$$

Sean A, B dos $C(K)$-espacios (o espacios $L^p([0,1])$) y sea $T : A \to B$ una aplicación lineal. Diremos que T es *separadora* si satisface la propiedad

$$fg = 0 \Longrightarrow T(f)T(g) = 0.$$

El citado párrafo de Banach afirma precisamente que una isometría sobreyectiva entre espacios $L^p([0,1])$ es separadora. Del mismo modo, el Teorema de Banach-Stone implica que toda isometría lineal sobreyectiva entre espacios $C(K)$ es separadora.

Definición 1.1.2 *Sean K_1, K_2 espacios compactos Hausdorff y sea $T : C(K_1) \to C(K_2)$ una aplicación lineal. Diremos que T es un* operador de composición con peso, *si existe una función continua $h \in C(K_2)$ y $\varphi : K_2 \to K_1$ continua en el conjunto $\{t : h(t) \neq 0\}$ tales que*

$$T(f)(s) = h(s)f(\varphi(s)),$$

para cualesquiera $f \in C(K_1), s \in K_2$. $\square$

Toda isometría lineal sobreyectiva entre espacios $C(K)$ es un operador de composición con peso. Además, es fácil comprobar que todo operador composición con peso es una aplicación separadora. Las isometrías lineales sobreyectivas entre espacios $L^p([0,1])$ (con $1 \leq p \neq 2$) también son operadores de composición con peso (con la salvedad de extender a los $L^p([0,1])$ las definiciones dadas anteriormente). Cabe mencionar que este último resultado de Banach fue generalizado por J. Lamperti a espacios de medida con una medida σ-finita arbitraria y también para $p < 1$ en [126]. El hecho de que las isometrías sobreyectivas son separadoras es importante en las pruebas de Banach y Lamperti. Tanto es así que las aplicaciones separadoras han sido también denominadas por muchos autores como *operadores de Lamperti*.

Desde su aparición, los trabajos de Banach y Stone han inspirado a muchos autores que dedicaron sus esfuerzos a obtener teoremas de tipo Banach-Stone en ambientes más generales, como por ejemplo: los retículos de Banach, los espacios de funciones continuas vector-valuadas, las álgebras de Banach, las álgebras de Jordan-Banach o los JB*-triples (ver por ejemplo [112], [149], [187], [162], [96], [119], [65] y [66]).

En vista de las contribuciones de Banach, Stone y Lamperti, entre otras, los investigadores en varios ámbitos del Análisis Funcional notaron que la propiedad de "ser aplicación separadora" tenía una gran importancia. A partir de la década de 1970 expertos en retículos de Banach empezaron un estudio sistemático de aquellas aplicaciones lineales que tienen dicha propiedad. Recordamos que un retículo de Banach es un retículo vectorial real, $(E, \|.\|)$,con una norma completa que tiene la siguiente propiedad adicional:

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

donde $|x| = \max\{x, -x\}$. Dos elementos x, y en un retículo de Banach E son *disjuntos* (notado mediante el símbolo $x \perp y$) si $\min\{x, y\} = 0$.

Un aplicación lineal $T : E \to F$ entre retículos de Banach se dice *separadora* si $T(x) \perp T(y)$ siempre que $x \perp y$ en E. Los operadores de composición con peso (definidos apropiadamente dependiendo del ambiente en el que se trabaje) son el prototipo de aplicaciones lineales separadoras. Así, los investigadores se preguntaron si toda aplicación lineal (y continua) entre retículos de Banach que es separadora se puede representar como un operador de composición con peso. Y.A. Abramovich, A.I. Veksler y A.V. Koldunov prueban en [2] que este es el caso, entre otros, cuando la T es biyectiva, separadora y su inversa también lo es.

Otro importante problema es el estudio de la continuidad automática, esto es, si bajo ciertas hipótesis se puede probar que una aplicación separadora es continua. Abramovich, Veksler y Koldunov ya probaron en [2] que si T es biyectiva, separadora y T^{-1} es separadora, entonces T es continua. En vista de este resultado surge la pregunta de si se pueden relajar un poco las hipótesis sobre T, por ejemplo, exigiendo solamente que ésta sea biyectiva y preserve ortogonalidad para obtener su continuidad de forma automática. Otra cuestión que surge de manera natural es si, en este caso, se puede demostrar que también T^{-1} es separadora. Como veremos a lo largo de esta introducción, estas cuestiones han dado lugar a una vasta literatura, no sólo en el ambiente de los retículos de Banach. Destacamos que Y.A. Abramovich y A.K. Kitover dieron un ejemplo de aplicación separadora biyectiva cuya inversa no es separadora (ver [1]).

De particular importancia para nuestros intereses son los trabajos de E. Beckenstein, L. Narici y A.R. Todd sobre aplicaciones separadoras entre espacios $C(K)$ (ver [23]). En estos trabajos dichos autores introducen una herramienta de gran utilidad para el estudio de estas aplicaciones lineales: la *función soporte* asociada a un aplicación separadora. Usando esta función soporte consiguen obtener varios resultados de continuidad automática.

En [100], K. Jarosz explota estas ideas y consigue obtener una descripción general de las aplicaciones lineales separadoras entre espacios de funciones continuas. Como consecuencia, prueba que si una tal aplicación es biyectiva, entonces es automáticamente continua y un operador de composición con peso.

Teorema 1.1.3 [K. Jarosz, Canadian J., 1990] *Consideremos una aplicación lineal y separadora $T : C(K_1) \to C(K_2)$. Entonces, existen subconjuntos disjuntos dos a dos Z_1, Z_2 y Z_3 de K_2 con $K_2 = Z_1 \cup Z_2 \cup Z_3$, Z_2 abierto y Z_3 cerrado, una función acotada que no se anula y es continua $h : Z_1 \to \mathbb{C}$, y una función continua $\varphi : Z_1 \cup Z_2 \to K_1$ tales que*

$$T(f)(s) = h(s)f(\varphi(s)), \text{ para cualesquiera } f \in C(K_1), s \in Z_1$$

y $T(f)(s) = 0$, para cualesquiera $f \in C(K_1), s \in Z_3$.

Además $\varphi(Z_2)$ es finito y todos los funcionales de la forma $\delta_s T$, para algún s en Z_2, son discontinuos. □

Merece la pena destacar que si $T : C(K_1) \to C(K_2)$ es una biyección lineal separadora, entonces T^{-1} también es separadora. Los resultados de Jarosz fueron generalizados por J.S. Jeang y N.C. Wong en [103] al ambiente de los espacios $C_0(L)$ (funciones continuas en un espacio localmente compacto Hausdorff que *se anulan en infinito*).

Este tipo de problemas se puede plantear en un ambiente más general, como el de las funciones continuas vector-valuadas o las álgebras de Banach. Nosotros nos centraremos en el segundo ambiente.

Recordamos que un *álgebra de Banach* es un álgebra asociativa, A, dotada de una norma completa tal que $\|ab\| \leq \|a\|\|b\|$, para cualesquiera a, b en A.

Definición 1.1.4 *Sea $T : A \to B$ un aplicación lineal entre álgebras de Banach. Diremos que T* preserva productos cero *si $ab = 0$ implica $T(a)T(b) = 0$.*

Si A y B son espacios $C(K)$ o $C_0(L)$, entonces las aplicaciones lineales entre A y B que preservan productos cero son precisamente las separadoras.

En toda álgebra asociativa se puede definir otro producto (no asociativo, en general) llamado *producto de Jordan*, definido mediante $a \circ b = \frac{1}{2}(ab + ba)$. Una aplicación lineal $T : A \to B$ entre álgebras de Banach se dice que es un *homomorfismo de Jordan* si verifica $T(a \circ b) = T(a) \circ T(b)$, para cualesquiera a, b en A.

En esta introducción usaremos con frecuencia la palabra operador para designar a una aplicación lineal y continua. Los operadores entre álgebras de Banach que preservan productos cero han sido estudiados por muchos autores en los últimos 20 años. El prototipo de operador que preserva productos cero es un múltiplo de un homomorfismo de Jordan $S : A \to B$ por un elemento de B que verifica ciertas propiedades de conmutatividad con los elementos de la imagen de S (ver por ejemplo [42], [43], [67], [192], [185], [124] y [5]). Sin embargo, no es, en general, posible describir estos operadores. Para obtener una descripción de los mismos suelen necesitarse hipótesis adicionales sobre las álgebras de Banach en las que actúan o sobre el propio operador (típicamente sobreyectividad).

Cuando la estructura de las álgebras de Banach en las que actúan los operadores es más rica se pueden obtener mejores descripciones de los mismos. Éste es el caso de las C*-álgebras. Recordemos que una C*-álgebra es un álgebra de Banach compleja, $(A, \|.\|)$, dotada de una involución $^* : A \to A$ que satisfacen la condición (conocida como axioma de Gelfand-Naimark):

$$\|aa^*\| = \|a\|^2 \quad (a \in A).$$

Dado en elemento a de una C*-álgebra, diremos que es autoadjunto si $a = a^*$. Denotaremos por A_{sa} al conjunto de los elemento autoadjuntos de A.

Sea A una C*-álgebra y sean a, b elementos de A. Diremos que a y b son *ortogonales*, y lo denotaremos con el símbolo $a \perp b$, si $ab^* = b^*a = 0$.

Definición 1.1.5 *Una aplicación lineal $T : A \to B$ entre C*-algebras* preserva ortogonalidad *cuando $a \perp b$ implica que $T(a) \perp T(b)$.*

En virtud del Teorema de Gelfand conmutativo toda C*-álgebra abeliana es *-isomorfa a un espacio $C_0(L)$, para un cierto espacio topológico localmente compacto Hausdorff L (compacto si ésta es unital). Teniendo en cuenta esto último y el hecho de que la ortogonalidad y el producto cero en una C*-álgebra abeliana coinciden, los resultados de Jarosz, y Jeang-Wong permiten describir los operadores que preservan ortogonalidad entre C*-álgebras abelianas.

Teorema 1.1.6 *Sea $T : A \to B$ un operador que preserva ortogonalidad entre C*-álgebras abelianas. Entonces existen un homomorfismo de Jordan $S : A \to B$ y un elemento h en B tales que $T = hS$.* $\square$

El Capítulo 3 de esta memoria está dedicado a la descripción de los operadores que preservan ortogonalidad entre C*-álgebras (no necesariamente abelianas).

En 1994, M. Wolff estudia aquellos operadores entre C*-álgebras (unitales) que preservan ortogonalidad y son *simétricos*, esto es, verifican la identidad adicional $T(a^*) = T(a)^*$, para todo a en A. Llamaremos **-homomorfismos de Jordan* a aquellos homomorfismos de Jordan que además son simétricos, en el sentido que acabamos de definir.

En [183], Wolff obtiene la siguiente descripción de estos operadores:

Teorema 1.1.7 [M. Wolff, Arch. Math., 1994] *Sean A, B C*-álgebras unitales, $T : A \to B$ un operador simétrico que preserva ortogonalidad y sea $h = T(1)$. Entonces h conmuta con todos los elementos de $T(A)$ y existe un *-homomorfismo de Jordan $S : A \to B^{**}$ tal que $T = hS$.* □

Conviene señalar que hemos reformulado el resultado de Wolff para no tener que entrar en demasiado detalle.

Los resultados de Wolff fueron generalizados por M.A. Chebotar, W.F. Ke, P.H. Lee y N.C. Wong en [42, Theorem 4.6]. Destacamos que un operador linear y simétrico que preserva productos cero preserva también ortogonalidad. Sin embargo, si el operador no es simétrico ésto deja de ser cierto (en general). Así, para generalizar los resultados de Wolff existen dos opciones: bien considerar operadores que preservan productos cero, o bien considerar operadores que preservan ortogonalidad. En [42] Chebotar, Ke, Lee y Wong optan por la primera opción. En este trabajo consideran operadores (no necesariamente simétricos) entre C*-álgebras que preservan productos cero. En dicho artículo, consiguen dar una descripción similar a la de Wolff bajo ciertas hipótesis adicionales (como sobreyectividad). Sin embargo, como ellos mismos observan, "una descripción de estos operador como múltiplos de un homomorfismo de Jordan no es, en general, posible" (ver [42, Ejemplo 4.8]).

En una C*-álgebra A podemos definir un *producto triple* $\{.,.,.\} : A \times A \times A \to A$ mediante la expresión $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$.

La ortogonalidad en una C*-álgebra se puede caracterizar en términos de este producto triple. Efectivamente, dos elementos a, b en A son ortogonales si, y sólo si, $\{a,b,c\} = 0$, para todo c en A (cf. [34, Lemma 1]).

Sea $T : A \to B$ un operador entre C*-álgebras. Diremos que T es un *triple homomorfismo*, si T preserva el producto triple, esto es, si $T(\{a,b,c\}) = \{T(a), T(b), T(c)\}$ para cualesquiera a, b, c en A.

A un elemento u de una C*-álgebra que verifique $\{u,u,u\} = u$ se le denomina *isometría parcial*. En base a lo antes mencionado, es claro que todo triple homomorfismo preserva ortogonalidad. Además, se puede comprobar que el elemento $h = T^{**}(1)$ verifica $\{h,h,h\} = h$ (esto es, h es una isometría parcial).

El recíproco de este enunciado se debe a N.C. Wong (ver [184]).

Teorema 1.1.8 [N.C. Wong, Southeast J. Asian Bull. Math., 2005] *Un operador $T : A \to B$ entre C*-algebras es un triple homomorfismo si, y sólo si, preserva ortogonalidad y $T^{**}(1)$ es una isometría parcial.* $\square$

El problema de dar una descripción general de los operadores que preservan ortogonalidad entre C*-algebras permaneció abierto hasta 2008. En este año, y en colaboración con los profesores M. Burgos, F.J. Fernández-Polo, J. Martínez y A. M. Peralta conseguimos en [34] determinar los operadores que preservan ortogonalidad entre C*-álgebras, sin más hipótesis que la continuidad. Para ello resultan de gran utilidad herramientas como las formas sesquilineales ortogonales o los polinomios ortogonalmente aditivos.

Recordemos que una forma sesquilineal $\Phi : A \times A \to \mathbb{C}$ sobre una C*-álgebra es llamada *ortogonal* si $\Phi(a,b) = 0$ para todo a, b en A con $a \perp b$. Una descripción general de estas formas fue obtenida por S. Goldstein en [83].

Teorema 1.1.9 [S. Goldstein, J. Funct. An., 1993] *Sea A una C*-álgebra y sea $\Phi : A \times A \to \mathbb{C}$ una forma sesquilinear ortogonal y continua. Entonces existen ψ_1, ψ_2 en A^* tales que*

$$\Phi(a,b) = \psi_1(ab^*) + \psi_2(b^*a),$$

para cualesquiera a, b en A. □

Sean A una C*-álgebra y X un espacio de Banach. Por un *polinomio n-homogéneo* X-valuado entenderemos una aplicación X-valuada y continua $P : A \to X$ tal que existe un operador n-lineal $T : A \times \ldots \times A \to X$ que satisface $P(x) = T(x, \ldots, x)$, para todo x en A. Diremos que un polinomio n-homogéneo es *ortogonalmente aditivo* (respectivamente, ortogonalmente aditivo en A_{sa}) si $P(a+b) = P(a) + P(b)$ siempre que $a \perp b$ en A (respectivamente, en A_{sa}).

Los polinomios n-homogéneos ortogonalmente aditivos fueron en primer lugar estudiados en el ambiente de los retículos de Banach por Y. Benyamini, S. Lassalle y J.G. Llavona y en el de las C*-álgebras abelianas (i.e. $C(K)$ espacios) por D. Pérez e I. Villanueva (ver [25] y [155], respectivamente).

La descripción de Pérez y Villanueva fue generalizada para C*-álgebras no necesariamente abelianas por C. Palazuelos, A.M. Peralta e I. Villanueva en [148].

Teorema 1.1.10 [C. Palazuelos, A.M. Peralta, I. Villanueva, Quart. J. Math. Oxford, 2008] *Sean A una C*-álgebra, X un espacio de Banach y $P : A \to X$ un polinomio n-homogéneo* *ortogonalmente aditivo. Entonces existe un operador $F : A \to X$ tal que*

$$P(a) = F(a^n),$$

para todo a en A. □

Otra de las herramientas fundamentales que nos permite obtener una descripción completa de los operadores que preservan ortogonalidad entre C*-álgebras es la estructura de *JB*-triple* asociada, de forma natural, a toda C*-álgebra.

Recordemos que un álgebra de Jordan es un álgebra (no necesariamente asociativa), $(J, \circ)$, cuyo producto es conmutativo y verifica la propiedad

$$a \circ (a^2 \circ b) = a^2 (a \circ b),$$

para cualesquiera a, b en J. Una *JB*-álgebra* es un álgebra de Jordan J dotada de una involución y una norma completa que satisfacen los axiomas:

$$\|a \circ b\| \leq \|a\| \|b\| \text{ y } \|U_a(a^*)\| = \|a\|^3,$$

para cualesquiera a, b en J (donde $U_a(b) = 2a(a \circ b) - b \circ a^2$).

Las C*-álgebras y las JB*-álgebras (complejas) pertenecen a una clase más general de espacios de Banach, conocidos como JB*-triples.

Recordemos que *sistema triple de Jordan normado* (o simplemente triple normado) es un espacio vectorial (real o complejo) normado, E, dotado de un producto triple $\{.,.,.\} : E \times E \times E \to E$, que es lineal y simétrico en las variables exteriores y conjugado lineal en la variable interior (trilineal si E es un espacio vectorial real) que es norma-continuo y además satisface la llamada identidad de Jordan:

$$L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y),$$

donde $L(a,b)$ es el operador en E que viene dado por $L(a,b)x = \{a,b,x\}$. Si además E es completo entonces se dice que E es un *sistema triple de Jordan-Banach*.

Un *tripotente* en un triple de Jordan E es un elemento e de E verificando tal que $\{e,e,e\} = e$. Todo tripotente da lugar a una descomposición de E, conocida como *descomposición de Peirce* de E asociada a e, esto es,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

donde para cada $i = 0, 1, 2$, $E_i(e)$ es el espacio propio asociado al valor propio $\frac{i}{2}$ del operador $L(e,e)$. Los espacios $E_i(e), i = 0, 1, 2$ son conocidos como *subespacios de Peirce* asociados al tripotente e.

El espacio de Peirce $E_2(e)$ puede ser dotado de estructura de algebra de Jordan con el producto $x \bullet_e y := \{x, e, y\}$. Además la aplicación $x^{\sharp_e} := \{e, x, e\}$ es una involución en $E_2(e)$.

Un JB*-triple es un sistema triple de Jordan-Banach complejo E, que satisface los axiomas adicionales:

(a) $L(a,a)$ es hermítico con espectro no negativo,

(b) $\|L(a,a)\| = \|a\|^2$,

para todo a en A.

Dado un elemento a de un JB*-triple E, existe un

Si E es un JB*-triple y e un tripotente de E, entonces $E_2(e)$ es una JB*-álgebra con el producto e involución definidos anteriormente (cf. [29]).

Algunos ejemplos particulares de JB*-triples fueron inicialmente estudiados en trabajos precursores de O. Loos y K. McCrimmon (ver [136]) y L.A. Harris en [90]. Sin embargo, la definición general de JB*-triple fue introducida por W. Kaup en [119].

En dicho trabajo, Kaup prueba que la categoría de los JB*-triples es equivalente a la de los *dominios simétricos acotados en espacios de Banach complejos*.

Aunque la motivación inicial para introducir los JB*-triples fue el estudio de la holomorfía en dimensión infinita. Estas estructuras algebraico-topológicas rápidamente cobraron relevancia por sí mismas y empezaron a ser estudiadas desde el punto de vista del Análisis Funcional y el álgebra.

Toda C*-álgebra (respectivamente, toda JB*-álgebra) es un JB*-triple para el producto

$$\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*a)$$

(respectivamente, $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

Una de las ventajas de utilizar la estructura de JB*-triple en una C*-álgebra es la teoría local de JB*-triples. Recordemos que si A es una C*-álgebra y a un elemento normal de A, entonces la C*-subálgebra de A generada por a es *-isomorfa a un espacio $C_0(L)$, para un cierto espacio topológico localmente compacto Hausdorff L. Este hecho es utilizado por M. Wolff para probar su descripción de los operadores simétricos que preservan ortogonalidad (entre C*-álgebras unitales). El hecho de que, en general, la C*-subálgebra generada por un elemento no necesariamente simétrico de una C*-álgebra no pueda ser descrita como un espacio $C_0(L)$ dificulta el estudio de los operadores que preservan ortogonalidad cuando $T(1)$ no es simétrico. La teoría local es más satisfactoria cuando consideramos *JB*-subtriples* en lugar de C*-subálgebras.

Sea E un JB*-triple e I un subespacio de I. Diremos que I es un *subtriple* de E si $\{I, I, I\} \subseteq I$. Dado un elemento x en E, el subtriple generado por x, E_x, es el menor subtriple norma-cerrado de E que contiene a x. El subtriple generado por un elemento de un JB*-triple siempre se puede identificar con un espacio $C_0(L)$, para un cierto espacio topológico localmente compacto Hausdorff $L \subseteq [0, \|x\|]$, tal que $L \cup \{0\}$ es compacto (ver [118, 4.8], [119, 1.15] y [70]).

Este hecho permite definir un *cálculo funcional triple* en todo elemento de un JB*-triples. Así, dado un elemento x de un JB*-triple E, existe un único elemento $y \in E_x$ que satisface $\{y, y, y\} = x$. El elemento y, que denotaremos $x^{[\frac{1}{3}]}$, es denominado *raíz cúbica* de x. Definimos inductivamente, $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. La sucesión $(x^{[\frac{1}{3^n}]})$ converge en la topología débil* de E^{**} a un tripotente que denotaremos por $r(x)$ y llamaremos el *tripotente rango* de x. El tripotente rango $r(x)$ es el menor tripotente $e \in E^{**}$ tal que x es positivo en la JBW*-álgebra $E_2^{**}(e)$ (ver [56, Lemma 3.3]).

Dos elementos a y b de un JB*-triple son llamados *ortogonales* si $L(a, b) = 0$. Como ya hemos mencionado con anterioridad, si A es una C*-álgebra entonces $ab^* = b^*a = 0$ si, y sólo si, $L(a, b) = 0$. Es decir, el concepto de ortogonalidad en una C*-álgebra coincide con el que ésta hereda de su estructura de JB*-triple.

Sea J un álgebra de Jordan y a en J. Definimos el operador de multiplicación $M_a : J \to J$ mediante $M_a(b) = a \circ b$. Diremos que dos elementos a, b de J *conmutan como operadores* si $M_a M_b = M_b M_a$.

Resultados sobre formas sesquilineales ortogonales y polinomios ortogonalmente aditivos, así como la teoría de JB*-triples fueron herramientas cruciales gracias a las que fuimos capaces de describir los operadores que preservan ortogonalidad en C*-álgebras en [34]. La caracterización que a continuación presentamos generaliza los resultados de Wolff y Wong mencionados anteriormente.

Teorema 1.1.11 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Applic., 2008] *Sea $T : A \to B$ un operador que preserva ortogonalidad entre dos C^*-álgebras y sea $h = T^{**}(1)$. Entonces*

a) $h^*T(z) = T(z^*)^*h, hT(z^*)^* = T(z)h^*$,

b) $r(h)^*T(z) = T(z^*)^*r(h)$, *y* $r(h)T(z^*)^* = T(z)r(h)^*$.

Además, existe un triple homomorfismo $S : A \to B^{**}$ tal que

$$T(z) = L(h, r(h))S(z) = \frac{1}{2}(hr(h)^*S(z) + S(z)r(h)^*h)$$

para todo $a \in A$. □

Sea $T : E \to F$ una aplicación lineal entre JB*-triples. Diremos que T preserva triples productos cero si para $x, y, z \in E$, $\{x, y, z\} = 0$ implica $\{T(x), T(y), T(z)\} = 0$. Es claro que todo operador que preserva productos triples cero preserva también ortogonalidad. Recíprocamente, si el operador actúa entre C*-álgebras el Teorema 1.1.11 garantiza que T preserva también productos triples cero.

Corolario 1.1.12 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Applic., 2008] *Sea $T : A \to B$ un operador entre dos C^*-álgebras. Entonces T preserva ortogonalidad si, y sólo si, T preserva productos triples cero.* □

Es fácil comprobar que para que un operador entre álgebras de Banach sea un homomorfismo de Jordan, es suficiente que éste preserve cuadrados (cuadrados de elementos simétricos si se trata de C*-álgebras o JB*-álgebras). El lector podría preguntarse si un resultado similar es cierto para triples homomorfismos entre

C*-álgebras. En la prueba del Teorema 1.1.8 N.C. Wong afirma que esto es cierto, aunque no da una referencia de este hecho. En [34] demostramos esta afirmación. Conviene destacar que para probar este resultado, el mero uso de identidades algebraicas parece no ser suficiente (contrariamente a lo que ocurre con los homomorfismos de Jordan).

Corolario 1.1.13 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Applic., 2008] *Sean A una C*-algebra, E un JB*-triple y $T : A \to E$ un operador. Las siguientes afirmaciones son equivalentes:*

1. *T es un triple homomorfismo.*
2. *$T(\{a, a, a\}) = \{T(a), T(a), T(a)\}$, para todo a en A_{sa}.*
3. *T preserva ortogonalidad en A_{sa} y $T^{**}(1)$ es una isometría parcial.* $\square$

En [34] también estudiamos los operadores que preservan ortogonalidad entre una JB*-álgebra y un JB*-triple y conseguimos describirlos asumiendo algunas hipótesis adicionales sobre el elemento $T^{**}(1)$. Sin embargo, una descripción general quedaría como problema abierto.

Un poco más tarde, en colaboración con M. Burgos, F.J. Fernández-Polo y A.M. Peralta, resolvimos el problema general en [35]. En este trabajo demostramos además que el uso del álgebra de multiplicadores permite asumir, en el estudio de polinomios n-homogéneos ortogonalmente aditivos u operadores que preservan ortogonalidad, que el álgebra de partida es unital. Gracias a esto damos una prueba simplificada de los resultado de Palazuelos, Peralta y Villanueva sobre polinomios ortogonalmente aditivos.

En cuanto a los operadores que preservan ortogonalidad, conseguimos generalizar la descripción al ambiente de las JB*-álgebras con el siguiente resultado:

Teorema 1.1.14 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Applic., 2008] *Sea $T : J \to E$ un operador de una JB*-álgebra J y un JB*-triple E y sea $h = T^{**}(1)$. Las siguientes afirmaciones son equivalentes:*

a) *T preserva ortogonalidad.*

b) *Existe un *-homomorfismo de Jordan unital $S : M(J) \to E_2^{**}(r(h))$ tal que $S(x)$ y h conmutan como operadores y*

$$T(x) = \{h, r(h), S(x)\} = h \bullet_{r(h)} S(x),$$

para todo $x \in J$. □

Es claro que como consecuencia del teorema anterior todo operador que preserva ortogonalidad preserva productos triples cero.

Corolario 1.1.15 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Applic., 2008] *Sea $T : J \to E$ un operador de una JB*-álgebra J en un JB*-triple E. Entonces T preserva ortogonalidad si, y sólo si, preserva productos triples cero.* □

Una vez descritos los operadores que preserva ortogonalidad nos interesamos en problemas de continuidad automática. El Capítulo 4 de esta memoria está dedicado a exponer varios resultados de continuidad automática para aplicaciones lineales que preservan ortogonalidad obtenidos en colaboración con M. Burgos y A.M. Peralta en [38] y [39].

Como ya hemos mencionado, es conocido que en algunos ambientes toda biyección que preserva ortogonalidad o es separadora y cuya inversa tiene la misma propiedad (llamadas aplicaciones que preservan ortogonalidad en ambos sentidos o biseparadoras, respectivamente), es automáticamente continua. En algunos casos, como los espacios $C(K)$, es suficiente que la aplicación lineal preserve ortogonalidad y sea biyectiva, como demostró K. Jarosz en [100]. Estos resultados han dado lugar a la conjetura que afirma que toda aplicación que preserva ortogonalidad en ambos sentidos o es biseparadora (considerando el concepto de ortogonalidad adecuado al ambiente en que se trabaje) debe ser automáticamente continua. Esta conjetura ha sido estudiada y confirmada en muchos casos particulares.

En [12], J. Araujo y K. Jarosz demuestran que toda aplicación biseparadora entre álgebras estándar de operadores (esto es, subálgebras de $L(X)$ que contienen a los operadores de rango finito y la identidad, siendo X un espacio de Banach) es automáticamente continua. En dicho artículo Araujo y Jarosz conjeturan que toda aplicación biseparadora entre C*-álgebras es automáticamente continua.

Sea $T : A \to B$ una aplicación lineal entre C*-álgebras. Diremos que T *preserva ortogonalidad en ambas direcciones* si tiene la propiedad

$$a \perp b \Longleftrightarrow T(a) \perp T(b).$$

La pregunta es, claro está, si toda biyección lineal que preserva ortogonalidad en ambas direcciones (ya sea entre C*-álgebras, JB*-álgebra o JB*-triples) es automáticamente continua. Es fácil comprobar que toda aplicación lineal que preserva ortogonalidad en ambas direcciones es inyectiva, así que esta hipótesis es de hecho superflua.

En [38] estudiamos aquellas aplicaciones entre C*-álgebras que preservan ortogonalidad en ambos sentidos.

Sean A un álgebra de Banach y a un elemento de A. Decimos que a es *compacto* si el operador $x \mapsto axa$ es un operador compacto. Un álgebra de Banach es *compacta* si todos sus elementos son compactos. Las *C*-álgebras compactas* fueron descritas por J.C. Alexander en [7] en la forma que exponemos a continuación. Dado un espacio de Hilbert complejo, denotamos por $K(H)$ al espacio de los operadores compactos en H. Si A es una C*-álgebra compacta, entonces existe una familia de espacios de Hilbert complejos (H_λ) tal que $A \cong \bigoplus_\lambda^{c_0} K(H_\lambda)$.

En [38] probamos que toda aplicación lineal entre C*-álgebras compactas que preserva ortogonalidad en ambas direcciones y es sobreyectiva es automáticamente continua.

Teorema 1.1.16 [M. Burgos, J. Garcés, A.M. Peralta, J. Math. Ann. Appl., 2010] *Toda aplicación lineal y sobreyectiva entre C*-álgebras compactas que preserva ortogonalidad en ambas direcciones es continua.* □

Ejemplos de C*-álgebras en las que todo elemento puede escribirse como una combinación lineal finita proyecciones ha sido descritos en [82], [138], [139] y[150]. Sorprendentemente, toda aplicación lineal que preserva ortogonalidad desde una de estas C*-álgebras (siempre que ésta sea unital) en otra C*-álgebra cualquiera es automáticamente continua.

Teorema 1.1.17 [M. Burgos, J. Garcés, A.M. Peralta, J. Math. Ann. Appl., 2010] *Sea A una C*-álgebra unital en la que todo elemento puede expresarse como una combinación lineal finita de proyecciones. Entonces toda aplicación lineal desde A en otra C*-álgebra que preserva ortogonalidad es continua.* □

Recordemos que un *álgebra de von Neumann* es una C*-álgebra que es un espacio de Banach dual. Es bien conocido que toda álgebra de von Neumann es unital.

Dos proyecciones p, q en un álgebra de von Neumann A son *Murray-von Neumann equivalentes* si existe una isometría parcial $u \in A$ tal que $u^*u = p$ and $uu^* = q$. Denotaremos este hecho por $p \sim q$. Si en cambio p es equivalente a una proyección $q_1 \leq q$, entonces escribiremos $p \lesssim q$.

Diremos que una proyección q es *finita* si $p \sim q \leq p$ implica $p = q$. Un álgebra de von Neumann es finita si su unidad lo es.

Proposition 1.1.1 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Applic., 2010] *Toda aplicación lineal y sobreyectiva entre álgebras de von Neumann, una de las cuales es finita, que preserva ortogonalidad en ambas direcciones es continua.* □

Usando el Teorema 1.1.17, la Proposición 1.1.1, la descomposición de Murray-von Neumann de un álgebra de von Neumann, así como la descripción de operadores que preservan ortogonalidad entre C*-álgebras conseguimos el siguiente resultado de continuidad automática en el ambiente de las álgebras de von Neumann:

Teorema 1.1.18 [M. Burgos, J. Garcés, A.M. Peralta, J. Math. Ann. Appl., 2010] *Todo aplicación lineal y sobreyectiva entre álgebras de von Neumann que preserva ortogonalidad en ambas direcciones es automáticamente continua.* □

Posteriormente estudiamos continuidad automática en algunos casos particulares de JB*-triples (ver [39]).

Un elemento x de un JB*-triple E se dice *débilmente compacto* si el operador $Q(x) : E \to E$, dado por $Q(x)y = \{x, y, x\}$ es débilmente compacto. Un JB*-triple es débilmente compacto si todos sus elementos son débilmente compactos. Los JB*-triples débilmente compactos fueron descritos por L. Bunce y C.H. Chu en [31]. Éstos son c_0-sumas de un tipo especial de JB*-triples llamados *JB*-triples elementales*. Un JB*-triple elemental es el espacio de los elementos débilmente compactos de algún “factor de Cartan” (ver Capítulo 4 para una descripción detallada de los mismos).

Conviene señalar que todo espacio de Hilbert complejo es un factor de Cartan (y un JB*-triple elemental). Además, su rango (el cardinal del mayor subconjunto de H en el que sus elementos son mutuamente ortogonales) es uno, así que toda aplicación lineal en H preserva ortogonalidad. Es claro que si H tiene dimensión infinita podemos encontrar una biyección lineal discontinua en H, por tanto en este caso no es cierto que toda aplicación lineal desde H en un JB*-triple que preserve ortogonalidad en ambas direcciones sea continua.

Teorema 1.1.19 [M. Burgos, J. Garcés, A.M. Peralta, Studia Math., 2011] *Toda aplicación lineal entre JB*-triples débilmente compactos (que no contengan sumandos de rango 1) que preserva ortogonalidad en ambas direcciones es continua.* □

Un JBW*-triple, esto es, un JB*-triple que es un espacio de Banach dual, es un *factor* si no contiene ideales (triples) propios débil*-cerrados. Los factores de Cartan se pueden clasificar (salvo isomorfismos) en 6 tipos diferentes (ver [122],[72] o el Capítulo 4, donde éstos se describen detalladamente).

En este trabajo también consideramos operadores que preservan ortogonalidad entre JBW*-triples atómicos. Recordemos que todo JBW*-triple atómico es una l_∞-suma de factores de Cartan (ver [72]).

Es bien sabido que el predual de $L(H)$ (donde H es un espacio de Hilbert complejo) coincide con los llamados operadores clase-traza. Un resultado de independiente interés obtenido en este trabajo es la descripción del predual de los factores de Cartan de tipo 1, 2 y 3 (ver [39, Proposition 5.1]). Un tripotente e en un JB*-triple E es llamado *minimal* si $E_2(e) \cong \mathbb{C}e$.

Proposición 1.1.20 [M. Burgos, J. Garcés, A.M. Peralta, Studia Math., 2011] *Sea C un factor de Cartan de dimensión infinita y de tipo 1, 2 ó 3. Para cada $\varphi \in C_*$, existen una sucesión $(\lambda_n) \in l_1$ y una sucesión (u_n) de tripotentes minimales mutuamente ortogonales en C tales que*

$$\|\varphi\| = \sum_{n=1}^{\infty} |\lambda_n| \quad \textit{and } \varphi(x) = \sum_n \lambda_n \varphi_n(x) \ (x \in C)$$

donde para cada $n \in \mathbb{N}$, $\varphi_n(x)u_n = P_2(u_n)(x)$ $(x \in C)$, donde $P_2(u_n)$ es la proyección de E en $E_2(u_n)$. □

Los resultados de continuidad para aplicaciones lineales entre JB*-triples débilmente compactos que preservan ortogonalidad en ambas direcciones, así como la anteriormente mencionada descripción de los preduales de los factores de Cartan de tipo 1,2 y 3 son algunas de las herramientas que nos permiten probar el siguiente resultado:

Teorema 1.1.21 [M. Burgos, J. Garcés, A.M. Peralta, Studia Math., 2011] *Toda aplicación lineal y sobreyectiva entre JBW*-triples atómicos (que no tengan sumandos de rango 1) que preserva ortogonalidad en ambas direcciones es continua.* □

El Capítulo 5 de esta memoria está dedicado al Teorema de Kaplasnsky en JB*-triples. Merece la pena destacar que este resultado además de ser importante por sí mismo, permitirá (como se expone en el Capítulo 6) obtener caracterizaciones de los triples homomorfismos débilmente compactos, claves para la descripción de los operadores que preservan ortogonalidad y son débilmente compactos.

Los antecedentes del Teorema de Kaplansky se remontan a 1940, cuando M. Eidelheit demostró que $L(X)$ (siendo X un espacio de Banach) tiene una única norma completa que lo convierte en un álgebra de Banach (ver [59]). Es en 1949 cuando I. Kaplansky obtiene el famoso resultado que lleva su nombre.

Teorema 1.1.22 [I. Kalplansky, Duke Math., 1949] *Sea* $\|.\|$ *un norma en* $C(K)$ *con la propiedad* $\|fg\| \leq \|f\|\|g\|$, *para cualesquiera* $f, g \in C(K)$. *Entonces* $\|.\|_\infty \leq \|.\|$, *donde* $\|.\|_\infty$ *denota a la norma del supremo en* $C(K)$. □

Como consecuencia del Teorema de Kaplansky, toda norma multiplicativa que sea $\|.\|_\infty$-continua, es equivalente a $\|.\|_\infty$.

Posteriormente W.G. Bade y P.C. Curtis o C.E. Rickart dan varios ejemplos de álgebras de Banach con esta propiedad (ver [16] y [159]). Uno de los resultados más importantes es el obtenido por B.E. Johnson en [105], donde prueba que toda álgebra de Banach semisimple tiene una única norma de álgebra de Banach.

Es fácil comprobar que el Teorema de Kaplansky es equivalente al siguiente enunciado: *Todo monomorfismo en* $C(K)$ *está acotado inferiormente.*

Un álgebra de Banach A, con norma $\|.\|$ tiene la *propiedad de minimalidad de la topología de norma* (MOANT), si para cualquier otra norma multiplicativa $\|.\|_2$ en A tal que $\|.\|_2 \leq \|.\|$ se tiene que $M\|.\| \leq \|.\|_2$, para algún $M > 0$. Si además $\|.\|_2 = \|.\|$, diremos que A tiene la propiedad de minimalidad de la norma.

Como consecuencia del Teorema de Kaplansky, $C(K)$ tiene la propiedad de minimalidad de la topología de la norma.

Una generalización del Teorema de Kaplansky para C*-álgebras (no necesariamente abelianas) fue obtenida por S. Clevenland en [46]. La correspondiente versión en el ámbito de las JB*-álgebras se debe a A. Bensebah [24]. Este autor además deja abierto el problema de si las JB*-álgebras tienen la propiedad de minimalidad de la norma.

Un respuesta afirmativa para esta pregunta fue dada por J. Pérez, L. Rico y A. Rodríguez-Palacios en [154] (de hecho, este resultado es probado en el ambiente más general de las JB*-álgebras no conmutativas). S. Hejazian y A. Nikman dieron también una demostración alternativa del Teorema de Kaplansky para JB*-álgebras en [92].

Sea E un sistema triple de Jordan normado con norma $\|.\|$. Diremos que E tiene la propiedad de minimalidad de la topología de la norma triple (MTNT), si para toda norma triple $\|.\|_1$ en E (esto es, para toda norma que verifique $\|\{x,y,z\}\|_1 \leq \|x\|_1\|y\|_1\|z\|_1$) tal que $\|.\|_1 \leq \|.\|$ se tiene que ésta es equivalente a la norma de E. Equivalentemente, E tiene la propiedad MT-NT si todo triple monomorfismo continuo T de E en otro triple normado está acotado inferiormente (esto es, existe $M > 0$ tal que $M\|x\| \leq \|T(x)\|, \forall x \in E$).

K. Bouhya y A. Fernández demostraron que todo JB*-triple (complejo) tiene la propiedad de minimalidad de la topología de la norma triple (ver [28]). En [62] damos una versión más general del Teorema de Kaplansky para JB*-triples eliminando algunas de las hipótesis que imponían Bouhya y Fernández y extendiendo su resultado al caso de los JB*-triples reales.

Recordemos un *JB*-triple real* es un subtriple real (es decir, un subespacio real que es además un subtriple) de un JB*-triple complejo (ver [95]).

Un *J*B-triple* es un sistema triple de Jordan Banach real cuya norma satisface la propiedad $\|\{a,a,a\}\| = \|a\|^3$ y los axiomas adicionales:

(J^*B1) $\|\{x,y,z\}\| \leq \|x\|\|y\|\|z\|$;

(J^*B2) $\sigma^{\mathbb{C}}_{L(E)}(L(x,x)) \subset [0,+\infty)$ para todo $x \in E$;

(J^*B3) $\sigma^{\mathbb{C}}_{L(E)}(L(x,y) - L(y,x)) \subset i\mathbb{R}$ para cualesquiera $x, y \in E$.

La clase de los J*B-triples incluye a la de los JB*-triples reales y complejos.

En uno de los resultados principales de [62] demostramos que todo J*B-triple tiene la propiedad MTNT. Sin embargo, recordemos que el Teorema de Kaplansky aseguraba que la norma de $C(K)$ tiene una propiedad más fuerte, y es que toda norma multiplicativa $\|.\|$ en $C(K)$ (no necesariamente $\|.\|_\infty$-continua) verifica que $M\|.\|_\infty \leq \|.\|$, para algún real positivo M. Equivalentemente, todo triple monomorfismo (no necesariamente continuo) de $C(K)$ en un triple normado está acotado inferiormente.

En [62] demostramos que los J*B-triples también tienen esta propiedad. Para ello usamos una estrategia clásica: los espacios separantes.

Teorema 1.1.23 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Proc. AMS, 2012] *Sea $T : E \to F$ un triple monomorfismo de un JB*-triple complejo o un J*B-triple real en un triple normado. Entonces T está acotado inferiormente.* □

En el Capítulo 6 de esta memoria volvemos al estudio de los operadores que preservan ortogonalidad. En este caso nos proponemos describir los operadores que preservan ortogonalidad y tienen la propiedad adicional de ser débilmente compactos.

Para estudiar los operadores débilmente compactos que preservan ortogonalidad necesitamos en primer lugar estudiar los triples homomorfismos. Es bien sabido que un homomorfismo desde una C*-álgebra es débilmente compacto si, y sólo si, su imagen tiene dimensión finita (ver [78] y [140]).

En [63] generalizamos los resultados sobre homomorfismos débilmente compactos desde una C*-álgebra al ámbito de los triples homomorfismos desde un JB*-triple. Una de las herramientas que nos permiten caracterizar los triples homomorfismos débilmente compactos desde un JB*-triple es precisamente el Teorema de Kaplansky para JB*-triples (o J*B-triples reales).

Teorema 1.1.24 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Sea T un triple homomorfismo de un JB*-triple real o complejo en un triple normado. Entonces la imagen de T es un triple normado reflexivo.* $\square$

Como consecuencia (aunque no inmediata), conseguimos demostrar que la imagen de un triple homomorfismo débilmente compacto desde una C*-álgebra es también finito dimensional. Sin embargo, existen JB*-álgebras y JB*-triples reflexivos de dimensión infinita, por tanto la imagen de un triple homomorfismo desde una JB*-álgebra o un JB*-triple no es, en general, finito dimensional.

Sea $T : A \to B$ un operador que preserva ortogonalidad entre C*-álgebras. Puesto que T es, esencialmente, un múltiplo de un triple homomorfismo, podríamos pensar que si T es débilmente compacto, entonces debería tener imagen finito dimensional. En [63] mostramos con un ejemplo que esto no es, en general, cierto.

Parece que el primero en consider los operadores débilmente compactos que preservan ortogonalidad entre C*-álgebras fue M. Wolff (cf. [183]). Aunque no consigue una descripción de éstos sí comenta al final de [183] artículo que su caracterización (ver Teorema 1.1.7) podría, en principio, usarse para determinar la forma de un operador simétrico que preserva ortogonalidad, para a continuación afirmar que esto podría ser "algo engorroso".

En el caso abeliano los operadores débilmente compactos que preservan ortogonalidad fueron descritos satisfactoriamente por Y.F. Lin y Ng.-Ch. Wong en [133].

Teorema 1.1.25 [Y.F. Lin, N.C. Wong, Math. Nachr., 2009] *Sea $T : C_0(L_1) \to C_0(L_2)$ un operador que preserva ortogonalidad. Las siguientes afirmaciones son equivalentes:*

1. *T es completamente continuo.*
2. *T es débilmente compacto.*
3. *T es compacto.*
4. *Existe una sucesión (a lo sumo numerable) $\{x_n\}$ de puntos de L_2 y una sucesión $\{h_n\}$ en $C_0(L_1)$ de funciones dos a dos ortogonales tales que*

$$Tf = \sum_n f(x_n)h_n, \text{ para toda } f \in C_0(L_1).$$

En caso de haber un número infinito de puntos $\{x_n\}$ y funciones $\{h_n\}$, entonces $\|h_n\| \to 0$. □

Es interesante mencionar que el resultado de Lin y Wong es una generalización al caso no unital de un precedente establecido por H. Kamowitz en [115].

En [63] conseguimos generalizar esta descripción al ámbito de las C*-álgebras y las JB*-álgebras. Para obtener la caracterización que a continuación presentamos, las herramientas fundamentales son los anteriormente mencionados resultados sobre triples homomorfismo débilmente compactos, el Teorema de Kaplansky para JB*-triples y la caracterización de los operadores que preservan ortogonalidad entre C*-álgebras obtenida en [34].

Teorema 1.1.26 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Sean A una C*-álgebra, E un JB*-triple, $T : A \to E$ un operador débilmente compacto que preserva ortogonalidad. Sea $r = r(h)$ el tripotente rango de $h = T^{**}(1) \in E$. Entonces existe una familia a lo sumo numerable, $\{I_n\}$, de ideales C* mutuamente ortogonales en A^{**}, una familia $\{S_n : A^{**} \to E_2^{**}(r)\}$ de *-homomorfismos de Jordan y una sucesión $\{x_n\}$ de elementos de E mutuamente ortogonales tales que:*

(a) *Cada I_n es un factor von Neumann de tipo I finito;*

(b) *$\|x_n\| \to 0$ y $h = \sum_n x_n$;*

(c) *$S_n|_{I_n}$ es un *-monomorfismo, $S_n|_{I_n^\perp} = 0$, S_n y S_m tienen imágenes ortogonales siempre que $n \neq m$;*

(d) *Para cada x en A^{**}, x_n y $S_m(x)$ conmutan como operadores, para cualesquiera n y m;*

y

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x), \qquad (1.1)$$

para todo $x \in A$. $\square$

Puesto que todo factor de von Neumann de tipo I irreducible en $C_0(L)^{**}$ es isomorfo a $\mathbb{C}$, es claro que la descripción dada por el Teorema anterior generaliza la establecida por Lin y Wong en [134]. En este trabajo Lin y Won prueban también que estos operadores factorizan a través de c_0.

Conviene destacar que, como consecuencia de nuestro resultado se puede ver que, en general, un operador débilmente compacto que preserva ortogonalidad entre C*-álgebras no necesariamente factoriza a través de c_0.

Teorema 1.1.27 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Sea T un operador que preserva ortogonalidad desde una C*-álgebra en un JB*-triple. Las siguientes afirmaciones son equivalentes:*

1. *T es compacto.*
2. *T es débilmente compacto.*
3. *T admite una factorización a través de una c_0-suma de la forma*
$$\bigoplus_{n}^{c_0} M_{m_n}(\mathbb{C}),$$
donde (m_n) es una sucesión de números naturales. □

En [63] también caracterizamos los operadores débilmente compactos que preservan ortogonalidad entre una JB*-álgebra y un JB*-triple. Una prueba similar a la del caso de las C*-álgebras, y el uso de la caracterización de los operadores que preservan ortogonalidad desde una JB*-álgebra, así como los resultados antes expuestos sobre triples homomorfismos y también el Teorema de Kaplansky para JB*-triples permiten establecer:

Teorema 1.1.28 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Sean A una JB*-álgebra, E un JB*-triple, $T : A \to E$ un operador débilmente compacto que preserva ortogonalidad y $r = r(h)$ el tripotente rango de $h = T^{**}(1) \in E$.*

Entonces existe una familia a lo sumo numerable, $\{I_n\}$, de JB*-ideales debil*-cerrados y mutuamente ortogonales en A^{**}, una familia $\{S_n : A^{**} \to E_2^{**}(r)\}$ de *-homomorfismos de Jordan y una sucesión (x_n) de elementos de E mutuamente ortogonales verificando:

(a) *Cada I_n es un factor debil*-cerrado reflexivo;*

(b) *$\|x_n\| \to 0$ y $h = \sum_n x_n$;*

(c) *$S_n|_{I_n}$ es un *-monomorfismo, $S_n|_{I_n^\perp} = 0$, S_n y S_m tienen imágenes ortogonales siempre que $n \neq m$;*

(d) *Para cada x en A^{**}, x_n y $S_m(x)$ conmutan como operadores, para cualesquiera n y m;*

y

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x),$$

para todo $x \in A$. $\square$

En este caso, tenemos que un tal operador factoriza a través de una c_0-suma de JBW*-triples factores reflexivos.

El Capítulo 8 está dedicado a los triples homomorfismos y las derivaciones generalizadas. En colaboración con A.M. Peralta estudiamos en [76] la continuidad automática de estas aplicaciones.

Sea $T : A \to B$ una aplicación lineal entre álgebras de Banach. Se dice que T es un *homomorfismo generalizado* si existe un $\varepsilon > 0$ tal que

$$\|T(ab) - T(b)T(a)\| \leq \varepsilon \|a\| \|b\|,$$

para cualesquiera a, b en A.

Los homomorfismos generalizados (también conocidos como homomorfismos aproximados o casi homomorfismos) fueron estudiados por K. Jarosz en [101]. En este trabajo Jarosz prueba que todo homomorfismo generalizado de un álgebra de Banach en un $C(K)$-espacio es automáticamente continuo.

B.E. Johnson también se interesó por los homomorfismos generalizados. En [107] estudia la continuidad automática de estas aplicaciones lineales.

Otro importante problema relacionado con estas aplicaciones es el de la estabilidad, esto es, ¿cúando un homomorfismo generalizado está cerca de un homomorfismo?. K. Jarosz y B.E. Johnson abordaron esta cuestión en [101] y [108], respectivamente (en el Capítulo 10 de esta memoria se describe más detalladamente este problema).

Sean E y F dos sistemas triples de Jordan normados y $T : E \to F$ una aplicación lineal. Diremos que T es un *triple homomorfismo generalizado* si existe un real positivo ε tal que

$$\|T\{a, b, c\} - \{T(a), T(b), T(c)\| \leq \varepsilon \|a\| \|b\| \|c\|,$$

para cualesquiera a, b, c en E.

La cuestión que nos planteamos en [76] es cuándo un triple homomorfismo generalizado es automáticamente continuo.

En primer lugar exploramos las conexiones entre los homomorfismos generalizados y los triples homomorfismos generalizados.

Sea A un álgebra de Banach. Definimos en A el producto triple (elemental) $\{a, b, c\} = \frac{1}{2}(abc + bca)$. Claramente todo homomorfismo es también un triple homomorfismo para el producto triple elemental. Es natural preguntarse si también todo homomorfismo generalizado es un triple homomorfismo generalizado. En [76] probamos que ésto es siempre cierto.

Una aplicación lineal entre *-álgebras de Banach es un *-homomorfismo generalizado si es un homomorfismo generalizado y además la aplicación lineal $a \mapsto T(a^*)^* - T(a)$ es continua. Destacamos que todo *-homomorfismo generalizado entre C*-álgebras es automáticamente continuo (ver [107]).

En una *-álgebra de Banach podemos definir el producto triple $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. En [76] también probamos que todo *-homomorfismo generalizado entre *-álgebras de Banach es un triple homomorfismo generalizado para el producto triple que acabamos de definir.

En [158, Page 70], C.E. Rickart introduce los *espacios separadores* como herramienta para el estudio de problemas relaciones continuidad automática en álgebras de Banach. Sea $T : X \to Y$ una aplicación lineal entre espacio de Banach. El *espacio separador*, $\sigma_Y(T)$, de T en Y se define como el conjunto de todos aquellos z en Y para los que existe una sucesión $(x_n) \subseteq X$ tal que $x_n \to 0$ y $T(x_n) \to z$. El espacio separador $\sigma_Y(T)$ es un subespacio de Y. Además, como consecuencia del Teorema de la gráfica cerrada T es continua si, y sólo si, $\sigma_Y(T) = \{0\}$ (c.f. [46, Proposition 4.5]).

Sea $T : A \to B$ es un homomorfismo generalizado entre álgebras de Banach, B.E. Johnson prueba en [107] que el espacio separador $\sigma_B(T)$ es un ideal de la subálgebra norma-cerrada de B generada por $T(A)$. En [76] también seguimos esta estrategia, sin embargo la dificultad para obtener el resultado análogo en ambiente triple es considerablemente más alta.

Para abordar este problema introducimos en [76] los monomios impares, que nos permiten dar una descripción más precisa del subtriple generado por un subconjunto. Gracias al uso de los monomios impares somos capaces de demostrar que, dado un triple homomorfismo generalizado $T : E \to F$, el espacio separador $\sigma_F(T)$ es un ideal (triple) del subtriple norma-cerrado de

F generado por $T(E)$.

Entre los resultados de continuidad automática que obtenemos en [76] destacamos los siguientes:

En primer lugar, demostramos que todo triple homomorfismo generalizado entre JB*-triples es automáticamente continuo. Conviene señalar que este resultado generaliza el resultado de continuidad automática de Johnson para *-homomorfismo generalizados.

El siguiente objetivo que nos planteamos es obtener una caracterización de la continuidad de los triples homomorfismos entre triples normados. Objetivo que conseguimos cuando en el dominio tenemos un JB*-triple.

Recordemos que, dado un triple normado E y $M \subseteq E$, se define el *anulador* de M en E, $Ann_E(M)$, como el conjunto

$$Ann_E(M) = \{a \in E : \{a, M, a\} = 0\}.$$

El *complemento ortogonal* the M, $M^{\perp}$, es el conjunto

$$\{a \in E : a \perp b, \forall b \in M\}.$$

Teorema 1.1.29 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Sea $T : E \to F$ un triple homomorfismo generalizado entre un JB*-triple y un triple normado y sea $J = T^{-1}(Ann_F(\sigma_F(T)))$. Equivalen:*

a) J es un ideal triple norma-cerrado de E y

$$\{Ann_F(\sigma_F(T)), Ann_F(\sigma_F(T)), \sigma_F(T)\} = 0.$$

b) T es continuo. □

Este resultado no es en general válido para JB*-triples reales salvo que estos sean "*JB*-triples reales reducidos*" (ver los comentarios posteriores al Lemma 16 en [76]).

En algunos casos concretos de JB*-triples el teorema anterior se puede mejorar considerablemente. Por ejemplo, probamos que todo triple homomorfismo generalizado desde un factor de Cartan de tipo I o un spin complejo en un triple normado anisotrópico (esto es, que no contiene elementos nilpotentes) es automáticamente continuo (ver [76, Lemma 15 y Lemma 16]).

Un importante resultado de J. Cuntz afirma que si $T : A \to X$ es una aplicación lineal de una C*-álgebra en un espacio de Banach, entonces T es continua si, y sólo si, su restricción a toda C*-subálgebra de A generada por un elemento simétrico de A es continua (ver [48]).

Sea $T : E \to X$ una aplicación lineal entre un JB*-triple y un espacio de Banach. En vista de lo probado por Cuntz, podría conjeturarse que si la restricción de T a todo subtriple de E generado por un elemento de E es continua, entonces T es continua. Lamentablemente esto no es, en general, cierto como mostramos en [76]. De hecho, este enunciado tampoco es cierto cuando, en lugar de considerar una aplicación lineal cualquiera de un JB*-triple en un espacio de Banach, consideramos un triple homomorfismo entre un JB*-triple y un triple normado.

Un ideal interno de un triple normado E es un subespacio I que verifica $\{I, E, I\} \subseteq I$. Dado un elemento a de un JB*-triple E, el *ideal interno* generado por a, $E(a)$, coincide con el cierre en norma del conjunto $\{a, E, a\}$ (ver [32, pp, 19-29]). Para evitar los contraejemplos dados al Teorema de Cuntz en JB*-triples, consideramos ideales internos generados por un elemento en lugar de subtriples generados por un elemento.

Las aplicaciones de la propiedad de factorización de Cohen para álgebras de Banach en el estudio de la continuidad automática son bien conocidas. En [76] también exploramos las consecuencias de la propiedad de factorización de Cohen en triples normados.

Sea E un triple normado. Se dice que E tiene la *propiedad de factorización de Cohen* (CFP) si para toda sucesión norma-nula (a_n) en E, existen x, y en E y una sucesión norma-nula (b_n) en E tales que $a_n = \{x, b_n, y\}, \forall n \in \mathbb{N}$.

Toda álgebra de Jordan con una identidad aproximada tiene la propiedad CFP (ver [3]). En consecuencia, las JB- y las JB*-álgebras tienen la propiedad de factorización de Cohen.

Recordemos que un sistema triple de Jordan es *anisotrópico* si dado un elemento a tal que $a^{[2n+1]}$ para algún natural n, entonces $a = 0$.

Teorema 1.1.30 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Sea $T : E \to F$ una aplicación lineal entre dos triples de Jordan-Banach y supongamos que una de las siguientes afirmaciones es cierta:*

1. *T es un triple homomorfismo generalizado y F es anisotrópico.*

2. *E tiene la propiedad de factorización de Cohen.*

Si la restricción de T a todo ideal interno norma-cerrado generado por un elemento de E es continua, entonces T es continua. □

En la sección final de [76] consideramos también las derivaciones triples generalizadas.

Una derivación en un JB*-triple es una aplicación conjugado lineal (lineal si es un JB*-triple real) tal que

$$\delta(\{a, b, c\}) = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}.$$

Las derivaciones en JB*-triples fueron inicialmente estudiadas por T. Barton y Y. Friedman en [20], donde prueban que toda derivación en un JB*-triple es continua. El resultado análogo para JB*-triples reales fue probado por T. Ho, A.M. Peralta y B. Russo en [93].

En [163], A.M. Peralta y B. Russo definen los módulos sobre un JB*-triple y estudian la continuidad de las derivaciones triples $\delta : E \to X$ de un JB*-triple E en un triple E-módulo X (ver [163], [76] o el Capítulo 7 de esta memoria para las definiciones de triple módulo y derivación triple). En el mencionado trabajo Peralta y Russo caracterizan la continuidad de las derivaciones triples. Como consecuencia de esta caracterización, demuestran que toda derivación triple de un JB*-triple (real o complejo) en sí mismo, o en su dual es automáticamente continua.

En [76] definimos las derivaciones triples generalizadas y damos una caracterización para la continuidad de las mismas. Como consecuencia, probamos que toda *derivación triple generalizada* de un JB*-triple en sí mismo, o en su dual es automáticamente continua.

Cuando el dominio es una C*-álgebra, los resultados se puede mejorar considerablemente. Efectivamente, haciendo uso del Teorema de Cuntz y de la caracterización de continuidad para derivaciones generalizadas, probamos que toda derivación triples generalizada desde una C*-álgebra A (vista como JB*-triple) en un Jordan-Banach triple A-módulo es automáticamente continua (generalizando así el resultado análogo para derivaciones obtenido previamente por Peralta y Russo en [163]).

En el Capítulo 8 de esta memoria, describimos los resultados sobre formas bilineales ortogonales y operadores que preservan ortogonalidad entre C*-álgebras reales abelianas obtenidos en colaboración con A.M. Peralta en [77].

Recordemos que una C*-álgebra real es una *-álgebra de Banach real cuya norma satisface el axioma de Gelfand-Naimark y la propiedad adicional de que $1 + aa^*$ es invertible, para todo a en A. Dada una C*-álgebra real, con complexificación B, existe un *-automorfismo conjugado lineal de cuadrado la identidad $\tau : B \to B$ tal que $A = B^\tau = \{x \in B : \tau(x) = x\}$ (ver [130, Proposition 5.1.3] o [158, Lemma 4.1.13], y [84, Corollary 15.4]).

Una vez caracterizados los operadores que preservan ortogonalidad desde una C*-álgebra o una JB*-álgebra en un JB*-triple complejo, es natural tratar de hacerlo también en el ambiente real. Hasta donde nosotros sabemos, este problema parece no haber sido considerado aún por otros autores.

Como conocemos por el caso complejo, las formas ortogonales son una herramienta muy útil para el estudio de los operadores que preservan ortogonalidad. Es por ello que resulta también natural abordar su estudio en el ambiente real.

Sean A una C*-álgebra real y $V : A \times A \to \mathbb{R}$ una forma bilineal. Diremos que V es ortogonal si $V(a, b^*) = 0$ para cualesquiera a, b en A tales que $a \perp b$. En la primer parte de [77] estudiamos la formas bilineales ortogonales sobre una C*-álgebra real. Para ello probamos primero algunos resultados previos de extensión de formas multilineales a los biduales. En particular, demostramos que toda forma multilineal en una C*-álgebra real admite una única extensión, que denotaremos por T^{**}, a A^{**} que es separadamente débil*-continua (ver [77]). Además, la restricción de T^{**} al álgebra de los multiplicadores de A, $M(A)$, es una forma bilineal ortogonal.

En una primera aproximación al problema logramos de describir la restricción de V a A_{sa} probando la existencia de un ψ en A^* tal que $V(a, b) = \psi(a \circ b)$, para cualesquiera a, b simétricos.

Una descripción general o incluso el comportamiento de V sobre los elementos antisimétricos de A permanece como problema abierto. Sin embargo, en el caso abeliano conseguimos describir satisfactoriamente cualquier forma bilineal ortogonal.

Sean A una C*-álgebra real abeliana y unital y sea $V : A \times A \to \mathbb{R}$ una forma bilineal ortogonal. Como consecuencia de la teoría de Gelfand conmutativa y el Teorema de Banach-Stone, existe un compacto K tal que $A \cong C(K)$ y un homeomorfismo $\sigma : K \to K$, con $\sigma^2(t) = t, \forall t \in K$, tales que

$$\tau(a)(t) = \overline{a(\sigma(t))},$$

para cualesquiera $a \in C(K)$ y $t \in K$.

En lo que sigue, dada un en espacio de tipo $C(K)^\tau$, usaremos el símbolo σ para el homeomorfismo $\sigma : K \to K$ dado por el Teorema de Banach-Stone aplicado a la isometría lineal sobreyectiva τ.

Nuestra estrategia consiste en extender V al álgebra de Borel de K, $B(K)$, y aprovechar la abundancia de proyecciones en esta C*-álgebra real.

Como resultado de importancia en sí mismo, obtenemos una resolución espectral para elementos antisimétricos muy útil para nuestros propósitos.

Lema 1.1.31 [J.J. Garcés, A.M. Peralta, Linear and Multilinear algebra, 2013] *Sean a, b elementos de $B(K)$, con a simétrico y b antisimétrico. Denotamos por F el conjunto de puntos fijos de σ. Entonces se verifican: :*

a) $b_{|F} = 0$;

b) *Para cada $\varepsilon > 0$, existen subconjuntos borelianos $B_1, \ldots, B_m \subseteq$ dos a dos disjuntos tales que $B_i \cap \sigma(B_i) = \emptyset, \forall i = 1, \ldots, m$,* *y números reales* $\lambda_1, \ldots, \lambda_m$ *tales que*

$$\left\| b - \sum_{j=1}^{m} i\ \lambda_j(\chi_{B_j} - \chi_{\sigma(B_j)}) \right\| < \varepsilon;$$

c) *Para cada* $\varepsilon > 0$*, existen borelianos dos a dos disjuntos* $C_1, \ldots, C_m \subset K$ *y números reales* $\mu_1, \ldots, \mu_m$ *tales que* $\sigma(C_j) = C_j$*, que verifican* $\left\| a - \sum_{j=1}^{m} \mu_j \chi_{C_j} \right\| < \varepsilon.$ □

La anunciada descripción de las formas bilineales ortogonales en C*-álgebras reales abelianas y unitales es la siguiente:

Teorema 1.1.32 [J.J. Garcés, A.M. Peralta, Linear and Multilinear algebra, 2013] *Sea* $V : A \times A \to \mathbb{R}$ *una forma bilineal y ortogonal en una C**-*álgebra real abeliana unital. Entonces existen* φ_1 *y* φ_2 *en* A^* *tales que*

$$V(x, y) = \varphi_1(xy) + \varphi_2(xy^*),$$

para todo $x, y \in A$. □

El lector podría preguntarse si habría sido posible obtener este resultado extendiendo V a la complexificación de A y aplicando el Teorema de Goldstein. En vista de la forma que tiene V, es claro que la extensión de una forma bilineal ortogonal a la complexificación de A no es, en general, una forma ortogonal.

Otro de los problemas que tratamos en [77] es el estudio de aplicaciones lineales que preservan ortogonalidad entre C*-algebras abelianas. Puesto que consideramos aplicaciones lineales no necesariamente continuas, no podemos en este aplicar los resultados sobre forma bilineales ortogonales previamente probados.

Sean $A = C(K)^\tau$ una C*-álgebra abeliana unital, $F = \{t \in K : \sigma(t) = t\}$ y $N = \{t \in K : \sigma(t) \neq t\}$. En [77] demostramos que F es cerrado y existe un abierto $\mathcal{O}$ maximal con respecto a la propiedad $\mathcal{O} \cap \sigma(\mathcal{O}) = \emptyset$. Además $N = \mathcal{O} \cup \sigma(\mathcal{O})$.

Sea $T : C(K_1)^{\tau_1} \to C(K_2)^{\tau_2}$ una aplicación lineal que preserva ortogonalidad. Definimos $L_i = F_i \cup \mathcal{O}_i$ y $C_r(L_i)$ como el conjunto de funciones continuas $f : L_i \to \mathbb{C}$ que toman valores reales en F_i. La aplicación que envía cada f en $C(K_i)^{\tau_i}$ a su restricción al conjunto L_i es un C*-isomorfismo. Esta observación permite reducir el estudio de las aplicaciones lineales que preservan ortogonalidad entre espacios $C(K)^\tau$ al estudio de las aplicaciones lineales que preservan ortogonalidad entre espacios $C_r(L)$.

Sea $T : C_r(L_1) \to C_r(L_2)$ una aplicación lineal que preserva ortogonalidad. Siguiendo las ideas de E. Beckenstein, L. Narici y A.R. Todd [23] K. y Jarosz [100] asociamos a T una *función soporte*. Esta técnica nos permite obtener una versión real del los famosos resultados de Jarosz sobre aplicaciones lineales (no necesariamente continuas) entre $C(K)$-espacios que preservan ortogonalidad.

Teorema 1.1.33 [J.J. Garcés, A.M. Peralta, Linear and Multilinear algebra, 2013] *Sea $T : C_r(L_1) \to C_r(L_2)$ una aplicación lineal que preserva ortogonalidad. Entonces L_2 descompone como unión de tres subconjuntos dos a dos disjuntos Z_1, Z_2, y Z_3, con Z_2 abierto y Z_3 cerrado, y existe una función soporte continua $\varphi : Z_1 \cup Z_2 \to L_1$, y una función acotada $T(i) : L_2 \to \mathbb{C}$ que es continua en el conjunto $\varphi^{-1}(\mathcal{O}_1)$ tales que:*

$$T(i)(s) \in \mathbb{R}, \ \forall s \in F_2,$$

$$T(i)(s) = 0, \ \forall s \in Z_3 \cup Z_2 \text{ y } \forall s \in Z_1 \text{ tal que } \varphi(s) \in F_1,$$

$$|T(1)(s)| + |T(i)(s)| \neq 0, \ (\forall s \in Z_1), \tag{1.2}$$

$$T(f)(s) = T(1)(s)\ \Re e f(\varphi(s)) + T(i)(s)\ \Im m f(\varphi(s)), \qquad (1.3)$$

para todo $s \in Z_1, f \in C_r(L_1)$, *y*

$$T(f)(s) = 0, \ (\forall s \in Z_3, f \in C_r(L_1)).$$

Además, dado $s \in L_2$, *la aplicación* $C_r(L_1) \to \mathbb{C}$, $f \mapsto T(f(s))$, *es discontinua si, y sólo si,* $s \in Z_2$ *y el conjunto* $\varphi(Z_2)$ *es finito.* □

Como en el caso complejo, cuando se asumen hipótesis adicionales sobre T, se obtienen propiedades adicionales sobre la función soporte. Sin embargo, el hecho de que T sea biyectiva no garantiza que la función soporte sea un homeomorfismo. Aun así obtenemos el deseado resultado de continuidad automática.

Corolario 1.1.34 [J.J. Garcés, A.M. Peralta, Linear and Multilinear algebra, 2013] *Toda biyección lineal que preserva ortogonalidad entre* C^**-álgebras reales abelianas unitales es automáticamente continua.* □

Encontramos aquí una gran diferencia con lo que ocurre en el caso complejo, y es que el hecho de que T sea biyectiva no garantiza que T^{-1} preserve ortogonalidad (ver ejemplo 3.7 en [77]). Surge así la pregunta de cuándo una biyección que preserva ortogonalidad preserva ortogonalidad en ambos sentidos. En [77] también damos respuesta a esta pregunta.

Teorema 1.1.35 [J.J. Garcés, A.M. Peralta, Linear and Multilinear algebra, 2013] *Sea* $T : C_r(L_1) \to C_r(L_2)$ *una aplicación. Las siguientes afirmaciones son equivalentes:*

(a) *T es sobreyectiva y preserva ortogonalidad en ambas direcciones;*

(*b*) *Existen un homeomorfismo* $\varphi : L_2 \to L_1$ *tal que* $\varphi(\mathcal{O}_2) = \mathcal{O}_1$, *una función* $a_1 = \gamma_1 + i\gamma_2$ *en* $C_r(L_2)$ *con* $a_1(s) \neq 0$ *para todo* $s \in L_2$, *y una función* $a_2 = \eta_1 + i\eta_2 : L_2 \to \mathbb{C}$ *que es continua en* $\mathcal{O}_2$ *y satisface la propiedad*

$$0 < \inf_{s \in \mathcal{O}_2} \left| \det \begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix} \right|$$

$$\leq \sup_{s \in \mathcal{O}_2} \left| \det \begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix} \right| < +\infty,$$

tales que

$$T(f)(s) = a_1(s)\ \Re e f(\varphi(s)) + a_2(s)\ \Im m f(\varphi(s))$$

para cualesquiera $s \in L_2$ *y* $f \in C_r(L_1)$. $\square$

En el Capítulo 9 de esta memoria exponemos los resultados sobre derivaciones locales en C*-álgebras obtenidos recientemente en colaboración con M. Burgos, F.J. Fernández-Polo y A.M. Peralta (ver [36]).

Las derivaciones locales aparecen por primera vez en el trabajo de R.V. Kadison [113]. Sean A un álgebra de Banach y X un A-bimódulo. Diremos que una aplicación lineal $T : A \to X$ es una *derivación local* si para cada a en A, existe una derivación $\delta_a : A \to X$ tal que $T(a) = \delta_a(a)$. En el trabajo que acabamos de mencionar, Kadison prueba que toda derivación local continua de un álgebra de Von Neumann W en un bimódulo dual es una derivación (ver [113, Theorem A]). Este resultado fue más tarde generalizado por B.E. Johnson, quien demuestra, en [110], que toda derivación local de una C*-álgebra en un bimódulo es una derivación.

Sea E un JB*-triple. Diremos que un aplicación conjugado lineal $T : E \to E$ es una *derivación triple local* si, para cada a en E existe una derivación triple $\delta_a : E \to E$ tal que $T(a) = \delta_a(a)$.

Las derivaciones locales triples fueron por primera vez consideradas por M. Mackey en [137]. Este autor demuestra que toda derivación triple local en un JBW*-triple es una derivación (ver [137]). Conviene señalar que, aunque ésto no es observado por el Mackey, sus argumentos pueden adaptarse para demostrar que toda derivación triple local en JB*-triple débilmente compacto es una derivación.

En [36] consideramos derivaciones triples locales en C*-álgebras (y JB*-álgebras) unitales. También estudiamos la conexión existente entre las derivaciones generalizadas introducidas por J. Li y Zh. Pan en [132].

Como principales resultados obtenidos en este trabajo probamos que toda triple derivación local en una C*-álgebra (o una JB*-álgebra) unital es una derivación triple.

Teorema 1.1.36 [M.Burgos, F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Comm. Alg., 2013] *Sea A una C*-álgebra (o una JB*-álgebra) unital. Entonces toda derivación triple local T : $A \to A$ es una derivación triple.* □

Finalmente, en el último Capítulo de esta memoria exponemos una serie de problemas abiertos relacionados con la temática de esta tesis. El estudio de estos problemas permitirá continuar la labor iniciada en estos años de doctorado que han culminado con la realización de esta memoria.

Incluimos también como anexo todos los artículos cuyos resultados han hemos utilizado para escribir esta memoria.

Chapter 2

Introduction

According to the rules governing the official Ph.D. Studies and Doctorate from the University of Granada, approved by the Governing Council of the University of Granada on May 2, 2012, "the thesis may consist in the regrouping, in a memoir, research papers published by the doctoral student in relevant scientific Journals in their field of knowledge." This memoir has been prepared as a compilation of nine papers, all of them published in journals of international importance in the field of Mathematical Analysis, referenced journals included in the list by the Journal Citations Reports and databases like MathSciNet (American Mathematical Society) and Zentralblatt für Mathematik (European Mathematical Society). We have opted for a memory which resumes the results obtained during the last five years. The original published results are in a series of papers attached at the end of this memoir. Thus, we shall survey the results, forerunners and previous contributions without paying attention to the detailed proofs of these results. We shall highlight the most important contributions and the context in which they were obtained (precedents, motivations, references and difficulties). All details can be found in the original papers enclosed after the final chapter and in the references.

2.1. Basic notions: C*-algebras, Function Algebras and JB*-triples

In general, we shall introduce basic concepts and definitions just before they are needed. As exception, in this section, we recall some basic results in the theory of C*-algebras, JB*-algebras and JB*-triples that will be needed in this memory. For further results on the theory of C*-algebras we recommend [152], [165] and [178], from where we have borrowed the results and definitions presented here.

By an *involution* on a complex algebra we mean a conjugate linear mapping $^* : A \to A$ verifying:

$$a^{**} := (a^*)^* = a, \ (ab)^* = b^* a^*,$$

for every a in A.

The term *Banach algebra* will stand for an associative complete normed algebra. A linear mapping between algebras is said to be a *homomorphism* when it preserves products. A Banach algebra endowed with an involution is called a Banach *-algebra. A **-homomorphism* (respectively, a *Jordan* **-homomorphism*) between two *-algebras A and B, is an homomorphism (respectively, Jordan homomorphism for the Jordan product $a \circ b := \frac{1}{2}(ab + ba)$) $T : A \to B$ which satisfies $T(a^*) = T(a)^*$, for every a in A. Two *-algebras are said to be *-isomorphic if there is a bijective *-homomorphism, called a *-isomorphism, between them.

Let A be a Banach algebra with identity 1 and let a be an element in A. The *spectrum* of a relative to A, denoted by $\sigma_{_A}(a)$, is the set of all complex numbers λ such that $a - \lambda 1$ is not invertible. When no confusion can arise we shall simply write $\sigma(a)$.

If A is a Banach algebra without identity, then there exists a norm in the unital algebra $\widetilde{A} = \mathbb{C}1 \oplus A$, such that $\widetilde{A}$ is a Banach algebra (we call $\widetilde{A}$ the *unitization* of A). Let a be an element in A. We define the spectrum of a to be the spectrum of a relative to $\widetilde{A}$, that is, $\sigma_{A}(a) := \sigma_{\widetilde{A}}(a)$.

Theorem 2.1.1 [Gelfand-Mazur theorem, Gelfand, [80]] *If A is a complex Banach algebra, then the spectrum of any element of A is a non-empty compact set.*

An important class of Banach *-algebras is that of *C***-algebras*.

Definition 2.1.2 *A C*-algebra is a complex Banach *-algebra satisfying the so-called* ***Gelfand Naimark axiom****:*

$$\|aa^*\| = \|a\|^2,$$

for every a in A.

Next, we give some examples of C*-algebras.

Let L be a locally compact Hausdorff space. The space $C_0(L)$, of all complex valued continuous functions on L vanishing at infinity, endowed with the supreme norm and the usual product is a Banach algebra. Further, it is easy to check that the assignment $^* : C_0(L) \to C_0(L)$, $f \mapsto f^*$, given by $f^*(t) = \overline{f(t)}$, is an involution which makes $C_0(L)$ into a Banach *-algebra. Morevorer, the norm in $C_0(L)$ also satisfies the Gelfand-Naimark axiom. Thus $C_0(L)$ is a (commutative) C*-algebra.

Let H be a complex Hilbert space and let $L(H)$ denote the space of continuous linear mappings on H. Then $L(H)$ with the operator norm and product given by composition is a Banach algebra. It is well know that, for each $T \in L(H)$ there exists a unique $T^* \in L(H)$ such that $(T(x)|y) = (x|T^*(y)), \forall x, y \in H$. The mapping $^* : L(H) \to L(H), T \mapsto T^*$ is an involution on $L(H)$. Furthermore, the above involution and product also satisfy the Gelfand-Naimark axiom, that is,

$$\|TT^*\| = \|T\|^2,$$

for every T in $L(H)$.

Now let A be a norm closed subalgebra of $L(H)$ which is also *-invariant (called a self-adjoint subalgebra of $L(H)$). Clearly, A also is a C*-algebra.

The origins of C*-algebras can be placed in the foundations of quantum mechanics, with the contributions by E. Heissenberg, P. Jordan and J. von Neumann.

The abstract characterisation of C*-algebras was given by I. Gelfand and M. Naimark in 1943 (see [81]), however they imposed the following extra condition: $1 + aa^*$ is invertible, for every a in A. Abstract C*-algebras (that is, those that satisfy the axioms given by Gelfand and Naimark) where called B*-algebras by C. E. Rickart in 1947, while those C*-algebras which are norm closed *-invariant subalgebras of some $L(H)$ were called C*-algebras, although nowadays the term B*-algebra is not in use anymore.

In 1943, Gelfand and Naimark proved that every C*-algebra (with the extra assumption of $1 + aa^*$ being invertible for every a in A), can be realised as a norm closed self adjoint subalgebra of some $L(H)$. They actually conjectured that this extra condition was superfluous, a conjecture which was proved in 1951 by Fukamiya (see [73]).

Theorem 2.1.3 [*Gelfand-Naimark, 1943*] *Every C*-algebra is C*-isomorphic to a norm-closed selfadjoint subalgebra of $L(H)$, for some complex Hilbert space H.*

Let A be an abelian C*-algebra. The spectrum of A, $\Omega(A)$, is the set non-zero homomorphisms from A to the complex numbers (called *characters*). It is well known that $\Omega(A)$ is contained in the unit ball of A^*, and $\Omega(A) \cup \{0\}$ is weak*-compact. If A is unital then $\Omega(A)$ is weak*-compact.

The *Gelfand transform* is the homomorphism from A into $C_0(\Omega(A))$ given by $x \mapsto \hat{x}$, $\hat{x}(t) = t(x)$, for all x in A and t in $\Omega(A)$.

Theorem 2.1.4 [Abelian Gelfand-Naimark theorem][165, Theorem 1.2.1 and Corollary 1.2.2] *If A is an abelian C*-algebra then the Gelfand transform is a surjective isometric *-homomorphism. Furthermore, when A is unital $\Omega(A)$ is compact and A is C*-isomorphic to $C(\Omega(A))$.*

Let a be an element in a C*-algebra A. We denote by A_a (respectively, $A_{1,a}$ if A is unital) the C*-subalgebra of A generated by a (respectively, by a and 1) that is, the smallest C*-subalgebra of A containing a (respectively, a and 1).

We say that an element a in A is *normal* (respectively, *self-adjoint*) if $a^*a = aa^*$ (respectively, $a^* = a$). Clearly a self-adjoint element is normal. We denote by A_{sa} the set of all self-adjoint elements in A. An element p in A is said to be a projection whenever $p^2 = p = p^*$. We recall that a *partial isometry* in A is an element e satisfying that ee^* (and e^*e) is a projection in A, or equivalently $ee^*e = e$.

Let A be a unital C*-algebra, and let a be a normal element in A. Then $A_{1,a}$ is an abelian C*-algebra. We further have:

Theorem 2.1.5 [165, Corollary 1.2.3] *Let a be a normal element in a unital C^*-algebra A. Then $A_{1,a}$ is C^*-isomorphic to $C(\sigma_A(a) \cup \{0\})$.*

If A is not unital, then we have the following:

Theorem 2.1.6 [165, Corollary 1.2.3] *Let a be a normal element in a C^*-algebra A. Then A_a is *-isomorphic to $C_0(\sigma(a) \cup \{0\})$, where $C_0(\sigma(a) \cup \{0\})$ stands for the space of all continuous functions on $\sigma(a) \cup \{0\})$ vanishing at 0.*

This representation theorem gives raise to the so-called *continuous functional calculus* of a normal element. We briefly describe how it works.

Given a normal element a in a C*-algebra, A, let

$$F : A_a \longrightarrow C_0(\sigma_A(a) \cup \{0\}),$$

denote the Gelfand representation. Given a continuous function $f \in C_0(\sigma(a) \cup \{0\})$, we define $f(a) \in A_a$ to be the unique element in A_a such that $F(f(a)) = f$. The mapping $C_0(\sigma(a) \cup \{0\}) \to A_a$, $f \mapsto f(a)$ is called the (continuous) functional calculus associated to the element a. This functional calculus enjoys the following properties:

$$(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a), \ (fg)(a) = f(a)g(a), \ \overline{f}(a) = f(a)^*,$$

$$\sigma(f(a)) = f(\sigma(a)), \text{ and } \|f(a)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(a)\}.$$

We say that a self-adjoint element a in a C*-algebra A is *positive* if $\sigma_A(a) \subset \mathbb{R}_0^+$. We shall denote by A^+ the set of positive elements in the C*-algebra A.

Two elements a, b in a C*-algebra A are said to be *orthogonal*, denoted $a \perp b$, if $ab^* = b^*a = 0$.

As an application of the functional calculus, it is easy to see that a self-adjoint element a can be uniquely written as the difference of two orthogonal positive elements a_+, a_-. The positive element $a_+ + a_-$ is called de *absolute value* of a and is denoted by $|a|$.

A *von Neumann algebra* is a C*-algebra which is also a dual Banach space. By a celebrated result due to Sakai, every von Neumann algebra has an unique (isometric) predual, its involution is weak*-continuous and its product is separately weak*-continuous (cf. [165, §1.7]).

The functional calculus also allows us to compute powers of a normal element, and roots of a positive element a. That is, for each natural n we are able to find an element z in A_a such that $z^n = a$. This element will be denoted be $z^{\frac{1}{n}}$ and will be called de n-th root of a.

A *Jordan algebra* is an abelian (but non-necessarily associative) algebra whose product satisfies the so-called *Jordan identity*

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2.$$

Every associative algebra is a Jordan algebra when endowed with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Let A be a Jordan algebra and let a be an element in A, we define the multiplication operator $M_a : A \to A$, by $M_a(b) = a \circ b$, while the quadratic operator $U_a : A \to A$ is given by $U_a(b) = 2a \circ (a \circ b) - b \circ a^2$.

It can be deduced from the Jordan identity that Jordan algebras are power associative, that is, if A is a Jordan algebra and $a \in A$ then for $m, n \geq 1$ we have

$$a^{m+n} = a^m \circ a^n,$$

for every a in A (compare [89, Lemma 2.4.5]).

A linear mapping $T : A \to B$ between Jordan algebras is said to be a *Jordan homomorphism* if it preserves the Jordan product, that is, if

$$T(a \circ b) = T(a) \circ T(b),$$

for every a, b in A.

A normed Jordan algebra is a Jordan algebra such that

$$\|a \circ b\| \leq \|a\| \|b\|,$$

for every a, b in A. If A is complete as a Banach space then we say that A is a *Jordan-Banach algebra*.

A *JB-algebra* is a real Jordan-Banach algebra A in which the norm satisfies the following two additional conditions for all a, b in A:

a) $\|a^2\| = \|a\|^2$;

b) $\|a^2\| \leq \|a^2 + b^2\|$.

Jordan algebras were first studied by P. Jordan, J. von Neumann and E. Winger in the decade of 1930 as a suitable setting for quantum formalism (see [111]).

Special cases of JB-algebras were studied by E. Stormer and D.M. Topping in [177] and [180], although general JB-algebras were defined and studied by E.M. Alfsen, F.W. Schultz and E. Stormer in [8]. For the basic results on the theory of JB-algebras we refer to [89].

Let A be a C*-algebra and denote by A_{sa} the set of all self-adjoint elements of A. We notice that A_{sa} is not, in general, an associative subalgebra of A. However it is easy to see that if we endow A with the Jordan product, then A_{sa} is a real Jordan subalgebra of A. Furthermore, A_{sa} is a JB-algebra.

A complex Jordan-Banach *-algebra is a complex Jordan-Banach algebra, A, endowed with a continuous involution * : $A \to A$. A Jordan *-homomorphism between Jordan-Banach *-algebras is a Jordan homomorphism $T : A \to B$ such that $T(a^*) = T(a)^*$, for all $a \in A$.

A *JB*-algebra* is a complex Jordan-Banach *-algebra satisfying the additional axiom

$$\|U_a(a^*)\| = \|a\|^3.$$

This axiom is the Jordan version of the Gelfand-Naimark axiom for C*-algebras. Actually, when a C*-algebra is endowed with the Jordan product and its natural involution, it becomes a JB*-algebra.

JB*-algebras are, in some sense, the complex version of JB-algebras. They were first considered by I. Kaplansky, who presented them at a lecture for the Edinburgh Mathematical Society in 1976. It is easy to see that the set of self-adjoint elements of a JB*-algebra is a JB-algebra. Conversely, J.D.M. Wright proved a milestone result in the Jordan theory showing that the complexification of a JB-algebra is a JB*-algebra (see [186]).

An associative JB*-algebra A is clearly an abelian C*-algebra, as a consequence, there exists a locally compact Hausdorff space L such that A is *-isomorphic to $C_0(L)$.

If A is an associative JB-algebra, then its complexification is an associative JB*-algebra (see [89, Theorem 3.2.2]) and thus $*$-isomorphic to some $C_0(L)$. It is easy to see that $C_0(L)_{sa}$ coincides with $C_0(L,\mathbb{R})$, and A is *-isomorphic to $C_0(L,)$. The set L is compact if, and only if, A is unital.

Let A be a unital JB*-algebra and a an element in A_{sa}. We denote by $A_{1,a}$ the JB*-subalgebra of A generated by a and 1. By power associativity, $A_{1,a}$ is associative, and hence a C*-algebra.

We define the spectrum of a, $\sigma(a)$, as the spectrum of a relative to $A_{1,a}$.

Theorem 2.1.7 [89, 3.2.4] *Let a be a self-adjoint element in a unital JB*-algebra A. Then $A_{\{1,a\}}$ is *-isomorphic to $C(\sigma(a) \cup \{0\})$. If A is not unital, then A_a is *-isomorphic to $C_0(\sigma(a) \cup \{0\})$.*

A continuous functional calculus on self-adjoint elements on a JB*-algebra can be analogously defined as in the case of C*-algebras.

A *JBW-algebra* A (respectively, a *JBW*-algebra*) is a JB-algebra (respectively, a JB*-algebra) which is a dual Banach space, that is, there exists some Banach space B such that $B^* = A$ (such a Banach space is called a predual of A). Every JBW-algebra (respectively, JBW*-algebra) has a unique isometric predual, which we call the predual of A, and denoted by A_* (cf. [89, Theorem 4.4.16]). It is also known that the Jordan product of a JBW- or a JBW*-algebra is separately weak*-continuous ([89, Corollary 4.1.6]).

C*-algebras and JB*-algebras belong to a more general class of (complex) Banach spaces known under the name of (complex) JB*-triples. We recall that a real (respectively, complex) *Jordan-Banach triple* is a real (respectively, complex) Banach space, E, together with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is trilinear (respectively, conjugate linear in the middle variable and bilinear in the outer variables) and symmetric in the outer variables satisfying the so-called *Jordan identity*,

$$L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y),$$

where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$.

A complex Jordan Banach triple E is said to be a *(complex) JB*-triple* if it satisfies the following additional axioms axioms:

(a) $L(a,a)$ is an hermitian operator with non-negative spectrum;

(b) $\| \{a,a,a\} \| = \|a\|^3$.

We observe that axiom (b) is the appropriate Jordan triple version of the Gelfand-Naimark axiom.

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by $y \mapsto Q(x)y = \{x,y,x\}$.

An element e in a Jordan triple E is said to be a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E gives raise to the so-called *Peirce decomposition* of E associated with e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i-j+k \in \{0,1,2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The corresponding *Peirce projections* from E onto $E_i(e)$ are denoted by $P_i(e) : E \to E_i(e)$, $(i = 0,1,2)$. The Peirce space $E_2(e)$ is a Jordan algebra with product $x \bullet_e y := \{x,e,y\}$ and involution $x^{\sharp_e} := \{e,x,e\}$. If E is a JB*-triple then $E_2(e)$ is a JB*-algebra (cf. [29]).

Every C*-algebra (respectively, every JB*-algebra) is a JB*-triple with respect to

$$\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*a)$$

(respectively, $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

A *JBW*-triple* is a JB*-triple which is also a dual Banach space (with a unique isometric predual [15]). It is known that the triple product of a JBW*-triple is separately weak*-continuous [15]. The second dual of a JB*-triple E is a JBW*-triple with a product extending the product of E [53].

Let I be a subspace of a Jordan triple E. We shall say that I is a *subtriple* of E if $\{I, I, I\} \subseteq I$. Given be a subset M of a JB*-triple E, we denote by $E_{_M}$ the norm-closed subtriple of E generated by M, that is, the smallest norm-closed subtriple of E containing M. When $M = \{x\}$, for some x in E, we shall simply write E_x instead of $E_{_M}$.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \left\{x, x, x^{[2n-1]}\right\}$, $(n \in \mathbb{N})$. It is known that, for every n, m, k in $\mathbb{N}$, $\{x^{[2n-1]}, x^{[2m-1]}, x^{[2k-1]}\} = x^{[2(n+m+k)-3]}$ that is, JB*-triples are power associative.

It is also known that, that for each element x, E_x is JB*-triple isomorphic (and hence isometric) to $C_0(L)$ for some locally compact Hausdorff space L contained in $(0, \|x\|]$, such that $L \cup \{0\}$ is compact and there exists a triple isomorphism Ψ from E_x onto $C_0(L)$, such that $\Psi(x)(t) = t$ $(t \in L)$ (cf. [118, 4.8], [119, 1.15] and [70]). The set $L \cup \{0\} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. This local theory provides us a *continuous triple functional calculus* as in the case of C*-algebras and JB*-algebras.

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying that $\{y, y, y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define, $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak*-topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying that x is positive in the JBW*-

algebra $E_2^{**}(e)$ (compare [56, Lemma 3.3]).

Complex JB*-triples were introduced by W. Kaup in the study of bounded symmetric domains in complex Banach spaces (see [118], [119]), although particular forerunners of JB*-triples were studied before by O. Loos and K.M Crimmon (see [136]) and by L.A. Harris in [90]. Although the initial motivation to study the JB*-triples was holomorphic theory, the theory of JB*-triples has motivated an area of independent interest and they are nowadays studied by many authors from the point of view of algebra and functional analysis. In these structures (the JB*-triples) the algebraic, topological, holomorphic and geometric structures have a particularly good interaction, and as consequence of this interaction, frequently purely algebraic hypothesis determine topological and geometric properties, and reciprocally.

Chapter 3

Characterisation of orthogonality preservers

3.1. Historical overview

The main subject of this thesis is the study of different properties of a type of linear operators called “orthogonality preserving operators”. We have tracked this problem and it seems to have its origins in the description of isometries in some classical Banach spaces, such as the $C(K)$-spaces or the L^p-spaces.

Let L be a locally compact Hausdorff space and $\mathbb{K}$ denote the field of real or complex numbers. A continuous function $f : L \to \mathbb{K}$ *vanishes at infinite* if for each $\varepsilon > 0$ the set

$$\{s \in L : |f(s)| \geq \varepsilon\}$$

is compact. We denote by $C_0(L, \mathbb{K})$ the set of all continuous functions on L vanishing at infinite.

Unless specified, we shall always endow the space $C_0(L, \mathbb{K})$ with the *sup norm*, which is a complete norm in $C_0(L, \mathbb{K})$. When

L is compact Hausdorff $C_0(L, \mathbb{K})$ coincides with $C(L, \mathbb{K})$, the space of $\mathbb{K}$-valued continuous functions on L. When the setting is clear we shall simply write $C_0(L)$ or $C(L)$ if L is compact Hausdorff.

In the seventh chapter of [17], Stefan Banach studies the structure of surjective linear isometries between $C(K, \mathbb{R})$-spaces.

Theorem 3.1.1 [S.Banach, 1932]Let K_1, K_2 be compact metric spaces and let $T : C(K_1, \mathbb{R}) \to C(K_2, \mathbb{R})$ be a (linear) surjective linear isometry. Then there exist an homeomorphism $\varphi : K_2 \to K_1$ and a continuous function $h \in C(K_2)$, with $h(s) \in \{+1, -1\}, \forall s \in K_2$, such that

$$T(f)(s) = h(s)f(\varphi(s)),$$

for every $f \in C(K_1), s \in K_2$.

We notice that the linearity assumption is superfluous since Banach himself proved that surjective linear isometries are *automatically* linear.

This important result is the origin of the study of isometries between various classical Banach spaces (and structures that generalise them) but also of a huge number of different (although related) problems in the vast area of linear preservers.

For each function f in $C(K)$ we define its *cozero set* as the set

$$\operatorname{coz}(f) = \{s \in K : f(s) \neq 0\},$$

that is, the complementary in K of the set of zeros of f.

Let's look a bit deeper on the properties of surjective linear isometries between $C(K)$-spaces. Let us fix a surjective linear isometry $T : C(K_1) \to C(K_2)$ and f, g in $C(K_1)$ such that $fg = 0$ (this means that $\operatorname{coz}(f)$ and $\operatorname{coz}(g)$ are disjoint sets). Then, we clearly have $T(f)T(g) = 0$. So T sends functions with disjoint cozero sets to functions with disjoint cozero sets. In other words, T sends disjoint functions to disjoint functions. Linear mappings that enjoy this property are usually said to be *disjointness preserving* or *separating*.

Now we observe that we can consider operators having a similar form to that of surjective linear isometries but with weaker assumptions on the functions h and φ, still preserving disjoint functions. Indeed, any linear mapping with the form $T(f) = h(f \circ \varphi)$ for some $h \in C(K_1)$ and $\varphi : K_2 \to K_1$ continuous on $\{t : h(t) \neq 0\}$, also is separating (or disjointness preserving).

Definition 3.1.2 *Let K_1, K_2 be two compact Hausdorff spaces and let $T : C(K_1) \to C(K_2)$ be a linear mapping. If there exists $h \in C(K_2)$ and $\varphi : K_2 \to K_1$ continuous on $\{t : h(t) \neq 0\}$ such that*

$$T(f)(s) = h(s)f(\varphi(s)),$$

for all $f \in C(K_1), s \in K_2$. Then T is said to be a weighted composition operator*.*

As proved by Banach, every surjective linear isometry is a weighted composition operator (it is easy to see that not every weighted composition operator is an isometry). As we have already observed, every weighted composition operator is separating. This key observation was not explicitly made by Banach, at least not for the case of isometries between $C(K)$-spaces. However, just a few pages after proving Theorem 3.1.1, Banach also studies surjective linear isometries between $L^p([0,1])$-spaces, (for $1 \leq p \neq 2$) (and also between l_p-spaces). We find in page 175 what might be the first reference in the literature to the disjointness preserving property (extracted from french version):

"Etant donée une rotation $y = U(x)$ de $(L^{(p)})$, oú $1 \leq p \neq 2$, autour de 0, si on a pour un couple $x_1(t), x_2(t)$ des fonctions appartenat á $(L^{(p)})$

$$x_1(t)x_2(t) = 0, \textit{persque partout dans } [0,1],$$

alors pour le couple $y_1(t), y_2(t)$, oú $y_1(t) = U(x_1)$ et $y_2 = U(x_2)$, on a également

$$y_1(t)y_2(t) = 0, \textit{persque partout dans } [0,1].''$$

That is, in our terminology, Banach proves that a surjective linear isometry between $L^p([0,1])$-spaces (for $1 \leq p \neq 2$) is separating (or disjointness preserving). Banach also characterises surjective linear isometries between $L^p([0,1])$-spaces, by proving that they are also weighted composition operators (in this case φ must be a bijection of $[0,1]$ such that both φ and φ^{-1} preserve measurable sets, and h satisfies the property that $h(t) = \text{lím}_{\delta\to 0^+} (\frac{m(\varphi([t,t+\delta]))}{\delta})^{\frac{1}{p}}, t \in [0,1]$).

This result was generalised by J. Lamperti in 1958 (see [126]) to arbitrary σ-finite measure spaces and also for $p < 1$. Lamperti also proves that surjective linear isometries are separating, a property that plays an important role in the description of this kind of operators (that might be the reason for which both Banach and Lamperti observed this property for isometries between L^p-spaces but not in the case of $C(K)$-spaces).

In 1937, M. Stone proves a version of Banach's theorem for complex $C(K)$-spaces (where K is a compact (non necessarily metric) Hausdorff space).

Theorem 3.1.3 [M. Stone, [176]] *Let K_1, K_2 be compact Hausdorff spaces and $T : C(K_1, \mathbb{K}) \to C(K_2, \mathbb{K})$ a surjective linear isometry. Then there exist a continuous function $h \in C(K_2)$*

with $|h(s)| = 1, \forall s \in K_2$*, and an homeomorphism* $\varphi : K_2 \to K_1$ *such that*

$$T(f)(s) = h(s)f(\varphi(s)),$$

for all $f \in C(K_1), s \in K_2$.

This theorem is nowadays known as *Banach-Stone theorem*. All the results trying to generalise the Banach-Stone theorem to some other setting are known as Banach-Stone type theorems (that is the case, for example, of the just quoted Banach's description of surjective linear isometries on $L^p([0,1])$. The study of different types of Banach-stone theorems in wider settings (Banach lattices, $C(K,X)$-spaces, Banach algebras, Jordan structures, etc.) has attracted the attention of many mathematician during the last 80 years, still being a very active area of research (see for instance [112], [149], [187], [162], [96], [119], [65] and [66]). We shall, however, focus our attention in another, but not less important, problem that also has its origin in the seminal work by Banach and Stone. We refer to the study of separating or disjointness preserving operators.

The $C(K)$-spaces or L^p-spaces can be replaced with spaces in some wider classes of Banach spaces, one of them being the category of *Banach lattices*. A Banach lattice is a real vector lattice with a complete norm which satisfies the additional property:

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

where $|x| = \text{máx}\{x, -x\}$. Two elements x, y in a Banach lattice E are said to be *disjoint*, denoted by $x \perp y$, if $\text{mín}\{x, y\} = 0$.

In [143, page 187], H. Nakano proves that order isomorphisms between $C_0(L)$-spaces are weighted composition operators (and hence disjointness preserving). An interesting fact is that continuity is not assumed, that is, Nakano also proves that order isomorphisms between $C_0(L)$-spaces are automatically continuous. This result can be considered as the starting point in the study of separating (disjointness preserving) mappings in Banach lattices.

As we have seen, many particular examples of separating linear mappings were already known by the decade of 1950, however a systematic study of these linear mappings had to wait until late 70's. It seems that the study of disjointness preserving linear mappings has brought first attention of experts in the field of Banach lattices. The special case of *band preserving* operators was first studied.

Definition 3.1.4 *A linear mapping $T : X \to Y$ between Banach lattices is said to be* band preserving *or to* preserve bands *if*

$$x \perp y \textit{ implies } T(x) \perp y.$$

Mathematicians were mainly concerned with questions like automatic continuity, representability (as weighted composition operators), spectral theory and the inverses of disjointness preserving linear mappings. Concerning automatic continuity, band preserving operators and lattice isomorphisms where known to be automatically continuous, however the situation for general disjointness preserving linear mappings was not clear.

In 1979, Y.A. Abramovich, A.I. Veksler and A.V. Koldunov prove that a surjective linear mapping $T : X \to Y$ between Banach lattices with the property that $x_1 \perp x_1$ if, and only if, $T(x_1) \perp T(x_2)$ is continuous. They also give a description of such a mapping as an appropriate weighted composition operator (see [2]).

Definition 3.1.5 *A linear mapping $T : X \to Y$ between Banach lattices with the property that*

$$x_1 \perp x_1 \textit{ if, and only if, } T(x_1) \perp T(x_2)$$

is said to be biseparating *or a* d-isomorphism.

As proved by Abramovich, Veksler and Koldunov, every biseparating linear surjection between Banach lattices is automatically continuous. Here we observe an interesting fact, a property that depends only on the order and linear structure (being linear and biseparating) determines a topological property (being continuous).

A question that naturally arises is whether the biseparating hypothesis can be dropped, that is, whether a separating linear bijection is automatically continuous and, in that case, whether its inverse is also a separating. This question turned out to be very difficult. Some examples where this is true were known, but it's not until 2000 when a counterexample is given by Y.A. Abramovich and A.K. Kitover (see [1]).

Back to the $C(K)$-spaces, in [23] E. Beckenstein, L. Narici and A.R. Todd studied separating linear maps in this setting. One of the interesting novelties found in their contribution is the use of the so-called *support* of a separating linear mapping. In their own words, this support "is based on an analog of the notion of support of a linear functional". The support function allows to fully understand separating linear mappings between $C(K)$-spaces, even when they are not continuous. In the just quoted paper they are able, under some additional hypothesis (such as $T(1)$ being invertible and T being surjective), to describe separating linear mappings (they are weighted composition operators, as expected) and give some automatic continuity results for separating linear bijections (although some additional hypothesis on the compact set K were also needed).

Later, in [100], K. Jarosz gives a general description of a separating linear mapping. We sketch here this description: Let $T : C(K_1) \to C(K_2)$ be a separating linear mapping between

$C(K)$-spaces. The compact set K_2 can be decomposed as a disjoint union $K_2 = Z_1 \cup Z_2 \cup Z_3$, where

$$Z_3 = \{s \in K_2 : \delta_s \circ T = 0\},\ Z_2 = \{s \in K_2 : \delta_s \circ T \text{ is unbounded}\},$$

and

$$Z_1 = K_2 \setminus Z_2 \cup Z_3.$$

For $s \in K_2$ the support of $\delta_s T$, denoted $supp(\delta_s T)$, is defined to be the set of all $t \in K_1$ such that for every open neighborhood U of t there exists $f \in C(K_1)$ with coz $\subset U$ and $T(f)(s) \neq 0$.

It can be proved that $supp(\delta_s T)$ is a single point whenever s lies in $Z_1 \cup Z_2$ and is empty if, and only if, $s \in Z_3$. This allows to define a continuous function $\varphi : Z_1 \cup Z_2 \to K_1$, given by $\varphi(s) = supp\{\delta_s T\}$. We recall that the set Z_1 is the set of those $s \in K_2$ where $\delta_s T$ is a nonzero continuous functional. Jarosz proved (among other things) that the restriction of T to the set Z_1 acts as a weighted composition operator.

Theorem 3.1.6 [K. Jarosz, Canadian J., 1990] *Let us consider a separating linear mapping $T : C(K_1) \to C(K_2)$. Under the above notation, Z_2 is open, Z_3 is closed and there exists a bounded, non-vanishing, continuous function $h : Z_1 \to \mathbb{C}$, such that*

$$T(f)(s) = h(s)f(\varphi(s)),$$

for every $f \in C(K_1), s \in Z_1$, $T(f)(s) = 0$, for every $s \in Z_3$ and every $f \in C(K_1)$, and the set $\varphi(Z_2)$ is finite.

When additional hypothesis on T are assumed (injectivity and/or surjectivity), additional properties on the support function can be obtained. When T is bijective Jarosz proves the following:

Corollary 3.1.7 [K. Jarosz, Canadian J., 1990] *Let T be a separating linear bijection from $C(K_1)$ to $C(K_2)$. Then T is continuous (and the mapping φ given in Theorem 3.1.6 is a homeomorphism).*

In the hypothesis of the above corollary, T^{-1} is also separating. Therefore, the inverse of a separating linear bijection between $C(K)$-spaces is separating.

This results were later generalised to the case of separating linear mappings between $C_0(L)$-spaces by J.S. Jeang and N.C. Wong in [103].

Another class of Banach spaces containing the $C(K)$-spaces is the class of vector valued continuous functions. Let K be a compact Hausdorff space and X a Banach space. The symbol $C(K, X)$ stands for the space of all continuous functions $f : K \to X$. The space $C(K, X)$ endowed with the supreme norm is again a Banach space.

Definition 3.1.8 *Two functions $f, g \in C(K, X)$ are said to be* disjoint*, denoted $fg = 0$, if*

$$\|f(t)\|\|g(t)\| = 0,$$

for every t in K.

Associated with this concept of disjointness, naturally arises the problem of studying disjointness preserving linear mappings between $C(K, X)$-spaces.

Definition 3.1.9 *Let K_1, K_2 be compact Hausdorff spaces and let X_1, X_2 Banach spaces. A linear mapping $T : C(K_1, X_1) \to C(K_1, X_2)$ is said to be* separating *(or* disjointness preserving*) if $T(f)T(g) = 0$, whenever $fg = 0$.*

A linear bijection $T : C(K_1, X_1) \to C(K_1, X_2)$ is said to be *biseparating* if both T and T^{-1} are separating.

In [79], H.L. Gau, J.S. Jeang and N.C. Wong studied biseparating linear mappings between $C(K,X)$-spaces. The main result of the just quoted paper states that these linear mappings are continuous and can be represented as an appropriate composition operator (see also [13]).

$C(K)$-spaces also belong to another class of Banach spaces with additional algebraic and geometric properties. We refer to the class of *Banach algebras*.

We say that two elements a, b in a Banach algebra A are *disjoint* (or have *zero product*) whenever $ab = 0$. Having zero product seems to be a natural generalisation of disjointness in this setting, however, we shall see later that, when the algebra has a richer structure, there are other choices.

Definition 3.1.10 *A linear mapping T between two Banach algebras A and B is said to be* disjointness preserving *or to* preserve zero-products *if*

$$ab = 0 \textit{ implies } T(a)T(b) = 0.$$

Clearly, every algebra homomorphism preserves zero-products.

Two elements a, b in a Banach algebra A are said to *commute* if $ab = ba$. Let $R \subseteq A$, the *commutant* of R, denoted by R', is the set of all elements of A that commute with all elements in R, that is $R' = \{a \in A : ab = ba, \forall b \in R\}$. The center of A is $Z(A) = A'$. An element in the center of A is called a *central element*. Of course, $Z(A) = A$ if, and only if, A is abelian.

Let A, B be Banach algebras, $S : A \to B$ be a homomorphism and let $h \in Z(B)$. It is easy to check that the linear mapping $T = hS$ preserves zero products. If h is not central, the operator

$T = hS$ might be zero product preserving, provided that h lies in $S(A)'$, for instance. Such a linear mapping seems to be the appropriate generalisation of a weighted composition operator to the new setting. Notice that, if A and B are $C(K)$-spaces and T is a weighted composition operator given by $T(f) = h(f \circ \varphi)$, then the mapping $S : A \to B$ given by $S(f) = f \circ \varphi$ is a homomorphism (and the element h trivially lies in in the commutant of $S(A)$).

The natural question that arises now is whether a zero product preserving linear mapping between Banach algebras, say T, can be represented as a multiple of an homomorphism by an element that verifies certain commutativity relations with all elements in the range of T.

Let A be a Banach algebra. We recall, once again, that the *Jordan product* of A is defined as

$$a \circ b = \frac{1}{2}(ab + ba).$$

The Jordan product is commutative, however, it is not, in general, associative. A linear mapping $T : A \to B$ between Banach algebras is said to be a *Jordan homomorphism* if

$$T(a \circ b) = T(a) \circ T(b),$$

while T is said to preserve Jordan-zero products if

$$T(a) \circ T(b) = 0 \text{ whenever } a \circ b = 0.$$

Obviously, Jordan homomorphisms and multiples of Jordan homomorphisms by a central element (sometimes *called weighted Jordan homomorphisms*) are examples of Jordan-zero product preserving linear mappings.

Zero-product and Jordan-zero product preservers have been studied by many authors during the last 20 years (see for instance [42], [43], [67], [192], [185], [124] and [5]). An hypothesis that is usually assumed on a zero-product or a Jordan-zero product preserving linear mapping when trying to describe it, is surjectivity (although some hypothesis on the algebra are needed). However, the desired description of zero product preserving operators between general Banach algebras (without assuming extra hypothesis on the operator) seems to be hopeless (see [42]). Fortunately, when the algebra has a richer structure, a better knowledge of these operators can be obtained. That will be the case for the class of C^*-algebras.

As a special case of Banach algebras, the study of zero product preservers makes perfect sense in this new setting. However, another notion of disjointness arises naturally in the class of C*-algebras.

Let H be a complex Hilbert space and let S, T be elements in $L(H)$. Since, for x, y in H, $(T(x)|S(y)) = (x|T^*S(y))$, we see that the ranges of T and S are orthogonal if, and only if, $T^*S = 0$. If we want this orthogonality relation to be symmetric, we should also require $ST^* = 0$.

Definition 3.1.11 *We say that two elements a, b in a C^*-algebra A are* orthogonal, *denoted $a \perp b$, if $ab^* = b^*a = 0$.*

It is easy to see that if A is a $C_0(L)$-space then two elements in A have zero product if, and only if, they are orthogonal.

With this new notion of disjointness a new problem arises: the study of those linear mappings between C*-algebras preserving orthogonal elements.

Definition 3.1.12 *Let $T : A \to B$ be a linear mapping between C^*-algebras. We say that T is* orthogonality preserving *or that*

T preserves orthogonality *if*

$$a \perp b \ \ \textit{implies} \ T(a) \perp T(b).$$

When $T(a) \perp T(b)$ in B if and only if $a \perp b$ in A, we say that T is *biorthogonality preserving*.

The study of orthogonality preserving operators between C*-algebras begins with the work of W. Arendt [14] in the setting of $C(K)$-spaces. More concretely, the author proved that for every orthogonality preserving operator (originally termed *Lamperti* operator in [14]), $T : C(K) \to C(K)$, there exists $h \in C(K)$ and a mapping $\varphi : K \to K$ being continuous on $\{t \in K : h(t) \neq 0\}$ satisfying that

$$T(f)(t) = h(t)f(\varphi(t)),$$

for all $f \in C(k)$, $t \in K$. The study was latter extended by K. Jarosz [100] and J.S. Jeang and N.C. Wong [103] to the setting of orthogonality preserving operators between $C_0(L)$-spaces, where L is a locally compact Hausdorff space.

From the results on disjointness preserving linear mappings between $C_0(L)$-spaces (see [103]) and the commutative Gelfand theory we already know that continuous linear orthogonality preservers between abelian C*-algebras are weighted composition operators and that orthogonality preserving linear bijections between abelian C*-algebras are automatically continuous (see [100]). The description given in [103] and [100] can be reformulated as follows:

Theorem 3.1.13 *Let A, B be abelian C*-algebras and let $T : A \to B$ be an orthogonality preserving linear mapping. The following statements hold:*

a) If T is continuous, then there exist h in B and a homomorphism $S : A \to B$ such that

$$T(f) = hS(f),$$

for every f in $C(K)$.

b) *If T is bijective then T is continuous and T^{-1} is orthogonality preserving.* □

Notice that every homomorphism between Banach algebras is a Jordan homomorphism.

For a result on orthogonality preservers between non necessarily commutative C*-algebras we have to wait until 1994. In [183], M. Wolff studies continuous linear mappings between unital C*-algebras that preserve orthogonality of self adjoint elements and are *symmetric*, that is, operators mapping self adjoint elements to self adjoint elements.

Definition 3.1.14 *Let A, B be C*-algebras and let $T : A \to B$ be a linear operator. We say that T* preserves orthogonality on *A_{sa} if T satisfies*

$$T(a) \perp T(b) \text{ whenever } a \perp b \text{ in } A_{sa}.$$

Following Wolff's terminology we say that T is *disjointness preserving* on A_{sa} if T preserves orthogonality on A_{sa} and is symmetric.

In a first step to describe disjointness preserving symmetric linear operators between unital C*-algebras, Wolff deals first with the special case in wich $T(1) = 1$ (see Lemma 3.3 in [183]). He proves that, in this case, T is a Jordan *-homomorphism. We notice that, in order to prove this statements, it is enough to prove that $T(a^2) = T(a)^2$, for every a in A_{sa}. Wolff is able to prove that by studying the restriction of T to the subalgebra generated by a single self adjoint element and the unit of A.

Proposition 3.1.15 [M. Wolff, Arch. Math., 1994] *Let S be a disjointness preserving operator between unital C*-algebras satisfying $S(1) = 1$. Then S is a Jordan *-homomorphism.* □

When the element $h = T(1)$ is invertible, Wolff proves that the linear operator $S = h^{-1}T$ is a Jordan *-homomorphism and $T = hS$ (see the proof of Theorem 3.5 in [183]).

Finally, in [183, Theorem 3.5] Wolff characterises general symmetric disjointness preserving linear operators between unital C*-algebras. To obtain this characterisation, the element h being self-adjoint seems to be crucial, since the commutative Gelfand theory is again used to describe $\mathcal{C}_{1,h}$, for a certain C*-subalgebra $\mathcal{C}$ of B^{**}.

We shall slightly reformulate Wolff's original result to avoid technical details. Let $T : A \to B$ be a disjointness preserving linear operator between unital C*-algebras and $h = T(1)$. Then h lies in $\{T(A)\}'$, the commutator of $T(A)$, and there exists a sequence $S_n : A \to B^{**}$ of Jordan *-homomorphisms such that w*-$\text{lím}_n\, hS_n(a) = T(a)$, for every a in A. Furthermore, if we take a free ultrafilter $\mathcal{U}$ then the assignment

$$a \mapsto S(a) = w^* - \underset{\mathcal{U}}{\text{lím}}\, S_n(a),$$

defines a Jordan *-homomorphism from A into B^{**} and $T = hS(a)$.

The following is a reformulation of the main theorem in [183].

Theorem 3.1.16 [M. Wolff, Arch. Math., 1994] *Let A, B be unital C* algebras, $T : A \to B$ a disjointness preserving symmetric operator and let $h = T(1)$. Then h lies in $\{T(A)\}'$ and there exists a Jordan *-homomorphism $S : A \to B^{**}$ such that $T = hS$.* □

Looking at Wolff's proof one is tempted to think that when h is normal it might be posible to generalise this result following the same ideas. If we want to drop the hypothesis of T being symmetric then we have two choices: first, considering zero product preservers, and second, considering orthogonality preservers.

In [42, Theorem 4.6], M.A. Chebotar, W.F. Ke, P.H. Lee and N.C. Wong studied not-necessarily symmetric zero-product preservers between C*-algebras. However they required additional hypothesis.

Theorem 3.1.17 [Chebotar, Ke, Lee, Wong, Monaths. Math., 2003] *Let T be a surjective bounded linear mapping from a unital C*-algebra to a C*-algebra B that preserves zero products of self adjoint elements. Then B is unital, $T(1)$ is invertible and lies in the center of B and there exists a Jordan homomorphism $S : A \to B$ such that*

$$T(a) = hS(a),$$

for every a in A. □

It should be noticed here that the Jordan homomorphism J in the above theorem need not be, in general, a Jordan *-homomorphism. The just quoted authors claim that "a general result without assuming that $T(1)$ is invertible might not be posible to obtain" (see the comments before Theorem 4.7 in [42]). In [42, Theorem 4.7] the same authors partially generalise Wolff's Theorem 3.1.16 by proving the following:

Theorem 3.1.18 [Chebotar, Ke, Lee, Wong, Monaths. Math., 2003] *Let $T : A \to B$ be a surjective bounded linear mapping from a unital C*-algebra to a C*-algebra B that preserves zero products of self adjoint elements and such that $h = T(1)$ is a normal element. Then there exists a sequence of bounded Jordan homomorphisms $S_n : A \to B^{**}$ such that for each a in A, $T(1)S_n(a)$ converges in the weak*-topology to $T(a)$.* □

However, in the above result, a representation of T as a weighted composition operator is not, in general, posible. More concretely, in [42, Example 4.8] an example is given of a linear operator satisfying the hypothesis of Theorem 3.1.18 that cannot be written in the form $T = T(1)S$, for any Jordan homomorphism $S : A \to B^{**}$. It is worth to notice that in the just quoted example, the sequence of Jordan homorphisms (S_n) such that $w^*\text{-lím}_n\, hS_n(a) = T(a), \forall a \in A$, satisfies $\|S_n\| \geq 2n, \forall n \in \mathbb{N}$, that is, the sequence (S_n) is unbounded.

If T is symmetric then the Jordan operators in the sequence (S_n) given by Theorem 3.1.18 [42] (see also the proof of the main Theorem in [183]) are not mere Jordan homomorphisms but also *-homomorphism. This might be the reason because Wolff succeeded in describing what he calls disjointness preserving operators, since Jordan *-homomorphisms are contractive, and thus the sequence (S_n) allows us to obtain a Jordan *-homomorphism S such that $T = hS$. The cause for (S_n) being Jordan *-homomorphisms and not only Jordan homomorphisms is that T preserves orthogonality instead of zero products.

In next section we shall see how Wolff's description for disjointness preserving linear mappings can be generalised. If T is not symmetric or h is not normal or A is not unital, then we have the problem that we cannot use local theory of C*-algebras. This problem will be avoided by using the triple structure which is naturally associated with every C*-algebra.

3.2. New Progress

Before giving a characterisation of orthogonality preserving linear mappings we need to give some basic results and definitions on Jordan-Banach structures that will be needed during this section.

Definition 3.2.1 *Two elements a, b in a JB^*-triple are said to be* orthogonal*, denoted $a \perp b$, if $L(a,b) = 0$ (equivalently $L(b,a) = 0$).*

When A is a C*-algebra, then $L(a,b) = 0$ if, and only if, $ab^* = b^*a = 0$. That is, the concept of orthogonality on a C*-algebra is equivalent to the concept of orthogonality inherited from its triple structure.

In [34], M. Burgos. F.J. Fernández-Polo, A.M. Peralta, J. Martínez-Moreno and the author of this thesis introduce the triple structure of a C*-algebra in the study of orthogonality preservers. The triple structure of a C*-algebra had already revealed to be a key tool for the solution of some important problems. For instance, it appears in the study of the Banach-Stone theorem for C*-algebras. In [112], R.V. Kadison proves that a surjective linear isometry between (unital) C*-algebras is a Jordan *-isomorphism multiplied by an unitary. Such a mapping need not be a *-homomorphism but it is always a triple homomorphism. This result was generalised for non-unital C*-algebras by A.L. Paterson and A.M. Sinclair in [149]. Banach-Stone type theorems for JB*-algebras and JB*-triples have also been obtained (see [187], [96], [119], [51] and [64]). A bijective linear mapping between JB*-triples is an isometry if and only if it is a triple isomorphism (cf. [119]).

Another problem where the triple structure of C*-algebras plays an important role is the contractive projection problem. The range of a contractive projection on a C*-algebra is a not, in general, a subalgebra but a subtriple (see [58], [68], [69], [71], and [120] for the mentioned result and generalisations).

Definition 3.2.2 *A linear mapping $T : E \to F$ between JB*-triples is a* triple homomorphism *if it preserves the triple product, that is, if it satisfies*

$$T\{x, y, z\} = \{T(x), T(y), T(z)\},$$

for all x, y, z in E.

Concerning orthogonality preservers, N.C. Wong proved in [184], the following result, which characterizes triple homomorphisms between C*-algebras in terms of orthogonality preserving properties.

Theorem 3.2.3 [N.C. Wong, Southeast J. Asian Bull. Math., 2005] *Let $T : A \to B$ be a linear operator between C*-algebras. Then T is a triple homomorphism if, and only if, T preserves orthogonality and $T^{**}(1)$ is a tripotent.* □

We shall see now how the triple structure of a C*-algebra also plays a crucial role in the study of orthogonality preservers. This new point of view together with recent results on orthogonal bilinear forms and orthogonality additive polynomials on C*-algebras allow us to give a complete description of orthogonality preserving linear operators between C*-algebras.

Let A be a C*-algebra. A sesquilinear form $\Phi : A \times A \to \mathbb{C}$ is called *orthogonal* if $\Phi(a, b) = 0$, whenever $a \perp b$ in A_{sa}. Orthogonal forms where first studied by K. Ylinen in 1975 and later by R. Jajte and A. Paszkiewicz in 1978 (see [189] and [98], respectively). However, a general description was not given until 1993, when S. Goldstein proved the following:

Theorem 3.2.4 [S. Goldstein, J. Funct. An., 1993] *Let A be a C*-algebra and let $\Phi : A \times A \to \mathbb{C}$ be a continuous orthogonal sesquilinear form. There exist ψ_1, ψ_2 in A^* such that*

$$\Phi(a, b) = \psi_1(ab^*) + \psi_2(b^*a),$$

for every a, b in A. $\square$

This result was also reproved by U. Haagerup and N.J. Lautsen using different techniques in [88].

Let us take a C*-algebra A and a Banach space X. By an X-valued *n-homogeneous polynomial* on A we mean a continuous X-valued mapping $P : A \to X$ such that there exists a continuous and symmetric n-linear mapping $T : A \times \ldots \times A \to X$ satisfying $P(x) = T(x, \ldots, x)$, for every x in A. An n-homogeneous polynomial is said to be *orthogonally additive* (respectively, orthogonally additive on A_{sa}) if $P(a+b) = P(a) + P(b)$, whenever $a \perp b$ in A (respectively, $a \perp b$ in A_{sa}.).

The study of n-homogeneous orthogonally additive polynomials in operators algebras is very recent. In [155], D. Perez and I. Villanueva describe the (scalar valued) orthogonally additive n-homogeneous polynomials on $C(K)$-spaces (see also [25] for a more general result by Y. Benyamini, S. Lassalle and J. G. Llavona in the category of Banach Lattices). Roughly speaking, these mappings are of the form $f \mapsto \phi(f^n)$, for some $\phi \in C(K)^*$, as it was expected. It seems natural to ask whether a similar result can expected for general C*-algebras.

The above description of orthogonally additive n-homogeneous polynomials on $C(K)$-spaces was later generalised to arbitrary C*-algebras by C. Palazuelos, A.M. Peralta and I. Villanueva in [148]. They describe the vector-valued n-homogeneous orthogonally additive polynomials from on a C*-algebra.

Theorem 3.2.5 [C. Palazuelos, A.M. Peralta, and I. Villanueva, Quart. J. Math., 2008] *Let A be a C*-algebra, X a Banach space and $P : A \to X$ an n-homogeneous orthogonally additive polynomial. Then there exists a bounded operator $F : A \to X$ satisfying*

$$P(x) = F(x^n),$$

for every x in A. □

We remind that in Wolff's description of symmetric orthogonality preserving operator the special case $T(1)$ being invertible was considered first. We shall next study orthogonality preserving operators from a C*-algebra to a JB*-triple.

Let $T : A \to E$ be an orthogonality preserving operator form a C*-algebra to a JB*-triple. The element $h = T^{**}(1)$ will play an important role in the description of T. Before attacking the general problem we shall first describe T with additional assumptions on the element h.

We recall that, in general, a JB*-triple does not have a unit element (unless it is a JB*-algebra) and thus the concept of invertibility dos not make sense.

Let A be a Banach algebra. An element a in A is said to be *regular* if there exists b in A such that $a = aba$ and $b = bab$. Regular elements in C*-algebras where first studied by R. Harte and M. Mbekhta (see [91]). Regularity seems to be the appropriate alternative for invertibility in Jordan triples. An element a in a JB*-triple E is said to be *von Neumann regular* if there exists $b \in E$ such that $Q(a)(b) = a$. The element b is called the *generalised inverse* of a.

We observe that every tripotent e in a JB*-triple E is von Neumann regular whose generalised inverse is itself. We refer to [135], [61], [123] and [40] for basics facts and results on von Neumann regularity. It is shown in [123, Lemma 3.2] (see also the proof of [40, Theorem 3.4]) that for each von Neumann regular element $a \in E$, there exists a tripotent $e \in E$ satisfying that a is a symmetric and invertible element in the JB*-algebra $E_2(e)$. Moreover, e coincides with the range tripotent of a. It is also known that an element a in E is von Neumann regular if, and only if, it is von Neumann regular in any JB*-subtriple F containing a (compare [121]).

We shall also need an appropriate alternative of commutativity in Jordan algebras. Two elements a and b in a Jordan algebra A are said to *operator commute* in A if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if, and only if, $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in A.

Self-adjoint elements a and b in a JB*-algebra A generate a JB*-subalgebra that can be realised as a JC*-subalgebra of some $B(H)$ (compare [186]) and, in this realisation, a and b commute in the usual sense whenever they operator commute in A (see [180, Proposition 1]). Similarly, two elements a and b of A_{sa} operator commute if, and only if, $a^2 \circ b = U_a(b)$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in A$ we use the symbol $\{b\}'$ to denote the set of elements in A that operator commute with b. (This corresponds to the usual notation in von Neumann algebras).

Let A be a C*-algebra, E be a JB*-triple and $T : A \to E$ be an orthogonality preserving linear operator. Before attacking the general case we study the case when $d = T^{**}(1)$ is von Neumann regular. We consider the linear operator $S = L(b, r(b))T$, where b is the generalised inverse of d. It is clear that $S^{**}(1)$ is a tripotent. Since T is orthogonality preserving, one would expect S to be so, however this is not easy to check. In [34], we apply the previously mentioned results on orthogonal forms and orthogonally additive n-homogeneous polynomials to study orthogonality preserving linear mappings. These application together with techniques from Jordan theory allow us to prove that all elements in the range of T verify certain commutativity relations with d, as a consequence we prove that S is a Jordan *-homomorphism (and that T is, in some sense, a multiple of S.)

We notice that no local theory is used to obtain this result.

Theorem 3.2.6 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let A be a C^*-algebra, E a JB^*-triple and $T : A \to E$ an orthogonality preserving linear operator. Let us assume that $h = T^{**}(1)$ is a von Neumann regular element. Then $T(A) \subseteq E_2^{**}(r(h))$, $T(A) \subset \{h\}'$ and there exists a Jordan *-homomorphism $S : A \to E_2^{**}(r(h))$ such that*

$$T = L(h, r(h))S.$$

□

Since every tripotent is a von Neumann regular element and a Jordan *-homomorphism is always a triple homomorphism, we obtain, as a direct consequence, a generalisation of Wong's result (Theorem 3.2.3).

Corollary 3.2.7 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let A be a C^*-algebra, E a JB^*-triple and $T : A \to E$ a linear operator. Then T is a triple homomorphism if, and only if, T preserves orthogonality and $T^{**}(1)$ is a tripotent.* □

We were also able to generalise, in the just quoted paper [34], the above Theorem 3.2.6 to the setting of orthogonality preserving linear mappings from a JB*-algebra to a JB*-triple.

Theorem 3.2.8 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let J be a JB^*-algebra, E a JB^*-triple and $T : J \to E$ an orthogonality preserving linear operator. Let us assume that $h = T^{**}(1)$ is a von Neumann regular element. Then $T(A) \subseteq E_2^{**}(r(h))$,*

$T(A) \subset \{h\}'$ and there exists a Jordan *-homomorphism $S : A \to E_2^{**}(r(h))$ such that

$$T = L(h, r(h))S.$$

□

The problem of describing general orthogonality preserving linear operators from a JB*-algebra to a JB*-triple remained open for some time. For the announced characterisation of orthogonality preservers between C*-algebras we take advantage of local theory of JB*-triples. We recall that if a is an element in a C*-algebra which is not normal, then a description of the C*-subalgebra generated by a as a $C_0(L)$-space cannot be obtained. Fortunately, if we consider "subtriples" the situation is different, as we have seen in chapter 2

Let $T : A \to B$ be an orthogonality preserving linear operator between two C*-algebras and let $h = T^{**}(1)$. Again, we use orthogonal forms and orthogonality additive polynomials, together with the triple functional calculus on the element h in B to prove that certain commutativity relations between $h, h^*, r(h), r(h)^*$ and all elements in the range of T hold. Local theory is then used to find a sequence $T_n : A \to B^{**}$ of orthogonality preserving linear operators such that, for each natural n, $T_n(1)$ is von Neumann regular, and w*-$\text{lím}_n T_n(a) = T(a)$, for every a in A. By Theorem 3.2.6 above, for each natural n, the operator $S_n = L(b_n, r(h_n))T_n$ is a Jordan *-homomorphism (and hence contractive). Since the sequence (S_n) is bounded, then the assignment $z \mapsto S(z) = w^* - \text{lím}\, S_n(z)$ defines a Jordan *-homomorphism such that $T = L(h, r(h))S$.

Theorem 3.2.9 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let*

$T : A \to B$ be an orthogonality preserving linear operator between two C^*-algebras and let $h = T^{**}(1)$. Then

a) $h^*T(z) = T(z^*)^*h, hT(z^*)^* = T(z)h^*$,

b) $r(h)^*T(z) = T(z^*)^*r(h)$, *and* $r(h)T(z^*)^* = T(z)r(h)^*$.

Furthermore, there exists a triple homomorphism $S : A \to B^{**}$ such that

$$T(z) = L(h, r(h))S(z) = \frac{1}{2}(hr(h)^*S(z) + S(z)r(h)^*h)$$

for all $a \in A$. □

Definition 3.2.10 *Let $T : E \to F$ be a linear operator between JB^*-triples. We say that T* preserves zero-triple products *if $\{T(x), T(y), T(z)\} = 0$ in F whenever $\{x, y, z\} = 0$ in E.*

It is easy to see that every zero-triple product preserving operator is orthogonality preserving. Conversely, by Theorem 3.2.9 every orthogonality preserving linear operator between C*-algebras preserves zero-triple products.

Corollary 3.2.11 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let $T : A \to E$ be a linear operator from a C^*-algebra to a JB^*-triple. Then T preserves orthogonality if, and only if, T preserves zero-triple products.* □

The reader may be wondering if there is some relation between zero-product preserving and orthogonality preserving linear operators. It is not hard to find a example of a linear operator that preserves orthogonality but does not preserves zero products. Indeed, let $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the operator given by

$T(x) = ux, (x \in M_2(\mathbb{C}))$, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Clearly T is a triple homomorphism and hence orthogonality preserving, but taking $x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, we have $xy = yx = 0$ and $T(y)T(x) \neq 0$.

Inspired by an assertion contained in the proof in the main theorem in [184] for which we were not able to find a reference, we also studied, in the final section of [34], those linear operators between C*-algebras that preserve cubes of self adjoint elements.

Corollary 3.2.12 [M. Burgos, F.J. Fernández-Polo, J. Garcés, J. Martínez-Moreno, A.M. Peralta, J. Math. Ann. Appl., 2008] *Let J be a JB*-algebra, E a JB*-triple and $T : J \to E$ a linear operator. The following statements are equivalent:*

a) T is a triple homomorphism;

b) $T(a^{[3]}) = T(a)^{[3]}$, por every a in J_{sa};

c) T preserves orthogonality on J_{sa} and $T^{**}(1)$ is a tripotent. □

In [35], M. Burgos, F.J. Fernández-Polo, A.M. Peralta and the author of this thesis solved one of the problems that remained open in [34], this problem is a general description of orthogonality preservers from a JB*-algebra to a JB*-triple. One of the novelties we introduce in [35] is the use of the *multiplier algebra* in the study of problems related with orthogonality.

Let A be a C*-algebra (respectively, a JB*-algebra). The *multiplier algebra* of A, $M(A)$, is the set of all elements x in A^{**}, such that $xA, Ax \subset A$ (respectively, $x \circ A \subset A$). It is well known that $M(A)$ is a unital C*-algebra (respectively, a JB*-algebra).

Two orthogonal elements in the multiplier algebra can be approximated (in the weak*-topology) by orthogonal nets in A, that is, for a, b in $M(A)$ with $a \perp b$ there exist nets $(a_\lambda), (b_\mu)$ such that $a_\lambda \perp b_\mu, \forall \lambda, \mu$ and a_λ (respectively b_μ) converges in the weak*-topology to a (respectively, to b). This property makes $M(A)$ the suitable unital C*-algebra (or JB*-algebra) to extend orthogonality preserving operators (or orthogonal forms, or orthogonally additive polynomials) in such a way that the extension keeps the property of being orthogonality preserving (respectively, an orthogonal form, or an orthogonally additive polynomial). If a and b lie in $M(A)_{sa}$, then the nets $(a_\lambda), (b_\lambda)$ can be found so that they lie in A_{sa} (see the proof of Proposition 3.1 in [35]).

Let $T : A \times \ldots \times A \to \mathbb{C}$ be a symmetric n-linear form such that the polynomial $P(x) = T(x, \ldots, x)$ is orthogonally additive on A_{sa} and denote by $T^{**} : A^{**} \times \ldots \times A^{**} \to \mathbb{C}$ the Aron-Berner (o Arens) extension of T. We prove in [35, Proposition 3.1] that the polynomial $R : M(A) \to \mathbb{C}, x \mapsto R(x) = T^{**}(x, \ldots, x)$ is orthogonally additive on $M(A)_{sa}$. We use this fact to give a simplified proof of Theorem 3.2.5 (compare [35, Section 3]).

In a similar way, we prove that the extension of an orthogonality preserving operator from a JB*-algebra J, to its multiplier algebra is orthogonality preserving.

The following result ([35, Corollary 4.1]) is the key tool to generalise the characterisation of orthogonality preservers between C*-algebras to the current setting of JB*-algebras:

Proposition 3.2.13 [M. Burgos, F.J. Fernández-Polo, J. Garcés, A.M. Peralta, Asian-European J. Math., 2009] *Let J be a JB*-algebra, E a JB*-triple and $T : J \to E$ an orthogonality preserving operator. Then for $h = T^{**}(1)$, the following assertions hold:*

a) $\{T(x), h, h\} = \{h, T(x^*), h\}$, *for all* $x \in J$;

b) $T(J_{sa}) = E_2^{**}(r(h))_{sa}$;

c) *For each* $a \in J_{sa}$, $T(a)$ *and* h *operator commute in* $E_2^{**}(r(h))$;

d) *When* h *is a tripotent, then* $T : A \to E_2^{**}(r(h))$ *is a Jordan***-homomorphism, in particular* T *is a triple homomorphism.* □

Finally, we generalise Theorem 3.2.9 to the Jordan setting.

Theorem 3.2.14 [M. Burgos, F.J. Fernández-Polo, J. Garcés, A.M. Peralta, Asian-European J. Math., 2009] *Let* $T : J \to E$ *be an operator from a JB***-algebra to a JB***-triple and let* $h = T^{**}(1)$. *Then the following are equivalent:*

a) T *is orthogonality preserving.*

b) *There exists a (unital) Jordan* **-homomorphism* $S : M(J) \to E_2^{**}(r(h))$ *such that* $S(x)$ *and* h *operator commute and* $T(x) = \{h, r(h), S(x)\}$, *for every* $x \in J$. □

As a consequence of the above theorem, every orthogonality preserving operator from a JB*-algebra to a JB*-triple preserves zero-triple products. We actually have:

Corollary 3.2.15 [M. Burgos, F.J. Fernández-Polo, J. Garcés, A.M. Peralta, Asian-European J. Math., 2009] *Let* $T : J \to E$ *be an operator from a JB***-algebra to a JB***-triple. Then* T *is orthogonality preserving if, and only if,* T *preserves zero-triple products.*

Chapter 4

Automatic continuity

Results assuring automatic continuity of certain classes of linear mappings between Banach algebras have focussed the attention of experts during the last 60 years. In these kind of problems the goal is to find algebraic conditions for a map that guarantee that it is (*automatically*) continuous.

It is not easy to locate exactly the origins of the automatic continuity problems, however, one of the first results ensuring automatic continuity for homomorphisms from a certain class of Banach algebras is probably that obtained by M. Eidelheit in 1940 (see [59] 2.5.10), who showed that every monomorphism from $L(X)$ (where X is a Banach space) into a Banach algebra is automatically continuous. In this result we see how a purely algebraic property (being an injective homomorphism from $L(X)$) determines a topological one (being continuous). Subsequent studies by I.M. Gelfand and C.E. Rickart pointed out again the connections between algebraic and topological structures (compare [80] and [159], respectively). One of the most important results in this area is that due to B.E. Johnson stating that *Any homomorphism onto a semisimple Banach algebra is automatically continuous* (see [105]).

In the class of C*-algebras, results on automatic continuity of *-homomorphisms are in the origin of the theory itself. In 1943, I.M. Gelfand and M.A. Naimark [81] prove that every *-isomorphism between C*-algebras is automatically isometric (this result would be later generalised to the setting of JB*-triples by W. Kaup in [119]). A few years later, in 1947, Segal proves that every *-representation of a C*-algebra (that is, every *-homomorphism into $L(H)$, where H is a complex Hilbert space) is automatically continuous (see [166]).

The monographs [174], [50] and the surveys [49], [182] provide more information on automatic continuity in various settings.

As we have already mentioned in Chapter 3, automatic continuity results for orthogonality preserving linear mappings have been also explored. There is the conjecture, supported by many affirmative answers in particular cases (in the settings of abelian C*-algebras, Banach lattices, or $C(K, E)$-spaces, for instance) that a biorthogonality preserving (or biseparating) linear bijection should be automatically continuous.

A *standard operator algebra* is a norm-closed subalgebra of $L(X)$ (where X is a Banach space) containing all rank-one operators. In [12], J. Araujo and K. Jarosz proved that linear bijections between standard operator algebras that preserve zero products in both directions are automatically continuous. In the same paper they pose the conjecture affirming that a linear mapping between C*-algebras preserving zero products in both directions must be automatically continuous. The papers [42], [43], [103], [128] and [179] give partial answers to the conjecture posed by Araujo and Jarosz. We also consider this conjecture.

4.2. Biorthogonality preservers on C*-algebras

In [38] M. Burgos, A. M. Peralta and the author of this thesis study biorthogonality preserving linear mappings between C*-algebras.

Definition 4.2.1 *Let $T : A \to B$ be a linear mapping between C*-algebras. We say that T is* biorthogonality preserving *if T satisfies the following property:*

$$T(a) \perp T(b) \Longleftrightarrow a \perp b.$$

Every biorthogonality preserving linear mapping between C*-algebras is injective. Indeed, let $T : A \to B$ be a biorthogonality preserving linear mapping and let a be an element in A such that $T(a) = 0$. Then $T(a) \perp T(b)$, for every b in A. Since T is biorthogonality preserving we have that $a \perp b$, for every b in A. In particular, $a \perp a$, equivalently $aa^* = a^*a = 0$, and hence $a = 0$.

4.2.1. The case of dual C*-algebras

K. Vala proves in [181], the following characterisation of compact operators on $L(X)$, where X is a Banach space.

Theorem 4.2.2 [K. Vala, Ann. Acad. Sci. Fenn. Ser., 1967] *Let X be a Banach space and T, T' nonzero elements in $L(X)$. The operator $S \mapsto TST'$ is compact if, and only if, both T and T' are compact.* $\square$

Inspired by this characterisation J.C. Alexander, introduces in [7] the following concept of *compact element* in a Banach algebra.

Definition 4.2.3 *Let A be a Banach algebra and let a be an element in A. Then a is said to be* compact *if the mapping $U_a : A \to A$, $x \mapsto U_a(x) := axa$ is compact.*

We say that a Banach algebra A is compact if all its elements are compact elements . It was J.C. Alexander who gave a characterisation of compact C*-algebras in [7]. We recall that given a complex Hilbert space H, we denote by $K(H)$ the space of all compact operators on H.

Theorem 4.2.4 [Alexander, Proc. London Math. Soc., 1968] *Let A be a compact C*-algebra, then there exists a family of complex Hilbert spaces $(H_\lambda)_\lambda$ such that A is C*-isomorphic to the c_0-sum*

$$A = \bigoplus_{\lambda}^{c_0} K(H_\lambda).$$

□

We recall that a projection p (respectively, a tripotent e) in a C*-algebra (respectively, in a JB*-triple) A is said to be minimal when $pAp = \mathbb{C}p$ (respectively, $A_2(e) = \mathbb{C}e$). The *socle* of A, soc(A), is defined as the linear span of all minimal tripotents in A (it coincides with the linear span of minimal projections in A when the latter is a C*-algebra).

It is also due to J.C. Alexander that compact C*-algebras can be characterised as those C*-algebras with dense socle (compare [7, Corollary 8.3]), and the latter are precisely the so-called *dual C*-algebras* considered by Kaplansky (see [117]).

The *ideal of compact elements* in A, $K(A)$, is defined as the norm closure of $\mathrm{soc}(A)$. Of course, $K(A)$ is a compact C*-algebra. Compact elements in C*-algebras were also studied by K. Ylinen in [188]. The following generalisation of the spectral theorem for self-adjoint compact operators an a Hilbert space is due to K. Ylinen (compare [188, Theorem 3.11]):

Theorem 4.2.5 [K. Ylinen, Ann. Acad. Sci. Fenn. Ser., 1968] *Let a be a self-adjoint compact element in a C*-algebra A. Then there exist a sequence $(\alpha_n) \in c_0$ and a family of mutually orthogonal minimal projections (p_n) such that*

$$a = \sum_n \alpha_n p_n.$$

□

In [35], in collaboration with M. Burgos and A. M. Peralta we study biorthogonality preserving linear mappings between dual C*-algebras.

In a first step, we provide a characterisation of all bounded biorthogonality preserving linear mappings between von Neumann algebras (it can be essentially deduced from Theorem 3.2.9).

Corollary 4.2.6 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] *Let $T : A \to B$ be a bounded linear operator between von Neumann algebras. For $h = T(1)$ and $r = r(h)$ the following assertions are equivalent:*

a) T is a biorthogonality preserving linear surjection.

b) *h is invertible and there exists a unique triple isomorphism $S : A \to B$ such that $h^*S(z) = S(z^*)h, hS(z^*)^* = S(z)h^*$, and*

$$T(z) = \frac{1}{2}(hr(h)^*S(z) + S(z)r(h)^*h)$$
$$= hr(h)^*S(z) = S(z)r(h)^*h,$$

for all z in A.

c) *h is positive and invertible in $B_2(r)$ and there exists a unique Jordan *-isomorphism $S : A \to B_2(r) = b$ satisfying $S(1) = r$, $h \in Z(B)$ and $T(z) = h \circ_r S(z)$ for all $z \in A$.* □

Now, let $T : A \to B$ be a biorthogonality preserving linear surjection between C*-algebras. In [38, Theorem 5] we prove that such a T maps minimal projections in A (if any) to scalar multiples of minimal tripotents, and thus it maps the socle of A into the socle of B.

After various lemmas we are able to establish the following result, which provides a necessary and sufficient condition to assure continuity of the restriction of a biorthogonality preserving linear surjection to the ideal of compact elements of A:

Proposition 4.2.7 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Applic., 2010] *Let T be a biorthogonality preserving linear surjection between C*-algebras. Then $T_{|K(A)}$ is continuous if and only if the set*

$$\{\|T(p)\| : p \text{ is a minimal projection in } A\}$$

is bounded. □

We recall that the C*-algebra generated by two non orthogonal minimal projections coincides with $M_2(\mathbb{C})$ (compare [151] or [157]).

Let p, q be two minimal projections that are not orthogonal. Since $A_{p,q} \cong M_2(\mathbb{C})$, $T_{|A_{p,q}} : A_{p,q} \to T(A_{p,q})$ is a biorthogonality preserving linear surjection and $Z(M_2(\mathbb{C})) = \mathbb{C}$, then $T_{|A_{p,q}}$ must be a scalar multiple of an isometry (by Corollary 4.2.6). Thus $\|T(p)\| = \|T(q)\|$.

Under these conditions, if the set

$$\{\|T(p)\| : p \text{ is a minimal projection in } A\}$$

is unbounded, we can find a sequence of mutually orthogonal minimal projections (p_n) in A such that $\|T(p_n)\| > n$. The element $a = \sum_m \frac{1}{\sqrt{m}} p_n$ lies in $K(A)$ and satisfies $\|T(a)\| \geq \sqrt{m}, \forall m \in \mathbb{N}$, which is a contradiction. We therefore have:

Theorem 4.2.8 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] *Every biorthogonality preserving linear surjection between compact C*-algebras is automatically continuous.* $\square$

4.2.2. The case of von Neumann algebras

From [82], [138], [139] and [150] it is known that there exist many examples of C*-algebras where every element is a finite linear combination of projections. Recall that a unital C*-algebra is *properly infinite* if it contains two orthogonal projections equivalent to the identity (i.e. it contains two isometries with mutually orthogonal range projections). It follows by [133, Corollary 2.2] (see also [150]) and [82, Theorem 2.2.(a)] that every element in a properly infinite C*-algebra or in a von Neumann algebra of type II$_1$. can be expressed as a finite linear combination of projections. Surprisingly, every orthogonality preserving linear mapping from such a C*-algebra (whenever it is unital) is automatically continuous.

Theorem 4.2.9 [*M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] Let $T : A \to B$ be an orthogonality preserving linear mapping between C^*-algebras, where A is unital. Suppose that every element in A is a finite linear combination od projections, then T is continuous.* $\square$

Two projections p, q in a von Neumann algebra A are said to be *Murray-von Neumann equivalent* if there exists a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* = q$. We denote this fact by $p \sim q$. If a projection q is equivalent to a projection $q_1 \leq q$ then we say that p is subequivalent to q, denoted by $p \lesssim q$. If $p \lesssim q$ but p is not equivalent to q then we write $p \prec q$. Clearly, the relation $\sim$ is an equivalence relation. The modular theory classifies von Neumann algebras in terms of this relations.

A projection q in a von Neumann algebra is said to be *finite* if $p \sim q \leq p$ implies $p = q$. Otherwise, it is said to be *infinite*. A projections p is said to be *purely infinite* if there is no nonzero finite projection $q \leq p$ in A. If zp is infinite for every central projection $z \in A$ with $zp \neq 0$, then p is said to be *properly infinite*. If pAp is abelian then p is said to be abelian.

A von Neumann algebra is said to be *finite*, *infinite*, *properly infinite*, or *purely infinite* according to the property of the identity projection 1.

A von Neumann algebra A is said to be of *type I* if every nonzero central projection in A majorizes a nonzero abelian projection in A. If there is no nonzero finite projection in A, that is, if A is purely infinite, the it is said to be of type III. If A has no nonzero abelian projection and if every nonzero central projection in A majorizes a nonzero finite projection A, then it is said to be of type II. If A is finite and of type II, then it is said to be of type II_1. If A is of type II and has no nonzero central finite projection, the A is said to be of type II_∞.

Every von Neumann algebra can be uniquely decomposed into an orthogonal sum of summands type I, type II_1, type II_∞, and type III.

Every element in a properly infinite C*-algebra or a type II_1 von Neumann algebra can be expressed as a finite linear combination (compare [133, Corollary 2.2] or [150] and [82, Theorem 2.2 (a)], repsectively). As a consequence of Theorem 4.2.9 we have the following:

Corollary 4.2.10 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] *Let A be a properly infinite unital C*-algebra or a type II_1 von Neumann algebra. Every orthogonality preserving linear map from A to another C*-algebra is continuous.* □

Let A be a von Neumann algebra, then by the Murray-von Neumann decomposition, A decomposes as

$$A = A_{I_{fin}} \oplus^\infty A_{I_\infty} \oplus^\infty A_{II_1} \oplus^\infty A_{II_\infty} \oplus^\infty A_{III}.$$

The summands $A_{I_{fin}}$ and A_{II_1} are finite von Neumann algebras, while the summands A_{I_∞}, and A_{III} are properly infinite C*-algebras. Let $A_1 = A_{I_{fin}} \oplus^\infty A_{II_1}$, $A_{p\infty} = A_{I_\infty} \oplus^\infty A_{III}$.

The finite part deserved its own argument. The following result is part of [38, Proposition 18].

Proposition 4.2.11 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Applic., 2010] *Every biorthogonality preserving linear surjection between von Neumann algebras one of which is finite is continuous.*

Now, let $T : A \to B$ be a biorthogonality preserving linear surjection. By Corollary 4.2.10, $T|_{A_{II_1}} : A_{II_1} \to B$ and $T|_{A_{p\infty}}$:

$A_{p\infty} \to B$ are continuous. Let $A_2 = A_{II_1} \oplus^\infty A_{p\infty}$, and $B_2 = B_{II_1} \oplus^\infty B_{p\infty}$, where B_{II_1} and $B_{p\infty}$, are defined from de Murray-von Neumann decompostion of B analogously as we have done for A. Although it is not easy, it can be proved that $T(A_2) = B_2$, and thus, by Proposition 4.2.11, $T_{A_{fin}} : A_{fin} \to B_{fin}$ is continuous.

Theorem 4.2.12 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] *Every biorthogonality preserving linear surjection between von Neumann algebras is continuous.*

Let $T : A \to B$ be a linear mapping surjection such that T and T^{-1} are preserve zero products. It is not hard to see that if T is symmetric then T is biorthogonality preserving. As a consequence of the previous theorem, we could give a partial affirmative answer to the conjecture posed by J. Araujo and K. Jarosz in [12].

Corollary 4.2.13 [M. Burgos, J.J. Garces and A.M. Peralta, J. Math. Ann. Appl., 2010] *Let $T : A \to B$ be a biseparating symmetric linear map between von Neumann algebras. Then T is continuous.* □

4.3. Biorthogonality preservers on atomic JBW*-triples

In collaboration with M. Burgos and A.M Peralta we successfully explored biorthogonality preserving linear surjections between "weakly compact JB*-triples" and also between "atomic JBW*-triples".

A subspace I of a JB*-triple E is said to be a triple ideal if $\{E, E, I\} + \{E, I, E\} \subseteq I$. By Proposition 1.3 in [31], I is a triple ideal if, and only if, $\{E, E, I\} \subseteq I$.

An important class of JB*-triples is given by the Cartan factors. A JBW*-triple is called a *factor* if it contains no proper weak*-closed ideals. The Cartan factors can be characterised as those factors containing a minimal tripotent (compare [122]). These come, up to isomorphisms, in the following six types (see [122] and [72]):

Type 1 Cartan factors

A Cartan factor of type I, denoted by $I_{n,m}$, is a JB*-triple of the form $L(H,H')$, where $L(H,H')$ stands for the space of bounded linear operators between two complex Hilbert spaces H, H' of dimension n and m, respectively (we note that n and m can be infinite). $L(H,H')$ is endowed with the triple product defined by $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$.

Type 2 and type 3 Cartan factors

We recall that given a conjugation j on a n-dimensional (where n can be infinite) complex Hilbert space H (i.e., a conjugate linear mapping j on H such that $j^2 = Id_H$), we define the linear involution $x \mapsto x^t := jx^*j$ on $L(H)$. A Cartan factor of type 2 (respectively, type 3), denoted by II_n (respectively, III_n), is the subtriple of $L(H)$ whose elements are the t-skew-symmetric (respectively, t-symmetric) operators. It is well known that II_n and III_n are, up to isomorphisms, independent of the conjugation j on H.

Type 4 Cartan factors

A Cartan factor of type 4, denoted IV_n (also called a *complex spin factor*), is an n-dimensional complex Hilbert space endowed with an conjugation $x \mapsto \overline{x}$, where the triple product and norm are given by

$$\{x,y,z\} = (x|y)z + (z|y)x - (x|\overline{z})\overline{y}$$

and $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\overline{x})|^2}$, respectively.

Before defining Cartan factors of type 5 and 6 (known as exceptional Cartan factors) we need to introduce the algebra of complex octonions.

Given a field $\mathbb{K}$, the so-called *Cayley-Dickson "split" algebras* (with divisors of zero) $C(\mathbb{K})$, over $\mathbb{K}$, can be represented in the following matricial form:

Matrices of the form $\left(\begin{array}{cc}\alpha & x\\ y & \beta\end{array}\right)$ where $\alpha, \beta \in \mathbb{K}$ and x, y lie in $\mathbb{K}^3$. The sum and product by scalar are the usual for matrices and the product is given by:

$$\left(\begin{array}{cc}\alpha & x\\ y & \beta\end{array}\right)\left(\begin{array}{cc}\gamma & z\\ t & \delta\end{array}\right) = \left(\begin{array}{ll}\alpha\gamma + (x \mid t) & \alpha z + \delta x - y \times t\\ \gamma y + \beta t + x \times z & \beta\delta + (y \mid z)\end{array}\right)$$

where

$$((x_1, x_2, x_3) \mid (y_1, y_2, y_3)) := x_1\ y_1 + x_2\ y_2 + x_3\ y_3$$

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The algebra of *complex octonions*, denoted by $\mathbb{O}^{\mathbb{C}}$, is the Cayley-Dickson split algebra over $\mathbb{C}$ (cf. [194, Theorem 2.7]).

We can endow $C(\mathbb{C})$ with the linear involution, $-$, given by

$$\overline{\left(\begin{array}{cc}\alpha & x\\ y & \beta\end{array}\right)} = \left(\begin{array}{cc}\beta & -x\\ -y & \alpha\end{array}\right)$$

When $\mathbb{K} = \mathbb{C}$ another involution on $C(\mathbb{K})$ can be defined by

$$\left(\begin{array}{cc}\alpha & x\\ y & \beta\end{array}\right)^* = \left(\begin{array}{cc}\overline{\alpha} & \overline{y}\\ \overline{x} & \overline{\beta}\end{array}\right)$$

where by $\overline{x}$ we mean the mapping that acts by conjugating componentwise.

We denote by $H_3(\mathbb{O}^{\mathbb{C}})$ the set of all those 3×3 matrices with coefficients onver $\mathbb{O}^{\mathbb{C}}$ which are symmetric for the (linear) involution $(a_{i,j})^t := (\overline{a_{j,i}})$. The algebra $H_3(\mathbb{O}^{\mathbb{C}})$ is a Jordan algebra when endowed with the Jordan product $a \circ b := \frac{1}{2}(ab + ba)$. Furthermore, the involution $(a_{i,j})^* := (a^*_{j,i})$ makes $H_3(\mathbb{O}^{\mathbb{C}})$ into a unital JB*-algebra (see [89, Remark 3.1.8], [8], [169], [186]).

The *Cartan factor of type 6*, which we denote by **VI** is the algebra $H_3(\mathbb{O}^{\mathbb{C}})$.

Finally, the *Cartan factor of type 5*, denoted V, is the Jordan triple $M_{1,2}(\mathbb{O}^{\mathbb{C}})$, which is a subtriple of the Cartan factor **VI**, via the monomorphism

$$(a, b) \mapsto \begin{pmatrix} 0 & a & b \\ \overline{a} & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix} \quad \text{for every } a, b \in \mathbb{O}^{\mathbb{C}}.$$

We recall that the Cartan factors of type $\mathbf{I_{n,n}}$, $\mathbf{II_n}$, with n even or infinite, $\mathbf{III_n}$ and $\mathbf{IV_n}$, can be seen as JB*-algebras (they have an unitary element).

The Cartan factors are the building blocks of the so-called atomic JB*-triples. A JB*-triple is said to be *atomic* if it coincides with the weak*-closed ideal generated by its minimal tripotents. Every atomic JB*-triple is an l_∞-sum of Cartan factors (see [72]).

In [31], L.J. Bunce and C.H. Chu describe the so-called "compact and weakly compact JB*-triples". These are the natural generalisation to the class of JB*-triples of the notion of dual C*-algebras (also to that of "dual JB-algebras" previously studied by L.J. Bunce in [30]).

Let E be a JB*-triple. An element x in E said to be *weakly compact* (respectively, *compact*) if the mapping $Q(x) : E \to E$ is weakly compact (respectively, compact). We denote by $K(E)$ the JB*-subtriple of E generated by its minimal tripotents. It is proved in Proposition 4.7 in [31] that $K(E)$ is a norm-closed triple ideal of E and that it coincides with the set of weakly compact elements of E.

An important subclass of weakly compact JB*-triples is the one formed by the so-called *elementary JB*-triples*. The elementary JB*-triples play the same role in the description of weakly compact JB*-triples that Cartan factors played in the description atomic JB*-triples.

For a Cartan factor C we define the *elementary* JB*-triple of the corresponding type to be $K(C)$. Thus, the elementary JB*-triples K_i (for $i = 1, \ldots, 6$) are defined as follows: $K_1 = K(H, H')$ (that is, the compact operators between the complex Hilbert spaces H and H'); $K_i = C_i \cap K(H)$ for $i = 2, 3$ and $K_i = C_i$ for $i = 4, 5, 6$.

If follows from [31, Lemma 3.3 and Theorem 3.4] that a JB*-triple E is weakly compact if and if one of the following statements holds:

a) $K(E^{**}) = K(E)$.

b) $K(E) = E$.

c) E is a c_0-sum of elementary JB*-triples.

Let E be a JB*-triple and $S \subseteq E$. The set S is said to be orthogonal if $0 \notin S$ and $x \perp y$ for every x, y in S. The minimal cardinal number r satisfying $\mathrm{card}(S) \leq r$ for every orthogonal set $S \subseteq E$ is called the rank of E (denoted $r(E)$).

In Remark 4.13 of [39] we give an example of an unbounded biorthogonality preserving linear bijection between rank-one JB*-triples. Thus, when we study automatic continuity of biorthogonality preserving linear surjections we shall assume that they do not have rank-one summands.

The following generalisation of Corollary 4.2.6 follows from the characterisation of orthogonality preserving operator described in Theorem 3.2.14 (see [39]).

Theorem 4.3.1 [M. Burgos, J.J. Garces and A.M. Peralta, Studia Math., 2011] *Let $T : J \to E$ be a surjective linear operator from a JB*-algebra onto a JBW*-triple and let h denote $T(1)$. Then T is biorthogonality preserving if, and only if, $r(h)$ is a unitary tripotent in E, h is an invertible element in the JBW*-algebra $E = E_2(r(h))$, and there exists a Jordan *-isomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ_{r(h)} S$. Further, if J is a factor (i.e. $Z(J) = \mathbb{C}1$), then T is a scalar multiple of a triple isomorphism.* □

In [39, Theorem 4.11] we generalise Proposition 4.2.7 by proving the following:

Proposition 4.3.2 [*M.Burgos, J.J. Garces and A.M. Peralta, Studia Math., 2011] Let $T : E \to F$ be a biorthogonality preserving linear surjection between JB*-triples, where E is weakly compact. Then T is continuous if, and only if, the set*

$$\mathcal{T} = \{\|T(e)\| : e \text{ minimal tripotent in } E\}$$

is bounded. □

If J is a weakly compact JB*-triple which is a JB*-algebra, then in the above proposition it is enough to prove that the set $\mathcal{T} = \{\|T(p)\| : p \text{ minimal projection in } J\}$ is bounded.

The idea behind the proofs is the same as in the case of C*-algebras, however to achieve it requires more work. If $\mathcal{T} = \{\|T(e)\| : e \text{ minimal tripotent in } E\}$ is unbounded, then there exists a sequence of minimal tripotents (e_n) such that $\|T(e_n)\|$ diverges.

Let us assume that we have proved that for any two minimal tripotents e_1, e_2 which are not orthogonal (notice that this means that they lie in the same summand), we have $\|T(e_1)\| = \|T(e_2)\|$. Then we can assume that the elements in the sequence (e_n) must be mutually orthogonal. Then the element $z = \sum_n \|T(e_n)\|^{-\frac{1}{2}} e_n$ lies in E and $\|T(z)\| \geq \sqrt{\|T(e_m)\|}, \forall m \in \mathbb{N}$, which is imposible.

We shall proved the desired property. If E is a JB*-algebra, then it is enough to prove that if p and q are two minimal projections which are not orthogonal, then $\|T(p)\| = \|T(q)\|$.

We need to distinguish the following cases:

1. E is finite dimensional:

2. E is a type 1 Cartan factor;

3. E is a weakly compact JB*-algebra.

Of course, in the first case T is continuous, and a scalar multiple of a triple isomorphism.

Let $T : E \to F$ be a biorthogonality preserving from a factor of type $I_{n,m}$, where $n, m \geq 2$. If e_1, e_2 are two non-orthogonal tripotents, we prove that the inner ideal generated by e_1 and e_2, $E(e_1, e_2)$, is isomorphic to $M_2(\mathbb{C})$ and that $T(E(e_1, e_2))$ is a subtriple of F. As a consequence $T_{|E(e_1,e_2)}$ is a scalar multiple of an isometry (this follows from Theorem 4.3.1 and the Banach-stone Theorem for JB*-triples) and $\|T(e_1)\| = \|T(e_2)\|$. By the comments after Proposition 4.2.7 we deduce that T is continuous.

Actually, T must be an scalar multiple of a triple isomorphism (see Theorem 4.2 and Corollary 4.7 in [39]).

Let J be a JB*-algebra. In [39, Lemma 4.8], we prove that the subtriple of J generated by two non orthogonal projections p, q, denoted by $J_{p,q}$, is *-isomorphic to $S_2(\mathbb{C})$, the type 3 Cartan factor of all symmetric operators on a two dimensional complex Hilbert space.

Let $T : J \to E$ be a biorthogonality preserving linear bijection from a weakly compact JB*-algebra onto a JB*-triple. Let p and q be two minimal projections which are not orthogonal and $J_{p,q} \equiv S_2(\mathbb{C})$ the subtriple generated by them. We prove that $T(J_{p,q})$ is a subtriple of F (see the proof of [39, Theorem 4.9]) and $T_{|J_{p,q}}$ is a scalar multiple of an isometry. Thus $\|T(p)\| = \|T(q)\|$ and T is continuous.

Any two minimal tripotents e, f in E which are not orthogonal lie in the same summand of E. As a consequence of a), b) and c) we have $\|T(e)\| = \|T(f)\|$, which gives the required statement.

Theorem 4.3.3 [M.Burgos, J.J. Garces and A.M. Peralta, Studia Math., 2011] *Every biorthogonality preserving linear surjection between weakly compact JB*-triples containing no rank-one summands is continuous.*

We also prove that a biorthogonality preserving linear surjection form a Cartan factor (of rank greater than 1) onto a JB*-triple is a scalar multiple of an isometry.

In the final section of [39] we deal with atomic JB*-triples.

It is well known that the predual of $L(H)$ coincides with the so-called *trace class* operators. In [39, Proposition 5.1] we describe the predual of the Cartan factors of type 1, 2 and 3.

Proposition 4.3.4 [M. Burgos, J.J. Garces and A.M. Peralta, Studia Math., 2011] *Let C be an infinite dimensional Cartan factor of type 1, 2 or 3. For each $\varphi \in C_*$, there exists a sequence $(\lambda_n) \in l_1$ and a sequence (u_n) of mutually orthogonal minimal tripotents in C such that*

$$\|\varphi\| = \sum_{n=1}^{\infty} \text{ and } \varphi(x) = \sum_n \lambda_n \varphi_n(x) \ (x \in C)$$

where for each $n \in \mathbb{N}$, $\varphi_n(x) u_n = P_2(u_n)(x)$ $(x \in C)$.

Finally, we use the description of biorthogonality preserving linear mappings between factors and Proposition 4.3.4 above to prove that biorthogonality preserving linear bijections between atomic JBW*-triples with no rank-one summands are also continuous.

Theorem 4.3.5 [M.Burgos, J.J. Garces and A.M. Peralta, Studia Math., 2011] *Every biorthogonality preserving linear surjection between atomic JBW*-triples containing no rank-one summands is continuous.*

Chapter 5

Minimality of triple norm topology and a Kaplansky Theorem for JB*-triples

In 1949, I. Kaplansky proves that if $\|.\|$ is a norm making $C(K)$ a normed algebra, then $\|.\| \geq \|.\|_\infty$ (where $\|.\|_\infty$ denotes the *sup* norm), equivalently, every (non necessarily continuous) monomorphism from $C(K)$ into a normed algebra is automatically *bounded bellow*.

Related to Kaplansky's Theorem we find the results by C.E. Rickart on uniqueness of norm topology in certain Banach algebras (see [159]). In the just quoted paper examples of Banach algebras enjoying the property that every complete multiplicative norm is equivalent to the original norm are given. One of the most important results in this subject is that by B.E. Johnson, who proved in [105] that every semisimple Banach algebra has a unique Banach algebra norm. We note that from Johnsons's result, every monomorphism (with closed range) from a semisimple

Banach algebra is both continuous and bounded bellow.

Under the additional assumption of continuity, versions of Kaplansky's Theorem have been obtained in different settings.

Definition 5.1.1 *Let A be a Banach (or a Jordan) algebra with norm $\|.\|$. We say that A has* minimality of algebraic norm topology *(abbreviated by MOANT) if for any multiplicative norm $\|.\|_2$ (i.e. $\|ab\|_2 \leq \|a\|_2\,\|b\|_2$), with $\|.\|_2 \leq \|.\|$ there exists $M > 0$ such that $M\|.\|_2 \geq \|.\|$.*

We say that A has minimality of the norm if $\|.\|_2 \leq \|.\|$ implies $\|.\|_2 = \|.\|$.

Of course, minimality of the norm implies minimality of the norm topology.

Let $T : A \to B$ be an homomorphism between Banach algebras. The assignment

$$\|.\|_2 : A \to \mathbb{R}^+, a \mapsto \|a\|_2 := \|T(a)\|,$$

defines a seminorm on A. Since T is an homomorphism then $\|.\|_2$ also satisfies the property $\|ab\|_2 \leq \|a\|_2\|b\|_2$, that is, $\|.\|_2$ is a multiplicative seminorm. Thus every homomorphism from A defines a multiplicative seminorm on A.

It is clear that $\|.\|_2$ is a norm if, and only if, T is a monomorphism. If A has a MOANT then every continuous monomorphism from A to a Banach algebra is bounded bellow.

Let us assume that every monomorphism from A to a Banach algebra is bounded bellow and let $\|.\|_2$ be a multiplicative and $\|.\|$-continuous norm on A. Then the identity mapping from $(A, \|.\|)$ to $(A, \|.\|_2)$ is a continuous monomorphism and thus bounded bellow, as a consequence there exists $M > 0$ such that $M\|.\| \leq \|.\|_2$.

We have seen that A having MOANT is equivalent to say that every continuous monomorphism from A to a normed algebra is bounded bellow.

The aforementioned Kaplansky's Theorem assures that an abelian unital C*-algebra has minimality of the norm topology (actually a stronger property).

In the setting of general C*-algebras, a noncommutative version of Kaplansky's Theorem was given by S. Cleveland in [46]:

Theorem 5.1.2 [S. Cleveland, Pacific J. Math., 1963] *Let $T : A \to B$ be a monomorphism between C*-algebras, then there exists $M > 0$ such that $M\|T(a)\| \geq \|a\|, \forall a \in A$.* □

A. Bensebah proved in [24] that JB*-algebras have MOANT. This author also posed the question whether every JB*-algebra has minimality of the norm (see [24]).

In [154], J. Pérez, L. Rico and A. Rodriguez-Palacios studied MOANT in the more general setting of "non-commutative JB*-algebras", they also give an affirmative answer to the question posed by Bensebah on the uniqueness of the Jordan norm (see [154, Proposition 11]). More recently, S. Hejazian and A. Nikman obtained an alternative proof of Kaplansky's Theorem for JB*-algebras in [92].

5.1.1. Kaplansky Theorem for JB*-triples

Recently, in collaboration with F.J. Fernández Polo and A. M. Peralta we have obtained a generalisation of Kaplansky's Theorem to the setting of JB*-triples (see [62]).

Let E be a Jordan triple with norm $\|.\|$. We say that the norm $\|.\|$ is multiplicative or a triple norm if $(E, \|.\|)$ is a normed Jordan triple , that is, if there exists $M > 0$ such that $\|\{x, y, z\}\| \leq$

$M\|x\|\|y\|\|z\|$, for every x, y, z in E. Given a multiplicative norm on E we let $N(E)$ or $N(E, \|.\|)$ be the supreme of the set

$$\{\|\{x, y, z\}\| : \|x\|, \|y\|, \|z\| \leq 1 \, x, y, z \in E\}.$$

It is easy to see that, in this case, we can find a norm $\|.\|_2$, equivalent to $\|.\|$, such that $N(E, \|.\|') \leq 1$. We notice that if E is a JB*-triple with norm $\|.\|$, then we have $N(E, \|.\|) = 1$.

We shall say that a normed Jordan triple E has *minimality of the triple norm topology (MOTNT)* if any other (non-necessarily complete) triple norm, dominated by the norm of E, defines an equivalent topology.

We notice that MOTNT admits the following reformulation:

Every bounded triple monomorphism from E to a Jordan-Banach triple is bounded bellow.

We say that E has minimality of the norm if, for every triple norm $\|.\|_2$ on E such that $\|.\|_2 \leq \|.\|$ we have $\|.\|_2 = \|.\|$.

Remark 5.1.3 *In order to study minimality of the norm we can always assume that $N(E, \|.\|_2) = 1$. Indeed, if $N(E, \|.\|_2) > 1$, we define $\|.\|_2' = \frac{1}{N(E, \|.\|_2)} \|.\|_2$, then $\|.\|_2' \leq \|.\|$ and $N(E, \|.\|_2') = 1$. Clearly. $\|.\|_2'$ and $\|.\|_2$ are equivalent triple norms on E that coincide whenever E has minimality of the norm.*

Let A be an associate normed algebra. We denote by $A^{(+)}$ the normed Jordan algebra A equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm. It is easy to see that if $A^{(+)}$ have MOANT then A has MOANT. However, we do not know if the reciprocal statement is, in general, true. By [45, Proposition 3], there exists an associative normed algebra $\mathcal{B}$ such that there exists a norm $\|.\|_1$ on $\mathcal{B}$ for which the Jordan product is continuous but the associative product is discontinuous. In particular, $(\mathcal{B}^{(+)}, \|.\|_1)$ doesn't have MOANT.

In [62, Section 2] we explore the relation between MOANT and MOTNT. Let J be a Jordan algebra. If we endow J with the triple product $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$, it is easy to see that MOANT implies MOTNT and the reciprocal statement holds whenever J is unital.

When A is simple and has a unit, every norm on A making the Jordan product continuous also makes continuous the associative product (compare [45, Theorem 3]). Under this additional hypothesis, we have

$$(A^{(+)}, \|.\|) \text{ has MOANT } \iff (A, \|.\|) \text{ has MOANT.}$$

As we have already mentioned, I. Kaplansky proved that abelian C*-algebras have MOANT. It is natural to ask whether they have MOTNT. The complex statement in the following result was established by K. Bouhya and A. Fernández López in [28, Proposition 13]. In [62] we prove the aforementioned result for real or complex $C_0(L)$-spaces.

Lemma 5.1.4 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let $L \subset \mathbb{R}_0^+$ be a subset of non-negative real numbers satisfying that $L \cup \{0\}$ is a compact. Let $C_0(L)$ denote the Banach algebra of all real or complex valued continuous functions on $L \cup \{0\}$ vanishing at zero (equipped with the supremum norm $\|.\|_\infty$). Suppose that $\|.\|_2$ is a $\|.\|_\infty$-continuous norm on $C_0(L)$ under which $C_0(L)$ is a normed triple system. Then $\|.\|_2$ is equivalent to an algebra norm on $C_0(L)$, and consequently $\|.\|_\infty$ and $\|.\|_2$ are equivalent norms. More concretely, writing $M = \sup\{\|x\|_2 : \|x\|_\infty \leq 1\}$ we have $\|a\|_\infty \leq MN(C_0(L), \|.\|_2)\ \|a\|_2$, for all $a \in C_0(L)$.* □

We remark that the above lemma also shows that real and complex $C_0(L)$-spaces have minimality of the triple norm (see Remark 5.1.3).

As we have already mentioned, S.B. Cleveland applied Kaplansky's Theorem to prove that every continuous monomorphism from a C*-algebra to a normed algebra is bounded below [46, Lemma 5.3], equivalently, every C*-algebra has MOANT. It follows as a consequence of [24, Theorem 1] or [154, Theorem 10] or [92], that JB*-algebras have MOANT. In the setting of (complex) JB*-triples, K. Bouhya and A. Fernández López proved the following result:

Proposition 5.1.5 [K. Bouhya, A. Fernández-Lopez, Proc. London Math. Soc., 1994] [28, Corollary 14] *Let $T : E \to F$ be a continuous triple monomorphism from a JB*-triple to a normed complex Jordan triple. Then T is bounded below. That is, every JB*-triple has MOTNT.* □

In [62] we give a version of the above result to the more general setting of (real) J*B-triples.

We recall that a *real JB*-triple* is a norm-closed real subtriple of a complex JB*-triple (compare [95]). A *J*B-triple* is a real Banach space E equipped with a structure of a real Banach Jordan triple which satisfies $\|\{a, a, a\}\| = \|a\|^3$ and the following additional axioms:

(J^*B1) $N(E) = 1$;

(J^*B2) $\sigma^{\mathbb{C}}_{L(E)}(L(x, x)) \subset [0, +\infty)$ for all $x \in E$;

(J^*B3) $\sigma^{\mathbb{C}}_{L(E)}(L(x, y) - L(y, x)) \subset i\mathbb{R}$ for all $x, y \in E$.

Every closed subtriple of a J*B-triple is a J*B-triple (c.f. [54, Remark 1.5]). The class of J*B-triples includes real (and complex) C*-algebras and real (and complex) JB*-triples. Moreover, in [54, Proposition 1.4] it is shown that complex JB*-triples are precisely those complex Jordan-Banach triples whose underlying real Banach space is a J*B-triple.

T. Dang and B. Russo established a Gelfand theory for J*B-triples in [54, Theorem 3.12]. This Gelfand theory can be refined to show that given an element a in a J*B-triple E, there exists a bounded set $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact such that the smallest (norm) closed subtriple of E containing a, E_a, is isometrically isomorphic to

$$C_0(L, \mathbb{R}) := \{f \in C_0(L), f(L) \subseteq \mathbb{R}\},$$

(see [41, Page 14]).

Let $T : E \to F$ be a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple. Take an arbitrary element a in E. Then, as we have already mentioned, E_a coincides $C_0(L, \mathbb{R})$ for some locally compact Hausdorff space L such that $L \cup \{0\}$ is compact. The assignment $\|x\|_2 = \|T(x)\|$ defines a $\|.\|_\infty$-continuous triple norm on $C_0(L, \mathbb{R})$. Since $N(E_a, \|.\|_2) \leq N(F)$ and

$$\sup\{\|x\|_2 : x \in E_a, \|x\|_\infty\} \leq \|T\|,$$

then Lemma 5.1.4 assures that $\|a\| \leq N(F)\|T\|\|a\|_2$, for every a in A.

As a consequence, every J*B-triple have minimality of the triple norm. Indeed, if $\|.\|_2$ is a norm such that $N(E, \|.\|_2) = 1$ and $\|.\|_2 \leq \|.\|$ then, by the above paragraph, $\|.\| \leq \|.\|_2$.

Proposition 5.1.6 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Am. Math. Soc., 2012] *Let $T : E \to F$ be a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple. Then T is bounded below. Equivalently, every J*B-triple has MOTNT.* $\square$.

A real JB*-algebra is a closed *-invariant real subalgebra of a (complex) JB*-algebra. Real C*-algebras (i.e., closed *-invariant real subalgebras of C*-algebras), equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, are examples of real JB*-algebras.

The following corollaries are immediate consequences of Proposition 5.1.6.

Corollary 5.1.7 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Every continuous triple homomorphism from a (real) J*B-triple to a normed Jordan triple has closed range. In particular, every continuous triple homomorphism from a real or complex C*-algebra to a normed Jordan triple has closed range.* □

Corollary 5.1.8 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let A be a real JB*-algebra and let B be a real Jordan Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

Corollary 5.1.9 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let A be a real or complex C*-algebra and let B be a real Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

We have seen that real J*B-triples have minimality of triple norm topology. However, we recall that Kaplansky's Theorem actually proves that $C(K)$-spaces enjoy a stronger property. If K is a compact Hausdorff space, it is not hard to see that every (non-necessarily $\|.\|_\infty$-continuous) norm $\|.\|_2$ in $C(K)$ that makes the triple product continuous, is equivalent to $\|.\|_\infty$ (see [62, remark 4]).

C*-algebras and JB*-algebras also satisfy this stronger property: when A is a C*-algebra (respectively, a JB*-algebra) every non-necessarily continuous monomorphism from A to a Banach algebra (respectively, a Jordan Banach algebra) is bounded below (compare [46, Theorem 5.4] and [24, Theorem 1] or [154, Theorem 10] or [92]).

The question clearly is whether every non-necessarily continuous triple monomorphism from a complex JB*-triple (respectively, from a real J*B-triple) to a normed Jordan triple is bounded below. In [62] we provide a positive answer to this question. Following a classical strategy, we shall study the *separating ideals* associated with a triple homomorphism.

Under additional geometric assumptions, triple homomorphisms are automatically continuous. More concretely, every triple homomorphism between two JB*-triples is automatically continuous (compare [19, Lemma 1]). In this setting the problem reduces to the question of minimality of triple norm topology that the we have just treated. However, when the codomain space is not a JB*-triple, the continuity of a triple homomorphism does not follow automatically. We shall derive a new strategy without any additional geometric hypothesis on the codomain space.

The following definitions and results are inspired by classical ideas developed by C. Rickart [158], B. Yood [191], W.G. Bade and P.C. Curtis [16] and S. Cleveland [46]. Let $T : X \to Y$ be a linear mapping between two normed spaces. Following [158, Page 70], the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. The *separating space*, $\sigma_X(T)$, of T in X is defined by $\sigma_X(T) := T^{-1}(\sigma_Y(T))$. For each element y in Y, $\Delta(y)$

is defined as the infimum of the set $\{\|x\| + \|y - T(x)\| : x \in X\}$. The mapping $x \mapsto \Delta(x)$, called the *separating* function of T, satisfies the following properties:

a) $\Delta(y_1 + y_2) \leq \Delta(y_1) + \Delta(y_2)$,

b) $\Delta(\lambda y) = |\lambda| \ \Delta(y)$,

c) $\Delta(y) \leq \|y\|$ and $\Delta(T(x)) \leq \|x\|$,

for every y, y_1 and y_2 in Y, x in X and λ scalar (compare [158, Page 71] or [46, Proposition 4.2]).

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$ (c.f. [46, Proposition 4.5]).

It is not hard to see that $\sigma_Y(T) = \{y \in Y : \Delta(y) = 0\}$, while $\sigma_X(T) = \{x \in X : \Delta(T(x)) = 0\}$. Therefore $\sigma_X(T)$ and $\sigma_Y(T)$ are closed linear subspaces of X and Y, respectively. The assignment

$$x + \sigma_X(T) \mapsto \widetilde{T}(x + \sigma_X(T)) = T(x) + \sigma_Y(T)$$

defines an injective linear operator from $X/\sigma_X(T)$ to $Y/\sigma_Y(T)$. Moreover, $\widetilde{T}$ is continuous whenever X and Y are Banach spaces.

It is not hard to see that if $T : E \to F$ is triple homomorphism between Jordan triples, then $\sigma_E(T)$ is a norm-closed triple ideal of E, while $\sigma_F(T)$ is a norm-closed triple ideal in the completion of the subtriple of F generated by $T(E)$ (see Lemma 10 in [62]).

From the preceding comments it is clear that if we factor the separating spaces out we obtain a continuous triple monomorphism.

Proposition 5.1.10 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Am. Math. Soc., 2012] *Let $T : E \to F$ be a non-necessarily continuous triple homomorphism between two Jordan-Banach triples. Then the mapping*

$$\widetilde{T} : E/\sigma_E(T) \to F/\sigma_F(T),$$

$$\widetilde{T}(a + E/\sigma_E(T)) = T(a) + F/\sigma_F(T)$$

is a continuous triple monomorphism. □

By Corollary 5.1.7, if E is a real or complex JB*-triple, then the continuous triple monomorphism $\widetilde{T}$ is bounded bellow, that is, there exists $M > 0$ such that

$$M\|x + \sigma_E(T)\| \leq \|T(x + \sigma_F(T))\| \leq \|T(x)\|,$$

for all $x \in E$. However that does not guarantee that T is bounded bellow. We shall see that $\sigma_E(T) = 0$.

First, we obtain a triple version of the "*main boundedness theorem*". The first version of the main boundedness Theorem is due to Badé and Curtis (compare [16, Theorem 2.1]), where these authors study continuity of homomorphisms into commutative semisimple Banach algebras. A non-commutative version of the main boundedness Theorem was obtained by S. Cleveleland in [46, Theorem 3.1].

Theorem 5.1.11 [F.J. Fernández-Polo, J.J. Garcs and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let $T : E \to F$ a non-necessarily continuous triple homomorphism between Jordan- Banach triples and let (x_n), (y_n) be two sequences of non-zero elements in E such that $x_n \perp x_m, y_m$ for every $n \neq m$, then*

$$\sup\left\{\frac{\|T(\{x_n, x_n, y_n\})\|}{\|x_n\|^2\|y_n\|}, n \in \mathbb{N}\right\} < \infty.$$

As a consequence of the main boundedness Theorem, it can be proved that if $T : E \to E$ is a triple monomorphism from a JB*-triple (or a real J*B-triple) and (x_n) is a sequence of mutually orthogonal elements in $\sigma_E(T)$, then $T(x_n) = 0$, except for finitely many n in $\mathbb{N}$ (see [62, Lemma 13]). Let us suppose that $\sigma_E(T) \neq 0$ and a is a nonzero element in $\sigma_E(T)$, we prove that if $E_a \cong C_0(L)$, then L must be finite and hence a is an algebraic element (compare see [62, Lemma 15]). But for every tripotent e in $\sigma_E(T)$ we have $T(e) = 0$ (compare [62, Lemma 14]), thus $T(a) = 0$ and hence $a = 0$.

Proposition 5.1.12 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Am. Math. Soc., 2012] *Let $T : E \to F$ be a non-necessarily continuous triple monomorphism from a (complex) JB*-triple (respectively, a (real) J*B-triple) to a Jordan-Banach triple. Then the linear mapping $\widetilde{T} : E \to F/\sigma_F(T)$, $\widetilde{T}(a) = T(a) + F/\sigma_F(T)$, is a continuous triple monomorphism.* □

Finally, we obtained the announced result:

Theorem 5.1.13 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let $T : E \to F$ be a non-necessarily continuous triple monomorphism from a (complex) JB*-triple (respectively, a (real) J*B-triple) to a normed Jordan triple. Then T is bounded below.* □

The following corollary is the desired generalisation of a result due to B. Yood [190] and S. Cleveland [46] (see also [16, Theorem 4.3]).

Corollary 5.1.14 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Proc. Amer. Math. Soc., 2012] *Let $T : E \to F$ be a non-necessarily continuous triple monomorphism from a (complex)*

JB^*-triple (respectively, a (real) J^*B-triple) to a normed Jordan triple. Then the norm closure of $T(E)$ in the canonical completion of F decomposes as the direct sum of $T(E)$ and $\sigma_F(T)$. $\square$

Chapter 6

Weakly compact orthogonality preservers

It is well known that every reflexive C*-algebra is finite dimensional (compare [164, Proposition 2]). In other words, the identity mapping on a C*-algebra A is weakly compact if and only if A is finite dimensional. Actually, an algebraic homomorphism from a C*-algebra to a normed algebra is weakly compact if and only if it has finite dimensional range (cf. [74], [78], [140]).

Suppose that $S : A \to B$ is a Jordan *-homomorphism between two C*-algebras. In this case, $S^{**} : A^{**} \to B^{**}$ is a Jordan *-isomorphism between von Neumann algebras. It follows, from Kadison's theorem (see [112, Theorem 10]), that there exist weak*-closed ideals I_1 and I_2 in A^{**} and J_1 and J_2 in B^{**} satisfying that $A^{**} = I_1 \oplus^\infty I_2$, $B^{**} = J_1 \oplus^\infty J_2$, $S^{**}|_{I_1} : I_1 \to J_1$ is an *-isomorphism, and $S^{**}|_{I_2} : I_2 \to J_2$ is a *-anti-isomorphism. Thus, S is weakly compact if and only if it has finite dimensional range.

We shall see that the above conclusion remains true for every triple homomorphism from a C*-algebra to a normed Jordan triple (in particular, for every Jordan homomorphism from a C*-algebra to a normed algebra). This statement will follow from a result which is valid in a more general setting.

The proof of the following Theorem combines the argument given by M. Mathieu in [140] with a recent Kaplansky theorem for JB*-triples presented in the previous chapter.

Theorem 6.1.1 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Let $S : E \to F$ be a weakly compact (respectively, compact) triple homomorphism from a real or complex JB*-triple to a normed Jordan triple. Then $E/\ker(S)$ and $S(E)$ are reflexive (respectively, finite dimensional) Jordan Banach triples.*

It is well known that a linear operator between Banach spaces is weakly compact if, and only if, it factors through a reflexive Banach space (compare [52]). In [74] J.E. Gale, T.J. Ransford and C. White study weakly compact homomorphisms between Banach algebras. They pose the following problem:

Problem 6.1.2 *Let $T : A \to B$ be a weakly compact homomorphism between Banach algebras. Do there exist a reflexive Banach algebra C and continuous homomorphisms $\varphi : A \to C$ and $\psi : C \to B$ such that $T = \psi\varphi$?*

As observed by Gale, Ransford and White "the interpolation method used in [52] does not respect Banach algebras".

Although it is not explicitly stated, our characterisation of weakly compact triple homomorphisms from a JB*-triple gives a partial affirmative answer to the above problem in the triple setting. Indeed, let $S : E \to F$ be a weakly compact homomorphism from a JB*-triple, let $R : E \to E/Ker(S)$ be the quotient map and let $T : E/Ker(S) \to F$ be the mapping given by $T(x + ker(S)) = S(x)$. Then $E/ker(S)$ is reflexive and $S = RT$.

An element h of a unital Banach algebra A is called *hermitian* if $\|e^{ith}\| = 1$ for all t in $\mathbb{R}$. Let H be the set of hermitian elements in A and A_H be the subalgebra of A generated by H. In [74, Theorem 3.2], it is proved that all elements in $\overline{H + iH}$ are algebraic of degree at most n, for some $n \in \mathbb{N}$. These authors also pose the question wether every element in $\overline{T(A_H)}$ is algebraic.

The characterisation that we next present contains a partial affirmative answer to the analogue of the above question in the triple setting.

Given a real or complex JB*-triple E,a necessary and sufficient requirement for E to be reflexive is that E has the Radon-Nikodym property, or equivalently, E is isomorphic to a Hilbert space or E has finite rank ([44, Theorem 6] and [21, Theorems 2.3 and 3.1]).Thus, every element in the range of a weakly compact triple homomorphism from a JB*-triple is algebraic.

Corollary 6.1.3 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Let $S : E \to F$ be a continuous triple homomorphism from a real or complex JB*-triple to a normed Jordan triple. The following statements are equivalent:*

(a) *S is weakly compact.*

(b) *There exists a triple isomorphism from $S(E)$ to a reflexive JB*-triple.*

(c) *There exists a triple isomorphism from $S(E)$ to a finite rank JB*-triple.*

(d) *$S(E)$ is isomorphic as normed space to a Hilbert space.*

(*e*) $E/\ker(S)$ *is a reflexive or finite rank JB***-triple.* $\square$

Weakly compact Jordan *-homomorphisms between JB*-algebras were also charactersied by J. Perez, L. Rico and A. Rodriguez (compare [154, Remark 14]) and J. Gale [75]. Since, as it is well known by the reader at this point, every JB*-algebra is a JB*-triple, the above result generalises the just quoted contributions. We notice that there exist infinite dimensional reflexive JB*-algebras, so the range of a weakly compact Jordan homomorphism from a JB*-algebra need not to be finite dimensional (take for instance the identity mapping on a complex spin factor).

Corollary 6.1.4 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Every weakly compact Jordan homomorphism from a JB***-algebra has reflexive range.* $\square$

When particularised to the setting of C*-algebras, the above result reads as follows.

Corollary 6.1.5 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Every weakly compact triple homomorphism from a C***-algebra to a normed Jordan triple has finite dimensional range.* $\square$

6.2. Weakly compact orthogonality preservers from a C*-algebra

As seen in previous sections, an orthogonalty preserving linear mapping between C*-algebras is, in some sense, a multiple of a triple homomorphism. By Corollary 6.1.5 every triple homomorphism between C*-algebras has finite dimensional range.

The reader might wonder whether a weakly compact orthogonality preserving operator between C*-algebras must also have finite dimensional range.

Let $K := \mathbb{N} \cup \{\infty\}$, $\varphi = id_K$ and define u un K by $u(n) = \frac{1}{n}$. The composition operator $T : C(K) \to C(K)$, $f \mapsto T(f)(n) = u(n) f \circ \varphi(n)$ is weakly compact, however it is easy to see that its range is not finite dimensional (cf. Remark 13 in [63]).

This is example also shows that an analogous to Problem 6.1.2 for orthogonality preserving operator between C^*-algebras does not have an affirmative answer, that is, a weakly compact orthogonality preserving operator between C^*-algebras might not factorise through a reflexive C*-algebra.

Weakly compact orthogonality preserving operators between abelian C*-algebras were described by Y.F. Lin and N.C. Wong in [134].

Theorem 6.2.1 [Y.F. Lin, N.C. Wong, Math. Nachr., 2009] *Let $T : C_0(L_1) \to C_0(L_2)$ be a bounded disjointness preserving operator. The following assertions are equivalent:*

1. *T is completely continuous.*
2. *T is weakly compact.*
3. *T is completely continuous.*
4. *There are at most countably many distinct points $\{x_n\}$ in X and mutually orthogonal disjoint functions $\{h_n\}$ in $C_0(L_1)$ such that*

$$Tf = \sum_n f(x_n) h_n, \text{ for all } f \in C_0(L_1).$$

In case there are infinitely many such $\{x_n\}$ and h_n, we have $\|h_n\| \to 0$ and thus the sum converges uniformly. □

The above result is a generalization of a theorem due to H. Kamowitz for Weakly compact orthogonality preserving operators between unital abelian C*-algebras (cf. [114]).

M. Wolff seems to have ventured to describe symmetric weakly compact orthogonality preserving operators between (non-necessarily abelian) C*-algebras, as the final remark in [183] suggests: *" The proof of Theorem 2.3 (cf. step III of 3.5) enables us in principal to characterize those disjointness preserving operators which are compact. For if $T(1_A)$ is invertible then S has to be compact and since Jordan *-homomorphisms are open onto their range S has to be of finite rank. Unfortunately it is cumbersome to chracterize such an operator in the non-commutative case contrary to the commutative case. So we are not able to generalize the results of Kamowitz [114] in any reasonable manner".*

In collaboration with F. Fernández-Polo and A.M. Peralta we succeeded in describing weakly compact orthogonality preserving operators even in a more general setting. Previous results on weakly compact triple homomorphisms and the characterisation of orthogonality preserving operators obtained in previous Chapters (see [34, 35]) were crucial tools needed to obtain this description.

Theorem 6.2.2 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Let A be a C*-algebra, E a JB*-triple, $T : A \to E$ a weakly compact orthogonality preserving operator and let $r = r(h)$ be the range tripotent of the element $h = T^{**}(1)$ in E^{**}. Then there exists a countable family $\{I_n\}_{n\in\mathbb{N}}$ of mutually orthogonal weak*-closed C*-ideals in A^{**}, a family $\{S_n : A^{**} \to E_2^{**}(r) : n \in \mathbb{N}\}$ of continuous Jordan *-homomorphisms and a sequence (x_n) of mutually orthogonal elements in E satisfying:*

(a) Each I_n is a finite type I von Neumann factor;

(b) $\|x_n\| \to 0$ *and* $h = \sum_{n=1}^{\infty} x_n$*;*

(c) $S_n|_{I_n}$ *is a Jordan* **-monomorphism,* $S_n|_{I_n^{\perp}} = 0$*,* S_n *and* S_m *have orthogonal ranges for* $n \neq m$*;*

(d) *For each* x *in* A^{**}*,* x_n *and* $S_m(x)$ *operator commute for every* n *and* m*;*

and

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x), \qquad (e)$$

for every $x \in A$*. Moreover, the Jordan* **-homomorphism* $S : A \to E_2^{**}(r)$ *given in Theorem 3.2.9, b), satisfies that* $S(z) = \sum_{n=1}^{\infty} S_n(z)$*, for each* z *in* A*, where the series converges in the weak* topology of* $E_2^{**}(r)$*.* □

Since every irreducible finite type I von Neumann factor in $C_0(L)^{**}$ is isomorphic to $\mathbb{C}$, then the description of weakly compact disjointness preserving operators between commutative C*-algebras given by Lin and Wong follows now as a consequence of Theorem 6.2.2. Lin and Wong also proved that, for an orthogonality preserving operator between abelian C*-algebras, being compact is equivalent to being weakly compact, and as consequence of Theorem 6.2.1 the latter is also equivalent to T admitting a compact factorisation through the whole c_0. We were also able to generalise this characterisation.

Theorem 6.2.3 [F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Math. Z., 2012] *Let* T *be a continuous orthogonality preserving operator from a JB*-algebra to a JB*-triple. The following are equivalent:*

1. *T is compact.*

2. *T is weakly compact.*

3. *T admits a compact factorisation through a c_0-sum of the form*
$$\bigoplus_{n}^{c_0} M_{m_n}(\mathbb{C}),$$
where (m_n) is a sequence of natural numbers. $\square$

As noticed in [63] the C*-algebra $\bigoplus_n^{c_0} M_{m_n}(\mathbb{C})$ is not, in general, isomorphic to c_0 (see comments after Corollary 10 in [63]), thus a weakly compact orthogonality preserving operator between C*-algebras does not admit in general a compact factorisation through c_0.

By using results on weakly compact Jordan homomorphisms from a JB*-algebra to a normed Jordan triple (see Corollary 6.1.3) and the characterisation of orthogonality preserving operators from a JB*-algebra to a JB*-triple described in Chapter 4 (see also [35]), we are also able to generalise Theorem 6.2.2 to this setting.

Theorem 6.2.4 [F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Let J be a C*-algebra, let E be a JB*-triple, $T : J \to E$ a weakly compact orthogonality preserving operator and let $r = r(h)$ be the range tripotent of the element $h = T^{**}(1)$ in E^{**}. Then there exists an at most countably family $\{I_n\}$ of mutually orthogonal weak*-closed ideals in J^{**}, a family $\{S_n : J^{**} \to E_2^{**}(r)\}$ of continuous Jordan *-homomorphisms and a set $\{x_n\}$ of mutually orthogonal elements in E satisfying:*

(a) *Each I_n is reflexive JBW*-factor;*

(b) *$\|x_n\| \to 0$ and $h = \sum_n x_n$;*

(c) *$S_n|_{I_n}$ is a Jordan *-monomorphism, $S_n|_{I_n^\perp} = 0$, S_n and S_m have orthogonal ranges for $n \neq m$;*

(d) *For each x in J^{**}, x_n and $S_m(x)$ operator commute for every n and m;*

and

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x),$$

for every $x \in A$. $\square$

In this case equivalence $a) \iff b)$ in Corollay 6.2.3 is no longer true. Indeed, the identity mapping on an infinite dimensional spin factor is weakly compact and orthogonality preserving but it's not a compact operator. Similar construction to that Corollary 6.2.3 c), allow to prove that in this case T factorises through a c_0-sum of reflexive JBW*-factors.

Finally, Theorems 6.2.2 and 6.2.4 allow to charactersie those C*-algebras (respectively, JB*-algebras) which admit a weakly compact orthogonality preserving operator.

Corollary 6.2.5 [F.J. Fernández Polo, J.J. Garcés, A.M. Peralta, Math. Z., 2012] *Let A be a C*-algebra (respectively, JB*-algebra). There exists a weakly compact orthogonality preserving operator from A to a JB*-triple if and only if A^{**} contains a non-zero finite dimensional weak*-closed C*-ideal (respectively, A^{**} contains a non-zero reflexive weak*-closed JB*-ideal).*

Chapter 7

Generalised triple homomorphisms and derivations

We have already commented that automatic continuity of homomorphisms in associative and non-associative context has been the subject of many studies, contributions begin with the already mentioned results by Eidelheit, Gelfand and Kaplansky. We recommend the surveys [174] and [182], which cover many of the main results in this area.

In [99], K. Jarosz considered those linear mappings that "almost preserve products" an proves that any such a mapping from a Banach algebra to a $C(K)$-space is automatically continuous. Further results on automatic continuity of these kind of mappings were added by B.E. Johnson in [107].

We recall the basic notions. Let $T : A \to B$ be a linear mapping between Banach algebras. A is said to be a *generalised (associative) homomorphism* if there exists $\varepsilon > 0$ such that

$$\|T(a)T(b) - T(ab)\| \leq \varepsilon\|a\|\|b\|,$$

for every a, b in A. Equivalently, T is a generalised associative homomorphism if and only if the bilinear mapping

$$\check{T} : A \times A \longrightarrow B,$$

$$\check{T}(a, b) = T(a)T(b) - T(ab)$$

is (jointly) continuous.

B.E. Johnson proved in [107, Theorem 1] that every generalised homomorphism from a Banach algebra onto a semisimple Banach algebra is continuous. This can be seen as a generalisation of his celebrated result on minimality of norm topology of semisimple Banach algebras (indeed, every homomorphism is clearly a generalised homomorphism).

In [76] A.M. Peralta and the author of this thesis studied those linear mappings between Jordan triples that "almost preserve the tripe product". We shall describe in this chapter the main results obtained in this line.

Definition 7.1.1 *Let $T : E \to F$ be a linear mapping between normed Jordan triples. T is said to be a* generalised triple homomorphism *if there exists $\varepsilon > 0$ such that*

$$\|\{T(a), T(b), T(c)\} - T(\{a, b, c\})\| \leq \varepsilon \|a\|\|b\|\|c\|,$$

for every a, b, c in E.

Let $T : E \to F$ be a linear mapping between normed Jordan triples. We define $\check{T} : E \times E \times E \to E$ by the rule

$$\check{T}(a, b, c) := T(\{a, b, c\}) - \{T(a), T(b), T(c)\}.$$

The mapping $\check{T}$ is linear in the outer variables and conjugate in the middle one (trilinear when E is a real Jordan triple). It can be seen that T is a generalised triple homomorphism if and only if $\check{T}$ is (jointly) continuous.

Let A be a Banach algebra. We shall refer to the triple product $\{a,b,c\} = \frac{1}{2}(abc+cba)$ as the "*elemental*" triple product of A. When A is a Banach *-algebra we can also consider the triple product $\{a,b,c\} = \frac{1}{2}(ab^*c+cb^*a)$, and we shall refer to it as the natural Jordan triple product of A.

Let $T : A \to B$ be a generalised homorphism between Banach *-algebras. We shall say that T is a *generalised *-homomorphism* if the mapping

$$a \mapsto S(a) = T(a^*)^* - T(a)$$

is continuous. Generalised *-homomoprhisms where also considered by Johnson in [107], where he proved the following:

Theorem 7.1.2 [B.E. Johnson, Bull. London Math., 1987] *Every generalised *-homomorphism between C^*-algebras is continuous.*

It is natural to ask whether there is a relation between generalised homomorphisms (respectively, generalised *-homomorphisms) between Banach algebras (respectively, Banach *-algebras) when they are endowed with the elemental triple product (respectively, natural Jordan triple product). The answer is affirmative as the following result shows:

Proposition 7.1.3 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Let A, B be Banach algebras. Every generalised homomorphism $T : A \to B$ is a generalised triple homomorphism when A and B are equipped with the elemental triple product.*

When A and B are Banach *-algebras and T is a generalised *-homomorphism, then T is a generalised triple homomorphism with respect to the triple product $2\{a,b,c\} = ab^*c + cb^*a$. $\square$

The aim of [76] is generalising Johnson's results to the setting of generalised triple homomorphisms between normed Jordan triples. To study automatic continuity of generalised triple homomorphisms between normed Jordan triples we make use of the separating spaces (see Chapter 5 for definitions).

Let $T : A \to B$ be a generalised homomorphism between Banach algebras. It is not hard to see that the separating space, $\sigma_F(T)$, is a two-sided ideal of the closed subalgebra of B generated by $T(A)$ (compare [107, Lemma 1]).

One would expect the separating space $\sigma_F(T)$ of a generalised triple homomorphism $T : E \to F$ between Jordan triples to be a triple ideal of the closed subtriple of F generated by $T(E)$. This statement, although true, is not easy to check. In order to prove it we first need a precise description of the subtriple generated by a subset. With this aim we introduce in [76] the "odd triple monomials".

Let $x_1, x_2, \ldots$ be a sequence of indeterminates. Then a triple monomial is a term that can be obtained by the following procedure:

1. Every indeterminate x_k is a triple monomial of degree 1.

2. If V_1, V_2 and V_3 are triple monomials of degrees d_1, d_3 and d_3 respectively, then $V := \{V_1, V_2, V_3\}$ is a triple monomial of degree $d_1 + d_2 + d_3$ where $\{.,.,.\}$ is a "formal triple product" in three variables.

If the triple monomial V does not contain any indeterminate x_j with $j > 2n-1$, we also write $V = V(x_1, \ldots, x_{2n-1})$. In that case, for every JB*-triple E and every $a = (a_1, \ldots, a_{2n-1})$ in E^{2n-1} the element $V(a) = V(a_1, \ldots, a_{2n-1}) \in E$ is well defined-just specialize every x_k to a_k and the "formal triple product" to the concrete triple product of E. In this sense V induces a polynomial map $E^{2n-1} \to E$ which is denoted by V (or by V_E to avoid confusion). Now, for a fixed natural $n \geq 1$, denote $\mathcal{OP}^{2n-1}$ the set of all triple monomials V of degree $2n-1$ in which every x_k with $1 \leq k \leq n$ occurs precisely once. Then $V = V(x_1, \ldots, x_n)$ and the induced map $V_E : E^n \to E$ is multilinear for ever JB*-triple E.

The symbol $\mathcal{OP}^{2m+1}(E)$ will stand for the set of all multilinear mappings of the form V_E, where V runs in $\mathcal{OP}^{2m+1}$, while $\mathcal{OP}(E)$ the set of all odd triple monomials of any degree on E. It should be here noticed that when F is another Jordan triple, each triple monomial V in $\mathcal{OP}^{2m+1}(E)$ by just replacing the triple product of E in the definition of V by the corresponding triple product of F.

The following Lemma is the key to prove that the desired property of the seprating space associated with a generalised triple homomorphism.

Lemma 7.1.4 [J.J. Garcés and A.M. Peralta, Canadian J. Math., 2013] *Let $T : E \to F$ be a triple homomorphism between normed Jordan triples and m a natural number. Let V be an odd triple monomial of degree $2m+1$, which can be regarded as an element in $\mathcal{OP}^{2m+1}(E)$ or in $\mathcal{OP}^{2m+1}(F)$ indistinclty. Suppose that V is of the form $V = \{., W, P\}$ (respectively, $V = \{W, ., P\}$) and let $j = deq(W)$. Then*

$$\lim_{n\to\infty} V(T(x_n), T(a_1), \ldots, T(a_{2m})) - T(x_n, a_1, \ldots, a_{2m}) = 0,$$

(respectively,

$$\lim_{n\to\infty} V(T(a_1), \ldots, T(a_j), T(x_n), T(a_{j+1}), \ldots, T(a_{2m}))$$
$$-T(V(a_1, \ldots, a_j, x_n, a_{j+1}, \ldots, a_{2m})) = 0,$$

for every $a_1, \ldots, a_{2m}$ in E. □

As a first application, the triple monomials can be used to obtain a more precise description of the subtriple generated by a subset. Let E be a normed Jordan triple and $S \subseteq E$. The norm-closed Jordan subtriple of E generated by S is the smallest norm-closed subtriple of E containing S, and will be denoted by E_S. It is not hard to see that E_S coincides with the norm closure of the linear span of the set

$$\mathcal{OP}(S) := \left\{ V(a_1, \ldots, a_{2m+1}) : \begin{array}{c} m \in \mathbb{N}, V \in \mathcal{OP}^{2m+1}(E), \\ a_1, \ldots, a_{2m+1} \in S \end{array} \right\}.$$

Let us recall that $I \subset E$ is a triple ideal of E if $\{E, E, I\} + \{E, I, E\} \subseteq I$. Let $S \subseteq E$ and $I \subseteq E_S$. By the continuity of the triple product, in orther to prove that I is an ideal of E_S it is enough to show that

$$V(I, S, \ldots, S) + V'(S, \ldots, S, I, S, \ldots, S) \subseteq I$$

holds, for arbitrary odd triple monomials V and V' of the form $\{W, ., P\}$ and $\{., W', P'\}$, respectively.

This fact, together with Lemma 7.1.4 allow us to prove the following.

Proposition 7.1.5 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Let $T : E \to F$ be a generalised triple homomorphism between Jordan-Banach triples. Let I and $\widetilde{F}$ denote $\sigma_F(T)$ and $F_{T(E)}$, respectively. Then we have the following:*

1. *I is a (closed) triple ideal of $\widetilde{F}$.*

2. *$I^{\perp}_{\widetilde{F}}$ contains all the elements of the form $\check{T}(a, b, c)$. Further, if J is a closed triple ideal of $\widetilde{F}$ containing $I^{\perp}_{\widetilde{F}}$ then $\pi \circ T$ is a triple homomorphism, where π is the quotient map $\widetilde{F} \to \widetilde{F}/J$.* □

If F is a JB*-triple then we have that $I_F^{\perp} := \{a \in F : a \perp I\}$ contains $\check{T}(E, E, E)$.

The following property have been very useful in the study of automatic continuity of homomorphism, and will prove itself to be useful for the case of generalised triple homomorphisms as well.

Remark 7.1.6 *Let $T : X \to Y$ be a linear mapping between Banach spaces. A useful property of the separating space $\sigma_F(T)$ asserts that for every bounded linear map R from Y to another Banach space Z, the composition RT is continuous if and only if $\sigma_F(T) \subseteq ker(R)$. It is also known that $\sigma(RT) = \overline{R(\sigma(T))}^{\|\cdot\|}$ (see [174, Lemma 1.3]).*

Remark 7.1.6 together with Lemma 7.1.4 and the well known fact that triple homomorphisms between JB*-triples are continuous (compare [19]) allow to prove that every generalised triple homomorphism between JB*-triples is continuous.

Theorem 7.1.7 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Every generalised triple homomorphism between JB*-triples is continuous.* $\square$

As a consequence we obtain Johnson's result on generalised *-homomorphisms (compare [107]).

Corollary 7.1.8 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Every generalised *-homomorphism between C*-algebras is continuous.*

A general characterisation for continuity of generalised triple homomorphisms form a JB*-triple is also abtained in [76].

The following Lemma can be seen, in some sense, as a main boundedndess Theorem for generalised triple homomorhpisms:

Lemma 7.1.9 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Let $T : E \to F$ be a generalised triple homomorphism between real Jordan-Banach triples and let $(x_n), (y_n)$ be sequences of elements in E such that*

$$Q(y_n)Q(x_n) = Q(x_n) \text{ and } Q(y_n)Q(x_m) = 0,$$

for $n \neq m$. Then $Q(T(x_n))T$ and $TQ(x_n)$ are continuous for all but a finite number of n. □

As a consequence of the above Lemma we have

$$\sup\left\{\frac{\|T(\{x_n, y_n, x_n\})\|}{\|x_n\|^2\|y_n\|}, n \in \mathbb{N}\right\} < \infty,$$

where $(x_n), (y_n)$ are sequences as in Lemma 7.1.9.

Let B be a subset of a Jordan-Banach triple E. We define its *quatratic annihilater*, $Ann_F(B)$, to be the set $\{a \in F : Q(a)(B) = \{a, B, a\} = 0$. It is not hard to see that the inclusion $B_F^{\perp} \subseteq Ann_F(B)$ holds, however the quadratic annihilator does not coincide, in general, with the orthogonal complement.

The set $J = T^{-1}(Ann_F(\sigma_F(T)))$ plays a crucial role in the study of continuity of T. Indeed, as a consequence of Remark 7.1.6 it can be seen that J coincides with the set $\{a \in E : Q(T(a))T$ is continuous $\}$. Since T is a generalised triple homomorphism it coincides also with the set

$$\{a \in E : T(Q(a)) \text{ is continuous } \}.$$

It can be proved, as a consequence of Lemma 7.1.9 that J enjoys additional properties.

Proposition 7.1.10 [J.J. Garcés and A.M. Peralta, Canad. J. Math., 2013] *Let $T : E \to F$ be a generalised triple homomorphism from a real JB*-triple to a Jordan-Banach triple. The following statements hold:*

1. *For every norm-closed triple ideal I of E containing the set $T^{-1}(Ann_F(\sigma_F(T)))$, then E/I is algebraic of bounded degree.*

2. *Let K be a triple ideal of E. The linear mapping*

$$x \in E \mapsto \{T(a), T(x), T(a)\}$$

is continuous if, and only if, K is contained in the set $T^{-1}(Ann_F(\sigma_F(T)))$.

$\square$

Let us suppose that $J = T^{-1}(Ann_F(\sigma_F(T)))$ is a norm-closed triple ideal. Then the mapping $x \in E \mapsto \{T(a), T(x), T(a)\}$ is continuous. However, this is not enough to prove that $T_{|J}$ is continuous (unless E has cohen's factorisation property, as we shall prove in Theorem 7.1.14). If J also satisfies the property

$$\{Ann_F(\sigma_F(T)), Ann_F(\sigma_F(T)), \sigma_F(T)\} = 0$$

the we are also able to prove that, for every $a, b \in E$ the mapping

$$x \in E \mapsto \{T(a), T(b), T(x)\}$$

is continuous. Then an application of the uniform boundedness principle and the triple functional calculus allow to prove that $T_{|J}$ is continous. By Proposition 7.1.10 E/J is algebraic of bounded degree. To prove that $E/J = 0$ we borrow some ideas from the proof of [163, Proposition 12].

Theorem 7.1.11 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Let $T : E \to F$ be a generalised triple homomorphism from a JB*-triple and let $J = T^{-1}(Ann_F(\sigma_F(T)))$. The following statements are equivalent:*

a) J is a norm closed triple ideal of E and

$$\{Ann_F(\sigma_F(T)), Ann_F(\sigma_F(T)), \sigma_F(T)\} = 0.$$

b) T is continuous.

$\square$

We notice here that Theorem 7.1.11 is valid for complex JB*-triples and "reduced real JB*-triples" (see comments after Lemma 16 in [76]). The characterisation of continuity of generalised triple homomorphism from an arbitrary real JB*-triple to a real Jordan-Banach triple remains as an open problem.

Some automatic continuity results in particular cases are also obtained. In [76, Lemma 15 and 16] we prove that every generalised triple homomorphism from a type I Cartan factor or from a complex spin factor to an annistropic Jordan triple is continuous.

In [161], J.R Ringrose proves that a linear functional on a von Neumann algebra A is continuous whenever its restriction to every maximal abelian C*-subalgebra of A is continuous. Later B.A. Barnes improved Ringrose's result, by showing that it is enough to prove that the functional is continuous on every C*-subalgebra generated by single hermitian element is continuous. Barnes also proved that dual C*-algebras enjoy this property (compare [18]). In [48], J. Cuntz obtained the definitive version of Ringorse's result, by proving that a linear mapping from a C*-algebra to a Banach space is continuous whenever its restriction to every C*-subalgebra generated by single hermitian element is continuous.

Let H be a complex Hilbert space regarded as a JB*-triple. It is easy to check that every norm-one element in H is a tripotent. Therefore the JB*-subtriple generated by a single element

a coincides with H. Let X be a Banach space and $T : H \to X$ be a linear mapping. Then T is continuous when restricted to any singly generated subtriple of H. When H is infinite dimensional, we can easily find a discontinuous linear mapping from H to X.

The above comment shows that a triple version of Cuntz's Theorem does not hold.

Earlier versions of Cuntz's Theorem were given by J.D.Stein Jr. and A.M. Sinclair for homomorphisms form a C*-algebra to a Banach algebra (see[175] and [173], respectively). In particular, Sinclair proved a similar automatic continuity result for homomorphisms from a C*-algebra. From Remark 1.6 in [163] it is easy to give an example of a triple discontiuous triple homomorphism from a JB*-triple to a Jordan triple which is continuous when restricted to any singly generated JB*-triple (apply to construction of Θ_δ described in next section to the mentiones example in Remark 1.6). Thus, an automatic continuity result for JB*-triples similar to Cuntz's Theorem does not hold, not even for the case of triple homomorphisms to a Jordan triple.

In [76] we explore assumptions that avoid the previous counterexamples. We replace the subtriple generated by the inner ideal generated by a single element. We remind that the inner ideal generated by a single element a, $E(a)$, coincides with norm closure of the set $\{a, E, a\}$ (see [32, pp, 19-29]). If H is a complex Hilbert space, then $H(a) = H$, for every norm-one element a in A.

Let $T : E \to F$ be a generalised triple homomorphism between Jordan-Banach triples, such that T is continuous when restricted to any inner ideal generated by a single element of E. Then all elements in $\sigma_F(T)$ are nilpotent (see comment preceding [76, Theorem 6]). If F is anisotropic then $\sigma_F(T) = 0$, and hence T is continuous.

Let $T : A \to B$ be a linear mapping between Banach algebras and a in A. Let us suppose that the linear mapping $(x, y) \mapsto T(xy)$ is continuous. If A has an approximate identity, then by Cohen's factorisation property ([104]), for a norm-null sequence (x_n) in A there exist $a \in A$ and a norm-null sequence (y_n) in A, such that $x_n = ay_n$. As a consequence $\text{lím}_n T(x_n) = \text{lím}_n T(ay_n) = 0$. By the closed graph theorem T is continuous. Cohen's factorisation Theorem is very useful in the study of automatic continuity of homomorphisms and other linear preservers (see for instance [104], [107] or [106]).

In [76] we also explore the applications of Cohen's factorisation property in the triple setting.

Definition 7.1.12 *A Jordan-Banach triple E has* Cohen's factorisation property (CFP) *if given a norm-null sequence (a_n) in E there exist elements x, y in E and a norm-null sequence (b_n) such that $a_n = \{x, b_n, y\}, \forall n \in \mathbb{N}$.*

Definition 7.1.13 *Let J be a Jordan-Banach algebra. An* approximate identity *in J is a bounded net (e_λ) satisfying:*

1. $\text{lím}_\lambda\, a \circ e_\lambda = a$, *and*

2. $\text{lím}_\lambda\, U_{e_\lambda}(a) = 0$, *for every a in J.*

Every Jordan-Banach algebra with an approximate identity has CFP (see [3]). As a consequence, every JB*-algebra has CFP (see [89, Proposition 3.5.4]).

Theorem 7.1.14 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Let $T : E \to F$ be a linear mapping between two Jordan-Banach triples and suppose that one of the following statements hold:*

1. *T is a generalised triple homomoprhism and F is anisotropic.*

2. *E has Cohen's factorisation property.*

If the restriction of T to any closed inner ideal generated by a single element is continuous, then T is continuous. □

7.2. Triple modules and derivations

Let A be an associative algebra. Le us recall that an A-bimodule is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(x,a) \mapsto xa$ form $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, a(xb) = (ax)b, \text{ and } (xa)b = x(ab)$$

for every $a, b \in A$ and $x \in X$.

Let J be a Jordan algebra. A *Jordan J-module* is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ form $J \times X \to X$, satisfying

$$a \circ x = x \circ a, a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every a, b in A and x in X.

In [163], A.M. Peralta and B. Russo introduced the so-called Jordan triple modules which we next present. These are the Jordan triple version of the bimodules and Jordan modules.

Let E be a complex (respectively, real) Jordan triple. A *Jordan triple E-module* (also called *triple E-module*) is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \{.,.,.\}_2 : E \times X \times E \to X$$

$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

satisfying the following axioms:

($JTM1$) $\{x, a, b\}_1$ is linear in a and x and conjugate linear in b (respectively, trilinear), $\{a, b, x\}_3$ is linear in b and x and conjugate linear in a (respectively, trilinear) and $\{a, x, b\}_2$ is conjugate linear in a, b, x (respectively, trilinear)

($JTM2$) $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$.

($JTM3$) Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

holds whenever one of the elements a, b, c, d, e is in X and the rest are in E.

When E is a Jordan-triple and X a triple E-module which is also a Banach space, we will say that X is a *Banach (Jordan) triple E-module* when the products $\{.,.,.\}_1, \{.,.,.\}_2$ and $\{.,.,.\}_3$ are (jointly) continuous. From now on, the products $\{.,.,.\}_1, \{.,.,.\}_2$ and $\{.,.,.\}_3$ will be simply denoted by $\{.,.,.\}$.

Every real or complex associative algebra A (respectively Jordan algebra J) is a real Jordan triple with respect to $\{a, b, c\} =$

$\frac{1}{2}(abc+cba), a,b,c \in A$ (respectively, $\{a,b,c\} = (a\circ b)\circ c + (c\circ b)\circ a - (a\circ x)\circ b), a,b,c \in J$). It is easy to see that an A-bimodule becomes a triple A-module when endowed with the triple products $\{a,x,b\}_2 = \frac{1}{2}(axb+bxa)$ and $\{a,b,x\}_3 = abx + xba$, and that every Jordan module X is a triple J-module with respect to the products

$$\{a,x,c\}_2 = (a\circ x)\circ c + (c\circ x)\circ a - (a\circ c)\circ x) \text{ and}$$

$$\{a,b,x\}_3 = (a\circ b)\circ x + (x\circ b)\circ a - (a\circ x)\circ b).$$

Let E be a real or complex Jordan-Banach triple. The dual space of E, E^*, is a triple E-module when endowed with the product:

$$\{a,b,\varphi\}(x) = \{\varphi,b,a\}(x) := \varphi\{b,a,x\}$$

and

$$\{a,\varphi,b\}(x) := \overline{\varphi\{a,x,b\}}.$$

Let $\delta : A \to X$ be a linear mapping from a Banach algbera to a Banach A-bimodule. Then δ is said to be a *derivation* if

$$\delta(ab) = a\delta(b) + \delta(a)b,$$

for every a,b in A. Jordan derivations are similarly defined.

As in the case of homomorphisms between Banach algebras, automatic continuity of derivations in associative and non associative context has a focused the interest of many authors (see for instance the survey [182]). It is due to J.R. Ringrose that every derivation from a C*-algebra to a Banach A-bimodule is continuous (cf. [160]).

Let E be a real (respectively, complex) be a Jordan triple a and X a triple E-module. A linear (respectively, conjugate linear) mapping $\delta : E \to X$ is said to be a *triple derivation* if it satisfies

$$\delta(\{a,b,c\}) = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}.$$

Ringore's result on automatic continuity of derivations from a C*-algebra is not true in this setting, as Remark 16 in [163] shows.

Automatic continuity of triple derivations on JB*-triples was first studied by T. Barton and Y. Friedman. These authors proved that every triple derivation from a complex JB*-triple to itself is continuous [20]. This result was extended to the real setting by T. Ho, A.M. Peralta and B. Russo in [93].

In [163] B. Russo and A.M. Peralta give a charactersiation for continuity of triple derivations. As a consequence they obtain an alternative prove for the aforementioned results on automatic continuity of triple derivations on a real or complex JB*-triple. Further, they prove that every derivation from a JB*-triple to its dual is automatically continuous.

Theorem 7.2.1 [A.M. Peralta, B. Russo, Preprint, 2010] *Let E be a real or complex JB*-triple. The following statements hold:*

1. *Every triple derivation $\delta : E \to E$ is continuous.*
2. *Every triple derivation $\delta : E \to E^*$ is continuous.* □

In the case of derivations from a C^*-algebra further automatic continuity results can be obtained. In [163] Peralta and Russo prove that every triple derivation from an abelian C*-algebra to a real Jordan-Banach triple A-module is continuous. Then, an application of Cuntz's theorem allow them to prove the following:

Theorem 7.2.2 [A.M. Peralta, B. Russo, Preprint, 2010] *Let A be a C^*-algebra. Then every triple derivation from A (respectively, from A_{sa}) into a complex (respectively, real) Jordan-Banach triple A-module is continuous.* □

7.3. Generalised triple derivations

It seems natural now to consider those linear or conjugate linear mappings that are "almost derivations". This problem seems not to have been considered before. In the last section of [76] we introduce the concept of generalised triple derivation.

Definition 7.3.1 *Let $\delta : E \to X$ be a linear (respectively, conjugate linear) mapping from a real (respectively, complex) Jordan-Banach triple into a Jordan-Banach triple E-module. Then δ is said to be a* generalised triple derivation *if the mapping*

$$\check{\delta} : E \times E \times E :\to X,$$

$$\check{\delta}(a,b,c) := \delta\{a,b,c\} - \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}.$$

To prove the desired automatic continuity result for generalised triple derivations we argue as in [163], by associating to a generalised derivation a generalised triple homomorphism into a certain Jordan-Banach triple module.

Let E be a Jordan-Banach triple and X a Jordan-Banach triple E-module. The linear space $E \oplus X$ equipped with the l_1-norm and the product

$$\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} := \{a_1, a_2, a_3\} + \{x_1, a_2, a_3\} +$$

$$\{a_1, x_2, a_3\} + \{a_1, a_2, x_3\}$$

is a Jordan-Banach triple, which we shall call the *triple split null extension* of E and X.

Let $\delta : E \to X$ be a generalised triple derivation. We define the mapping

$$\Theta_\delta : E \to E \oplus X,$$
$$a \mapsto a + \delta(a).$$

It is clear that δ is continuous if and only if Θ_δ is continuous. Furthermore, straightforward calculations show that

$$\check{\delta}(a, b, c) = \check{\Theta}_\delta(a, b, c),$$

for every (a, b, c) in E. Thus, Θ_δ is a generalised triple homomorphism. The fact that $J_\delta := \Theta_\delta^{-1}(Ann_{E\oplus X}(\sigma_{E\oplus X}(\Theta_\delta)))$ coincides with $Ann_E(\sigma_X(\delta)$ is proved in [76, page 22].

If E is a JB*-triple and X coincides E or E^*, the latter yields that J_δ is a is a norm-closed triple ideal of E and

$$\{Ann_{E\oplus X}(\sigma_{E\oplus X}(\Theta_\delta)), Ann_{E\oplus X}(\sigma_{E\oplus X}(\Theta_\delta)), \sigma_{E\oplus X}(\Theta_\delta)\} = 0.$$

Now, Theorem 7.1.11 proves that Θ_δ is continuous.

Theorem 7.3.2 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Let E be a real or complex JB*-triple and $\delta : E \to X$ be a generalised triple derivation, where $X = E$ or E^*. Then T is continuous.* □

Since every triple derivation is a generalised triple derivation, the above Theorem generalises the aforementioned automatic continuity results on automatic continuity of triple derivations by Peralta and Russo [163].

In the final part of [76] we also benefit from Cuntz Theorem and follow the ideas of [163] to prove that every generalised triple derivation from a C^*-algebra A to a Jordan-Banach triple A-module is continuous.

Theorem 7.3.3 [J.J. Garcés, A.M. Peralta, Canad. J. Math., 2013] *Every generalised triple derivation from a real or complex C^*-algebra to a Jordan-Banach triple A-module is continuous.* $\square$

Chapter 8

Orthogonality preservers on real C*-algebras

Once the structure of an orthogonality preserving operator between complex C*-algebras has been described, one might wonder about the structure orthogonality preserving operators between real C*-algebras. As we have seen in Chapters 3 and 4, orthogonal sesquilinear forms play an important role in the description of orthogonality preservers in the complex setting. Thus, it seems to be a worth problem that of describing orthogonal bilinear forms in the real setting. A little or nothing was known about orthogonality preserving operators (nor about orthogonal bilinear forms) on real C*-algebras when we (Antonio M. Peralta and the author of this thesis) studied them in [77].

Before presenting the results obtained in [77], we shall recall some background on real C*-algebras.

A *real C*-algebra* is a real Banach *-algebra A which satisfies the standard C*-identity, $\|a^*a\| = \|a\|^2$, and which also has the property that $1 + a^*a$ is invertible in the unitization of A for every $a \in A$. It is known that a real Banach *-algebra, A, is a real C*-algebra if, and only if, it is isometrically *-isomorphic to a norm-closed real *-subalgebra of the space of all bounded operators on a real Hilbert space (cf. [130, Corollary 5.2.11]).

Clearly, every (complex) C*-algebra is a real C*-algebra when scalar multiplication is restricted to the real field. If A is a real C*-algebra whose algebraic complexification is denoted by $B = A \oplus iA$, then there exists a C*-norm on B extending the norm of A. It is further known that there exists an involutive conjugate-linear *-automorphism τ on B such that $A = B^{\tau} := \{x \in B : \tau(x) = x\}$ (compare [130, Proposition 5.1.3] or [158, Lemma 4.1.13], and [84, Corollary 15.4]). The dual space of a real or complex C*-algebra A will be denoted by A^*. Let $\widetilde{\tau} : B^* \to B^*$ denote the map defined by

$$\widetilde{\tau}(\phi)(b) = \overline{\phi(\tau(b))} \qquad (\phi \in B^*,\, b \in B).$$

Then $\widetilde{\tau}$ is a conjugate-linear isometry of period 2 and the mapping

$$(B^*)^{\widetilde{\tau}} \to A^*$$

$$\varphi \mapsto \varphi|_A$$

is a surjective linear isometry. We shall identify $(B^*)^{\widetilde{\tau}}$ and A^* without making any explicit mention.

Now, we focus our attention on orthogonal bilinear forms. Let $V : A \times A \to \mathbb{R}$ be a bounded bilinear form on a real C*-algebra. We say that V is *orthogonal* if $V(a, b^*) = 0$, whenever $a \perp b$ in A.

As we see in Chapter 3, orthogonal sesquilinear forms on complex C*-algebra were described by S. Goldstein in [83] (see Theorem 3.2.4).

Let $V : A \times A \to \mathbb{R}$ be a bounded bilinear form from a real C*-algebra. At first look, one is tempted to extend V to a bilinear form on $A \oplus iA$ and when the latter is orthogonal, apply Goldstein Theorem. However, this extension need not to be be orthogonal (see [77, Example 2.7]). Thus, we need to develop a different strategy. Our approach will consist in extending V to a bilinear form on A^{**}. To that end we proved some preliminary results on extension of multilinear operator on real C*-algebras (compare [77, pp. 3-4]).

Lemma 8.1.1 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let $A_1, \ldots, A_k$ be real C*-algebras and let T be a multilinear continuous operator from $A_1 \times \ldots \times A_k$ to a real Banach space X. Then T admits a unique Arens extension $T^{**} : A_1^{**} \times \ldots \times A_k^{**} \to X^{**}$ which is separately weak* continuous.* $\square$

Given a real or complex C*-algebra, A, the *multiplier algebra* of A, $M(A)$, is the set of all elements $x \in A^{**}$ such that, for each element $a \in A$, xa and ax both lie in A. We notice that $M(A)$ is a C*-algebra and contains the unit element of A^{**}. It should be recalled here that $A = M(A)$ whenever A is unital. The following property allows us to restrict ourselves to the study of orthogonal bilinear forms on unital real C*-algebras.

Proposition 8.1.2 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let A be a real C*-algebra. Suppose that $V : A \times A \to \mathbb{R}$ is an orthogonal bounded bilinear form. Then the continuous bilinear form*

$$\tilde{V} : M(A) \times M(A) \to \mathbb{R}, \quad \tilde{V}(a, b) := V^{**}(a, b)$$

is orthogonal. $\square$

Using two different approaches we were able to describe the behavior of an orthogonal bilinear form on the set of self-adjoint elements, however we are not able to give a complete description of its form on the whole real C*-algebras.

Proposition 8.1.3 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let A be a unital C*-algebra with unit* 1. *Suppose that $V : A \times A \to \mathbb{R}$ is an orthogonal, symmetric, bounded bilinear form. Then defining $\phi(x) := V(x, 1)$ $(x \in A)$, we have $V(a, b) = \phi_1(a \circ b)$, for every a, b in A_{sa}.* $\square$

A description of such a form V on the whole A remains open for general real C*-algebras. Nevertheless, we were able to describe bounded orthogonal forms on abelian real C*-algebras.

8.2. Orthogonal bilinear forms on abelian real C*-algebras

Throughout this section, A will denote a unital, abelian, real C*-algebra whose complexification will be denoted by B. Clearly B is a unital abelian C*-algebra. It is known that there exists a period 2 conjugate-linear *-automorphism $\tau : B \to B$ such that $A = B^\tau := \{x \in B : \tau(x) = x\}$ (cf. [158, 4.1.13] and [84, 15.4] or [130, §5.2]).

By the commutative Gelfand theory, there exists a compact Hausdorff space K such that B is C*-isomorphic to the C*-algebra $C(K)$ of all complex valued continuous functions on K. The Banach-Stone Theorem implies the existence of a homeomorphism $\sigma : K \to K$ such that $\sigma^2(t) = t$, and

$$\tau(a)(t) = \overline{a(\sigma(t))},$$

for all $t \in K$, $a \in C(K)$. Real function algebras of the form $C(K)^\tau$ have been studied by its own right and are interesting in some other settings (see, for example, [125]).

Henceforth, the symbol $\mathfrak{B}$ will stand for the σ-algebra of all Borel subsets of K, $S(K)$ will denote the space of $\mathfrak{B}$-simple scalar functions defined on K, while the *Borel algebra over* K, $B(K)$, is defined as the completion of $S(K)$ under the supremum norm. It is known that $B = C(K) \subset B(K) \subset C(K)^{**}$. The mapping $\tau^{**} : C(K)^{**} \to C(K)^{**}$ is a period 2 conjugate-linear *-automorphism on $B^{**} = C(K)^{**}$. It is easy to see that $\tau^{**}(B(K)) = B(K)$, and hence $\tau^{**}|_{B(K)} : B(K) \to B(K)$ defines a period 2 conjugate-linear *-automorphism on $B(K)$. By an abuse of notation, the symbol τ will denote τ, τ^{**} and $\tau^{**}|_{B(K)}$ indistinctly. It is clear that, for each Borel set $B \in \mathfrak{B}$, $\tau(\chi_{_B}) = \chi_{\sigma(B)}$.

Let V be an orthogonal bilinear form on an abelian real C*-algebra. First of all, we observe that, by Proposition 8.1.2, we can assume that A is unital. Our strategy shall consist in extending V to a bilinear form on $B(K)$. The main reason to do that is the abundance of projections in $B(K)$. We recall that such an extension exists and is unique by Lemma 8.1.1. However, it is not clear, at least initially, that this extension, that we shall also call V, is orthogonal.

Before studying bilinear forms we study spectral resolutions of self-adjoint and skew symmetric elements in A. It is well known that a self-adjoint element in A can be approximated in norm by finite sums of the form $\sum_{k=1}^{n} \lambda_k \chi_{_{B_k}}$, where $\{B_k\}_{k=1}^{n}$ is a family of mutually disjoint borel sets such that $\sigma(B_k) = B_k$ for every k.

If b is a *-skew-symmetric element in A, then it is not hard to see that b can be approximated in norm by sums of the form

$$\sum_{k=1}^{n} \lambda_k i \alpha_k (\chi_{_{E_k}} - \chi_{\sigma(E_k)}),$$

where $(E_k)_{k=1}^{n}$ is a family of mutually disjoint borel sets. However, this "spectral resolution" is no as good as one could wish, since for $k \neq j$ the elements $i\alpha_k(\chi_{E_k} - \chi_{\sigma(E_k)})$ and $i\alpha_j(\chi_{E_j} - \chi_{\sigma(E_j)})$ might not be orthogonal.

The following Lemma is the key to obtain a more useful spectral resolution for *-skew-symmetric elements in A.

Lemma 8.2.1 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let A be a unital, abelian, real C^*-algebra whose complexification is denoted by $B = C(K)$, for a suitable compact Hausdorff space K. Let $\tau : B \to B$ be a period 2 conjugate-linear *-automorphism satisfying $A = B^\tau$ and $\tau(a)(t) = \overline{a(\sigma(t))}$, for all $t \in K$, $a \in C(K)$, where $\sigma : K \to K$ is a period 2 homeomorphism. Then the set $N = \{t \in K : \sigma(t) \neq t\}$ is an open subset of K, $F = \{t \in K : \sigma(t) = t\}$ is a closed subset of K and there exists an open subset $\mathcal{O} \subset K$ maximal with respect to the property $\mathcal{O} \cap \sigma(\mathcal{O}) = \emptyset$.* □

It should be noticed here that, in Lemma 8.2.1, $\mathcal{O} \cup \sigma(\mathcal{O}) = N$, an equality which follows from the maximality of $\mathcal{O}$.

From Lemma 8.2.1 and the comments preceding it, the following "spectral resolution" of a *-skew-symmetric element in $B(K)^\tau$ can be deduced:

Lemma 8.2.2 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *In the notation of Lemma 8.2.1, let $B(A) = B(K)^\tau$, let $a \in B(K)^\tau_{sa}$, and let b be an element in $B(A)_{skew}$. Then the following statements hold:*

a) $b|F = 0$;

b) *For each $\varepsilon > 0$, there exist mutually disjoint Borel sets $B_1, \ldots, B_m \subset \mathcal{O}$ and real numbers $\lambda_1, \ldots, \lambda_m$ satisfying*

$$\left\| b - \sum_{j=1}^{m} i\ \lambda_j(\chi_{B_j} - \chi_{\sigma(B_j)}) \right\| < \varepsilon;$$

c) *For each $\varepsilon > 0$, there exist mutually disjoint Borel sets C_1, $\ldots$, $C_m \subset K$ and real numbers $\mu_1, \ldots, \mu_m$ satisfying $\sigma(C_j) = C_j$, and* $\left\| a - \sum_{j=1}^{m} \mu_j \chi_{C_j} \right\| < \varepsilon$.

We shall keep the notation of Lemma 8.2.1 throughout the section. Henceforth, for each $C \subseteq \mathcal{O}$ we shall write $u_{C} = i\,(\chi_{C} - \chi_{\sigma(C)})$. The symbol u_0 will stand for the element $u_{\mathcal{O}}$. It is easy to check $1 = \chi_{F} + u_0 u_0^*$, where 1 is the unit element in $B(K)^\tau$. By Lemma 8.2.2 a), for each $b \in B(K)^\tau_{skew}$ we have $b \perp \chi_{F}$, and so $b = b u_0 u_0^*$.

The separate weak*-continuity of V^{**} together with successive applications of the Urysohn Lemma, and regularity of elements in $B(K)^\tau$ allow us to prove the following properties of V^{**}:

Proposition 8.2.3 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let K be a compact Hausdorff space, τ a period 2 conjugate-linear isometric *-homomorphism on $C(K)$, $A = C(K)^\tau$, and $V : A \times A \to \mathbb{R}$ be an orthogonal bounded bilinear form whose Arens extension is denoted by $V^{**} : A^{**} \times A^{**} \to \mathbb{R}$. Let $\sigma : K \to K$ be a period 2 homeomorphism satisfying $\tau(a)(t) = \overline{a(\sigma(t))}$, for all $t \in K$, $a \in C(K)$. Then the following assertions hold for all Borel subsets D, B, C of K with $\sigma(B) \cap B = \sigma(C) \cap C = \emptyset$:*

a) $V(\chi_{D}, u_{B}) = V(u_{B}, \chi_D) = 0$, *whenever $A \cap B = \emptyset$;*

b) $V(u_{B}, u_{C}) = 0$, *whenever $B \cap C = \emptyset$;*

c) $V((u_0 u_0^* - u_{C} u_{C}^*) u_{B}, u_{C}) = V(u_{C}, (u_0 u_0^* - u_{C} u_{C}^*) u_{B}) = 0.$

□

The identities proved in the above proposition together with the spectral resolution in Lemma 8.2.2 led us to describe all continuous orthogonal forms on real C*-algebras.

Theorem 8.2.4 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let $V : A \times A \to \mathbb{R}$ be a continuous orthogonal form on a commutative real C*-algebra, then there exist φ_1, and φ_2 in A^* satisfying*

$$V(x,y) = \varphi_1(xy) + \varphi_2(xy^*),$$

for every $x, y \in A$. □

An immediate consequence of the above theorem is that the Arens extension of an orthogonal form on A is again an orthogonal form (compare [77, Corollary 2.6]).

We notice that the functionals φ_1 and φ_2 appearing in the above Theorem 8.2.4 need not to be unique, as it is shown in [77, Remark 2.5].

Clearly, the statement of the above Theorem 8.2.4 doesn't hold for bilinear forms on a commutative (complex) C*-algebra. The real version presented here is completely independent from the result proved by K. Ylinen for commutative complex C*-algebras in [189, Theorem 6.11] and from [83]. It seems natural to ask whether the real result follows from the complex one by a mere argument of complexification. The answer is, in general, negative. In [77, Example 2.7] we provide an example of an orthogonal bilinear form on a commutative real C*-algebra whose (canonical) extension to the complexification is not an orthogonal form.

The reader may be wondering which orthogonal bilinear forms can be extended to a bilinear form on the complexification. The following result answers this question:

Corollary 8.2.5 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let $V : A \times A \to \mathbb{R}$ be a continuous orthogonal form on a commutative real C*-algebra, let B denote the complexification of A and let $\widetilde{V} : B \times B \to \mathbb{R}$ be the (complex) bilinear extension of V. Then the form $\widetilde{V}$ is orthogonal if, and only if, V writes in the form $V(x, y) = \varphi_1(xy)$ $(x, y \in A)$, where φ_1 is a functional in A^*.* □

8.3. Orthogonality preservers

Once the the structure of orthogonal forms on abelian real C*-algebras is known, it is natural to try to describe orthogonality preserving operators between unital abelian real C*-algebras. Our approach in this study will consist on associating to an orthogonality preserving linear mapping a "support function" that will enjoy similar properties to those of the support of an orthogonality preserving linear mapping between complex $C(K)$-spaces.

The results presented here are independent innovations and extensions of those proved by Beckenstein, Narici and Todd, and Jarosz for $C(K)$-spaces.

Let $T : C(K_1)^{\tau_1} \to C(K_2)^{\tau_2}$ be an orthogonality preserving linear mapping. Keeping in mind the notation in the previous section, we write $L_i := \mathcal{O}_i \cup F_i$, where $\mathcal{O}_i$ and F_i are the subsets of K_i given by Lemma 8.2.1. The map sending each f in $C(Ki)^{\tau_i}$ to its restriction to L_i is a C*-isomorphism (and hence a surjective linear isometry) from $C(Ki)^{\tau_i}$ onto the real C*-algebra $C_r(L_i)$ of all continuous functions $f : L_i \to \mathbb{C}$ taking real values on F_i. Thus, studying orthogonality preserving linear maps between $C(K)^\tau$ spaces is equivalent to study orthogonality preserving linear mappings between the corresponding $C_r(L)$-spaces.

Henceforth, we consider an orthogonality preserving (not necessarily continuous) linear map $T : C_r(L_1) \to C_r(L_2)$, where L_1 and L_2 are two compact Hausdorff spaces and each F_i is a closed subset of L_i.

We consider a partition of the set L_2 :

$$Z_1 = \{s \in L_2 : \delta_s T \text{ is a non-zero bounded real-linear mapping}\},$$

$$Z_3 = \{s \in L_2 : \delta_s T = 0\}, \text{ and } Z_2 = L_2 \backslash (Z_1 \cup Z_3).$$

It is easy to see that Z_3 is closed.

As in the complex case, we can define a continuous support map $\varphi : Z_1 \cup Z_2 \to L_1$. More concretely, for each $s \in Z_1 \cup Z_2$, we write $\text{supp}(\delta_s T)$ for the set of all $t \in L_1$ such that for each open set $U \subseteq L_1$ with $t \in U$ there exists $f \in C_r(L_1)$ with $\text{coz}(f) \subseteq U$ and $\delta_s(T(f)) \neq 0$.

Following a standard argument, it can be shown that, for each $s \in Z_1 \cup Z_2$, $\text{supp}(\delta_s T)$ is non-empty and reduces exactly to one point $\varphi(s) \in L_1$, and the assignment $s \mapsto \varphi(s)$ defines a continuous map from $Z_1 \cup Z_2$ to L_1. Furthermore, the value of $T(f)$ at every $s \in Z_1$ depends strictly on the value $f(\varphi(s))$. More precisely, for each $s \in Z_1$ with $\varphi(s) \notin F_1$, the value $T(g)(s)$ is the same for every function $g \in C_r(L_1)$ with $g \equiv i$ on a neighborhood of $\varphi(s)$. Thus, defining $T(i)(s) := 0$ for every $s \in Z_3 \cup Z_2$ and for every $s \in Z_1$ with $\varphi(s) \in F_1$, and $T(i)(s) := T(g)(s)$ for every $s \in Z_1 \cup Z_2$ with $\varphi(s) \notin F_1$, where g is any element in $C_r(L_1)$ with $g \equiv i$ on a neighborhood of $\varphi(s)$, we get a (well-defined) mapping $T(i) : L_2 \to \mathbb{C}$. It should be noticed that "$T(i)$" is just a symbol to denoted the above mapping and not an element in the image of T.

The next theorem is the desired generalisation of the classical results on complex $C(K)$-spaces presented in Chapter 3.

Theorem 8.3.1 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *In the notation above, let* $T : C_r(L_1) \to C_r(L_2)$ *be an orthogonality preserving linear mapping. Then* L_2 *decomposes as the union of three mutually disjoint subsets* $Z_1, Z_2,$ *and* Z_3*, where* Z_2 *is open and* Z_3 *is closed, there exist a continuous support map* $\varphi : Z_1 \cup Z_2 \to L_1$*, and a bounded mapping* $T(i) : L_2 \to \mathbb{C}$ *which is continuous on* $\varphi^{-1}(\mathcal{O}_1)$ *satisfying:*

$$T(i)(s) \in \mathbb{R} \ (\forall s \in F_2),$$

$$T(i)(s) = 0 \ (\forall s \in Z_3 \cup Z_2 \textit{ and } \forall s \in Z_1 \textit{ with } \varphi(s) \in F_1),$$

$$|T(1)(s)| + |T(i)(s)| \neq 0, \ \ (\forall s \in Z_1), \tag{8.1}$$

$$T(f)(s) = T(1)(s) \ \Re e f(\varphi(s)) + T(i)(s) \ \Im m f(\varphi(s)), \tag{8.2}$$

for all $s \in Z_1$, $f \in C_r(L_1)$,

$$T(f)(s) = 0, \ \ (\forall s \in Z_3, f \in C_r(L_1)),$$

and for each $s \in L_2$*, the mapping* $C_r(L_1) \to \mathbb{C}$, $f \mapsto T(f(s))$*, is unbounded if, and only if,* $s \in Z_2$*. Furthermore, the set* $\varphi(Z_2)$ *is finite.* □

As in the complex case, when further hypothesis on T are assumed, more properties on φ are obtained. When T is bijective we were able to prove that T is continuous, however contrary to the result in the complex case, φ need not to be, in general, a homeomorphism, and T^{-1} need not be orthogonality preserving (see Example 3.7 in [77]). However, a result of automatic continuity can be derived.

Theorem 8.3.2 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Every orthogonality preserving linear bijection between unital commutative real* C^**-algebras is* (*automatically*) *continuous.*

As we have already mentioned, the inverse of an orthogonality preserving linear bijection between $C_r(L)$-spaces need not be orthogonality preserving. In [77], we also characterised biorthogonality preserving operators between $C_r(L)$-spaces.

Theorem 8.3.3 [J.J. Garcés and A.M. Peralta, Linear and Multilinear algebra, 2013] *Let $T : C_r(L_1) \to C_r(L_2)$ be a mapping. The following statements are equivalent:*

(a) *T is a biorthogonality preserving linear surjection;*

(b) *There exists a (surjective) homeomorphism $\varphi : L_2 \to L_1$ with $\varphi(\mathcal{O}_2) = \mathcal{O}_1$, a function $a_1 = \gamma_1 + i\gamma_2$ in $C_r(L_2)$ with $a_1(s) \neq 0$ for all $s \in L_2$, and a function $a_2 = \eta_1 + i\eta_2 : L_2 \to \mathbb{C}$ continuous on $\mathcal{O}_2$ with the property that*

$$0 < \inf_{s\in\mathcal{O}_2} \left|\det\begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix}\right|$$

$$\leq \sup_{s\in\mathcal{O}_2} \left|\det\begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix}\right| < +\infty,$$

such that

$$T(f)(s) = a_1(s)\, \Re e f(\varphi(s)) + a_2(s)\, \Im m f(\varphi(s))$$

for all $s \in L_2$ and $f \in C_r(L_1)$. $\square$

Chapter 9

Local triple derivations

According to a chronological order, one of the latest problems we have explored is the problem of local triple derivations on C*-algebras, treated in collaboration with M. Burgos, F.J. Fernández-Polo and A.M. Peralta in [36].

Local (associative) derivations on a Banach algebra where introduced by R. Kadison in 1990 (see [160]) in the following sense: Let A be an associative Banach algebras and X an A-bimodule. A linear mapping $T : A \to X$ is said to be a *local (associative) derivation* if for each a in A, there is a derivation $D_a : A \to X$ such that $T(a) = D_a(a)$. R. Kadison proved that each norm-continuous local derivation of a von Neumann algebra W into a dual W-bimodule is a derivation (cf. [113, Theorem A]). In a remarkable paper, B.E. Johnson extended the above result proving that every (continuous) local derivations from any C*-algebra B into any Banach B-bimodule is a derivation (see [110, Theorem 5.3]). In [110], B.E. Johnson also gave an automatic continuity result, showing that local derivations on C*-algebras are automatically continuous.

The above results were highly stimulating for a multitude of studies on local derivations on C*-algebras (see, for example, [4, 5, 47, 86, 87, 94, 127, 132, 131, 171] and [193]). The above theorems also led to the study of local automorphisms, local isometries and 2-local derivations and automorphisms (see for instance [167, 141, 85, 142, 102] and [168]).

In [36] M. Burgos, F.J. Fernández-Polo, A.M. Peralta and the author of this Thesis study local triple derivations on C*-algebras. We also explore the connections of local triple derivations and the generalised derivations introduced in [132] by J. Li and Z. Pan (we aware that this concept of generalised derivation does not, in general, coincide with the concept of generalised derivation introduced in [76]).

Although our main results are obtained in the setting of C*-algebras we also give some of the definitions and preliminary results in a more general setting.

Definition 9.1.1 *Let E be a JB*-triple and let X be a Jordan-Banach triple E-module. A conjugate linear mapping $\delta : E \to X$ is said to be a* local triple derivation *if for each a in E, there exits a triple derivation $\delta_a : E \to X$ such that $\delta(a) = \delta_a(a)$. A local triple derivation on E is a linear mapping $T : E \to E$ satisfying that for each $a \in E$, there exists a triple derivation $\delta_A : E \to E$ such that $T(a) = \delta_a(a)$.*

The problem we are interested in is whether every local triple derivation on a C*-algebra or on a JB*-triple is a triple derivations.

Local triple derivations on a JB*-triple were introduced by M. Mackey in [137]. In the just quoted paper, M. Mackey gave a partial affirmative answer the above problem by proving that every continuous local triple derivation on a JBW*-triple is a triple derivation. The proofs and techniques applied by M. Mackey in this result depend heavily in the particular structure of a JBW*-triple and the abundance of tripotent elements in this setting (for this reason, they are also valid to prove that every local triple derivation on a compact JB*-triple is a triple derivation). Mackey's theorem is an appropriate triple version of the aforementioned Kadison's theorem. The corresponding JB*-triple version of Johnson's theorem was an open problem.

Problem 9.1.2 *Is every (continuous) local triple derivation on a JB*-triple E (or more generally, every local triple derivation from E into a Jordan Banach triple E-module) a triple derivation?*

Let a, b be elements in E. Then the conjugate linear mapping $\delta(a, b) := L(a, b) - L(b, a)$ is a derivation. A triple derivations is said to be *inner* if it can be written as a finite sum of derivations of the form $\delta = \delta(a, b) := L(a, b) - L(b, a)$.

Throughout this chapter A will denote a unital C*-algebra and B a subalgebra of A containing the unit of A. It is not hard to see that if $\delta : A \to B$ is a local triple derivation, then $\delta(1)^* = -\delta(1)$. The mapping $\delta(T(1), 1)$ is a triple derivation with $\delta(T(1), 1)1 = 2T(1)$. Therefore

$$\widetilde{T} = T - \frac{1}{2}\delta(T(1), 1)$$

is a local triple derivation with $\widetilde{T}(1) = 0$.

Following the terminology employed by J. Li and Zh. Pan in [132], a linear mapping D from a unital C*-algebra A to a (unital) Banach A-bimodule X is called a *generalised derivation* whenever the identity

$$D(ab) = D(a)b + aD(b) - aD(1)b$$

holds for every a, b and c in A.

Let a be an element in a C*-algebra B. It is easy to see that the operator $G_a : B \to B$, $x \mapsto G_a(x) := ax + xa$, is an example of a generalised derivation on B. Since, in the case of B being unital, $G_a(1) = 2a$, it follows that G_a is not a local ternary derivation whenever $a^* \neq -a$.

We shall say that D is a *generalised Jordan derivation* whenever $D(a \circ b) = D(a) \circ b + a \circ D(b) - U_{a,b}D(1)$, for every a, b in A, where the Jordan product is given by $a \circ b := \frac{1}{2}(ab + ba)$ and $U_{a,b}(x) := (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x$. Every generalised (Jordan) derivation $D : A \to X$ with $D(1) = 0$ is a (Jordan) derivation, and every generalised derivation is a generalised Jordan derivation. The reciprocal statement is not clear at this stage (see [36, Remark 9.6] for completeness).

Let us note that the above notions of generalised derivations and generalised Jordan derivations are not related to the concept of generalised triple derivations used in Chapter 7. Unfortunately, the names are similar but the concept are not related.

It is due to B.E. Johnson that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation (cf. [109]). It is also known that every Jordan derivation from a C*-algebra A to a Banach A-module or to a Jordan Banach A-module is continuous (cf. [163, §1]). Therefore, every generalised Jordan derivation D from a unital C*-algebra A to a Banach A-bimodule with $D(1) = 0$ is a bounded Jordan derivation and hence a continuous derivation.

We shall explore now the connections between generalised (Jordan) derivations and triple derivations from A to B. Let $\delta : A \to B$ be a triple derivation. Since $\delta(1)^* = -\delta(1)$, we have

$$\delta(a \circ b) = \delta \{a, 1, b\} = \{\delta(a), 1, b\} + \{a, 1, \delta(b)\} + \{a, \delta(1), b\}$$

$$= \delta(a) \circ b + a \circ \delta(b) + U_{a,b}(\delta(1)^*) = \delta(a) \circ b + a \circ \delta(b) - U_{a,b}(\delta(1)),$$

for every a, b in A, which shows that δ is a generalised Jordan derivation.

Direct computations allow to prove the following property for local triple derivations:

Lemma 9.1.3 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *Let E be a JB*-subtriple of a JB*-triple F, where the latter is regarded as a Jordan Banach triple E-module with respect to its natural triple product. Let $T : E \to F$ be a bounded local triple derivation. Then the products of the form $\{a, T(b), c\}$ vanish for every a, b, c in A with $a \perp b \perp c$.* □

The next result, whose proves makes use of Lemma 9.1.3 and Goldstein's description of orthogonal sesquilinear forms, shows that every local triple derivation from a commutative unital C*-algebra is also a generalised Jordan derivation.

Proposition 9.1.4 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *Let A be a commutative C*-subalgebra of a C*-algebra B. Suppose A and B are unital, A contains the unit, 1, of B and the latter is regarded as a Jordan Banach triple A-module with respect to its natural triple product. Let $T : A \to B$ be a bounded local triple derivation. Then T is a generalised Jordan derivation.* □

During the prove of the above Proposition, we showed that the identity

$$\{x, T(ys), t\} = \{x, T(y), s^*t\} + \{xy^*, T(s), t\} - \{xy^*, T(1), s^*t\}$$

holds for every x, y, s, t in A. An application of Goldstine's Theorem and the separate weak*-continuity of the triple product in A^{**} and the weak*-continuity of T^{**} allow to prove that the equality

$$\{x, T^{**}(ys), t\} = \{x, T^{**}(y), s^*t\} + \{xy^*, T^{**}(s), t\}$$
$$-\{xy^*, T(1), s^*t\}$$

holds for every x, y, s, t in A^{**}. Taking $x = t = 1$ we see that T^{**} also is a generalised Jordan derivation.

The above observation establishes a stronger version of Proposition 9.1.4, which is a subtle variant of [113, Sublemma 5] and [132, Proposition 1.1].

Proposition 9.1.5 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *In the hypothesis of Proposition 9.1.4, let $T : A \to B$ be a bounded local triple derivation. Then for each $a, b, c \in A$ with $ab = bc = 0$ we have*

$$aT(b)^*c = aT(b^*)^*c = 0.$$

$\square$

One of the main results established by J. Li and Z. Pan in [132, Corollary 2.9] implies that a bounded linear operator $T : A \to B$ is a generalised derivation if, and only if, $aT(b)c = 0$, whenever $ab = bc = 0$.

Let us suppose that, in the above hypothesis, A is commutative and $T : A \to B$ is a local triple derivation. Proposition 9.1.5 assures that $aT(b^*)^*c = 0$, for every $ab = bc = 0$ in A, and consequently, the mapping $x \mapsto T(x^*)^*$ is a generalised derivation, and thus,

$$T(a^*b^*)^* = T(a^*)^*b + aT(b^*)^* - aT(1)^*b,$$

or equivalently,

$$T(ba) = T(ab) = bT(a) + T(b)a - bT(1)a,$$

showing that T is actually a generalised derivation. We have therefore proved the following:

Corollary 9.1.6 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *In the hypothesis of Proposition 9.1.4, every local bounded triple derivation T from A to B is a generalised derivation. Moreover, taking $\widetilde{T} = T - \frac{1}{2}\delta(T(1), 1) = T - \delta\left(\frac{1}{2}T(1), 1\right)$, it follows that $\widetilde{T}$ is a local triple derivation with $\widetilde{T}(1) = 0$ and hence $\widetilde{T}$ is a derivation.* □

The statement concerning $\widetilde{T}$ in the above corollary could be also derived applying the previously mentioned Johnson's theorem on the equivalence of Jordan derivations and (associative) derivations (cf. [109, Theorem 6.3]).

Associative derivations from A to B are not far away from triple derivation. It is not hard to check that, in our setting, a bounded linear operator $\delta : A \to B$ is a triple derivation and $\delta(1) = 0$ if, and only if, it is a *-*derivation*, that is, it is a derivation and $\delta(a^*) = \delta(a)^*$. The next result assures that local triple derivations also are symmetric operators.

Lemma 9.1.7 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *Let B be a unital C*-algebra, and let $T : B \to B$ be a bounded local triple derivation with $T(1) = 0$. Then T is a symmetric operator, that is, $T(a^*) = T(a)^*$, for every $a \in B$.* □

In [36, Theorem 10] we were finally able to prove that, in our setting, every local triple derivation is actually a triple derivation.

Theorem 9.1.8 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *Let B be a unital C^*-algebra. Every bounded local triple derivation on B is a triple derivation.* □

In the final section of [36] we generalise the above Theorem 9.1.8 to the setting of unital JB*-algebras.

Theorem 9.1.9 [M. Burgos, F.J. Fernández-Polo, J.J. Garcés and A.M. Peralta, Commun. Alg., 2013] *Let J be a unital JB^*-algebra. Every bounded local triple derivation from J to J is a triple derivation.* □

Chapter 10

Conclusions and open problems

We believe that the relevance of the problems considered in this thesis has been proved. As we have seen, these problems have been treated in past by many important authors.

We also believe that our contributions have their importance since, in many cases, we gave a final solution to problems that had been open for a long time (that is the case, for example, of the description of orthogonality preservers between C*-algebras or the characterization of weakly compact orthogonality preserving operators between C*-algebras). Some problems remain open after our research. In these cases our contribution doesn't give a definitive answer to the problem we treated, but our results improve considerably what was known before on the subject. As it can be contrasted in the list of references, our results have been published in well reputed journals in our area.

We conclude this memory by explaining some current research lines that we are considering at the moment or that we want to consider in the future.

10.1. Orthogonality preserves on real C*-algebras

As we have seen in Chapter 3, the structure of an orthogonality preserving operator between complex C*-algebras (also between JB*-algebras) is already known. In Chapter 8 we initiated the study of orthogonality preservers on real C*-algebras.

Using classical techniques (the support function) we are able to determine orthogonality preserving linear mappings between abelian unital real C*-algebras. However, the non-unital and the non-associative case remain open.

Problem 10.1.1 *Goal: To study orthogonality preserving linear mappings between abelian real C*-algebras.*

Problem 10.1.2 *Is every orthogonality preserving linear bijection between abelian real C*-algebras continuous?*

We notice that under continuity assumption the problem can be reduced to the unital case (via an extension of the mapping to the multiplier algebra) where the structure of an orthogonality preserving operator is already known. However in Problem 10.1.1 we do not assume continuity.

More generally, we can pose the problem of describing orthogonality preserving operators between real C*-algebras.

Problem 10.1.3 *To describe orthogonality preserving operators between real C*-algebras.*

If we observe the techniques used to determine orthogonality preserving operators in the complex setting, we see that the orthogonal forms and the orthogonality additive n-homogeneous polynomials were important tools to obtain this description. Thus, these notions might also play a role in the solution of Problem 10.1.3.

Problem 10.1.4 *To describe orthogonal bilinear forms on real C*-algebras.*

We recall that, by Proposition 8.1.3, if $V : A \times A \to \mathbb{R}$ is an orthogonal bilinear form on a real C*-algebra A, the there exists ϕ in A^* such that

$$V(a,b) = \phi(a \circ b),$$

for every a, b in A_{sa}. However, the behavior of V on A_{skew} remains unknown.

Let $P : A \to \mathbb{R}$ be an orthogonally additive 2-homogeneous polynomial on an abelian (unital) real C*-algebra. We define $V : A \times A \to \mathbb{R}$ by $V(a,b) = P(a+b) - P(a) - P(b)$. Clearly, V is an orthogonal bilinear form. By Theorem 8.2.4 there exist ψ, φ in A^* such that

$$V(a,b) = \psi(ab) + \varphi(ab^*),$$

for every a, b in A. We then have that $P(a+b) - P(a) - P(b) = \psi(ab) + \varphi(ab^*)$, for every a, b in A. Taking $a = b$ we prove that $P(a) = \frac{1}{2}\psi(a^2) + \frac{1}{2}\varphi(aa^*)$, for every a in A.

Let us fix n in $\mathbb{N}$ and ψ, φ in A^*. For $k \leq n$, it is not hard to see that the assignment

$$P_{\{k,\psi,\varphi\}}(a) = \psi(a^n) + \varphi(a^k (a^*)^{n-k})$$

defines an n-homogeneous polynomials on A which is clearly orthogonally additive.

Problem 10.1.5 *Is every n-homogeneous polynomial on a commutative (unital) real C*-algebra A a finite sum of polynomials of the form $P_{\{k,\psi,\varphi\}}$?*

More generally, What is the form of an n-homogeneous orthogonally additive polynomial on a (not necessarily commutative) real C-algebra?*

Another problem we have considered, once the structure of an orthogonality preserving operator is known, is that of studying the special case in which the operator also is weakly compact.

Let $T : A \to B$ be an orthogonality preserving operator between abelian unital C*-algebra. From Theorems 8.3.1 and Theorem 6.2.2 (see also 6.2.1) one might conjecture the existence of norm-null sequences $(a_n), (b_n)$ in B, where $a_n \perp a_m$ and $b_n \perp b_m$ for $n \neq m$, and a a sequence of points mutually distinct points (t_n) in K_2 such that

$$T(f) = \sum_n a_n \, \Re\mathrm{e} f(\varphi(t_n)) + b_n \, \Im\mathrm{m} f(\varphi(t_n))$$

for every f in A (where φ stands for the support function of T).

However we should be careful since, as we have already seen, frequently, statements which are true in the complex setting need not be true in the real setting.

Problem 10.1.6 *To describe weakly compact orthogonality preserving operators between (abelian) real C*-algebras.*

10.2. Automatic continuity

Concerning automatic continuity of orthogonality preservers between C*-algebras our main results in [38] (see also Chapter 4) shows that every biorthogonality preserving linear surjection between dual C*-algebras or von Neumann algebras is continuous. As a consequence, a partial affirmative answer to the a conjecture posed by J. Araujo and K. Jarosz in [12] can be given (compare Corollary 4.2.13). Indeed, if T is a symmetric zero-product preserving operator, then T preserves orthogonality. Thus, every symmetric and bijective linear mapping between dual C*-algebras or von Neumann algebras which preserves zero-products in both directions is biorthogonality preserving and hence continuous.

In [129], C.W. Leung, N.C. Tsai, N.C. Wong prove that every linear bijection between von Neumann algebras which preserves zero-products in both directions is automatically continuous.

The problem for general C*-algebras remains open, for both biorthogonality preserving linear surjections and linear bijections preserving zero-products in both senses.

Problem 10.2.1 *Is every biorthogonality preserving linear surjection between C*-algebras (automatically) continuous?*

We can also consider biorthogonality preserving linear surjections on JB*-algebras. From the automatic continuity results proved in [39], we know that every biorthogonality preserving linear surjection between dual JB*-algebras or atomic JBW*-algebras is continuous.

JBW*-algebras are, in some sense, non-associative Jordan analogues of von Neumann algebras. Thus, it is natural to ask whether every biorthogonality preserving linear surjection between JBW*-algebras is continuous.

Problem 10.2.2 *Is every biorthogonality preserving linear surjection between JBW*-algebras (automatically) continuous?*

Let J be a unital JB*-algebra and let $\mathcal{P}(J)$ denote the set of projections of J. Let $T : J \to E$ be a non-zero orthogonality preserving linear mapping from J to a JB*-triple. Take a projection p in $\mathcal{P}(J)$, the JB*-subalgebra generated by 1 and p, $J_{1,p}$, is a two dimensional JB*-algebra. Let $r = r(h)$, where $h = T(1)$. By Theorem 3.2.14, there exists a (unital) Jordan *-homomorphism $S_p : J_{1,p} \to E_2^{**}(r)$, such that $S(J_{1,p}) \subseteq \{h\}'$ and

$$T(x) = h \circ_r S_p(x),$$

for every x in $J_{1,p}$. In particular, there exists a unique projection $S_p(p)$ in $E_2^{**}(r)$ such that $T(p) = h \circ_r S_p(p)$.

We define the mapping

$$S : \operatorname{span}(\mathcal{P}(J)) \to E_2(r),$$

given by

$$S\left(\sum_{k=1}^{n} \lambda_k p_k\right) := \sum_{k=1}^{n} \lambda_k S_{p_k}(p_k).$$

Let us suppose that $\sum_{k=1}^{n} \lambda_k p_k = \sum_{j=1}^{m} \alpha_j q_j$ then, since

$$T\left(\sum_{k=1}^{n} \lambda_k p_k\right) = \sum_{k=1}^{n} \lambda_k T(p_k) = \sum_{j=1}^{m} \alpha_j T(q_j) = T\left(\sum_{j=1}^{m} \alpha_j q_j\right),$$

we have

$$h \circ_r \left(\sum_{k=1}^{n} \lambda_k S_{p_k}(p_k)\right) = h \circ_r \left(\sum_{j=1}^{m} \alpha_j S_{q_j}(q_j)\right).$$

Now, it follows from [35, Lemma 4.1] that the multiplication operator $M_h : E_2^{**}(r) \to E_2^{**}(r)$ is injective. The latter implies that $\sum_{k=1}^{n} \lambda_k S_{p_k}(p_k) = \sum_{j=1}^{m} \alpha_j S_{q_j}(q_j)$, and thus the mapping S is well-defined. Actually, it is not hard to see that S is a Jordan *-homomorphism.

Proposition 10.2.3 *Let $T : J \to E$ be an orthogonality preserving linear mapping from a unital JB*-algebra to a JB*-triple. Then there exists a (unital) Jordan *-homomorphism*

$$S : span(\mathcal{P}(J)) \to E_2^{**}(r(h))$$

such that $S(span(\mathcal{P}(J))) \subseteq \{h\}'$ and

$$T(x) = h \circ_r S(x),$$

for every x in $span(\mathcal{P}(J))$. □

If $J = \text{span}(\mathcal{P}(J))$ then S is continuous and therefore T is continuous too.

Theorem 10.2.4 *Let J be a unital JB*-algebra linearly spanned by its projections. Then every orthogonality preserving linear mapping from J to a JB*-triple is continuous.* □

The following generalisation of Theorem 4.2.9 follows as a direct consequence:

Corollary 10.2.5 *Let A be a unital C*-algebra linearly spanned by its projections. Then every orthogonality preserving linear mapping form A to a JB*-triple is continuous.* □

Theorem 10.2.4 justifies the interest of the following problem:

Problem 10.2.6 *To find examples of JB*-algebras linearly spanned by projections.*

10.3. Stability

Let A and B be Banach algebras and let $T : A \to B$ be a bounded generalised homoeomorphism, that is, a mapping for which there exists $\varepsilon > 0$ such that

$$\|T(ab) - T(a)T(b)\| \leq \varepsilon \|a\| \|b\|,$$

for every $a, b \in A$.

Since, in some sense, T almost preserves the product, one might wonder whether T is far from being an homomorphism or not. In [108] studies this problem. More concretely, Johnson studies when a generalised homomorphism is near an homomorphism.

We shall denote by $M(A, B)$ the set of homomorphisms form a into B. Given a generalised homomorphism T in $L(A, B)$ we define

$$d(T) = \text{ínf}\{\|T - S\| : S \in M(A, B)\}.$$

The constant $d(T)$ can be seen as a measure of how products are preserved by T.

Clearly, T is an homomorphism if, and only if, $d(T) = 0$. The problem considered by Johnson in [108] is whether $\|\check{T}\|$ being small implies $d(T)$ being small.

Definition 10.3.1 *We say that a pair of Banach algebras* (A, B) *is* AMNM (almost multiplicative are near multiplicative maps) *if for each positive* ε *and* K *there is a positive* δ *such that if* $T \in L(A, B)$ *with* $\|T\| < K$ *and* $\|\check{T}\| < \delta$ *then* $d(T) < \varepsilon$.

Definition 10.3.1 admits the following reformulation:

Proposition 10.3.2 [108, PROPOSITION 1.4] *The following are equivalent:*

1. *(A, B) is AMNM.*

2. *For any bounded sequence $\{T_n\}$ in $L(A, B)$, with $\check{T}_n \to 0$ there is a sequence $\{S_n\}$ in $M(A, B)$ with $T_n - S_n \to 0$.* □

A Banach algebra is said to be AMNM if $(A, \mathbb{C})$ is AMNM (is Jarosz's terminology being AMNM is called f-stable [101]).

Many examples of AMNM pairs of Banach algebras are provided by Johnson in [108]. Perhaps one of the more important ones is the following:

Theorem 10.3.3 [108, THEOREM 3.1] *Let A and B be amenable Banach algebras and suppose that B is a Banach algebra such that there is a Banach B-bimodule B_* so that B is isomorphic as a B-bimodule to $(B_*)^*$. The (A, B) is AMNM.* □

In the same paper, Johnson also considers stability of generalised *-homomorphisms.

Definition 10.3.4 *We shall say that a pair of Banach *-algebras (A, B) is* AMNM* *if for each positive ε and K there is a positive δ such that if $T \in L(A, B)$ with $\|T\| < K$ and $\|\check{T}\| < \delta$ and $\|T(a^*)^* - T(a)\| \leq \delta\|a\|$, then there is a *-homomorphism $S : A \to B$ such that $\|T - S\| < \varepsilon$.*

Every pair of Banach star algebras satisfying the conditions of Theorem 10.3.3 is an AMNM* pair (compare [108, Theorem 7.2]). The motivation to study stability of homorphisms seems to be in the deformation theory of Banach algebras (see the monograph [99] for more on this subject).

Examples of Banach algebras not being AMNM can be found in [172] and [108].

The study of stability of homomorphisms naturally leads to the study of stability of disjointness preservers.

Definition 10.3.5 *Let $T : A \to B$ be a linear mapping between Banach algebras. We say that T* approximately preserves zero products (*or that T is* almost zero-product preserving) *if there is $\varepsilon \geq 0$ such that $\|T(ab)\| \leq \varepsilon\|a\|\|b\|$, whenever $ab = 0$.*

In [55], G. Dolinar proves the following stability result:

Theorem 10.3.6 *Let K_1 and K_2 be compact Hausdorff spaces and $T : C(K_1) \to C(K_2)$ an almost zero-product preserving mapping with $\|T(ab)\| \leq \varepsilon\|a\|\|b\|$, whenever $ab = 0$. Then there is an homomorphism $S : C(K_1) \to C(K_2)$ such that*

$$\|T(f) - S(f)\| \leq 20\sqrt{\varepsilon}\|f\|$$

for all f in $C(K_1)$. □

See [9], [10] and [11] for further results on stability in $C(K)$-spaces by J. Araujo and J.J. Font.

Stability of zero-product preserving linear mapping between Banach algebras has also being considered by J. Alaminos, J. Extremera and A.R. Villena in [6].

Let A, B be Banach algebras. For a surjective linear mapping $T \in L(A, B)$ a positive constant $M > 0$ is called an openness constant for T if given $y \in Y$, there exists $x \in X$ with $T(x) = y$ and $\|x\| \leq M\|y\|$. The *openness index* of T, denoted $op(T)$, is defined as the infimum of the set $\{M > 0 : M \text{ is an openness constant for } T\}$. On the other hand, a measure of how zero-products are, in some sense, preserved by T is given by the constant $zmult(T)$ given by

$$zmult(T) = \sup\{\|T(a)T(b)\| : a, b \in A, ab = 0, \|a\| = \|b\| = 1\}.$$

The following stability result for zero-product preservers between Banach algebras was obtained by J. Alaminos, J. Extremera and A. Villena.

Theorem 10.3.7 *Let A be either the group algebra $L^1(G)$, for some locally compact group G, or a C^*-algebra and let B be a Banach algebra. Suppose that both A and B are amenable and that there is a Banach B-bimodule X so that $M(B)$ is isomorphic as a B-bimodule to X^*. Let $\varepsilon, K, M > 0$. Then there exists delta > 0 such that if $T \in L(A, B)$ is surjetive with $\|T\| < K$, $op(T) \leq M$ and $zmult(T) < \delta$, then there are an invertible element υ in in the zentroid $Z(M(B))$ and a continuous epimorphism $\Phi : A \to B$ with $\|T - \upsilon\Phi\| < \varepsilon$.* $\square$

As long as we know stability of triple homomorphisms or orthogonality preservers has not been studied yet.

Let E, F be JB*-triples. We denote by $\mathrm{THom}(E, F)$ the set of triple homomorphism between E and F, while $\mathrm{OrthP}(E, F)$ stands for the set of orthogonality preserving operators between E and F. If E and F are JB*-algebras the symbol $\mathrm{J^*Hom}(E, F)$ will stand for the set of Jordan *-homomorphisms between E and F.

We shall also somehow measure how an operator preserves triple products (respectively Jordan products or orthogonality). Let $T : E \to F$ be a linear operator, we define the following constants

$$Tpres(T) = \sup\{\|\check{T}(a, b, c)\| : \|a\| = \|b\| = \|c\| = 1\},$$

$$Orthp(T) = \sup\{\{T(a), T(b), z\} : a \perp b, \|a\| = \|b\| = \|z\| = 1\}\},$$

and $J^*pres(T)$ is defined as the maximum of

$$\sup\{\|T(a \circ b) - T(a) \circ T(b)\| : \|a\| = \|b\| = 1\}$$

and

$$\sup\{\|T(a^*)^* - T(a)\| : \|a\| = 1\}.$$

The question clearly is whether these constants measure the distance from T to the set of those operators preserving the defining property of the corresponding constant.

The answer to the above problem might depend on the pair of JB*-triples E and F. The following definitions seem to be the appropriate generalisations of the AMNM pairs considered by Johnson.

Definition 10.3.8 *The pair of JB*-triples (E, F) is said to be* ATHNTH (*respectively, AOPNOP*) *if for each $\varepsilon, K > 0$ there is a $\delta > 0$ such that if $T \in L(A, B)$ with $\|T\| < K$ and $\|\check{T}\| < \delta$ (respectively, $Orthp(T) < \delta$) then there is a triple homomorphism* (*respectively, an orthogonality preserving operator*) *$S : E \to F$ such that $\|T - S\| < \varepsilon$.*

If E, F are JB*-algebras then AJHNJH* pairs are analogously defined.

Problem 10.3.9 *Goal: To find examples of ATHNTH, AOPNOP and AJHNJH pairs of JB*-triples and JB*-algebras.*

A simpler problem to begin with might be the case when $F = \mathbb{C}$. Following Johnson we shall say that a JB*-triple E is ATHNTH if $(E, \mathbb{C})$ is a ATHNTH pair. Analogously we define AJHNJH and AOPNOP JB*-triples and JB*-algebras.

Since an orthogonality preserving linear operator between C*-algebras (or JB*-algebras) is essentially a multiple of a triple homomorphism, the following question arises naturally.

Problem 10.3.10 *What is the relation* (*if any*) *between being ATHNTH* (*or AJHNJH*) *and being AOPNOP?*

Similarly, stability of generalised derivation can also be considered.

10.4. Local triple homomorphisms

In Chapter 9 of this memory we presented local triple derivations and the results we obtained in this subject (see [36] for more details). Our main results state that every local triple derivation on a unital C*-algebra (or a unital JB*-algebra) is actually a triple derivation.

The problem for non-unital C*-algebras and JB*-algebras (as well as for general JB*-triples) was left open. This problem has been recently solve in full generality by M. Burgos, F.J. Fernandez-Polo and A.M. Peralta in [37], where they also prove that every local triple derivation between JB*-triples is automatically continuous (compare [37, Theorem 2.8]).

Theorem 10.4.1 [M.J. Burgos, F.J. Fernández-Polo, and A.M. Peralta, preprint, 2013] *Every (non necessarily continuous) local triple derivation on a JB*-triple is a triple derivation.* □

The study of local (associative) derivation gave way to the study of local automorphisms, local isometries and 2-local derivations and automorphisms (see for instance [167, 141, 85, 142, 102] and [168]).

Once the problem of local triple derivations is solved, it seems natural to consider the triple version of the above mentioned problems.

Definition 10.4.2 *A linear operator $T : E \to F$ between JB*-triples is said to be a* local triple homomorphism *if for each a in E there is a triple homomorphism $T_a : E \to F$ such that $T(a) = T_a(a)$.*

Since triple homomorphisms are contractive, if T is a local triple homomorphism then $\|T(a)\| = \|T_a(a)\| \leq \|a\|$, thus local triple homomorphism are not merely continuous but also contractive.

The following question is clear (cf. [137, Remark 6.3]).

Problem 10.4.3 *Is every local triple homomorphism a triple homomorphism?*

The question if every local triple derivation on a real JB*-triple is a triple derivation remains as an open question (cf. [37]).

10.5. M-norms

A geometric notion of orthogonality (called M-orthogonality) can be given in an arbitrary Banach space. Two elements x, y in a Banach space X are said to be *M-orthogonal*, denoted $x \perp_M y$, if $\|a \pm b\| = \text{máx}\{\|a\|, \|b\|\}$. The elements x, y are said to be *semi-M-orthogonal*, written $x \perp_{SM} y$, if $\|x \pm y\| \geq \text{máx}\{\|x\|, \|y\|\}$.

When the Banach space has additional algebraic structure there exists a connection between algebraic orthogonality and M-orthogonality. Indeed, let a and b be elements in a C*-algebra A. It is known that $a \perp_M b$ whenever $a \perp b$. In general, two M-orthogonal elements in a C*-algebra need not be (algebraically) orthogonal.

As we have mentioned, the C*-norm of a C*-algebras satisfies the following property

$$\text{if } a \perp b \textit{ then } \|a \pm b\| = \text{máx}\{\|a\|, \|b\|\}.$$

A norm $\|.\|_1$ on A is said to be an M-norm (respectively, a *semi-M-norm*) if for every a, b in A with $a \perp b$ we have $\|a \pm b\|_1 = \text{máx}\{\|a\|_1, \|b\|_1\}$ (respectively, $\|a \pm b\|_1 \geq \text{máx}\{\|a\|_1, \|b\|_1\}$). M- and semi-M-norms were introduced and studied in [146], by T.

Oikhberg, A.M. Peralta and M.I. Ramírez, a paper where they posed the following problem:

Problem 10.5.1 [146] *Is every complete semi-M-norm on a C*-algebra automatically continuous with respect to the original C*-norm?*

In the aforementioned paper, T. Oikhberg, A.M. Peralta and M.I. Ramírez give an affirmative answer to the above problem for von Neumann algebras and for compact C*-algebras (compare [146] Corollaries 4.6 and 5.9). However, the problem in full generality remains open.

Problem 10.5.1 has a connection with the study of automatic continuity of orthogonality preserving linear mapping between C*-algebras. Indeed, if $T : A \to B$ is an injective orthogonality preserving linear mapping between C*-algebras which has closed range, then the assignment

$$\|a\|_1 := \|T(a)\|$$

defines an M-norm on A. If $\|.\|_1$ is continuous then T is continuous.

More generally, linar mappings $T : A \to X$ from a C*-algebra to a Banach space sending (algebraically) orthogonal elements to M-orthogonal or semi-M-orthogonal elements in X can be considered.

Connections between M-orthogonality and algebraic orthogonality on JB*-triples have been explore by C.M. Edwards and G.T. Ruttimann in [57]. In this setting orthogonal elements also are M-orthogonal. The reciprocal statement is not true in general, but it holds for tripotents (compare Theorem 5.3 and Lemma 5.5 in [57]).

It would be interesting to study M-norms in the more general setting of JB*-triples.

Problem 10.5.2 *Is every complete semi-M-norm on a JB*-triple automatically continuous with respect to the original JB*-norm?*

As we already mentioned, an affirmative answer to the above problem would yield an automatic continuity result for orthogonality preserving linear mappings between JB*-triples. Perhaps, the particular cases of M-norms on JBW*-algebras or weakly compact JB*-triples should be explored before considering the problem in its full generality.

Not so much is known yet about the structure of an orthogonality preserving linear mapping between real C*-algebras (only linear orthogonality preservers between unital abelian real C*-algebras have been studied). Nevertheless, the following problem seems to be a natural problem too.

Problem 10.5.3 *Is every complete semi-M-norm on a real C*-algebra automatically continuous with respect to the original C*-norm?*

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Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples ☆

María Burgos, Francisco J. Fernández-Polo, Jorge J. Garcés, Juan Martínez Moreno, Antonio M. Peralta *

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

We study orthogonality preserving operators between C*-algebras, JB*-algebras and JB*-triples. Let $T : A \to E$ be an orthogonality preserving bounded linear operator from a C*-algebra to a JB*-triple satisfying that $T^{**}(1) = d$ is a von Neumann regular element. Then $T(A) \subseteq E_2^{**}(r(d))$, every element in $T(A)$ and d operator commute in the JB*-algebra $E_2^{**}(r(d))$, and there exists a triple homomorphism $S : A \to E_2^{**}(r(d))$, such that $T = L(d, r(d))S$, where $r(d)$ denotes the range tripotent of d in E^{**}. An analogous result for A being a JB*-algebra is also obtained. When $T : A \to B$ is an operator between two C*-algebras, we show that, denoting $h = T^{**}(1)$ then, T orthogonality preserving if and only if there exists a triple homomorphism $S : A \to B^{**}$ satisfying $h^* S(z) = S(z^*)^* h$, $hS(z^*)^* = S(z)h^*$, and

$$T(z) = L\big(h, r(h)\big)\big(S(z)\big) = \frac{1}{2}\big(hr(h)^* S(z) + S(z)r(h)^* h\big).$$

This allows us to prove that a bounded linear operator between two C*-algebras is orthogonality preserving if and only if it preserves zero-triple-products.

1. Introduction

Two elements a, b in a C*-algebra A are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. A linear operator T between two C*-algebras A and B is *orthogonality preserving* or *disjointness preserving* if $T(a) \perp T(b)$, for all $a \perp b$ in A. If A_{sa} denotes the self-adjoint part of A, we shall say that T is *orthogonality preserving on* A_{sa}, if $T(a) \perp T(b)$, whenever $a \perp b$ in A_{sa}. We observe that when T is symmetric, that is $T(A_{sa}) \subseteq B_{sa}$, then T is orthogonality preserving on A_{sa} if and only if T preserves zero-products on A_{sa}.

The study of orthogonality preserving operators between C*-algebras begins with the work of W. Arendt [4] in the setting of $C(K)$-spaces. More concretely, the author proved that for every orthogonality preserving operator (originally termed *Lamperti* operator in [4]), $T : C(K) \to C(K)$, there exists $h \in C(K)$ and a mapping $\varphi : K \to K$ being continuous on $\{t \in K:\ h(t) \neq 0\}$ satisfying that

$$T(f)(t) = h(t) f\big(\varphi(t)\big),$$

for all $f \in C(K)$, $t \in K$. The study was latter extended by K. Jarosz [23] and J.-S. Jeang and N.-C. Wong [24] to the setting of orthogonality preserving operators between $C_0(L)$-spaces, where L is a locally compact Hausdorff space.

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* Corresponding author.
E-mail addresses: mariaburgos@ugr.es (M. Burgos), pacopolo@ugr.es (F.J. Fernández-Polo), jgarces@correo.ugr.es (J.J. Garcés), jmmoreno@ugr.es (J.M. Moreno), aperalta@ugr.es (A.M. Peralta).

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It should be noticed here that a bounded linear operator between two abelian C*-algebras is orthogonality preserving if and only if it preserves zero-products (i.e., it sends zero-products to zero-products). This equivalence does not hold for operators between general C*-algebras (see Section 4). There are many contributions to the study of zero-products preserving operators between Banach algebras and C*-algebras, see for example [1–3,10–12,19,23,24,32,39].

M. Wolff gave a complete characterisation of all symmetric orthogonality preserving operators between general C*-algebras in [37, Theorem 2.3]. More precisely, if $T : A \to B$ is a symmetric orthogonality preserving bounded linear operator between two C*-algebras with A unital, then denoting $T(1) = h$ the following assertions hold:

(a) $T(A)$ is contained in the norm closure of $h\{h\}'$, where $\{h\}'$ denotes the commutator of $\{h\}$.
(b) There exists a Jordan *-homomorphism $S : A \to B^{**}$ such that $T(z) = hS(z)$, for all $z \in A$.

In [38, Theorem 3.2], N.-C. Wong established that a bounded linear operator T between two C*-algebras is a triple homomorphism if and only if T is orthogonality preserving and $T^{**}(1)$ is a partial isometry (tripotent).

Every C*-algebra belongs to a more general class of Banach spaces known under the name of JB*-triples. JB*-triples were introduced by W. Kaup in [26]. A JB*-triple is a complex Banach space E together with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying that

(a) (Jordan Identity) $L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;
(b) $L(a,a)$ is a hermitian operator with non-negative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by the law $y \mapsto Q(x)y = \{x,y,x\}$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = \left(x \circ y^*\right) \circ z + \left(z \circ y^*\right) \circ x - (x \circ z) \circ y^*.$$

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [6]). It is known that the triple product of a JBW*-triple is separately weak*-continuous [6] (see also [31]). The second dual of a JB*-triple E is a JBW*-triple with a product extending the product of E [13].

Let E be a JB*-triple. Following [28], we call two elements a,b in E *orthogonal* and write $a \perp b$ if $L(a,b) = 0$ (or equivalently $L(b,a) = 0$) holds. Two sets $R, S \subseteq E$ are said to be orthogonal if for every $a \in S$, $b \in R$, we have $a \perp b$.

On every C*-algebra A we can consider its structure of JB*-triple and its natural structure of C*-algebra. We have, a priory, two notions of orthogonality in A. However, it can be checked that two elements $a,b \in A$ are orthogonal for the C*-algebra product if and only if they are orthogonal when A is considered as a JB*-triple (compare Lemma 1).

Let E and F be JB*-triples. A linear operator $T : E \to F$ is said to be *orthogonality preserving* if $T(a) \perp T(b)$ whenever $a \perp b$ in E. This concept extends the usual definition of orthogonality preserving linear operator between C*-algebras.

Despite of the vast literature on zero-products preserving and orthogonality preserving operators between C*-algebras, no attention has yet been paid to those orthogonality preserving operators from a C*-algebra or a JB*-algebra to a JB*-triple. In Section 3, we shall study orthogonality preserving operators $T : A \to E$, in the case of A being a C*-algebra, a JB*-algebra or a JB*-triple and E being a JB*-triple.

Theorems 6 and 10 establish the following description: Let A be a C*-algebra or a JB*-algebra, E a JB*-triple and $T : A \to E$ an orthogonality preserving bounded linear operator. Suppose that $T^{**}(1) = d$ is a von Neumann regular element in E^{**}. Then the following statements hold:

(a) $T(A)$ is contained in the Peirce subspace $E_2^{**}(r(d))$, where $r(d)$ denotes the range tripotent of d. Moreover, when $E_2^{**}(r(d))$ is regarded as a JB*-algebra, then $T(A)$ is in the commutator of d.
(b) There exists a triple homomorphism $S : A \to E_2^{**}(r(d))$, satisfying that $T = L(d, r(d))S$.

Since every tripotent in a JB*-triple is von Neumann regular, we shall establish, in Corollaries 7 and 11, that whenever A is a C*-algebra (or a JB*-algebra) and E is a JB*-triple, then a bounded linear operator $T : A \to E$ is a triple homomorphism if and only if T is orthogonality preserving and $T^{**}(1)$ is a tripotent. Recalling that for a C*-algebra, A, tripotents and partial isometries in A coincide, then the main result in [38] follows now as a consequence.

Section 4 is completely devoted to the study of orthogonality preserving operators between C*-algebras. As a novelty, we consider the natural JB*-triple structure associated to each C*-algebra. This new point of view allows us to apply new techniques based on the triple spectrum and the triple functional calculus to establish a definite description of orthogonality preserving operators between C*-algebras (Theorem 17). More precisely, we prove the following: Let $T : A \to B$ be an operator between two C*-algebras. For $h = T^{**}(1)$ the following assertions are equivalent:

(a) T is orthogonality preserving.
(b) There exists a triple homomorphism $S : A \to B^{**}$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and

$$T(z) = L\big(h, r(h)\big)\big(S(z)\big) = \frac{1}{2}\big(hr(h)^*S(z) + S(z)r(h)^*h\big) = hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$.

The main consequence of the above characterisation assures that when T is a bounded linear operator between two C*-algebras then T is orthogonality preserving if and only if T preserves zero-triple-products (see Corollary 18). It could be said that the appropriate structure to characterise the orthogonality preserving operators between C*-algebras is the natural structure of JB*-triple associated to each one of them.

In the last section we shall prove another generalisation of the main result in [38]. Indeed, for each operator T from a C*-algebra to a JB*-triple the following three statements are equivalent:

(a) T is a triple homomorphism.
(b) $T(a^3) = \{T(a), T(a), T(a)\}$, for all $a \in A_{sa}$.
(c) T is orthogonality preserving (on A_{sa}) and $T^{**}(1)$ is a tripotent.

2. Notation and preliminaries

Given Banach spaces X and Y, $L(X, Y)$ will denote the space of all bounded linear mappings from X to Y. The space $L(X, X)$ will be denoted by $L(X)$. Throughout the paper the word "operator" will always mean bounded linear mapping. The dual space of a Banach space X is denoted by X^*.

When A is a JB*-algebra or a C*-algebra then, A_{sa} will denote the set of all self-adjoint elements in A. We shall make use of standard notation in C*-algebra and JB*-triple theory.

An element e in a JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. Each tripotent e in E gives raise to the so-called *Peirce decomposition* of E associated to e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2,$ $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e, e)$. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\big\{E_i(e), E_j(e), E_k(e)\big\} \subseteq E_{i-j+k}(e), \quad \text{if } i - j + k \in \{0, 1, 2\}, \text{ and is zero otherwise.}$$

In addition,

$$\big\{E_2(e), E_0(e), E\big\} = \big\{E_0(e), E_2(e), E\big\} = 0.$$

The corresponding *Peirce projections* are denoted by $P_i(e) : E \to E_i(e)$ $(i = 0, 1, 2)$. The Peirce space $E_2(e)$ is a JB*-algebra with product $x \bullet y := \{x, e, y\}$ and involution $x^\sharp := \{e, x, e\}$.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \{x, x, x^{[2n-1]}\}$ $(n \in \mathbb{N})$. The symbol E_x will stand for the JB*-subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [25, Corollary 4.8], [26, Corollary 1.15] and [18]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. We should note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$, whenever $0 \notin \mathrm{Sp}(x)$.

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying that $\{y, y, y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define, $x^{[\frac{1}{3^n}]} = (x^{[\frac{1}{3^{n-1}}]})^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak*-topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying that x is positive in the JBW*-algebra $E_2^{**}(e)$ (compare [14, Lemma 3.3]).

The *symmetrized Jordan triple product* in a JB*-triple E is defined by the expression

$$\langle x, y, z\rangle := \frac{1}{3}\big(\{x, y, z\} + \{y, z, x\} + \{z, x, y\}\big).$$

A subspace I of a JB*-triple E is said to be an *inner ideal* of E if $\{I, E, I\} \subseteq I$. Given an element x in E, let $E(x)$ denote the norm closed inner ideal of E generated by x. It is known that $E(x)$ coincides with the norm-closure of the set $Q(x)(E)$. Moreover $E(x)$ is a JB*-subalgebra of $E_2^{**}(r(x))$ and contains x as a positive element (compare [7, Proposition 2.1, p. 19]).

The following result characterises the relation of orthogonality between elements in a JB*-triple.

Lemma 1. *Let a and b be two elements in a* JB*-*triple E. The following assertions are equivalent*:

(a) $a \perp b$;
(b) $\{a, a, b\} = 0$;

(c) $a \perp r(b)$;
(d) $r(a) \perp r(b)$;
(e) $E_2^{**}(r(a)) \perp E_2^{**}(r(b))$;
(f) $E(a) \perp E(b)$;
(g) $E_a \perp E_b$.

Proof. The implication (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c). The condition $L(a,a)(b) = 0$, together with the Jordan Identity assure that $L(a,a)(b^{[3]}) = 0$. We deduce, by mathematical induction, that $L(a,a)(b^{[2n-1]}) = 0$ ($n \in \mathbb{N}$). Since the "odd polynomials" in b form a norm dense subset of E_b, we have $L(a,a)(E_b) = \{0\}$.

Notice that, for each natural n, $b^{[\frac{1}{3^n}]}$ lies in E_b. Thus, by assumptions, $L(a,a)(b^{[\frac{1}{3^n}]}) = 0$. The separate weak*-continuity of the triple product implies that

$$L(a,a)r(b) = L(a,a)\big(w^* - \lim b^{[\frac{1}{3^n}]}\big) = w^* - \lim L(a,a)\big(b^{[\frac{1}{3^n}]}\big) = 0.$$

This proves, via [18, Lemma 1.5] or via [8, Proposition 2.4], that a lies in $E_0^{**}(r(b))$, and hence $a \perp r(b)$ by the Peirce arithmetic.

(c) $\Rightarrow$ (d). Since $r(b) \perp a$. We deduce from (a) $\Rightarrow$ (b) $\Rightarrow$ (c) that $r(b) \perp r(a)$.

(d) $\Rightarrow$ (e) follows by Peirce arithmetic.

The implications (e) $\Rightarrow$ (f), (f) $\Rightarrow$ (g) and (g) $\Rightarrow$ (a) are clear. □

Let E be a JB*-triple. Suppose that a and b are two elements in a JB*-subtriple, F, of E. The equivalence (a) $\Leftrightarrow$ (b) in Lemma 1, implies that $a \perp b$ in F if and only if a and b are orthogonal in E.

The following lemma provides another useful characterisation of orthogonality in JB*-triples.

Lemma 2. *Let e and x be two elements in a* JB**-triple E. Suppose that e is a tripotent. The following are equivalent*:

(a) $e \perp x$;
(b) $(e \pm x)^{[3]} = e \pm x^{[3]}$.

Proof. (a) $\Rightarrow$ (b). Since $L(e,e)(x) = 0$, we have $x \in E_0(e)$. Therefore, by the Peirce rules, $(e \pm x)^{[3]} = e \pm x^{[3]}$.

(b) $\Rightarrow$ (a). The equality

$$(e \pm x)^{[3]} = e \pm x^{[3]} + 2\{x,x,e\} + \{x,e,x\} \pm 2\{e,e,x\} \pm \{e,x,e\},$$

implies that $(e+x)^{[3]} - (e-x)^{[3]} = 4\{e,e,x\} + 2\{e,x,e\} + 2x^{[3]}$. On the other hand, by hypothesis we also have

$$(e+x)^{[3]} - (e-x)^{[3]} = e + x^{[3]} - e + x^{[3]} = 2x^{[3]},$$

and consequently $2\{e,e,x\} + \{e,x,e\} = 0$. In particular $x = P_0(e)x$, and hence $x \perp e$. □

An element a in a JB*-triple E is said to be *von Neumann regular* if there exists $b \in E$ such that $Q(a)(b) = a$ and $Q(b)(a) = b$. The element b is called the *generalised inverse* of a. We observe that every tripotent e in E is von Neumann regular and its generalised inverse coincides with it. We refer to [9,15,29,30] for basics facts and results on von Neumann regularity. It is shown in [29, Lemma 3.2] (see also the proof of [9, Theorem 3.4]) that for each von Neumann regular element $a \in E$, there exists a tripotent $e \in E$ satisfying that a is a symmetric and invertible element in the JB*-algebra $E_2(e)$. Moreover, e coincides with the range tripotent of a. It is also known that an element a in E is von Neumann regular if and only if it is von Neumann regular in any JB*-subtriple F containing a (compare [27]).

We shall make use of the *triple functional calculus*. We recall that if x is an element in a JB*-triple then the JB*-subtriple generated by the element x is JB*-triple isomorphic (and hence isometric) to $C_0(\mathrm{Sp}(x))$, where $\mathrm{Sp}(x)$ is the *triple spectrum* of x. To avoid possible confusion below, given a function $f \in C_0(\mathrm{Sp}(x))$, $f(x)$ shall have its usual meaning when E_x is regarded as an abelian C*-algebra and $f_t(x)$ shall denote the same element of E_x when the latter is regarded as a JB*-subtriple of E. Thus, for any odd polynomial, $P(\lambda) = \sum_{k=0}^n \alpha_k \lambda^{2k+1}$, we have $P_t(x) = \sum_{k=0}^n \alpha_k x^{[2k+1]}$.

Lemma 3. *Let a and b be two orthogonal elements in a* JB**-triple E.*

(a) *a and b are tripotents whenever $a+b$ is.*
(b) *a and b are von Neumann regular whenever $a+b$ is.*

Proof. (a) Let us suppose that $a+b$ is a tripotent, that is $(a+b)^{[3]} = a+b$. Since $a \perp b$ it follows that

$$a^{[3]} + b^{[3]} = (a+b)^{[3]} = a+b.$$

Lemma 1 assures that

$$a^{[3]} = a, \qquad b^{[3]} = b.$$

(b) By Lemma 1, E_a and E_b are orthogonal subtriples of E, and hence, the JB*-subtriple of E generated by $\{a, b\}$ coincides with the direct sum $E_a \oplus^\infty E_b$. The subtriple E_{a+b} is a JB*-subtriple of $E_a \oplus^\infty E_b$.

The von Neumann regularity of $a+b$ in E and hence in $E_a \oplus E_b$ assures the existence of an element $c \in E_a \oplus E_b$ satisfying that $Q(c)(a+b) = c$ and $Q(a+b)(c) = a+b$. Since $c = c_1 + c_2$, with $c_1 \in E_a$ and $c_2 \in E_b$, it follows, via Lemma 1, that $Q(c_1)(a) = c_1$, $Q(a)(c_1) = a$, $Q(b)(c_2) = b$ and $Q(c_2)(b) = c_2$, which give the desired statement. □

3. Orthogonality preserving operators between JB*-triples

In this section we shall study orthogonality preserving operators between JB*-triples. In a first step we shall focus our attention on orthogonality preserving operators from a C*-algebra to a JB*-triple. We shall generalise several results on orthogonality preserving operators between C*-algebras obtained by M. Wolff [37], N.C. Wong [38] and M.A. Chebotar, W.-F. Ke, P.H. Lee and N.C. Wong [11]. New trends in the study of "orthogonal" sesquilinear forms on C*-algebras (see [20,21]) and orthogonally additive polynomials on C*-algebras (compare [33]) will allow us to introduce new techniques in the study of orthogonality preserving operators between JB*-triples. The results obtained in [11,37,38] will follow as a consequence.

Let A be a C*-algebra. A sesquilinear form $\Phi : A \times A \to \mathbb{C}$ is called *orthogonal* if $\Phi(a, b) = 0$, whenever $a \perp b$ in A_{sa}. S. Goldstein proved in [20] that for every orthogonal sesquilinear form Φ, there exist functionals $\varphi, \psi \in A^*$ satisfying that $\Phi(x, y) = \varphi(xy^*) + \psi(y^*x)$, for all $x, y \in A$.

Proposition 4. *Let $T : A \to E$ be an orthogonality preserving operator from a C*-algebra to a JB*-triple. Then for each z in E (or in E^{**}) the equality*

$$\{T^{**}(x), T^{**}(y), z\} = \{T^{**}(y), T^{**}(x), z\}$$

holds for every $x, y \in A^{**}$ satisfying $[x, y^*] = 0$ and $x \circ y^* = x^* \circ y$.

Proof. Let us fix $z \in E^{**}$ and let $V : A \times A \to E^{**}$ be the sesquilinear operator defined by

$$V(x, y) := \{T(x), T(y), z\}.$$

For each $\varphi \in E^*$, $V_\varphi(x, y)$ will stand for $\varphi V(x, y)$.

Since T is orthogonality preserving, for each $x \perp y$ in A, we have $T(x) \perp T(y)$ in E. Since E is a JB*-subtriple of E^{**} it follows that $T(x) \perp T(y)$ in E^{**}. We then have $L(T(x), T(y))(c) = 0$ for all $c \in E^{**}$, and thus $V_\varphi(x, y) = \varphi\{T(x), T(y), z\} = 0$. This shows that $V_\varphi : A \times A \to \mathbb{C}$ is an orthogonal sesquilinear form on A.

Theorem 1.10 in [20] (see also [21, Theorem 3.1]) assures the existence of two functionals $\omega_1, \omega_2 \in A^*$ satisfying that

$$V_\varphi(x, y) = \omega_1(xy^*) + \omega_2(y^*x),$$

for all $x, y \in A$. Denoting $\phi = \omega_1 + \omega_2$ and $\psi = \omega_1 - \omega_2$, we have

$$V_\varphi(x, y) = \phi(x \circ y^*) + \psi([x, y^*]),$$

for all $x, y \in A$, where $\circ$ and $[.,.]$ denote the usual Jordan and Lie product of A, respectively (concretely, $a \circ b := \frac{1}{2}(ab + ba)$, $[a, b] := \frac{1}{2}(ab - ba)$). Since the product of A^{**} and the triple product of E^{**} are separately weak*-continuous, the weak*-denseness of A in A^{**} assures that

$$\varphi\{T^{**}(x), T^{**}(y), z\} = V_\varphi(x, y) = \phi(x \circ y^*) + \psi([x, y^*]), \tag{1}$$

for all $x, y \in A^{**}$.

It follows from (1) that

$$\varphi\{T^{**}(x), T^{**}(y), z\} = V_\varphi(x, y) = V_\varphi(y, x) = \varphi\{T^{**}(y), T^{**}(x), z\}, \tag{2}$$

whenever $[x, y^*] = 0$ and $x \circ y^* = x^* \circ y$. Since φ was arbitrarily chosen in E^*, the Hahn–Banach theorem together with equality (2) give the desired statement. □

If X is a complex Banach space, then an *X-valued n-homogeneous polynomial* on A is a continuous X-valued mapping $P : A \to X$, for which there exists a continuous n-linear operator $T : A \times \cdots \times A \to X$ satisfying that $P(x) = T(x, \ldots, x)$, for every x in X. An n-homogeneous polynomial P on A is said to be *orthogonally additive* (respectively orthogonally additive on A_{sa}) if $P(x + y) = P(x) + P(y)$ for all $x, y \in A$ (respectively $x, y \in A_{sa}$) with $x \perp y$.

In [33, Theorem 2.8 and Corollary 3.1] C. Palazuelos, A.M. Peralta and I. Villanueva showed that whenever $P : A \to X$ is an n-homogeneous polynomial satisfying that P is orthogonally additive on A_{sa}, then there exists an operator $F : A \to X$ satisfying that $P(x) = F(x^n)$, for all x in A.

Our next theorem will generalise [38, Theorem 3.2], [37, Lemma 3.3] and [11, Theorem 4.3] to the setting of orthogonality preserving operators from a C*-algebra to a JB*-triple.

We firstly recall that two elements a and b in a JB*-algebra J are said to *operator commute* in J if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in J. Self-adjoint elements a and b in J generate a JB*-subalgebra that can be realised as a JC*-subalgebra of some $B(H)$ [40] and, in this realisation, a and b commute in the usual sense whenever they operator commute in J [36, Proposition 1]. Similarly, two elements a and b of J_{sa} operator commute if and only if $a^2 \circ b = \{a, b, a\}$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in J$ we use $\{b\}'$ to denote the set of elements in J that operator commute with b. (This corresponds to the usual notation in von Neumann algebras).

The arguments given in the above paragraph allow us to establish the following result.

Lemma 5. *Let J be a JB*-algebra. Suppose that $a, b \in J_{sa}$ operator commute, then $a \circ b$ and b operator commute.*

We can now state the main result of this section.

Theorem 6. *Let A be a C*-algebra, E a JB*-triple and $T : A \to E$ an orthogonality preserving operator satisfying that $T^{**}(1) = d$ is a von Neumann regular element in E^{**}. Then $T(A) \subseteq E_2^{**}(r(d))$, $T(A) \subseteq \{d\}'$ and there exists a Jordan *-homomorphism $S : A \to E_2^{**}(r(d))$, satisfying that $T = L(d, r(d))S$.*

Proof. Since $d = T^{**}(1)$ is von Neumann regular in E^{**}, then d is positive and invertible in $E_2^{**}(r(d))$. Let b denote the inverse of d in $E_2^{**}(r(d))$. It is also known that $L(d, b) = L(b, d) = L(r(d), r(d))$ and $r(d) = r(b)$ (compare [29, Lemma 3.2]).

Proposition 4 guarantees that

$$\{T^{**}(a), T^{**}(1), b\} = \{T^{**}(1), T^{**}(a), b\},$$

for every $a \in A_{sa}^{**}$. In particular, the identity

$$L(r(d), r(d))(T^{**}(a)) = \{T^{**}(a), d, b\} = \{d, T^{**}(a), b\} \quad \left(\in E_2^{**}(r(d))\right) \tag{3}$$

holds for all $a \in A_{sa}^{**}$. This implies that

$$T(A_{sa}) \subseteq E_2^{**}(r(d)) \oplus E_0^{**}(r(d)). \tag{4}$$

Consider now the mapping $P_3 : A \to E$, $P_3(x) = \{T(x), T(x^*), T(x)\}$. It is clear that P_3 is a 3-homogeneous polynomial on A. Since T is orthogonality preserving, P_3 is orthogonally additive on A_{sa}. By [33, Corollary 3.1] there exists an operator $F_3 : A \to E$ satisfying that

$$P_3(x) = F_3(x^3),$$

for all x in A. If $S_3 : A \times A \times A \to E$ is the (unique) symmetric 3-linear operator associated to P_3, we have

$$F_3(\langle x, y, z\rangle) = S_3(x, y, z) = \langle T(x), T(y), T(z)\rangle, \tag{5}$$

for all $x, y, z \in A_{sa}$. The separate weak*-continuity of the triple product together with the weak* density of A_{sa} in A_{sa}^{**} assure that the above equality (5) remains valid for all $x, y, z \in A_{sa}^{**}$. Therefore, taking $x = a^* = a \in A$ and $y = z = 1$ in (5), we deduce that

$$F_3(a) = \langle T(a), d, d\rangle = \frac{2}{3}\{T(a), d, d\} + \frac{1}{3}\{d, T(a), d\}.$$

Thus, for each $a \in A_{sa}$ we have

$$\{T(a), T(a), T(a)\} = F_3(a^3) = \langle d, d, T(a^3)\rangle. \tag{6}$$

Now, (4), (6) and the Peirce arithmetic show that

$$T(A_{sa}) \subseteq E_2^{**}(r(d)) \cap E. \tag{7}$$

The law $x \mapsto P_2(x) = \{T(x), T(x^*), b\}$, defines a 2-homogeneous polynomial $P_2 : A \to E^{**}$, which is orthogonally additive on A_{sa}. Corollary 3.1 in [33] implies the existence of an operator $F_2 : A \to E$ satisfying that

$$\frac{1}{2}\left(\{T^{**}(x), T^{**}(y^*), b\} + \{T^{**}(y), T^{**}(x^*), b\}\right) = F_2^{**}(x \circ y), \tag{8}$$

for all $x, y \in A^{**}$.

Expressions (3) and (7), imply that

$$T(a) = \{T(a), d, b\} = \{d, T(a), b\}, \tag{9}$$

for all a in A_{sa}, and hence it follows from (8) that

$$F_2(x) = \{T(x), d, b\} = \{T(x), b, d\} = \{d, T(x^*), b\} = T(x),$$

for all $x \in A$.

$E_2^{**}(r(d))$ is a JB*-algebra with Jordan product and involution given by $x \bullet y = \{x, r(d), y\}$ and $x^\sharp = \{r(d), x, r(d)\} = Q(r(d))(x)$, respectively. The triple product in $E_2^{**}(r(d))$ is also determined by the expression

$$\{x, y, z\} = (x \bullet y^\sharp) \bullet z + (z \bullet y^\sharp) \bullet x - (x \bullet z) \bullet y^\sharp.$$

Since d is invertible in $E_2^{**}(r(d))$, with inverse b, $Q(d, b) = Q(r(d))$. Thus Eq. (9) gives $T(a)^\sharp = T(a)$, for all $a \in A_{sa}$.

Having the above facts in mind, then Eq. (8) and $T = F_2$ guarantee that

$$T(x \circ y) = (T(x) \bullet T(y)) \bullet b. \tag{10}$$

For each $a \in A_{sa}$, Proposition 4 also implies that

$$\{T(a), T^{**}(1), d\} = \{T^{**}(1), T(a), d\},$$

and hence

$$(d \bullet d) \bullet T(a) = \{T(a), d, d\} = \{d, T(a), d\} = 2(d \bullet T(a)) \bullet d - (d \bullet d) \bullet T(a),$$

which assures that $T(a)$ and d operator commute in the JB*-algebra $E_2^{**}(r(d))$. Thus $T(A) \subseteq \{d\}'$. Noticing that b is the inverse of d in $E_2^{**}(r(d))$, then $T(A) \subseteq \{d\}'$ implies that $T(A) \subseteq \{b\}'$.

Finally, (10) and Lemma 5 guarantee that

$$\begin{aligned} b \bullet T(x \circ y) &= ((T(x) \bullet T(y)) \bullet b) \bullet b = M_b M_b M_{T(x)}(T(y)) = M_b M_{T(x)} M_b(T(y)) = ((T(y) \bullet b) \bullet T(x)) \bullet b \\ &= M_b M_{T(y)\bullet b}(T(x)) = M_{T(y)\bullet b} M_b(T(x)) = (T(x) \bullet b) \bullet (T(y) \bullet b), \end{aligned}$$

which assures that $S = M_b T = L(b, r(b))T : A \to E_2^{**}(r(d))$ is a Jordan *-homomorphism and $T = M_d S = L(d, r(d))S$. □

Recalling that every tripotent e in a JB*-triple is von Neumann regular with $r(e) = e$, Theorem 6 gives the following generalisation of [38, Theorem 3.2].

Corollary 7. *Let A be a C*-algebra, E a JB*-triple and $T : A \to E$ an orthogonality preserving operator satisfying that $T^{**}(1) = e$ is a tripotent element in E^{**}. Then T is a triple homomorphism. More concretely, $T(A) \subseteq E_2^{**}(e) \cap E$ and when $E_2^{**}(e)$ is considered as a JB*-algebra, then $T^{**} : A^{**} \to E_2^{**}(e)$ is a unital Jordan *-homomorphism.*

The main result of [38] follows now as a corollary of the above theorem.

Corollary 8. *Let $T : A \to B$ be an operator between two C*-algebras. Then T is a triple homomorphism if and only if T is orthogonality preserving and $T^{**}(1)$ is a tripotent (partial isometry).*

We shall now consider orthogonality preserving operators from a JB*-algebra to a JB*-triple.

The notions of compact, open and closed tripotents (respectively projections) in JB*-triple biduals (respectively JB*-algebra biduals) studied and developed in [16,17] will allow us to extend the above Theorem 6 to orthogonally operators from a JB*-algebra to a JB*-triple.

At this point we need to recall the definition of the Strong*-topology on von Neumann algebras and JBW*-triples. The Strong*-topology in a JBW*-triple was introduced by T.J. Barton and Y. Friedman in [5] and is defined in the following way: Given a JBW*-triple W, a norm-one element φ in W_* and a norm-one element z in W such that $\varphi(z) = 1$, it follows from [5, Proposition 1.2] that the law

$$(x, y) \mapsto \varphi\{x, y, z\}$$

defines a positive sesquilinear form on W. Moreover, for every norm-one element w satisfying $\varphi(w) = 1$, we have $\varphi\{x, y, z\} = \varphi\{x, y, w\}$, for all $x, y \in W$. The law $x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{\frac{1}{2}}$, defines a prehilbertian seminorm on W. The Strong*-topology (noted by $S^*(W, W_*)$) is the topology on W generated by the family $\{\|\cdot\|_\varphi\colon \varphi \in W_*,\ \|\varphi\| = 1\}$. When a von Neumann algebra M is regarded as a JBW*-triple, then $S^*(M, M_*)$ coincides with the so-called "C*-algebra Strong*-topology" of M, namely the topology on M generated by the family of seminorms of the form $x \mapsto \sqrt{\xi(xx^* + x^*x)}$, where ξ is any positive functional in M_* (compare [35, Proposition 3]).

The Strong*-topology is compatible with the duality (W, W_*) (see [5, Theorem 3.2]). Many other properties of the Strong*-topology have been revealed in [34,35]. In particular, the triple product of every JBW*-triple is jointly Strong*-continuous on bounded sets (see [34,35]).

The next result is partially based on a generalised Urysohn's lemma for JB*-triples and JB*-algebras established in [17, Theorem 1.10].

Lemma 9. *Let J be a* JB**-algebra. Suppose that x is a positive element in J. Let $r(x) \in J^{**}$ be the range (tripotent) projection of x. Then there exist two sequences $(y_n) \subseteq J$ and $(p_n) \subseteq J^{**}$ satisfying that*

$$S^*\big(J^{**}, J^*\big) - \lim_n y_n = r(x) = S^*\big(J^{**}, J^*\big) - \lim_n p_n,$$

for each natural n, p_n is a closed projection in J^{**}, $(1 - p_n) \perp y_n$ and there exists a net $(z_\lambda^n) \in J_2^{**}(1 - p_n) \cap J$ with $S^*(J^{**}, J^*) - \lim_\lambda z_\lambda^n = 1 - p_n$.

Proof. Let $x \in J$ with $x \geqslant 0$. We may assume $\|x\| = 1$. Let J_x denote the JB*-subalgebra of J generated by x. It is known that J_x is JB*-algebra isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $[0, 1]$, such that $\Omega \cup \{0\}$ is compact. Moreover, if we denote by Ψ the JB*-algebra isomorphism from J_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. [22, 3.2.4]).

For each natural $n \geqslant 2$, define $p_n := \chi_{\Omega \cap [\frac{1}{2n}, 1]}$. $y_n \in J_x$ is defined by

$$y_n(t) := \begin{cases} 1 & \text{if } \frac{2}{n} \leqslant t \leqslant 1, \\ 0 & \text{if } 0 \leqslant t \leqslant \frac{1}{n}, \\ \text{affine} & \text{if } \frac{1}{n} \leqslant t \leqslant \frac{2}{n}. \end{cases}$$

It is clear that $S^*(J^{**}, J^*) - \lim y_n = r(x) = S^*(J^{**}, J^*) - \lim p_n$.

Since p_n is a closed projection, $J_0^{**}(p_n) \cap J = J_2^{**}(1 - p_n) \cap J$ is $S^*(J^{**}, J^*)$-dense in $J_2^{**}(1 - p_n) = J_0^{**}(p_n)$ (compare [16, §2]). Thus there exists a net $(z_\lambda^n) \in J_2^{**}(1 - p_n) \cap J$ satisfying that $S^*(J^{**}, J^*) - \lim_\lambda z_\lambda^n = 1 - p_n$. $\square$

The corresponding version of Theorem 6 for operators from a JB*-algebra to a JB*-triple can be established now.

Theorem 10. *Let J be a* JB**-algebra, E a* JB**-triple and $T : J \to E$ an orthogonality preserving operator satisfying that $T^{**}(1) = d$ is a von Neumann regular element in E^{**}. Then $T(J) \subseteq E_2^{**}(r(d)) \cap E$, $T(J) \subseteq \{d\}'$ and there exists a Jordan *-homomorphism $S : J \to E_2^{**}(r(d))$, satisfying that $T = L(d, r(d))S$.*

Proof. Let b denote the inverse of d in $E_2^{**}(r(d))$ and let us take a norm-one positive element x in J. J_x will denote the JB*-subalgebra of J generated by x. We have already commented that J_x is JB*-algebra isomorphic to an abelian C*-algebra. The unit element of J_x^{**} coincides with $r(x)$.

By Lemma 9 above there exist two sequences $(y_n) \subseteq J$ and $(p_n) \subseteq J^{**}$ satisfying that

$$S^*\big(J^{**}, J^*\big) - \lim_n y_n = r(x) = S^*\big(J^{**}, J^*\big) - \lim_n p_n,$$

for each natural n, p_n is a closed projection in J^{**}, $(1 - p_n) \perp y_n$ and there exists a net $(z_\lambda^n) \in J_2^{**}(1 - p_n) \cap J$ with $S^*(J^{**}, J^*) - \lim_\lambda z_\lambda^n = 1 - p_n$.

Fix a natural n. Since T is orthogonality preserving, we have $T(y_n) \perp T(z_\lambda^n)$ $(\forall \lambda)$, i.e., $L(T(y_n), T(z_\lambda^n))(z) = 0$ for all $z \in E^{**}$. Since T^{**} is $S^*(J^{**}, J^*)$-to-$S^*(E^{**}, E^*)$-continuous (compare [34, §4] or [35, Corollary 3]), taking limit in λ, we have

$$L\big(T^{**}(y_n), T^{**}(1 - p_n)\big)(z) = 0 \quad \big(\forall z \in E^{**}\big).$$

When we take limit in $n \to \infty$, we deduce that

$$L\big(T^{**}\big(r(x)\big), T^{**}\big(1 - r(x)\big)\big)(z) = 0 \quad \big(\forall z \in E^{**}\big).$$

That is, $T^{**}(r(x))$ and $T^{**}(1 - r(x))$ are orthogonal. Since

$$d = T^{**}(1) = T^{**}\big(r(x)\big) + T^{**}\big(1 - r(x)\big)$$

is von Neumann regular, it follows, via Lemma 3(a), that $d_x = T^{**}(r(x))$ and $T^{**}(1 - r(x))$ are von Neumann regular in E^{**}. We further have $r(d_x) \leqslant r(d) = r(T^{**}(1))$ (compare Lemma 1).

The mapping $T|_{J_x} : J_x \to E$ is orthogonality preserving and its bitranspose sends the unit of J_x^{**} to a von Neumann regular element in E^{**}. Theorem 6 assures that

$$T(J_x) \subseteq E_2^{**}\big(r(d_x)\big) \subseteq E_2^{**}\big(r(d)\big),$$
$$T\big((J_x)_{sa}\big) \subseteq \big(E_2^{**}\big(r(d_x)\big)\big)_{sa} \subseteq \big(E_2^{**}\big(r(d)\big)\big)_{sa}, \qquad T(x) \in \{d_x\}',$$
$$S_x = M_{b_x} T|_{J_x} = L\big(b_x, r(b_x)\big)T|_{J_x} : J_x \to E_2^{**}\big(r(d_x)\big) \subseteq E_2^{**}\big(r(d)\big)$$

is a Jordan *-homomorphism and

$$T|_{J_x} = L\big(d_x, r(d_x)\big)S_x = L\big(d, r(d)\big)S_x,$$

where b_x denotes the generalised inverse of d_x.

We claim that $S = L(b, r(b))T : J \to E_2^{**}(r(d))$ is a Jordan *-homomorphism. Indeed, let $\bullet$ and $\sharp$ denote the Jordan product and the involution of $E_2^{**}(r(d))$, respectively. For each positive $x \in J$, we have

$$S(x)^{\sharp} = \big\{r(d), S(x), r(d)\big\} = \big\{r(d_x), S_x(x), r(d_x)\big\} = S_x(x) = S(x)$$

and

$$S\big(x^2\big) = S_x\big(x^2\big) = \big\{S_x(x), r(d_x), S_x(x)\big\} = S_x(x) \bullet S_x(x) = S(x) \bullet S(x).$$

Every $x \in J_{sa}$ can be written in the form $x = x^+ - x^-$, where x^+ and x^- are two orthogonal positive elements in J. Since T is orthogonality preserving, $T(x^+) \perp T(x^-)$. The weak*-continuity of T^{**}, together with Lemma 1, give $d_{x^+} = T^{**}(r(x^+)) \perp T^{**}(r(x^-)) = d_{x^-}$. For $\sigma \in \{\pm 1\}$, we have

$$S\big(x^\sigma\big) = S_{x^\sigma}\big(x^\sigma\big) \in E_2^{**}\big(r\big(d_{x^\sigma}\big)\big),$$

and hence $S(x^+) \perp S(x^-)$ (compare Lemma 1). Therefore

$$S\big(x^2\big) = S\big(\big(x^+\big)^2\big) + S\big(\big(x^-\big)^2\big) = S\big(x^+\big) \bullet S\big(x^+\big) + S\big(x^-\big) \bullet S\big(x^-\big) = S(x) \bullet S(x),$$

which shows that S is a Jordan *-homomorphism.

Finally, the equality $T = L(d, r(d))S$, can be easily checked. □

Corollary 11. *Let J be a* JB**-algebra, E a* JB**-triple and $T : J \to E$ an orthogonality preserving operator satisfying that $T^{**}(1) = e$ is a tripotent element in E^{**}. Then T is a triple homomorphism. More concretely, $T(J) \subseteq E_2^{**}(e) \cap E$ and when $E_2^{**}(e)$ is considered as a* JB**-algebra, then $T^{**} : J^{**} \to E_2^{**}(e)$ is a unital Jordan *-homomorphism.*

The following corollary is a generalisation of [37, Lemma 3.3] to the setting of JB*-algebras. Here the operator is not assumed to be symmetric. We observe that [37, Lemma 3.3] is established for symmetric operators between C*-algebras.

Corollary 12. *Let $T : J_1 \to J_2$ be an operator between two* JB**-algebras (respectively two* C**-algebras). Suppose that T is orthogonality preserving and T^{**} is unital, then T is a Jordan *-homomorphism.*

Remark 13. It should be noticed that Theorem 6 and Corollary 7 (respectively Theorem 10 and Corollary 11) remain true when T is assumed to be orthogonality preserving only on the self-adjoint part of the C*-algebra A (respectively of the JB*-algebra J). In fact, the proofs of the above results remain valid under the weaker hypothesis of $T(a) \perp T(b)$ whenever $a \perp b$ and $a = a^*$, $b = b^*$.

The following corollary also follows from Theorem 10 above.

Corollary 14. *Let $T : E \to F$ be an orthogonality preserving operator between two* JB**-triples. Let x be a norm-one element in E.*

(a) *If $T^{**}(r(x))$ is a tripotent, then $T|_{E(x)} : E(x) \to F$ is a triple homomorphism.*
(b) *If $T^{**}(r(x)) = d$ is von Neumann regular, then $T(E(x)) \subseteq E_2^{**}(r(d)) \cap E$, $T(E(x)) \subseteq \{d\}'$ and there exists a Jordan *-homomorphism $S : E(x) \to E_2^{**}(r(d))$, satisfying that $T|_{E(x)} = L(d, r(d))S$.*

Remark 15. Let E be a JB*-triple. We call a subset $S \subset E$ *orthogonal* if $0 \notin S$ and $x \perp y$, for every $x \neq y$ in S. Denote by $r = r(E)$ the minimal cardinal number satisfying $\operatorname{card}(S) \leqslant r$, for every orthogonal subset $S \subset E$ and call it the *rank* of E. The existence of JB*-triples having rank one points out that Corollary 14 above is, in some sense, an optimal result for orthogonality preserving operators between JB*-triples. More concretely, every complex Hilbert space H is a rank one JB*-triple with respect to the triple product $2\{x, y, z\} = (x/y)z + (z/y)x$, where $(./.)$ denotes the inner product of H. This implies that every operator T from H to another JB*-triple F is orthogonality preserving. In this case, every norm-one element $e \in H$ is a tripotent. However, the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$ represents an operator $T : \ell_2^3 \to \ell_2^3$, which is not a triple homomorphism but for $e = (1, 0, 0)$, $T(e) = e$ is a tripotent.

4. Operators between C*-algebras

When particularised to orthogonality preserving operators between C*-algebras, the techniques developed in the previous section will allow us to get some generalisations of the results obtained by M. Wolff [37, Theorem 2.3], N.C. Wong [38, Theorem 3.2] and M.A. Chebotar, W.-F. Ke, P.H. Lee and N.C. Wong [11, Theorems 4.3, 4.6 and 4.7]. When A and B are two C*-algebras, we shall describe the orthogonality preserving operators $T : A \to B$. Compared with those previous results established in [11,37,38], and the previous section, here the operator T is not assumed to be self-adjoint and $T^{**}(1)$ need not be invertible, nor normal, nor a partial isometry nor von Neumann regular. The use of the triple spectral resolution will play an important role and an advantage in our results.

When a C*-algebra, A, is considered as a JB*-triple, then tripotents and partial isometries in A coincide. Given a tripotent e in A, then we have $A_2(e) = ee^*Ae^*e$, $A_0(e) = (1 - ee^*)A(1 - e^*e)$, and $A_1(e) = ee^*A(1 - e^*e) \oplus (1 - ee^*)Ae^*e$.

The following technical result will provide us the necessary tools for the main theorem of this section.

Proposition 16. *Let h be an element in a* C**-algebra A. Then the following statements hold*:

(a) $\mathrm{Sp}(h) = \mathrm{Sp}(h^*)$.
(b) $r(h) = r(h^*)^*$.
(c) *If z is any element in A satisfying that zh^* and z^*h are self-adjoint elements in A, then z lies in $A_2^{**}(r(h)) \oplus A_0^{**}(r(h))$. Moreover, for each $f \in C_0(\mathrm{Sp}(h))$, the following relations hold*

$$f_t(h^*)z = z^* f_t(h) \quad \textit{and} \quad f_t(h)z^* = zf_t(h^*).$$

In particular

$$r(h^*)z = z^*r(h) \quad \textit{and} \quad r(h)z^* = zr(h^*).$$

Proof. (a) Since the mapping $x \mapsto x^*$ is a conjugate-linear JB*-triple isomorphism on A, the triple spectrum is preserved by the canonical involution of A.

(b) Let $c \in A$ satisfy $cc^*c = c^{[3]} = h$. Then $c^*cc^* = h^*$, which according to our terminology, means $(h^{[\frac{1}{3}]})^* = (h^*)^{[\frac{1}{3}]}$. It follows, by mathematical induction, that

$$\left(h^{[\frac{1}{3^n}]}\right)^* = \left(h^*\right)^{[\frac{1}{3^n}]}, \quad \forall n \in \mathbb{N}.$$

Taking weak*-limits in the above expression we have $r(h)^* = r(h^*)$.

(c) Take an element $z \in A$, satisfying that $zh^*, z^*h \in A_{sa}$ (i.e., $hz^* = zh^*$ and $h^*z = z^*h$). It can be easily seen, by mathematical induction, that

$$\left(h^*\right)^{[2n-1]}z = z^*h^{[2n-1]} \quad \text{and} \quad h^{[2n-1]}z^* = z\left(h^*\right)^{[2n-1]},$$

for all $n \in \mathbb{N}$. The above relations guarantee that, for each odd polynomial $P(\lambda)$ we have

$$P_t\left(h^*\right)z = z^* P_t(h) \quad \text{and} \quad P_t(h)z^* = zP_t\left(h^*\right).$$

We can easily check, by the classical Stone–Weierstrass theorem, that

$$f_t\left(h^*\right)z = z^* f_t(h) \quad \text{and} \quad f_t(h)z^* = zf_t\left(h^*\right), \tag{11}$$

for all $f \in C_0(\mathrm{Sp}(h))$.

Let us fix an arbitrary natural n. Taking $f(t) := \sqrt[3^n]{t}$ $(t \in \mathrm{Sp}(h))$, Eq. (11) assures that

$$\left(h^*\right)^{[\frac{1}{3^n}]}z = z^*h^{[\frac{1}{3^n}]} \quad \text{and} \quad h^{[\frac{1}{3^n}]}z^* = z\left(h^*\right)^{[\frac{1}{3^n}]}.$$

When in the above expressions we take weak*-limits, or $S^*(A^{**}, A^*)$-limits, for $n \to \infty$, we have

$$r\left(h^*\right)z = z^*r(h) \quad \text{and} \quad r(h)z^* = zr\left(h^*\right).$$

Now, from (b) and the above commutativity relations, we show that

$$r(h)r(h)^*z = r(h)r\left(h^*\right)z = r(h)z^*r(h) = zr\left(h^*\right)r(h) = zr(h)^*r(h),$$

which proves the first statement in (c). $\square$

We can now establish the main result of this section.

Theorem 17. *Let $T : A \to B$ be an operator between two* C**-algebras. For $h = T^{**}(1)$ the following assertions are equivalent*:

(a) *T is orthogonality preserving.*
(b) *There exists a triple homomorphism $S : A \to B^{**}$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and*

$$T(z) = L\big(h, r(h)\big)\big(S(z)\big) = \frac{1}{2}\big(hr(h)^*S(z) + S(z)r(h)^*h\big) = hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$.

Proof. (a) $\Rightarrow$ (b). The symbol B^{**}_h will stand for the JB*-subtriple of B^{**} generated by h. There is no loss of generality in assuming $\|h\| = 1$. We define a sesquilinear operator $V : A \times A \to B$ by

$$V(x, y) := T(x)T(y)^*.$$

For each $\varphi \in B^*$, $V_\varphi(x, y)$ will stand for $\varphi V(x, y)$.

Since T is orthogonality preserving, for all $x \perp y$ in A, we have $T(x)T(y)^* = 0$, which implies that $V_\varphi : A \times A \to \mathbb{C}$ is an orthogonal sesquilinear form on A.

Similar arguments to those applied in the proof of Proposition 4 will imply that

$$T^{**}(x)\big(T^{**}(y)\big)^* = V(x, y) = V(y, x) = T^{**}(y)\big(T^{**}(x)\big)^*,$$

for all $x, y \in A^{**}$ with $[x, y^*] = 0$ and $x \circ y^* = x^* \circ y$. In particular, for each $x \in A_{sa}$ we have

$$T(x)h^* = hT(x)^*. \tag{12}$$

When the above argument is applied to the sesquilinear operator $W : A \times A \to A$, $W(x, y) := T(y)^*T(x)$, we deduce that

$$h^*T(x) = T(x)^*h, \tag{13}$$

for all $x \in A_{sa}$.

Proposition 16(c) together with (12) and (13) now imply that

$$\begin{aligned} &T(A_{sa}) \subseteq B^{**}_2\big(r(h)\big) \oplus B^{**}_0\big(r(h)\big),\\ &f_t\big(h^*\big)T(x) = T(x)^* f_t(h), \qquad f_t(h)T(x)^* = T(x) f_t\big(h^*\big),\\ &r(h)^*T(x) = T(x)^*r(h), \quad \text{and} \quad r(h)T(x)^* = T(x)r(h)^*, \end{aligned}$$

for every $x \in A_{sa}$, and $f \in C_0(\mathrm{Sp}(h))$. Consequently, by linearity,

$$T(A) \subseteq B^{**}_2\big(r(h)\big) \oplus B^{**}_0\big(r(h)\big), \tag{14}$$

$$f_t\big(h^*\big)T(z) = T\big(z^*\big)^* f_t(h), \qquad f_t(h)T\big(z^*\big)^* = T(z) f_t\big(h^*\big), \tag{15}$$

$$r(h)^*T(z) = T\big(z^*\big)^*r(h), \quad \text{and} \quad r(h)T\big(z^*\big)^* = T(z)r(h)^*, \tag{16}$$

for every $z \in A$, and $f \in C_0(\mathrm{Sp}(h))$.

The law $x \mapsto T(x)T(x^*)^*T(x)$, defines an orthogonally additive 3-homogeneous polynomial from A to B. Corollary 3.1 in [33] assures the existence of an operator $F_3 : A \to B$ satisfying that

$$T(x)^{[3]} = T(x)T(x)^*T(x) = F\big(x^3\big), \tag{17}$$

for all $x \in A_{sa}$. It can be easily checked that

$$F(x) = \frac{1}{3}\big(2\big\{h, h, T(x)\big\} + \big\{h, T(x), h\big\}\big), \tag{18}$$

for all $x \in A_{sa}$. Now, it follows from Peirce arithmetic and (14) that

$$\big(P_2\big(r(h)\big)\big(T(x)\big)\big)^{[3]} + \big(P_0\big(r(h)\big)\big(T(x)\big)\big)^{[3]} = T(x)^{[3]} = \frac{1}{3}\big(2\big\{h, h, T\big(x^3\big)\big\} + \big\{h, T\big(x^3\big), h\big\}\big) \in B^{**}_2\big(r(h)\big),$$

for all $x \in A_{sa}$. Therefore, $P_0(r(h))(T(A)) = \{0\}$, and hence

$$T(A) \subseteq B^{**}_2\big(r(h)\big). \tag{19}$$

For each natural n, let e_n (respectively f_n) be the closed tripotent (partial isometry) in A_h^{**} whose representation in $C_0(\mathrm{Sp}(h))^{**}$ is the characteristic function $\chi_{(\mathrm{Sp}(h)\cap[\frac{1}{n},1])}$ (respectively $\chi_{(\mathrm{Sp}(h^*)\cap[\frac{1}{n},1])}$). We notice that (e_n) and (f_n) converge to $r(h)$ and $r(h^*)$, respectively, in the $S^*(B^{**},B^*)$-topology of B^{**}. We also have $e_n^*=f_n$.

The separate weak*-continuity of the product of B^{**}, together with (15), show that

$$f_nT(z)=T\big(z^*\big)^*e_n \quad \text{and} \quad e_nT\big(z^*\big)^*=T(z)f_n, \tag{20}$$

for all $n\in\mathbb{N}$. Therefore, the mapping

$$T_n:A\to B^{**}$$
$$T_n(z)=\big\{e_n,e_n,T(z)\big\}=e_ne_n^*T(z)=T(z)e_n^*e_n=e_nT\big(z^*\big)^*e_n$$

is orthogonality preserving. Since $T_n^{**}(1)=\{e_n,e_n,h\}=h_n$ is a von Neumann regular element in B^{**}, with generalised inverse denoted by b_n, Theorem 6 implies that $T_n=L(h_n,r(h_n))\circ S_n$, where

$$S_n=L\big(b_n,r(h_n)\big)\circ T_n:A\to B^{**}$$

is a triple homomorphism and $r(h_n)=e_n$. In particular $\|S_n\|\leqslant 1$, for all $n\in\mathbb{N}$.

Let us take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By the Banach–Alaoglu theorem, any bounded set in B^{**} is relatively weak*-compact and hence the law $z\mapsto S(z):=w^*-\lim_{\mathcal{U}}S_n(z)$ defines an operator $S:A\to B^{**}$.

Let us observe that, by (20) and (16), we have

$$S_n(z)=L(b_n,e_n)T_n(z)=\frac{1}{2}\big(b_ne_n^*L(e_n,e_n)T(z)+L(e_n,e_n)T(z)e_n^*b_n\big)=\frac{1}{2}\big(b_ne_n^*e_ne_n^*T(z)+T(z)e_n^*e_ne_n^*b_n\big)$$
$$=\frac{1}{2}\big(b_ne_n^*T(z)+T(z)e_n^*b_n\big)=b_ne_n^*T(z)=T(z)e_n^*b_n \quad (z\in A).$$

Therefore, we deduce, by orthogonality, that

$$L\big(h,r(h)\big)S_n(z)=\frac{1}{2}\big(hr(h)^*b_ne_n^*T(z)+T(z)e_n^*b_nr(h)^*h\big)=\frac{1}{2}\big(he_n^*b_ne_n^*T(z)+T(z)e_n^*b_ne_n^*h\big)=\big\{h,\{e_n,b_n,e_n\},T(z)\big\}$$
$$=L(h,b_n)\big(T(z)\big)=L(h_n,b_n)\big(T(z)\big)=L(e_n,e_n)\big(T(z)\big)=T_n(z) \quad (z\in A).$$

Now, by the separate weak*-continuity of the triple product of B^{**} we deduce, from the above equality, that

$$L\big(h,r(h)\big)S(z)=w^*-\lim_{\mathcal{U}}e_ne_n^*T(z)=T(z).$$

It can be also seen from the above formulae that $h^*S_m(z)=S_m(z^*)^*h$, $hS_m(z^*)^*=S_m(z)h^*$, for all $m\in\mathbb{N}$, $z\in A$, which in turn gives $h^*S(z)=S(z^*)^*h$, $hS(z^*)^*=S(z)h^*$, for all $z\in A$.

We claim that S preserves orthogonality. Indeed, let $x\perp y$ in A. In this case $T(x)\perp T(y)$ because T is orthogonality preserving. It follows from the equality $S_n(z)=b_ne_n^*T(z)=T(z)e_n^*b_n$, that

$$S_n(x)S_m(y)^*=b_ne_n^*T(x)T(y)^*e_mb_m^*=0$$

and

$$S_m(y)^*S_n(x)=b_m^*e_mT(y)^*T(x)e_n^*b_n=0,$$

for every $n,m\in\mathbb{N}$. Taking $w^*-\lim_{n,\mathcal{U}}$ in the above expressions, we deduce that

$$S(x)S_m(y)^*=0=S_m(y)^*S(x),$$

for all $m\in\mathbb{N}$, which, by the same reasons, gives

$$S(x)S(y)^*=0=S(y)^*S(x).$$

Finally, since S is orthogonality preserving and

$$S(1)=w^*-\lim_{\mathcal{U}}S_n(z)=w^*-\lim_{\mathcal{U}}h_n=r(h)$$

is a tripotent, Corollary 8 guarantees that S is a triple homomorphism.

(b) $\Rightarrow$ (a). By hypothesis $h^*S(z)=S(z^*)^*h$, $hS(z^*)^*=S(z)h^*$, for all $z\in A$. Applying Proposition 16(c), we get $(h^*)^{[\frac{1}{3^n}]}S(z)=S(z^*)^*(h)^{[\frac{1}{3^n}]}$, and $(h)^{[\frac{1}{3^n}]}S(z^*)^*=S(z)(h^*)^{[\frac{1}{3^n}]}$. It follows from the separate weak*-continuity of the product and Proposition 16(b), that

$$r(h)^*S(z) = r\big(h^*\big)S(z) = S\big(z^*\big)^*r(h),$$
$$r(h)S\big(z^*\big)^* = S(z)r\big(h^*\big) = S(z)r(h)^*.$$

Since S is a triple homomorphism, the above commutativity relations show that

$$T(z) = L\big(h, r(h)\big)\big(S(z)\big) = hr(h)^*S(z) = S(z)r(h)^*h \quad (z \in A)$$

is orthogonality preserving. $\square$

Let $T : E \to F$ be an operator between two JB*-triples. We shall say that T *preserves zero-triple-products* if $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$. We recall that an operator T between two C*-algebras is said to be *zero-products preserving* if $T(x)T(y) = 0$ whenever $xy = 0$.

It is clear that when T is symmetric (i.e., $T(x^*) = T(x)^*$), then T is orthogonality preserving on A_{sa} if and only if T preserves zero-products on A_{sa}. However, not every orthogonality preserving operator sends zero-products to zero-products. Consider for example $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $T(x) = ux$, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Clearly T is a triple homomorphism and hence orthogonality preserving, but taking $x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, we have $xy = yx = 0$ and $T(x)T(y) \neq 0$.

Any operator between two JB*-triples is orthogonality preserving whenever it preserves zero-triple-products. Indeed, let $T : E \to F$ be an operator between two JB*-triples. Suppose that T preserves zero-triple-products. If $x \perp y$ in E then $L(x, x)(y) = \{x, x, y\} = 0$ and hence $L(T(x), T(x))(T(y)) = 0$. Lemma 1 shows that $T(x) \perp T(y)$.

If T is an orthogonality preserving operator between two C*-algebras, then Theorem 17 implies that T preserves zero-triple-products. We therefore have

Corollary 18. *Let $T : A \to B$ be an operator between two* C*-*algebras. Then T is orthogonality preserving if and only if T preserves zero-triple-products.*

Remark 19. The following description of all zero-products preserving operators between C*-algebras was established in [11, Theorem 4.7].

Let T be an operator from a unital C*-algebra A to a C*-algebra B. Suppose that T preserves zero-products and $T(1)$ is a normal element. Then there exists a sequence of continuous Jordan homomorphisms $J_n : A \to B^{**}$ such that $T(1)J_n(z)$ converges strongly to $T(z)$, for all $z \in A$.

The sequence J_n given by the above result need not be, in general bounded (compare [11, Example 4.8]). The same example given in the just quoted reference assures the existence of a zero-products preserving operator T between two unital C*-algebras A and B satisfying that T cannot be written in the form $T = T(1)J$ for any Jordan homomorphism J.

Roughly speaking, the above Corollary 18 confirms that the appropriate structure to characterise the orthogonality preserving operators between C*-algebras is the natural structure of JB*-triple associated to each one of them.

5. Operators preserving cubes of self-adjoint elements

This section is inspired by the following statement placed by N.C. Wong in [38, Proof of Theorem 3.2]: if $T : A \to B$ is a self-adjoint operator between two C*-algebras, then T is a triple homomorphism if and only if $T(a^3) = T(a)^3$, for all $a = a^*$ in A. There is no direct argument to show, algebraically, that the identities $T((a \pm b)^3) = (T(a) \pm T(b))^3$ $(\forall a, b \in A_{sa})$ imply that $T(aba) = T(a)T(b)T(a)$ $(\forall a, b \in A_{sa})$, and hence T is a triple homomorphism. However, the following theorem shows that the above statement is true even in a more general setting.

Theorem 20. *Let J be a* JB*-*algebra, E be a* JB*-*triple and let $T : J \to E$ be an operator. The following assertions are equivalent*:

(a) *T is a triple homomorphism.*
(b) *$T(a^3) = T(a)^{[3]}$, for all $a \in J_{sa}$.*
(c) *T is orthogonality preserving on J_{sa} and $T^{**}(1)$ is a tripotent.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious, while (c) $\Rightarrow$ (a) follows by Corollary 11 (see also Remark 13).

We shall prove (b) $\Rightarrow$ (c). We assume that $T(a^3) = T(a)^{[3]}$, for all $a \in J_{sa}$. We claim that, for each $a \in J_{sa}$, $T^{**}(r(a)) = r(T(a))$. Indeed, having in mind the uniqueness of the cubic root of an element in a JB*-triple, it follows that, for each $a \in J_{sa}$, $T(a^{[\frac{1}{3}]})^{[3]} = T(a)$, which implies $T(a^{[\frac{1}{3}]}) = T(a)^{[\frac{1}{3}]}$. By mathematical induction we have $T(a^{[\frac{1}{3^n}]}) = T(a)^{[\frac{1}{3^n}]}$, $\forall n \in \mathbb{N}$. Since T^{**} is Strong*-continuous and $r(a) = S^*(E^{**}, E^*) - \lim a^{[\frac{1}{3^n}]}$, we deduce the desired conclusion.

Let $a \perp b$ in J_{sa}. By Lemma 1 we have that $b \perp J_a$, which in particular implies that $b \perp a^{[\frac{1}{3^n}]}$, $\forall n \in \mathbb{N}$, and consequently

$$b^3 \pm \big(a^{[\frac{1}{3^n}]}\big)^3 = \big(b \pm a^{[\frac{1}{3^n}]}\big)^3 \quad \text{(see Lemma 1).}$$

Therefore, by our assumptions $T(b)^{[3]} \pm (T(a^{[\frac{1}{3^n}]}))^{[3]} = (T(b) \pm T(a^{[\frac{1}{3^n}]}))^{[3]}$. Now, the joint Strong*-continuity of the triple product (on bounded sets) allows us to prove that $T(b)^{[3]} \pm (T(r(a)))^{[3]} = (T(b) \pm T(r(a)))^{[3]}$.

Another application of Lemmas 2 and 1 gives $T(a) \perp T(b)$, which shows that T is orthogonality preserving on J_{sa}.

Finally, since 1 is always an open projection in J^{**}, there exists a net $(x_\lambda) \subset J_{sa}$ satisfying that $S^*(J^{**}, J^*) - \lim_\lambda x_\lambda = 1$. It follows that $T^{**}(1) = S^*(E^{**}, E^*) - \lim T(x_\lambda^3) = S^*(E^{**}, E^*) - \lim(T(x_\lambda))^{[3]} = T^{**}(1)^{[3]}$. □

Corollary 21. *Let $T : A \to B$ be a symmetric operator between two* C**-algebras. The following assertions are equivalent*:

(a) *T is a triple homomorphism.*
(b) *$T|_{A_{sa}} : A_{sa} \to B_{sa}$ is a (real) triple homomorphism.*
(c) *$T(a^3) = T(a)^3$, for all $a \in A_{sa}$.*
(d) *T is orthogonality preserving (on A_{sa}) and $T^{**}(1)$ is a tripotent.*

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ORTHOGONALITY PRESERVERS REVISITED*

María Burgos[†], Francisco J. Fernández-Polo[‡], Jorge J. Garcés[§] and Antonio M. Peralta[¶]
Departamento de Análisis Matemático, Universidad de Granada,
Facultad de Ciencias 18071, Granada, Spain
[†]*mariaburgos@ugr.es,* [‡]*pacopolo@ugr.es,* [§]*jgarces@correo.ugr.es,* [¶]*aperalta@ugr.es*

To the memory of Issac Kantor

We obtain a complete characterization of all orthogonality preserving operators from a JB*-algebra to a JB*-triple. If $T : J \to E$ is a bounded linear operator from a JB*-algebra (respectively, a C*-algebra) to a JB*-triple and h denotes the element $T^{**}(1)$, then T is orthogonality preserving, if and only if, T preserves zero-triple-products, if and only if, there exists a Jordan *-homomorphism $S : J \to E_2^{**}(r(h))$ such that $S(x)$ and h operator commute and $T(x) = h \bullet_{r(h)} S(x)$, for every $x \in J$, where $r(h)$ is the range tripotent of h, $E_2^{**}(r(h))$ is the Peirce-2 subspace associated to $r(h)$ and $\bullet_{r(h)}$ is the natural product making $E_2^{**}(r(h))$ a JB*-algebra.

This characterization culminates the description of all orthogonality preserving operators between C*-algebras and JB*-algebras and generalizes all the previously known results in this line of study.

Keywords: Orthogonality preserving operators; orthogonally additive mappings; C*-algebras; JB*-algebras; JB*-triples

AMS Subject Classification: 17C65, 46L05, 46L40, 46L70, 46B04, 47B48

1. Introduction

From a historical point of view, the study of orthogonality preserving operators between C*-algebras started with the paper [1], where W. Arendt initiated the study of all operators preserving disjoint (or orthogonal) functions between $C(K)$-spaces. It was established there that for each orthogonality preserving operator $T : C(K) \to C(K)$, there exist $h \in C(K)$ and a mapping $\varphi : K \to K$ being continuous on the set $\{t \in K : h(t) \neq 0\}$ satisfying that

$$T(f)(t) = h(t)f(\varphi(t)),$$

for all $f \in C(k)$, $t \in K$. K. Jarosz [16] and J.-S. Jeang and N.-C. Wong [17] proved later that the description remains valid for all orthogonality preserving operators

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between $C_0(L)$-space, where L is a locally compact Hausdorff space.

$C(K)$ and $C_0(L)$ spaces are examples of abelian C*-algebras. In fact, the Gelfand theory assures that every abelian C*-algebra is C*-isomorphic to a $C_0(L)$-space. Therefore, the just quoted results by Jarosz and Jeang-Wong provide a complete description of all orthogonality preserving operators between abelian C*-algebras.

In the setting of a general C*-algebra A, two elements a and b in A are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. A linear operator T between two C*-algebras A and B is called *orthogonality preserving* or *disjointness preserving* if $T(a) \perp T(b)$, for all $a \perp b$ in A. The description of all orthogonality preserving operators between two C*-algebras raised as an important problem studied by many authors.

When the problem is considered only for symmetric operators between general C*-algebras, M. Wolff established a full description in [26]. More precisely, if $T : A \to B$ is a symmetric orthogonality preserving bounded linear operator between two C*-algebras with A unital, then denoting $T(1) = h$ the following assertions hold:

a) $T(A)$ is contained in the norm closure of $h\{h\}^{'}$, where $\{h\}^{'}$ denotes the commutator of $\{h\}$.
b) There exists a Jordan *-homomorphism $S : A \to B^{**}$ such that $T(z) = hS(z)$, for all $z \in A$.

On every C*-algebra A we can also consider a triple product defined by $\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x)$. This triple product has been shown as an important tool to characterize orthogonal elements in a C*-algebra. In fact, two elements a and b in A are orthogonal if and only if $\{a, a, b\} = 0$ (compare Lemma 1 in [7]). In particular, every triple homomorphism between two C*-algebras preserves orthogonal elements. Theorem 3.2 in [27] shows that a bounded linear operator T between two C*-algebras is a triple homomorphism if and only if T is orthogonality preserving and $T^{**}(1)$ is a partial isometry (tripotent).

There exists a wider class of complex Banach spaces containing all C*-algebras in which the notion of orthogonality makes sense and extends the concept given for C*-algebras. We refer to the class of JB*-triples. A JB*-triple is a complex Banach space E, equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, satisfying suitable algebraic and geometric conditions (see definition in §2). Every C*-algebra is a JB*-triple for the triple product given above.

Two elements a and b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$, where $L(a, b)$ is the linear operator on E defined by $L(a, b)(x) := \{a, b, x\}$. It is known that two elements in a C*-algebra A are orthogonal for the C*-algebra product if and only if they are orthogonal when A is considered as a JB*-triple (compare the introduction of §4).

Techniques in JB*-triple theory were revealed as a powerful tool in the description of all orthogonality preserving operators between two C*-algebras established in

[7]. Concretely, for every operator T between two C*-algebras, denoting $h = T^{**}(1)$, the following assertions are equivalent:

a) T is orthogonality preserving.
b) There exists a triple homomorphism $S : A \to B^{**}$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and

$$T(z) = L(h, r(h))(S(z)) = \frac{1}{2}\left(hr(h)^*S(z) + S(z)r(h)^*h\right)$$

$$= hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$, where $r(h)$ denotes the range tripotent of h.
c) T preserves zero-triple-products (that is, $\{T(a), T(b), T(c)\} = 0$ whenever $\{a, b, c\} = 0$).

Reference [7] also contains the following generalization of the main result in [27]: Let T be an operator from a JB*-algebra J to a JB*-triple E. Then T is a triple homomorphism if and only if T is orthogonality preserving and $T^{**}(1)$ is a tripotent. This result is in fact a consequence of a complete description of all orthogonality preserving operators from J to E whose second adjoint maps the unit of J^{**} to a von Neumann regular element. It seems natural to ask whether the condition of $T^{**}(1)$ being von Neumann regular can be omitted.

This paper culminates with the characterization of all orthogonality preserving operators from a JB*-algebra to a JB*-triple. Theorem 4.1 and Corollary 4.2 show that for a bounded linear operator T from a JB*-algebra J to a JB*-triple E the following are equivalent:

a) T is orthogonality preserving.
b) There exists a (unital) Jordan *-homomorphism $S : M(J) \to E_2^{**}(r(h))$ such that $S(x)$ and h operator commute and $T(x) = h \bullet_{r(h)} S(x)$, for every $x \in J$, where $M(J)$ is the multiplier algebra of J, $r(h)$ is the range tripotent of h, $E_2^{**}(r(h))$ is the Peirce-2 subspace associated to $r(h)$ and $\bullet_{r(h)}$ is the natural product making $E_2^{**}(r(h))$ a JB*-algebra.
c) T preserves zero-triple-products.

The proofs presented here are partially based on techniques developed in JB*-triple theory. The arguments do not depend on those results previously obtained by Arendt [1], Wolff [26], Wong [27] and Burgos, Fernández-Polo, Garcés, Martínez and Peralta [7]. We shall actually show that all of them are direct consequences of the main result here.

A useful tool applied in the proof of the main result of this paper is the characterization of all orthogonally additive n-homogeneous polynomials on a general C*-algebra. This characterization has been recently obtained in [20]. Section 3 presents a shorter and simplified proof of this description.

2. Notation and preliminaries

Given Banach spaces X and Y, $L(X,Y)$ will denote the space of all bounded linear mappings from X to Y. We shall write $L(X)$ for the space $L(X,X)$. Throughout the paper the word "operator" (respectively, multilinear or sesquilinear operator) will always mean bounded linear mapping (respectively bounded multilinear or sesquilinear mapping). The dual space of a Banach space X is always denoted by X^*.

When A is a JB*-algebra or a C*-algebra then, A_{sa} will stand for the set of all self-adjoint elements in A. We shall make use of standard notation in C*-algebra and JB*-triple theory.

C*-algebras and JB*-algebras belong to a more general class of Banach spaces known under the name of JB*-triples. JB*-triples were introduced by W. Kaup in [19]. A JB*-triple is a complex Banach space E together with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying that,

(JB1) $L(a,b)L(x,y) = L(x,y)L(a,b)+L(L(a,b)x,y)-L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;
(JB2) $L(a,a)$ is a hermitian operator with non-negative spectrum;
(JB3) $\|L(a,a)\| = \|a\|^2$.

For each x in a JB*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by the law $y \mapsto Q(x)y = \{x,y,x\}$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [4]). It is known that the triple product of a JBW*-triple is separately weak*-continuous [4]. The second dual of a JB*-triple E is a JBW*-triple with a product extending that of E (compare [9]).

An element e in a JB*-triple E is said to be a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E gives raise to the so-called *Peirce decomposition* of E associated to e, that is,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0,1,2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The corresponding *Peirce projections* are denoted by $P_i(e) : E \to E_i(e)$, $(i = 0, 1, 2)$. The Peirce space $E_2(e)$ is a JB*-algebra with product $x \bullet_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

For each element x in a JB*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \left\{x, x, x^{[2n-1]}\right\}$, $(n \in \mathbb{N})$. The symbol E_x will stand for the JB*-subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$, such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. Corollary 4.8 in [18], Corollary 1.15 in [19] and [12]).

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying that $\{y, y, y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define, $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak*-topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The element $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying that x is positive in the JBW*-algebra $E_2^{**}(e)$ (compare [11], Lemma 3.3).

A subspace I of a JB*-triple E is said to be an *inner ideal* of E if $\{I, E, I\} \subseteq I$. Given an element x in E, let $E(x)$ denote the norm closed inner ideal of E generated by x. It is known that $E(x)$ coincides with the norm-closure of the set $Q(x)(E) = \{x, E, x\}$. Moreover $E(x)$ is a JB*-subalgebra of $E_2^{**}(r(x))$ and contains x as a positive element (compare page 19 and Proposition 2.1 in [6]).

The *symmetrized Jordan triple product* in a JB*-triple E is defined by the expression

$$< x, y, z >:= \frac{1}{3} \left(\{x, y, z\} + \{y, z, x\} + \{z, x, y\}\right).$$

Given a C*-algebra (respectively, a JB*-algebra), A, the *multiplier algebra* of A, $M(A)$, is the set of all elements $x \in A^{**}$ such that for each elements $a \in A$, xa and ax (respectively, $x \circ a$) also lie in A. We notice that $M(A)$ is a C*-algebra (respectively, a JB*-algebra) and contains the unit element of A^{**}.

3. Orthogonally additive polynomials on C*-algebras: The role played by the multiplier algebra

One of the most useful tools used in the study of orthogonality preserving operators between C*-algebras is the description of all orthogonally additive n-homogeneous polynomials on a C*-algebra, obtained in [20]. We present here a shorter and simplified proof of the main results established in the just quoted paper.

Let A be a C*-algebra and let X be a complex Banach space. A mapping $f : A \to X$ is said to be *orthogonally additive* (respectively, *orthogonally additive on* A_{sa}) if for every $a, b \in A$ (respectively, $a, b \in A_{sa}$) with $a \perp b$ we have $f(a+b) = f(a)+f(b)$.

We shall say that f is *additive on elements having zero-product* if for every $a, b \in A$ with $ab = 0 = ba$ we have $f(a + b) = f(a) + f(b)$. When f behaves additively only on self-adjoint elements having zero-product, we shall say that f is *additive on elements having zero-product on* A_{sa}.

An X*-valued* n*-homogeneous polynomial* between two Banach spaces Y and X is a *continuous* X-valued mapping P on Y for which there exists a continuous (and symmetric) n-linear operator $T : Y \times \cdots \times Y \longrightarrow X$ satisfying $P(x) = T(x, \ldots, x)$, for every x in X. The following polarization formula

$$T(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_n \ P\left(\sum_{i=1}^{n} \varepsilon_i x_i\right), \tag{3.1}$$

holds for all $x_1, \ldots, x_n \in Y$.

Given two Banach spaces X and Y, the symbol $\mathcal{P}^n(X, Y)$ will stand for the Banach space of all n-homogeneous polynomials from X to Y and we write $\mathcal{P}^n(X) := \mathcal{P}^n(X, \mathbb{K})$.

D. Pérez and I. Villanueva prove in [23] that for every compact Hausdorff space K and every orthogonally additive n-homogeneous polynomial P from $C(K)$ to a Banach space X, there exists an operator $T : C(K) \to X$ satisfying that $P(f) = T(f^n)$, for all $f \in C(K)$. The proof remains valid when $C(K)$-spaces are replaced with $C_0(L)$ spaces, where L is a locally compact Hausdorff space.

Let $X_1, \ldots, X_n$, and X be Banach spaces, $T : X_1 \times \cdots \times X_n \to X$ a (continuous) n-linear operator, and $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation. It is known that there exists a unique n-linear extension $AB(T)_\pi : X_1^{**} \times \cdots \times X_n^{**} \to X^{**}$ such that for every $z_i \in X_i^{**}$ and every net $(x^i_{\alpha_i}) \in X_i$ $(1 \leq i \leq n)$, converging to z_i in the weak* topology we have

$$AB(T)_\pi(z_1, \ldots, z_n) = \text{weak*-}\lim_{\alpha_{\pi(1)}} \cdots \text{weak*-}\lim_{\alpha_{\pi(n)}} T(x^1_{\alpha_1}, \ldots, x^n_{\alpha_n}).$$

Moreover, $AB(T)_\pi$ is bounded and has the same norm as T. The extensions $AB(T)_\pi$ coincide with those constructed by Aron and Berner for polynomials in [2], and are usually termed the *Aron-Berner extensions* of T (see also Proposition 3.1 in [22]).

If every operator from X_i to X_j^* is weakly compact $(i \neq j)$, the Aron-Berner extensions of T defined above do not depend on the chosen permutation π (see [3], and Theorem 1 in [5]). In particular, this happens when every X_i has Pelczynski's property (V) (if all of the X_i's satisfy property (V), then their duals, X_i^*, have no copies of c_0, therefore every operator from X_i to X_j^* is unconditionally converging, and hence weakly compact by (V), see [21]). When all the Aron-Berner extensions of T coincide, the symbol $AB(T)$ will stand for any of them. It is also known that, $AB(T)$ is symmetric whenever T is.

When $P : X \to Y$ is the n-homogeneous polynomial defined by T, $AB(P) : X^{**} \to Y^{**}$ will denote the n-homogeneous polynomial whose associated symmetric n-linear operator is $AB(T)$.

We should note at this point that every C*-algebra satisfies property (V) (see Corollary 6 in [24]).

The original proof presented in [20] relies on the following technical result: for every symmetric and continuous n-linear form T on a C*-algebra A such that the n-homogeneous polynomial $P(x) = T(x, \ldots, x)$, $(\forall x \in A)$ is orthogonally additive on A_{sa}, the $(n-1)$-homogeneous polynomial $R(x) = AB(T)(1, x, \ldots, x)$, $(\forall x \in A)$ is orthogonally additive on A_{sa}, where 1 denotes the unit of A^{**}. The proof exhibited in this paper avoids the use of the above technical tool. Instead of using the Aron-Berner extension on the $A^{**} \times \ldots \times A^{**}$ we shall focus our attention on its restriction to the Cartesian product $M(A) \times \ldots \times M(A)$, where $M(A)$ denotes the multiplier algebra of A in A^{**}.

The following result, whose proof is essentially algebraic, is inspired by Proposition 2.4 in [23].

Lemma 3.1. *Let $P : A \to \mathbb{K}$ be an element in $\mathcal{P}^n(A)$ and let $T : A \times \cdots \times A \to \mathbb{K}$ be its associate symmetric n-linear operator. Suppose that P is orthogonally additive on A_{sa}. Then for every $1 < s < n$ and every $a_1, \ldots, a_s$, $b_1, \ldots, b_{n-s}$ in A_{sa} such that, for each i and j, a_i and b_j are orthogonal we have*

$$T\left(a_1, \ldots, a_s, b_1, \ldots, b_{n-s}\right) = 0.$$

Proof. Let $1 < s < n$. We claim that for every a and b in A_{sa} with $a \perp b$ we have

$$T\left(a, \overset{s}{\ldots}, a, b, \overset{(n-s)}{\ldots}, b\right) = 0. \tag{3.2}$$

Indeed, the equation

$$\lambda^n T\left(a, \ldots, a\right) + \mu^n T\left(b, \ldots, b\right) = \lambda^n P(a) + \mu^n P(b) = P(\lambda a + \mu b)$$

$$= \sum_{\substack{0 \le k_1, k_2 \le n \\ k_1 + k_2 = n}} \frac{n!}{k_1! k_2!} \lambda^{k_1} \mu^{k_2} \; T\left(a, \overset{k_1}{\ldots}, a, b, \overset{k_2}{\ldots}, b\right) \text{ (by the symmetry of } T\text{)},$$

holds for every λ and μ in $\mathbb{R}$. Therefore,

$$\sum_{\substack{0 < k_1, k_2 < n \\ k_1 + k_2 = n}} \frac{n!}{k_1! k_2!} \lambda^{k_1} \mu^{k_2} \; T\left(a, \overset{k_1}{\ldots}, a, b, \overset{k_2}{\ldots}, b\right) = 0,$$

for all λ and μ in $\mathbb{R}$, which in particular gives (3.2).

Let $a_1, \ldots, a_s$, $b_1, \ldots, b_{n-s}$ in A_{sa} be such that, for each i and j, a_i and b_j are orthogonal. Having in mind that whenever we fix s variables of T we have another symmetric and continuous multilinear form, the polarization formula (3.1) yields

$$T\left(a_1, \ldots, a_s, \sum_{j=1}^{n-s} \varepsilon_j b_j, \ldots, \sum_{j=1}^{n-s} \varepsilon_j b_j\right)$$

$$= \frac{1}{2^s (s)!} \sum_{\sigma_j = \pm 1} \sigma_1 \cdots \sigma_s \; T\left(\sum_{k=1}^{s} \sigma_k a_k, \ldots, \sum_{k=1}^{s} \sigma_k a_k, \sum_{j=1}^{n-s} \varepsilon_j b_j, \ldots, \sum_{j=1}^{n-s} \varepsilon_j b_j\right) = 0, \tag{3.3}$$

where in the last equality we applied (3.2) and the fact that $\sum_{k=1}^{s} \sigma_k a_k$ and $\sum_{j=1}^{n-s} \varepsilon_j b_j$ are orthogonal. Finally, the formula (3.3) gives

$$T\left(a_1, \ldots, a_s, b_1, \ldots, b_{n-s}\right)$$

$$= \frac{1}{2^{n-s}(n-s)!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{n-s} \; T\left(a_1, \ldots, a_s, \sum_{j=1}^{n-s} \varepsilon_j b_j, \ldots, \sum_{j=1}^{n-s} \varepsilon_j b_j\right) = 0. \qquad \square$$

Proposition 3.1. *Let A be a C^*-algebra. Suppose that $T : A \times \ldots \times A \to \mathbb{C}$ is a symmetric and continuous n-linear form on A such that the n-homogeneous polynomial $P(x) = T(x, \ldots, x)$, $\forall x \in A$, is orthogonally additive on A_{sa}. Then the polynomial $R : M(A) \to \mathbb{C}$, $R(x) := AB(T)(x, \ldots, x)$ is orthogonally additive on $M(A)_{sa}$.*

Proof. Let a and b be two orthogonal elements in $M(A)_{sa}$. Since $a^{\frac{1}{3}}$ and $b^{\frac{1}{3}}$ are orthogonal, we deduce that, for each pair x, y in A, $a^{\frac{1}{3}} x a^{\frac{1}{3}}$ and $b^{\frac{1}{3}} y b^{\frac{1}{3}}$ also are orthogonal elements in A. The hypothesis of P being orthogonally additive assures, via Lemma 3.1, that

$$T(a^{\frac{1}{3}} x_1 a^{\frac{1}{3}} + b^{\frac{1}{3}} y_1 b^{\frac{1}{3}}, \ldots, a^{\frac{1}{3}} x_n a^{\frac{1}{3}} + b^{\frac{1}{3}} y_n b^{\frac{1}{3}}) = T(a^{\frac{1}{3}} x_1 a^{\frac{1}{3}}, \ldots, a^{\frac{1}{3}} x_n a^{\frac{1}{3}})$$

$$+T(b^{\frac{1}{3}} y_1 b^{\frac{1}{3}}, \ldots, b^{\frac{1}{3}} y_n b^{\frac{1}{3}}), \text{ for all } x_1, \ldots x_n, y_1, \ldots, y_n \in A. \tag{3.4}$$

Now, Goldstine's theorem (cf. Theorem V.4.2.5 in [10]) guarantees that the closed unit ball of A_{sa} is weak*-dense in the closed unit ball of A_{sa}^{**}. Therefore there exist two bounded nets (x_λ) and (y_μ) in A_{sa}, converging in the weak*-topology of A^{**} to $a^{\frac{1}{3}}$ and $b^{\frac{1}{3}}$, respectively. In our setting the Aron-Berner extension of T is separately weak*-continuous. Thus, by replacing, in equation (3.4), x_1 and y_1 with (x_λ) and (y_μ), respectively, and taking weak*-limits, we have:

$$AB(T)(a + b, a^{\frac{1}{3}} x_2 a^{\frac{1}{3}} + b^{\frac{1}{3}} y_2 b^{\frac{1}{3}}, \ldots, a^{\frac{1}{3}} x_n a^{\frac{1}{3}} + b^{\frac{1}{3}} y_n b^{\frac{1}{3}})$$

$$= AB(T)(a, a^{\frac{1}{3}} x_2 a^{\frac{1}{3}}, \ldots, a^{\frac{1}{3}} x_n a^{\frac{1}{3}}) + AB(T)(b, b^{\frac{1}{3}} y_2 b^{\frac{1}{3}}, \ldots, b^{\frac{1}{3}} y_n b^{\frac{1}{3}}),$$

for all $x_2, \ldots x_n, y_1, \ldots, y_n \in A$. When the above argument is repeated for $x_2, y_2, \ldots, x_n, y_n$ we derive

$$R(a+b) = AB(T)(a+b, , \ldots, a+b)$$

$$= AB(T)(a, \ldots, a) + AB(T)(b, \ldots, b) = R(a) + R(b),$$

which finishes the proof. □

We observe that $M(A)$ is always unital, so Proposition 3.1 allows us to apply the final argument in the proof of Theorem 2.8 in [20] but avoiding some technical and laborious results needed in its original proof.

Theorem 3.1. [20] *Let A be a C^*-algebra, $n \in \mathbb{N}$ and P an n-homogeneous scalar polynomial on A. The following are equivalent.*

(a) There exists $\varphi \in A^*$ such that, for every $x \in A$,

$$P(x) = \varphi(x^n).$$

(b) P is additive on elements having zero-products.

(c) P is orthogonally additive on A_{sa}.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. To see that $(c) \Rightarrow (a)$ we proceed by induction on n. When $n = 1$ the result is trivial. We suppose that the statement is true for $n-1$.

Let $T : A \times \ldots \times A \to \mathbb{C}$ be the unique symmetric and continuous n-linear form on A associated to P. Proposition 3.1 guarantees that the polynomial $AB(P)$ associated to $AB(T)$ is orthogonally additive on $M(A)_{sa}$.

Let θ be defined by $\theta(x_2, \ldots, x_n) := AB(T)(1, x_2, \ldots, x_n), (x_2, \ldots, x_n \in M(A))$. We claim that the polynomial R associated to θ is orthogonally additive on $M(A)_{sa}$. Indeed, let a and b be two orthogonal elements in $M(A)_{sa}$ and let C denote C*-subalgebra of $M(A)$ generated by a, b and 1. Clearly C is a unital abelian C*-algebra and $P|_C$ is orthogonally additive. Thus, Theorem 2.1 in [23] assures the existence of a functional $\psi_{_C} \in C^*$ such that

$$AB(T)|_C(y_1, \ldots, y_n) = \psi_{_C}(y_1 \ldots y_n)$$

for all $y_1, \ldots, y_n \in C_x$. In particular

$$R(a+b) = \theta(a+b, \ldots, a+b) = AB(T)|_C(1, a+b, \ldots, a+b)$$

$$= \psi_{_C}((a+b)^{n-1}) = \psi_{_C}(a^{n-1} + b^{n-1}) = \psi_{_C}(a^{n-1}) + \psi_{_C}(b^{n-1})$$

$$= AB(T)|_C(1, a, \ldots, b) + AB(T)|_C(1, b, \ldots, b) = R(a) + R(b),$$

which proves the claim.

By the induction hypothesis, there exists $\varphi \in M(A)^*$ such that

$$R(x) = \varphi(x^{n-1})$$

for all $x \in M(A)$.

On the other hand, for every $x \in M(A)_{sa}$, let C_x be the abelian C*-subalgebra of $M(A)$ generated by 1 and x, and let $T_{|C_x} : C_x \times \ldots \times C_x \to \mathbb{C}$ be the restriction of T. Clearly the polynomial associated to $T_{|C_x}$ also is orthogonally additive. Therefore, Theorem 2.1 of [23] guarantees the existence of a measure $\psi_x \in (C_x)^*$ with $\|\psi_x\| = \|T_{|C_x}\|$ such that

$$T_{|C_x}(y_1, \ldots, y_n) = \psi_x(y_1 \ldots y_n)$$

for all $y_1, \ldots, y_n \in C_x$.

Now, we claim that, for every $x \in M(A)_{sa}$, $\psi_x = \varphi_{|C_x}$. Indeed, let us fix $x \in M(A)_{sa}$ and pick a positive element $z \in C_x$. There is no loss of generality in assuming that $\|z\| = 1$. The positivity of z implies the existence of a positive norm-one element $y \in C_x$ satisfying $y^{n-1} = z$.

We therefore have

$$\psi_x(z) = \psi_x(y^{n-1}) = AB(T_{|C_x})(1, y, \ldots, y) = AB(T)(1, y, \ldots, y)$$

$$= \theta(y, \ldots, y) = R(y) = \varphi(y^{n-1}) = \varphi(z).$$

Since z is an arbitrary positive norm-one element in C_x we deduce, by linearity, that $\psi_x = \varphi_{|C_x}$.

Thus, for each $x \in M(A)_{sa}$, we have

$$AB(P)(x) = AB(T)(x, \ldots, x) = \psi_x(x^n) = \varphi(x^n).$$

The polarization formula given in (3.1) applies to prove that $AB(P)(x) = \varphi(x^n)$ for all $x \in M(A)$. $\square$

The following vector-valued version of the above theorem was established in [20], Corollary 3.1.

Theorem 3.2. [20] *Let A be a C*-algebra, X a complex Banach space, $n \in \mathbb{N}$ and $P : A \to X$ an n-homogeneous polynomial. The following are equivalent.*

(a) There exists an operator $T : A \to X$ such that, for every $x \in A$,

$$P(x) = T(x^n).$$

(b) P is additive on elements having zero-products.
(c) P is orthogonally additive on A_{sa}.

4. Orthogonality preservers between C*-algebras and JB*-algebras

Let J be an arbitrary JB*-algebra. One of the main results stated in [7] describes the orthogonality preserving operators from J to a JB*-triple whose second transpose maps the unit in A^{**} to a tripotent in E^{**}. This section contains most of the novelties in this paper. We shall present a complete description of all orthogonality preserving operators from a JB*-algebra to a JB*-triple, without assuming any additional condition.

We recall that two elements a, b in a JB*-triple are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. Lemma 1 in [7] shows that $a \perp b$ if and only if one of the following statements holds:

$$\begin{array}{ccc} \{a, a, b\} = 0; & a \perp r(b); & r(a) \perp r(b); \\ E_2^{**}(r(a)) \perp E_2^{**}(r(b)); & r(a) \in E_0^{**}(r(b)); & a \in E_0^{**}(r(b)); \\ b \in E_0^{**}(r(a)); & E_a \perp E_b & \{b, b, a\} = 0. \end{array} \tag{4.1}$$

The Jordan identity (JB1) and the above reformulations assure that

$$a \perp \{x, y, z\} \text{ whenever } a \text{ is orthogonal to } x,\ y \text{ and } z. \tag{4.2}$$

If A is a C*-algebra, it can be checked from the above reformulations, that two elements a, b in A are orthogonal for the C*-algebra product (i.e. $ab^* = 0 = b^*a$) if and only if they are orthogonal when A is considered as a JB*-triple.

The equivalent reformulations of orthogonality given in (4.1) admit another equivalent statement in the setting of JB*-algebra when one of the elements is positive.

Lemma 4.1. *Let h and x be elements in a JB*-algebra J with h positive. Then $x \perp h$ if and only if $h \circ x = 0$.*

Proof. Having in mind that $h \circ x = \{1, h, x\}$, where 1 denotes the unit element in J^{**}, it is clear that $h \circ x = 0$ whenever $h \perp x$. We shall show that $x \perp h$ whenever $h \circ x = 0$. Given a positive element h in J, there exists another positive element b satisfying $b^2 = h$. Since the triple product $\{b, b, x\}$ coincides with $b^2 \circ x = h \circ x = 0$, the equivalent reformulations of orthogonality given in (4.1) guarantee that $b \perp x$, or equivalently, $x \in J_0^{**}(r(b))$. It is not hard to check that for a positive b, the range tripotents $r(b)$ and $r(b^2) = r(h)$ both coincide with the range projection of b in J^{**} and hence $r(b) = r(b^2) = r(h)$. Again, the equivalences stated in (4.1) assure that $x \perp h$. □

Let E and F be JB*-triples. An operator $T : E \to F$ is said to be *orthogonality preserving* if $T(a) \perp T(b)$ whenever $a \perp b$ in E. This concept extends the usual definition of orthogonality preserving linear operator between C*-algebras.

Lemma 4.2. *Let $T : J \to E$ be an orthogonality preserving operator from a JB*-algebra to a JB*-triple, then $T^{**}|_{M(J)} : M(J) \to E^{**}$ is orthogonality preserving.*

Proof. Let $a, b \in M(J)$. By (4.1), $a^{[\frac{1}{3}]}$ and $b^{[\frac{1}{3}]}$ are orthogonal elements in $M(J)$. Thus, we deduce that for each pair x, y in J, $Q(a^{[\frac{1}{3}]})x$ and $Q(b^{[\frac{1}{3}]})y$ are two orthogonal elements in J. Now, Goldstine's theorem guarantees that the closed unit ball of J is weak*-dense in the closed unit ball of J^{**}. Therefore there exist two bounded nets (x_λ) and (y_μ) in J, converging in the weak*-topology of J^{**} to $a^{[\frac{1}{3}]}$ and $b^{[\frac{1}{3}]}$, respectively.

Since the triple product of any JBW*-triple is separately weak* continuous ([4]) and T^{**} is weak*-continuous, we deduce that, for each x, y in J, the net $0 = \left\{T(Q(a^{[\frac{1}{3}]})x_\lambda), T(Q(a^{[\frac{1}{3}]})x), T(Q(b^{[\frac{1}{3}]})y)\right\}$ converges to $\left\{T^{**}(a), T(Q(a^{[\frac{1}{3}]})x), T(Q(b^{[\frac{1}{3}]})y)\right\}$ in the weak*-topology of E^{**}. Therefore

$$\left\{T^{**}(a), T(Q(a^{[\frac{1}{3}]})x), T(Q(b^{[\frac{1}{3}]})y)\right\} = 0,$$

for all $x, y \in J$. Similarly, $\left\{T^{**}(a), T^{**}(a), T(Q(b^{[\frac{1}{3}]})y)\right\} = 0$, for all $y \in J$.

Finally, $0 = \left\{T^{**}(a), T^{**}(a), T(Q(b^{[\frac{1}{3}]})y_\mu)\right\} \to \{T^{**}(a), T^{**}(a), T^{**}(b)\}$, in the weak*-topology of E^{**}, and hence $T^{**}(a) \perp T^{**}(b)$. $\square$

Let A be a C*-algebra and let X be a complex Banach space. A continuous sesquilinear mapping $\Phi : A \times A \to X$ is said to be *orthogonal* if $\Phi(a, b) = 0$ for every $a, b \in A$ such that $a \perp b$. By a celebrated result due to S. Goldstein [13] (see [14] for an alternative proof), for every continuous sesquilinear orthogonal form $V : A \times A \to \mathbb{C}$, there exist two functionals $\omega_1, \omega_2 \in A^*$ satisfying that

$$V(x, y) = \omega_1(xy^*) + \omega_2(y^*x),$$

for all $x, y \in A$. Denoting $\phi = \omega_1 + \omega_2$ and $\psi = \omega_1 - \omega_2$, we have

$$V(x, y) = \phi(x \circ y^*) + \psi([x, y^*]),$$

for all $x, y \in A$, where $a \circ b := \frac{1}{2}(ab + ba)$, $[a, b] := \frac{1}{2}(ab - ba)$. In particular, $V(x, y) = V(y, x)$ whenever $[x, y^*] = 0$ and $x \circ y^* = x^* \circ y$. The following lemma follows straightforwardly from the above remarks and the Hahn-Banach theorem.

Lemma 4.3. *Let A be a C*-algebra, X a Banach space and $\Phi : A \times A \to X$ a continuous sesquilinear orthogonal operator. Then $\Phi(x, y) = \Phi(y, x)$ whenever $[x, y^*] = 0$ and $x \circ y^* = x^* \circ y$.*

Let us recall that two elements a and b in a JB*-algebra J are said to *operator commute* in J if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operators commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in J. Self-adjoint elements a and b in J generate a JB*-subalgebra that can be realized as a JC*-subalgebra of some $B(H)$, [29], and, in this identification, a and b commute in the usual sense whenever the operators commute in J (compare Proposition 1 in [25]). Similarly, two elements a and b of J_{sa} operator commute if and only if $a^2 \circ b = \{a, b, a\}$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in J$ we use $\{b\}^{'}$ to denote the set of elements in J that operator commute with b. (This corresponds to the usual notation in von Neumann algebras.)

Proposition 4.1. *Let A be a C^*-algebra, E a JB^*-triple and $T : A \to E$ an orthogonality preserving operator. Then for $h = T^{**}(1)$, the following assertions hold:*

a) $\{T(x), h, h\} = \{h, T(x^*), h\}$, *for all $x \in A$.*
b) $T(A_{sa}) \subset E_2^{**}(r(h))_{sa}$.
c) *For each $a \in A$, $T(a)$ and h operator commute in the JB^*-algebra $E_2^{**}(r(h))$.*
d) *When h is a tripotent, then $T : A \to E_2^{**}(r(h))$ is a Jordan *-homomorphism, in particular T is a triple homomorphism.*

Proof. *a*) By Lemma 4.2, $T^{**}|_{M(A)} : M(A) \to E^{**}$ is orthogonality preserving. Therefore, the assignment $(x, y) \mapsto \{T^{**}(x), T^{**}(y), h\}$, defines a continuous sesquilinear orthogonal operator on $M(A) \times M(A)$. Lemma 4.3, applied to $x \in A_{sa}$ and $y = 1$ gives $\{T(x), h, h\} = \{h, T(x), h\}$. The desired statement follows by linearity.

b) Let $a \in A_{sa}$. By the Peirce arithmetic and *a*) we have

$$\{P_2(r(h))T(a), h, h\} + \{P_1(r(h))T(a), h, h\} = \{T(a), h, h\}$$

$$= \{h, T(a), h\} = \{h, P_2(r(h))T(a), h\},$$

which implies that $\{P_1(r(h))T(a), h, h\} = 0$, and hence $P_1(r(h))T(a) \perp h$. The equivalences in (4.1) imply that $P_1(r(h))T(a) \in E_0^{**}(r(h))$, which gives

$$T(A_{sa}) \subseteq E_2^{**}(r(h)) \oplus E_0^{**}(r(h)). \tag{4.3}$$

Consider now the mapping $P_3 : M(A) \to E^{**}$,

$$P_3(x) = \{T^{**}(x), T^{**}(x^*), T^{**}(x)\}.$$

It is clear that P_3 is a 3-homogeneous polynomial on $M(A)$. Since, by Lemma 4.2, $T^{**}|_{M(A)}$ is orthogonality preserving, P_3 is orthogonally additive on $M(A)_{sa}$. By Corollary 3.1 in [20] or Theorem 3.2, there exists an operator $F_3 : M(A) \to E^{**}$ satisfying that

$$P_3(x) = F_3(x^3),$$

for all x in $M(A)$. If $S_3 : M(A) \times M(A) \times M(A) \to E^{**}$ is the (unique) symmetric 3-linear operator associated to P_3, we have

$$F_3(< x, y, z >) = S_3(x, y, z) =< T^{**}(x), T^{**}(y), T^{**}(z) >, \tag{4.4}$$

for all $x, y, z \in M(A)_{sa}$. Now, taking $a \in M(A)_{sa}$ and $y = z = 1$ in (4.4), we deduce that

$$F_3(a) =< T^{**}(a), h, h >= \frac{2}{3}\{T^{**}(a), h, h\} + \frac{1}{3}\{h, T^{**}(a), h\}. \tag{4.5}$$

Thus, for each $a \in M(A)_{sa}$ we have

$$\{T^{**}(a), T^{**}(a), T^{**}(a)\} = F_3(a^3) =< h, h, T^{**}(a^3) > . \tag{4.6}$$

Now, (4.3), (4.6) and the Peirce arithmetic show that

$$T(A_{sa}) \subseteq E_2^{**}(r(h)) \cap E.$$

We shall finally prove that T is symmetric for the involution in $E_2^{**}(r(h))$. In order to simplify notation, we shall write $r(h) = r$. Let us recall that $E_2^{**}(r)$ is a JB*-algebra with Jordan product and involution given by $x \bullet_r y = \{x, r, y\}$ and $x^{\sharp_r} = \{r, x, r\} = Q(r)(x)$, respectively. The triple product in $E_2^{**}(r)$ is also determined by the expression

$$\{x, y, z\} = (x \bullet_r y^{\sharp_r}) \bullet_r z + (z \bullet_r y^{\sharp_r}) \bullet_r x - (x \bullet_r z) \bullet_r y^{\sharp_r}.$$

Lemma 4.3 applied to the form $\Phi(x, y) = \{T^{**}(x), T^{**}(y), z\}$ guarantees that

$$\{T^{**}(x), h, z\} = \{h, T^{**}(x), z\}$$

for every $x \in M(A)_{sa}$ and $z \in E^{**}$. Let us fix $x = a \in A_{sa}$. By taking $z = r$, the above identity gives $h \bullet_r T(a)^{\sharp_r} = h \bullet_r T(a)$, that is, $h \bullet_r \frac{T(a)-T(a)^{\sharp_r}}{2i} = 0$. Lemma 4.1 now applies to give $\left(T(a) - T(a)^{\sharp_r}\right) \perp h$, and hence $T(a) - T(a)^{\sharp_r}$ lies in $E_2^{**}(r) \cap E_0^{**}(r) = \{0\}$ (compare (4.1)). This implies $T(A_{sa}) \subset E_2^{**}(r)_{sa}$.

c) It follows by *b*) that $T(A_{sa}) \subset E_2^{**}(r)_{sa}$ and hence the triple product in $T(A_{sa})$ is determined by the Jordan product of $E_2^{**}(r)_{sa}$. By *a*), for each $a \in A_{sa}$, we have $\{h, h, T(a)\} = \{h, T(a), h\}$. Thus, $h^2 \bullet_r T(a) = 2(h \bullet_r T(a)) \bullet_r h - h^2 \bullet_r T(a)$, which gives the desired statement.

d) Let us assume that h is a tripotent. In this case $h = r(h) = r$. Statement *b*) assures that $T(A_{sa}) \subset E_2^{**}(r)_{sa}$. Thus, equation (4.5) guarantees that $F_3(a) = \{T^{**}(a), h, h\} = \{h, T^{**}(a), h\} = T^{**}(a)$, for all $a \in M(A)_{sa}$. Now, the formula established in (4.4) implies that

$$< T^{**}(a), T^{**}(b), T^{**}(c) >= F_3(< a, b, c >) = T^{**}(< a, b, c >),$$

for all $a, b, c \in M(A)_{sa}$. Taking $c = 1$ in the above equation, we have

$$T^{**}(a) \bullet_r T^{**}(b) = \{T^{**}(a), T^{**}(b), r\} = T^{**}(\{a, b, 1\}) = T^{**}(a \circ b),$$

for all $a, b \in M(A)_{sa}$. We have then shown that $T^{**}|_{M(A)} : M(A) \to E_2^{**}(r)$ is a unital Jordan *-homomorphism, which proves *d*). $\square$

It should be noticed that the main result in [27] is a direct consequence of statement *d*) in the above proposition.

Let $T : J \to E$ be an orthogonality preserving operator from a JB*-algebra to a JB*-triple and let h denote $T^{**}(1)$. Lemma 4.2 assures that $T^{**}|_{M(J)} : M(J) \to E^{**}$ also is orthogonality preserving. Since for each self-adjoint element $a \in M(J)$, the JB*-subalgebra $C_{\{1,a\}}$ of $M(J)$ generated by a and 1 is JB*-isomorphic to an abelian C*-algebra (compare Theorem 3.2.4 in [15]), the mapping $T^{**}|_{C_{\{1,a\}}} : C_{\{1,a\}} \to E^{**}$ satisfies the hypothesis of Proposition 4.1 above. Therefore, $T^{**}(a) \in E_2^{**}(r(h))_{sa}$, $T^{**}(a)$ and h operator commute in the JB*-algebra $E_2^{**}(r(h))$ and if h is a tripotent then, $T^{**}(a^2) = T^{**}(a) \bullet_{r(h)} T^{**}(a)$. We have proved the following result.

Corollary 4.1. *Let J be a JB*-algebra, E a JB*-triple and $T : J \to E$ an orthogonality preserving operator. Then for $h = T^{**}(1)$, the following assertions hold:*

a) $\{T(x), h, h\} = \{h, T(x^*), h\}$, *for all* $x \in J$.
b) $T(J_{sa}) \subset E_2^{**}(r(h))_{sa}$.
c) For each $a \in J$, $T(a)$ and h operator commute in the JB-algebra $E_2^{**}(r(h))$.*
*d) When h is a tripotent, then $T : J \to E_2^{**}(r(h))$ is a Jordan *-homomorphism, in particular T is a triple homomorphism.*

The result describing orthogonality preserving operators from a JB*-algebra to a JB*-triple can be now stated.

Theorem 4.1. *Let $T : J \to E$ be an operator from a JB*-algebra to a JB*-triple and let $h = T^{**}(1)$. The following are equivalent:*

a) T is orthogonality preserving.
*b) There exists a (unital) Jordan *-homomorphism $S : M(J) \to E_2^{**}(r(h))$ such that $S(x)$ and h operator commute and $T(x) = h \bullet_{r(h)} S(x)$, for every $x \in J$.*

Proof. The implication *b)*$\Rightarrow$ *a)* is clear.

a) $\Rightarrow$ *b)* Let C denote the JB*-subalgebra of $E_2^{**}(r(h))$ generated by h and $r(h)$.

Let $\sigma(h) \subseteq (0, \|h\|]$ denote the spectrum of h in $E_2^{**}(r(h))$. It is known that $\sigma(h) \cup \{0\}$ is compact and C is JB*-isomorphic to $C(\sigma(h) \cup \{0\})$, and under this identification h corresponds to the function $t \mapsto t$ (compare Theorem 3.2.4 in [15]). For each natural n, let p_n be the projection in $\overline{C}^{w^*}$ whose representation in $C(\sigma(h) \cup \{0\})^{**}$ is the characteristic function $\chi_{\left((\sigma(h)\cup\{0\})\cap[\frac{1}{n},1]\right)}$, and let $h_n = p_n \bullet_{r(h)} h$. We notice that (p_n) converges to $r(h)$ in the $\sigma(E^{**}, E^*)$-topology of E^{**}.

By Corollary 4.1 $T^{**}(M(J)_{sa}) \subset E_2^{**}(r(h))_{sa}$ and $T^{**}(M(J)) \subseteq \{h\}'$. The separate weak*-continuity of the product of $E_2^{**}(r(h))$ implies that y and $T^{**}(x)$ operator commute for all $y \in \overline{C}^{w^*}$ and $x \in M(J)$. In particular, for each natural n, p_n and $T^{**}(x)$ operator commute, for all $x \in M(J)$. Thus, the mapping $S_n : M(J) \to E_2^{**}(r(h))$, $S_n(x) := h_n^{-1} \bullet_{r(h)} T^{**}(x)$ is an orthogonality preserving operator between two JB*-algebras satisfying that $S_n(1) = p_n$ is a tripotent. Corollary 4.1 assures that S_n is a Jordan *-homomorphism and hence $\|S_n\| \leq 1$, for all $n \in \mathbb{N}$.

Let us take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By the Banach-Alaoglu Theorem, any bounded set in $E_2^{**}(r(h))$ is relatively weak*-compact and hence the assignment $z \mapsto S(z) := w^* - \lim_{\mathcal{U}} S_n(z)$ defines an operator $S : J \to E_2^{**}(r(h))$.

For each natural n, and each $x \in M(J)$, $h \bullet_{r(h)} S_n(x) = h \bullet_{r(h)} (h_n^{-1} \bullet_{r(h)} T^{**}(x)) = p_n \bullet_{r(h)} T^{**}(x)$. Since $r(h) = w^* - \lim_n p_n$, it follows from the separate weak*-continuity of the Jordan product of $E_2^{**}(r(h))$, that $h \bullet_{r(h)} S(x) = T^{**}(x)$, for all $x \in M(J)$. We have already seen that h_n^{-1}, h and $T^{**}(x)$ pairwise operator commute for every $x \in M(J)$. Therefore, $S_n(x)$ and h operator commute for every natural n. The separate weak*-continuity of the product assures that h and $S(x)$ operator commute for all $x \in M(J)$.

Finally, let $a \in M(J)_{sa}$. For each natural n, $S_n(a) \in E_2^{**}(r(h))_{sa}$ and $S_n(a^2) = S_n(a) \bullet_{r(h)} S_n(a)$. Being $E_2^{**}(r(h))_{sa}$ weak*-closed, it is clear that $S(a) \in E_2^{**}(r(h))_{sa}$. Let n and m be two natural numbers. Since h_n^{-1}, h_m^{-1}, and $T^{**}(a)$ are pairwise operator commuting, we have
$S_n(a) \bullet_{r(h)} S_m(a) = h_n^{-1} \bullet_{r(h)} h_m^{-1} \bullet_{r(h)} T^{**}(a) \bullet_{r(h)} T^{**}(a) = S_{\min(n,m)}(a)^2$
$= S_{\min(n,m)}(a^2)$.
For a fixed natural m, taking $w^* - \lim_{n \geq m, \mathcal{U}}$ in the above expressions, we deduce that

$$S(a) \bullet_{r(h)} S_m(a) = S_m(a^2),$$

for all $m \in \mathbb{N}$. The same argument gives

$$S(a) \bullet_{r(h)} S(a) = S(a^2). \qquad \square$$

The description provided by the above Theorem generalizes Theorems 6 and 10 in [7]. Concretely, the just quoted theorems make use of the hypothesis of $T^{**}(1)$ being von Neumann regular, and this assumption is completely removed in Theorem 4.1.

We recall that an operator T between two JB*-triples *preserves zero-triple-products* if $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$. While an operator T between two C*-algebras is said to be *zero-products preserving* if $T(x)T(y) = 0$ whenever $xy = 0$.

The papers [8], [26], and [28] give a complete description of zero-product preserving bounded linear maps between C*-algebras.

The equivalent reformulations of orthogonality stated in (4.1) together with Theorem 4.1 above, give the following generalization of Corollary 18 in [7].

Corollary 4.2. *Let $T : J \to E$ be an operator from a JB*-algebra to a JB*-triple. Then T is orthogonality preserving if and only if T preserves zero-triple-products.*

Example 4.1. Let T be a bounded linear operator between two C*-algebras. It was already noticed in [7] that in the case of T being symmetric (i.e., $T(x^*) = T(x)^*$), then T is orthogonality preserving on A_{sa} if and only if T preserves zero-products on A_{sa}. However, not every orthogonality preserving operator sends zero-products to zero-products. Consider, for example, $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, $T(x) = ux$, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Clearly T is a triple homomorphism and hence orthogonality preserving, but taking $x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, we have $xy = yx = 0$ and $T(x)T(y) \neq 0$.

Theorem 17 in [7] follows now as a consequence of Theorem 4.1.

Corollary 4.3. *Let $T : A \to B$ be an operator between two C^*-algebras. For $h = T^{**}(1)$ the following assertions are equivalent:*

a) T is orthogonality preserving.
b) There exists a triple homomorphism $S : A \to B^{**}$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and

$$T(z) = L(h, r(h))(S(z)) = \frac{1}{2}\left(hr(h)^*S(z) + S(z)r(h)^*h\right)$$

$$= hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$.

Proof. The implication *b*)$\Rightarrow$ *a*) is clear.

a) $\Rightarrow$ *b*) By Theorem 4.1 there exists a (unital) Jordan *-homomorphism $S : M(A) \to B_2^{**}(r(h))$ such that $S(x)$ and h operator commute in $B_2^{**}(r(h))$ and $T(x) = h \bullet_{r(h)} S(x)$, for every $x \in A$. In order to simplify notation we shall write $r = r(h)$. Notice that r is a partial isometry in B^{**}, with left and right projections given by rr^* and r^*r, respectively. It is well known that $B_2^{**}(r) = rr^*B^{**}r^*r$.

It can be easily checked that $L_{r^*} : B_2^{**}(r) \to B_2^{**}(r^*r)$, $x \mapsto r^*x$, is a unital Jordan *-homomorphism and $B_2^{**}(r^*r)$ is a C*-subalgebra of B^{**} because r^*r is a projection.

Take an element $a \in A_{sa}$. Since $S(a)$ and h operator commute in $B_2^{**}(r(h))_{sa}$, $L_{r^*}(h) = r^*h$ and $L_{r^*}(S(a)) = r^*S(a)$ operator commute in B_{sa}^{**}. Equivalently, r^*h and $r^*S(a)$ are two commuting elements in B^{**}. Therefore

$$h^*S(a) = h^*rr^*S(a) = (r^*h)^*(r^*S(a)) = (r^*h)(r^*S(a)) = (r^*S(a))(r^*h)$$

$$= (r^*S(a))^*(r^*h) = S(a)^*rr^*h = S(a)^*h,$$

and similarly $hS(a)^* = S(a)h^*$. The proof concludes by a linear argument. $\square$

The general description of all orthogonality preserving operators between two JB*-triples remains open. We can only prove the following local property.

Corollary 4.4. *Let $T : E \to F$ be an orthogonality preserving operator between two JB*-triples. Let x be a norm-one element in E and let $h = T^{**}(r(x))$. Then there exists a Jordan *-homomorphism $S : E(x) \to F_2^{**}(r(h))$, satisfying that $T|_{E(x)} = L(h, r(h))$.*

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Automatic continuity of biorthogonality preservers between compact C*-algebras and von Neumann algebras ☆

María Burgos, Jorge J. Garcés, Antonio M. Peralta *

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

We prove that every biorthogonality preserving linear surjection between two dual or compact C*-algebras or between two von Neumann algebras is automatically continuous.

1. Introduction and preliminaries

Two elements a, b in a C*-algebra A are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. A linear mapping $T : A \to B$ between two C*-algebras is called *orthogonality preserving* if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping T is *biorthogonality preserving* whenever the equivalence

$$x \perp y \quad \Leftrightarrow \quad T(x) \perp T(y)$$

holds for all x, y in A.

It can easily be seen that every biorthogonality preserving linear surjection, $T : A \to B$ between two C*-algebras is injective. Indeed, for each $x \in A$, the condition $T(x) = 0$ implies that $T(x) \perp T(x)$, and hence $x \perp x$, which gives $x = 0$.

The study of orthogonality preserving operators between C*-algebras begins with the work of W. Arendt [3] in the setting of unital abelian C*-algebras. His main result establishes that every orthogonality preserving bounded linear mapping $T : C(K) \to C(K)$ is of the form

$$T(f)(t) = h(t) f\big(\varphi(t)\big) \quad \big(f \in C(K),\ t \in K\big),$$

where $h \in C(K)$ and $\varphi : K \to K$ is a mapping which is continuous on $\{t \in K\colon\ h(t) \neq 0\}$. Several years later, K. Jarosz [16] extended the study to the setting of orthogonality preserving (not necessarily bounded) linear mappings between abelian C*-algebras.

A linear mapping $T : A \to B$ between two C*-algebras is said to be *symmetric* if $T(x)^* = T(x^*)$, equivalently, T maps the self-adjoint part of A into the self-adjoint part of B.

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* Corresponding author.
E-mail addresses: mariaburgos@ugr.es (M. Burgos), jgarces@ugr.es (J.J. Garcés), aperalta@ugr.es (A.M. Peralta).

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The study of orthogonality preservers between general C*-algebras was started in [31]. M. Wolff proved in [31, Theorem 2.3] that every orthogonality preserving bounded linear and symmetric mapping between two C*-algebras is a multiple of an appropriate Jordan *-homomorphism.

F.J. Fernández-Polo, J. Martínez Moreno and the authors of this note studied and described orthogonality preserving bounded linear maps between C*-algebras, JB*-algebras and JB*-triples in [7] and [8]. New techniques developed in the setting of JB*-algebras and JB*-triples were a fundamental tool to establish a complete description of all orthogonality preserving bounded linear (non-necessarily symmetric) maps between two C*-algebras. We recall some background results before stating the description obtained.

Each C*-algebra A admits a Jordan triple product defined by the expression $\{a,b,c\} := \frac{1}{2}(ab^*c + cb^*a)$. Fixed points of the triple product are called *partial isometries* or *tripotents*. Every partial isometry e in A induces a decomposition of A in the form

$$A = A_2(e) \oplus A_1(e) \oplus A_0(e),$$

where $A_2(e) := ee^*Ae^*e$, $A_1(e) := (1-ee^*)Ae^*e \oplus ee^*A(1-e^*e)$, and $A_0(e) := (1-ee^*)A(1-e^*e)$. This decomposition is termed the *Peirce decomposition*. The subspace $A_2(e)$ also admits a structure of unital JB*-algebra with product and involution given by $x \circ_e y := \{x,e,y\}$ and $x^{\sharp_e} := \{e,x,e\}$, respectively (compare [15]). The element e acts as the unit element of $A_2(e)$ (we refer to [15] and [30] for the basic results on JB- and JB*-algebras).

Recall that two elements a and b in a JB*-algebra J are said to *operator commute* in J if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in J. Self-adjoint elements a and b in J generate a JB*-subalgebra that can be realised as a JC*-subalgebra of some $B(H)$ (cf. [32]), and, in this realisation, a and b commute in the usual sense whenever they operator commute in J [30, Proposition 1]. Similarly, two self-adjoint elements a and b in J operator commute if and only if $a^2 \circ b = \{a,b,a\}$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in J$ we use $\{b\}'$ to denote the set of elements in J that operator commute with b. (This corresponds to the usual notation in von Neumann algebras.)

For each element a in a von Neumann algebra W there exists a unique partial isometry $r(a)$ in W such that $a = r(a)|a|$, and $r(a)^*r(a)$ is the support projection of $|a|$, where $|a| = (a^*a)^{\frac{1}{2}}$. It is also known that $r(a)a^*r(a) = \{r(a), a, r(a)\} = a$. We refer to [28, §1.12] for a detailed proof of these results. The element $r(a)$ will be called the *range partial isometry* of a.

The characterisation of all orthogonality preserving bounded linear maps between C*-algebras reads as follows:

Theorem 1. *(See [7, Theorem 17 and Corollary 18].) Let $T : A \to B$ be a bounded linear mapping between two C*-algebras. For $h = T^{**}(1)$ and $r = r(h)$ the following assertions are equivalent:*

a) *T is orthogonality preserving.*
b) *There exists a unique triple homomorphism $S : A \to B^{**}$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and*

$$T(z) = \frac{1}{2}\big(hr(h)^*S(z) + S(z)r(h)^*h\big) = hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$.
c) *There exists a unique Jordan *-homomorphism $S : A \to B_2^{**}(r)$ satisfying that $S^{**}(1) = r$, $T(A) \subseteq \{h\}'$ and $T(z) = h \circ_r S(z)$ for all $z \in A$.*
d) *T preserves zero triple products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x,y,z\} = 0$.*

The problem of automatic continuity of those linear maps preserving zero-products between C*-algebras has inspired many papers in the last twenty years. A linear mapping between abelian C*-algebras is zero-products preserving if and only if it is orthogonality preserving, however the equivalence doesn't hold for general C*-algebras (compare [7, comments before Corollary 18]). K. Jarosz proved the automatic continuity of every linear bijection preserving zero-products between $C(K)$ spaces (see [16, Corollary]). In 2003, M.A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong showed that every zero-products preserving linear bijection from a properly infinite von Neumann algebra into a unital ring is automatically continuous [9, Theorem 4.2]. In the same year, J. Araujo and K. Jarosz proved that a linear bijection which preserves zero-products in both directions between algebras $L(X)$, of continuous linear maps on a Banach space X, is automatically continuous and a scalar multiple of an algebra isomorphism [2]. The same authors conjectured that every linear bijection between two C*-algebras preserving zero-products in both directions is automatically continuous (see [2, Conjecture 1]).

In this paper we study the problem of automatic continuity of biorthogonality preserving linear surjections between C*-algebras. In Section 2 we prove that every biorthogonality preserving linear surjection between two compact C*-algebras is continuous. In Sections 3 and 4, we establish, among many other results, that every biorthogonality preserving linear surjection between two von Neumann algebras is automatically continuous. It follows as a consequence that a symmetric linear bijection between two von Neumann algebras preserving zero-products in both directions is automatically continuous. This provides a partial answer to the conjecture posed by Araujo and Jarosz.

1.1. Preliminary results

A subspace J of a C*-algebra A is said to be an *inner ideal* of A if $\{J, A, J\} \subseteq J$. Inner ideals in C*-algebras were completely described by M. Edwards and G. Rüttimann in [13].

Given a subset M of A, we write $M_A^{\perp}$ for the *annihilator of* M (in A) defined by

$$M_A^{\perp} := \{y \in A\colon\ y \perp x,\ \forall x \in M\}.$$

When no confusion can arise, we shall write $M^{\perp}$ instead of $M_A^{\perp}$. The following properties can be easily checked.

Lemma 2. *Let M be a subset of a C*-algebra A. The following statements hold*:

a) *$M^{\perp}$ is a norm closed inner ideal of A. When A is a von Neumann algebra, then $M^{\perp}$ is weak* closed.*
b) *$M \cap M^{\perp} = \{0\}$, and $M \subseteq M^{\perp\perp}$.*
c) *$S^{\perp} \subseteq M^{\perp}$ whenever $M \subseteq S \subseteq A$.*
d) *$M^{\perp}$ is closed for the product of A.*
e) *$M^{\perp}$ is $*$-invariant whenever M is.*

The next lemma describes the annihilator of a projection.

Lemma 3. *Let p be a projection in a* (*non-necessarily unital*) *C*-algebra A. The following assertions hold*:

a) *$\{p\}_A^{\perp} = (1-p)A(1-p)$, where* 1 *denotes the unit of A^{**}*;
b) *$\{p\}_A^{\perp\perp} = pAp$.*

Proof. Statement a) follows straightforwardly.

b) It is clear from a) that $\{p\}_A^{\perp\perp} \supseteq pAp$. To show the opposite inclusion, let $a \in \{p\}_A^{\perp\perp}$. Goldstine's theorem (cf. Theorem V.4.2.5 in [12]) guarantees that the closed unit ball of A is weak*-dense in the closed unit ball of A^{**}. Thus, there exists a net (x_λ) in the closed unit ball of A, converging in the weak*-topology of A^{**} to $1-p$. Noticing that $((1-p)x_\lambda(1-p)) \subset \{p\}_A^{\perp}$, we deduce that

$$(1-p)x_\lambda(1-p)a^* = a^*(1-p)x_\lambda(1-p) = 0, \tag{1}$$

for all λ. Since the product of A^{**} is separately weak*-continuous (compare [28, Theorem 1.7.8]), taking weak*-limits in (1), we have $(1-p)a^* = a^*(1-p) = 0$, which shows that $pa = ap = a$. $\square$

We shall also need some information about the norm closed inner ideal generated by a single element. Let a be an element in a C*-algebra A. Then $r(a)A^{**}r(a) \cap A = r(a)r(a)^*A^{**}r(a)^*r(a) \cap A$ is the smallest norm closed inner ideal in A containing a and will be denoted by $A(a)$. Further, the weak* closure of $A(a)$ coincides with $r(a)r(a)^*A^{**}r(a)^*r(a)$ (cf. [13, Lemma 3.7 and Theorem 3.10 and its proof]). Since $\{a\}^{\perp\perp}$ is an inner ideal containing a, we deduce that $A(a) \subseteq \{a\}^{\perp\perp}$.

It is well known that $\|\lambda a + \mu b\| = \max\{\|\lambda a\|, \|\mu b\|\}$, whenever $a \perp b$ and $\lambda, \mu \in \mathbb{C}$. For every family $(A_i)_i$ of C*-algebras, the direct sum $\bigoplus^{\infty} A_i$ is another C*-algebra with respect to the pointwise product and involution. In this case, for each $i \neq j$, A_i and A_j are mutually orthogonal C*-subalgebras of $\bigoplus_i^{\infty} A_i$.

Proposition 4. *Let A_1, A_2 and B be C*-algebras* (*respectively, von Neumann algebras*)*. Let us suppose that $T : A_1 \oplus^{\infty} A_2 \to B$ is a biorthogonality preserving linear surjection. Then $T(A_1)$ and $T(A_2)$ are norm closed* (*respectively, weak* closed*) *inner ideals of B, $B = T(A_1) \oplus^{\infty} T(A_2)$, and for $j = 1, 2$, $T|_{A_j} : A_j \to T(A_j)$ is a biorthogonality preserving linear surjection. Further, if T is symmetric then $T(A_1)$ and $T(A_2)$ are norm closed* (*respectively, weak* closed*) *ideals of B.*

Proof. Let us fix $j \in \{1, 2\}$. Since $A_j = A_j^{\perp\perp}$ and T is a biorthogonality preserving linear surjection, we deduce that $T(A_j) = T(A_j^{\perp\perp}) = T(A_j)^{\perp\perp}$. Lemma 2 guarantees that $T(A_j)$ is a norm closed inner ideal of B (respectively, a weak* closed subalgebra of B whenever A_1, A_2 and B are von Neumann algebras). The rest of the proof follows from Lemma 2(e)), and the fact that B coincides with the orthogonal sum of $T(A_1)$ and $T(A_2)$. $\square$

2. Biorthogonality preservers between dual C*-algebras

A projection p in a C*-algebra A is said to be *minimal* if $pAp = \mathbb{C}p$. A partial isometry e in A is said to be minimal if ee^* (equivalently, e^*e) is a minimal projection. The *socle* of A, $\mathrm{soc}(A)$, is defined as the linear span of all minimal projections in A. The *ideal of compact elements* in A, $K(A)$, is defined as the norm closure of $\mathrm{soc}(A)$. A C*-algebra is said to be *dual* or *compact* if $A = K(A)$. We refer to [19, §2], [1] and [33] for the basic references on dual C*-algebras.

The following theorem proves that biorthogonality preserving linear surjections between C*-algebras send minimal projections to scalar multiples of minimal partial isometries.

Theorem 5. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between two C*-algebras and let p be a minimal projection in A. Then $\|T(p)\|^{-1}\ T(p) = e_p$ is a minimal partial isometry in B. Further, T satisfies that $T(pAp) = e_p e_p^* B e_p^* e_p$ and $T((1-p)A(1-p)) = (1-e_p e_p^*)B(1-e_p^* e_p)$.*

Proof. Since T is a biorthogonality preserving linear surjection, the equality

$$T\big(S_A^{\perp}\big) = T(S)_B^{\perp}$$

holds for every subset S of A. For each minimal projection p in A, $\{T(p)\}_B^{\perp\perp} = T(\{p\}_A^{\perp\perp})$ is a norm closed inner ideal in B. Since $\{p\}_A^{\perp\perp} = pAp = \mathbb{C}p$, it follows that $\{T(p)\}_B^{\perp\perp}$ is a one-dimensional subspace of B. Having in mind that $\{T(p)\}_B^{\perp\perp}$ contains the inner ideal of B generated by $T(p)$, we deduce that $B(T(p))$ must be one-dimensional. In particular $(r(T(p))r(T(p))^*)B^{**}(r(T(p))^*r(T(p))) = \overline{B(T(p))}^{w^*}$ has dimension one, and hence $B(T(p)) = (r(T(p))r(T(p))^*)B^{**}(r(T(p))^*r(T(p))) = \mathbb{C}r(T(p))$. This implies that $\|T(p)\|^{-1}\ T(p) = e_p$ is a minimal partial isometry in B.

The equality $T(pAp) = e_p e_p^* B e_p^* e_p$ has been proved. Finally,

$$T\big((1-p)A(1-p)\big) = T\big((pAp)_A^{\perp}\big) = \big(T(pAp)\big)_B^{\perp} = \big(e_p e_p^* B e_p^* e_p\big)_B^{\perp} = \big(1-e_p e_p^*\big)B\big(1-e_p^* e_p\big). \qquad \square$$

Let a and b be two elements in a C*-algebra A. It is not hard to see that $a \perp b$ if and only if $r(a)$ and $r(b)$ are two orthogonal partial isometries in A^{**} (compare [7, Lemma 1]).

We shall make use of the following result which is a direct consequence of Theorem 1. The proof is left for the reader.

Corollary 6. *Let $T : A \to B$ be a bounded linear operator between two von Neumann algebras. For $h = T(1) \in B$ and $r = r(h)$ the following assertions are equivalent:*

a) *T is a biorthogonality preserving linear surjection.*
b) *h is invertible and there exists a unique triple isomorphism $S : A \to B$ satisfying $h^*S(z) = S(z^*)^*h$, $hS(z^*)^* = S(z)h^*$, and*

$$T(z) = \frac{1}{2}\big(hr(h)^*S(z) + S(z)r(h)^*h\big) = hr(h)^*S(z) = S(z)r(h)^*h,$$

for all $z \in A$.

c) *h is positive and invertible in $B_2(r)$ and there exists a unique Jordan *-isomorphism $S : A \to B_2(r) = B$ satisfying that $S(1) = r$, $T(A) \subseteq \{h\}'$ and $T(z) = h \circ_r S(z)$ for all $z \in A$.*

Further, in any of the previous statements, when A is a factor, then h is a multiple of the unit element in B.

We deal now with dual C*-algebras.

Remark 7. Given a sequence $(\mu_n) \subset c_0$ and a bounded sequence (x_n) in a Banach space X, the series $\sum_k \mu_k x_k$ needs not be, in general, convergent in X. However, when (x_n) is a bounded sequence of mutually orthogonal elements in a C*-algebra, A, the equality

$$\left\| \sum_{k=1}^{n} \mu_k x_k - \sum_{k=1}^{m} \mu_k x_k \right\| = \max\big\{|\mu_{n+1}|, \ldots, |\mu_m|\big\} \sup_{n+1 \leqslant k \leqslant m} \big\{\|x_k\|\big\},$$

holds for every $n < m$ in $\mathbb{N}$. It follows that $(\sum_{k=1}^{n} \mu_k x_k)$ is a Cauchy sequence and hence convergent in A. Alternatively, noticing that $\sum_k x_k$ defines a *w.u.C.* series in the terminology of [11], the final statement also follows from [11, Theorem V.6].

Lemma 8. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between two C*-algebras and let $(p_n)_n$ be a sequence of mutually orthogonal minimal projections in A. Then the sequence $(\|T(p_n)\|)$ is bounded.*

Proof. By Theorem 5, for each natural n, there exist a minimal partial isometry $e_n \in B$ and $\lambda_n \in \mathbb{C} \setminus \{0\}$ such that $T(p_n) = \lambda_n e_n$, and $\|T(e_n)\| = \lambda_n$. Note that, by hypothesis, (e_n) is a sequence of mutually orthogonal minimal partial isometries in B.

Let (μ_n) be any sequence in c_0. Since the p_n's are mutually orthogonal, the series $\sum_{k\geqslant 1} \mu_k p_k$ converges to an element in A (compare Remark 7). For each natural n, $\sum_{k\geqslant 1}^{\infty} \mu_k p_k$ decomposes as the orthogonal sum of $\mu_n p_n$ and $\sum_{k\neq n}^{\infty} \mu_k p_k$, therefore

$$T\left(\sum_{k\geqslant 1}^{\infty} \mu_k p_k\right) = \mu_n \lambda_n e_n + T\left(\sum_{k\neq n}^{\infty} \mu_k p_k\right),$$

with $\mu_n\lambda_n e_n \perp T(\sum_{k\neq n}^{\infty}\mu_k p_k)$, which in particular implies

$$\left\| T\left(\sum_{k\geqslant 1}^{\infty}\mu_k p_k\right)\right\| = \max\left\{|\mu_n||\lambda_n|, \left\| T\left(\sum_{k\neq n}^{\infty}\mu_k p_k\right)\right\|\right\} \geqslant |\mu_n||\lambda_n|.$$

This establishes that for each (μ_n) in c_0, $(\mu_n\lambda_n)$ is a bounded sequence, which proves the statement. □

Lemma 9. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between two C^*-algebras, (μ_n) a sequence in c_0 and let $(p_n)_n$ be a sequence of mutually orthogonal minimal projections in A. Then the sequence $(T(\sum_{k\geqslant n}^{\infty}\mu_k p_k))_n$ is well defined and converges in norm to zero.*

Proof. By Theorem 5 and Lemma 8 it follows that $(T(p_n))$ is a bounded sequence of mutually orthogonal elements in B. Let $M = \sup\{\|T(p_n)\|\colon n \in \mathbb{N}\}$. For each natural n, Remark 7 assures that the series $\sum_{k\geqslant n}^{\infty}\mu_k p_k$ converges.

Let us define $y_n := T(\sum_{k\geqslant n}^{\infty}\mu_k p_k)$. We claim that (y_n) is a Cauchy sequence in B. Indeed, given $n < m$ in $\mathbb{N}$, we have

$$\|y_n - y_m\| = \left\| T\left(\sum_{k\geqslant n}^{m-1}\mu_k p_k\right)\right\| = \left\|\sum_{k\geqslant n}^{m-1}\mu_k T(p_k)\right\| \overset{(*)}{\leqslant} M\max\bigl\{|\mu_n|, \ldots, |\mu_{m-1}|\bigr\}, \tag{2}$$

where at $(*)$ we apply the fact that $(T(p_n))$ is a sequence of mutually orthogonal elements. Consequently, (y_n) converges in norm to some element y_0 in B. Let z_0 denote $T^{-1}(y_0)$.

Let us fix a natural m. By hypothesis, for each $n > m$, p_m is orthogonal to $\sum_{k\geqslant n}^{\infty}\mu_k p_k$. This implies that $T(p_m) \perp y_n$, for every $n > m$, which, in particular, gives $T(p_m)^* y_n = y_n T(p_m)^* = 0$, for every $n > m$. Taking limits when n tends to ∞ we have $T(p_m)^* y_0 = y_0 T(p_m)^* = 0$. This shows that $y_0 = T(z_0) \perp T(p_m)$, and hence $p_m \perp z_0$. Since m was arbitrarily chosen we deduce that, for each natural n, z_0 is orthogonal to $\sum_{k\geqslant n}^{\infty}\mu_k p_k$. Therefore, $(y_n) \subset \{y_0\}_B^{\perp}$, and hence y_0 belongs to the norm closure of $\{y_0\}_B^{\perp}$, which implies that $y_0 = 0$. □

Proposition 10. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between C^*-algebras. Then $T|_{K(A)}$ is continuous if and only if the set $\{\|T(p)\|\colon p$ minimal projection in $A\}$ is bounded.*

Proof. The necessity being obvious. Suppose that

$$M = \sup\bigl\{\bigl\|T(p)\bigr\|\colon p \text{ minimal projection in } A\bigr\} < \infty.$$

Each nonzero self-adjoint element x in $K(A)$ can be written as a norm convergent (possibly finite) sum $x = \sum_n \lambda_n p_n$, where p_n are mutually orthogonal minimal projections in A, and $\|x\| = \sup\{|\lambda_n|\colon n \in \mathbb{N}\}$ (compare [1]). If the series $x = \sum_n \lambda_n p_n$ is finite then

$$\bigl\|T(x)\bigr\| = \left\|\sum_{n=1}^{m}\lambda_n T(p_n)\right\| \overset{(*)}{=} \max\bigl\{\bigl\|\lambda_n T(p_n)\bigr\|\colon n = 1, \ldots, m\bigr\} \leqslant M\|x\|,$$

where at $(*)$ we apply the fact that $(T(p_n))$ is a finite set of mutually orthogonal elements in B. When the series $x = \sum_n \lambda_n u_n$ is infinite we may assume that $(\lambda_n) \in c_0$.

It follows from Lemma 9 that the sequence $(T(\sum_{k\geqslant n}^{\infty}\lambda_k p_k))_n$ is well defined and converges in norm to zero. We can find a natural m such that $\|T(\sum_{k\geqslant m}^{\infty}\lambda_k p_k)\| < M\|x\|$. Since the elements $\lambda_1 p_1, \ldots, \lambda_{m-1}p_{m-1}$, $\sum_{k\geqslant m}^{\infty}\lambda_k p_k$ are mutually orthogonal, we have

$$\bigl\|T(x)\bigr\| = \max\left\{\bigl\|T(\lambda_1 p_1)\bigr\|, \ldots, \bigl\|T(\lambda_{m-1}p_{m-1})\bigr\|, \left\|T\left(\sum_{k\geqslant m}^{\infty}\lambda_k p_k\right)\right\|\right\} \leqslant M\|x\|.$$

We have established that $\|T(x)\| \leqslant M\|x\|$, for all $x \in K(A)_{sa}$, and by linearity $\|T(x)\| \leqslant 2M\|x\|$, for all $x \in K(A)$. □

Theorem 11. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between two C^*-algebras. Then $T|_{K(A)} : K(A) \to K(B)$ is continuous.*

Proof. Theorem 5 implies $T(\mathrm{soc}(A)) = \mathrm{soc}(B)$ (compare [33, Theorem 5.1]). By Proposition 10 it is enough to show the boundedness of the set

$$\mathcal{P} = \bigl\{\bigl\|T(p)\bigr\|\colon p \text{ minimal projection in } A\bigr\}.$$

Suppose, on the contrary, that $\mathcal{P}$ is unbounded. We shall show by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in A such that $\|T(p_n)\| > n$.

The case $n = 1$ is clear. The induction hypothesis guarantees the existence of mutually orthogonal minimal projections $p_1, \ldots, p_n$ in A with $\|T(p_k)\| > k$ for all $k \in \{1, \ldots, n\}$.

By assumption, there exists a minimal projection $q \in A$ satisfying $\|T(q)\| > \max\{\|T(p_1)\|, \ldots, \|T(p_n)\|, n+1\}$. We claim that q must be orthogonal to each p_j. Suppose, on the contrary, that for some j, p_j and q are not orthogonal. Let C denote the C*-subalgebra of A generated by q and p_j. We conclude from Theorem 1.3 in [27] (see also [26, §3]) that there exist $0 < t < 1$ and a *-isomorphism $\Phi : C \to M_2(\mathbb{C})$ such that $\Phi(p_j) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Phi(q) = \left(\begin{smallmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{smallmatrix}\right)$. Since $T|_C : C \cong M_2(\mathbb{C}) \to T(C)$ is a continuous biorthogonality preserving linear bijection, Theorem 1 (see also Corollary 6) guarantees the existence of a scalar $\lambda \in \mathbb{C}\backslash\{0\}$ and a triple isomorphism $\Psi : C \to T(C)$ such that $T(x) = \lambda\Psi(x)$ for all $x \in C$. In this case, $\|T(p_j)\| = |\lambda|\|\Psi(p_j)\|$ implies that

$$\left\|T(p_j)\right\| < \left\|T(q)\right\| = |\lambda|\left\|\Psi(q)\right\| = |\lambda|\left\|\Psi(p_j)\right\| = \left\|T(p_j)\right\|,$$

which is a contradiction. Therefore $q \perp p_j$, for every $j = 1, \ldots, n$.

It follows by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in A such that $\|T(p_n)\| > n$. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} p_n$ defines an element a in A (compare Remark 7). For each natural m, a decomposes as the orthogonal sum of $\frac{1}{\sqrt{m}} p_m$ and $\sum_{n\neq m}^{\infty} \frac{1}{\sqrt{n}} p_n$, therefore

$$T(a) = \frac{1}{\sqrt{m}} T(p_m) + T\left(\sum_{n\neq m}^{\infty} \frac{1}{\sqrt{n}} p_n\right),$$

with $\frac{1}{\sqrt{m}} T(p_m) \perp T(\sum_{n\neq m}^{\infty} \frac{1}{\sqrt{n}} p_n)$. This argument implies that

$$\left\|T(a)\right\| = \max\left\{\frac{1}{\sqrt{m}}\left\|T(p_m)\right\|, \left\|T\left(\sum_{n\neq m}^{\infty} \frac{1}{\sqrt{n}} p_n\right)\right\|\right\} > \sqrt{m}.$$

Since m was arbitrarily chosen, we have arrived at our desired contradiction. □

The following result is an immediate consequence of the above theorem.

Corollary 12. *Let $T : A \to B$ be a biorthogonality preserving linear surjection between two dual C*-algebras. Then T is continuous.*

Given a complex Hilbert space H, it is well known that $\mathrm{soc}(L(H))$ coincides with the space of all finite rank operators on H. The ideal $K(L(H))$ agrees with the ideal $K(H)$ of all compact operators on H.

Corollary 13. *Let $T : K(H) \to K(H)$ be a biorthogonality preserving linear surjection, where H is a complex Hilbert space. Then T is continuous.*

3. C*-algebras linearly spanned by their projections

In a large number of C*-algebras every element can be expressed as a finite linear combination of projections: the Bunce–Deddens algebras; the irrotational rotation algebras; simple, unital AF C*-algebras with finitely many extremal states; UHF C*-algebras; unital, simple C*-algebras of real rank zero with no tracial states; properly infinite C*- and von Neumann algebras; von Neumann algebras of type II_1 ... (see [21–23,25,20] and the references therein).

Theorem 14. *Let $T : A \to B$ be an orthogonality preserving linear map between C*-algebras, where A is unital. Suppose that every element of A is a finite linear combination of projections, then T is continuous.*

Proof. Let p be a projection in A. As $p \perp (1-p)$ then $T(p) \perp T(1) - T(p)$, that is $T(p)T(1)^* = T(p)T(p)^*$ and $T(1)^*T(p) = T(p)^*T(p)$. In particular, $T(p)T(1)^* = T(1)T(p)^*$ and $T(1)^*T(p) = T(p)^*T(1)$. Since every element in A coincides with a finite linear combination of projections, it follows that

$$T(x)T(1)^* = T(1)T\left(x^*\right)^*, \tag{3}$$

for all $x \in A$.

Let now p, q be two projections in A. The relation $qp \perp (1-q)(1-p)$ implies that $T(qp) \perp T(1-q-p+qp)$. Therefore

$$T(qp)T(1)^* - T(qp)T(q)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0. \tag{4}$$

Similarly, since $q(1-p) \perp (1-q)p$, we have $T(q-qp) \perp T(p-qp)$, and hence

$$T(q)T(p)^* - T(q)T(qp)^* - T(qp)T(p)^* + T(qp)T(qp)^* = 0. \tag{5}$$

From (4) and (5), we get

$$T(qp)T(1)^* - T(qp)T(q)^* = T(q)T(p)^* - T(q)T(qp)^*.$$

Being A linearly spanned by its projections, the last equation yields to

$$T(qx)T(1)^* - T(qx)T(q)^* = T(q)T\big(x^*\big)^* - T(q)T\big(qx^*\big)^*, \tag{6}$$

for all $x \in A$, and $q = q^* = q^2 \in A$.

By replacing, in (6), q with $1-q$, we get

$$T(q-qx)T(1)^* - T(x-qx)T(1-q)^* = T(1-q)T\big(x^*\big)^* - T(1-q)T\big(x^* - qx^*\big)^*.$$

Having in mind (3) we obtain

$$T(x)T(q)^* - T(qx)T(q)^* = T(1)T\big(qx^*\big)^* - T(q)T\big(qx^*\big)^*. \tag{7}$$

From Eqs. (6) and (7), we deduce that

$$T(qx)T(1)^* - T(x)T(q)^* = T(q)T\big(x^*\big)^* - T(1)T\big(qx^*\big)^*,$$

for every x in A and every projection q in A. Again, the last equation and the hypothesis on A prove:

$$T(yx)T(1)^* - T(x)T\big(y^*\big)^* = T(y)T\big(x^*\big)^* - T(1)T\big(y^*x^*\big)^*,$$

for all $x, y \in A$. Since, by (3), $T(1)T(y^*x^*)^* = T(xy)T(1)^*$, we can write the above equation as:

$$T(yx+xy)T(1)^* = T(y)T\big(x^*\big)^* + T(x)T\big(y^*\big)^*, \tag{8}$$

for all $x, y \in A$.

Let $h = T(1)$. It follows from (8) that

$$T\big(x^2\big)h^* = T(x)T\big(x^*\big)^* \quad (x \in A).$$

We claim that the linear mapping $S : A \to B$, $S(x) := T(x)h^*$, is positive, and hence continuous. Indeed, given $a \in A^+$, there exists $x \in A_{sa}$ such that $a = x^2$. Then $S(a) = T(x^2)h^* = T(x)T(x)^* \geqslant 0$.

Finally, for any $x \in A_{sa}$,

$$\big\|T(x)\big\|^2 = \big\|T(x)T(x)^*\big\| = \big\|S\big(x^2\big)\big\| \leqslant \|S\|\|x\|^2,$$

which implies that T is bounded on self-adjoint elements, and thus T is continuous. □

We have actually proved the following:

Proposition 15. *Let $T : A \to B$ be a linear map between C*-algebras, where A is unital and every element in A is a finite linear combination of projections. Suppose that T satisfies one of the following statements*:

a) $ab^* = 0 \Rightarrow T(a)T(b)^* = 0$;
b) $b^*a = 0 \Rightarrow T(b)^*T(a) = 0$.

Then T is continuous.

Recall that a unital C*-algebra is *properly infinite* if it contains two orthogonal projections equivalent to the identity (i.e. it contains two isometries with mutually orthogonal range projections). Zero product preserving linear mappings from a properly infinite von Neumann algebra to a unital ring were studied and described in [9, Theorem 4.2]. In this paper we consider a wider class of C*-algebras. Let A be a properly infinite C*-algebra or a von Neumann algebra of type II_1. It follows by [20, Corollary 2.2] (see also [25]) and [14, Theorem 2.2.(a)] that every element in A can be expressed as a finite linear combination of projections. Our next result follows immediately from Theorem 14.

Corollary 16. *Let A be a properly infinite unital C*-algebra or a von Neumann algebra of type II_1. Every orthogonality preserving linear map from A to another C*-algebra is automatically continuous.*

The following corollary is, in some sense, a generalisation of [24, Theorem 2] and [6, Theorem 1].

Corollary 17. *Let A be a properly infinite unital C^*-algebra or a von Neumann algebra of type II_1 and let $T : A \to B$ be a linear map from A to another C^*-algebra. Suppose that T satisfies one of the following statements*:

a) $ab^* = 0 \Rightarrow T(a)T(b)^* = 0$;
b) $b^*a = 0 \Rightarrow T(b)^*T(a) = 0$.

Then T is continuous.

4. Biorthogonality preservers between von Neumann algebras

Let us recall some fundamental results derived from the Murray–von Neumann dimension theory. Two projections p and q in a von Neumann algebra A are (Murray–von Neumann) *equivalent* (written $p \sim q$) if there exists a partial isometry $u \in A$ with $u^*u = p$ and $uu^* = q$. We write $q \precsim p$ when $q \leqslant p$ and $p \sim q$. A projection p in A is said to be *finite* if $q \precsim p$ implies $p = q$. Otherwise, it is called *infinite*. A von Neumann algebra is said to be finite or infinite according to the property of its identity projection. A projection p in A is called *abelian* if pAp is a commutative von Neumann algebra (compare [29, §V.1]).

A von Neumann algebra A is said to be of *type* I if every nonzero central projection in A majorizes a nonzero abelian projection. If there is no nonzero finite projection in A, that is, if A is purely infinite, then it is of *type* III. If A has no nonzero abelian projection and if every nonzero central projection in A majorizes a nonzero finite projection of A, then it is of *type* II. If A is finite and of type II (respectively, type I), then it is said to be of type II_1 (respectively, type I_{fin}). If A is of type II and has no nonzero central finite projection, then A is said to be of type II_∞. Every von Neumann algebra is uniquely decomposable into the direct (orthogonal) sum of weak* closed ideals of type I, type II_1, type II_∞, and type III (this decomposition is usually called, the *Murray–von Neumann* decomposition).

Proposition 18. *Let $T : A \to B$ be a surjective linear mapping from a unital C^*-algebra onto a finite von Neumann algebra. Suppose that for each invertible element x in A we have $\{T(x)\}^\perp \subseteq T(\{x\}^\perp)$. Then T is continuous. Further, if A is a von Neumann algebra, then $T(x) = T(1)S(x)$ ($x \in A$), where $S : A \to B$ is a Jordan homomorphism. In particular, every biorthogonality preserving linear surjection between two von Neumann algebras one of which is finite is continuous.*

Proof. Let $T : A \to B$ be a surjective linear mapping satisfying the hypothesis. We claim that T preserves invertibility. Let z be an invertible element in A and let $r = r(T(z))$ denote the range partial isometry of $T(z)$ in B. In this case

$$\left(1 - rr^*\right)B\left(1 - r^*r\right) = \left\{T(z)\right\}^\perp \subseteq T\left(\{z\}^\perp\right) = \{0\}.$$

Since B is a finite von Neumann algebra, $1 - rr^*$ and $1 - r^*r$ are equivalent projections in B (compare [18, Exercise 6.9.6]). Thus, there exists a partial isometry w in B such that $ww^* = 1 - rr^*$ and $w^*w = 1 - r^*r$, and hence $ww^*Bw^*w = \{0\}$. It follows that $1 - rr^* = ww^* = 0 = w^*w = 1 - r^*r$.

Since $1 = r^*r$ (respectively, $1 = rr^*$) is the support projection of $T(z)^*T(z)$ (respectively, $T(z)T(z)^*$), we deduce that $T(z)T(z)^*$ and $T(z)^*T(z)$ are invertible elements in B, and therefore $T(z)$ is invertible.

Having in mind that T sends invertible elements to invertible elements, we deduce from [10, Corollary 2.4] (see also [4, Theorem 5.5.2]) that T is continuous. If A is a von Neumann algebra, then it is well known that T is a Jordan homomorphism multiplied by $T(1)$ (cf. [5, Theorem 1.3] or [10, Corollary 2.4]). □

It is obvious that every *-isomorphism between two von Neumann algebras preserves the summands appearing in the Murray–von Neumann decomposition. However, it is not so clear that every Jordan *-isomorphism between two von Neumann algebras also preserves the Murray–von Neumann decomposition. The justification follows from an important result due to R. Kadison [17]. If $T : A \to B$ is a Jordan *-isomorphism between von Neumann algebras, then there exist weak* closed ideals A_1 and A_2 in A and B_1 and B_2 in B satisfying that $A = A_1 \oplus^\infty A_2$, $B = B_1 \oplus^\infty B_2$, $T|_{A_1} : A_1 \to B_1$ is a *-isomorphism, and $T|_{A_2} : A_2 \to B_2$ is a *-anti-isomorphism (see [17, Theorem 10]). It follows that every Jordan *-isomorphism preserves the Murray–von Neumann decomposition.

Theorem 19. *Every biorthogonality preserving linear surjection between von Neumann algebras is automatically continuous.*

Proof. Let $T : A \to B$ be a biorthogonality preserving linear surjection between von Neumann algebras.

It is well known that every von Neumann algebra is uniquely decomposed into a direct sum of five algebras of types I_{fin}, I_∞, II_1, II_∞ and III, respectively, where I_{fin} is a finite type I von Neumann algebra, II_1 is a finite type II von Neumann algebra and the direct sum of those summands of types I_∞, II_∞ and III is a properly infinite von Neumann algebra (compare [29, Theorem V.1.19]). Therefore, A and B decompose in the form $A = A_{I_{fin}} \oplus^\infty A_{II_1} \oplus^\infty A_{p\infty}$, $B = B_{I_{fin}} \oplus^\infty B_{II_1} \oplus^\infty B_{p\infty}$, where

$A_{I_{fin}}$ and $B_{I_{fin}}$ are finite type I von Neumann algebras, A_{II_1} and B_{II_1} are type II_1 von Neumann algebras, and $A_{p\infty}$ and $B_{p\infty}$ are properly infinite von Neumann algebras.

Corollary 16 guarantees that $T|_{A_{p\infty}} : A_{p\infty} \to B$ and $T|_{A_{II_1}} : A_{II_1} \to B$ are continuous linear mappings. In order to simplify notation we denote $A_1 = A_{I_{fin}}$, $A_2 = A_{II_1} \oplus^\infty A_{p\infty}$, $B_1 = B_{I_{fin}}$, and $B_2 = B_{II_1} \oplus^\infty B_{p\infty}$. According to this notation, $T|_{A_2} : A_2 \to B$ is continuous. Theorem 1 assures the existence of a Jordan *-homomorphism $S_2 : A_2 \to B_2(r_2)$ satisfying that $S_2(1_2) = r_2$, $T(A_2) \subseteq \{h_2\}'$ and $T(z) = h_2 \circ_{r_2} S_2(z)$ for all $z \in A_2$, where 1_2 is the unit of A_2 and r_2 is the range partial isometry of $T(1_2) = h_2$. We notice that, for each $z \in T(A_2)$, $r_2 r_2^* z + z r_2^* r_2 = 2z$, and we therefore have $r_2 r_2^* z = z r_2^* r_2 = z$.

Proposition 4 implies that $T(A_1)$ and $T(A_2)$ are orthogonal weak* closed inner ideals of B, whose direct sum is B. Thus, the unit of B decomposes in the form $1_B = v + w$, where $v \in T(A_1)$ and $w \in T(A_2)$. Since v and w are orthogonal we have $1_B = 1_B 1_B^* = (v+w)(v^*+w^*) = vv^* + ww^*$ and $1_B = v^*v + w^*w$, which shows that vv^* and ww^* (respectively, v^*v and w^*w) are two orthogonal projections in B whose sum is 1_B. It follows that $vv^*y = yv^*v = y$ for every element y in $T(A_1)$.

It can be checked that $u = v + r_2$ is a unitary element in B and the mapping

$$\Phi : (B, \circ_u, \sharp_u) \to (B, \circ, *), \quad x \mapsto xu^*$$

is a Jordan *-isomorphism. By noticing that $T(A_2)$ is a weak* closed inner ideal of B, $r_2 \in T(A_2)$ and $B_2(r_2)$ is the smallest weak* closed inner ideal containing r_2, we have $T(A_2) = B_2(r_2)$. Since B decomposes in the form

$$B_2(v) \oplus^\infty B_2(r_2) = B = T(A_1) \oplus^\infty T(A_2),$$

we deduce that $v \in T(A_1) \subseteq B_2(v)$, which gives $T(A_1) = B_2(v)$. We also have

$$B = \Phi\big(B_2(v)\big) \oplus^\infty \Phi\big(B_2(r_2)\big) = B_2\big(vv^*\big) \oplus^\infty B_2\big(r_2 r_2^*\big) = vv^* B vv^* \oplus^\infty r_2 r_2^* B r_2 r_2^*.$$

The mapping $\Phi|_{B_2(r_2)} S_2$ is a Jordan *-isomorphism from A_2 onto $B_2(r_2 r_2^*)$. Since $B_2(r_2 r_2^*)$ is a weak* closed ideal of B and every Jordan *-isomorphism preserves the Murray–von Neumann decomposition we deduce that

$$\Phi|_{B_2(r_2)} S_2(A_2) \subseteq B_2 = B_{II_1} \oplus^\infty B_{p\infty},$$

that is, $\Phi(B_2(r_2)) \subseteq B_2$. We can similarly prove that $\Phi'(B_2(r_2)) \subseteq B_2$, where $\Phi'(x) := u^*x$. Having in mind that B_2 is a weak* closed ideal of B we have

$$\begin{aligned} T(A_2) = h_2 \circ_{r_2} S_2(A_2) &\subseteq \frac{1}{2}\big(h_1 r_2^* S(A_2) + S(A_2) r_2^* h_2\big) \\ &= \frac{1}{2}\big(h_1 \Phi'\big(B_2(r_2)\big) + \Phi\big(B_2(r_2)\big) h_2\big) \subseteq \frac{1}{2}(h_1 B_2 + B_2 h_2) \subseteq B_2. \end{aligned}$$

It follows that $T^{-1}(B_2) \subseteq A_2$, and hence $T(A_2) = B_2$. Thus

$$T(A_1) = T\big(A_1^{\perp\perp}\big) = T\big(A_1^{\perp}\big)^{\perp} = T(A_2)^{\perp} = B_2^{\perp} = B_1.$$

Finally, since A_1 and B_1 are finite type I von Neumann algebras, Proposition 18 proves that $T|_{A_1} : A_1 \to B_1$ is continuous, which shows that T enjoys the same property. $\square$

Remark 20. Let A be a properly infinite or a (finite) type II_1 von Neumann algebra. Corollary 16 shows that every orthogonality preserving linear map from A into a C*-algebra is continuous. We shall present an example showing that a similar statement doesn't hold when A is replaced with a finite type I von Neumann algebra. In other words, the hypothesis of T being surjective cannot be removed in Theorem 19.

It is well know that a von Neumann algebra A is type I and finite if and only if A decomposes in the form

$$A = \bigoplus_{i \in I}^{\ell_\infty} C\big(\Omega_i, M_{m_i}(\mathbb{C})\big),$$

were the Ω_i's are hyperstonean compact Hausdorff spaces and (m_i) is a family of natural numbers (cf. [29, Theorem V.1.27]). In particular, every abelian von Neumann algebra is type I and finite.

Let K be an infinite (hyperstonean) compact set. By [16, Example in page 142], there exists a discontinuous orthogonality preserving linear map $\varphi : C(K) \to \mathbb{C}$. Let $T : C(K) \to C(K) \oplus^\infty \mathbb{C}$ be the linear mapping defined by $T(f) := (f, \varphi(f))$ $(f \in C(K))$. It is easy to check that T is discontinuous and biorthogonality preserving but not surjective.

Following [2] and [7], a linear map T between algebras A, B is called *separating* or *zero-product preserving* if $ab = 0$ implies $T(a)T(b) = 0$, for all a, b in A; it is called *biseparating* if $T^{-1} : B \to A$ exists and is also separating. J. Araujo and K. Jarosz conjectured in [2, Conjecture 1] that every biseparating map between C*-algebras is automatically continuous. We can now give a partial positive answer to this conjecture.

Let $T : A \to B$ be a symmetric linear mapping between two C*-algebras. Suppose that T is separating. Then for every a, b in A with $a \perp b$, we have $T(a)T(b)^* = T(a)T(b^*) = 0$, because T is separating. We can similarly prove that $T(b)^*T(a) = 0$, which shows that T is orthogonality preserving. The following result follows now as a consequence of Theorem 19.

Corollary 21. *Let $T : A \to B$ be a biseparating symmetric linear map between von Neumann algebras. Then T is continuous.*

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Automatic continuity of biorthogonality preservers between weakly compact JB^*-triples and atomic JBW^*-triples

by

María Burgos, Jorge J. Garcés and
Antonio M. Peralta (Granada)

Abstract. We prove that every biorthogonality preserving linear surjection from a weakly compact JB^*-triple containing no infinite-dimensional rank-one summands onto another JB^*-triple is automatically continuous. We also show that every biorthogonality preserving linear surjection between atomic JBW^*-triples containing no infinite-dimensional rank-one summands is automatically continuous. Consequently, two atomic JBW^*-triples containing no rank-one summands are isomorphic if and only if there exists a (not necessarily continuous) biorthogonality preserving linear surjection between them.

1. Introduction and preliminaries. Studies on the automatic continuity of linear surjections between C^*-algebras and von Neumann algebras preserving orthogonality relations in both directions constitute the latest variant of a problem initiated by W. Arendt in the early eighties.

We recall that two complex-valued continuous functions f and g are said to be *orthogonal* whenever they have disjoint supports. A mapping T between $C(K)$-spaces is called *orthogonality preserving* if it maps orthogonal functions to orthogonal functions. The main result established by Arendt states that every orthogonality preserving bounded linear mapping $T : C(K) \to C(K)$ is of the form

$$T(f)(t) = h(t)f(\varphi(t)) \quad (f \in C(K),\, t \in K),$$

where $h \in C(K)$ and $\varphi : K \to K$ is a mapping which is continuous on $\{t \in K : h(t) \neq 0\}$.

The hypothesis of T being continuous was relaxed by K. Jarosz in [24]. In fact, Jarosz obtained a complete description of all orthogonality preserving (not necessarily continuous) linear mappings between $C(K)$-spaces.

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A consequence of his description is that an orthogonality preserving linear surjection between $C(K)$-spaces is automatically continuous.

Two elements a, b in a general C^*-algebra A are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. When $a = a^*$ and $b = b^*$, we have $a \perp b$ if and only if $ab = 0$. A mapping T between two C^*-algebras A, B is called *orthogonality preserving* if $T(a) \perp T(b)$ for every $a \perp b$ in A. When $T(a) \perp T(b)$ in B if and only if $a \perp b$ in A, we say that T is *biorthogonality preserving*. Under continuity assumptions, orthogonality preserving bounded linear operators between C^*-algebras are completely described in [10, §4]. This last paper is a culmination of the studies developed by W. Arendt [2], K. Jarosz [24], M. Wolff [34], and N.-C. Wong [35], among others, on bounded orthogonality preserving linear maps between C^*-algebras.

C^*-algebras belong to a wider class of complex Banach spaces in which orthogonality also makes sense. We refer to the class of (complex) JB^*-triples (see §2 for definitions). Two elements a, b in a JB^*-triple E are said to be *orthogonal* (denoted by $a \perp b$) if $L(a, b) = 0$, where $L(a, b)$ is the linear operator in E given by $L(a, b)x = \{a, b, x\}$. A linear mapping $T : E \to F$ between two JB^*-triples is called *orthogonality preserving* if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping T is *biorthogonality preserving* whenever the equivalence $x \perp y \Leftrightarrow T(x) \perp T(y)$ holds for all x, y in E.

Most of the novelties introduced in [10] consist in studying orthogonality preserving bounded linear operators from a C^*-algebra or a JB^*-algebra to a JB^*-triple to take advantage of the techniques developed in JB^*-triple theory. These techniques were successfully applied in the subsequent paper [11] to obtain a description of such operators (see §2 for a detailed explanation).

Despite the vast literature on orthogonality preserving bounded linear operators between C^*-algebras and JB^*-triples, just a few papers have considered the problem of automatic continuity of biorthogonality preserving linear surjections between C^*-algebras. Besides Jarosz [24], mentioned above, M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong proved in [13, Theorem 4.2] that every zero products preserving linear bijection from a properly infinite von Neumann algebra into a unital ring is a ring homomorphism followed by left multiplication by the image of the identity. J. Araujo and K. Jarosz showed that every linear bijection between algebras $L(X)$, of continuous linear maps on a Banach space X, which preserves zero products in both directions is automatically continuous and a multiple of an algebra isomorphism [1]. These authors also conjectured that every linear bijection between two C^*-algebras preserving zero products in both directions is automatically continuous (see [1, Conjecture 1]).

The authors of this note proved in [12] that every biorthogonality preserving linear surjection between two compact C^*-algebras or between two von Neumann algebras is automatically continuous. One of the consequences of this result is a partial answer to [1, Conjecture 1]. Concretely, every surjective and symmetric linear mapping between von Neumann algebras (or compact C^*-algebras) which preserves zero products in both directions is continuous.

In this paper we study the problem of automatic continuity of biorthogonality preserving linear surjections between JB^*-triples, extending some of the results obtained in [12]. Section 2 contains the basic definitions and results used in the paper. Section 3 is devoted to the structure and properties of the (orthogonal) annihilator of a subset M in a JB^*-triple, focusing on the annihilators of single elements. In Section 4 we prove that every biorthogonality preserving linear surjection from a weakly compact JB^*-triple containing no infinite-dimensional rank-one summands to a JB^*-triple is automatically continuous. In Section 5 we show that two atomic JB^*-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them, a result which follows from the automatic continuity of every biorthogonality preserving linear surjection between atomic JB^*-triples containing no infinite-dimensional rank-one summands.

2. Notation and preliminaries. Given Banach spaces X and Y, $L(X,Y)$ will denote the space of all bounded linear mappings from X to Y. The symbol $L(X)$ will stand for the space $L(X,X)$. Throughout the paper the word "operator" will always mean bounded linear mapping. The dual space of a Banach space X is denoted by X^*.

JB^*-triples were introduced by W. Kaup in [26]. A JB^*-*triple* is a complex Banach space E together with a continuous triple product $\{\cdot,\cdot,\cdot\}$: $E\times E\times E\to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables, and satisfies:

(a) $L(a,b)L(x,y)=L(x,y)L(a,b)+L(L(a,b)x,y)-L(x,L(b,a)y)$, where $L(a,b)$ is the operator on E given by $L(a,b)x=\{a,b,x\}$;
(b) $L(a,a)$ is an hermitian operator with nonnegative spectrum;
(c) $\|L(a,a)\|=\|a\|^2$.

For each x in a JB^*-triple E, $Q(x)$ will stand for the conjugate linear operator on E defined by the assignment $y\mapsto Q(x)y=\{x,y,x\}$.

Every C^*-algebra is a JB^*-triple via the triple product given by

$$2\{x,y,z\}=xy^*z+zy^*x,$$

and every JB^*-algebra is a JB^*-triple under the triple product

$$\{x,y,z\}=(x\circ y^*)\circ z+(z\circ y^*)\circ x-(x\circ z)\circ y^*. \tag{2.1}$$

The so-called *Kaup–Banach–Stone theorem* for JB^*-triples states that a bounded linear surjection between JB^*-triples is an isometry if and only if it is a triple isomorphism (cf. [26, Proposition 5.5], [5, Corollary 3.4] or [18, Theorem 2.2]). It follows, among many other consequences, that when a JB^*-algebra is a JB^*-triple for a suitable triple product, then the latter coincides with the one defined in (2.1).

A JBW^*-*triple* is a JB^*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the triple product of a JBW^*-triple is separately weak* continuous [3]. The second dual of a JB^*-triple E is a JBW^*-triple with a product extending the product of E [15].

An element e in a JB^*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. Each tripotent e in E gives rise to the decomposition

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $i/2$-eigenspace of $L(e, e)$ (cf. [28, Theorem 25]). The natural projection of E onto $E_i(e)$ will be denoted by $P_i(e)$. This decomposition is termed the *Peirce decomposition* of E with respect to the tripotent e. The Peirce decomposition satisfies certain rules known as *Peirce arithmetic*:

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e)$$

if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The Peirce space $E_2(e)$ is a JB^*-algebra with product $x \circ_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

A tripotent e in E is called *complete* (resp., *unitary*) if $E_0(e) = 0$ (resp., $E_2(e) = E$). When $E_2(e) = \mathbb{C}e \neq \{0\}$, we say that e is *minimal*.

For each element x in a JB^*-triple E, we shall denote $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \{x, x, x^{[2n-1]}\}$ ($n \in \mathbb{N}$). The symbol E_x will stand for the JB^*-subtriple generated by x. It is known that E_x is JB^*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \|x\|]$ such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism Ψ from E_x onto $C_0(\Omega)$ satisfying $\Psi(x)(t) = t$ ($t \in \Omega$) (cf. [25, Corollary 4.8], [26, Corollary 1.15] and [20]). The set $\overline{\Omega} = \mathrm{Sp}(x)$ is called the *triple spectrum* of x. Note that $C_0(\mathrm{Sp}(x)) = C(\mathrm{Sp}(x))$ whenever $0 \notin \mathrm{Sp}(x)$.

Therefore, for each $x \in E$, there exists a unique element $y \in E_x$ such that $\{y, y, y\} = x$. The element y, denoted by $x^{[1/3]}$, is termed the *cubic root* of x. We can inductively define $x^{[1/3^n]} = (x^{[1/3^{n-1}]})^{[1/3]}$, $n \in \mathbb{N}$. The sequence $(x^{[1/3^n]})$ converges in the weak* topology of E^{**} to a tripotent denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ such that x is positive in the JBW^*-algebra $E_2^{**}(e)$ (cf. [16, Lemma 3.3]).

A subspace I of a JB^*-triple E is a *triple ideal* if $\{E,E,I\}+\{E,I,E\}\subseteq I$. By Proposition 1.3 in [7], I is a triple ideal if and only if $\{E,E,I\}\subseteq I$. We shall say that I is an *inner ideal* of E if $\{I,E,I\}\subseteq I$. Given an x in E, let $E(x)$ denote the norm closed inner ideal of E generated by x. It is known that $E(x)$ coincides with the norm closure of the set $Q(x)(E)$. Moreover $E(x)$ is a JB^*-subalgebra of $E_2^{**}(r(x))$ and contains x as a positive element (cf. [8]). Every triple ideal is, in particular, an inner ideal.

We recall that two elements a,b in a JB^*-triple E are said to be *orthogonal* (written $a\perp b$) if $L(a,b)=0$. Lemma 1 in [10] shows that $a\perp b$ if and only if one of the following nine statements holds:

$$\begin{aligned}
&\{a,a,b\}=0; \quad a\perp r(b); \quad r(a)\perp r(b);\\
&E_2^{**}(r(a))\perp E_2^{**}(r(b)); \quad r(a)\in E_0^{**}(r(b)); \quad a\in E_0^{**}(r(b));\\
&b\in E_0^{**}(r(a)); \quad E_a\perp E_b; \quad \{b,b,a\}=0.
\end{aligned}\tag{2.2}$$

The Jordan identity and the above reformulations ensure that

$$a\perp\{x,y,z\}\quad\text{whenever}\quad a\perp x,y,z.\tag{2.3}$$

An important class of JB^*-triples is given by the Cartan factors. A JBW^*-triple E is called a *factor* if it contains no proper weak* closed ideals. The *Cartan factors* are precisely the JBW^*-triple factors containing a minimal tripotent [27]. These can be classified in six different types (see [21] or [27]).

A Cartan factor *of type 1*, denoted by $I_{n,m}$, is a JB^*-triple of the form $L(H,H')$, where $L(H,H')$ denotes the space of bounded linear operators between two complex Hilbert spaces H and H' of dimensions n,m respectively, with the triple product defined by $\{x,y,z\}=\frac{1}{2}(xy^*z+zy^*x)$.

We recall that given a conjugation j on a complex Hilbert space H, we can define the linear involution $x\mapsto x^t:=jx^*j$ on $L(H)$. A Cartan factor *of type 2* (respectively, *type 3*), denoted by II_n (respectively, III_n), is the subtriple of $L(H)$ formed by the t-skew-symmetric (respectively, t-symmetric) operators, where H is an n-dimensional complex Hilbert space. Moreover, II_n and III_n are, up to isomorphism, independent of the conjugation j on H.

A Cartan factor *of type 4*, IV_n (also called a *complex spin factor*), is an n-dimensional complex Hilbert space provided with a conjugation $x\mapsto\overline{x}$, where the triple product and norm are given by

$$\{x,y,z\}=(x|y)z+(z|y)x-(x|\overline{z})\overline{y}\tag{2.4}$$

and $\|x\|^2=(x|x)+\sqrt{(x|x)^2-|(x|\overline{x})|^2}$, respectively.

The Cartan factor *of type 6* is the 27-dimensional exceptional JB^*-algebra $VI=H_3(\mathbb{O}^{\mathbb{C}})$ of all symmetric 3×3 matrices with entries in the complex octonions $\mathbb{O}^{\mathbb{C}}$, while the Cartan factor *of type 5*, $V=M_{1,2}(\mathbb{O}^{\mathbb{C}})$, is the subtriple of $H_3(\mathbb{O}^{\mathbb{C}})$ consisting of all 1×2 matrices with entries in $\mathbb{O}^{\mathbb{C}}$.

Remark 2.1. Let E be a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. It is not hard to check (and part of the folklore of JB^*-triple theory) that an element w in E is a minimal tripotent if and only if $(w|\overline{w}) = 0$ and $(w|w) = 1/2$. For every minimal tripotent w in E we have $E_2(w) = \mathbb{C}w$, $E_0(w) = \mathbb{C}\overline{w}$ and $E_1(w) = \{x \in E : (x|w) = (x|\overline{w}) = 0\}$. Therefore, every minimal tripotent $w_2 \in E$ satisfying $w \perp w_2$ can be written in the form $w_2 = \lambda\overline{w}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

3. Biorthogonality preservers. Let M be a subset of a JB^*-triple E. We write $M_E^{\perp}$ for the *(orthogonal) annihilator* of M defined by

$$M_E^{\perp} := \{y \in E : y \perp x, \forall x \in M\}.$$

When no confusion can arise, we shall write $M^{\perp}$ instead of $M_E^{\perp}$.

The next result summarises some basic properties of the annihilator. The reader is referred to [17, Lemma 3.2] for a detailed proof.

Lemma 3.1. *Let M a nonempty subset of a JB^*-triple E.*

(a) *$M^{\perp}$ is a norm closed inner ideal of E.*
(b) *$M \cap M^{\perp} = \{0\}$.*
(c) *$M \subseteq M^{\perp\perp}$.*
(d) *If $B \subseteq C$ then $C^{\perp} \subseteq B^{\perp}$.*
(e) *$M^{\perp}$ is weak* closed whenever E is a JBW^*-triple.* ∎

As illustration of the main identity (axiom (a) in the definition of a JB^*-triple) we shall prove statement (a). For a, a' in $M^{\perp}$, b in M, and c, d in E we have $\{c, a, \{d, a', b\}\} = \{\{c, a, d\}, a', b\} - \{d, \{a, c, a'\}, b\} + \{d, a', \{c, a, b\}\}$, which shows that $\{a, c, a'\} \perp b$.

Let e be a tripotent in a JB^*-triple E. Clearly, $\{e\} \subseteq E_2(e)$. Therefore, by Peirce arithmetic and Lemma 3.1,

$$E_2(e)^{\perp} \subseteq \{e\}^{\perp} = E_0(e) \subseteq E_2(e)^{\perp},$$

and hence

$$E_2(e)^{\perp} = \{e\}^{\perp} = E_0(e). \tag{3.1}$$

The next lemma describes the annihilator of an element in an arbitrary JB^*-triple. Its proof follows directly from the reformulations of orthogonality in (2.2) (see also [10, Lemma 1]).

Lemma 3.2. *Let x be an element in a JB^*-triple E. Then*

$$\{x\}_E^{\perp} = E_0^{**}(r(x)) \cap E.$$

Moreover, when E is a JBW^*-triple we have

$$\{x\}_E^{\perp} = E_0(r(x)). \ ∎$$

PROPOSITION 3.3. *Let e be a tripotent in a JB^*-triple E. Then*

$$E_2(e) \oplus E_1(e) \supseteq \{e\}_E^{\perp\perp} = E_0(e)^\perp \supseteq E_2(e).$$

Proof. It follows from (3.1) that $\{e\}^{\perp\perp} = \{e\}_E^{\perp\perp} = (E_0(e))^\perp \supseteq E_2(e)$. Now select $x \in (E_0(e))^\perp$. For each $i \in \{0,1,2\}$ we write $x_i = P_i(e)(x)$, where $P_i(e)$ denotes the Peirce i-projection with respect to e. Since $x \in (E_0(e))^\perp$, x must be orthogonal to x_0 and so $\{x_0, x_0, x\} = 0$. This equality, together with Peirce arithmetic, shows that $\{x_0, x_0, x_0\} + \{x_0, x_0, x_1\} = 0$, which implies that $\|x_0\|^3 = \|\{x_0, x_0, x_0\}\| = 0$. ∎

REMARK 3.4. For a tripotent e in a JB^*-triple E, the equality $\{e\}_E^{\perp\perp} = E_0(e)^\perp = E_2(e)$ does not hold in general. Let H_1 and H_2 be two infinite-dimensional complex Hilbert spaces and let p be a minimal projection in $L(H_1)$. We define E as the orthogonal sum $pL(H_1) \oplus^\infty L(H_2)$. In this example $\{p\}_E^\perp = L(H_2)$ and $\{p\}_E^{\perp\perp} = pL(H_1) \neq \mathbb{C}p = E_2(p)$.

However, if E is a Cartan factor and e is a noncomplete tripotent in E, then the equality $\{e\}^{\perp\perp} = E_0(e)^\perp = E_2(e)$ always holds (cf. Lemma 5.6 in [27]).

COROLLARY 3.5. *Let x be an element in a JB^*-triple E. Then*

$$E(x) \subseteq E_2^{**}(r(x)) \cap E \subseteq \{x\}_E^{\perp\perp}.$$

Proof. Clearly, $E(x) = \overline{Q(x)(E)} \subseteq E_2^{**}(r(x)) \cap E$. Pick y in $E_2^{**}(r(x)) \cap E$. Then $y \in E_2^{**}(r(x)) \subseteq \{x\}_{E^{**}}^{\perp\perp}$. Since $\{x\}_E^\perp \subset \{x\}_{E^{**}}^\perp$, we conclude that $y \in \{x\}_{E^{**}}^{\perp\perp} \cap E \subseteq (\{x\}_E^\perp)_{E^{**}}^\perp \cap E = \{x\}_E^{\perp\perp}$. ∎

In the setting of C^*-algebras the following conditions describing the first and second annihilator of a projection were established in [12, Lemma 3].

LEMMA 3.6. *Let p be a projection in a (not necessarily unital) C^*-algebra A. The following assertions hold:*

(a) $\{p\}_A^\perp = (1-p)A(1-p)$, *where* 1 *denotes the unit of* A^{**};
(b) $\{p\}_A^{\perp\perp} = pAp$. ∎

Let x be an element in a JB^*-triple E. We say that x is *weakly compact* (respectively, *compact*) if the operator $Q(x) : E \to E$ is weakly compact (respectively, compact). A JB^*-triple is *weakly compact* (respectively, *compact*) if every element in E is weakly compact (respectively, compact).

Let E be a JB^*-triple. If we denote by $K(E)$ the Banach subspace of E generated by its minimal tripotents, then $K(E)$ is a (norm closed) triple ideal of E and it coincides with the set of weakly compact elements of E (see Proposition 4.7 in [7]). For a Cartan factor C we define the *elementary JB^*-triple* of the corresponding type to be $K(C)$. Consequently, the elementary JB^*-triples K_i $(i = 1, \ldots, 6)$ are defined as follows: $K_1 = K(H, H')$ (the compact operators between complex Hilbert spaces H and H'); $K_i = C_i \cap K(H)$ for $i = 2, 3$, and $K_i = C_i$ for $i = 4, 5, 6$.

It follows from [7, Lemma 3.3 and Theorem 3.4] that a JB^*-triple E is weakly compact if and only if one of the following statement holds:

(a) $K(E^{**}) = K(E)$.
(b) $K(E) = E$.
(c) E is a c_0-sum of elementary JB^*-triples.

Let E be a JB^*-triple. A subset $S \subseteq E$ is said to be *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $\operatorname{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E (and will be denoted by $r(E)$).

For every orthogonal family $(e_i)_{i \in I}$ of minimal tripotents in a JBW^*-triple E the weak* convergent sum $e := \sum_i e_i$ is a tripotent, and we call $(e_i)_{i \in I}$ a *frame* in E if e is a *maximal tripotent* in E (i.e., e is a complete tripotent and $\dim(E_1(e)) \leq \dim(E_1(\widetilde{e}))$ for every complete tripotent $\widetilde{e}$ in E). Every frame is a maximal orthogonal family of minimal tripotents; the converse is not true in general (see [4, §3] for more details).

Proposition 3.7. *Let e be a minimal tripotent in a JB^*-triple E. Then $\{e\}_E^{\perp\perp}$ is a rank-one norm closed inner ideal of E.*

Proof. Let F denote $\{e\}_E^{\perp\perp}$. Since e is a minimal tripotent (i.e. $E_2(e) = \mathbb{C}e$), the set of states on $E_2(e)$, $\{\varphi \in E^* : \varphi(e) = 1 = \|\varphi\|\}$, reduces to one point φ_0 in E^*. Proposition 2.4 and Corollary 2.5 in [9] imply that the norm of E restricted to $E_1(e)$ is equivalent to a Hilbertian norm. More precisely, in the terminology of [9], the norm $\|\cdot\|_e$ coincides with the Hilbertian norm $\|\cdot\|_{\varphi_0}$ and is equivalent to the norm of $E_1(e)$.

Proposition 3.3 guarantees that F is a norm closed subspace of $E_2(e) \oplus E_1(e) = \mathbb{C}e \oplus E_1(e)$, and hence F is isomorphic to a Hilbert space.

We deduce, by Proposition 4.5(iii) in [7] (and its proof), that F is a finite orthogonal sum of Cartan factors $C_1, \ldots, C_m$ which are finite-dimensional, or infinite-dimensional spin factors, or of the form $L(H, H')$ for suitable complex Hilbert spaces H and H' with $\dim(H') < \infty$. Since F is an inner ideal of E (and hence a JB^*-subtriple of E) and e is a minimal tripotent in E, we can easily check that e is a minimal tripotent in $F = \bigoplus_{j=1,\ldots,m}^{\ell_\infty} C_j$. If we write $e = e_1 + \cdots + e_m$, where each e_j is a tripotent in C_j and $e_j \perp e_k$ whenever $j \neq k$, then since $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_1 \subseteq F_2(e) = \mathbb{C}e$, we deduce that there exists a unique $j_0 \in \{1, \ldots, m\}$ satisfying $e_j = 0$ for all $j \neq j_0$ and $e = e_{j_0} \in C_{j_0}$.

For each $j \neq j_0$, we have $C_j \subseteq \{e\}_E^{\perp}$, and hence

$$\bigoplus_{j=1,\ldots,m}^{\ell_\infty} C_j = F = \{e\}^{\perp\perp} \subseteq C_j^{\perp}.$$

This implies that $C_j \perp C_j$ (or equivalently $C_j = 0$) for every $j \neq j_0$. We consequently have $F = \{e\}_E^{\perp\perp} = C_{j_0}$.

Finally, if $r(F) \geq 2$, then we deduce, via Proposition 5.8 in [27], that there exist minimal tripotents $e_2, \ldots, e_r$ in F such that $e, e_2, \ldots, e_r$ is a frame in F. For each $i \in \{2, \ldots, r\}$, e_i is orthogonal to e and lies in $F = \{e\}_E^{\perp\perp}$, which is impossible. ■

Let $T : E \to F$ be a linear map between two JB^*-triples. We shall say that T is *orthogonality preserving* if $T(x) \perp T(y)$ whenever $x \perp y$. The mapping T is said to be *biorthogonality preserving* whenever the equivalence

$$x \perp y \;\Leftrightarrow\; T(x) \perp T(y)$$

holds for all x, y in E.

It can be easily seen that every biorthogonality preserving linear mapping $T : E \to F$ between JB^*-triples is injective. Indeed, for each $x \in E$, the condition $T(x) = 0$ implies that $T(x) \perp T(x)$, and hence $x \perp x$, which gives $x = 0$.

Orthogonality preserving bounded linear maps from a JB^*-algebra to a JB^*-triple were completely described in [11].

Before stating the result, let us recall some basic definitions. Two elements a and b in a JB^*-algebra J are said to *operator commute* in J if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in J. Self-adjoint elements a and b in J generate a JB^*-subalgebra that can be realised as a JC^*-subalgebra of some $B(H)$ [36], and, in this realisation, a and b commute in the usual sense whenever they operator commute in J [33, Proposition 1]. Similarly, two self-adjoint elements a and b in J operator commute if and only if $a^2 \circ b = \{a, a, b\} = \{a, b, a\}$ (i.e., $a^2 \circ b = 2(a \circ b) \circ a - a^2 \circ b$). If $b \in J$ we use $\{b\}'$ to denote the set of elements in J that operator commute with b. We shall write $Z(J) := J'$ for the center of J (this agrees with the usual notation in von Neumann algebras).

THEOREM 3.8 ([11, Theorem 4.1]). *Let $T : J \to E$ be a bounded linear mapping from a JB^*-algebra to a JB^*-triple. For $h = T^{**}(1)$ and $r = r(h)$ the following assertions are equivalent:*

(a) *T is orthogonality preserving.*
(b) *There exists a unique Jordan *-homomorphism $S : J \to E_2^{**}(r)$ such that $S^{**}(1) = r$, $S(J)$ and h operator commute, and $T(z) = h \circ_r S(z)$ for all $z \in J$.*
(c) *T preserves zero triple products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$.* ■

The above characterisation proves that the bitranspose of an orthogonality preserving bounded linear mapping from a JB^*-algebra onto a JB^*-triple is also orthogonality preserving.

The following theorem was essentially proved in [11]. We include here a sketch of proof for completeness.

THEOREM 3.9. *Let $T : J \to E$ be a surjective linear operator from a JBW^*-algebra onto a JBW^*-triple and let h denote $T(1)$. Then T is biorthogonality preserving if and only if $r(h)$ is a unitary tripotent in E, h is an invertible element in the JB^*-algebra $E = E_2(r(h))$, and there exists a Jordan *-isomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ_{r(h)} S$. Further, if J is a factor (i.e. $Z(J) = \mathbb{C}1$) then T is a scalar multiple of a triple isomorphism.*

Proof. The sufficiency is clear. We shall prove the necessity. To this end let $T : J \to E$ be a surjective linear operator from a JBW^*-algebra onto a JBW^*-triple and let $h = T(1) \in E$. We have already seen that every biorthogonality preserving linear mapping between JB^*-triples is injective. Therefore T is a linear bijection.

From Corollary 4.1(b) in [11] and its proof, we deduce that

$$T(J_{\mathrm{sa}}) \subseteq E_2(r(h))_{\mathrm{sa}}, \quad \text{and hence} \quad E = T(J) \subseteq E_2(r(h)) \subseteq E.$$

This implies that $E = E_2(r(h))$, which ensures that $r(h)$ is a unitary tripotent in E. Since the range tripotent of h, $r(h)$, is the unit of $E_2(r(h))$, and h is a positive element in the JBW^*-algebra $E_2(r(h))$, we can easily check that h is invertible in $E_2(r(h))$. Furthermore, $h^{1/2}$ is invertible in $E_2(r(h))$ with inverse $h^{-1/2}$.

The proof of [11, Theorem 4.1] can be literally applied here to show the existence of a Jordan *-homomorphism $S : J \to E = E_2(r(h))$ such that $S(J) \subseteq \{h\}'$ and $T = h \circ_{r(h)} S$. Since, for each $x \in J$, h and $S(x)$ operator commute and $h^{1/2}$ lies in the JB^*-subalgebra of $E_2(r(h))$ generated by h, we can easily check that $S(x)$ and $h^{1/2}$ operator commute. Thus,

$$T = h \circ_{r(h)} S = U_{h^{1/2}} S,$$

where $U_{h^{1/2}} : E_2(r(h)) \to E_2(r(h))$ is the linear mapping defined by

$$U_{h^{1/2}}(x) = 2(h^{1/2} \circ_{r(h)} x) \circ_{r(h)} h^{1/2} - (h^{1/2} \circ_{r(h)} h^{1/2}) \circ_{r(h)} x.$$

It is well known that $h^{1/2}$ is invertible if and only if $U_{h^{1/2}}$ is an invertible operator and, in this case, $U_{h^{1/2}}^{-1} = U_{h^{-1/2}}$ (cf. [22, Lemma 3.2.10]). Therefore, $S = U_{h^{-1/2}} T$. It follows from the bijectivity of T that S is a Jordan *-isomorphism.

Finally, when $Z(J) = \mathbb{C}1$, the center of $E_2(r(h))$ also reduces to $\mathbb{C}r(h)$, and since h is an invertible element in the center of $E_2(r(h))$, we deduce that T is a scalar multiple of a triple isomorphism. ∎

PROPOSITION 3.10. *Let E_1, E_2 and F be three JB^*-triples (respectively, JBW^*-triples). Let $T : E_1 \oplus^\infty E_2 \to F$ be a biorthogonality preserving linear surjection. Then $T(E_1)$ and $T(E_2)$ are norm closed (respectively, weak* closed) inner ideals of F, $B = T(A_1) \oplus^\infty T(A_2)$, and for $j = 1, 2$, $T|_{A_j} : A_j \to T(A_j)$ is a biorthogonality preserving linear surjection.*

Proof. Fix $j \in \{1, 2\}$. Since $E_j = E_j^{\perp\perp}$ and T is a biorthogonality preserving linear surjection, we deduce that $T(E_j) = T(E_j^{\perp\perp}) = T(E_j)^{\perp\perp}$. Lemma 3.1 guarantees that $T(E_j)$ is a norm closed inner ideal of F (respectively, a weak* closed inner ideal of F whenever E_1, E_2 and F are JBW^*-triples). The rest of the assertion follows from Lemma 3.1 and the fact that F coincides with the orthogonal sum of $T(E_1)$ and $T(E_2)$. ∎

4. Biorthogonality preservers between weakly compact JB^*-triples. The following theorem generalises [12, Theorem 5] by proving that biorthogonality preserving linear surjections between JB^*-triples send minimal tripotents to scalar multiples of minimal tripotents.

THEOREM 4.1. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples and let e be a minimal tripotent in E. Then $\|T(e)\|^{-1}T(e) = f_e$ is a minimal tripotent in F. Further, $T(E_2(e)) = F_2(f_e)$ and $T(E_0(e)) = F_0(f_e)$.*

Proof. Since T is a biorthogonality preserving surjection, the equality

$$T(S_E^\perp) = T(S)_F^\perp$$

holds for every subset S of E. Lemma 3.1 ensures that for each minimal tripotent e in E, $\{T(e)\}_F^{\perp\perp} = T(\{e\}_E^{\perp\perp})$ is a norm closed inner ideal in F. By Proposition 3.7, $\{e\}_E^{\perp\perp}$ is a rank-one JB^*-triple, and hence $\{T(e)\}_F^{\perp\perp}$ cannot contain two nonzero orthogonal elements. Thus, $\{T(e)\}_F^{\perp\perp}$ is a rank-one JB^*-triple.

The arguments given in the proof of Proposition 3.7 above (see also Proposition 4.5.(iii) in [7] and its proof or [4, §3]) show that the inner ideal $\{T(e)\}_F^{\perp\perp}$ is a rank-one Cartan factor, and hence a type 1 Cartan factor of the form $L(H, \mathbb{C})$, where H is a complex Hilbert space, or a type 2 Cartan factor II_3 (it is known that II_3 is a JB^*-triple isomorphic to a 3-dimensional complex Hilbert space). This implies that $\|T(e)\|^{-1}\, T(e) = f_e$ is a minimal tripotent in F and $T(e) = \lambda_e f_e$ for a suitable $\lambda_e \in \mathbb{C} \setminus \{0\}$.

The equality $T(E_2(e)) = F_2(f_e)$ has been proved. Concerning the Peirce zero subspace we have

$$T(E_0(e)) = T(E_2(e)_E^\perp) = T(E_2(e))_F^\perp = F_2(f_e)_F^\perp = F_0(f_e). \ ∎$$

Let H and H' be complex Hilbert spaces. Given $k \in H'$ and $h \in H$, we define $k \otimes h$ in $L(H, H')$ by $k \otimes h(\xi) := (\xi|h)k$. Then every minimal tripotent in $L(H,H')$ can be written in the form $k\otimes h$, where h and k are norm-one elements in H and H', respectively. It can be easily seen that two minimal tripotents $k_1\otimes h_1$ and $k_2\otimes h_2$ are orthogonal if and only if $h_1\perp h_2$ and $k_1\perp k_2$.

THEOREM 4.2. *Let $T:E\to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where E is a type $I_{n,m}$ Cartan factor with $n,m\geq 2$. Then there exists a positive real number λ such that $\|T(e)\| = \lambda$ for every minimal tripotent e in E.*

Proof. Let H,H' be complex Hilbert spaces such that $E=L(H,H')$. Let $e_1:=k_1\otimes h_1$ and $e_2:=k_2\otimes h_2$ be two minimal tripotents in E. We write $H_1=\operatorname{span}(\{h_1,h_2\})$ and $H_1'=\operatorname{span}(\{k_1,k_2\})$. The tripotents $k_1\otimes h_1$ and $k_2\otimes h_2$ can be identified with elements in $L(H_1,H_1')$. By Theorem 4.1, $T(e_1)=\alpha_1 f_1$ and $T(e_2)=\alpha_2 f_2$, where f_1 and f_2 are two minimal tripotents in F.

If $\dim(H_1)=\dim(H_1')=2$, then the norm closed inner ideal E_{e_1,e_2} of E generated by e_1 and e_2 identifies with $L(H_1,H_1')$, which is JB^*-isomorphic to $M_2(\mathbb{C})$ and coincides with the inner ideal generated by the orthogonal minimal tripotents $g_1=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $g_2=\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, where g_1+g_2 is the unit element in $E_{e_1,e_2}\cong M_2(\mathbb{C})$.

By Theorem 4.1, $w_1:=\frac{1}{\|T(g_1)\|}T(g_1)$ and $w_2:=\frac{1}{\|T(g_2)\|}T(g_2)$ are orthogonal minimal tripotents in F. The element $w=w_1+w_2$ is a rank-2 tripotent in F and coincides with the range tripotent of the element $h=T(g_1+g_2)=\|T(g_1)\|w_1+\|T(g_2)\|w_2$. By Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(E_{e_1,e_2})\subseteq F_2(w)$. It is not hard to see that h is invertible in $F_2(w)$ with inverse $h^{-1}=\frac{1}{\|T(g_1)\|}w_1+\frac{1}{\|T(g_2)\|}w_2$.

The inner ideal E_{e_1,e_2} is finite-dimensional, $T(E_{e_1,e_2})$ is norm closed and $T|_{E_{e_1,e_2}}:E_{e_1,e_2}\to F$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan *-homomorphism $S:E_{e_1,e_2}\cong M_2(\mathbb{C})\to F_2(w)$ such that $S(g_1+g_2)=w$, $S(E_{e_1,e_2})$ and h operator commute and

$$T(z)=h\circ_w S(z)\quad\text{for all } z\in E_{e_1,e_2}. \tag{4.1}$$

It follows from the operator commutativity of h^{-1} and $S(E_{e_1,e_2})$ that $S(z)=h^{-1}\circ_w T(z)$ for all $z\in E_{e_1,e_2}$. The injectivity of T implies that S is a Jordan *-monomorphism.

Lemma 2.7 in [19] shows that $F_2(w)=F_2(w_1+w_2)$ coincides with $\mathbb{C}\oplus^{\ell_\infty}\mathbb{C}$ or with a spin factor. Since $4=\dim(T(E_{e_1,e_2}))\leq\dim(F_2(w))$, we deduce that $F_2(w)$ is a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x\mapsto\overline{x}$. From Remark 2.1, we may assume, without loss of generality, that $(w_1|w_1)=1/2$, $(w_1|\overline{w}_1)=0$, and $w_2=\overline{w}_1$.

Now, we take $g_3 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $g_4 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in E_{e_1,e_2}. The elements $w_3 := S(g_3)$ and $w_4 := S(g_4)$ are orthogonal minimal tripotents in $F_2(w)$ with $\{w_i, w_i, w_j\} = \frac{1}{2}w_j$ for every $(i,j),(j,i) \in \{1,2\} \times \{3,4\}$. Applying again Remark 2.1, we may assume that $(w_3|w_3) = 1/2$, $(w_3|\overline{w}_3) = 0$, $w_4 = \overline{w}_3$, and $(w_3|w_1) = (w_3|w_2) = 0$. Applying the definition of the triple product in a spin factor given in (2.4) we can check that $(w_1, w_3, w_2 = \overline{w}_1, w_4 = \overline{w}_3)$ are four minimal tripotents in $F_2(w)$ with $w_1 \perp w_2$, $w_3 \perp w_4$, $\{w_i, w_i, w_j\} = \frac{1}{2}w_j$ for every $(i,j),(j,i) \in \{1,2\}\times\{3,4\}$, $\{w_1, w_3, w_2\} = -\frac{1}{2}w_4$, $\{w_3, w_2, -w_4\} = \frac{1}{2}w_1$, $\{w_2, -w_4, w_1\} = \frac{1}{2}w_3$, and $\{-w_4, w_1, w_3\} = \frac{1}{2}w_2$. Thus, denoting by M the JB^*-subtriple of $F_2(w)$ generated by w_1, w_3, w_2, and w_4, we have shown that M is a JB^*-triple isomorphic to $M_2(\mathbb{C})$.

Combining (4.1) and (2.4) we get

$$T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_3,$$
$$T(g_4) = h \circ_w S(g_4) = \{h, w, w_4\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_4.$$

Since $T(g_1) = \|T(g_1)\|w_1$, $T(g_2) = \|T(g_2)\|w_2$, and E_{e_1,e_2} is linearly generated by g_1, g_2, g_3 and g_4, we deduce that $T(E_{e_1,e_2}) \subseteq M$ with $4 = \dim(T(E_{e_1,e_2})) \le \dim(M) = 4$. Thus, $T(E_{e_1,e_2}) = M$ is a JB^*-subtriple of F.

The mapping $T|_{E_{e_1,e_2}} : E_{e_1,e_2} \cong M_2(\mathbb{C}) \to T(E_{e_1,e_2})$ is a continuous biorthogonality preserving linear bijection. Theorem 3.9 implies that $T|_{E_{e_1,e_2}}$ is a (nonzero) scalar multiple of a triple isomorphism, and hence $\|T(e_1)\| = \|T(e_2)\|$.

If $\dim(H_1') = 1$, then $L(H_1, H_1')$ is a rank-one JB^*-triple. Since $n, m \ge 2$, we can find a minimal tripotent e in E such that the norm closed inner ideals of E generated by $\{e, e_1\}$ and $\{e, e_2\}$ both coincide with $M_2(\mathbb{C})$. The arguments in the above paragraph show that $\|T(e_1)\| = \|T(e)\| = \|T(e_2)\|$.

Finally, the case $\dim(H_1) = 1$ follows from the same arguments. ∎

Remark 4.3. Given a sequence $(\mu_n) \subset c_0$ and a bounded sequence (x_n) in a Banach space X, the series $\sum_k \mu_k x_k$ need not be, in general, convergent in X. However, when (x_n) is a bounded sequence of mutually orthogonal elements in a JB^*-triple E, the equality

$$\Big\| \sum_{k=1}^{n} \mu_k x_k - \sum_{k=1}^{m} \mu_k x_k \Big\| = \max\{|\mu_{n+1}|, \ldots, |\mu_m|\} \sup\{\|x_n\|\}$$

holds for every $n < m$ in $\mathbb{N}$. It follows that $(\sum_{k=1}^{n} \mu_k x_k)$ is a Cauchy sequence and hence converges in E.

The following three results generalise [12, Lemmas 8, 9 and Proposition 10] to the setting of JB^*-triples.

LEMMA 4.4. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples and let (e_n) be a sequence of mutually orthogonal minimal tripotents in E. Then there exist positive constants $m \le M$ satisfying $m \le \|T(e_n)\| \le M$ for all $n \in \mathbb{N}$.*

Proof. We deduce from Theorem 4.1 that, for each natural n, there exist a minimal tripotent f_n and a scalar $\lambda_n \in \mathbb{C} \setminus \{0\}$ such that $T(e_n) = \lambda_n f_n$, where $\|T(e_n)\| = \lambda_n$. Note that T being biorthogonality preserving implies (f_n) is a sequence of mutually orthogonal minimal tripotents in F.

Let (μ_n) be any sequence in c_0. Since the e_n's are mutually orthogonal the series $\sum_{k\ge1} \mu_k e_k$ converges to an element in E (cf. Remark 4.3). For each natural n, $\sum_{k\ge1} \mu_k e_k$ decomposes as the orthogonal sum of $\mu_n e_n$ and $\sum_{k\ne n} \mu_k e_k$, therefore

$$T\Big(\sum_{k\ge1} \mu_k e_k\Big) = \mu_n \lambda_n f_n + T\Big(\sum_{k\ne n} \mu_k e_k\Big)$$

with $\mu_n\lambda_n f_n \perp T\big(\sum_{k\ne n}^{\infty} \mu_k e_k\big)$, which in particular implies

$$\Big\|T\Big(\sum_{k\ge1} \mu_k e_k\Big)\Big\| = \max\Big\{|\mu_n|\,|\lambda_n|, \Big\|T\Big(\sum_{k\ne n} \mu_k e_k\Big)\Big\|\Big\} \ge |\mu_n|\,|\lambda_n|.$$

This establishes that, for each (μ_n) in c_0, $(\mu_n\lambda_n)$ is a bounded sequence, which in particular implies that (λ_n) is bounded.

Finally, since T is a biorthogonality preserving linear surjection and $T^{-1}(f_n) = \lambda_n^{-1} e_n$, we can similarly show that (λ_n^{-1}) is also bounded. ∎

LEMMA 4.5. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, (μ_n) a sequence in c_0, and (e_n) a sequence of mutually orthogonal minimal tripotents in E. Then the sequence $(T(\sum_{k\ge n} \mu_k e_k))_n$ is well defined and converges in norm to zero.*

Proof. From Theorem 4.1 and Lemma 4.4 it follows that $(T(e_n))$ is a bounded sequence of mutually orthogonal elements in F. Let M denote a bound of the above sequence. For each natural n, Remark 4.3 ensures that the series $\sum_{k\ge n} \mu_k e_k$ converges.

Define $y_n := T(\sum_{k\ge n} \mu_k e_k)$. We claim that (y_n) is a Cauchy sequence in F. Indeed, given $n < m$ in $\mathbb{N}$, we have

$$\begin{aligned} \|y_n - y_m\| &= \Big\|T\Big(\sum_{k\ge n}^{m-1} \mu_k e_k\Big)\Big\| = \Big\|\sum_{k\ge n}^{m-1} \mu_k T(e_k)\Big\| \\ &\le M \max\{|\mu_n|, \dots, |\mu_{m-1}|\}, \end{aligned} \tag{4.2}$$

where in the last inequality we have used the fact that $(T(e_n))$ is a sequence of mutually orthogonal elements. Consequently, (y_n) converges in norm to some element y_0 in F. Let z_0 denote $T^{-1}(y_0)$.

Fix a natural m. By hypothesis, for each $n > m$, e_m is orthogonal to $\sum_{k\geq n}\mu_k e_k$. This implies that $T(e_m) \perp y_n$ for every $n > m$, which in particular implies $\{T(e_m), T(e_m), y_n\} = 0$ for every $n > m$. Letting n tend to ∞ we have $\{T(e_m), T(e_m), y_0\} = 0$. This shows that $y_0 = T(z_0)$ is orthogonal to $T(e_m)$, and hence $e_m \perp z_0$. Since m was arbitrary, we deduce that z_0 is orthogonal to $\sum_{k\geq n}\mu_k e_k$ for every n. Therefore, $(y_n) \subset \{y_0\}^{\perp}$, and hence y_0 belongs to the norm closure of $\{y_0\}^{\perp}$, which implies $y_0 = 0$. ∎

PROPOSITION 4.6. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where E is weakly compact. Then T is continuous if and only if the set $\mathcal{T} := \{\|T(e)\| : e$ a minimal tripotent in $E\}$ is bounded. Moreover, in that case $\|T\| = \sup(\mathcal{T})$.*

Proof. The necessity being obvious, suppose that

$$M = \sup\{\|T(e)\| : e \text{ a minimal tripotent in } E\} < \infty.$$

Since E is weakly compact, each nonzero element x of E can be written as a norm convergent (possibly finite) sum $x = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents of E, and $\|x\| = \sup\{|\lambda_n| : n \geq 1\}$ (cf. Remark 4.6 in [7]). If the series $x = \sum_n \lambda_n u_n$ is finite then

$$\|T(x)\| = \Big\|\sum_{n=1}^{m} \lambda_n T(u_n)\Big\| \overset{(*)}{=} \max\{\|\lambda_n T(u_n)\| : n = 1, \ldots, m\} \leq M\|x\|,$$

where at $(*)$ we apply the fact that $(T(u_n))$ is a finite set of mutually orthogonal tripotents in F. When the series $x = \sum_n \lambda_n u_n$ is infinite we may assume that $(\lambda_n) \in c_0$.

It follows from Lemma 4.5 that the sequence $(T(\sum_{k\geq n}\lambda_k u_k))_n$ is well defined and converges in norm to zero. We can find a natural m such that $\|T(\sum_{k\geq m}\lambda_k u_k)\| < M\|x\|$. Since the elements $\lambda_1 u_1, \ldots, \lambda_{m-1}u_{m-1}$, $\sum_{k\geq m}\lambda_k u_k$ are mutually orthogonal, we have

$$\begin{aligned}\|T(x)\| &= \max\Big\{\|T(\lambda_1 u_1)\|, \ldots, \|T(\lambda_{m-1}u_{m-1})\|, \Big\|T\Big(\sum_{k\geq m}\lambda_k u_k\Big)\Big\|\Big\} \\ &\leq M\|x\|. \ \blacksquare\end{aligned}$$

Let E be an elementary JB^*-triple of type 1 (that is, an elementary JB^*-triple such that E^{**} is a type 1 Cartan factor), and let $T : E \to F$ be a biorthogonality preserving linear surjection from E onto another JB^*-triple. Then by Theorem 4.2 and Proposition 4.6, T is continuous. Further, we claim that T is a scalar multiple of a triple isomorphism. Indeed, let us see that $S = (1/\lambda)T$ is a triple isomorphism, where $\lambda = \|T(e)\| = \|T\|$ for some (and hence any) minimal tripotent e in E (cf. Theorem 4.2). Let $x \in E$. Then $x = \sum_n \lambda_n e_n$ for a suitable $(\lambda_n) \in c_0$ and a family of mutually orthogonal minimal tripotents (e_n) in E [7, Remark 4.6]. Then by observing that T is continuous we have

$$\|S(x)\| = \frac{1}{\lambda}\|T(x)\| = \frac{1}{\lambda}\Big\|T\Big(\sum_n \lambda_n e_n\Big)\Big\| = \frac{1}{\lambda}\Big\|\sum_n \lambda_n T(e_n)\Big\|$$
$$= \frac{1}{\lambda}\sup_n |\lambda_n|\,\|T(e_n)\| = \frac{1}{\lambda}\sup_n |\lambda_n|\lambda = \sup_n |\lambda_n| = \|x\|.$$

This proves that S is a surjective linear isometry between JB^*-triples, and hence a triple isomorphism (see [26, Proposition 5.5], [5, Corollary 3.4], [18, Theorem 2.2]). We have thus proved the following result:

COROLLARY 4.7. *Let $T : E \to F$ a biorthogonality preserving linear surjection from a type 1 elementary JB^*-triple of rank greater than one onto another JB^*-triple. Then T is a scalar multiple of a triple isomorphism.* ∎

Let p and q be two minimal projections in a C^*-algebra A with $q \neq p$. It is known that the C^*-subalgebra of A generated by p and q is isometrically isomorphic to $\mathbb{C} \oplus^\infty \mathbb{C}$ when p and q are orthogonal, and isomorphic to $M_2(\mathbb{C})$ otherwise. More concretely, by [31, Theorem 1.3] (see also [29, §3]), denoting by $C_{p,q}$ the C^*-subalgebra of A generated by p and q, we have the following statements:

(a) If $p \perp q$ then there exists an isometric C^*-isomorphism $\Phi : C_{p,q} \to \mathbb{C} \oplus^\infty \mathbb{C}$ such that $\Phi(p) = (1,0)$ and $\Phi(q) = (0,1)$.
(b) If p and q are not orthogonal then there exist $0 < t < 1$ and an isometric C^*-isomorphism $\Phi : C_{p,q} \to M_2(\mathbb{C})$ such that

$$\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

In the setting of JB^*-algebras we have:

LEMMA 4.8. *Let p and q be two minimal projections in a JB^*-algebra J with $q \neq p$ and let $J_{p,q}$ denote the JB^*-subalgebra of J generated by p and q.*

(a) *If $p \perp q$ then there exists an isometric JB^*-isomorphism $\Phi : J_{p,q} \to \mathbb{C} \oplus^\infty \mathbb{C}$ such that $\Phi(p) = (1,0)$ and $\Phi(q) = (0,1)$.*
(b) *If p and q are not orthogonal then there exist $0 < t < 1$ and an isometric JB^*-isomorphism $\Phi : C \to S_2(\mathbb{C})$ such that*

$$\Phi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \Phi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix},$$

where $S_2(\mathbb{C})$ denotes the type 3 Cartan factor of all symmetric operators on a two-dimensional complex Hilbert space.

Moreover, the JB^*-subtriple of J generated by p and q coincides with $J_{p,q}$.

Proof. Statement (a) is clear. Now assume that p and q are not orthogonal. The Shirshov–Cohn theorem (see [22, Theorem 7.2.5]) ensures that $J_{p,q}$ is a JC^*-algebra, that is, a Jordan *-subalgebra of some C^*-algebra A. The symbol $C_{p,q}$ will stand for the (associative) C^*-subalgebra of A generated by p and q. Set

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

We have already mentioned that there exist $0 < t < 1$ and an isometric C^*-isomorphism $\Phi : C_{p,q} \to M_2(\mathbb{C})$ such that $\Phi(p) = P$ and $\Phi(q) = Q$.

Since $J_{p,q}$ is a Jordan *-subalgebra of $C_{p,q}$, $J_{p,q}$ can be identified with the Jordan *-subalgebra of $M_2(\mathbb{C})$ generated by the matrices P and Q. It can be easily checked that

$$P \circ Q = \begin{pmatrix} t & \frac{1}{2}\sqrt{t(1-t)} \\ \frac{1}{2}\sqrt{t(1-t)} & 0 \end{pmatrix},$$

$$2P \circ Q - 2tP = \begin{pmatrix} 0 & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 0 \end{pmatrix},$$

$$Q - (2P \circ Q - 2tP) - tP = \begin{pmatrix} 0 & 0 \\ 0 & 1-t \end{pmatrix}.$$

These identities show that $J_{p,q}$ contains the generators of the JB^*-algebra $S_2(\mathbb{C})$, and hence identifies with $S_2(\mathbb{C})$.

In order to prove the last assertion, let $E_{p,q}$ denote the JB^*-subtriple of J generated by p and q. As $J_{p,q}$ is itself a subtriple containing p and q, we have $E_{p,q} \subseteq J_{p,q}$. If $p \perp q$ then it can easily be seen that $E_{p,q} \cong \mathbb{C} \oplus^\infty \mathbb{C} \cong J_{p,q}$. Now assume that p and q are not orthogonal.

From Proposition 5 in [20], $E_{p,q}$ is a JB^*-triple isometrically isomorphic to $M_{1,2}(\mathbb{C})$ or $S_2(\mathbb{C})$. If $E_{p,q}$ is a rank-one JB^*-triple, that is, $E \cong M_{1,2}(\mathbb{C})$, then $P_0(p)(q)$ must be zero. Thus, according to the above representation, we have $1 - t = 0$, which is impossible. ∎

A JB^*-algebra which is a weakly compact JB^*-triple will be called *weakly compact* or *dual* (see [6]). Every positive element x in a weakly compact JB^*-algebra J can be written in the form $x = \sum_n \lambda_n p_n$ for a suitable $(\lambda_n) \in c_0$ and a family (p_n) of mutually orthogonal minimal projections in J (see Theorem 3.3 in [6]).

Our next theorem extends [12, Theorem 11].

THEOREM 4.9. *Let $T : J \to E$ be a biorthogonality preserving linear surjection from a weakly compact JB^*-algebra onto a JB^*-triple. Then T is continuous and $\|T\| \leq 2 \sup\{\|T(p)\| : p$ a minimal projection in $J\}$.*

Proof. Since J is a JB^*-algebra, it is enough to show that T is bounded on positive norm-one elements. In this case, it suffices to prove that the set

$$\mathcal{P} = \{\|T(p)\| : p \text{ a minimal projection in } J\}$$

is bounded (cf. the proof of Proposition 4.6).

Suppose, on the contrary, that $\mathcal{P}$ is unbounded. We shall show by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in J such that $\|T(p_n)\| > n$.

The case $n = 1$ is clear. The induction hypothesis guarantees the existence of mutually orthogonal minimal projections $p_1, \dots, p_n$ in J with $\|T(p_k)\| > k$ for all $k \in \{1, \dots, n\}$.

By assumption, there exists a minimal projection $q \in J$ satisfying

$$\|T(q)\| > \max\{\|T(p_1)\|, \dots, \|T(p_n)\|, n+1\}.$$

We claim that q must be orthogonal to each p_j. If that is not the case, there exists j such that p_j and q are not orthogonal. Let C denote the JB^*-subtriple of J generated by q and p_j. We conclude from Lemma 4.8 that C is isomorphic to the JB^*-algebra $S_2(\mathbb{C})$.

Let $g_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $g_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $g_1 + g_2$ is the unit element in $C \cong S_2(\mathbb{C})$. By Theorem 4.1, $w_1 := \frac{1}{\|T(g_1)\|} T(g_1)$ and $w_2 := \frac{1}{\|T(g_2)\|} T(g_2)$ are two orthogonal minimal tripotents in E. The element $w = w_1 + w_2$ is a rank-2 tripotent in E and coincides with the range tripotent of the element $h = T(g_1 + g_2) = \|T(g_1)\| w_1 + \|T(g_2)\| w_2$. Furthermore, h is invertible in $E_2(w)$, and by Theorem 3.8 (see also [11, Corollary 4.1(b)]), $T(C) \subseteq E_2(w)$.

The rest of the argument is parallel to the argument in the proof of Theorem 4.2.

The finite-dimensionality of the JB^*-subtriple C ensures that $T(C)$ is norm closed and $T|_C : C \cong S_2(\mathbb{C}) \to E$ is a continuous biorthogonality preserving linear operator. Theorem 3.8 guarantees the existence of a Jordan *-homomorphism $S : C \to E_2(w)$ such that $S(g_1 + g_2) = w$, $S(C)$ and h operator commute and

$$T(z) = h \circ_w S(z) \quad \text{for all } z \in C. \tag{4.3}$$

It follows from the operator commutativity of h^{-1} and $S(C)$ that $S(z) = h^{-1} \circ_w T(z)$ for all $z \in C$. The injectivity of T implies that S is a Jordan *-monomorphism.

Lemma 2.7 in [19] shows that $E_2(w) = E_2(w_1 + w_2)$ coincides with $\mathbb{C} \oplus^{\ell_\infty} \mathbb{C}$ or with a spin factor. Since $3 = \dim(T(C)) \leq \dim(E_2(w))$, we deduce that $E_2(w)$ is a spin factor with inner product $(\cdot|\cdot)$ and conjugation $x \mapsto \overline{x}$. We may assume, by Remark 2.1, that $(w_1|w_1) = 1/2$, $(w_1|\overline{w}_1) = 0$, and $w_2 = \overline{w}_1$.

Now, taking $g_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in C \cong S_2(\mathbb{C})$, the element $w_3 := S(g_3)$ is a tripotent in $E_2(w)$ with $\{w_i, w_i, w_3\} = \frac{1}{2} w_3$ for every $i \in \{1, 2\}$. Remark 2.1 implies that $(w_3|w_1) = (w_3|w_2) = 0$. Let M denote the JB^*-subtriple of $E_2(w)$ generated by w_1, w_2, and w_3. The mapping $S : C \cong S_2(\mathbb{C}) \to M$ is a Jordan *-isomorphism.

Combining (4.3) and (2.4) we get

$$T(g_3) = h \circ_w S(g_3) = \{h, w, w_3\} = \frac{\|T(g_1)\| + \|T(g_2)\|}{2} w_3.$$

Since $T(g_1) = \|T(g_1)\| w_1$, $T(g_2) = \|T(g_2)\| w_2$, and C is linearly generated by g_1, g_2 and g_3, we deduce that $T(C) \subseteq M$ with $3 = \dim(T(C)) \leq \dim(M) = 3$. Thus, $T(C) = M$ is a JB^*-subtriple of E.

The mapping $T|_C : C \cong S_2(\mathbb{C}) \to T(C)$ is a continuous biorthogonality preserving linear bijection. Theorem 3.9 guarantees the existence of a scalar $\lambda \in \mathbb{C}\backslash\{0\}$ and a triple isomorphism $\Psi : C \to T(C)$ such that $T(x) = \lambda \Psi(x)$ for all $x \in C$. Since p_j and q are projections, $\|\Psi(q)\| = \|\Psi(p_j)\| = 1$. Hence $\|T(p_j)\| = |\lambda|$ and $\|T(q)\| = |\lambda|$, contradicting the induction hypothesis. Therefore $q \perp p_j$ for every $j = 1, \ldots, n$.

It follows by induction that there exists a sequence (p_n) of mutually orthogonal minimal projections in J such that $\|T(p_n)\| > n$. The series $\sum_{n=1}^{\infty} (1/\sqrt{n}) p_n$ defines an element a in J (cf. Remark 4.3). For each natural m, a decomposes as the orthogonal sum of $(1/\sqrt{m}) p_m$ and $\sum_{n \neq m} (1/\sqrt{n}) p_n$, therefore

$$T(a) = \frac{1}{\sqrt{m}} T(p_m) + T\bigg(\sum_{n \neq m} \frac{1}{\sqrt{n}} p_n \bigg),$$

with orthogonal summands. This argument implies that

$$\|T(a)\| = \max\bigg\{ \frac{1}{\sqrt{m}} \|T(p_m)\|, \ \bigg\| T\bigg(\sum_{n \neq m} \frac{1}{\sqrt{n}} p_n \bigg) \bigg\| \bigg\} > \sqrt{m}.$$

Since m was arbitrary, we have arrived at the desired contradiction. ∎

By Proposition 2 in [23], every Cartan factor of type 1 with $\dim(H) = \dim(H')$, every Cartan factor of type 2 with $\dim(H)$ even or infinite, and every Cartan factor of type 3 is a JBW^*-algebra factor for a suitable Jordan product and involution. In the case of C being a Cartan factor which is also a JBW^*-algebra, the corresponding elementary JB^*-triple $K(C)$ is a weakly compact JB^*-algebra.

Corollary 4.10. *Let K be an elementary JB^*-triple of type 1 with* $\dim(H) = \dim(H')$, *or of type 2 with* $\dim(H)$ *even or infinite, or of type 3. Suppose that $T : K \to E$ is a biorthogonality preserving linear surjection from K onto a JB^*-triple. Then T is continuous. Further, since K^{**} is a*

JBW^*-algebra factor, Theorem 3.9 *ensures that T is a scalar multiple of a triple isomorphism.* ∎

THEOREM 4.11. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between JB^*-triples, where E is weakly compact containing no infinite-dimensional rank-one summands. Then T is continuous.*

Proof. Since E is a weakly compact JB^*-triple, the statement follows from Proposition 4.6 as soon as we prove that the set

$$\mathcal{T} := \{\|T(e)\| : e \text{ a minimal tripotent in } E\}$$

is bounded.

We know that $E = \bigoplus_{\alpha\in\Gamma}^{c_0} K_\alpha$, where $\{K_\alpha : \alpha \in \Gamma\}$ is a family of elementary JB^*-triples (see Lemma 3.3 in [7]). Now, Lemma 3.1 guarantees that $T(K_\alpha) = T(K_\alpha^{\perp\perp}) = T(K_\alpha)^{\perp\perp}$ is a norm closed inner ideal for every $\alpha \in \Gamma$.

For each $\alpha \in \Gamma$, K_α is finite-dimensional, or a type 1 elementary JB^*-triple of rank greater than one, or a JB^*-algebra. It follows, by Corollary 4.7 and Theorem 4.9, that $T|_{K_\alpha} : K_\alpha \to T(K_\alpha)$ is continuous.

Suppose that $\mathcal{T}$ is unbounded. Having in mind that every minimal tripotent in E belongs to a unique factor K_α, by Proposition 4.6, there exists a sequence (e_n) of mutually orthogonal minimal tripotents in E such that $\|T(e_n)\|$ diverges to $+\infty$. The element $z := \sum_{n=1}^{\infty} \|T(e_n)\|^{-1/2} e_n$ lies in E and hence $\|T(z)\| < \infty$. We fix an arbitrary natural m. Since $z - \|T(e_m)\|^{-1/2} e_m$ and $\|T(e_m)\|^{-1/2} e_m$ are orthogonal, we have

$$T(z - \|T(e_m)\|^{-1/2} e_m) \perp T(\|T(e_m)\|^{-1/2} e_m),$$

and hence

$$\begin{aligned}\|T(z)\| &= \|T(z - \|T(e_m)\|^{-1/2} e_m)) + T(\|T(e_m)\|^{-1/2} e_m)\| \\ &= \max\{\|T(z - \|T(e_m)\|^{-1/2} e_m)\|, \|T(e_m)\|^{-1/2} \|T(e_m)\|\} \geq \sqrt{\|T(e_m)\|},\end{aligned}$$

which contradicts that $\|T(e_m)\|^{1/2} \to +\infty$. Therefore $\mathcal{T}$ is bounded. ∎

COROLLARY 4.12. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between two JB^*-triples, where $K(E)$ contains no infinite-dimensional rank-one summands. Then $T|_{K(E)} : K(E) \to K(F)$ is continuous.*

Proof. Pick $x \in K(E)$. It can be written in the form $x = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents of E, and $\|x\| = \sup\{|\lambda_n| : n \geq 1\}$ (cf. Remark 4.6 in [7]). For each natural m we define $y_m := T(\sum_{n\geq m+1} \lambda_n u_n)$. Theorem 4.1 guarantees that $T(x_m) = T(\sum_{n=1}^{m} \lambda_n u_n)$ defines a sequence in $K(F)$.

Since, by Lemma 4.5, $y_m \to 0$ in norm, we deduce that $T(x_m) = T(x) - y_m$ tends to $T(x)$ in norm. Therefore $T(K(E)) = K(F)$ and $T|_{K(E)} : K(E) \to K(F)$ is a biorthogonality preserving linear surjection between weakly compact JB^*-triples. The result now follows from Theorem 4.11. ∎

REMARK 4.13. In Remark 15 of [10] it was already pointed out that the conclusion of Theorem 4.11 is no longer true if we allow E to have infinite-dimensional rank-one summands. Indeed, let $E = L(H) \oplus^\infty L(H, \mathbb{C})$, where H is an infinite-dimensional complex Hilbert space. We can always find an unbounded bijection $S : L(H, \mathbb{C}) \to L(H, \mathbb{C})$. Since $L(H, \mathbb{C})$ is a rank-one JB^*-triple, S is a biorthogonality preserving linear bijection and the mapping $T : E \to E$ given by $x + y \mapsto x + S(y)$ has the same properties.

COROLLARY 4.14. *Two weakly compact JB^*-triples containing no rank-one summands are isomorphic if and only if there exists a biorthogonality preserving linear surjection between them.*

5. Biorthogonality preservers between atomic JBW^*-triples. A JBW^*-triple E is said to be *atomic* if it coincides with the weak* closed ideal generated by its minimal tripotents. Every atomic JBW^*-triple can be written as an ℓ_∞-sum of Cartan factors [21].

The aim of this section is to study when the existence of a biorthogonality preserving linear surjection between two atomic JBW^*-triples implies that they are isomorphic (note that continuity is not assumed). We shall establish an automatic continuity result for biorthogonality preserving linear surjections between atomic JBW^*-triples containing no rank-one factors.

Before dealing with the main result, we survey some results describing the elements in the predual of a Cartan factor. We make use of the description of the predual of $L(H)$ in terms of the *trace class* operators (cf. [32, §II.1]). The results, included here for completeness, are direct consequences of this description but we do not know an explicit reference.

Let $C = L(H, H')$ be a type 1 Cartan factor. Lemma 2.6 in [30] ensures that each φ in C_* can be written in the form $\varphi := \sum_{n=1}^\infty \lambda_n \varphi_n$, where (λ_n) is a sequence in ℓ_1^+ and each φ_n is an extreme point of the closed unit ball of C_*. More concretely, for each natural n there exist norm-one elements $h_n \in H$ and $k_n \in H'$ such that $\varphi_n(x) = (x(h_n)|k_n)$ for every $x \in C$, that is, for each natural n there exists a minimal tripotent e_n in C such that $P_2(e_n)(x) = \varphi_n(x) e_n$ for every $x \in C$ (cf. [20, Proposition 4]).

We now consider (infinite-dimensional) type 2 and type 3 Cartan factors. Let j be a conjugation on a complex Hilbert space H, and consider the linear involution on $L(H)$ defined by $x \mapsto x^t := jx^*j$. Let $C_2 = \{x \in L(H) :$

$x^t = -x\}$ and $C_3 = \{x \in L(H) : x^t = x\}$ be Cartan factors of type 2 and 3, respectively.

Noticing that $L(H) = C_2 \oplus C_3$, it is easy to see that every element φ in $(C_2)_*$ (respectively, $(C_3)_*$) admits an extension of the form $\widetilde{\varphi} = \varphi\pi$, where π denotes the canonical projection of $L(H)$ onto C_2 (respectively, C_3). Making use of [32, Lemma 1.5], we can find an element $x_{\widetilde{\varphi}} \in K(H)$ satisfying

$$(x_{\widetilde{\varphi}}(h)|k) = \widetilde{\varphi}(h \otimes k) \quad (h, k \in H). \tag{5.1}$$

Since, for each $x \in L(H)$, $\widetilde{\varphi}(x) = \frac{1}{2}\widetilde{\varphi}(x - x^t)$, we can easily check, via (5.1), that $x_{\widetilde{\varphi}}^t = -x_{\widetilde{\varphi}}$. Therefore $x_{\widetilde{\varphi}} \in K_2 = K(C_2)$. From [7, Remark 4.6] it may be deduced that $x_{\widetilde{\varphi}}$ can be (uniquely) written as a norm convergent (possibly finite) sum $x_{\widetilde{\varphi}} = \sum_n \lambda_n u_n$, where u_n are mutually orthogonal minimal tripotents in K_2 and $(\lambda_n) \in c_0$ (notice that u_n is a minimal tripotent in C_2 but it need not be minimal in $L(H)$; in any case, either u_n is minimal in $L(H)$ or it can be written as a convex combination of two minimal tripotents in $L(H)$). For each $(\beta_n) \in c_0$, $z := \sum_n \beta_n u_n \in K_2$ and, by (5.1), $\sum_n \lambda_n \beta_n = \widetilde{\varphi}(z) = \varphi(z) < \infty$. Thus, $(\lambda_n) \in \ell_1$, and another application of (5.1) shows that $\varphi(x) = \sum_n \lambda_n \varphi_n(x)$ for all $x \in C_2$, where φ_n lies in $(C_2)_*$ and satisfies $P_2(u_n)(x) = \varphi_n(x) u_n$. A similar reasoning remains true for C_3.

We have thus proved:

PROPOSITION 5.1. *Let C be an infinite-dimensional Cartan factor of type 1, 2 or 3. For each φ in C_*, there exist a sequence $(\lambda_n) \in \ell_1$ and a sequence (u_n) of mutually orthogonal minimal tripotents in C such that*

$$\|\varphi\| = \sum_{n=1}^{\infty} |\lambda_n| \quad \textit{and} \quad \varphi(x) = \sum_n \lambda_n \varphi_n(x) \quad (x \in C),$$

where for each $n \in \mathbb{N}$, $\varphi_n(x) u_n = P_2(u_n)(x)$ $(x \in C)$. ∎

Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic JBW^*-triples, where E contains no rank-one Cartan factors. In this case $K(E)$ and $K(F)$ are weakly compact JB^*-triples with $K(E)^{**} = E$ and $K(F)^{**} = F$. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. This is not, a priori, enough to guarantee that T is continuous. In fact, for each nonreflexive Banach space X there exists an unbounded linear operator $S : X^{**} \to X^{**}$ such that $S|_X : X \to X$ is continuous. The main result of this section establishes that a mapping T as above is automatically continuous.

THEOREM 5.2. *Let $T : E \to F$ be a biorthogonality preserving linear surjection between atomic JBW^*-triples, where E contains no rank-one Cartan factors. Then T is continuous.*

Proof. Corollary 4.12 ensures that $T|_{K(E)} : K(E) \to K(F)$ is continuous. By Lemma 3.3 in [7], $K(E)$ decomposes as a c_0-sum of all elementary triple ideals of E, that is, if $E = \bigoplus^{\ell_\infty} C_\alpha$, where each C_α is a Cartan factor, then $K(E) = \bigoplus^{c_0} K(C_\alpha)$. By Proposition 3.10, for each α, $T(K_\alpha)$ (respectively, $T(C_\alpha)$ is a norm closed (respectively, weak* closed) inner ideal of $K(F)$ (respectively, F) and $K(F) = \bigoplus^{c_0} T(K(C_\alpha))$ (respectively, $F = \bigoplus^{c_0} T(C_\alpha)$).

For each α, C_α is either finite-dimensional, or an infinite-dimensional Cartan factor of type 1, 2 or 3. Corollaries 4.7 and 4.10 prove that the operator $T|_{K(C_\alpha)} : K(C_\alpha) \to T(K(C_\alpha))$ is a scalar multiple of a triple isomorphism. We claim that, for each α and each φ_α in the predual of $T(C_\alpha)$, $\varphi_\alpha T$ is weak* continuous. There is no loss of generality in assuming that C_α is infinite-dimensional.

Each minimal tripotent f in F lies in a unique elementary JB^*-triple $T(K(C_\alpha))$. Since $T|_{K(C_\alpha)} : K(C_\alpha) \to T(K(C_\alpha))$ is a scalar multiple of a triple isomorphism, there exist a nonzero scalar λ_α and a minimal tripotent e satisfying $T^{-1}(f) = \lambda_\alpha e$, $|\lambda_\alpha| \leq \|(T|_{K(C_\alpha)})^{-1}\| \leq \|(T|_{K(E)})^{-1}\|$, and

$$T(K(C_\alpha)_i(e)) = T(K(C_\alpha))_i(f) \tag{5.2}$$

for every $i = 0, 1, 2$. Theorem 4.1 shows that $T((C_\alpha)_i(e)) = T(C_\alpha)_i(f)$ for every $i = 0, 2$. Since $K(E)$ is an ideal of E and e is a minimal tripotent, $(C_\alpha)_1(e) = E_1(e) = K(E)_1(e) = K(C_\alpha)_1(e)$. It follows from (5.2) that

$$T((C_\alpha)_i(e)) = T((C_\alpha))_i(f)$$

for every $i = 0, 1, 2$. Consequently, $P_2(f)T = \lambda_\alpha^{-1} P_2(e) \in (C_\alpha)_*$, and $|\lambda_\alpha^{-1}| \leq \|T|_{K(C_\alpha)}\| \leq \|T|_{K(E)}\|$.

Since f was an arbitrary minimal tripotent in F (equivalently, in $T(K(C_\alpha))$), Proposition 5.1 ensures that $\varphi_\alpha T \in E_*$ with $\|\varphi_\alpha T\| \leq \|T|_{K(E)}\|$ for every $\varphi_\alpha \in (T(C_\alpha))_*$. Therefore, T is bounded with

$$\|T\| \leq \|T|_{K(E)}\| \leq \|T\|. \ \blacksquare$$

COROLLARY 5.3. *Two atomic JBW^*-triples containing no rank-one summands are isomorphic if and only if there is a biorthogonality preserving linear surjection between them.* $\blacksquare$

The conclusion of Theorem 5.2 does not hold for atomic JBW^*-triples containing rank-one summands.

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María Burgos, Jorge J. Garcés, Antonio M. Peralta
Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: mariaburgos@ugr.es
jgarces@ugr.es
aperalta@ugr.es

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A KAPLANSKY THEOREM FOR JB*-TRIPLES

FRANCISCO J. FERNÁNDEZ-POLO, JORGE J. GARCÉS, AND ANTONIO M. PERALTA

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Abstract. Let $T : E \to F$ be a not necessarily continuous triple homomorphism from a (complex) JB*-triple (respectively, a (real) J*B-triple) to a normed Jordan triple. The following statements hold:

(1) T has closed range whenever T is continuous.
(2) T is bounded below if and only if T is a triple monomorphism.

This result generalises classical theorems of I. Kaplansky and S.B. Cleveland in the setting of C*-algebras and of A. Bensebah and J. Pérez, L. Rico and A. Rodríguez Palacios in the setting of JB*-algebras.

1. Introduction

A celebrated result of I. Kaplansky (cf. [13, Theorem 6.2]) establishes that any algebra norm on a commutative C*-algebra dominates the C*-norm. Subsequently, S.B. Cleveland (see [8]) generalised this result to the noncommutative case by showing that every (not necessarily complete nor continuous) algebra norm on a C*-algebra generates a topology stronger than the topology of the C*-norm. In other words, every not necessarily continuous monomorphism from a C*-algebra to an associative normed algebra is bounded below. Alternative proofs to Cleveland's result were given by H.G. Dales [9] and A. Rodríguez Palacios [22] (see also [16, Theorem 6.1.16]).

The arguments presented by A. Rodríguez Palacios in [22] were adapted by A. Bensebah [3] and J. Pérez, L. Rico and A. Rodríguez Palacios [17] to extend Kaplansky's theorem to the more general setting of JB*-algebras. The results established in [3] and [17] show that every not necessarily continuous Jordan monomorphism from a JB*-algebra to a normed Jordan algebra is bounded below. This result was proved again by S. Hejazian and A. Niknam in [11].

Every C*-algebra, A, admits a triple product defined by

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a). \tag{1}$$

Let us suppose that $\|.\|_2$ is another (not necessarily complete nor continuous) norm on A which makes continuous the triple product of A. It is natural to ask whether this norm generates a topology stronger than the topology of the C*-norm.

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Every C*-algebra, A, equipped with its C*-norm and the triple product defined in (1) can be regarded as an element in the wider category of (complex) JB*-triples (see §2 for the detailed definitions). The question posed in the above paragraph also makes sense in the (larger) categories of (complex) JB*-triples and real J*B-triples. In this setting, the problem can be reformulated in the following terms:

Problem (P). *Let E be a (complex) JB*-triple or a (real) J*B-triple whose norm is denoted by $\|.\|$, and let $\|.\|_2$ be another (not necessarily complete nor $\|.\|$-continuous) norm on the vector space E which makes continuous the triple product of E. Does $\|.\|_2$ generate a topology stronger than the topology generated by the JB*-triple norm $\|.\|$?*

Equivalently, is every (not necessarily continuous) triple monomorphism T from E to a normed Jordan triple bounded below?

Under the additional hypothesis of T being $\|.\|$-continuous (resp., $\|.\|_2$ being $\|.\|$-continuous), Problem (P) was solved by K. Bouhya and A. Fernández López in the case of (complex) JB*-triples [4, Corollary 14].

In this paper we solve Problem (P) without any additional assumptions on the triple monomorphism T (resp., on $\|.\|_2$). When particularized to C*-algebras, our main result shows that every not necessarily continuous triple monomorphism from a real or complex C*-algebra to a normed Jordan triple is bounded below.

Section 2 is devoted to presenting the basic facts and definitions needed in the paper. We shall also survey the results on the property of minimality of norm topology in the setting of Banach algebras and Jordan-Banach triples. We shall adapt the arguments given by K. Bouhya and A. Fernández López [4] to obtain their result in the setting of real J*B-triples.

In Section 3 we present our main results (Theorem 17 and Corollary 18). This section contains a deep study of the separating spaces associated with a triple homomorphism between normed Jordan triples. Among the tools developed here, we mention a main boundedness theorem type for Jordan-Banach triples (see Theorem 12), which is the Jordan triple version of a classical result in the setting of Banach algebras due to W.G. Bade and P.C. Curtis [1].

2. Minimality of norm topology for JB*-triples

A normed algebra A has *minimality of algebraic norm topology (MOANT)* if any other (not necessarily complete) algebra norm dominated by the given norm yields an equivalent topology. It is part of the folklore that C*-algebras have MOANT (compare [8, Lemma 5.3]).

In this section, we study the minimality of norm topology in the setting of normed Jordan triples. We recall that a complex (resp., real) *normed Jordan triple* is a complex (resp., real) normed space E equipped with a nontrivial, continuous triple product

$$E \times E \times E \to E,$$
$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$. If E is complete with respect to the norm (i.e. if E is a Banach space), then it is called a complex (resp., real) *Jordan-Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan-Banach triple. Unless specified otherwise, the term "normed Jordan triple" (resp., "Jordan-Banach triple") will always mean a real or complex normed Jordan triple (resp., "Jordan-Banach triple").

For each Jordan-Banach triple E, the constant $N(E)$ or $N(E, \|.\|)$ will denote the supremum of the set $\{\| \{x, y, z\} \| : \|x\|, \|y\|, \|z\| \leq 1\}$.

A real (resp., complex) *Jordan algebra* is a (not necessarily associative) algebra over the real (resp., complex) field whose product is abelian and satisfies $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. A normed Jordan algebra is a Jordan algebra A equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$, a, b in A. A *Jordan-Banach algebra* is a normed Jordan algebra whose norm is complete.

Every real or complex associative Banach algebra (resp., Jordan Banach algebra) is a real Jordan-Banach triple with respect to the product $\{a, b, c\} = \frac{1}{2}(abc + cba)$ (resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$).

A JB*-algebra is a complex Jordan Banach algebra A equipped with an algebra involution * satisfying that $\| \{a, a^*, a\} \| = \|2(a \circ a^*) \circ a - a^2 \circ a^*\| = \|a\|^3$, a in A.

Every JB*-algebra has MOANT (compare [17, Theorem 10]).

We shall say that a normed Jordan triple E has *minimality of triple norm topology (MOTNT)* if any other (not necessarily complete) triple norm dominated by the norm of E defines an equivalent topology.

Remark 1. Let A be a real or complex associative normed algebra whose norm is denoted by $\|.\|$. The symbol A^+ will stand for the normed Jordan algebra A equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm. Let $\|.\|_1$ be a norm on the space A. Since the Jordan product is $\|.\|_1$-continuous whenever the associative product is, we deduce:

$$(A^+, \|.\|) \text{ has MOANT } \Longrightarrow (A, \|.\|) \text{ has MOANT.}$$

However, we do not know if the reciprocal statement is, in general, true. By [7, Proposition 3], there exists an associative normed algebra $\mathcal{B}$ such that there exists a norm $\|.\|_1$ on $\mathcal{B}$ for which the Jordan product is continuous but the associative product is discontinuous. In particular, $(\mathcal{B}^+, \|.\|_1)$ doesn't have MOANT.

When A is simple and has a unit, every norm on A making the Jordan product continuous also makes continuous the associative product (compare [7, Theorem 3]). Under this additional hypothesis, we have

$$(A^+, \|.\|) \text{ has MOANT } \Longleftrightarrow (A, \|.\|) \text{ has MOANT.}$$

Suppose that J is a real or complex normed Jordan algebra, whose norm is denoted by $\|.\|$. When J is regarded as a real or complex normed Jordan triple with respect to the product $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$, every Jordan algebra norm on J makes continuous the triple product. Therefore J has MOANT whenever it has MOTNT.

When J has a unit, the Jordan and the triple product of J are mutually determined, and hence

$$(J, \|.\|) \text{ has MOANT } \Longleftrightarrow (J, \|.\|) \text{ has MOTNT.}$$

A *(complex) JB*-triple* is a complex Jordan-Banach triple E satisfying the following axioms:

(JB*1) For each a in E the map $L(a,a)$ is a hermitian operator on E with nonnegative spectrum.

(JB*2) $\|\{a,a,a\}\| = \|a\|^3$ for all a in A.

The following theorem is a celebrated result of I. Kaplansky (see [13, Theorem 6.2] or [23, Theorem 1.2.4]).

Theorem 2. *Let A be a commutative C*-algebra with a norm $\|.\|$ and let $\|.\|_1$ be another norm on A under which A is a normed algebra. Then $\|a\| \leq \|a\|_1$, for every a in A. Further, for any algebra norm, $\|.\|_1$, on A_{sa}, the inequality $\|a\| \leq \|a\|_1$ holds for every a in A_{sa}.* □

Every C*-algebra is a JB*-triple with respect to the product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. It seems natural to ask whether in the above Theorem 2 the norm $\|.\|_1$ can be replaced with another norm $\|.\|_2$ under which A is a normed Jordan triple. The complex statement in the following result was established by K. Bouhya and A. Fernández López in [4, Proposition 13]. A detailed proof is included here for completeness.

Lemma 3. *Let $L \subset \mathbb{R}_0^+$ be a subset of nonnegative real numbers satisfying that $L \cup \{0\}$ is compact. Let $C_0(L)$ denote the Banach algebra of all real- or complex-valued continuous functions on $L \cup \{0\}$ vanishing at zero (equipped with the supremum norm $\|.\|_\infty$). Suppose that $\|.\|_2$ is a $\|.\|_\infty$-continuous norm on $C_0(L)$ under which $C_0(L)$ is a normed Jordan triple. Then $\|.\|_2$ is equivalent to an algebra norm on $C_0(L)$, and consequently $\|.\|_\infty$ and $\|.\|_2$ are equivalent norms. More concretely, writing $M = \sup\{\|x\|_2 : \|x\|_\infty \leq 1\}$ we have $\|a\|_\infty \leq MN(C_0(L), \|.\|_2)\ \|a\|_2$, for all $a \in C_0(L)$.*

Proof. Since $\|.\|_2$ is $\|.\|_\infty$-continuous, there exists a positive M such that $\|x\|_2 \leq M\|x\|_\infty$, for all $x \in C_0(L)$.

When L is compact, $C_0(L)$ coincides with $C(L)$, the C*-algebra of all complex-valued continuous functions on L or with the selfadjoint part of that C*-algebra. Let 1 denote the unit element in $C(L)$. Take a, b in $C(L)$. Applying the fact that $\|.\|_2$ is a triple norm we have

$$\begin{aligned} \|a\ b\|_2 = \|\ \{a,1,b\}\ \|_2 &\leq N(C_0(L), \|.\|_2)\ \|a\|_2\ \|1\|_2\ \|b\|_2 \\ &\leq N(C_0(L), \|.\|_2)\ M\ \|a\|_2\ \|b\|_2. \end{aligned}$$

This shows that $\|.\|_2$ is equivalent to $MN(C_0(L), \|.\|_2)\ \|.\|_2$, and the latter is an algebra norm on $C_0(L)$.

Suppose that L is not compact. Take a and b in $C_0(L)$. For each natural n, let p_n, a_n and b_n be the functions in $C_0(L)$ defined by

$$a_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}] \cap L; \\ a(t), & \text{if } t \in [\frac{1}{n}, \infty) \cap L; \end{cases} \qquad b_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}] \cap L; \\ b(t), & \text{if } t \in [\frac{1}{n}, \infty) \cap L; \end{cases}$$

$$\text{and } p_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{4n}, \frac{1}{2n}] \cap L; \\ 1, & \text{if } t \in [\frac{1}{2n}, \infty) \cap L. \end{cases}$$

Since $a_n\ b_n = \{a_n, p_n, b_n\}$ and $\|p_n\|_\infty \leq 1$, we deduce that

$$\begin{aligned}\|a_n\ b_n\|_2 = \|\ \{a_n, p_n, b_n\}\ \|_2 &\leq N(C_0(L), \|.\|_2)\ \|a_n\|_2\ \|p_n\|_2\ \|b_n\|_2 \\ &\leq N(C_0(L), \|.\|_2)\ M\ \|a_n\|_2\ \|b_n\|_2.\end{aligned}$$

Having in mind that $\|a_n - a\|_\infty \to 0$, $\|b_n - b\|_\infty \to 0$, it follows, from the $\|.\|_\infty$-continuity of the norm $\|.\|_2$, that

$$\|a\ b\|_2 \leq N(C_0(L), \|.\|_2)\ M\ \|a\|_2\ \|b\|_2,$$

which shows that $\|.\|_2$ is equivalent to $MN(C_0(L), \|.\|_2)\ \|.\|_2$, and the latter is an algebra norm on $C_0(L)$. The final statement is a direct consequence of Kaplansky's theorem (see Theorem 2). □

Remark 4. Let K be a compact Haussdorff space. Suppose that $\|.\|_2$ is a norm on $C(K)$ under which $C(K)$ is a normed Jordan triple ($\|.\|_\infty$-continuity of $\|.\|_2$ is not assumed). Let us write $N = N(C(K), \|.\|_2)$. Following the argument given in the proof of Lemma 3, we deduce that

$$\|a\ b\|_2 = \|\ \{a, 1, b\}\ \|_2 \leq N\ \|1\|_2\ \|a\|_2\ \|b\|_2,$$

for all $a, b \in C(K)$, which shows that $\|.\|_2$ is equivalent to $\|1\|_2\ N\ \|.\|_2$, and the latter is an algebra norm on $C(K)$. It follows by Kaplansky's theorem, that $\|a\|_\infty \leq \|1\|_2\ N\ \|a\|_2$, for all $a \in C(K)$.

S.B. Cleveland applied Kaplansky's theorem to prove that every continuous monomorphism from a C*-algebra to a normed algebra is bounded below (cf. [8, Lemma 5.3]); equivalently, every C*-algebra has MOANT. It follows as a consequence of [3, Theorem 1] or [17, Theorem 10] or [11] that JB*-algebras have MOANT. In the setting of (complex) JB*-triples, K. Bouhya and A. Fernández López proved the following result:

Proposition 5 ([4, Corollary 14]). *Let $T : E \to F$ be a continuous triple monomorphism from a JB*-triple to a normed complex Jordan triple. Then T is bounded below. That is, every JB*-triple has MOTNT.* □

We shall complete this section by proving a generalization of the above result to the setting of (real) J*B-triples.

We recall that a *real JB*-triple* is a norm-closed real subtriple of a complex JB*-triple (compare [12]). A *J*B-triple* is a real Banach space E equipped with a structure of a real Banach-Jordan triple which satisfies (JB*2) and the following additional axioms:

(J*B1) $N(E) = 1$;
(J*B2) $\sigma^{\mathbb{C}}_{L(E)}(L(x,x)) \subset [0, +\infty)$ for all $x \in E$;
(J*B3) $\sigma^{\mathbb{C}}_{L(E)}(L(x,y) - L(y,x)) \subset i\mathbb{R}$ for all $x, y \in E$.

Every closed subtriple of a J*B-triple is a J*B-triple (cf. [10, Remark 1.5]). The class of J*B-triples includes real (and complex) C*-algebras and real (and complex) JB*-triples. Moreover, in [10, Proposition 1.4] it is shown that complex JB*-triples are precisely those complex Jordan-Banach triples whose underlying real Banach space is a J*B-triple.

T. Dang and B. Russo established a Gelfand theory for J*B-triples in [10, Theorem 3.12]. This Gelfand theory can be refined to show that given an element a in a J*B-triple E, there exists a bounded set $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact such that the smallest (norm) closed subtriple of E containing a, E_a, is isometrically isomorphic to

$$C_0(L,\mathbb{R}) := \{f \in C_0(L), f(L) \subseteq \mathbb{R}\},$$

(see [6, page 14]). The argument given in the proof of Corollary 14 in [4] can be adapted to prove the following result. The proof is included here for completeness.

Proposition 6. *Let $T : E \to F$ be a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple. Then T is bounded below. Equivalently, every J*B-triple has MOTNT.* □.

Proof. Take an arbitrary element a in E. Let E_a denote the smallest (norm) closed subtriple of E containing a. By [6, page 14], there exists a subset $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact satisfying that E_a is isometrically J*B-triple isomorphic to $C_0(L,\mathbb{R})$, when the latter is equipped with the supremum norm $\|.\|_\infty$. We shall identify E_a and $C_0(L,\mathbb{R})$. The mapping $T|_{E_a} : E_a \cong C_0(L,\mathbb{R}) \to F$ is a continuous triple monomorphism. Therefore the mapping $x \mapsto \|x\|_2 := \|T(x)\|$ defines a $\|.\|_\infty$-continuous norm on $C_0(L,\mathbb{R})$ under which $C_0(L,\mathbb{R})$ is a normed Jordan triple.

Noticing that $N(E_a, \|.\|_2) \leq N(F)$ and

$$M = \sup\{\|x\|_2 : x \in E_a, \|x\|_\infty \leq 1\} \leq \|T\|,$$

Lemma 3 assures that $\|a\| \leq N(F)\ \|T\|\ \|T(a)\|$, for every $a \in E$. □

We recall that a subspace I of a normed Jordan triple E is a *triple ideal* if $\{E,E,I\} + \{E,I,E\} \subseteq I$. The quotient of a normed Jordan triple by a closed triple ideal is a normed Jordan triple. It is also known that the quotient of a JB*-triple (resp., a J*B-triple) by a closed triple ideal is a JB*-triple (resp., a J*B-triple) (compare [14]).

Let $T : E \to F$ be a continuous triple homomorphism from a (real) J*B-triple to a normed Jordan triple. The kernel of T, $ker(T)$, is a norm-closed triple ideal of E and the linear mapping $\widetilde{T} : E/ker(T) \to F$ given by $\widetilde{T}(a + ker(T)) = T(a)$ is a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple and $\widetilde{T}(E) = T(E)$. Proposition 6 assures that $\widetilde{T}$ is bounded below, and hence it has closed range.

A real JB*-algebra is a closed *-invariant real subalgebra of a (complex) JB*-algebra. Real C*-algebras (i.e., closed *-invariant real subalgebras of C*-algebras), equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, are examples of real JB*-algebras.

Corollary 7. *Every continuous triple homomorphism from a (real) J*B-triple to a normed Jordan triple has closed range. In particular, every continuous triple homomorphism from a real or complex C*-algebra to a normed Jordan triple has closed range.* □

Corollary 8. *Let A be a real JB*-algebra and let B be a real Jordan Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

Corollary 9. *Let A be a real or complex C*-algebra and let B be a real Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

3. Separating spaces for triple homomorphisms

We have seen in the previous section that real and complex C*-algebras and real and complex JB*-algebras have MOTNT and MOANT. Equivalently, if A denotes a real or complex C*-algebra (resp., a real or complex JB*-algebra) every continuous (triple) monomorphism T from A to a Banach algebra (resp., a Jordan-Banach algebra) is bounded below. C*-algebras and JB*-algebras satisfy a stronger property: when A is a C*-algebra (resp., a JB*-algebra) every not necessarily continuous monomorphism from A to a Banach algebra (resp., a Jordan-Banach algebra) is bounded below (compare [8, Theorem 5.4] and [3, Theorem 1] or [17, Theorem 10] or [11]).

The question clearly is whether every not necessarily continuous triple monomorphism from a complex JB*-triple (resp., from a real J*B-triple) to a normed Jordan triple is bounded below. In this section we provide a positive answer to this question. Following a classical strategy, we shall study the *separating ideals* associated with a triple homomorphism.

Under additional geometric assumptions, triple homomorphisms are automatically continuous. More concretely, every triple homomorphism between two JB*-triples is automatically continuous (compare [2, Lemma 1]). In this setting the problem reduces to the question of minimality of triple norm topology treated in Section 2. However, when the codomain space is not a JB*-triple, the continuity of a triple homomorphism does not follow automatically. We shall derive a new strategy to solve Problem (P) without any additional geometric hypothesis on the codomain space.

The following definitions and results are inspired by classical ideas developed by C. Rickart [19], B. Yood [26], W.G. Bade and P.C. Curtis [1] and S.B. Cleveland [8]. Let $T : X \to Y$ be a linear mapping between two normed spaces. Following [20, page 70], the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. The *separating space*, $\sigma_X(T)$, of T in X is defined by $\sigma_X(T) := T^{-1}(\sigma_Y(T))$. For each element y in Y, $\Delta(y)$ is defined as the infimum of the set $\{\|x\|+\|y-T(x)\| : x \in X\}$. The mapping $x \mapsto \Delta(x)$, called the *separating* function of T, satisfies the following properties:

a) $\Delta(y_1 + y_2) \leq \Delta(y_1) + \Delta(y_2)$,
b) $\Delta(\lambda y) = |\lambda|\ \Delta(y)$,
c) $\Delta(y) \leq \|y\|$ and $\Delta(T(x)) \leq \|x\|$,

for every y, y_1 and y_2 in Y, x in X and λ scalar (compare [20, page 71] or [8, Proposition 4.2]).

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$ (cf. [8, Proposition 4.5]).

It is not hard to see that $\sigma_Y(T) = \{y \in Y : \Delta(y) = 0\}$, while $\sigma_X(T) = \{x \in X : \Delta(T(x)) = 0\}$. Therefore $\sigma_X(T)$ and $\sigma_Y(T)$ are closed linear subspaces of X and Y, respectively. The assignment

$$x + \sigma_X(T) \mapsto \widetilde{T}(x + \sigma_X(T)) = T(x) + \sigma_Y(T)$$

defines an injective linear operator from $X/\sigma_X(T)$ to $Y/\sigma_Y(T)$. Moreover, $\widetilde{T}$ is continuous whenever X and Y are Banach spaces.

The separating subspaces of a triple homomorphism enjoy additional algebraic structure.

Lemma 10. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two normed Jordan triples. Then $\sigma_E(T)$ is a norm-closed triple ideal of E and $\sigma_F(T)$ is a norm-closed triple ideal of the norm closure of $T(E)$ in the completion of F.*

Proof. Let us fix $z \in \sigma_F(T)$. In this case there exists a sequence $(x_n) \subseteq E$ with $x_n \to 0$ and $T(x_n) \to z$. Given x, y in E, the sequences $(\{x_n, x, y\})$ and $(\{x, x_n, y\})$ are norm-null,

$$T(\{x_n, x, y\}) = \{T(x_n), T(x), T(y)\} \to \{z, T(x), T(y)\}$$

and

$$T(\{x, x_n, y\}) = \{T(x), T(x_n), T(y)\} \to \{T(x), z, T(y)\}.$$

This shows that $\sigma_F(T)$ is a norm-closed triple ideal of $\overline{T(E)}^{\|\cdot\|}$.

We have already proved that

$$\{\sigma_F(T), T(E), T(E)\} \subseteq \sigma_F(T)$$

and

$$\{T(E), \sigma_F(T), T(E)\} \subseteq \sigma_F(T).$$

This implies that

$$T(\{\sigma_E(T), E, E\}) \subseteq \{\sigma_F(T), T(E), T(E)\} \subseteq \sigma_F(T)$$

and

$$T(\{E, \sigma_E(T), E\}) \subseteq \{T(E), \sigma_F(T), T(E)\} \subseteq \sigma_F(T),$$

which shows that $\{\sigma_E(T), E, E\}, \{E, \sigma_E(T), E\} \subseteq \sigma_E(T)$. □

The following result follows from Lemma 10 and the basic properties of the separating spaces.

Proposition 11. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples. Then the mapping $\widetilde{T} : E/\sigma_E(T) \to F/\sigma_F(T)$, defined by $\widetilde{T}(a + E/\sigma_E(T)) = T(a) + F/\sigma_F(T)$, is a continuous triple monomorphism.* □

Let E be a normed Jordan triple. Two elements a and b in E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = L(b, a) = 0$. A direct application of the Jordan identity yields that, for each x in E,

$$a \perp \{b, x, b\} \text{ whenever } a \perp b. \tag{2}$$

The following theorem is a "*main boundedness theorem*" type result for Jordan-Banach triples (compare [1, Theorem 2.1]; see also [8, Theorem 3.1]).

Theorem 12. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between Jordan-Banach triples and let (x_n), (y_n) be two sequences of nonzero elements in E such that $x_n \perp x_m, y_m$ for every $n \neq m$. Then*

$$\sup\left\{ \frac{\|T(\{x_n, x_n, y_n\})\|}{\|x_n\|^2 \|y_n\|}, n \in \mathbb{N} \right\} < \infty.$$

Proof. Suppose that $\sup\left\{\frac{\|T(\{x_n,x_n,y_n\})\|}{\|x_n\|^2\|y_n\|}, n\in\mathbb{N}\right\}=\infty$. Under this assumption, we may find a subsequence $(a_{p,q})_{p,q\in\mathbb{N}}$ of (x_n) formed by mutually orthogonal elements such that

$$\|T\{a_{p,q},a_{p,q},b_{p,q}\}\| > 4^p8^q\|a_{p,q}\|^2\|b_{p,q}\|, \quad p,q\in\mathbb{N},$$

where $b_{p,q}=y_m$ whenever $a_{p,q}=x_m$. Now, for each $p\in\mathbb{N}$, we define

$$z_p=\sum_{k=1}^{\infty}\frac{a_{p,k}}{2^k\|a_{p,k}\|}.$$

It is easy to see that, for each natural q, $b_{l,q}\perp z_p$ whenever $l\neq p$. The equality

$$\{z_p,z_p,b_{p,q}\}=\frac{1}{4^q\|a_{p,q}\|^2}\{a_{p,q},a_{p,q},b_{p,q}\},\ q\in\mathbb{N},$$

follows from the (joint) continuity of the triple product and the orthogonality hypothesis. Thus, $T(z_p)\neq 0,\ \forall p\in\mathbb{N}$.

For each p in $\mathbb{N}$ choose $n(p)$ in $\mathbb{N}$ with $2^{n(p)}>\|T(z_p)\|^2$ and define $y=\sum_{k=1}^{\infty}\frac{b_{k,n(k)}}{2^k\|b_{k,n(k)}\|}$. It follows that

$$\{z_p,z_p,y\}=\frac{1}{2^p4^{n(p)}\|b_{p,n(p)}\|\|a_{p,n(p)}\|^2}\ \{a_{p,n(p)},a_{p,n(p)},b_{p,n(p)}\}.$$

Therefore,

$$N(F)\ \|T(y)\|\|T(z_p)\|^2>\|T(\{z_p,z_p,y\})\|>2^p2^{n(p)}>2^p\|T(z_p)\|^2.$$

This implies that $N(F)\ \|T(y)\|>2^p$ for every positive integer p, which is impossible. $\square$

Given an element a in a normed Jordan triple E, we denote $a^{[1]}=a$, $a^{[3]}=\{a,a,a\}$ and $a^{[2n+1]}:=\left\{a,a^{[2n-1]},a\right\}$ $(\forall n\in\mathbb{N})$. The Jordan identity implies that $a^{[5]}=\left\{a,a,a^{[3]}\right\}$, and by induction, $a^{[2n+1]}=L(a,a)^n(a)$ for all $n\in\mathbb{N}$. The element a is called *nilpotent* if $a^{[2n+1]}=0$ for some n.

A Jordan-Banach triple E for which the vanishing of $\{a,a,a\}$ implies that a itself vanishes is said to be *anisotropic*. It is easy to check that E is anisotropic if and only if zero is the unique nilpotent element in E.

Let a and b be two elements in an anisotropic normed Jordan triple E. If $L(a,b)=0$, then, for each x in E, the Jordan identity implies that

$$\{L(b,a)x,L(b,a)x,L(b,a)x\}=0,$$

and hence $L(b,a)=0$. Therefore $a\perp b$ if and only if $L(a,b)=0$.

In the setting of (complex) JB*-triples, every element admits 3rd- and 5th- square roots. In fact, a continuous functional calculus can be derived from the Gelfand representation for abelian JB*-triples (cf. [14, §1]). Let a be an element in a JB*-triple E. Denoting by E_a the JB*-subtriple generated by the element a, it is known that E_a is JB*-triple isomorphic (and hence isometric) to $C_0(L)=C_0(L,\mathbb{C})$ for some locally compact Hausdorff space $L\subseteq(0,\|a\|]$, such that $L\cup\{0\}$ is compact. It is also known that there exists a triple isomorphism Ψ from E_a onto $C_0(L)$ satisfying $\Psi(a)(t)=t$ $(t\in L)$ (compare [14, Lemma 1.14] or [15, Proposition 3.5]). Having in mind this identification we can always find a (unique) element z in E_a such that $z^{[5]}=a$. The element z will be denoted by $a^{[\frac{1}{5}]}$.

When E is a (real) J*B-triple, we have already commented that the norm closed subtriple generated by a single element a is triple isomorphic (and isometric) to

$C_0(L,\mathbb{R}) := \{f \in C_0(L), f(L) \subseteq \mathbb{R}\}$, for some locally compact subset $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact. Therefore there exists a (unique) element $z = a^{[\frac{1}{5}]}$ in E_a such that $z^{[5]} = a$.

It should be noticed here that, in the setting of J*B-triples (resp., JB*-triples) orthogonality is a "local concept" (compare Lemma 1 in [5] whose proof remains valid for J*B-triples). Indeed, two elements a and b in a J*B-triple E are orthogonal if and only if one of the following equivalent statements holds:

$$\text{(a) } \{a,a,b\} = 0, \quad \text{(b) } E_a \perp E_b, \quad \text{(c) } \{b,b,a\} = 0,$$
$$\text{(d) } a \perp b \text{ in a subtriple of } E \text{ containing both elements.}$$

It can be easily seen that $a \perp b$ if and only if $a^{[\frac{1}{5}]} \perp b^{[\frac{1}{5}]}$.

Lemma 13. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples and let (x_n) be a sequence of mutually orthogonal norm-one elements in $\sigma_E(T)$. Then, except for a finite number of values of n, $T(x_n)^{[5]} = 0$. Further, if E is a JB*-triple or a (real) J*B-triple or F is an anisotropic Jordan-Banach triple, then $T(x_n) = 0$, except for finitely many $n \in \mathbb{N}$.*

Proof. We shall argue by contradiction, supposing that $T(x_n)^{[5]} \neq 0$ for infinitely many n in $\mathbb{N}$. By passing to a subsequence, we may assume $T(x_n)^{[5]} \neq 0$ for every $n \in \mathbb{N}$. Since (x_n) is a sequence in $\sigma_E(T)$, for each $n \in \mathbb{N}$, there is a sequence $(a_{n,k})_k \subseteq E$ such that $\lim_k a_{n,k} = 0$ and $\lim_k T(a_{n,k}) = T(x_n)$. Thus, for each n in $\mathbb{N}$, $\lim_k \{x_n, a_{n,k}, x_n\} = 0$. The continuity of the triple product in F implies that

$$\begin{aligned}&\lim_k T(\{x_n, x_n, \{x_n, a_{n,k}, x_n\}\})\\ &\qquad = \lim_k \{T(x_n), T(x_n), \{T(x_n), T(a_{n,k}), T(x_n)\}\} = T(x_n)^{[5]} \neq 0.\end{aligned}$$

We observe that, for each $n \in \mathbb{N}$, the set

$$\{k \in \mathbb{N} : \{x_n, a_{n,k}, x_n\} \neq 0\}$$

is infinite. Passing to a subsequence of $(a_{n,k})$ we may assume that

$$\{x_n, a_{n,k}, x_n\} \neq 0, \forall (n,k) \in \mathbb{N} \times \mathbb{N}.$$

Therefore,

$$\lim_k \frac{\|T(\{x_n, x_n, \{x_n, a_{n,k}, x_n\}\})\|}{\|\{x_n, a_{n,k}, x_n\}\|} = \infty.$$

For each positive integer n, pick $m(n)$ such that

$$\frac{\|T(\{x_n, x_n, \{x_n, a_{n,m(n)}, x_n\}\})\|}{\|\{x_n, a_{n,m(n)}, x_n\}\|} > n\|x_n\|^2. \tag{3}$$

Writing $y_n = \{x_n, a_{n,m(n)}, x_n\}$, it follows by (2) that $y_n \perp x_m$ for $n \neq m$. The inequality (3) yields $\frac{\|T(x_n,x_n,y_n)\|}{\|x_n\|^2\|y_n\|} > n, \forall n \in \mathbb{N}$, which contradicts the main boundeness theorem (compare Theorem 12).

If E is a JB*-triple (resp., a J*B-triple), by Lemma 10, $\sigma_E(T)$ is a norm closed ideal of E and hence a JB*-triple (resp., a J*B-triple). Therefore, the sequence of mutually orthogonal elements $(z_n) = (x_n^{[\frac{1}{5}]})$ lies in $\sigma_E(T)$. Since $T(z_n)^{[5]} = T(z_n^{[5]}) = T(x_n)$, we have $T(x_n) = 0$ except for finitely many n in $\mathbb{N}$.

Finally, when F is anisotropic the final statement follows straightforwardly. $\square$

An element e in a normed Jordan triple E is called *tripotent* if $\{e,e,e\} = e$. Every tripotent e induces a decomposition $E = E_2(e) \oplus E_1(e) \oplus E_0(e)$ into the corresponding *Peirce spaces* where $E_j(e)$ is the $\frac{j}{2}$ eigenspace of $L(e,e)$. Furthermore, the following Peirce rules are satisfied:

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0,$$
$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

where $E_{i-j+k}(e) = 0$ whenever $i-j+k \notin \{0,1,2\}$ (compare [24, Proposition 21.9]). The projection $P_j(e) : E \to E_j(e)$ is called the *Peirce-j* projection induced by e.

The Peirce-2 subspace, $E_2(e)$, associated with a tripotent e is a normed Jordan *-algebra with respect to the product and involution defined by $x \circ_e y := \{x,e,y\}$ and $x^{\sharp_e} := \{e,x,e\}$, respectively (compare [24, Lemma 21.11]).

Lemma 14. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples. Then for each tripotent e in $\sigma_E(T)$ we have $T(e) = 0$.*

Proof. Suppose that there exists a tripotent e in $\sigma_E(T)$ with $T(e) \neq 0$. The linear mapping $T_{|E_2(e)} : E_2(e) \to F_2(T(e))$ is a (unital) triple homomorphism between (unital) Jordan-Banach algebras. Then T is a (unital) Jordan homomorphism. Let (x_n) be a sequence in E such that $x_n \to 0$ and $T(x_n) \to T(e)$. Then $P_2(e)(x_n) \to 0$ and $T(P_2(e)(x_n)) = P_2(T(e))(T(x_n)) \to T(e)$. Thus, e is an idempotent in $\sigma_{E_2(e)}(T_{|E_2(e)})$ with $T(e) \neq 0$, which contradicts Theorem 3.12 or Corollary 3.13 in [18]. □

Lemma 15. *Let $T : E \to F$ be a not necessarily continuous triple monomorphism from a JB*-triple (resp., a J*B-triple) to a Jordan-Banach triple. Then $\sigma_E(T) = 0$.*

Proof. Suppose that $\sigma_E(T) \neq 0$. Then, by Lemma 10, $\sigma_E(T)$ is a norm-closed triple ideal of E, and hence a JB*-triple (resp., a J*B-triple). Suppose that a is a nonzero element in $\sigma_E(T)$. We have already seen that E_a is isometrically triple isomorphic to $C_0(L)$ for some subset $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact.

We claim that L is finite. Indeed, if L were infinite, we could find, via Urysohn's lemma, a sequence of mutually orthogonal norm-one elements $(x_n)_{n\in\mathbb{N}} \subseteq E_a \subseteq \sigma_E(T)$. Since T is injective we have $T(x_n) \neq 0, \forall n \in \mathbb{N}$, which contradicts Lemma 13. Therefore L must be finite.

Let $t \in L$. Since L is finite, the function e defined by $e(t) = 1$ and $e(L \setminus \{t\}) = 0$ lies in $C_0(L)$. The element e is a tripotent in $E_a \subseteq \sigma_E(T)$ with $T(e) \neq 0$, which, by Lemma 14, is impossible. □

The following proposition is a direct consequence of Lemma 15 and Proposition 11.

Proposition 16. *Let $T : E \to F$ be a not necessarily continuous triple monomorphism from a (complex) JB*-triple (resp., a (real) J*B-triple) to a Jordan-Banach triple. Then the linear mapping $\widetilde{T} : E \to F/\sigma_F(T)$, $\widetilde{T}(a) = T(a) + F/\sigma_F(T)$, is a continuous triple monomorphism.* □

Theorem 17. *Let $T : E \to F$ be a not necessarily continuous triple monomorphism from a (complex) JB*-triple (resp., a (real) J*B-triple) to a normed Jordan triple. Then T is bounded below.*

Proof. We may assume, without loss of generality, that F is a Jordan-Banach triple; otherwise we can replace F with its canonical completion.

Let π denote the canonical projection of F onto $F/\sigma_F(T)$. Proposition 16 assures that the linear mapping $\widetilde{T} : E \to F/\sigma_F(T)$, $x \mapsto \pi(T(x))$, is a continuous triple monomorphism. By Propositions 6 and 5, there exists a positive constant M satisfying

$$M \ \|x\| \leq \|\widetilde{T}(x)\| = \|\pi(T(x))\| \leq \|T(x)\|, \ x \in E,$$

which shows that T is bounded below. □

The following corollary is the desired generalisation of a result due to B. Yood [25] and S.B. Cleveland [8].

Corollary 18. *Let $T : E \to F$ be a not necessarily continuous triple monomorphism from a (complex) JB^*-triple (resp., a (real) J^*B-triple) to a normed Jordan triple. Then the norm closure of $T(E)$ in the canonical completion of F decomposes as the direct sum of $T(E)$ and $\sigma_F(T)$.*

Proof. Let b be an element in the norm closure of $T(E)$ in the completion of F. By assumptions, there exists a sequence (x_n) in E such that $b = \lim T(x_n)$.

Since, by Theorem 17, T is bounded below, the sequence (x_n) is a Cauchy sequence in E. Therefore there exists x_0 in E satisfying $\lim x_n - x_0 = 0$ and $\lim T(x_n - x_0) = b - T(x_0)$. This shows that $b = T(x_0) + (b - T(x_0))$, where $b - T(x_0) \in \sigma_F(T)$. Finally, $T(E) \cap \sigma_F(T) = T(\sigma_E(T)) = \{0\}$, by Lemma 15. □

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
E-mail address: pacopolo@ugr.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
E-mail address: jgarces@correo.ugr.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
E-mail address: aperalta@ugr.es

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Weakly compact orthogonality preservers on C*-algebras

Francisco J. Fernández-Polo · Jorge J. Garcés · Antonio M. Peralta

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Abstract We establish a complete description of weakly compact orthogonality preserving operators from a C*-algebra A (or a JB*-algebra) to a JB*-triple E. We prove that any such operator is the sum of a series of mutually orthogonal triple homomorphisms from A^{**} to a certain Peirce-2 subspace of E^{**} multiplied by a mutually orthogonal norm-null sequence in E which enjoy certain operator commutativity relations.

1 Introduction

A weighted endomorphism of the C*-algebra, $C(K)$, of all complex-valued continuous functions on a compact Hausdorff space K is a linear operator $T : C(K) \to C(K)$ satisfying $T(f)(t) = u(t)f(\varphi(t))$, where u is an element in $C(K)$ and φ is a continuous mapping from K to K. The operator T is usually denoted by uC_φ. H. Kamowitz described compact weighted endomorphisms of $C(K)$ in [23]. The characterisation established by Kamowitz assures that a weighted endomorphism, uC_φ, is compact if and only if for each connected component C of $\{t \in K : u(t) \neq 0\}$ there exists an open set $V \supset C$ such that φ is constant on V. This description essentially says that any such operator is the sum of a series of one-dimensional weighted endomorphisms. It is well known that every compact homomorphism between $C(K)$-spaces has finite dimensional range; however, when K is infinite, for

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F. J. Fernández-Polo · J. J. Garcés · A. M. Peralta (✉)
Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
e-mail: aperalta@ugr.es

F. J. Fernández-Polo
e-mail: pacopolo@ugr.es

J. J. Garcés
e-mail: jgarces@ugr.es

example [0, 1], we can find examples of compact weighted endomorphisms of $C(K)$ with infinite dimensional ranges (see Remark 13).

Weighted endomorphisms of $C(K)$ belong to the class of those continuous linear operators on $C(K)$ which are called separating or disjointness preserving or Lamperti operators (compare [2, Example 2.2.1]). We recall that a linear mapping T between two normed algebras is said to be *separating* or *zero-product preserving* if $T(a)T(b) = 0$ whenever $ab = 0$. We say that a linear mapping T between two C*-algebras is *orthogonality preserving* if $T(a) \perp T(b)$ whenever $a \perp b$, where two elements a, b in a C*-algebra are said to be *orthogonal* (written $a \perp b$) whenever $ab^* = b^*a = 0$. When A and B are two abelian C*-algebras, a linear mapping $T : A \to B$ is separating if and only if it is orthogonality preserving. Every homomorphism between two normed algebras is separating. The class of orthogonality preserving operators on $C(K)$ is strictly bigger than the class of weighted endomorphisms of $C(K)$. Actually, a bounded linear operator $T : C(K) \to C(K)$ is orthogonality preserving if and only if there exists u in $C(K)$ and a mapping $\varphi : K \to K$ which is continuous on $\{t \in K : u(t) \neq 0\}$ such that $T(f)(t) = (uC_\varphi)(f)(t) = u(t)\, f(\varphi(t))$ (compare [2, Example 2.2.1]).

Lin and Wong [27] considered weakly compact orthogonality preserving operators between abelian C*-algebras (i.e. $C_0(L)$-spaces). Given two locally compact Hausdorff spaces L_1 and L_2, Lin and Wong proved that a bounded orthogonality preserving linear operator from $C_0(L_1)$ to $C_0(L_2)$ is weakly compact if and only if it is compact, if and only if, it can be represented as a (possibly finite) norm convergent countable sum of the form $T = \sum_n \delta_{t_n} \otimes h_n$, (i.e. $T(f) = \sum_n f(t_n)\, h_n$) where (t_n) is a sequence of distinct points in L_1 and (h_n) is a norm-null sequence of mutually disjoint functions in $C_0(L_2)$.

Orthogonality preserving operators between general C*-algebras have been intensively studied by many researchers (compare [1,2,6–8,21,35,36] and [9]). The studies on continuous orthogonality preserving operators between general C*-algebras culminate with the following description obtained in [6] and [7] (see §2 for the detailed definitions and concepts).

Theorem 1 ([6, Theorem 17 and Corollary 18] and [7, Theorem 4.1]) *Let $T : A \to E$ be a bounded linear mapping from a C*-algebra (resp., a JB*-algebra) to a JB*-triple. For $h = T^{**}(1)$ and $r = r(h)$ the following assertions are equivalent.*

(*a*) *T is orthogonality preserving.*
(*b*) *There exits a (unique) Jordan *-homomorphism $S : A \to E_2^{**}(r)$ such that $S^{**}(1) = r$, $S(x)$ and h operator commute and $T(x) = h \bullet_r S(x)$, for every $x \in A$.*
(*c*) *T preserves zero-triple-products, that is, $\{T(x), T(y), T(z)\} = 0$ whenever $\{x, y, z\} = 0$.* □

Based on the above description, we shall prove here a complete characterisation of weakly compact orthogonality preserving operators between general C*-algebras (or between JB*-algebras). The characterisation actually follows from a more general theorem which determines the general form of a weakly compact orthogonality preserving operator from a C*-algebra to a JB*-triple (see Theorems 8 and 11). More concretely, we prove that for every weakly compact orthogonality preserving operator T from a C*-algebra, A, to a JB*-triple, E, denoting by $r = r(h)$ the range tripotent of the element $h = T^{**}(1)$ in E^{**}, there exist a countable family $\{I_n\}_{n\in\mathbb{N}}$ of mutually orthogonal weak*-closed C*-ideals in A^{**}, a family $\{S_n : A^{**} \to E_2^{**}(r) : n \in \mathbb{N}\}$ of continuous Jordan *-homomorphisms and a sequence (x_n) of mutually orthogonal elements in E satisfying the following statements:

(*a*) Each I_n is a finite type I von Neumann factor;
(*b*) $\|x_n\| \to 0$ and $h = \sum_{n=1}^{\infty} x_n$;

(*c*) $S_n|_{I_n}$ is a Jordan *-monomorphism, $S_n|_{I_n^\perp} = 0$, S_n and S_m have orthogonal ranges for $n \neq m$;
(*d*) For each x in A^{**}, x_n and $S_m(x)$ operator commute for every n and m; and
(*e*)

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x),$$

for every $x \in A$. Moreover, the Jordan *-homomorphism $S : A \to E_2^{**}(r)$ given in Theorem 1, (*b*), satisfies that $S(z) = \sum_{n=1}^{\infty} S_n(z)$, for each z in A, where the series converges in the weak* topology of $E_2^{**}(r)$.

Among the consequences of this result, it follows that for each orthogonality preserving operator T from a C*-algebra to a JB*-triple the following are equivalent:

(*a*) T is compact;
(*b*) T is weakly compact;
(*c*) T admits a compact factorisation through a c_0-sum of the form

$$\bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C}),$$

where (m_n) is a sequence of natural numbers.

These results generalise the previous studies due to Kamowitz [23] and Lin and Wong [27].

Before dealing with our main results, we study in section §3 the structure of weakly compact triple homomorphisms from a JB*-triple to a normed Jordan triple. As a corollary, we establish that every weakly compact triple homomorphism from a C*-algebra to a normed Jordan triple has finite range, a result which generalises a previous contribution due to Mathieu [30].

2 Preliminaries

In this preliminary section we review the basic facts and definitions needed in this paper. We begin by recalling that a complex (resp., real) *normed Jordan triple* is a complex (resp., real) normed space E equipped with a continuous triple product

$$E \times E \times E \to E$$
$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$ $(z \in E)$. If E is complete with respect to the norm (i.e. if E is a Banach space), then it is called a complex (resp., real) *Jordan-Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan-Banach triple. Unless otherwise specified, the term "normed Jordan triple" (resp., "Jordan-Banach triple") will always mean a real or complex normed Jordan triple (resp., a real or complex Jordan-Banach triple).

A real (resp., complex) *Jordan algebra* is a (non-necessarily associative) algebra over the real (resp., complex) field whose product is abelian and satisfies $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. A normed Jordan algebra is a Jordan algebra A equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$, $a, b \in A$. A *Jordan-Banach algebra* is a normed Jordan algebra whose norm is complete.

Every real or complex associative Banach algebra (resp., Jordan-Banach algebra) is a real Jordan-Banach triple with respect to the product $\{a, b, c\} = \frac{1}{2}(abc + cba)$ (resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$).

A linear mapping T between two Jordan triples is said to be a *triple homomorphism* if $T\{a, b, c\} = \{T(a), T(b), T(c)\}$.

A JB-algebra is a real Jordan-Banach algebra A in which the norm satisfies the following two additional conditions for a, b in A:

(a) $\|a^2\| = \|a\|^2$;
(b) $\|a^2\| \leq \|a^2 + b^2\|$.

A JB*-algebra is a complex Jordan-Banach algebra A equipped with an algebra involution * satisfying

$$\|2(a \circ a^*) \circ a - a^2 \circ a^*\| = \|a\|^3,$$

for every $a \in A$. JB-algebras can be identified with the self adjoint parts of JB*-algebras (c.f. [18, §3] or [34]).

C*- and JB*-algebras belong to a more general class of (complex) Banach spaces known under the name of (complex) JB*-triples. We recall that a complex Jordan-Banach triple E is said to be a (*complex*) *JB*-triple* if it satisfies the following axioms:

(1) The map $L(a, a) : E \to E, x \mapsto \{a, a, x\}$ is an hermitian operator with non negative spectrum for all a in E;
(2) $\|\{a, a, a\}\| = \|a\|^3$ for all a in E.

Every C*-algebra (resp., every JB*-algebra) is a JB*-triple with respect to $\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a)$ (resp., $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

Complex JB*-triples were introduced by Kaup in the study of Bounded Symmetric Domains in complex Banach spaces (see [24], [25]). *Real JB*-triples* were introduced by Isidro et al. (c.f. [20]) as norm-closed real subtriples of complex JB*-triples.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [3]). It is known that the triple product of a JBW*-triple is separately weak* continuous (c.f. [3] or [29]). The second dual of a JB*-triple E is a JBW*-triple with a product extending the product of E [14].

An element e in a JB*-triple E is said to be a *tripotent* if $\{e, e, e\} = e$. Each tripotent e in E gives rise to the following decomposition of E

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e, e)$ (compare [28, Theorem 25]). This decomposition is called the *Peirce decomposition* of E with respect to the tripotent e.

The Peirce space $E_2(e)$ is a JB*-algebra with product and involution defined by $x \bullet_e y := \{x, e, y\}$ and $x^{\sharp_e} := \{e, x, e\}$, respectively.

For each element x in a JB*-triple E, the symbol E_x will stand for the JB*-subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(L)$ for some locally compact Hausdorff space L contained in $(0, \|x\|]$, such that $L \cup \{0\}$ is compact, where $C_0(L)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(L)$, then $\Psi(x)(t) = t (t \in L)$ (cf. [24, Corollary 4.8] and [25, Corollary 1.15]).

Consequently, for each $x \in E$, there exists a unique element $y \in E_x$ satisfying $\{y, y, y\} = x$. The element y, denoted by $x^{[\frac{1}{3}]}$, is termed the *cubic root* of x. We can inductively define, $x^{[\frac{1}{3^n}]} = \left(x^{[\frac{1}{3^{n-1}}]}\right)^{[\frac{1}{3}]}$, $n \in \mathbb{N}$. The sequence $(x^{[\frac{1}{3^n}]})$ converges in the weak* topology of E^{**} to a tripotent in E^{**}, denoted by $r(x)$ and called the *range tripotent* of x. The tripotent $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying x is positive in the JBW*-algebra $E_2^{**}(e)$ (see, for example, [15, Lemma 3.3]).

Another central concept in this paper is orthogonality in C*-algebras and JB*-triples. Two elements a, b in a JB*-triple, E, are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. Lemma 1 in [6] shows that $a \perp b$ if and only if one of the following nine statements holds:

$$\begin{gathered} \{a, a, b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b); \\ E_2^{**}(r(a)) \perp E_2^{**}(r(b)); \quad r(a) \in E_0^{**}(r(b)); \quad a \in E_0^{**}(r(b)); \\ b \in E_0^{**}(r(a)); \quad E_a \perp E_b \quad \{b, b, a\} = 0. \end{gathered}$$

On every C*-algebra, A, we can consider its structure of JB*-triple and its natural structure of C*-algebra. It was already remarked in [6] that the two notions of orthogonality in A coincide, i.e. two elements a, b in A are orthogonal for the C*-algebra product if and only if they are orthogonal when A is considered as a JB*-triple.

Given a subset M in a JB*-triple E, the *annihilator of* M, $M^\perp$, is the set

$$\{y \in E : y \perp x, \forall x \in M\}.$$

If (x_n) is a bounded sequence in a Banach space X, the series $\sum_k \mu_k x_k$ need not be, in general, convergent in X for every $(\mu_n) \subset c_0$. Remark 13 in [9] points out that for each mutually orthogonal bounded sequence (x_n) in a JB*-triple, E, the series $\sum_k \mu_k x_k$ converges in E for every $(\mu_n) \subset c_0$.

The last ingredient needed in this paper is the notion of operator commutativity. Two elements a and b in a JB*-algebra A are said to *operator commute* in A if the multiplication operators M_a and M_b commute, where M_a is defined by $M_a(x) := a \circ x (x \in A)$. That is, a and b operator commute if and only if $(a \circ x) \circ b = a \circ (x \circ b)$ for all x in A. Two self-adjoint elements a and b in A generate a JB*-subalgebra that can be realised as a JC*-subalgebra of some $B(H)$, [34], and, in this realisation, a and b commute in the usual sense whenever they operator commute in A (see [33, Proposition 1]).

Remark 2 Let a, b be two elements in a JB*-algebra A with $a = a^*$. Suppose that a and b operator commute. Then it is easy to see, via Macdonald's theorem (cf. [18, Theorem 2.4.13]), that for each z in the JB*-subalgebra (resp., JB*-subtriple) of A generated by a, the elements z and b operator commute. When A is a JBW*-algebra, the same statement holds for every z in the JBW*-subalgebra (resp., JBW*-subtriple) of A generated by a.

The following technical result will be needed later.

Lemma 3 *Let a and b be two orthogonal elements in a JB-algebra A with $a \geq 0$. Let h be an element in A such that h and b operator commute and $a \circ h \geq 0$. Then $a \circ h \perp b$.*

Proof Since $a \circ h \geq 0$, Lemma 4.1 in [7] affirms that $a \circ h \perp b$ if and only if $b \circ (a \circ h) = 0$. Applying that h and b operator commute we have

$$b \circ (a \circ h) = h \circ (a \circ b) = 0,$$

where in the last equality we apply that $a \geq 0$ and $a \perp b$. □

3 Weakly compact triple homomorphisms

It is well known that every reflexive C*-algebra is finite dimensional (compare [31, Proposition 2]). In other words, the identity mapping on a C*-algebra A is weakly compact if and only if A is finite dimensional. Actually, an algebraic homomorphism from a C*-algebra to a normed algebra is weakly compact if and only if it has finite dimensional range (see [17], [30]).

Suppose that $S : A \to B$ is a Jordan *-isomorphism between two C*-algebras. In this case, $S^{**} : A^{**} \to B^{**}$ is a Jordan *-isomorphism between von Neumann algebras. It follows, from Kadison's theorem (see [22, Theorem 10]), that there exist weak* closed ideals I_1 and I_2 in A^{**} and J_1 and J_2 in B^{**} satisfying $A^{**} = I_1 \oplus^\infty I_2$, $B^{**} = J_1 \oplus^\infty J_2$, $S^{**}|_{I_1} : I_1 \to J_1$ is an *-isomorphism, and $S^{**}|_{I_2} : I_2 \to J_2$ is an *-anti-isomorphism. Thus, S is weakly compact if and only if it has finite dimensional range.

We shall see that the above conclusion remains true for every triple homomorphism from a C*-algebra to a normed Jordan triple (in particular, for every Jordan homomorphism from a C*-algebra to a normed algebra). This statement will follow from a result which is valid in a more general setting.

We recall that a subspace I of a normed Jordan triple E is a *triple ideal* if $\{E, E, I\} + \{E, I, E\} \subseteq I$. The quotient of a normed Jordan triple by a closed triple ideal is a normed Jordan triple. It is also known that the quotient of a JB*-triple (resp., a J*B-triple) by a closed triple ideal is a JB*-triple (resp., a J*B-triple) (compare [25]).

The proof of the following theorem combines the argument given by Mathieu [30] with a recent Kaplansky theorem for real and complex JB*-triples obtained in [16].

Theorem 4 *Let $S : E \to F$ be a weakly compact (resp., compact) triple homomorphism from a real or complex JB*-triple to a normed Jordan triple. Then $E/\ker(S)$ and $S(E)$ are reflexive (resp., finite dimensional) Jordan-Banach triples.*

Proof The mapping $\widetilde{S} : E/\ker(S) \to S(E)$, $\widetilde{S}(x + \ker(S)) := S(x)$ is a well-defined weakly compact triple monomorphism. By Proposition 6 in [16], there exists $M > 0$ such that $M\|x\| \leq \|\widetilde{S}(x)\| \leq \|S\|\|x\|$, for every $x \in E$. This shows that the mapping $\widetilde{S}^{-1} : S(E) \to E/\ker(S)$, $S(x) \mapsto x + \ker(S)$ is a bounded linear operator.

Finally, since the identity mapping $Id : \widetilde{S}^{-1} \circ \widetilde{S} : E/\ker(S) \to E/\ker(S)$ is weakly compact, it follows that $E/\ker(S)$ is a reflexive JB*-triple. The statement follows because $E/\ker(S)$ and $S(E)$ are isomorphic as normed Jordan (Banach) triples. □

Now, let E be a real or complex JB*-triple. A subset $S \subseteq E$ is said to be *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $card(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E.

Given a real or complex JB*-triple E, a necessary and sufficient requirement for E to be reflexive is that E has the Radon-Nikodym property, or equivalently, E is isomorphic to a Hilbert space or E has finite rank ([11, Theorem 6] and [4, Theorems 2.3 and 3.1]).

Corollary 5 *Let $S : E \to F$ be a continuous triple homomorphism from a real or complex JB*-triple to a normed Jordan triple. The following statements are equivalent:*

(a) *S is weakly compact.*
(b) *There exists a triple isomorphism from $S(E)$ to a reflexive JB*-triple.*
(c) *There exists a triple isomorphism from $S(E)$ to a finite rank JB*-triple.*
(d) *$S(E)$ is homeomorphic to a Hilbert space.*

(*e*) $E/\ker(S)$ *is a reflexive (or finite rank) JB***-triple.* □

When specialized to the setting of C*-algebras, the above result reads as follows:

Corollary 6 *Every weakly compact triple homomorphism from a C***-algebra to a normed Jordan triple has finite dimensional range.*

Proof Let $S : A \to E$ be a weakly compact triple homomorphism from a C*-algebra to a normed Jordan triple. The mapping $S^{**} : A^{**} \to E^{**}$ is a weak*-continuous weakly compact triple homomorphism with $S^{**}(A^{**}) \subseteq E$.

Since $K = \ker(S^{**})$ is a weak*-closed triple ideal of A^{**}, there exists a weak*-closed triple ideal J of A^{**} such that $A^{**} = J \oplus^{\infty} K$ (compare [19, Theorem 4.2]). In this case the unit element in A^{**} decomposes as an orthogonal sum $1 = a + b$ where $a \in K$ and $b \in J$. It follows that a and b are projections in A^{**} with $az = za = z$, for every $z \in K$ (resp., $bz = zb = z, \forall z \in J$). Thus, $K = aA^{**}a$ and $J = bA^{**}b$ are weak*-closed C*-ideals of A^{**}. Since the identity mapping $S^{**}|_J \circ S^{**}|_J^{-1} : J \to J$ is weakly compact, J is a reflexive C*-algebra and hence finite dimensional, which shows that $S^{**}(A^{**})$ and $S(A)$ are finite dimensional. □

Corollary 7 *Let* $S : J \to F$ *be a weakly compact triple homomorphism from a JB***-algebra to a normed Jordan triple. Then* $J/\ker(S)$ *is a reflexive JB*-algebra.* □

4 Weakly compact orthogonality preserving operators on C*-algebras

The aim of this section is to establish a complete description of those orthogonality preserving operators between C*-algebras which enjoy the additional property of being weakly compact.

We have already commented that continuous linear orthogonality preserving operators between C*-algebras (resp., JB*-algebras) have been recently characterised by Burgos, Martínez and the authors of this note in [6] and [7]. The characterization obtained in the just quoted papers was stated in Theorem 1.

An element a in a JB*-triple E is said to be *von Neumann regular* if there exists $b \in E$ such that $Q(a)(b) = a$ and $Q(b)(a) = b$. The element b is called the *generalised inverse* of a. We observe that every tripotent e in E is von Neumann regular and its generalised inverse is e itself. Lemma 3.2 in [26] (see also the proof of [10, Theorem 3.4]) shows that, for each von Neumann regular element $a \in E$, there exists a tripotent $e \in E$ satisfying a is a symmetric and invertible element in the JB*-algebra $E_2(e)$. Moreover, e coincides with the range tripotent of a. It is also known that an element a in E is von Neumann regular if and only if it is von Neumann regular in any JB*-subtriple containing a.

Given a weak*-closed subalgebra, B, of a von Neumann algebra A, we shall say that B is a *factor* of A if A decomposes as the orthogonal (ℓ_∞)-sum of B and another weak*-closed ideal of A.

We can now deal with the desired description of weakly compact orthogonality preserving operators from a C*-algebra to a JB*-triple.

Theorem 8 *Let A be a C***-algebra, E a JB***-triple,* $T : A \to E$ *a weakly compact orthogonality preserving operator and let* $r = r(h)$ *be the range tripotent of the element* $h = T^{**}(1)$ *in* E^{**}*. Then there exists a countable family* $\{I_n\}_{n\in\mathbb{N}}$ *of mutually orthogonal weak***-closed C***-ideals in* A^{**}*, a family* $\{S_n : A^{**} \to E_2^{**}(r) : n \in \mathbb{N}\}$ *of continuous Jordan *-homomorphisms and a sequence* (x_n) *of mutually orthogonal elements in E satisfying:*

(*a*) *Each I_n is a finite type I von Neumann factor;*
(*b*) $\|x_n\| \to 0$ *and* $h = \sum_{n=1}^{\infty} x_n$;
(*c*) $S_n|_{I_n}$ *is a Jordan* **-monomorphism,* $S_n|_{I_n^{\perp}} = 0$, S_n *and* S_m *have orthogonal ranges for* $n \neq m$;
(*d*) *For each* x *in* A^{**}, x_n *and* $S_m(x)$ *operator commute for every* n *and* m*; and*
(*e*)

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x),$$

for every $x \in A$*. Moreover, the Jordan* **-homomorphism* $S : A \to E_2^{**}(r)$ *given in Theorem 1,* (*b*)*, satisfies that* $S(z) = \sum_{n=1}^{\infty} S_n(z)$*, for each* z *in* A*, where the series converges in the weak** *topology of* $E_2^{**}(r)$.

Proof Let 1 denote the unit of A^{**}, and let $S : A \to E_2^{**}(r)$ be the (unital, i.e. $S^{**}(1) = r$) Jordan *-homomorphism given by Theorem 1, *(b)*, that is, denoting $h = T^{**}(1)$ and $r = r(h)$, it follows that $S(x)$ and h operator commute in $E_2^{**}(r)$ and $T(x) = L(h, r)S = h \bullet_r S(x)$, for every $x \in A$.

We claim that for each natural n, there exist a finite set $\mathcal{F}_n \subset \mathbb{N}$, a finite rank projection q_n in A^{**}, a family of mutually orthogonal finite type I von Neumann factors, $M_{m_{n,i}}(\mathbb{C})$, in the atomic part of A^{**}, with $m_{n,i} \in \mathbb{N}$, and a set $\{x_{n,i} : i \in \mathcal{F}_n\}$ of mutually orthogonal elements in E, satisfying:

(*i*) $\left\| T^{**}\left(1 - \sum_{i=1}^{n} q_i\right)\right\| \leq \frac{\|h\|}{n+1}$;
(*ii*) q_n coincides with the unit of the $\ell_\infty - sum\ I_n := \bigoplus_{i \in \mathcal{F}_n} M_{m_{n,i}}(\mathbb{C})$;
(*iii*) I_n is a weak*-closed C*-ideal of A^{**};
(*iv*) If $p_{n,i}$ denotes the unit in $M_{m_{n,i}}(\mathbb{C})$, we have $x_{n,i} = T^{**}(p_{n,i})$, and, for each $i \in \mathcal{F}_n$, $\|x_{n,i}\| \leq \frac{\|h\|}{n}$;
(*v*) $\{q_1, \ldots, q_n\}$ and $\cup_{k=1}^{n}\{x_{k,i} : i \in \mathcal{F}_k\}$ are subsets of mutually orthogonal elements in A^{**} and E, respectively;
(*vi*) $S_{n,i} := S|_{M_{m_{n,i}}(\mathbb{C})} : M_{m_{n,i}}(\mathbb{C}) \to E^{**}$ is a triple monomorphism and $L(x_{n,i}, r)S = L(x_{n,i}, r)S_{n,i}$.

We shall proceed by induction.

Since T is weakly compact, $T^{**}(A^{**}) \subseteq E$, and thus $h = T^{**}(1)$ lies in E. Let us write E_h for the JB*-subtriple of E generated by h. We have already commented that there exists a locally compact space L such that $L \cup \{0\}$ is compact and E_h is JB*-triple isomorphic to $C_0(L)$, and under this identification h corresponds to the function $t \mapsto t$ (cf. [24, Corollary 4.8], [25, Corollary 1.15]).

Let e_1 be the tripotent in $\overline{E_h}^{w^*}$ whose triple representation in $C_0(\sigma(h))^{**}$ is the characteristic function of the set $\left(\sigma(h) \cap]\frac{\|h\|}{2}, \|h\|]\right)$, and let $a_1 = L(e_1, r)h \in E^{**}$. It is clear that a_1 is a von Neumann regular element in E^{**} whose generalized inverse will be denoted by b_1.

We define $R_1 : A \to E^{**}$ by $R_1 = L(b_1, r)T$. We notice that $R_1^{**}(1) = e_1$ is a tripotent in E^{**}. Following the argument in the proof of [7, Theorem 4.1] (see also Theorem 6 in [6]), we deduce that R_1 is a triple homomorphism which is clearly weakly compact. R_1^{**} is again a weakly compact triple homomorphism whose range lies in E^{**}. Since R_1^{**} actually is a weak*-continuous triple homomorphism, $\ker(R_1^{**})$ is a weak*-closed triple ideal of A^{**}, and hence there exists a weak*-closed triple ideal I_1 of A^{**} such that $A^{**} = \ker(R_1) \oplus^{\infty} I_1$ and $R_1^{**}|_{I_1}$ is a triple monomorphism. Corollary 6 (and its proof) applies now to show that I_1 is a finite dimensional C*-ideal of A^{**}. Thus, there exists a finite subset $\mathcal{F}_1 = \{1, \ldots, k_1\}$ and a finite family of mutually orthogonal finite type I von Neumann factors, $M_{m_{1,i}}(\mathbb{C})$, in the atomic part of A^{**}, with $m_{1,i} \in \mathbb{N}$, such that

$$I_1 = \bigoplus_{i\in\mathcal{F}_1}^{\ell_\infty} M_{m_{1,i}}(\mathbb{C})$$

(compare [32, Theorem I.11.2]). For each $i \in \mathcal{F}_1$, we denote by $\pi_{1,i}$ the projection of A^{**} onto $M_{m_{1,i}}(\mathbb{C})$ and we define $p_{1,i} = \pi_{1,i}(1)$, $q_1 = \sum_{i\in\mathcal{F}_1} p_{1,i}$ and $x_{1,i} = T^{**}(p_{1,i})$. Since $\{p_{1,i}\}_{i\in\mathcal{F}_1}$ are mutually orthogonal projections and T^{**} is orthogonality preserving and E-valued, the elements in the set $\{x_{1,i} : i \in \mathcal{F}_1\}$ are mutually orthogonal and lie in E.

Having in mind that, for each $x \in A$ and $z \in \overline{E_h}^{w^*}$, $T(x)$ (resp., $S(x)$) and z operator commute in $E_2^{**}(r)$ (compare Remark 2 or [7, §4]), we have

$$\begin{aligned} L(a_1, r)R_1 &= L(a_1, r)L(b_1, r)T = L(e_1, r)T \\ &= L(e_1, r)L(h, r)S = L(a_1, r)S, \end{aligned}$$

which gives

$$\begin{aligned} L(a_1, r)S(q_1) &= L(a_1, r)R_1(q_1) = L(a_1, r)R_1(1) \\ &= L(a_1, r)S(1) = L(a_1, r)(r) = a_1. \end{aligned}$$

Denoting by $y_1 = \sum_{i\in\mathcal{F}_1} x_{1,i} = T^{**}(q_1)$ we obtain that

$$\begin{aligned} h - y_1 &= T^{**}(1 - q_1) = h - T^{**}(q_1) = h - L(h, r)S(q_1) \\ &= h - L(h - a_1, r)S(q_1) - L(a_1, r)S(q_1) \\ &= h - L(h - a_1, r)S(q_1) - a_1 \\ &= L(h - a_1, r)(r - S(q_1)). \end{aligned}$$

Since $\|h - a_1\| \leq \frac{\|h\|}{2}$ and $\|r - S(q_1)\| \leq 1$ then we obtain

$$\left\| T^{**}(1 - q_1) \right\| = \|h - y_1\| \leq \frac{\|h\|}{2}.$$

We shall prove now that $S_{1,i} := S|_{M_{m_{1,i}}(\mathbb{C})} : M_{m_{1,i}}(\mathbb{C}) \to E_2^{**}(r)$ is a Jordan *-monomorphism. Suppose that $S_{1,i}(x) = S(x) = 0$, for some $x \in M_{m_{1,i}}(\mathbb{C}) \subseteq A^{**}$. Since

$$R_1(x) = L(b_1, r)T(x) = L(b_1, r)L(h, r)S(x) = 0$$

we have $x = 0$. Now, we notice that, $A^{**} = M_{m_{1,i}}(\mathbb{C}) \oplus^\infty M_{m_{1,i}}(\mathbb{C})^\perp$, and for each self adjoint element x in $M_{m_{1,i}}(\mathbb{C})^\perp$, $S(x) \perp S(p_{1,i})$. The element h (resp., $S(1) = r$) decomposes as the orthogonal sum of $T^{**}(p_{1,i}) = x_{1,i}$ and $T^{**}(1 - p_{1,i})$ (resp., $S(p_{1,i})$ and $S(1 - p_{1,i})$). Therefore $S(p_{1,i})$ and $h \circ S(p_{1,i})$ are positive elements in the self-adjoint part of $E_2^{**}(r)$. Since h and $S(x)$ operator commute, Lemma 3 implies that

$$x_{1,i} = T^{**}(p_{1,i}) = h \bullet_r S(p_{1,i}) = L(h, r)S(p_{1,i}) \perp S(x).$$

By linearity $x_{1,i} \perp S(x)$, for every $x \in M_{m_{1,i}}(\mathbb{C})^\perp$, and hence $r(x_{1,i}) \perp S(x)$, for every $x \in M_{m_{1,i}}(\mathbb{C})^\perp$ (compare [6, Lemma 1]). Moreover, r also decomposes as the orthogonal sum of the range of $x_{1,i}$ and the range of $T^{**}(1 - p_{1,i})$, thus $L(x_{1,i}, r)S(x) = 0$, which gives $L(x_{1,i}, r)S = L(x_{1,i}, r)S_{1,i}$.

Suppose now, in our inductive step, that for each $k = 1, \ldots, n$, the finite rank projection q_k and the families $\mathcal{F}_k$, $\{x_{k,i} : i \in \mathcal{F}_k\}$, and $\{M_{m_{k,i}}(\mathbb{C}) : i \in \mathcal{F}_k\}$, have been defined satisfying the corresponding induction hypothesis.

The von Neumann algebra A^{**} decomposes as the orthogonal sum

$$A^{**} = A_0^{**}\left(\sum_{i=1}^{n} q_i\right) \oplus A_2^{**}\left(\sum_{i=1}^{n} q_i\right),$$

where, by the induction hypothesis, $A_2^{**}\left(\sum_{i=1}^{n} q_i\right)$ coincides with the orthogonal sum of $I_1, \ldots, I_n$.

Let $h_{n+1} = T^{**}(1 - \sum_{i=1}^{n} q_i)$. From the induction hypothesis we have

$$\|h_{n+1}\| = \left\| T^{**}\left(1 - \sum_{i=1}^{n} q_i\right)\right\| \leq \frac{\|h\|}{n+1}.$$

Let e_{n+1} be the tripotent in $\overline{E_{h_{n+1}}}^{w^*}$ whose representation in $\overline{E_{h_{n+1}}}^{w^*} = \overline{C_0(\sigma(h_{n+1}))}^{w^*}$ is the characteristic function of the set $\left(\sigma(h_{n+1}) \cap]\frac{\|h\|}{n+2}, \|h\|]\right)$, and let $a_{n+1} = L(e_{n+1}, r)h$. It is easy to check that h decomposes as the orthogonal sum of h_{n+1} and $T^{**}(\sum_{i=1}^{n} q_i)$, which, in particular, gives $a_{n+1} = L(e_{n+1}, r)h_{n+1} = L(e_{n+1}, e_{n+1})h_{n+1}$.

In case $a_{n+1} = 0$ we skip to step $n+2$. Let us suppose that $a_{n+1} \neq 0$. Since $a_{n+1} = L(e_{n+1}, e_{n+1})h_{n+1}$, a_{n+1} is a von Neumann regular element in E^{**} whose generalized inverse is denoted by b_{n+1}. Let $R_{n+1} : A \to E^{**}$ be the operator defined by $R_{n+1} = L(b_{n+1}, e_{n+1})T$.

The same reasoning given in the case $n = 1$ ascertains that R_{n+1} is a weakly compact triple homomorphism, and hence there exist a finite set $\mathcal{F}_{n+1} = \{1, \ldots, k_{n+1}\}$, a family of mutually orthogonal finite type I von Neumann factors, $\left(M_{m_{n+1,i}}(\mathbb{C})\right)_{i \in \mathcal{F}_{n+1}}$, in the atomic part of A^{**}, such that $I_{n+1} = \bigoplus_{i \in \mathcal{F}_{n+1}}^{\infty} M_{m_{n+1,i}}(\mathbb{C})$ is a weak*-closed C*-ideal of A^{**}, $A^{**} = \ker(R_{n+1}^{**}) \bigoplus^{\infty} I_{n+1}$, and $R_{n+1}^{**}|_{I_{n+1}} : I_{n+1} \to E^{**}$ is a triple monomorphism.

For each $i \in \mathcal{F}_{n+1}$, let $p_{n+1,i}$ denote $\pi_{n+1,i}(1)$, where $\pi_{n+1,i}$ stands for the canonical projection of A^{**} onto $M_{m_{n+1,i}}(\mathbb{C})$, and $x_{n+1,i} = T^{**}(p_{n+1,i})$. Having in mind that $\{p_{n+1,i}\}_{i \in \mathcal{F}_{n+1}}$ are mutually orthogonal (and T is weakly compact and orthogonality preserving) we deduce that $\{x_{n+1,i}\}_{i \in \mathcal{F}_{n+1}}$ is a set of mutually orthogonal elements in E. We have already commented that $h_{n+1} \perp T^{**}(q_k)$, for each $k = 1, \ldots, n$. Therefore $R_{n+1}^{**}(I_k) = \{0\}$, for every $k = 1, \ldots, n$, which implies that

$$\bigoplus_{i=1,\ldots,n}^{\ell_\infty} I_k = A_2^{**}\left(\sum_{k=1}^{n} q_k\right) \subseteq \ker(R_{n+1}^{**}).$$

Since $p_{n+1,i} \leq 1 - \sum_{k=1}^{n} q_k$, h_{n+1} can be written as the orthogonal sum of $T^{**}\left(\sum_i p_{n+1,i}\right)$ and $T^{**}\left(1 - \sum_{k=1}^{n} q_k - \sum_i p_{n+1,i}\right)$. By the induction hypothesis,

$$\begin{aligned} \sup_{i \in \mathcal{F}_{n+1}} \|x_{n+1,i}\| &= \left\| T^{**}\left(\sum_i p_{n+1,i}\right)\right\| \\ &\leq \|h_{n+1}\| = \left\| h - T^{**}\left(\sum_{i=1}^{n} q_i\right)\right\| \leq \frac{\|h\|}{n+1}. \end{aligned}$$

We define $q_{n+1} = \sum_{i \in \mathcal{F}_{n+1}} p_{n+1,i}$ and $y_{n+1} = T^{**}(q_{n+1})$.

Since $\bigoplus_{i=1,\ldots,n}^{\ell_\infty} I_k = A_2^{**}(\sum_{k=1}^n q_k) \subseteq \ker(R_{n+1}^{**})$, it is easy to check that $L(a_{n+1}, r) R_{n+1} = L(a_{n+1}, r)S$ and hence $L(a_{n+1}, r)S(q_{n+1}) = L(a_{n+1}, r)S(1)$. Therefore,

$$\begin{aligned} h - y_{n+1} &= T^{**}\left(1 - \sum_{i=1}^{n+1} q_i\right) = T^{**}\left(1 - \sum_{i=1}^{n} q_i - q_{n+1}\right) \\ &= h_{n+1} - T^{**}(q_{n+1}) = h_{n+1} - L(a_{n+1}, r)S(q_{n+1}) \\ &= h_{n+1} - L(h_{n+1} - a_{n+1}, r)S(q_{n+1}) - L(a_{n+1}, r)S(q_{n+1}) \\ &= L(h_{n+1} - a_{n+1}, r)(r - S(q_{n+1})). \end{aligned}$$

Since $\|h_{n+1} - a_{n+1}\| \leq \frac{\|h\|}{n+2}$ and $\|r - S(q_{n+1})\| \leq 1$ then we obtain

$$\left\| T^{**}\left(1 - \sum_{i=1}^{n+1} q_i\right)\right\| = \|h - y_{n+1}\| \leq \frac{\|h\|}{n+2}.$$

By virtue of the aforementioned fact that $\bigoplus_{i=1,\ldots,n}^{\ell_\infty} I_k = A_2^{**}(\sum_{k=1}^n q_k) \subseteq \ker(R_{n+1}^{**}) \perp I_{n+1}$, it can be easily deduced that $q_{n+1} \perp q_k, \forall k = 1, \ldots, n$, which guarantees that $I_1, \ldots, I_{n+1}$ are mutually orthogonal weak*-closed ideals of A^{**} and $\cup_{k=1}^{n+1}\{x_{k,i} : i \in \mathcal{F}_k\}$ is a set of mutually orthogonal elements in E.

The same argument given in the case $n = 1$ proves that

$$S_{n+1,i} := S|_{M_{m_{n+1,i}}(\mathbb{C})} : M_{m_{n+1,i}}(\mathbb{C}) \to E_2^{**}(r)$$

is a triple monomorphism and $L(x_{n+1,i}, r)S = L(x_{n+1,i}, r)S_{n+1,i}$. This statement completes the proof of our claim.

Finally, the properties proved in the claim show that $h = \sum_{n=1}^{\infty} \sum_{i\in\mathcal{F}_n} x_{n,i}$. Therefore

$$T = L(h, r)S = \sum_{n=1}^{\infty} \sum_{i\in\mathcal{F}_n} L(x_{n,i}, r)S = \sum_{n=1}^{\infty} \sum_{i\in\mathcal{F}_n} L(x_{n,i}, r)S_{n,i}.$$

We shall finally prove that $S(z) = \sum_{n=1}^{\infty} S_n(z)$, for each z in A. We define $\widetilde{S} : A^{**} \to E_2^{**}(r)$ by the assignment $\widetilde{S}(z) := \sum_{n=1}^{\infty} \sum_{i\in\mathcal{F}_n} S_{n,i}(z)$, $z \in A^{**}$, where the sum is taken in the weak*-topology. It easy to check, from the above properties, that $\widetilde{S}$ is a Jordan *-homomorphism and $T(x) = L(h, r)\widetilde{S}(x)$, for every $x \in A$. The uniqueness of the Jordan *-homomorphism given in Theorem 1, (b), shows that $S = \widetilde{S}|_A$. □

When we consider weakly compact orthogonality preserving operators between C* -algebras we obtain the following description.

Remark 9 Let A and B be two C*-algebras, $T : A \to B$ a weakly compact orthogonality preserving operator and let $r = r(h)$ be the range tripotent of the element $h = T^{**}(1)$ in B^{**}. Then there exists a countable family $\{I_n : n \in \mathbb{N}\}$ of mutually orthogonal weak*-closed C*-ideals in A^{**}, a family $\{S_n : A^{**} \to B_2^{**}(r) : n \in \mathbb{N}\}$ of continuous Jordan * -homomorphisms and a sequence (x_n) of mutually orthogonal elements in B satisfying statements (a)-(d) in Theorem 8, and in this particular setting

$$T(x) = \sum_{n=1}^{\infty} x_n\, r^*\, S_n(x) = \sum_{n=1}^{\infty} S_n(x)\, r^*\, x_n, \qquad (e')$$

for every $x \in A$ (compare the argument given in the proof of [7, Corollary 4.3]).

Whenever A is an abelian C*-algebra, every irreducible finite type I von Neumann factor in A^{**} is isomorphic to $\mathbb{C}$. Thus, the description of the weakly compact disjointness preserving linear operators between commutative C*-algebras obtained by Lin and Wong [27] follows now as consequence of our Theorem 8 (see also Remark 9).

Let T be an orthogonality preserving bounded linear operator between two abelian C*-algebras. Lin and Wong proved in [27, Theorem 2.6] that T being compact is equivalent to T being weakly compact. On the other hand, every compact operator between Banach spaces factorises compactly through some closed subspace of c_0 (compare [13, Exercise 6, (iii), Page 15]). Corollary 4.1 in [27] shows that a compact orthogonality preserving operator between two abelian C*-algebras actually admits a compact factorisation through the whole c_0. Our next corollary extends these results to the setting of general C*-algebras.

Corollary 10 *Let T be a a continuous orthogonality preserving operator from a C*-algebra to a JB*-triple. The following are equivalent:*

(a) *T is compact;*
(b) *T is weakly compact;*
(c) *T admits a compact factorisation through a c_0-sum of the form*

$$\bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C}),$$

where (m_n) is a sequence of natural numbers.

Proof The implication $(a)\Rightarrow(b)$ is clear. Since the limit of a norm convergent series of finite rank operators always defines a compact operator, the proof of $(c)\Rightarrow(a)$ follows straightforwardly. We shall prove $(b)\Rightarrow(c)$. Let us suposse that T is weakly compact. Let $h = T^{**}(1)$ and let r denote the range tripotent of h in E^{**}. By Theorem 8, there exists an at most countably family $\{I_n\}$ of mutually orthogonal weak*-closed C*-ideals in A^{**}, a family $\{S_n : A^{**} \to E_2^{**}(r)\}$ of continuous Jordan *-homomorphisms and a set $\{x_n\}$ of mutually orthogonal elements in E satisfying statements $(a)-(e)$ in the referred Theorem. We may assume that $x_n \neq 0$, for every n in $\mathbb{N}$. Since each I_n is a finite type I von Neumann factor, there exists a sequence (m_n) in $\mathbb{N}$ such that $I_n = M_{m_n}(\mathbb{C})$, $\forall n \in \mathbb{N}$.

We define $U : A \to \bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C})$, by the assignment

$$z \mapsto \left(\sqrt{\|x_n\|}\pi_n(z)\right)_n,$$

where π_n is the natural projection of A^{**} onto $I_n = M_{m_n}(\mathbb{C})$ and A is identified with a subspace of A^{**}. We notice that U is a countable sum of finite rank operators, which assures that U is compact. Let

$$R : \bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C}) \to E_2^{**}(r),$$

be the linear operator defined by $R((a_n)) := \sum_{n=1}^{\infty} \|x_n\|^{-\frac{1}{2}}\, x_n \bullet_r S_n(a_n)$, for every $(a_n) \in \bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C})$. It is easy to see that $T = RU$. □

Given a sequence (m_n) in $\mathbb{N}$, the space $X := \bigoplus_{n\in\mathbb{N}}^{c_0} M_{m_n}(\mathbb{C})$ needs not be, in general, isomorphic to c_0. Actually, these two spaces are isomorphic if and only if (m_n) is bounded (when (m_n) is unbounded, the space X^{**} doesn't have the Dunford-Pettis property while c_0^{**} always satisfies this property, compare [12, Theorem 3] or [5]). In the setting of general C*

-algebras, we cannot guarantee that a compact orthogonality preserving operator factorises through the whole c_0.

When in the proof of Theorem 8 above, Corollary 6 is replaced with Corollary 7, then the same arguments given there apply to obtain the following result.

Theorem 11 *Let A be a JB*-algebra, E a JB*-triple, $T : A \to E$ a weakly compact orthogonality preserving operator and let $r = r(h)$ be the range tripotent of the element $h = T^{**}(1)$ in E^{**}. Then there exist a countable family $\{I_n\}_{n\in\mathbb{N}}$ of mutually orthogonal weak*-closed JBW*-ideals in A^{**}, a family $\{S_n : A^{**} \to E_2^{**}(r) : n \in \mathbb{N}\}$ of continuous Jordan *-homomorphisms and a sequence (x_n) of mutually orthogonal elements in E satisfying:*

(*a*) *Each I_n is a reflexive JBW*-factor;*
(*b*) *$\|x_n\| \to 0$ and $h = \sum_n x_n$;*
(*c*) *$S_n|_{I_n}$ is a Jordan *-monomorphism, $S_n|_{I_n^\perp} = 0$, S_n and S_m have orthogonal ranges for $n \neq m$;*
(*d*) *For each x in A^{**}, x_n and $S_m(x)$ operator commute for every n and m; and*
(*e*)

$$T(x) = \sum_{n=1}^{\infty} L(x_n, r) S_n(x) = \sum_{n=1}^{\infty} x_n \bullet_r S_n(x) = h \bullet_r \left(\sum_{n=1}^{\infty} S_n(x) \right),$$

for every $x \in A$.

The existence of infinite dimensional reflexive JB*-algebras shows that the equivalences established in Corollary 10 do not hold in the setting of JB*-algebras. For example, the identity mapping on an infinite dimensional spin factor (see e.g. [18, §6]) is weakly compact and orthogonality preserving but it is not compact.

The description given in Theorem 8 allows us to characterize the class of C*-algebras that admit weakly compact orthogonality preservers.

Corollary 12 *Let A be a C*-algebra (resp., a JB*-algebra). There exists a weakly compact orthogonality preserving operator from A to a JB*-triple if, and only if, A admits a finite dimensional irreducible representation, or equivalently, A^{**} contains a non-zero finite dimensional weak*-closed C*-ideal (reps., A^{**} contains a non-zero reflexive weak*-closed JB*-ideal).* $\square$

Remark 13 In the third paragraph of the Introduction in [27], the authors claim that "... Kamowitz showed that every compact algebraic endomorphism, and indeed every compact disjointness preserving operator, of $C(K)$ is of finite rank. For disjointness preserving operators on locally compact spaces, however, the theory is much richer". We would like to note that the statement concerning compact disjointness preserving or orthogonality preserving operators is not true. Actually, there are examples of compact weighted endomorphisms of $C(K)$ which do not have finite dimensional range. For example, let $K := [0, 1]$. Choose a sequence of pairwise disjoint open intervals $I_n \subset K$ together with a continuous map $\varphi : K \to K$ satisfying $\varphi(I_n) \subset I_n$ and $\varphi|_{I_n}$ constant for every n. Choose furthermore for every n a continuous function h_n on K with $\|h_n\| = 1/n$ having compact support in I_n. Then for $u := \sum h_n$ the operator $T := uC_\varphi$ is compact and has every h_n in its range.

Or even simpler, let $K := \mathbb{N} \cup \{\infty\}$, $\varphi := id_K$ and define u on K by $u(n) := 1/n$. Then every function on K with finite support in $\mathbb{N}$ is in the range of uC_φ.

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Generalized Triple Homomorphisms and Derivations

Jorge J. Garcés and Antonio M. Peralta

Abstract. We introduce generalized triple homomorphisms between Jordan–Banach triple systems as a concept that extends the notion of generalized homomorphisms between Banach algebras given by K. Jarosz and B. E. Johnson in 1985 and 1987, respectively. We prove that every generalized triple homomorphism between JB*-triples is automatically continuous. When particularized to C^*-algebras, we rediscover one of the main theorems established by Johnson. We will also consider generalized triple derivations from a Jordan–Banach triple E into a Jordan–Banach triple E-module, proving that every generalized triple derivation from a JB*-triple E into itself or into E^* is automatically continuous.

1 Introduction

During the last seventy years, a multitude of studies have been published proving that a homomorphism T between Banach algebras (*i.e.*, a linear map with $T(ab) = T(a)T(b)$ for all a, b) must be, under some additional conditions, continuous (*cf.* [9], [10] and [28]). For example, it follows from Gelfand's original theory that every homomorphism from a Banach algebra to a commutative, semisimple Banach algebra is automatically continuous. It is well known that every $*$-homomorphism between C^*-algebras is continuous. It is due to B. E. Johnson that if a unital C^*-algebra has no closed cofinite ideals (*e.g.*, $L(H)$, where H is an infinite dimensional Hilbert space), then each homomorphism from it into a Banach algebra is continuous (*cf.* [19]).

Johnson and K. Jarosz considered *generalized homomorphisms* (also called *ε-multiplicative linear maps* or *ε-isomorphisms*) between Banach algebras in [18], [21] and [20]. Let A and B be Banach algebras. A linear mapping $T\colon A \to B$ is a generalized homomorphism if there exists $\varepsilon > 0$ satisfying $\|T(ab) - T(a)T(b)\| \leq \varepsilon\|a\|\,\|b\|$, for every $a, b \in A$. The first result in this line is due to Jarosz, who proved that every generalized homomorphism from a Banach algebra into a unital abelian C^*-algebra is necessarily continuous (*cf.* [18, Proposition 5.5]). Johnson established in [20, Theorem 4] that a generalized homomorphism T between C^*-algebras is continuous if and only if the mapping $a \mapsto T(a^*)^* - T(a)$ is continuous. A *generalized $*$-homomorphism* between Banach $*$-algebras A and B is a generalized homomorphism $T\colon A \to B$ for which the mapping $a \mapsto T(a^*)^* - T(a)$ is continuous.

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Every Banach algebra A can be regarded as an element in the class of Jordan–Banach triples with respect to the product

$$\{a, b, c\} := \frac{1}{2}(abc + cba). \tag{1}$$

JB*-triples constitute a subclass of the Jordan–Banach triples which contains all C^*-algebras and plays a similar role of that played by the latter inside the class of Banach algebras (see definitions in Section 2). However, according to our knowledge, the automatic continuity of triple homomorphisms between Jordan–Banach triples (*i.e.*, linear mappings T satisfying $T(\{a, b, c\}) = \{T(a), T(b), T(c)\}$, for every a, b, c) have not been deeply studied. The forerunners in this line reduce to a work of T. J. Barton, T. Dang, and G. Horn, where these authors prove the automatic continuity of every triple homomorphism between JB*-triples (see [3, Lemma 1]).

In Section 3 we define a *generalized triple homomorphism* between Jordan–Banach triples E and F as a linear mapping $T\colon E \to F$ for which there exists $\varepsilon > 0$ satisfying

$$\|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| \leq \varepsilon \|a\| \, \|b\| \, \|c\|,$$

for all a, b, c in E. We show that every generalized homomorphism between Banach algebras A and B is a generalized triple homomorphism when A and B are equipped with the product defined in (1). We further prove that every generalized $*$-homomorphism between Banach $*$-algebras A and B is a generalized triple homomorphism when A and B are equipped with the product $\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a)$ (see Proposition 1). In this section we also study the basic properties of the separating space of a generalized triple homomorphism T between Jordan–Banach triples E and F, proving that the separating space $\sigma_F(T)$ is a closed triple ideal of the closed subtriple of F generated by $T(E)$ (compare Proposition 3).

In Section 4 we establish some theorems of automatic continuity of generalized triple homomorphisms between Jordan–Banach triples. One of the main results states that every generalized triple homomorphism between JB*-triples is automatically continuous (see Theorem 8). Since every generalized $*$-homomorphism between C^*-algebras is a generalized triple homomorphism, the aforementioned result of Johnson (see [20, Theorem 4]) follows as a direct consequence. Theorem 14 provides necessary and sufficient conditions, in terms of the quadratic annihilator of the separating space, to characterize when a generalized triple homomorphism from a JB*-triple to a Jordan–Banach triple is continuous. We also prove that every generalized triple homomorphism from a Hilbert space, regarded as a type I Cartan factor, or from a spin factor into an anisotropic Jordan–Banach triple is automatically continuous (*cf.* Lemmas 15 and 16).

In the last section we consider *generalized triple derivations* from a Jordan–Banach triple E to a Jordan–Banach triple E-module X. A conjugate linear mapping $\delta\colon E \to X$ is said to be a *generalized derivation* when there exists $\varepsilon > 0$ satisfying:

$$\|\delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}\| \leq \varepsilon \|a\| \, \|b\| \, \|c\|,$$

for every a, b, c in E. In a recent paper, B. Russo and the second author prove that every triple derivation from a real or complex JB*-triple, E, into its dual space E^* (*i.e.*, a conjugate linear map $\delta\colon E \to E^*$ satisfying $\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}$) is automatically continuous (compare [25, Corollary 15]). We complement this result by proving that every generalized triple derivation from a real or complex JB*-triple E into itself or into E^* is automatically continuous (see Theorem 18). When specialized to C^*-algebras, we show that every generalized triple derivation from a C^*-algebra A to a Jordan–Banach triple A-module is automatically continuous (compare Theorem 22). Our results are not mere generalizations of those forerunners due to Johnson [20] and A. M. Peralta and Russo [25], the proofs are completely independent and the theorems presented here are novelties of independent interest even in the category of C^*-algebras.

2 Preliminaries

We recall that a complex (resp., real) *(normed) Jordan triple* is a complex (resp., real) (normed) space E equipped with a continuous triple product

$$E \times E \times E \to E(xyz) \mapsto \{x,y,z\}$$

that is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called "*Jordan Identity*",

$$L(a,b)L(x,y) - L(x,y)L(a,b) = L\big(L(a,b)x,y\big) - L\big(x,L(b,a)y\big),$$

for all a,b,x,y in E, where $L(x,y)z := \{x,y,z\}$. If E is complete with respect to the norm (*i.e.*, if E is a Banach space), then it is called a complex (resp., real) *Jordan–Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan–Banach triple. Unless otherwise stated, the term "normed Jordan triple" (resp., "Jordan–Banach triple") will always mean a real or complex normed Jordan triple (resp., "Jordan–Banach triple").

For each element a in a Jordan triple E, $Q(a)$ will denote the mapping defined by $Q(a)(x) := \{a,x,a\}$.

Given an element a in a Jordan triple E and a natural number n, we denote $a^{[1]} = a$, and $a^{[2n+1]} := Q(a)^n(a)$. The Jordan identity implies that $a^{[5]} = \big\{a,a,a^{[3]}\big\}$, and by induction, $a^{[2n+1]} = L(a,a)^n(a)$ for all $n \in \mathbb{N}$. The element a is called *nilpotent* if $a^{[2n+1]} = 0$ for some n. Jordan triples are power associative, that is, $\big\{a^{[k]},a^{[l]},a^{[m]}\big\} = a^{[k+l+m]}$.

A Jordan triple E for which the vanishing of $\{a,a,a\}$ implies that a itself vanishes is said to be *anisotropic*. It is easy to check that E is anisotropic if and only if zero is the unique nilpotent element in E.

A real (resp., complex) *Jordan algebra* is a (non-necessarily associative) algebra over the real (resp., complex) field whose product $\circ$ is abelian and satisfies $(a\circ b)\circ a^2 = a\circ(b\circ a^2)$. A normed Jordan algebra is a Jordan algebra A equipped with a norm, $\|\cdot\|$, satisfying $\|a\circ b\| \le \|a\|\,\|b\|$, $a,b \in A$. A *Jordan–Banach algebra* is a normed Jordan algebra whose norm is complete.

Every real or complex associative Banach algebra (resp., Jordan Banach algebra) is a real Jordan–Banach triple with respect to the product $\{a,b,c\} = \frac{1}{2}(abc+cba)$ (resp., $\{a,b,c\} = (a\circ b)\circ c + (c\circ b)\circ a - (a\circ c)\circ b$).

A JB*-algebra is a complex Jordan–Banach algebra A equipped with an algebra involution $*$ satisfying $\| \{a, a^*, a\} \| = \|2(a \circ a^*) \circ a - a^2 \circ a^*\| = \|a\|^3$, $a \in A$.

A *(complex)* JB**-triple* is a complex Jordan–Banach triple E satisfying the following axioms:

(JB* 1) For each a in E the map $L(a, a)$ is an hermitian operator on E with nonnegative spectrum.

(JB* 2) $\|\{a, a, a\}\| = \|a\|^3$ for all a in A.

We recall that a *real* JB**-triple* is a norm-closed real subtriple of a complex JB*-triple (see [17]).

We also recall that a subspace I of a normed Jordan triple E is a *triple ideal* (resp., a *subtriple*) if $\{E, E, I\} + \{E, I, E\} \subseteq I$ (resp., $\{I, I, I\} \subseteq I$). The quotient of a normed Jordan triple by a closed triple ideal is a normed Jordan triple. It is also known that the quotient of a JB*-triple (resp., a real JB*-triple) by a closed triple ideal is a JB*-triple (resp., a real JB*-triple) (compare [22]).

A real JB*-algebra is a closed $*$-invariant real subalgebra of a (complex) JB*-algebra. Real C^*-algebras (*i.e.*, closed $*$-invariant real subalgebras of C^*-algebras) equipped with the Jordan product $a \circ b = \frac{1}{2}(ab+ba)$ are examples of real JB*-algebras.

3 The Separating Space of a Generalized Triple Homomorphism

Let $T\colon E \to F$ be a (not necessarily continuous) linear mapping between normed Jordan triples. We define $\check{T}\colon E \times E \times E \to F$ by the rule

$$(a, b, c) \mapsto \check{T}(a, b, c) = T(\{a, b, c\}) - \{T(a), T(b), T(c)\}.$$

The mapping $\check{T}$ is symmetric and linear in the outer variables and conjugate linear in the middle one (trilinear when E is a real Jordan triple). The mapping T is said to be a *generalized triple homomorphism* if $\check{T}$ is (jointly) continuous, equivalently, if there exists $C > 0$ such that

$$\|\check{T}(a, b, c)\| = \|T(\{a, b, c\}) - \{T(a), T(b), T(c)\}\| \leq C\|a\| \, \|b\| \, \|c\|.$$

Let A, B be Banach algebras. We have already mentioned that a linear mapping $T\colon A \to B$ is a generalized homomorphism when the bilinear mapping

$$(a, b) \to T(ab) - T(a)T(b)$$

is continuous. Every Banach algebra is a Jordan–Banach triple when endowed with the triple product

$$2\{a, b, c\} = abc + cba. \tag{2}$$

We will refer to this product as the *elemental* (Jordan) triple product of A.

A richer structure on the Banach algebra A provides richer ternary products. For example, when A is a $*$-algebra we can consider the Jordan triple product given by

$$2\{a, b, c\} = ab^*c + cb^*a. \tag{3}$$

Let A and B be Banach $*$-algebras. A linear mapping $T\colon A \to B$ is said to be a *generalized $*$-homomorphism* if T is a generalized homomorphism and the mapping

$$a \mapsto S(a) = T(a^*)^* - T(a)$$

is continuous. Generalized $*$-homomorphisms were already considered by Johnson in [20, Theorem 4].

Our next result explores the connections between generalized ($*$-) homomorphisms and generalized triple homomorphisms between Banach ($*$-)algebras.

Proposition 1 *Let A, B be Banach algebras. Every generalized homomorphism $T\colon A \to B$ is a generalized triple homomorphism when A and B are equipped with the elemental triple product $2\{a,b,c\} = abc + cba$.*

When A and B are Banach $*$-algebras and T is a generalized $*$-homomorphism, then T is a generalized triple homomorphism with respect to the triple product $2\{a,b,c\} = ab^*c + cb^*a$.

Proof We start proving the first statement. Let $T\colon A \to B$ be a generalized homomorphism between Banach algebras. We will show that T is a generalized triple homomorphism when A and B are equipped with the triple product (2).

Throughout this proof, $\tilde{T}$ will denote the continuous bilinear mapping from $A\times A$ into B defined by $\tilde{T}(a,b) := T(ab) - T(a)T(b)$.

First, let us see that the (real) trilinear mapping $(a,b,c) \mapsto T(a)\tilde{T}(b,c)$ is continuous. Applying the uniform boundedness principle we see that a trilinear mapping from the cartesian product of three Banach spaces to another Banach space is (jointly) continuous if, and only if, it is continuous whenever we fix two variables. Since $\tilde{T}$ is continuous, the desired statement will follow as soon as we prove that the linear mapping $x \mapsto T(x)\tilde{T}(b,c)$ is continuous whenever we fix b and c in A. Let (x_n) be a norm-null sequence in A, then

$$\begin{aligned}\lim_n T(x_n)\tilde{T}(b,c) &= \lim_n T(x_n)T(bc) - T(x_n)T(b)T(c)\\ &= \lim_n \tilde{T}(x_n,b)T(c) + \tilde{T}(x_nb,c) - \tilde{T}(x_n,bc) = 0,\end{aligned}$$

which proves the desired continuity.

Now, the identity

$$T(abc) - T(a)T(b)T(c) = \tilde{T}(a,bc) + T(a)\tilde{T}(b,c)$$

implies that the assignment $(a,b,c) \mapsto T(abc) - T(a)T(b)T(c)$ defines a (jointly) continuous trilinear mapping. It follows that the assignment

$$\begin{aligned}(a,b,c) &\mapsto T(\{a,b,c\}) - \{T(a),T(b),T(c)\}\\ &= \frac{1}{2}\big(T(abc) + T(cba) - T(a)T(b)T(c) - T(c)T(b)T(a)\big)\\ &= \frac{1}{2}\big(T(abc) - T(a)T(b)T(c)\big) + \frac{1}{2}\big(T(cba) - T(c)T(b)T(a)\big)\end{aligned}$$

defines a continuous trilinear mapping, which gives the first statement.

Let us suppose now that T is a generalized $*$-homomorphism between Banach $*$-algebras A and B. By the first part of the proof, T is a generalized triple homomorphism when A and B are equipped with the triple product (2). We have actually shown that the mapping

$$(a, b, c) \mapsto T(abc) - T(a)T(b)T(c) \tag{4}$$

is continuous. We will see that T is a generalized triple homomorphism when A and B are endowed with the product defined in (3).

Let us write $S(x) = T(x^*)^* - T(x)$. Fix two elements a, c in A. We claim that the (real) linear mapping

$$x \mapsto T(ax^*c + cx^*a) - T(a)T(x)^*T(c) - T(c)T(x)^*T(a) \tag{5}$$

is continuous. Clearly, it is enough to check that the restriction to A_{sa} is continuous. Let x be a self-adjoint element in A, then

$$\begin{aligned}
T(axc) &- T(a)T(x)^*T(c) \\
&= T(axc) - T(a)T(x)T(c) - T(a)T(x)^*T(c) + T(a)T(x)T(c) \\
&= T(axc) - T(a)T(x)T(c) - T(a)\big(T(x)^* - T(x)\big)T(c),
\end{aligned}$$

and hence

$$\begin{aligned}
T(axc + cxa) &- T(a)T(x)^*T(c) - T(c)T(x)^*T(c) \\
&= \big(T(axc + cxa) - T(a)T(x)T(c) - T(c)T(x)T(a)\big) \\
&\quad - \big(T(a)S(x)T(c) + T(c)S(x)T(a)\big),
\end{aligned}$$

which proves the claim.

Now, we fix a, b in A and claim that the linear mapping

$$x \mapsto T(ab^*x) - T(a)T(b)^*T(x) \tag{6}$$

is continuous. To this end, let (x_n) be a norm null sequence in A. Then by (4),

$$\begin{aligned}
\lim_n T(ab^*x_n) &- T(a)T(b)^*T(x_n) \\
&= \lim_n \big(T(ab^*x_n) - T(a)T(b^*)T(x_n)\big) \\
&\qquad + \big(T(a)T(b^*)T(x_n) - T(a)T(b)^*T(x_n)\big) \\
&= \lim_n \big(T(ab^*x_n) - T(a)T(b^*)T(x_n)\big) \\
&\qquad + \lim_n T(a)\tilde{T}(x_n^*, b)^* + T(a)T(b)^*S(x_n) \\
&\qquad - T(a)\tilde{T}(b^*, x_n) - T(a)S(b^*x_n) = 0.
\end{aligned}$$

Similarly, for every b, c in A the linear mapping

$$x \mapsto T(xb^*c) - T(x)T(b)^*T(c) \tag{7}$$

is continuous.

Combining (5), (6), and (7) with the uniform boundedness principle we deduce that the (real) trilinear mapping $(x, y, x) \mapsto T(xy^*z) - T(x)T(y)^*T(z)$ is jointly continuous, and hence, T is a generalized triple homomorphism for the product defined in (3). ■

The separating space of a linear mapping played an important role in many problems of automatic continuity (compare [2, 7, 13, 25, 26, 29], among others). Let $T\colon X \to Y$ be a linear mapping between two normed spaces. We recall that the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. It is well known that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$.

When $T\colon A \to B$ is a generalized homomorphism between Banach algebras and $z \in \sigma_Y(T)$ it is clear that $T(a)z$ and $zT(a)$ lie in $\sigma_Y(T)$, for every $a \in A$. This was actually noticed and applied by Johnson to show that the separating space of T is a closed two-sided ideal of the closed subalgebra of B generated by $T(A)$ (compare [20, Lemma 1]).

We are interested in the properties of the separating space of a generalized triple homomorphism T between Jordan–Banach triples E and F. Clearly, the image of a generalized triple homomorphism $T\colon E \to F$ and the image of $\check{T}$ are both contained in the subtriple of F generated by $T(E)$. However, $T(E)$ and $\check{T}(E \times E \times E)$ need not be Jordan subtriples of F. Moreover, it is not so easy to check that the separating space of T is a closed triple ideal of the closed subtriple of F generated by the image of T. The difficulties in the triple setting grow seriously. For this reason, we will require an appropriate description of the subtriple of F generated by a subset.

In the following we need the notion of a *triple monomial* or an *odd triple monomial*. Let $x_1, x_2, \ldots$ be a sequence of indeterminates. Then a triple monomial is a term that can be obtained by the following recursive procedure:

(i) Every indeterminate x_k is a triple monomial of degree 1.

(ii) If V_1, V_2, and V_3 are triple monomials of degrees d_1, d_2, and d_3 respectively, then $V := \{V_1, V_2, V_3\}$ is a triple monomial of degree $d_1 + d_2 + d_3$, where $\{\cdot, \cdot, \cdot\}$ is a "formal triple product" in three variables.

Notice that this procedure is neither commutative nor associative in general, and the degrees of triple monomials are always odd numbers. If the triple monomial V does not contain any indeterminate x_j with $j > n$, we also write $V = V(x_1, \ldots, x_n)$. In that case, for every JB*-triple E and every $a = (a_1, \ldots, a_n) \in E^n$ the element $V(a) = V(a_1, \ldots, a_n) \in E$ is well defined—just specialize every x_k to a_k and the "formal triple product" to the concrete triple product of E. In this sense V induces a polynomial map $E^n \to E$ which is denoted by the same symbol (or by V_E to avoid confusion). Now, for each fixed odd integer $n \geq 1$, denote by $\mathcal{OP}^n$ the set of all triple monomials V of degree n in which every x_k with $1 \leq k \leq n$ occurs precisely once.

Then $V = V(x_1, \ldots, x_n)$ and the induced map $V_E \colon E^n \to E$ is multilinear for every JB*-triple E.

The symbol $\mathcal{OP}^{2m+1}(E)$ will stand for the set of all multilinear mappings of the form V_E, where V runs in $\mathcal{OP}^{2m+1}$, while $\mathcal{OP}(E)$ will denote the set of all odd triple monomials of any degree on E. It should be noted here that when F is another Jordan triple, each triple monomial V in $\mathcal{OP}^{2m+1}$ induces an element V_F in $\mathcal{OP}^{2m+1}(F)$ by just replacing the triple product of E in the definition of V with the corresponding triple product on F.

Lemma 2 *Let $T \colon E \to F$ be a generalized triple homomorphism between normed Jordan triples and m a natural number. Let V be an odd triple monomial of degree $2m+1$, which can be regarded as an element in $\mathcal{OP}^{2m+1}(E)$ or in $\mathcal{OP}^{2m+1}(F)$ indistinctly. Suppose V of the form $V = \{\cdot, W, P\}$ (resp.,$V = \{W, \cdot, P\}$), and let $j = \deg(W)$. Then*

$$\lim_{n\to\infty} V\big(T(x_n), T(a_1), \ldots, T(a_{2m})\big) - T\big(V(x_n, a_1, \ldots, a_{2m})\big) = 0,$$

$$\text{(resp., } \lim_{n\to\infty} V\big(T(a_1), \ldots, T(a_j), T(x_n), T(a_{j+1}), \ldots, T(a_{2m})\big)$$

$$- T\big(V(a_1, \ldots, a_j, x_n, a_{j+1}, \ldots, a_{2m})\big) = 0\text{)},$$

for every norm-null sequence (x_n) and $a_1, \ldots, a_{2m}$ in E.

Proof We will proceed by induction on m. Since T is a generalized triple homomorphism, the statement trivially holds for every odd triple monomial of degree 3. Now, let us suppose that the statement is true for odd triple monomials of degree less or equal than $2m-1$.

Let V be an odd triple monomial of degree $2m+1$. We will assume $V = \{\cdot, W, P\}$, the case $V = \{W, \cdot, P\}$ follows similarly. Pick a norm-null sequence (x_n) and $a_1, \ldots, a_{2m}$ in E. The odd triple monomials W and P can be written in the form $W = \{W_1, W_2, W_3\}$ and $P = \{P_1, P_2, P_3\}$ for some odd triple monomials P_i, W_i, $i = 1, 2, 3$. Clearly $1 \leq \deg(W_i), \deg(P_i) < 2m-1$.

Applying the Jordan identity we have

(8)

$$\begin{aligned}
&V(T(x_n), T(a_1), \ldots, T(a_{2m})) = \{T(x_n), W(T(a_i)), P(T(a_j))\} \\
&= \big\{ T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), P_3(T(a_{j_3}))\}\big\} \\
&= \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\
&- \big\{ P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\big\} \\
&+ \big\{ P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\big\}.
\end{aligned}$$

We will treat the summands in the right-hand side independently. We claim that

(9)

$$\begin{aligned}
&\lim_n \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\
&\qquad - T\big(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\}\big) = 0.
\end{aligned}$$

Indeed, consider the monomial $Q = \big\{ \big\{ \cdot, \{W_1, W_2, W_3\}, P_1 \big\}, P_2, P_3 \big\}$. It is clear that $\deg(Q) \leq 2m - 1$, and

$$\begin{aligned}(10)\qquad & \big\{ T(x_n), \big\{ W_1\big(T(a_{i_1})\big), W_2\big(T(a_{i_2})\big), W_3\big(T(a_{i_3})\big) \big\}, P_1 T\big((a_{j_1})\big) \big\} \\ &= Q\big(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})\big).\end{aligned}$$

Taking limits in $n \to \infty$ and applying the induction hypothesis we get

$$(11)\quad \lim_n Q\big(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})\big) - T\big(Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1})\big) = 0.$$

Let $z_n := Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1})$. It follows from the continuity of the triple product that (z_n) is a norm-null sequence in E.

Consider now the monomial $Q' = \{\cdot, P_2, P_3\}$. Since $\deg(Q') \leq 2m - 1$ we can apply the induction hypothesis to prove

$$\begin{aligned}(12)\qquad & \lim_n \big\{ T(z_n), P_2\big(T(a_{j_2})\big), P_3\big(T(a_{j_3})\big) \big\} - T\big(\{z_n, P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ &= \lim_n Q'\Big(T(z_n), T\big((a_{j_2}), T(a_{j_3})\big)\Big) - T\big(Q'(z_n, a_{j_2}, a_{j_3})\big) = 0.\end{aligned}$$

Combining (10), (11), and (12) we have

$$\begin{aligned}& \lim_n \big\{ \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\ &\qquad - T\big(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ &= \lim_n \big\{ Q(T(x_n), T(a_{i_1}), T(a_{i_2}), T(a_{i_3}), T(a_{j_1})), P_2(T(a_{j_2})), P_3(T(a_{j_3}))\big\} \\ &\qquad - T\big(\{Q(x_n, a_{i_1}, a_{i_2}, a_{i_3}, a_{j_1}), P_2(a_{j_2}), P_3(a_{j_3})\}\big) \\ &= \lim_n \{T(z_n), P_2(T(a_{j_1})), P_3(T(a_{j_2}))\} - T\big(\{z_n, P_2(a_{j_1}), P_3(a_{j_2})\}\big) = 0,\end{aligned}$$

which proves the claim (9).

We can similarly prove that

$$\begin{aligned}(13)\qquad & \lim_n \{P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\} \\ &\qquad - T\big(\{P_1(a_{j_1}), \{\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, T(x_n), P_2(a_{j_2})\}, P_3(a_{j_3})\}\big) = 0\end{aligned}$$

and

$$\begin{aligned}(14)\qquad & \lim_n \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\} \\ &\qquad - T\big(\{P_1(a_{j_1}), P_2(a_{j_2}), \{T(x_n)\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_3(a_{j_3})\}\}\big) = 0.\end{aligned}$$

Finally, from (8), (9), (13), and (14) we obtain

$$\lim_n V\big(T(x_n), T(a_1), \ldots, T(a_{2m})\big) - T\big(V(x_n, a_1, \ldots, a_{2m})\big) = \text{(from the Jordan identity)}$$

$$\lim_n \{\{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_1(T(a_{j_1}))\}, P_2(T(a_{j_2})), P_3(T(a_{j_3}))\}$$

$$- T(\{\{x_n, \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_1(a_{j_1})\}, P_2(a_{j_2}), P_3(a_{j_3})\})$$

$$- \{P_1(T(a_{j_1})), \{\{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, T(x_n), P_2(T(a_{j_2}))\}, P_3(T(a_{j_3}))\}$$

$$+ T(\{P_1(a_{j_1}), \{\{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, T(x_n), P_2(a_{j_2})\}, P_3(a_{j_3})\})$$

$$+ \{P_1(T(a_{j_1})), P_2(T(a_{j_2})), \{T(x_n), \{W_1(T(a_{i_1})), W_2(T(a_{i_2})), W_3(T(a_{i_3}))\}, P_3(T(a_{j_3}))\}\}$$

$$- \lim_n T(\{P_1(a_{j_1}), P_2(a_{j_2}), \{T(x_n), \{W_1(a_{i_1}), W_2(a_{i_2}), W_3(a_{i_3})\}, P_3(a_{j_3})\}\}) = 0,$$

as we desired. ∎

We recall that two elements a and b in a Jordan–Banach triple E are said to be *orthogonal* (written $a \perp b$) if $L(a,b) = L(b,a) = 0$. A direct application of the Jordan identity yields that, for each c in E,

$$a \perp \{b, c, b\} \quad \text{whenever} \quad a \perp b.$$

When E is anisotropic, $a \perp b$ if and only if $L(a,b) = 0$. In case E is a real or complex JB*-triple, the relation of being orthogonal admits several equivalent reformulations (*cf.* [6, Lemma 1]).

Given a subset M of a Jordan–Banach triple, E, we write $M_E^{\perp}$ for the *(orthogonal) annihilator of* M, defined by

$$M_E^{\perp} := \{y \in E : y \perp x, \forall x \in M\}.$$

When no confusion arises, we will write $M^{\perp}$ instead of $M_E^{\perp}$.

Let E be a Jordan–Banach triple and $S \subseteq E$. The norm-closed Jordan subtriple of E generated by S is the smallest norm-closed subtriple of E containing S and will be denoted by E_S. Clearly, E_S coincides with the norm-closure of the linear span of the set

$$\mathcal{OP}_E(S) := \{V(a_1, \ldots, a_{2m+1}) : m \in \mathbb{N}, V \in \mathcal{OP}^{2m+1}(E), a_1, \ldots, a_{2m+1} \in S\}.$$

When a is an element in E, we write E_a instead of $E_{\{a\}}$.

Proposition 3 *Let $T \colon E \to F$ be a generalized triple homomorphism between two Jordan–Banach triples. Let I and $\widetilde{F}$ denote $\sigma_F(T)$ and the norm-closed subtriple of F generated by $T(E)$, respectively. Then we have the following:*

(i) *I is a (closed) triple ideal of $\widetilde{F}$.*
(ii) *$I_{\widetilde{F}}^{\perp}$ contains all the elements of the form $\check{T}(a,b,c)$.*

Further, if J is a closed triple ideal of $\widetilde{F}$ containing $I_{\widetilde{F}}^{\perp}$, then $\pi \circ T$ is a triple homomorphism, where π is the quotient map $\widetilde{F} \to \widetilde{F}/J \cap \widetilde{F}$.

Proof (i) Since I is a closed linear subspace of F, we only have to prove that $\{\widetilde{F},\widetilde{F},I\}+\{\widetilde{F},I,\widetilde{F}\}\subseteq I$. Since $\mathcal{OP}_F\big(T(E)\big)$ is dense in $\widetilde{F}$, it is enough to show that

$$V\big(I,T(E),\dots,T(E)\big)+V'\big(T(E),\dots,T(E),I,T(E),\dots,T(E)\big)\subseteq I,$$

where V and V' are arbitrary odd triple monomials of the form $\{W,\cdot,P\}$ and $\{\cdot,W',P'\}$, respectively.

Let z be an element in I, then there exists a norm-null sequence (z_n) in E such that $z=\lim_n T(z_n)$. Now let $V=\{W,\cdot,P\}$ and $V'=\{\cdot,W',P'\}$ be odd triple monomials of degree $2m_1+1$ and $2m_2+1$, respectively, with $j=\deg(W)$. Let us fix $a_1,\dots,a_{2m_1},b_1,\dots,b_{2m_2}$ in E. By Lemma 2,

$$\begin{aligned}V'\big(z,T(a_1),\dots,T(a_{2m_1})\big)&=\lim_n V'\big(T(z_n),T(a_1),\dots,a_{2m_1}\big)\\&=\lim_n T\big(V'(z_n,a_1,\dots,a_{2m_1})\big),\end{aligned}$$

and

$$\begin{aligned}&V\big(T(b_1),\dots,T(b_j),z,T(b_{j+1}),\dots,T(b_{2m_2})\big)\\&\quad=\lim_n V\big(T(b_1),\dots,T(b_j),T(z_n),T(b_{j+1}),\dots,T(b_{2m_2})\big)\\&\quad=\lim_n T\big(V(b_1,\dots,b_j,z_n,b_{j+1},\dots,b_{2m_2})\big).\end{aligned}$$

By the continuity of the Jordan triple product $x_n=V'(z_n,a_1,\dots,a_{2m_1})$ and $y_n=V(b_1,\dots,b_j,z_n,b_{j+1},\dots,b_{2m_2})$ are norm-null sequences in E, and thus

$$V'\big(z,T(a_1),\dots,T(a_{2m_1})\big)=\lim_n T(x_n)\in I$$

and

$$V\big(T(b_1),\dots,T(b_j),z,T(b_{j+1}),\dots,T(b_{2m_2})\big)=\lim_n T(y_n)\in I.$$

(ii) In order to see that $I^{\perp}_{\widetilde{F}}\supseteq\check{T}(E,E,E)$, we will show that

$$L\big(I,\check{T}(a,b,c)\big)\big|_{\widetilde{F}}=L\big(\check{T}(a,b,c),I\big)\big|_{\widetilde{F}}=0,\quad\forall a,b,c\in E.$$

Let $z=\lim T(z_n)$ in I, where (z_n) is a norm-null sequence in E, V and odd triple monomial of degree $2m+1$ and $a,b,c,a_1,\dots,a_{2m+1}$ in E. Then

$$\begin{aligned}&L\big(z,\check{T}(a,b,c)\big)\Big(V\big(T(a_1),\dots,T(a_{2m+1})\big)\Big)\\&\quad=\lim_n\big\{T(z_n),\check{T}(a,b,c),V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}\\&\quad=\lim_n\big\{T(z_n),T(\{a,b,c\}),V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}\\&\qquad\quad-\big\{T(z_n),\{T(a),T(b),T(c)\},V\big(T(a_1),\dots,T(a_{2m+1})\big)\big\}=\text{(by Lemma 2)}\\&\quad=\lim_n T\big(\big\{z_n,\{a,b,c\},V(a_1,\dots,a_{2m+1})\big\}\big)\\&\qquad\quad-T\big(\big\{z_n,\{a,b,c\},V(a_1,\dots,a_{2m+1})\big\}\big)=0.\end{aligned}$$

We can similarly show that $L\big(\check{T}(a,b,c),z\big)\,\big(V(a_1,\ldots,a_{2m+1})\big) = 0$. Therefore, it follows from the density of $\mathcal{OP}_F\big(T(E)\big)$ in $\widetilde{F}$ and the continuity of the triple product that $L\big(I,\check{T}(a,b,c)\big)\big|_{\widetilde{F}} = L\big(\check{T}(a,b,c),I\big)\big|_{\widetilde{F}} = 0$, which proves (ii).

Finally, to see the last statement we observe that, since $I^{\perp}_{\widetilde{F}}$ contains all the elements of the form $\check{T}(a,b,c)$, we have

$$\begin{aligned} 0 = \pi\big(\check{T}(a,b,c)\big) &= \pi\big(T(\{a,b,c\}) - \{T(a),T(b),T(c)\}\big)\\ &= \pi\big(T(\{a,b,c\})\big) - \pi\big(\{T(a),T(b),T(c)\}\big), \quad \forall a,b,c\in E, \end{aligned}$$

so $\pi\circ T$ is a triple homomorphism. ■

Let us suppose that, in the hypothesis of Proposition 3 above, F is assumed to be a JB*-triple. In this setting two elements a,b in F are orthogonal if and only if $\{a,a,b\} = 0$ (*cf.* [6, Lemma 1]). Under these assumptions, let z be an element in I and pick arbitrary a,b,c in E. Since there exists a null sequence (z_n) in E such that $z = \lim_n T(z_n)$, by Lemma 2 and the uniform boundedness principle, we have

$$\begin{aligned} \{z,z,\check{T}(a,b,c)\} = \lim_n \big\{\, T(z_n), T(z_n), T(\{a,b,c\}\big\}&\\ - \big\{\,T(z_n),T(z_n),\{T(a),T(b),T(c)\}\big\} &= 0, \end{aligned}$$

which implies $I^{\perp}_{F} \supseteq I^{\perp}_{\widetilde{F}} \supseteq \check{T}(E,E,E)$.

4 Automatic Continuity

4.1 Generalized Triple Homomorphisms Between Jordan–Banach Triples

A celebrated result of J. Cuntz states that a linear mapping $T\colon A \to X$ from a C^*-algebra to a Banach space is continuous if and only if its restriction to any C^*-subalgebra of A generated by a single hermitian element is continuous (*cf.* [8]). Some years before A. M. Sinclair [27] established that a similar automatic continuity result holds for homomorphism from a C^*-algebra to a Banach algebra. At this point, the reader should be tempted to ask if a similar statement holds for linear mappings whose domain is a JB*-triple (by replacing C^*-subalgebras generated by a single hermitian element by JB*-subtriples generated by a single element). Unfortunately, we will see next that the answer to this question is negative.

Example 4 A complex Hilbert space H becomes a JB*-triple when endowed with the triple product defined by $\{a,b,c\} = \frac{1}{2}\big((a|b)c + (c|b)a\big)$, where $(\cdot|\cdot)$ denotes the inner product of H. It can be easily seen that every norm-one element e in E is tripotent (*i.e.*, $\{e,e,e\} = e$). Therefore, the JB*-subtriple of E generated by a single element a coincides with $\mathbb{C}a$. This implies that, for each Banach space X, the restriction of any linear mapping $T\colon H \to X$ to any JB*-subtriple of H generated by a single element is continuous. When H is infinite-dimensional, we can easily find a discontinuous linear mapping from H into a Banach space. We can similarly consider a JB*-triple E of infinite dimension with finite rank (*e.g.*, all E_a have finite dimensions, see [4, Section 3]).

The above example shows that a simple translation to the setting of JB*-triples of the hypotheses assumed by Cuntz in [8] is not enough to guarantee that a linear mapping from a JB*-triple to a Banach space is automatically continuous. Finding an assumption to avoid the previous counterexample, we will replace the subtriple generated by a single element by the norm-closed inner ideal generated by a single element. We recall that a subspace J of a JB*-triple E is said to be an *inner ideal* if $\{J, E, J\}$ is contained in J. Let a be an element in E and let $E(a)$ denote the norm closure of $\{a, E, a\}$ in E. It is known that $E(a)$ coincides with the norm-closed inner ideal of E generated by a (*cf.* [5, pp. 19–20]). Let us notice that in the previous Example 4, $H(a) = H$ for every norm-one element $a \in H$.

Let $T\colon E \to F$ be a generalized triple homomorphism between Jordan–Banach triples and suppose that T is continuous when restricted to any norm-closed inner ideal generated by a single (norm-one) element. Let z be an element in $\sigma_F(T)$. Then there exists a norm-null sequence (z_n) in E such that $z = \lim_n T(z_n)$. Pick a norm-one element a in E. Then

$$\begin{aligned}\{T(a), z, T(a)\} &= \lim_n \{T(a), T(z_n), T(a)\} = \lim_n T(\{a, z_n, a\}) - \check{T}(a, z_n, a)\\ &= \lim_n T_{|E(a)}(\{a, z_n, a\}) - \check{T}(a, z_n, a) = 0,\end{aligned}$$

since $\{a, z_n, a\}$ is a norm-null sequence in $E(a)$ and $\check{T}$ and $T_{|E(a)}$ are continuous by hypothesis. Therefore $\{T(z_n), z, T(z_n)\} = 0$, for every natural n, and hence $z^{[3]} = \lim_n \{T(z_n), z, T(z_n)\} = 0$, which affirms that all elements in $\sigma_F(T)$ are nilpotents.

Definition 5 A Jordan–Banach triple E has *Cohen's factorization property* (CFP) if given a norm-null sequence (a_n) in E there exist a norm-null sequence (b_n) and two elements x, y in E such that $a_n = \{x, b_n, y\}$, $\forall n \in \mathbb{N}$.

Every Jordan–Banach algebra with a bounded approximate identity has Cohen's factorisation property (compare [1]). In particular, JB and JB*-algebras have Cohen factorisation property (see [16, Proposition 3.5.4]). It follows from [5, pp. 19–20] (see also [12, Lemma 3.2]) that for every norm-one element a in a JB*-triple E, $E(a)$ satisfies CFP.

Our next result is an extension of Sinclair's result [27, Corollary 4.3].

Theorem 6 *Let $T\colon E \to F$ be a linear mapping between two Jordan–Banach triples and suppose that one of the following statements holds:*

(i) *T is a generalized triple homomorphism and F is anisotropic;*

(ii) *E has Cohen's factorisation property.*

If the restriction of T to any closed inner ideal generated by a single element is continuous, then T is continuous.

Proof The proof under hypothesis (i) was already given in the paragraph preceding Definition 5. Suppose E satisfies CFP. Let (y_n) be a norm-null sequence in E and let $a \in E$. Since $T|_{E(a)}$ is continuous, we have $\lim_n T\{a, y_n, a\} = 0$. Since a was arbitrarily chosen, we deduce that

$$\lim_n T(\{a, y_n, b\}) = 0, \tag{15}$$

for every $a, b \in E$.

Let us pick $z \in \sigma_F(T)$ and a norm-null sequence (z_n) in E satisfying $T(z_n) \to z$. By hypothesis, there exist a, b in E and a norm-null sequence $(y_n) \subseteq E$ such that $z_n = \{a, y_n, b\}$. In such a case, by (15),

$$z = \lim_n T(z_n) = \lim_n T(\{a, y_n, b\}) = 0. \qquad \blacksquare$$

Remark 7 Let $T\colon E \to F$ be a linear mapping between Banach spaces. A useful property of the separating space $\sigma_F(T)$ asserts that for every bounded linear map R from F to another Banach space Z, the composition RT is continuous if and only if $\sigma_F(T) \subseteq \ker(R)$. It is also known that $\sigma(RT) = \overline{R(\sigma(T))}^{\|\cdot\|}$ (see [28, Lemma 1.3]).

Based on the Commutative Gelfand Theory established by W. Kaup (*cf.* [22]), T. J. Barton, T. Dang, and G. Horn proved the automatic continuity of triple homomorphisms between JB*-triples (see [3, Lemma 1]). The natural extension of this automatic continuity property to the setting of generalized triple homomorphisms is contained in our next result.

Theorem 8 *Every generalized triple homomorphism between* JB**-triples is continuous.*

Proof Let $T\colon E \to F$ be a generalized triple homomorphism between JB*-triples. The norm closed subtriple of F generated by $T(E)$ will be again denoted by $\widetilde{F}$, while the symbol I will stand for the separating space $\sigma_F(T)$. Since $\widetilde{F}$ is a norm-closed subtriple of F, then $\widetilde{F}$ is a JB*-triple itself. Proposition 3 (i) assures that I is a closed ideal of $\widetilde{F}$, and by [25, Lemma 4] $I^{\perp}_{\widetilde{F}}$ is a norm-closed triple ideal of $\widetilde{F}$.

The final statement in Proposition 3 guarantees that the linear mapping $\pi \circ T\colon E \to \widetilde{F}/I^{\perp}_{\widetilde{F}}$ is a triple homomorphism. Since the quotient $\widetilde{F}/I^{\perp}_{\widetilde{F}}$ is a JB*-triple, the triple homomorphism $\pi \circ T$ is continuous (*cf.* [3, Lemma 1]). By Remark 7, we have $I = \sigma_F(T) \subseteq \ker(\pi) = I^{\perp}_{\widetilde{F}}$, and the latter implies that $I = \sigma_F(T) = 0$. $\blacksquare$

Since every C^*-algebra, endowed with the triple product given in (3), is a JB*-triple, Theorem 8, together with Proposition 1, allows us to rediscover the following result, which is originally due to Johnson [20, Theorem 4].

Corollary 9 ([20, Theorem 4]) *Every generalized $*$-homomorphism between C^*-algebras is continuous.* $\blacksquare$

Our next goal is to explore the automatic continuity of a generalized triple homomorphism from a JB*-triple to a Jordan–Banach triple. To this end we will require some additional concepts and tools.

Let E be a real or complex Jordan–Banach triple system. We will say that E is *algebraic* if all singly generated (norm-closed) subtriples of E are finite-dimensional. If in fact there exists $m \in \mathbb{N}$ such that single-generated subtriples of E have dimension $\leq m$, then E is said to be of *bounded degree*, and the minimum of such an m will be called the degree or the rank of E. For real and complex JB*-triples algebraic and bounded degree are the same (*cf.* [4, Section 3]).

Our next result owes much to the proof given in [25, Proposition 12] by Russo and the second author.

Theorem 10 *Let $T\colon E \to X$ be a linear mapping from a* JB*-triple to a Banach space. Let $J_T := \{a \in E : T \circ Q(a), T \circ L(a,a)$ are continuous$\}$. Suppose that J_T has the following properties:*

(i) $J_T + J_T \subseteq J_T$.

(ii) $\{E, E, J_T\} + \{E, J_T, E\} \subseteq J_T$.

(iii) *If I is a norm-closed triple ideal containing J_T, then E/I is algebraic of bounded degree.*

Then T is continuous if and only if J_T is norm-closed.

Proof When T is continuous, J_T coincides with E and nothing has to be proved. Suppose now that J_T is norm-closed. It follows from (i) and (iii) that J_T is a norm-closed triple ideal of E. We claim that the restriction of T to J_T is continuous. Indeed, the assignment $(a, b, c) \mapsto W(a, b, c) = T(\{a, b, c\})$ defines a (real) trilinear mapping $W\colon J_T \times J_T \times J_T \to F$. From (i) and the definition of J_T, W is separately continuous whenever we fix two variables. An application of the uniform boundedness principle implies that W is jointly continuous. Therefore, there exists a positive constant M such that $\|T\{a, b, c\}\| \leq M\|a\|\,\|b\|\,\|c\|$, for every a, b, c in J_T. Since J_T is a JB*-subtriple of E, for each a in J_T there exists b in J_T such that $b^{[3]} = a$. In this case

$$\|T(a)\| = \|T(\{b, b, b\})\| \leq M\|b\|^3 = M\|\{b, b, b\}\| = M\|a\|,$$

which shows that $T|_{J_T}$ is continuous.

Finally, let us prove that $J_T = E$. By hypothesis (iii), E/J_T is algebraic of bounded degree m. Thus, for each element $a + J_T$ in E/J_T there exist mutually orthogonal minimal tripotents $e_1 + J_T, \ldots, e_k + J_T$ in E/J_T and $0 < \lambda_1 \leq \cdots \leq \lambda_k$ with $k \leq m$ such that $a + J_T = \sum_{j=1}^{k} \lambda_k e_k + J_T$. We will show that $e_1, \ldots, e_k \in J_T$, and hence, $a \in J_T$, which proves $E = J_T$.

Let $e + J_T$ be a minimal tripotent in E/J_T. Henceforth, $\pi\colon E \to E/J_T$ will denote the canonical projection. Take an arbitrary norm-null sequence (a_n) in E. For each natural n, there exists a scalar $\mu_n \in \mathbb{C}$ such that $\pi\big(Q(e)(a_n)\big) = \mu_n(e + J_T)$. The continuity of π and the Peirce projection $P_2(e + J_T)$ assure that $\mu_n \to 0$. It follows that $Q(e)(a_n) - \mu_n e$ lies in J_T and tends to zero in norm. Since, by hypothesis, $T|_{J_T}$ is continuous we have

$$T\big(Q(e)(a_n)\big) = T\big(Q(e)(a_n) - \mu_n e\big) + \mu_n T(e) \to 0.$$

The arbitrarity of (a_n) guarantees that $T \circ Q(e)$ is continuous, or equivalently, e lies in J_T. ∎

The following auxiliary lemmas will be needed later.

Lemma 11 *Let E be a real* JB*-triple and J a subset of E satisfying that whenever we have two sequences $(x_n), (y_n)$ in E such that $Q(y_n)Q(x_n) = Q(x_n)$ and $Q(y_n)Q(x_m) = 0$ for $n \neq m$, then the x_n lie in J except (perhaps) for finitely many n. Suppose I is a norm-closed triple ideal of E containing J then E/I is algebraic of bounded degree.*

Proof Since I contains J then I also has the property assumed in the hypothesis.

Let us write $F = E/I$. As noticed in the proof of Corollary 8 in [25] for $\overline{a} = a + I$ we have $F_{\overline{a}} = E_a/(E_a \cap I)$.

The commutative JB*-triple E_a is triple isomorphic to some $C_0(L)$ (*cf.* [22, Section 1]). We will identify E_a with $C_0(L)$. It is known that $F_{a+I} \cong C_0(\Gamma)$ where

$$\Gamma = \{t \in L : b(t) = 0, \forall b \in E_a \cap I\}.$$

We claim that Γ is finite. Otherwise, there exists an infinite sequence (t_n) in Γ and a sequence of open disjoint sets $\{U_n\}_n$. By local compactness we can find open sets V_n, W_n with $\overline{V_n}$ and $\overline{W_n}$ compact, such that $t_n \in V_n \subseteq \overline{V_n} \subseteq W_n \subseteq \overline{W_n} \subseteq U_n$.

By Urysohn's lemma, for each natural n, we can find $f_n \in C_0(L)$ with $t_n \in \operatorname{supp}(f_n) \subseteq W_n$ and $g_n \in C_0(L)$ such that $g_n \equiv 1$ in $\overline{W_n}$ and vanishing outside U_n. Since $f_n(t_n), g_n(t_n) \neq 0, \forall n \in \mathbb{N}$, then $f_n, g_n \notin I, \forall n \in \mathbb{N}$. In this case the sequences $(f_n), (g_n)$ verify that $Q(g_n)Q(f_n) = Q(f_n)$ and $Q(g_n)Q(f_m) = 0$ for $n \neq m$, and they do not lie in I, which is a contradiction.

It follows that Γ is finite and therefore F_{a+I} is finite dimensional. Since $a + I$ was arbitrary chosen, the statement of the lemma follows from [4, Theorem 3.8]. ∎

Lemma 12 *Let $T\colon E \to F$ be a generalized triple homomorphism between real Jordan–Banach triples, and let $(x_n), (y_n)$ be sequences of elements in E such that $Q(y_n)Q(x_n) = Q(x_n)$ and $Q(y_n)Q(x_m) = 0$ for $n \neq m$. Then $Q(T(x_n))T$ and $TQ(x_n)$ are continuous for all but a finite number of n.*

Proof Let us suppose that $Q\big(T(x_n)\big)\,T$ is discontinuous for infinitely many n in $\mathbb{N}$. By passing to a subsequence if necessary, we can assume that $Q\big(T(x_n)\big)\,T$ is discontinuous for all n in $\mathbb{N}$. We observe that, since T is a generalized triple homomorphism the identity

$$\{T(x_n), T(b), T(x_n)\} = T(\{x_n, b, x_n\}) - \check{T}(x_n, b, x_n),$$

holds for every $b \in E$ and $n \in \mathbb{N}$. It is then clear that $Q\big(T(x_n)\big)\,T$ is continuous if and only if $TQ(x_n)$ is. So, we may assume that $TQ(x_n)$ is discontinuous for all n in $\mathbb{N}$. Choose (a_n) in E such that $\|a_n\| \leq 2^{-n}\|x_n\|^{-2}$ and

$$\|TQ(x_n)(a_n)\| \geq 2^n\big(1 + \|T(y_n)\|^2\big) + \|\check{T}\|\,\|y_n\|^2.$$

Let $a = \sum_{m\geq 1}\{x_m, a_m, x_m\}$. Since $\{y_n, a, y_n\} = \{x_n, a_n, x_n\}$ we have

$$\begin{aligned}
2^n\big(1 + \|T(y_n)\|^2\big) &+ \|\check{T}\|\,\|y_n\|^2 \leq \|TQ(x_n)(a_n)\| \\
&= \|TQ(y_n)(a)\| = \Big\|\,Q\big(T(y_n)\big)\,\big(T(a)\big) + \check{T}(y_n, a, y_n)\Big\| \\
&\leq \|T(y_n)\|^2\,\|T(a)\| + \|\check{T}\|\,\|y_n\|^2\,\|a\| \leq \big(1 + \|T(y_n)\|^2\big)\,\|T(a)\| + \|\check{T}\|\,\|y_n\|^2.
\end{aligned}$$

So we have that $\|T(a)\| \geq 2^n, \forall n \in \mathbb{N}$, which is impossible. ∎

Let $T\colon E \to F$ be a generalized triple homomorphism between Jordan–Banach triples. Following the notation employed in Proposition 3, the symbol $\widetilde{F}$ will denote the norm-closed subtriple of F generated by $T(E)$.

According to the notation defined in [25], for each subset B of a Jordan–Banach triple F, we define its *quadratic annihilator*, $\mathrm{Ann}_F(B)$, as the set

$$\{a \in F : Q(a)(B) = \{a, B, a\} = 0\}.$$

The quadratic annihilator will be used later in a more general setting.

If we set $J := T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$, it not hard to see, from the basic properties of the separating space, that J coincides with the set $\big\{a \in E : Q\big(T(a)\big)T$ is continuous$\big\}$ (compare Remark 7), and since T is a generalized triple homomorphism, the latter equals $\{a \in E : TQ(a)$ is continuous$\}$ (compare the proof of Lemma 12). The following result follows straightforwardly from Lemmas 12 and 11 and the above comments.

Proposition 13 *Let $T\colon E \to F$ be a generalized triple homomorphism from a real* JB**-triple to a Jordan–Banach triple. The following statements hold:*

(i) *If I is a norm-closed triple ideal containing $T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$, then E/I is algebraic of bounded degree.*

(ii) *Let K be a triple ideal of E. The linear mapping*

$$x \in E \mapsto \{T(a), T(x), T(a)\}$$

is continuous for all a in K if, and only if, K is contained in $T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$.

We can establish now the main result of this section.

Theorem 14 *Let $T\colon E \to F$ be a generalized triple homomorphism from a* JB**-triple to a Jordan–Banach triple and let $J = T^{-1}\big(\mathrm{Ann}_F\big(\sigma_F(T)\big)\big)$. The following statements are equivalent:*

(i) *J is a norm-closed triple ideal of E and*

$$\big\{\mathrm{Ann}_F\big(\sigma_F(T)\big), \mathrm{Ann}_F\big(\sigma_F(T)\big), \sigma_F(T)\big\} = 0.$$

(ii) *T is continuous.*

Proof The implication (ii) $\Rightarrow$ (i) is clear. We will prove (i) $\Rightarrow$ (ii). We already know, by Proposition 13 (ii), that for each element a in J, the linear mapping

$$x \in E \mapsto \{T(a), T(x), T(a)\}$$

is continuous. Let us fix a, b in J. Since J is a linear subspace of E, then $a + b$ also lies in J, that is, the mapping $x \mapsto \{T(a+b), T(x), T(a+b)\}$ is continuous. The identity

$$\begin{aligned} 2\{T(a), T(x), T(b)\} = {} & \{T(a+b), T(x), T(a+b)\} \\ & - \{T(a), T(x), T(a)\} - \{T(b), T(x), T(b)\}, \end{aligned}$$

guarantees that the mapping $x \mapsto \{T(a), T(x), T(b)\}$ is continuous, or equivalently (because T is a generalized triple homomorphism), $TQ(a,b)$ is continuous.

Since $\left\{ \mathrm{Ann}_F\big(\sigma_F(T)\big), \mathrm{Ann}_F\big(\sigma_F(T)\big), \sigma_F(T)\right\} = 0$, the linear mapping

$$x \in E \mapsto \{T(a), T(b), T(x)\}$$

is continuous for every $a, b \in J$. Applying that T is a generalized triple homomorphism, we deduce that the linear mapping $x \in E \mapsto T(\{a,b,x\})$ also is continuous for every $a, b \in J$. This shows that the trilinear mapping $W\colon E \times E \times E$, given by $(a,b,c) \mapsto W(a,b,c) = T(\{a,b,c\})$ is continuous whenever we fix two variables in J. An application of the uniform boundedness principle proves that $W|_{J\times J\times J}$ is jointly continuous. Following the argument given in the proof of Theorem 10, we show that $T|J\colon J \to F$ is continuous.

Proposition 13 (i) implies that E/J is algebraic of bounded degree. The proof concludes applying the argument given in the final part of the proof of Theorem 10. ■

The above Theorem 14 admits a more detailed statement in the particular setting of some Cartan factors. We recall that a complex Hilbert space H can be regarded as a *type I Cartan factor* with its natural norm and the product given by

$$2\{a,b,c\} := (a|b)c + (c|b)a, \quad (a,b,c \in H),$$

where $(\cdot|\cdot)$ denotes the inner product of H.

Lemma 15 *Let H be a complex Hilbert space regarded as a type I Cartan factor, F an anisotropic Jordan–Banach triple and $T\colon H \to F$ a generalized triple homomorphism. Then T is continuous.*

Proof Let $\widetilde{F}$ denote the norm-closed subtriple of F generated by $T(E)$. It is enough to prove that $T\colon H \to \widetilde{F}$ is continuous. Replacing F with $\widetilde{F}$, we may assume, by Proposition 3, that $\sigma_F(T)$ is a norm-closed triple ideal of F and F is generated by $T(E)$. It follows from our hypothesis that the mapping

$$\check{T}(a,b,c) = \frac{1}{2}\big((a|b)T(c) + (c|b)T(a)\big) - \{T(a), T(b), T(c)\}, (a,b,c \in H),$$

is continuous. Let z be an element in $\sigma_F(T)$, there exists a norm-null sequence $(x_n) \subset H$ such that $T(x_n) \to z$. If we fix two arbitrary elements a, c in H, by the continuity of $\check{T}$ and the triple product of F we have

$$0 = \lim_n \frac{1}{2}\big((a|x_n)T(c) + (c|x_n)T(a)\big) - \{T(a), T(x_n), T(c)\} = -\{T(a), z, T(c)\}.$$

It follows from the arbitrariness of a and c that $\{T(E), \sigma_F(T), T(E)\} = 0$. Similarly, let V and W be odd triple monomials of degree $2m_1 + 1$ and $2m_2 + 1$, respectively, and let us fix $a_1, \dots, a_{2m_1}, b_1, \dots, b_{2m_2}$ in H. By Lemma 2,

$$\begin{aligned}
&\left\{V\left(T(a_1), \dots, T(a_{2m_1+1})\right), z, W\left(T(b_1), \dots, T(b_{2m_2+1})\right)\right\} \\
&\quad = \lim_n \left\{V\left(T(a_1), \dots, T(a_{2m_1+1})\right), T(x_n), W\left(T(b_1), \dots, T(b_{2m_2+1})\right)\right\} \\
&\quad = \lim_n T\left(\left\{V(a_1, \dots, a_{2m_1+1}), x_n, W(b_1, \dots, b_{2m_2+1})\right\}\right) \\
&\quad = \lim_n \frac{1}{2}\left(V(a_1, \dots, a_{2m_1+1})|x_n\right) T\left(W(b_1, \dots, b_{2m_2+1})\right) \\
&\qquad + \frac{1}{2}\left(W(b_1, \dots, b_{2m_2+1})|x_n\right) T\left(V(a_1, \dots, a_{2m_1+1})\right) = 0.
\end{aligned}$$

Since we have assumed that F is the Jordan–Banach triple generated by $T(E)$, it follows by linearity and from the continuity of the product of F that $\{F, \sigma_F(T), F\} = 0$. Finally, F being anisotropic implies that $\sigma_F(T) = 0$ and hence T is continuous. ■

A *(complex) spin factor* is a complex Hilbert space S provided with a conjugation (*i.e.*, a conjugate linear isometry of period 2) $x \mapsto \bar{x}$, triple product

$$\{a, b, c\} = \frac{1}{2}\left((a|b)c + (c|b)a - (a|\bar{c})\bar{b}\right),$$

and norm given by $\|a\|^2 = \frac{1}{2}(a|a) + \frac{1}{2}\sqrt{(a|a)^2 - |(a|\bar{a})|^2}$, for every $a, b, c \in S$.

Lemma 16 *Let S be a (complex) spin factor, F an anisotropic Jordan–Banach triple and $T\colon S \to F$ a generalized triple homomorphism. Then T is continuous.*

Proof Let S be a spin factor. The corollary in [11, p. 313] and the proof of the proposition on p. 312 in the just-quoted paper assure that S is the norm closed linear span of a "*spin grid*" $\{u_i, v_i, u_0\}_{i\in\Gamma}$, where $(u_i|u_j) = (v_i|v_j) = (u_i|v_j) = (u_i|v_i) = (u_0|u_i) = (u_0|v_i) = 0$, $\|u_i\| = 1$, $\|v_i\| = 1$, $\|u_0\| = 1$ or 0, $\overline{u_i} = v_i$, and $\overline{u_0} = u_0$, for every $i \neq j$ in Γ. Let S_1 (resp., S_2) denote the norm-closed subspace of S generated by $\{u_i : i \in \Gamma\}$ (resp., $\{v_i : i \in \Gamma\}$). Clearly $S = S_1 \oplus S_2 \oplus \mathbb{C}u_0$. It is easy to see that S_1 and S_2 are norm-closed subtriples of S (*i.e.*, $\{S_i, S_i, S_i\} \subset S_i$) and $\{a, b, c\} = \frac{1}{2}\left((a|b)c + (c|b)a\right)$, for every a, b, c in S_i ($i = 1, 2$). Therefore S_1 and S_2 are Hilbert spaces equipped with structure of type I Cartan factors. Lemma 15 shows that $T|_{S_i}\colon S_i \to F$ is continuous for every $i = 1, 2$. Finally, the continuity of the natural projections of S onto S_1, S_2 and $\mathbb{C}u_0$ assures that T is continuous. ■

According to the comments given before Proposition 17 in [25], the proof of Theorem 10 (and hence the proof of Theorem 14) is only valid for complex JB*-triples, the reason being that, in the real setting, a minimal tripotent e in a real JB*-triple E need not satisfy that $E_2(e) = \mathbb{R}e$. Actually, there exist examples of minimal tripotents e for which $E_2(e)$ is infinite dimensional. The extension of Theorem 14 to the real setting is not a trivial consequence of the result proved in the complex case and constitutes a result of independent interest which remains open in this paper. However, there exists a subclass of real JB*-triples for which the statements of Theorems 10 and 14 remain true. A real JB*-triple E is called *reduced* whenever

$E_2(e) = \mathbb{R}e$ (equivalently, $E^{-1}(e) = 0$) for every minimal tripotent $e \in E$. Reduced real JB*-triples were considered in [24], [23], [14] and [25]. We note that the proof of Theorem 14 is valid for reduced real JB*-triples.

4.2 Generalized Triple Derivations from a JB*-Triple

Russo and the second author carried out in [25] a pioneer study on automatic continuity of ternary derivations from a JB*-triple E into a Jordan–Banach triple E-module. The concept of Jordan–Banach triple module is introduced in the just-quoted paper, where it is also established that every triple derivation from a real or complex JB*-triple into its dual space or into itself is automatically continuous. It seems natural, at this stage, to consider generalized triple derivations in the context of JB*-triples, studying the automatic continuity of these mappings.

Jordan triple modules over Jordan triples were introduced as appropriate extensions of bimodules over associative algebras and Jordan modules over Jordan algebras (*cf.* [25]). The concrete definition reads as follows: Let E be a complex (resp., real) Jordan triple, a *Jordan triple E-module* (also called a *triple E-module*) is a vector space X equipped with three mappings

$$\{\cdot,\cdot,\cdot\}_1 \colon X \times E \times E \to X, \quad \{\cdot,\cdot,\cdot\}_2 \colon E \times X \times E \to X,$$
$$\text{and} \quad \{\cdot,\cdot,\cdot\}_3 \colon E \times E \times X \to X$$

satisfying the following axioms:

(JTM1) $\{x,a,b\}_1$ is linear in a and x and conjugate linear in b (resp., trilinear), $\{a,b,x\}_3$ is linear in b and x and conjugate linear in a (resp., trilinear) and $\{a,x,b\}_2$ is conjugate linear in a,b,x (resp., trilinear).

(JTM2) $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$.

(JTM3) Denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$, and $\{\cdot,\cdot,\cdot\}_3$, the identity

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\},$$

holds whenever one of the elements a,b,c,d,e is in X and the rest are in E.

When E is a Jordan–Banach triple and X is a triple E-module which is also a Banach space, we will say that X is a *Banach (Jordan) triple E-module* when the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ are (jointly) continuous. From now on, the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ will be simply denoted by $\{\cdot,\cdot,\cdot\}$.

Every real or complex associative algebra A (resp., Jordan algebra J) is a real Jordan triple with respect to $\{a,b,c\} := \frac{1}{2}(abc + cba)$, $a,b,c \in A$ (resp., $\{a,b,c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b)$, $a,b,c \in J$). It is not hard to see that every A-bimodule X is a real triple A-module with respect to the products $\{a,b,x\}_3 := \frac{1}{2}(abx + xba)$ and $\{a,x,b\}_2 = \frac{1}{2}(axb + bxa)$, and that every Jordan module X over a Jordan algebra J is a real triple J-module with respect to the products

$$\{a,b,x\}_3 := (a \circ b) \circ x + (x \circ b) \circ a - (a \circ x) \circ b \quad \text{and}$$
$$\{a,x,b\}_2 := (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x.$$

The dual space, E^*, of a complex (resp., real) Jordan–Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}},$$

$\forall \varphi \in E^*$, $a, b, x \in E$ (*cf.* [25]).

Given a triple E-module X over a Jordan triple E, the space $E \oplus X$ can be equipped with a structure of real Jordan triple with respect to the product

$$\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} = \{a_1, a_2, a_3\} + \{x_1, a_2, a_3\} + \{a_1, x_2, a_3\} + \{a_1, a_2, x_3\}.$$

The Jordan triple $E \oplus X$ will be called the *triple split null extension* of E and X.

Let X be a Jordan triple E-module over a Jordan triple E. A *triple derivation* from E to X is a conjugate linear map $\delta\colon E \to X$ satisfying $\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}$.

Let E be a real (resp., complex) Jordan–Banach triple and let X be a Jordan–Banach triple E-module. A (conjugate) linear mapping $\delta\colon E \to X$ is said to be a *generalized derivation* when the mapping $\check{\delta}\colon E \times E \times E \to X$,

$$(a, b, c) \mapsto \check{\delta}(a, b, c) := \delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}$$

is (jointly) continuous.

Arguing as in [25], we will associate with each generalized derivation from a JB*-triple E into a Jordan–Banach triple E-module a generalized triple homomorphism, in such a a way that the continuity of these two mappings is mutually determined.

Let $\delta\colon E \to X$ be a generalized derivation. The symbol $E \oplus X$ will stand for the triple split null extension of E and X equipped with the ℓ_1-norm. We define the mapping

$$\begin{aligned}\Theta_\delta\colon E &\to E \oplus X,\\ a &\mapsto a + \delta(a).\end{aligned}$$

It is clear that δ is continuous if and only if Θ_δ is continuous. Furthermore, the identity

$$\begin{aligned}\check{\delta}(a, b, c) &= \delta\{a, b, c\} - \{\delta(a), b, c\} - \{a, \delta(b), c\} - \{a, b, \delta(c)\}\\ &= \Theta_\delta\{a, b, c\} - \{\Theta_\delta(a), \Theta_\delta(b), \Theta_\delta(c)\} = \check{\Theta}_\delta(a, b, c),\end{aligned}$$

shows that Θ_δ is a generalized triple homomorphism. According to this notation, we set $\Delta := \Theta_\delta(E) = \{a + \delta(a) : a \in E\}$. Let $(E \oplus X)_\Delta$ be the norm closed subtriple of $E \oplus X$ generated by Δ. Since Θ_δ is a generalized triple homomorphism, by Lemma 3, the separating space $\sigma_{E\oplus X}(\Theta_\delta)$ is a triple ideal of $(E \oplus X)_\Delta$. It is not hard to see that $\sigma_{E\oplus X}(\Theta_\delta)$ coincides with $\{0\} \times \sigma_X(\delta)$.

A subspace S of a triple E-module X is said to be a *Jordan triple submodule* or a *triple submodule* if $\{E, E, S\} \subseteq S$ and $\{E, S, E\} \subseteq S$. Every triple ideal J of E is a Jordan triple E-submodule of E.

Let $a+x$ and $b+y$ be elements in $(E \oplus X)_\Delta$ and $z \in \{0\} \times \sigma_X(\delta) = \sigma_{E\oplus X}(\Theta_\delta)$. By the definition of the triple product in $E \oplus X$ and the just-quoted fact that $\sigma_{E\oplus X}(\Theta_\delta)$ is a triple ideal of $(E \oplus X)_\Delta$ we have

$$\{a, b, z\} = \{a + x, b + y, z\} \tag{16}$$

and

$$\{a, z, b\} = \{a + x, z, b + y\} \tag{17}$$

Since $(E \oplus X)_\Delta$ contains Δ, it follows from (16) and (17) that $\{E, E, \sigma_X(\delta)\} \subseteq \sigma_X(\delta)$ and $\{E, \sigma_X(\delta), E\} \subseteq \sigma_X(\delta)$. Since $\sigma_X(\delta)$ is always a linear subspace, it is also a triple E-submodule of X.

For each subset A of a triple E-module X, we define its *quadratic annihilator*, $\mathrm{Ann}_E(A)$, as the set $\{a \in E : Q(a)(A) = \{a, A, a\} = 0\}$.

We will also make use of the following equality:

$$\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X.$$

Remark 17 The quadratic annihilator of a submodule S of a triple module X need not be, in general, a linear subspace (*cf.* [25]). However, it is known that when E is a JB*-triple and $X = E$ or $X = E^*$ then, for each submodule S of X, $\mathrm{Ann}_E(S)$ is a linear subspace, and hence a norm-closed triple ideal of E (see Lemma 1 and Proposition 2 in [25]). Further, Proposition 2 (or Remark 3) in [25] shows that, in this case, $\{\mathrm{Ann}_E(S), \mathrm{Ann}_E(S), S\} = 0$ in the triple split null extension $E \oplus X$.

From now on, we assume that E is a JB*-triple and X denotes E or E^*. In this case, Remark 17 and the fact that $\sigma_X(\delta)$ is a triple E-submodule of X prove that $\mathrm{Ann}_E\big(\sigma_X(\delta)\big)$ is a norm-closed triple ideal of E.

The strategy for obtaining results on automatic continuity for generalized triple derivations will consist in applying Theorem 14 to the generalized triple homomorphism Θ_δ. In order to do this, we will first check that

$$J := \Theta_\delta^{-1}\Big(\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big)\Big)$$

is a norm-closed triple ideal of E. It is not hard to see that $\mathrm{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X$ and

$$\Theta_\delta^{-1}\Big(\mathrm{Ann}_E\big(\sigma_X(\delta)\big) \oplus X\Big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big).$$

This proves that J is a norm-closed triple ideal of E (see Remark 17). On the other hand,

$$\begin{aligned}&\left\{\operatorname{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big), \operatorname{Ann}_{E\oplus X}\big(\sigma_{E\oplus X}(\Theta_\delta)\big), \sigma_{E\oplus X}(\Theta_\delta)\right\}\\ &\quad= \left\{\operatorname{Ann}_E\big(\sigma_X(\delta)\big), \operatorname{Ann}_E\big(\sigma_X(\delta)\big), \sigma_X(\delta)\right\} = 0\end{aligned}$$

(compare the final statement in Remark 17). Theorem 14 proves the continuity of Θ_δ and hence the continuity of δ.

Theorem 18 *Let E be a real or complex* JB**-triple and $\delta\colon E \to X$ a generalized triple derivation, where $X = E$ or E^*. Then δ is continuous.* ■

The statement concerning real JB*-triples can be derived from the complex case applying Remark 14 in [25].

Since every triple derivation is a generalized triple derivation we get the following.

Corollary 19 ([25, Corollary 15]) *Let E be a real or complex* JB**-triple and let $\delta\colon E \to X$ be a triple derivation, where $X = E$ or E^*. Then δ is continuous.* ■

4.3 Generalized Triple Derivations Whose Domain is a C^*-algebra

We have already mentioned that every C^*-algebra belongs to the class of JB*-triples. We will conclude this paper by applying some of the previous results to C^*-algebras. The results obtained this way are interesting by themselves.

Lemma 20 *Let $T\colon A_{sa} \to X$ be a linear mapping from the self-adjoint part, A_{sa}, of an abelian C^*-algebra, A, to a Banach space. Suppose that $J_T := \{a \in A_{sa} : TQ(a)$ is continuous $\}$ is a norm-closed subset of A_{sa} with $\{a, A_{sa}, a\} \in J_T$, for every $a \in J_T$. Then J_T is a triple ideal of A_{sa}.*

Proof It is easy to see that every norm-closed inner ideal of the selfadjoint part of an abelian C^*-algebra A is a triple ideal in A_{sa} (norm-closed by assumption). Therefore, we only have to prove that J_T is a linear subspace. To this end, it is enough to show that $a + b \in J_T$ whenever $a, b \in J_T$.

Let a and b be two elements in A_{sa}. First we observe that, since A_{sa} is abelian, $L(a + b) = Q(a + b)$. Obviously, the linear mapping $L_b\colon A_{sa} \to A_{sa}, c \mapsto cb = bc$ is continuous. Since A_{sa} is abelian we have $L(a^2, b) = Q(a)L_b = L_bQ(a)$. Therefore $TL(a^2, b) = TQ(a)L_b$ is continuous for every $a \in J_T, b \in A_{sa}$.

Let us pick $a \in J_T$. We write a in the form $a = a_1 - a_2$ where a_1, a_2 are orthogonal positive elements in A_{sa}. Since $Q(a)A_{sa} \in J_T$, a_1^3 lies in J_T, and hence $a_1^6 A_{sa} = Q(a_1^3)A_{sa} \subseteq J_T$. This implies that J_T contains the norm-closed ideal of A_{sa} generated by a_1^6, which guarantees that J_T contains a_1 and $a_1^{\frac{1}{2}}$. Similarly, we show that J_T contains a_2 and $a_2^{\frac{1}{2}}$. Now

$$TL(a, b) = TL(a_1, b) - TL(a_2, b) = TL\big((a_1^{\frac{1}{2}})^2, b\big) - TL\big((a_2^{\frac{1}{2}})^2, b\big),$$

and thus $TL(a,b)$ is continuous for every $b \in A_{sa}$. Finally, the equality

$$TQ(a+b) = TL(a+b) = TL(a,a) + TL(b,b) + 2TL(a,b)$$

shows that $TQ(a+b)$ is continuous for every $a, b \in J_T$. ■

Proposition 21 *Let $\delta\colon A \to X$ be a generalized derivation from an abelian C^*-algebra to a Jordan–Banach triple A-module. Then δ is continuous.*

Proof We will only prove that $\delta_{|A_{sa}}$ is continuous. Let $\Theta_{\delta_0}\colon A_{sa} \to A_{sa} \oplus X$ be the generalized triple homomorphism associated to $\delta_0 := \delta|_{A_{sa}}$. We have already shown that $J = \Theta_{\delta_0}^{-1}\big(\mathrm{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big)\big)$ coincides with $\mathrm{Ann}_{A_{sa}}\big(\sigma_X(\delta_0)\big)$ (see the comments prior to Theorem 18). Therefore, J is the quadratic annihilator of a closed submodule of X, and hence J is norm closed and satisfies $\{a, A_{sa}, a\} \in J$, for every $a \in J$ (*cf.* [25, Section 2.3]).

It is easy to see that J coincides with $\{a \in A_{sa} : \Theta_{\delta_0} Q(a) \text{ is continuous}\}$. Now, Lemma 20 proves that $J = \Theta_{\delta_0}^{-1}\big(\mathrm{Ann}_{A_{sa}\oplus X}\big(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\big)\big)$ is a norm-closed triple ideal of A_{sa}, and since A is abelian,

$$\begin{aligned}&\left\{\mathrm{Ann}_{A_{sa}\oplus X}\left(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\right), \mathrm{Ann}_{A_{sa}\oplus X}\left(\sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\right), \sigma_{A_{sa}\oplus X}(\Theta_{\delta_0})\right\}\\ &\quad = \left\{\mathrm{Ann}_{A_{sa}}\left(\sigma_{A_{sa}}(\delta_0)\right), \mathrm{Ann}_{A_{sa}}\left(\sigma_{A_{sa}}(\delta_0)\right), \sigma_{A_{sa}}(\delta_0)\right\} = 0.\end{aligned}$$

Having in mind that A_{sa} is a reduced real JB*-triple and the validity of Theorem 14 for reduced real JB*-triples, we conclude that $\delta|_{A_{sa}}$ is continuous. ■

A celebrated result of J. Cuntz (see [8]) establishes that a linear mapping from a C^*-algebra A to a Banach space is continuous if and only if its restriction to each subalgebra of A generated by a single hermitian element is continuous. We finish this note with a consequence of Cuntz' theorem and Proposition 21.

Theorem 22 *Every generalized triple derivation from a real or complex C^*-algebra A to a Jordan–Banach triple A-module is continuous.* ■

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain
e-mail: jgarces@ugr.es aperalta@ugr.es

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Orthogonal forms and orthogonality preservers on real function algebras

Jorge J. Garcés and Antonio M. Peralta*

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, Granada, Spain

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We initiate the study of orthogonal forms on a real C*-algebra. Motivated by previous contributions, due to Ylinen, Jajte, Paszkiewicz and Goldstein, we prove that for every continuous orthogonal form V on a commutative real C*-algebra, A, there exist functionals φ_1 and φ_2 in A^* satisfying

$$V(x, y) = \varphi_1(xy) + \varphi_2(xy^*),$$

for every x, y in A. We describe the general form of a (not-necessarily continuous) orthogonality preserving linear map between unital commutative real C*-algebras. As a consequence, we show that every orthogonality preserving linear bijection between unital commutative real C*-algebras is continuous.

Keywords: Orthogonal form; real C*-algebra; orthogonality preservers; disjointness preserver; separating map

AMS Subject Classifications: Primary 46H40; 4J10; Secondary 47B33; 46L40; 46E15; 47B48.

1. Introduction and preliminaries

Elements a and b in a real or complex C*-algebra, A, are said to be *orthogonal* (denoted by $a \perp b$) if $ab^* = b^*a = 0$. A bounded bilinear form $V : A \times A \to \mathbb{K}$ is called *orthogonal* (resp., *orthogonal on self-adjoint elements*) whenever $V(a, b^*) = 0$ for every $a \perp b$ in A (resp., in the self-adjoint part of A). All the forms considered in this paper are assumed to be continuous. Motivated by the seminal contributions by Ylinen [1] and Jajte and Paszkiewicz [2], Goldstein proved that every orthogonal form V on a (complex) C*-algebra, A, is of the form

$$V(x, y) = \phi(xy) + \psi(xy) \quad (x, y \in A),$$

where ϕ and ψ are two functionals in A^* (cf. [3, Theorem 1.10]). A simplified proof of Goldstein's theorem was published by Haagerup and Laustsen in [4]. This characterization

*Corresponding author. Email: aperalta@ugr.es

has emerged as a very useful tool in the study of bounded linear operators between C*-algebras which are orthogonality or disjointness preserving (see, for example, [5,6]).

The first aim of this paper is to study orthogonal forms on the wider class of real C*-algebras. Little or nothing is known about the structure of an orthogonal form V on a real C*-algebra. At first look, one is tempted to consider the canonical complex bilinear extension of V to a form on the complexification, $A_{\mathbb{C}} = A \oplus i A$, of A and, when the latter is orthogonal, to apply Goldstein's theorem. However, the complex bilinear extension of V to $A_{\mathbb{C}} \times A_{\mathbb{C}}$, need not be, in general, orthogonal (see Example 2.7). The study of orthogonal forms on real C*-algebras requires a completely independent strategy; surprisingly, the resulting forms will enjoy a different structure to that established by Goldstein in the complex setting.

In section 2, we establish some structure results for orthogonal forms on a general real C*-algebra, showing, among other properties, that every orthogonal form on a real C*-algebra extends to an orthogonal form on its multiplier algebra (see Proposition 1.3). It is also proved that, for each orthogonal and symmetric form V on a real C*-algebra, A, there exists a functional $\phi \in A^*$ satisfying $V(a, b) = \phi(ab + ba)$, for every $a, b \in A$ with $a = a^*$, $b^* = b$ (cf. Proposition 1.5). In the real setting, the skew-symmetric part of a real C*-algebra, A, is not determined by the self-adjoint part of A, so the information about the behaviour of V on the rest of A is very limited.

Section 3 contains one of the main results of the paper: the characterization of all orthogonal forms on a commutative real C*-algebra. Concretely, we prove that a form V on a commutative real C*-algebra A is orthogonal if, and only if, there exist functionals φ_1 and φ_2 in A^* satisfying

$$V(x, y) = \varphi_1(xy) + \varphi_2(xy^*),$$

for every $x, y \in A$ (see Theorem 2.4). Among the consequences, it follows that the complex bilinear extension of V to the complexification of A is orthogonal if, and only if, we can take $\varphi_2 = 0$ in the above representation.

We recall that a mapping $T : A \to B$ between real or complex C*-algebras is said to be *orthogonality or disjointness preserving* (also called *separating*) whenever $a \perp b$ in A implies $T(a) \perp T(b)$ in B. The mapping T is *bi-orthogonality preserving* whenever the equivalence

$$a \perp b \Leftrightarrow T(a) \perp T(b)$$

holds for all a, b in A. As noticed in [7], every bi-orthogonality preserving linear surjection, $T : A \to B$ between two C*-algebras is injective.

The study of orthogonality preserving operators between C*-algebras started with the work of Arendt [8] in the setting of unital abelian C*-algebras. Subsequent contributions by Jarosz [9] extended the study to the setting of orthogonality preserving (not necessarily bounded) linear mappings between abelian C*-algebras. The first study on orthogonality preserving symmetric (bounded) linear operators between general (complex) C*-algebras is originally due to Wolff (cf. [10]). Orthogonality preserving bounded linear maps between C*-algebras, JB*-algebras and JB*-triples were completely described in [5,6].The pioneer works of Beckenstein et al. in [11] and [12] (see also [13]) were applied by Jarosz to prove that every orthogonality preserving linear bijection between $C(K)$-spaces is (automatically) continuous (see [9]). More recently, Burgos and the authors of this note proved in [7] that every bi-orthogonality preserving linear surjection between two von Neumann algebras (or between two compact C*-algebras) is automatically continuous (compare [14,15] for recent additional generalisations).

The main goal of section 4 is to describe the orthogonality preserving linear mappings between unital commutative real C*-algebras (see Theorem 3.2). As a consequence, we shall prove that every orthogonality preserving linear bijection between unital commutative real C*-algebras is automatically continuous. We shall exhibit some examples illustrating that the results in the real setting are completely independent from those established for complex C*-algebras. We further give a characterization of those linear mappings between real forms of $C(K)$-spaces which are bi-orthogonality preserving.

1.1. *Preliminary results*

Let us now introduce some basic facts and definitions required later. A *real C*-algebra* is a real Banach *-algebra A which satisfies the standard C*-identity, $\|a^*a\| = \|a\|^2$, and which also has the property that $1 + a^*a$ is invertible in the unitization of A for every $a \in A$. It is known that a real Banach *-algebra, A, is a real C*-algebra if, and only if, it is isometrically *-isomorphic to a norm-closed real *-subalgebra of bounded operators on a real Hilbert space (cf. [16, Corollary 5.2.11]).

Clearly, every (complex) C*-algebra is a real C*-algebra when scalar multiplication is restricted to the real field. If A is a real C*-algebra whose algebraic complexification is denoted by $B = A \oplus iA$, then there exists a C*-norm on B extending the norm of A. It is further known that there exists an involutive conjugate-linear *-automorphism τ on B such that $A = B^\tau := \{x \in B : \tau(x) = x\}$ (compare [16, Proposition 5.1.3] or [17, Lemma 4.1.13], and [18, Corollary 15.4]). The dual space of a real or complex C*-algebra A will be denoted by A^*. Let $\widetilde{\tau} : B^* \to B^*$ denote the map defined by

$$\widetilde{\tau}(\phi)(b) = \overline{\phi(\tau(b))} \qquad (\phi \in B^*,\ b \in B).$$

Then $\widetilde{\tau}$ is a conjugate-linear isometry of period 2 and the mapping

$$\begin{aligned} (B^*)^{\widetilde{\tau}} &\to A^* \\ \varphi &\mapsto \varphi|_A \end{aligned}$$

is a surjective linear isometry. We shall identify $(B^*)^{\widetilde{\tau}}$ and A^* without making any explicit mention.

When A is a real or complex C*-algebra, then A_{sa} and A_{skew} will stand for the set of all self-adjoint and skew-symmetric elements in A, respectively. We shall make use of standard notation in C*-algebra theory.

Given Banach spaces X and Y, $L(X, Y)$ will denote the space of all bounded linear mappings from X to Y. We shall write $L(X)$ for the space $L(X, X)$. Throughout the paper, the word 'operator' (respectively, multilinear or sesquilinear operator) will always mean bounded linear mapping (respectively bounded multilinear or sesquilinear mapping). The dual space of a Banach space X is always denoted by X^*.

Let us recall that a series $\sum_n x_n$ in a Banach space is called *weakly unconditionally Cauchy (w.u.C.)* if there exists $C > 0$ such that for any finite subset $F \subset \mathbb{N}$ and $\varepsilon_n = \pm 1$ we have $\left\| \sum_{n\in F} \varepsilon_n x_n \right\| \leq C$. A (linear) operator $T : X \longrightarrow Y$ is *unconditionally converging* if for every w.u.C. series $\sum_n x_n$ in X, the series $\sum_n T(x_n)$ is unconditionally convergent in Y, that is, every subseries of $\sum_n T(x_n)$ is norm converging. It is known that $T : X \to Y$ is unconditionally converging if, and only if, for every w.u.C. series $\sum_n x_n$ in X, we have $\|T(x_n)\| \to 0$ (compare, for example, [19, p. 1257]).

Let us also recall that a Banach space X is said to have *Pełczyński's property* (V) if, for every Banach space Y, every unconditionally converging operator $T : X \to Y$ is weakly compact.

The proof of the following elementary lemma is left to the reader.

LEMMA 1.1 *Let X be a complex Banach space, $\tau : X \to X$ a conjugate-linear period-2 isometry. Then the real Banach space $X^\tau := \{x \in X : \tau(x) = x\}$ satisfies property (V) whenever X does.*

We shall require, for later use, some results on extensions of multilinear operators. Let $X_1, \ldots, X_n$, and X be Banach spaces, $T : X_1 \times \cdots \times X_n \to X$ a (continuous) n-linear operator, and $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation. It is known that there exists a unique n-linear extension $AB(T)_\pi : X_1^{**} \times \cdots \times X_n^{**} \to X^{**}$ such that for every $z_i \in X_i^{**}$ and every net $(x^i_{\alpha_i}) \in X_i$ $(1 \leq i \leq n)$, converging to z_i in the weak* topology we have

$$AB(T)_\pi(z_1, \ldots, z_n) = \text{weak*} - \lim_{\alpha_{\pi(1)}} \cdots \text{weak*} - \lim_{\alpha_{\pi(n)}} T(x^1_{\alpha_1}, \ldots, x^n_{\alpha_n}).$$

Moreover, $AB(T)_\pi$ is bounded and has the same norm as T. The extensions $AB(T)_\pi$ coincide with those considered by Arens in [20,21] and by Aron and Berner for polynomials in [22]. The n-linear operators $AB(T)_\pi$ are usually called the *Arens* or *Aron-Berner extensions* of T.

Under some additional hypothesis, the Arens extension of a multilinear operator also is separately weak* continuous. Indeed, if every operator from X_i to X_j^* is weakly compact $(i \neq j)$ the Arens extensions of T defined above do not depend on the chosen permutation π and they are all separately weak* continuous (see [23], and Theorem 1 in [24]). In particular, the above requirements always hold when every X_i satisfies Pelczynski's property (V) (in such case X_i^* contains no copies of c_0, therefore every operator from X_i to X_j^* is unconditionally converging, and hence weakly compact by property (V), see [25]). When all the Arens extensions of T coincide, the symbol $AB(T) = T^{**}$ will denote any of them.

We should note at this point that every C*-algebra satisfies property (V) (cf. Corollary 6 in [26]). Since every real C*-algebra is, in particular, a real form of a (complex) C*-algebra, it follows from Lemma 1.1 that every real C*-algebra satisifes property (V). We therefore have:

LEMMA 1.2 *Let $A_1, \ldots, A_k$ be real C*-algebras and let T be a multilinear continuous operator from $A_1 \times \ldots \times A_k$ to a real Banach space X. Then T admits a unique Arens extension $T^{**} : A_1^{**} \times \ldots \times A_k^{**} \to X^{**}$ which is separately weak* continuous.*

Given a real or complex C*-algebra, A, the *multiplier algebra* of A, $M(A)$, is the set of all elements $x \in A^{**}$ such that, for each element $a \in A$, xa and ax both lie in A. We notice that $M(A)$ is a C*-algebra and contains the unit element of A^{**}. It should be recalled here that $A = M(A)$ whenever A is unital.

PROPOSITION 1.3 *Let A be a real C*-algebra. Suppose that $V : A \times A \to \mathbb{R}$ is an orthogonal bounded bilinear form. Then the continuous bilinear form*

$$\tilde{V} : M(A) \times M(A) \to \mathbb{R}, \quad \tilde{V}(a, b) := V^{**}(a, b)$$

is orthogonal.

Proof Let a and b be two orthogonal elements in $M(A)$. Let $a^{[\frac{1}{3}]}$ (resp., $b^{[\frac{1}{3}]}$) denote the unique element z in $M(A)$ satisfying $zz^*z = a$ (resp., $zz^*z = b$). We notice that $a^{[\frac{1}{3}]}$ and $b^{[\frac{1}{3}]}$ are orthogonal, so, for each pair x, y in A, $a^{[\frac{1}{3}]}xa^{[\frac{1}{3}]}$ and $b^{[\frac{1}{3}]}yb^{[\frac{1}{3}]}$ are orthogonal elements in A. Since V is orthogonal, we have

$$V(a^{[\frac{1}{3}]}xa^{[\frac{1}{3}]}, (b^{[\frac{1}{3}]})^*y(b^{[\frac{1}{3}]})^*) = 0$$

for every $x, y \in A$.

Goldstine's theorem (cf. Theorem V.4.2.5 in [27]) guarantees that the closed unit ball of A is weak*-dense in the closed unit ball of A^{**}. Therefore, we can pick two bounded nets (x_λ) and (y_μ) in A, converging in the weak* topology of A^{**} to $(a^{[\frac{1}{3}]})^*$ and $b^{[\frac{1}{3}]}$, respectively.

We have already mentioned that $V^{**} : A^{**} \times A^{**} \to \mathbb{R}$ is separately weak* continuous. Since $0 = V(a^{[\frac{1}{3}]}x_\lambda a^{[\frac{1}{3}]}, (b^{[\frac{1}{3}]})^* y_\mu (b^{[\frac{1}{3}]})^*)$, for every λ and μ, taking limits, first in λ and subsequently in μ, we deduce that

$$V^{**}(a^{[\frac{1}{3}]}(a^{[\frac{1}{3}]})^*a^{[\frac{1}{3}]}, (b^{[\frac{1}{3}]})^*b^{[\frac{1}{3}]}(b^{[\frac{1}{3}]})^*)) = \tilde{V}(a, b^*) = 0,$$

which shows that $\tilde{V}$ is orthogonal. $\square$

Since the multiplier algebra of a real or complex C*-algebra always has a unit element, Proposition 1.3 allows us to restrict our study on orthogonal bilinear forms on a real C*-algebra A to the case in which A is unital.

A *real von Neumann algebra* is a real C*-algebra which is also a dual Banach space (cf. [28] or [16, §6.1]). Clearly, the self-adjoint part of a real von Neumann algebra is a JW-algebra in the terminology employed in [29], so every self-adjoint element in a real von Neumann algebra W can be approximated in norm by a finite real linear combination of mutually orthogonal projections in W (cf. [29, Proposition 4.2.3]). We shall explore now the validity in the real setting of some of the results established by Goldstein in [3].

LEMMA 1.4 *Let A be a real von Neumann algebra with unit* 1. *Suppose that $V : A \times A \to \mathbb{R}$ is a bounded bilinear form. The following are equivalent:*

(a) *V is orthogonal on A_{sa};*
(b) *$V(p, q) = 0$, whenever p and q are two orthogonal projections in A;*
(c) *$V(a, b) = V(ab, 1)$ for every $a, b \in A_{sa}$ with $ab = ba$.*

If any of the above statements holds and V is symmetric, then defining $\phi_1(x) := V(x, 1)$ ($x \in A$), we have $V(a, b) = \phi_1(\frac{ab+ba}{2})$, for every $a, b \in A_{sa}$.

Proof Applying the existence of spectral resolutions for self-adjoint elements in a real von Neumann algebra, the argument given by Goldstein in [3, Proposition 1.2] remains valid to prove the equivalence of (a), (b) and (c).

Suppose now that V is symmetric. Let $a = \sum_{j=1}^m \lambda_j p_j$ be an algebraic element in A_{sa}, where the λ_j's belong to $\mathbb{R}$ and $p_1, \ldots, p_m$ are mutually orthogonal projections in A. Since V is orthogonal, for every projection $p \in A$, we have

$$V(p, 1) = V(p, 1 - p) + V(p, p) = V(p, p).$$

Thus,

$$V(a,a) = \sum_{j=1}^{m} \lambda_j^2 V(p_j, p_j) = \sum_{j=1}^{m} \lambda_j^2 V(p_j, 1) = V\left(\sum_{j=1}^{m} \lambda_j^2 p_j, 1\right) = V(a^2, 1).$$

The (norm) density of algebraic elements in A_{sa} and the continuity of V imply that $V(a,a) = V(a^2, 1)$, for every $a \in A_{sa}$. Finally, applying that V is symmetric we have

$$V(a^2, 1) + V(b^2, 1) + V(ab + ba, 1) = V((a+b)^2, 1)$$

$$= V(a+b, a+b) = V(a,a) + V(b,b) + 2V(a,b),$$

for every $a, b \in A_{sa}$, and hence $V(a,b) = V\left(\frac{ab+ba}{2}, 1\right)$, for all $a, b \in A_{sa}$. □

The above result holds for every monotone σ-complete unital real C*-algebra A (that is, each upper bounded, monotone increasing sequence of self-adjoint elements of A has a least upper bound).

Surprisingly, the final conclusion of the above Lemma can be established for unital real C*-algebras with independent basic techniques.

Proposition 1.5 *Let A be a unital real C*-algebra with unit 1. Suppose that $V : A \times A \to \mathbb{R}$ is an orthogonal, symmetric, bounded, bilinear form. Then defining $\phi_1(x) := V(x, 1)$ $(x \in A)$, we have $V(a,b) = \phi_1(\frac{ab+ba}{2})$, for every $a, b \in A_{sa}$.*

Proof Let a be a self-adjoint element in A. The real C*-subalgebra, C, of A generated by 1 and a is isometrically isomorphic to the space $C(K, \mathbb{R})$ of all real-valued continuous functions on a compact Hausdorff space K. The restriction of V to $C \times C$ is orthogonal, therefore, the mapping $x \mapsto V(x,x)$ is a 2-homogeneous orthogonally additive polynomial on C. The main result in [30] implies the existence of a functional $\varphi_a \in C^*$ such that $V(x,x) = \varphi_a(x^2)$, for every $x \in C$. It is clear that $\varphi_a(x) = V(x, 1)$ for every $x \in C$. In particular

$$V(a,a) = \varphi_a(a^2) = V(a^2, 1).$$

The argument given at the end of the proof of Lemma 1.4 gives the desired statement. □

The above proposition shows that we can control the form of a symmetric orthogonal form on the self adjoint part of a (unital) real C*-algebra. The form on the skew-symmetric part remains out of control for the moment.

2. Orthogonal forms on abelian real C*-algebras

Throughout this section, A will denote a unital, abelian, real C*-algebra whose complexification will be denoted by B. It is clear that B is a unital, abelian C*-algebra. It is known that there exists a period-2 conjugate-linear *-automorphism $\tau : B \to B$ such that $A = B^\tau := \{x \in B : \tau(x) = x\}$ (cf. [17, 4.1.13] and [18, 15.4] or [16, §5.2]).

By the commutative Gelfand theory, there exists a compact Hausdorff space K such that B is C*-isomorphic to the C*-algebra $C(K)$ of all complex, valued continuous functions on K. The Banach–Stone Theorem implies the existence of a homeomorphism $\sigma : K \to K$ such that $\sigma^2(t) = t$, and

$$\tau(a)(t) = \overline{a(\sigma(t))},$$

for all $t \in K$, $a \in C(K)$. Real function algebras of the form $C(K)^\tau$ have been studied by its own right and are interesting in some other settings (cf. [31]).

Henceforth, the symbol $\mathfrak{B}$ will stand for the σ-algebra of all Borel subsets of K, $S(K)$ will denote the space of $\mathfrak{B}$-simple scalar functions defined on K, while the *Borel algebra over* K, $B(K)$, is defined as the completion of $S(K)$ under the supremum norm. It is known that $B = C(K) \subset B(K) \subset C(K)^{**}$. The mapping $\tau^{**} : C(K)^{**} \to C(K)^{**}$ is a period-2 conjugate-linear *-automorphism on $B^{**} = C(K)^{**}$. It is easy to see that $\tau^{**}(B(K)) = B(K)$, and hence $\tau^{**}|_{B(K)} : B(K) \to B(K)$ defines a period-2 conjugate-linear*-automorphism on $B(K)$. By an abuse of notation, the symbol τ will denote τ, τ^{**} and $\tau^{**}|_{B(K)}$ indistinctly. It is clear that, for each Borel set $B \in \mathfrak{B}$, $\tau(\chi_B) = \chi_{\sigma(B)}$.

Let a be an element in $B(K)$. For each $\varepsilon > 0$, there exist complex numbers $\lambda_1, \ldots, \lambda_r$ and disjoint Borel sets $B_1, \ldots, B_r$ such that $\left\| a - \sum_{k=1}^r \lambda_k \chi_{B_k} \right\| < \varepsilon$. When $a \in A$ is τ-symmetric (i.e. $\tau(a) = a$) then, since $a = \frac{1}{2}(a + \tau(a))$, we have

$$\left\| a - \frac{1}{2} \sum_{k=1}^r \lambda_k \chi_{B_k} + \overline{\lambda_k} \chi_{\sigma(B_k)} \right\| \leq \frac{1}{2} \left\| a - \sum_{k=1}^r \lambda_k \chi_{B_k} \right\| + \frac{1}{2} \left\| a - \sum_{k=1}^r \overline{\lambda_k} \chi_{\sigma(B_k)} \right\|$$

$$\leq \frac{1}{2} \left\| a - \sum_{k=1}^r \lambda_k \chi_{B_k} \right\| + \frac{1}{2} \left\| \tau \left(a - \sum_{k=1}^r \lambda_k \chi_{B_k} \right) \right\| < \varepsilon.$$

Consequently, every element in $B(K)^\tau$ can be approximated in norm by finite linear combinations of the form $\sum_k \alpha_k \chi_{B_k} + \overline{\alpha_k} \chi_{\sigma(B_k)}$, where $\alpha_1, \ldots, \alpha_n$ are complex numbers and $B_1, \ldots, B_n$ are mutually disjoint Borel sets. Having in mind that, for each Borel set $B \in \mathfrak{B}$ and each $\alpha \in \mathbb{C}$, $\left(\alpha \chi_B + \overline{\alpha} \chi_{\sigma(B)} \right)^* = \overline{\alpha} \chi_B + \alpha \chi_{\sigma(B)}$, we have

$$\left(\alpha \chi_B + \overline{\alpha} \chi_{\sigma(B)} \right) + \left(\alpha \chi_B + \overline{\alpha} \chi_{\sigma(B)} \right)^* = 2\Re\mathrm{e}(\alpha) \left(2\chi_{\sigma(B)\cap B} + \chi_{\sigma(B)\setminus B} + \chi_{B \setminus \sigma(B)} \right)$$

$$= 2\Re\mathrm{e}(\alpha) \left(2\chi_{\sigma(B)\cap B} + \chi_{(\sigma(B)\setminus B) \cup \sigma(\sigma(B)\setminus B)} \right),$$

and

$$\left(\alpha \chi_B + \overline{\alpha} \chi_{\sigma(B)} \right) - \left(\alpha \chi_B + \overline{\alpha} \chi_{\sigma(B)} \right)^* = 2i \Im\mathrm{m}(\alpha) \left(\chi_{B \setminus \sigma(B)} - \chi_{\sigma(B) \setminus B} \right).$$

Suppose now that $a \in B(K)^\tau$ is *-symmetric (i.e. $a^* = a$). It follows from the above that a can be approximated in norm by linear combinations of the form $\sum_{k=1}^r \alpha_k \chi_{E_k}$, where $\alpha_k \in \mathbb{R}$ and $E_1, \ldots, E_r$ are mutually disjoint Borel subsets of K with $\sigma(E_i) = E_i$. Let b be an element in $B(K)^\tau$ satisfying $b^* = -b$. Similar arguments to those given for *-symmetric elements, allow us to show that b can be approximated in norm by finite linear combinations of the form $\sum_{k=1}^r i\, \alpha_k (\chi_{E_k} - \chi_{\sigma(E_k)})$, where $\alpha_k \in \mathbb{R}$ and $E_1, \ldots, E_r$ are mutually disjoint Borel subsets of K with $\sigma(E_i) \cap E_i = \emptyset$.

LEMMA 2.1 *Let A be a unital, abelian, real C^*-algebra whose complexification is denoted by $B = C(K)$, for a suitable compact Hausdorff space K. Let $\tau : B \to B$ be a period-2 conjugate-linear*-automorphism satisfying $A = B^\tau$ and $\tau(a)(t) = \overline{a(\sigma(t))}$, for all $t \in K$, $a \in C(K)$, where $\sigma : K \to K$ is a period-2 homeomorphism. Then the set $N = \{t \in K : \sigma(t) \neq t\}$ is an open subset of K, $F = \{t \in K : \sigma(t) = t\}$ is a closed subset of K and there exists an open subset $\mathcal{O} \subset K$ maximal with respect to the property $\mathcal{O} \cap \sigma(\mathcal{O}) = \emptyset$.*

Proof That F is closed follows easily from the continuity of σ, and consequently, $N = K/F$ is open.

Let $\mathcal{F}$ be the family of all open subsets $O \subseteq K$ such that $O \cap \sigma(O) = \emptyset$ ordered by inclusion. Let $S = \{O_\lambda\}_\lambda$ be a totally ordered subset of $\mathcal{F}$. We shall see that $O = \bigcup_\lambda O_\lambda$ is an open set which also lies in $\mathcal{F}$, that is, $O \cap \sigma(O) = \emptyset$.

Let us suppose, on the contrary, that there exists $t \in O \cap \sigma(O) \neq \emptyset$. Then there exist λ, β such that $t \in O_\lambda$ and $t \in \sigma(O_\beta)$. Since S is totally ordered, $O_\lambda \subseteq O_\beta$ or $O_\beta \subseteq O_\lambda$. We shall assume that $O_\lambda \subseteq O_\beta$. Then t lies in $O_\beta \cap \sigma(O_\beta) = \emptyset$, which is a contradiction. Finally, Zorn's Lemma gives the existence of a maximal element $\mathcal{O}$ in $\mathcal{F}$. $\square$

It should be noticed here that, in Lemma 2.1, $\mathcal{O} \cup \sigma(\mathcal{O}) = N$, an equality which follows from the maximality of $\mathcal{O}$.

Our next lemma analyses the 'spectral resolution' of a *-skew-symmetric element in $B(K)^\tau$.

LEMMA 2.2 *In the notation of Lemma 2.1, let $B(A) = B(K)^\tau$, let $a \in B(K)^\tau_{sa}$, and let b be an element in $B(A)_{skew}$. Then the following statements hold:*

(a) *$b|F = 0$;*
(b) *For each $\varepsilon > 0$, there exist mutually disjoint Borel sets $B_1, \ldots, B_m \subset \mathcal{O}$ and real numbers $\lambda_1, \ldots, \lambda_m$ satisfying $\left\| b - \sum_{j=1}^m i\, \lambda_j (\chi_{B_j} - \chi_{\sigma(B_j)}) \right\| < \varepsilon$;*
(c) *For each $\varepsilon > 0$, there exist mutually disjoint Borel sets $C_1, \ldots, C_m \subset K$ and real numbers $\mu_1, \ldots, \mu_m$ satisfying $\sigma(C_j) = C_j$, and $\left\| a - \sum_{j=1}^m \mu_j \chi_{C_j} \right\| < \varepsilon$.*

Proof (a) Since $b^* = -b$, we have $Re(b(t)) = 0, \forall t \in K$. Now, let $t \in F$, applying $\sigma(t) = t$ and $\tau(b) = b$ we get $\overline{b(t)} = \overline{b(\sigma(t))} = b(t)$, and hence $\Im\mathrm{m}(b(t)) = 0$.

Statements (b) and (c) follow from the comments prior to Lemma 2.1 and the maximality of $\mathcal{O}$ in that Lemma. $\square$

It is clear that in a commutative real (or complex) C*-algebra, A, two elements a, b are orthogonal if and only if they have zero-product, that is, $ab = 0$. Therefore, $V(a, b^*) = 0 = V(a, b)$ whenever $V : A \times A \to \mathbb{R}$ is an orthogonal bilinear form on an abelian real C*-algebra and a, b are two orthogonal elements in A. We shall make use of this property without an explicit mention.

We shall keep the notation of Lemma 2.1 throughout the section. Henceforth, for each $C \subseteq \mathcal{O}$ we shall write $u_C = i\,(\chi_C - \chi_{\sigma(C)})$. The symbol u_0 will stand for the element $u_{\mathcal{O}}$. It is easy to check $1 = \chi_F + u_0 u_0^*$, where 1 is the unit element in $B(K)^\tau$. By Lemma 2.2 a), for each $b \in B(K)^\tau_{skew}$ we have $b \perp \chi_F$, and so $b = b u_0 u_0^*$.

PROPOSITION 2.3 *Let K be a compact Hausdorff space, τ a period-2 conjugate-linear isometric *-homomorphism on $C(K)$, $A = C(K)^\tau$, and $V : A \times A \to \mathbb{R}$ be an orthogonal bounded bilinear form whose Arens extension is denoted by $V^{**} : A^{**} \times A^{**} \to \mathbb{R}$. Let $\sigma : K \to K$ be a period-2 homeomorphism satisfying $\tau(a)(t) = \overline{a(\sigma(t))}$, for all $t \in K$, $a \in C(K)$. Then the following assertions hold for all Borel subsets D, B, C of K with $\sigma(B) \cap B = \sigma(C) \cap C = \emptyset$ and $\sigma(D) = D$:*

(*a*) $V(\chi_D, u_B) = V(u_B, \chi_D) = 0$, *whenever* $D \cap B = \emptyset$;
(*b*) $V(u_B, u_C) = 0$, *whenever* $B \cap C = \emptyset$;
(*c*) $V((u_0 u_0^* - u_C u_C^*)u_B, u_C) = V(u_C, (u_0 u_0^* - u_C u_C^*)u_B) = 0.$

Proof By an abuse of notation, we write V for V and V^{**}.

Let K_1, K_2 be compact subsets of K such that K_1, K_2 and $\sigma(K_2)$ are mutually disjoint. By regularity and Urysohn's Lemma, there exist nets $(f_\lambda)_\lambda$, $(g_\gamma)_\gamma$ in $C(K)^+$ such that $\chi_{K_1} \leq f_\lambda \leq \chi_{K\setminus(K_2\cup\sigma(K_2))}$, $\chi_{K_2} \leq g_\gamma \leq \chi_{K\setminus(K_1\cup\sigma(K_1)\cup\sigma(K_2))}$, $(f_\lambda)_\lambda$ (respectively, $(g_\gamma)_\gamma$) converges to χ_{K_1} (resp., to χ_{K_2}) in the weak* topology of $C(K)^{**}$.

The nets $\widetilde{f_\lambda} = \frac{1}{2}(f_\lambda + \tau(f_\lambda))$ and $\widetilde{g}_\gamma = i(g_\gamma - \tau(g_\gamma))$ lie in $C(K)^\tau$ and converge in the weak* topology of $C(K)^{**}$ to $\frac{1}{2}(\chi_{K_1} + \chi_{\sigma(K_1)})$ and u_{K_2}, respectively. It is also clear that $f_\lambda \perp g_\gamma$, $\tau(f_\lambda) \perp g_\gamma$, and hence $\widetilde{f_\lambda} \perp \widetilde{g}_\gamma$, for every λ, γ.

By the separate weak* continuity of $V^{**} \equiv V$, we have

$$V\left(\frac{1}{2}(\chi_{K_1} + \chi_{\sigma(K_1)}), u_{K_2}\right) = w^* - \lim_{\lambda}\left(w^* - \lim_{\gamma} V\left(\widetilde{f_\lambda}, \widetilde{g}_\gamma\right)\right) = 0, \tag{1}$$

and

$$V\left(u_{K_2}, \frac{1}{2}(\chi_{K_1} + \chi_{\sigma(K_1)})\right) = 0.$$

We can similarly prove that

$$V\left(u_{K_1}, u_{K_2}\right) = 0, \tag{2}$$

whenever K_1 and K_2 are two compact subsets of K such that $K_1, K_2, \sigma(K_1)$ and $\sigma(K_2)$ are pairwise disjoint.

(a) Let now D, B be two disjoint Borel subsets of K such that $\sigma(D) = D$ and $B \subseteq \mathcal{O}$. By inner regularity, there exist nets of the form $(\chi_{K_\lambda^D})_\lambda$ and $(\chi_{K_\gamma^B})_\gamma$ such that $(\chi_{K_\lambda^D})_\lambda$ and $(\chi_{K_\gamma^B})_\gamma$ converge in the weak* topology of $C(K)^{**}$ to χ_D and χ_B, respectively, where each $K_\lambda^D \subseteq D$ and each $K_\gamma^B \subseteq B$ is a compact subset of K. By the assumptions made on D and B, we have that $K_\lambda^D \cap K_\gamma^B = K_\lambda^D \cap \sigma(K_\gamma^B) = \emptyset$ and $K_\gamma^B \subseteq \mathcal{O}$ for all λ and γ. By (1) and the separate weak* continuity of V, we have

$$V(\chi_D, u_B) = w^* - \lim_{\lambda}\left(w^* - \lim_{\gamma} V\left(\frac{\chi_{K_\lambda^D} + \chi_{\sigma(K_\lambda^D)}}{2}, u_{K_\gamma^B}\right)\right) = 0, \tag{3}$$

and

$$V(u_B, \chi_D) = 0. \tag{4}$$

A similar argument, but replacing (1) with (2), applies to prove (b).

To prove the last statement, we observe that

$$(u_0 u_0^* - u_c u_c^*)u_B = (\chi_{\mathcal{O}} + \chi_{\sigma(\mathcal{O})} - \chi_C - \chi_{\sigma(C)})u_B = (\chi_{\mathcal{O}\setminus C} + \chi_{\sigma(\mathcal{O}\setminus C)})u_B = u_{(\mathcal{O}\setminus C)\cap B},$$

and hence the statement (c) follows from (b). $\square$

We can now establish the description of all orthogonal forms on a commutative real C*-algebra.

THEOREM 2.4 *Let $V : A \times A \to \mathbb{R}$ be a continuous orthogonal form on a commutative real C^*-algebra, then there exist φ_1, and φ_2 in A^* satisfying*

$$V(x, y) = \varphi_1(xy) + \varphi_2(xy^*),$$

for every $x, y \in A$.

Proof We may assume, without loss of generality, that A is unital (compare Proposition 1.3). Let B denote the complexification of A. In this case B identifies with $C(K)$ for a suitable compact Hausdorff space K and $A = C(K)^\tau$, where τ is a conjugate-linear period-2 *-homomorphism on $C(K)$. We shall follow the notation employed in the rest of this section.

The form $V : A \times A \to \mathbb{R}$ extends to a continuous form $V^{**} : A^{**} \times A^{**} \to \mathbb{R}$ which is separately weak* continuous (cf. Lemma 1.2). The restriction $V^{**}|_{B(K)^\tau \times B(K)^\tau} : B(K)^\tau \times B(K)^\tau \to \mathbb{R}$ also is a continuous extension of V. We shall prove the statement for $V^{**}|_{B(K)^\tau \times B(K)^\tau}$. Henceforth, the symbol V will stand for V, V^{**} and $V^{**}|_{B(K)^\tau \times B(K)^\tau}$ indistinctly.

Let us first take two self-adjoint elements a_1, a_2 in $B(K)^\tau$. By Proposition 1.5,

$$V(a_1, a_2) = V(a_1 a_2, 1). \tag{5}$$

To deal with the skew-symmetric part, let D, B, C be Borel subsets of K with, $D = \sigma(D)$ and $B, C \subseteq \mathcal{O}$. From Proposition 2.3 (a), we have

$$\begin{aligned} V(\chi_D, u_B) &= V(\chi_D, u_B(1 - \chi_D + \chi_D)) = V(\chi_D, u_{B\cap(K\setminus D)}) + V(\chi_D, u_B\chi_D) \\ &= V(\chi_D - 1 + 1, u_B\chi_D) = V(-\chi_{(K\setminus D)} + 1, u_{(B\cap D)}) = V(1, u_B\chi_D). \end{aligned} \tag{6}$$

Similarly,

$$V(u_B, \chi_D) = V(u_B\chi_D, 1). \tag{7}$$

Now, Proposition 2.3 (b) and (c), repeatedly applied give:

$$\begin{aligned} V(u_B, u_C) &= V(u_B(\chi_F + u_0u_0^*), u_C) = V(u_Bu_0u_0^*, u_C) \\ &= V(u_B(u_0u_0^* + u_Cu_C^* - u_Cu_C^*), u_C) = V(u_Bu_Cu_C^*, u_C) \\ &= V(u_Bu_Cu_C^*, u_C - u_0 + u_0) = V(u_{(B\cap C)}, -u_{(\mathcal{O}\setminus C)} + u_0) = V(u_{(B\cap C)}, u_0) \\ &= V(u_Bu_C(u_C^* - u_0^* + u_0^*), u_0) = V(u_Bu_Cu_0^*, u_0). \end{aligned}$$

Thus, we have

$$V(u_B, u_C) = V(u_Bu_Cu_0^*, u_0), \tag{8}$$

and similarly

$$V(u_B, u_C) = V(u_0, u_Bu_Cu_0^*). \tag{9}$$

Let $a_l = \sum_{j=1}^{m_l} \mu_{l,j}\chi_{D_j^l}$, $b_l = \sum_{k=1}^{p_l} \lambda_{l,k}u_{B_k^l}$ $(l \in \{1, 2\})$ be two simple elements in $B(K)^\tau_{sa}$ and $B(K)^\tau_{skew}$, respectively, where $\lambda_{l,k}, \mu_{l,j} \in \mathbb{R}$, for each $l \in \{1, 2\}$, $\{D_1^l, \ldots, D_{m_l}^l\}$ and $\{B_1^l, \ldots, B_{p_l}^l\}$ are families of mutually disjoint Borel subsets of K with $\sigma(D_j^l) = D_j^l$ and $B_i^l \subseteq \mathcal{O}$. By (5), (6), (7) and (8), we have

$$V(a_1 + b_1, a_2 + b_2) = V(a_1a_2, 1) + \sum_{j=1}^{m_1}\sum_{k=1}^{p_2} \mu_{1,j}\lambda_{2,k} V\left(\chi_{D_j^1}, u_{B_k^2}\right)$$

$$+\sum_{k=1}^{p_1}\sum_{j=1}^{m_2}\mu_{2,j}\lambda_{1,k}V\left(u_{B^1_k},\chi_{D^2_j}\right)+\sum_{k=1}^{p_1}\sum_{k=1}^{p_2}\lambda_{2,k}\lambda_{1,k}V\left(u_{B^1_k},u_{B^2_k}\right)$$

$$=V(a_1a_2,1)+\sum_{j=1}^{m_1}\sum_{k=1}^{p_2}\mu_{1,j}\lambda_{2,k}V\left(1,\chi_{D^1_j}u_{B^2_k}\right)$$

$$+\sum_{k=1}^{p_1}\sum_{j=1}^{m_2}\mu_{2,j}\lambda_{1,k}V\left(u_{B^1_k}\chi_{D^2_j},1\right)+\sum_{k=1}^{p_1}\sum_{k=1}^{p_2}\lambda_{2,k}\lambda_{1,k}V\left(u_{B^1_k}u_{B^2_k}u_0^*,u_0\right)$$

$$=V(a_1a_2,1)+V(1,a_1b_2)+V(b_1a_2,1)+V\left(b_1b_2u_0^*,u_0\right)$$

$$=\psi_1(a_1a_2)+\psi_2(a_1b_2)+\psi_1(b_1a_2)+\psi_4(b_1b_2),$$

where ψ_1, ψ_2 and ψ_4 are the functionals in A^* defined by $\psi_1(x)=V(x,1)$, $\psi_2(x)=V(1,x)$, and $\psi_4(x)=V(xu_0^*,u_0)$, respectively. Since, by Proposition 2.2, simple elements of the above form are norm-dense in $B(K)^\tau_{sa}$ and $B(K)^\tau_{skew}$, respectively, and V is continuous, we deduce that

$$V(a_1+b_1,a_2+b_2)=\psi_1(a_1a_2)+\psi_2(a_1b_2)+\psi_1(b_1a_2)+\psi_4(b_1b_2),$$

for every $a_1,a_2\in B(K)^\tau_{sa}$, $b_1,b_2\in B(K)^\tau_{skew}$.

Now, taking $\phi_1=\frac{1}{4}(2\psi_1+\psi_2+\psi_4)$, $\phi_2=\frac{1}{4}(2\psi_1-\psi_2-\psi_4)$, $\phi_3=\frac{1}{4}(\psi_2-\psi_4)$ and $\phi_4=\frac{1}{4}(\psi_4-\psi_2)$, we get

$$V(a_1+b_1,a_2+b_2)=\phi_1((a_1+b_1)(a_2+b_2))+\phi_2\left((a_1+b_1)(a_2+b_2)^*\right)$$

$$+\phi_3\left((a_1+b_1)^*(a_2+b_2)\right)+\phi_4\left((a_1+b_1)^*(a_2+b_2)^*\right),$$

for every $a_1,a_2\in B(K)^\tau_{sa}$, $b_1,b_2\in B(K)^\tau_{skew}$.

Finally, defining $\varphi_1(x)=\phi_1(x)+\phi_4(x^*)$ and $\varphi_2(x)=\phi_2(x)+\phi_3(x^*)$ $(x\in A)$, we get the desired statement. □

Remark 2.5 The functionals φ_1 and φ_2 appearing in Theorem 2.4 need not be unique. For example, let (φ_1,φ_2) and (ϕ_1,ϕ_2) be two couples of elements in the dual of a commutative real C*-algebra A. It is not hard to check that

$$\varphi_1(xy)+\varphi_2(xy^*)=\phi_1(xy)+\phi_2(xy^*),$$

for every $x,y\in A$ if, and only if, $\varphi_1+\varphi_2=\phi_1+\phi_2$, $(\varphi_1-\varphi_2)(z)=(\phi_1-\phi_2)(z)$ and $(\varphi_1-\varphi_2)(zw)=(\phi_1-\phi_2)(zw)$, for every $z,w\in A_{skew}$. These conditions are not enough to guarantee that $\phi_i=\varphi_i$. Take, for example, $A=\mathbb{R}\oplus^\infty\mathbb{C}_\mathbb{R}$, $\phi_1(a,b)=a+\Re\mathrm{e}(b)+\Im\mathrm{m}(b)$, $\phi_2(a,b)=0$, $\varphi_1(a,b)=\frac{a}{2}+\Re\mathrm{e}(b)+\Im\mathrm{m}(b)$, and $\varphi_2(a,b)=\frac{a}{2}$.

Corollary 2.6 *Let $V:A\times A\to\mathbb{R}$ be a continuous orthogonal form on a commutative real C*-algebra, then its (unique) Arens extension $V^{**}:A^{**}\times A^{**}\to\mathbb{R}$ is an orthogonal form.* □

Clearly, the statement of the above Theorem 2.4 does not hold for bilinear forms on a commutative (complex) C*-algebra. The real version established in this paper is completely independent to the result proved by Ylinen for commutative complex C*-algebras in [1] and [3]. It seems natural to ask whether the real result follows from the complex one by a mere argument of complexification. Our next example shows that the (canonical) extension of an orthogonal form on a commutative real C*-algebra need not be an orthogonal form on the complexification.

Example 2.7 Let $K = \{t_1, t_2\}$. We define $\sigma : K \to K$ by $\sigma(t_1) = t_2$. Let $A = C(K)^\tau$ be the real C*-algebra whose complexification is $C(K)$ and let $V : A \times A \to \mathbb{R}$, be the orthogonal form defined by $V(x, y) = \phi_{t_1}(xy^*) = \Re\mathrm{e}(x(t_1)\overline{y(t_1)}) = \Re\mathrm{e}(x(t_1)y(t_2))$, where $\phi_{t_1} = \Re\mathrm{e}(\delta_{t_1})$. In this case, the canonical complex bilinear extension $\widetilde{V} : C(K) \times C(K) \to \mathbb{C}$ is given by $\widetilde{V}(x, y) = \phi_{t_1}(x\tau(y)^*) = x(t_1)y(t_2)$ $(x, y \in C(K))$. It is clear that $\chi_{t_1} \perp \chi_{t_2}$ in $C(K)$, however $\widetilde{V}(\chi_{t_1}, \chi_{t_2}) = 1 \neq 0$, which implies that $\widetilde{V}$ is not orthogonal.

The (complex) bilinear extension of an orthogonal form V on a real C*-algebra to its complexification is orthogonal precisely when V satisfies the generic form of an orthogonal form on a (complex) C*-algebra given by the main result in [3].

COROLLARY 2.8 *Let $V : A \times A \to \mathbb{R}$ be a continuous orthogonal form on a commutative real C*-algebra, let B denote the complexification of A and let $\widetilde{V} : B \times B \to \mathbb{R}$ be the (complex) bilinear extension of V. Then the form $\widetilde{V}$ is orthogonal if, and only if, V writes in the form $V(x, y) = \varphi_1(xy)$ $(x, y \in A)$, where φ_1 is a functional in A^*.*

Proof Let τ be the period-2 *-automorphism on B satisfying that $B^\tau = B$ and let $\widetilde{\tau} : B^* \to B^*$ be the involution defined by $\widetilde{\tau}(\phi)(b) = \overline{\phi(\tau(b))}$.

Suppose $\widetilde{V}$ is orthogonal. By the main result in [3] (see also [1]), there exists $\phi \in B^*$ satisfying $\widetilde{V}(x, y) = \phi(xy)$, for every $x, y \in B$. Since $\widetilde{V}$ is an extension of V, we get $V(a, b) = \Re\mathrm{e}\phi(ab) = \phi(ab)$, for every $a, b \in A$. In particular, $\phi(a) \in \mathbb{R}$, for every $a \in A$ and hence $\widetilde{\tau}(\phi) = \phi$ lies in $(B^*)^{\widetilde{\tau}} \equiv A^*$.

Let us assume that V writes in the form $V(x, y) = \varphi_1(xy)$ $(x, y \in A)$, where φ_1 is a functional in A^*. The functional φ_1 can be regarded as an element in B^* satisfying $\widetilde{\tau}(\varphi_1) = \varphi_1$. It is easy to check that $\widetilde{V}(x, y) = \varphi_1(xy)$, for every $x, y \in B$. $\square$

3. Orthogonality preservers between commutative real C*-algebras

Throughout this section, $A_1 = C(K_1)^{\tau_1}$ and $A_2 = C(K_2)^{\tau_2}$ will denote two unital commutative real C*-algebras, K_1 and K_2 will be two compact Hausdorff spaces and τ_i will denote a conjugate-linear period-2 *-automorphism on $C(K_i)$ given by $\tau_i(f)(t) = \overline{f(\sigma_i(t))}$ $(t \in K_i, f \in C(K_i))$, where $\sigma_i : K_i \to K_i$ is a period-2 homeomorphism. We shall write $B_1 = C(K_1)$ and $B_2 = C(K_2)$ for the corresponding complexifications of A_1 and A_2, respectively.

By the Banach–Stone theorem, every surjective isometry $T : C(K_1) \to C(K_2)$ is a composition operator, that is, there exist a unitary element u in $C(K_2)$ and a homeomorphism $\sigma : K_2 \to K_1$ such that $T(f)(t) = (uC_\sigma)(f)(t) := u(t)\, f(\sigma(t))$ $(t \in K_2, f \in C(K_1))$. This result led to the study of the so-called Banach–Stone theorems in different classes of Banach spaces containing $C(K)$-spaces, in which their algebraic and geometric properties are mutually determined. That is the case of general C*-algebras (Kadison [32] and Paterson and Sinclair [33]), JB- and JB*-algebras (Wright and Youngson [34] and Isidro and Rodríguez [35]), JB*-triples (Kaup [36] and Dang et al. [37]), real C*-algebras (Grzesiak [38], Kulkarni and Arundhathi [39], Kulkarni and Limaye [31] and Chu et al. [40]) and real JB*-triples (Isidro et al. [41], Kaup [42] and Fernández-Polo et al. [43]). In what concerns us, we highlight that any surjective linear isometry $T : C(K_1)^{\tau_1} \to C(K_2)^{\tau_2}$ is a composition operator given by a homeomorphism $\phi : K_2 \to K_1$ which satisfies $\sigma_1 \circ \phi = \phi \circ \sigma_2$ (cf. [38] or [39] or [31, Corollary 5.2.4]).

The class of orthogonality preserving (continuous) operators between $C(K)$-spaces is strictly bigger than the class of surjective isometries. Actually, a bounded linear operator $T : C(K_1) \to C(K_2)$ is orthogonality preserving (equivalently, disjointness preserving) if, and only if, there exist u in $C(K_2)$ and a mapping $\varphi : K_2 \to K_1$ which is continuous on $\{t \in K_2 : u(t) \neq 0\}$ such that $T(f)(t) = (uC_\varphi)(f)(t) = u(t)\ f(\varphi(t))$ (compare [8, Example 2.2.1]).

Developing ideas given by Beckenstein et al. in [11] and [12] (see also [13]), Jarosz showed, in [9], that the above hypothesis of T being continuous can be, in some sense, relaxed. More concretely, for every orthogonality preserving linear mapping $T : C(K_1) \to C(K_2)$, there exists a disjoint decomposition $K_2 = S_1 \cup S_2 \cup S_3$ (with S_2 open, S_3 closed), and a continuous mapping φ from $S_1 \cup S_2$ into K_1 such that $T(f)(s) = \chi(s) f(\varphi(s))$ for all $s \in S_1$ (where χ is a continuous, bounded, non-vanishing, scalar-valued function on S_1), $T(f)(s) = 0$ for all $s \in S_3$, $\varphi(S_2)$ is finite and, for each $s \in S_2$, the mapping $f \mapsto T(f)(s)$ is not continuous. As a consequence, every orthogonality preserving linear bijection between $C(K)$-spaces is (automatically) continuous. More recently, Burgos and the authors of this note prove, in [7], that every bi-orthogonality preserving linear surjection between two von Neumann algebras (or between two compact C*-algebras) is automatically continuous (compare [14,15] for recent additional generalisations).

The main goal of this section is to describe the orthogonality preserving linear mappings between $C(K)^\tau$-spaces. Among the consequences, we establish that every orthogonality preserving linear bijection between unital commutative real C*-algebras is automatically continuous. We shall provide an example of an orthogonality preserving linear bijection between $C(K)^\tau$-spaces which is not bi-orthogonality preserving and give a characterisation of bi-orthogonality preserving linear maps.

We shall borrow and adapt some of the ideas developed in those previously mentioned papers (cf. [9,11,12]). In order to have a good balance between completeness and conciseness, we just give some sketch of the refinements needed in our setting. In any case, the results presented here are independent innovations and extensions of those proved by Beckenstein, Narici, and Todd and Jarosz for $C(K)$-spaces.

Let $T : C(K_1)^{\tau_1} \to C(K_2)^{\tau_2}$ be an orthogonality preserving linear mapping. Keeping in mind the notation in the previous section, we write $L_i := \mathcal{O}_i \cup F_i$, where $\mathcal{O}_i$ and F_i are the subsets of K_i given by Lemma 2.1. The map sending each f in $C(Ki)^{\tau_i}$ to its restriction to L_i is a C*-isomorphism (and hence a surjective linear isometry) from $C(Ki)^{\tau_i}$ onto the real C*-algebra $C_r(L_i)$ of all continuous functions $f : L_i \to \mathbb{C}$ taking real values on F_i. Thus, studying orthogonality preserving linear maps between $C(K)^\tau$ spaces is equivalent to study orthogonality preserving linear mappings between the corresponding $C_r(L)$-spaces.

Henceforth, we consider an orthogonality preserving (not necessarily continuous) linear map $T : C_r(L_1) \to C_r(L_2)$, where L_1 and L_2 are two compact Hausdorff spaces and each F_i is a closed subset of L_i. Let us consider the sets

$$Z_1 = \{s \in L_2 : \delta_s T \text{ is a non-zero bounded real-linear mapping}\},$$

$$Z_3 = \{s \in L_2 : \delta_s T = 0\}, \text{ and } Z_2 = L_2 \backslash (Z_1 \cup Z_3).$$

It is easy to see that Z_3 is closed. Following a very usual technique (see, for example, [9,11,12,44,45]), we can define a continuous support map $\varphi : Z_1 \cup Z_2 \to L_1$. More concretely, for each $s \in Z_1 \cup Z_2$, we write $\operatorname{supp}(\delta_s T)$ for the set of all $t \in L_1$ such that for each open set $U \subseteq L_1$ with $t \in U$ there exists $f \in C_r(L_1)$ with $\operatorname{coz}(f) \subseteq U$ and $\delta_s(T(f)) \neq 0$. Actually, following a standard argument, it can be shown that, for each $s \in Z_1 \cup Z_2$, $\operatorname{supp}(\delta_s T)$ is non-empty and reduces exactly to one point $\varphi(s) \in L_1$, and the assignment $s \mapsto \varphi(s)$ defines a continuous map from $Z_1 \cup Z_2$ to L_1. Furthermore, the value of $T(f)$ at every $s \in Z_1$ depends strictly on the value $f(\varphi(s))$. More precisely, for each $s \in Z_1$ with $\varphi(s) \notin F_1$, the value $T(g)(s)$ is the same for every function $g \in C_r(L_1)$ with $g \equiv i$ on a neighbourhood of $\varphi(s)$. Thus, defining $T(i)(s) := 0$ for every $s \in Z_3 \cup Z_2$ and for every $s \in Z_1$ with $\varphi(s) \in F_1$, and $T(i)(s) := T(g)(s)$ for every $s \in Z_1 \cup Z_2$ with $\varphi(s) \notin F_1$, where g is any element in $C_r(L_1)$ with $g \equiv i$ on a neighbourhood of $\varphi(s)$, we get a (well-defined) mapping $T(i) : L_2 \to \mathbb{C}$. It should be noticed that '$T(i)$' is just a symbol to denoted the above mapping and not an element in the image of T. In this setting, the identity

$$T(f)(s) = T(1)(s)\, \Re\mathrm{e} f(\varphi(s)) + T(i)(s)\, \Im\mathrm{m} f(\varphi(s)),$$

holds for every $s \in Z_1$. Clearly, $T(1)(s), T(i)(s) \in \mathbb{R}$, for every $s \in F_2$ and $|T(1)(s)| + |T(i)(s)| \neq 0$, for every $s \in Z_1$.

The following property also follows from the definition of φ by standard arguments: Under the above conditions, let s be an element in $Z_1 \cup Z_2$, then

$$\delta_s T(f) = 0 \text{ for every } f \in C_r(L_1) \text{ with } \varphi(s) \notin \overline{\operatorname{coz}(f)}. \tag{10}$$

Lemma 3.1 *The mapping $T(i)$ is bounded on the set $\varphi^{-1}(\mathcal{O}_1)$. Furthermore, the inequality*

$$|T(f)(s)| \leq \|T(1)\| + \sup_{\widetilde{s} \in \varphi^{-1}(\mathcal{O}_1)} |T(i)(\widetilde{s})|$$

holds for all $s \in Z_1$ and all $f \in C_r(L_1)$ with $|\Re e(f)|, |\Im m(f)| \leq 1$.

Proof Arguing by contradiction, we suppose that, for each natural n, there exists $s_n \in \varphi^{-1}(\mathcal{O}_1)$ such that $|T(i)(s_n)| > n^3$. The elements $s_n' s$ can be chosen so that $\varphi(s_n) \neq \varphi(s_m)$ for $n \neq m$, and consequently we can find a sequence of pairwise disjoint open subsets (U_n) of $\mathcal{O}_1$ with $\varphi(s_n) \in U_n$. It is easily seen that we can define a function $g = \sum_{n=1}^{\infty} i\, g_n \in C_r(L_1)$ with $\operatorname{coz}(g_n) \subset U_n$, $0 \leq g_n \leq \frac{1}{n^2}$, and $g_n \equiv \frac{1}{n^2}$ on a neighbourhood of s_n, for all n. By the form of g, and since T is orthogonality preserving, we have $|T(g)(s_n)| = n^2 |T(i)(s_n)| > n$ for all n, which is absurd. $\square$

We can easily show now that Z_2 is an open subset of L_2. With this aim, we consider an element s_0 in Z_2. We can pick a function $f \in C_r(L_1)$ such that $\|f\| \leq 1$ and

$$|T(f)(s_0)| > 1 + \|T(1)\| + \sup_{\widetilde{s} \in \varphi^{-1}(\mathcal{O}_1)} |T(i)(\widetilde{s})|.$$

Since $|T(f)(s)| \leq \|T(1)\| + \sup_{\widetilde{s} \in \varphi^{-1}(\mathcal{O}_1)} |T(i)(\widetilde{s})| < |T(f)(s_0)| - 1$, for every $s \in Z_1 \cup Z_3$, we conclude that there exists an open neighbourhood of s_0 contained in Z_2.

The next theorem resumes the above discussion.

THEOREM 3.2 *In the notation above, let $T : C_r(L_1) \to C_r(L_2)$ be an orthogonality preserving linear mapping. Then L_2 decomposes as the union of three mutually disjoint subsets Z_1, Z_2, and Z_3, where Z_2 is open and Z_3 is closed, there exist a continuous support map $\varphi : Z_1 \cup Z_2 \to L_1$, and a bounded mapping $T(i) : L_2 \to \mathbb{C}$ which is continuous on $\varphi^{-1}(\mathcal{O}_1)$ satisfying:*

$$T(i)(s) \in \mathbb{R}\ (\forall s \in F_2),\ T(i)(s) = 0,\ (\forall s \in Z_3 \cup Z_2 \text{ and } \forall s \in Z_1 \text{ with } \varphi(s) \in F_1),$$

$$|T(1)(s)| + |T(i)(s)| \neq 0,\ (\forall s \in Z_1), \tag{11}$$

$$T(f)(s) = T(1)(s)\, \Re e\, f(\varphi(s)) + T(i)(s)\, \Im m\, f(\varphi(s)),\ (\forall s \in Z_1,\ f \in C_r(L_1)), \tag{12}$$
$$T(f)(s) = 0,\ (\forall s \in Z_3,\ f \in C_r(L_1)),$$

and for each $s \in L_2$, the mapping $C_r(L_1) \to \mathbb{C}$, $f \mapsto T(f(s))$, is unbounded if, and only if, $s \in Z_2$. Furthermore, the set $\varphi(Z_2)$ is finite.

Proof Everything has been substantiated except perhaps the statement concerning the set $\varphi(Z_2)$. Arguing by contradiction, we assume the existence of a sequence (s_n) in Z_2 such that $\varphi(s_n) \neq \varphi(s_m)$ for every $n \neq m$. Find a sequence (U_n) of mutually disjoint open subsets of L_1 satisfying $\varphi(s_n) \in U_n$ and a sequence $(f_n) \subseteq C_r(L_1)$ such that $\|f_n\| \leq \frac{1}{n}$, $\operatorname{coz}(f_n) \subseteq U_n$ and $|\delta_{s_n} T(f_n)| > n$, for every $n \in \mathbb{N}$. The element $f = \sum_{n=1}^{\infty} f_n$ lies in $C_r(L_1)$, and for each natural n_0, $f_{n_0} \perp \sum_{n=1, n\neq n_0}^{\infty} f_n$. Thus, $|\delta_{s_{n_0}} T(f)| \geq |\delta_{s_{n_0}} T(f_{n_0})| > n_0$, which is impossible. □

Remark 3.3 The mapping $T(i) : L_2 \to \mathbb{C}$ has been defined to satisfy $T(i)(s) = 0$, for all $s \in Z_3 \cup Z_2$ and for all $s \in Z_1$ with $\varphi(s) \in F_1$. It should be noticed here that the value $T(i)(s)$ is uniquely determined only when $s \in Z_1$ and $\varphi(s) \notin F_1$. There are some other choices for the values of $T(i)(s)$ at $s \in Z_3 \cup Z_2$ and at $s \in Z_1$ with $\varphi(s) \in F_1$ under which conditions (11) and (12) are satisfied.

Remark 3.4 We shall now explore some of the consequences derived from Theorem 3.2. Let $T : C_r(L_1) \to C_r(L_2)$ be an orthogonality preserving linear mapping.

(*a*) The set Z_3 is empty whenever T is surjective.

(*b*) $Z_3 = \emptyset$ implies that $Z_1 = L_2 \backslash Z_2$ is a compact subset of L_2.

(*c*) $\varphi(Z_2)$ is a finite set of non-isolated points in L_1. Indeed, if $\varphi(s_0) = t_0$ is isolated for some $s_0 \in Z_2$, then we can find an open set $U \subseteq L_1$ such that $U \cap K_1 = \{t_0\}$. Therefore, for each $f \in C_r(L_1)$ with $f(t_0) = 0$ we have $\delta_{s_0} T(f) = 0$. Pick an arbitrary $h \in C_r(L_1)$. Clearly, $\chi_{t_0} \in C_r(L_1)$, while $i\chi_{t_0}$ lies in $C_r(L_1)$ if, and only if, $t_0 \notin F_1$. Therefore,

$$h_0 = \Re\mathrm{e}(h(t_0))\chi_{t_0} + \Im\mathrm{m}(h(t_0))\, i\chi_{t_0}$$

lies in $C_r(L_1)$ and $(h - h_0)(t_0) = 0$.

Assume first that $t_0 \notin F_1$. Denoting $\lambda_0 = \delta_{s_0} T(\chi_{t_0})$ and $\mu_0 = \delta_{s_0} T(i\chi_{t_0})$, we have

$$\begin{aligned}\delta_{s_0} T(h) &= \delta_{s_0} T(h_0) = \lambda_0 \Re\mathrm{e}(h(t_0)) + \mu_0 \Im\mathrm{m}(h(t_0)) \\ &= \frac{\lambda_0 - i\mu_0}{2}\delta_{t_0}(h) + \frac{\lambda_0 + i\mu_0}{2}\overline{\delta_{t_0}}(h).\end{aligned}$$

This shows that $\delta_{s_0} T = \frac{\lambda_0 - i\mu_0}{2}\delta_{t_0} + \frac{\lambda_0 + i\mu_0}{2}\overline{\delta_{t_0}}$ is a continuous mapping from $C_r(L_1)$ to $\mathbb{C}$, which is impossible.

When $t_0 \in F_1$ we have $\delta_{s_0} T = \lambda_0 \delta_{t_0}$ is a continuous mapping from $C_r(L_1)$ to $\mathbb{R}$, which is also impossible.

(d) T surjective implies $\varphi(Z_1 \cap \mathcal{O}_2) \subseteq \mathcal{O}_1$. Suppose, on the contrary that there exists $s_0 \in Z_1 \cap \mathcal{O}_2$ with $\varphi(s_0) \in F_1$. By (12),

$$T(f)(s_0) = T(1)(s_0)\Re e f(\varphi(s_0)),$$

for every $f \in C_r(L_1)$. It follows from the surjectivity of T, together with the condition $s_0 \in \mathcal{O}_2$, that for every complex number ω there exists a real λ satisfying $\omega = T(1)(s_0)\lambda$, which is impossible.

(e) Suppose T is surjective and fix $s_0 \in Z_1 \cap \mathcal{O}_2$. The mapping $\delta_{s_0} T$ is a bounded real-linear mapping from $C_r(L_1)$ onto $\mathbb{C}$. On the other hand, by (12),

$$\delta_{s_0} T(f) = T(1)(s_0)\Re e f(\varphi(s_0)) + T(i)(s_0)\Im m f(\varphi(s_0)), \ (\forall f \in C_r(L_1)).$$

Thus, T being surjective implies that the space $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$ is linearly spanned by the elements $T(1)(s_0)$ and $T(i)(s_0)$. Therefore, for each $s_0 \in Z_1 \cap \mathcal{O}_2$, the set $\{T(1)(s_0), T(i)(s_0)\}$ is a basis of $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$. Consequently, when T is surjective and $s_0 \in Z_1 \cap \mathcal{O}_2$, the condition $T(f)(s_0) = 0$ implies $f(\varphi(s_0)) = 0$. For any other $s_1 \in Z_1 \cap \mathcal{O}_2$ with $\varphi(s_0) = \varphi(s_1)$, we have:

$$T(f)(s_0) = 0 \Rightarrow f(\varphi(s_0)) = 0 \Rightarrow T(f)(s_1) = 0.$$

The fact that $C_r(L_2)$ separates points implies that $s_1 = s_0$. Thus, φ is injective on $Z_1 \cap \mathcal{O}_2$.

We can now state the main result of this section which affirms that every orthogonality preserving linear bijection between unital commutative real C*-algebras is (automatically) continuous.

THEOREM 3.5 *Every orthogonality preserving linear bijection between unital commutative (real) C*-algebras is (automatically) continuous.*

Proof Since T is surjective, $Z_3 = \emptyset$, and hence $Z_1 = L_2 \backslash Z_2$ is a compact subset of L_2. It is also clear that $\varphi(L_2)$ is compact. We claim that $\varphi(L_2) = L_1$. Otherwise, there would exist a non-zero function $f \in C_r(L_1)$ with $\overline{\mathrm{coz}(f)} \subseteq L_1 \backslash \varphi(L_2)$. Thus, by (10), $T(f) = 0$, contradicting the injectivity of T. By Remark 3.4 (c), $\varphi(Z_1) = \overline{\varphi(Z_1)} = \varphi(L_2) = \varphi(Z_1) \cup \varphi(Z_2) = L_1$.

We next see that $Z_2 = \emptyset$. Otherwise we can take $g \in C_r(L_2)$ with $\emptyset \neq coz(g) \subset Z_2$. Let $h = T^{-1}(g)$. Obviously $Th(s) = 0$ whenever $s \in Z_1$. We claim that $h(t) = 0$, for every $t \in \varphi(Z_1) \setminus \varphi(Z_2)$. Let us fix $t \in \varphi(Z_1) \setminus \varphi(Z_2)$. Since $\varphi(Z_2)$ is a finite set there are disjoint open sets U_1, U_2 such that $t \in U_1, \varphi(Z_2) \subset U_2$. Let $f \in C(L_1, \mathbb{R})$ be such that $f(t) \neq 0$ and $\overline{coz(f)} \subset U_1$. We see that $T(fh) = 0$. Indeed, let $s \in L_2 = Z_1 \cup Z_2$. If s lies in Z_1, then the maps fh and $f(\varphi(s))h$ lie in $C_r(L_1)$ and coincide at $\varphi(s)$. Since T is linear over $\mathbb{R}$ and f takes real values, we deduce, by (12), that $T(fh)(s) = f(\varphi(s))Th(s) = 0$. If $s \in Z_2$ then, since $\varphi(s) \notin \overline{coz(fh)}$, then $\delta_s T(fh) = T(fh)(s) = 0$.

We have shown that $T(fh) = 0$. Thus, since T is injective, $fh = 0$ and therefore $h(t) = 0$. We have therefore proved that $coz(h) \subset \varphi(Z_2)$ which is a finite set. This means that h must be a finite linear combination of characteristic function on points of $\varphi(Z_2)$ and these points must be isolated which is impossible, since by (c) in Remark 3.4 no point in $\varphi(Z_2)$ can be isolated. We have proved that $Z_2 = \emptyset$. Now the fact that T is continuous follows easily. □

The above theorem is the first step toward extending, to the real setting, those results proved in [7,9,14,46–48] for (complex) C*-algebras.

Orthogonality preserving linear bijections enjoy an interesting additional property.

PROPOSITION 3.6 *In the notation of this section, let $T : C_r(L_1) \to C_r(L_2)$ be an orthogonality preserving linear bijection. Then T^{-1} preserves invertible elements, that is, $T^{-1}(g)$ is invertible whenever g is an invertible element in $C_r(L_2)$.*

Proof Take an invertible element $g \in C_r(L_2)$. Let f be the unique element in $C_r(L_1)$ satisfying $T(f) = g$. Theorem 3.2 implies that

$$0 \neq g(s) = T(f)(s) = T(1)(s)\, \Re\mathrm{e} f(\varphi(s)) + T(i)(s)\, \Im\mathrm{m} f(\varphi(s)),$$

for every $s \in Z_1$. This assures that $f(\varphi(s)) \neq 0$, for every $s \in Z_1$, and since $\varphi(Z_1) = L_1$, $f = T^{-1}(g)$ must be invertible in $C_r(L_1)$. □

In the setting of complex Banach algebras, it follows from the Gleason–Kahane–Żelazko theorem that a linear transformation ϕ from a unital, commutative, complex Banach algebra A into $\mathbb{C}$ satisfying $\phi(1) = 1$ and $\phi(a) \neq 0$ for every invertible element a in A is multiplicative, that is, $\phi(ab) = \phi(a)\phi(b)$ (see [49,50]). Although, the Gleason–Kahane–Żelazko theorem fails for real Banach algebras, Kulkarni found in [51] the following reformulation: a linear map ϕ from a real unital Banach algebra A into the complex numbers is multiplicative if $\varphi(1) = 1$ and $\phi(a)^2 + \phi(b)^2 \neq 0$ for every $a, b \in A$ with $ab = ba$ and $a^2 + b^2$ invertible. It is not clear that statement (b) in the above proposition can be improved to get the hypothesis of Kulkarni's theorem. The structure of orthogonality preserving linear mappings between $C_r(L)$-spaces described in Theorem 3.2 invites us to affirm that they are not necessarily multiplicative.

3.1. *Bi-orthogonality preservers*

As a consequence of the description of orthogonality preserving linear maps given in [9], it can be shown that an orthogonality preserving linear bijection between (complex) $C(K)$-spaces is bi-orthogonality preserving. It is natural to ask wether every orthogonality preserving linear bijection between commutative (unital) real C*-algebras is bi-orthogonality preserving.

This is known to be true in two cases: first, between spaces $C_{\mathbb{R}}(K)$ of real (and also complex) valued functions on a compact Hausdorff space K, as it is well known; second, between spaces of the type $C_{\mathbb{R}}(K; \mathbb{R}^n)$ (compare [44, Section 3]). Spaces like those we are dealing with in this paper need not satisfy this property, that is, there exists an orthogonality preserving linear bijection $T : C_r(L_1) \to C_r(L_2)$ which is not bi-orthogonality preserving (and even L_1 and L_2 are not homeomorphic either).

Example 3.7 Let $L_1 = \{t_1, t_2, t_3\}$ $L_2 = \{s_1, s_2, s_3, s_4\}$ with $\mathcal{O}_1 = \{t_1, t_3\}$, $\mathcal{O}_2 = \{s_1\}$, $F_1 = \{t_2\}$ and $F_2 = \{s_2, s_3, s_4\}$. Define $\varphi : L_2 \to L_1$ by $\varphi(s_i) = t_i$, for $i = 1, 2$, and

$\varphi(s_i) = t_3$, for $i = 3, 4$. It is easy to check that $T(f)(s_i) = f(\varphi(s_i))$ if $i = 1, 2$, and $T(f)(s_3) = \Re e f(t_3)$, $T(f)(s_4) = \Im m f(t_3)$ is an orthogonality preserving linear bijection, but T^{-1} is not orthogonality preserving.

In the above example, $\varphi^{-1}(\mathcal{O}_1) \cap F_2$ is non-empty. Our next result shows that a topological condition on F_2 assures that an orthogonality preserving linear bijection between unital commutative real C*-algebras is bi-orthogonality preserving.

Proposition 3.8 *In the notation of this section, let $T : C_r(L_1) \to C_r(L_2)$ be an orthogonality preserving linear bijection (not assumed to be bounded). The following statements hold:*

(a) *If T is bi-orthogonality preserving then $\varphi : L_2 \to L_1$ is a (surjective) homeomorphism, $\varphi(F_2) = F_1$, and $\varphi(\mathcal{O}_2) = \mathcal{O}_1$. In particular, $\varphi^{-1}(\mathcal{O}_1) \cap F_2 = \emptyset$.*
(b) *If F_2 has empty interior then T is biorthogonality preserving.*

Proof (a) If T is bi-orthogonality preserving, it can be easily seen that $\varphi : L_2 \to L_1$ is a homeomorphism, and for each $s \in L_2$, $\operatorname{supp}(\delta_{\varphi(s)} T^{-1}) = \{s\}$. By Remark 3.4($d$), applied to T and T^{-1}, we have $\varphi(F_2) = F_1$ and $\varphi(\mathcal{O}_2) = \mathcal{O}_1$. Then $\varphi^{-1}(\mathcal{O}_1) \cap F_2 = \emptyset$. So, a) is clear.

(b) Let us assume that F_2 has empty interior. Arguing by contradiction we suppose that T^{-1} is not orthogonality preserving. Then there exist $f_1, f_2 \in C_r(L_1)$ with $f_1 f_2 \neq 0$, but $T(f_1) \perp T(f_2)$. Thus $U := \operatorname{coz}(f_1) \cap \operatorname{coz}(f_2)$ is a non-empty open subset of L_1. Keeping again the notation of Theorem 3.2 for T, we recall that, by Theorem 3.5 and Remark 3.4, $Z_3 = \emptyset$, $Z_2 = \emptyset$, $\varphi(L_2) = L_1$, $\varphi(\mathcal{O}_2) \subset \mathcal{O}_1$, $\varphi|_{\mathcal{O}_2}$ is injective, and for each $s \in \mathcal{O}_2$, and $\{T(1)(s), T(i)(s)\}$ is a basis of $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$.

By the form of T, there are no points of $\varphi(\mathcal{O}_2)$ in $U = \operatorname{coz}(f_1) \cap \operatorname{coz}(f_2)$ (because for each $s \in \mathcal{O}_2$, $T(f)(s) \neq 0$ when $f(\varphi(s)) \neq 0$). Now, let k be a non-zero element in $C(L_1, \mathbb{R})$, with $\operatorname{coz}(k) \subseteq \operatorname{coz}(f_1) \cap \operatorname{coz}(f_2)$. By Theorem 3.2 (12), it is clear that $\varphi(\operatorname{coz}(T(k))) \subseteq \operatorname{coz}(k)$, and hence, since $\varphi(\mathcal{O}_2) \subseteq \mathcal{O}_1$, $\operatorname{coz}(T(k))$ is a non-empty subset of F_2, against our hypotheses. □

As we have already seen, an orthogonality preserving linear bijection between $C_r(L)$-spaces needs not to be biorthogonality preserving. Example 3.7 also shows that, unlike in the complex case, the existence of an orthogonality preserving linear bijection between $C_r(L)$-spaces does not guarantee that the corresponding compacts spaces are homeomorphic. We next provide a characterization of those (linear) mappings which are bi-orthogonality preserving. As a consequence, we shall see that if there exists a bi-orthogonality preserving linear map $T : C_r(L_1) \to C_r(L_2)$ then L_1 and L_2 are homeomorphic.

Theorem 3.9 *Let $T : C_r(L_1) \to C_r(L_2)$ be a mapping. The following statements are equivalent:*

(a) *T is a bi-orthogonality preserving linear surjection;*
(b) *There exists a (surjective) homeomorphism $\varphi : L_2 \to L_1$ with $\varphi(\mathcal{O}_2) = \mathcal{O}_1$, a function $a_1 = \gamma_1 + i\gamma_2$ in $C_r(L_2)$ with $a_1(s) \neq 0$ for all $s \in L_2$, and a function $a_2 = \eta_1 + i\eta_2 : L_2 \to \mathbb{C}$ continuous on $\mathcal{O}_2$ with the property that*

$$0 < \inf_{s \in \mathcal{O}_2} \left| \det \begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix} \right| \leq \sup_{s \in \mathcal{O}_2} \left| \det \begin{pmatrix} \gamma_1(s) & \eta_1(s) \\ \gamma_2(s) & \eta_2(s) \end{pmatrix} \right| < +\infty,$$

such that

$$T(f)(s) = a_1(s)\,\Re e f(\varphi(s)) + a_2(s)\,\Im m f(\varphi(s))$$

for all $s \in L_2$ *and* $f \in C_r(L_1)$.

Proof $(a) \Rightarrow (b)$ Since every bi-orthogonality preserving linear mapping is injective, we can assume that $T : C_r(L_1) \to C_r(L_2)$ is a bi-orthogonality preserving linear bijection. We keep the notation given in Theorem 3.2. We have already shown that $Z_3 = \emptyset$, $Z_2 = \emptyset$, $\varphi : L_2 \to L_1$ is a surjective homeomorphism, $\varphi(\mathcal{O}_2) = \mathcal{O}_1$, and for each $s \in \mathcal{O}_2$, $\{T(1)(s), T(i)(s)\}$ is a basis of $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$ (compare Theorem 3.5, Remark 3.4 and Proposition 3.8). Taking $a_1 = T(1) = \gamma_1 + i\gamma_2$ and $a_2 = T(i) = \eta_1 + i\eta_2$, we only have to show that

$$0 < \inf_{s\in\mathcal{O}_2}\left|\det\begin{pmatrix}\gamma_1(s) & \eta_1(s)\\ \gamma_2(s) & \eta_2(s)\end{pmatrix}\right| \leq \sup_{s\in\mathcal{O}_2}\left|\det\begin{pmatrix}\gamma_1(s) & \eta_1(s)\\ \gamma_2(s) & \eta_2(s)\end{pmatrix}\right| < +\infty.$$

Let us denote $M_s = \begin{pmatrix}\gamma_1(s) & \eta_1(s)\\ i\gamma_2(s) & i\eta_2(s)\end{pmatrix}$. Clearly $\det(M_s) \neq 0$, for every $s \in \mathcal{O}_2$ and $T(f)(s) = M_s \cdot \begin{pmatrix}\Re\mathrm{e} f(\varphi(s))\\ \Im\mathrm{m} f(\varphi(s))\end{pmatrix}$, for every $f \in C_r(L_1)$, $s \in L_2$. By the boundedness of $T(1) : L_2 \to \mathbb{C}$ and $T(i)|_{\mathcal{O}_2} : \mathcal{O}_2 \to \mathbb{C}$ (see Lemma 3.1) there exists $M > 0$ such that $|det(M_s)| \leq M$ for all $s \in \mathcal{O}_2$.

Applying the above arguments to the mapping T^{-1} we find a surjective homeomorphism $\psi = \varphi^{-1} : L_1 \to L_2$, a mapping $T^{-1}(i) : L_1 \to L_2$ and $m > 0$, such that $\psi(\mathcal{O}_1) = \mathcal{O}_2$, for each $t \in \mathcal{O}_1$, $\{T^{-1}(1)(t), T^{-1}(i)(t)\}$ is a basis of $\mathbb{C}_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$, $T^{-1}(g)(t) = N_t \cdot \begin{pmatrix}\Re\mathrm{e} g(\psi(t))\\ \Im\mathrm{m} g(\psi(t))\end{pmatrix}$ $(g \in C_r(L_2), t \in L_1)$, $|\det(N_t)| \leq m$, for all $t \in \mathcal{O}_1$, where $N_t = \begin{pmatrix}\Re\mathrm{e}T^{-1}(1)(t) & \Re\mathrm{e}T^{-1}(i)(t)\\ i\Im\mathrm{m}T^{-1}(1)(t) & i\Im\mathrm{m}T^{-1}(i)(t)\end{pmatrix}$. It can be easily seen that, for each $s \in \mathcal{O}_2$, $N_{\varphi(s)} = M_s^{-1}$, which shows that $|det(M_s)| \geq \frac{1}{m}$, for all $s \in \mathcal{O}_2$.

$(b) \Rightarrow (a)$ Let $T : C_r(L_1) :\to C_r(L_2)$ be a mapping satisfying the hypothesis in (b). Clearly, T is linear, and since $\varphi(F_2) = F_1$, $Tf(s) \in \mathbb{R}$ for all $s \in F_2$ and $f \in C_r(L_1)$ (that is, $T(f) \in C_r(L_2)$). We can easily check that, under these hypothesis, T is injective and preserves orthogonality.

We shall now prove that T is surjective. Indeed, for each $s \in \mathcal{O}_2$

$$T(f)(s) = \begin{pmatrix}\Re\mathrm{e} g(s)\\ \Im\mathrm{m} g(s)\end{pmatrix} = \begin{pmatrix}\gamma_1(s) & \eta_1(s)\\ i\gamma_2(s) & i\eta_2(s)\end{pmatrix}\cdot\begin{pmatrix}\Re\mathrm{e} f(\varphi(s))\\ \Im\mathrm{m} f(\varphi(s))\end{pmatrix}$$

$$= M_s \cdot \begin{pmatrix}\Re\mathrm{e} f(\varphi(s))\\ \Im\mathrm{m} f(\varphi(s))\end{pmatrix},$$

thus,

$$\begin{pmatrix}\Re\mathrm{e} f(\varphi(s))\\ \Im\mathrm{m} f(\varphi(s))\end{pmatrix} = M_s^{-1}\cdot\begin{pmatrix}\Re\mathrm{e} g(s)\\ \Im\mathrm{m} g(s)\end{pmatrix}.$$

We define $b_1(t) : L_1 \to \mathbb{C}$ and $b_2 : \mathcal{O}_1 \to \mathbb{C}$ by $b_1(t) = \widetilde{\gamma}_1(t) + i\widetilde{\gamma}_2(t)$ and $b_2 = \widetilde{\eta}_1(t) + i\widetilde{\eta}_2(t)$ $(t \in \mathcal{O}_1)$, where $M^{-1}_{\varphi^{-1}(t)} = \begin{pmatrix}\widetilde{\gamma}_1(t) & \widetilde{\eta}_1(t)\\ i\widetilde{\gamma}_2(t) & i\widetilde{\eta}_2(t)\end{pmatrix}$, and $b_1(t) = \frac{1}{\gamma_1(\varphi^{-1}(t))}$, for every $t \in F_1$. Then $S : C_r(L_2) \to C_r(L_1)$, defined by $S(g)(t) = b_1(t)\,\Re\mathrm{e} g(\varphi^{-1}(t)) +$

$b_2(t)\,\Im\mathrm{mg}(\varphi^{-1}(t))$, is linear, preserves orthogonality and it is easy to check that $S = T^{-1}$. It follows that T is bi-orthogonality preserving. $\square$

Let T be a bi-orthogonality preserving linear mapping with associated homeomorphism $\varphi : L_2 \to L_1$. Clearly, the operator $S : C_r(L_1) \to C_r(L_2)$, $S(f)(s) := f(\varphi(s))$ is a *-isomorphism. Having in mind that a linear mapping $T : A \to B$ between real C*-algebras is a *-isomorphism if, and only if, the complex linear extension $\widetilde{T} : A \oplus iA \to B \oplus iB$, $\widetilde{T}(a + ib) = T(a) + iT(b)$ is a *-isomorphism, we get the following corollary.

Corollary 3.10 *The following statements are equivalent:*

(a) *There exists a bi-orthogonality preserving linear bijection $T : C_r(L_1) \to C_r(L_2)$;*
(b) *There exists a C*-isomorphism $S : C_r(L_1) \to C_r(L_2)$;*
(c) *There exists a C*-isomorphism $\widetilde{S} : C(L_1) \to C(L_2)$;*
(d) *L_1 and L_2 are homeomorphic.*

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LOCAL TRIPLE DERIVATIONS ON C*-ALGEBRAS

MARIA BURGOS, FRANCISCO J. FERNÁNDEZ-POLO, JORGE J. GARCÉS, AND ANTONIO M. PERALTA

ABSTRACT. We prove that every bounded local triple derivation on a unital C*-algebra is a triple derivation. A similar statement is established in the category of unital JB*-algebras.

1. Introduction

In a pioneering work, R. Kadison started, in 1990, the study of local derivations from an associative algebra $\mathcal{R}$ into an $\mathcal{R}$-bimodule $\mathcal{M}$ (cf. [16]). We recall that a linear mapping $D : \mathcal{R} \to \mathcal{M}$ is a *derivation* or an *associative derivation* whenever $D(ab) = D(a)b + aD(b)$, for every $a, b \in \mathcal{R}$. In words of Kadison "The defining property of a linear mapping $T : \mathcal{R} \to \mathcal{M}$ to be a *local (associative) derivation* is that for each a in $\mathcal{R}$, there is a derivation $D_a : \mathcal{R} \to \mathcal{M}$ such that $T(a) = D_a(a)$". R. Kadison proved that each norm-continuous local derivation of a von Neumann algebra W into a dual W-bimodule is a derivation (cf. [16, Theorem A]). B.E. Johnson extended the above result proving that every (continuous) local derivations from any C*-algebra B into any Banach B-bimodule is a derivation (see [15, Theorem 5.3]). Concerning the hypothesis of continuity in the above result, let us briefly notice that J.R. Ringrose proved that every (associative) derivation from a C*-algebra B to a Banach B-bimodule X is continuous (cf. [24]). In [15], B.E. Johnson also gave an automatic continuity result, showing that local derivations on C*-algebras are continuous even if not assumed a priori to be so.

The above results motivated a multitude of studies on local derivations on C*-algebras. There exists a rich list of references revisiting, rediscovering and extending Kadison-Johnson theorem in many directions (see, for example, [4, 8, 9, 10, 18, 19, 20, 25] and [26]).

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C*-algebras belong to a more general class of Banach spaces, called JB*-triples, which is defined in terms of algebraic, topological and geometric axioms mutually interplaying (see section 2 for more details). Originally introduced by W. Kaup in the classification of *bounded symmetric domains* on arbitrary complex Banach spaces (cf. [17]), JB*-triples now have their own importance in Functional Analysis and Geometry of Banach spaces. A *triple* or *ternary* derivation on a JB*-triple E is a linear mapping $\delta : E \to E$ satisfying:

$$\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}, \tag{1}$$

for every $a,b,c \in E$. In the setting of JB*-triples, J.T. Barton and Y. Friedman proved that every triple derivation on a JB*-triple is continuous (cf. [1], see also [11]). A *local triple derivation* on E is a linear map $T : E \to E$ such that for each a in E there exists a triple derivation δ_a on E satisfying $T(a) = \delta_a(a)$.

Quite recently, Jordan Banach triple modules over a JB*-triple and triple derivations from a JB*-triple E to a Jordan Banach triple E-module X were introduced by B. Russo and the fourth author of this note. In [23] these authors provide necessary and sufficient conditions under which a derivation from E into X is continuous. We refer to [23] and [12] for the basic definitions and results on JB*-triples, Jordan Banach triple modules and triple derivations not included in this note. Following [23], a conjugate linear mapping $\delta : E \to X$ is a *triple or ternary derivation* whenever it satisfies the above identity (1). In particular, the dual, E^*, of a JB*-triple E, is a Jordan Banach triple E-module and every triple derivation from E into E^* is continuous (see [23, Corollary 15]). Furthermore, every triple derivation from a C*-algebra B to a Banach triple B-module is automatically continuous [23, Theorem 20]. A bounded conjugate linear operator $T : E \to X$ is said to be a *local triple derivation* if for each $a \in E$, there exists a triple derivation $\delta_a : E \to X$ satisfying $T(a) = \delta_a(a)$. Clearly, every triple derivation is a local triple derivation, while the reciprocal implication is an open problem.

Problem 1. *Is every local triple derivation on a JB*-triple E (or more generally, every local triple derivation from E into a Jordan Banach triple E-module) a triple derivation?*

In a Conference held in Hong-Kong in April 2012, M. Mackey announced a result establishing that, for each von Neumann algebra (and more generally, for every JBW*-triple, i.e. a JB*-triple which is also a dual Banach space), W, every local triple derivation $T : W \to W$ is a triple derivation (see [21, Theorem 5.11]). Actually, the arguments provided by Mackey are also valid to prove that every local triple derivation on a compact JB*-triple is a triple derivation. The proofs and techniques applied by M. Mackey in this result depend heavily on the particular structure of a JBW*-triple and the abundance of tripotent elements in this setting. Mackey's theorem is an appropriate version of the aforementioned Kadison's theorem. The corresponding JB*-triple version of Johnson's theorem is an open problem. Part of the above Problem 1 appears in [21, Conjecture 6.2 (C1) and (C3)].

Every C*-algebra B is a JB*-triple with product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. Triple and local triple derivations on B make sense in this setting without any need to appeal to the above general concepts on JB*-triple setting. The following C*-version of the above Problem 1 is interesting by itself.

Problem 2. *Is every local triple derivation on a C*-algebra B a triple derivation?*

In this paper we focus on Problem 2. Our main result shows that every local triple derivation on a unital C*-algebra is a triple derivation (Theorem 10). Section 3 contains a similar statement for local triple derivations on a unital JB*-algebra. The results presented here connect local triple derivations on a unital C*-algebra with generalised derivations, a concept studied by J. Li and Zh. Pan in [19]. We recall that a linear mapping D from a unital C*-algebra A to a (unital) Banach A-bimodule X is called a *generalised derivation* whenever the identity

$$D(ab) = D(a)b + aD(b) - aD(1)b$$

holds for every a, b in A. We shall say that D is a *generalised Jordan derivation* whenever $D(a \circ b) = D(a) \circ b + a \circ D(b) - U_{a,b}D(1)$, for every a, b in A, where the Jordan product is given by $a \circ b := \frac{1}{2}(ab + ba)$ and $U_{a,b}(x) := \frac{1}{2}(axb + bxa)$. Every generalised (Jordan) derivation $D : A \to X$ with $D(1) = 0$ is a (Jordan) derivation. Let A be a C*-subalgebra of a C*-algebra B. Suppose B is unital and A contains the unit, 1, of B. The C*-algebra B can be regarded as A-bimodule with respect to its original product and as a (complex) Jordan Banach triple A-module with respect to $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. In a first approach we prove that every (linear) local triple derivation from A to B is a generalised derivation. The main result establishes that every local triple derivation on a unital C*-algebra is an associative derivation plus an inner triple derivation (see Theorem 10 and Corollary 7).

Although the proofs and results contained in this paper are developed only with techniques of C*-algebra theory, at some stage we have opted for a more general result in the setting of JB*-triples and to pose Problem 1 in the more general context.

2. Local triple derivations on unital C*-algebras

We recall that a *JB*-triple* is a complex Banach space E equipped with a continuous triple product $\{.,.,.\} : E \times E \times E \to E$, which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies:

(a) The mapping $\delta(a,b) := L(a,b) - L(b,a)$ is a triple derivation on E, where $L(a,b)$ is the operator on E given by $L(a,b)x = \{a,b,x\}$;

(b) $L(a,a)$ is an hermitian operator with non-negative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

Every C*-algebra is a JB*-triple with respect $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$.

A triple or ternary derivation δ on E is said to be *inner* if it can be written as a finite sum of derivations of the form $\delta(a,b)$ $(a,b \in E)$.

Throughout this section, A will denote a C*-subalgebra of a unital C*-algebra, B, containing the unit of B. The self-adjoint part of a C*-algebra $\mathcal{B}$ will be denoted by $\mathcal{B}_{sa}$. The C*-algebra B can be regarded as A-bimodule with respect to its original product and as a (complex) Jordan Banach triple A-module with respect to $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. By an abuse of notation, a linear map $\delta : A \to B$ is called a triple derivation whenever it satisfies identity (1) (beware that this is not exactly the definition introduced in [12]). A bounded linear operator $T : A \to B$ is a local triple derivation if for each a in A there exists a (linear) triple derivation $\delta_a : A \to B$ satisfying $\delta_a(a) = T(a)$.

Lemma 3. [11, Lemma 1] *Let $T : A \to B$ be a local triple derivation. Then $T(1)^* = -T(1)$.*

Proof. Take a triple derivation $\delta_1 : A \to B$ satisfying $T(1) = \delta_1(1)$. In this case,

$$\begin{aligned} T(1) = \delta_1\{1,1,1\} &= 2\{\delta_1(1),1,1\} + \{1,\delta_1(1),1\} \\ &= 2\delta_1(1) + \delta_1(1)^* = 2T(1) + T(1)^*, \end{aligned}$$

which implies that $T(1)^* = -T(1)$. □

The above lemma was also established in [11, Proof of Lemma 1] and rediscovered in [21, Lemma 3.1], the proof is included here for completeness reasons.

We shall deduce now some consequences of the above Lemma 3. In the setting above, the mapping $\delta(T(1),1) = L(T(1),1) - L(1,T(1)) : A \to B$ is a triple or ternary derivation and $\delta(T(1),1)(1) = \{T(1),1,1\} - \{1,T(1),1\} = T(1) - T(1)^* = 2T(1)$. Thus,

$$\widetilde{T} = T - \frac{1}{2}\delta(T(1),1) = T - \delta\left(\frac{1}{2}T(1),1\right) \tag{2}$$

is a local triple derivation and $\widetilde{T}(1) = T(1) - T(1) = 0$.

We can exhibit now some examples of generalised derivations which are not local triple derivation. A basic example is described as follows: let a be an element in a C*-algebra B, the mapping $\mathrm{adj}_a : B \to B$, $x \mapsto \mathrm{adj}_a(x) := ax - xa$, is an example of an associative derivation on B. It is easy to see that the operator $G_a : B \to B$, $x \mapsto G_a(x) := ax + xa$, is an example of a generalised derivation on B. Since, in the case of B being unital, $G_a(1) = 2a$, it follows that G_a is not a local ternary derivation whenever $a^* \neq -a$.

The next lemma is established in the general setting of JB*-triples although we shall only require the corresponding version for C*-algebras.

Previously, we recall that elements a, b in a JB*-triple, E, are said to be *orthogonal* ($a \perp b$ for short) if $L(a, b) = 0$. By Lemma 1 in [2] we know that

$$a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0.$$

When a C*-algebra $\mathcal{B}$ is regarded as a JB*-triple, it is known that elements a, b in $\mathcal{B}$ are orthogonal if, and only if, $ab^* = 0 = b^*a = 0$ (cf. [3, §4]). When $\mathcal{B}$ is commutative, $a \perp b$ if, and only if, $ab = 0$.

Lemma 4. *Let E be a JB*-subtriple of a JB*-triple F, where the latter is seen as a Jordan Banach triple E-module with respect to its natural triple product. Let $T : E \to F$ be a local triple derivation. Then the products of the form $\{a, T(b), c\}$ vanish for every a, b, c in E with $a \perp b \perp c$.*

Proof. Let us take a, b, c in E satisfying $a \perp b \perp c$, and a triple derivation $\delta_b : E \to F$ such that $\delta_b(b) = T(b)$. The identity

$$\{a, T(b), c\} = \{a, \delta_b(b), c\} = \delta_b \{a, b, c\} - \{\delta_b(a), b, c\} - \{a, b, \delta_b(c)\} = 0,$$

proves the statement. □

It is due to B.E. Johnson that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation (cf. [14]). It is also known that every Jordan derivation from a C*-algebra A to a Banach A-bimodule or to a Jordan Banach A-module is continuous (cf. [23, §1]). Therefore, every generalised Jordan derivation D from a unital C*-algebra A to a Banach A-bimodule with $D(1) = 0$ is a bounded Jordan derivation and hence a continuous associative derivation.

We shall explore now the connections between generalised (Jordan) derivations and triple derivations from A to B. Let $\delta : A \to B$ be a triple derivation. By Lemma 3, $\delta(1)^* = -\delta(1)$, and hence

$$\delta(a \circ b) = \delta \{a, 1, b\} = \{\delta(a), 1, b\} + \{a, 1, \delta(b)\} + \{a, \delta(1), b\}$$
$$= \delta(a) \circ b + a \circ \delta(b) + U_{a,b}(\delta(1)^*) = \delta(a) \circ b + a \circ \delta(b) - U_{a,b}(\delta(1)),$$

for every a, b in A, which shows that δ is a generalised Jordan derivation (compare also [21, Lemma 3.1]).

We shall focus now our attention on local triple derivations on a commutative unital C*-algebra.

Proposition 5. *Let us assume that A is commutative. Then every local triple derivation $T : A \to B$ is a generalised Jordan derivation.*

Proof. Let us fix an arbitrary $\varphi \in B^*$ and define $W_\varphi : A \times A \times A \to \mathbb{C}$ a mapping given by $W_\varphi(a, b, c) := \varphi(\{a, T(b), c\})$. Clearly, W_φ is linear and symmetric in a and c and conjugate linear in b. Lemma 4 assures that

$$W_\varphi(a, b, c) = \varphi \{a, T(b), c\} = \frac{1}{2}\varphi(aT(b)^*c + cT(b)^*a) = 0, \tag{3}$$

whenever $a \perp b \perp c$ (or equivalently, $ab = bc = 0$). Fix $a, b \in A$ with $ab = 0$. The form $V(s, t) := W_\varphi(a, bs, t)$ is linear in t and conjugate linear in s and

$V(s,t) = 0$ for every $s,t \in A$ with $st = 0$. That is, V an orthogonal form in the terminology of Goldstein in [6]. It follows from [6, Theorem 1.10] (see also [7] or [22]) that there exists $\phi \in A^*$ satisfying

$$V(s,t) = \phi(s^*t), \ \forall s,t \in A. \tag{4}$$

It follows from (4) that

$$W_\varphi(a, bs, t) = V(s,t) = V(1, s^*t) = W_\varphi(a, b, s^*t) \tag{5}$$

for every $s,t,a,b \in A$ with $ab = 0$. Fix $s,t \in A$, the above equation (5) shows that the form $V_2(x,y) := W_\varphi(x, ys, t) - W_\varphi(x, y, s^*t)$ is orthogonal. Again Goldstein's theorem shows the existence of $\phi_1 \in A^*$ satisfying $V_2(x,y) = \phi_1(xy^*)$, for every $x,y \in A$. Consequently, $V_2(x,y) = V_2(1, x^*y) = V_2(xy^*, 1)$, for all $x,y \in A$. We have therefore proved that

$$W_\varphi(x, ys, t) - W_\varphi(x, y, s^*t) = W_\varphi(xy^*, s, t) - W_\varphi(xy^*, 1, s^*t),$$

or equivalently,

$$\varphi\left(\{x, T(ys), t\} - \{x, T(y), s^*t\} - \{xy^*, T(s), t\} + \{xy^*, T(1), s^*t\}\right) = 0,$$

for every $x,y,s,t \in A$, $\varphi \in B^*$. The arbitrariness of φ and the Hahn-Banach theorem give

$$\{x, T(ys), t\} = \{x, T(y), s^*t\} + \{xy^*, T(s), t\} - \{xy^*, T(1), s^*t\}. \tag{6}$$

Finally, taking $x = t = 1$, we have

$$T(ys)^* = T(y)^* \circ s^* + y^* \circ T(s)^* - U_{y^*,s^*}\left(T(1)^*\right),$$

which shows that T is a generalised Jordan derivation. $\square$

Let us make some observations to the proof of the above proposition. The identity (6) holds for every x,y,s,t in A. Moreover, since, by Goldstine's theorem, the unit ball of A is weak*-dense in the unit ball of A^{**}, by Sakai's theorem, the products of A^{**} and of B^{**} are separately weak*-continuous, and T^{**} is weak*-continuous, the equality

$$\{x, T^{**}(ys), t\} = \{x, T^{**}(y), s^*t\} + \{xy^*, T^{**}(s), t\} - \{xy^*, T(1), s^*t\}. \tag{7}$$

holds for every x,y,s,t in A^{**}, and hence T^{**} also is a generalised Jordan derivation.

We can prove now a stronger version of Proposition 5 which is a subtle variant of [16, Sublemma 5] and [19, Proposition 1.1].

Proposition 6. *In the hypothesis of Proposition 5, let $T : A \to B$ be a local triple derivation. Then for each $a,b,c \in A$ with $ab = bc = 0$ we have*

$$aT(b)^*c = aT(b^*)^*c = 0.$$

Proof. Fix $a,b,c \in A$ with $ab = bc = 0$. Let us identify A with some $C(K)$ for a suitable compact Hausdorff space K. Let p denote the range projection of b in A^{**}, that is $p = \chi_{S(b)}$, where $S(b) := \{t \in K : b(t) \neq 0\}$ is the co-zero set of b. Observe that $ap = 0 = pc$ and $pb = bp = b$.

By (7), we have

$$(1-p)T(b)^*(1-p) = \{1-p, T(b), 1-p\}$$
$$= \{1-p, T(bp), 1-p\} = \{1-p, T(b), p(1-p)\} + \{(1-p)b^*, T^{**}(p), 1-p\}$$
$$- \{(1-p)b^*, T(1), p(1-p)\} = 0.$$

Therefore, $aT(b)^*c = a(1-p)T(b)^*(1-p)c = 0$. $\square$

One of the main results established by J. Li and Zh. Pan in [19, Corollary 2.9] implies that a bounded linear operator $T : A \to B$ is a generalised derivation if, and only if, $aT(b)c = 0$, whenever $ab = bc = 0$. Let us suppose that, in the above hypothesis, A is commutative and $T : A \to B$ is a local triple derivation. Proposition 6 assures that $aT(b^*)^*c = 0$, for every $ab = bc = 0$ in A, and consequently, the mapping $x \mapsto T(x^*)^*$ is a generalised derivation, and thus,

$$T(a^*b^*)^* = T(a^*)^*b + aT(b^*)^* - aT(1)^*b,$$

or equivalently,

$$T(ba) = T(ab) = bT(a) + T(b)a - bT(1)a,$$

showing that T is actually a generalised derivation. We have therefore proved the following:

Corollary 7. *Let us assume that A is commutative. Then every local triple derivation from A to B is a generalised derivation. Moreover, taking $\widetilde{T} = T-\frac{1}{2}\delta(T(1),1) = T-\delta\left(\frac{1}{2}T(1),1\right)$, it follows that $\widetilde{T}$ is a local triple derivation with $\widetilde{T}(1) = 0$, and hence $\widetilde{T}$ is a (Jordan) derivation.* $\square$

The statement concerning $\widetilde{T}$ in the above corollary could be also derived applying the previously mentioned Johnson's theorem on the equivalence of Jordan derivations and (associative) derivations (cf. [14, Theorem 6.3]).

Remark 8. The argument given in the proof of Proposition 6 is also valid to show that, under the same hypothesis, a generalised Jordan derivation $T : A \to B$ satisfies that $aT(b)c = 0$, for every $a, b, c \in A$ with $ab = bc = 0$. Combining Goldstine's theorem with the separate weak*-continuity of the product of A^{**} and B^{**} we guarantee that T^{**} is a generalised Jordan derivation too. Let p denote the range projection of b in A^{**}. In this case

$$T(b) = T(p \circ b) = T(p) \circ b + p \circ T(b) - U_{p,b}T(1)$$
$$= \frac{1}{2}\Big(bT(p) + T(p)b + pT(b) + T(b)p - pT(1)b - bT(1)p\Big),$$

which implies that $(1-p)T(b)(1-p) = 0$, and hence $aT(b)c = 0$, for every $a, b, c \in A$ with $ab = bc = 0$. By [19, Corollary 2.9], T is a generalised derivation. This shows that every generalised Jordan derivation on a unital C*-algebra is a generalised derivation.

Associative derivations from A to B are not far away from triple derivation. It is not hard to check that, in our setting, a bounded linear operator $\delta : A \to B$ is a triple derivation and $\delta(1) = 0$ if, and only if, it is a *-derivation, that is, it is a derivation and $\delta(a^*) = \delta(a)^*$.

Lemma 9. *Let B be a unital C*-algebra, and let $T : B \to B$ be a bounded local triple derivation with $T(1) = 0$. Then T is a symmetric operator, that is, $T(a^*) = T(a)^*$, for every $a \in B$.*

Proof. Let A denote the abelian C*-subalgebra generated by a normal element a and the unit of B. Since $T|_A : A \to B$ is a local triple derivation and $T(1) = 0$, by Corollary 7 and the subsequent comments, $T|_A$ is an associative derivation. Let u be a unitary element in A. Since $T|_A$ is a derivation, we have $0 = T(1) = T(uu^*) = uT(u^*) + T(u)u^*$, so

$$T(u) = -uT(u^*)u.$$

Now, having in mind that T is a local triple derivation, there exists a triple derivation δ_u such that $T(u) = \delta_u(u)$, we deduce that $T(u) = \delta_u(u) = \delta_u(uu^*u) = \delta_u\{u, u, u\} = 2\{u, u, T(u)\} + \{u, T(u), u\} = 2T(u) + uT(u)^*u$, which gives

$$T(u) = -uT(u)^*u.$$

Combining these two equations we have $uT(u^*)u = uT(u)^*u$, and hence $T(u^*) = T(u)^*$.

Finally, by the Russo-Dye theorem, $T(b^*) = T(b)^*$, for every b in A. The arbitrariness of the normal element a implies that $T(b)^* = T(b)$, for every $b \in B_{sa}$, which gives the statement of the lemma. □

We can state now the main result.

Theorem 10. *Let B be a unital C*-algebra. Every local triple derivation from B to B is a triple derivation.*

Proof. Let A denote the abelian C*-subalgebra generated by a normal element a and the unit of B. Since $T|_A : A \to B$ is a local triple derivation, we can apply Corollary 7 and the comments following it to assure that $T|_A$ is a triple derivation and $\widetilde{T}|_A = \left(T - \frac{1}{2}\delta(T(1), 1)\right)|_A$ is an associative derivation. It follows that $\widetilde{T}(a^2) = \widetilde{T}(a)a + a\widetilde{T}(a)$. Since a was arbitrarily chosen, we can affirm that

$$\widetilde{T}((a + b)^2) = \widetilde{T}(a + b)(a + b) + (a + b)\widetilde{T}(a + b),$$

for every $a, b \in B_{sa}$, which implies that

$$\widetilde{T}(a \circ b) = \widetilde{T}(a) \circ b + a \circ \widetilde{T}(b),$$

for every $a, b \in A_{sa}$. It is easy to check that $\widetilde{T}$ is a Jordan derivation, and hence an associative derivation by [14, Theorem 6.3]. Now, Lemma 9 assures that $\widetilde{T}$ is a symmetric operator and thus a triple derivation, which concludes the proof. □

We shall conclude this section with a result on "automatic continuity" for generalised derivations. The following construction is inspired by the results in [5, §4] (see also [23]). Let $D : B \to X$ be a generalised Jordan derivations from a unital C*-algebra to a Banach B-module. We regard X as a Jordan Banach triple B-module with the triple products defined by $\{x, b, a\} = \{a, b, x\} := (a \circ b) \circ x (x \circ b) \circ a - (a \circ x) \circ b$, and $\{a, x, b\} := (a \circ x) \circ b (x \circ b) \circ a - (a \circ b) \circ x$, where for each $a \in B$ and $x \in X$, $a \circ x := \frac{1}{2}(ax + xa)$.

Fix $a, b, c \in B_{sa}$. The identity

$$D(\{a, b, c\}) - \{D(a), b, c\} - \{a, D(b), c\} - \{a, b, D(c)\} =$$

$$= -U_{a,b}(D(1)) \circ c - U_{a \circ b, c}(D(1)) - U_{c,b}(D(1)) \circ a - U_{c \circ b, a}(D(1))$$

$$+U_{a,c}(D(1)) \circ b + U_{c \circ a, b}(D(1)),$$

shows that the mapping $\check{D}|_{B^3_{sa}} : B_{sa} \times B_{sa} \times B_{sa} \to X$, $\check{D}(a, b, c) := D(\{a, b, c\}) - \{D(a), b, c\} - \{a, D(b), c\} - \{a, b, D(c)\}$ is a continuous trilinear operator and hence D is a "*generalised triple derivation*" in the terminology employed in [5, §4]. It follows from [5, Proposition 21] (see also [5, Theorem 22]) that $D|_{B_{sa}}$ is continuous. The continuity of D follows straightforwardly.

Proposition 11. *Every generalised (Jordan) derivation, not assumed a priori to be continuous, from a unital C*-algebra B into a Banach B-bimodule is continuous.* $\square$

Despite the automatic continuity of generalised derivations, in the results included in this section, local triple derivations, generalised derivations and generalised Jordan derivations are assumed to be continuous, and these assumptions are needed in the arguments. The results established by J. Li and Zh. Pan in [19, Proposition 1.1 and Corollary 2.9] on generalised derivations need to assume hypothesis of continuity.

Problem 12. [21, Conjecture 6.2 (C2)] *Is every local triple derivation, not assumed a priori to be continuous, on a C*-algebra or on a JB*-triple E continuous?*

3. Local triple derivations on unital JB*-algebras

Every JB*-algebra J can be equipped with a structure of JB*-triple with respect to the product

$$\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

A Jordan derivation on J is a linear mapping $d : J \to J$ satisfying $d(a \circ b) = d(a) \circ b + a \circ d(b)$, for every $a, b \in J$. Given a Jordan-Banach triple J-module X, a conjugate linear mapping $\delta : J \to X$ is said to be a triple derivation whenever the identity

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\},$$

holds for every $a, b, c \in J$.

According to what we did in the setting of C*-algebras, given a unital JB*-algebra J and a JB*-subalgebra, A, containing the unit of J, J can be regarded as a Jordan-Banach J-module and a Jordan-Banach triple A-module with respect to its natural Jordan product and its natural triple product, respectively. By a little abuse of notation, a linear mapping $\delta : A \to J$ satisfying $\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}$, for every $a,b,c \in A$, is said to be a *triple derivation.* A *local triple derivation* from A to J is bounded linear operator $T : A \to B$ such that for each $a \in A$ there exists a triple derivation $\delta_a : A \to J$ satisfying $T(a) = \delta_a(a)$.

Arguing as in the previous section, we have:

Lemma 13. [11, Lemma 1] *Let A be a JB*-subalgebra of a unital JB*-algebra J containing the unit of J, and let $T : A \to J$ be a local triple derivation. Then $T(1)^* = -T(1)$.* □

As in the C*-setting, the mapping $\delta(T(1),1) = L(T(1),1) - L(1,T(1)) : A \to J$ is an inner triple or ternary derivation, $\delta(T(1),1)(1) = 2T(1)$, and $\widetilde{T} = T - \frac{1}{2}\delta(T(1),1)$ is a local triple derivation with $\widetilde{T}(1) = 0$.

Motivated by the definitions made in the associative setting, a linear mapping $D : A \to J$ is a *generalised Jordan derivation* whenever $D(a \circ b) = D(a) \circ b + a \circ D(b) - U_{a,b}D(1)$, for every a,b in A. Every generalised Jordan derivation $D : A \to J$ with $D(1) = 0$ is a Jordan derivation and every triple derivation $\delta : A \to J$ is a generalised Jordan derivation.

The proof of Proposition 5 remains valid in the following sense:

Proposition 14. *Let A be the (associative) JB*-subalgebra of a unital JB*-algebra J generated by a self-adjoint element a and the unit of J. Suppose $T : A \to J$ is a local triple derivation, then T is a generalised Jordan derivation.* □

Since the proof of Lemma 9 remains valid in the Jordan setting, the reasoning given in Corollary 7 and Theorem 10 can be rephrased to prove the following:

Theorem 15. *Let J be a unital JB*-algebra. Every local triple derivation from J to J is a triple derivation.* □

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E-mail address: `maria.burgos@uca.es`

Departamento de Matematicas, Facultad de Ciencias Sociales y de la Educacion, Universidad de Cadiz, 11405, Jerez de la Frontera, Spain.

E-mail address: `pacopolo@ugr.es`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

E-mail address: `jgarces@correo.ugr.es`

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

E-mail address: aperalta@ugr.es

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.

ugr Universidad de Granada