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Population Dynamics from a Topological Point of View

Tesis doctoral

Editor: Editorial de la Universidad de Granada
Autor: Alfonso Ruiz Herrera
D.L.: GR 2231-2013
ISBN: 978-84-9028-642-5

Tesis doctoral dirigida por Dr. Rafael Ortega Ríos

Dr. Rafael Ortega Ríos, doctor en Matemáticas y catedrático en la Facultad de Ciencias de la Universidad de Granada, manifiesta que la presente memoria titulada "Population dynamics from a topological point of view" presentada por Alfonso Ruiz Herrera para optar al grado de doctor en Matemáticas ha sido realizada bajo mi supervisión en el departamento de Matemática Aplicada de la Universidad de Granada.

Fdo: Rafael Ortega Ríos

Memoria presentada por Alfonso Ruiz Herrera

Fdo: Alfonso Ruiz Herrera

Con el fin de obtener la mención internacional en el título de doctor se han cumplido los siguientes requisitos:

- *La tesis está redactada en inglés con resumen y conclusiones en español.*
- *Uno de los miembros del tribunal propuesto es de una universidad extranjera.*
- *La defensa se hará en inglés con resumen y conclusiones en español.*
- *Dos informes de expertos extranjeros han avalado esta tesis.*
- *Una parte de la tesis se ha realizado en la universidad de Udine (Italia) bajo la supervisión del profesor F. Zanolin.*

Me gustaría agradecer a las personas que me han ayudado en este periodo, en especial a E. Liz, R. Ortega, P. J. Torres y F. Zanolin.

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Chapter 1

Summary

Mathematical models are important in many different disciplines since from the knowledge of an initial data it is possible to determine how the system varies in the time. In the last decades, an infinity of models have appeared to describe different biological situations. This growing interest has generated a recent discipline, Math-Biology. Math-Biology has a character multidisciplinary involving mathematicians, physicists, engineers, biologists... The importance of this topic is reflected on hundreds of analytical, numerical and experimental papers that can be consulted in the bibliographies of the monographs [32] and [7].

Throughout this thesis we focus our attention on population dynamics. Roughly speaking, the main aim is to use mathematical models in order to study the interaction of different species sharing the same environment. More precisely, given an initial data, i.e. the number of individuous of each species at time $t = 0$, our purpose is to determine the evolution of the number of individuous in the time. It would be very desirable to determine, in a explicit way, this evolution. However, in most of models, it is impossible to obtain such an analytic expression from the initial data.

An important family of models in population dynamics is known as Kolmogorov systems and has the form

$$x_i(N+1) = T(x(N)) = x_i(N)f_i(x_1(N), ..., x_n(N)), \quad i = 1, ..., n \tag{1.1}$$

where $f_i : \mathbb{R}^n_+ := \{(x_1, ..., x_n) : x_i \geq 0\} \longrightarrow]0, +\infty[$ is a continuous function. This system is used to model the evolution of n species sharing the same environment. $x_i(N)$ is the time-varying population density of the i-th species at the period N. The function f_i is the so-called growth rate of the i-th species and represents the dependence of the density of the different species on the population of the i-th species.

This thesis is focused on three different topics. Exclusion, Dominance and Permanence. The purpose of this introductory section is twofold: On the one hand, we define these notions from a mathematical and biological point of view. On the other hand, we discuss how to find the presence of these concepts in (1.1).

From a biological point of view, a system is permanent if for all initial data so that the number of individuous of each species is non zero, none species goes to extinction. In an informal way, mathematically, system (1.1) is permanent if there is a compact set in $Int\mathbb{R}^n_+$ "absorbing" the dynamics of (1.1) in $Int\mathbb{R}^n_+$. More precisely,

Definition 1.0.1 *System (1.1) is permanent if there is a compact set K satisfying that*

$$K \subset Int\mathbb{R}^n_+$$

and for all $x(0) \in Int\mathbb{R}^n_+$ there is $N_0 = N_0(x(0))$ with

$$x(N) \in K$$

for all $N \geq N_0$.

Our aim will be to characterize the notion of permanent system in case of two species. For it we introduce the following framework.

Definition 1.0.2 *There is logistic growth in the i-th species if*

$$x_i(N+1) = x_i(N) f_i(x_i(N) e_i) \tag{1.2}$$

has a global attractor $x_i^* > 0$ in $]0, +\infty[$. We employ the notation $\{e_1, ..., e_n\}$ to denote the usual basis in $\mathbb{R}^n$

Definition 1.0.3 *System (1.1) is dissipative if there is a constant $M > 0$ so that*

$$\limsup_{N \longrightarrow \infty} |x_i(N)| < M$$

for all i, where $x(N) = (x_1(N), ..., x_n(N))$ is the sequence from (1.1) with initial condition $x(0) = (x_1(0), ..., x_n(0))$.

The notion of dissipative system is usually used in population dynamics to model the limitations of the environment.

Theorem 1.0.4 *For $n = 2$, assume that (1.1) is dissipative and the origin does not attract points of $Int\mathbb{R}^2_+$. In addition, assume that there is logistic growth in both species with attractor x_i^* (respectively) and*

$$0 < |\frac{\partial}{\partial x_i} T(x_i^* e_i)| < 1$$

for $i = 1, 2$. *Then (1.1) is permanent if and only if*

$$index_{\mathbb{R}^2_+}(T, x_1^* e_1) = index_{\mathbb{R}^2_+}(T, x_2^* e_2) = 0.$$

This result characterizes the permanence in our system in case of two species. In general, using the geometrical flavour of the index, to decide if our system is permanent, it is enough to draw two "concrete curves" and to count the number of laps around the origin of these curves.

From a biological point of view, there is exclusion if for every initial condition, some species goes to extinction. That is, it is impossible that all the species survive for all the time. Note that, in general, the species which goes to extinction depends on the initial data. The notion of dominant species avoids this last situation. Indeed, the i-th species is dominant if apart from this species, all the species go to extinction. Clearly, the notion of dominant species is stronger than the notion of exclusion. In a completely analogous way, the notion of dominated species is defined. From a mathematical point of view, there is exclusion in system (1.1) if for all initial condition $x(0) \in \mathbb{R}^n_+$, there is an index j so that

$$x_j(N) \longrightarrow 0.$$

It is clear that the notion of exclusion implies the non existence of fixed points of T in $Int\mathbb{R}^n_+$. Our aim will be to show that it is also a necessary condition when there are 3-competing species. For modelling a competitive interaction, we assume:

C1) If $x \prec y$ then $f_i(y) < f_i(x)$ for all $i = 1, ..., n$. ($\prec$ denotes the usual ordering in $\mathbb{R}^n$)

C2) There is logistic growth in each axis.

C3) $T(p) \prec T(q)$ then $p_i < q_i$ for all $i = 1, ..., n$ provided $q_i \neq 0$.

Additionally, for our result we introduce this condition

C4) T has a finite number of fixed points on $\partial\mathbb{R}^n_+$.

Theorem 1.0.5 *For* $n = 3$, *assume that (1.1) satisfies* **C1), C2), C3), C4)**. *Then the following statements are equivalent:*

i) T *has no fixed points in* $Int\mathbb{R}^n_+$.

ii) *There is exclusion for system (1.1).*

To prove the previous result we will use that under **C1), C2), C3)** the dynamics of (1.1) is essentially in $\mathbb{R}^2$. After that, we use a variant of Massera's theorem developed in [5].

The mathematical translation of the notion of dominant species is the following: The j-th species is dominant if there is $\delta > 0$ so that for every initial condition $x(0) \in \mathbb{R}^n_+$ with $x_j(0) > 0$,

$$\lim_{N \longrightarrow \infty} x_i(N) = 0$$

for all $i \neq j$ and

$$\liminf_{N \longrightarrow \infty} x_j(N) > \delta > 0.$$

To detect the notion of dominant species we use the following condition:

Definition 1.0.6 *The species j verifies the F-Y condition if*

$$\bigcup_{i \in \{1,2,\ldots,n\} \setminus \{j\}} D_i^+ \subset D_j^*$$

where $D_j^* = \{x \in \mathbb{R}^n_+ : f_j(x) > 1\}$ *and* $D_i^+ = \{x \in \mathbb{R}^n_+ : f_i(x) \geq 1\}$.

The biological interpretation of the previous definition is as follows: if some species different from the species j does not decrease its size, then the species j increases strictly. Let us remark that

- the F-Y condition for the species j does not imply, in general, $f_j(x) > f_i(x)$ for all $i \neq j$,
- the F-Y condition does not ensure the presence of dominant species, (see [15], [57]).

Theorem 1.0.7 *Assume that system (1.1) satisfies* **C1), C2), C3)**. *If the species j has the F-Y condition then the species j is dominant.*

Chapter 2

Index on convex sets and stability

The index on convex sets is usually used to prove the existence of positive solutions of certain equations or systems. In contrast with this point of view, we will see that this topological tool can be naturally linked with the problem of local stability of fixed points. This approach has as an advantage that, being of topological nature, non-hyperbolic situations can be treated easily. Moreover, the geometrical character of the index allows us to understand perfectly the dynamical behavior in a small neighborhood of a fixed point by looking at a curve. Along this chapter, this informal discussion is studied in detail. For it, in Sections 2.1 and 2.2 we recall some elementary notions on degree theory. In Section 2.3 some classical results about linearization are presented. Finally, in Section 2.4 we properly study the connection between index and stability.

2.1 Degree, index, and index on convex sets

In this section we present some basic results about degree theory. In general terms, the degree of a continuous map $f:\overline{\Omega}\longrightarrow\mathbb{R}^d$ will provide us with a "count" of the solutions of

$$f(x)=0$$

with $x\in\Omega$. We want a "count" being stable under perturbations and easy to use in concrete examples. With these aims in our mind, we proceed to define rigorously the degree of a continuous map. Assume that $\Omega\subset\mathbb{R}^d$ is a non-empty, bounded, and open set. Consider

$$f:\overline{\Omega}\longrightarrow\mathbb{R}^d$$

a continuous map with $f(x)\neq 0$ for all $x\in\partial\Omega$. In this setting, we define the degree of f in Ω, in the sequel $deg(f,\Omega)$, after three steps.

Step 1 f is of class $\mathcal{C}^1$ and all its zeros are simple, (0 is a regular value)

In this case,

$$deg(f,\Omega):=\sum_{i=1}^{n} sign(\det(f'(\xi_i)))$$

with $f^{-1}(0)=\{\xi_1,...,\xi_n\}$. We use the convention $\sum_{\emptyset}=0$.

Step 2 f is of class $\mathcal{C}^1$ but its zeros are not necessarily simple.

Under these conditions, by Sard's Theorem, we can take a sequence of vectors $\{v_n\}$ so that

- $\{v_n\} \longrightarrow 0$,
- $f_n = f - v_n$ has all its zeros simple.

Then we can apply Step 1 to the map $f_n(x) = f(x) - v_n$ and define

$$deg(f, \Omega) := \lim_{n\to\infty} deg(f_n, \Omega).$$

Observe that $f_n(x) \neq 0$ for all $x \in \partial\Omega$ with n large. In this definition, we must prove that the sequence $deg(f_n, \Omega)$ becomes eventually constant and also that the limit is independent of the choice of v_n. These properties can be found in [10].

Step 3 (General case) f is continuous.

In this setting, we approximate f by a sequence of functions f_n of class $\mathcal{C}^1$ such that $f_n \longrightarrow f$ uniformly in $\overline{\Omega}$. After that we use the previous step and define

$$deg(f, \Omega) := \lim_{n\to\infty} deg(f_n, \Omega).$$

It must be proven that this limit exists and is independent of the choice of f_n. Again this property can be found in [10].

Following the previous steps can be a strategy to compute the degree of a concrete map in a determined domain. However, this mechanism is not easy to apply in many concrete examples. For this problem, the next properties can be useful, (assume that f and Ω are as above).

1. $deg(id, \Omega) = 1$ if $0 \in \Omega$.

2. $deg(f, \Omega) \neq 0$ implies that there exists $x \in \Omega$ such that $f(x) = 0$.

3. **(Excision)** If $\Omega_1, ..., \Omega_k$ are disjoint and open subsets of Ω and $f(x) \neq 0$ for

$$x \in \Omega \setminus \bigcup_{i=1}^{k} \Omega_i$$

then

$$deg(f, \Omega) = \sum_{i=1}^{k} deg(f, \Omega_i).$$

4. (**Homotopy invariance**) Let $F : \overline{\Omega}\times[0,1] \longrightarrow \mathbb{R}^d$ be a continuous map and assume that

$$F_t : \overline{\Omega} \longrightarrow \mathbb{R}^d$$

$$x \mapsto F(t,x)$$

satisfies that $F_t(x) \neq 0$ if $x \in \partial\Omega$ and $t \in [0,1]$. Then $deg(F_t,\Omega)$ is independent of t.

5. (**Product**) If $\Delta \subset \mathbb{R}^m$ is a bounded open set and $g : \overline{\Delta} \longrightarrow \mathbb{R}^m$ is continuous and $g(x) \neq 0$ for all $x \in \partial\Delta$ then

$$deg((f,g),\Omega\times\Delta) = deg(f,\Omega)\cdot deg(g,\Delta).$$

The proofs of these properties can be deduced from the definition of the degree, see [10]. At this moment we recall that there is a unique "degree" satisfying the previous properties, that is , if $deg_1(f,\Omega)$ and $deg_2(f,\Omega)$ are two maps having properties $1-4$ then $deg_1(f,\Omega) = \mathrm{deg}_2(f,\Omega)$ for all Ω and f as above, see [3].
Next we present the notion of index and index on convex sets. Indeed, from the definition of the degree and using the excision property, given $x_0 \in \Omega$ an isolated fixed point for a continuous map $g : \Omega \longrightarrow \mathbb{R}^d$, we define the index of g at the point p as

$$index(g,p) := \lim_{\varepsilon\to 0^+} deg(id - g, B(p,\varepsilon))$$

where $B(p,\varepsilon)$ is the Euclidean ball with center at p and radius ε.
To finish this section we study the notion of index on convex sets. For it, we consider an open and convex set $V \subset \mathbb{R}^d$ and assume that $f : \overline{V} \longrightarrow \overline{V}$ is a continuous map with $p \in \partial V$ an isolated fixed point for f. In this framework, we define

$$index_V(f,p) := index(\widehat{f},p)$$

where $\widehat{f} : \mathbb{R}^d \longrightarrow \overline{V}$ is any extension of f taking values in $\overline{V}$, that is $\widehat{f}|_{\overline{V}} = f$ and $\widehat{f}(\mathbb{R}^d) \subset \overline{V}$. It can be proven that the previous definition does not depend on the extension $\widehat{f}$, see [10]. It is important to note that the fixed point index considered here is a particular case of the index in ENR's considered by Dold [13], see also [27].

2.2 Degree in $\mathbb{R}^2$ and Examples

In this section, we give a geometrical point of view of the degree in the plane in order to simplify its computation in concrete examples. To this end we recall the definition of winding number associated to a closed curve. Consider $\alpha : [0,1] \longrightarrow \mathbb{R}^2 \backslash \{0\}$ a continuous curve with $\alpha(0) = \alpha(1)$. The winding number of α is defined as

$$\frac{\theta(1) - \theta(0)}{2\pi}$$

where $\theta(t)$ is a continuous argument of $\alpha(t)$. Next we link this notion with the degree. Assume that Ω is a Jordan domain in $\mathbb{R}^2$, this means that Ω is the bounded connected component of $\mathbb{R}^2 - \Gamma$, where Γ is a Jordan curve. Also suppose that $\alpha : [0,1] \longrightarrow \mathbb{R}^2$ is a positive (counter-clockwise) parametrization of Γ, that is

$$\alpha(0) = \alpha(1),$$

$$\alpha|_{[0,1)} \text{ is one} - \text{to} - \text{one},$$

$$\alpha([0,1]) = \Gamma.$$

Then given $f : \overline{\Omega} \longrightarrow \mathbb{R}^2$ a continuous map with

$$f(x) \neq 0$$

for all $x \in \partial\Omega = \Gamma$, we can check that $deg(f, \Omega)$ is the winding number of $f \circ \alpha : [0,1] \longrightarrow \mathbb{R}^2 \backslash \{0\}$.

Now we give a concrete example to see the difference between the usual index and the index on convex sets. For it, we consider the map $f(x,y) = (2x, 2y)$, clearly by the definition of the degree and Step 1 in the construction we know that

$$index(f, 0) = 1.$$

However, for the convex set $V = \{(x,y) : y > 0\}$,

$$index_V(f, 0) = 0.$$

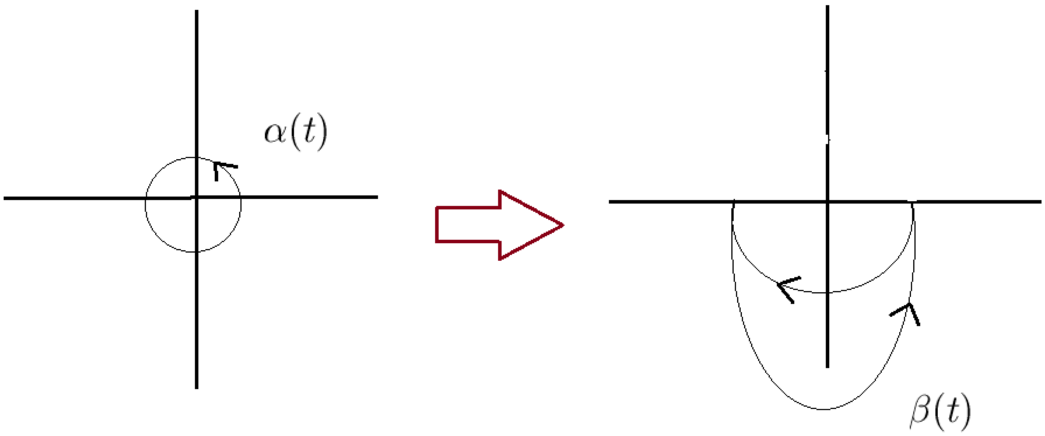

Figure 2.1: Pictorial explanation of the degree in $\mathbb{R}^2_+$.

To see this claim we take $\widehat{f}(x,y) = f(x,|y|)$ and consider the curve $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$. We observe that the winding number of $\beta(t) = \alpha(t) - \widehat{f}(\alpha(t))$ is 0, see Figure 2.1.

2.3 Dynamics in a neighbourhood of a hyperbolic or partially hyperbolic fixed point

Along this section we recall some results about stability theory taken mainly from [31] and [46]. Given $P : U \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a diffeomorphism of class $\mathcal{C}^1$ having a fixed point at p where U is an open set, we say that p is hyperbolic if the Jacobian matrix $P'(p)$ has no eigenvalues in $\mathbb{S}^1$. We understand that $G : V \longrightarrow \mathbb{R}^d$ is conjugate to P if there exists a homeomorphism $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $G = \phi^{-1} \circ P \circ \phi$. Under this last definition, G has "essentially" the same dynamical behavior as P. Next we state the classical Hartman-Groβman theorem.

Theorem 2.3.1 *Consider $P : U \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a diffeomorphism of class $\mathcal{C}^1$ so that p is a hyperbolic fixed point. Then there exist open sets V_1, V_2 such that $p \in V_1$, $0 \in V_2$ and a homeomorphism $\phi : V_1 \longrightarrow V_2$ such that $\phi(P(x)) = L(\phi(x))$ if $x, P(x) \in V_1$ with $L = P'(p)$.*

Our next aim is to derive some consequences of Theorem 2.3.1 in connection with the stable and unstable manifold. Rigorously, given $P : U \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a diffeomorphism of class $\mathcal{C}^1$ with a fixed point p, we define the stable and unstable manifold as

$$\mathcal{W}^s(p) := \{q \in U : P^n(q) \to p \;\text{ as } n \to +\infty\},$$

$$\mathcal{W}^u(p) := \{q \in U : P^{-n}(q) \to p \;\text{ as } n \to +\infty\}.$$

As a direct consequence of the definition we check that

- $p \in \mathcal{W}^s(p) \cap \mathcal{W}^u(p)$,
- $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$ are invariant under P.

After that, we determine the stable and unstable manifold in some easy examples:

1. Take P a lineal map in $\mathbb{R}^d$ with no eigenvalues in $\mathbb{S}^1$. In this case, $\mathcal{W}^s(0) = E_s$ and $\mathcal{W}^u(0) = E_u$ where E_s (resp. E_u) is the vector space generated by the eigenvectors associated to eigenvalues with modulus less (resp. greater) than 1.
2. Take P as rotation. In this case, $\mathcal{W}^s(0) = \mathcal{W}^u(0) = \{0\}$.

Next, we study the stable manifold in a more complicated situation. Indeed, assume that p is a hyperbolic fixed point for P. In advance, we know by Theorem 2.3.1 that there are two neighbourhoods V_1 of p, V_2 of 0 and a homeomorphism $\phi : V_1 \longrightarrow V_2$ satisfying that

$$L\xi = \phi \circ P \circ \phi^{-1}(\xi)$$

for all $\xi \in V_2$ and $L\xi \in V_2$ where $L = P'(p)$. Let E_s and E_u be the stable and unstable subspaces of L respectively. Clearly, we can find a neighbourhood of 0, namely U_2, such that if $\xi \in U_2 \cap E_s$ then $L^n(\xi) \in V_2$ for all $n \geq 0$. As a consequence,

$$L^n(\xi) = \phi \circ P^n \circ \phi^{-1}(\xi)$$

and so

$$P^n(\phi^{-1}(\xi)) \to \phi^{-1}(0) = p.$$

From this discussion we see that if $\xi \in V_2 \cap E_s$ then $\phi^{-1}(\xi) \in \mathcal{W}^s(p)$. Therefore we have found a set homeomorphic to $E_s \cap V_2$ contained in $\mathcal{W}^s(p)$. With this construction in our

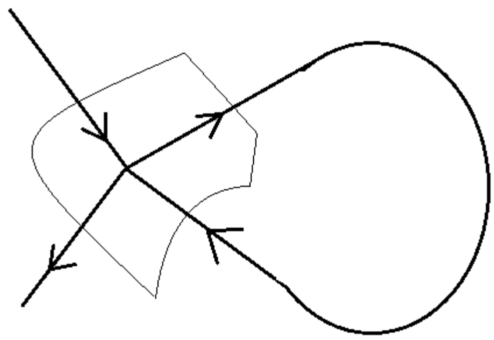

Figure 2.2: Illustration of a1 and a2.

mind, let us determine all the stable manifold. Indeed, given U_1 a small neighbourhood of p contained in $\phi^{-1}(U_2)$, define

$$\Lambda = \{q \in U_1 : P^n(q) \in U_1, for\ all\ n \geq 2\}.$$

Observe that, by the previous arguments, $\Lambda \subset \mathcal{W}^s(p)$ and $P(\Lambda) \subset \Lambda$. In addition, if $r \in \mathcal{W}^s(p)$ then $P^{n_0}(r) \in \Lambda$ for some $n_0 \in \mathbb{N}$. Putting all the information together we can construct $\mathcal{W}^s(p)$ as

$$\bigcup_{n=0}^{\infty} P^{-n}(\Lambda).$$

Remark 2.3.2 *We can repeat the analogous construction for the unstable manifold.*

There are two comments to be made concerning the stable manifold:

a1 In general, $\phi^{-1}(E_s \cap V_2)$ is not the intersection of $\mathcal{W}^s(p)$ with V_1. See figure 4.1

a2 $\mathcal{W}^s(p)$ is not necessarily a sub-manifold of $\mathbb{R}^n$. see figure 4.1.

Now, we go back to the Hartman-Groβman Theorem. From this theorem, the following question arises: what happens if the hyperbolic behavior of $P'(p)$ is dropped? To answer this question we recall some results in [31]. In the remainder of the section we assume, without further mention, the following conditions for P. Assume that 0 is a fixed point and P has an expression of the type

$$P(z, y) = (f(z, y), Ly + Y(z, y))$$

with $f : \mathbb{R}^r \times \mathbb{R}^t \longrightarrow \mathbb{R}^r$, $Y : \mathbb{R}^r \times \mathbb{R}^t \longrightarrow \mathbb{R}^t$, and L a $t \times t$ expansion matrix (i.e. there is $l < 1$ such that $|L^{-1}y| \leq l|y|$ for all $y \in \mathbb{R}^r$). In addition, we suppose that

- Y is bounded,
- P has inverse,
- f and Y are Lipschitz continuous with

$$|f(z, y) - f(\widetilde{z}, \widetilde{y})| \leq k_{zz}|z - \widetilde{z}| + k_{zy}|y - \widetilde{y}|$$

$$|Y(z, y) - Y(\widetilde{z}, \widetilde{y})| \leq k_{yz}|z - \widetilde{z}| + k_{yy}|y - \widetilde{y}|$$

  for all $z, \widetilde{z} \in \mathbb{R}^d$, $y, \widetilde{y} \in \mathbb{R}^t$

- hypothesis **H** holds, namely, there is $\rho > 1$ such that

$$\alpha < \rho < \beta,$$

$$k_{zy}k_{yz} < (\beta - \rho)(\rho - \alpha)$$

  where

$$\alpha = k_{zz}$$
$$\beta = \frac{1}{l} - k_{yy}.$$

Theorem 2.3.3 *There is a map $G : (z, g) \in \mathbb{R}^r \times \mathbb{R}^t \longrightarrow G(z, g) \in \mathbb{R}^t$, continuous with respect to z, g, lipschitzian with respect to z, $G(z, g) - g$ bounded, such that*

$$S(g) = \{(z, G(z, g))|z \in \mathbb{R}^r\}.$$

Moreover

$$P(S(g)) = S(L(g)) \text{ and } S(g) = P^{-1}(S(L(g)))$$

and so $S(0)$ is invariant under P.

Theorem 2.3.4 *The map P is topologically conjugate to*

$$\overline{P}(z,y) = (f(z, G(z,0)), Ly).$$

2.4 Degree and stability

In this section we properly study the connection between the index on convex sets and the stability theory. This result is taken from [47].
Let $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the closed positive cone in $\mathbb{R}^2$. Along this section, we consider $P = (P_1, P_2) : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ a map of class $\mathcal{C}^1$ and assume that the axes and $Int\mathbb{R}^2_+$ are positively invariant under P, that is

$$P(Int\mathbb{R}^2_+) \subset Int\mathbb{R}^2_+$$

$$P(\{(x,0) : x \geq 0\}) \subset \{(x,0) : x \geq 0\}$$

$$P(\{(0,y) : y \geq 0\}) \subset \{(0,y) : y \geq 0\}.$$

In this setting we say that a fixed point of P of the type $(0, p_2)$ is a repeller if we can take $\varepsilon > 0$ such that, for all $x_0 = (x,y) \in \mathbb{R}^2_+$ with $\|(x,y) - (0,p_2)\| < \varepsilon$ and $x > 0$, there is $N = N(x,y) > 0$ satisfying that $P_1^N(x,y) > \varepsilon$. Analogously, we say that $(0,p_2)$ is an attractor if we can take $\varepsilon > 0$ such that, for all $(x,y) \in \mathbb{R}^2_+$ with $\|(x,y) - (0,p_2)\| < \varepsilon$, $P^n(x,y) \to (0,p_2)$. Our purpose is to study these behaviors via the index on the convex set $\mathbb{R}^2_+$.

Theorem 2.4.1 *Let $P : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ be as above. Assume that $(0,p_2)$ is a fixed point of P and*

$$0 < |\frac{\partial P_2(0,p_2)}{\partial y}| < 1.$$

Then $(0,p_2)$ is a repeller if and only if $(0,p_2)$ is an isolated fixed point of P and

$$index_{\mathbb{R}^2_+}(P, (0,p_2)) = 0.$$

Remark 2.4.2 *Condition*

$$0 < |\frac{\partial P_2(0,p_2)}{\partial y}|$$

in the previous theorem can be dropped. Nevertheless the proof is much more technical.

Remark 2.4.3 *Although we work with* $(0, p_2)$*, the analogous result can be obtained in working with* $(p_1, 0)$.

Before starting the proof of Theorem 2.4.1 we need some preliminary results. Using that $P_1(0, y) = 0$ and $P_1(x, y) > 0$, $P_2(x, y) > 0$ for all $x, y > 0$, we can define the following extension to the second quadrant

$$\widehat{P}(x,y) = \begin{cases} P(x,y) & \text{if} \quad x \geq 0 \\ s \circ P \circ s(x,y) & \text{if} \quad x \leq 0 \end{cases} \tag{2.1}$$

where s is the symmetry respect to the y-axis. For this extension we note the following properties:

- If for some $U \subset \mathbb{R}^2_+$, $P\mid_U$ is a homeomorphism then $\widehat{P}\mid_{U \cup s(U)}$ is also a homeomorphism where $P\mid_U$ denotes the restriction of P to U. Moreover, in this case, $(\widehat{P})^{-1} = \widehat{P^{-1}}$.
- $s \circ \widehat{P} \circ s = \widehat{P}$.

To compute $index_{\mathbb{R}^2_+}(P, (0, p_2))$, we will use the map

$$\overline{P}(x, y) = P(|x|, |y|).$$

To prove Theorem 2.4.1, the following result will be useful.

Lemma 2.4.4 *Assume the conditions of Theorem 2.4.1.*

1. *If* $\partial_x P_1(0, p_2) > 1$ *then* $index_{\mathbb{R}^2_+}(P, (0, p_2)) = 0$.
2. *If* $\partial_x P_1(0, p_2) < 1$ *then* $index_{\mathbb{R}^2_+}(P, (0, p_2)) = 1$.

Remark 2.4.5 *Notice that* $\partial_x P_1(0, p_2) \geq 0$.

Proof. First we assume that $\partial_x P_1(0, p_2) > 1$. Then it is clear that there exist $\epsilon > 0$ and D a disk centered at $(0, p_2)$ such that $\partial_x P_1(x, y) > 1 + \epsilon$ for all $(x, y) \in D$. From these comments and using that $P_1(0, y) = 0$, we prove that

$$(1 + \epsilon)x < P_1(x, y) \tag{2.2}$$

for all $x > 0$ with $(x, y) \in D$. Next, we define the homotopy

$$H : [0,1] \times \overline{D} \longrightarrow \mathbb{R}^2$$

$$H(t,(x,y)) = t\overline{P}(x,y) + (1-t)\widetilde{P}(x,y)$$

where

$$\widetilde{P}(x,y) = ((1+\epsilon)|x|, P_2(0,y)).$$

First, let us prove that H is an admissible homotopy, i.e. $H(t,(x,y)) \neq (x,y)$ for all $(x,y) \in \partial D$. Indeed, consider $(x_0, y_0) \in \partial D$ with $x_0 \neq 0$. From (2.2), we deduce that

$$(1-t_0)(1+\epsilon)|x_0| + t_0 P_1(|x_0|, y_0) \geq (1+\epsilon)|x_0|$$

holds. For $x_0 = 0$, we check that H has no fixed points in ∂D using that $|\frac{\partial P_2}{\partial y}(0,p_2)| < 1$. Finally by properties 4 and 5 of the degree, we conclude that

$$index_{\mathbb{R}^2_+}(P,(0,p_2)) := index(\overline{P},(0,p_2)) = index(\widetilde{P},(0,p_2)) = 0.$$

To prove the second statement, consider ϵ and D analogous to the previous case and define the map

$$\widetilde{P}(x,y) = ((1-\epsilon)x, P_2(0,y)).$$

Proof of Theorem 2.4.1. Firstly we compute the Jacobian matrix of P at $(0, p_2)$, namely

$$J_P(0,p_2) = \begin{pmatrix} \partial_x P_1(0,p_2) & 0 \\ \partial_x P_2(0,p_2) & \underbrace{\partial_y P_2(0,p_2)}_{\eta} \end{pmatrix}.$$

From this expression we see that the eigenvalues of $J_P(0,p_2)$ are

$$\{\partial_x P_1(0,p_2), \eta\}.$$

By the assumption of Theorem 2.4.1 we know that $0 < |\eta| < 1$. On the other hand, since $P(Int\mathbb{R}^2_+) \subset Int\mathbb{R}^2_+$ we obtain that $\partial_x P_1(0,p_2) \geq 0$. At this moment we can ensure that if $\partial_x P_1(0,p_2) < 1$, $(0,p_2)$ is an attractor and if $\partial_x P_1(0,p_2) > 1$, $(0,p_2)$ is a repeller. From these comments together with the previous lemma we obtain that

$$index_{\mathbb{R}^2_+}(P,(0,p_2)) = \begin{cases} 0 & \text{if } \partial_x P_1(0,p_2) > 1 \\ 1 & \text{if } \partial_x P_1(0,p_2) < 1. \end{cases}$$

The rest of the proof consists of studying the case $\partial_x P_1(0,p_2) = 1$. Indeed, using the expression of the Jacobian matrix, we can check that there exists P^{-1} in a neighbourhood of $(0,p_2)$ and, in that neighbourhood,

$$P^{-1}(x,y) = (g(x,y), \frac{1}{\eta}y + Y(x,y))$$

where $\partial_x g(0,p_2) = 1$, $\partial_y g(0,p_2) = 0$, $\partial_y Y(0,p_2) = 0$ and Y is bounded. From these comments, we can prove that in a neighbourhood of $(0,p_2)$, P^{-1} satisfies the hypothesis **H** considered in the previous section. To check this, we consider the Lipschitz constant on $D_r \cap \mathbb{R}^2_+$, where D_r is the disk centered at $(0,p_2)$ of radius r. It is clear that

$$k_{xx} \longrightarrow 1, k_{xy} \longrightarrow 0, k_{yy} \longrightarrow 0$$

as $r \longrightarrow 0$ while k_{xy} remains bounded. After that, we pick a constant $\rho > 1$ lying between α and β for r small enough. Then, $(\beta-\rho)(\rho-\alpha)$ is a fixed quantity and the product $k_{xy}k_{yx}$ tends to zero.
Now we consider $\widehat{P^{-1}}(x,y) = (\widehat{g}(x,y), \frac{1}{\eta}y + \widehat{Y}(x,y))$ where

$$\widehat{g}(x,y) = \begin{cases} g(x,y) & \text{if } x \geq 0 \\ -g(-x,y) & \text{if } x \leq 0 \end{cases}$$

and $\widehat{Y}(x,y) = Y(|x|,y)$. The map $\widehat{P^{-1}}$ is not necessarily $\mathcal{C}^1$ but it still satisfies hypothesis **H**. Therefore, applying Theorem 2.3.4 (after a translation), we prove that $\widehat{P^{-1}}$ is topologically conjugate to

$$G^{-1}(x,y) = (\widehat{g}(x,h(x)), \frac{1}{\eta}y)$$

where $h :]-\epsilon,\epsilon[\longrightarrow \mathbb{R}$ is a Lipschitz continuous function with $h(0) = p_2$ and $\widetilde{M} = \{(x,h(x)) : x \in]-\epsilon,\epsilon[\}$ is a local invariant manifold associated to $\widehat{P^{-1}}$. We notice that

$\{(x, h(x)) : x \geq 0\}$ is also invariant under P^{-1}. Therefore $\widehat{P}$ is topologically conjugate to

$$G(x, y) = (\widehat{P}_1(x, h(x)), \eta y).$$

It is posible to see that the previous map is the inverse of G^{-1} in a neighbourhood of $(0, p_2)$ from

$$G(\widehat{g}(x, h(x)), \frac{1}{\eta} y) = (\widehat{P_1}(\widehat{g}(x, h(x)), h(\widehat{g}(x, h(x)))), y) =$$
$$= (\widehat{P_1}(\widehat{P^{-1}}(x, h(x))), y) = (x, y)$$

where first we have used that $\widetilde{M} = \{(x, h(x)) : x \in] - \epsilon, \epsilon[\}$ is a local invariant manifold and in the second equality, $(\widehat{P^{-1}})^{-1} = \widehat{P}$. On the other hand, the homeomorphism of conjugation ϕ which is built in the proof Theorem 2.3.4 satisfies that $\phi(\{(x, y) : x = 0\}) \subset \{(x, y) : x = 0\}$ (See page 43 in [31]). Hence using that $s \circ \widehat{P} \circ s = \widehat{P}$, we can take ϕ satisfying that $s \circ \phi = \phi \circ s$. Finally, since

$$\overline{P}(x, y) = \begin{cases} \widehat{P}(x, y) = P(x, y) & \text{if } x \geq 0 \\ \widehat{P} \circ s(x, y) & \text{if } x \leq 0, \end{cases}$$

we see that $\overline{P}$ is topologically conjugate to

$$\widetilde{G}(x, y) = \begin{cases} (P_1(x, h(x)), \eta y) & \text{if } \ x \geq 0 \\ (P_1(-x, h(-x)), \eta y) & \text{if } \ x \leq 0. \end{cases}$$

All these comments enable us to conclude that $(0, p_2)$ is a repeller for P if and only if $x < P_1(x, h(x))$ for $x > 0$. Notice that if $(0, p_2)$ is an isolated fixed point, then either

$$x < P_1(x, h(x))$$

or

$$x > P_1(x, h(x)).$$

In addition, when $x < P_1(x, h(x))$, there is $\delta > 0$ so that for all $(x, y) \in U \cap Int\mathbb{R}^2_+$ there exists $N_0 = N_0(x, y)$ with $P_1^{N_0}(x, y) > \delta$.
To finish the proof, we use the invariance of the index by conjugation (See Remark 14

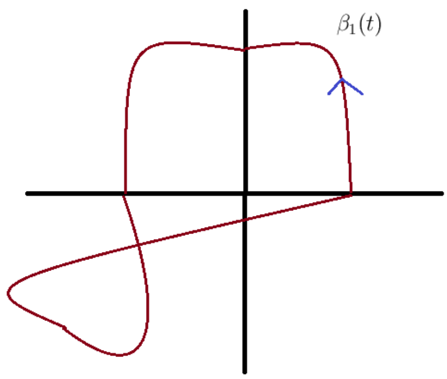

Figure 2.3: Illustration of β_1

[10]) in order to see that

$$index_{\mathbb{R}^2_+}(F,(0,p_2)) := index_{\mathbb{R}^2}(\overline{F},(0,p_2)) = index_{\mathbb{R}^2}(\widetilde{G},(0,0)) = 0.$$

□

Now we apply the previous theorem in a concrete example. Consider

$$P(x,y) = (x\exp(0.5 - x - 4y(y-1)), y\exp(1.5 - 3x - y))$$

A simple study on the axes enables us to deduce that $(0.5, 0)$ is a fixed point in the x-axis satisfying that

$$0 < |\frac{\partial P_1(0.5,0)}{\partial x}| < 1.$$

Therefore, we have just to compute $index_{\mathbb{R}^2_+}(P,(0.5,0))$. Apart from the fixed points on the axes, $(\frac{5}{36},\frac{13}{12})$ is the unique fixed point of our map in $Int(\mathbb{R}^2_+)$. After this remark, we draw the curve $\beta_1(t) = \alpha_1(t) - \overline{P}(\alpha_1(t))$ for $\alpha_1(t) = (0.5 + 0.1\cos(2\pi t), 0.1\sin(2\pi t))$, and $\overline{P}(x,y) = P(|x|,|y|)$, (see figure 2.3) Now we can see that $index_{\mathbb{R}^2_+}(P,(0.5,0)) = 1$ and so $(0.5, 0)$ is not a repeller. In fact, we can deduce from the proof that there is a local attractor. Notice 1 is an eigenvalue of the Jacobian matrix of P at $(0.5, 0)$.

Chapter 3

Translation Arcs

Convergence of all solutions to equilibria is the simplest asymptotic behavior of a dynamical system. In planar flows, Poincaré-Bendixson's theory can be used to derive criteria ensuring this simple behavior. In particular, this dynamics occurs when there are no closed orbits or poly-cycles. In planar discrete-time dynamical systems, however, the situation is more delicate since chaotic behavior can appear. This fact has motivated a broad literature dealing with criteria of global attraction for planar systems using different tools such as the theory of monotone systems, the notion of translation arcs developed by Brouwer, Carathéodory's prime ends just to mention a few different approaches (see, for instance [1], [4], [37], [42] for some significant examples and applications).
The purpose of this chapter is to establish a new criterion of trivial dynamics for planar discrete-time dynamical systems. Roughly speaking, we will say that there is trivial dynamics *if the omega limit set of any bounded orbit (in the future) is a connected set contained in the fixed point set.* After this definition we properly explain our criterion. Indeed, consider

$$p_{n+1} = h(p_n), \tag{3.1}$$

where $h : M \longrightarrow M$ is an orientation preserving embedding (not necessarily onto) and the phase space M is a simply connected two dimensional manifold with boundary ∂M. Assume that every fixed point in the interior of M can be connected with a fixed point on ∂M through an invariant arc. Under these conditions, h has trivial dynamics.
The prototype of manifold M is the first quadrant of $\mathbb{R}^2$, that is

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}.$$

Figure 3.1 illustrates a hypothetical situation where our result can be applied. Throughout this chapter we will study in detail this criterion and its proof. For it, we present the notion of translation arc and prove the classical Brouwer's lemma, (this result is taken from [44]).

3.1 Orientation preserving embeddings

Given $M \subset \mathbb{R}^2$ a simply connected two dimensional manifold with boundary ∂M, a map $h : M \longrightarrow M$ which is continuous and injective is called an embedding. In contrast with a homeomorphism, it is important to see that an embedding is not necessarily onto. In the class of embeddings there are two important groups, namely the preserving orienta-

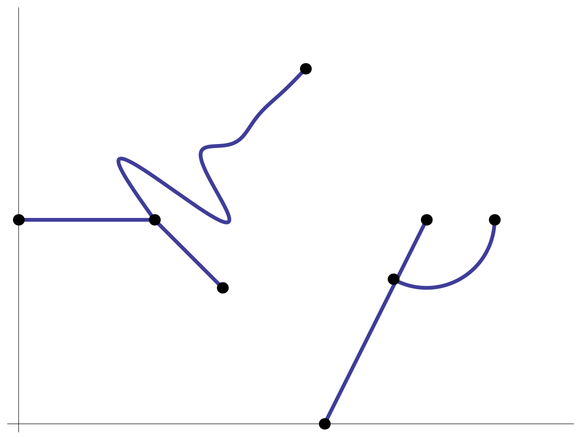

Figure 3.1: Example where we can use our criterion. The blue arcs are invariant under h and the black points represent the fixed points of h.

tion embeddings and the reversing orientation embeddings. Rigorously, an embedding h preserves the orientation if

$$deg(h - q_0, U) = 1 \tag{3.2}$$

where $q_0 = h(p_0)$ and U is any bounded and open neighbourhood of p_0. This condition intuitively says that if we consider Γ a positive parametrization of a Jordan curve around p_0 then $h(\Gamma)$ is other positive parametrization of a Jordan curve with $h(p_0)$ in the interior of the domain limited by $h(\Gamma)$.

In connection with (3.2) we point out that to deduce if a concrete embedding preserves the orientation, it is enough to know the behavior in any open set. More in detail, if g, f are two embeddings, g preserves the orientation and $g = f$ in some non-empty open set then f also preserves the orientation.

Along this chapter we employ the notation $\mathcal{E}(M)$ and $\mathcal{E}_*(M)$ to denote the class of embeddings and orientation preserving embeddings defined on the manifold M.

To finish this introductory section we give some elementary properties of orientation preserving embeddings. For $g, h \in \mathcal{E}_*(M)$ and $\phi : M \longrightarrow M$ a homeomorphism, we have that

- $g \circ h \in \mathcal{E}_*(M)$,

- $\phi^{-1} \circ g \circ \phi \in \mathcal{E}_*(M)$.

3.2 Limit sets and Trivial dynamics

In this section we recall the usual definition of ω limit set and some elementary properties. Given $h \in \mathcal{E}(M)$ and $p \in M$, the usual ω limit set of the point p is defined as

$$\omega(h,p) := \{q \in M : h^{\sigma(n)}(p) = p_{\sigma(n)} \to q \text{ for some } \sigma(n) \to \infty\}.$$

When $\{p_n\}_{n\in\mathbb{N}}$ is bounded, the omega limit set is non-empty, compact and invariant under h. In general, the omega limit set is not necessarily connected. However we have a similar property.

Lemma 3.2.1 *Assume that $\{p_n\}_{n\in\mathbb{N}}$ is bounded and there exist two disjoint compact subsets A and B of $\mathbb{R}^2$ satisfying that*

- $\omega(p,h) = A \cup B$,
- $A \cap B = \emptyset$,
- $h(A) \subset A$.

Then either $A = \emptyset$ or $B = \emptyset$.

Proof. Assume by contradiction that both sets are non-empty and let A_ε and B_ε open neighbourhoods of A and B respectively with $A_\varepsilon \cap B_\varepsilon = \emptyset$. It is easy to prove that

$$p_n \in A_\varepsilon \cup B_\varepsilon \tag{3.3}$$

for large n. The neighbourhoods A_ε and B_ε can be chosen small enough so that if $x \in A_\varepsilon$ then $h(x) \notin B_\varepsilon$. Here one uses that A is positively invariant under h and also that h is uniformly continuous on A_ε. From assumptions, we know that $\{p_n\}_{n\in\mathbb{N}}$ has to enter infinitely many times in A_ε and B_ε. This fact implies the existence of a subsequence $\{p_{\sigma(n)}\}_{n\in\mathbb{N}}$ lying in A_ε and such that $h(p_{\sigma(n)}) \notin A_\varepsilon$. By construction we know that $h(p_{\sigma(n)})$ cannot belong to B_ε and this is incompatible with (3.3). □

After this lemma, we can deduce some deep consequences. For instance, if $\{p_n\}_{n\in\mathbb{N}}$ is a bounded orbit of h and

$$\omega(p,h) \subset Fix(h) \tag{3.4}$$

then $\omega(p,h)$ is a connected set. This fact motivates the definition of trivial dynamics for an embedding h. Specifically, we say that h has trivial dynamics if for every bounded orbit $\{p_n\}_{n\in\mathbb{N}}$, $\omega(p,h)$ is a connected set contained in the fixed point set of h. Observe that by the previous comments, it is enough to check inclusion (3.4).

3.3 Definition of translation arcs and Brouwer's lemma

In this section we give the precise definition of translation arc and the statement of Brouwer's lemma. For the proof of this powerful lemma we need the results of Sub-Sections 3.3.1-3.3.3.

By an oriented arc we understand a subset of the plane which is homeomorphic to a compact interval and such that the end points are ordered. An oriented arc will be denoted as $\alpha = \widehat{pq}$ where p and q are the end points. Given $h \in \mathcal{E}(M)$, $\alpha = \widehat{pq}$ with $\alpha \subset IntM$ is a translation arc if

$$h(\alpha\backslash\{q\}) \cap (\alpha\backslash\{q\}) = \emptyset.$$

As a direct consequence of this definition we deduce that a translation arc does not contain fixed points. Observe that $h(q)$ can belong to $\alpha\backslash\{q\}$.

The notion of translation arcs is important in dynamical systems by the following result.

Theorem 3.3.1 *(**Brouwer's Lemma**) Assume that $h \in \mathcal{E}_*(M)$ and α is a translation arc with*

$$h^n(\alpha) \cap \alpha \neq \emptyset$$

for some $n \geq 2$. Then there exists a Jordan curve $\Gamma \subset IntM\backslash Fix(h)$ such that

$$deg(id - h, R_i(\Gamma)) = 1,$$

where $R_i(\Gamma)$ is the bounded connected component of $\mathbb{R}^2\backslash\Gamma$.

As mentioned before, the proof of this theorem is a consequence of the results developed in the following subsections. This proof is essentially taken from [44].

3.3.1 Compression of translation arcs

Given two maps $h, g \in \mathcal{E}_*(M)$, we say that h and g are strongly equivalent if there exists a topological disk $D \subset IntM$, in the sequel we refer to this disk as disk of modification, such that

$$h(D) \cap D = \emptyset \text{ and } h = g \text{ on } M \backslash D.$$

Observe that this notion ensures that the set of fixed points is the same for both embeddings. Moreover, as a direct consequence of this fact together with the excision property we conclude that

$$deg(id - h, \Omega) = deg(id - g, \Omega) \tag{3.5}$$

for Ω any bounded and open subset of $IntM$ with $\partial\Omega \cap Fix(h) = \emptyset$. The maps $h, g \in \mathcal{E}(M)$ are freely equivalent if there exists a chain $h = h_1, h_2, ..., h_k = g$ in $\mathcal{E}(\mathbb{R}^2)$ such that h_i and h_{i+1} are strongly equivalent for each $i = 1, ..., k-1$. Reasoning in a similar way as above, the notion of freely equivalent preserves the set of fixed points and property (3.5). After these notions, we illustrate how to modify, via free modifications, a translation arc. This kind of construction will be useful in the following section. Along these results, $\alpha = \widehat{pq}$ is an arc and p^* is a point in $\dot{\alpha} = \alpha \backslash \{p, q\}$. This point splits the arc α into two sub-arcs $\alpha' = \widehat{pp^*}$ and $\alpha'' = \widehat{p^*q}$.

Proposition 3.3.2 *Take α a translation arc for some $h \in \mathcal{E}(M)$ and U a neighbourhood of α'. Then there exists $g \in \mathcal{E}(M)$ which is strongly equivalent to h with disk of modification contained in U and such that*

$$g(\alpha'') = h(\alpha), \;\; g(p^*) = q, \;\; g(q) = h(q).$$

It is important to observe that α'' is a translation arc for g. Before giving the proof of this proposition we state an obvious result.

Lemma 3.3.3 *Let D be a disk with $\alpha' \subset int(D)$. Then there exists ϕ a homeomorphism with*

$$\phi(p) = p^*, \; \phi(q) = q, \; \phi(\alpha) = \alpha'', \; \phi = id \text{ outside } D.$$

Proof of Proposition 3.3.2. Since α is a translation arc for h we know that $h(\alpha') \cap \alpha' = \emptyset$. This allows us to find a disk D with $\alpha' \subset int(D)$, $D \subset U$ and $h(D) \cap D = \emptyset$. This disk can be obtained by inflating α'. The previous lemma produces a homeomorphism ϕ contracting α to α''. The embedding $g = h \circ \phi^{-1}$ is strongly equivalent to h with D as a disk of modification and so it is the searched map. $\square$

In the next result we give an analogous construction to Proposition 3.3.2.

Proposition 3.3.4 *Assume that α is a translation arc for $h \in \mathcal{E}(M)$ and $h(q) \notin \alpha$. Given V a neighbourhood of α'' there exists $g \in \mathcal{E}(M)$ strongly equivalent to h , with a disk of modification contained in V and such that*

$$g(\alpha) = h(\alpha'),\ g(p) = q,\ g(q) = h(q^*).$$

Proof. Take $\Delta \subset V$ a disk containing α'' in its interior. Clearly, $h(\Delta) = D$ is also a disk with $h(\alpha'') \subset int(D)$. The arcs α'' and $h(\alpha'')$ are disjoint and so we can select D small enough so that $D \cap \Delta = \emptyset$. We apply Lemma 3.3.3 and find ϕ compressing $h(\alpha)$ onto $h(\alpha')$ and such that $\phi = id$ outside D. The searched map is $g = \phi \circ h$. □

3.3.2 Reduction to periodic points

Assume that $\alpha = \widehat{pq}$ is a translation arc for $h \in \mathcal{E}(M)$. For this arc we define the index $\nu = \nu(\alpha)$ as

- $\nu = 2$ whenever $h(q) \in \alpha$,
- $3 \leq \nu < \infty$ whenever $h(q) \notin \alpha$ and

$$h^k(\alpha) \cap \alpha = \emptyset, \quad \text{if } 2 \leq k < \nu - 1,$$

$$h^\nu(\alpha) \cap \alpha \neq \emptyset$$

- $\nu = \infty$ whenever $h^k(\alpha) \cap \alpha = \emptyset$ for all k.

Proposition 3.3.5 *Assume that α is a translation arc for $h \in \mathcal{E}(M)$ and $\nu < \infty$. Then there exists $g \in \mathcal{E}(M)$, which is freely equivalent to h, a periodic orbit with minimum period ν,*

$$P_0, P_1 = g(P_0), ..., P_\nu = g^\nu(P_0) = P_0,$$

and a translation arc for g denoted by $\beta = \widehat{P_0P_1}$ such that

$$\Gamma = \beta \cup g(\beta) \cup ... \cup g^{\nu-1}(\beta)$$

is a Jordan curve contained in $\alpha \cup h(\alpha) \cup ... \cup h^{\nu-1}(\alpha)$.

We first state a preliminary obvious result.

Lemma 3.3.6 *Assume that $a_0, ..., a_{r-1}$ are $r \geq 3$ different points which are connected by arcs, namely,*

$$\gamma_0 = \widehat{a_0 a_1}, \gamma_1 = \widehat{a_1 a_2}, ..., \gamma_{r-1} = \widehat{a_{r-1} a_0}$$

and these arcs satisfy that

$$\gamma_0 \cap \gamma_1 = \{a_1\}, ..., \gamma_{r-2} \cap \gamma_{r-1} = \{a_{r-1}\}, \gamma_{r-1} \cap \gamma_0 = \{a_0\}$$

with

$$\gamma_j \cap \gamma_k = \emptyset \text{ if } 2 \leq |j-k| < r-1.$$

Then $\Gamma = \gamma_0 \cup \gamma_1 ... \cup \gamma_{r-1}$ is a Jordan curve.

Proof of Proposition 3.3.5. Assume that $\nu = 2$ so that the point $p^* = h(q)$ lies in $\alpha \backslash \{q\}$. If $p^* = p$ the proof is complete. Assume now that $p^* \in \dot{\alpha}$, then $\Gamma = \alpha'' \cup h(\alpha)$ is a Jordan curve with $\alpha'' = \widehat{p^* q}$. We apply Proposition 3.3.2 in order to compress α and α''. In this way, we obtain g, strongly equivalent to h so that g has the periodic orbit $\{p^*, q\}$ and the translation arc $\beta = \alpha''$.
Assume from now on that $3 \leq \nu < \infty$. We define the arcs

$$\alpha_0 = \alpha, \alpha_1 = h(\alpha), ..., \alpha_{\nu-1} = h^{\nu-1}(\alpha).$$

Observe that they satisfy that

$$\alpha_j \cap \alpha_k = \emptyset \text{ if } 2 \leq |j-k| < \nu - 1$$

and

$$\alpha_0 \cap \alpha_1 = \{h(p)\},\ \alpha_1 \cap \alpha_2 = \{h^2(p)\},\ ...,\ \alpha_{\nu-2} \cap \alpha_{\nu-1} = \{h^{\nu-1}(p)\}.$$

For these properties we use that α is a translation arc, h is injective and the definition of ν.
At this moment, we observe that α_0 and $\alpha_{\nu-1}$ can intersect many times. We select p^* as the first point in the arc $\alpha_{\nu-1}$ finding α_0. In principle the point p^* can be anywhere on $\alpha \backslash \{q\}$. In particular, it could coincide with p. The same can be said about the location of p^* with respect to $\alpha_{\nu-1} \backslash \{h^{\nu-1}(p)\}$. From now on we assume that

$$p^* \neq p, h^{\nu}(p).$$

Consider the sub-arc of α from p^* to q,

$$\alpha_0'' = \widehat{p^* q}$$

and the sub-arc of $\alpha_{\nu-1}$ from $h^{\nu-1}(p)$ to p^*,

$$\alpha_{\nu-1}' = \widehat{h^{\nu-1}(p) p^*}.$$

By construction the arcs $\alpha_0'', \alpha_1, ..., \alpha_{\nu-2}, \alpha_{\nu-1}'$ are in the conditions of Lemma 3.3.6 and

$$\Gamma = \alpha_0'' \cup \alpha_1 ... \cup \alpha_{\nu-2} \cup \alpha_{\nu-1}'$$

is a Jordan curve. The successive images of α are also translation arcs for h. In particular $\alpha_{\nu-2}$ is a translation arc with end points $h^{\nu-2}(p)$ and $h^{\nu-2}(q)$. Since $\nu > 2$ we know that $h(q) \notin \alpha$ and this implies that $h(h^{\nu-2}(q)) \notin \alpha_{\nu-2}$. Now we are in a position to apply Proposition 3.3.4 with $\alpha = \alpha_{\nu-2}$. We employ the notation $\alpha_{\nu-2}' = \widehat{h^{\nu-2}(p) r}$, $\alpha_{\nu-2}'' = \widehat{r h^{\nu-1}(p)}$ where r is such that $h(r) = p^*$. In this way we obtain $g_1 \in \mathcal{E}(M)$ strongly equivalent to h and such that $g_1(\alpha_{\nu-2}) = \alpha_{\nu-1}'$. Observe that we can assume that the disk of modification contains $\alpha_{\nu-2}''$ in its interior and does not intersect $\alpha_0 \cup \alpha_1 ... \cup \alpha_{\nu-3}$. The maps h and g_1 coincide in these last arcs and so

$$g_1(\alpha_0) = \alpha_0, ..., g_1(\alpha_{\nu-3}) = \alpha_{\nu-2}.$$

The arc α is also a translation arc for g_1 and we can apply Proposition 3.3.4 to obtain $g \in \mathcal{E}(M)$ strongly equivalent to g_1 and such that

$$g(\alpha_0'') = \alpha_1.$$

The disk of modification is chosen in a neighbourhood of α_0' which does not intersect $\alpha_1 \cup ... \cup \alpha_{\nu-2}$. This leads to

$$g(\alpha_1) = \alpha_2, ..., g(\alpha_{\nu-2}) = \alpha_{\nu-1}'.$$

This completes the proof when

$$p^* \neq p, h^\nu(p).$$

The other cases are completely analogous.

3.3.3 Lemmas on isotopies

The exterior of the unit disk is denoted by

$$\mathbb{E} = \{p \in \mathbb{R}^2 : \|p\| \geq 1\}.$$

The boundary of $\mathbb{E}$ is the unit circle $\partial\mathbb{E} = \mathbb{S}^1$. The class of mappings $h : \mathbb{E} \longrightarrow \mathbb{E}$ which are continuous and injective will be denoted by $\mathcal{E}(\mathbb{E})$. Two maps $h_0, h_1 \in \mathcal{E}(\mathbb{E})$ are isotopic relative to $\partial\mathbb{E}$ if $h_0 = h_1$ on $\partial\mathbb{E}$ and there exists a continuous map

$$H : [0,1] \times \mathbb{E} \longrightarrow \mathbb{E}$$

such that

- $H_0 = h_0$, $H_1 = h_1$,
- $H_1 \in \mathcal{E}(\mathbb{E})$ for each $t \in [0,1]$,
- $H_t(p) = h_0(p) = h_1(p)$ if $p \in \partial\mathbb{E}$ and $t \in [0,1]$.

In a completely analogous way, we can define the notion of isotopy in $\mathcal{E}(U)$ relative to an arc $\alpha \subset U$ for U an open and simply connected set.

Lemma 3.3.7 *Assume that $h \in \mathcal{E}(\mathbb{E})$ and $h = id$ on $\partial\mathbb{E}$. Then h is isotopic to id relative to $\partial\mathbb{E}$.*

Proof. Define

$$H_t(p) = \begin{cases} \frac{1}{t}h(tp) & \text{if } \ t\|p\| \geq 1 \\ p & \text{if } \ t\|p\| \leq 1. \end{cases}$$

The required properties are a consequence of the definition of H. □

This lemma is useful for the following result.

Lemma 3.3.8 *Let α be an arc contained in $IntM$ and $h \in \mathcal{E}_*(M)$ with $h = id$ on α. Given any U open and simply connected set satisfying that $\alpha \subset U$, then $h|_U$ is isotopic to id relative to α.*

Proof. First of all we take ϕ a homeomorphism mapping α onto the segment $\beta = [0,1]\times\{0\}$. It is enough to prove the result for $g = \phi\circ h\circ\phi^{-1}$ and the arc β. Next we cut the segment to form a disk. This means that we are going to distinguish the two sides of β in $U\backslash\beta$. To this end we consider the topological space

$$X = (U\backslash\dot{\beta}) \cup \{(r,+) : r\in\dot{\beta}\} \cup \{(r,-) : r\in\dot{\beta}\}.$$

The points $p_* = (0,0)$, $q_* = (1,0)$ are the points of β. The definition of the topology in X is natural if it must produce the continuity of the projection $\pi : X \longrightarrow U$ and to make X homeomorphic to $\mathbb{E}$. We just notice that a sequence $\{p_n\}$ in $U\backslash\beta$ converges to $(r,+)$ in X if and only if $p_n \longrightarrow r$ and $y_n > 0$ for n large enough, were $p_n = (x_n, y_n)$.
Given $r\in\dot{\beta}$ we consider the disk $D = \{p\in\mathbb{R}^2 : \|p-r\|\le\varepsilon\}$. We can choose ε small enough so that $g(D)$ is contained in the vertical strip $\beta\times\mathbb{R}$. This is just a consequence of the continuity of g at r. Define

$$H^+ = \{(x,y) : y>0\}, \quad H^- = \{(x,y) : y<0\}$$

$$D^+ = D\cap H^+, \quad D^- = D\cap H^-.$$

We claim that one of the possibilities below holds:

- $g(D^+)\subset H^+$, $g(D^-)\subset H^-$
- $g(D^+)\subset H^-$, $g(D^-)\subset H^+$.

To prove this claim we first notice that $g(D^+)$ and $g(D^-)$ cannot intersect the axis $y=0$. This is a consequence of $g(D)\subset\beta\times\mathbb{R}$ and $g = id$ on $\beta\times\{0\}$. The sets $g(D^+)$ and $g(D^-)$ are connected and contained in $H^+\cup H^-$. Each of them must be either in H^+ or in H^-. It remains to prove that both of them cannot lie in the same half-plane. To this end we proceed by contradiction. If $g(D^+)$ and $g(D^-)$ were in the same plane, then r should belong to the boundary of the disk $g(D)$. But g is an open map and so $r\in int(D)$ should imply $r\in g(r)\in g(int(D)) = int(g(D))$. Then r would be simultaneously on the boundary and in the interior of the disk $g(D)$ and this is a contradiction. Once the claim has been proven it is possible to define a continuous and injective map

$$\widehat{g} : X \longrightarrow X$$

satisfying that $\pi \circ \widehat{g} = g \circ \pi$. Specifically, $g = \widehat{g}$ on $U \backslash \dot{\beta}$ and $\widehat{g}(r, \pm) = (r, \pm)$ in the first case, $\widehat{g}(r, \pm) = (r, \mp)$ in the second case.
In the space X we consider the arcs

$$C_{\pm} = \{(r, \pm) : r \in \dot{\beta}\} \cup \{p_*, q_*\}.$$

The Jordan curve $\Gamma = C_+ \cup C_-$ is invariant under $\widehat{g}$. In the first case, all the points in Γ are fixed. Since the pairs $(\mathbb{E}, \mathbb{S}^1)$ and (X, Γ) are homeomorphic, we can apply Lemma 3.3.7 in order to obtain an isotopy $\widehat{H_t} : X \longrightarrow X$ between id and $\widehat{g}$. This isotopy is relative to Γ and so induces an isotopy H_t in U between id and g. This fact completes the proof in the first case. Our next aim is to rule out the second case. Indeed, first we see that $g = \phi \circ h \circ \phi^{-1}$ is an orientation preserving embedding. We notice that the symmetry $S(x, y) = (x, -y)$ induces a homeomorphism of X which will be denoted by $\widehat{S}$ and satisfying that

$$S \circ \pi = \pi \circ \widehat{S}.$$

If g is in the second case, then $\widehat{S} \circ \widehat{g} = id$ on Γ. We apply again Lemma 3.3.7 to $\widehat{S} \circ \widehat{g}$ and find an isotopy $\widehat{H}_t : X \longrightarrow X$ between id and $\widehat{S} \circ \widehat{g}$ relative to Γ. It induces an isotopy $H_t : U \longrightarrow \mathbb{R}^2$ between id and $S \circ g$ which is relative to β. Given any open and bounded neighbourhood V of the origin we observe that $deg(S \circ H_t, V)$ is independent of t. This is a contradiction because

$$deg(g, V) = -1$$

and on the other hand g is an orientation preserving embedding. □

3.3.4 Proof of Brouwer's lemma

As we will see, the proof of Brouwer's lemma is a direct consequence of the following result.

Proposition 3.3.9 *Assume that $h \in \mathcal{E}_*(M)$ and γ is an arc in $IntM$ such that*

- *$Fix(h) \cap \gamma = \emptyset$,*
- *$\Gamma = \gamma \cup h(\gamma)$ is a Jordan curve.*

Then $Fix(h) \cap \Gamma = \emptyset$ and

$$deg(id - h, R_i(\Gamma)) = 1$$

Notice that by using that M is simply connected we deduce that $R_i(\Gamma) \subset IntM$.
Before giving the proof of this proposition we prove Brouwer's lemma.
Proof of Theorem 3.3.1. Take $h \in \mathcal{E}_*(M)$ and α a translation arc with $h^n(\alpha) \cap \alpha \neq \emptyset$ for some $n \geq 2$. This means that the index of this arc is finite, say $\nu \geq 2$. We apply Proposition 3.3.5 and find $g \in \mathcal{E}_*(M)$, freely equivalent to h, and a translation arc $\widehat{P_0P_1}$ such that $\Gamma = \beta \cup g(\beta) \cup ... \cup g^{\nu-1}(\beta)$ is a Jordan curve contained in $\alpha \cup h(\alpha) ... \cup h^{\nu-1}(\alpha)$. Next we apply Proposition 3.3.9 to the map g and the curve $\gamma = \beta \cup g(\beta) ... \cup g^{\nu-2}(\beta)$. We observe that $Fix(g) \cap \gamma = \emptyset$ because γ is a translation arc. Also $\gamma \cup g(\gamma) = \Gamma$ is a Jordan curve and so we can conclude that $deg(id - g, R_i(\Gamma)) = 1$. The property of equivalent embedding allows us to conclude the proof. $\square$

Next we focus our attention on the proof of Proposition 3.3.9. With this purpose in our mind, we give the following two lemmas.

Lemma 3.3.10 *Assume that $h \in \mathcal{E}(M)$ and $\Gamma \subset IntM$ is an invariant curve without fixed points. Then*

$$deg(id - h, R_i(\Gamma)) = 1.$$

Proof. The disk $\Gamma \cup R_i(\Gamma)$ is transformed onto the unit disk $\mathbb{D}$ via some ϕ homeomorphism in $\mathbb{R}^2$. The embedding $\widehat{h} = \phi \circ h \circ \phi^{-1}$ has the unit circle as an invariant curve and $Fix(\widehat{h}) \cap \partial\mathbb{D} = \emptyset$. We define the homotopy

$$H(p, t) = p - t\widehat{h}(p).$$

Observe that for all $t \in [0, 1]$, $H(t, p) \neq 0$ for all $\|p\| = 1$. The conclusion of this lemma is now clear. $\square$

Lemma 3.3.11 *Assume that $h \in \mathcal{E}(M)$ has an invariant curve contained in $IntM$ without fixed points. Then h preserves the orientation.*

The proof is a bit technical and does not provide us with any idea and so we omit it.
Proof of Proposition 3.3.9. Clearly we know that h has no fixed points in Γ. Therefore we can construct a homeomorphism

$$k : \Gamma \longrightarrow \Gamma$$

without fixed points and such that $k = h$ on γ. Next we extend k to a homeomorphism ϕ defined on $\mathbb{R}^2$. Clearly ϕ preserves the orientation and so $h \circ \phi^{-1}$ as well. The map $h \circ \phi^{-1}$ is the identity on γ. Therefore, we can apply Lemma 3.3.8 and find an isotopy $\{G_t\}$ with $G_0 = h \circ \phi^{-1}$, $G_1 = id$ and $G_t = id$ on γ. The family $H_t = G_t \circ \phi$ defines an isotopy between h and ϕ without fixed points in Γ. In particular,

$$deg(id - h, R_i(\Gamma)) = deg(id - \phi, R_i(\Gamma))$$

and so by Lemma 3.3.11 we deduce that the previous degree is 1. $\square$

3.4 Construction of translation arcs

The goal of this section will be to derive criteria ensuring the presence of translation arcs. With this respect, we have the following result.

Proposition 3.4.1 *Assume that $h \in \mathcal{E}(M)$ and $D \subset IntM$ is a topological disk with D and $h(D)$ lying on the same connected component of $M \backslash Fix(h)$. In addition, assume that*

$$h(D) \cap D = \emptyset.$$

Then, given points $w_1, ..., w_n \in D$, there exists a translation arc $\alpha = \widehat{pq}$ with

$$w_1, ..., w_n \in \dot{\alpha} = \alpha \backslash \{p, q\}.$$

There are some remarks to be made concerning the previous proposition.

- D and $h(D)$ always lie on the same component of $M \backslash Fix(h)$ if h preserves the orientation.

- In general, we cannot fix the end of the arc α.

The rest of the section is devoted to the proof of Proposition 3.4.1. For it we need the following definitions: a family of topological disks $\{D_t\}_{t \in [0,1]}$ will be admissible if

$$D_t = \bigcap_{s>t} D_s \quad \text{for all } t \in [0,1[\ \text{ and } \ int(D_t) = \bigcup_{s<t} D_s \ \text{ for } t \in]0,1].$$

Admissible families are monotone in the sense that

$$D_s \subset int(D_t)$$

if $s < t$. We also observe that these families are preserved by homeomorphisms. This means that $\{\phi(D_t)\}$ is another admissible family if ϕ is any homeomorphism. Next we give two results about admissible families.

Lemma 3.4.2 *Let G be an open and connected subset of $\mathbb{R}^2$ containing the disjoint disks D and Δ. Then there exists an admissible family of disks $\{D_t\}$ satisfying that*

$$D_t \subset G \; for \; each \; t \in [0,1], D_0 = D, D_1 \cap \Delta \neq \emptyset.$$

Proof. In the proof we use the following result: *given a disk D and an arc $\alpha = \widehat{pq}$ with $p \in \partial D$ and $(\alpha \backslash \{p\}) \cap D = \emptyset$, then there exists a homeomorphism ϕ such that*

$$\phi(\mathbb{D}) = D, \; \phi(\mathbb{A}) = \alpha, \; \phi(1,0) = p, \phi(2,0) = q,$$

where $\mathbb{D}$ is the Euclidean unit disk and $\mathbb{A} = [1,2] \times \{0\}$.
See [44] for a proof. Once this result is accepted, we observe that G is arc-wise connected and so we can find an arc $\beta = \widehat{rs}$ lying in G and such that $r \in D$, $s \in \Delta$. The arc β is parameterized as $\beta = \beta(t)$, $t \in [0,1]$. After that consider

$$\tau := \sup\{t \in [0,1] : \beta(t) \in D\}, \sigma := \{t \in [\tau, 1] : \beta(t) \in \Delta\}.$$

If we define $p = \beta(\tau)$, $q = \beta(\sigma)$ then it is easy to prove that the sub-arc $\alpha = \widehat{pq}$ satisfies $p \in \partial D$, $q \in \partial \Delta$ and $\dot{\alpha} \subset G \backslash (\Delta \cup D)$. Let ϕ be the homeomorphism mapping $\mathbb{D} \cap \mathbb{A}$ onto $D \cap \alpha$ as indicated above. Given a continuous and strictly increasing function $m : [0,1] \longrightarrow \mathbb{R}$ with $m(0) = 0$, we consider the families of disks

$$E_t = \{z \in \mathbb{R}^2 : \|z\| \leq 1 + m(t)\} \cup ([0, 1+t] \times [-m(t), m(t)]), \;\; D_t = \phi(E_t).$$

By construction $D_0 = D$ and q is a point in $D_1 \cap \Delta$.

Lemma 3.4.3 *Assume that $\{D_t\}$ and $\{\Delta_t\}$ are admissible families of disks with*

$$D_0 \cap \Delta_0 = \emptyset, \;\; D_1 \cap \Delta_1 \neq \emptyset.$$

Then, for some $\tau \in]0,1]$,

$$int(D_\tau) \cap int(\Delta_0) = \emptyset, \;\; \partial D_\tau \cap \partial \Delta_\tau \neq \emptyset.$$

Proof. Define

$$\tau = \sup\{t \geq 0 : D_t \cap \Delta_t = \emptyset\}$$

We first prove that $D_\tau \cap \Delta_\tau \neq \emptyset$. This is clear from the assumption if $\tau = 1$. If $\tau < 1$, we use the first condition for admissibility to deduce that

$$D_\tau \cap \Delta_\tau = \bigcap_{s>t} (D_s \cap \Delta_s).$$

Cantor's lemma on decreasing sequences of compact sets can be applied to conclude that $D_\tau \cap \Delta_\tau$ is non-empty. In particular, $\tau > 0$. Both families of disks are increasing and so the second condition for admissibility leads to

$$int(D_\tau) \cap int(\Delta_\tau) = \bigcup_{s<\tau} (D_s \cap \Delta_s).$$

The definition of the number τ now implies that $int(D_\tau) \cap int(\Delta_\tau) = \emptyset$. As D_τ and Δ_τ are disks we can conclude that $\partial D_\tau \cap \partial \Delta_\tau \neq \emptyset$. □

Proof of Proposition 3.4.1. We enlarge the disk D inside $\mathbb{R}^2 \backslash Fix(h)$ until it touches its image. This means that we can apply Lemma 3.4.2 with $\Delta = h(D)$ and G the connected component of $\mathbb{R}^2 \backslash Fix(h)$ containing D and $h(D)$. The families of disks $\{D_t\}$ and $\{h(D_t)\}$ are admissible and we can find $\tau > 0$ according to Lemma 3.4.3 and $\Delta_t = h(D_t)$. Now we take w and z in ∂D_τ and such that $h(w) = z$. Therefore, $z \in \partial h(D_\tau)$. After that we select an arc contained in D_0 with ends at z, w and joining $w_1, ..., w_n$. Observe that

$$(\alpha \backslash \{z\}) \cap h(\alpha \backslash \{z\}) \subset [int(D_\tau) \cup \{w\}] \cap [int(h(D_\tau)) \cup \{z\}] = \emptyset.$$

This condition says that α is a translation arc. □

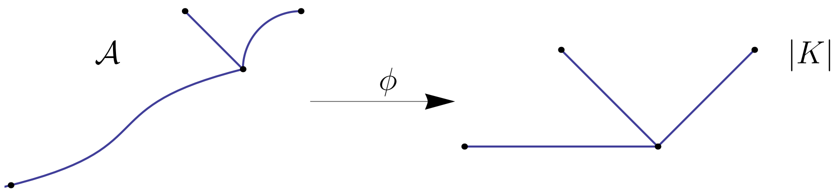

Figure 3.2: Example of topological linear graph.

3.5 Embeddings and Topological Linear Graphs

Given two different points $p, q \in \mathbb{R}^2$, we can define the 1-simplex with vertices at p, q as

$$\{tp + (1-t)q : t \in [0,1]\}.$$

A point $\{p\}$ will be a 0-simplex. A **linear graph** is a finite collection K of 0 or 1 simplices in $\mathbb{R}^2$ with the following properties:

- K contains all vertices of all 1 simplices of K.
- If $\sigma, \tau \in K$ are two different 1 simplices with $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a vertex of both of them.

Given K a linear graph we will say that the dimension of K is 1 if K contains some 1-simplex and is 0 otherwise. A triplet $(\mathcal{A}, K, \phi)$ is a **topological linear graph** if $\mathcal{A} \subset \mathbb{R}^2$, K is a linear graph and $\phi : \mathcal{A} \longrightarrow |K|$ is a homeomorphism where $|K|$ denotes the union of the simplices of K. In such a case we define the topological 1 or 0 simplices of $(\mathcal{A}, K, \phi)$ in a natural way. See figure 3.2 for a pictorial definition of topological linear graph. The concept of topological linear graph allows us to introduce a natural notion of invariance for these sets. Specifically, given $g : \Xi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a continuous map, we will say that a topological linear graph $(\mathcal{A}, K, \phi)$ is **graph invariant** under g if $\mathcal{A} \subset \Xi$ and every topological simplex of $(\mathcal{A}, K, \phi)$ is invariant under g. Clearly, if $(\mathcal{A}, K, \phi)$ is graph invariant then $\mathcal{A}$ is invariant. Next we give an example to illustrate that the converse is false. Indeed, consider $g : [0,1] \longrightarrow [0,1]$ a continuous map satisfying that $Fix(g) = \{0, \frac{1}{2}, 1\}$, $g(\frac{1}{4}) = \frac{3}{4}$ and $g(\frac{3}{4}) = \frac{1}{4}$. In this case $\mathcal{A} = [0,1]$ is invariant but $(\mathcal{A}, K, \phi)$ with $K = \{[0, \frac{1}{2}], [\frac{1}{2}, 1], \{0\}, \{1\}, \{\frac{1}{2}\}\}$ and $\phi(x) = x$ is not graph invariant since the topological 1-simplices are not invariant. In the following result we study the notion of graph invariance when g is one-to-one. Notice that this last property does not hold in the previous example.

Lemma 3.5.1 *Assume that $g : \Xi \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is continuous, one-to-one and $(\mathcal{A}, K, \phi)$ is a topological linear graph with $\mathcal{A} \subset \Xi$. Then $(\mathcal{A}, K, \phi)$ is graph invariant if and only if the set of vertices of $(\mathcal{A}, K, \phi)$ are fixed points of g and $\mathcal{A}$ is invariant.*

Proof. Firstly we observe that by definition of topological linear graph, $(\mathcal{A}, K, \phi)$ does not have two different 1-simplices with the same vertices. This fact enables us to conclude that if $\mathcal{A}$ is invariant and $V \subset Fix(g)$ then the connected components of $\mathcal{A} \backslash V$ are invariant under g where V denotes the set of vertices of $(\mathcal{A}, K, \phi)$. The proof follows from these comments. $\square$

Let $M \subset \mathbb{R}^2$ be a simply connected two dimensional manifold with boundary, and consider map $H : M \longrightarrow M$

$$p_{n+1} = H(p_n). \tag{3.6}$$

An equilibrium $p = (p_1, p_2) \in \mathbb{R}^2$ of (3.6) is a global attractor in a set $D \subset \mathbb{R}^2$ if $\lim_{N \to \infty} H^N(z) = p$ for all $z \in D$.
Now we give the main result of this section.

Theorem 3.5.2 *Let $M \subset \mathbb{R}^2$ be a simply connected two dimensional manifold with boundary and consider $H : M \longrightarrow M$ so that $H \in \mathcal{E}_*(M)$. Moreover we assume that there exists a family of connected and disjoint topological linear graphs $(\mathcal{A}_1, K_1, \phi_1)$,...,$(\mathcal{A}_n, K_n, \phi_n)$ with $\mathcal{A}_1, ..., \mathcal{A}_n \subset M$ and satisfying the following properties:*

- *$Int(M) \backslash (\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ is connected and $Fix(H) \cap Int(M) \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_n$.*
- *For all $i = 1, ..., n$, $\mathcal{A}_i \cap \partial M$ is a non empty subset of $Fix(H)$.*
- *For all $i = 1, ..., n$, $(\mathcal{A}_i, K_i, \phi_i)$ is graph invariant under H.*

Then H has trivial dynamics.

Under the conditions of the previous theorem, clearly if the fixed point set is totally disconnected then for each $z \in M$ with $\{H^n(z) : n \in \mathbb{N}\}$ bounded,

$$\omega(z, H) = \{p\} \subset Fix(H). \tag{3.7}$$

Our following aim will be to guarantee (3.7) in a more general setting.

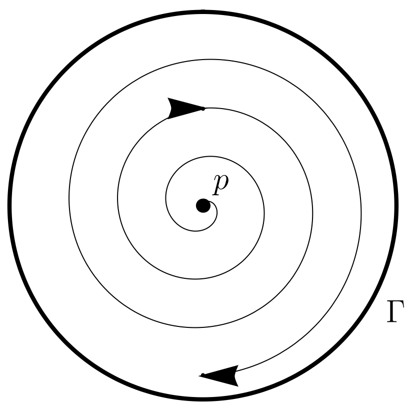

Figure 3.3: Dynamics of Ψ.

Theorem 3.5.3 *Let M, H and $(\mathcal{A}_1, K_1, \phi_1), ..., (\mathcal{A}_n, K_n, \phi_n)$ be as in Theorem 3.5.2. Assume that $H \in \mathcal{C}^1(M)$ and every non isolated fixed point p is partially hyperbolic i.e. $H'(p)$ has an eigenvalue with modulus different from 1. Then for each $z \in M$, $\omega(z, H)$ is a unique fixed point of H, (depending on z).*

There are some remarks to be made concerning the previous theorems. We say that H is of class $\mathcal{C}^1$ if there is an open set $U \supset M$ and an extension of H defined on U, namely $\widetilde{H}$, such that $\widetilde{H}$ is of class $\mathcal{C}^1$ in U. The previous results are not true in higher dimensions, (see Example 3 in [5]). The condition of manifold with boundary is essential for the validity of the theorems. For instance, if we replace M, $Int(M)$, ∂M in the previous theorems by $\overline{\Omega}$, Ω, $\partial\Omega$ with Ω an open and simply connected set, the previous results are false. Indeed, consider any continuous flow in the plane Ψ with the dynamics illustrated in figure 3.3, (Γ is a limit cycle and p is an equilibrium). Define $\Omega = Int(\widetilde{D})\backslash(\{\Psi(t;q) : t \in \mathbb{R}\} \cup \{p\})$ where $\widetilde{D}$ is the topological disk limited by Γ and q is any point in $Int(\widetilde{D})\backslash\{p\}$. Observe that $\partial\Omega = \{\Psi(t;q) : t \in \mathbb{R}\} \cup \{p\} \cup \Gamma$ and $\overline{\Omega} = \widetilde{D}$. Next take two points r, s such that

- $r \in \Omega$,
- $s \in \Gamma$,
- $\Psi(\sigma(n), r) \longrightarrow s$ for some strictly increasing sequence $\{\sigma(n)\}_{\mathbb{N}} \subset \mathbb{N}$,
- $\Psi(\frac{1}{n_0}, s) \neq s$ for some $n_0 \in \mathbb{N}$.

Finally, we consider $H = \Psi(\frac{1}{n_0}, \cdot)$ and the topological graph $\mathcal{A}_1 = (\{p\}, \{p\}, id)$. Clearly, the conditions of Theorem 3.5.2 hold, and $\omega(r, H) \not\subset Fix(H)$.
After this example we study the condition used in Theorem 3.5.2. As mentioned before, a possible setting where we can apply the previous theorem is illustrated in Figure 3.1. Another interesting situation appears when $\emptyset \neq Fix(H) \subset \partial\mathcal{D}$. In this case we pick p a fixed point on the boundary of $\mathcal{D}$ and apply our results with the topological linear graph $(\mathcal{A}, K, \phi)$ where $\mathcal{A} = K = \{p\}$ and $\phi = id$. It is important to see that if $Fix(H) = \emptyset$ then we can directly deduce that there is trivial dynamics since in this case, all the orbits are unbounded (and so for all $z \in M$, $\omega(z, H) = \emptyset$). This fact for homeomorphisms can be found in [4] and for embeddings in [42, 44]. Next we collect these comments in the next result.

Corollary 3.5.4 *Let $M \subset \mathbb{R}^2$ be a simply connected two dimensional manifold with boundary and consider $H : M \longrightarrow M$ with $H \in \mathcal{E}_*(M)$ and $Fix(H) \subset \partial M$. Under these conditions, H has trivial dynamics.*

The previous result for homeomorphisms and M a topological disk (i.e. a simply connected and compact two dimensional manifold with boundary) was obtained in [5]. Notice that in [5], these two conditions are used in the proofs.
To finish this section we present the following result of trivial dynamics.

Theorem 3.5.5 *Suppose that $H \in \mathcal{E}_*(\mathbb{R}^2)$ and there exist disjoint sets $\gamma_1, ..., \gamma_n \subset \mathbb{R}^2$ with the following properties:*

- *for all $i = 1, ..., n$, $H(\gamma_i) \subset \gamma_i$,*
- *for all $i = 1, ..., n$, $\gamma_i = \Phi_i([0, +\infty[)$ where $\Phi_i : [0, +\infty[\longrightarrow \Phi_i([0, +\infty[) \subset \mathbb{R}^2$ is a homeomorphism with $\Phi_i(0) \in Fix(H)$ and $\lim_{t\to\infty} |\Phi_i(t)| = \infty$.*

Then, H has trivial dynamics.

3.6 Proofs

This section is devoted to prove the previous theorems.
Proof of Theorem 3.5.2. Take $z \in M$ so that $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. Firstly we prove that $\omega(z, H)$ is contained in $Fix(H)$. We distinguish three different situations:

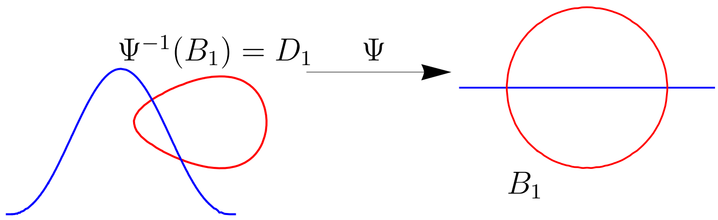

Figure 3.4: Illustration of Theorem 8 in [41].

- $z \in Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$. Assume, by contradiction, that there is $p \in \omega(z, H)$ such that $p \notin Fix(H)$. Under this condition we can take a topological disk D_1 satisfying that

  - $p \in Int(D_1)$,
  - $D_1 \cap H(D_1 \cap M) = \emptyset$,
  - $D_1 \cap (M\backslash\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ has a finite number of connected components.

  This fact is clear if $p \in \partial M$ by using the notion of manifold with boundary. In the case $p \in Int(M)\backslash\mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, the existence of D_1 is clear. Finally for the case $p \in \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, firstly we observe that p belongs to the interior of a 1-simplex since the vertices of $(\mathcal{A}_1, K_i, \phi_i)$ are fixed points of H. Next we apply Theorem 8 in [41] in order to obtain a homeomorphism $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ so that $\Psi(\mathcal{A}_i) = |K_i|$. After that we define $\Psi^{-1}(B_1) = D_1$ where B_1 is a ball centered at $\Psi(p)$ satisfying that $B_1\backslash\Psi(\mathcal{A}_i)$ has exactly two connected components. We illustrate the previous argument with the figure 3.4. Notice that this argument is genuinely two dimensional since in three dimensions we can have wild arcs, (see Section 4 in [41]).

  After this discussion, by using that $p \in \omega(z, H)$, $z \in Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ and $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ is positively invariant, we can take a connected component K_1 of $D_1\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ so that $H^{n_1}(z), H^{n_2}(z) \in Int(K_1)$ with $n_2 > n_1$. Here we have used that the number of connected components is finite. At this moment we consider a closed topological disk $\widetilde{D_1} \subset K_1$ such that $H^{n_1}(z), H^{n_2}(z) \in \widetilde{D_1}$. The construction of $\widetilde{D_1}$ is as follows. Using that $Int(K_1)$ is arcwise connected we can take an arc β joining $H^{n_1}(z), H^{n_2}(z)$ such that $\beta \subset Int(K_1)$. Finally we inflate β without getting out from $Int(K_1)$. Once this reasoning has been done, we apply Lemma 3.4.1 to $\widetilde{D_1}$, $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ and H, in order to conclude that there exists $\alpha \subset$

$Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$ a translation arc passing through $H^{n_1}(z)$ and $H^{n_2}(z)$, (it is important to realize that, by standard topological arguments, $Int(M)\backslash(\mathcal{A}_1\cup...\cup\mathcal{A}_n)$ is simply connected). This fact is a contradiction. Indeed we know in advance that H does not have any fixed point in $Int(M)\backslash(\mathcal{A}_1 \cup ... \cup \mathcal{A}_n)$. On the other hand, $H^{n_2-n_1}(\alpha)\cap\alpha \neq \emptyset$ and by Lemma 3.3.1, H has a fixed point in $Int(M)\backslash(\mathcal{A}_1\cup...\cup\mathcal{A}_n)$. This contradiction implies that for all $z \in Int(M)\backslash(\mathcal{A}_1\cup...\cup\mathcal{A}_n)$, $\omega(z,H) \subset Fix(H)$.

- $z \in \partial M$. Assume by contradiction that $\omega(z,H) \not\subset Fix(H)$. Under this condition, we see that $z \notin Fix(H)$ and so $z \notin \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$, (see second condition of the theorem). Consequently we realize that $H^N(z) \notin \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$ since $\mathcal{A}_1 \cup ... \cup \mathcal{A}_n$ is invariant. After this discussion, clearly, if for some $n \in \mathbb{N}$, $H^n(z) \in Int(M)$, the conclusion is clear by the previous reasoning. Therefore we have to study the case when $H^n(z) \in \partial M$ for all $n \in \mathbb{N}$. Indeed, take $p \in \omega(z,H)$ and assume that $H(p) \neq p$. In this setting, we can take a topological disk D_1 such that $D_1 \cap H(D_1 \cap M) = \emptyset$, $D_1 \cap Int(M)$ is simply connected and $p \in Int(D_1)$. Clearly, using that $p \in \omega(z,H)$, there exist $q \in Int(D_1) \cap Int(M)$ and n_0 with $H^{n_0}(q) \in Int(D_1) \cap Int(M)$. We reason as above (from the construction of $\widetilde{D_1}$) in order to obtain a contradiction.

- Assume now that $z \in \mathcal{A}_i$. In this case we have that $\omega(z,H) \subset Fix(H)$ by using the definition of graph invariant together with some elementary notions of dynamics in $\mathbb{R}$.

To finish the proof of this theorem we use lemma 3.2.1 and (3.4) ensuring that the omega limit set is connected provided $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. $\square$

Proof of Theorem 3.5.3. The proof of this theorem is a direct consequence of the previous proof together with Proposition 3 in [6]. $\square$

Proof of Theorem 3.5.5. Again, take a point $z \in \mathbb{R}^2$ such that $\{H^n(z) : n \in \mathbb{N}\}$ is bounded. Assume by contradiction that there is $q \in \mathbb{R}^2\backslash Fix(H)$ so that $q \in \omega(z,H)$. In this situation we can take a topological disk D satisfying that $q \in D$ and $D\backslash(\gamma_1 \cup ... \cup \gamma_n)$ has at most two connected components. Indeed, if $q \notin \gamma_1 \cup ... \cup \gamma_n$ the construction is clear. Otherwise the construction of D is as follows. Assume that $q \in \gamma_j$. Consequently there exists $t_0 > 0$ with $\Phi_j(t_0) = q$. By Theorem 8 in [41] we can take $\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ homeomorphism so that

$$\Psi(\{\Phi_j(t) : t \in [0, 2t_0]\}) = \{(x, 0) : x \in [0, T]\}.$$

It is clear that $\lim_{t\to\infty} |\Psi \circ \Phi_j(t)| = \infty$ (Ψ is a homeomorphism) and therefore, we can take $\delta > 0$ such that

$$B = B(\Psi(q), \delta) \not\ni \Psi(\Phi_j(t)) \quad \text{for all } t \geq 2t_0$$

and $B(\Psi(q), \delta) \backslash \Psi(\{\Phi_j(t) : t \in [0, 2t_0]\})$ has two connected components. Finally consider $D = \Psi^{-1}(B)$. The remainder of the proof is the same as the proof of Theorem 3.5.2. $\square$

Chapter 4

Stability, Permanence, Dominance and Exclusion

4.1 Introduction to Kolmogorov systems: Some classical examples

Mathematical formulation of discrete models of population dynamics leads to consider difference equations or matrix systems, which are nonlinear if changes in response to population density are taken into account. In this thesis, we focus our attention on discrete time Kolmogorov systems for the interaction of n-species sharing the same environment. More in detail, consider the systems of the type

$$x_i(N+1) = x_i(N)f_i(x_1(N), ..., x_n(N)), \tag{4.1}$$

where $f_i : \mathbb{R}^n_+ := \{(x_1, ..., x_n) : x_i \geq 0\} \longrightarrow]0, +\infty[$ is a continuous function. In (4.1), $x_i(N)$ is the time-varying population density of the i-th species. The function f_i is the so-called growth rate of the i-th species and represents the dependence of the density of the different species on the population of the i-th species. Notice that f_i only takes strictly positive values. This fact implies that there is no extinction at finite time. This property is typical in Kolmogorov systems but is not shared in many biological situations. For instance, to study the interaction of host-parasitoid type, a classical model due to May is

$$\begin{cases} H_{n+1} = H_n \exp(-\lambda H_n)\exp(-aP_n) \\ P_{n+1} = bH_n(1 - \exp(-aP_n)) \end{cases} \tag{4.2}$$

Clearly, in the absence of the host, $(H = 0)$, the parasitoid directly goes to extinction.
A classical model of Kolmogorov type is due to Leslie-Gower and Berveton-Holt

$$x_i(N+1) = \frac{A_i x_i(N)}{1 + B_{i1}x_1(N) + \cdots + B_{in}x_n(N)}, \tag{4.3}$$

In this system we study the evolution of n-competing species sharing the same habitat. The parameter $A_i > 0$ is the intrinsic rate of growth of the i-th species and $B_{ij} > 0$ is the competition rate of the j-th species on the i-th species.
Another important model is due to May-Oster and has the form

$$x_i(N+1) = x_i(N)\exp(A_i + B_{i1}x_1(N) + \cdots + B_{in}x_n(N)). \tag{4.4}$$

The interpretation of the parameters in (4.3) is completely analogous to (4.4).
In some real and mathematical situations, some authors have mixed growth rates of May-Oster and Leslie-Gower types in the same model. In general terms, these systems are much more complicated to study than the previous ones (see [30], [2], [14]).

4.2 Trichotomy for orientation preserving embeddings

As a direct consequence of Theorem 3.5.2 we can obtain a version of Theorem 5.2 in [53] for the class of orientation-preserving embeddings. Specifically, we have.

Theorem 4.2.1 *Let* $J = [0, a] \times [0, b] \subset \mathbb{R}^2$ *with* $0 < a, b$ *or* $J = \mathbb{R}^2_+$ *be and let* $P : J \longrightarrow J$ *be a continuous map with the following properties:*

A1 $P(0) = 0$ *and 0 is a repeller, that is, there is* $\delta > 0$ *such that for all* $\xi \in J \backslash \{0\}$*, there exists* $N := N(\xi) > 0$ *so that* $\|P^n(\xi)\| > \delta$ *for all* $n > N(\xi)$*.*

A2 $Fix(P) \cap \partial J = \{0, \widehat{u}e_1, \widehat{v}e_2\}$ *with* $0 < \widehat{u} < a, 0 < \widehat{v} < b$*.*

A3 $P \in \mathcal{E}_*(J)$ *and for all* $z \in J$*,* $\{P^n(z) : n \in \mathbb{N}\}$ *is bounded.*

Then given $z_0 \in Int(J)$*, one of the following holds:*

1. *There exists a fixed point* E_* *of* P *in* $Int(J)$*.*

2. $P^n(z_0) \longrightarrow \widehat{u}e_1$*.*

3. $P^n(z_0) \longrightarrow \widehat{v}e_2$*.*

Proof. Assume that $Fix(P) \cap Int(J) = \emptyset$, otherwise the proof is complete. After that apply Corollary 3.5.4 with $M = J$ and $H = P$ in order to deduce that P has trivial dynamics. Observe that, by **A3**, the notion of trivial dynamics says that the omega limit set of any orbit is a connected set of $Fix(P)$. □

The previous theorem for competitive systems in the framework of Banach spaces can be deduced using [53] (also see [11] and [17]). The advantage of our result is that we do not need any property of monotony. However it must be noted that our proof only works in the Euclidean plane.

4.3 Partially competitive maps

Consider the system of difference equations

$$\begin{cases} x_{n+1} = x_n f_1(x_n, y_n) \\ y_{n+1} = y_n f_2(y_n, x_n) \end{cases} \tag{4.5}$$

where f_1, f_2 are strictly positive functions and the map $F = (F_1, F_2) : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+^2$ defined by the right-hand side of (4.5) is of class $\mathcal{C}^1$.
The aim of this section will be to derive a criterion of global attraction for a fixed point lying in $Int(\mathbb{R}_+^2)$. In this direction we can find interesting results developed by Smith in [53]. Namely if we assume that

i) $\det F'(x, y) > 0$ for all $(x, y) \in \mathbb{R}_+^2$,

ii) $F'(x, y)$ is a competitive matrix for all $(x, y) \in Int(\mathbb{R}_+^2)$, (by a competitive matrix we understand a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$),

iii) for all $z \in \mathbb{R}_+^2$, $\{F^N(z) : N \in \mathbb{N}\}$ is bounded,

then system (4.5) has trivial dynamics, (see Proposition 2.1, Theorem 4.2 and Lemma 4.3 in [53]). Next we prove that condition **ii)** can be refined. Specifically, it is enough to impose **ii)** in a smaller set.

Theorem 4.3.1 *Assume that F satisfies **i)**, **iii)**, and the following conditions:*

- *$Fix(F) \cap Int(\mathbb{R}_+^2) = p$,*
- *$F'(x, y)$ is a competitive matrix for all $(x, y) \in C = \{(z_1, z_2) : z_1 \leq p_1, z_2 \leq p_2\} \cap Int(\mathbb{R}_+^2)$, ($p = (p_1, p_2)$).*

Then p is a global attractor in $Int(\mathbb{R}_+^2)$ if and only if $W^s(q) \cap Int(\mathbb{R}_+^2) = \emptyset$ for all $q \in Fix(F) \cap \partial\mathbb{R}_+^2$.

In the previous result $W^s(q)$ is defined as

$$W^s(q) = \{z \in \mathbb{R}_+^2 : \lim_{N \longrightarrow \infty} F^{\sigma(N)}(z) = q \text{ with } \{\sigma(N)\}_{N \in \mathbb{N}} \subset \mathbb{N}\}.$$

Proof. Firstly we notice that F is one-to-one in $\mathbb{R}^2_+$. For it, we use that $F^{-1}(\{0\}) = \{0\}$ together with the following elementary result.

Lemma 4.3.2 *(Lemma 2.3.4 in [8]) Assume that $K \subset \mathbb{R}^n$ is a compact set and*

$$f : K \longrightarrow f(K)$$

is a local homeomorphism. Then for all $y \in f(K)$, the cardinal of $f^{-1}(y)$ is finite. If $f(K)$ is also connected then there exists a constant r so that the cardinal of $f^{-1}(y)$ is exactly r for all $y \in f(K)$.

At this moment, using **i)** we know that $F \in \mathcal{E}_*(\mathbb{R}^2_+)$. After that, we prove that $C \subset F(C)$. By using that $F'(x, y)$ is a competitive matrix in C, we deduce that

$$F_1(p_1, t) \geq p_1 = F_1(p_1, p_2) \quad 0 \leq t \leq p_2$$

$$F_2(t, p_2) \geq p_2 = F_2(p_1, p_2) \quad 0 \leq t \leq p_1.$$

These inequalities and $F_1(0, p_2) = 0 = F_2(p_1, 0)$ imply that $C \subset F(C)$. Consequently

$$F(\overline{\mathbb{R}^2_+ \backslash C}) \subset \overline{\mathbb{R}^2_+ \backslash C}$$

and

$$Fix(F) \cap Int(\overline{\mathbb{R}^2_+ \backslash C}) = \emptyset.$$

Now we apply Corollary 3.5.4 to F, $\overline{\mathbb{R}^2_+ \backslash C}$, in order to obtain that for all $z \in \overline{\mathbb{R}^2_+ \backslash C}$, $\omega(z, F)$ is a connected set contained in $Fix(F)$. Notice that this behavior also holds if $z \in C$ and for some $j \in \mathbb{N}$, $F^j(z) \in \overline{\mathbb{R}^2_+ \backslash C}$. Finally we take $z \in C$ so that $F^j(z) \in C$ for all $j \in \mathbb{N}$. In such a case, $\omega(z, F)$ is a fixed point by applying Proposition 2.1, Theorem 4.2, Lemma 4.3 in [53]. The proof follows from the previous comments. □

4.4 Permanence and Index

In this section we consider the population model given by (4.5). Understanding the conditions which guarantee the coexistence of both species for all initial condition is a crucial problem in Ecology. In this direction, several mathematical concepts of coexistence of species have been developed. In our work, we study the notion of permanent system.

Specifically, we say that system (4.5) is permanent if there is a compact set K satisfying that

$$K \subset Int\mathbb{R}^2_+ = \{(x,y) : x > 0, y > 0\}$$

and for all $(x,y) \in Int\mathbb{R}^2_+$ there is $N_0 = N_0(x,y)$ with

$$(x_n, y_n) \in K$$

for all $n \geq N_0$.

It is well known that the dynamics in the absence of x_1 and x_2, respectively, plays a crucial role in the problem of permanence. In our results we will see that a good knowledge of these dynamics together with some conditions on indices enable us to understand perfectly the notion of permanence in (4.5). Specifically, the goal of this section is to link the notion of permanent system with the index on convex sets. For this question we need some preliminary notions.

Consider the map associated to the right-hand side of system (4.5), namely

$$F : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+,$$

$$(x,y) \mapsto (xf_1(x,y), yf_2(x,y)).$$

We assume that this map is of class $\mathcal{C}^1$ with $f_1(x,y) > 0, f_2(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2_+$. From these conditions, we deduce that, for each initial condition $(x_0, y_0) \in \mathbb{R}^2_+$, the system of difference equations (4.5) has a well defined forward solution $\{(x_n, y_n)\}_{n\geq 0}$ lying in $\mathbb{R}^2_+$. Moreover the sets $Int\mathbb{R}^2_+$, $\{(x,y) : x = 0\}$ and $\{(x,y) : y = 0\}$ are positively invariant.

We always impose that system (4.5) is **dissipative** i.e. there exists a constant $M > 0$ such that

$$\limsup_n \|(x_n, y_n)\| \leq M$$

for any sequence in $\mathbb{R}^2_+$ obtained from (4.5), ($\|\cdot\|$ is a concrete norm in $\mathbb{R}^2$). This assumption is usual in populations models since it reflects the limitations of the environment. A second condition concerns the behavior of the system near the origin. Specifically, we suppose that there exists V, neighborhood of the origin in $\mathbb{R}^2_+$ such that for each $(x,y) \in Int\mathbb{R}^2_+$, there exists $N_0 = N_0(x,y) \geq 0$ with $(x_{N_0}, y_{N_0}) \notin V$. When this happens we will say that the **origin does not attract interior points**. In particular this condition excludes the possibility of simultaneous extinction of both species. Observe that if

$f_1(0,0) > 1$ and $f_2(0,0) > 1$, we can guarantee that the origin does not attract interior points. Finally we work with conditions on the dynamics in the absence of one of the species. For the difference equation

$$x_{n+1} = F_1(x_n, 0) \tag{4.6}$$

we assume that there exists a fixed point x_* satisfying $0 < |\partial_x F_1(x_*, 0)| < 1$ and $x_n \longrightarrow x_*$ for every positive solution $\{x_n\}_{n\geq 0}$ of (4.6). In this case we will say that x_* is a **hyperbolic attractor on the x-axis**. For the equation on the y-axis

$$y_{n+1} = F_2(0, y_n) \tag{4.7}$$

we can impose an analogous condition and say that $y_* > 0$ is a hyperbolic attractor on the y-axis. An alternative condition for (4.7) will be the nonexistence of positive fixed points. Notice that the dissipativity together with the invariance of the y-axis imply that, in the second case, $y_n \longrightarrow 0$ for every solution $\{y_n\}_n$ of (4.7).

Now we are in a position to give the main results.

Theorem 4.4.1 *Assume that (4.5) is dissipative and the origin does not attract interior points. In addition there exist $x_* > 0$ and $y_* > 0$ hyperbolic global attractors for (4.6) and (4.7) respectively. Then (4.5) is permanent if and only if*

$$index_{\mathbb{R}^2_+}(F, (x_*, 0)) = index_{\mathbb{R}^2_+}(F, (0, y_*)) = 0.$$

Observe that if the fixed points are not isolated then the system is not permanent.

In many models, one of the species cannot survive in the absence of the other. This is the case of some prey-predator models. This motivates us to consider the following theorem.

Theorem 4.4.2 *Assume all the conditions of the previous theorem excepting that there is no positive fixed point on the y-axis. Then (4.5) is permanent if and only if*

$$index_{\mathbb{R}^2_+}(F, (x_*, 0)) = 0.$$

4.4.1 Proofs

The proof of Theorem 4.4.1 is separated into two parts. In the first part we use the results in Chapter 1 to determine the local behavior in a neighbourhood of the fixed points. In the second part, we will study the global behaviour using that our system is dissipative. For this last part it is essential the following result:

Lemma 4.4.3 ***(Lemma 2.1 in [23])*** *Let $F : X \longrightarrow X$ be continuous map where X is a metric space and assume that K is a compact set satisfying that for all $x \in X$ there exists $N = N(x)$ so that $x_N \in K$. Then there exists k_0 such that*

$$\widetilde{K} = \bigcup_{j=0}^{k_0} F^j(K)$$

is compact and positively invariant.

Part 1: Local behavior.

Observe that if the fixed points $(x_*, 0)$ and $(0, y_*)$ are not isolated (as fixed points) then system (4.5) is not permanent. Therefore, if we assume this additional condition, we know by Theorem 2.4.1 that

$$index_{\mathbb{R}^2_+}(F, (x_*, 0)) = 0 \Longleftrightarrow \text{if } (\mathrm{x}_*, 0) \text{ is a repellor,}$$

$$index_{\mathbb{R}^2_+}(F, (0, y_*)) = 0 \Longleftrightarrow \text{if } (0, \mathrm{y}_*) \text{ is a repellor.}$$

Clearly, if either $(x_*, 0)$ or $(0, y_*)$ is an attractor, then system (4.5) is not permanent.

Part 2: Conclusion of Theorem 4.4.1.

Using that (4.5) is dissipative, $F(Int\mathbb{R}^2_+) \subset Int\mathbb{R}^2_+$ and the origin does not attract interior points, we deduce the existence of two constants R, r so that for all $(x, y) \in \mathbb{R}^2_+ \backslash \{0\}$ there exists $N_0 = N_0(x, y)$ with

$$F^{N_0}(x, y) = (x_{N_0}, y_{N_0}) \in K = (\overline{B_R(0)} \backslash B_r(0)) \cap \mathbb{R}^2_+.$$

Then applying Lemma 4.4.3 with $X = \mathbb{R}^2 \backslash \{0\}$, we conclude that there is $m_0 \in \mathbb{N}$ such that

$$\bigcup_{j=0}^{m_0} F^j(K) = K_1$$

is compact and positively invariant (notice that $0 \notin K_1$). From this moment, we concentrate on K_1 to study the dynamics. Using that $(x_*, 0)$ is a repeller in a small neighbourhood, we can take V a neighbourhood of $(x_*, 0)$ and $\delta > 0$ so that for all $(x, y) \in V$ with $y > 0$, there is $N_1 = N_1(x, y)$ satisfying that $(x_{N_1}, y_{N_1}) \notin V$ with $y_{N_1} > \delta$. On the other hand, since $K_1 \cap \{(x, y) : y = 0\}$ is compact and $(x_*, 0)$ is a global attractor in the x-axis, we can take N_2 such that for all $(x, 0) \in K_1$, there exists an index $j \leq N_2$ with $(x_j, 0) \in V$. From the continuity of F, we can choose $\delta_1 > 0$ with $\delta > \delta_1$ so that for all $(x, y) \in K_1$ with $y \leq \delta_1$, there exists an index $j \leq N_2$ verifying that $(x_j, y_j) \in V$. After that, taking $\widetilde{K_1} = K_1 \cap \{(x, y) : y > \delta\}$ and again applying Lemma 4.4.3, we deduce that there exists s_0 such that

$$K_2 = \bigcup_{j=0}^{s_0} F(\widetilde{K_1})$$

is compact and positively invariant.
The result is concluded repeating the same argument with the y-axis and K_2.

4.4.2 Applications

The use of the index allows to deal with degenerate cases which can not be treated via hyperbolicity. However, even in the hyperbolic case, it has some interest since it replaces an algebraic computation of eigenvalues by the study of the winding number that can be done visually. To show this we start with a concrete example. Consider the model

$$\begin{cases} x_{n+1} = x_n \exp(0.5 - x_n - 4y_n(y_n - 1)) \\ y_{n+1} = y_n \exp(1.5 - 3x - y_n) \end{cases} \tag{4.8}$$

The function F satisfies all the assumptions in Theorem 4.4.1. To check the dissipativity we notice that $F(\mathbb{R}^2_+)$ is bounded. It is also clear that the origin does not attract interior points since the eigenvalues of the Jacobian matrix of F at $(0, 0)$ are $\{\exp 0.5, \exp 1.5\}$. A simple study on the axes enables us to deduce that $(0.5, 0)$ and $(0, 1.5)$ are hyperbolic attractors in the x-axis and y-axis respectively, (this study will be done in the next example). Therefore, we have just to compute $index_{\mathbb{R}^2_+}(F, (0.5, 0))$ and $index_{\mathbb{R}^2_+}(F, (0, 1.5))$. Apart from the fixed points on the axes, $(\frac{5}{36}, \frac{13}{12})$ is the unique fixed point of our system in $Int(\mathbb{R}^2_+)$. After this remark, we draw the curves $\beta_i(t) = \alpha_i(t) - \overline{F}(\alpha_i(t))$ for $\alpha_1(t) = (0.5 + 0.1\cos(2\pi t), 0.1\sin(2\pi t))$, $\alpha_2(t) = (0.2\cos(2\pi t), 1.5 + 0.2\sin(2\pi t))$ and $\overline{F}(x, y) = F(|x|, |y|)$. (see figure 4.1 and 2.3) From these pictures we can deduce that

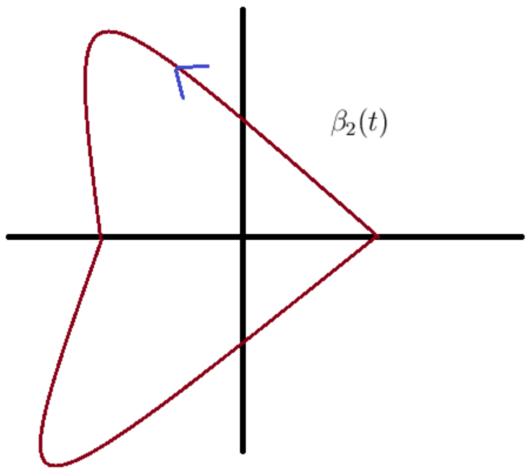

Figure 4.1: Illustration of β_2.

$index_{\mathbb{R}^2_+}(F,(0,1.5)) = index_{\mathbb{R}^2_+}(F,(0.5,0)) = 1$. By Theorem 4.4.1 we conclude that the system is not permanent. We notice 1 is an eigenvalue of the Jacobian matrix of F at $(0.5,0)$.

A general model

Next we consider

$$\begin{cases} x_{n+1} = x_n \exp(r_1 - x_n - f(y_n)) \\ y_{n+1} = y_n \exp(r_2 - g(x_n) - y_n) \end{cases} \tag{4.9}$$

where $r_i \in]0,2[\setminus\{1\}$ and the functions f,g satisfying that

- f,g are of class $\mathcal{C}^1$,
- $f(0) = g(0) = 0$,
- for some constant $M > 0$, $f(t), g(t) \geq 0$ for all $t \geq M$.

It is known that if $0 < r_1 < 1$ then $(x_n,0) \longrightarrow (r_1,0)$ for all $x_0 \in]0,+\infty[$ and the sequence $\{x_n\}$ is monotone. For $r_1 \in]1,2[$, we obtain the same conclusion but in this case the sequence is oscillating (see [9]). Let us apply Theorem 4.4.1 to characterize the permanence of system (4.9) under the hypothesis $r_i \in]0,2[\setminus\{1\}$ for $i=1,2$.

Again using that the map is bounded we deduce that the system is dissipative. On the other hand, using that $\{\exp(r_1), \exp(r_2)\}$ are the eigenvalues of the Jacobian matrix of F at $(0,0)$, we deduce that the origin does not attract interior points. Then by Theorem 4.4.1, our system is permanent if and only if

$$index_{\mathbb{R}^2_+}(F,(r_1,0)) = index_{\mathbb{R}^2_+}(F,(0,r_2)) = 0$$

Next we study when both indices are zero. Indeed, let us concentrated on $index_{\mathbb{R}^2_+}(F,(r_1,0))$, the analogous conclusion can be obtained in the other case. Using Lemma 2.4.4 in the x-axis, we deduce that for $r_2 - g(r_1) > 0$,

$$index_{\mathbb{R}^2_+}(F,(r_1,0)) = 0$$

and for $r_2 - g(r_1) < 0$,

$$index_{\mathbb{R}^2_+}(F,(r_2,0)) = 1.$$

From this moment we concentrate on the $index_{\mathbb{R}^2_+}(F,(r_1,0))$ when

$$r_2 = g(r_1). \tag{4.10}$$

In the remainder of the argument, assume that $1 - f'(0)g'(r_1) \neq 0$. This condition implies that $(r_1, 0)$ is an isolated fixed point for (4.9) since if (x_*, y_*) is a fixed point with $y_* > 0$, then

$$\begin{cases} r_1 = x_* + f(y_*) \\ r_2 = g(x_*) + y_*. \end{cases} \tag{4.11}$$

Now we consider the curve

$$\beta(t) = \alpha(t) - \overline{F}(\alpha(t))$$

where $\overline{F}(x,y) = F(|x|,|y|)$ and $\alpha(t) = (r_1 + \rho\cos 2\pi t, \rho \sin 2\pi t)$ for a sufficiently small $\rho > 0$ and $t \in [0,1]$. Observe that $\beta(t) \in \{(x,y) : y < 0\}$ for $t \in]\frac{1}{2}, 1[$. Since r_1 is an attractor in the x-axis, $\beta(0) = \beta(1) = (\xi_1, 0)$, $\beta(\frac{1}{2}) = (-\xi_2, 0)$ with $\xi_i > 0$. Next we concentrate on studying the behavior of $\beta(t)$ for $t \in]0, \frac{1}{2}[$. At this moment we distinguish two cases:

- $1 - f'(0)g'(r_1) > 0$,
- $1 - f'(0)g'(r_1) < 0$.

In the first case we can deduce that $G(y) = y + g(r_1 - f(y))$ is strictly increasing in a neighbourhood of $y = 0$. This condition implies that for $t \in]0, \frac{1}{2}[$, the curve $\beta(t)$ only cuts the y-axis in the positive part. Indeed, if for some $t_* \in]0, \frac{1}{2}[$,

$$r_1 - \alpha_1(t_*) - f(\alpha_2(t_*)) = 0$$

then $r_2 - g(\alpha_1(t_*)) - \alpha_2(t_*) = r_2 - (\alpha_2(t_*) + g(r_1 - f(\alpha_2(t_*)))) = r_2 - G(\alpha_2(t_*)) < r_2 - G(0) = r_2 - g(r_1) = 0$, here we have used that $\alpha_2(t_*) > 0$ and (4.10). In the second situation we have that for $t \in]0, \frac{1}{2}[$ the curve $\beta(t)$ only cuts the y-axis in the negative part. From this reasoning we can deduce that $index_{\mathbb{R}^2_+}(F, (r_1, 0)) = 1$ in the first case and $index_{\mathbb{R}^2_+}(F, (r_1, 0)) = 0$ in the second one. These conclusions are clear from the following result.

Lemma 4.4.4 *Let* $\beta : [0, 1] \longrightarrow \mathbb{R}^2 \backslash \{0\}$ *be a continuous map with* $\beta(0) = \beta(1)$ *and satisfying that,*

1. $\beta(0) = (\xi_1, 0)$ *with* $\xi_1 > 0$,

2. $\beta(t)$ *only cuts the* y*-axis in the positive part for* $t \in [0, \frac{1}{2}[$,

3. $\beta(\frac{1}{2}) = (-\xi_2, 0)$ *with* $\xi_2 > 0$,

4. $\beta(t) \in \{y < 0\}$ *for* $t \in]\frac{1}{2}, 1[$.

Then $\frac{\theta(1)-\theta(0)}{2\pi} = 1$ *where* $\theta(t)$ *is any continuous argument of* $\beta(t)$. *Moreover if we replace positive part by negative part in condition* 2), *we obtain* $\frac{\theta(1)-\theta(0)}{2\pi} = 0$.

Proof. Take $\theta : [0, 1] \longrightarrow \mathbb{R}$ a continuous argument for $\beta(t)$. Using condition 1), it is not restrictive to assume that $\theta(0) = 0$. After that, from condition 2) we deduce that $\theta(t) \in]\frac{-\pi}{2}, \frac{3\pi}{2}[$ for $t \in]0, \frac{1}{2}[$. In this situation condition 3) implies that $\theta(t) \in]\pi, 2\pi[$. Hence $\theta(t) \in]\pi, 2\pi[$ if $t \in]\frac{1}{2}, 1[$ what enables us to conclude that $\theta(1) = 2\pi$.

Finally we deduce the values of the index using the definition via the winding number.

4.5 Kolmogorov systems in higher dimensions

In 1976, S.Smale showed in [50] that an arbitrary smooth flow in the simplex $\Delta^{n-1} \subset \mathbb{R}^n$ spanned by the unit coordinate vectors can be embedded as an attractor in a system of differential equations of the type

$$\begin{cases} x_1' = x_1 g_1(x_1, x_2, ..., x_n) \\ x_2' = x_2 g_2(x_1, x_2, ..., x_n) \\ \vdots \\ x_n' = x_n g_n(x_1, x_2, ..., x_n). \end{cases} \tag{4.12}$$

with $\frac{\partial g_i}{\partial x_j}(x) > 0$. In several papers, M. W. Hirsch in [21], [20] tried to prove the converse implication of Smale's result. Specifically, he wanted to prove that every competitive system of the type (4.12) has a set homeomorphic to a $(n-1)$-simplex attracting all non trivial orbits. This aim was obtained in the famous paper [21] introducing the notion of carrying simplex. However, recently, he discovered that his proof was wrong (see [18]) and suggested possible theorems about the existence of a carrying simplex both in discrete and continuous framework. In this scenario, the aims of this chapter will be to prove the results suggested in [18] and to derive some implications in the problem of dominance and exclusion.

4.5.1 Notation and Retrotone maps

To simplify the notation, along this section the positive cone of $\mathbb{R}^n$ will be denoted by K. In coordinates,

$$K = \{x \in \mathbb{R}^n : \; x_i \geq 0\}.$$

The I' facet of K for I a subset of $\{1, ..., n\}$ will be

$$K_I = \{x \in K : x_j = 0 \text{ if } j \notin I\}.$$

According to the previous definition, the i-th positive coordinate axis will be denoted by $K_{\{i\}}$. After that we introduce the usual ordering in $\mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, we write $x \preceq y$ if $x_i \leq y_i$ for all $i = 1, ..., n$. If $x \preceq y$ and $x \neq y$, we write $x \prec y$. Given $a \preceq b$, we can define the closed order interval as

$$[a, b] = \{x \in \mathbb{R}^n : a \preceq x \preceq b\}.$$

An important concept in this section is the following.

Definition 4.5.1 *A map $F : K \longrightarrow K$ is retrotone in a subset $X \subset K$ if for $x, y \in X$ with $F(x) \succ F(y)$ we have that $x_i > y_i$ provided $x_i \neq 0$.*

In dimension 1 a map $F : [0, \infty[\longrightarrow [0, \infty[$ is retrotone if and only if it is monotone non-decreasing. Another example of retrotone map is the Poincaré map associated with the system

$$x_i' = x_i g_i(t, x) \quad \text{for all } i = 1, ..., n \tag{4.13}$$

where $g_i : \mathbb{R} \times K \longrightarrow \mathbb{R}$ is T-periodic in time and strictly decreasing in each variable, see [52]. Our next aim is to derive criteria ensuring that a map of the type

$$T(x_1, ..., x_n) = (x_1 f_1(x_1, ..., x_n), ..., x_n f_n(x_1, ..., x_n))$$

with $f_i > 0$ for all $i = 1, ..., n$ is retrotone in a set $C = [0, r]$ for $r \in IntK$. In this direction, we have the following results.

Proposition 4.5.2 *Consider U a neighbourhood of C. If $T \in \mathcal{C}^1(U)$ satisfies that for each $x \in C \backslash \{0\}$,*

$$[DT(x)]^{-1}_{i,j} > 0 \quad \textit{with } i, j \in I(x) = \{j : x_j \neq 0\}, \tag{4.14}$$

then T is retrotone and one-to-one in C.

Proof. Use that $T^{-1}(\{0\}) = \{0\}$ to conclude that $T : C \longrightarrow T(C)$ is a homeomorphism, see lemma 4.3.2. Finally, we use Proposition 2.1 in [28]. □

Next we present a criterion given in [18] to prove condition (4.14). First of all we compute the Jacobian matrix of T,

$$DT(x) = [F(x)]^{diag} + [x]^{diag} DF(x) \text{ then}$$

$$DT(x) = [F(x)]^{diag} (Id - M(x))$$

where $M(x) = -[\frac{x}{F(x)}]^{diag} DF(x)$ and $F(x) = (f_1(x), f_2(x), ..., f_n(x))$. If for all $x \in K \backslash \{0\}$, we assume that $DF(x)$ has strictly negative entries then $M(x)$ satisfies that

$$M_{i,j}(x) := -\frac{x_i}{f_i(x)} \frac{\partial f_i}{\partial x_j}(x) =$$

$$= -x_i \frac{\partial \log f_i(x)}{\partial x_j} > 0.$$

By the previous computations, we deduce that if the spectral radius $\rho(M(x))$ is less than 1 then the matrix $Id - M(x)$ is invertible and $[DT(x)]^{-1}_{i,j \in I(x)} > 0$. For it, we have to use that

$$DT(x)^{-1} = (\sum_{k=0}^{\infty} M^k(x))([F(x)]^{diag})^{-1}.$$

Therefore, in order to prove that T is retrotone we need only check that $\rho(M(x)) < 1$. Consequently, if

$$\max\{\sum_i M_{i,j}(x),\ j = 1, 2..., n\} < 1 \quad \text{or} \tag{4.15}$$

$$\max\{\sum_j M_{ij}(x),\ i = 1, ..., n\} < 1, \tag{4.16}$$

for all $x \in C$, then T is retrotone in C.

The following lemma determines the function $M_{ij}(x)$ in some concrete examples.

Lemma 4.5.3 *If $f_i(x_1, ..., x_n) = \exp(B_i - \sum_j (A_{ij}x_j))$ with $A_{ij} > 0$ then $M_{ij}(x) = A_{ij}x_i$. If $f_i(x_1, ..., x_n) = \frac{B_i}{1+\sum_j A_{ij}x_j}$ with $A_{ij} > 0$ then $M_{ij}(x) = \frac{A_{ij}x_i}{1+\sum_j A_{ij}x_j}$.*

4.5.2 Construction of a carrying simplex

Consider $C = [0, r]$ for $r \in IntK$ and $T : C \longrightarrow T(C) \subset C$ a continuous map with an expression of the type

$$T(x) = (T_1(x), ..., T_n(x)) = (x_1 f_1(x), x_2 f_2(x), ..., x_n f_n(x)) \tag{4.17}$$

with $f_i(x) > 0$ for all $i = 1, ..., n$. This kind of maps enjoys the following property:

$$T_j(x) > 0 \quad \text{if and only if} \quad x_j > 0. \tag{4.18}$$

Next we give the precise definition of a carrying simplex for (4.17).

Definition 4.5.4 *We will say that $T : C \longrightarrow T(C)$ admits a* ***carrying simplex*** *if there exists a subset $\Gamma^{n-1} \subset C \backslash \{0\}$ having the following properties:*

A1) *Γ^{n-1} is homeomorphic to a $n-1$-simplex (the definition of $n-1$ simplex is $\{(x_1, \ldots, x_n) : x_i \geq 0,\ \sum_i x_i = 1\}$),*

A2) Γ^{n-1} *is unordered, i.e. if* $x, y \in \Gamma^{n-1}$ *and* $x \succeq y$ *then* $x = y$,

A3) *for every* $x(0) \in C\backslash\{0\}$, *there exists* $y(0) \in \Gamma^{n-1}$ *so that* $\lim_{N\to+\infty}[x(N)-y(N)] = 0$,

A4) Γ^{n-1} *is invariant, i.e.* $T(\Gamma^{n-1}) = \Gamma^{n-1}$ *and* $T : \Gamma^{n-1} \longrightarrow \Gamma^{n-1}$ *is a homeomorphism.*

We employ the notation $x(N) = T^N(x(0))$ where $x(0) \in K$ denotes an initial condition. Once this definition has been introduced, the next step will be to give criteria to guarantee the existence of a carrying simplex for the map T. This motivates the following result.

Theorem 4.5.5 *Assume that* T *has the following conditions:*

1. $T\mid_{K_{\{i\}}}: C \cap K_{\{i\}} \longrightarrow T(C) \cap K_{\{i\}}$ *admits a fixed point* $q_i e_i$ *with* $q_i > 0$. *Moreover, we assume that* $q = (q_1, ..., q_n) \in IntC$,
2. T *is retrotone and locally one to one in* C,
3. *for* $x, y \in C$ *with* $T(x) \prec T(y)$, *we have that, for each* j, *either* $x_j = 0$ *or* $f_j(x) = \frac{T_j(x)}{x_j} > f_j(y) = \frac{T_j(y)}{y_j}$.

Then the map T *admits a carrying simplex.*

Remark 4.5.6 *Firstly we note that by (4.18), the third condition always makes sense. Moreover if we assume that* T *is retrotone in* C, *this condition is weaker than the condition below*

$$f_i(y) < f_i(x) \text{ for all } i = 1, ..., n \text{ provided } x \prec y.$$

Remark 4.5.7 *For* $n = 1$, *if* T *is retrotone and locally injective then* T *is strictly increasing.*

Throughout this section, we will always assume, without further mention, that conditions 1, 2, 3 of Theorem 4.5.5 hold. Next we point out two simple properties of the map T. First we observe that $T : C \longrightarrow T(C)$ is a homeomorphism. Indeed, by the theorem of invariance of the domain, we deduce that T is a local homeomorphism. To see that T is one-to-one, we use that $T^{-1}(\{0\}) = \{0\}$ together with Lemma 4.3.2. The second property of T is given in the following result.

Lemma 4.5.8 *For all* $\lambda_0 > 1$ *such that* $\lambda_0 q \in IntC$ *and for all* $x(0) \in C$, *there exists* $N_0(x(0)) := N_0 \in \mathbb{N}$ *so that* $T^N(x(0)) = x(N) \in [0, \lambda_0 q]$ *for all* $N \geq N_0$.

Proof. Firstly we prove that $[0, \lambda_0 q]$ is positively invariant. Indeed, given $x \in [0, \lambda_0 q]$, it is clear that $x_1 e_1$, $x_2 e_2$,...,$x_n e_n$ also belong to $[0, \lambda_0 q]$. Moreover, we can deduce that

$$T_i(x) \leq T_i(x_i e_i) \tag{4.19}$$

for all $i = 1, ..., n$. To prove these inequalities, we reason by contradiction and use that T is retrotone, $x_i e_i \preceq x$ and $T_j(x_i e_i) = 0$ for all $j \neq i$. Therefore, to conclude that $[0, \lambda_0 q]$ is positively invariant, we only need to prove that $T_i([0, \lambda_0 q_i e_i]) \subset [0, \lambda_0 q_i e_i]$. Let us now prove this fact. Since T is retrotone and locally one-to-one, the functions

$$h_i : [0, \lambda_0 q_i] \longrightarrow \mathbb{R}$$

$$h_i(x_i) := T_i(x_i e_i) = x_i f_i(x_i e_i)$$

are strictly increasing for all $i = 1, ..., n$, (see Remark 4.5.7). Next we use that h_i is strictly increasing together with condition 3 of Theorem 4.5.5, to obtain that $x_i \mapsto f_i(x_i e_i)$ is strictly decreasing. This property enables us to obtain that

$$\begin{cases} f_i(x_i e_i) > 1 & \text{if } x_i < q_i \\ f_i(x_i e_i) < 1 & \text{if } x_i > q_i \end{cases} . \tag{4.20}$$

Combining (4.20) with the strict monotonicity of h_i, we deduce that $T_i([0, \lambda_0 q_i e_i]) = h_i([0, \lambda_0 q_i]) = [0, h_i(\lambda_0 q_i)] \subset [0, \lambda_0 q_i[$ and so $[0, \lambda_0 q]$ is positively invariant.
In fact, using that f_i is strictly decreasing and (4.19), we prove that

S1) if $x_i(N_0) < \lambda_0 q_i$ then for all $N \geq N_0$, $x_i(N) < \lambda_0 q_i$,

S2) $x_i(N+1) < x_i(N)$ provided $x_i(N) > q_i$.

As a second step we prove that every orbit must enter into $[0, \lambda_0 q]$. By contradiction, suppose that there exists a $x(0) \in C$ such that for all $N \in \mathbb{N}$, $x(N)$ does not belong to $[0, \lambda_0 q]$. From **S1)**, we can take an index $i \in \{1, ..., n\}$ so that $x_i(N) > \lambda_0 q_i$ for all $N \in \mathbb{N}$. Then, by **S2)**, the sequence $\{x_i(N)\}$ is strictly decreasing. Hence there exists $\beta \geq \lambda_0 q_i$ with $x_i(N) \searrow \beta$. At this moment we have found the contradiction. Indeed, any point in $\omega(x(0), T)$ is of the type $\widetilde{x} = (\widetilde{x_1}, ..., \widetilde{x_{i-1}}, \beta, \widetilde{x_{i+1}}, ..., \widetilde{x_n}))$ with $T_i(\widetilde{x}) = \beta$. On the other hand, if we consider $\widetilde{z} = \beta e_i$, we obtain that

$$\beta = T_i(\widetilde{x}) = \beta f_i(\widetilde{x}) \overbrace{\leq}^{(4.19)} \beta f_i(\widetilde{z}) \underbrace{<}_{\mathbf{S2)}} \beta.$$

This contradiction ends the proof of this lemma. □

Next we present some useful lemmas for the proof of Theorem 4.5.5.

Lemma 4.5.9 *Consider $y(0) \in C$ satisfying that for all $N \in \mathbb{N}$, there exists $y(-N) = T^{-N}(y(0))$ and belongs to C. Then for $x(0) \in C$ with $x(0) \prec y(0)$, there exists $x(-N)$ for all $N \in \mathbb{N}$ and*

$$\lim_{N \longrightarrow +\infty} x(-N) = 0.$$

Proof. Firstly we prove the existence of $x(-N)$ for all $N \in \mathbb{N}$. This claim follows from the arguments of Proposition 2.1 in [28].

Next we prove that $x_j(-N) \longrightarrow 0$ for all $j = 1, ..., N$. Indeed, fix an index j. Using that T is retrotone, we can deduce that $x_i(-N) < y_i(-N)$ for all $i = 1, .., n$ with $y_i(0) \neq 0$. Assume that $x_j(0) \neq 0$ because otherwise $x_j(-N) = 0$ for all $N \in \mathbb{N}$, see (4.17). Next, define $\Delta_N^j = \frac{x_j(-N)}{y_j(-N)}$. Notice that $x_i(-N) < y_i(-N)$ provided $y_i(0) \neq 0$. This fact together with the third condition of Theorem 4.5.5 implies that Δ_N^j is strictly decreasing. Indeed,

$$\Delta_N^j := \frac{x_j(-N)}{y_j(-N)} = \frac{T_j(x(-N-1))}{T_j(y(-N-1))} = \frac{x_j(-N-1)f_j(x(-N-1))}{y_j(-N-1)f_j(y(-N-1))} > \Delta_{N+1}^j.$$

Hence, there exists $\beta \in [0, 1[$ such that $\Delta_N^j \searrow \beta$. If $\beta = 0$, the proof of the convergence is complete because $\Delta_j^N y_j(-N) = x_j(-N) \longrightarrow 0$. Now we prove that in the case $\beta > 0$, we also have $x_j(-N) \longrightarrow 0$. By contradiction, suppose that $\beta > 0$ and there exists $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ an increasing function so that $x_j(-\sigma(N)) > \delta > 0$. Under these assumptions, we consider the compact subset of $\mathbb{R}^N \times \mathbb{R}^N$

$$S = \overline{\{(x(-\sigma(n)), y(-\sigma(n))) : n \in \mathbb{N}\}}.$$

For each point $(\widetilde{x}, \widetilde{y}) \in S$, we have that $\widetilde{x} \preceq \widetilde{y}$. Actually we will prove that $\widetilde{x} \prec \widetilde{y}$. By contradiction, suppose that there is $(\widetilde{x}, \widetilde{y}) \in S$ so that $\widetilde{x} = \widetilde{y}$. In such a case, there exists an strictly increasing function $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ so that

$$x(-\sigma(\tau(N))) \longrightarrow \widetilde{x}$$

$$y(-\sigma(\tau(N))) \longrightarrow \widetilde{x}.$$

This is a contradiction since $\Delta^j_{\sigma(\tau(N))} \longrightarrow 1$. Besides this property, it is clear that for all $(x,y) \in S$,

$$x_j \geq \delta > 0. \tag{4.21}$$

Next we consider the compact set

$$S_1 = (T^{-1} \times T^{-1})(S) = \overline{\{(x(-\sigma(n)-1), y(-\sigma(n)-1) : n \in \mathbb{N}\}}.$$

Combining (4.21) and (4.18) we obtain that for all $(x,y) \in S_1$, there is $\delta_1 > 0$ satisfying that $x_j \geq \delta_1 > 0$. Therefore we deduce that $f_j(y) < f_j(x)$ for all $(x,y) \in S_1$. Here we have used that $x^0 \prec y^0$ for all $(x^0, y^0) \in S$ as well as the third condition of Theorem 4.5.5. Now it is clear that

$$\min_{(x,y)\in S_1} \frac{f_j(x)}{f_j(y)} = \eta > 1.$$

Finally one can see that

$$\Delta^j_{\sigma(N)+1} < \Delta^j_1 (\frac{1}{\eta})^N.$$

To prove this inequality, notice that $\Delta^j_{i+1} \leq \Delta^j_i$ for $i = 1, ..., \sigma(1)$ and $\Delta^j_{\sigma(1)+1} \leq \frac{1}{\eta}\Delta^j_{\sigma(1)}$. Hence $\Delta^j_N \longrightarrow 0 = \beta$. This contradiction ends the proof of this lemma. □

Next, we define the following sets:

$$\Sigma^n = \{x(0) \in C : \exists T^{-N}(x(0)) \in C \text{ for all } N \in \mathbb{N}\},$$

$$\Sigma^n_0 = \{x(0) \in \Sigma^n : x(-N) \longrightarrow 0 \text{ as } N \longrightarrow \infty\},$$

$$\Gamma^{n-1} = \Sigma^n \backslash \Sigma^n_0.$$

Given an integer $k = 1, 2, ..., n$ we define

$$\Sigma^k = \{p = (p_1, p_2, ..., p_k, 0, ..., 0) : p \in \Sigma^n\},$$

$$\Sigma^k_0 = \{p \in \Sigma^k : p \in \Sigma^n_0\},$$

$$\Gamma^{k-1} = \Sigma^k \backslash \Sigma^k_0.$$

These sets are clearly invariant under T. Next, we present some useful properties about these sets.

Lemma 4.5.10 Σ^n *is a compact set.*

Proof. By definition, Σ^n is contained in C and so it is bounded. Therefore it remains to prove that Σ^n is a closed set. Indeed, consider the sequence $\{z_N\}_N \subset \Sigma^n$ with $\{z_N\} \longrightarrow z_0$. According to the definition of Σ^n, we must to prove that $T^{-N}(z_0)$ exists and belongs to C for each $N \geq 1$. Using that C is compact, we conclude that there is a partial sequence verifying that $T^{-1}(z_{\sigma(N)}) \longrightarrow y_0 \in C$ and so $T(y_0) = z_0$. In this way we have proved the existence of $T^{-1}(z_0) \in C$. The proof is complete after an induction with respect to N. □

Lemma 4.5.11 Σ_0^n *is an open set (relative to C).*

Proof. We have already proven that $f_i(0) > 1$, (see proof of Lemma 4.5.8). Then, by continuity, we deduce that there exist $\delta > 0$ and a ball B centered at zero so that $f_i(x) > 1 + \delta$ for all $x \in B \cap K$. These inequalities complete this lemma. □

Remark 4.5.12 *Using that $f_i(0) > 1$ for all $i = 1, ..., n$, it is clear that the origin is a repellor for T.*

Lemma 4.5.13 *Suppose that there exist $x(0), y(0) \in C\backslash\{0\}$ so that $x(N) \prec y(N)$ for all $N \in \mathbb{N}$. Then,* $\lim_{N\longrightarrow\infty}[x(N) - y(N)] = 0$.

Proof. Using that the map T is retrotone, we can assume that $x_i(N) < y_i(N)$ for all $N \in \mathbb{N}$ provided $y_i(0) \neq 0$. Now, we fix an index j with $x_j(0) \neq 0$ and prove that

$$\lim_{N\to\infty} x_j(N) - y_j(N) = 0. \tag{4.22}$$

Indeed, consider $\Delta_j^N = \frac{x_j(N)}{y_j(N)}$. Reasoning in the same way as in Lemma 4.5.9, one checks that Δ_j^N is an increasing sequence. If $\Delta_j^N \nearrow 1$ we have finished since $y_j(N) - x_j(N) = y_j(N)(1-\Delta_j^N) \longrightarrow 0$. Next we prove that if $\Delta_j^N \nearrow \beta < 1$, the sequence $y_j(N)-x_j(N) \longrightarrow 0$. By contradiction, assume that $\Delta_j^N \nearrow \beta < 1$ and there is a partial sequence $\sigma(N)$ so that $y_j(\sigma(N)) - x_j(\sigma(N)) \longrightarrow \rho > 0$. In this case it is clear that there is $\eta > 0$ so that $x_j(\sigma(N)) \geq \eta > 0$, for otherwise $\Delta_j^N \longrightarrow 0$. Now consider

$$S = \overline{\{(x(\sigma(N)+1), y(\sigma(N)+1)) : N \in \mathbb{N}\}}$$

and reason as in Lemma 4.5.9 to obtain a contradiction. To finish the proof suppose that we can take an index k satisfying that $x_k(0) = 0$. In such a case we must prove that $y_k(N) \longrightarrow 0$. By contradiction, assume that there is a point z in $\omega(y(0),T)$ so that $z_k > 0$. Then, by (4.22), there exists a point $\widetilde{z} \in \omega(x(0),T)$ satisfying that $\widetilde{z}_j = z_j$ if $x_j(0) \neq 0$; $\widetilde{z}_j = 0 \leq z_j$ if $x_j(0) = 0$ and $0 = \widetilde{z}_k < z_k$. From the previous comments, we know that $\widetilde{z} \prec z$. Moreover by Remark 4.5.12 there exists an index j_0 with $0 < \widetilde{z}_{j_0} = z_{j_0}$. This contradicts the first part of the proof of this Lemma. Indeed, using that the map T is retrotone we obtain that $T_i^{-1}(\widetilde{z}) < T_i^{-1}(z)$ provided $z_i \neq 0$ and in particular, $T_{j_0}^{-1}(\widetilde{z}) < T_{j_0}^{-1}(z)$. □

Proof of Theorem 4.5.5. Firstly, we prove that there is a continuous map, strictly decreasing

$$\Psi : \Sigma^{n-1} \longrightarrow [0, q_n]$$

so that

$$\Sigma^n = \{(x,y) \in \Sigma^{n-1} \times [0,q_n] : 0 \leq y \leq \Psi(x)\},$$

$$\Gamma^{n-1} = \{(x, \Psi(x)) : x \in \Sigma^{n-1}\}.$$

Define

$$\Psi(x) = \max\{y : (x,y) \in \Sigma^n\}.$$

The function Ψ is strictly decreasing, that is $\Psi(x) < \Psi(y)$ if $y \prec x$. We prove this assertion by contradiction. Assume that $x \prec y$ with $\Psi(x) \leq \Psi(y)$. Then, by Lemma 4.5.9 we obtain that $(x, \Psi(x)) \in \Sigma_0^n$. This is impossible since Σ_0^n is open and this point lies on the boundary. It is important to notice that from this property, we obtain that Γ^{n-1} is unordered. Let us now prove that Ψ is continuous. Using that Ψ is bounded, it is sufficient to show that the graph of Ψ is closed. Indeed, consider the sequence $\{x_N\}_N \subset \Sigma^{n-1}$ with

$$\{(x_N, \Psi(x_N))\} \longrightarrow (x_0, y_0).$$

Using that Σ^n is compact, we deduce that $(x_0, y_0) \in \Sigma^n$. Assume that $(x_0, y_0) \in \Sigma_0^n$. In such a case, as Σ_0^n is an open set, one deduces that there exists $N \in \mathbb{N}$ so that $(x_N, \Psi(x_N)) \in \Sigma_0^n$. This contradiction proves that the graph of Ψ is closed.

The next step is to prove that Γ^{n-1} determines completely the dynamics of T. Indeed, for $p \in C \backslash \Gamma^{n-1}$, we prove that there exists $q \in \Gamma^{n-1}$ so that

$$\lim_{N \to \infty} [T^N(q) - T^N(p)] = 0. \tag{4.23}$$

We distinguish two cases: $p \in C\backslash\Sigma^n$ and $p \in \Sigma_0^n$. Suppose that we are in the first case. By lemma 4.5.13, it is sufficient to prove that there exists $q \in \Gamma^{n-1}$ verifying that $T^N(q) \preceq T^N(p)$ for all $N \in \mathbb{N}$. With this purpose, we define $\Gamma(N,p) = \{q \in \Gamma^{n-1} : T^N(q) \preceq T^N(p)\}$. Using that T is retrotone, we see that $\Gamma(N+1,p) \subset \Gamma(N,p)$. Next, we prove that $\Gamma(N,p)$ is non empty for all $N \in \mathbb{N}$. It is important to recall that $T^N : C \longrightarrow C$ is a homeomorphism onto its image and maps Γ^{n-1} and Σ^n to Γ^{n-1} and Σ^n respectively. Given $s = T^N(p)$, there exists $\lambda_0 < 1$ such that $\lambda_0 s \in \Gamma^{n-1}$. Here we are using that Σ_0^n is an open and Σ^n is compact. From the previous comments, we deduce that $q = T^{-N}(\lambda_0 s) \in \Gamma^{N-1}$ and so $q \in \Gamma(N,p)$. Finally, one checks that

$$\bigcap_{N=1}^{\infty} \Gamma(N,p) \neq \emptyset$$

by using that the sequence $\{\Gamma(N,p)\}$ is a decreasing sequence of compact sets. The other case can be proved similarly reversing the ordering. □

Corollary 4.5.14 *Assume that T satisfies:*

C1) *If $x \prec y$ then $f_i(y) < f_i(x)$ for all $i = 1,...,n$.*

C2) *$T|_{K_{\{i\}}}: K_{\{i\}} \cap K \longrightarrow K_{\{i\}} \cap K$ admits a fixed point $q_i e_i$ with $q_i > 0$.*

C3) *T is retrotone and locally one-to-one in $[0,q]$ where $q = (q_1,...,q_n)$.*

Then T admits carrying simplex.

Proof. This corollary is immediate from Theorem 4.5.5, (see Remark 4.5.6) and the following result.

Lemma 4.5.15 *Assume that C1), C2) and C3) hold. Given $\lambda > 1$, the set $C = [0, \lambda q]$ is positively invariant, i.e. $T(C) \subset C$ for all $N \in \mathbb{N}$ and for all $x(0) \in K$, there exists $N_0 \in \mathbb{N}$ so that $x(N_0) \in C$.*

Proof. The proof of this result is very similar to Lemma 4.5.8. For this reason we only give a sketch. Firstly, we prove that C is positively invariant. Using C1), we directly obtain (3.5) and (3.6). By C3), we have that there exists $\lambda_0 > 1$ such that T is retrotone and locally injective in $[0, \lambda_0 q]$ and so the functions

$$h_i : [0, q_i] \longrightarrow \mathbb{R}$$

$$h_i(x_i) = x_i f_i(x_i e_i)$$

are strictly increasing. The rest of the proof of this statement is the same as in Lemma 4.5.8. □

Remark 4.5.16 *If we replace **C1)** and **C3)** by the conditions*

C1') *$DF(x)$ has strictly negative entries,*

C3') *$\rho(M(x)) < 1$ for all $x \in [0, q]\backslash\{0\}$, $M(x)$ is defined in section 5.1*

we obtain Theorem 4 in [18].

Finally we compare our construction with [12]. Firstly, we can drop the conditions on regularity, global injectivity and hyperbolicity of fixed points. Moreover we can replace T retrotone in the whole domain K by conditions **C1)** and **C3)**. In applications, these assumptions are more suitable.

4.5.3 Criteria of Dominance and Exclusion

The aim of this section is to derive criteria of dominance and exclusion in system (4.17) Along this section we assume, without further mention, the following properties in (4.17)

C1) If $x \prec y$ then $f_i(y) < f_i(x)$ for all $i = 1, ..., n$,

C2) $T\mid_{K_{\{i\}}}: K_{\{i\}} \longrightarrow K_{\{i\}}$ admits a fixed point $(0, ...0, q_i, 0, ..., 0)$ with $q_i > 0$ for all $i = 1, ..., n$,

C3) T is retrotone and locally injective in a neighbourhood of $[0, q]$ where $q = (q_1, q_2, ..., q_n)$.

The biological interpretation of the previous conditions is as follows: condition **C2)** says that each species has a coexistence state in the absence of the other species. The conditions **C1)** and **C3)** imply that our system is competitive and enjoys an additional monotonicity in the past.

Definition 4.5.17 *We say that there is exclusion in system (4.17) if for all initial condition $x(0) \in K$ there is an index $j = j(x(0))$ so that*

$$T_j^N(x(0)) \longrightarrow 0 \text{ as } N \longrightarrow +\infty.$$

It is important to observe that under this definition, the index j depends on the initial condition. Next we present our exclusion criteria.

Theorem 4.5.18 *For $n = 2$, assume that (4.17) satisfies* **C1), C2), C3)**. *Then the following statements are equivalent:*

i) *T has no fixed points in $IntK$.*

ii) *There is exclusion for system (4.17).*

The previous result is well known, see for instance [53]. However this theorem motivates a similar result for $n = 3$. In this case we need to introduce an additional condition.

C4) T has a finite number of fixed points on ∂K.

Theorem 4.5.19 *For $n = 3$, assume that (4.17) satisfies* **C1), C2), C3), C4)**. *Then the following statements are equivalent:*

i) *T has no fixed points in $IntK$.*

ii) *There is exclusion for system (4.17).*

As a direct consequence of the previous theorem, we can obtain the following result of global stability.

Corollary 4.5.20 *For $n = 3$, assume that (4.17) satisfies* **C1), C2), C3)** *and* ***C4)*** *with $\{p_1, p_2, ..., p_r\} = Fix(T) \cap \partial K$. Then the following statements are equivalent:*

i) $\lim_{N \to \infty} x(N) = p_1$ *for all initial condition $x(0) \in IntK$.*

ii) *System (4.17) has no equilibria in $IntK$ and $W^s(p_n) \cap IntK = \emptyset$ for all $n \geq 2$.*

In the previous result $W^s(p)$ is defined as

$$W^s(p) = \{x(0) \in K : \lim_{N \longrightarrow \infty} x(N) = p\}.$$

We notice that the second condition of Corollary 4.5.20 is easy to check in many models. For instance, if we assume that $p = (0, y_2, y_3)$ is a fixed point of T with $f_1(0, y_2, y_3) > 1$ then $W^s(p) \cap IntK = \emptyset$. At this point it is important to notice that the previous results are not true for $n > 3$ since we can adapt Example 3 in [5] to our context, (see also the last section in [5]). On the other hand, if we do not assume **C4)**, we can only ensure that

$\omega(p,T)$ is a connected set contained in $Fix(T)$, (see Example 2 in [5]).
Next we study the problem of dominant species in (4.17).

Definition 4.5.21 *We understand that the species j is dominant in (4.17) if for all initial condition $x(0)$ in $IntK$, there is $\delta > 0$ so that*

$$\liminf T_j^N(x(0)) > \delta,$$

$$T_i^N(x(0)) \longrightarrow 0 \text{ for all } i \neq j.$$

In a completely similar way we define a dominated species in (4.17).
Next we give a criterion guaranteeing the presence of dominant species. Motivated by [15], [14], [16] and [34], we introduce the following concept.

Definition 4.5.22 *The species j verifies the F-Y condition if*

$$\bigcup_{i\in\{1,2,\ldots,n\}\setminus\{j\}} D_i^+ \subset D_j^*$$

where $D_j^* = \{x \in K : f_j(x) > 1\}$ and $D_i^+ = \{x \in K : f_i(x) \geq 1\}$.

The biological interpretation of the previous definition is as follows: if some species different from the species j does not decrease its size, then the species j increases strictly. Let us remark that

- the F-Y condition for the species j does not imply, in general, $f_j(x) > f_i(x)$ for all $i \neq j$,
- the F-Y condition does not ensure the presence of dominant species, (see [15], [57]).

Franke and Yakubu understand that a species is weakly dominant if it satisfies the F-Y condition. However this condition is far from being a necessary condition for the presence of dominant species. This will be shown with an example in the last section of this chapter.

Theorem 4.5.23 *Assume that system (4.17) satisfies* **C1), C2), C3)**. *If the species j has the F-Y condition then the species j is dominant.*

Finally we give a result ensuring the presence of dominated species in our system.

Theorem 4.5.24 *For n=3, assume that system (4.17) satisfies* **C1), C2), C3)**. *If* $D_1^+ \subset D_2^*$ *then the species 1 is dominated.*

4.5.4 Proofs

The aim of this section is to prove the previous results. The key ingredient will be the existence of a carrying simplex for system (4.17), i.e. the existence of a subset $\Gamma^{n-1} \subset [0, q]\backslash\{0\}$ having the following properties:

A1) Γ^{n-1} is homeomorphic to a $(n-1)$-simplex.

A2) Γ^{n-1} is unordered, i.e. if $x, y \in \Gamma^{n-1}$ and $x \succeq y$ then $x = y$.

A3) For every $x(0) \in K\backslash\{0\}$ the trajectory of $x(0)$ is asymptotic with the trajectory of some $y(0) \in \Gamma^{n-1}$, i.e. $\lim_{N\to\infty} x(N) - y(N) = 0$.

A4) $T(\Gamma^{n-1}) = \Gamma^{n-1}$ and $T : \Gamma^{n-1} \longrightarrow \Gamma^{n-1}$ is a homeomorphism.

By Corollary 4.5.14, we can deduce that conditions **C1), C2), C3)** guarantee that system (4.17) admits a carrying simplex. Now we proceed to prove our results.

Proof of Theorem 4.5.18.
ii) $\Rightarrow$ *i)*.
From the definition of exclusion we know that all solutions are attracted by the boundary of K. This excludes fixed points lying in $IntK$.
i) $\Rightarrow$ *ii)*.
Since system (4.17) admits carrying simplex,

$$T : \Gamma^1 \longrightarrow \Gamma^1$$

is a homeomorphism with Γ^1 homeomorphic to a closed interval. Now, using that T has no fixed points in $IntK \cap \Gamma^1$, we deduce that for all $(x, y) \in \Gamma^1 \cap IntK$ either $T^N(x, y) \to (q_1, 0)$ or $T^N(x, y) \to (0, q_2)$ where $(q_1, 0)$ and $(0, q_2)$ are the non trivial fixed points of T on the axes. Notice that we are dealing with monotone dynamics in one dimension. We conclude the proof by using that any orbit in $K\backslash\{0\}$ is asymptotic to an orbit in Γ^1, as stated in **A3)**. $\square$

Proof of Theorem 4.5.19. Firstly, we prove that $T|_{\Gamma^2}$ is an orientation-preserving homeomorphism. Indeed, on the boundary of the carrying simplex, we can define two orientations. Specifically if $\gamma : [0,1] \longrightarrow \partial\Gamma^2$ is a parametrization of $\partial\Gamma^2$ with $\gamma(0) = (q_1, 0, 0)$ then either

$$t_2 < t_3 \tag{4.24}$$

or

$$t_3 < t_2, \tag{4.25}$$

where $\gamma(t_2) = (0, q_2, 0)$ and $\gamma(t_3) = (0, 0, q_3)$. Now it is clear that $T \circ \gamma$ is a parametrization of $\partial\Gamma^2$ satisfies (4.24) (resp. (4.25)) provided γ satisfies (4.24) (resp (4.25)). From these comments, we deduce that T is an orientation preserving homeomorphism in Γ^2. An alternative proof of this fact can be found in [43].

By Corollary 3.5.4 and **C4)**, we deduce that for all $x(0) \in \Gamma^2$, $\omega(x(0), T) = \{p_i\}$ with $p_i \in Fix(T)$. The proof is complete because we know that for all $y(0) \in K\backslash\{0\}$, there exists $x(0) \in \Gamma^2$ so that $\omega(y(0), T) = \omega(x(0), T)$, (see **A3)**). $\square$

Proof of Corollary 4.5.20. By Theorem 4.5.19 (see paragraph above), we deduce that for all $x(0) \in IntK$, $\omega(x(0), T) = \{p_i\}$ for some $i = 1, ..., n$. We rule out the cases $i = 2, ..., n$ by using condition **ii)**. $\square$

To prove Theorem 4.5.23 we use some geometrical aspects of the carrying simplex. Specifically the key fact will be that Γ^{n-1} is an unordered manifold.

Proof of Theorem 4.5.23.

Using that the species j has the F-Y condition, we deduce that

$$Fix(T) \cap \{x_j \neq 0\} = \{(0, ..., 0, q_j, 0, ..., 0)\}$$

where $(0, ..., 0, q_j, 0, ..., 0)$ is the unique positive fixed point in $K_{\{j\}}$. After this remark we split the proof into two steps.

Step 1: Dominance in the carrying simplex.

In this step we prove that if $x(0) \in \Gamma^{n-1}$ with $x_j(0) \neq 0$ then

$$\lim_{N\to\infty} x(N) = (0, ..., 0, q_j, 0, ..., 0).$$

Indeed, take $x(0) \in \Gamma^{n-1}$ with $x(0) \neq (0, ..., 0, q_j, 0, ..., 0)$ and $x_j(0) \neq 0$. First of all, we prove that the sequence $\{x_j(N)\}_N$ is strictly increasing. Using that Γ^{n-1} is unordered and invariant, we can take an index i such that $x_i(0) < x_i(1)$. Now, by applying that the species j verifies the F-Y condition, we deduce that $x_j(0) < x_j(1)$. Hence, by repeating this argument we obtain that the sequence $\{x_j(N)\}_N$ is strictly increasing. At this point it is clear that there exists $\alpha > 0$ such that $x_j(N) \nearrow \alpha$. Let us prove that $\alpha = q_j$. By contradiction, assume that $\alpha < q_j$. If this were the case, there would exist an orbit contained in $\{x \in \Gamma^{n-1} : x_j = \alpha\}$. This is impossible since given $y(0) \in \Gamma^{n-1} \cap \{x \in K : x_j = \alpha\}$, the sequence $\{y_j(N)\}_N$ is strictly increasing. To prove this claim, use that there are not fixed points for T in $\{x \in \Gamma^{n-1} : x_j = \alpha\}$ together with the previous argument. Finally, as $\Gamma^{n-1} \cap \{(x_1, ..., x_n) : x_j = q_j\} = (0, ..., 0, q_j, 0, ..., 0)$ we conclude that

$$\lim_{N\to\infty} x(N) = (0, ..., 0, q_j, 0, ..., 0).$$

Step 2: $S = \Gamma^{n-1} \cap \{(x_1, ..., x_n) : x_j = 0\}$ *is a repellor.*
In this step we prove that there exists $\epsilon > 0$ so that for all $x(0) \in IntK$ with $dist(x(0), S) < \epsilon$, there exists $N_0 := N_0(x(0)) > 0$ such that

$$dist(x(N_0), S) > \epsilon.$$

Indeed, take $(x_1, x_2, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) \in \Gamma^{n-1}$and distinguish two cases:

Case 1: The point $(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$ is a fixed point of T.
In this case, there exists an index i different from j such that

$$f_i(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 1.$$

Therefore, using that the species j verifies the F-Y condition, we deduce that

$$f_j(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

Case 2: The point $(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$ is not a fixed point for T.
In this case using that $\Gamma^{n-1} \cap \{x \in K : x_j = 0\}$ is unordered, we deduce that there exists an index i such that

$$f_i(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

Then, as the species j has the F-Y condition, we obtain that

$$f_j(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) > 1.$$

In short, from the previous comments and the compactness of the set $S = \Gamma^{n-1} \cap \{x \in K : x_j = 0\}$, we deduce that there exist $\epsilon > 0$ and $\delta > 0$ so that

$$f_j(x) > 1 + \delta \tag{4.26}$$

for all $x \in K$ with $dist(x, S) < \epsilon$. Thus we obtain that $x_j(N) > x_j(0)(1 + \delta)^N$ whenever $dist(x(N), S) < \epsilon$ and $x(0) \in IntK$. The proof of the claim in Step 2 is complete.
From Step 2 and **A3)** we deduce that for all $x(0) \in IntK$ there exists $y(0) \in \Gamma^{n-1} \backslash \mathcal{S}$ so that $\omega(x(0), T) = \omega(y(0), T)$. To finish the proof we use the first step. □

Proof of Theorem 4.5.24.
From the hypotheses of Theorem 4.5.24 we deduce that T has no fixed points in $IntK$. Combining Theorem 3.5.4 and **A3)**, we deduce that given $p = (x_1, x_2, x_3) \in K$, $\omega(p, T)$ is a connected set contained in $Fix(T)$. Then we only need to prove that $A = Fix(T) \cap \{x \in \Gamma^2 : x_1 \neq 0\}$ is a repellor (in the same sense as in the previous proof). We proceed by steps:
Step 1: Fixed points in $\{x \in A, x_2 = 0\}$.
Firstly we observe that the set $F_2 = Fix(T) \cap \{x \in A : x_2 = 0\}$ is compact. Now we distinguish two cases:

- The fixed point $(0, 0, q_3)$ is not an accumulation point of F_2.
  In this case the set $\widetilde{F_2} = F_2 \backslash \{(0, 0, q_3)\}$ is compact and for all $(x_1, 0, x_3) \in \widetilde{F_2}$,

  $$f_1(x_1, 0, x_3) = 1.$$

  Thus, from $D_1^+ \subset D_2^*$, we deduce that

  $$f_2(x_1, 0, x_3) > 1$$

  for all $(x_1, 0, x_3) \in \widetilde{F_2}$. Finally we proceed as in the previous theorem in order to conclude that $\widetilde{F_2}$ is a repellor.

- The fixed point $(0,0,q_3)$ is an accumulation point of F_2.
  In this case we can deduce that for all $(x_1,0,x_3) \in \widetilde{F_2}$,

$$f_1(x_1,0,x_3) = 1.$$

Hence, by continuity, $f_1(0,0,q_3) = 1$. Applying $D_1^+ \subset D_2^*$, we deduce that

$$f_2(x_1,0,x_3) > 1$$

for all $(x_1,0,x_3) \in F_2$. From these facts we also deduce that F_2 is a repellor.

Step 2: Fixed points in $\{x \in \Gamma^2 : x_3 = 0\}$.
By $D_1^+ \subset D_2^*$, it is clear that $(0,q_2,0)$ and $(q_1,0,0)$ are the unique fixed points of T in $\{x_3 = 0\}$. Moreover $f_2(q_1,0,0)$ is greater than 1.
The previous steps allow us to conclude the proof, (reason as in the previous theorems).
□

4.5.5 Examples

The aim of this section is to illustrate our results with concrete examples. First we consider the classical May Oster model and obtain new results on exclusion. In contrast, the conclusions on dominance that we obtain are known. This has lead us to present Example 2. This model contains different types of growths, nevertheless our results on dominance apply. This is not the case for the results in [14]. Finally, in the third example we show that the F-Y condition is not a necessary condition for the existence of dominant species.

Example 1: May Oster model.
Consider the system

$$x_i(N+1) = x_i(N)\exp(B_i - A_{i1}x_1(N) - A_{i2}x_2(N) - A_{i3}x_3(N)) \tag{4.27}$$

for $i = 1,2,3$, where the coefficients satisfy that $B_i, A_{ij} > 0$. It is clear that **C1), C2)** hold. Moreover if

$$\frac{B_i}{A_{ii}}(A_{i1} + A_{i2} + A_{i3}) < 1$$

for $i, j = 1, 2, 3$, then **C3)** also holds, (see Lemma 4.5.3, (4.16) and Proposition 4.5.2). Therefore, under these assumptions we can apply our results. Indeed, if

$$A_{ii}A_{jj} - A_{ij}A_{ji} \neq 0$$

for all $i \neq j$ then **C4)** holds and so, according to Theorem 4.5.19 we deduce that there is exclusion for system (4.27) if and only if the linear system

$$B_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3$$

$$B_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3$$

$$B_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3$$

has no solutions in $IntK$. Next we apply our dominance results. Indeed, the species 1 verifies the F-Y condition if and only if

$$\frac{B_1}{A_{1i}} > \max\{\frac{B_2}{A_{2i}}, \frac{B_3}{A_{3i}}\} \quad \text{for } i = 1, 2, 3. \tag{4.28}$$

Thus if our system enjoys condition (4.28), by Theorem 4.5.23, the species 1 is dominant. If

$$\frac{B_1}{A_{1i}} > \frac{B_2}{A_{2i}} \quad \text{for } i = 1, 2, 3 \tag{4.29}$$

then $D_1^+ \subset D_2^*$ and so, by Theorem 4.5.24, the species 2 is dominated.
As mentioned before, the conclusions on exclusion are new but the conclusions on dominance can be obtained using [15] and [14]. Actually, for this model, the conditions required by [15] and [14] are less restrictive than ours.

Example 2: Mixing exponential and rational functions.

If the system has different types of growth functions, for instance $f_1(x) = \exp(B_1 - \sum_j A_{1j}x_j)$ and $f_2(x) = \frac{B_2}{1+\sum_j A_{2j}x_j}$, we cannot apply the results in [14]. Motivated by this fact we consider the system

$$x_1(N+1) = x_1(N)\exp(B_1 - A_{11}x_1(N) - A_{12}x_2(N) - A_{13}x_3(N)) \tag{4.30}$$

$$x_2(N+1) = \frac{B_2x_2(N)}{1 + A_{21}x_1(N) + A_{22}x_2(N) + A_{23}x_3(N)}$$

$$x_3(N+1) = \frac{B_3 x_3(N)}{1 + A_{31}x_1(N) + A_{32}x_2(N) + A_{33}x_3(N)},$$

with $B_1 > 0$, $B_2, B_3 > 1$ and $A_{ij} > 0$. To ensure that system (4.30) verifies **C3)**, assume that

$$\frac{B_1}{A_{11}}(A_{11} + A_{12} + A_{13}) < 1$$

$$\frac{(B_i - 1)}{A_{ii}B_i}(\sum_{j=1}^{3} A_{ij}) < 1 \quad i = 2, 3,$$

(see Lemma 4.5.3, (4.16) and Proposition 4.5.2). Now it is clear that $D_2^+ \subset D_1^*$ if and only if

$$\frac{B_1}{A_{1i}} > \frac{B_2 - 1}{A_{2i}} \quad \text{for } i = 1, 2, 3. \tag{4.31}$$

Therefore, if (4.31) holds, by Theorem 4.5.24, $x_2(N) \longrightarrow 0$ for all initial condition $x(0) \in IntK$.

If

$$\frac{B_1}{A_{1i}} > \max\{\frac{B_2 - 1}{A_{2i}}, \frac{B_3 - 1}{A_{3i}}\}$$

for $i = 1, 2, 3$ then the species 1 has the F-Y condition and so, by Theorem 4.5.24, $x_2(N), x_3(N) \longrightarrow 0$ for all initial condition $x(0) \in IntK$.

Example 3: Leslie Gower model, a concrete case.

The purpose of this example is to show the applicability of Corollary 4.5.20. Moreover we will see that the F-Y condition is not necessary for the existence of dominant species. Indeed, consider

$$x_1(N+1) = \frac{1.15x_1(N)}{1 + x_1(N) + x_2(N) + x_3(N)}$$

$$x_2(N+1) = \frac{1.1x_2(N)}{1 + x_1(N) + x_2(N) + 0.8x_3(N)}$$

$$x_3(N+1) = \frac{1.2x_3(N)}{1 + x_1(N) + 2.5x_2(N) + x_3(N)}.$$

This system has the properties **C1), C2), C3)** (see Lemma 4.5.3, (4.16) and Proposition 4.5.2) and has no fixed points in $IntK$. The fixed points on ∂K are $(0.15, 0, 0)$,$(0, 0.1, 0)$, $(0, 0, 0.2)$ and $(0, 0.06, 0.05)$ and satisfy $f_3(0.15, 0, 0)$, $f_1(0, 0.1, 0)$, $f_1(0, 0.06, 0.05) > 1$. Thus, by Corollary 4.5.20, the fixed point $(0, 0, 0.2)$ is a global attractor in $IntK$. On the other hand, it is clear that the species 3 does not have the F-Y condition. Furthermore, we observe that $D_3^* \not\supset D_1^+$ and $D_3^* \not\supset D_2^+$.

Chapter 5

Resumen

Los modelos matemáticos son importantes en muchas disciplinas puesto que a partir del conocimiento de unas condiciones iniciales es posible determinar cómo el sistema varía en el tiempo. En las últimas décadas, una infinidad de modelos han aparecido para describir diferentes situaciones biológicas. Este creciente interés ha generado una nueva disciplina, Biología Matemática. La biología matemática tiene un carácter multidisciplinar que involucra a matemáticos, físicos, ingenieros, biólogos...La importancia de este tema se refleja en cientos de artículos experimentales, analíticos y numéricos que pueden consultarse en las bibliografías de las monografías [32] and [7]. A lo largo de esta tesis nos concentramos en dinámica de poblaciones. Informalmente, el principal objetivo es utilizar modelos matemáticos con el fin de estudiar la interacción de diferentes especies que comparten el mismo ecosistema. Concretamente, dada una condición inicial, i.e. el número de animales de cada especie en $t = 0$, nuestro propósito es determinar el número de animales en cada instante de tiempo. Sería muy deseable determinar, de manera explícita, esta evolución. Sin embargo, en la mayoría de los modelos, es imposible obtener tal expresión a partir de los datos iniciales.

Una familia importante de modelos en dinámica de poblaciones está formada por los sistemas de Kolmogorov. Estos modelos tienen la forma

$$x_i(N+1) = T(x(N)) = x_i(N)f_i(x_1(N), ..., x_n(N)), \quad i = 1, ..., n \tag{5.1}$$

donde $f_i : \mathbb{R}^n_+ := \{(x_1, ..., x_n) : x_i \geq 0\} \longrightarrow]0, +\infty[$ es una función continua. Este sistema se usa para modelar la evolución de n especies que conviven en el mismo ecosistema. $x_i(N)$ es la densidad de población de la especie i en el periodo N. La función f_i es la llamada tasa de crecimiento de la especie i y representa la dependencia de la densidad de las diferentes especies en el crecimiento de la especie i.

Esta tesis se centra en tres cuestiones: Exclusión, Dominación y Permanencia. El propósito de esta sección introductoria es doble: Por un lado, definimos estos conceptos desde un punto de vista matemático y biológico. Por otro lado, discutimos cómo encontrar estos conceptos en nuestro modelos.

Desde un punto de vista biológico, un sistema es permanente si para toda condición inicial con número de animales de cada especie no cero, ninguna especie se extingue. De manera informal, matemáticamente, el sistema (5.1) es permanente si existe un conjunto compacto en $Int\mathbb{R}^n_+$ "absorbiendo" la dinámica de (5.1) en $Int\mathbb{R}^n_+$. Más precisamente,

Definición 5.0.25 *El sistema (5.1) es permanente si existe un conjunto compacto K satisfaciendo que*

$$K \subset Int\mathbb{R}^n_+$$

y para todo $x(0) \in Int\mathbb{R}^n_+$ hay $N_0 = N_0(x(0))$ con

$$x(N) \in K$$

para todo $N \geq N_0$.

Nuestro objetivo será caracterizar la noción de sistema permanente en el caso de dos especies. Para ello, introducimos el siguiente marco.

Definición 5.0.26 *Hay crecimiento logístico en la especie i si*

$$x_i(N+1) = x(N) f_i(x_i(N)e_i) \tag{5.2}$$

tiene un atractor global $x_i^* > 0$ en $]0, +\infty[$. Empleamos la notación $\{e_1, ..., e_n\}$ para denotar la base usual de $\mathbb{R}^n$

Definición 5.0.27 *El sistema (5.1) es disipativo si existe una constante $M > 0$ de manera que*

$$\limsup_{N \longrightarrow \infty} |x_i(N)| < M$$

para todo i, donde $x(N) = (x_1(N), ..., x_n(N))$ es la sucesión de (5.1) con condición inicial $x(0) = (x_1(0), ..., x_n(0))$.

La noción de sistema disipativo se usa normalmente en dinámica de poblaciones para modelar las limitaciones del medio ambiente.

Teorema 5.0.28 *Para $n = 2$, supongamos que (5.1) es disipativo y el origen no atrae puntos del interior de $\mathbb{R}^2_+$. Además, supongamos que hay crecimiento logístico para cada especie con atractores x_i^* (respectivamente) y*

$$0 < |\frac{\partial}{\partial x_i} T(x_i^* e_i)| < 1$$

para $i = 1, 2$. Entonces (5.1) es permanente si y sólo si

$$index_{\mathbb{R}^2_+}(T, x_1^* e_1) = index_{\mathbb{R}^2_+}(T, x_2^* e_2) = 0.$$

Este resultado caracteriza la permanencia en nuestro sistema en el caso de dos especies. En general, usando el carácter geométrico del índice, para decidir si nuestro sistema es permanente es suficiente dibujar dos curvas y contar el número de vueltas de éstas alrededor del origen.

Desde un punto de vista biológico, hay exclusión si para toda condición inicial, alguna especie se extingue. Esto es, es imposible que todas las especies sobrevivan durante todo el tiempo. Nótese que en general, la especie que se extingue depende de la condición inicial. La noción de especie dominante evita esta última situación. En efecto, la especie i es dominante si aparte de esta especie, todas las especies se extinguen. Claramente, esta noción es más fuerte que la noción de exclusión. De manera completamente análoga, definimos la noción de especie dominada. Desde un punto de vista matemático, hay exclusión en el sistema (5.1) si para toda condición inicial $x(0) \in \mathbb{R}^n_+$, existe un índice j de manera que

$$x_j(N) \longrightarrow 0.$$

Es claro que la noción de exclusión implica la no existencia de puntos fijos de T en $Int\mathbb{R}^n_+$. Nuestro objetivo será mostrar que es también una condición necesaria cuando hay tres especies que compiten. Para modelar una interacción competitiva, suponemos que :

C1) Si $x \prec y$ entonces $f_i(y) < f_i(x)$ para todo $i = 1, ..., n$. ($\prec$ denota el orden usual en $\mathbb{R}^n$)

C2) Hay crecimiento logístico en cada eje.

C3) $T(p) \prec T(q)$ then $p_i < q_i$ para todo $i = 1, ..., n$ siempre que $q_i \neq 0$.

Adicionalmente, para nuestro resultado introducimos la siguiente condición.

C4) T tiene un número finito de puntos fijos en $\partial\mathbb{R}^n_+$.

Teorema 5.0.29 *Para $n = 3$, supongamos que (5.1) satisface* **C1), C2), C3), C4)**. *Entonces las siguientes afirmaciones son equivalentes:*

i) *T no tiene puntos fijos en $Int\mathbb{R}^n_+$.*

ii) *Hay exclusión en (5.1).*

Para probar el resultado anterior, usaremos que bajo **C1), C2), C3)** la dinámico de (5.1) está esencialmente en $\mathbb{R}^2$. Después de esto, usaremos una variante del teorema de Massera.

La traducción matemática de la noción de especie dominante es la siguiente: La especie j es dominante si existe $\delta > 0$ de manera que para toda condición inicial $x(0) \in \mathbb{R}^n_+$ con $x_j(0) > 0$,

$$\lim_{N \longrightarrow \infty} x_i(N) = 0$$

para todo $i \neq j$ y

$$\liminf_{N \longrightarrow \infty} x_j(N) > \delta > 0.$$

Para detectar la noción de especie dominante vamos a usar el siguiente concepto.

Definición 5.0.30 *La especie j satisface la condición F-Y si*

$$\bigcup_{i \in \{1,2,\ldots,n\} \setminus \{j\}} D_i^+ \subset D_j^*$$

donde $D_j^* = \{x \in \mathbb{R}^n_+ : f_j(x) > 1\}$ y $D_i^+ = \{x \in \mathbb{R}^n_+ : f_i(x) \geq 1\}$.

La interpretación biológica del concepto anterior es la siguiente: Si alguna especie diferente de la especie j no decrece su tamaño (transcurrido un periodo de tiempo), entonces la especie j crece estrictamente. Vamos a remarcar que

- La condición F Y en la especie j no implica, en general, $f_j(x) > f_i(x)$ para todo $i \neq j$,
- La condición F-Y no asegura la presencia de especie dominante, (ver [15], [57]).

Teorema 5.0.31 *Supongamos que (5.1) satisface* **C1), C2), C3)**. *Si la especie j tiene la condición F-Y entonces la especie j es dominante.*

Chapter 6

Conclusiones

En esta tesis hemos visto que el uso de técnicas topológicas puede ser de gran utilidad para estudiar modelos de dinámica de poblaciones. Los principales logros conseguidos han sido:

- Extensión de la noción de "Order interval trichotomy" (Theorem 5.2 en [53]) sin suponer hipótesis de monotonía.
- Rebajamos las condiciones de monotonía en los resultados desarrollados en [53].
- Caracterizamos la permanencia en sistemas de dos especies usando la noción de índice.
- Damos una demostración completa para el resultado de Carrying simplex propuesto en [18].
- Usamos el concepto de carrying simplex para desarrollar criterios de exclusión y dominación.

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