Universidad de Granada

TESIS DOCTORAL

Ecuaciones elípticas con singularidades aisladas y superficies de curvatura constante.

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Editor: Editorial de la Universidad de Granada
Autor: María Asunción Jiménez Grande
D.L.: GR 3107-2012
ISBN: 978-84-9028-235-9

Ecuaciones elípticas con singularidades aisladas y superficies de curvatura constante.

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Granada, 23 de abril de 2012

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Agradecimientos

Estaré eternamente agradecida a José Antonio y a Pablo por todo lo que me han dado: su tiempo, sus conocimientos, su confianza y su cariño.

Gracias de corazón José Antonio, por ser amigo antes que nada (y antes de que esto nada fuera), por ser sincero conmigo aunque eso a veces no me gustara. Gracias por la oportunidad, gracias por tantas risas y experiencias, por ser un espejo en que mirarme y por contarme teoremas en servilletas de cafetería. Como te dije al otro lado del mundo: has marcado mi vida para siempre. Gracias por hacerme tita (gracias María), y gracias a Antonio por saber sacar lo mejor de su padre.

Un millón de gracias Pablo. Por ser mi poli bueno (aunque nunca existiera el malo), por tu pasión por las matemáticas algo contagiosa, por dejarme compartir contigo un paseo, unas compras, un viaje o los juegos de tus hijos mientras me hablas de matemáticas. Gracias por escucharme aunque no tuviera nada matemático que contar, y por escribirme en un papel que las cuentas me saldrían.

Gracias a los dos por cuidar de mí y preocuparos en la medida de lo posible de que las cosas me vayan bien. Mil gracias por enseñarme tanto, por vuestra paciencia y por ser tan buenas personas. Nunca sabré cómo compensar todo eso, así que prometo seguir sin quejarme cada vez que recordéis que una vez creí que las superficies se *exfoliaban*.

Os quiero mucho y espero que no dudéis en contar siempre conmigo para lo que sea.

Gracias Jose por no dejarme tirar la toalla. Sin ti esto no hubiera sido posible. Gracias por tus ánimos, por confiar en mí más que yo misma. Gracias por haberme sabido querer (y aguantar) en los malos momentos. Gracias por los buenos, tú has sido el responsable de muchos. Perdón por no haber estado, aunque nunca me haya ido del todo, espero que haya valido la pena. Te quiero mucho más que a un millón de tesis (que ya es decir).

Gracias a mi familia, sin ellos yo no sería quien soy. Gracias a mis abuelos Asunción y Diego que han sido más que mis padres y a los que sé que esto les haría más ilusión incluso que a mí. Gracias tío Diego por enseñarme el Teorema de Pitágoras, y tantas otras cosas, durante los veranos de mi infancia. Gracias a mi madre Cati y a mis hermanos Ana y Juan José. A vosotros dos últimos os diría que todo es posible con esfuerzo y que una tesis no es comparable con todo lo que podáis llegar a conseguir en vuestras vidas. Como hermana mayor me comprometo a daros ese *empujoncito* que os pueda llegar a hacer falta.

Porque también son familia para mi: gracias hermana Kenza por ser más que eso durante todo este tiempo, y gracias a la familia de Jose (mi familia de Granada): Mami, Carmen madre, Carmen hija, Susi, tito Jose y Concha, por acogerme como una más y aceptarme como soy.

Gracias a mis amigas de siempre (y ahora en la distancia), Ana, Meri y Pili. Y Belén, mi dedicatoria de esta tesis no es suficiente para agradecerte lo que me has enseñado. Por eso, y como tú ya no podrás tener ninguno, te cedo éste que es mi sueño. El recuerdo de tu lucha siempre me inspira y me da fuerzas.

Muchas gracias también a los amigos que he ido encontrando por el camino y que, de una forma u otra, siento partícipes de este periodo de mi vida. Gracias a mis compañeros de batalla (léase carrera y sus consecuencias), ¡Anaaaa!, José Luis y Elena por seguir ahí después de todos estos años. A mis compañeros de despacho y del tan necesario café o charla de pasillo, José María (mi tito-hermano matemático), Magdalena (una chica de mi edad), César (a quien nadie le prepara las tostadas como yo), Paco y Manzano (a quienes tanto interrumpí en el despacho 5)... y a Antonio y Ana, que se me han vuelto imprescindibles a pesar de que a veces me ganen a *los Colonos*. No se me olvidan los *sevillanos* que me encanta encontrarme aunque sea de congreso, Isa y Cañete.

Y gracias a mi *huevo frito*, Andrea, sólo a ti te perdonaría las meriendas inoportunas sin pasteles.

Quiero también aprovechar estas líneas para agradecer al Departamento de Geometría y Topología su acogida y su trato durante todos estos años, me siento realmente orgullosa de haber podido hacer mi tesis entre tanta gente brillante. Gracias también a Harold Rosenberg y Barbara Nelli por su magnífica acogida durante mis estancias en Rio y Roma.

En definitiva, gracias a toda la gente que ha puesto los granos de arena necesarios para conseguir dar forma a esta tesis que por momentos me pareció una montaña.

A mis abuelos
A Belén

Nunca perseguí la gloria,
ni dejar en la memoria
de los hombres mi canción;
yo amo los mundos sutiles,
ingrávidos y gentiles,
como pompas de jabón.

A. Machado

Contents

Resumen y conclusiones

Muchos problemas interesantes en Matemáticas se comprenden mejor cuando se consideran simultáneamente desde el punto de vista de diferentes teorías matemáticas. Un ejemplo típico de este hecho es la interacción entre la Geometría Diferencial y las Ecuaciones en Derivadas Parciales (EDPs), un tema que sienta las bases de la rama matemática del Análisis Geométrico. De hecho, existen muchos problemas naturales que pueden ser formulados a la vez de una forma puramente geométrica y desde el más estricto punto de vista de la teoría de EDPs, y cuyas soluciones se basan fundamentalmente en ambos enfoques. Este fenómeno está presente en muchos avances matemáticos importantes, tanto clásicos como recientes. Podemos mencionar por ejemplo la derivación de Lagrange de la EDP que describe las superficies mínimas en el espacio Euclídeo $\mathbb{R}^3$, los famosos problemas clásicos de Weyl, Christoffel o Minkowski sobre la existencia de superficies convexas con curvaturas predeterminadas en $\mathbb{R}^3$, los avances sobre los problemas de Yamabe y Kazdan-Warner, y los flujos geométricos, en particular, la solución de Perelman a la conjetura de Poincaré. El desarrollo de la teoría a lo largo de todos estos años ha dado lugar a técnicas poderosas y flexibles como el método "moving planes" para EDPs elípticas, cuyo equivalente geométrico es comúnmente conocido como el Principio de Reflexión de Alexandrov.

Un tema central en la teoría de EDPs no lineales elípticas es el estudio de singularidades aisladas. Algunas preguntas típicas sobre este problema serían bajo qué condiciones son dichas singularidades aisladas evitables, o, en caso de que existieran, cómo clasificar dichas singularidades no evitables. Para muchas ecuaciones cuasilineales elípticas, las singularidades aisladas son evitables si y sólo si la solución pertenece a un cierto espacio de Sobolev (ver [GiTr]). Por ejemplo, el Teorema de Bers [Ber] muestra que toda singularidad aislada de una superficie mínima es evitable, y este resultado ha sido ampliamente extendido a EDPs más generales. Pero existen ciertos tipos de ecuaciones de Monge-Ampère elípticas cuya solución u y sus primeras derivadas son acotadas en la singularidad y sin embargo u no se extiende de manera C^2 hasta ella. Esto es, existen ecuaciones de Monge-Ampère con soluciones acotadas en Ω y una singularidad no evitable en un punto de Ω.

Todo esto muestra que, incluso en dimensión dos, el problema de decidir si una EDP no lineal elíptica admite singularidades aisladas no evitables, y la clasificación de dichas soluciones singulares en el caso de que existan, es ciertamente no trivial y de un claro interés geométrico y analítico.

En esta memoria estudiaremos las soluciones de ciertas ecuaciones geométricas no lineales elípticas en dimensión dos en presencia de singularidades aisladas, tanto en puntos interiores como en puntos del borde.

Dividimos nuestra exposición en dos partes diferentes cuyos contenidos explicamos a continuación.

Parte I: La ecuación de Liouville

Dedicamos esta primera parte de la memoria, que consta de los tres primeros capítulos, al estudio de la famosa ecuación de Liouville

$$\Delta v + 2Ke^v = 0,$$

que bajo diferentes puntos de vista puede considerarse como la EDP no lineal elíptica conformemente invariante más sencilla, y cuyo estudio se remonta a los trabajos de Poincaré, Picard, y Liouville entre otros.

Ocurre que si consideramos un dominio plano $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$, entonces la métrica Riemanniana $e^v|dz|^2$, donde v es una solución de la ecuación de Liouville y z es una coordenada compleja en Ω, tiene curvatura constante K. Entonces, el problema de encontrar métricas conformes de curvatura positiva en un dominio puede reducirse a resolver la ecuación de Liouville. Además, podemos suponer salvo homotecia que $K \in \{-1, 0, 1\}$.

Nuestro **primer objetivo** en esta memoria es estudiar en detalle el llamado *problema de Neumann geométrico* para esta ecuación en un dominio plano Ω, posiblemente en presencia de singularidades aisladas en el borde del dominio. Este problema consiste en determinar la existencia de una métrica conforme de curvatura constante K en Ω tal que cada componente del borde $\partial\Omega$ tenga curvatura geodésica constante con respecto a dicha métrica. Por la invariancia conforme de la ecuación de Liouville podemos considerar en general que Ω es un dominio simétrico, por tanto en nuestro estudio nos centraremos en el caso en que Ω es el semiplano superior $\mathbb{R}^2_+$ o un anillo en $\mathbb{R}^2$.

Con este propósito, en el Capítulo 1 damos algunos preliminares sobre métricas conformes de curvatura constante. Definiremos el concepto de singularidad aislada de una métrica y, como caso particular, las de *tipo cónico* (ver Definición 1.5) ya que van a ser tratadas de manera especial en el Capítulo 2.

Después, explicaremos la relación de la ecuación de Liouville con el análisis complejo que será decisiva para obtener nuestros resultados en los Capítulos 2 y 3. Esta conexión viene dada por la función g, llamada *la aplicación desarrolladora*, en el Teorema de Liouville (Teorema 1.3):

> *Sea $v : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \longrightarrow \mathbb{R}$ una solución a la ecuación de Liouville en un dominio simplemente conexo Ω. Entonces existe una función meromorfa localmente inyectiva g (holomorfa tal que $1 + K|g|^2 > 0$ si $K \leq 0$) en Ω tal que*
>
> $$v = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}. \tag{1}$$
>
> *Recíprocamente, si g es una función meromorfa localmente inyectiva (holomorfa tal que $1 + K|g|^2 > 0$ si $K \leq 0$) en Ω, entonces la función v en* (1) *es una solución a la ecuación de Liouville en Ω.*

Con esto, deducimos que g es una isometría local $g : (\Omega, ds^2) \longrightarrow \mathcal{Q}(K) := (\Sigma_K, ds^2_K)$ donde $\Sigma_K = \mathbb{D}, \mathbb{C}, \overline{\mathbb{C}}$ respectivamente para $K = -1, 0, 1$, y ds^2_K es la métrica Riemanniana en Σ_K dada por

$$ds^2_K = \frac{4|d\zeta|^2}{(1 + K|\zeta|^2)^2}. \tag{2}$$

En el Capítulo 1 también explicamos cómo se puede generalizar el Teorema de Liouville para el caso en el que el dominio Ω no sea simplemente conexo, y mostramos el concepto de *derivada Schwartziana* que nos será de gran utilidad a lo largo de esta memoria. Ésta se denota por Q, y viene dada por las siguientes fórmulas, donde g es la aplicación desarrolladora de v:

$$Q := v_{zz} - \frac{1}{2} v_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2.$$

Posteriormente, discutiremos los principales aspectos y los ejemplos más simples de los problemas de valores iniciales para la ecuación de Liouville.

Finalmente, explicamos cómo la ecuación de Liouville también aparece en otros temas de teoría de superficies como superficies minimales en $\mathbb{R}^3$, superficies de curvatura media uno en el espacio hiperbólico 3-dimensional $\mathbb{H}^3$, o una clase más amplia de superficies de Weingarten.

En el Capítulo 2 presentamos nuestros resultados sobre la solución al problema de Neumann geométrico para la ecuación de Liouville en el semiplano superior con un número finito de singularidades borde. Analíticamente, este problema viene dado por:

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{en } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 :\ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_j\, e^{v/2} & \text{en } I_j \subset \mathbb{R} \equiv \partial\mathbb{R}^2_+, \qquad c_j \in \mathbb{R}, \end{cases} \tag{3}$$

donde $I_j := (q_j, q_{j+1})$, $j = 0, \dots, n-1$ y $q_0 = -\infty < q_1 < \cdots < q_{n-1} < q_n = \infty$. Con estas condiciones de Neumann sobre el borde estaríamos imponiendo diferentes valores de curvatura geodésica constante, $-\frac{c_j}{2}$, a lo largo de cada intervalo I_j.

Este problema fue estudiado en un principio sin singularidades borde y para una dimensión arbitraria en los trabajos [CLi1, CSF, GaMi3, HaWa, LiZh, LiZha, Ou, Zha]. Para el caso de una única singularidad borde (que puede considerarse situada en el origen), Jost, Wang y Zhou [JWZ] dieron una clasificación completa de las soluciones del problema anterior bajo las siguientes hipótesis:

1. La métrica $e^v|dz|^2$ tiene área finita en $\mathbb{R}^2_+$, es decir

$$\int_{\mathbb{R}^2_+} e^v < \infty. \tag{4}$$

2. El borde $\partial\mathbb{R}^2_+$ tiene longitud finita para la métrica $e^v|dz|^2$, esto es

$$\int_{\mathbb{R}_-} e^{v/2} + \int_{\mathbb{R}_+} e^{v/2} < \infty. \tag{5}$$

3. La métrica $e^v|dz|^2$ tiene una *singularidad borde cónica* en el origen (ver Definición 1.5), es decir, existe el límite $\lim_{z\to 0} |z|^{-2\alpha} e^v \neq 0$ para algún $\alpha > -1$.

4. $K = 1$.

Con estas hipótesis, en [JWZ, Teorema 1.2] los autores obtienen las fórmulas explícitas de las soluciones a (3) dando también su interpretación geométrica (ver Sección 2.1).

A lo largo del Capítulo 2 obtendremos varias mejoras al teorema de Jost-Wang-Zhou. Los resultados que presentaremos acerca de este problema de Neumann geométrico con singularidades en el borde pueden resumirse como sigue:

(i) Para el caso de una única singularidad que supondremos situada en el origen, obtenemos un teorema general de clasificación inspirado en [GaMi4] donde suprimimos las cuatro hipótesis extra de [JWZ] mencionadas anteriormente (ver Teorema 2.3). Para probar este resultado usamos las técnicas de análisis complejo explicadas en el Capítulo 1 y el hecho de que, por las condiciones de Neumann en (3), las imágenes por al aplicación desarrolladora g de los ejes $\mathbb{R}^+$ y $\mathbb{R}^-$ caen, respectivamente, sobre ciertas circunferencias en $\overline{\mathbb{C}}$.

(ii) Caracterizaremos el comportamiento asintótico de una solución al problema de Neumann geométrico alrededor de una singularidad borde cuando el área de la métrica $e^v|dz|^2$ es finita cerca de la singularidad. Es decir, supondremos que la singularidad se haya en el origen y consideraremos un cierto dominio $D_\varepsilon^+ = \{z \in \mathbb{R}_+^2 : \ |z| < \varepsilon\}$, $\varepsilon > 0$. Entonces, la condición de área finita en D_ε^+ es

$$\int_{D_\varepsilon^+} e^v < \infty. \tag{6}$$

Obtendremos así tres tipos diferentes de comportamientos asintóticos, uno de ellos correspondiendo a que la singularidad sea de tipo cónico (ver Corolario 2.1). Además, como consecuencia del comportamiento asintótico geométrico de dichas soluciones, obtendremos el siguiente corolario (Corolario 2.3):

Sea $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ una solución al problema de Neumann geométrico local que satisface la condición de energía finita (6)*, y sea $g : \overline{D_\varepsilon^+} \setminus \{0\} \longrightarrow \Sigma_K \subseteq \overline{\mathbb{C}}$ su aplicación desarrolladora, que es un difeomorfismo local. Entonces:*

(a) La imagen $g(\overline{D_\varepsilon^+} \cap \mathbb{R}^+)$ cae sobre una circunferencia $\mathcal{C}_1$, y la imagen $g(\overline{D_\varepsilon^+} \cap \mathbb{R}^-)$ cae sobre otra circunferencia $\mathcal{C}_2$, tales que $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ (posiblemente $\mathcal{C}_1 = \mathcal{C}_2$).

(b) *La curvatura geodésica de $\mathcal{C}_i$, parametrizada como $g(s,0)$, es constantemente $-c_i/2$ para la métrica ds_K^2 en* (2)*.*

(c) *g se extiende de forma continua al origen con $g(0) \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset \overline{\mathbb{C}}$.*

(d) *La aplicacicón Schwarziana $Q = v_{zz} - v_z^2/2$ se extiende de forma holomorfa a D_ε^* como $Q(\bar{z}) = \overline{Q(z)}$, y tiene en el origen a lo sumo un polo de orden dos.*

(e) *Si $K = 0$, entonces $g(0) \in \mathbb{C}$. Si $K = -1$ y $g(0) \in \partial\mathbb{D} \equiv \mathbb{S}^1$, entonces $\mathcal{C}_1$ y $\mathcal{C}_2$ son tangentes en $g(0)$, y no son arcos de horociclos.*

(iii) Para el problema global (3) con una singularidad borde, clasificamos todas las soluciones que cumplen la condición de área finita (4) y determinamos para qué valores de (K, c_1, c_2) existen dichas soluciones. Concretamente, primero describimos una familia de soluciones a (3) que llamaremos *soluciones canónicas* (ver Sección 2.2), dando sus fórmulas explícitas y explicando su interpretación geométrica. Entonces, probaremos que (Teorema 2.3):

Toda solución a (3) *con una singularidad borde en el origen y área finita es una* solución canónica.

Como consecuencia, recuperamos las soluciones obtenidas en [JWZ] sin imponer ninguna hipótesis adicional sobre el borde, el signo de K ni el tipo de singularidad.

(iv) Para el caso de un número arbitrario de singularidades, también describiremos todas las soluciones de área finita de (3) desde un punto de vista geométrico. Los ejemplos básicos de métricas conformes de curvatura constante con singularidades borde y curvatura geodésica constante a lo largo de cada componente del borde son los determinados por polígonos circulares en $\overline{\mathbb{C}}$, pero hay muchos más. Para obtener esta familia de métricas más general, consideramos polígonos circulares *inmersos* posiblemente con auto-intersecciones, y damos un criterio topológico-diferencial (embebimiento tipo Alexandrov) para que éstos generen dichas métricas (ver Definición 2.5). Con todo esto, el Corolario 2.4 prueba el recíproco:

Cualquier métrica conforme de área finita y curvatura constante en $\mathbb{R}_+^2$ (o equivalentemente en el disco unidad $\mathbb{D}$), con un *número finito de singularidades borde y curvatura geodésica constante a lo largo de cada componente del borde es una de dichas* métricas poligonales circulares *construidas a partir de polígonos Alexandrov-embebidos, con posibles autointersecciones.*

Analíticamente, dichas métricas no tendrán expresiones simples, aunque sí que es posible dar cierta información analítica sobre ellas. En el Corolario 2.5 describiremos el espacio moduli de dichas métricas para $K = 1$ a través de su parametrización en términos de sus aplicaciones Schwarzianas, que sí tienen una expresión más simple.

El Capítulo 3 está dedicado a alcanzar nuestro **segundo objetivo** en esta tesis, también relacionado con la ecuación de Liouville. Éste consiste en resolver el problema de Neumann geométrico en el caso en que Ω es un anillo que se corresponde analíticamente con el siguiente problema de Neumann no lineal:

$$\begin{cases} \Delta u + 2Ke^u = 0, & \text{in } \mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu_1} = c_1 e^{\frac{u}{2}} + 2, & \text{on } C_1 = \{z \in \mathbb{C} : \ |z| = 1\}, \\ \dfrac{\partial u}{\partial \nu_2} = c_2 e^{\frac{u}{2}} - 2e^{r\pi}, & \text{on } C_2 = \{z \in \mathbb{C} : \ |z| = e^{-r\pi}\}. \end{cases} \tag{7}$$

Aquí ν_i denota el normal interior a C_i, $i = 1, 2$ respectivamente, y $r > 0$ es una constante. Como en el caso del semiplano superior supondremos salvo homotecia que $K = \{-1, 0, 1\}$.

Observemos que, junto con el disco punteado, el anillo es el dominio no simplemente conexo más simétrico que podemos considerar. En este sentido, los resultados que obtenemos son una generalización natural a resultados previos en [HaWa, BHL, ChWa]. De hecho, puesto que valores diferentes de r proporcionan anillos que no son conformemente equivalentes, estamos considerando una familia de problemas que tampoco son conformemente equivalentes.

Primeramente presentamos una cierta familia de *anillos canónicos* que pueden ser construidos geométricamente de una forma muy sencilla.

(1) Consideremos la métrica inducida sobre cualquier anillo $\mathcal{A}'$ en $\mathcal{Q}(K)$ cuyo borde consista en dos circunferencias disjuntas. Observemos

que componiendo con una aplicación recubridora con un número finito de hojas de este anillo $\mathcal{A}'$ también obtendríamos una métrica en las mismas condiciones. La estructura conforme de dichas métricas dependerá del número de hojas del recubridor.

(2) Supongamos que $\mathcal{A}'$ es una anillo radialmente simétrico en $\mathcal{Q}(K)$. Esto es, su borde consiste en dos circunferencias C_1', C_2', y $\mathcal{A}'$ está foliado por arcos de geodésicas de $\mathcal{Q}(K)$ que cortan a C_1' y C_2' de forma ortogonal. Entonces, podemos considerar el sector de $\mathcal{A}'$ acotado por dos de estas geodésicas radiales que formen un cierto ángulo γ, posiblemente mayor que 2π. Tras identificar dichas geodésicas, el espacio cociente es una anillo topológico y la métrica ds_K^2 restringida a $\mathcal{A}'$ se proyecta en una métrica bien definida de curvatura constante K sobre dicho cociente. De esta forma obtenemos una métrica conforme bajo las condiciones requeridas.

También podemos hacer construcciones parecidas en las siguientes situaciones.

(3) Cuando $K = 0$: consideramos una banda en $\mathbb{R}^2$ en lugar de un anillo radialmente simétrico, e identificamos dos segmentos de recta ortogonales al borde de la banda.

(4) Cuando $K = -1$: consideramos la región de $\mathcal{Q}(-1) \equiv \mathbb{D}$ acotada por dos horociclos con el mismo punto ideal $p \in \mathbb{S}^1$, junto con dos arcos geodésicos $\mathcal{Q}(-1)$ partiendo de p, e identificamos dichos arcos.

(5) Cuando $K = -1$: consideramos la región de $\mathcal{Q}(-1) \equiv \mathbb{D}$ acotada por dos arcos de circunferencia con puntos extremos ideales en común $p_1,\ p_2 \in \mathbb{S}^1$, junto con dos arcos geodésicos de $\mathcal{Q}(-1)$ que cortan a las circunferencias anteriores ortogonalmente, e identificamos dichos arcos de geodésicos.

Posteriormente, obtenemos una clasificación completa con fórmulas explícitas de todas las soluciones a (7) (Teorema 3.1), y también deducimos los valores de (K, c_1, c_2) para los que existe solución a (7) (Corolario 3.1). Además, la descripción geométrica de dichas soluciones da lugar a nuestro resultado principal (Teorema 3.2):

Todas las soluciones a (7) *hacen que* $(\mathcal{A}, e^u|dz|^2)$ *sea isométrico a uno de los anillos descritos anteriormente.*

En este caso, aunque no consideremos singularidades borde, también tendremos que resolver problemas de periodos puesto que, además de la periodicidad natural que la solución ha de tener por estar definida en un anillo, la condición de Neumann en el borde interior del anillo hará aparecer una condición de periodicidad extra, necesaria para garantizar la buena definición de la solución.

Merece la pena resaltar que, aunque tanto la ecuación como el dominio y las condiciones iniciales en (7) son radialmente simétricas, existen soluciones a (7) que no son radialmente simétricas.

Parte II: Ecuaciones de Monge-Ampère

En esta segunda parte de la memoria, compuesta por los Capítulos 4 y 5, estudiaremos la clase de ecuaciones de Monge-Ampère elípticas, es decir, ecuaciones de la forma

$$\det\{D^2u + \mathcal{A}(.,u,Du)\} = f(.,u,Du), \tag{8}$$

en dominios $\Omega \subset \mathbb{R}^n$.

Aquí $\mathcal{A}$ y f son funciones $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathcal{M}_n(\mathbb{R})$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, y consideraremos soluciones $u \in C^2(\Omega)$.

En el Capítulo 4 explicamos brevemente cómo, originalmente, el estudio de esta EDP de segundo orden fue desarrollado para resolver problemas de transporte óptimo para una función coste dada. Sin embargo, la gran cantidad de aplicaciones de las ecuaciones de Monge-Ampère también incluyen la Geometría Diferencial. Por ejemplo, mostraremos cómo la ecuación de Hessiano uno $\det(D^2u) = 1$ en dimensión dos aparece en la descripción de superficies mínimas en $\mathbb{R}^3$, superficies llanas en el espacio hiperbólico $\mathbb{H}^3$, esferas afines impropias, o superficies Lagrangianas especiales entre otros ejemplos. Además, explicaremos cuál es la expresión exacta de la ecuación de Monge-Ampère que aparece en otros problemas geométricos como el problema del embebimiento local de métricas en $\mathbb{R}^3$, la construcción de grafos con curvatura predeterminada en espacios modelo, etc.

En el Capítulo 5 presentaremos nuestros resultados acerca de la ecuación de Monge-Ampère elíptica (8) en dimensión dos, la cual puede escribirse como

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{9}$$

donde $A = A(x, y, z, p, q), \ldots, E = E(x, y, z, p, q)$, y usamos la notación $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. Consideraremos sólo soluciones elípticas z en un disco punteado Ω tales que tanto la solución como su gradiente estén acotados alrededor de la singularidad, que podemos suponer en el origen de $\mathbb{R}^2$. Si la solución no se extiende de manera diferenciable al origen diremos que z es una solución no trivial *acotada y elíptica* de (9) (ver Definición 5.1).

Con todo esto, nuestro **tercer objetivo** será estudiar soluciones acotadas elípticas no triviales de (9) en el caso en que los coeficientes $A, ..., E$ sean analíticos reales y que la solución satisfaga la *condición de Heinz-Beyerstedt* (denotada de forma abreviada como *condición-HeB*), la cual se enuncia como sigue:

> A_p, $A_q + 2B_p$, $C_p + 2B_q$ *y* C_q, *vistas como funciones de* x *e* y *son Liptschitzianas en* $\overline{\Omega}$.

Cabe destacar que existe una noción natural de solución generalizada a ecuaciones de Monge-Ampère elípticas: las *soluciones de viscosidad*. No obstante, incluso los casos más simples de soluciones de ecuaciones de Monge-Ampère con singularidades aisladas no evitables no son soluciones de viscosidad alrededor de la singularidad. En este sentido, necesitamos un enfoque diferente para obtener nuestro resultado de clasificación.

Comenzamos explicando que cada solución tiene asociada una cierta métrica Riemanniana que admite parámetros conformes $(u, v) \in \Lambda$, donde $\Lambda \subset \mathbb{R}^2$ es conformemente equivalente a un disco punteado o a un anillo $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$. Dichos parámetros cumplen un sistema cuasileal con buenas propiedades analíticas. Además, se sabe (cf. [HeB]) que una solución acotada elíptica de (9) tal que se cumpla la condición-HeB es no trivial si y sólo si Λ es conforme a algún anillo $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

Definimos el *gradiente límite* de z en el origen como el conjunto $\gamma \subset \mathbb{R}^2$ de los puntos $\xi \in \mathbb{R}^2$ para los que existe una sucesión $\nu_n \to (0, 0)$ en Ω tal que $(p, q)(\nu_n) \to \xi$.

Entonces, usando el Teorema de Cauchy-Kowalevsky aplicado al sistema diferencial de primer orden con respecto a los parámetros (u, v) que cumple $\mathbf{z} = (x, y, z, p, q)$, obtenemos el siguiente resultado de existencia (Teorema 5.1):

> *Sea* γ *una curva de Jordan regular, real analítica y estrictamente convexa en* $\mathbb{R}^2$. *Entonces, dados ciertos coeficientes analíticos*

$A, ..., E$ en un abierto $\mathcal{U} \subset \mathbb{R}^5$ que contiene a $\{(0,0,0,\gamma)\}$ y tales que la ecuación (9) *para dichos coeficientes es elíptica en $\mathcal{U}$, existe una solución no trivial acotada y elíptica z de* (9) *cuyo gradiente límite en el origen es la curva de partida γ.*

Además, supongamos que los coeficientes $A, \ldots, E$ son tales que:

Las funciones A_p, $A_q + 2B_p$, $C_p + 2B_q$ y C_q no dependen de p y q en $\mathcal{U}$. $(\star)$

Entonces dicha solución z cumple la condición-HeB.

Por otro lado, usando técnicas de EDPs aplicadas al sistema cuasilineal de segundo orden que cumple $\mathbf{z} = (x, y, z, p, q)$, tales como un método de bootstrapping, el Lema de Hopf y un resultado de continuación analítica en [Mu2], podemos obtener el siguiente resultado de unicidad (Teorema 5.2):

Si z es una solución acotada elíptica no trivial de (9) *en un disco punteado tal que la condición-HeB se cumple, entonces el gradiente límite es una curva de Jordan regular, analítica real y estrictamente convexa.*

Dos soluciones acotadas elípticas no triviales de (9) *en un disco punteado con el mismo gradiente límite en el origen son iguales.*

Los resultados anteriores proporcionan una clasificación total del espacio de soluciones acotadas elípticas no triviales de (9) que cumplen la condición $(\star)$, en términos del espacio de curvas de Jordan regulares, analíticas reales, estrictamente convexas en $\mathbb{R}^2$ (ver Teorema 5.3).

Por tanto, generalizamos así teoremas previos sobre soluciones elípticas a ecuaciones de Monge-Ampère en presencia de singularidades aisladas como los mencionados en el Capítulo 4 para soluciones a (4.4) y (4.18), y también resultados más generales en [Bey1, Bey2, HeB].

Finalmente, nuestro **cuarto objetivo** será usar los resultados anteriores con el fin de obtener la clasificación de las singularidades aisladas que aparecen al tratar ciertos problemas geométricos explicados en el Capítulo 4. Como consecuencia, obtendremos interesantes resultados relacionados con:

(i) Grafos de curvatura predeterminada positiva en $\mathbb{R}^3$.

(ii) Grafos de curvatura predeterminada $K > -1$ en el espacio hiperbólico $\mathbb{H}^3$ y con curvatura $K > 1$ en $\mathbb{S}^3$.

(iii) Superficies espaciales con curvatura $K < 0$ en $\mathbb{L}^3$.

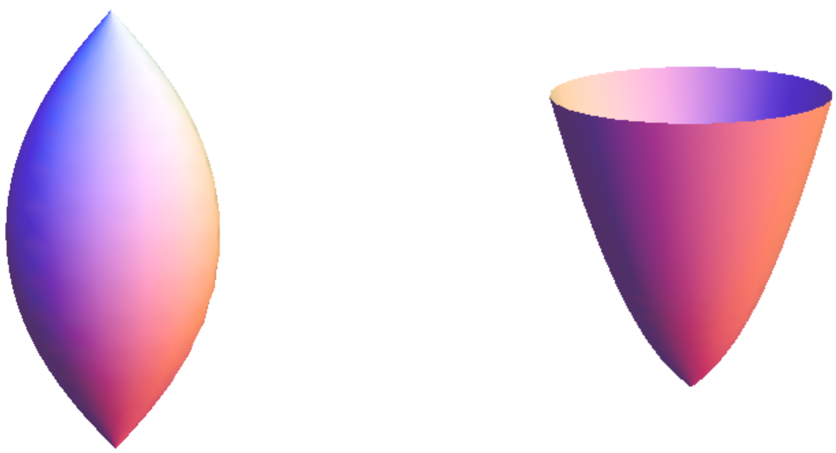
(iv) Superficies lineales de Weingarten de tipo elíptico en $\mathbb{R}^3$.

La mayoría de los resultados presentados en esta memoria forman parte de los artículos de investigación [GJM2, GJM3, Jim].

Introduction

Many interesting mathematical problems are better understood when they are considered simultaneously from the viewpoint of different mathematical theories. A typical example of this situation is the interplay between Differential Geometry and nonlinear Partial Differential Equations (PDEs), a topic which constitutes the basis of the mathematical branch of Geometric Analysis. Indeed, there are many natural problems that can be formulated both in a purely geometric way and from a strict PDE theory point of view, and whose solutions are based fundamentally on both approaches. This phenomenon is illustrated by many important mathematical advances, both classical and recent. We may quote for instance the Lagrange's derivation of the PDE describing minimal surfaces in Eulidean space $\mathbb{R}^3$, the famous classical problems by Weyl, Christoffel or Minkowski on the existence of convex surfaces with prescribed curvatures in $\mathbb{R}^3$, the advances regarding Yamabe and Kazdan-Warner problems, and geometric flows, in particular, Perelman's solution of the Poincaré conjecture. The development of the theory along all these years has produced powerful and flexible techniques such as the method of moving planes for elliptic PDEs, whose geometric counterpart is commonly known as the Alexandrov Reflection Principle.

A central topic in the theory of nonlinear elliptic PDEs is the study of isolated singularities. Typical questions about this problem include under which conditions are isolated singularities removable, or how to classify non-removable singularities of the equation if they exist. For many quasilinear elliptic equations, isolated singularities are removable if and only if the solution lies in a suitable Sobolev space, see [GiTr]. For instance, Bers' theorem [Ber] shows that any isolated singularity of a solution to the minimal surface equation is removable, and this result has been greatly extended to more general PDEs. But there exist certain elliptic Monge-Ampere equations whose solution u and its first derivatives are bounded around the singularity but u does not extend C^2 across it (see the figure below). That is, there exist Monge-Ampère equations with bounded solutions in Ω and a *non-removable* singularity at the puncture.



A rotational peaked sphere ($K = 1$) in $\mathbb{R}^3$ and a solution to $u_{xx}u_{yy} - u_{xy}^2 = 1$ with a conical singularity.

All of this shows that even in dimension two, the problem of deciding if a nonlinear elliptic PDE admits non-removable isolated singularities, and of classifying those singular solutions in the case that they exist, is certainly non-trivial and of clear analytic and geometric interest.

In the present memoir we will study the solutions of certain elliptic nonlinear geometric PDEs in dimension two in the presence of isolated singularities, both at interior points and boundary points. We divide our exposition into two different parts whose contents we explain next.

Part I: The Liouville equation

We devote the first part of the memoir, consisting of the first three chapters, to the study of the famous Liouville equation

$$\Delta v + 2Ke^v = 0,$$

which may be regarded in many ways as the simplest nonlinear conformally invariant elliptic PDE, and whose study traces back to works in the 19th century by Poincaré, Picard, Liouville and others. It happens that if we consider a planar domain $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$, then the Riemannian metric $e^v|dz|^2$, where v is a solution to the Liouville equation and z is a complex coordinate in Ω, has constant Gaussian curvature K. Then, the problem of finding conformal metrics of constant curvature in a domain can be reduced to solving the Liouville equation. Moreover we suppose up to scaling that $K \in \{-1, 0, 1\}$.

Our **first objective** in this memoir is to study in detail the so-called *geometric Neumann problem* for this equation on a planar domain Ω, possibly in the presence of isolated boundary singularities. This problem asks for the existence of a conformal metric of constant curvature K in Ω such that each component of the boundary $\partial\Omega$ has constant geodesic curvature with respect to that metric. By the conformal invariance of the Liouville equation we may consider Ω to be in general a symmetric domain, so in our study we will focus on the case that Ω is the upper half-plane $\mathbb{R}^2_+$ or an annulus in $\mathbb{R}^2$.

With this purpose, in Chapter 1 we give preliminaries about conformal metrics of constant curvature. We introduce the concept of isolated singularity of a metric and, as a particular case, those of *conical type* (see Definition 1.5) as they are going to be specially considered in Chapter 2. Later on, we explain the relation of the Liouville equation with complex analysis which will be decisive to obtain our results in Chapters 2 and 3. This link is provided by the function g, called *the developing map*, in the Liouville Theorem (Theorem 1.3):

> *Let $v : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \longrightarrow \mathbb{R}$ denote a solution to the Liouville equation in a simply connected domain Ω. Then there exists a locally univalent meromorphic function g (holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$) in Ω such that*
>
> $$v = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}. \tag{1}$$
>
> *Conversely, if g is a locally univalent meromorphic function (holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$) in Ω, then v defined as in* (1) *is a solution to the Liouville equation in Ω.*

With this, we have that g is a local isometry $g : (\Omega, ds^2) \longrightarrow (\Sigma_K, ds^2_K)$ where $\Sigma_K = \mathbb{D}, \mathbb{C}, \overline{\mathbb{C}}$ respectively for $K = -1, 0, 1$, and ds^2_K is the Riemannian metric on Σ_K given by

$$ds_K^2 = \frac{4|d\zeta|^2}{(1+K|\zeta|^2)^2}. \tag{2}$$

In Chapter 1 we also explain how Liouville Theorem can be generalized for the case when the domain Ω is not simply connected, and we introduce the concept of *Schwartzian derivative* that will be a very useful tool in our approach. It is denoted by Q, and it is given by the formulas below, where g is the developing map of v:

$$Q := v_{zz} - \frac{1}{2}\, v_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2.$$

After that, we discuss the main aspects and simplest examples of initial value problems for the Liouville equation.

Finally, we explain how the Liouville equation also appears in other topics of surface theory such as minimal surfaces in $\mathbb{R}^3$, surfaces of constant mean curvature one in the hyperbolic 3-space $\mathbb{H}^3$, flat surfaces in $\mathbb{H}^3$, or a large class of Weingarten surfaces.

In Chapter 2 we present our results on the solution of the geometric Neumann problem for the Liouville equation in the upper half plane with a finite number of singularities on the boundary. Analytically this problem is given by:

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 :\ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_j\, e^{v/2} & \text{on } I_j \subset \mathbb{R} \equiv \partial\mathbb{R}^2_+, \qquad c_j \in \mathbb{R}, \end{cases} \tag{3}$$

where $I_j := (q_j, q_{j+1})$, $j = 0, \ldots, n-1$ and $q_0 = -\infty < q_1 < \cdots < q_{n-1} < q_n = \infty$. With these Neumann boundary conditions, we are imposing possibly different values of constant geodesic curvature $-\frac{c_j}{2}$, along each interval I_j.

This problem was first studied without boundary singularities and arbitrary dimension in the works [CLi1, CSF, GaMi3, HaWa, LiZh, LiZha, Ou, Zha]. For the case of one boundary singularity (which can be supposed to be at the origin), Jost, Wang and Zhou [JWZ] gave a complete classification of the solutions to the above problem under the following assumptions:

1. The metric $e^v|dz|^2$ has finite area in $\mathbb{R}^2_+$, i.e.

$$\int_{\mathbb{R}^2_+} e^v < \infty. \tag{4}$$

2. The boundary $\partial\mathbb{R}^2_+$ has finite length for the metric $e^v|dz|^2$, i.e.

$$\int_{\mathbb{R}_-} e^{v/2} + \int_{\mathbb{R}_+} e^{v/2} < \infty. \tag{5}$$

3. The metric $e^v|dz|^2$ has a *boundary conical singularity* at the origin (see Definition 1.5), i.e. there exists $\lim_{z\to 0} |z|^{-2\alpha}e^v \neq 0$ for some $\alpha > -1$.

4. $K = 1$.

With these hypotheses, they obtain in [JWZ, Theorem 1.2] the explicit formulas for the solutions of (3) and provide their geometric interpretation (see Section 2.1).

Along Chapter 2 we will show several improvements of the Jost-Wang-Zhou theorem. The results we will present about this geometric Neumann problem in the presence of boundary singularities can be summarized as follows:

(i) For the case of only one boundary singularity, that we suppose at the origin, we obtain a general classification theorem in the spirit of [GaMi4] where we suppress the four extra hypotheses in [JWZ] mentioned above (see Theorem 2.3). To prove this result we use the techniques of complex analysis introduced in Chapter 1 and the fact that, by the Neumann conditions in (3), the images by the developing map g of the axes $\mathbb{R}^+$ and $\mathbb{R}^-$ lie, respectively, in certain circles in $\overline{\mathbb{C}}$.

(ii) We will characterize the asymptotic behavior of a solution to the geometric Neumann problem around a boundary singularity, when the area of the metric $e^v|dz|^2$ is finite around the singularity. That is, we suppose that the singularity is placed at the origin and we consider a certain domain $D^+_\varepsilon = \{z \in \mathbb{R}^2_+ : \ |z| < \varepsilon\}$, $\varepsilon > 0$. Then, the finite area condition in D^+_ε is

$$\int_{D^+_\varepsilon} e^v < \infty. \tag{6}$$

We obtain three different types of asymptotic behaviors, one of which corresponds to the singularity being conical (see Corollary 2.1). Moreover, as a consequence of the geometric asymptotic behavior of such solutions, we obtain the following corollary (Corollary 2.3):

Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ *be a solution to the local geometric Neumann problem that satisfies the finite energy condition* (6)*, and let* $g : \overline{D_\varepsilon^+} \setminus \{0\} \longrightarrow \Sigma_K \subseteq \overline{\mathbb{C}}$ *denote its developing map, which is a local diffeomorphism. Then:*

(a) The image $g(\overline{D_\varepsilon^+} \cap \mathbb{R}^+)$ *lies on a circle* $\mathcal{C}_1$*, and the image* $g(\overline{D_\varepsilon^+} \cap \mathbb{R}^-)$ *lies on another circle* $\mathcal{C}_2$*, such that* $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ *(possibly* $\mathcal{C}_1 = \mathcal{C}_2$*).*

(b) The geodesic curvature of $\mathcal{C}_i$*, when parameterized as* $g(s,0)$*, is constant of value* $-c_i/2$*, for the metric* ds_K^2 *in* (2)*.*

(c) g *extends continuously to the origin, with* $g(0) \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset \overline{\mathbb{C}}$*.*

(d) The Schwarzian map $Q = v_{zz} - v_z^2/2$ *extends holomorphically to* D_ε^* *with* $Q(\bar{z}) = \overline{Q(z)}$*, and has at the origin at most a pole of order two.*

(e) If $K = 0$*, then* $g(0) \in \mathbb{C}$*. If* $K = -1$ *and* $g(0) \in \partial\mathbb{D} \equiv \mathbb{S}^1$*, then* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are tangent at* $g(0)$*, and are* not *arcs of horocycles.*

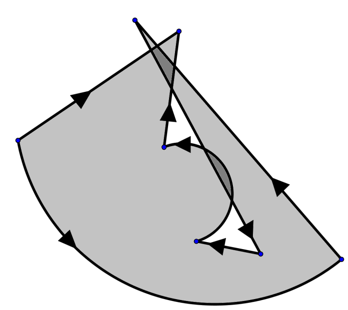
(iii) For the global problem (3) with one boundary singularity, we classify all the solutions that satisfy the finite area condition (4) and we determine for what values of (K, c_1, c_2) they do exist. Specifically, first we describe a family of solutions to (3) which we call the *canonical solutions* (see Section 2.2). We provide their explicit formulas and explain their geometric interpretation. Then, we prove that (Theorem 2.3):

Any solution to (3) *with one boundary singularity at the origin and finite area is a* canonical solution.

As a consequence, we recover the solutions in [JWZ] without additional hypotheses on the boundary, the sign of K and the shape of the singularity.

(iv) For the case of an arbitrary number of boundary singularities, we will also describe all the solutions to (3) with finite area from a geometric perspective. The basic examples of conformal metrics of constant curvature with boundary singularities and constant geodesic curvature along each boundary component are the ones determined by circular polygons in $\overline{\mathbb{C}}$, but there are many others. To obtain this larger family, we consider *immersed* circular polygons for which we allow self-intersections (see the picture below), and give a differential-topological criterion (Alexandrov embeddedness) for them to generate such metrics (see Definition 2.5). With all of this, Corollary 2.4 proves the converse:

> *Any conformal metric of finite area and constant curvature on $\mathbb{R}^2_+$ (or equivalently on the unit disc $\mathbb{D}$), with finitely many boundary singularities and constant geodesic curvature along each boundary component, is one of those* circular polygonal metrics *constructed from Alexandrov-embedded, possibly self-intersecting, circular polygons.*



An Alexandrov-embedded polygon.

Analytically, those metrics will not have simple explicit expressions, although one can still give some analytic information about them. In Corollary 2.5 we will describe for $K = 1$ the moduli space of these metrics, by parameterizing it in terms of their associated Schwarzian maps, which have simple explicit expressions.

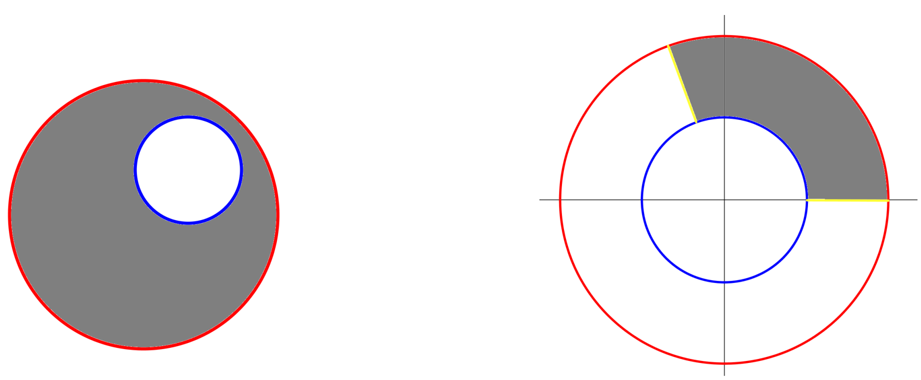
Chapter 3 is devoted to achieve our **second objective** in this thesis, also dealing with the Liouville equation. It is to solve the geometric Neumann problem in the case that Ω is an annulus, which corresponds to solving the following nonlinear Neumann problem:

$$\begin{cases} \Delta u + 2Ke^u = 0, & \text{in } \mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu_1} = c_1 e^{\frac{u}{2}} + 2, & \text{on } C_1 = \{z \in \mathbb{C} : \ |z| = 1\}, \\ \dfrac{\partial u}{\partial \nu_2} = c_2 e^{\frac{u}{2}} - 2e^{r\pi}, & \text{on } C_2 = \{z \in \mathbb{C} : \ |z| = e^{-r\pi}\}. \end{cases} \tag{7}$$

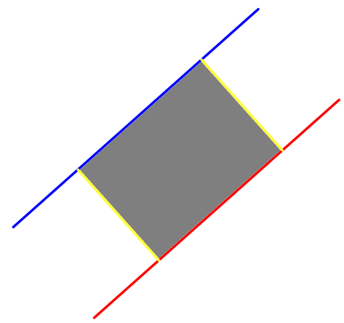
Here ν_i denotes the interior unit normal to C_i, $i = 1, 2$ respectively, and $r > 0$ is a constant. As in the case of the upper half plane we suppose up to dilation that $K = \{-1, 0, 1\}$.

Note that, together with the punctured disc, the annulus is the most symmetric non-simply connected domain we can consider. In this sense, the results we show constitute a natural generalization of previous results in [HaWa, BHL, ChWa]. In fact, since different values of r provide annuli that are not conformally equivalent, we are considering a family of problems that are also not conformally equivalent.

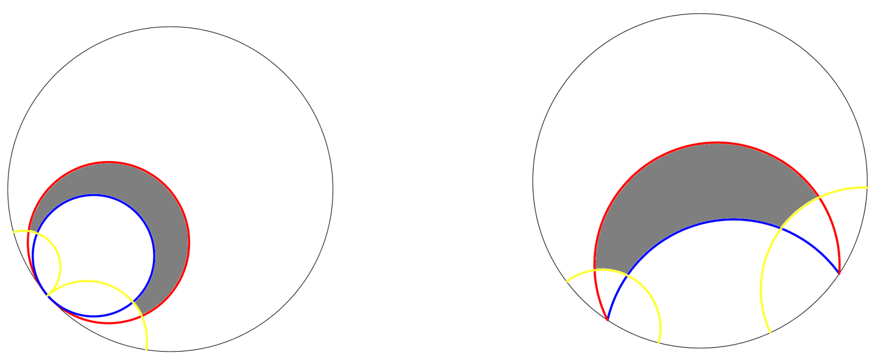
We first present a certain family of *canonical annuli* which are geometrically constructed in a very simple way. They correspond to the following pictures where the curves in yellow are geodesic arcs which must be identified.



A general topological annulus maybe composed with a finite-folded covering map, or a sector of a radially symmetric annulus bounded by two geodesic segments that we identify.



A portion of a strip (case $K = 0$).



A portion of the region between two tangent horocycles or between two curves with geodesic curvature $|k_g| < 1$ (case $K = -1$).

Later on, we obtain a complete classification by explicit formulas of all the solutions to (7) (Theorem 3.1), and we also deduce the values of (K, c_1, c_2) for which a solution of (7) exists (Corollary 3.1). Moreover, the geometric description of such solutions yields our main result (Theorem 3.2):

All the solutions to (7) *make* $(\mathcal{A}, e^u|dz|^2)$ *be isometric to one of the annuli described above.*

In this case, although we do not consider boundary singularities, we will also deal with period problems since, besides of the the natural periodicity that the solution must have as it is defined in an annulus, the

Neumann condition at the interior boundary will make an extra periodicity condition appear in order to guarantee that the solutions are well defined.

It is worth pointing out that, even though the equation, the domain and the initial conditions in (7) are all radially symmetric, there exist solutions to (7) that are not radially symmetric.

Part II: Monge-Ampère equations

In this second part of the memoir, composed by Chapters 4 and 5, we will consider the class of elliptic Monge-Ampère equations, i.e. equations of the form

$$\det\{D^2u + \mathcal{A}(., u, Du)\} = f(., u, Du), \tag{8}$$

in domains $\Omega \subset \mathbb{R}^n$.

Here $\mathcal{A}$ and f are functions $\mathcal{A} : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathcal{M}_n(\mathbb{R})$, $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$, and we will consider solutions $u \in C^2(\Omega)$.

In Chapter 4 we explain briefly how, originally, the study of this second order PDE was developed to solve optimal transportation problems for a given cost function. However, the wide range of applications of Monge-Ampère equations also includes Differential Geometry. For instance, we will show how the Hessian one equation $\det(D^2u) = 1$ in dimension two appears in the description of minimal surfaces in $\mathbb{R}^3$, flat surfaces in hyperbolic three-space, improper affine spheres or special Lagrangian surfaces, among other examples. Moreover, we will explain what is exactly the expression of the Monge-Ampère equation that appears in other geometric problems such as the local embedding problem for metrics in $\mathbb{R}^3$, the construction of graphs with prescribed curvature in model spaces, etc.

In Chapter 5 we will present our results on the elliptic Monge-Ampère equation (8) in dimension two, which we will write as

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{9}$$

where $A = A(x, y, z, p, q), \ldots, E = E(x, y, z, p, q)$, and we use the notation $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. We consider only elliptic solutions z on a punctured disc Ω such that both the solution and its gradient are bounded around the puncture, which can be assumed to be the origin in $\mathbb{R}^2$. If the solution does not extend smoothly across the origin, we will say that z is a non-trivial *bounded elliptic solution* to (9) (see Definition 5.1).

With all of this, our **third objective** will be to study the non-trivial bounded solutions to (9) in the case that the coefficients $A, ..., E$ are real analytic, and that the solution satisfies the *Heinz-Beyerstedt condition* (*HeB-condition* in short), which is stated as follows:

> A_p*,* $A_q + 2B_p$*,* $C_p + 2B_q$ *and* C_q *are Liptschitz continuous in* $\overline{\Omega}$ *considered as functions of* x *and* y*.*

Let us point out that there is a natural notion of generalized solution to elliptic Monge-Ampère equations: the *viscosity solutions*. However, even the simplest examples of solutions to elliptic Monge-Ampère equations with a non-removable isolated singularity are not solutions in the viscosity sense around the singularity. In this sense, a different approach is needed for our classification result.

First, we explain that the solution has attached a certain Riemannian metric that admits conformal parameters $(u, v) \in \Lambda$, where $\Lambda \subset \mathbb{R}^2$ is conformally equivalent to either the punctured disc $\mathbb{D}^*$ or an annulus $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$. Such parameters satisfy a quasilinear system with good analytic properties. Moreover, it is known (cf. [HeB]) that a bounded elliptic solution to (9) such that the HeB-condition holds is non-trivial if and only if Λ is conformally equivalent to some annulus $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

We define the *limit gradient* of z at the origin to be the set $\gamma \subset \mathbb{R}^2$ of points $\xi \in \mathbb{R}^2$ for which there is a sequence $\nu_n \to (0, 0)$ in Ω such that $(p, q)(\nu_n) \to \xi$.

Then, by means of the Cauchy-Kowalevsky Theorem applied to the first order differential system that $\mathbf{z} = (x, y, z, p, q)$ satisfies with respect to the parameters (u, v) we obtain the following existence result (Theorem 5.1):

> *Let* γ *be a real analytic, strictly convex regular Jordan curve in* $\mathbb{R}^2$*. Then, given some analytic coefficients* $A, ..., E$ *in an open subset* $\mathcal{U} \subset \mathbb{R}^5$ *that contains* $\{(0, 0, 0, \gamma)\}$ *and so that the equation* (9) *for these coefficients is elliptic on* $\mathcal{U}$*, there exists a non-trivial bounded elliptic solution* z *to* (9) *whose limit gradient at the origin is the curve* γ *we started with.*
>
> *Moreover, suppose that the coefficients* $A, \dots, E$ *are such that:*
>
> > *The functions* A_p*,* $A_q + 2B_p$*,* $C_p + 2B_q$ *and* C_q *do not depend on* p *and* q *in* $\mathcal{U}$*.* $(\star)$

Then the previous solution z satisfies the HeB-condition.

On the other hand, using techniques of PDEs applied to the second order quasilinear system satisfied by $\mathbf{z} = (x, y, z, p, q)$ such as a bootstrapping method, the Hopf Lemma and a continuation result in [Mu2], we can give the following uniqueness result (Theorem 5.2):

If z denotes a non-trivial bounded elliptic solution to (9) *in a punctured disc such that the HeB-condition holds, then the limit gradient γ is a regular real analytic strictly convex Jordan curve.*

Two non-trivial bounded elliptic solutions to (9) *in a punctured disc with the same limit gradient at the origin must coincide.*

The previous results provide a complete classification of the space of non-trivial bounded elliptic solutions to (9) that satisfy the condition $(\star)$, in terms of the space of real analytic, strictly convex, regular Jordan curves in $\mathbb{R}^2$ (see Theorem 5.3).

Thus, we generalize previous theorems on solutions to elliptic Monge-Ampère equations in the presence of isolated singularities as the ones mentioned in Chapter 4 for equations (4.4) and (4.18), and also more general results in [Bey1, Bey2, HeB].

Finally, our **fourth objective** will be to use the previous results in order to obtain a classification of the isolated singularities that appear when dealing with some geometric problems explained in Chapter 4. As a consequence, we will obtain interesting results related with:

(i) Graphs of prescribed positive curvature in the Euclidean space $\mathbb{R}^3$.

(ii) Graphs of prescribed curvature $K > -1$ in the hyperbolic three-space $\mathbb{H}^3$ and with curvature $K > 1$ in $\mathbb{S}^3$.

(iii) Spacelike surfaces with curvature $K < 0$ in $\mathbb{L}^3$.

(iv) Linear Weingarten surfaces of elliptic type in $\mathbb{R}^3$.

Most of the results presented in this memoir are a part of the scientific papers [GJM2, GJM3, Jim].

Part I

The Liouville equation

1

Conformal metrics of constant curvature

1.1 Conformal metrics on a surface

We start this section by giving some preliminaries about conformal metrics on a surface.

Definition 1.1. *We say that two Riemannian metrics ds^2 and $\widehat{ds^2}$ on a manifold M are* conformal *or equivalently, that they are in the same* conformal class*, if they coincide up to a conformal factor, that is, if there exists a global positive function $\lambda > 0$ such that $\widehat{ds^2} = \lambda ds^2$.*

Definition 1.2. *A diffeomorphism $\psi : (M, ds^2) \longrightarrow (N, \widehat{ds^2})$ between two Riemannian manifolds is said to be* conformal *if the pullback metric $\psi^*(\widehat{ds^2})$ is conformal to ds^2.*

Definition 1.3. *Let S be a Riemann surface and ds^2 a Riemannian metric in S. We say that ds^2 is a conformal metric in S if for every local complex parameter $z = s + it$ in a open set $\mathcal{U} \subset S$ we have that $ds^2 = \lambda|dz|^2$ for a certain positive function λ defined in $\mathcal{U}$.*

The previous definition is obviously independent of the local complex parameter that we consider on each open set $\mathcal{U} \subset S$.

On the other hand, if (S, ds^2) is an orientable surface endowed with a Riemannian metric, the process above can be reversed. The key point is that around any point $p \in S$ there exist local isothermal parameters, that is, parameters (s, t) defined in a neighborhood of p such that $ds^2 = \lambda(ds^2 + dt^2)$ for a certain positive function λ. More specifically, consider the atlas of S formed by all the local isothermal coordinate systems positively oriented in S. Then the changes of charts satisfy the Cauchy-Riemann equations and hence they define a structure of Riemann surface on S with respect to which the metric ds^2 is a conformal metric in the sense of Definition 1.3. We will call such a structure of Riemann surface on S *the conformal structure on S induced by the metric ds^2*. With all of this, we will say equivalently that (s, t) are isothermal parameters or that $z = s + it$ is a *complex conformal parameter* for the metric ds^2.

The classification of simply connected Riemann surfaces is given by the famous Uniformization Theorem.

Theorem 1.1. *Every simply connected Riemann surface is biholomorphic to the unit disc $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.*

Theorem 1.1 was first proved by Koebe and Poincaré independently in 1907. It is a classification theorem of all Riemann surfaces according to their universal covering spaces. It reduces many aspects of Riemann surfaces to the study of the disc, the plane, and the sphere. In this sense, it can be seen as a generalization of the Riemann mapping theorem from proper simply connected open subsets of the plane to arbitrary simply connected Riemann surfaces.

In particular, we have the following consequence. Let (S, ds^2) be an orientable compact Riemannian surface which can be seen as a Riemann surface considering on S the conformal structure induced by ds^2 as explained before. Let $\widetilde{S}$ be the universal cover of S that, by the Uniformization Theorem 1.1 is $\overline{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. Now, observe that each of these spaces admits a conformal metric of constant curvature K. Specifically, K is positive for $\widetilde{S} = \overline{\mathbb{C}}$, $K = 0$ when $\widetilde{S} = \mathbb{C}$ and K is negative if $\widetilde{S} = \mathbb{D}$. As a consequence, S also admits a conformal metric of constant curvature, that is, there exists a metric $\widehat{ds^2}$ on S which has constant curvature and is conformal to the original metric ds^2 (see [FaKr] for more details).

The extension of this property to the case of Riemannian metrics on n-dimensional manifolds is known as the classical Yamabe problem which was originally formulated by Hidehiko Yamabe [Ymb]. More specifically, given a smooth, compact, Riemannian manifold (M, ds^2) without boundary and dimension $n \geq 3$, Yamabe asked if there exists a metric conformal to ds^2 for which the scalar curvature is constant. This question has a positive answer given by Neil Trudinger, Thierry Aubin, and Richard Schoen in [Aub, Sch, Tdg]. Nevertheless, if we remove the compactness hypothesis the answer is not positive anymore, some counterexamples were given by Jin in [Jin].

Another related problem is the so called Kazdan-Warner problem (Nirenberg problem for the two dimensional case) that can be formulated as follows. Let $(\mathbb{S}^n, ds_0^2)$ be the standard n-dimensional sphere. Which functions $R : \mathbb{S}^n \longrightarrow \mathbb{R}$ are the scalar curvature of a metric ds^2 on $\mathbb{S}^n$ that is conformally equivalent to ds_0^2? This problem has received a lot of contributions in the last decades. As part of the works in this subject we can mention [AmMa, BaCo, BoEz, Cha, CY1, CY2, CLi2, CLi3, CLn, EsSc, KW1, KW2, Li1, Li2, Mo] and the survey [Li3] for more details. Yet, the characterization of those functions on $\mathbb{S}^n$ is still uncomplete.

The point is that all those geometric problems can be described in terms of non-linear elliptic partial differential equations. Specifically, a computation shows that if R_0 and R are the scalar curvatures of two conformal metrics ds_0^2 and ds^2 on a n-dimensional manifold M such that: $ds^2 = e^v ds_0^2$ if $n = 2$, and $ds^2 = v^{\frac{4}{n-2}} ds_0^2$ if $n > 2$, where $v \in \mathcal{C}^\infty(M)$ and $v > 0$ if $n > 2$, the following relation holds

$$\begin{cases} \Delta v = R_0 - Re^v & \text{if} \quad n = 2, \\ \Delta v = \dfrac{(n-2)}{4(n-1)}(R_0 v - R v^{\frac{n+2}{n-2}}) & \text{if} \quad n > 2. \end{cases} \tag{1.1}$$

Here Δ denotes the Laplacian with respect to the metric ds_0^2. Formula (1.1) is the so called *Liouville equation*.

Conversely, if we start with a metric ds_0^2 whose scalar curvature is R_0, a solution v of (1.1) where R is a function on M will give a metric ds^2 in the same conformal class of ds_0^2, whose scalar curvature is equal to R.

In particular, if we consider a domain in the Euclidean space $\Omega \subset \mathbb{R}^n$ and we look for a metric of constant scalar curvature R on Ω, then, the desired metric is *conformally flat* and equation (1.1) reduces to

$$\begin{cases} \Delta v = -Re^{v} & \text{if} \quad n = 2, \\ \Delta v = -\dfrac{(n-2)}{4(n-1)} R v^{\frac{n+2}{n-2}} & \text{if} \quad n > 2; \end{cases} \tag{1.2}$$

where now Δ denotes the Laplacian with respect to the usual flat metric on $\mathbb{R}^n$, $dx_1^2 + \cdots + dx_n^2$. Observe that, for the two-dimensional case, the metric $ds^2 = e^v(dx_1^2 + dx_2^2)$ will have Gauss curvature $K = R/2$.

Note also that, by construction, the Liouville equation is conformally invariant.

A natural step in the study of conformal metrics of constant (or prescribed) curvature is to consider metrics with singularities. Since a singularity on the conformal metric yields a singularity in a solution of the elliptic PDE (1.1), this type of problems also have interest from a purely analytic point of view. Concretely, we will consider metrics with isolated singularities on Riemann surfaces with and without boundary.

First we give some definitions.

Definition 1.4. *Let S be a Riemann surface with or without boundary. If ds^2 is a Riemannian metric on $S \setminus \{p\}$ we say that ds^2 has an isolated singularity (respectively, boundary singularity) at a point $p \in S$ (respectively, $p \in \partial S$). If ds^2 does not extend as a Riemannian metric to S the singularity is said non-removable, otherwise the singularity is removable.*

In this sense, we can talk about the *asymptotic behavior* of a conformal Riemannian metric $ds^2 = \lambda|dz|^2$ (where z is a complex parameter defined in S) close to the singularity at $p \in S$ as the local behavior of the conformal factor λ when we approach to p. In particular, we will be interested in the following type of singularities.

Definition 1.5. *Let $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$ be a planar domain and z a complex parameter in Ω. It is said that a conformal metric $ds^2 = \lambda|dz|^2$ on $\Omega \setminus \{p\}$ has a* conical singularity *at a point $p \in \Omega$ (respectively, a* boundary conical singularity *or* corner *at $p \in \partial\Omega$), if there exists $\lim_{z \to p} |z-p|^{-2\alpha}\lambda \neq 0$ for some $\alpha > -1$. In that case, we say that the conical angle of the singularity is $2\pi(\alpha+1)$ (respectively, $\pi(\alpha+1)$).*

Observe that Definition 1.5 does not depend on the conformal parameter in Ω that we consider.

Some examples of Riemannian manifolds with isolated singularities, according to Definition 1.5, are the following. In the case $-1 < \alpha < 0$, we can mention an American football that has two singularities of equal angle, and a teardrop that has only one. In the case $\alpha > 0$, the angle is larger than 2π, leading to a different geometric picture. Such singularities also appear in orbifolds and branched coverings. Geometrically, we could say that a manifold with a conical singularity at an interior point p has a tangent cone of conical angle $2\pi(\alpha + 1) > 0$. This geometric description will be extended for the case of metrics of constant curvature when the singularity happens at $p \in \partial S$ (see Subsection 2.2.2).

Its geometric interpretation, and the fact that they describe the asymptotic behavior of metrics under certain bounded energy conditions, make conical singularities be one of the most studied situations in this topic. In this sense, we may cite the works [BaTa, Bry, ChWa, GHM, GJM2, JWZ, Tro1].

Finally, to end this subsection, it also seems convenient to mention some details about an open problem related to this topic. We do that in the following remark.

Remark 1.1. *Let (S, ds^2) be a compact Riemann surface without boundary and conical singularities of order $\alpha_1, \alpha_2, \ldots, \alpha_n$ at $p_1, p_2, \ldots, p_n \in M$. Then, if there exists a conformal metric to ds^2, $\widehat{ds^2}$, with constant curvature $\widehat{K}$, the following restriction must be satisfied*

$$\frac{1}{2\pi}\int_S \widehat{K} d\widehat{A} = \chi(S) + \sum_{i=1}^{n} \alpha_i, \tag{1.3}$$

where $\chi(S)$ is the Euler characteristic of S. Such a restriction is a consequence of the Gauss-Bonnet formula, and becomes sufficient for the existence of $\widehat{ds^2}$ if $\widehat{K} \leq 0$ (see [Hei, McO, Tro2, Tro3]). In Troyanov's papers [Tro2, Tro3] the case of non-constant $\widehat{K}$ is also considered.

On the contrary, not much is known in the case $\widehat{K} > 0$. Troyanov, in [Tro1], considered the case of two points on the sphere and showed that the necessary and sufficient condition for the existence of solution in this case is $\alpha_1 = \alpha_2$.

However, for a general compact manifold, and three or more singularities, the problem is still open. Nevertheless, we may cite some partial results in this line as the obtained by F. Luo and G. Tian in [LuTi]. There, the authors show that (1.3) *and*

$$\chi(S) + \sum_{i=1}^{n} \alpha_i < \min\{2, 2\min(\alpha_i + 1)\}$$

become necessary and sufficient conditions for the existence and uniqueness of a metric of constant curvature 1 with $n \geq 3$ prescribed conical singularities under the restrictions (i) S is an sphere, and (ii) $\alpha_i \in (-1, 0)$. Moreover, also for the case when S is an sphere, the problem of finding such a unique convex metric when we prescribe exactly three singularities has been completely solved by A. Eremenko in [Er]. That is, their results are obtained without the restriction $\alpha_i \in (-1, 0)$. As a consequence, the author generalizes previous results in [UmYa] where is considered only the case when all the α_i are not integer.

1.2 The Liouville equation and the complex analysis

We have seen that finding a metric of prescribed curvature and conformal to a given metric on a manifold is equivalent to solving equation (1.1). In particular, if we want to find a conformal metric of constant scalar curvature on a certain domain in $\mathbb{R}^n$, we are led to solve (1.2). It is in this last case, for $n = 2$, when besides of the techniques of elliptic partial differential equations, it seems natural to exploit the geometric interpretation of the equation and its relation with complex analysis to solve it.

In this section, we show this classical relation between the Liouville equation

$$\Delta v + 2Ke^v = 0 \tag{1.4}$$

for constant Gauss curvature K (remind $K = R/2$ in (1.2)), and complex analysis. We will explain the tools and the main results in this subject that we will use along this memory.

From now on we will identify $\mathbb{R}^2$ and $\mathbb{C}$, and write $z = s + it \equiv (s, t)$ for points in the domain of a solution to the Liouville equation. As we are interested in metrics of constant curvature, we also assume up to dilation that $K \in \{-1, 0, 1\}$.

We denote by $\mathcal{Q}(K)$ the two-dimensional model space of constant curvature $K = \{-1, 0, 1\}$, that is, $\mathcal{Q}(0) = \mathbb{R}^2$ and

$$\mathcal{Q}(K) = \begin{cases} \mathbb{S}^2(K) = \left\{(x_0, x_1, x_2) : \ x_0^2 + x_1^2 + x_2^2 = \dfrac{1}{\sqrt{K}}\right\} & \text{if } K > 0, \\ \mathbb{H}^2(K) = \left\{(x_0, x_1, x_2) : \ -x_0^2 + x_1^2 + x_2^2 = -\dfrac{1}{\sqrt{-K}}, \ x_0 > 0\right\} & \text{if } K < 0. \end{cases} \tag{1.5}$$

The we have the following well-known result of E. Cartan (cf. [Ca])

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and ds^2 a Riemannian metric such that (Ω, ds^2) has constant curvature $K \in \{-1, 0, 1\}$. Then there exists a local isometry*

$$g : (\Omega, ds^2) \longrightarrow \mathcal{Q}(K).$$

If we consider the *sterographic projection* π from $\mathcal{Q}(K)$ into $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ defined by

$$\begin{aligned} \pi(x_0, x_1, x_2) &= \frac{x_1 + ix_2}{1 - Kx_0}, \quad \text{if } K = \pm 1, \\ \pi(x_1, x_2) &= x_1 + ix_2 \quad\quad \text{if } K = 0, \end{aligned} \tag{1.6}$$

then, by uniformization, $\mathcal{Q}(K)$ can be viewed as (Σ_K, ds_K^2) where

$$\Sigma_K = \begin{cases} \overline{\mathbb{C}} & \text{if } \ K = 1, \\ \mathbb{C} & \text{if } \ K = 0, \\ \mathbb{D} \subset \mathbb{C} & \text{if } \ K = -1, \end{cases} \tag{1.7}$$

and ds_K^2 is the conformal Riemannian metric on Σ_K given, for any $\zeta \in \Sigma_K$ by

$$ds_K^2 = \frac{4|d\zeta|^2}{(1 + K|\zeta|^2)^2}. \tag{1.8}$$

Observe that, if the metric ds^2 in Theorem 1.2 is conformal, then, the map g must be conformal (see Definition 1.2), i.e. it must be meromorphic or anti-meromorphic.

Moreover, we have the following classical result due to Liouville [Li].

Theorem 1.3. *Let $v : \Omega \subset \mathbb{R}^2 \equiv \mathbb{C} \longrightarrow \mathbb{R}$ denote a solution to* (1.4) *in a simply connected domain Ω. Then there exists a locally univalent meromorphic function g (holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$) in Ω such that*

$$v = \log \frac{4|g'|^2}{(1 + K|g|^2)^2}. \tag{1.9}$$

Conversely, if g is a locally univalent meromorphic function (holomorphic with $1+K|g|^2>0$ if $K\le 0$) in Ω, then (1.9) *is a solution to* (1.4) *in Ω.*

Observe that the function g in the above theorem, which is called the *developing map* of the solution, is unique up to Möbius transformations of the form

$$g \mapsto \frac{\alpha g - \bar\beta}{K\beta g + \bar\alpha}, \qquad |\alpha|^2 + K|\beta|^2 = 1, \tag{1.10}$$

which correspond to the orientation preserving isometries of the correspondent model space $\mathcal{Q}(K)$.

Remark 1.2. *The developing map g has a natural geometric interpretation: if $v\in C^2(\Omega)$ is a solution to* (1.4)*, then, its developing map $g:\Omega\longrightarrow\Sigma_K\subseteq\overline{\mathbb{C}}$ provides the local isometry in Theorem 1.2 from $(\Omega, e^v|dz|^2)$ into $\mathcal{Q}(K)\equiv(\Sigma_K, ds_K^2)$, where ds_K^2 is given by* (1.8).

The importance of Theorem 1.3 lies in the fact that it gives the way to produce solutions to the Liouville equation (1.4) in terms of holomorphic functions. Observe also that Theorem (1.3) implies Theorem 1.2.

Now, as a direct consequence of Theorem 1.3, we recall the following result concerning metrics of constant curvature over the whole complex plane $\mathbb{C}\equiv\mathbb{R}^2$. It was originally discovered in [CLi1] through the method of moving planes. The proof we give here is in [ChWa].

Theorem 1.4. *Let $ds^2=e^v|dz|^2$ be a conformal metric of constant curvature K in $\mathbb{R}^2\equiv\mathbb{C}$ with finite area, that is, such that $\int_{\mathbb{R}^2}e^v<\infty$. Then $K>0$ and*

$$ds^2 = \frac{4\lambda^2|dz|^2}{(1+K\lambda^2|z-x_0|^2)^2},$$

for some $\lambda>0$ and $x_0\in\mathbb{C}$.

Proof. By Theorem (1.3), we only need to find a conformal map g from $\mathbb{C}\equiv\mathbb{R}^2$ into the correspondent space $\mathcal{Q}(K)$ in order to obtain a metric of constant curvature K.

If $K>0$, by the finite area condition, g cannot have an essential singularity at infinity. Otherwise it would cover $\overline{\mathbb{C}}$ (possibly except one point) infinitely many times near infinity, which is impossible since $ds^2=g^*(ds_1^2)$ has finite area. Therefore, up to an isometry of $\mathcal{Q}(K)$, we can suppose that $\lim_{z\to\infty}g(z)=\infty$. Then g maps $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. Since $\mathbb{C}$ cannot cover $\overline{\mathbb{C}}$ (notice that $g'(z)\neq 0$ for all $z\in\mathbb{C}$), g does not have poles in $\mathbb{C}$. Thus, $g:\mathbb{C}\longrightarrow\mathbb{C}$

is a covering map and therefore it assumes the form $g(z) = az + b$ for some $a \neq 0$ and $b \in \mathbb{C}$. If we substitute this in (1.9) we obtain the desired conclusion.

In the case $K = 0$, we can do the same reasoning we did for the case $K > 0$ and deduce that, necessarily, the associated developing map must be $g(z) = az+b$ for some $a \neq 0$ and $b \in \mathbb{C}$. Then, the metric will have infinite area which is impossible.

Finally, for the case $K < 0$ we would have an entire function whose image lies in a disc. Thus, the classical Liouville Theorem for holomorphic functions gives that such a function must be constant, which is a contradiction. $\square$

Note that the integrability condition $\int_{\mathbb{R}^2} e^v < \infty$ in Theorem 1.4 can not be removed because, if we take any non-polinomial function whose derivative never vanishes (for example $g(z) = e^z$), then the metric $\frac{4|g'|^2}{(1+|g|^2)^2}|dz|^2$ has constant curvature one and infinite area. Furthermore, by Theorem 1.4, we deduce that the unique conformal metric of curvature one and finite area in the plane is the usual metric on $\mathbb{S}^2 \setminus \{\infty\}(\equiv \mathbb{C})$. Observe that, in particular, as the developing map is $g(z) = az + b$ they are radial with respect to the point $x_0 = -b/a \in \mathbb{C}$.

Next, we define another relevant invariant attached to any solution v of the Liouville equation (1.4). We will denote it by Q, and it is given by the formulas below, where g is the developing map of v:

$$Q := v_{zz} - \frac{1}{2}v_z^2 = \{g, z\} := \left(\frac{g_{zz}}{g_z}\right)_z - \frac{1}{2}\left(\frac{g_{zz}}{g_z}\right)^2. \tag{1.11}$$

Here, $v_z = (v_s - iv_t)/2$ (and $g_z = g'$), and $\{g, z\}$ is the classical *Schwarzian derivative* of the meromorphic function g with respect to z. The second equality in (1.11) follows from an easy computation in (1.9). In particular, it implies that the Schwarzian derivative Q does not depend on the choice of the developing map g, in the sense that a transformation as in (1.10) does not change the value of Q. In fact, a more general result is true; it is easy to see that if f is a conformal map and $w = g(z)$, then

$$\{f \circ g, z\} = \{f, w\}g'(z) + \{w, z\}. \tag{1.12}$$

In particular, if $f = \mathcal{M}$ is any Möbius transformation, then $\{\mathcal{M}(g), z\} = \{g, z\}$.

Moreover the following property is satisfied.

Lemma 1.1. *Q is holomorphic*

Proof. First we observe that v satisfies $-4v_{z\bar{z}} = 2Ke^v$, hence $-4v_{zz\bar{z}} = 2Ke^v v_z = -4v_z v_{z\bar{z}}$. This implies $(v_{zz} - \frac{1}{2}v_z^2)_{\bar{z}} = 0$, that is, $v_{zz} - \frac{1}{2}v_z^2$ is a holomorphic function. □

Alternatively, Lemma (1.1) also follows from the second equality in (1.11).

The Schwarzian derivative is an important tool in complex analysis that will be extremely useful in Chapters 2 and 3 to classify the solutions to the geometric Neumann problem. Moreover, as we will explain in Section 1.5, when the Liouville equation appears in other problems in surface theory, the associated Schwarzian derivative always inherits a geometric interpretation. Furthermore, it can be also introduced from the point of view of the theory of second-order differential equations by means of the following result [Hil].

Theorem 1.5. *Let $Q(z)$ be a holomorphic function defined in a simply connected domain $\Omega \subset \mathbb{C}$. Consider two linearly independent solutions y_1 and y_2 of the complex ODE*

$$y''(z) + \frac{Q(z)}{2}y(z) = 0 \tag{1.13}$$

in Ω. Then, the meromorphic function

$$g(z) := \frac{y_1(z)}{y_2(z)}$$

satisfies the equation

$$\{g, z\} = Q(z) \tag{1.14}$$

in Ω. Conversely, if $g(z)$ is a meromorphic solution of (1.14) in Ω, then there are two linearly independent solutions $p(z)$ and $q(z)$ of (1.13) defined in Ω such that

$$g(z) = \frac{p(z)}{q(z)}.$$

Here p and q are uniquely determined if we fix $q(z_0) = 1$ for some $z_0 \in \Omega$.

Theorem 1.5 gives a simple way, at least when the expression of $Q(z)$ is not too complicated, to obtain the solutions to (1.14). This property is important since, in such a case we would obtain, up to a Möbius transformation, the developing map of a solution to (1.4) whose Schwarzian derivative is $Q(z)$. We will see several applications of this fact along this memory.

To finish this section, we must remark that the Liouville Theorem 1.3 is only valid in simply connected domains. Hence, it seems convenient to present here a generalization of Theorem 1.3 which allows to solve the Liouville equation also in non-simply connected domains. The following result in [BHL] is a direct consequence of the definition of covering map and the chain rule.

Proposition 1.1. *If $\Phi : \Omega_2 \longrightarrow \Omega_1$ is a conformal map between two planar domains $\Omega_1, \Omega_2 \subset \mathbb{C}$, the solutions of the Liouville equation* (1.4) *in Ω_2 are given by*

$$v = u \circ \Phi + 2 \log |\Phi'|, \tag{1.15}$$

where u is a solution to (1.4) *in Ω_1. Moreover, their developing maps, which are possibly multivalued, are also related. Specifically, the developing map associated to v can be written as $\widetilde{g}(w) = g(\Phi(w))$ where g is the developing map of u.*

Observe that, in general, if Φ is a covering map, g is multivalued unless Ω_1 is simply connected.

Due to the complexity of the covering map, when we deal with an arbitrary non-simply connected domain, not too much is known in this subject. In the simpler case of the punctured disc $\mathbb{D}^*$ we can mention the results [Bry, ChWa, Hei, Nit1, War], and the work of F. Brito, J. Hounie and M. L. Leite in [BHL] in the case of a general non-simply connected domain.

We will use Proposition 1.1 to avoid period problems when solving the geometric Neumann problem for the Liouville equation in chapters 2 and 3.

1.3 Initial value problems

Many important problems in PDE theory are formulated by fixing some restrictions that the solutions of the PDE should verify. These problems often comes as initial value problems, being the main three of those, the Dirichlet problem, the Cauchy problem and the Neumann problem. This section is devoted to explain what is known about the last two of these problems when the PDE is the Liouville equation.

1.3.1
The Cauchy problem for the Liouville equation

Given a second order PDE on an n-dimensional domain $\Omega \subset \mathbb{R}^n$, and some smooth (n-1)-dimensional submanifold $\Gamma \subset \Omega$, the *Cauchy problem* for that equation consists on finding all solutions to the PDE that have prescribed values up to first order along Γ.

Under some assumptions on the initial data and the coefficients of the PDE, such as analyticity, the Cauchy-Kowalevsky Theorem asserts that the solution to the Cauchy problem always exists and is unique.

We will consider next the Cauchy problem for the Liouville equation. If we denote $\phi = e^v$, where v is a solution to (1.4) in a planar domain $\Omega \subset \mathbb{C}$, this problem can be formulated as follows:

$$\begin{cases} \Delta(\log\phi) = -2K\phi, \\ \phi(s,0) = a(s), \\ \phi_t(s,0) = d(s). \end{cases} \tag{1.16}$$

Here $a(s)$ and $d(s)$ are real analytic functions and $a(s)$ is positive.

The solutions to problem (1.16) are completely described in the following result, see [GaMi1, Theorem 3].

Theorem 1.6. *The unique solution to the Cauchy problem for the Liouville equation* (1.16) *is*

$$\phi(s,t) = \frac{4|g'(z)|^2}{(1+K|g|^2)^2} \qquad z = s + it;$$

where $g(z)$ *is the meromorphic extension of* $g(s) = \pi(\alpha(s))$, *being* $\pi : \mathcal{Q}(K) \longrightarrow \overline{\mathbb{C}}$ *the stereographic projection defined in* (1.6), *and* $\alpha(s)$ *the unique curve in* $\mathcal{Q}(K)$ *with arclength parameter and geodesic curvature given respectively by*

$$u(s) = \int^s \sqrt{a}(r)dr \quad \text{and} \quad \kappa(s) = \frac{-d(s)}{2a(s)^{3/2}}.$$

Geometrically, this means that solving the Cauchy problem for (1.4) is equivalent to the problem of integrating the Frenet equations for curves in the standard two-dimensional Riemannian space model $\mathcal{Q}(K)$ of constant curvature $K \in \mathbb{R}$. Note that the particular choice $d(s) = 0$ corresponds to geodesics in $\mathcal{Q}(K)$.

The following Corollary is just a version of Theorem 1.6 in the case that those curves are of constant geodesic curvature, see [GaMi3, Theorem 2].

Corollary 1.1. *Let $f(s) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ denote a real analytic function, and $K \in \{-1, 0, 1\}$. The unique solution to the Cauchy problem*

$$\begin{cases} \Delta v + 2Ke^{u} = 0, \\ v(s,0) = f(s), \\ v_t(s,0) = ce^{f(s)/2}, \quad c \in \mathbb{R}, \end{cases} \tag{1.17}$$

is given by

$$v = \log \frac{4|g'(z)|^2}{(1 + K|g(z)|^2)^2} \quad z = s + it;$$

where here $g(z)$ is the meromorphic extension to an open set $\Omega \subset \mathbb{C}$ containing I of $g(s) = \pi(\alpha(s))$, being $\alpha(s)$ a curve in $\mathcal{Q}(K)$ with constant curvature $-c/2$ and arclength parameter $v(s) = \int^s e^{f(r)/2} dr$.

Corollary 1.1 can be proved just by writing $\phi = e^v$, $a(s) = e^{f(s)}$ and $d(s) = ce^{3f(s)/2}$ in Theorem 1.6. Note that this substitution also yields that the metric $e^v|dz|^2$, where v is a solution to (1.17), make the real axis to have constant geodesic curvature $-c/2$.

We remark here that if we choose two different curves α_1, α_2 in $\mathcal{Q}(K)$ with constant curvature $-c/2$ and the same arclength parameter, they differ by an orientation-preserving isometry of $\mathcal{Q}(K)$. Thus, $g_i(s) = \pi(\alpha_i(s))$ differ only by a linear fractional transformation as in (1.10), which does not affect the solution v to problem (1.17).

As a matter of fact, the curves with constant curvature and prescribed arclength parameter in $\mathcal{Q}(K)$ are well-known. Specifically, up to isometries preserving orientations, if $K = 1$, the only curve in $\mathbb{S}^2 \subset \mathbb{R}^3$ with arclength parameter $\nu(s)$ and constant geodesic curvature κ is

$$\alpha(s) = (\sqrt{1 - \varrho^2}, \varrho \cos(\nu(s)/\varrho), \varrho \sin(\nu(s)/\varrho), \quad \varrho := \frac{\mathrm{sg}(\kappa)}{\sqrt{1 + \kappa^2}}. \tag{1.18}$$

The curve $\alpha(s)$ in the above conditions in $\mathcal{Q}(0) = \mathbb{R}^2$ is

$$\alpha(s) = (\frac{1}{\kappa} \cos(\kappa\nu(s)), \frac{1}{\kappa} \sin(\kappa\nu(s))) \quad \text{or} \quad \alpha(s) = (\nu(s), 0), \tag{1.19}$$

depending if $\kappa \neq 0$ or $\kappa = 0$.

And finally, when $K = -1$, the only (up to isometries preserving orientations) curve $\alpha(s)$ in $\mathbb{H}^2$ with arclength parameter $\nu(s)$ and geodesic curvature κ is

$$\alpha(s) = \begin{cases} (\sqrt{1+\varrho^2}, \varrho\cos(\nu(s)/\varrho), \varrho\sin(\nu(s)/\varrho), & \varrho := \dfrac{\mathbf{sg}(\kappa)}{\sqrt{\kappa^2-1}} \quad \text{if } |\kappa| > 1, \\ (1+\nu(s)^2/2, \nu(s)^2/2, -\mathbf{sg}(\kappa)\nu(s)) & \text{if } |\kappa| = 1, \\ (\varrho\cosh(\nu(s)/\varrho), \varrho\sinh(\nu(s)/\varrho), -\mathbf{sg}(\kappa)\sqrt{\varrho^2-1}), & \varrho := \dfrac{1}{\sqrt{1-\kappa^2}} \quad \text{if } |\kappa| < 1. \end{cases} \tag{1.20}$$

Thereby, by means of (1.18)-(1.20), Corollary 1.1 will yield an explicit formula for the solution to the Cauchy problem (1.17). Also, it constitutes the key point to solve the geometric Neumann problem in the upper half plane that we will explain in the following subsection.

1.3.2 The Neumann problem for the Liouville equation

Another classical initial value problem in PDE theory is the so-called Neumann problem. This problem asks for the solution of a certain PDE in the interior of a domain with boundary Ω, such that the derivative of the solution in the normal direction is prescribed along $\partial\Omega$.

The natural *geometric Neumann problem* attached to the Liouville equation (1.4) comes from the following question:

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\partial\Omega$. *What are the conformal Riemannian metrics on Ω having constant curvature K, and constant geodesic curvature along each boundary component of $\partial\Omega$?*

From now on, this will be called *the geometric Neumann problem* for the Lioville equation.

We must remark that although the Liouville equation is conformally invariant, the boundary condition fixing the value of the geodesic curvature along the boundary depends on the boundary itself. Hence, this geometric Neumann problem will be modeled by different equations on the boundary depending on the domain we are considering.

Because of such a conformal invariance, it is not very restrictive to consider only simple symmetric domains Ω, such as discs, half-planes or annuli.

The Neumann problem for the Liouville equation has been widely studied when Ω is a half space and the metric extends smoothly to the whole boundary. The first studies were done in dimension $n \geq 3$, where the analogous problem can be written as

$$\begin{cases} \Delta v = -2Kv^{\frac{n+2}{n-2}} \quad v > 0 & \text{in } \mathbb{R}^n_+ = \{(x_1, \dots, x_n) : \; x_n > 0\}, \\ \dfrac{\partial v}{\partial x_n} = cv^{\frac{n}{n-2}} & \text{on } \; \partial\mathbb{R}^n_+, \quad K, c \in \mathbb{R}. \end{cases}$$

This problem was solved for $K \geq 0$ by Y.Y. Li and M. Zhu in [LiZh] and completed for the case $K < 0$ in [CSF].

In the two dimensional case, the problem in $\mathbb{R}^2_+$ is formulated as follows:

$$\begin{cases} \Delta v = -2Ke^v & \text{in} \quad \mathbb{R}^2_+ = \{(s,t) : \; t > 0\}, \\ \dfrac{\partial v}{\partial t} = ce^{\frac{v}{2}} & \text{on} \quad \partial\mathbb{R}^2_+, \quad K, c \in \mathbb{R}. \end{cases} \tag{1.21}$$

In [Zha], L. Zhang classified the solutions to (1.21) under the additional assumption of finite area, i.e. under the integral condition

$$\int_{\mathbb{R}^2_+} e^v < +\infty. \tag{1.22}$$

He found out the pairs (K, c) for which the problem (1.21) has a solution under the hypothesis (1.22). Previous results on the problem (1.21) under alternative integral finiteness assumptions had been obtained in [LiZh, LiZha, Ou].

In [GaMi3], J. A. Gálvez and P. Mira solved completely problem (1.21) without any extra hypothesis. The key point of their method is the treatment of problem (1.21) as a Cauchy problem where the Dirichlet condition is suppressed.

We clarify this approach. By the Liouville Theorem 1.3, we know that the developing map g associated to the solution of (1.21) must map the real axis into a curve of constant geodesic curvature $-c/2$ in the correspondent model space $\mathcal{Q}(K)$. Hence, by Corollary 1.1, the problem reduces to finding the solutions to (1.17) for some $f(s)$ globally defined on $\mathbb{C}^+ \equiv \mathbb{R}^2_+$. In this sense, they classify the solutions to (1.21) by means of entire functions obtained as the extension of a real analytic function related with $f(s)$. They obtain the values of (K, c) for which solutions to (1.21) do exist and give explicit formulas for them (remind that the curves of constant curvature and arbitrary arclength are given explicitly by formulas (1.18)-(1.20)).

We refer to [GaMi3] for more details about the exact form of the solutions. However, for convenience, we give a brief proof of the following lemma which is a consequence of some arguments in [GaMi3]. It contains some basic local properties of a solution to the geometric Neumann problem for the Liouville equation along the boundary, that we will use frequently in chapters 2 and 3.

Lemma 1.2. *Let* $D_\varepsilon^+ = \{z \in \mathbb{C} : |z| < \varepsilon, \mathrm{Im}\ z > 0\}$, *and let* $v \in C^2(\overline{D_\varepsilon^+})$ *be a solution to*

$$\begin{cases} \Delta v + 2Ke^v = 0 & in \quad D_\varepsilon^+, \\ \dfrac{\partial v}{\partial t} = ce^{v/2} & on \quad I_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbb{R}, \end{cases}$$

where $K, c \in \mathbb{R}$. *Then:*

(i) The Schwarzian derivative map Q *of* v*, defined by* (1.11)*, takes real values along* I_ε*, and extends holomorphically to the whole disc* $D_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}$ *by* $Q(\bar{z}) = \overline{Q(z)}$.

(ii) The developing map g *of* v *can be extended to* D_ε *as a locally univalent meromorphic function.*

(iii) $g(s, 0) : I_\varepsilon \longrightarrow \overline{\mathbb{C}}$ *is a regular parametrization of a piece of a circle* $\mathcal{C}$ *in* $\overline{\mathbb{C}}$.

Proof. By the Neumann condition $v_t = ce^{v/2}$ along I_ε, we have

$$\mathrm{Im}\, Q(s, 0) = -\frac{1}{2}\left(\frac{c}{2}v'(s)e^{v(s)/2} - \frac{c}{2}v'(s)e^{v(s)/2}\right) = 0,$$

for every $s \in I_\varepsilon$. Thus, (i) holds immediately by Schwarzian reflection.

For (ii), we only need to recall that by Theorem 1.5, if $q(z)$ is a holomorphic function in a simply connected domain, then the equation $\{g, z\} = q(z)$ always has a locally univalent meromorphic solution g, which is unique up to linear fractional transformations. In our case, we have $\{g, z\} = Q$ on D_ε^+, and so (ii) follows from (i).

Finally, (iii) is clear from the fact that the developing map g defines a local isometry from $(\overline{D_\varepsilon^+}, e^v|dz|^2)$ into $\mathcal{Q}(K)$, and I_ε has constant curvature $-c/2$ for the metric $e^v|dz|^2$, by the Neumann condition $v_t = ce^{v/2}$. $\square$

Other results concerning the geometric Neumann problem for the Liouville euquation in simply connected domains such as the unit disc can be found in [HaWa]. In Section 1.4 we will show some results in this line in the simplest situations.

It is also interesting to study the geometric Neumann problem for the Liouville equation in non-simply connected domains.

In this sense, we may cite the classification of all the metrics in the punctured unit disc $\mathbb{D}^*$ that have constant curvature K and constant geodesic curvature $-c/2$ on the boundary in [GaMi3]. The authors obtain such a classification by finding all the solutions $u \in \mathcal{C}^2(\mathbb{D}^* \cup \mathbb{S}^1)$ to

$$\begin{cases} \Delta u + 2Ke^u = 0 & \text{in} \quad \mathbb{D}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\} \\ \dfrac{\partial u}{\partial \eta} = ce^{u/2} + 2 & \text{on} \quad \mathbb{S}^1 = \{z : \ |z| = 1\}, \ K, c \in \mathbb{R}, \end{cases} \tag{1.23}$$

where η is the unit inner normal to $\mathbb{S}^1$.

Moreover, they show that any solution to (1.23) such that $\int_{\mathbb{D}^*} e^u < \infty$ and which does not extend smoothly across the origin is radially symmetric, and describe all such solutions explicitly.

They use Proposition 1.1 to relate problem (1.23) with (1.21). Specifically, by means of the covering map $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{D}^*$, $\Phi(s,t) = (e^t \cos s, e^t \sin s)$ and formula (1.15), they obtain solutions u to (1.23) from solutions v to (1.21) such that $v(s,0)$ is 2π-periodic.

After the punctured disc $\mathbb{D}^*$, the most natural non-simply connected domain where we can try to model and solve the geometric Neumann problem is an annulus. That is, we can look for the metrics of constant curvature on an annulus with certain values for the geodesic curvature on each component of its boundary. We will devote Chapter 3 to explain the procedure to solve it.

1.4 A basic result

The aim of this section is to prove a basic result concerning the geometric Neumann problem for the Liouville equation. In doing so, we want to clarify how, by means of the tools of complex analysis we have shown before, we can obtain an interesting already known result whose proof would be maybe more complicated if it were given by techniques of PDE theory.

Specifically, we consider the geometric Neumann problem in a disc

$$\begin{cases} \Delta v = -2Ke^v & \text{in } \mathbb{D}, \\ \dfrac{\partial v}{\partial \eta} = ce^{v/2} + 2 & \text{on } \partial\mathbb{D}, \end{cases} \tag{1.24}$$

where as before, η denotes the inner unit normal of the domain along the boundary.

The following result can be found in [HaWa, Theorem 3.1], although only for the case $K = 1$. It can be also deduced by the classification of the solutions to the Neumann problem in the punctured disc in [GaMi3].

Theorem 1.7. *There exist solution to* (1.24) *only in the cases (i)* $K = 1$*, (ii)* $K = 0$ *and* $c < 0$ *or (iii)* $K = -1$ *and* $c < -2$*. Moreover, their explicit formulas are given by the following expressions.*

(i) If $K = 1$, $e^v = \frac{4\lambda^2}{(\lambda^2+|z-z_0|^2)^2}$, *where* $\lambda > 0$ *and* $z_0 \in \mathbb{C}$ *are such that* $c = -\frac{\lambda^2+|z_0|^2-1}{\lambda}$.

(ii) If $K = 0$ *and* $c > 0$*,*

- *(a)* $e^v = \lambda^2$, *where* $\lambda > 0$ *is such that* $c = -\frac{2}{\lambda}$, *or*
- *(b)* $e^v = \frac{4\lambda^2}{|z-z_0|^4}$, *where* $\lambda > 0$ *and* $z_0 \in \{z \in \mathbb{C} : |z| > 1\}$ *are such that* $c = \frac{1-|z_0|^2}{\lambda}$.

(iii) If $K = -1$ *and* $c < -2$*,*

- *(a)* $e^v = \frac{4\lambda^2}{(-\lambda^2+|z-z_0|^2)^2}$, *where* $\lambda > 0$ *and* $z_0 \in \{z \in \mathbb{C} : |z| > 1 + \lambda\}$ *are such that* $c = \frac{\lambda^2+1-|z_0|^2}{\lambda}$, *or*
- *(b)* $e^v = \frac{4}{(\lambda+2\mathrm{Re}(e^{i\theta_0}z))^2}$, *where* $\lambda, \theta_0 \in \mathbb{R}$ *are such that* $c = -|\lambda|$.

Proof. By Lemma 1.2 we know that the Schwarzian derivative Q associated to the solution of (1.24) is a holomorphic function. On the other hand, the boundary condition can be written as

$$-zv_z - \bar{z}v_{\bar{z}} = ce^{v/2} + 2 \quad \text{on } \partial\mathbb{D}.$$

Differentiating along the tangential direction which can be written as $i(z\partial_z - \bar{z}\partial_{\bar{z}})$ we get

$$-z^2 v_{zz} + \bar{z}^2 v_{\bar{z}\bar{z}} - z v_z + \bar{z} v_{\bar{z}} = \frac{c e^{v/2}}{2}(z v_z - \bar{z} v_{\bar{z}}).$$

If we use the boundary condition again we obtain

$$z^2(v_{zz} - \frac{1}{2}v_z^2) = \bar{z}^2(v_{\bar{z}\bar{z}} - \frac{1}{2}v_{\bar{z}}^2).$$

This means that the holomorphic function $z^2(v_{zz} - \frac{1}{2}v_z^2) = z^2 Q$ is real on $\partial\mathbb{D}$. Hence, as Q is holomorphic, it must vanish identically.

On the other hand, it is well-known that the solutions g to the equation $\{g, z\} = 0$ are Möbius transformations [Hil]. That is, $g(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$ that we suppose satisfying $ad - bc = 1$. Observe that as $\overline{\mathbb{D}}$ is compact, the metrics that we will construct by means of (1.9) will be of finite area whenever they are well defined. Substituting in (1.9) we have that

$$e^v = \frac{4}{(|cz + d|^2 + K|az + b|^2)^2}. \tag{1.25}$$

If $K = 1$ and we denote

$$\lambda = \frac{1}{|a|^2 + |c|^2} \qquad \text{and} \qquad z_0 = -\frac{\bar{a}b + \bar{c}d}{(|a|^2 + |c|^2)^2},$$

we easily deduce formula (i) from (1.25). Note that in this case, the conformal factor is well defined in $\overline{\mathbb{D}}$ for every value of the parameters λ and z_0.

If $K = 0$ and $c = 0$, formula (1.25) yields (ii.a) by denoting $\lambda = 2/|d|^2$. If $c \neq 0$ and we write $\lambda = 1/|c|^2$, and $z_0 = -d/c$ we obtain (ii.b) which is well defined only when $z_0 \in \{z \in \mathbb{C} : \ |z| > 1\}$.

If $K = -1$ and $|a|^2 \neq |c|^2$ we can denote

$$\lambda = \frac{1}{|a|^2 - |c|^2} \qquad \text{and} \qquad z_0 = -\frac{\bar{a}b - \bar{c}d}{(|a|^2 - |c|^2)^2}$$

and obtain then (iii.a). Observe that such a conformal factor is well defined in $\overline{\mathbb{D}}$ if and only if $z_0 \in \{z \in \mathbb{C} : \ |z| > 1 + \lambda\}$. On the other hand, if $|a|^2 = |c|^2$ and we write

$$\lambda = |b|^2 - |d|^2 \qquad \text{and} \qquad z_0 = a\bar{b} - c\bar{d},$$

we easily deduce that $z_0 = e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$ and thus formula (iii.b) holds. Observe that $2\mathrm{Re}(e^{i\theta_0}z) \in [-2, 2]$ if $z \in \overline{\mathbb{D}}$, hence the solution (iii.b) is well defined only when $|\lambda| > 2$.

Finally, a simple computation gives the exact value of c in each case. Observe that the restrictions in the values of the parameters in (i)-(iii.b) yield that there is no solution for the constants (K, c) if $K = 0$ and $c \geq 0$ or if $K = -1$ and $c \geq -2$ (see Figure 1.1). Conversely, for prescribed values of (K, c) as before, it is trivial that we can always find parameters as in the statement of the theorem and so we can obtain a solution by (i)-(iii.2), respectively. □

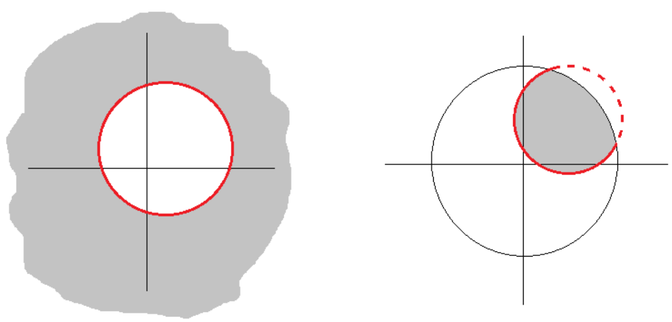


Figure 1.1: *The image of g in the cases $K = 0$ and $c \geq 0$, and $K = -1$ and $c \geq -2$, respectively.*

By the proof of Theorem 1.7, we deduce that the only possible metrics of constant curvature K in the unit disc with constant geodesic curvature $-c/2$ on $\mathbb{S}^1$ are, up to isometry, the pullback of the metric ds_K^2 restricted to discs in $\mathcal{Q}(K)$ bounded by a curve of constant geodesic curvature $-c/2$. In this sense, we can say that all the solutions to the Neumann problem in $\mathbb{D}$ are the canonical ones.

1.5 The Liouville equation in surface theory

Apart from the examples we have already shown, the Liouville equation is also related to other topics in surface theory. In this section, we explain its relation with minimal surfaces in $\mathbb{R}^3$ and their Lawson cousins, that is, surfaces of constant mean curvature one in the hyperbolic 3-space $\mathbb{H}^3$. We also show how the Liouville equation appears in the theory of flat surfaces in $\mathbb{H}^3$, and finally, we relate it to a large class of Weingarten surfaces which includes the constant mean curvature one surfaces and the flat surfaces in $\mathbb{H}^3$.

Along this section, we will denote by $\langle , \rangle$ the inner product in the Lorentz- Minkowski space $\mathbb{L}^4$, that is, $\mathbb{L}^4 = (\mathbb{R}^4, \langle , \rangle)$ where $\langle , \rangle = -dx_0^2 + \sum_{i=1}^3 dx_i^2$ in canonical coordinates.

1.5.1 Minimal surfaces in $\mathbb{R}^3$. Bryant Surfaces

The interrelation of the Liouville equation $\Delta v + 2e^v = 0$ with the study of minimal surfaces in $\mathbb{R}^3$ is well-known and can be found in books as [DHKW, Nit2, Oss]. Such a connection, that we will explain next, also extends to the case of Bryant surfaces.

Let us recall that a Bryant surface is an immersed surface of constant mean curvature $H = 1$ in the hyperbolic 3-space $\mathbb{H}^3$. The term *Bryant surface* was adopted after the paper by R.L. Bryant in 1987 [Bry], where a conformal representation for this type of surfaces was obtained. A previous local representation formula for $H = 1$ surfaces in $\mathbb{H}^3$ was already known to Bianchi [Bi].

The way to connect minimal surfaces in $\mathbb{R}^3$ with Bryant surfaces is given by the so-called Lawson correspondence (cf. [Law]). Specifically, if (I, II) denote the first and second fundamental forms of a simply connected minimal surface, then there exists a Bryant surface whose first and second fundamental forms are $(I, II + I)$. In particular, both surfaces are isometric. It is important to remark that, hence, both of them have the same Hopf differential, defined as the $(2, 0)$-part of their second fundamental forms, which is a holomorphic 2-form.

In contrast, the global geometry of these two classes of surfaces is completely different as we explain next. For more information on the topic see [GaMi1].

In both classes of surfaces, the Gauss curvature of the first fundamental form is non positive, so, when S is simply connected, we can consider a local conformal parameter z such that the first fundamental form is of the form $I = \lambda |dz|^2$, and the Hopf differential is written as $Q = q dz^2$.

Moreover, the Gauss equation in $\mathbb{H}^3$ leads to

$$(\log \lambda)_{z\bar{z}} = 2|q|^2/\lambda. \tag{1.26}$$

Hence, as q is holomorphic, we obtain from (1.26) that the function v defined by $e^v = 4|q|^2/\lambda$ satisfies the Liouville equation $\Delta v + 2e^v = 0$ out of umbilical points, where $q = 0$. As we have seen in Section 1.2, this equation models a conformal metric of curvature one. In our case, this metric is exactly $-KI = -K\lambda|dz|^2 = e^v|dz|^2$, where $K(\leq 0)$ is the Gauss curvature of the first fundamental form I.

As S is simply connected, Theorem 1.3 implies that there exists a meromorphic function $g : S \longrightarrow \overline{\mathbb{C}}$ such that $e^v = \frac{4|g'|^2}{(1+|g|^2)^2}$, where v is a solution to $\Delta v + 2e^v = 0$. Actually, for the minimal immersion $\psi_1 : S \longrightarrow \mathbb{R}^3$, such a g has a geometric interpretation: if $N : S \longrightarrow \mathbb{S}^2$ is the unit normal to $\psi_1(S)$, then $g = \pi \circ N$, where π is the stereographic projection defined in (1.6). That is, the developing map g coincides with the Gauss map of the minimal surface $\psi_1(S)$ and, therefore, it is well defined also in the case when S is non-simply connected because of its geometric nature.

The cousin Bryant surface $\psi_2 : S \longrightarrow \mathbb{H}^3$, which is isometric to $\psi_1 : S \longrightarrow \mathbb{R}^3$, also satisfies that $-KI = e^v|dz|^2$ is a pseudo-metric of constant curvature one. But the difference with the minimal case is that now the associated developing map g, which is referred to the *secondary Gauss map* of $\psi_2(S)$ does not have a geometric meaning. Hence, if S is non-simply connected, it will be well defined only in the universal cover $\widetilde{S}$ of S.

This fact makes the periods problem for Bryant surfaces much more complicated than in the minimal case, where it reduces to check that a certain holomorphic 1-form does not have real periods. Furthermore, although in both cases we can recover a surface in terms of the meromorphic developing map g and the holomorphic Hopf differential Q, in the minimal case, the immersion can be computed just by an integration, in contrast with the differential equation appearing in the case of Bryant surfaces (see [Bry, GaMi1]).

In [GaMi1], the authors avoided some of those difficulties by providing a back-and-forth construction connecting the Cauchy problem for the Liouville equation (1.16), and the Cauchy problem for Bryant surfaces. This Cauchy problem for Bryant surfaces was inspired by the classical Björling problem for minimal surfaces in $\mathbb{R}^3$, proposed by E. G. Björling in 1844 and solved by H. A. Schwarz in 1890. Geometrically, it can be stated as follows.

> Let $\beta : I \longrightarrow \mathbb{H}^3$ be a regular analytic curve, and let $V : I \longrightarrow \mathbb{S}^3_1$ be an analytic vector field along β such that $\langle \beta, V \rangle \equiv \langle \beta', V \rangle \equiv 0$. Find all Bryant surfaces containing β and with unit normal in $\mathbb{H}^3$ along β given by V.

In [GaMi1] it is shown that, given a pair of Björling data β, V , there is a unique solution to the Cauchy problem for Bryant surfaces with initial data β, V, which is constructed in terms of the solution to the Cauchy problem for the Liouville equation (1.16).

While the secondary Gauss map g does not have a geometric interpretation for Bryant's surfaces, there exists other holomorphic data which have a geometric meaning and so are well defined on S, even if it is non-simply connected. That is the case of the Hopf differential Q we have already mentioned, and the *hyperbolic Gauss map* G which is defined as follows. Let $\eta : S \longrightarrow \mathbb{S}^3_1$ be the unit normal to ψ_2 where $\mathbb{S}^3_1 = \{x \in \mathbb{L}^4 : \langle x, x \rangle = 1\}$ is the 3-dimensional de Sitter space. Then, the map $\psi_2 + \eta : S \longrightarrow \mathbb{L}^4$ takes values on the positive null cone $\mathbb{N}^3 = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{L}^4 : \langle x, x \rangle = 0, \ x_0 > 0\}$, and so we can define G as the projection of $\psi_2 + \eta$ into the quotient $\mathbb{N}^3/\mathbb{R}_+ \equiv \mathbb{S}^2_\infty$:

$$G = [\psi_2 + \eta] : S \longrightarrow \mathbb{S}^2_\infty.$$

Here $\mathbb{S}^2_\infty$ denotes the ideal boundary of $\mathbb{H}^3$. The geometric interpretation of G is the following: for each $p \in S$, $G(p)$ is the point at the ideal boundary of $\mathbb{H}^3$, $\mathbb{S}^2_\infty$, of the unique oriented geodesic of $\mathbb{H}^3$ that starts from $\psi_2(S)$ with initial velocity $\eta(p)$.

In fact, we must point out that the associated pseudometric of curvature one, $-KI$, can be written as $\langle d(\psi_2+\eta), d(\psi_2+\eta)\rangle$. That is, $\psi+\eta : S \longrightarrow \mathbb{N}^3$ is, at its regular points, an immersion of constant curvature 1.

Moreover, g, G and Q are related as follows (see [UmYa]):

$$\{g, z\} - \{G, z\} = -2Q$$

where $\{., z\}$ denotes the Schwarzian derivative defined in (1.11).

1.5.2
Flat surfaces in $\mathbb{H}^3$

As happens for the case of surfaces of constant mean curvature one, flat surfaces in $\mathbb{H}^3$ also admit a conformal representation that is closely related to the Liouville equation. Such a representation in terms of holomorphic data in the spirit of Bryant's representation can be found in [GMM1].

We explain next, as we did in Subsection 1.5.1, how some holomorphic data appearing in such a theory can be related with the Liouville equation.

Let $\psi : S \longrightarrow \mathbb{H}^3$ be a flat surface whose unit normal field is $\eta : S \longrightarrow \mathbb{S}^3_1$. The Gauss equation yields that the extrinsic curvature is constant $K_{\text{ext}} = 1$. Then, if we consider an orthonormal basis that diagonalizes the second fundamental form II of the immersion, the condition $K_{\text{ext}} = 1$ implies that II is definite. Therefore, we can suppose that the orientation given by the unit normal η is *canonical*, in the sense that it makes II be positive definite. That is, the second fundamental form II of a flat surface in $\mathbb{H}^3$, $\psi : S :\longrightarrow \mathbb{H}^3$, is a Riemannian metric that induces (as we saw in Section 1.1) a Riemann surface structure on S. Observe that, in contrast with the case of Bryant surfaces, such a structure is not the one induced by the metric $\langle d\psi, d\psi \rangle$. This structure is the one considered in [GMM1] to obtain the conformal representation for flat surfaces in $\mathbb{H}^3$.

The first data appearing in such a representation in [GMM1] is the *hyperbolic Gauss map* $G : S \longrightarrow \mathbb{S}^2_\infty$ which is defined as in Subsection 1.5.1. The map $\psi + \eta$ is conformal and so G is meromorphic (now with respect to the conformal structure of the second fundamental form). On the other hand, the $(2,0)$-part of the first fundamental form, which we denote as $Q_I = \langle \psi_z, \psi_z \rangle dz^2$, is a 2-form well defined on the whole surface S that vanishes at the umbilic points of $\psi(S)$. It then follows from the Gauss equation that Q_I is holomorphic since the immersion $\psi : S :\longrightarrow \mathbb{H}^3$ is flat. This differential Q_I is the second basic holomorphic data in the representation of flat surfaces.

We recall now that, for the case of a Bryant surface $\psi_2 : S \longrightarrow \mathbb{H}^3$, the conformal pseudo-metric $-KI = \langle d(\psi_2 + \eta), d(\psi_2 + \eta) \rangle$ had constant curvature one, and so the Liouville equation appeared. In the case of flat surfaces in $\mathbb{H}^3$ we have the following result ([GMM1]):

Theorem 1.8. *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a canonically oriented flat surface whose unit normal field is $\eta : S \longrightarrow \mathbb{S}^3_1$. Then $\langle d(\psi + \eta), d(\psi + \eta) \rangle$ is a flat pseudo-metric on S which is conformal to II.*

So, Theorem 1.8 automatically relates flat surfaces in $\mathbb{H}^3$ with the Liouville equation

$$\Delta \log \phi = 0,$$

where ϕ is defined by $\phi|dz|^2 = \langle d(\psi+\eta), d(\psi+\eta)\rangle$.

Moreover, as $\psi+\eta$ is conformal, there must exist by Theorem 1.8 a holomorphic 1-form ω such that $\langle d(\psi+\eta), d(\psi+\eta)\rangle = |\omega|^2$. These three holomorphic data, G, Q_I and ω, are enough to recover all the the information of the immersion ψ.

1.5.3 Linear Weingarten surfaces

Finally, we explain how to relate Liouville equation with a large class of Weingarten surfaces which includes the constant mean curvature one surfaces and flat surfaces in the hyperbolic 3-space. In this sense, the results we explain next unify the relations we have shown along this section. They can be found in [GMM4].

Specifically, the family of immersions we are going to consider are linear Weingarten immersions of Bryant type, (in short, BLW-surfaces). We say that $\psi : S \longrightarrow \mathbb{H}^3$ is a BLW-surface if the mean curvature H and the Gauss curvature K satisfy a linear relation of the form,

$$2a(H-1)+bK=0, \tag{1.27}$$

for some $a, b \in \mathbb{R}$, $a+b \neq 0$.

Observe that the values $b=0$ and $a=0$ in formula (1.27) yield, respectively, Bryant surfaces and flat surfaces. The immersions satisfying (1.27) with $a+b=0$, that is, the non elliptic case, are the ones with a constant principal curvature equal to 1. They are studied in [AlGa].

The following result can be found in [GMM4, Lemma 1].

Lemma 1.3. *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface. Then, we can consider that $|a+b|=1$,*

$$2a(H-1)+b(K_{ext}-1)=0$$

and $\sigma = aI + bII$ is a positive definite metric, where K_{ext} is the extrinsic curvature, and I and II, are the first and second fundamental forms of the immersion, respectively.

In this case, the conformal structure induced by σ on S will be the one used to give the conformal representation of the surface.

As for the case of Bryant surfaces and flat surfaces, we consider for any immersion $\psi : S \longrightarrow \mathbb{H}^3$ with Gauss map $\eta : S \longrightarrow \mathbb{S}^3_1$, its associated *hyperbolic Gauss map* given by $G := [\psi + \eta]$. Then, we have the following result ([GMM4, Theorem 1]).

Theorem 1.9. *Let $\psi : S \longrightarrow \mathbb{H}^3$ be a BLW-surface. Then $\psi+\eta$ is a conformal map with respect to the metric $\sigma = aI + bII$ and*

$$\Delta^\sigma(\psi + \eta) = \frac{2}{a+b}((H-1)\psi + (K-H)\eta)$$

where Δ^σ denotes the Laplacian of σ. In particular, its hyperbolic Gauss map G is conformal. Moreover, the immersion lies in a horosphere or $\langle d(\psi + \eta), d(\psi + \eta)\rangle$ is a pseudometric of constant curvature $\frac{a}{a+b}$.

That is, any BLW-surface has associated the Liouville equation

$$\Delta v + 2\frac{a}{a+b}e^v = 0$$

where v is such that $\langle d(\psi + \eta), d(\psi + \eta)\rangle = e^v|dz|^2$ and z is a conformal parameter with respect to the structure induced by σ. Furthermore, the developing map g associated to the metric $e^v|dz|^2$ and the hyperbolic Gauss map G will be the Weierstrass data of the immersion ψ.

Not only the Liouville equation, but also other related tools of complex analysis we have mentioned in this chapter, as the Schwarzian derivative have a great importance in the study of BLW-surface. We refer to [GMM4] for more details.

2

The Liouville equation in a half-space

2.1 The geometric Neumann problem in a halfplane

As explained in Chapter 1, the conformal invariance of the Liouville equation allows to consider simple symmetric domains Ω, such as discs, halfplanes or annuli.

In this sense, our aim will be to generalize the geometric Neumann problem introduced in Subsection 1.3.2, when the domain is $\Omega = \mathbb{R}^2_+$. More precisely, we will study conformal metrics $e^v|dz|^2$ with constant curvature K on $\mathbb{R}^2_+$ and constant geodesic curvature on $\partial\mathbb{R}^2_+$, in the case when the metric $e^v|dz|^2$ does not extend smoothly to the whole boundary and so boundary singularities appear.

Specifically, we will deal with the following Neumann problem for functions $v \in \overline{\mathbb{R}^2_+} \setminus \{0\}$:

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ \dfrac{\partial v}{\partial t} = c_2 e^{v/2} & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\}, \end{cases} \tag{P}$$

with $K \in \{-1, 0, 1\}$ and $c_1, c_2 \in \mathbb{R}$. Observe that, we are imposing possibly different values of constant geodesic curvature on the boundary $-\frac{c_i}{2}$, $i = 1, 2$. In principle, v presents a boundary singularity at the origin.

Recently, in [JWZ], Jost, Wang and Zhou gave a complete classification of the solutions to the above problem under the following assumptions:

1. The metric $e^v|dz|^2$ has finite area in $\mathbb{R}^2_+$, i.e.

$$\int_{\mathbb{R}^2_+} e^v < \infty. \tag{2.1}$$

2. The boundary $\partial\mathbb{R}^2_+$ has finite length for the metric $e^v|dz|^2$, i.e.

$$\int_{\mathbb{R}_-} e^{v/2} + \int_{\mathbb{R}_+} e^{v/2} < \infty. \tag{2.2}$$

3. The metric $e^v|dz|^2$ has a *boundary conical singularity* at the origin (see Definition 1.5), i.e. there exists $\lim_{z\to 0} |z|^{-2\alpha} e^v \neq 0$ for some $\alpha > -1$.

4. $K = 1$.

With these hypotheses, they showed in [JWZ] that any solution to (P) corresponds to the conformal metric associated to the sector of a sphere of radius one limited by two circles that intersect at exactly two points, or to the complement of a closed arc of circle in the sphere, possibly composed with an adequate branched covering of the Riemann sphere $\overline{\mathbb{C}}$. In particular, they provided explicit expressions for all these solutions.

Along this chapter we will show several improvements of the Jost-Wang-Zhou theorem. In Theorem 2.3, we will remove the last three hypotheses of the above list in the Jost-Wang-Zhou result, and prove that any solution to (P) of finite area is one of the *canonical solutions* that we will describe in Section 2.2.

For the case $K = 1$ we recover the solutions obtained in [JWZ], together with some new solutions corresponding to the case that the boundary singularity at the origin is not conical; we do not prescribe here any asymptotic behavior at the origin, nor the finite length condition (2.2).

Furthermore, as a consequence of Theorem 2.1, where we consider the local problem for an arbitrary half-disc $D_\varepsilon^+ \subset \mathbb{R}_+^2$, we obtain a general classification of all the solutions to (P), without any integral finiteness assumptions, in the spirit of [GaMi3].

In Theorem 2.2 we classify the solutions to the local problem with finite area and give a general procedure to construct all of them. In particular, we describe their asymptotic behavior at the origin. This is a generalization to the case of boundary singularities of the well-known results in [Bry, ChWa, Hei, Nit1, War] which describe the asymptotic behavior of metrics of constant curvature and finite area in the punctured disc $\mathbb{D}^*$.

Furthermore, in Section 2.6 we will study the extension of problem (P) for an arbitrary number of boundary singularities. Geometrically, we show that such metrics are in the class of a certain circular polygonal metrics constructed from Alexandrov-embedded, possibly self-intersecting, circular polygons. Analytically, those solutions will not have simple explicit expressions; yet, one can still give some analytic information about them in terms of their associated Schwarzian maps.

2.2 The canonical solutions

Our objective in this section is to describe, both analytically and geometrically, an explicit family of solutions to (P) satisfying the finite energy condition (2.1). We will prove in Theorem 2.3 that these are actually *all* the finite energy solutions to (P).

2.2.1 Analytic description

We start by giving the explicit formula for the canonical solutions to problem (P). Later on, we will deduce some properties from their analytic expression.

Definition 2.1. *A* canonical solution *is a function of one of the following types:*

1. $v_1 : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ *given by*

$$v_1 = \log \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{(K\lambda^2 + |z^\gamma - z_0|^2)^2} \tag{2.3}$$

where $\gamma, \lambda > 0$ *and* $z_0 \in \mathbb{C}$ *satisfy* $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ *for all* $z \in \overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$*.*

2. $v_2 : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ *given by*

$$v_2 = \log \frac{4\,\lambda^2}{|z|^2(K\lambda^2 + |\log z - z_0|^2)^2}, \tag{2.4}$$

where $\lambda > 0$ *and* $z_0 \in \mathbb{C}$*, satisfy* $K\lambda^2 + |\log z - z_0|^2 \neq 0$ *for all* $z \in \overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$*. Here,* $\log z = \ln|z| + i\arg(z)$*, where* $\arg(z) \in [0, \pi]$*.*

Let us observe some elementary properties of these *canonical solutions*, and explain for what choices of the constants γ, λ, z_0 and K they exist.

The function v_1 given by (2.3) is well defined in $\overline{\mathbb{R}^2_+} \backslash \{0\}$ if $K = 1$ for all $\gamma, \lambda > 0$, $z_0 \in \mathbb{C}$. However, if $K = 0, -1$, v_1 is well defined if and only if $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$. In other words, if and only if the distance from the point z_0 to the sector $\{z^\gamma : \ z \in \overline{\mathbb{R}^2_+} \backslash \{0\}\}$ is bigger than $-K\lambda^2$. A simple analysis shows that this happens:

- for $K = 0$ if and only if $z_0 = 0$, or $z_0 \neq 0$ and $\pi\gamma < \theta_0$ with $\theta_0 = \arg(z_0) \in [0, 2\pi)$.
- for $K = -1$ if and only if $\lambda \leq |z_0|$, $\pi\gamma < \theta_0 - \alpha_0$, and $|\operatorname{Im}(z)| > \lambda$ when $\operatorname{Re}(z) > 0$. Here, $\theta_0 = \arg(z_0) \in [0, 2\pi)$ and $\alpha_0 \in (0, \pi/2)$ with $\sin\alpha_0 = \lambda/|z_0|$.

Besides, if $K = 1$ the function v_1 satisfies the finite area condition

$$\int_{\mathbb{R}^2_+} e^{v_1} = \int_{\mathbb{R}^2_+} \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{(K\lambda^2 + |z^\gamma - z_0|^2)^2} < \infty$$

for every γ, λ, z_0.

In the other cases, if it holds $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$ (and not just for all $z \in \overline{\mathbb{R}^2_+} \setminus \{0\}$), then the metric trivially has finite area.

Otherwise, it means that $z_0 = 0$ if $K = 0$, or $|z_0| = \lambda$ when $K = -1$. But in these cases we clearly have infinite area at the origin.

As a consequence, v_1 is a well defined function in $\overline{\mathbb{R}^2_+}\backslash\{0\}$ with finite area if and only if $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$, which is the condition of Definition 2.1. Observe that $\gamma < 2$ when $K = 0, -1$.

Analogously the function v_2 given in (2.4) is well defined in $\overline{\mathbb{R}^2_+}\backslash\{0\}$ and has finite area if and only if $K\lambda^2 + |\log z - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$.

In particular, if $K = 1$ the condition $K\lambda^2 + |\log z - z_0|^2 \neq 0$ for all $z \in \overline{\mathbb{C}_+}$ is satisfied for every λ, z_0. However, in the other cases, we need to impose that the distance from the point z_0 to the strip $\{\log z : \ z \in \overline{\mathbb{R}^2_+}\backslash\{0\}\} = \{\zeta \in \mathbb{C} : 0 \leq \ \text{Im}\zeta \leq \pi\}$ is bigger than $-K\lambda^2$. This condition happens

- for $K = 0$ if and only if $\text{Im}(z_0) < 0$ or $\text{Im}(z_0) > \pi$.
- for $K = -1$ if and only if $\text{Im}(z_0) < -\lambda$ or $\text{Im}(z_0) > \pi + \lambda$.

This analysis together with a simple computation shows that these canonical solutions are indeed finite area solutions to problem (P).

Lemma 2.1. *Any canonical solution $v : \overline{\mathbb{R}^2_+} \setminus \{0\} \longrightarrow \mathbb{R}$ is a solution to the geometric Neumann problem (P) satisfying*

$$\int_{\mathbb{R}^2_+} e^v < \infty,$$

where the constants c_1, c_2 associated to the problem are given by the following expressions in terms of γ, λ and $z_0 := r_0 e^{i\theta_0}$:

1. *For v_1 as in* (2.3),
$$c_1 = 2\,\frac{r_0}{\lambda}\,\sin\theta_0, \qquad c_2 = -2\,\frac{r_0}{\lambda}\,\sin(\theta_0 - \pi\gamma). \tag{2.5}$$
2. *For v_2 as in* (2.4),
$$c_1 = \frac{2}{\lambda}\ \textit{Im}(z_0), \qquad c_2 = \frac{2}{\lambda}\,(\pi - \ \textit{Im}(z_0)). \tag{2.6}$$

Remark 2.1. *The canonical solutions given by formula* (2.3) *when $K = 1$ coincide, up to a reparametrization, with the solutions obtained by Jost-Wan-Zou (see Theorem 2.1 in [JWZ]). We will check in the proof of Theorem 2.3 that their asymptotic behavior at the origin correspond to a boundary conical singularity (see Definition 1.5). Moreover, we will also prove that the finite length condition on the boundary given by* (2.2) *automatically holds for a solution v to problem (P) if we impose the finite area condition* (2.1) *and the fact that the singularity of v at the origin is of conical type.*

2.2.2
Geometric description

Following the notation of Chapter 1, we consider $\mathcal{Q}^2(K)$ as the 2-dimensional space form of constant curvature $K \in \{-1, 0, 1\}$, which will be viewed as (Σ_K, ds_K^2) where Σ_K is given by formula (1.7) and ds_K^2 by formula (1.8).

Moreover, we know by formulas (1.18)-(1.20) that a regular curve in Σ_K has constant geodesic curvature if and only if its image is a piece of a circle in $\overline{\mathbb{C}}$.

Definition 2.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two different circles in $\overline{\mathbb{C}}$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$, and let $\mathcal{U} \subset \overline{\mathbb{C}}$ be any of the regions in which $\mathcal{C}_1$ and $\mathcal{C}_2$ divide $\overline{\mathbb{C}}$. Assume that $\overline{\mathcal{U}}$ is contained in Σ_K. Then $\mathcal{U}$ is called a* basic domain *of $\mathcal{Q}^2(K)$.*

Let now $\mathcal{U} \subset \Sigma_K$ be a basic domain equipped with the metric ds_K^2 in (1.8). Note that one can conformally parametrize $\mathcal{U}$ by a biholomorphism $g : \mathbb{C}_+ \longrightarrow \mathcal{U}$ such that $g(\infty)$ is a point $p \in \mathcal{C}_1 \cap \mathcal{C}_2$, and in the case that $\mathcal{C}_1 \cap \mathcal{C}_2$ is not a single point $g(0)$ is also some $q \in \mathcal{C}_1 \cap \mathcal{C}_2$.

It is then clear from this process that the pull-back metric $g^*(ds_K^2)$ produces a conformal metric of constant curvature K in $\overline{\mathbb{C}_+} \equiv \overline{\mathbb{R}^2_+}$, which has constant geodesic curvature along $\mathbb{R}_-$ and $\mathbb{R}_+$, and a singularity at the origin. Also, this metric trivially has finite area, so we have a solution to (P) that satisfies (2.1).

When $K = 1$ more solutions exist. First of all, a similar process can be done if $K = 1$, by considering $\mathcal{U}$ to be the complement of a closed arc of a circle in $\overline{\mathbb{C}}$. This would correspond in some sense to taking $\mathcal{C}_1 = \mathcal{C}_2$ in the above process.

Furthermore, if $K = 1$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p, q\}$ consists of two points , we can easily create other finite area solutions to (P), starting from the basic region $\mathcal{U} \subset \overline{\mathbb{C}}$. For that, it suffices to consider a finite-folded branched holomorphic covering of $\overline{\mathbb{C}}$, with branching points at p and q. If we denote this branched covering by Φ, and consider $\widehat{g} := \Phi \circ g$, the pullback metric of ds_K^2 via $\widehat{g}$ again describes as before a finite area solution to (P). This composition can be also considered in the case above when we start with a solution such that $\mathcal{C}_1 = \mathcal{C}_2$ and we take a closed arc bounded by two different points $p, q \in \mathcal{C}_1$.

This construction provides a geometric interpretation of the canonical solutions. Indeed, we have:

Fact: Let $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{0\})$ be a canonical solution. Then $e^v|dz|^2$ is the pullback metric on $\mathbb{R}^2_+$ of either:

a) some basic region $\mathcal{U}$ in $\mathcal{Q}^2(K)$, or

b) the complement in $\mathcal{Q}^2(1) \equiv \overline{\mathbb{C}}$ of a closed arc of a circle, possibly composed with a suitable branched covering of $\overline{\mathbb{C}}$ in case $K = 1$.

We do not give a direct proof of this fact here since it will become evident from the results we will explain later on (see Section 2.6).

2.3 The local problem

In this section we show that although the class of solutions to (P) is quite large, it can be described in terms of entire holomorphic functions satisfying some adequate properties. Moreover, we give such a result not only in $\mathbb{R}^2_+$ but also in an arbitrary half-disc $D^+_\varepsilon \subset \mathbb{R}^2_+$. That is, we consider the solutions $v \in C^2(\overline{D^+_\varepsilon} \setminus \{0\})$ to the local problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } D^+_\varepsilon = \{(s,t) \in \mathbb{R}^2 : \ s^2 + t^2 < \varepsilon^2, \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } I^+_\varepsilon = \{(s,0) \in \mathbb{R}^2 : \ 0 < s < \varepsilon\}, \\ \dfrac{\partial v}{\partial t} = c_2 e^{v/2} & \text{on } I^-_\varepsilon = \{(s,0) \in \mathbb{R}^2 : \ -\varepsilon < s < 0\}. \end{cases} \tag{L}$$

In the spirit of the main result in [GaMi3], we can prove the following classification theorem. We let D^*_ε denote $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$.

Theorem 2.1. *Let v be a solution of (L). Then there exists a meromorphic function $F : D^*_\varepsilon \longrightarrow \overline{\mathbb{C}}$ such that v can be computed from* (1.9) *for a locally univalent meromorphic function $g : \overline{D^+_\varepsilon} \setminus \{0\} \longrightarrow \overline{\mathbb{C}}$ given by one of the following expressions:*

(i) $g(z) = \psi(z^\gamma F(z))$, *with $\gamma \in [0,1)$ and $F(r) \in \mathbb{R} \cup \{\infty\}$ for any $r \in \mathbb{R} \cap D^*_\varepsilon$,*

(ii) $g(z) = \psi(F(z) + \log(z))$, *with $F(r) \in \mathbb{R} \cup \{\infty\}$ for any $r \in \mathbb{R} \cap D^*_\varepsilon$,*

(iii) $g(z) = \psi(z^{i\gamma} F(z))$, *with $\gamma < 0$ and $|F(r)| = 1$ for any $r \in \mathbb{R} \cap D^*_\varepsilon$.*

Here, ψ is a Möbius transformation and g is holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$.

Conversely, let $g : \overline{D_\varepsilon^+} \setminus \{0\} \longrightarrow \overline{\mathbb{C}}$ be a locally univalent meromorphic function, holomorphic with $1 + K|g|^2 > 0$ if $K \leq 0$, constructed from a meromorphic function $F : D_\varepsilon^* \longrightarrow \overline{\mathbb{C}}$ as in $(i) - (iii)$ above. Then, the function v given by (1.9) *is a solution of problem (L).*

Remark 2.2. *Theorem 2.1 also provides all the solutions of the global problem (P). For that, it is enough to consider $\varepsilon = \infty$ in the previous theorem, that is, to change D_ε^* by $\mathbb{C}^*$.*

For the proof of Theorem 2.1, we will need the following elementary lemma.

Lemma 2.2. *Let $\widetilde{\Omega} = \{w \in \mathbb{C} : \ a < \mathrm{Re}(w) < b\}$, with $-\infty \leq a < b \leq +\infty$, and let $h : \widetilde{\Omega} \longrightarrow \overline{\mathbb{C}}$ be a function such that $h(w + 2\pi i) = h(w)$. Then, there exists a well-defined function $f : \Omega \longrightarrow \mathbb{C}$ on the topological annulus $\Omega = \{z \in \mathbb{C} : a < \log|z| < b\}$ such that $h(w) = f(e^w)$ for all $w \in \widetilde{\Omega}$.*

Moreover, if h is a meromorphic function then so is f.

Proof. We only must note that the function $f : \Omega \to \mathbb{C}$ such that $f(z) = h(\log z)$ is well defined in Ω because of the condition $h(w + 2\pi i) = h(w)$. Thus, if we write $\log z = w$, we obtain that $h(w) = f(e^w)$ in $\widetilde{\Omega}$ as we wanted.

Moreover, by the way we have defined f in terms of h, if h is meromorphic, so is f. □

Proof of Theorem 2.1. Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ be a solution of problem (L), and consider an associated developing map g. As explained in Lemma 1.2, the Schwarzian map of v, Q, given by (1.11), extends holomorphically to the punctured disc D_ε^*. Consider now the covering map $w \mapsto e^w$, from $\widetilde{D_\varepsilon^*} = \{w \in \mathbb{C} : \ \mathrm{Re}(w) < \log \varepsilon\}$ to D_ε^*, which is a local biholomorphism. Then, in the region of $\widetilde{D_\varepsilon^*}$ such that $0 < \mathrm{Im}(w) < \pi$ we can take the meromorphic map $\widetilde{g}$ given by

$$\widetilde{g}(w) = g(e^w). \tag{2.7}$$

Moreover, by formula (1.12), the Schwarzian of $\widetilde{g}(w)$ satisfies

$$\{\widetilde{g}, w\} = e^{2w} Q(e^w) - \frac{1}{2}. \tag{2.8}$$

As $Q(e^w)$ is globally defined and holomorphic in $\widetilde{D_\varepsilon^*}$, we see by Theorem 1.5 that $\widetilde{g}(w)$ can be extended to a locally univalent meromorphic function globally defined on $\widetilde{D_\varepsilon^*}$.

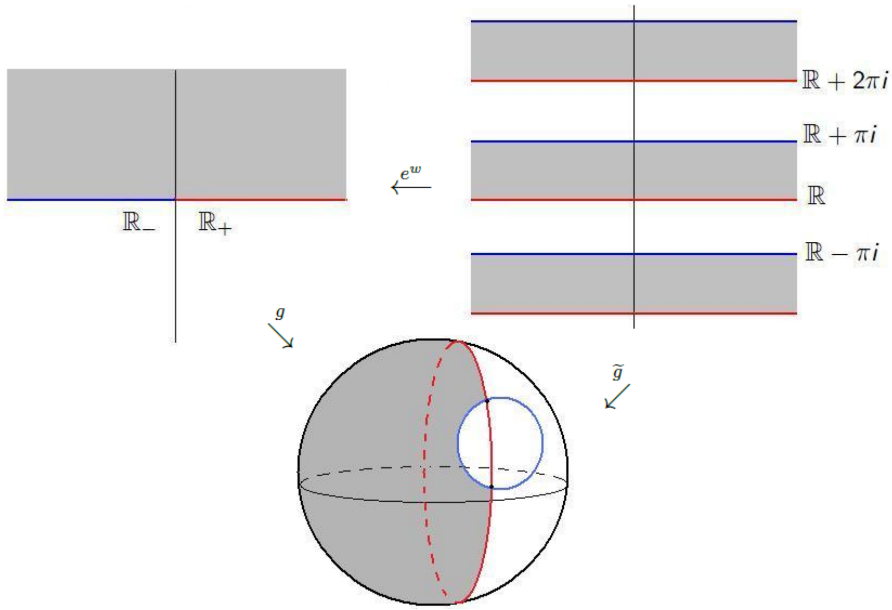


Figure 2.1: *Construction of* $\widetilde{g}$

In addition, since the right hand side of (2.8) is $2\pi i$-periodic, and since solutions to the Schwarzian equation $\{y, w\} = q(w)$ are unique up to Möbius transformations as we explained in Section 1.2, we see that the meromorphic function $\widetilde{g} : \widetilde{D_\varepsilon^*} \longrightarrow \overline{\mathbb{C}}$ satisfies

$$\widetilde{g}(w + 2\pi i) = \psi(\widetilde{g}(w)) \tag{2.9}$$

for a certain Möbius transformation ψ.

As explained in Lemma 1.2, $\widetilde{g}(w)$ lies on a circle $\mathcal{C}_1 \subset \overline{\mathbb{C}}$ for $\{w \in \widetilde{D_\varepsilon^*} : \mathrm{Im}(w) = 0\}$, and $\widetilde{g}(w)$ lies on another circle $\mathcal{C}_2 \subset \overline{\mathbb{C}}$ for $\{w \in \widetilde{D_\varepsilon^*} : \mathrm{Im}(w) = \pi\}$. We will study the behavior of g in terms of the relative position of both circles (see Figure 2.1).

Case 1: $\mathcal{C}_1$ intersects $\mathcal{C}_2$ in two points or they coincide.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ share at least two points, then we can consider a Möbius transformation φ such that $\varphi(\mathcal{C}_1)$ is the circle $\mathbb{R} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$ and $\varphi(\mathcal{C}_2)$ is the circle given by a straight line passing through the origin and $\infty \in \overline{\mathbb{C}}$. For that, observe that φ is the composition of a Möbius transformation which maps the previous two points of $\mathcal{C}_1 \cap \mathcal{C}_2$ into $\{0, \infty\}$, and a rotation with respect to the origin.

From (2.7) and (2.9), the new locally univalent meromorphic maps $G = \varphi \circ g$ and $\widetilde{G} = \varphi \circ \widetilde{g}$ satisfy

$$\widetilde{G}(w) = G(e^w) \tag{2.10}$$

and

$$\widetilde{G}(w + 2\pi i) = \Psi(\widetilde{G}(w)) \tag{2.11}$$

for a certain Möbius transformation Ψ.

For any real number $r \in (-\infty, \log \varepsilon)$ we have $\widetilde{G}(r) \in \varphi(\mathcal{C}_1) \subseteq \mathbb{R} \cup \{\infty\}$. Hence, by the Schwarz reflection principle,

$$\widetilde{G}(w) = \overline{\widetilde{G}(\overline{w})}, \qquad \text{for all } w \in \widetilde{D_\varepsilon^*}. \tag{2.12}$$

Thus, from (2.11) and (2.12),

$$\widetilde{G}(r + \pi i) = \Psi(\widetilde{G}(r - \pi i)) = \Psi\left(\overline{\widetilde{G}(r + \pi i)}\right), \qquad \text{for all } r \in (-\infty, \log \varepsilon). \tag{2.13}$$

And, since the set $\{\widetilde{G}(r + \pi i) : \ r \in (-\infty, \log \varepsilon)\}$ lies on the circle $\varphi(\mathcal{C}_2)$ and has no empty interior in $\varphi(\mathcal{C}_2)$, then

$$\zeta = \Psi(\overline{\zeta}), \qquad \text{for all } \zeta \in \varphi(\mathcal{C}_2).$$

But a Möbius transformation is determined by the image of three points, and $\varphi(\mathcal{C}_2)$ passes through 0 and ∞. So, if we take an arbitrary point $\zeta_0 \in \varphi(\mathcal{C}_2)\backslash\{0, \infty\}$ we easily obtain that

$$\Psi(\zeta) = \frac{\zeta_0}{\overline{\zeta_0}}\, \zeta, \qquad \text{for all } \zeta \in \overline{\mathbb{C}}.$$

Therefore, from (2.11), we get

$$\widetilde{G}(w + 2\pi i) = e^{i\theta_0}\ \widetilde{G}(w), \qquad w \in \widetilde{D_\varepsilon^*},$$

where $e^{i\theta_0} = \zeta_0/\overline{\zeta_0}$ for a real constant $\theta_0 \in [0, 2\pi)$. Finally, in order to obtain $\widetilde{G}$ we observe that the new meromorphic function

$$H(w) = e^{-\frac{\theta_0}{2\pi}\, w}\ \widetilde{G}(w) \tag{2.14}$$

satisfies

$$H(w + 2\pi i) = H(w), \qquad w \in \widetilde{D_\varepsilon^*}.$$

So, from Lemma 2.2, there exists a well defined meromorphic function $F(z)$ in the punctured disc D_ε^* such that

$$H(w) = F(e^w), \qquad w \in \widetilde{D_\varepsilon^*}.$$

Hence, (2.12) and (2.14) give

$$\widetilde{G}(w) = e^{\gamma w}\, F(e^w), \qquad w \in \widetilde{D_\varepsilon^*}, \tag{2.15}$$

with $\gamma = \theta_0/(2\pi) \in [0,1)$ and $F(z) = \overline{F(\overline{z})}$, $z \in D_\varepsilon^*$.

In particular, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have at least two common points is given, from (2.10) and (2.15), by

$$g(z) = \frac{A\, z^\gamma\, F(z) + B}{C\, z^\gamma\, F(z) + D}, \tag{2.16}$$

for certain complex constants A, B, C, D, with $AD - BC = 1$, which determine the Möbius transformation φ^{-1}.

Remark 2.3. *If $\mathcal{C}_1 = \mathcal{C}_2$, then $\zeta_0 \in \mathbb{R}$ and so $\gamma = 0$.*

Case 2: $\mathcal{C}_1$ intersects $\mathcal{C}_2$ in a unique point.

Let p_0 be the common point of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Then we consider a Möbius transformation φ that maps $\mathcal{C}_1$ to the circle $\mathbb{R} \cup \{\infty\} \subseteq \overline{\mathbb{C}}$ and maps $\mathcal{C}_2$ to the circle $\{z \in \mathbb{C} : \mathrm{Im}(z) = \pi\} \cup \{\infty\}$. For that, observe that φ can be seen as a Möbius transformation mapping $\mathcal{C}_1$ into $\mathbb{R} \cup \{\infty\}$ which maps p_0 to ∞, composed with a homothety.

As in the previous case, we define the new locally univalent meromorphic maps $G = \varphi \circ g$ and $\widetilde{G} = \varphi \circ \widetilde{g}$ which satisfy (2.10), (2.11), (2.12) and (2.13).

Since the set $\{\widetilde{G}(r + \pi i) : \ r \in (-\infty, \log \varepsilon)\}$ lies on the circle $\{z \in \mathbb{C} : \mathrm{Im}(z) = \pi\} \cup \{\infty\}$ and has no empty interior there, then

$$\zeta = \Psi(\overline{\zeta}), \qquad \text{for all } \zeta \in \{z \in \mathbb{C} : \mathrm{Im}(z) = \pi\} \cup \{\infty\}.$$

Therefore, $\Psi(\zeta) = \zeta + 2\pi i$, and so, from (2.11),

$$\widetilde{G}(w + 2\pi i) = \widetilde{G}(w) + 2\pi i, \qquad w \in \widetilde{D_\varepsilon^*}.$$

Now, the new meromorphic function $H(w) = \widetilde{G}(w) - w$ satisfies $H(w + 2\pi i) = H(w)$ for all $w \in \widetilde{D_\varepsilon^*}$. Hence, using Lemma 2.2 for the meromorphic function $H(w)$, there exists a well defined meromorphic function $F(z)$ in the punctured disc D_ε^* such that

$$\widetilde{G}(w) = F(e^w) + w, \qquad w \in \widetilde{D_\varepsilon^*}. \tag{2.17}$$

Moreover, from (2.12), $F(z) = \overline{F(\overline{z})}$, $z \in D_\varepsilon^*$.

With all of this, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have only one common point is given, from (2.10) and (2.17), by

$$g(z) = \frac{A\,(F(z) + \log z) + B}{C\,(F(z) + \log z) + D}, \tag{2.18}$$

for certain complex constants A, B, C, D, with $AD - BC = 1$, which determine the Möbius transformation φ^{-1}.

Case 3: $\mathcal{C}_1$ does not intersect $\mathcal{C}_2$.

In this case, it is well-known that there exists a Möbius transformation φ such that the image of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are the circles centered at the origin with radii 1 and $R > 1$, respectively.

We start by considering the locally univalent meromorphic maps $G = \varphi \circ g$ and $\widetilde{G} = \varphi \circ \widetilde{g}$, which satisfy again (2.10) and (2.11) for a certain Möbius transformation Ψ.

Given a real number $r \in (-\infty, \log \varepsilon)$ we have $|\widetilde{G}(r)| = 1$. So, from the Schwarz reflection principle

$$\widetilde{G}(w) = \frac{1}{\overline{\widetilde{G}(\overline{w})}}, \qquad w \in \widetilde{D_\varepsilon^*}. \tag{2.19}$$

In addition, from (2.11),

$$\widetilde{G}(r + \pi i) = \Psi(\widetilde{G}(r - \pi i)) = \Psi\left(\frac{1}{\overline{\widetilde{G}(r + \pi i)}}\right), \qquad r \in (-\infty, \log \varepsilon).$$

Thus, proceeding as in the previous cases, we have

$$\zeta = \Psi(\frac{1}{\overline{\zeta}}), \qquad \text{for } |\zeta| = R,$$

that is, $\Psi(\frac{1}{R}\,e^{i\theta}) = R\,e^{i\theta}$ for any $\theta \in \mathbb{R}$.

Therefore, $\Psi(\zeta) = R^2\zeta$, and so, from (2.11),

$$\widetilde{G}(w + 2\pi i) = R^2\,\widetilde{G}(w), \qquad w \in \widetilde{D_\varepsilon^*}.$$

Then we can apply Lemma 2.2 to the meromorphic function

$$H(w) = e^{-\frac{\log(R^2)}{2\pi i}\, w}\ \widetilde{G}(w).$$

Hence, there exists a well defined meromorphic function $F(z)$ in D_ε^* such that

$$\widetilde{G}(w) = e^{i\gamma w} F(e^w), \qquad w \in \widetilde{D_\varepsilon^*}$$

for the negative real constant $\gamma = -\frac{\log(R^2)}{2\pi}$. Moreover, from (2.19), $F(z)\,\overline{F(\overline{z})} = 1$, for any $z \in D_\varepsilon^*$.

As a consequence, the developing map g of any solution of the local problem (L) when $\mathcal{C}_1$ and $\mathcal{C}_2$ have no common points is given, from (2.10), by

$$g(z) = \frac{A\, z^{\gamma\, i} F(z) + B}{C\, z^{\gamma\, i} F(z) + D}, \tag{2.20}$$

for certain complex constants A, B, C, D, with $AD - BC = 1$, which determine the Möbius transformation φ^{-1}.

This completes the proof of the first part of Theorem 2.1. Also, the converse part of the theorem is just a straightforward computation, so we are done.

□

2.4 The local problem with finite area

We devote this section to study the hypotheses that make solutions to problem (L) given by Theorem 2.1 satisfy the finite area condition

$$\int_{D_\varepsilon^+} e^v < \infty. \tag{2.21}$$

In this sense, we can state the following result.

Theorem 2.2. *Let v be a solution of (L) that satisfies the finite energy condition* (2.21)*. Then, its developing map g is given by the cases (i) or (ii) in Theorem 2.1, and F does not have an essential singularity at the origin.*

In particular, g can be continuously extended to the origin, and the Schwarzian map $Q : D_\varepsilon^* \longrightarrow \mathbb{C}$ of v has at most a pole of order two at 0.

Proof. Let us start by explaining that it suffices to prove the result for the case $K = 1$. Indeed, let v be a solution of (L)–(2.21) for a constant $K = K_0 = -1, 0$ and g an associated developing map. Now, let us consider the function v_1 given by (1.9) for the map g and $K = 1$. Then, v_1 is also a solution of (L), but in this case for $K = 1$, and it also satisfies (2.21) since

$$\int e^{v_1}\,|dz| = \int \frac{4|g'(z)|^2}{(1+|g(z)|^2)^2}\,|dz| \leq \int \frac{4|g'(z)|^2}{(1+K_0\,|g(z)|^2)^2}\,|dz| = \int e^{v}\,|dz| < \infty.$$

In other words, if the result is true for $K = 1$, it will automatically be true for $K = -1, 0$, as claimed.

Thus, let g be the developing map of a solution to (L) for $K = 1$. First, let us prove that the lengths of the semicircles

$$C_r = \{z \in D_\varepsilon^* :\ |z| = r,\ \mathrm{Im}(z) \geq 0\},$$

for the metric $e^v|dz|^2$ of constant curvature $K = 1$, tend to zero when r goes to zero.

If we denote by $L(r)$ the length of the semicircle C_r for $e^v|dz|^2$, and write $z = r\,e^{i\theta}$, we have from (1.9)

$$L(r) = r\int_0^\pi e^{v/2}d\theta = r\int_0^\pi \frac{2|g'|}{1+|g|^2}d\theta \leq 2\pi \sup_{0\leq\theta\leq\pi}\left(r\,g^\sharp(r\,e^{i\theta})\right)$$

where $g^\sharp(z)$ is the spherical derivative of g with respect to z, that is,

$$g^\sharp(z) := \frac{|g'(z)|}{1+|g(z)|^2}.$$

Hence,

$$\limsup_{|z|\to 0} L(|z|) \leq 2\pi \limsup_{z\to 0}\left(|z|\,g^\sharp(z)\right). \tag{2.22}$$

Claim: in the above conditions, we have $\limsup_{z\to 0}\left(|z|\,g^\sharp(z)\right) = 0$.

Let us prove the claim above. Assume that $\limsup_{z\to 0}\left(|z|\,g^\sharp(z)\right) \neq 0$, and so $\limsup_{z\to 0} g^\sharp(z) = \infty$. Consider for some $r \in (\varepsilon/4, \varepsilon/2)$ the fixed domain

$$\Omega = \{z \in \mathbb{R}^2_+ :\ r \leq |z| \leq \varepsilon\},$$

and the family of functions

$$g_n(z) = g\left(\frac{z}{2^n}\right), \quad z \in \Omega.$$

That is, the functions of the family $\mathfrak{G} = \{g_n\}_{n\in\mathbb{N}}$ are nothing but the function g evaluated over a domain $\Omega_n \subset \mathbb{R}^2_+$ that gets smaller and closer to the origin as n increases. Moreover, by the choice of r, it holds $\Omega_n \cap \Omega_{n+1} \neq \emptyset$ and $\Omega_n \cap \Omega_{n+2} = \emptyset$. The Theorem of Marty (see [Ma]) characterizes a family of meromorphic functions $\mathfrak{G}$ as a normal family if and only if for every compact $K \subset \Omega$ there exists a constant $M(K)$ such that $g_n^\sharp(z) \leq M(K)$ on K for every function $g_n \in \mathfrak{G}$. Thus, as $\limsup_{z\to 0} g^\sharp(z) = \infty$, $\mathfrak{G}$ is not normal on Ω. On the other hand, the Theorem of Montel (see [Mon]) asserts that as $\mathfrak{G}$ is not normal there will be at least one function $g_{n_0} \in \mathfrak{G}$ that takes every complex value with at most two exceptions on Ω. Then, as $\mathfrak{G} \setminus \{g_{n_0}\}$ is not a normal family, we can iterate this argument and conclude that g assumes every complex value infinitely many times with at most two exceptions in a neighborhood of the origin in $\mathbb{R}^2_+$. This means that $g(z)$ covers an infinite area on $\{z \in D^*_\varepsilon : \ \mathrm{Im}(z) \geq 0\}$, which is a contradiction.

Thus, the claim is proved.

Observe that g is a local isometry from $\{z \in D^*_\varepsilon : \ \mathrm{Im}(z) \geq 0\}$ with the metric $e^v|dz|^2$ into the unit sphere $\overline{\mathbb{C}}$ with its standard metric. Also, we know from Lemma 1.2 that $g(D_\varepsilon \cap \mathbb{R}^+)$ lies on a circle $\mathcal{C}_1 \subseteq \overline{\mathbb{C}}$ and $g(D_\varepsilon \cap \mathbb{R}^-)$ lies on a circle $\mathcal{C}_2 \subseteq \overline{\mathbb{C}}$. So, $g(C_r)$ is a curve in the sphere $\overline{\mathbb{C}}$ with length $L(r)$, joining the point $g(r)$ of $\mathcal{C}_1$ and the point $g(-r)$ of $\mathcal{C}_2$.

Thus, the case (iii) in Theorem 2.1 cannot happen because this kind of solutions only occur when $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, which contradicts $\lim_{r\to 0} L(r) = 0$.

The solutions of type (ii) in Theorem 2.1 happen when $\mathcal{C}_1$ intersects $\mathcal{C}_2$ at a unique point $p_0 \in \overline{\mathbb{C}}$. Let us see that, in this case, $\lim_{z\to 0} g(z) = p_0$, what shows that the function $F(z)$ such that $g(z) = \psi(F(z) + \log(z))$ cannot have an essential singularity at the origin as we wanted to show.

Let $\{z_n\}$ be a sequence converging to 0, and $\delta > 0$ small enough. Now, let us prove that there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $d(g(z_n), p_0) < \delta$, where $d(,)$ denotes distance in $\overline{\mathbb{C}} \equiv \mathcal{Q}^2(1)$.

Let D be the open disc of radius $\delta/2$ centered at p_0 and

$$\widehat{\delta} = \min\{d(\mathcal{C}_1 \backslash D, \mathcal{C}_2), d(\mathcal{C}_1, \mathcal{C}_2 \backslash D)\}.$$

Since $\lim_{r\to 0} L(r) = 0$ there exists $r_0 > 0$ such that if $r < r_0$ then $L(r) < \min\{\delta, \widehat{\delta}\}$. Now, we choose n_0 such that $|z_n| < r_0$ for all $n \geq n_0$. Thus, $L(|z_n|) < \widehat{\delta}$ and so $g(|z_n|) \in \mathcal{C}_1 \cap D$ and $g(-|z_n|) \in \mathcal{C}_2 \cap D$. Hence, if $d(g(z_n), p_0) \geq \delta$ we would have

$$L(|z_n|) \geq d(g(|z_n|), g(z_n)) + d(g(z_n), g(-|z_n|)) > \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which is a contradiction. This proves that $\lim_{z\to 0} g(z) = p_0$.

Let us show now that the Schwarzian derivative map $Q(z)$, which is well defined in D^*_ε, has at most a pole of order two at the origin. Since the Schwarzian derivative is invariant under Möbius transformations, in this case we only need to do the computation for $g(z) = F(z) + \log z$. As

$$\frac{g''(z)}{g'(z)} = \frac{z^2 F''(z) - 1}{z\,(zF'(z) + 1)}$$

has a pole of order one at the origin, we get from (1.11) that $Q(z)$ has at most a pole of order two there.

Finally, we analyze the solutions of the case (i) in Theorem 2.1. They correspond to the situation in which $\mathcal{C}_1 \cap \mathcal{C}_2$ has exactly two points, or $\mathcal{C}_1 = \mathcal{C}_2$.

If $\mathcal{C}_1 = \mathcal{C}_2$ then from Remark 2.3 the associated developing map g is given by $g(z) = \psi(F(z))$, where $F(z)$ is a meromorphic function in D^*_ε. In addition, the meromorphic function $F(z)$ cannot have an essential singularity at the origin. Otherwise, since $F(\overline{z}) = \overline{F(z)}$, $g(z)$ would take every value of $\overline{\mathbb{C}}$, except at most two points, infinity many times in $\{z \in D^*_\varepsilon : \ \mathrm{Im}(z) \geq 0\}$ what would contradict the finite area condition.

If $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p_1, p_2\}$, a similar argument to the one we just used for the case that $\mathcal{C}_1 \cap \mathcal{C}_2 = \{p_0\} \subset \overline{\mathbb{C}}$ lets us prove that there exists a unique $i_0 \in \{1, 2\}$ such that $\lim_{z\to 0} g(z) = p_{i_0}$. Thus, g can be continuously extended to the origin with $g(0) = p_{i_0}$ and $F(z)$ does not have an essential singularity at 0. Let us outline this argument.

Take again a sequence $\{z_n\} \to 0$ and $\delta > 0$ satisfying $\delta < d(p_1, p_2)/3$. Consider D_i the open disc centered at p_i and radius $\delta/2$, and let

$$\widehat{\delta} = \min\{d(\mathcal{C}_1 \backslash (D_1 \cup D_2), \mathcal{C}_2), d(\mathcal{C}_1, \mathcal{C}_2 \backslash (D_1 \cup D_2))\}.$$

Arguing as before, we can show that there exists a unique $i_0 \in \{1, 2\}$ such that $g(|z_n|), g(-|z_n|) \in D_{i_0}$ for n sufficiently large.

Hence, every point z_n with n sufficiently large satisfies $d(g(z_n), p_{i_0}) < \delta$, since otherwise $d(g(z_n), p_{i_0}) \geq \delta$, and so $L(|z_n|) \geq \delta$, which contradicts that $\lim_{r\to 0} L(r) = 0$. Therefore, $\lim_{z\to 0} g(z) = p_{i_0}$.

In order to finish the proof, let us show that the Schwarzian derivative $Q(z)$ of $g(z)$ has at most a pole of order two at the origin. Again by the invariance of the Schwarzian derivative under Möbius transformations, we only need to do the computation for $g(z) = z^\gamma F(z)$. And since $F(z)$ has no essential singularity at the origin,

$$\frac{g''(z)}{g'(z)} = \frac{z^2F''(z) + 2\gamma z F'(z) + (\gamma-1)\gamma F(z)}{z\left(zF'(z) + \gamma F(z)\right)}$$

has at most a pole of order one there. Thus, from (1.11), $Q(z)$ has at most a pole of order two at the origin. This completes the proof of Theorem 2.2. $\square$

Theorem 2.2 shows that for classifying the solutions to (L)-(2.21), it suffices to determine when the functions v given by (1.9) in terms of a developing map g as in the statement of Theorem 2.2 verify (2.21). We do this next.

First, assume that g is given by case (i) in Theorem 2.1, where F has at 0 either a pole or a finite value. If we let A, B, C, D denote the coefficients of the Möbius transformations defining g, with $AD - BC = 1$, a computation gives

$$e^v = \frac{4|\gamma z^{\gamma-1}F(z) + z^\gamma F'(z)|^2}{(|Cz^\gamma F(z) + D|^2 + K|Az^\gamma F(z) + B|^2)^2}.$$

If F has a finite value (resp. a pole) at 0, and if $|D|^2 + K|B|^2 \neq 0$ (resp. $|C|^2 + K|A|^2 \neq 0$), it is easy to check that near 0 we have $e^v = |z|^{2\alpha} a(z)$ where $\alpha > -1$ and $a(z)$ is continuous at 0, with $a(0) \neq 0$. Thus (2.21) holds in these cases.

We suppose now that F has a finite value at 0 and $|D|^2 + K|B|^2 = 0$. If $K = 0$, then $D = 0$ and $e^v = a(z)|z|^{-2(\gamma+k+1)}$ with $a(z)$ continuous at 0 and $a(0) \neq 0$, where k is the order of the zero that F has at the origin (if $F(0) \neq 0$ then $k = 0$). Thus (2.21) does not hold. If $K = -1$, the conditions $AD - BC = 1$ and $|D|^2 = |B|^2$ give that $\delta = -A\overline{B} + C\overline{D}$ is such that $|\delta| = 1$, and so we can estimate

$$\begin{aligned}
\int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1}F(z) + z^\gamma F'(z)|^2}{|z|^{2\gamma}|F|^2\left((|C|^2 - |A|^2)|z|^\gamma|F(z)| + 2\mathrm{Re}(\delta\frac{F(z)z^\gamma}{|F(z)||z|^\gamma})\right)^2} \\
&\geq \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1}F(z) + z^\gamma F'(z)|^2}{|z|^{2\gamma}|F|^2\left(|\,|C|^2 - |A|^2\,|\,|z|^\gamma|F(z)| + 2\right)^2} \\
&\geq R\int_0^\varepsilon \frac{dr}{r} = \infty,
\end{aligned}$$

for a certain constant $R > 0$, where $r = |z|$. So, (2.21) does not hold in this case.

We consider now the situation in which F has a pole of order $k \geq 1$ and $|C|^2 + K|A|^2 = 0$. If $K = 0$, we easily check that close to the origin, $e^v = a(z)|z|^{2(\gamma-k-1)}$ with $a(z)$ continuous at 0 and $a(0) \neq 0$. So, (2.21) does not hold since $k > \gamma$. If $K = -1$ and $\delta := -A\overline{B} + C\overline{D}$, we also have $|\delta| = 1$, and since $k > \gamma$ we can estimate as before

$$\begin{aligned} \int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|\gamma z^{\gamma-1}F(z) + z^\gamma F'(z)|^2}{|F|^2|z|^{2\gamma}\left(\frac{|D|^2-|B|^2}{|F||z|^\gamma} + 2\mathrm{Re}(\delta\frac{F(z)z^\gamma}{|F||z|^\gamma})\right)^2} \\ &\geq R\int_0^\varepsilon \frac{dr}{r} = \infty, \end{aligned} \tag{2.23}$$

for a certain constant $R > 0$, where $r = |z|$, and so (2.21) does not hold.

Next, assume that g is given by case (ii) in Theorem 2.1, where again F has at 0 a pole or a finite value. Arguing as before, since

$$e^v = \frac{4|F'(z) + 1/z|^2}{(|C(F(z) + \log z) + D|^2 + K|A(F + \log z) + B|^2)^2}, \tag{2.24}$$

we see that if $|C|^2 + K|A|^2 \neq 0$ and F has a finite value (resp. a pole of order $k \geq 1$) at 0, then near 0 we have $e^v = |z|^{-2}(\ln|z|)^{-4}a(z)$ (resp. $e^v = |z|^{2(k-1)}a(z)$) where $a(z)$ is continuous at 0, with $a(0) \neq 0$. Again, (2.21) holds automatically.

Suppose now that $|C|^2 + K|A|^2 = 0$. If $K = 0$, it is immediate to see that (2.21) does not hold. So, we are left with the case $K = -1$ and $|A| = |C|$. Observe that this condition implies that $g(0) \in \mathbb{S}^1$. As g is given by case (ii) in Theorem 2.1, the images of $(0,\varepsilon)$ and $(-\varepsilon, 0)$ by g are two circle arcs meeting tangentially at one point of $\mathbb{S}^1$.

Assume first that F has a finite value at 0. If the constant $\delta = -A\overline{B}+C\overline{D}$ is such that $\mathrm{Re}\ \delta = 0$, then we easily check that close to the origin

$$e^v = \frac{4|F'(z) + 1/z|^2}{(|D|^2 - |B|^2 - 2\mathrm{Im}\ \delta(\mathrm{Im}\ F(z) + \arg z))^2} \geq R|F'(z) + 1/z|^2$$

for some $R > 0$. Thus (2.21) does not hold. On the other hand, if $\mathrm{Re}\ \delta \neq 0$ the asymptotic behavior of e^v is

$$e^v = \frac{|F' + 1/z|^2}{(\ln|z|)^2(\frac{|D|^2-|B|^2}{2\ln|z|} + \mathrm{Re}\ \delta(1 + \frac{\mathrm{Re}\ F(z)}{\ln|z|}) - \mathrm{Im}\ \delta\frac{(\mathrm{Im}\ F(z)+\arg z)}{\ln|z|})^2} = \frac{a(z)}{|z|^2(\ln|z|)^2},$$

where $a(z)$ is continuous at 0 and $a(0) \neq 0$. Then, we can easily check that (2.21) holds. Furthermore, a computation shows that $c_1 = -c_2 \in (-2, 2)$.

Now, assume that F has a pole at 0, $K = -1$ and $|A| = |C|$. If $\delta = -A\overline{B} + C\overline{D}$, we can proceed as we did before and estimate

$$\begin{aligned}\int_{D_\varepsilon^+} e^v &= \int_{D_\varepsilon^+} \frac{4|F'(z)+1/z|^2}{|F+\log z|^2\left(\frac{|D|^2-|B|^2}{|F+\log z|} + 2\mathrm{Re}(\delta(\frac{F(z)+\log z}{|F+\log z|})\right)^2} \\ &\geq R\int_0^\varepsilon \frac{dr}{r} = \infty,\end{aligned}$$

for a certain constant $R > 0$, where $r = |z|$. Again, (2.21) does not hold.

This completes the classification of all the solutions to (L)-(2.21). In particular, we have the following description of the asymptotic behavior of such solutions.

Corollary 2.1. *Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ be a solution to (L)-(2.21). There are three possible asymptotic behaviors for v at 0:*

1. $\lim_{z\to 0} |z|^{-2\alpha} e^v \neq 0$, *for some $\alpha > -1$, i.e. $e^v|dz|^2$ has at 0 a boundary conical singularity.*

2. $\lim_{z\to 0} |z|^2(\ln|z|)^4 e^v \neq 0$.

3. $\lim_{z\to 0} |z|^2(\ln|z|)^2 e^v \neq 0$.

Here, the last case happens only when $K = -1$ and the boundary has infinite length around 0. In this last situation, it holds $c_1 = -c_2 \in (-2, 2)$.

2.5 Global solutions with finite area

In Section 2.2 we described in detail a large family of explicit solutions to (P) with finite area, i.e. such that (2.1) holds: the *canonical solutions*. We prove next that, actually, these are the only solutions to (P) with finite area.

Theorem 2.3. *Any solution to (P) satisfying the finite energy condition* (2.1) *is a canonical solution.*

Proof. Let v be a solution of (P) such that

$$\int_{\mathbb{R}_+^2} e^v < \infty,$$

and let $Q(z)$ be the Schwarzian map associated to v. From Theorem 2.2, $Q(z)$ has at most a pole of order two at the origin.

We observe that the new meromorphic function $h(w) = g(z)$ with $z = -1/w$ is also the developing map of a solution $\widehat{v}$ to (P). Thus, since

$$\int_{\mathbb{R}^2_+} e^{\widehat{v}} = \int_{\mathbb{R}^2_+} e^{v} < \infty,$$

we obtain again from Theorem 2.2 that the Schwarzian derivative $\widehat{Q}(w)$ has at most a pole of order two at the origin.

By using that $w^4\,\widehat{Q}(w) = Q(z)$ (see formula (1.12)), we conclude that $Q(z)$ is a holomorphic function in $\mathbb{C}^*$ with at most a pole of order two at the origin and at least a zero of order two at infinity. Therefore,

$$Q(z) = \frac{c}{z^2}, \qquad z \in \mathbb{C}^*$$

for a certain constant $c \in \mathbb{C}$.

The solutions $g(z)$ of the Schwarzian equation (1.11) for $Q(z) = c/z^2$ are well-known. They are given by $g(z) = \psi(\log(z))$ if $2c = 1$ and by $g(z) = \psi(z^\gamma)$ if $2c = 1 - \gamma^2 \neq 1$, where ψ is an arbitrary Möbius transformation. In the latter case, if follows from Theorem 2.2 that in our situation γ must be a real constant. In fact, up to composition with the Möbius transformation $z \to 1/z$ if necessary, we can assume $\gamma > 0$.

Finally, in order to finish the proof, we compute the solutions v to our problem depending of the value of $g(z)$.

Let

$$g(z) = \psi(z^\gamma) = \frac{A\,z^\gamma + B}{C\,z^\gamma + D}$$

with $AD - BC = 1$.

Then, from (1.9),

$$e^v = \frac{4\gamma^2|z|^{2(\gamma-1)}}{(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})z^\gamma + (K\overline{A}B + \overline{C}D)\overline{z}^\gamma + (K|A|^2 + |C|^2)|z|^{2\gamma})^2}\,.$$

If $K|A|^2 + |C|^2 = 0$, an argument as in (2.23) proves that $\int_{\mathbb{R}^2_+} e^v = \infty$. On the other hand, if $K|A|^2 + |C|^2 \neq 0$ we can take

$$\lambda = \frac{1}{|\,K|A|^2 + |C|^2\,|}, \qquad z_0 = -\frac{K\overline{A}B + \overline{C}D}{K|A|^2 + |C|^2} \tag{2.25}$$

and so we have

$$e^v = \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{(K\lambda^2 + |z^\gamma - z_0|^2)^2},$$

that is, we obtain the canonical solution (2.3), as wished.

Now, let us consider the case

$$g(z) = \psi(\log z) = \frac{A\,\log z + B}{C\,\log z + D}. \tag{2.26}$$

with $AD - BC = 1$.

Then, from (1.9),

$$\begin{aligned} e^v \;=\; & 4/(|z|^2(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})\log z + (K\overline{A}B + \overline{C}D)\log\overline{z} + \\ & +(K|A|^2 + |C|^2)|\log z|^2)^2). \end{aligned}$$

If $K|A|^2 + |C|^2 = 0$, the function v has infinite area in $\mathbb{R}^2_+$. This follows directly from our discussion after Theorem 2.2 except when $\mathrm{Re}\,\delta \neq 0$ and $K = -1$, where here $\delta := -A\overline{B} + C\overline{D}$. But in that case, using that $|A| = |C|$, the condition that the map g in (2.26) must satisfy $|g(z)| < 1$ for every $z \in \mathbb{C}_+$ leads to the inequality

$$2\mathrm{Re}\,\delta\ln|z| > |B|^2 - |D|^2 + 2\mathrm{Im}\,\delta\arg\,z,$$

which cannot hold since $\ln|z| : \mathbb{C}_+ \longrightarrow \mathbb{R}$ is surjective. This proves the claim.

If $K|A|^2 + |C|^2 \neq 0$ the function v can be rewritten as

$$e^v = \frac{4\,\lambda^2}{|z|^2(K\lambda^2 + |\log z - z_0|^2)^2},$$

where λ and z_0 are chosen as in (3.15). This completes the proof of Theorem 2.3. $\square$

As a consequence, we get:

Corollary 2.2. *Given $K \in \{-1, 0, 1\}$, $c_1, c_2 \in \mathbb{R}$, there exists a solution to the problem (P) with finite area (i.e. so that condition* (2.1) *holds) if and only if*

- $K = 1$, *or*
- $K = 0$ *and* $c_i < 0$ *for some* $i \in \{1, 2\}$, *or*

- $K = -1$ *and one of these conditions are satisfied:*

$$c_1 < -2, \qquad \textit{or} \qquad c_2 < -2, \qquad \textit{or} \qquad c_1 + c_2 < 0.$$

Proof. It suffices to prove the result for canonical solutions. Consider first of all a canonical solution given by (2.3) and write $z_0 = r_0 e^{i\theta_0}$. We already explained in Section 2.2 the restrictions between the parameters appearing in (2.3) for the solution to be well defined. Also, the relationship between the constants c_i and these parameters is given in Lemma 2.1. From this, we see directly that there are always canonical solutions with $K = 1$ for every $c_1, c_2 \in \mathbb{R}$, of the form (2.3).

In the cases $K = 0, -1$, the situation is more restrictive. Label $x = \theta_0$, $y = \theta_0 - \pi\gamma$ and $R_0 = 2r_0/\lambda$. From Lemma 2.1 we see that

$$c_1 = R_0 \sin(x) \qquad c_2 = -R_0 \sin(y). \tag{2.27}$$

By our analysis in Section 2.2, we have:

- if $K = -1$, then $0 < \alpha_0 < y < x < 2\pi - \alpha_0$ where $\alpha_0 \in (0, \pi/2)$ and $R_0 = 2/\sin(\alpha_0) > 2$. So, if $\alpha_0 < y \leq \pi/2$ (resp. $3\pi/2 \leq x < 2\pi - \alpha_0$) we get $c_2 < -2$ (resp. $c_1 < -2$), and if $\pi/2 < y < x < 3\pi/2$ we have $c_1 + c_2 < 0$. Conversely, assume that c_1 and c_2 satisfy the restrictions above, choose $R_0 > 2$ such that $c_1/R_0, c_2/R_0 \in [-1, 1]$, and call $\sin(\alpha_0) = 2/R_0$. Then, it is clear that some choices of $x = \arcsin(c_1/R_0)$ and $y = \arcsin(-c_2/R_0)$ satisfy $\alpha_0 < y < x < 2\pi - \alpha_0$. So we can find z_0 and λ as in Section 2.2, such that (2.5) holds.

- If $K = 0$, then $0 < y < x < 2\pi$. Thus, a simple analysis shows that c_1 and c_2 can not be positive simultaneously. The converse is analogous to the previous $K = -1$ case.

Finally, assume that the canonical solution is given by (2.4). Again, there are no restrictions if $K = 1$. On the other hand, from (2.6) in the cases $K = 0$ and $K = -1$, the condition $c_1 + c_2 > 0$ must be satisfied. Moreover, because of the restriction we had for this kind of solutions, at least one c_i must be strictly negative (strictly less than -2 in the case $K = -1$). The converse is trivial. $\square$

Remark 2.4. *As a consequence of Theorem 2.3 and the discussion about the geometry of the* canonical solutions *in Subsection 2.2.2, we can deduce that if* $K = 0$ *or* $K = -1$, g *is a conformal diffeomorphism from* $\overline{\mathbb{R}^2_+} \cup \{\infty\}$ *into* *a compact domain in $\mathbb{C}$ or in the unit disc, respectively. Such a domain will be bounded by two circle arcs intersecting at two points if the singularity is of conical type (case* (2.3)*) or at a unique point otherwhise (case* (2.4)*). In Figure 2.2 we show two examples of the image of the developing map g for the finite area case when $K = -1$.*

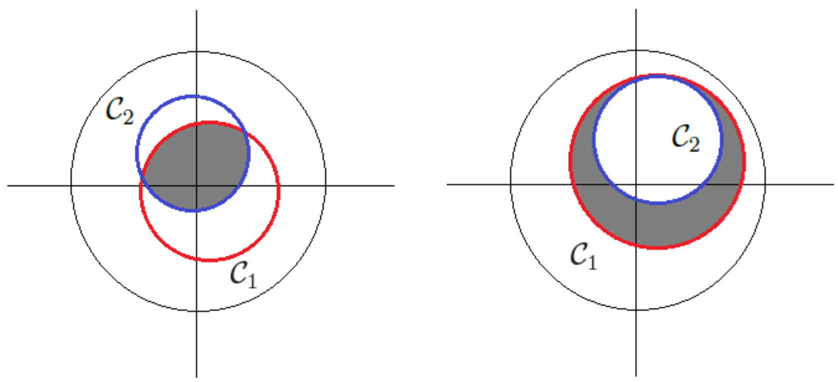


Figure 2.2: *Examples of the image of g (in grey) when $K = -1$. Singularities of conical and non conical type, respectively.*

2.6 The extended problem. Uniqueness of polygonal circular metrics

In this section we extend our study of problem (P) by allowing the solution to have an arbitrary number of boundary singularities. In this sense, Corollary 2.4 is a generalization of Theorem 2.3 that solves a problem posed in [JWZ], under milder hypotheses.

Let us start by stating the following result, which follows from the proof of Theorem 2.2 and the subsequent discussion.

Corollary 2.3. *Let $v \in C^2(\overline{D_\varepsilon^+} \setminus \{0\})$ be solution to (L) that satisfies the finite energy condition (2.21), and let $g : \overline{D_\varepsilon^+} \setminus \{0\} \longrightarrow \Sigma_K \subseteq \overline{\mathbb{C}}$ denote a developing map of v (which is always a local diffeomorphism). Then:*

1. *The image $g(I_\varepsilon^+)$ lies on a circle $\mathcal{C}_1$, and the image $g(I_\varepsilon^-)$ lies on another circle $\mathcal{C}_2$, such that $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ (possibly $\mathcal{C}_1 = \mathcal{C}_2$).*

2. *The geodesic curvature of $\mathcal{C}_i$, when parameterized as $g(s,0)$, is constant of value $-c_i/2$, for the metric ds_K^2 in (1.8).*

3. *g extends continuously to the origin, with $g(0) \in \mathcal{C}_1 \cap \mathcal{C}_2 \subset \overline{\mathbb{C}}$.*

4. *The Schwarzian map $Q = v_{zz} - v_z^2/2$ extends holomorphically to D_ε^* with $Q(\bar{z}) = \overline{Q(z)}$, and has at the origin at most a pole of order two.*

5. *If $K = 0$, then $g(0) \in \mathbb{C}$. If $K = -1$ and $g(0) \in \partial\mathbb{D} \equiv \mathbb{S}^1$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent at $g(0)$, and are* not *arcs of horocycles (specifically, they correspond to item 3 in Corollary 2.1).*

From this result, it is not difficult to classify the conformal metrics of constant curvature K and finite area on $\mathbb{R}^2_+$ that have a finite number of boundary singularities on the real axis, and constant geodesic curvature along each boundary arc. From an analytical point of view, this corresponds to classifying the solutions $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ with $\int_{\mathbb{R}^2_+} e^v < \infty$ to the Neumann problem

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2 : \ t > 0\}, \\ \dfrac{\partial v}{\partial t} = c_j \, e^{v/2} & \text{on } I_j \subset \mathbb{R} \equiv \partial \mathbb{R}^2_+, \qquad c_j \in \mathbb{R}, \end{cases} \tag{2.28}$$

where $I_j := (q_j, q_{j+1})$, $j = 0, \dots, n-1$ and $q_0 = -\infty < q_1 < \dots < q_{n-1} < q_n = \infty$.

There are obvious examples of this type of conformal metrics on $\mathbb{R}^2_+$. To see this, we denote

$$\widetilde{\Sigma_K} = \begin{cases} \Sigma_K & \text{if } K = 1, 0, \\ \overline{\Sigma_K} \equiv \overline{\mathbb{D}} & \text{if } K = -1, \end{cases}$$

and we consider a polygon in $\widetilde{\Sigma_K} \subseteq \overline{\mathbb{C}}$ whose edges are circular arcs, and a conformal mapping from $\mathbb{R}^2_+$ into the region bounded by it (there are two such regions if $\widetilde{\Sigma_K} = \overline{\mathbb{C}}$). In the case that $K = -1$ we will allow that these polygons have some vertices at $\mathbb{S}^1 \equiv \partial\widetilde{\Sigma_{-1}}$, as long as the edges common to any of such vertices are tangent at the vertex, and are not pieces of horocycles. Then, the induced metric on $\mathbb{R}^2_+$ from ds^2_K via this conformal mapping gives a metric in the above conditions. (That the area is finite when $K = -1$ and the polygon has ideal vertices is proved in the discussion after Theorem 2.2).

Also, in the case $K = 1$ (and so $\widetilde{\Sigma_K} = \overline{\mathbb{C}}$) we may compose with a suitable branched covering of $\overline{\mathbb{C}}$ to obtain other conformal metrics with the desired properties, as we explained in Subsection 2.2.2.

Still, there exist many other conformal metrics on $\mathbb{R}^2_+$ of finite area, constant geodesic curvature on the boundary, and a finite number of boundary singularities. In order to explain how to construct them, we give first some definitions.

Definition 2.3. *By a* piecewise regular closed curve *in $\widetilde{\Sigma_K}$ we mean a continuous map $\alpha : \mathbb{S}^1 \longrightarrow \widetilde{\Sigma_K}$ such that α is smooth and regular everywhere except at a finite number of points $\theta_1, \dots, \theta_n \in \mathbb{S}^1$. By a* piecewise regular parametrization *of α we mean a composition $\beta = \alpha \circ \phi : \mathbb{S}^1 \longrightarrow \widetilde{\Sigma_K}$, where ϕ is a diffeomorphism of $\mathbb{S}^1$.*

Let $A_j \subset \mathbb{S}^1$, $j \in \{1, \dots, n\}$, be the arc between θ_j and θ_{j+1} (we define $\theta_{n+1} := \theta_1$). Then α will be called an immersed circular polygon *in $\widetilde{\Sigma_K}$ if each regular open arc $\alpha|_{A_j}$ has constant geodesic curvature in Σ_K, and in the case that $K = -1$ and $\alpha(\theta_j) \in \mathbb{S}^1$ the arcs $\alpha|_{A_j}$ and $\alpha|_{A_{j-1}}$ are tangent at $\alpha(\theta_j)$ and are not pieces of horocycles.*

Observe that we allow the curve α to have self-intersections, even along each regular arc $A_j \subset \mathbb{S}^1$. We now introduce a concept from differential topology.

Definition 2.4. *A piecewise regular closed curve $\alpha : \mathbb{S}^1 \longrightarrow \widetilde{\Sigma_K}$ is* Alexandrov embedded *(or simply* A-embedded*) if there exists a continuous map $G : \overline{\mathbb{D}} \longrightarrow \widetilde{\Sigma_K}$ such that $G \in C^2(\overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\})$ for some $p_1, \dots, p_n \in \mathbb{S}^1$, and:*

1. *For every $z \in \mathbb{D}$ it holds that $G(z) \in \Sigma_K$ and G is a local diffeomorphism around z.*
2. *$G|_{\mathbb{S}^1} : \mathbb{S}^1 \longrightarrow \widetilde{\Sigma_K}$ is a piecewise regular parametrization of α.*

Example 1. *Any circular polygon without self-intersections in $\widetilde{\Sigma_K}$ is A-embedded. Also, given two points $p, q \in \mathbb{C}$, if γ_1, (resp. γ_2) is an oriented* *geodesic arc from p to q (resp. q to p), then $\gamma_1 \cup \gamma_2$ is a circular polygon, which is not A-embedded in $\mathbb{C}$, but it is A-embedded in $\overline{\mathbb{C}}$. Two further examples are given in Figure 2.3.*

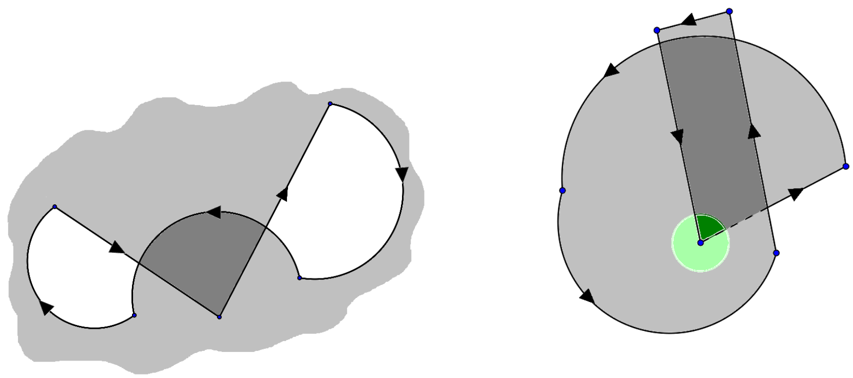


Figure 2.3: *Two examples of circular polygonal contours that are not embedded. The first one is A-embedded in $\overline{\mathbb{C}}$ but not A-embedded in $\mathbb{C}$. The second one is A-embedded in $\mathbb{C}$ and $\overline{\mathbb{C}}$. The green circle indicates that the angle at that vertex is bigger than 2π.*

We can now associate a conformal metric of constant curvature K in $\mathbb{R}^2_+$ to any immersed circular polygon that is A-embedded. Indeed, let $d\sigma^2$ denote the metric $d\sigma^2 = G^*(ds^2_K)$ induced on $\Gamma := \overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\}$. It is then clear that Γ with the complex structure induced by $d\sigma^2$ is conformally equivalent to $\overline{\mathbb{D}} \setminus \{\theta_1, \dots, \theta_n\}$ for some $\theta_1, \dots, \theta_n \in \mathbb{S}^1$, or alternatively, to $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ for some $q_1 < \cdots < q_{n-1}$. Consequently, $d\sigma^2$ produces on $\mathbb{R}^2_+$ a conformal metric $ds^2 = e^v |dz|^2$ on $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ that has finite area, constant geodesic curvature on each boundary component, and $n-1$ boundary singularities along the real axis (we have n singularities if we also count the one placed at ∞).

Definition 2.5. *Any such metric ds^2 on $\mathbb{R}^2_+$ will be called a* circular polygonal metric *on $\mathbb{R}^2_+$.*

Remark 2.5. *The metrics on $\mathbb{R}^2_+$ that we considered after equation* (2.28)*, starting from a polygon in $\widetilde{\Sigma_K}$ whose edges are circle arcs, are examples of circular polygonal metrics. In that situation, the map G is given by an adequate conformal equivalence from $\mathbb{D}$ into the region bounded by this polygon. Note that if $K = 1$, the freedom of composing with suitable branched* *coverings of $\overline{\mathbb{C}}$ only gives different choices of G associated to the Alexandrov embedded polygonal boundary. Thus, even the metrics involving such branched coverings are trivially circular polygonal metrics as in Definition 2.5.*

Once here, we have the following consequence of Corollary 2.3:

Corollary 2.4. *Let $ds^2 = e^v|dz|^2$ be a conformal metric of constant curvature K and finite area in $\mathbb{R}^2_+$. Assume that ds^2 extends smoothly to $\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\}$ for some $q_1 < \cdots < q_{n-1} \in \mathbb{R}$, so that the geodesic curvature of each boundary arc in $\mathbb{R} \equiv \partial\mathbb{R}^2_+$ is constant. Then ds^2 is a circular polygonal metric in $\mathbb{R}^2_+$.*

Proof. Let $g : \mathbb{R}^2_+ \equiv \mathbb{C}_+ \longrightarrow \Sigma_K \subseteq \overline{\mathbb{C}}$ be the developing map of v, let $\Psi : \mathbb{D} \longrightarrow \mathbb{C}_+$ be a Möbius transformation giving a conformal equivalence, and define $G := g \circ \Psi : \mathbb{D} \longrightarrow \Sigma_K$. Then, by Corollary 2.3 G extends continuously to $\overline{\mathbb{D}}$, and is a local diffeomorphism around each $z \in \mathbb{D}$. Again by Corollary 2.3, it is clear that $G|_{\mathbb{S}^1}$ is a piecewise regular parametrization of an immersed circular polygon α in $\widetilde{\Sigma_K}$, and that $G \in C^2(\overline{\mathbb{D}} \setminus \{p_1, \dots, p_n\})$, where the points $p_j \in \mathbb{S}^1$ are given by $\Psi(p_j) = q_j$ for $j = 1, \dots, n-1$, and $\Psi(p_n) = \infty$. Hence, α is Alexandrov-embedded.

Finally, since g is a local isometry (see Remark 1.2), we conclude that $ds^2 = e^v|dz|^2$ is indeed a circular polygonal metric on $\mathbb{R}^2_+$. □

Remark 2.6. *Corollary 2.4 together with Theorem 2.3 explain the geometric interpretation of the canonical solutions that we pointed out without proof in Subsection 2.2.2.*

Corollary 2.4 provides a satisfactory geometric classification of all the finite area solutions to problem (2.28). Let us now describe such solutions from an analytic point of view, using for that Corollary 2.3 and some classical arguments of the conformal mapping problem from the upper half-plane to a circular polygonal domain in $\mathbb{C}$, see [Neh] for instance. We shall focus on the $K = 1$ case, although many of the next statements also hold when $K \leq 0$.

Corollary 2.5. *Let $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ be a solution to (2.28) for $K = 1$ that satisfies $\int_{\mathbb{R}^2_+} e^v < \infty$. Then the Schwarzian map $Q = v_{zz} - v_z^2/2$ of v is given by*

$$Q = \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{(z - q_i)^2} + \frac{\beta_i}{z - q_i} \right) \tag{2.29}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ *with* $\alpha_i \leq 1/2$*,* $i = 1, \dots, n-1$*, satisfy the following conditions:*

$$\sum_{i=1}^{n-1} \beta_i = 0, \qquad \sum_{i=1}^{n-1} (\alpha_i + q_i \beta_i) \leq 1/2. \tag{2.30}$$

Conversely, if Q *is as in* (2.29)-(2.30)*, then there is a solution* $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ *to problem* (2.28) *for* $K = 1$ *that satisfies* $\int_{\mathbb{R}^2_+} e^v < \infty$ *(where the values of* c_i *are not prescribed). Moreover, the family of such solutions with the same* Q *is generically three-dimensional.*

Proof. Let v be a solution to problem (2.28) satisfying $\int_{\mathbb{R}^2_+} e^v < \infty$. By Corollary 2.3, the Schwarzian map Q of v is holomorphic on $\mathbb{C} \setminus \{q_1, \dots, q_{n-1}\}$, and has at most a pole of order two at each q_i. So, clearly Q is of the form

$$Q = \sum_{i=1}^{n-1} \left(\frac{\alpha_i}{(z - q_i)^2} + \frac{\beta_i}{z - q_i} \right) + p(z)$$

for some $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, n-1$, and for some polynomial $p(z)$ with real coefficients.

Let now $g : \mathbb{C}_+ \longrightarrow \overline{\mathbb{C}}$ denote the developing map of v, which satisfies $\{g, z\} = Q$. As v has finite area around each q_i, by Theorem 2.2 we know that g is a Möbius transformation of a function of the form $(z - q_i)^\lambda F(z)$ or $F(z) + \log(z - q_i)$ near each q_i, where F is holomorphic on a punctured neighborhood of q_i, and has at worst a pole at q_i. Noting that the Scwharzian derivative is invariant by Möbius transformations, a simple computation shows that the coefficient α_i in (2.29) satisfies $\alpha_i \leq 1/2$.

The rest of restrictions come from the fact that, by finite area, the holomorphic quadratic differential Qdz^2 has at ∞ at most a pole of order two. If we let $w = -1/z$, then by conformal invariance (see formula (1.12)) $Q(z)dz^2 = \widehat{Q}(w)dw^2$ where

$$\widehat{Q}(w) = \frac{1}{w^4} Q(-1/w).$$

So, again by Theorem 2.2 and the previous computation, the finite area condition at infinity implies that there exists $\lim_{w \to 0} w^2 \widehat{Q}(w) = \alpha_n$ for some $\alpha_n \in (-\infty, 1/2]$. By computing the first terms in the Taylor expansion of $Q(-1/w)$, we easily see that this happens if and only if $p = 0$ and α_i, β_i satisfy the conditions (2.30). This completes the first part of the proof.

Conversely, let Q be as in (2.29)-(2.30), and let g be a solution to $\{g, z\} = Q$ in $\mathbb{C}_+$. By construction, g is a locally injective meromorphic function on $\mathbb{C}_+$, unique up to Möbius transformations, and which extends smoothly to $\overline{\mathbb{C}_+} \setminus \{q_1, \dots, q_{n-1}, \infty\}$. As Q is real on the real axis, we deduce from the equation $\{g, s\} = Q(s)$ on $\mathbb{R}$ that $g(s)$ lies on a circle in $\overline{\mathbb{C}}$ for each interval in $\mathbb{R} \setminus \{q_1, \dots, q_{n-1}\}$. All of this shows that the map $v \in C^2(\overline{\mathbb{R}^2_+} \setminus \{q_1, \dots, q_{n-1}\})$ given by

$$e^v = \frac{4|g'|^2}{(1+|g|^2)^2}$$

is a solution to (2.28) for $K = 1$. We only have left to show that $\int_{\mathbb{R}^2_+} e^v < \infty$, for what we only need to prove this condition around each q_i and around ∞.

Let us fix q_i, $i \in \{1, \dots, n-1\}$, and consider the complex ODE $y'' + \frac{1}{2}Qy = 0$. As Q has at worst a pole of order two at q_i, it is a classical result that a fundamental system of solutions (y_1, y_2) of this equation around q_i is

$$y_1(z) = (z - q_i)^{\lambda_1}\, a_1(z), \qquad y_2(z) = (z - q_i)^{\lambda_2}\, a_2(z) + k y_1(z) \log(z - q_i),$$

where $k \in \mathbb{C}$, $a_1(z), a_2(z)$ are holomorphic on a neighborhood of q_i with $a_i(0) \neq 0$ for $i = 1, 2$, and λ_1, λ_2 are solutions of the *indicial equation*

$$\lambda^2 - \lambda + \frac{\alpha_i}{2} = 0.$$

Here α_i is the coefficient of Q in q_i given by (2.29). Note that from $\alpha_i \leq 1/2$ we deduce that $\lambda_1, \lambda_2 \in \mathbb{R}$, and we may assume that $\lambda_1 \leq \lambda_2$. Therefore, $k \neq 0$ if and only if $\lambda_2 - \lambda_1 \in \mathbb{N}$.

Then, Theorem 1.5 asserts that y_2/y_1 provides a solution to $\{g, z\} = Q$, that is, a developing map for the solution v. Thus, depending on whether $\lambda_2 - \lambda_1 \in \mathbb{N}$ or not, g is of the form

$$g(z) = F(z) + \log(z - q_i) \qquad \text{or} \qquad g(z) = (z - q_i)^{\lambda_2 - \lambda_1} F(z)$$

for some meromorphic function F around q_i such that $F(\bar{z}) = \overline{F(z)}$. By Theorem 2.2 and its subsequent discussion, we see then that $\int e^v < \infty$ on the half-disc $D^+(q_i, \varepsilon) \subset \mathbb{R}^2_+$ for $\varepsilon > 0$ small enough (note that we are assuming that $K = 1$).

The same argument can be done at ∞, this time using the additional conditions (2.30) and the conformal change $w = -1/z$, as we did before. This concludes the proof of existence.

Finally, observe that the solution g to $\{g, z\} = Q$ is unique up to Möbius transformations, so there is a real 6-parameter family of possible choices for g. As the developing map of a solution v to (1.21) is defined up to the change (1.10), we obtain generically a 3-parameter family of solutions to (2.28) for $K = 1$ with the same Q (in this family, the coefficients c_i are not prescribed). This concludes the proof of the result.

$\square$

3

The Liouville equation in an annulus

3.1 The geometric Neumann problem in an annulus

In this section we present the formulation of the geometric Neumann problem for the Liouville equation in an annulus.

We explained in Section 1.2 how the conformal invariance of the Liouville equation allows to transfer classification results between conformally equivalent domains. Specifically, we already mentioned how to use this conformal invariance to relate the geometric Neumann problem in the punctured disc with the Neumann problem in $\mathbb{R}^2_+$ (see Subsection 1.3.2). In this sense, we know that solutions to the Neumann problem in $\mathbb{R}^2_+$ provide solutions to the associated problem in $\mathbb{D}^*$ if and only if they satisfy a certain periodicity condition which avoids period problems.

The approach we give in the case of the annulus is inspired in such a procedure. The difference with the case of the punctured disc is that, besides the natural periodicity that the solution must have from being defined in an annulus, the Neumann condition at the interior boundary will make an extra periodicity condition appear in order for the solution to be well defined (see Section 3.2).

Analytically, we will deal with the following elliptic Neumann problem on an annulus $\mathcal{A}$ in $\mathbb{R}^2 \equiv \mathbb{C}$:

$$\begin{cases} \Delta u + 2Ke^u = 0, & \text{in } \mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}, \\ \dfrac{\partial u}{\partial \nu_1} = c_1 e^{\frac{u}{2}} + 2, & \text{on } C_1 = \{z \in \mathbb{C} : \ |z| = 1\}, \\ \dfrac{\partial u}{\partial \nu_2} = c_2 e^{\frac{u}{2}} - 2e^{r\pi}, & \text{on } C_2 = \{z \in \mathbb{C} : \ |z| = e^{-r\pi}\}. \end{cases} \qquad (P_A)$$

Here ν_i denotes the interior unit normal to C_i, $i = 1, 2$ respectively, and $r > 0$ is a constant. As in previous chapters we suppose up to scaling of u that $K = \{-1, 0, 1\}$.

We know by Section 1.2 that the solutions to (P_A) provide conformal metrics $e^u|dz|^2$ on $\mathcal{A}$ of constant curvature K. Moreover, it will be clear after the proof of Lemma 3.1 that the Neumann conditions in (P_A) make the metric $e^u|dz|^2$ to have constant geodesic curvature $-c_i/2$ on $C_i \subset \partial\mathcal{A}$ for $i = 1, 2$. And conversely, if Σ is a compact surface diffeomorphic to a closed annulus, and ds^2 is a Riemannian metric of constant curvature K on Σ and constant geodesic curvature on each boundary component of $\partial\Sigma$, then (Σ, ds^2) is isometric to $(\overline{\mathcal{A}}, e^u|dz|^2)$ for some solution u to (P_A) with adequate constants c_1, c_2 and r.

In fact, since different values of r provide annuli that are not conformally equivalent, we are considering a family of problems that are also not conformally equivalent.

In Theorem 3.1 we show the explicit formulae for all the solutions to (P_A). As a consequence, we deduce necessary conditions on the values of (K, c_1, c_2) for such solutions to exist (see Lemma 3.2) and prove the existence part in Corollary 3.1. Obviously, the conformal structure of the annulus, that is, the constant r, will be involved in the restrictions on the values of (K, c_1, c_2) that provide a solution.

From a geometric point of view, there are several ways to produce constant curvature annuli with constant geodesic curvature on each boundary component, as we explain next. We illustrate these examples in figures 3.1-3.3, where the curves in yellow are geodesic arcs which must be identified.

(1) First of all, one has the induced metric of any annulus $\mathcal{A}'$ in $\mathcal{Q}(K)$ whose boundary consists of two disjoint circles. Observe that by composing with a finite-folded covering map of this annulus $\mathcal{A}'$ we also obtain conformal metrics in the same conditions. The conformal structure of such metrics depends on the number of sheets of the covering.

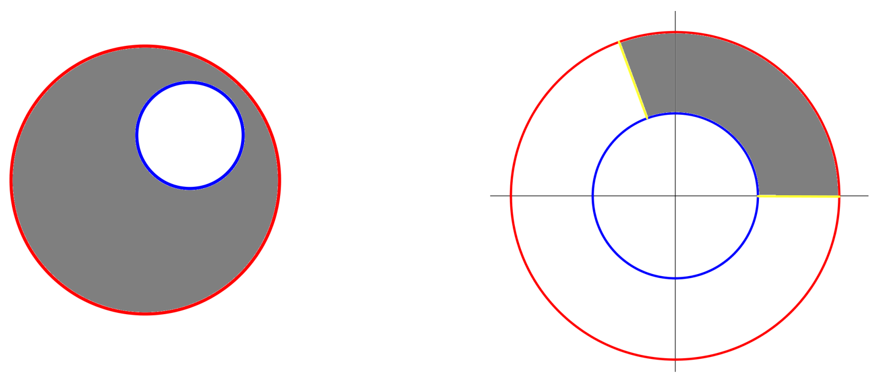


Figure 3.1: *Annuli of type (1) and (2)*

(2) Secondly, assume that $\mathcal{A}'$ is a radially symmetric annulus in $\mathcal{Q}(K)$. That is, its boundary consists of two circles C_1', C_2', and $\mathcal{A}'$ is foliated by geodesic arcs in $\mathcal{Q}(K)$ that meet both C_1' and C_2' orthogonally. Then, we may consider the sector of $\mathcal{A}'$ bounded by two of these radial geodesics, which make some angle γ, possibly greater than 2π. After identifying those geodesics, the quotient space is a topological annulus and the metric ds_K^2 restricted to $\mathcal{A}'$ projects to a well-defined metric of constant curvature K on this quotient. Hence, we obtain a conformal metric satisfying the desired conditions.

One can make similar constructions in the following cases.

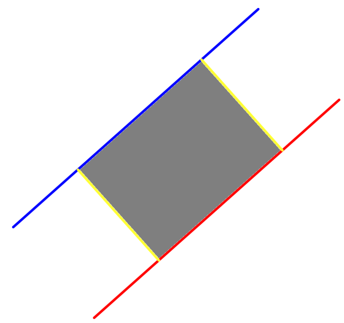


Figure 3.2: *Annulus of type (3)*

(3) When $K = 0$: we consider a strip in $\mathbb{R}^2$ instead of a radially symmetric annulus, and we identify two given different line segments orthogonal to the boundary of the strip.

(4) When $K = -1$: we consider the region of $\mathcal{Q}(-1) \equiv \mathbb{D}$ bounded by two horocycles with the same ideal point $p \in \mathbb{S}^1$, together with two geodesic arcs in $\mathcal{Q}(-1)$ starting at p, and we identify these arcs.

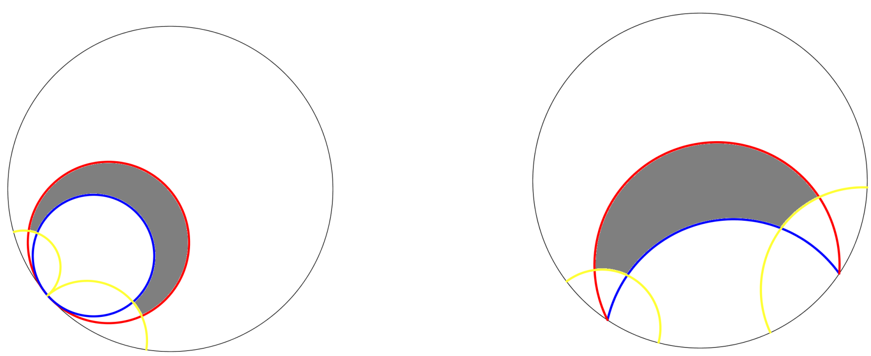


Figure 3.3: *Annuli of type (4) and (5)*

(5) When $K = -1$: we consider the region of $\mathcal{Q}(-1) \equiv \mathbb{D}$ bounded by two circle arcs with common ideal endpoints $p_1,\ p_2 \in \mathbb{S}^1$, together

with two geodesic arcs in $\mathcal{Q}(-1)$ which meet the previous two circles orthogonally, and we identify those geodesic arcs.

Actually, we will show in Theorem 3.2 that all the solutions to (P_A) correspond to some of the *canonical* geometric situations described before. Such a result is the analogous, in the case of the annulus, to the classification of discs of constant curvature with constant geodesic curvature on the boundary (see Theorem 1.7).

The geometric classification of the solutions to (P_A) will be deduced from the analytic expression of the solutions obtained in Theorem 3.1.

3.2 Canonical solutions

Our objective in this section is to describe analytically the solutions to (P_A). In the spirit of the study of the Neumann problem in $\mathbb{R}^2_+$ in Subsection 2.2.1, we obtain such a description from the proof of Theorem 3.1.

3.2.1 Analytic description

The following result gives the explicit formulas for all the solutions to (P_A).

Theorem 3.1. *Any solution to (P_A) is given by one of the following expressions, where $z = Re^{i\arg z}$.*

1. $$e^u = \frac{4\gamma^2\lambda^2R^{2(\gamma-1)}}{(K\lambda^2 + |R^\gamma e^{i\gamma\arg z} - z_0|^2)^2} \tag{3.1}$$

   with $\gamma > 0$, $\lambda > 0$ and $z_0 \in \mathbb{C}$ such that (i) if $K = 0$ and $z_0 \neq 0$ then $|z_0| \notin [e^{-r\gamma\pi}, 1]$ and $\gamma \in \mathbb{N}$; (ii) if $K = -1$ and $z_0 \neq 0$, then $|z_0| \notin [e^{-r\pi\gamma} - \lambda, 1 + \lambda]$ and $\gamma \in \mathbb{N}$, and (iii) if $K = -1$ and $z_0 = 0$, then $\lambda \notin [e^{-r\gamma\pi}, 1]$.

2. *If $K = 0$,*
   $$e^u = 4\lambda^2R^{2(\gamma-1)}, \tag{3.2}$$
   for some $\lambda > 0$, $\gamma \geq 0$.

3. If $K = -1$*,*

$$e^u = \frac{4}{R^2(\lambda + 2\log R)^2}, \tag{3.3}$$

where $\lambda \notin [0, 2\pi r]$*, or*

$$e^u = \frac{\gamma^2}{R^2(\cos(\theta - \gamma \log R))^2} \tag{3.4}$$

where $0 < \gamma < 1/r$ *and* $\theta \in \mathbb{R}$ *is such that* $\pi/2 + k\pi \notin [\theta, \theta + \gamma r \pi]$ $\forall k \in \mathbb{Z}$ *and* $\cos(\theta) > 0$*, or*

$$e^u = \frac{4\gamma^2 R^{2(\gamma-1)}}{(\lambda + 2R^\gamma \cos(\theta + \gamma \arg z))^2} \tag{3.5}$$

with $\gamma \in \mathbb{N}$ *and* $\lambda \notin [-2, 2]$*.*

As in previous cases, the main tool to prove Theorem 3.1 will be Theorem 1.3. Since it is only valid for simply connected domains, we will apply formula (1.9) by passing to the universal cover of $\mathcal{A}$. The way to do this will be shown in Lemma 3.1. It is a consequence of Proposition 1.1 that explained how to extend Liouville Theorem to non-simply connected domains.

Lemma 3.1. *Solving problem* (P_A) *is equivalent to obtaining the solutions of*

$$\begin{cases} \Delta v + 2Ke^v = 0 & \text{in } \Gamma = \{w = s + it \in \mathbb{C} : \ 0 < \mathrm{Im} w < \pi\}, \\ \dfrac{\partial v}{\partial t} = c_1 e^{v/2} & \text{on } \mathbb{R}, \\ \dfrac{\partial v}{\partial t} = -c_2 e^{v/2} & \text{on } \mathbb{R} + \pi i, \end{cases} \tag{$\widetilde{P_A}$}$$

that are $(2\pi/r)$*-periodic. Specifically, the solutions of* (P_A) *are given by formula* (1.9)*, where* $g = \widetilde{g} \circ \Phi^{-1}$*, with* $\Phi : \Gamma \to \mathcal{A}$ *given by* $\Phi(w) = e^{irw}$*, and* $\widetilde{g}$ *being the developing map associated to* $(\widetilde{P_A})$*.*

Proof. It is clear that Φ defines a conformal diffeomorphism between $\mathcal{A}$ and the quotient $\Gamma/\sim$, where $w \sim w' \Leftrightarrow w' = w + \frac{2\pi}{r}\mathbb{Z}$.

On the other hand, by Proposition 1.1, we must only check that if u is a solution of (P_A) then

$$v(s, t) = u(\Phi(s, t)) + 2\log r - 2rt \tag{3.6}$$

is a solution of $(\widetilde{P_A})$ which is $(2\pi/r)$-periodic. Conversely, if v is a $(2\pi/r)$-periodic solution of $(\widetilde{P_A})$, then

$$u(x,y) = v(\Phi^{-1}(x,y)) - 2\log r - \log(x^2+y^2) \tag{3.7}$$

is a solution of (P_A). But this is a simple computation, taking into account the following facts.

(i) Formula (3.6) comes from (1.15), and (3.7) is just its inverse.

(ii) Φ is $(2\pi/r)$-periodic and Φ^{-1} is multivalued in the following way: $\Phi^{-1}(x,y) = \Phi^{-1}(x,y) + 2\pi/r$.

(iii) It holds that

$$\frac{\partial u}{\partial \nu_1} = -xu_x - yu_y \quad \text{on } C_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu_2} = e^{\pi r}(xu_x + yu_y) \quad \text{on } C_2. \tag{3.8}$$

□

Remark 3.1. *We already knew that the boundary restrictions in* $(\widetilde{P_A})$ *correspond geometrically to the constant geodesic curvature condition (see Corollary 1.1). Hence, from* (3.8) *and* (3.6) *or* (3.7) *we deduce that this is also the geometric condition we are imposing in problem* (P_A).

Proof of Theorem 3.1. Let $v \in \mathcal{C}^2(\overline{\Gamma})$ be a $(2\pi/r)$-periodic solution of $(\widetilde{P_A})$. Then, its associated Schwarzian derivative $\widetilde{Q} = v_{ww} - \frac{1}{2}v_w^2$ will be a holomorphic function, also $(2\pi/r)$-periodic. Moreover, because of the boundary conditions in $(\widetilde{P_A})$, we have that

$$\begin{aligned} \operatorname{Im}\widetilde{Q}(s,0) &= -\tfrac{1}{2}\left(\tfrac{c_1}{2}v_s(s,0)e^{v(s,0)/2} - \tfrac{c_1}{2}v_s(s,0)e^{v(s,0)/2}\right) = 0, \\ \operatorname{Im}\widetilde{Q}(s,\pi) &= -\tfrac{1}{2}\left(-\tfrac{c_2}{2}v_s(s,\pi)e^{v(s,\pi)/2} + \tfrac{c_2}{2}v_s(s,\pi)e^{v(s,\pi)/2}\right) = 0. \end{aligned} \tag{3.9}$$

So, by the Schwarz reflection principle for harmonic functions, we can extend $\operatorname{Im}\widetilde{Q}$ from $\overline{\Gamma}$ to $\mathbb{C}$ as a $(2\pi i)$-periodic function. As $\operatorname{Im}\widetilde{Q}$ is also $(2\pi/r)$-periodic, we deduce from (3.9) that $\operatorname{Im}\widetilde{Q} = 0$. Consequently, $\widetilde{Q} = c =$ constant for a certain $c \in \mathbb{R}$.

It is well-known [Hil] that the solutions of the Schwarzian equation $\{\widetilde{g}, w\} = c$ are of the form $\widetilde{g}(w) = \psi(w)$, if $c = 0$, or $\widetilde{g}(w) = \psi(e^{\sqrt{-2c}w})$ if $c \neq 0$, where ψ is a Möbius transformation.

Thus, by Lemma 3.1, the developing map $g : \mathcal{A} \longrightarrow \overline{\mathbb{C}}$ associated to the solutions of (P_A) can be written as

$$g(z) = \frac{Az^{-i\frac{\sqrt{-2c}}{r}} + B}{Cz^{-i\frac{\sqrt{-2c}}{r}} + D}, \qquad \text{if } c \neq 0, \tag{3.10}$$

or

$$g(z) = \frac{-Ai \log z/r + B}{-Ci \ \log z/r + D}, \qquad \text{if } c = 0, \tag{3.11}$$

for some $A, B, C, D \in \mathbb{C}$ with $AD - BC = 1$. Here, $\psi(\xi) = \frac{A\xi + B}{C\xi + D}$.

We will denote $\gamma = \sqrt{-2c}/r$ if $c < 0$, or $i\gamma = \sqrt{-2c}/r$ if $c > 0$. So, from (3.10) and (3.11), we obtain the following expressions.

If $c > 0$,

$$e^u = \frac{4\gamma^2 |z^{\gamma-1}|^2}{(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})z^\gamma + (K\overline{A}B + \overline{C}D)\bar{z}^\gamma + (K|A|^2 + |C|^2)|z|^{2\gamma})^2}, \tag{3.12}$$

if $c < 0$,

$$e^u = \frac{4\gamma^2 |z^{-i\gamma-1}|^2}{(K|B|^2 + |D|^2 + (KA\overline{B} + C\overline{D})z^{-i\gamma} + (K\overline{A}B + \overline{C}D)\bar{z}^{i\gamma} + (K|A|^2 + |C|^2)|z^{-i\gamma}|^2)^2}, \tag{3.13}$$

and, if $c = 0$,

$$\begin{aligned} e^u = 4/(r^2|z|^2(&K|B|^2 + |D|^2 - i(KA\overline{B} + C\overline{D}) \log z/r + i(K\overline{A}B + \overline{C}D) \log \bar{z}/r \\ &+(K|A|^2 + |C|^2)|\log z|^2/r^2)^2). \end{aligned} \tag{3.14}$$

We determine now which of them are valid solutions in terms of the constants A, B, C, D.

Assume first of all that $K|A|^2 + |C|^2 \neq 0$. Then, we can take

$$\lambda = \frac{1}{|\, K|A|^2 + |C|^2 \,|}, \qquad z_0 = -\frac{K\overline{A}B + \overline{C}D}{K|A|^2 + |C|^2}, \tag{3.15}$$

so (3.12)-(3.14) yield, respectively,

$$e^u = \frac{4\gamma^2\lambda^2|z^{\gamma-1}|^2}{(K\lambda^2 + |z^\gamma - z_0|^2)^2}, \tag{3.16}$$

$$e^u = \frac{4\gamma^2\lambda^2|z^{-i\gamma-1}|^2}{(K\lambda^2 + |z^{-i\gamma} - z_0|^2)^2}, \tag{3.17}$$

$$e^u = \frac{4\lambda^2}{r^2|z|^2(K\lambda^2 + |-\frac{i}{r}\log z - z_0|^2)^2}. \tag{3.18}$$

Observe that, due to the behavior of the function $\arg(z)$ in $\mathcal{A}$, $z^{-i\gamma} = z^{-i\gamma}e^{2\pi\gamma}$. Thus, in (3.17), the multivaluation of the numerator cannot be compensated with the multivaluation of the denominator, and so this metric is excluded. In the same way, it is easy to see that (3.18) is never well defined in $\mathcal{A}$. Hence we also exclude it. On the other hand, (3.16) is well defined only when we are in one of the following cases.

- If $z_0 = 0$ and $K\lambda^2 + |z^\gamma|^2 \neq 0$. The last condition is always satisfied in $\mathcal{A}$ if $K = 1, 0$. When $K = -1$, it is equivalent to the condition $\lambda \notin [e^{-\pi r\gamma}, 1]$. Such solutions are always radially symmetric.

- If $z_0 \neq 0$, $\gamma \in \mathbb{N}$ and $K\lambda^2 + |z^\gamma - z_0|^2 \neq 0$. The last condition is always satisfied if $K = 1$. However, if $K = 0$, it is equivalent to the condition $|z_0| \notin [e^{-r\gamma\pi}, 1]$, and, if $K = -1$, it reduces to $|z_0| \notin [e^{-r\gamma\pi} - \lambda, 1 + \lambda]$. These solutions are not radially symmetric.

Hence, we have obtained all the solutions of the first type as stated in Theorem 3.1.

Let us consider now the case $K|A|^2 + |C|^2 = 0$, and so it must hold that $K = 0, -1$.

Thus, writing

$$KA\overline{B} + C\overline{D} = d, \tag{3.19}$$

(3.12) can be simplified to

$$e^u = \frac{4\gamma^2|z|^{2(\gamma-1)}}{(K|B|^2 + |D|^2 + 2|d||z|^\gamma\cos(\arg d + \gamma\arg z))^2}.$$

Because of the condition $K|A|^2 + |C|^2 = 0$ we have that $d = 0$ if $K = 0$ and $|d| = 1$ if $K = -1$. Thus, if $K = 0$, e^u is well defined if and only if $D \neq 0$. We then obtain the solutions (3.2) (for $\gamma > 0$). When $K = -1$, we have to impose that $\gamma \in \mathbb{N}$ and $|D|^2 - |B|^2 \notin [-2, 2]$ for e^u to be well defined. This solution corresponds to the non-radially symmetric solution (3.5).

Now, if d is as in (3.19) then (3.13) can be written as

$$e^u = \frac{4\gamma^2 e^{2(\gamma\arg z - \log|z|)}}{(K|B|^2 + |D|^2 + 2|d|e^{\gamma\arg(z)}\cos(\arg(d) - \gamma\log|z|))^2}.$$

Then it is easy to see that, if $K = 0$, and so $d = 0$, the function e^u is not well defined. If $K = -1$ we need that $|B|^2 = |D|^2$ and $\pi/2 + k\pi \notin [\arg(d), \arg(d) + \gamma r\pi]$ $\forall k \in \mathbb{Z}$ (in particular $\gamma < 1/r$) in order that e^u is well defined in $\mathcal{A}$. As the condition $g(1) = \psi(1) \in \overline{\mathbb{D}}$ is necessary, we have that $\cos(\arg(d)) > 0$. Then, denoting $\theta = \arg(d)$, we obtain the radially symmetric solutions in (3.4).

Finally, from (3.19), the expression of (3.14) reduces to

$$e^u = \frac{4}{|z|^2(r(K|B|^2 + |D|^2) + 2|d|(\sin(\arg d)\log|z| + \arg z\cos(\arg d)))^2}.$$

If $K = 0$, as $d = 0$, this conformal factor is well defined provided that $D \neq 0$. Thus, calling $\lambda^2 = 1/(r|D|)^2$, we obtain the solutions in (3.2) (for $\gamma = 0$). If $K = -1$, we need to impose that $\arg d = \pi/2 + k\pi$ for some $k \in \mathbb{Z}$, that is, $d = (-1)^k i$, and that $(-1)^k(|D|^2 - |B|^2) \notin [0, 2\pi]$. Calling $r(|D|^2 - |B|^2)(-1)^k = \lambda$, we obtain the solutions in (3.3). This concludes the proof of Theorem 3.1.

3.2.2 Necessary and sufficient conditions for existence

Next, we show the exact value of the constants c_1 and c_2 in (P_A) depending on the parameters appearing in the solutions given by Theorem 3.1. Moreover, we give the counterpart of this result by proving the existence of solutions for the correspondent values of (K, c_1, c_1).

The following lemma follows from a simple computation that we omit.

Lemma 3.2. *Let $u \in \mathbb{C}^2(\overline{\mathcal{A}})$ be a solution to (P_A) given by one of the expressions* (3.1)-(3.5) *in Theorem 3.1. Then, its associated constants $c_1, c_2 \in \mathbb{R}$ are given as follows.*

- *For u as in* (3.1),

$$c_1 = S\frac{-K\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = S\frac{e^{r\pi\gamma}(K\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda},$$

where

$$S = \begin{cases} \textit{sign}(1-\lambda) & \textit{if } K = -1,\ z_0 = 0 \\ 1 & \textit{otherwise} \end{cases}$$

- *For u as in* (3.2), $c_1 = -\dfrac{\gamma}{\lambda}$ *and* $c_2 = \dfrac{e^{\pi r\gamma}\gamma}{\lambda}$.
- *For u as in* (3.3), $c_1 = 2$ *sign*(λ), *and* $c_2 = -2$ *sign*(λ).
- *For u as in* (3.4), $c_1 = 2\sin(\theta)$, *and* $c_2 = -2\sin(\theta + r\pi\gamma)$.
- *For u as in* (3.5), $c_1 = -|\lambda|$, *and* $c_2 = |\lambda|e^{\pi r\gamma}$.

Now, we use Lemma 3.2 to deduce for which values of K, c_1 and c_2 does a solution of (P_A) exist.

Corollary 3.1. *Given $c_1, c_2 \in \mathbb{R}$, there exists a solution to problem (P_A) if and only if*

- $K = 1$ *and* $c_1 + c_2 > 0$*;*
- $K = 0$ *and (i)* $c_1 + c_2 > 0$ *with* $c_i < 0$ *for some* $i = \{1, 2\}$ *, or (ii)* $c_1 = 0 = c_2$*;*
- $K = -1$ *and (i)* $c_1 + c_2 > 0$ *with* $c_1 < -2$ *and* $c_2 > 2$ *(or with* $c_1 > 2$ *and* $c_2 < -2$*), or (ii)* $c_1 = \pm 2$ *and* $c_2 = \mp 2$*, or (iii)* $c_1 + c_2 < 0$ *with* $0 \leq |c_i| < 2$ *for both* $i = \{1, 2\}$*.*

Proof. When $K = 1$, all the solutions are given by (3.1). Then, since

$$c_1 = \frac{-\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}(\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda}, \tag{3.20}$$

a simple computation shows that $c_1 + c_2 > 0$. Conversely, if we consider c_1 and c_2 such that $c_1 + c_2 > 0$, taking $z_0 = 0$ it is easy to obtain two constants $\lambda, \gamma > 0$ such that formula (3.20) is satisfied. Thus, for $K = 1$, the condition $c_1 + c_2 > 0$ is also sufficient for the existence of a solution. Moreover, these constants are completely determined by the conformal structure of $\mathcal{A}$ (given by r).

Observe that, if the solution is given by (3.1), for all the values $K = 1, 0, -1$ we had the restriction $\gamma \in \mathbb{N}$ if $z_0 \neq 0$. Therefore, in such cases the choice of c_1 and c_2 will have another technical restriction in terms of the conformal structure, in order to obtain a solution. Note that condition $z_0 \neq 0$ makes the solution be non-radially symmetric, hence we lose one degree of freedom in the choice of the parameters.

When $K = 0$, we have more possibilities, since the solutions to (P_A) are given by either (3.1) or (3.2). If the solution is given by (3.1), then

$$c_1 = \frac{-|z_0|^2 + 1}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}|z_0|^2 - e^{-r\pi\gamma}}{\lambda}, \tag{3.21}$$

and because of the restrictions in Theorem 3.1 we deduce that $c_1 + c_2 > 0$ and that (i) if $|z_0| > 1$, then $c_1 < 0$ and $c_2 > 0$, and (ii) if $|z_0| < e^{-r\pi\gamma}$ then $c_1 > 0$ and $c_2 < 0$. Conversely, if we have c_1 and c_2 such that $c_1 + c_2 > 0$, $c_1 < 0$ and $c_2 > 0$, then we can chose $z_0 = 0$ and find $\lambda, \gamma > 0$ (unique for each fixed $r > 0$) such that (3.21) holds.

On the other hand, if the solution is given by (3.2) and $\gamma = 0$, as $c_1 = 0 = c_2$, the metric always exists, given any parameter $\lambda > 0$, for any conformal structure. If $\gamma \neq 0$ in (3.2), then

$$c_1 = \frac{-\gamma}{\lambda}, \qquad c_2 = \frac{e^{r\pi\gamma}\gamma}{\lambda}, \tag{3.22}$$

and so $c_1 + c_2 > 0$, $c_1 < 0$ and $c_2 > 0$. Hence, given c_1 and c_2 under these assumptions, we trivially find $\lambda, \gamma > 0$ such that (3.22) is satisfied. Looking at the solution (3.2) and the solution (3.1) when $z_0 = 0$, we see that they differ by an inversion; that is, the roles of c_1 and c_2 in (3.22) and (3.21) are interchanged.

If $K = -1$, the solution u to (P_A) can be given by formulas (3.1) or (3.3)-(3.5). If u is given by (3.1), then from Lemma 3.2 we know that

$$c_1 = S\frac{\lambda^2 - |z_0|^2 + 1}{\lambda}, \qquad c_2 = S\frac{e^{r\pi\gamma}(-\lambda^2 + |z_0|^2) - e^{-r\pi\gamma}}{\lambda}, \tag{3.23}$$

and a simple computation shows that $c_1 + c_2 > 0$. Moreover, from the restrictions in Theorem 3.1, we have that

(i) if $z_0 \neq 0$, then $|z_0| > 1 + \lambda$, and so $c_1 < -2$ and $c_2 > 2$, or $|z_0| < e^{-r\pi\gamma} - \lambda$, and therefore $c_1 > 2$ and $c_2 < -2$;

(ii) if $z_0 = 0$, then either $\lambda > 1$, and so $c_1 < -2$ and $c_2 > 2$, or $\lambda < e^{-r\pi\gamma}$, and then $c_1 > 2$ and $c_2 < -2$.

Conversely, consider c_1 and c_2 such that $c_1 + c_2 > 0$ and $c_1 < -2$ (respectively, $c_1 > 2$) and $c_2 > 2$ (respectively, $c_2 < -2$). Then, at least for the case $z_0 = 0$, we can always find (just by solving a second-order equation) parameters $\gamma > 0$ and $\lambda > 1$ (respectively, $\lambda < e^{-r\pi\gamma}$) such that (3.23) holds.

By means of Lemma 3.2, if $c_1 = \pm 2$ and $c_2 = \mp 2$, we can always obtain a solution of type (3.3) for a convenient choice of $\lambda \notin [0, 2\pi r]$ provided that the equalities $c_1 = 2\ \mathrm{sign}(\lambda)$ and $c_2 = -2\ \mathrm{sign}(\lambda)$ are satisfied.

If the solution u is given by (3.4), we have

$$c_1 = 2\sin(\theta), \quad c_2 = -2\sin(\theta + r\pi\gamma), \tag{3.24}$$

for a certain $\theta \in \mathbb{R}$ and $\gamma > 0$ under the restrictions

$$\pi/2 + k\pi \notin [\theta, \theta + \gamma r\pi],\ \forall k \in \mathbb{Z}, \quad \cos(\theta) > 0. \tag{3.25}$$

Thus it is easy to deduce that $c_1 + c_2 < 0$ and that $0 \leq |c_i| < 2$ for both $i = \{1, 2\}$. Conversely, because of the behavior of the $\sin$ and $\cos$ functions, given c_1 and c_2 under these assumptions we can always find $\theta \in \mathbb{R}$ and $\gamma > 0$ such that (3.25) and (3.24) are satisfied.

Finally, if the solution is given by (3.5), as

$$c_1 = -|\lambda|, \quad c_2 = |\lambda| e^{\pi r \gamma}, \tag{3.26}$$

we are led again to the relation $c_1 + c_2 > 0$; and since in this case $\lambda \notin [-2, 2]$, then $c_1 < -2$ and $c_2 > 2$. But it is easy to see from (3.26) that we have a restriction involving the conformal structure. Only when $c_1/c_2 = -e^{\pi r \gamma}$ for a certain $\gamma \in \mathbb{N}$ we will obtain solutions of type (3.5). Therefore, not all the conformal structures are admissible for the existence of such solutions. □

3.3 Classification of constant curvature annuli

As explained in Section 3.1, the solutions to (P_A) provide Riemannian annuli of constant curvature with constant geodesic curvature on each boundary component, and vice versa. We also showed five possible ways to construct such type of annuli geometrically.

In this section we give the geometric counterpart to Theorem 3.1. The following result, which is a direct consequence of the proof of Theorem 3.1, asserts that those five types of Riemannian annuli provide *all* possible conformal metrics of constant curvature K in an annulus $\mathcal{A}$, with constant geodesic curvature on $\partial\mathcal{A}$.

Theorem 3.2. *Let (Σ, ds^2) be a Riemannian surface diffeomorphic to a closed annulus. Assume that ds^2 has constant curvature on Σ, and constant geodesic curvature along each boundary component of $\partial\Sigma$. Then, (Σ, ds^2) is isometric to one of the five examples of Riemannian annuli described in Section 3.1.*

Proof. Up to a conformal change of coordinates, we may view (Σ, ds^2) as $(\overline{\mathcal{A}}, e^u|dz|^2)$ where $\mathcal{A} = \{z \in \mathbb{C} : \ e^{-r\pi} < |z| < 1\}$ for some $r > 0$, and u is a solution to (P_A) for some adequate constants K, c_1 and c_2. We now analyze from a geometric point of view the possible expressions for u, as given by Theorem 3.1.

We consider first the solutions given by (3.1). In this case we know by the proof of Theorem 3.1 that the associated developing map is $g(z) = \psi(z^\gamma)$ with $\gamma > 0$ and $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ a certain Möbius transformation.

If $z_0 \neq 0$, we have the restriction $\gamma \in \mathbb{N}$. Thus, g is single valued on $\mathcal{A}$. Moreover, $g(\mathcal{A})$ is a topological annulus $\mathcal{A}'$ in $\mathcal{Q}(K)$ whose boundary consists of two circles, and the map g defines a γ-folded covering map from $\mathcal{A}$ into $\mathcal{A}'$. Thus, by Remark 1.2, we see that $(\mathcal{A}, e^u|dz|^2)$ is isometric to the annulus $\mathcal{A}'$ endowed with the metric ds_K^2, covered a number $\gamma \in \mathbb{N}$ of times. That is, $(\mathcal{A}, e^u|dz|^2)$ is isometric to the first type of *canonical* annuli of constant curvature defined in Section 3.1.

Now, suppose we are in the case when $z_0 = 0$, and so γ is not necessarily an integer. The multivalued function z^γ maps $\mathcal{A}$ into a piece of the annulus $\mathcal{B} = \{\xi \in \mathbb{C} : \ e^{-r\pi\gamma} < |\xi| < 1\}$ bounded by the segment $[e^{-r\pi\gamma}, 1]$ and $R_{2\pi\gamma}([e^{-r\pi\gamma}, 1])$, where R_t denotes from now on the rotation by angle t. These two segments correspond to the splitting by z^γ of the segment $[e^{-r\pi}, 1]$. Because of the condition $z_0 = 0$, we have from (3.15) that $K\overline{A}B + \overline{C}D = 0$. Then it is easy to prove that, for each $\theta \in \mathbb{R}$, $\psi \circ R_\theta \circ \psi^{-1} = \phi$ where ϕ is an isometry of $\mathcal{Q}(K)$ described in (1.10). That is, for each $\theta \in \mathbb{R}$, $g(e^{i\theta}z) = \phi_\theta(g(z))$ for a certain isometry of $\mathcal{Q}(K)$, ϕ_θ. On the other hand, observe that any metric of this kind coincides with one of the *canonical* solutions that solve the Newmann problem in $\mathbb{R}^2_+$, given by formula (2.3). Remember that the developing map of those metrics mapped the positive part of the real axis into a curve of constant curvature. Thus, we have from Lemma 2.1 that since $[e^{-r\pi}, 1] \subset \mathbb{R}_+$ and $z_0 = 0$, then $g([e^{-r\pi}, 1])$ is a geodesic arc in $\mathcal{Q}(K)$. Hence, $g(\mathcal{A})$ is a piece of an annulus $\mathcal{A}'$ which is radially symmetric, that is, foliated by geodesic arcs meeting $\partial\mathcal{A}'$ orthogonally. These geodesic arcs are the image of the segments orthogonal to $\partial\mathcal{A}$ that foliate $\mathcal{A}$. Such a piece, $g(\mathcal{A})$, is bounded by two of those geodesic arcs that make an angle $2\pi\gamma > 0$, where γ can be greater than 1. As explained before, they correspond to the splitting of the segment $[e^{-r\pi}, 1]$. We see then that $(\mathcal{A}, e^u|dz|^2)$ is isometric to the domain $g(\mathcal{A})$, where the the extremal geodesic arcs are identified, endowed with the projection of ds_K^2. Thus, these solutions correspond to *canonical* annuli of type (2) mentioned in Section 3.1.

Consider now the case $K=0$, when the solution is given by (3.2) with $\gamma \neq 0$. We have again, by the proof of Theorem 3.1, that $g(z)=\psi(z^\gamma)$, where now the Möbius transformation $\psi(\xi)=\frac{A\xi+B}{C\xi+D}$ satisfies that $C=0$, that is, it is the composition of a dilation with an isometry of $\mathcal{Q}(0)\equiv\mathbb{R}^2$. Thus, $g(\mathcal{A})$ lies in an annulus $\mathcal{A}'$, which is radially symmetric in $\mathcal{Q}(0)$, and the image of the segments orthogonal to $\partial\mathcal{A}$ that foliate $\mathcal{A}$ are segments (and so geodesic arcs) in $\mathcal{A}'$ which are orthogonal to $\partial\mathcal{A}'$. As in the case of the solutions of type (3.1), $g(\mathcal{A})$ is a portion of such an annulus $\mathcal{A}'$ delimited by two of those segments which correspond to the split of the segment $[e^{-r\pi},1]$ and that make an angle $2\pi\gamma$, possibly greater than 2π. Hence, we are led again to the case of an annulus of type (2).

If $\gamma=0$ in formula (3.2), we know that $g(z)=\psi(-\frac{i}{r}\log z)$ where, as before, ψ is the composition of a dilation with an isometry of $\mathcal{Q}(0)$. The multivalued function $-\frac{i}{r}\log z$ maps $\mathcal{A}$ into the strip $\Gamma=\{w\in\mathbb{C}:\ 0<\operatorname{Im} w<\pi\}$, where the segment $[e^{-\pi r},1]$ splits into the vertical segments $S_1=\{\xi\in\overline{\Gamma}:\ \operatorname{Re}\xi=0\}$ and $S_2=\{\xi\in\overline{\Gamma}:\ \operatorname{Re}\xi=2\pi/r\}$. So, $g(\mathcal{A})$ is a piece of the strip $\psi(\Gamma)$ bounded by the segments $\psi(S_1)$ and $\psi(S_2)$. This solution makes $(\mathcal{A},e^u|dz|^2)$ isometric to the domain $g(\mathcal{A})$, where $\psi(S_1)$ and $\psi(S_1)$ are identified, endowed with the projection of ds_0^2. Thus, it corresponds to an annulus of type (3) described before.

Assume next that $K=-1$ and that the solution is given by formula (3.3). Then, the developing map associated to it is $g(z)=\psi(-\frac{i}{r}\log z)$, where $\psi(\xi)=\frac{A\xi+B}{C\xi+D}$ is a Möbius transformation satisfying $|A|=|C|$, i.e. it maps the point of infinity into a point $p=\frac{A}{C}\in\partial\mathbb{D}$. Thus, we deduce that $g(\mathcal{A})$ lies in $\psi(\Gamma)$, which is the region limited by two horocycles $\mathcal{C}_1$ and $\mathcal{C}_2$ that are tangent at p. Observe also that the image by ψ of the vertical segments foliating Γ will be arcs of curves that start at p and which are orthogonal to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Hence they are geodesic arcs that foliate the region between $\mathcal{C}_1$ and $\mathcal{C}_2$. Two of those geodesic arcs, corresponding to the splitting of the segment $[e^{-r\pi},1]$, are identified to obtain the quotient which, with the projected metric ds_{-1}^2, is isometric to $(\mathcal{A},e^u|dz|^2)$. These solutions correspond with annuli of type (4) in Section 3.1.

If $K=-1$ and the solution is given by the formula (3.4), then the associated developing map is $g(z)=\psi(z^{-i\gamma})$ where $\gamma<1/r$ and the Möbius transformation $\psi(\xi)=\frac{A\xi+B}{C\xi+D}$ is such that $|A|=|C|$ and $|B|=|D|$. Note that the multivalued function $z^{-i\gamma}$ maps C_1 into the segment $S_1=[1,e^{2\pi\gamma}]$ and C_2 into its rotated $S_2=R_{\pi\gamma r}([1,e^{2\pi\gamma}])$. And the two circle arcs centered at the origin with radii 1 and $e^{2\pi\gamma}$, respectively, that join the endpoints of

S_1 and S_2, correspond to the splitting by the function $z^{-i\gamma}$ of the segment $[e^{-r\pi}, 1]$. Thus $\mathcal{A}$ is mapped by $z^{-i\gamma}$ into the region delimited by S_1, S_2 and these two circle arcs. On the other hand, it is easy to check that ψ maps the line passing through the origin corresponding to the arguments $\pi/2 - \theta$ and $-\pi/2 - \theta$ (where θ is the parameter appearing in (3.4)) into $\partial\mathbb{D}$. As a consequence, all the circles centered at the origin (since they are orthogonal to such a line) will be mapped by ψ into geodesics of $\mathcal{Q}(-1) \equiv \mathbb{D}$ that will foliate $g(\mathcal{A})$. By all the arguments above, we deduce that g maps $\mathcal{A}$ into the region bounded by (i) two circle arcs (the image of C_1 and C_2) that meet at two points $p_1 = A/C, p_2 = B/D \in \partial\mathbb{D}$ with angle $\pi\gamma r$ and, (ii) two geodesic arcs orthogonal to them. These geodesic arcs that we identify correspond to the splitting by g of the segment $[e^{-r\pi}, 1]$. Hence, $(\mathcal{A}, e^u|dz|^2)$ is isometric to this quotient of $g(\mathcal{A})$ endowed with the projection of the metric ds^2_{-1}. It is then a Riemannian annulus of type (5).

Finally, if $K = -1$ and the solution is given by formula (3.5), we have again $g(z) = \psi(z^\gamma)$. Now, $\gamma \in \mathbb{N}$, and $\psi(\xi) = \frac{A\xi+B}{C\xi+D}$ is such that $|A| = |C|$. Thus we can deduce as before that g is a γ-folded covering map from $\mathcal{A}$ into an annulus $\mathcal{A}' \subset \mathcal{Q}(-1)$. In this case, the boundary of $\mathcal{A}'$ is intersected orthogonally by two curves with common ideal point $p = \frac{A}{C} \in \partial\mathbb{D}$. These curves are the images by g of the real and imaginary axes. Thus, we are led again to the case of annuli of type (1).

□

Part II

Monge-Ampère equations

4

Monge-Ampère equations and surface theory

4.1 The general Monge-Ampère equation

A Monge-Ampère equation in its more general version is a fully nonlinear, second order partial differential equation of the form,

$$\det\{D^2u + \mathcal{A}(., u, Du)\} = f(., u, Du), \tag{4.1}$$

in domains Ω in the Euclidean n-space $\mathbb{R}^n$.

Here $\mathcal{A}$ and f are functions $\mathcal{A} : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathcal{M}_n(\mathbb{R})$, $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$, and we will consider solutions $u \in C^2(\Omega)$

As usual, Du and D^2u denote the gradient and the Hessian matrix of second derivatives of u, respectively. We will assume also that the matrix $\mathcal{A}$ is symmetric.

Definition 4.1. *The equation* (4.1) *is* elliptic *with respect to* u *if the matrix*

$$D^2u + \mathcal{A}(., u, Du) \tag{4.2}$$

is positive definite. In the same way, it is said to be hyperbolic *or* parabolic *w.r.t.* u *when the matrix* (4.2) *is negative definite or indefinite, respectively.*

In the above cases, u *is called an elliptic (respectively hyperbolic or parabolic) solution of* (4.1).

We will be interested in elliptic solutions of (4.1). Observe that this ellipticity condition implies that $f(., u, Du) > 0$.

Equation (4.1) models the original problem studied by Gaspard Monge in the eighteenth century, and later by Andre-Marie Ampère: *the optimal transportation problem*

> Can we find an optimal mapping from one mass distribution to another such that a cost functional is minimized among all measure preserving mappings?.

In the original approach of Monge in [Mg], the mass distribution was considered in $\mathbb{R}^3$ and the cost function for moving one point to another was proportional to the distance between these points. Monge noted that optimal transport should follow straight lines which are orthogonal to a family of surfaces (formally worked out by Appel [Ap]). In this particular case, the problem has taken two centuries to be completely solved [CFMc, TrWa] (see also [Amb, Ev, EvGa]). When the cost function is strictly convex, it was proved [Caf4, GaMc] that there exists a unique optimal mapping determined by potential functions. But all these advances would not have been possible without the aid of the relaxed formulation of the problem given by Kantorovich, who received a Nobel prize for related work in economics [Kan].

In some sense, part of the great amount of contributions in this so called Monge-Kantorovich optimal mass transportation theory is due to the fact that its modeling Monge-Ampère equation (4.1) has widespread applications. It describes certain problems not only in differential geometry as we explain in Section 4.4, but also in infinite-dimensional linear programming, functional analysis, mathematical economics and probability and statistics.

Moreover, in the case of dealing with the transport of a supply measure into a demand measure minimizing an associated cost, we can mention natural applications as optimal water distribution in irrigation channel systems, optimal urban planning, traffic network planning in cities, internet traffic optimization, optimal branching in the growth of trees, structuring of arteries in leaves, branching of rivers systems and other fluid dynamics, etc. We refer to [Be, BuSt, Ev] for more examples and details and to the survey [McGui] for an overview of the problem.

Next we explain briefly how equation (4.1) appears in Monge-Kantorovich optimal mass transportation theory. The function u in (4.1) corresponds to the potential function given by the approach of Kantorovich [Kan, MaTrWa], and the matrix $\mathcal{A}$ and the right hand side f are given by

$$\mathcal{A}(x, Du) = D_x^2 c(x, T_u(x)), \quad f = |\det\{D_{xy}^2 c\}| \frac{\rho}{\rho^* \circ T_u},$$

where $c(.,.)$ is the cost function, $T_u(x) = y$ is the optimal mapping determined by $Du(x) = D_x c(x, y)$, and ρ and ρ^* are the mass distributions in the initial domain Ω and the target domain $\Omega^* = T_u(\Omega)$, respectively. Moreover, it is known (see for instance [GaMc]) that under certain restrictions on the cost function $c(.,.)$, the optimal map T_u is uniquely determined by the solution to (4.1). Therefore, in those cases, the study of the regularity of solutions to (4.1) yields the study of the optimal mapping.

The most studied types of cost functions are $c(x, y) = |x - y|$, originally considered by Monge, $c(x, y) = |x - y|^2$ and $c(x, y) = xy$, where (maybe up to the adding of $|x|^2$) equation (4.1) becomes the standard Monge-Ampère equation

$$\det\{D^2 u\} = f(., u, Du). \tag{4.3}$$

In the particular case $c(x, y) = |x - y|^2$, various regularity results have been obtained in [Caf1, Caf2, Caf3, Del, Ur]. On the other hand, the regularity of the optimal maps in the case of a general cost function is an important open problem as is pointed out in [Caf5, Vi]. Some partial results in this topic can be found in [Liu1, Liu2, LiTrWa1, LiTrWa2].

4.2 The Hessian one equation

In this section we will study the most simple type of Monge-Ampère equation in the two-dimensional case. We consider *the Hessian one equation*

$$u_{xx}u_{yy} - u_{xy} = 1, \qquad (x, y) \in \Omega, \tag{4.4}$$

where $\Omega \subset \mathbb{R}^2$ is a planar domain.

In this casesolutions to (4.4) have associated some geometric holomorphic functions with respect to a certain conformal structure that relates the problem with complex analysis. Indeed, solutions to (4.4) can be recovered in terms of holomorphic data. Next we explain an sketch of the approach in the survey [GaMi4] to obtain such a representation.

Observe that in (4.4) we can suppose $u_{xx} > 0$, and then

$$d\sigma^2 = u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2 \tag{4.5}$$

defines a Riemannian metric on Ω. Moreover, if we consider the graph of u, $S = \{(x, y, u(x, y))\} \subset \mathbb{R}^3$, it turns out that S is a convex regular surface in $\mathbb{R}^3$ whose second fundamental form is conformal to the metric $d\sigma^2$.

Definition 4.2. *We define* the underlying conformal structure *of the solution u to* (4.4) *to the Riemann surface structure induced on S by $d\sigma^2$, as explained in Section 1.1.*

As we said before, there are some complex functions associated to any solution to (4.4) in Ω that are holomorphic with respect to the underlying conformal structure. Specifically, consider the function $G(x, y) : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{C}$ given by

$$G(x, y) = s(x, y) + it(x, y) := x + u_x(x, y) + i(y + u_y(x, y)). \tag{4.6}$$

Then, (4.4) yields that the Jacobian of the mapping $(x, y) \mapsto (s, t)$ is ≥ 2, and also that the associated metric $d\sigma^2$ can be written as

$$d\sigma^2 = \frac{1}{2 + u_{xx} + u_{yy}}(ds^2 + dt^2).$$

Therefore, $G = s + it$ is a conformal parameter on Ω for the underlying conformal structure associated to u. That is, G is holomorphic for this Riemann surface structure.

Moreover, the function $F : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{C}$ given by

$$F(x, y) = x - u_x(x, y) + i(-y + u_y(x, y)) \tag{4.7}$$

satisfies the Cauchy-Riemann equations with respect to the conformal parameters (s, t) and so it is holomorphic for the underlying conformal structure. Also, we have that

$$2(x + iy) = G + \overline{F}. \tag{4.8}$$

Hence, we deduce that x and y are harmonic functions, and the Jacobian of $2(x+iy)$ with respect to (s,t) is $|dG|^2 - |dF|^2 > 0$. Furthermore, we have that

$$4(dx^2+dy^2) = |d(G+F)|^2 \leq (|dF|+|dG|)^2 \leq 4|dG|^2,$$

that is,

$$dx^2+dy^2 \leq |dG|^2. \tag{4.9}$$

These holomorphic functions F and G provide the conformal representation of the solutions to (4.4). This representation can be deduced from the formulas above and some integrability arguments, and is given by the following result [FMM].

Theorem 4.1. *Let S denote a Riemann surface. Let $F, G : S \longrightarrow \mathbb{C}$ be two holomorphic functions on S satisfying $|dF/dG| < 1$ and $dG \neq 0$. Then the map $\psi : S \longrightarrow \mathbb{R}^3 \equiv \mathbb{C}\times\mathbb{R}$ given by*

$$\psi = \left(G+\overline{F}, \frac{1}{4}\{|G|^2 - |F|^2 + 2\boldsymbol{Re}(FG)\} - \boldsymbol{Re}\int FdG\right) \tag{4.10}$$

is the graph of a solution to the Hessian one equation (4.4) *as long as:*

- *$G+\overline{F}$ is one-to-one, and*
- *the integral $\int FdG$ has no real periods on S.*

Conversely, every graph in $\mathbb{R}^3$ of a solution to (4.4) *over a planar domain $\Omega \subset \mathbb{R}^2$ is recovered by formula* (4.10) *from the holomorphic functions F, G given by* (4.6) *and* (4.7).

This representation formula was used classically by Blaschke and Jörgens [Jor1, Jor2] and generalized by Ferrer, Martínez and Milán [FMM]. Furthermore, we will see how it is used to classify all the entire solutions, that is C^2 solutions globally defined in $\mathbb{R}^2$, of the Monge-Ampère equation (4.4). This is a result of Jörgens [Jor1].

Theorem 4.2. *Any entire solution to the Hessian one equation* (4.4) *is a quadratic polynomial.*

Proof. Let $u(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^2 entire solution to (4.4). Then, by Theorem 1.1, it has the underlying conformal structure S of the disc $\mathbb{D}$ or the complex plane $\mathbb{C}$. However, (4.9) shows that $|dG|^2$ is a complete flat conformal metric on S, and hence S is conformally the complex plane $\mathbb{C}$. But as $|dF/dG| < 1$, classical Liouville Theorem for holomorphic functions gives that the function dF/dG must be constant. As $dG \neq 0$, we can reparametrize locally S around any point, so that $G(\xi) = \xi$, where ξ is the new conformal parameter. Hence $F(\xi) = a\xi + b$ for some $a, b \in \mathbb{C}$, and from (4.10) we deduce that the surface is the graph of a quadratic polynomial in our local domain. By analyticity, the same holds globally.

□

Being (4.4) the most simple case of elliptic Monge-Ampère equation we can consider, it seems convenient to examine some aspects of its solutions which will be developed later in more general situations. That is the case of the local behavior of a solution u to (4.4) in a neighborhood of a non-removable isolated singularity. Although the results we will show in this topic do not extend directly to the general case studied in Chapter 5, or to the case of prescribed curvature graphs in Section 4.4, some of the techniques we are about to explain will be adapted to those situations.

The following result [HeB, GMMi] gives a criterion to determine, in terms of the underlying conformal structure, when a singularity at a point $q \in \Omega$ is non-removable.

Lemma 4.1. *Let $u(x, y) : \Omega \setminus \{q\} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth solution to* (4.4). *Then u extends smoothly to q if and only if its underlying conformal structure around the singularity is that of a punctured disc.*

We must remark that, by Lemma 4.1, the underlying conformal structure at a non-removable singularity is that of an annulus. Moreover, by the convexity of the graph G, the gradient (u_x, u_y) extends to the interior boundary of such an annulus as a closed convex curve in $\mathbb{R}^2$. In other words, there exists a limit tangent cone at the singularity, which is generated by a closed convex curve. In this sense, we will say that these singularities are of *conical type*.

We consider two solutions to (4.4) as equivalent if they agree on an open set around the singularity. Observe that, if this is the case, both solutions agree everywhere on their common domain as they are real analytic by formula (4.10). Furthermore, we can suppose that u has the singularity at the origin and that, up to a translation, $u(0, 0) = 0$ (note that Euclidean translations preserve solutions to (4.4)).

Let $\mathfrak{A}$ denote the space of solutions to (4.4) having the origin as a non-removable singularity, modulo equivalence as explained before. Then $\mathfrak{A}$

can be described in terms of their limit tangent cones at the singularity. This relation is given by the following result in [ACG]. Some of the underlying ideas are similar to those in [GaMi2], where isolated singularities of flat surfaces in the hyperbolic 3-space $\mathbb{H}^3$ where studied.

Theorem 4.3. *There exists a bijective explicit correspondence between the class* $\mathfrak{A}$ *and the class of regular, real analytic, strictly convex Jordan curves in* $\mathbb{R}^2$.

The next natural step in the study of isolated singularities of solutions to (4.4) is to characterize them globally. The quadratic polynomials are the simplest solutions to (4.4), and by Theorem 4.2 they are the only ones that are defined in $\mathbb{R}^2$. The simplest solution other than these polynomials is the rotational one

$$u(x,y) = \frac{1}{2}(r\sqrt{1+r^2} + \sinh^{-1}(r)), \qquad r := \sqrt{x^2+y^2}, \tag{4.11}$$

which is C^2 in $\mathbb{R}^2 \setminus \{0\}$ and has a non-removable singularity at the origin. Indeed, Jörgens proved in [Jor2] that it is essentially the unique solution to (4.4) globally defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ with a non-removable singularity at the origin. To clarify this uniqueness we must take into account that equation (4.4) is invariant under equiaffine transformations of the Euclidean space $\mathbb{R}^3$:

Lemma 4.2. *Let* $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *be an equiaffine transformation of the form*

$$\Phi\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad a_{11}a_{22} - a_{21}a_{12} = 1. \tag{4.12}$$

Then if $u(x,y)$ *is a solution to* (4.4)*, then* $u^*(x^*,y^*)$ *given by*

$$\Phi(x,y,u(x,y)) = (x^*, y^*, u^*(x^*,y^*))$$

is also a solution to (4.4) *in terms of the variables* x^**,* y^**.*

It is natural then, by Lemma 4.2, to consider solutions *modulo equiaffine transformations*, i.e. two solutions of (4.4) will be considered equivalent if they differ by a equiaffine transformation as in (4.12). In this sense, the uniqueness result given by Jörgens in [Jor2] can be stated as follows.

Theorem 4.4. *Any C^2 solution to* (4.4) *globally defined on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *is an equiaffine transformation of the rotational example* (4.11) *(or a quadratic polynomial).*

In fact, there is a more general result given by Gálvez, Martínez and Mira in [GMMi]. They show that any solution to (4.4) in a finitely punctured plane is uniquely determined (up to equiafifne transformations) by its underlying conformal structure. Conversely, any circular domain in $\mathbb{C}$ is the conformal structure of some global solution to (4.4) in a finitely punctured plane. In particular, the moduli space of global solutions to (4.4) with $n > 1$ singularities is, modulo equiaffine transformations, a $(3n-4)$-dimensional family. We refer to [GMMi] for the details of the proof of this result that generalizes Theorem 4.4.

To finish this section we show how equation (4.4) is related to several topics in surface theory.

- Solutions to (4.4) are related to improper affine spheres in the affine 3-space in the following sense: Any solution to (4.4) has the property that its graph $S = \{(x, y, u(x, y))\}$ is an improper affine sphere in the 3-dimensional affine space, with constant affine normal $(0, 0, 1)$. Conversely, any improper affine sphere with affine normal $(0, 0, 1)$ is locally the graph of a solution to (4.4). In fact, the representation formula (4.10) has been used to describe the global behavior of improper affine spheres, with or without singularities. We refer to [ACG, FMM, Mar] for more details about this topic.

- Equation (4.4) also appears when dealing with special Lagrangian immersions in $\mathbb{C}^2$, since these immersions are related to improper affine spheres as we explain next (see [ACG, Mar, Wo] for instance). Let us denote by $\mathbb{C}^2 \equiv \mathbb{R}^4$ the complex 2-plane and let (x_1, x_2, x_3, x_4) be canonical coordinates in $\mathbb{R}^4$. By taking the complex coordinates $y_1 = x_1 + ix_2$, $y_2 = x_3 + ix_4$, we consider the usual metric g, the symplectic form ω and the complex 2-form Ω given by

$$g = |dy_1|^2 + |dy_2|^2, \quad \omega = \frac{i}{2}(dy_1 \wedge \overline{dy_1} + dy_2 \wedge \overline{dy_2}), \quad \Omega = dy_1 \wedge dy_2.$$

  Then, if S is a Riemann surface and $\psi = (\psi_1, \psi_2, \psi_3) : S \longrightarrow \mathbb{R}^3$ is an improper affine sphere with affine normal $(0, 0, 1)$ and affine conormal $N = (N_1, N_2, 1)$, then the map $L_\psi : S \longrightarrow \mathbb{C}^2$ given by $(\psi_1 + i\psi_2, N_1 +$

$iN_2)$ is a special Lagrangian immersion with respect to the calibration $\mathrm{Re}(i\Omega)$. Moreover, the metric $d(\psi_1)^2+d(\psi_2)^2+d(N_1)^2+d(N_2)^2$ induced on S by L_ψ is conformal to the affine metric of ψ. Conversely, if $L: S \longrightarrow \mathbb{C}^2$, given by $L(z) = (y_1(z), y_2(z))$, $z \in S$ is a special Lagrangian immersion with respect to the calibration $\mathrm{Re}(i\Omega)$, then

$$\psi_L(z) = \left(\mathrm{Re}(y_1(z)), \mathrm{Im}(y_1(z)), -\int_{z_0}^{z} \langle dy_1, y_2\rangle\right)$$

is a (possibly multivalued) improper affine sphere at its regular points, where z_0 is a fixed point and $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^2$. Indeed, we obtain the direct relation with equation (4.4) since the immersion is locally a graph $(x_1, x_2, u(x_1, x_2))$ whose affine conormal and affine metric can be defined as

$$N_L(z) = (\mathrm{Re}(y_2(z)), \mathrm{Im}(y_2(z)), 1), \qquad \langle d\psi_L, -dN_L\rangle.$$

In fact, singularities of improper affine spheres are studied by means of this characterization in [ACG].

- It is also classical the local correspondence given by Calabi [Ca] between solutions to (4.4) and minimal surfaces in $\mathbb{R}^3$. Here we sketch the approach given in [Sp, pag. 265]. It is well-known that $\{(x, y, f(x, y))\} \subset \mathbb{R}^3$ is a minimal graph in the Euclidean 3-space if and only if f satisfies the PDE

$$(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0 \tag{4.13}$$

in a certain planar domain Ω. Then, if we define $W = \sqrt{1+f_x^2+f_y^2}$, a computation using formula (4.13) shows that if Ω is simply connected, there exist maps α and β defined on Ω satisfying:

$$\begin{aligned} \alpha_x &= \frac{1+f_x^2}{W}, & \alpha_y &= \frac{f_xf_y}{W}, \\ \beta_x &= \frac{f_xf_y}{W}, & \beta_y &= \frac{1+f_y^2}{W}. \end{aligned} \tag{4.14}$$

Hence, the integrability conditions for the existence of a function $\phi : \Omega \longrightarrow \mathbb{R}$ such that

$$\phi_x = \alpha \quad \text{and} \quad \phi_y = \beta$$

are trivially satisfied by (4.14). Moreover, formula (4.14) also implies that ϕ is a solution to (4.4). Conversely, if we start with a solution to (4.4) and we denote by p and q the functions satisfying

$$\phi_{xx} = \frac{1+p^2}{\sqrt{1+p^2+q^2}}, \qquad \phi_{xy} = \frac{pq}{\sqrt{1+p^2+q^2}}, \qquad \phi_{yy} = \frac{1+q^2}{\sqrt{1+p^2+q^2}},$$

then, the equality $\phi_{xxy} = \phi_{xyx}$ (or equivalently $\phi_{xyy} = \phi_{yyx}$) yields that if there exists a function $f \in C^2(\Omega)$ satisfying $p = f_x$ and $q = f_y$, it must be a solution to (4.13).

This relation between minimal graphs and solutions to (4.4) has been used for instance to prove Bernstein's theorem in $\mathbb{R}^3$ by means of Jörgens theorem 4.2. More precisely, as entire solutions to (4.4) are quadratic polynomials, it follows by the previous computation that if $\{(x, y, f(x, y))\}$ is an entire minimal graph, then

$$\frac{1+f_x^2}{W}, \qquad \frac{f_x f_y}{W}, \qquad \text{and} \qquad \frac{1+f_y^2}{W}$$

are constants and so $\{(x, y, f(x, y))\}$ must be a plane.

- Now, consider $\psi : S \longrightarrow \mathbb{H}^3$ a flat surface in the hyperbolic space $\mathbb{H}^3$. The Gauss equation of an immersion in the hyperbolic space is $K = K_{\text{ext}} - 1$, where K_{ext} denotes from now on the extrinsic curvature. Hence, if we take local coordinates (x, y) on S such that its first fundamental form is written as $I = dx^2 + dy^2$, the Codazzi equation implies that there exists a solution $u(x, y)$ to the Hessian one equation (4.4) such that the second fundamental form of the surface is given by $II = u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2$. The use of the underlying conformal structure and the holomorphic data is crucial in this topic to give a representation formula as was explained in Section 1.5. Furthermore, this correspondence is also useful in the study of isolated singularities. In this line we may cite [GaMi2] where the authors solve completely the local problem of classifying the isolated singularities of flat surfaces in $\mathbb{H}^3$ in terms of the behavior of the limit unit normal at the singularity. In [CMM], the correspondence is used to classify the complete embedded flat surfaces in $\mathbb{H}^3$ with exactly two singularities and one end.

  However, there are some important global problems in the theory of flat surfaces in $\mathbb{H}^3$ that still remain open (see for instance [GaMi4]).

- The way to relate flat surfaces in $\mathbb{S}^3$ with solutions to Monge-Ampère equations is, as happens for flat surfaces in $\mathbb{H}^3$, by means of the Gauss equation of the immersion, which now yields $K = K_{\text{ext}} + 1$. Thus, if we consider $\psi : S \longrightarrow \mathbb{S}^3$ a flat immersion such that in local coordinates $I = dx^2 + dy^2$ then, by the Codazzi equation, the second fundamental form can be written as $II = u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2$ where u is a solution to the hyperbolic Monge-Ampère equation

  $$u_{xx}u_{yy} - u_{xy}^2 = -1.$$

  The difference with the elliptic case is that, now, the holomorphic representation is no longer true, and in fact, the surfaces are not analytic. Moreover, some problems about uniqueness of the solution to the Cauchy problem for these surfaces appear.

  Nevertheless, in [AGM] the authors give a procedure to solve the Cauchy problem by means of a geometric version of D'Alembert formula for the homogeneous wave equation, which is a PDE that naturally appears in this theory.

- Finally, note that if $u(x, y)$ is a solution to (4.4) over a planar domain Ω, then the map $\Phi : \Omega \longrightarrow \mathbb{R}^2$ given by $\Phi(x, y) = (u_x, u_y)$ satisfies that its Jacobian is Jac$(\Phi) = 1$ and hence it defines an area-preserving diffeomorphism between Ω and the planar domain $\Phi(\Omega)$. The relation of area-preserving diffeomorphisms with Lagrangian immersions, which in the case of planar domains is given by means of the Monge-Ampere equation, is indeed more general. It is true that if $f : S_1 \longrightarrow S_2$ is an area-preserving diffeomorphism between compact Riemann surfaces of constant curvature, then the graph of f can be viewed as a Lagrangian submanifold in $S_1 \times S_2$. For more details we refer to [Wo, Wa] for instance.

4.3 The local embedding problem

A classical problem in geometry, first posed in 1873 by Schlaefli [Sc], is to determine when a smooth 2-dimensional Riemannian manifold admits a smooth local isometric embedding into $\mathbb{R}^3$. We will check that this problem is equivalent to solving a certain Monge-Ampère equation, the *Darboux equation*, under some restrictions.

First note that a metric g defined in a domain $\Omega \subset \mathbb{R}^2$ is isometrically immersed in $\mathbb{R}^3$ if there exists a map $X = (X_1, X_2, X_3) : \Omega \longrightarrow \mathbb{R}^3$ such that its first fundamental form is given by

$$dX_1^2 + dX_2^2 + dX_3^2 = g.$$

Moreover, every isometric immersion is locally an embedding.

Now, suppose that $g = g_{ij}dx_i dx_j$, $i, j = 1, 2$, is a metric in an open set $\Omega \subset \mathbb{R}^2$ and that $X = (X_1, X_2, X_3) : \Omega \longrightarrow \mathbb{R}^3$ is an isometric embedding of g. Then, by the fundamental equations of the immersion, we have that

$$\nabla_{ij} X = h_{ij} N \tag{4.15}$$

where N is the unit normal to the immersion, h_{ij} is its second fundamental form, and $\nabla_{ij} X$ is the Hessian matrix of X with respect to g, i.e.,

$$\nabla_{ij} X = D^2 X - \Gamma_{ij}^k \partial_k X.$$

Here, as usual, D^2 denotes the Hessian with respect to the usual flat metric and the subscripts in ∂ denote partial derivatives. Now, fix a unit vector $e \in \mathbb{R}^3$ and set

$$u = X \cdot e.$$

If we take the scalar product in (4.15) and use the expression of the Gauss curvature of the immersion, which is given by $K = \det(h_{ij})/\det(g_{ij})$, we obtain

$$\det(\nabla_{ij} u) = K \det(g_{ij})(N \cdot e)^2.$$

Observe that $N \cdot e = 1 - |\nabla u|^2$, where ∇ denotes the gradient with respect to g. Therefore, if we write as g^{ij} the inverse matrix of g, a simple computation yields the following expression which seems more convenient in order to relate it with (4.1):

$$\det(D^2 u - \Gamma_{ij}^k \partial_k u) = K \det(g_{ij})(1 - g^{ij} \partial_i u \partial_j u). \tag{4.16}$$

The Monge-Ampère equation (4.16) is called the *Darboux equation*. Obviously, each component X_i of X satisfies (4.16) by taking e as the unit vector in the correspondent axis.

An important fact is that the Darboux equation determines the existence of the local isometric embedding of a metric in the following sense.

Lemma 4.3. *Let g be a C^α metric in a domain $\Omega \subset \mathbb{R}^2$ for some $\alpha \in [2, \infty]$. If there exists a C^β solution to* (4.16)*, z, with $|\nabla z| < 1$ for some $\beta \in [2, \alpha]$, then (Ω, g) admits a C^β isometric immersion in $\mathbb{R}^3$.*

Proof. Suppose that z is a function in the conditions of Lemma 4.3. Then, a simple computation using formula (4.16) yields that $\widetilde{g} = g - dz^2$ is a flat metric. Therefore, as Ω is simply connected, there exists a C^β immersion $(x, y) : \Omega \longrightarrow \mathbb{R}^2$ such that $dx^2 + dy^2 = \widetilde{g}$, i.e.,

$$dx^2 + dy^2 + dz^2 = g.$$

This provides a C^β immersion $X = (x, y, z)$ of g. $\square$

From Lemma 4.3 we obtain the desired correspondence between the existence of a local isometric embedding of a given metric and the existence of solutions to the Monge-Ampère equation (4.16) with the restriction $|\nabla u| < 1$. Observe that if we start with a given immersion $X = (x, y, z)$ there always exists a coordinate with respect to which X is a graph and so the hypothesis of Lemma 4.3 hold.

Remark 4.1. *We must mention that the strategy of obtaining a flat metric from a given immersion was first used by J. Weingarten [We]. Later, in a joint work of the author with J. A. Gálvez and P. Mira [GJM1], this approach was generalized to obtain a correspondence between immersed graphs in Riemannian and Lorentzian warped products.*

Some classical results on this problem of the local embedding of metrics in $\mathbb{R}^3$ are that a solution always exists when K is analytic or when K does not vanish in the domain Ω. They can be found in [Ja], and [Po1, Po2]. In the case that K is sufficiently smooth and satisfy $K \leq 0$, or if $K(p) = 0$ and $\nabla K(p) \neq 0$ for some point $p \in \Omega$, C. S. Lin provides the desired local embedding in [Lin1] and [Lin2]. It have been also proved [HHL] that if $K \geq 0$ and ∇K possesses a certain nondegeneracy, a solution always exists. Other partial results and related topics can be found respectively in [Ha, Kh2, Kh3] and the survey [HaHo].

On the other hand, A. V. Pogorelov constructed in [Pog] a $C^{2,1}$ metric with no C^2 isometric embedding in $\mathbb{R}^3$, and other examples of local nonsolvability can be found in [NaYu, Kh1]. We must point out that the general question of determining whether a metric can be locally embedded in $\mathbb{R}^3$ remains open.

4.4
Graphs with prescribed curvature

In this section we explain how Monge-Ampère equations appear in the classical problem in differential geometry of finding surfaces of prescribed curvature in 3-dimensional spaces. We will focus on the case of graphs of prescribed Gaussian curvature in model spaces and in $\mathbb{L}^3$. Also we will show how in the case of graphs in $\mathbb{R}^3$, equation (4.1) appears when dealing with a certain family of Weingarten surfaces that generalizes the constant curvature case. Moreover, we explain how in those cases holomorphic data associated to the immersion appear.

In the Monge-Ampère equations appearing in this section we will consider the curvature function K depending on (x, y, z, p, q) where $p = z_x$ and $q = z_y$.

4.4.1
Prescribed curvature in model spaces

We recall that, given an isometric immersion $\psi : S \longrightarrow M$ where $M = \mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3$ is one of the model spaces of constant curvature $\varepsilon = 0, -1, 1$, respectively, the Gauss equation gives

$$K = K_{\text{ext}} + \varepsilon \tag{4.17}$$

where K is the Gauss curvature and K_{ext} is the extrinsic curvature of the immersion. Hence, to prescribe the Gauss curvature is equivalent to prescribe the extrinsic curvature of the immersion.

First, we consider the situation in which we have a graph $z = z(x, y)$ in $\mathbb{R}^3$ over a planar domain Ω. In that case, (4.17) yields

$$rt - s^2 = K(1 + p^2 + q^2)^2 \tag{4.18}$$

where, as usual, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$ and $t = z_{yy}$. The graphs of the solutions to (4.18) provide the graphs in $\mathbb{R}^3$ of prescribed curvature function K.

We can generalize this notion of graph in $\mathbb{R}^3$ to the other model spaces of constant curvature $\mathbb{S}^3$ and $\mathbb{H}^3$ by considering graphs over other natural domains.

Let us begin with graphs in the hyperbolic 3-dimensional space. We must emphasize that the equation for a graph of prescribed curvature will depend on the base surface that we consider.

First, we consider the model $\mathbb{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : -x_0^2+x_1^2+x_2^2+x_3^2 = -1\} \subset \mathbb{L}^4$. It is well-known that the central projection $(x_0, x_1, x_2, x_3) \mapsto (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})$ is an isometry between that model of $\mathbb{H}^3$ and the Klein model $(\mathbb{B}^3, g_{\mathcal{K}})$, where $\mathbb{B}^3 \subset \mathbb{R}^3$ is the unit ball. Here, geodesics in $\mathbb{H}^3$ correspond to segments in $\mathbb{B}^3 \subset \mathbb{R}^3$. Now, suppose that we have a graph $\{(x, y, z(x, y))\} \subset (\mathbb{B}^3, g_{\mathcal{K}})$. As the restriction of $g_{\mathcal{K}}$ to the disc $\mathbb{B}^3 \cap \{(a, b, c) \in \mathbb{R}^3 : \ c = 0\}$ yields a totally geodesic surface (a hyperbolic disc in this Klein model), we would be in a similar situation as when considering graphs over a planar domain in $\mathbb{R}^3$. Furthermore, a computation using the isometry between those models of $\mathbb{H}^3$ gives that, the prescribed curvature equation for such a graph is in the form

$$rt - s^2 = (K + 1)\frac{(-1 + z^2 + (-1 + y^2)q^2 + 2xypq + (-1 + x^2)p^2 - 2z(yp + xq))^2}{(-1 + x^2 + y^2 + z^2)^2}. \tag{4.19}$$

Observe that (4.19) corresponds to a Monge-Ampère equations of type (4.3). Moreover, by Definition 4.1, the value of the curvature will determine the character of the PDE. That is, it will be elliptic only when $K > -1$.

Apart from graphs over totally geodesic domains, there exists another natural type of graphs in $\mathbb{H}^3$ that also yields a Monge-Ampère equation when prescribing its curvature. Let $\mathbb{H}^3 = (\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 > 0\}, \frac{1}{x_3^2}(dx_1^2 + dx_2^3 + dx_3^2))$ be the half-space model of the hyperbolic 3-space. As it is well-known, in this model, the horizontal planes $\{x_3 = \text{constant}\}$ are horospheres of $\mathbb{H}^3$. Hence, we can fix for instance the horosphere $H = \{x_3 = 1\}$ and take a domain $\Omega \subset H$. Then, if we consider a graph $z = z(x, y)$, $(x, y) \in \Omega \subset H$, a computation using (4.17) yields [AlCu]

$$\begin{aligned}(2q^2 + e^{-2z})r + (2p^2 + e^{-2z})t - 4pqs + rt - s^2 = \ & (K + 1)e^{-4z}(1 + e^{2z}(p^2 + q^2))^2 \\ & -e^{-4z}(1 + 2e^{2z}(p^2 + q^2)).\end{aligned} \tag{4.20}$$

As before, we deduce from (4.20) that only those graphs with Gaussian curvature $K > -1$ will yield an elliptic PDE. Some results in this topic can be found in [RoSp].

In the case of surfaces of prescribed curvature in $\mathbb{S}^3$, we can also make some computations in the same spirit as we did for the case of the hyperbolic 3-space. Now, the projection $(x_0, x_1, x_2, x_3) \mapsto (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})$, maps

$\mathbb{S}^3_+ = \mathbb{S}^3 \cap \mathbb{R}^4_+$ into an isometric model $(\mathbb{R}^3, g_S)$ which becomes very useful in the study of convex surfaces in $\mathbb{S}^3$. Specifically, by means of such a map, geodesics in $\mathbb{S}^3_+$ correspond to lines in $\mathbb{R}^3$, and planes with the induced metric become totally geodesic surfaces. Then, if we consider a graph over a domain Ω in such a totally geodesic surface, $z = z(x, y)$ satisfies the following equation in Ω:

$$rt - s^2 = (K-1)\frac{(1 + z^2 + (1+y^2)q^2 + 2xypq + (1+x^2)p^2 - 2z(yp+xq))^2}{(1+x^2+y^2+z^2)^2}. \tag{4.21}$$

Next, we focus on the case when all the graphs mentioned before have constant Gaussian curvature (and so constant extrinsic curvature). We will see that then, holomorphic data associated to the immersion appear. Those holomorphic data have a geometric meaning and in some cases yield a representation formula for the surface. These representation formulas also have an analytic interest, as they provide an approach to solve the Monge-Ampère equation associated to the immersion. First, we state the following result which is a direct consequence of the Codazzi equation of an immersion.

Proposition 4.1. *Let $\psi : S \longrightarrow M = \mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3$, be an isometric immersion such that the second fundamental form II is positive definite. Then, the $(2,0)$-part of the first fundamental form I is holomorphic with respect to the conformal structure induced on S by II if and only if the extrinsic curvature of the immersion K_{ext} is constant.*

For a surface ψ in the conditions of Proposition 4.1 we will call *the extrinsic conformal structure* to the conformal structure induced on the surface by II, as was explained in Section 1.1. We must remark that Proposition 4.1 is the key to prove Liebmann's theorem in $\mathbb{R}^3$ and its generalization to $\mathbb{S}^3$ and $\mathbb{H}^3$. That is, from Proposition 4.1 we can deduce that complete surfaces with positive constant Gaussian curvature in $M = \mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ ($K > 1$ if $M = \mathbb{S}^3$) are totally umbilical.

We have already mentioned the there exists a representation formula for flat surfaces in $\mathbb{H}^3$ (see Subsection 1.5.2). Next we show how, in the particular case of surfaces of constant positive curvature in $\mathbb{R}^3$, a representation formula is possible in terms of the Gauss map.

Observe first of all that if $\psi : S \longrightarrow \mathbb{R}^3$ is an immersion of a surface with constant curvature $K > 0$ we can assume up to dilation $K = 1$. Then, when the surface is a graph (note that this happens always locally) it satisfies (4.18) for $K = 1$, and its extrinsic conformal structure agrees with the underlying conformal structure of the solution introduced in Definition 4.2 since $d\sigma^2$ and II are conformal. In this sense, if (u, v) are conformal coordinates on S with respect to the conformal extrinsic structure, and $N : S \longrightarrow \mathbb{S}^2$ denotes the unit normal of the surface, then $\langle \psi_u, N_u \rangle = \langle \psi_v, N_v \rangle < 0$ and $\langle \psi_u, N_v \rangle = 0$ where here $\langle , \rangle$ denotes the usual inner product in $\mathbb{R}^3$. The condition $K = 1$ implies

$$\psi_u = N \times N_v \qquad \text{and} \qquad \psi_v = -N \times N_u, \tag{4.22}$$

which in particular yields that $N : S \longrightarrow \mathbb{S}^2$ is a harmonic map into $\mathbb{S}^2$, and that the immersion ψ satisfies the equation

$$\psi_{uu} + \psi_{vv} = 2\psi_u \times \psi_v, \tag{4.23}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$. We must point out that, since II is definite, N is a local diffeomorphism, and ψ is uniquely determined by N up to translations by means of the representation formula (4.22).

In [GHM] the authors analyze the behavior of a solution to equation (4.18) at an isolated singularity. Specifically, they prove that the following assertions are equivalent: (i) the isolated singularity is removable, (ii) the mean curvature of the graph is bounded around the singularity, and (iii) the graph has around the singularity the underlying conformal structure of a punctured disc. Moreover, by means of equation (4.23) and an extension result for that type of PDE in [Jac], they show that when the singularity is non-removable, the unit normal can be extended to the boundary of the annulus that parametrizes S (remember that the conformal structure of the punctured disc is forbidden) and give a classification result in the following sense: the limit unit normal of a solution to (4.18) at a non-removable isolated singularity is a real-analytic, regular, strictly convex Jordan curve in $\mathbb{S}^2$; and conversely, any such curve arises as the limit unit normal of exactly one solution to (4.18) with an isolated singularity. They also extend the above results on isolated singularities of solutions to the constant curvature equation (4.18) to the case where the surface is not a graph anymore. The classification result they obtain in this case is similar to the one above, but with the difference that the limit unit normal is a real-analytic, immersed closed curve in $\mathbb{S}^2$ that is locally convex at regular points, but that may have a certain type of singular points of cuspidal type.

Also, in [GHM] authors deal with the global problem of classifying peaked spheres in $\mathbb{R}^3$. By definition, a peaked sphere is a closed convex surface of constant curvature K that is everywhere regular except at a finite number of points. A regular peaked sphere is a round sphere, and it can be deduced by Alexandrov reflection principle that a peaked sphere with exactly two singularities is a rotational sphere and that there are no peaked spheres with exactly one singularity. Using some results by Troyanov, Luo-Tian, Alexandrov and Pogorelov, they explain that the space of peaked spheres with $n > 2$ singularities is a $(3n-6)$-dimensional family.

Some of these techniques used to recover solutions to (4.18) with isolated singularities when K is constant will inspire the approach explained in Chapter 5. In fact, although the general Monge-Ampère equation we will study does not have an associated holomorphic data, it is possible to associate to any solution of it, a natural conformal structure playing the role of II in Proposition 4.1, and the role of $d\sigma^2$ in (4.5) for the Hessian one equation (4.4). Moreover, in terms of such a conformal structure, the coordinates of the graph to the solution satisfy a quasilinear elliptic system with good analytic properties.

4.4.2 Prescribed negative curvature in $\mathbb{L}^3$

Next we explain how equation (4.3) also appears when dealing with immersed spacelike graphs of prescribed negative curvature in the 3-dimensional Lo-rentz-Minkowski space, which we view as $\mathbb{L}^3 = (\mathbb{R}^3, \langle , \rangle = -dx_0^2 + dx_1^2 + d_3^2)$. If we consider a spacelike immersion $\psi : S \longrightarrow \mathbb{L}^3$ with negative Gaussian curvature K with respect to the induced Riemannian metric I, the Gauss equation yields that $K_{\text{ext}} = -K > 0$. Observe that the spacelike character of ψ guarantees that it can be seen as a graph $(x, y, z(x, y))$ over a domain in the (x, y)-plane, with $|\nabla z| < 1$. Then, the Monge-Ampère equation that models such a graph is, with the usual notation for the derivatives of the height function h,

$$rt - s^2 = -K(1 - p^2 - q^2)^2, \qquad p^2 + q^2 < 1. \tag{4.24}$$

Suppose then that we chose the normal map of the immersion $N : S \longrightarrow \mathbb{H}^2$ in order that the second fundamental form is positive definite. Then, if (u, v) are conformal coordinates on S with respect to the conformal extrinsic structure, a computation by means of the Gauss equation as in the case of $\mathbb{R}^3$ gives that

$$\psi_u = -\frac{1}{\sqrt{-K}} N \times N_v \qquad \text{and} \qquad \psi_v = \frac{1}{\sqrt{-K}} N \times N_u. \tag{4.25}$$

Here $\times$ denotes the vector product in $\mathbb{L}^3$, that is, if $\langle , \rangle$ is the usual metric in $\mathbb{L}^3$ and $a, b \in \mathbb{L}^3$ then $a \times b$ is the unique vector such that $\langle a \times b, c\rangle = -\det(a, b, c)$ for all $c \in \mathbb{L}^3$.

Moreover, if we suppose that $K = -1$, we deduce from (4.25) that the Gauss map N is harmonic into the hyperbolic plane and that the immersion ψ satisfies the equation

$$\psi_{uu} + \psi_{vv} = -2\psi_u \times \psi_v. \tag{4.26}$$

By (4.26) we obtain, as in Proposition 4.1 for the case of model spaces, that the $(2, 0)$-part of the complex expression of I is holomorphic.

Conversely, if S is a simply connected Riemann surface and $N : S \longrightarrow \mathbb{H}^2$ is a harmonic local diffeomorphism then there exists, up to translations, a unique immersion with constant Gaussian curvature $K = -1$ given by (4.25) such that N (or -N) is its Gauss map and its conformal structure is induced by the second fundamental form of the immersion.

Finally, we state the following result in [GMM2] that links the theory of harmonic diffeomorphisms with the existence of solutions of Monge-Ampère equations of type (4.3) in convex domains. It is obtained by a reformulation of equation (4.24) which is used to characterize complete spacelike surfaces with constant negative Gaussian curvature.

Theorem 4.5. *There exists a bijective correspondence between harmonic diffeomorphisms from the unit disc or the complex plane onto the hyperbolic plane (up to conformal equivalences in the domain) and the solutions of the Monge-Ampere equation*

$$\phi_{uu}\phi_{vv} - \phi_{uu} = \frac{1}{1 - u^2 - v^2}, \qquad u^2 + v^2 < 1. \tag{4.27}$$

Proof. Let $S = \mathbb{D}, \mathbb{C}$, and consider $N : S \longrightarrow \mathbb{H}^2$ a surjective harmonic diffeomorphism. We already know that there exists a unique, up to translations, spacelike immersion $\psi : S \longrightarrow \mathbb{L}^3$ with constant negative Gaussian curvature -1 such that N is its Gauss map and the Riemann structure on S is given by the second fundamental form. Then, we consider the Legendre transform (see [LSZ, pag. 89])

$$\phi(u, v) = ux + vy - z,$$

where $\psi = (x, y, z)$ is locally a graph on the (x, y)-plane, and $(u, v) \in \mathbb{D}$ are given by $N = (N_1, N_2, N_3) \mapsto (u, v) = (N_1/N_3, N_2/N_3)$, which is a diffeomorphism from $\mathbb{H}^2$ onto the unit disc. Thus, (u, v) are new global (non-conformal) parameters on S and a straightforward computation yields $\phi_u = x$, $\phi_v = y$ and

$$\mathcal{D}\phi_{uu} = t, \qquad \mathcal{D}\phi_{uv} = s, \qquad D\phi_{vv} = r,$$

where $\mathcal{D} = rt - s^2$ and, as usual, r, t and s denote the second derivatives of z with respect to (x, y). Therefore, from (4.24) we deduce (4.27). Conversely, given a solution to (4.27), the harmonic map N can be recovered by the inverse Legendre transformation of the graph $\{(u, v, \phi(u, v))\} \subset \mathbb{R}^3$. That is, the immersion given by $\{(\phi_u, \phi_v, u\phi_u + v\phi_v - \phi)\}$ agrees with ψ, hence, it is spacelike with constant curvature $K = -1$ and its Gauss map N will be a harmonic diffeomorphism. $\square$

4.4.3 Elliptic Weingarten graphs

We explain next how the general Monge-Ampère equation (4.1) also appears when dealing with linear Weingarten surfaces in $\mathbb{R}^3$. We focus on the elliptic case as its underlying Monge-Ampère equation becomes a particular case of the one that will be studied in Chapter 5.

Let $\psi : S \longrightarrow \mathbb{R}^3$ be an immersion of an orientable surface with Gauss map $N : S \longrightarrow \mathbb{S}^2$. It is said that ψ is a linear Weingarten immersion if there exist three real numbers a, b, c not all zero, such that

$$2aH + bK = c, \tag{4.28}$$

where K and H are respectively the Gauss curvature and the mean curvature of the immersion. It is said that ψ is an elliptic linear Weingarten surface when

$$a^2 + bc > 0. \tag{4.29}$$

Observe that, in such a case, we can suppose up to a change of sign in (4.28) that $c \geq 0$. This family of surfaces contains surfaces of constant mean curvature, when $b = 0$, and surfaces with positive constant Gauss curvature, when $a = 0$. As we are interested in Monge-Ampère equations we will suppose that $b \neq 0$. Now, if we consider a graph

$\psi(S) = (x, y, z(z, y)) \subset \mathbb{R}^3$ and we impose that it is an elliptic linear Weingarten surface, we easily deduce that

$$rt - s^2 + \frac{a}{b}\sqrt{1+p^2+q^2}((1+p^2)t - 2pqs + (1+q^2)r) = \frac{c}{b}(1+p^2+q^2)^2. \quad (4.30)$$

Observe that the ellipticity condition (4.29) yields that equation (4.30) is an elliptic PDE by Definition 4.1.

For this type of surfaces there also exists a special conformal structure which provides holomorphic data associated to the immersion. Specifically, it can be proved (see [GMM3]) that $\sigma = aI + bII$ is positive definite, where I and II are the first and the second fundamental form of ψ. It is also showed in [GMM3] that

$$\Delta^\sigma \psi = \frac{c + bK}{a^2 - bc} N, \qquad \Delta^\sigma N = 2\frac{aK - cH}{a^2 - bc} N, \quad (4.31)$$

where Δ^σ denotes the Laplacian with respect to the metric σ. From (4.31) it can be deduced that N is a harmonic map from (S, σ) into $\mathbb{S}^2$. Then, it turns out that the $(2,0)$-parts of I and II are holomorphic with respect to the conformal structure induced on S by σ. Furthermore, in [GMM3] the authors give a representation formula in terms of N that unifies the one given by Kentmotsu [Ken] in terms of the Gauss map and the mean curvature, and the one explained before, in the case of surfaces of constant Gaussian curvature.

Remark 4.2. *It is important to remark that by normal translations of constant mean curvature surfaces in $\mathbb{R}^3$ we can obtain elliptic linear Weingarten surfaces (possibly with singularities).*

In this sense, it is natural that some local aspects of both classes of surfaces could be translated from one class to another. For instance, this is the case of the fact that the Gauss map (shared by both surfaces) is harmonic, with respect to the first fundamental form in the case of constant mean curvature surfaces, or with respect to the correspondent translated metric σ for the case of linear Weingarten surfaces.

Nevertheless, for these classes of surfaces the study of global aspects as completeness or embeddedness differs from one class to another. Note for instance that by the process of translating the surface some singularities may appear. Actually, the nature of isolated singularities in both theories is completely different since elliptic linear Weingarten surfaces may admit bounded isolated singularities which are non-removable as we will see in Chapter 5.

5

Isolated singularities of Monge-Ampère equations

5.1 Non-trivial bounded elliptic solutions. Main results

In this chapter we study isolated singularities of the general equation of Monge-Ampère type in dimension two that we introduced in Chapter 4:

$$\det\{D^2 z + \mathcal{A}(x,y,z,Dz)\} = f(x,y,z,Dz),$$

where here $Dz, D^2 z$ denote respectively the gradient and the Hessian of z, and $\mathcal{A}(x,y,z,Dz) \in \mathcal{M}_2(\mathbb{R})$ is symmetric. This equation can be rewritten as

$$Ar + 2Bs + Ct + rt - s^2 = E, \tag{5.1}$$

where $A = A(x,y,z,p,q), \ldots, E = E(x,y,z,p,q)$, and we use the previous notation $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. With this reformulation, and following Definition 4.1, a solution $z \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^2$, to (5.1) is elliptic provided

$$D := AC - B^2 + E > 0 \tag{5.2}$$

holds on the set

$$\mathfrak{H} := \{(x, y, z(x, y), p(x, y), q(x, y)) : (x, y) \in \Omega\}.$$

Definition 5.1. *Let $z \in C^2(\Omega)$ be a solution to* (5.1). *We say that z is a* bounded elliptic solution *of* (5.1) *if there exists a bounded open set $\mathcal{U} \subset \mathbb{R}^5$ such that:*

1. $\overline{\mathfrak{H}} \subset \mathcal{U}$.
2. *The functions $A = A(x, y, z, p, q), \dots, E = E(x, y, z, p, q)$ are of class $C^{1,\mu}$ for some $\mu \in (0, 1)$ on $\mathcal{U} \subset \mathbb{R}^5$.*
3. *The ellipticity condition* (5.2) *holds on $\mathcal{U}$.*

From now on, and unless we explicitly state it otherwise, we let Ω be a punctured disc, $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho^2\}$.

In the case that the coefficients $A, \dots, E$ are real analytic on $\mathcal{U}$, the solution z is also real analytic on Ω. Also, in the conditions of the definition, $z(x, y)$ extends *continuously* to $(0, 0)$. To see this, observe that the ellipticity condition (5.2) is equivalent to

$$(A + t)(C + r) - (B - s)^2 > 0.$$

We may suppose then that $C + r > 0$ and $A + t > 0$ in Ω (otherwise we can proceed in a similar way). Consequently,

$$p(x, y) + \int_0^x C(\tau, y, \dots, q(\tau, y))d\tau \qquad \text{and} \qquad q(x, y) + \int_0^y A(x, \tau, \dots, q(x, \tau))d\tau$$

are strictly increasing in x and y, respectively. Thus p and q are bounded and this proves the assertion. We will assume for simplicity that $z(0, 0) = 0$.

There is a trivial example of bounded elliptic solutions to (5.1), namely the case in which z extends C^2 across the origin. We shall call these solutions *trivial solutions* to (5.1). On the other hand, when z is a non-trivial solution, it presents a non-removable isolated singularity at the origin.

Our main objective in this chapter is to study a general classification of the non-trivial, analytic bounded solutions to elliptic Monge-Ampère equations in a punctured disc. We will also study the non-analytic case.

In order to state our main results, we need to introduce some terminology. Let $z \in C^2(\Omega)$ be a bounded elliptic solution to (5.1). It follows then from the ellipticity condition (5.2) that the expression

$$ds^2 = (r + C)dx^2 + 2(s - B)dxdy + (t + A)dy^2 \tag{5.3}$$

is a Riemannian metric on Ω. This metric is conformal to the second fundamental form of the graph when $A = B = C = 0$ and, in this case, it was used for studying the equations (4.4) and (4.18). We will see that the *underlying conformal structure* that it induces on Ω constitutes an important tool in this problem. From now on, we will assume that z as well as the coefficients $A, \dots, E$ of (5.1) are real analytic (see Section 5.4 for a discussion about the non-analytic case).

It is a well-known fact (cf. [Ve]) that ds^2 admits conformal coordinates $w := u + iv$ such that

$$ds^2 = \frac{\sqrt{D}}{u_x v_y - u_y v_x}|dw|^2. \tag{5.4}$$

That is, there exists a real analytic diffeomorphism

$$\Phi : \Omega \longrightarrow \Lambda := \Phi(\Omega) \subset \mathbb{R}^2, \qquad (x, y) \mapsto \Phi(x, y) = (u(x, y), v(x, y)) \tag{5.5}$$

satisfying

$$x_u y_v - x_v y_u > 0, \tag{5.6}$$

and the Beltrami system

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{1}{\sqrt{D}} \begin{pmatrix} s - B & -(C + r) \\ A + t & -(s - B) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}. \tag{5.7}$$

Here, Λ is a domain in $\mathbb{R}^2 \equiv \mathbb{C}$ which is conformally equivalent to either the punctured disc $\mathbb{D}^*$ or an annulus $\mathbb{A}_R = \{z \in \mathbb{C} : 1 < |z| < R\}$.

From now on, we will often consider all the functions depending on the parameters (u, v) via $(x, y) = \Phi^{-1}(u, v)$. For simplicity, we keep the same notation.

We may observe that system (5.7) yields the following equations (see for example [Bey1, Mu1]):

$$\begin{aligned} p_u &= \sqrt{D}y_v + By_u - Cx_u,\\ p_v &= -\sqrt{D}y_u + By_v - Cx_v,\\ q_u &= -\sqrt{D}x_v + Bx_u - Ay_u,\\ q_v &= \sqrt{D}x_u + Bx_v - Ay_v. \end{aligned} \tag{5.8}$$

Moreover we have that

$$\begin{aligned} z_v &= px_v + qy_v\\ &= -\frac{p}{\sqrt{D}}(q_u - Bx_u + Ay_u) + \frac{q}{\sqrt{D}}(p_u - By_u + Cx_u)\\ &= \frac{1}{\sqrt{D}}(x_u(Bp + Cq) - y_u(Ap + Bq) + qp_u - pq_u). \end{aligned}$$

Observe that the computation above make sense as we are assuming that (5.2) is satisfied. So, the following system holds:

$$\begin{pmatrix} x\\ y\\ z\\ p\\ q \end{pmatrix}_v = \widetilde{M}\begin{pmatrix} x\\ y\\ z\\ p\\ q \end{pmatrix}_u, \qquad \widetilde{M} = \frac{1}{\sqrt{D}}\begin{pmatrix} B & -A & 0 & 0 & -1\\ C & -B & 0 & 1 & 0\\ Bp + Cq & -Ap - Bq & 0 & q & -p\\ 0 & -E & 0 & B & C\\ E & 0 & 0 & -A & -B \end{pmatrix}. \tag{5.9}$$

In this situation, motivated by [HeB] we introduce the following definition.

Definition 5.2. *A solution z to* (5.1) *satisfies the* Heinz-Beyerstedt condition*, in short* HeB-condition*, if A_p, $A_q + 2B_p$, $C_p + 2B_q$ and C_q are Liptschitz continuous in $\overline{\Omega}$ considered as functions of x and y.*

Lemma 5.1. *Suppose that the coefficients $A, \dots, E : \mathcal{U} \subset \mathbb{R}^5 \longrightarrow \mathbb{R}$ in* (5.1) *satisfy the condition:*

The functions A_p, $A_q + 2B_p$, $C_p + 2B_q$ and C_q do not depend on p and q in $\mathcal{U}$. $(\star)$

Then the solution $z(x, y)$ to (5.1) *satisfy the HeB-condition in Definition 5.2.*

Proof. We denote by $F(x,y,z)$ any of the functions in the statement of the condition $(\star)$. Then, F can be seen as a function $\widetilde{F}$ depending on the variables (x,y) using the composition $\widetilde{F}(x,y) = (F\circ G)(x,y)$ where $G(x,y) = (x,y,z(x,y))$. On the other hand, observe that by the regularity of $\widetilde{F}$ away from the origin, to prove that the HeB-condition holds we only need to show that for any $(x,y)\in\Omega$:

$$|\widetilde{F}(x,y) - \widetilde{F}(0,0)| \leq c|(x,y)|, \qquad \text{for some } c>0. \tag{5.10}$$

Since z is a bounded elliptic solution its partial derivatives p and q are bounded in Ω and so $z \in C^{0,1}(\overline{\Omega})$ (see [GiTr, pag. 154]). Then, as F is analytic in $\mathcal{U}$ we deduce:

$$|F(x,y,z) - F(0,0,0)| \leq c_1|(x,y,z)| \leq c_1(|(x,y)| + |z|) \leq c_1(|(x,y)| + c_2|(x,y)|),$$

for certain constants $c_1, c_2 > 0$. That is, (5.10) holds. $\square$

Note that the analyticity of the coefficients $A,\dots,E$ is a sufficient condition in the previous proof but it is not necessary. If we suppose just that the coefficients are of class $C^{1,1}(\mathcal{U})$, the function F as above is Lipschitz continuous in $\mathcal{U}$ and we can proceed in the same way.

Moreover, a simple computation yields that the condition $(\star)$ implies that the expression of the coefficients A,B,C must be:

$$\begin{aligned} A &= a_1 + a_2 p + a_3 q + \alpha q^2,\\ B &= b_1 + b_2 p + b_3 q - \alpha p q,\\ C &= c_1 + c_2 p + c_3 q + \alpha p^2, \end{aligned}$$

where $a_1,\dots,c_3$ and α are certain functions depending only on (x,y,z).

A simple observation is that the condition $(\star)$ is trivially satisfied when the coefficients in (5.1) are such that $A=B=C=0$, that is, for the classical Monge-Ampère equation (4.3).

The HeB-condition is used in [HeB, Lemma 3.3] and provides the following classification result:

Lemma 5.2. *Suppose that the HeB-condition is satisfied. Then, a bounded elliptic solution to* (5.1) *is non-trivial (i.e. it has a non-removable singularity at the origin) if and only if Λ is conformally equivalent to some annulus $\mathbb{A}_R$.*

Thus, in order to study non-trivial solutions to (5.1) when the coefficients satisfy the HeB-condition we may consider $\Lambda = \mathbb{A}_R$. If we denote $\Sigma_r := \{z : 0 < \operatorname{Im}(z) < r\}$, then $\mathbb{A}_R$ is conformally equivalent to

$\Gamma_r := \Sigma_r/(2\pi\mathbb{Z})$ for $r = \log R$. So, composing with this conformal equivalence we will suppose that the map Φ in (5.5) is a diffeomorphism from Ω into Γ_r; in particular, Φ is 2π-periodic and (u,v) will denote the canonical coordinates of the strip Σ_r.

Let $G = \{(x,y,z(x,y)) : (x,y) \in \Omega\} \subset \mathbb{R}^3$ be the graph of $z(x,y)$. By using the parameters (u,v), we may parametrize G as a map

$$\psi(u,v) = (x(u,v), y(u,v), z(u,v)) : \Gamma_r \longrightarrow G \subset \mathbb{R}^3 \tag{5.11}$$

such that ψ extends continuously to $\mathbb{R}$ with $\psi(u,0) = (0,0,0)$.

Finally, we define the *limit gradient* of z at the origin to be the set $\gamma \subset \mathbb{R}^2$ of points $\xi \in \mathbb{R}^2$ for which there is a sequence $\nu_n \to (0,0)$ in Ω such that $(p,q)(\nu_n) \to \xi$.

With all of this, our first main result is the following one, which provides a very general existence theorem for analytic bounded elliptic solutions to (5.1) with a non-removable isolated singularity at the origin.

In what follows, we will denote $\Sigma_r := \{z : 0 < \operatorname{Im} z < r\}$, $\widehat{\Sigma_r} := \{z : -r < \operatorname{Im} z < r\}$, $\Gamma_r := \Sigma_r/(2\pi\mathbb{Z})$ and $\widehat{\Gamma_r} := \widehat{\Sigma_r}/(2\pi\mathbb{Z})$.

Theorem 5.1. *Let $\gamma(u) = (\alpha(u), \beta(u))$ be a real analytic, 2π-periodic curve, and assume that the $A,\dots,E : \mathcal{U} \subset \mathbb{R}^5 \longrightarrow \mathbb{R}$ are real analytic functions that satisfy the ellipticity condition* (5.2) *on an open set $\mathcal{U} \subset \mathbb{R}^5$ that contains $(0,0,0,\gamma(\mathbb{R}))$.*

Then, there exists a real analytic map $\psi : \widehat{\Gamma_r} \longrightarrow \mathbb{R}^3$ such that:

1. *$\psi(u,0) = (0,0,0)$ for every $u \in \mathbb{R}$.*

2. *There exists a map $(p,q) : \widehat{\Gamma_r} \longrightarrow \mathbb{R}^2$ such that $(p,q)(u,0) = \gamma(u)$ for every $u \in \mathbb{R}$ and $(\psi,p,q)(\Gamma_r) \subset \mathcal{U}$. Moreover, the map $N(u,v) : \widehat{\Gamma_r} \longrightarrow \mathbb{S}^2$ defined by*
$$N(u,v) = \frac{(-p,-q,1)}{\sqrt{1+p^2+q^2}}(u,v)$$
satisfies that $\langle \psi_u, N\rangle = \langle \psi_v, N\rangle = 0$ in Γ_r.

3. *Assume that the map $(x(u,v), y(u,v))$ is an orientation preserving local diffeomorphism at some point $(u_0,v_0) \in \Gamma_r$. Then, the image of ψ around that point is the graph $G \subset \mathbb{R}^3$ of some real analytic, elliptic solution $z = z(x,y)$ to* (5.1) *for the coefficients $A,\dots,E$.*

4. *If the curve $\gamma(u)$ is regular, negatively oriented, and strictly convex (i.e. both $-|\gamma'(u)|$ and the curvature of $\gamma(u)$ are strictly negative for every*

u), then for $r > 0$ small enough, $\psi(\Gamma_r)$ is the graph of a non-trivial, real analytic, bounded elliptic solution to (5.1)*. Moreover, the limit gradient of this solution is the curve $\gamma = \gamma(\mathbb{R})$.*

Remark 5.1. *Observe that, by continuity, in order to guarantee that the ellipticity condition* (5.2) *holds on an open set $\mathcal{U} \subset \mathbb{R}^5$ that contains $(0, 0, 0, \gamma(\mathbb{R}))$ it is enough to impose that* (5.2) *is satisfied along $(0, 0, 0, \gamma(\mathbb{R}))$, i.e. that*

$$D(0, 0, 0, \gamma(\mathbb{R})) > 0. \tag{5.12}$$

Theorem 5.2 provides a converse statement to Theorem 5.1. In particular, it shows that the limit gradient of a non-trivial, analytic, bounded elliptic solution $z(x, y)$ to (5.1) is a regular, analytic strictly convex curve that determines $z(x, y)$ uniquely whenever the HeB-condition is satisfied.

Theorem 5.2. *Let $z(x, y)$ be a non-trivial bounded elliptic solution to* (5.1) *in a punctured disc $\Omega \subset \mathbb{R}^2$. And assume that the coefficients $A, \ldots, E$ are real analytic in $\mathcal{U} \subset \mathbb{R}^5$ and that the HeB-condition is satisfied. Then:*

1. *The limit gradient of $z(x, y)$ is a regular, strictly convex Jordan curve γ in $\mathbb{R}^2$, which is real analytic.*

2. *If (u, v) denote conformal coordinates on Σ_r for the metric ds^2 as explained previously, and $p = z_x$, $q = z_y$ are viewed as functions of (u, v), then those functions extend analytically to $\Sigma_r \cup \mathbb{R}$ and $\gamma(u) := (p(u, 0), q(u, 0))$ is an analytic, 2π-periodic, negatively oriented parametrization of γ such that $\gamma'(u) \neq (0, 0)$ for all $u \in \mathbb{R}$.*

3. *The graph G of $z(x, y)$ can be constructed following the procedure described in Theorem 5.1, in terms of the parameterized limit gradient $\gamma(u)$.*

4. *If $z'(x, y)$ is another non-trivial bounded elliptic solution to* (5.1) *in Ω for the coefficients $A, \ldots, E$, and with the same limit gradient γ, then the graphs of z and z' agree on an open set containing the origin.*

Observe that by Theorem 5.1, Theorem 5.2 and Lemma 5.1 we obtain a classification of the non-trivial analytic bounded elliptic solutions of (5.1), in the case that the condition $(\star)$ is satisfied, in terms of regular strictly convex Jordan curves in $\mathbb{R}^2$ that satisfy condition (5.12). This fact will be explained in detail in Section 5.4.

5.2
Existence of solutions: Proof of Theorem 5.1

In this section we prove Theorem 5.1. So, let $\gamma(u) = (\alpha(u), \beta(u))$ be a real analytic, 2π-periodic curve in $\mathbb{R}^2$, and assume that $A, \dots, E$ are real analytic functions on an open set $\mathcal{U} \subset \mathbb{R}^5$ that contains $(0,0,0,\gamma(\mathbb{R}))$ such that the ellipticity condition (5.2) is satisfied. We wish to construct a non-trivial bounded elliptic solution to (5.1) on a punctured disc $\Omega \subset \mathbb{R}^2$ for these coefficients, so that the limit gradient map of the solution at the singularity is exactly $\gamma(\mathbb{R})$.

Let us now consider the 2π-periodic initial data $(0,0,0,\alpha(u),\beta(u))$ along the axis $v = 0$ in the (u,v)-plane for the system (5.9). By the Cauchy-Kowalevsky theorem and the periodicity of $(0,0,0,\alpha(u),\beta(u))$, there exists a unique real analytic solution (x,y,z,p,q) to (5.9), defined on a neighborhood $\widehat{\Sigma_r} = \{(u,v) : -r < v < r\}$ of the axis $v = 0$, such that

$$(x,y,z,p,q)(u,0) = (0,0,0,\alpha(u),\beta(u)).$$

Observe that, by the uniqueness of the solution to (5.9), $\Psi := (x,y,z,p,q) : \widehat{\Sigma_r} \longrightarrow \mathbb{R}^5$ is 2π-periodic with respect to u, i.e. Ψ well defined on the quotient $\widehat{\Gamma_r} := \widehat{\Sigma_r}/(2\pi\mathbb{Z})$.

It is clear from the way we obtained system (5.9), or alternatively by a direct computation, that p and q satisfy (5.8), and also that $z_v = px_v + qy_v$. Besides, a computation from (5.8) proves the relation

$$p_v x_u + q_v y_u = p_u x_v + q_u y_v, \tag{5.13}$$

which is exactly the integrability condition needed for the existence of some smooth function z_0 on Σ_r, unique up to an additive constant, such that

$$(z_0)_u = px_u + qy_u, \qquad (z_0)_v = px_v + qy_v.$$

Note that $(z_0)_v = z_v$ and so $z(u,v) = z_0(u,v) + f(u)$ for a certain real valued function f. Also, observe that the initial conditions that the solution (x,y,z,p,q) to (5.9) satisfies imply that $z(u,0) \equiv 0$ and $(z_0)_u(u,0) \equiv 0$. Thus, $f(u)$ must be constant, and as z_0 was defined up to additive constants we may assume that $z(u,v) = z_0(u,v)$. In particular, it holds

$$z_u = px_u + qy_u, \qquad z_v = px_v + qy_v. \tag{5.14}$$

Defining now

$$\psi(u,v) := (x(u,v), y(u,v), z(u,v)) : \widehat{\Gamma}_r \longrightarrow \mathbb{R}^3 \tag{5.15}$$

and

$$N(u,v) := \frac{(-p(u,v), -q(u,v), 1)}{\sqrt{1+p(u,v)^2+q(u,v)^2}} : \widehat{\Gamma}_r \longrightarrow \mathbb{S}^2 \tag{5.16}$$

we see that the first two items of Theorem 5.1 hold.

To prove item 3, suppose now that the map $(x(u,v), y(u,v))$ is an orientation preserving local diffeomorphism at some point $(u_0, v_0) \in \Sigma_r$, i.e. the condition

$$\omega := x_u y_v - x_v y_u > 0 \tag{5.17}$$

holds at this point. By the Inverse Function Theorem we may write z, p, q in terms of the coordinates x, y. Thus, around (u_0, v_0) the image of the map $\psi(u,v)$ is the graph G in $\mathbb{R}^3$ of the real analytic function $z = z(x,y)$, and from formula (5.14) the relations $z_x = p$ and $z_y = q$ hold. We prove next that $z(x,y)$ is a solution to (5.1) for the coefficients $A, \dots, E$ we started with.

If we recall the notation $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, then using (5.8) and working in terms of the (u,v) coordinates we obtain

$$\sqrt{D} y_v = p_u - B y_u + C x_u = (C+r) x_u - (B-s) y_u,$$

and working similarly,

$$\begin{aligned} \sqrt{D} y_u &= -(C+r) x_v + (B-s) y_v, \\ \sqrt{D} x_v &= (B-s) x_u - (A+t) y_u, \\ \sqrt{D} x_u &= -(B-s) x_v + (A+t) y_v. \end{aligned}$$

After the change of coordinates $(u,v) \mapsto (x,y)$, these expressions yield

$$\begin{aligned} u_x &= \frac{(C+r) v_y + (B-s) v_x}{\sqrt{D}}, \quad & v_x &= \frac{-(C+r) u_y - (B-s) u_x}{\sqrt{D}}, \\ u_y &= \frac{-(B-s) v_y - (A+t) v_x}{\sqrt{D}}, \quad & v_y &= \frac{(B-s) u_y + (A+t) u_x}{\sqrt{D}}. \end{aligned} \tag{5.18}$$

We deduce then from the second and fourth equations in (5.18) that the system

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \mathfrak{M}_1 \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{5.19}$$

holds, where

$$\mathfrak{M}_1 = \frac{1}{\sqrt{D}} \begin{pmatrix} -(B-s) & -(C+r) \\ A+t & B-s \end{pmatrix}.$$

Similarly, from the first and the third equations in (5.18) we get

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \mathfrak{M}_2 \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \tag{5.20}$$

where

$$\mathfrak{M}_2 = \frac{1}{\sqrt{D}} \begin{pmatrix} B-s & C+r \\ -(A+t) & -(B-s) \end{pmatrix}.$$

Clearly, $\mathfrak{M}_1$ is proportional to $\mathfrak{M}_2^{-1}$, i.e. $\mathfrak{M}_1\mathfrak{M}_2 = \lambda(x,y)\mathrm{Id}$ for some function λ. Hence, from (5.19) and (5.20) we obtain $\lambda = 1$, i.e. $\mathfrak{M}_1\mathfrak{M}_2 = \mathrm{Id}$, and so

$$(A+t)(C+r) - (B-s)^2 = D.$$

That is, $z(x,y)$ is a solution to (5.1), as we wanted to show. This completes the proof of item 3.

To prove item 4, assume that $\gamma(u) = (\alpha(u), \beta(u))$ is also regular, negatively oriented and strictly convex, i.e. $\alpha''(u)\beta'(u) - \beta''(u)\alpha'(u) > 0$ for every u. If we let ω be the function in (5.17), then $\omega(u,0) = 0$ for every u, and a computation from (5.8) yields

$$\begin{aligned} \omega_v(u,0) &= (x_{uv}y_v - x_v y_{uv})(u,0) \\ &= \left(\left(-\frac{\beta'}{\sqrt{D}} \right)_u \frac{\alpha'}{\sqrt{D}} + \frac{\beta'}{\sqrt{D}} \left(\frac{\alpha'}{\sqrt{D}} \right)_u \right)(u,0) \\ &= \frac{-1}{D^{3/2}} \Big((\beta''\sqrt{D} - \beta'(\sqrt{D})_u)\alpha' - \beta'(\alpha''\sqrt{D} - \alpha'(\sqrt{D})_u) \Big)(u,0) \\ &= \frac{1}{D}(\beta'(u)\alpha''(u) - \beta''(u)\alpha'(u)) > 0. \end{aligned} \tag{5.21}$$

Consequently, since ω is 2π-periodic, there is some $r > 0$ such that $\omega > 0$ on Γ_r. In particular, the map $\psi(u,v) : \widehat{\Gamma_r} \longrightarrow \mathbb{R}^3$ given by (5.15) satisfies:

1. The projection $(x(u,v), y(u,v)) : \Gamma_r \longrightarrow \mathbb{R}^2$ is a local diffeomorphism.

2. $\psi(u,0)=0$ for every $u\in\mathbb{R}$.
3. The upwards-pointing unit normal $N:\Gamma_r \longrightarrow \mathbb{S}^2_+$ to ψ restricted to Γ_r extends analytically to $\widehat{\Gamma}_r$, and (5.16) holds.

We need to prove now that for $r>0$ small enough, $\psi(\Gamma_r)$ is a graph of a function $z=z(x,y)$ over a punctured disc $\Omega\subset\mathbb{R}^2$. For this purpose, we first prove the following lemma.

Lemma 5.3. *Let $z\in C^2(\Omega)$ be a bounded elliptic solution to* (5.1)*, where Ω is an open domain of $\mathbb{R}^2$ (not necessarily a punctured disc). Then, for sufficiently large constants $a,c>0$, the function*

$$z^*(x,y)=z(x,y)+\frac{a}{2}x^2+\frac{c}{2}y^2 \tag{5.22}$$

satisfies that its graph $\{(x,y,z^*(x,y)):(x,y)\in\Omega\}$ is a locally convex surface in $\mathbb{R}^3$.

Proof. As the coefficients $A,\dots,E$ are bounded on $\mathfrak{h}$, we can find constants $a,c>0$ such that

$$c-C>0,\qquad a-A>0,\qquad (c-C)(a-A)-B^2>0,$$

i.e. so that the matrix

$$\mathcal{N}=\begin{pmatrix} c-C & B\\ B & a-A\end{pmatrix} \tag{5.23}$$

is positive definite. Now, we may use the fact that ds^2 in (5.3) is positive definite to conclude that the symmetric bilinear form

$$ds^2+(dx,dy)\mathcal{N}(dx,dy)^T=(dx,dy)\begin{pmatrix} r+c & s\\ s & t+a\end{pmatrix}(dx,dy)^T$$

is also positive definite, that is, it is a Riemannian metric on Ω.

On the other hand, a straightforward computation shows that the matrix of the second fundamental form of the graph of $z^*(x,y)$ in (5.22) is given by

$$II^*=\frac{1}{\sqrt{1+(p+cx)^2+(q+ay)^2}}\begin{pmatrix} r+c & s\\ s & t+a\end{pmatrix},$$

which we have just proved is positive definite on Ω. In particular, the graph of $z^*(x,y)$ has positive curvature at every point, which proves the assertion. $\square$

Consider next the new map $\psi^* : \widehat{\Gamma}_r \longrightarrow \mathbb{R}^3$ given by

$$\psi^*(u,v) = \left(x(u,v), y(u,v), z(u,v) + \frac{a}{2} x(u,v)^2 + \frac{c}{2} y(u,v)^2 \right),$$

where $a, c \in \mathbb{R}$ are as in Lemma 5.3. Clearly, $\psi^*(u,0) = 0$ for every $u \in \mathbb{R}$. Also, it follows from the proof of item 3 and Lemma 5.3 that ψ^* restricted to Γ_r is a regular, strictly convex surface in $\mathbb{R}^3$ whose projection to $\mathbb{R}^2$ is a local diffeomorphism. Moreover, a direct computation shows that the unit normal to ψ^* in Γ_r is

$$N^*(u,v) = \frac{1}{\sqrt{1 + (p+cx)^2 + (q+ay)^2}} (-p - cx, -q - ay, 1),$$

where x, y, p, q are evaluated at (u, v). We remark that

$$N^*(u,0) = N(u,0) = \frac{(-\alpha(u), -\beta(u), 1)}{\sqrt{1 + \alpha(u)^2 + \beta(u)^2}}.$$

Consider now the *Legendre transform* of $\psi^*(u,v)$, given by (see [LSZ, pag. 89])

$$\mathcal{L}(u,v) = \left(-\frac{N_1^*}{N_3^*}, -\frac{N_2^*}{N_3^*}, -x\frac{N_1^*}{N_3^*} - y\frac{N_2^*}{N_3^*} - z^* \right) : \Gamma_r \longrightarrow \mathbb{R}^3,$$

where we are denoting $N^* = (N_1^*, N_2^*, N_3^*)$ and $\psi^* = (x, y, z^*)$. It is well known that, since ψ^* is a regular, locally convex surface in $\mathbb{R}^3$ whose projection to the (x, y)-plane is a local diffeomorphism, then so is $\mathcal{L}$. Its upwards-pointing unit normal is

$$\mathcal{N}_{\mathcal{L}} = \frac{(-x, -y, 1)}{\sqrt{1 + x^2 + y^2}} : \Gamma_r \longrightarrow \mathbb{S}^2_+, \tag{5.24}$$

where x, y are evaluated at (u, v). In particular $\mathcal{L}(u,0) = (\alpha(u), \beta(u), 0)$ is a regular, strictly convex Jordan curve in $\mathbb{R}^2$ and $\mathcal{N}_{\mathcal{L}}(u,0) = (0,0,1)$. Therefore, for $r > 0$ small enough, $\mathcal{L}(\Gamma_r)$ lies on the upper half-space of $\mathbb{R}^3$. Moreover, the intersection of $\mathcal{L}(\Gamma_r)$ with each plane $z = \varepsilon$ for ε small enough is a regular strictly convex Jordan curve in that plane. In particular, the piece of the surface $\mathcal{L}(\Gamma_r)$ lying between two of those parallel planes associated to $0 < \varepsilon_1 < \varepsilon_2$ is strictly convex, and bounded by two regular strictly convex Jordan curves, one on each plane. In these conditions, the unit normal of this piece of $\mathcal{L}$ defines a global diffeomorphism onto some annular domain of $\mathbb{S}^2_+$. Letting $\varepsilon_2 \to 0$ we conclude that there exists some $r > 0$ small enough such that the unit normal (5.24) to $\mathcal{L}$ restricted to Γ_r is a diffeomorphism onto a domain of $\mathbb{S}^2$. But now, in the view of the expression (5.24), this means that the map $(x(u,v), y(u,v))$ restricted to this domain Γ_r is a global diffeomorphism onto its image. Thus, both $\psi(\Gamma_r)$ and $\psi^*(\Gamma_r)$ are graphs of functions $z(x,y)$ and $z^*(x,y)$ over a punctured disc $\Omega \subset \mathbb{R}^2$.

Observe that by item 3, the function $z(x,y)$ is a non-trivial bounded elliptic solution to (5.1). Moreover, it is clear from the construction process we have followed that its limit gradient at the singularity is the curve γ we started with. This concludes the proof of item 4 and so the proof of Theorem 5.1.

5.3 Uniqueness of solutions: Proof of Theorem 5.2

Along this section, $z(x,y)$ will denote a non-trivial bounded elliptic solution to (5.1) on a punctured disc Ω, with real analytic coefficients $A, \dots, E$. Moreover, we will suppose that HeB-condition holds which by Lemma 5.2 shows that the conformal structure induced by the metric ds^2 in (5.4) is that of an annulus. That is, we can parametrize the graph $G \subset \mathbb{R}^3$ of z as (5.11) for some $r > 0$, so that (u,v) are conformal parameters in Γ_r for the metric ds^2 in (5.3). Observe that ψ extends to $\mathbb{R}$ as $\psi(u,0) = (0,0,0)$ for all u.

If we also parametrize $p = z_x$, $q = z_y$ in terms of u, v, then

$$\mathbf{z}(u,v) := (x(u,v), y(u,v), z(u,v), p(u,v), q(u,v)) : \Gamma_r \longrightarrow \mathbb{R}^5$$

is a solution to system (5.9). The following proposition provides a boundary regularity result for $\mathbf{z}(u,v)$:

Proposition 5.1. *In the above conditions, $\mathbf{z}(u,v)$ extends as a real analytic map to $\Gamma_r \cup \mathbb{R}$.*

Proof. The first part of the proof follows a bootstrapping method. Consider an arbitrary point of $\mathbb{R}$, which we will suppose without loss of generality to be the origin. Also, consider for $0<\delta<r$ the domain $\mathbb{D}^+=\{(u,v):0<u^2+v^2<\delta^2\}\cap\Gamma_r$.

From (5.8) it follows that (cf. [HeB])

$$\begin{aligned}\Delta x &= h_1(x_u^2+x_v^2)+h_2(x_uy_u+x_vy_v)+h_3(y_u^2+y_v^2)+h_4(x_uy_v-x_vy_u)\\ \Delta y &= \widetilde{h}_1(x_u^2+x_v^2)+\widetilde{h}_2(x_uy_u+x_vy_v)+\widetilde{h}_3(y_u^2+y_v^2)+\widetilde{h}_4(x_uy_v-x_vy_u)\end{aligned} \tag{5.25}$$

where the coefficients $h_1=h_1(x,y,z,p,q),\dots,\widetilde{h}_4=\widetilde{h}_4(x,y,z,p,q)$ are

$$\begin{aligned}h_1 &= B_q-\tfrac{1}{2D}(D_x+D_zp-D_pC+D_qB),\\ h_2 &= -A_q-B_p-\tfrac{1}{2D}(D_y+D_zq+D_pB-D_qA),\\ h_3 &= A_p,\\ h_4 &= \tfrac{1}{\sqrt{D}}(A_x+B_y+A_zp+B_zq-A_pC+(A_q+B_p)B-B_qA-\tfrac{1}{2}D_p),\\ \widetilde{h}_1 &= C_q,\\ \widetilde{h}_2 &= -B_q-C_p-\tfrac{1}{2D}(D_x+D_zp-D_pC+D_qB),\\ \widetilde{h}_3 &= B_p-\tfrac{1}{2D}(D_y+D_zq+D_pB-D_qA),\\ \widetilde{h}_4 &= \tfrac{1}{\sqrt{D}}(C_y+B_x+C_zq+B_zp-B_pC+(B_q+C_p)B-C_qA-\tfrac{1}{2}D_q),\end{aligned}$$

all of them evaluated at $\mathbf{z}(u,v)$. In particular we have that

$$\widetilde{h}_1=C_q,\qquad h_1-\widetilde{h}_2=C_p+2B_q,\qquad h_2-\widetilde{h}_3=-A_q-2B_p,\qquad h_3=A_p. \tag{5.26}$$

On the other hand, observe that the inequalities

$$(x_u-y_v)^2+(x_v+y_u)^2\ge 0,\qquad (x_u-y_u)^2+(x_v-y_v)^2\ge 0$$

lead, respectively, to $x_uy_v-x_vy_u\le\frac{1}{2}(|\nabla x|^2+|\nabla y|^2)$ and $x_uy_u+x_vy_v\le\frac{1}{2}(|\nabla x|^2+|\nabla y|^2)$.

Hence, if we denote $Y=(x,y):\mathbb{D}^+\longrightarrow\Omega$, formula (5.25) and the fact that $h_1,\dots,\widetilde{h}_4$ are bounded yield

$$|\Delta Y|\le c(|\nabla x|^2+|\nabla y|^2) \tag{5.27}$$

for a certain constant $c>0$.

Observe that $Y\in C^2(\mathbb{D}^+)\cap C^0(\overline{\mathbb{D}^+})$ with $Y(u,0)=(0,0)$ for all u. Hence, we can apply Heinz's Theorem in [He] to deduce that $Y\in C^{1,\alpha}(\overline{\mathbb{D}^+_\varepsilon})$ for all $\alpha\in(0,1)$, where $\mathbb{D}^+_\varepsilon=\mathbb{D}^+\cap B(0,\varepsilon)$ for a certain $0<\varepsilon<\delta$.

Now, the right hand side terms in (5.8) are bounded in $\overline{\mathbb{D}^+_\varepsilon}$ and so $p,q\in W^{1,\infty}(\overline{\mathbb{D}^+_\varepsilon})$. Hence $p,q\in C^{0,1}(\overline{\mathbb{D}^+_\varepsilon})$ (cf. [GiTr, pag. 154]). In particular, p and q are Hölder continuous of any order in $\overline{\mathbb{D}^+_\varepsilon}$.

Taking into account (5.14), we obtain $z \in C^{1,\alpha}(\overline{\mathbb{D}_\varepsilon^+})\ \forall \alpha \in (0,1)$. Then, the right hand side functions in (5.8) are Hölder continuous of any order in $\overline{\mathbb{D}_\varepsilon^+}$. That is, $p, q \in C^{1,\alpha}(\overline{\mathbb{D}_\varepsilon^+})\ \forall \alpha \in (0,1)$.

With this, we have from (5.25) that ΔY is Hölder continuous in $\overline{\mathbb{D}_\varepsilon^+}$. Then, a standard potential analysis argument (cf. [GiTr, Lemma 4.10]) ensures that $x, y \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$. Again, by formula (5.8) we have that $p, q \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$ and so, from (5.14) that $z \in C^{2,\alpha}(\overline{\mathbb{D}_{\varepsilon/2}^+})$.

At this point, we may apply the same argument to Y_u and Y_v, in order to obtain that $x, y, z, p, q \in C^{3,\alpha}(\overline{\mathbb{D}_{\varepsilon/4}^+})$. A recursive process leads to the fact that $\mathbf{z} = (x, y, z, p, q)$ is C^∞ at the origin. As we can do the same argument for all points of $\mathbb{R}$ and not just the origin, we conclude that $\mathbf{z}(u, v) \in C^\infty(\Gamma_r \cup \mathbb{R})$.

Now, a computation in the same spirit of formula (5.25) shows that the Laplacians of z, p, q are given by:

$$
\begin{aligned}
\Delta p &= (\sqrt{D \circ \mathbf{z}})_u y_v - (\sqrt{D \circ \mathbf{z}})_v y_u + (B \circ \mathbf{z})_u y_u + (B \circ \mathbf{z})_v y_v \\
&\quad + (B \circ \mathbf{z})\Delta y - (C \circ \mathbf{z})\Delta x - (C \circ \mathbf{z})_u x_u - (C \circ \mathbf{z})_v x_v, \\[2ex]
\Delta q &= -(\sqrt{D \circ \mathbf{z}})_u x_v + (\sqrt{D \circ \mathbf{z}})_v x_u + (B \circ \mathbf{z})_u x_u + (B \circ \mathbf{z})_v x_v \\
&\quad + (B \circ \mathbf{z})\Delta x - (A \circ \mathbf{z})\Delta y - (A \circ \mathbf{z}_u y_u - (A \circ \mathbf{z})_v y_v, \\[2ex]
\Delta z &= p_u x_u + p_v x_v + q_u y_u + q_v y_v + p\Delta x + q\Delta y,
\end{aligned}
\tag{5.28}
$$

where all quantities are evaluated at $(u, v) \in \Gamma_r$. Therefore, as A, B, C, E are analytic (note that the analyticity of the coefficients had not been used in this section up to now), $\mathbf{z}(u, v)$ satisfies

$$
\Delta \mathbf{z} = h(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v) \tag{5.29}
$$

where $h : \mathcal{O} \subset \mathbb{R}^{15} \longrightarrow \mathbb{R}^5$ is a real analytic function on an open set $\mathcal{O}$ of $\mathbb{R}^{15}$ containing the closure of the bounded set $\{(\mathbf{z}, \mathbf{z}_u, \mathbf{z}_v)(u, v) : (u, v) \in \Gamma_r\}$. Moreover, if we write

$$
\mathbf{z}(u, v) = (\psi(u, v), \phi(u, v)) : \Gamma_r \longrightarrow \mathbb{R}^3 \times \mathbb{R}^2 \equiv \mathbb{R}^5
$$

where $\psi(u, v)$ is given by (5.11) and $\phi(u, v) = (p(u, v), q(u, v))$, then we see that $\mathbf{z}(u, v)$ is a solution to (5.29) that meets the mixed initial conditions

$$
\begin{cases}
\psi(u, 0) = (0, 0, 0), \\[1ex]
\phi_v(u, 0) = \begin{pmatrix} -C(0,0,0,\phi(u,0)) & B(0,0,0,\phi(u,0)) & 0 \\ B(0,0,0,\phi(u,0)) & -A(0,0,0,\phi(u,0)) & 0 \end{pmatrix} \psi_v(u, 0)^T.
\end{cases}
$$

As $\mathbf{z} \in C^\infty(\Gamma_r \cup \mathbb{R})$, we are in the conditions to apply Theorem 3 in [Mu2] to $\mathbf{z}$ around every point in $\mathbb{R}$. Thus, we conclude that $\mathbf{z}$ is real analytic in $\Gamma_r \cup \mathbb{R}$, which concludes the proof of Proposition 5.1. $\square$

It follows from Proposition 5.1 that the functions $p(u,v)$ and $q(u,v)$ extend analytically to $\Gamma_r \cup \mathbb{R}$, so that $(\alpha(u), \beta(u)) := (p(u,0), q(u,0))$ is a real analytic, 2π-periodic map. Let now $\gamma \subset \mathbb{R}^2$ denote the limit gradient of $z(x,y)$. Then, clearly $\gamma = \{(\alpha(u), \beta(u)) : u \in \mathbb{R}\}$, and so we get that γ is a closed curve in $\mathbb{R}^2$, possibly with singularities, that can be parameterized as a 2π-periodic function as $\gamma(u) = (\alpha(u), \beta(u))$ in terms of the conformal parameters (u,v) associated to the solution $z(x,y)$.

In Theorem 5.1 we proved that a sufficient condition for the possibly multivalued solution to (5.1) constructed there in terms of $\gamma(u)$ to be actually an bounded elliptic solution to (5.1) on Ω is that $\gamma(u)$ is regular, negatively oriented and strictly convex. We show next that under the hypothesis that the HeB-condition holds these conditions on $\gamma(u)$ are also necessary. The following proposition completes the proof of items 1 and 2 of Theorem 5.2.

Proposition 5.2. *Let $\gamma \subset \mathbb{R}^2$ be the limit gradient of $z(x,y)$, and let $\gamma(u) : \mathbb{R}/(2\pi\mathbb{Z}) \longrightarrow \mathbb{R}^2$ be its parametrization in terms of the conformal coordinates $(u,v) \in \Gamma_r \cup \mathbb{R}$. Then $\langle \gamma''(u), J\gamma'(u) \rangle < 0$ for all $u \in \mathbb{R}$. That is, $\gamma(u)$ is a regular analytic negatively oriented Jordan curve which, in particular, is strictly convex and bounds a convex set in $\mathbb{R}^2$.*

Proof. That $\gamma(u)$ is real analytic was proved in Proposition 5.1.

Consider next the analytic map ω defined on $\Gamma_r \cup \mathbb{R}$ by (5.17), which vanishes on $\mathbb{R}$ and is positive on Γ_r.

We can use (5.25) to compute the Laplacian of ω in (5.17) at points in the real axis. We get

$$
\begin{aligned}
\Delta\omega(u,0)=\omega_{vv}(u,0)= \;& (x_{uvv}y_v - x_v y_{uvv} + 2(x_{uv}y_{vv} - x_{vv}y_{uv}))(u,0)\\
=\;& [(h_1)_u x_v^2 y_v + 2h_1 x_v y_v x_{vu}\\
& +(h_2)_u x_v y_v^2 + h_2 x_{vu} y_v^2 + h_2 x_v y_v y_{vu}\\
& +(h_3)_u y_v^3 + 2h_3 y_v^2 y_{vu}\\
& -(\widetilde{h}_1)_u x_v^3 - 2\widetilde{h}_1 x_v^2 x_{vu}\\
& -(\widetilde{h}_2)_u x_v^2 y_v - \widetilde{h}_2 x_{vu} y_v x_v - \widetilde{h}_2 x_v^2 y_{vu}\\
& -(\widetilde{h}_3)_u y_v^2 x_v - 2\widetilde{h}_3 y_v x_v y_{vu}\\
& +2x_{uv}(\widetilde{h}_1 x_v^2 + \widetilde{h}_2 x_v y_v + \widetilde{h}_3 y_v^2)\\
& -2y_{uv}(h_1 x_v^2 + h_2 x_v y_v + h_3 y_v^2)](u,0)\\
=\;& [(h_1-\widetilde{h}_2)_u x_v^2 y_v + (h_2-\widetilde{h}_3)_u y_v^2 x_v + (h_3)_u y_v^3 - (\widetilde{h}_1)_u x_v^3\\
& +(x_v(\widetilde{h}_2+2h_1) + y_v(h_2+2\widetilde{h}_3))\omega_v](u,0).
\end{aligned}
\tag{5.30}
$$

Observe that since HeB-condition holds, the functions in (5.26) are constant along the real axis. With this, and using that x_v and y_v are bounded from above along the real axis by (5.8), we deduce from (5.30) that there exists a constant $c>0$ such that

$$\Delta\omega \leq c\,\omega_v \tag{5.31}$$

for all $(u,0)\in\mathbb{R}^2$. Therefore, (5.31) holds in some Γ_ρ with $r>\rho>0$. As $\omega(u,0)=0$ and $\omega(u,v)>0$ in Γ_r, Hopf's Lemma (see [GiTr, Lemma 3.4]) implies that $\omega_v(u,0)>0$. Using now (5.21) we get that $\langle\gamma''(u), J\gamma'(u)\rangle<0$ at every point. Hence, $\gamma'(u)\neq 0$ for all u and the curve is negatively oriented and strictly convex. Thus, the proof of Proposition 5.2 is finished. □

Now we are in conditions to finish the proof of Theorem 5.2. Observe that the map $\mathbf{z}(u,v)$ can be recovered in terms of a real analytic, 2π-periodic curve $\gamma(u)=(\alpha(u),\beta(u))$ as the unique solution to the Cauchy problem for the system (5.9) with the initial condition

$$\mathbf{z}(u,0)=(0,0,0,\alpha(u),\beta(u)). \tag{5.32}$$

In other words, any non-trivial analytic bounded elliptic solution to (5.1) on a punctured disc Ω, whenever the HeB-condition is satisfied, can be recovered through the process described in Theorem 5.1 to construct solutions to (5.1) with an isolated singularity at the origin. This shows that item 3 in Theorem 5.2 holds.

Also, item 4 is a direct consequence of the previous facts and the uniqueness of the solution to the Cauchy problem for system (5.9).

Remark 5.2. *Observe that the parameters* $(u, v) \in \Gamma_r$ *associated to the solution* z *of* (5.1) *are defined up to* 2π*-periodic conformal changes of* Γ_r *that simply yield a reparametrization of the limit gradient* γ*. Hence, the graph constructed in Section 5.2 is uniquely determined by the initial data* γ*, independently of its parametrization.*

This finishes the proof of Theorem 5.2.

5.4 Further results and comments

We devote this section to explain in detail how Theorem 5.1 and Theorem 5.2 together with Lemma 5.1 yield a classification result in terms of Jordan curves for bounded elliptic non-trivial solutions to (5.1) in the spirit of [GHM]. We will also explain which of our results hold in the non-analytic case.

Firstly, it is worth to point out some remarks about the approach we have explained in the previous sections. In what follows, we will say equivalently that a solution to (5.1) z is non-trivial (bounded elliptic) or that it has a non-trivial (bounded elliptic) singularity.

Remark 5.3. *By the process we explained in the proof of Theorem 5.1 we can construct possibly multivalued solutions to* (5.1) *from a closed analytic curve. Observe that we can only guarantee that those solutions provide actually a graph over the* (x, y)*-plane under the assumption that the curve is regular, negatively oriented and strictly convex.*

Remark 5.4. *All those solutions, multivalued or not, have associated the conformal structure of an annulus even without imposing the HeB-condition. That is, given a non-trivial bounded elliptic solution to* (5.1)*, the HeB-condition is sufficient to determine that the conformal structure defined by the parameters* (u, v) *is that of an annulus (see Lemma 5.2), but it is not necessary. It is then an open problem to obtain solutions of* (5.1) *with a bounded non-trivial singularity and the conformal structure of a punctured disc for which, by Lemma 5.2, the HeB-condition does not hold. Moreover, it would be also interesting to find a less restrictive hypothesis than the HeB-condition that characterize the conformal structure for isolated singularities of* (5.1).

5.4.1 A classification result

It is important to point out that the HeB-condition, in general, not only depends on the coefficients $A, \ldots, E$ but also may depend on the solution z to (5.1). In this sense, restricting ourselves to the case when the coefficients $A, \ldots, E$ satisfy the condition $(\star)$ provides a simpler way to obtain a characterization of non-trivial bounded elliptic singularities in terms of Jordan curves in the spirit of [GHM].

Observe that if we were under the hypothesis $(\star)$ in the construction process of Section 5.2 we would not need any extra assumption for the initial curve γ more than γ be a regular analytic and strictly convex Jordan curve satisfying (5.12) in order to recover a solution to (5.1) for the coefficients $A, \ldots, E$.

Nevertheless, we still need to be more precise prescribing the domain $\mathcal{U}$ where the coefficients $A, \ldots, E$ are defined to guarantee that our desired classification result makes sense.

Consider any set of analytic functions $A, \ldots, E : \mathcal{U} \subset \mathbb{R}^5 \longrightarrow \mathbb{R}$ defined in an open domain $\mathcal{U} \subset \mathbb{R}^5$ such that $H := \mathcal{U} \cap \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : (x_1, x_2, x_3) = (0, 0, 0)\} \neq \emptyset$ and such that the ellipticity condition (5.2) and the property $(\star)$ hold.

Then, we denote by $\mathcal{A}_1$ the class of solutions $z = z(x, y)$ to (5.1) for the coefficients $A, \ldots, E$ as above such that, the correspondent solutions z are defined in an punctured neighborhood of the origin Ω and have a non-trivial bounded elliptic singularity at the origin with $z(0, 0) = 0$. Moreover, by restricting Ω if necessary, we consider as equal two solutions that coincide in a neighborhood of the origin.

On the other hand, we denote as $\mathcal{A}_2$ the class of regular analytic strictly convex Jordan curves γ in H such that they satisfy the condition (5.12).

With all of this, it is easy to realize that Lemma 5.1, Theorem 5.1 and Theorem 5.2 yield the following classification result.

Theorem 5.3. *The map $z(x, y) \mapsto (p(x, y), q(x, y))$ gives an explicit one-to-one correspondence between the classes $\mathcal{A}_1$ and $\mathcal{A}_2$.*

5.4.2
The non-analytic case

Along the chapter we have supposed that the coefficients $A, \ldots, E$ in (5.1) are real analytic and so is the solution z. Now we study what results can be still obtained and which of them fail if we suppress such a hypothesis.

- First, we observe that if we do not suppose that the coefficients $A, \ldots, E$ are analytic, the results about the conformal structure around a non-removable singularity still remain true. Specifically, in [HeB] the authors work just under the hypotheses in Definition 5.1 and the HeB-condition and they obtain the conformal parameters (u, v) given by (5.7) and the characterization of the non-removable singularities given by Lemma 5.2.

- The bootstrapping method in the proof of Theorem 5.2 is also valid except for the final part when we need the analyticity hypothesis to deduce that $\mathbf{z} = (x, y, z, p, q)$ extends analytically to $\mathbb{R}$. Specifically, by the same process, just assuming that the coefficients $A, \ldots, E$ are of class $C^{1,\mu}(\mathcal{U})$ for some $\mu \in (0, 1)$ and that the HeB-condition holds we can obtain that $(x, y, z) \in C^{3,\mu}(\Gamma_r \cup \mathbb{R})$ and $(p, q) \in C^{2,\mu}(\Gamma_r \cup \mathbb{R})$. Furthermore, we do not need that the coefficients of the equation are analytic to deduce that the limit tangent of z at the singularity $\gamma(u) = (p(u), q(u))$ becomes a regular strictly convex Jordan curve in $\mathbb{R}^2$ (now of class $C^{2,\mu}$). This fact is a direct consequence of the condition (5.21) which still can be deduced from (5.30) and Hopf Lemma. Observe that now, the HeB-condition makes the functions (5.26) be continuous along the real axis and so formula (5.30) makes sense.

- Without the analyticity assumption, given a non-trivial bounded elliptic solution z to (5.1), we still have that system (5.9) holds since it is deduced just from (5.1) and (5.7).

- On the contrary, given a regular non-analytic strictly convex Jordan curve γ we can not assure in general the existence of solutions to (5.9) for the initial data $(0, 0, 0, \gamma(u))$ since we are not in the conditions of Cauchy-Kowalevsky Theorem. Hence, we can not assert that Theorem 5.1 holds in this case.

5.5 Geometric applications

Next we will see how our theorems can be applied to obtain results in several geometric situations explained in Chapter 4 where a Monge-Ampère equation of type (5.1) appears.

1. Remind that the problem of prescribing positive Gaussian curvature for graphs in $\mathbb{R}^3$ over a planar domain was modeled by the equation (4.18) which corresponds to (5.1) for the case $A = B = C = 0$. Hence, in this case the condition $(\star)$ is trivially satisfied. Moreover, observe that if the function $K = K(x, y, z, p, q)$ in (4.18) is positive and analytic then the ellipticity condition (5.2) holds independently of the solution that we consider. So any solution to (4.18) with a non-trivial singularity at a certain point of the domain is in fact a non-trivial bounded elliptic solution. In this situation, we may apply Theorem 5.3 to classify such singular solutions. In particular, when $K = K(x, y)$ we obtain the following classification result.

   Theorem 5.4. *Let $K(x, y)$ be an analytic positive function defined in a planar domain $\mathcal{O} \subset \mathbb{R}^2$, and consider a point $p_0 \in \mathcal{O}$. Then, the set of graphs of curvature $K(x, y)$ over a punctured neighborhood of p_0 in $\mathcal{O}$ whose height function has a non-trivial singularity at p_0 is in one-to-one correspondence with the class of regular real analytic strictly convex Jordan curves in $\mathbb{R}^2$.*

   Theorem 5.4 generalizes the results for graphs in [GHM] where the authors obtain the same type of classification result for the particular case of constant positive Gaussian curvature.

2. Moreover, the same approach extends to the case of graphs of prescribed curvature $K = K(x, y, z, p, q) > -1$ in the hyperbolic space $\mathbb{H}^3$, or graphs with $K = K(x, y, z, p, q) > 1$ in $\mathbb{S}^3$.

   When we consider the Klein model of $\mathbb{H}^3$ and the base surface is totally geodesic we are lead to the elliptic Monge-Ampère equation (4.19) which is of the same type as (5.1) when $A = B = C = 0$. On the other hand, graphs over a horosphere in the upper-half space model of $\mathbb{H}^3$ have (4.20) as its prescribed curvature equation. In this last case, the associated coefficients in (5.1) are

$$A = 2q^2 + e^{-2z}, \qquad B = -2pq, \qquad C = 2p^2 + e^{-2z},$$

$$E = (K+1)e^{-4z}(1 + e^{2z}(p^2+q^2))^2 - e^{-4z}(1 + 2e^{2z}(p^2+q^2)).$$

Something similar happens when we are given a graph with $K = K(x,y,z,p,q) > 1$ over a totally geodesic surface in $\mathbb{S}^3$ as explained in Section 4.4. Now we are lead to (4.21) which is an elliptic Monge-Ampère equation of type (5.1) with $A = B = C = 0$. Hence, in these three cases the condition $(\star)$ is trivially satisfied since the involved functions vanish identically.

As happens in the Euclidean case, if we start with an analytic curvature function $K = K(x,y,z,p,q)$ in the case of $\mathbb{H}^3$ (respectively of $\mathbb{S}^3$) such that $K = K(x,y,z,p,q) > -1$ (respectively $K = K(x,y,z,p,q) > 1$) then the ellipticity condition (5.2) holds and any solution to (4.19) or (4.20) (respectively of (4.21)) with a non-trivial singularity at a certain point of the domain is in fact a non-trivial bounded elliptic solution. Thus, Theorem 5.3 yields a characterization of graphs of prescribed curvature K under the above conditions over domains in the correspondent surfaces in $\mathbb{H}^3$ or $\mathbb{S}^3$. For the particular case when $K = K(x,y)$ the following classification results hold.

Theorem 5.5. *Let $K(x,y) > -1$ be an analytic function defined in a domain $\mathcal{O}$ of a totally geodesic surface in the Klein model of $\mathbb{H}^3$, or in an horosphere in the upper-space model, and consider a point $p_0 \in \mathcal{O}$. Then, the set of graphs of curvature $K(x,y)$ over a punctured neighborhood of p_0 in $\mathcal{O}$ whose height function (in the sense that it is considered in Section 4.4) has a non-trivial singularity at p_0 is in one-to-one correspondence with the class of regular real analytic strictly convex Jordan curves in $\mathbb{R}^2$.*

Theorem 5.6. *Let $K(x,y) > 1$ be an analytic function defined in a domain $\mathcal{O}$ of a totally geodesic surface in $\mathbb{S}^3$, and consider a point $p_0 \in \mathcal{O}$. Then, the set of graphs of curvature $K(x,y)$ over a punctured neighborhood of p_0 in $\mathcal{O}$ whose height function (in the sense it is considered in Section 4.4) has a non-trivial singularity at p_0 is in one-to-one correspondence with the class of regular real analytic strictly convex Jordan curves in $\mathbb{R}^2$.*

3. When we consider a spacelike graph over a planar domain in $\mathbb{L}^3$ with prescribed negative curvature $K = K(x, y, z, p, q) < 0$, we obtain the elliptic equation (4.24). Again, we deduce that the condition $(\star)$ is trivially satisfied.

   On the other hand, observe that if we impose that K is a negative analytic function then the ellipticity condition (5.2) will be guaranteed by the the spacelike character of the immersion. Hence, by virtue of Remark 5.1, the ellipticity condition (5.12) for the limit tangent $\gamma(u)$ at a non-trivial bounded elliptic singularity will correspond to the fact that $|\gamma(u)| < 1$ for all $u \in \mathbb{R}$.

   Moreover, if z is a solution to (4.24) in Ω with a non-trivial elliptic singularity at $p_0 \in \Omega$, and we choose $\varepsilon > 0$ small enough, then it holds that the maximum c_0 of $p^2 + q^2$ in $\partial B(p_0, \varepsilon) \subset \Omega$ is less than 1. And we deduce by convexity that such a maximum is less or equal than c_0 in $\overline{B(p_0, \varepsilon)} \setminus \{p_0\} \subset \Omega$. In particular, the ellipticity condition (5.2) will also hold at the singularity and it will be in fact bounded elliptic in the sense of Definition 5.1.

   With all of this we can deduce the following characterization from Theorem 5.3.

   Theorem 5.7. *Let $K(x, y)$ be a negative analytic function defined in a domain $\mathcal{O}$ of a spacelike plane in $\mathbb{L}^3$, and consider $p_0 \in \mathcal{O}$. Then, the set of spacelike graphs of curvature $K(x, y)$ over a punctured neighborhood of p_0 in $\mathcal{O}$ such that its height function has a non-trivial elliptic singularity at the puncture is in one-to-one correspondence with the class of regular real analytic strictly convex Jordan curves in the unit disc $\mathbb{D} \subset \mathbb{R}^2$.*

4. Our results have also a link with surface theory when we study elliptic linear Weingarten graphs in $\mathbb{R}^3$. The associated Monge-Ampère equation of these graphs is (4.30) which corresponds with (5.1) if we write

$$\begin{aligned} A &= \frac{a}{b}\sqrt{1+p^2+q^2}(1+q^2), \quad B = \frac{a}{b}\sqrt{1+p^2+q^2}(-pq) \\ C &= \frac{a}{b}\sqrt{1+p^2+q^2}(1+p^2), \quad E = \frac{c}{b}(1+p^2+q^2)^2. \end{aligned} \tag{5.33}$$

   In this case, the HeB-condition depends strongly on the solution to (4.30) and so Theorem 5.3 does not provide a direct classification result as in previous items. In fact, by the procedure explained in Section 5.2 we can give a way to construct linear Weingarten graphs with the underlying conformal structure of an annulus in terms of regular analytic strictly convex Jordan curves. However, we must emphasize that there are elliptic linear Weingarten surfaces that may have singularities which can not be recovered by this process as they are not bounded in the sense of Definition 5.1 (see Figure 5.1).

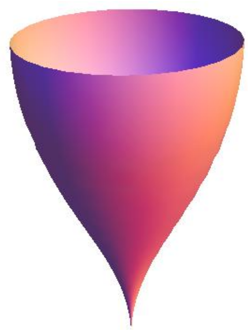


Figure 5.1: *Non-trivial isolated singularity which is non-bounded since its gradient tends to infinity at the singularity.*

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