UNIVERSIDAD DE GRANADA

FACULTAD DE CIENCIAS

Tesis presentada a través del

Departamento de Matemática Aplicada

HARMONIC ANALYSIS AND QUANTUM FIELD THEORY ON THE CONFORMAL GROUP

ANÁLISIS ARMÓNICO Y TERÍA CUÁNTICA DE CAMPOS SOBRE EL GRUPO CONFORME

TESIS DOCTORAL

Presentada por:

EMILIO PÉREZ ROMERO

Febrero de 2012

Editor: Editorial de la Universidad de Granada
Autor: Emilio Pérez Romero
D.L.: GR 2230-2012
ISBN: 978-84-9028-107-9

MANUEL CALIXTO MOLINA, Profesor Titular de Universidad en el Departamento de Matemática Aplicada de la Universidad de Granada.

CERTIFICO:

Que la presente Memoria **"Análisis Armónico y Teoría Cuántica de Campos sobre el Grupo Conforme"** ha sido realizada bajo mi dirección en el Departamento de Matemática Aplicada de la Universidad de Granada por **Emilio Pérez Romero**, y constituye su Tesis para optar al grado de Doctor en Ciencias.

Y para que así conste, en cumplimiento de la legislación vigente, presento ante la Comisión de Doctorado de la Universidad de Granada la referida Tesis.

Granada, a 31 de Enero de 2012.

Fdo. Manuel Calixto Molina.

Fdo. Emilio Pérez Romero

Deseo agradecer a D. Manuel Calixto Molina su entusiasmo, constancia y dedicación aportados para la puesta en marcha, desarrollo y finalización de esta tesis. También quiero agradecer a D. Víctor Aldaya Valverde su colaboración y apoyo incondicional.

Esta tesis se la dedico a todas aquellas personas a las que no les he podido dedicar tanto tiempo como me hubiese gustado. En especial a Silvia, Emilio, Ana, Ignacio, Felisa y Emilio (padre).

Índice general

Parte I

RESUMEN DE LA TESIS

Capítulo 1

Aspectos Generales

1.1. Publicaciones Derivadas de la Tesis e Indicios de Calidad

Publicaciones en Revistas de Impacto (JCR)

1. *AUTORES:* M. Calixto and E. Pérez-Romero
   TÍTULO: Extended MacMahon-Schwinger's Master Theorem and Conformal Wavelets in Complex Minkowski Space
   REVISTA: Applied and Computational Harmonic Analysis **31** (2011) 143-168
   INDICIOS DE CALIDAD:

   - Revista recogida en "JOURNAL CITATION REPORTS"
   - PARÁMETRO DE IMPACTO: 3.211 (2010)
   - RANKING: 1/54 en "Physics, Mathematical" y 3/236 en "Mathematics, Applied"
   - Recogida en "MathSciNet" de "American Mathematical Society" con código MR 2795879 y en "Zentralblatt-MATH" de "European Mathematical Society" con código Zbl pre05907601.

2. *AUTORES:* M. Calixto and E. Pérez-Romero
   TÍTULO: Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in U(2,2) and Born's Reciprocity
   REVISTA: International Journal of Geometric Methods in Modern Physics **8** (2011) 587-619
   INDICIOS DE CALIDAD:

   - Revista recogida en "JOURNAL CITATION REPORTS"

- PARÁMETRO DE IMPACTO: 1.612 (2009)
- RANKING: 18/47 en "Physics, Mathematical"
- Recogida en "MathSciNet" de "American Mathematical Society" con código MR 2807118 y en "Zentralblatt-MATH" de "European Mathematical Society" con código Zbl 1222.81253 .

3. *AUTORES:* M. Calixto, E. Pérez-Romero and V. Aldaya
TÍTULO: Coherent States of Accelerated Relativistic Quantum Particles, Vacuum Radiation and the Spontaneous Breakdown of the Conformal SU(2,2) Symmetry
REVISTA: Journal of Physics (Mathematical & Theoretical) **A45** (2012) 015202 (16pp)
INDICIOS DE CALIDAD:

- Revista recogida en "JOURNAL CITATION REPORTS"
- PARÁMETRO DE IMPACTO: 1.641 (2010)
- RANKING: 17/54 en "Physics, Mathematical" y 23/80 en "Physics, Multidisciplinary"

Proceedings en Congresos

4. *AUTORES:* M. Calixto, E. Pérez-Romero and V. Aldaya
TÍTULO: Group-Theoretical Revision of the Unruh Effect
CONGRESO: "XXVIII International Colloquium on Group-Theoretical Methods in Physics: Physical and Mathematical Aspects of Symmetry". Newcastle, UK. Julio 26-30, 2010
REVISTA: Journal of Physics Conference Series **284** (2011) 012014 9pags. (IOP). ISSN: 1742-6588
INDICIOS DE CALIDAD: Revista arbitrada (revisión por pares). Publicación recogida en "ISI WEB OF KNOWLEDGE".

5. *AUTORES:* M. Calixto, E. Pérez-Romero and V. Aldaya
TÍTULO: Unruh Effect Revisited: Poincaré θ-Vacua as Coherent States of Conformal Zero Modes
CONGRESO: "Spanish Relativity Meeting 2010: Gravity as a Crossroad in Physics". Granada, Spain. Septiembre 6-10, 2010
REVISTA: Journal of Physics Conference Series **314** (2011) 012117 4pags. (IOP). ISSN: 1742-6588
INDICIOS DE CALIDAD: Revista arbitrada (revisión por pares). Publicación recogida en "ISI WEB OF KNOWLEDGE".

6. *AUTORES:* M. Calixto, V. Aldaya, F.F. López, E. Pérez-Romero
TÍTULO: Non-linear Sigma Models: Discretization and Perturbative Solutions
CONGRESO: "NoLineal 2010". Cartagena, España. Junio 8-11, 2010
PUBLICACIÓN: Servicio de Publicaciones Universidad de Murcia, pag. 49. ISBN: 978-84-693-2271-0

7. *AUTORES:* M. Calixto and E. Pérez-Romero
TÍTULO: Extended MacMahon-Schwinger's Master Theorem and Relativistic Orthogonal Polynomials in Complex Minkowski Space
CONGRESO: "Workshop on Generalized Special Functions of Mathematical Physics". Granada, España. Enero 26-27, 2012

1.2. Resumen

Esta tesis se basa fundamentalmente en los resultados obtenidos en las tres primeras publicaciones de la sección anterior. El objetivo fundamental es entender mejor la dinámica cuántica de partículas relativistas aceleradas y los fenómenos de radiación de vacío (efecto Fulling-Unruh-Davies [1, 2, 3]) que subyacen en estos sistemas, análisis que se hace en el tercer artículo. Estos fenómenos de radiación de vacío bajo aceleraciones resultan estar íntimamente relacionados con la radiación de Hawking [4] de agujeros negros y el fenómeno cosmológico de "desplazamiento al rojo" (*redshift* o ley de Hubble) debido a la expansión del universo. Veremos cómo las transformaciones conformes específicas (cuadri-aceleraciones) dan cuenta de ambos fenómenos (parte espacial y parte temporal, respectivamente) de una forma muy especial que no ha sido analizada hasta la fecha. La explicación tradicional de estos fenómenos de radiación radica fundamentalmente en las llamadas transformaciones de Bogoliubov, las cuales encuentran una explicación natural en el marco teórico de los llamados "Estados Coherentes" [5, 6, 7].

Para conseguir nuestros objetivos, hemos tenido que desarrollar primeramente toda una teoría matemática de Estados Coherentes (también llamados *Wavelets* en el contexto del Análisis Armónico Aplicado y Multirresolución) del grupo conforme en 1+3 dimensiones: $SU(2,2)$ (localmente isomorfo a $SO(4,2)$), siendo éste el tema del primer artículo. La teoría tradicional de *Wavelets* ("Ondículas") y el Análisis Multiresolución encuentra múltiples aplicaciones en muchas ramas de la Ciencia y la Ingeniería, a destacar su papel como herramienta indispensable en la compresión de imágenes y tratamiento de la señal. Véase el trabajo pionero de Grossmann, Morlet y Paul [8],

Para nosotros, el vacío cuántico, visto desde un sistema de referencia relativista acelerado, resulta ser un Estado Coherente del grupo conforme, el cual incorpora al grupo de Poincaré de la relatividad especial, junto con cambios de escala y transiciones a sistemas relativistas uniformemente acelerados. El vacío cuántico acelerado se asemeja a una colectividad estadística. Para obtener funciones de distribución, valores medios, etc, ha sido fundamental la demostración de una extensión uniparamétrica λ ("escala conforme o dimensión de masa") del Teorema Maestro de MacMahon (que a su vez generaliza la fórmula de Schwinger), un resultado fundamental en Combinatoria. La demostración de este teorema se encuentra en el primer artículo y la identidad asociada ha resultado ser una herramienta matemática muy útil para nuestros objetivos, proporcionando también una función generatriz de polinomios ortogonales relativistas holomorfos en 4 variables (espacio-temporales) en un dominio del espacio de Minkowski compactificado $\mathbb{C}^4$. Una vez desarrollado el Análisis Armónico sobre el grupo conforme, en el segundo artículo se propone una teoría cuántica de partículas invariantes conforme sobre el espacio-tiempo de Minkowski como una cuantización de un Modelo Sigma No Lineal Invariante Gauge sobre $SU(2,2)$. La cuantización de dicho modelo se lleva a cabo mediante una generalización del método de Dirac para la cuantización de sistemas singulares (invariantes gauge) o con ligaduras. También se propone una adaptación del Principio de Reciprocidad de Born a la relatividad conforme y se discute sobre la existencia o no de una aceleración máxima. Toda esta construcción matemática sirve como pilar sobre el que se asienta nuestra Teoría Cuántica de Campos Relativista Invariante Conforme, marco a su vez para el estudio del análisis cuántico de sistemas de referencia relativistas uniformemente acelerados, estudiado en el tercer y último artículo.

Resumiendo, el orden cronológico de aparición de las publicaciones coincide con el orden lógico de desarrollo de la tesis. Es por ello que consideramos adecuado presentar la tesis como un compendio de artículos, cada uno de ellos autocontenido, pero enlazados entre ellos y con un objetivo final común.

Seguidamente ofrecemos una exposición más detallada de los temas tratados en cada uno de los artículos que componen los capítulos 2, 3 y 4 de esta tesis. Los capítulos 5 y 6 se corresponden con resultados anteriores al capítulo 4 y que fueron publicados en los *proceedings* de conferencias internacionales.

- En el Capítulo 2 construimos la Transformada *Wavelet* ("ondícula") Continua (*Continuous Wavelet Transform* CWT) en el espacio homogéneo (dominio de Cartan) $\mathbb{D}_4 = SO(4,2)/(SO(4) \times SO(2))$ del grupo conforme $SO(4,2)$ (localmente isomorfo a $SU(2,2)$) en 1+3 dimensiones. A través de una transformación de Cayley, se puede establecer una biyección entre la variedad $\mathbb{D}_4$ y el dominio tubo hacia el futuro $\mathbb{C}^4_+$ del espacio de Minkowski complejo $\mathbb{C}^4$, donde se han propuesto otro tipo de ondículas conformes de tipo electromagnético en la literatura. Estudiamos las representaciones unitarias e irreducibles del grupo conforme sobre los espacios de Hilbert $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ y $L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)$ de funciones holomorfas cuadrado integrables con dimensión de escala λ y espectro de masa continuo; probamos un isomorfismo (equivariancia) entre ambos espacios de Hilbert, condiciones de admisibilidad y de *tight frame*, proporcionamos fórmulas de reconstrucción y una base ortonormal de polinomios homogéneos y discutimos propiedades de simetría y el límite Euclídeo de dichas ondículas conformes. Para este propósito, primeramente exponemos y demostramos una extensión uniparamétrica λ del Teorema Maestro de Schwinger, el cual resulta ser una herramienta matemática útil para nosotros, particularmente como una función generatriz para las funciones base de la representación unitaria del grupo conforme y la obtención del kernel reproductor de Bergman de $L^2_h(\mathbb{D}_4, d\nu_\lambda)$. El Teorema Maestro de Schwinger está relacionado con el Teorema Maestro de MacMahon, del cual ofrecemos también una extensión en términos de armónicos sólidos de Louck de $SU(N)$. También estudiamos condiciones de convergencia.

- En el Capítulo 3 revisamos el uso del dominio complejo de Cartan $\mathbb{D}_4$ como una base para la construcción de una teoría cuántica (de campos) invariante conforme, bien como espacio de fases o de configuración. Seguimos una aproximación Lagrangiana invariante gauge de tipo "Modelo Sigma No Lineal" y usamos una generalización del método de Dirac para la cuantización de sistemas con ligaduras, el cual recuerda en algunos aspectos a la aproximación estándar para cuantizar órbitas coadjuntas de un grupo G. Calculamos explícitamente funciones de onda físicas, medidas de Haar, bases ortonormales y núcleos reproductores (*reproduccing Bergman kernels*) en una imagen holomorfa en estos dominios de Cartan para partículas cuánticas escalares y con espín arbitrario. También se discuten similitudes y diferencias con otros resultados en la literatura. También proponemos una adaptación del Principio de Reciprocidad de Born a la relatividad conforme, la sustitución del espacio-tiempo tetra-dimensional por el espacio de fases $\mathbb{D}_4$ de ocho dimensiones a cortas distancias (escala de Planck) y discutimos sobre la existencia o no de una aceleración máxima.

- En el Capítulo 4 damos una descripción mecanico-cuántica de partículas relativistas aceleradas en el marco de Estados Coherentes del grupo conforme en 3+1 dimensiones $SU(2,2)$, donde las transformaciones conformes específicas juegan el papel de aceleraciones (parte espacial) y de "*redshift* cinemático" (parte temporal), como contraposición al *redshift* (desplazamiento al rojo) gravitatorio expresado por la ley de Hubble. El estado fundamental acelerado $\tilde{\varphi}_0$ de la primera cuantización es un Estado Coherente del grupo conforme. Calculamos la función de distribución que da el número de ocupación de cada nivel energético en $\tilde{\varphi}_0$ y, con ella, la función de partición Z, la energía media E y la entropía S, la cual recuerda a la del “Sólido de Einstein” en Mecánica Estadística. Se puede asignar una temperatura efectiva a este “colectivo acelerado” a través de la expresión termodinámica dE/dS, la cual lleva a una relación no lineal entre aceleración y temperatura diferente de la fórmula de Unruh (lineal). Entonces construimos la correspondiente teoría invariante conforme $SU(2,2)$ de segunda cuantización (muchas partículas) y su rotura espontánea cuando se selecciona un θ-vacío degenerado invariante Poincaré (a saber, un Estado Coherente de modos cero del conforme). La transformaciones conformes específicas (aceleraciones) desestabilizan el vacío de Poincaré y lo hacen radiar. Esto proporciona una explicación alternativa, desde un punto de vista grupo-teórico, del fenómeno de radiación de vacío en sistemas acelerados (denominado efecto Fulling-Unruh-Davies).

1.3. Introducción

Las transformaciones conformes (preservan ángulos) han jugado y juegan un papel muy importante en muchas ramas de la Física y la Matemática. Quizás las aplicaciones físicas más conocidas de las transformaciones conformes se dan en *dos* dimensiones, donde ayudan a resolver problemas en, por ejemplo:

- *Flujo de fluidos irrotacionales e incompresibles no viscosos.* Aquí el campo de velocidades deriva de un potencial complejo, que satisface la ecuación de Laplace (es armónico) en las regiones donde el flujo es irrotacional e incompresible. Las transformaciones conformes son de especial ayuda en la deteminación de flujos alrededor de obstáculos.

- *Electrostática.* Aquí el campo eléctrico deriva de un potencial complejo electrostático.

- *Flujo de calor.* Aquí el flujo de calor deriva de una *temperatura compleja*. La ecuación de Laplace describe el estado estacionario y las trasformaciones conformes permiten cambiar de unas geometrías (fronteras) a otras.

- *Materia Condensada.* En transiciones de fase, se tiene invariancia conforme (o de escala) cerca del punto crítico.

- *Teorías de campos invariantes conforme: Gravedad bidimensional, Teoría de Cuerdas, Agujeros negros, etc.* A nivel más fundamental, las transformaciones conformes en dos dimensiones aparecen como simetría (gauge) de numerosas teorías de campo bidimensionales

En dimensiones realistas (D=3+1), el número de transformaciones conformes se reduce drásticamente de infinito (para el caso D=1+1) a un número finito de ellas (15 en el caso de D=3+1). Ya Cunningham [9] y Bateman [10] reconocieron el grupo conforme 15-dimensional $SO(4,2) = SU(2,2)/\mathbb{Z}_4$ como una simetría de las ecuaciones de Maxwell del electromagnetismo sin fuentes. Actualmente, en la formulación de Teorías de Unificación de la Física, también se parte de modelos con simetría conforme (sin masa), simetría que luego se restringe al subgrupo de Poincaré por un proceso de "Rotura Espontánea de la Simetría" en la que entra en juego un campo escalar introducido "a mano" (campo o partícula de Higgs) cuya existencia está aun por demostrar.

Nosotros también partiremos de un modelo de partículas cuánticas relativistas con invariancia conforme (modelo que se construye en el Capítulo 3) y romperemos dicha simetría a Poincaré seleccionando un pseudo-vacío degenerado que resulta ser un Estado Coherente de modos cero del campo invariante conforme.

Antes de ello, hemos tenido que analizar, demostrar y obtener propiedades útiles de los Estados Coherentes del grupo conforme $SU(2,2)$ que, al contener dilataciones y traslaciones espaciotemporales (junto con boosts, rotaciones y aceleraciones), constituyen una extensión o generalización de las tradicionales Wavelets [8, 11, 12] (basadas en el grupo afín de la recta real) a variedades más generales (Minkowski compactificado). Véase a ese respecto [6, 13] para revisiones generales y [14, 15] para recientes trabajos sobre Transformada *Wavelet* en variedades homogéneas.

Para construir la Transformada *Wavelet* Continua sobre el grupo conforme (Capítulo 2), nos hemos ayudado de ejemplos en esferas $\mathbb{S}^{N-1}$, por medio de una adecuada representación unitaria del grupo de Lorentz en $N+1$ dimensiones $SO(N,1)$ [16, 17, 18], y en hiperboloides $\mathbb{H}^2$ [19], o su proyección sobre el disco unidad abierto.

$$\mathbb{D}_1 = SO(1,2)/SO(2) = SU(1,1)/U(1) = \{z \in \mathbb{C} : 1 - |z|^2 < 1\}$$

El ingrediente básico en todas estas construtiones es un grupo de transformaciones G que contiene dilationes y movimientos en una variedad $\mathbb{X}$, junto con una acción transitiva de G en $\mathbb{X}$.

Hay que decir que ya existen en la literatura otras construcciones de *Wavelets* Conformes [20, 21, 22] que proporcionan un análisis espacio-tiempo-escala local de las ondas electromagnéticas de la misma manera que las *Wavelets* tradicionales proporcionan un análisis tiempo-escala de las señales temporales. En estos trabajos, las ondas electromagnéticas están analíticamente continuadas o extendidas desde el espacio-tiempo real $\mathbb{R}^4$ al complejo $\mathbb{C}^4$ y se obtienen a partir de una única wavelet *madre* mediante aplicación de transformaciones conformes de espacio y tiempo. Las ondas electromagnéticas pueden ser descritas entonces como superposiciones de un conjunto particular de *wavelets* conformes.

En el Capítulo 2 nos ocuparemos de otro tipo de ondículas conformes, aunque vamos a trabajar en el espacio-tiempo complejo $\mathbb{C}^4$ también. Además de las anteriores representaciones de masa nula de $SO(4,2)$ sobre el campo electromagnético, el grupo conforme tiene otras representaciones con espectro de masa continuo caracterizadas (indexadas o etiquetadas) por las representaciones del subgrupo compacto maximal $SO(4) \times SO(2)$: dos números cuánticos de *spin* $s_1, s_2 \in \mathbb{N}/2$ y la *dimensión de escala* $\lambda \in \mathbb{N}$ del campo correspondiente [23]. En el Capítulo 2 nos restringiremos a campos escalares ($s_1 = s_2 = 0$) por simplicidad. El caso de spin arbitrario será tratado en el Capítulo 3. Después de un recordatorio de estas representaciones, proporcionamos las condiciones de admisibilidad, *tight wavelet frames* y fórmulas de reconstrucción para las funciones en el dominio de Cartan complejo o bola de Lie (ver [24] para una exposición general sobre estos dominios complejos clásicos):

$$\begin{aligned}\mathbb{D}_4 &= SO(4,2)/(SO(4) \times SO(2)) = SU(2,2)/S(U(2) \times U(2)) \\ &= \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^\dagger > 0\}\end{aligned}$$

que es el análogo de cuatro dimensiones del disco abierto unidad $\mathbb{D}_1$ mencionado anteriormente. Se puede establecer un isomorfismo entre este dominio y el "dominio tubo futuro" (*forward/future tube domain*):

$$\mathbb{C}^4_+ \equiv \{W = X + iY \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \, Y > 0\}$$

con X e Y Hermíticas, que es el análogo en cuatro dimensiones del semiplano superior hiperbólico de Poincaré/Lobachevsky $\mathbb{C}_+$ del espacio complejo de Minkowski. Tal isomorfismo viene dado por la transformación de Cayley (y su inversa)

$$\begin{aligned} W &\rightarrow Z(W) = (I - iW)^{-1}(I + iW) = (I + iW)(I - iW)^{-1}, \\ Z &\rightarrow W(Z) = i(I - Z)(I + Z)^{-1} = i(I + Z)^{-1}(I - Z), \end{aligned}$$

Ofrecemos una isometría (equivariancia) entre los espacios de Hilbert de funciones holomorfas sobre $\mathbb{D}_4$ y $\mathbb{C}^4_+$ (donde disponemos de más intuición física).

Con el fin de probar condiciones de admisibilidad y fórmulas de reconstrucción, demostraremos primero una extensión λ de la fórmula de Schwinger [25], Teorema que se mostrará como una herramienta matemática útil para nosotros. La fórmula del producto interior de Schwinger resulta ser esencialmente equivalente al Teorema Maestro de MacMahon [26], que es uno de los resultados fundamentales en análisis combinatorio. Un análogo cuántico del Teorema Maestro de MacMahon también ha sido elaborado en relación con una generalización cuántica de la correspondencia bosón-fermión en Física [27]. Por otra parte, una extensión del Teorema Maestro de MacMahon clásica [26] fue demostrada en [28] utilizando el grupo de permutaciones. La unificación de la fórmula de Schwinger y del Teorema Maestro de MacMahon mediante el uso de las propiedades de los denominados armónicos sólidos $SU(N)$ [29, 30, 31] (una generalización de las matrices $\mathcal{D}$ de Wigner de $SU(2)$; ver por ejemplo, [32]), fué realizada por Louck en [29]. La culminación del Teorema Maestro de MacMahon-Schwinger proporciona una función generatriz para los elementos diagonales, la traza, y las funciones de representación de las llamadas representaciones unitarias totalmente simétricas del grupo unitario compacto $U(N)$ [29, 30, 31].

En el Capítulo 2 vamos a presentar y demostrar una extensión uniparamétrica λ del Teorema Maestro de MacMahon-Schwinger mediante el uso de los mencionados armónicos sólidos de $SU(N)$ de [30, 31]. Esta extensión λ del Teorema Maestro de MacMahon-Schwinger resultará ser útil como una función generatriz para las funciones de la representación unitaria del grupo especial pseudo-unitario no compacto $SU(N,N)$ y para el cálculo del nucleo reproductor (*reproducing kernel*) de Bergman. Nos concentraremos primeramente en el caso $N=2$, es decir, en el grupo conforme $SU(2,2)$ (el caso general $N \geq 2$ se analiza en el Apéndice del Capítulo 2), que será esencial en el desarrollo de las ondículas o estados coherentes conformes para campos con un espectro continuo de masa.

Variedades complejas y, en particular, dominios clásicos de Cartan como $\mathbb{D}_4$, han sido estudiados durante muchos años por matemáticos y físicos teóricos (ver e.g. [24] y sus referencias para una revisión). En el Capítulo 3 consideraremos $\mathbb{D}_4$ como el espacio de fases de partículas masivas conformes. Existe un interés renovado en la cuantización de estos espacios de fases (ver por ejemplo [33] y sus referencias). La presentación seguida en la literatura es de tipo geométrico (*twistor* [34, 35] o descripciones de tipo Konstant-Kirillov-Souriau [36, 37]) y de naturaleza teórico-representacional [38, 23]. Aquí adoptaremos un enfoque Lagrangiano (tipo "Modelo Sigma no Lineal") para el tema y utilizaremos un método de Dirac generalizado para la cuantización de sistemas con restricciones, que se asemeja en algunos aspectos al enfoque estándar de cuantización de órbitas coadjuntas de un grupo G, desarrollado hace tiempo en [39] (ver también [40] y [41] y ejemplos interesantes en $G = SU(3)$).

Compartimos con muchos autores (a saber, [24, 33, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]) la creencia de que el uso del espacio complejo de Minkowski 8-dimensional como espacio de fases para la construcción de una teoría cuática de campos es no sólo útil desde un punto de vista técnico sino que también puede ser de gran importancia física. Realmente, como se sugiere en [44], el dominio conforme $\mathbb{D}_4$ podría ser considerado como el sustituto del espacio-tiempo a cortas distancias (en la "microescala"). Esta interpretación se basa en el Principio de Reciprocidad de Born (BRP) [45, 46], originalmente destinado a unificar la teoría cuántica y la relatividad general. La simetría de reciprocidad entre coordenadas x_μ y momentos p_μ afirma que las leyes de la naturaleza son (o deberían ser) invariantes bajo las transformaciones

$$(x_\mu, p_\mu) \to (\pm p_\mu, \mp x_\mu). \tag{1.1}$$

La palabra “reciprocidad” se usa en analogía con la red teórica de cristales, donde algunos fenómenos físicos (como la teoría de difracción) se describen algunas veces mejor en el espacio de “momentos” por medio de la red recíproca (Bravais). El argumento utilizado es que la reciprocidad de Born implica que debe haber un principio de relatividad recíprocamente conjugado de acuerdo con el que la tasa de variación del momento (fuerza) debería estar acotada por una constante universal b, del mismo modo que el principio de relatividad usual implica una acotación de la tasa de variación de la posición (velocidad) mediante la velocidad de la luz c. Como consecuencia del BRP, debe existir una longitud mínima (a saber, la londgitud de Planck) $\ell_{\min} = \sqrt{\hbar c/b}$.

Esta simetría llevó a Born a conjeturar que el espasio físico básico subyacente es el 8-dimensional $\{x_\mu, p_\mu\}$ y a reemplazar el elemento de línea invariante Poincaré $d\tau^2 = dx_\mu dx^\mu$ por la métrica de tipo Finsleriana (ver [47, 48] para una extension a espacios de fases de Born-Clifford)

$$d\tilde{\tau}^2 = dx_\mu dx^\mu + \frac{\ell_{\min}^4}{\hbar^2} dp_\mu dp^\mu. \tag{1.2}$$

Desde el punto de vista del BRP, las teorías de campos locales (versus no locales o extensas), como la de Klein-Gordon, representan el “límite de partícula puntual” $\ell_{\min} \to 0$, para las que la simetría recíproca está rota. También, el espacio-tiempo de Minkowski se interpreta o como una versión local ($\ell_{\min} \to 0$) o como un límite de alta trasferencia de energía-momento ($b \to \infty$) de este dominio o espacio de fases 8-dimensional. Además, poniendo $dp_\mu/d\tau = md^2x_\mu/d\tau^2 = ma_\mu$, con $m = b\ell_{\min}/c^2$ una masa (por ejemplo, la de Planck) y a_μ la aceleración propia (con $a^2 \leq 0$, espacial), se puede escribir el elemento de línea modificado previo como

$$d\tilde{\tau} = d\tau\sqrt{1 - \frac{|a^2|}{a_{\max}^2}}, \tag{1.3}$$

que naturalmente conduce a una *aceleración máxima (propia)* $a_{\max} = c^2/\ell_{\min}$. La existencia de una aceleración máxima y sus consecuencias físicas fueron ya deducidas por Caianiello [49]. Muchos trabajos han sido publicados en los últimos años (véase, por ejemplo [50] y sus referencias), cada uno introduce la aceleración máxima partiendo de motivaciones diferentes y de diferentes esquemas teóricos. Entre la larga lista de aplicaciones de física del modelo de Caianiello, nos gustaría destacar el que en cosmología evita una singularidad inicial, mientras que preserva la inflación. Además, un principio relativista de máxima aceleración conduce a una “constante” de estructura fina variable, α [50], según el cual α podría haber sido muy pequeña (cero) en el Universo primitivo y toda la materia del Universo podría haber surgido por el efecto

Fulling-Davies-Unruh-Hawking (radiación de vacío por la aceleración) [1, 3, 2, 4], del cual ya hemos hablado anteriormente.

Han habido revisiones grupo-teóricas del BRP, como [51, 52], reemplazando el grupo de Poincaré por el Canónico (o *Quapléctico*) de la relatividad recíproca, que goza de una estructura más rica que la de Poincaré. En el capítulo 3 buscamos otra reformulación del BRP como una simetría natural dentro del grupo conforme $SO(4,2)$ y la sustitución del espacio-tiempo por el dominio conforme 8-dimensional $\mathbb{D}_4$ o $\mathbb{C}^4_+$ para distancias pequeñas. Creemos que todavía quedan por descubrir nuevos fenómenos físicos interesantes dentro de esta construcción. En realidad, en el Capítulo 4, vamos a discutir una revisión grupo-teórica del efecto Unruh [2] como una ruptura espontánea de la simetría conforme en este dominio $\mathbb{D}_4$. También, una transformación *wavelet* en el dominio tubo $\mathbb{C}^4_+$, basada en el grupo conforme, podría proporcionar una manera de analizar los paquetes de onda localizados tanto en espacio como en tiempo. Importantes desarrollos en esta dirección se han realizado en [20, 21] para señales electromagnéticas (sin masa) y para campos con un espectro de masa continua.

Por ultimo, en el Capítulo 4, abordaremos el análisis cuántico de los sistemas de referencia acelerados. Éste ha sido estudiado principalmente en relación con la Teoría Cuántica de Campos (QFT) en el espacio-tiempo curvo. Por ejemplo, el caso de la cuantización de un campo de Klein-Gordon en coordenadas de Rindler implica una mutilación global del espacio-tiempo plano, con la aparición de un horizonte de sucesos y conduce a una cuantización no equivalente a la cuantización estándar de Minkowski. Físicamente se dice que, mientras que el vacío Poincaré-invariante (Minkowskiano) $|0\rangle$ en QFT es el mismo para cualquier observador inercial (es decir, es estable bajo las transformaciones de Poincaré), éste se convierte en un baño térmico de radiación con temperatura:

$$T = \frac{\hbar a}{2\pi c k_B} \tag{1.4}$$

al pasar a un sistema de referencia uniformemente acelerado. Esto se conoce como el efecto Fulling-Davies-Unruh , que comparte algunas características con el efecto Hawking. Su explicación se basa en las transformaciones de Bogoliubov, que encuentra una explicación natural en el marco de los Estados Coherentes.

También nosotros nos acercamos al análisis cuántico de los sistemas de referencia acelerados desde un punto de vista de los Estados Coherentes, pero el esquema es bastante diferente, aunque comparte algunas características con el enfoque estándar que se ha comentado anteriormente. La situación será similar en algunos aspectos a la que se presenta en los sistemas cuánticos de muchos cuerpos en materia condensada, por ejemplo, superfluidez y superconductividad, donde el estado fudamental juega el papel de vacío cuántico en muchos aspectos y las cuasipartículas (partículas como excitaciones por encima del estado fundamental) el papel de la materia.

Vamos a ampliar el grupo de simetrías del grupo de Poincaré $\mathcal{P}$, con álgebra de Lie

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \\ [P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho, \quad [P_\mu, P_\nu] = 0, \end{aligned} \tag{1.5}$$

para dar cabida a nuevas simetrías como las dilataciones $x^\mu \to \kappa x^\mu$ y a las transformaciones conformes específicas

$$x^\mu \to x'^\mu = \frac{x^\mu + \mathrm{a}^\mu x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2 x^2}, \tag{1.6}$$

que pueden ser interpretadas como las transformaciones a sistemas de observadores relativistas uniformemente acelerados. Los generadores infinitesimales respectivos, D y K_μ, cierran nuevas relaciones de conmutación con los generadores de Poincaré:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \\ [P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho, \quad [P_\mu, P_\nu] = 0, \\ [K_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho, \quad [K_\mu, K_\nu] = 0, \\ [D, P_\mu] &= -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_{\mu\nu}] = 0, \\ [K_\mu, P_\nu] &= 2(\eta_{\mu\nu} D + M_{\mu\nu}). \end{aligned}$$

Una vez ampliado el grupo de Poincaré al conforme, romperemos espontáneamente la simetría conforme anterior de nuevo a Poincaré mediante la selección adecuada de un pseudo-vacío degenerado estable bajo $\mathcal{P}$. Desde el punto de vista de simetría conforme, el vacío invariante Poincaré está formado por estados coherentes de *modos cero* del conforme, que se puede decir que son "indetectables" por observadores inerciales, pero inestables bajo transformaciones conformes específicas o aceleraciones.

Hay que decir que el vacío cuántico realmente no está "vacío" para todos los observadores, ya que en realidad está lleno de fluctuaciones de punto cero de los campos cuánticos. Entonces la acción de las transformaciones de la simetría rota (aceleraciones) desestabilizará y excitará el vacío haciéndolo *radiar*. Nosotros trataremos este vacío acelerado como un colectivo estadístico, con una función de distribución determinada, y calcularemos propiedades termodinámicas como energía media, entropía y temperatura, dependientes todas de la aceleración. Referente a la imagen del vacío acelerado como una colectividad estadística, cabe hacer la siguiente aclaración. Para sistemas estadísticos canónicos, los niveles de energía (discreta) E_n de un sistema cuántico en contacto con un baño térmico a una temperatura T están "poblados" siguiendo la función de distribución de Boltzmann $f_n(T) \sim e^{-E_n/k_BT}$. Para otros focos externos o interacciones (como, por ejemplo, de campos electromagnéticos que actúan sobre una partícula cargada) también se puede calcular (en principio) la función de distribución que da la población de cada nivel energético. En realidad, si se pudiera implementar la interacción de forma unitaria en el sistema cuántico original, entonces se podría deducir la función de distribución de población de cada nivel energético a partir de primeros principios de la Mecánica Cuántica. Esto es precisamente lo que hemos hecho con la aceleración uniforme de partículas cuánticas relativistas invariantes-Poincaré, proporcionando la transformación unitaria que da la población de cada nivel de energía E_n^λ en el estado fundamental acelerado. Veremos entonces el vacío acelerado (Estado Coherente) como un colectivo estadístico ("colectivo acelerado").

Un intento preliminar previo para analizar sistemas de referencia cuánticos acelerados desde una perspectiva del grupo conforme se hizo en la referencia [53] (véase también [54]), donde se desarrolló un sofisticado "formalismo de segunda cuantización sobre un grupo G" y se aplicó a la (parte finita del) grupo conforme en 1+1 dimensiones, $SO(2,2) \simeq SO(2,1) \times SO(2,1)$, que está formado por dos copias del grupo pseudo-ortogonal especial $SO(2,1)$ (modos izquierdos y derechos, respectivamente). Aquí vamos a utilizar métodos más convencionales de cuantización y vamos a trabajar en dimensiones realistas (3+1), utilizando el grupo conforme $SO(4,2) \simeq SU(2,2)/\mathbb{Z}_4$. Nuevas consecuencias de este enfoque grupo-teórico se encontrarán aquí, con respecto a una similitud entre el estado fundamental acelerado y el "sólido de Einstein", el cálculo de la entropía y una desviación de la fórmula de Unruh (1.4).

Nos gustaría mencionar también que la simetría conforme en *dos* dimensiones en la cercanía de un horizonte de sucesos también ha desempeñado un papel fundamental en la descripción microscópica del efecto Hawking. De hecho, hay fuertes indicios de que las teorías de campos invariantes conformes proporcionan una descripción universal (independiente de los detalles del modelo particular de gravedad cuántica utilizado) de la entropía de agujeros negros de baja energía, la cual sólo se resuelve bajo argumentos de simetría (véase, por ejemplo [55, 56]). En este caso, el álgebra de Virasoro se convierte en el subálgebra relevante de las deformaciones del horizonte de un agujero negro y constituye el principio general gauge que indica la densidad de estados. Sin embargo, en 3+1 dimensiones, la invariancia conforme es necesariamente global (finito-(15)-dimensional). En el Capítulo 4 vamos a estudiar los efectos gravitatorios de orden cero en teoría cuántica (aceleraciones uniformes). Para tener en cuenta efectos de orden superior (como aceleraciones no constantes) en un marco grupo-teórico se debe, en primer lugar, ampliar la simetría conforme SO(4,2) a una simetría mayor de infinitas dimensiones. Esto no es una tarea fácil, aunque algunos pasos se han dado en esta dirección [57, 58, 54, 59, 60].

1.4. Justificación y Objetivos

Aunque existe una amplia literatura científica, tanto en Física como en Matemáticas, sobre la simetría conforme y su papel relevante en ambas ciencias, en esta tesis ponemos de manifiesto que aún quedan numerosos aspectos matemáticos y fenómenos físicos interesantes por estudiar ligados a esta simetría. En particular:

1. Entender mejor la dinámica cuántica de observadores acelerados desde una perspectiva grupo-teórica y sus implicaciones físicas, en particular, *cosmológicas*.

2. Construir una teoría matemática de *Ondículas Conformes* para analizar campos relativistas a diferentes escalas espacio-temporales.

3. Aprovechar el papel esencial del Teorema de MacMahon y su extensión λ para analizar las representaciones de $SU(N,N)$.

4. Analizar las consecuencias físicas de una extensión del Principio de Reciprocidad de Born a la Relatividad Conforme.

Pensamos que el grupo conforme debe sustituir al grupo de Poincaré (subgrupo suyo) como simetría fundamental de todas las interacciones. La introducción de masa no debe ser un argumento para descartar la simetría conforme en favor de la simetría de Poincaré, máxime cuando el mecanismo de generación de masa habitual (mecanismo de Higgs) no está aun corroborado experimentalmente. De hecho, en esta tesis se trata con representaciones de masa no nula (y espectro de masas contínuo) del grupo conforme. Veremos cómo los generadores de las traslaciones espaciotemporales P_μ y los de las aceleraciones K_μ son *conjugados* (no pueden ser medidos simultaneamente) y cómo el generador de las dilataciones D se puede tomar como Hamiltoniano asociado a un *tiempo propio*. El Principio de Reciprocidad de Born aludio anteriormente se manifiesta aquí en el sentido de que, la transformación:

$$P_\mu \to K_\mu,\ K_\mu \to P_\mu,\ D \to -D,$$

deja las relaciones de conmutación del conforme inalteradas. Esta simetría puede verse también en el operador de Casimir cuadrático

$$C_2 = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu), \tag{1.7}$$

que generaliza al Casimir de Poincaré $P^2 = P_\mu P^\mu$ que, para campos escalares ϕ, lleva a la ecuación de Klein-Gordon $P^2\phi = m_0^2\phi$, con m_0^2 el cuadrado de la masa en reposo. El hecho de que $[D, P^2] = -2P^2$ implica que los campos conformes tienen que tener o masa nula o un espectro contínuo de masas (ver por ejemplo [61] y [62]). En realidad, al igual que la masa invariante Poincaré m_0 comprende un continuo de masas "galileanas" m, una masa invariante conforme m_{00} se puede definir mediante el Casimir (1.7), que comprende un continuo de masas de Poincaré m_0. La ecuación de valores propios $C_2\phi = m_{00}^2\phi$ se puede ver como una ecuación de Klein-Gordon *generalizada*, donde D reemplaza a P_0 como el generador de evolución del tiempo (propio) (ver Capítulo 3 para más información) y m_{00} reemplaza a m_0 (ver [61] para la formulación de otras ecuaciones de movimiento conformemente invariantes de campo masivo en el espacio de Minkowski generalizado). Algunos autores definen una masa invariante conforme m_{00}, a partir de la masa invariante Lorentz m_0, de una forma parecida a como se define una masa invariante Lorentz a partir de una masa invariante de Galileo m. Esto es "m" de Galileo no es invariante Lorentz; esto no quiere decir que el grupo de Lorentz no sea una simetría de la física, sino que para que lo sea hay que redefinir el concepto de masa y se define la masa en reposo m_0 que sí es un invariante Lorentz (al igual que Poincaré). De igual manera se puede extender el grupo de simetrías de la física al grupo conforme, restringiendo el concepto de masa en reposo (Lorentz) a masa conforme. La masa conforme se definirá a partir de la masa en reposo de la siguiente manera: $m_{00} \equiv m_0/\kappa$, donde κ es el factor de escala que se introduce al aumentar las unidades de longitud en el sistema de referencia (desde el punto de vista de transformación pasiva) con las dilataciones, siendo el cambio de longitudes $x' \equiv \kappa x$.

Para terminar, queremos aclarar una especie de controversia existente en la literatura sobre si las transformaciones conformes específicas se corresponden o no con transiciones a un sistema de referencia relativista uniformemente acelerado. La interpretación de las transformaciones conformes específicas (1.6) como transiciones de un sistema de referencia inercial a otro sistema relativista uniformemente acelerado, fué identificado hace muchos años por [63, 64, 65]. Una forma simple de verlo es tomando $\mathrm{a}^\mu = (0, 0, 0, \mathrm{a})$ en la dirección del eje z, y el camino temporal $x^\mu = (t, 0, 0, 0)$. Entonces la transformación (1.6) se simplifica a:

$$t' = \frac{t}{1 - \mathrm{a}^2 t^2}\,, \qquad z' = \frac{\mathrm{a}t^2}{1 - \mathrm{a}^2 t^2}. \tag{1.8}$$

Escribiendo z' en términos de t' resulta la fórmula común para el movimiento uniformemente acelerado (hiperbólico) relativista:

$$z' = \frac{1}{\mathrm{g}}(\sqrt{1 + \mathrm{g}^2 t'^2} - 1) \tag{1.9}$$

con $\mathrm{g} = 2\mathrm{a}$.

Hill [66] también advirtió una relación interesante procedente de la componente temporal ($\mu = 0$) de (1.6) generado por K_0. Tomando ahora $\mathrm{a}^\mu = (\mathrm{a}^0, 0, 0, 0)$ y escribiendo como $\vec{v} = \frac{d\vec{x}}{dx^0}$ y $\vec{v}' = \frac{d\vec{x}'}{dx'^0}$ a las velocidades en ambos sistemas de referencia, la ecuación (1.6) nos lleva a la fórmula de la velocidad

$$\vec{v}' = \vec{v} - 2\mathrm{a}^0\vec{x} + O((\mathrm{a}^0)^2), \tag{1.10}$$

que, en primer orden de aproximación, se asemeja a la ley de Hubble de desplazamiento al rojo cuando identificamos $H_0 = -2\mathrm{a}^0$ (la constante de Hubble). De hecho, el término añadido es una velocidad radial con una magnitud proporcional a la distancia $\vec{x}$ desde el observador. Hay que tener en cuenta que la derivación anterior es puramente cinemática y no recurre a la teoría relativista de la gravitación. Las implicaciones físicas de esta fórmula (y creemos que de las transformaciones conformes (1.6), en general) han sido pasados por alto durante casi 65 años. Este *redshift* cinemático podría dar lugar a una ambigüedad en la interpretación actual del desplazamiento hacia el rojo en cosmología. Recientemente, Wulfman [67] ha propuesto varios experimentos, basados en un análisis del corrimiento anómalo de frecuencia descubierto en las misiones espaciales Pioneer 10 y 11, intentando determinar el valor del parámetro de grupo a^0, y por lo tanto eliminar la posible ambigüedad en la fórmula del Hubble.

Con estas discusiones no queremos más que justificar la necesidad de un estudio más profundo de las transformaciones conformes. Esta tesis pretende ser un paso más en este sentido.

1.5. Discusión Conjunta de los Resultados Obtenidos

Como se ha dicho anteriormente, esta tesis se basa fundamentalmente en los resultados obtenidos en las tres primeras publicaciones de la sección 1.1 que aparecen recogidas en los Capítulos 2, 3 y 4, respectivamente. El orden cronológico de aparición de las publicaciones coincide con el orden lógico del desarrollo de la tesis. Es por ello que consideramos adecuado presentar la tesis como un compendio de artículos, cada uno de ellos autocontenido, pero enlazados entre ellos y con un objetivo final común. Como se ha dicho, el objetivo fundamental es entender mejor la dinámica cuántica de partículas relativistas aceleradas y los fenómenos de radiación de vacío.

Para conseguir este objetivo final, hemos tenido que desarrollar progresivamente la teoría matemática subyacente. En el Capítulo 2 se desarrolla el Análisis Armónico sobre el grupo conforme y se definen las "Ondículas (*Wavelets*) o Estados Coherentes Conformes", que resultan ser de gran utilidad en teoría de campos invariante conforme. En el Capítulo 3 se construye una teoría cuántica de partículas relativistas invariante conforme a partir de la cuantización de un Lagrangiano, de tipo "Modelo sigma no lineal", invariante gauge $SU(2,2)$, que se cuantiza de acuerdo a un método generalizado de Dirac para sistemas con ligaduras. Una vez construída la teoría cuántica invariante conforme y desarrolladas las herramientas matemáticas necesarias, en el Capítulo 4 se estudia la dinámica cuántica de partículas aceleradas desde una perspectiva grupo-teórica, con el papel de las aceleraciones jugado por las transformaciones conformes específicas y el papel del tiempo propio jugado por las dilataciones. El vacío acelerado resulta ser un estado coherente del grupo conforme, estudiado en el primer Capítulo. Nosotros trataremos este vacío acelerado como un colectivo estadístico, con una función de distribución determinada, y calcularemos propiedades termodinámicas como energía media, entropía y temperatura, dependientes todas de la aceleración. Para el cálculo de todas estas magnitudes resulta esencial, como herramienta matemática indispensable, la extensión λ del Teorema Maestro de MacMahon y la teoría de Estados Coherentes desarrollada en el Capítulo 2.

1.6. Conclusiones y Perspectivas

Aunque la literatura científica sobre el grupo conforme y su papel fundamental en Física y en Matemáticas es amplísima, creemos firmemente que quedan aún por desvelar muchos más fenómenos físicos interesantes dentro de la teoría cuántica de campos invariante conforme. Como dijo hace tiempo Hill en [66]:

> "...a more complete analysis of the physical interpretation of the full conformal group of transformations will be required before all of its implications can be appreciated...".
>
> (E.L. Hill)

Pensamos que la construcción matemática de *Conformal Wavelets* que hacemos en el Capítulo 2 allanará el camino hacia a una nueva herramienta de análisis de campos en el espacio-tiempo complejo de Minkowski, con un espectro continuo de masas en términos de ondículas conformes. Es común en Física de Partículas Relativistas analizar campos o señales (por ejemplo, partículas elementales) en el espacio de Fourier (energía-momento). Sin embargo, de la misma manera que no hay sonidos de infinita duración, las partículas se crean y se destruyen. Una transformada *wavelet* basada en el grupo conforme, proporciona una forma de analizar los paquetes de onda localizados en ambos: espacio y tiempo. Importantes desarrollos en esta dirección también se han hecho en [20, 21, 22] para señales electromagnéticas (sin masa).

En el camino, hemos dado con (y demostrado) una extensión λ de la fórmula de Schwinger. Esta extensión ha resultado ser una herramienta matemática útil para nosotros, especialmente como función generatriz de las funciones de la representación unitaria de $SU(2,2)$, para la derivación del kernel reproductor de Bergman de $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ y la demostración de las condiciones de admisibilidad y de *tight frame*. La generalización de este teorema para matrices X de tamaño $N \geq 2$ sigue similares directrices y los detalles particulares se analizan en el Apéndice A del Capítulo 2, utilizando los armónicos sólidos de $SU(N)$ $\mathcal{D}^p_{\alpha\alpha}(X)$ introducidos por Louck [31]. Este resultado puede servir para el estudio de las series discretas (infinito-dimensionales) de representaciones del grupo pseudo-unitario $SU(N,N)$ no-compacto.

El siguiente paso debe ser el problema de la discretización. Las referencias [68, 69, 70] nos dan las directrices generales para la construcción de frames discretos en la esfera y el hiperboloide y [71] en el grupo de Poincaré. El grupo conforme es mucho más complicado, aunque en principio se aplica el mismo esquema.

En busca de nuevas aplicaciones potenciales de las ondículas conformes construidas en esta tesis, pensamos que podrían ser útiles en el análisis de los problemas de renormalización en Teoría Cuántica de Campos Relativistas. Al describir el espacio y el tiempo como un continuo, ciertas construcciones en Mecánica Estadística y Cuántica están mal definidas. Con el fin de definirlas adecuadamente, el límite al contínuo debe ser tomado con cuidado partiendo de un enfoque discreto. Hay un conjunto de técnicas utilizadas para hacer un límite al continuo, usualmente denominadas como “reglas de renormalización”, que determinan la relación entre los parámetros de la teoría a pequeña y gran escala. Las reglas de renormalización fallan al definir una teoría cuántica finita de la Relatividad General de Einstein, uno de los principales avances en Física Teórica. La sustitución del espacio-tiempo clásico (conmutativo) por un espacio-tiempo cuántico (no conmutativo) promete restaurar la finitud a la gravedad cuántica a altas energías y pequeñas escalas (Planck), donde la geometría se convierte también en *cuántica* (no conmutativa) [72]. Las ondículas conformes también podrían ser aquí de importancia fundamental como herramienta de análisis.

En el Capítulo 3 hemos revisado la utilización del espacio de Minkowski complejo 8-dimensional (más precisamente, los dominios $\mathbb{D}_4$ y $\mathbb{C}^4_+$) como una base para la construcción de una teoría cuántica de campos invariante conforme, ya sea como un espacio de fases o como espacio de configuración (el último caso relacionado con Lagrangianos lineales en las velocidades). Hemos seguido un enfoque Lagrangiano invariante gauge (de tipo modelo sigma no lineal) y hemos utilizado un método generalizado de tipo Dirac para la cuantización de sistemas con ligaduras, que se asemeja en algunos aspectos al enfoque particular de cuantizar las órbitas coadjuntas de un grupo G desarrollado, por ejemplo, en [39].

Uno podría pensar en estos dominios 8-dimensionales como el sustituto del espacio-tiempo a cortas distancias o en estados de transferencia de alta energía-impulso, hecho que está implícito en el Principio de Reciprocidad de Born original [45, 46]. La teoría de la relatividad estándar es entonces el límite $\ell_{\min} \to 0$ (longitud mínima nula). Revisiones grupo-teóricas del Principio de Reciprocidad de Born, sustituyendo el grupo de Poincaré por el Canónico (o Quapléctico) de la relatividad recíproca, se habían propuesto ya en [51, 52]. En esta tesis proponemos un Principio de Reciprocidad de Born (PRB) "conforme", como una simetría natural dentro del grupo $SO(4,2)$, y la sustitución del espacio-tiempo por el dominio 8-dimensional conforme $\mathbb{D}_4$ o $\mathbb{C}^4_+$ para pequeñas escalas. En realidad, nos sentimos tentados a establecer una conexión entre *holomorficidad*↔*quiralidad* y PRB↔simetría CPT, dentro del grupo conforme. En realidad, la realización de P_μ y K_μ en términos de matrices 4×4 está vinculada a los proyectores de quiralidad derecha e izquierda $(1+\gamma^5)/2$ y $(1-\gamma^5)/2$, respectivamente. De acuerdo con la simetría de tipo PRB (conforme), la física conforme es simétrica bajo intercambio $P_\mu \leftrightarrow K_\mu$, a la vez que realizamos una inversión del tiempo propio $D \to -D$. Por otro lado, $P_\mu \leftrightarrow K_\mu$ implica un intercambio de quiralidad $(1+\gamma^5)/2 \leftrightarrow (1-\gamma^5)/2$, una conjugación compleja de la función de onda $\psi_\lambda(g) \leftrightarrow \check{\psi}_\lambda(g) = \overline{\psi_\lambda(g)}$ y una inversión de paridad $\sigma_\mu \leftrightarrow \check{\sigma}_\mu = \sigma^\mu$. Sin embargo, ahora mismo, una conexión PRB↔CPT dentro del grupo conforme es sólo una conjectura y es todavía prematuro dibujar algunas conclusions físicas basadas en ello.

Hemos mencionado la conexión del PRB con la existencia de una escala de longitud mínima. Nótese que las nuevas combinaciones (vector y pseudo-vector)

$$\tilde{P}_\mu \equiv \frac{1}{2}(P_\mu + K_\mu), \quad \tilde{K}_\mu \equiv \frac{1}{2}(P_\mu - K_\mu),$$

tienen relaciones de conmutación:

$$\left[\tilde{P}_\mu, \tilde{K}_\nu\right] = \eta_{\mu\nu} D, \ \left[\tilde{P}_\mu, \tilde{P}_\nu\right] = M_{\mu\nu}, \ \left[\tilde{K}_\mu, \tilde{K}_\nu\right] = -M_{\mu\nu}.$$

En un escenario de geometría no conmutativa [72], el que estos conmutadores no se anulen puede ser interpretado como un signo de la granularidad (no conmutatividad) del espacio-tiempo en las teorías invariantes conforme, junto con la existencia de una longitud mínima.

Referente al Capítulo 4, y como ya hemos comentado anteriormente, las teorías de campo conformes también proporcionan una descripción universal de la termodinámica de agujeros negros a baja energía, que sólo se establece mediante argumentos de simetría (ver [55, 56] y sus referencias). De hecho, la temperatura de Unruh

$$T = \frac{\hbar a}{2\pi c k_B}$$

coincide con la temperatura de Hawking

$$T = \frac{\hbar c^3}{8\pi M k_B G} = \frac{2\pi G M \hbar}{\Sigma c k_B}$$

($\Sigma = 4\pi r_g^2 = 8\pi G^2 M^2/c^4$ representa la superficie del horizonte de sucesos) cuando la aceleración es la de un obsevador en caída libre en la superficie Σ, es decir, $a = c^4/(4GM) = GM/r_g^2$. En este caso, el álgebra de Virasoro demuestra tener importancia como subálgebra del álgebra de gauge de deformaciones de la superficie que dejan el horizonte invariante. Por lo tanto, los campos en la superficie deben de transformarse de acuerdo con las representaciones irreducibles del álgebra de Virasoro, que es el principio general de simetría que rige la densidad de estados microscópicos. Por lo tanto, en el efecto Hawking, el cálculo de cantidades termodinámicas, vinculadas al problema mecano-estadístico de contar estados microscópicos, se reduce al estudio de la teoría de representación del grupo conforme.

Aunque nuestro enfoque para el análisis cuántico de sistemas de referencia acelerados comparte con la descripción anterior, de la termodinámica de agujeros negros, la existencia de un principio subyacente de invariancia conforme, no hay que confundir los dos esquemas. La invariancia conforme en el efecto Hawking se manifiesta como un algebra de gauge infinito-dimensional de una superficie bidimensional de deformaciones. Sin embargo, el carácter infinito-dimensional de la simetría conforme parece ser patrimonio exclusivo de la física en dos dimensiones, y la invariancia conforme en (3+1) dimensiones es finita (15)-dimensional, lo que explica las transiciones a sistemas de referencia uniformemente acelerados solamente. Para tener en cuenta los efectos gravitatorios de orden superior en la teoría cuántica de campos a partir de un punto de vista grupo-teórico, se deben considerar difeomorfismos más generales (Lie) álgebras. Se han propuesto análogos altodimensionales del grupo conforme en dos dimensiones en [57, 58, 54, 59, 60]. Creemos que estas simetrías infinitas pueden jugar un papel fundamental en modelos de gravedad cuántica, como un principio de simetría gauge.

Parte II

ANEXO DE ARTÍCULOS PUBLICADOS

Capítulo 2

Teorema de MacMahon Extendido y Ondículas Conformes

Appl. Comput. Harmon. Anal. 31 (2011) 143–168

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Extended MacMahon–Schwinger's Master Theorem and conformal wavelets in complex Minkowski space

M. Calixto [a,b,*], E. Pérez-Romero [b]

[a] Departamento de Matemática Aplicada, Universidad de Granada, Facultad de Ciencias, Campus de Fuentenueva, 18071 Granada, Spain
[b] Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado Postal 3004, 18080 Granada, Spain

ARTICLE INFO

Article history:
Received 27 April 2010
Revised 19 November 2010
Accepted 23 November 2010
Available online 25 November 2010
Communicated by Guy Battle

Keywords:
Continuous wavelet transform
Non-commutative harmonic analysis
Conformal group
Complex domains
Unitary holomorphic representations
Schwinger's Master Theorem

ABSTRACT

We construct the Continuous Wavelet Transform (CWT) on the homogeneous space (Cartan domain) $\mathbb{D}_4 = SO(4,2)/(SO(4)\times SO(2))$ of the conformal group $SO(4,2)$ (locally isomorphic to $SU(2,2)$) in $1+3$ dimensions. The manifold $\mathbb{D}_4$ can be mapped one-to-one onto the future tube domain $\mathbb{C}^4_+$ of the complex Minkowski space through a Cayley transformation, where other kind of (electromagnetic) wavelets have already been proposed in the literature. We study the unitary irreducible representations of the conformal group on the Hilbert spaces $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ and $L^2_h(\mathbb{C}^4_+, d\tilde{\nu}_\lambda)$ of square integrable holomorphic functions with scale dimension λ and continuous mass spectrum, prove the isomorphism (equivariance) between both Hilbert spaces, admissibility and tight-frame conditions, provide reconstruction formulas and orthonormal basis of homogeneous polynomials and discuss symmetry properties and the Euclidean limit of the proposed conformal wavelets. For that purpose, we firstly state and prove a λ-extension of Schwinger's Master Theorem (SMT), which turns out to be a useful mathematical tool for us, particularly as a generating function for the unitary-representation functions of the conformal group and for the derivation of the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$. SMT is related to MacMahon's Master Theorem (MMT) and an extension of both in terms of Louck's $SU(N)$ solid harmonics is also provided for completeness. Convergence conditions are also studied.

1. Introduction

Since the pioneer work of Grossmann, Morlet and Paul [1], several extensions of the standard Continuous Wavelet Transform (CWT) on $\mathbb{R}$ (traditionally based on the affine group of time translations and dilations, see e.g. [2,3]) to general manifolds $\mathbb{X}$ have been constructed (see e.g. [4,5] for general reviews and [6,7] for recent papers on WT and Gabor systems on homogeneous manifolds). Particular interesting examples are the construction of CWT on spheres $\mathbb{S}^{N-1}$, by means of an appropriate unitary representation of the Lorentz group in $N+1$ dimensions $SO(N,1)$ [8–10], and on the upper sheet $\mathbb{H}^2_+$ of the two-sheeted hyperboloid $\mathbb{H}^2$ [11], or its stereographical projection onto the open unit disk

$$\mathbb{D}_1 = SO(1,2)/SO(2) = SU(1,1)/U(1). \tag{1}$$

The basic ingredient in all these constructions is a group of transformations G which contains dilations and motions on $\mathbb{X}$, together with a transitive action of G on $\mathbb{X}$.

In this article we shall consider the 15-parameter conformal group $G = SO(4,2)$ in $1+3$ dimensions and its natural action on the Minkowski space–time. The fact that the conformal group contains space–time dilations and translations

* Corresponding author at: Departamento de Matemática Aplicada, Universidad de Granada, Facultad de Ciencias, Campus de Fuentenueva, 18071 Granada, Spain.

E-mail addresses: calixto@ugr.es, Manuel.Calixto@upct.es (M. Calixto).

1063-5203/$ – see front matter
doi:10.1016/j.acha.2010.11.004

leads to a natural generalization of the standard CWT for signals on the real line to a higher dimensional manifold. Actually, the conformal group $SO(4,2)$ consists of Poincaré transformations (space–time translations and Lorentz relativistic rotations and boosts), augmented by dilations and relativistic uniform accelerations, which can also be seen as a sort of local (point-dependent) scale transformations (see later on Section 5).

The conformal group $SO(4,2)$ (or its four-covering $SU(2,2)$) has been recognized as a symmetry of Maxwell theory of electromagnetism without sources since [12,13]. Electromagnetic waves turn out to be written as superpositions of a particular set of *conformal wavelets* [14–16]. Thus, conformal wavelets provide a local space–time-scale analysis of electromagnetic waves in much the same way as standard wavelets provide a time-scale analysis of time signals. In these works, electromagnetic waves are analytically continued or extended from real to complex space–time and they are obtained from a single *mother* wavelet by applying conformal transformations of space and time.

Here we shall deal with a different type of conformal wavelets, although we shall work in complex space–time too. Besides the above massless representations of $SO(4,2)$ on the electromagnetic field, the conformal group has other representations with continuous mass spectrum labelled by the representations of the stability subgroup $SO(4) \times SO(2)$: the two *spins* $s_1, s_2 \in \mathbb{N}/2$ and the *scale dimension* $\lambda \in \mathbb{N}$ of the corresponding field [17]. We shall restrict ourselves to scalar fields ($s_1 = s_2 = 0$) for the sake of simplicity. After a reminder of these representations, we provide admissibility conditions, tight wavelet frames and reconstruction formulas for functions on the complex Cartan domain or Lie ball (see [18] for a general discussion on these classical complex domains)

$$\mathbb{D}_4 = SO(4,2)/\big(SO(4) \times SO(2)\big) = SU(2,2)/S\big(U(2) \times U(2)\big), \tag{2}$$

which is the four-dimensional analogue of the open unit disk $\mathbb{D}_1$ abovementioned. This domain can be mapped one-to-one onto the forward/future tube domain $\mathbb{C}^4_+$ (the four-dimensional analogue of the Poincaré/Lobachevsky/hyperbolic upper half-plane $\mathbb{C}_+$) of the complex Minkowski space through a Cayley transformation. For completeness, we also provide an isometric (equivariant) map between the Hilbert spaces of holomorphic functions on $\mathbb{D}_4$ and $\mathbb{C}^4_+$, where we enjoy more physical intuition.

In order to prove admissibility conditions and reconstruction formulas, an extension of the traditional Schwinger's Master Theorem (SMT) [19] will show up as a useful mathematical tool for us. Schwinger's inner product formula turns out to be essentially equivalent to MacMahon's Master Theorem (MMT) [20], which is one of the fundamental results in combinatorial analysis. A quantum analogue of the MMT has also been constructed [21] and related to a quantum generalization of the boson–fermion correspondence of Physics. Moreover, an extension of the classical MMT [20] was proved in [22] by using the permutation group. The unification of SMT and MMT into a single form by using properties of the so-called $SU(N)$ solid harmonics [23–25] (a generalization of Wigner's $\mathcal{D}$-matrices for $SU(2)$, see e.g. [26]), was pointed out by Louck in [23]. The combined MacMahon–Schwinger's Master Theorem provides a generating function for the diagonal elements, the trace, and the representation functions of the so-called totally symmetric unitary representations of the compact unitary group $U(N)$ [23–25].

In this article we shall state and prove a λ-extension of the SMT by using the abovementioned $SU(N)$ solid harmonics of [24,25]. This λ-extension of the SMT will appear to be useful as a generating function for the unitary-representation functions of the non-compact special pseudo-unitary group $SU(N,N)$ and for the computation of the reproducing (Bergman) kernel. We shall concentrate on the $N = 2$ case, i.e., on the conformal group $SU(2,2)$ (the general case $N \geqslant 2$ is discussed in Appendix A), which will be essential in the development of conformal wavelets for fields with continuum mass spectrum.

The paper is organized as follows. In Section 2 we remind Schwinger and MacMahon's Master Theorems and state and prove a λ-extension of Schwinger's formula. The generalization to matrices X of size $N \geqslant 2$ is also discussed for completeness in Appendix A. In order to be as self-contained as possible, in Section 3 we present the group-theoretical backdrop and leave for Appendix B a succinct exposition of the CWT on a general manifold $\mathbb{X}$, collecting the main definitions used in this paper. In Section 4 we briefly remind the CWT on $\mathbb{R}$ and extend it to the Lobachevsky plane $\mathbb{C}_+$ and the open unit disk $\mathbb{D}_1$. The action of the affine group on $\mathbb{C}_+$ extends naturally to the conformal group $SO(1,2)$ of the time axis $\mathbb{R}$. This will serve us to introduce and establish a parallelism between standard and conformal wavelets in complex Minkowski space in Section 5. We shall eventually work in the Cartan domain $\mathbb{D}_4$, although we shall provide in Section 5.3 an (intertwiner) isometry between the Hilbert spaces of holomorphic functions on $\mathbb{D}_4$ and the future tube domain $\mathbb{C}^4_+$. The λ-extended SMT turns out to be a valuable mathematical tool inside $\mathbb{D}_4$ for proving admissibility and tight-frame conditions, reconstruction formulas and reproducing (Bergman) kernels in Section 5.2. We also discuss symmetry properties and comment on the Euclidean limit of the proposed wavelets in Sections 5.4 and 5.5, respectively. Section 6 is devoted to convergence considerations and Section 7 to conclusions and outlook. In Appendix C we prove orthonormality properties of a basis of homogeneous polynomials introduced in Section 5.

2. Schwinger's Master Theorem: an extension

Schwinger's inner product formula [19] can be stated as follows:

Theorem 2.1 *(SMT). Let X be any 2×2 matrix X and $Y = tI$, where t is an arbitrary parameter and I stands for the 2×2 identity matrix. Let us denote by*

$$\mathcal{D}^j_{q_1,q_2}(X)=\sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}}\sum_{k=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)}\binom{j+q_2}{k}\binom{j-q_2}{k-q_1-q_2}x_{11}^k x_{12}^{j+q_1-k}x_{21}^{j+q_2-k}x_{22}^{k-q_1-q_2}, \tag{3}$$

the Wigner's $\mathcal{D}$*-matrices for* $SU(2)$ [26], *where* $j\in\mathbb{N}/2$ (*the spin*) *runs on all non-negative half-integers and* $q_1,q_2=-j,-j+1,\dots,j-1,j$. *Then the following identity holds*:

$$e^{(\partial_u:X:\partial_v)}e^{(u:Y^T:v)}\big|_{u=v=0}=\sum_{j\in\mathbb{N}/2}t^{2j}\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)=\det(I-tX)^{-1} \tag{4}$$

where we denote by $(u:X:v)\equiv uXv^T=\sum_{i,j=1}^N u_i x_{ij} v_j$ *and* $\partial_{u_i}\equiv\partial/\partial u_i$.

This formula turns out to be essentially equivalent to MacMahon's Master Theorem:

Theorem 2.2 *(MMT). Let* X *be an* $N\times N$ *matrix of indeterminates* x_{ij}, *and* Y *be the diagonal matrix* $Y\equiv\mathrm{diag}(y_1,y_2,\dots,y_N)$. *Then the coefficient of* $y^\alpha\equiv y_1^{\alpha_1}y_2^{\alpha_2}\dots y_N^{\alpha_N}$ *in the expansion of* $\det(I-XY)^{-1}$ *equals the coefficient of* y^α *in the product*

$$\prod_{i=1}^N(x_{i1}y_1+x_{i2}y_2+\cdots+x_{iN}y_N)^{\alpha_i}. \tag{5}$$

These abovementioned coefficients can be written in terms of the so-called $SU(N)$ solid harmonics $\mathcal{D}^p_{\alpha\beta}(X)$ (see [24,25] and Appendix A for a general definition). $SU(N)$ solid harmonics (103) are a natural generalization of the standard Wigner's $\mathcal{D}$-matrices (3) to matrices X of size $N\geqslant 2$. In fact, replacing tX with XY in (4) and using the multiplication property

$$\sum_{q'=-j}^{j}\mathcal{D}^j_{qq'}(X)\mathcal{D}^j_{q'q''}(Y)=\mathcal{D}^j_{qq''}(XY) \tag{6}$$

and the transpositional symmetry

$$\mathcal{D}^j_{qq'}(Y)=\mathcal{D}^j_{q'q}\big(Y^T\big), \tag{7}$$

we can restate MMT for $N=2$ as:

$$\sum_{j\in\mathbb{N}/2}\sum_{q,q'=-j}^{j}\mathcal{D}^j_{qq'}(X)\mathcal{D}^j_{qq'}\big(Y^T\big)=\det(I-XY)^{-1}. \tag{8}$$

Actually, MMT preceded Schwinger's result by many years. Schwinger re-discovered the MMT in the context of his generating function approach to the angular momentum theory of many-particle systems. The unification into a single form by using properties of the $SU(N)$ solid harmonics was established by Louck in [23–25].

Wigner matrices $\mathcal{D}^j_{qq'}(X)$ are homogeneous polynomials of degree $2j$ in x_{kl}. Inspired by Euler's theorem, we shall define the following differential operator:

$$D_\lambda f(t)\equiv\left(\lambda+t\frac{\partial}{\partial t}\right)f(t),\quad \lambda\in\mathbb{N} \tag{9}$$

which will be useful in the sequel. Now we are in condition to state and prove an extension of SMT 2.1. For the sake of completeness, a generalization for matrices X of size $N\geqslant 2$ is also given in Appendix A.

Theorem 2.3 *(λ-Extended SMT). For every* $\lambda\in\mathbb{N}$, $\lambda\geqslant 2$ *and every* 2×2 *matrix* X, *the following identity holds*:

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}t^{2j+2n}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}\det(X)^n\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)=\det(I-tX)^{-\lambda}. \tag{10}$$

Proof. We start from the basic SMT 2.1 and apply the operator D_1 on both sides of Eq. (4):

$$\sum_{j\in\mathbb{N}/2}(2j+1)t^{2j}\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)=\frac{1-t^2\det(X)}{\det(I-tX)^2}. \tag{11}$$

Here we have used that

$$\det(I-tX)=1-\mathrm{tr}(tX)+\det(tX)$$

and that $\mathrm{tr}(X)$ and $\det(X)$ are homogeneous polynomials of degrees 1 and 2, respectively. Making use of the expansion:

$$\frac{1}{1-t^2\det(X)}=\sum_{n=0}^{\infty}t^{2n}\det(X)^n, \tag{12}$$

the expression (11) can be recast as:

$$\sum_{j\in\mathbb{N}/2}(2j+1)\sum_{n=0}^{\infty}t^{2j+2n}\det(X)^n\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)=\frac{1}{\det(I-tX)^2}. \tag{13}$$

This identity is a particular case of (10) for $\lambda=2$. Now we shall proceed by induction on λ. Assuming that (10) is valid for every $\lambda\geqslant 2$ and applying the operator D_λ on both sides of Eq. (10), we arrive at:

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}(\lambda+2j+2n)t^{2j+2n}\det(X)^n\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)=\lambda\frac{1-t^2\det(X)}{\det(I-tX)^{\lambda+1}},$$

where we have made use again of (12). Considering (12) one more time, we can assemble the previous expression as:

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{n,m=0}^{\infty}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}(\lambda+2j+2n)t^{2j+2(n+m)}\det(X)^{n+m}\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)$$
$$=\frac{\lambda}{\det(I-tX)^{\lambda+1}}. \tag{14}$$

Rearranging series:

$$\sum_{n,m=0}^{\infty}a_nb^{n+m}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}a_m\right)b^n, \tag{15}$$

the identity (14) can be recast in the form:

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}(\lambda+2j+2m)t^{2j+2n}\det(X)^n\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X)$$
$$=\frac{\lambda}{\det(I-tX)^{\lambda+1}}. \tag{16}$$

It remains to prove the following combinatorial identity:

$$\frac{1}{\lambda-1}\sum_{m=0}^{n}\binom{m+\lambda-2}{\lambda-2}\binom{m+p+\lambda-1}{\lambda-2}(\lambda+p+2m)=\binom{n+\lambda-1}{\lambda-1}\binom{n+p+\lambda}{\lambda-1}. \tag{17}$$

We shall proceed by induction on n. Let us define both sides of the previous equality as the two sequences:

$$F(n)=\frac{1}{\lambda-1}\sum_{m=0}^{n}\binom{m+\lambda-2}{\lambda-2}\binom{m+p+\lambda-1}{\lambda-2}(\lambda+p+2m),$$
$$G(n)=\binom{n+\lambda-1}{\lambda-1}\binom{n+p+\lambda}{\lambda-1}.$$

It is easy to verify that $F(0)=G(0)$. Assuming that $F(n)=G(n)$, we ask whether $F(n+1)=G(n+1)$. Indeed, on the one hand

$$\begin{aligned}F(n+1)&=\frac{1}{\lambda-1}\sum_{m=0}^{n+1}\binom{m+\lambda-2}{\lambda-2}\binom{m+p+\lambda-1}{\lambda-2}(\lambda+p+2m)\\&=F(n)+\frac{\lambda+p+2(n+1)}{\lambda-1}\binom{n+\lambda-1}{\lambda-2}\binom{n+p+\lambda}{\lambda-2}\\&=F(n)+\frac{(\lambda+p+2n+2)(\lambda-1)^2}{(\lambda-1)(n+1)(n+p+2)}G(n)\\&=\frac{(n+1)(n+p+2)+(\lambda+p+2n+2)(\lambda-1)}{(n+1)(n+p+2)}G(n)\end{aligned}$$

and, on the other hand

$$G(n+1) = \binom{n+\lambda}{\lambda-1}\binom{n+p+\lambda+1}{\lambda-1} = \frac{(n+p+\lambda+1)(n+\lambda)}{(n+1)(n+p+2)}G(n).$$

Realizing that

$$(n+1)(n+p+2) + (\lambda+p+2n+2)(\lambda-1) = (n+p+\lambda+1)(n+\lambda)$$

we arrive at $F(n+1) = G(n+1)$, which proves (17). Finally, inserting (17) in (16), we conclude that (10) is valid for $\lambda+1$, thus completing the proof. □

3. The group-theoretical backdrop

The usual CWT on the real line $\mathbb{R}$ is derived from the natural unitary representation of the affine or similitude group $G = SIM(1)$ in the space of finite energy signals $L^2(\mathbb{R}, dx)$ (see Section 4 for a reminder). The same scheme applies to the CWT on a general manifold $\mathbb{X}$, subject to the transitive action, $x \to gx$, $g \in G$, $x \in \mathbb{X}$, of some group of transformations G which contains dilations and motions on $\mathbb{X}$. We address the reader to Refs. [4,5] for a nice and thorough exposition on this subject with multiple examples. For the sake of self-containedness, we also collect in Appendix B some basic definitions which are essential for our construction of conformal wavelets.

As already said in Section 1, the CWT on spheres $\mathbb{X} = \mathbb{S}^{N-1}$ has been constructed in [8–10] by means of an appropriate unitary representation of the Lorentz group in $N+1$ (space–time) dimensions $G = SO(N,1)$. The case of $G = SO(2,1)$ is particularly interesting as it encompasses wavelets on the circle $\mathbb{S}^1$ and on the real line $\mathbb{R}$, associated to the continuous and discrete series representations, respectively (see [27] for a unified group-theoretical treatment of both type of wavelets inside $SL(2,\mathbb{R}) \simeq SO(1,2)$). The group $SO(1,2)$ (the conformal group in $0+1$ dimensions) has also been used to construct wavelets on the upper sheet $\mathbb{H}^2_+$ of the two-sheeted hyperboloid $\mathbb{H}^2$ [11], or its stereographical projection onto the open unit disk (1).

The (angle-preserving) conformal group in N (space–time) dimensions is finite-dimensional except for $N = 2$. For $N \neq 2$, the conformal group $SO(N,2)$ consists of Poincaré [space–time translations $b^\mu \in \mathbb{R}^N$ and restricted Lorentz $\Lambda^\mu_\nu \in SO^+(N-1,1)$] transformations augmented by dilations ($a \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations $c^\mu \in \mathbb{R}^N$) which, in N-dimensional Minkowski space–time, have the following realization:

$$\begin{aligned} x'^\mu &= x^\mu + b^\mu, \qquad x'^\mu = \Lambda^\mu_\nu(\omega)x^\nu, \\ x'^\mu &= ax^\mu, \qquad x'^\mu = \frac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2}, \end{aligned} \tag{18}$$

respectively. We are using the Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, \overbrace{\cdots}^{N-1}, -1)$ to rise and lower space–time indices and the Einstein summation convention $cx = c_\mu x^\mu$. The new ingredients with regard to the affine group $SIM(1)$ are the extension from time-translations by b^0 to N-translations by b^μ, the addition of Lorentz transformations Λ^μ_ν (rotations and boosts) and accelerations by c^μ. Special conformal transformations can be seen as a sequence of inversions and translations by c^μ of the form:

$$x^\mu \xrightarrow{\text{inv}} \frac{x^\mu}{x^2} \xrightarrow{c_\mu} \frac{x^\mu + x^2c^\mu}{x^2} \xrightarrow{\text{inv}} \frac{(x^\mu + x^2c^\mu)/x^2}{(x^\mu + x^2c^\mu)^2/x^4} = \frac{x^\mu + c^\mu x^2}{1+2cx+c^2x^2}. \tag{19}$$

They can also be interpreted as point-dependent (generalized/gauge) dilations in the sense that, while standard dilations change the space–time interval $ds^2 = dx^\mu dx_\mu$ globally as $ds^2 \to a^2 ds^2$, special conformal transformations scale the space–time interval point-to-point as $ds^2 \to \sigma(x)^{-2} ds^2$, with $\sigma(x) = 1+2cx+c^2x^2$. The same happens with the squared mass m^2, thus forcing a continuous mass spectrum unless $m = 0$, as for the electromagnetic field.

The infinitesimal generators of the transformations (18) are easily deduced:

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial x^\mu}, \qquad M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \\ D &= x^\mu \frac{\partial}{\partial x^\mu}, \qquad K_\mu = -2x_\mu x^\nu \frac{\partial}{\partial x^\nu} + x^2 \frac{\partial}{\partial x^\mu}, \end{aligned} \tag{20}$$

and they close into the conformal Lie algebra

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}, \\ &[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \qquad [P_\mu, P_\nu] = 0, \\ &[K_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho, \qquad [K_\mu, K_\nu] = 0, \\ &[D, P_\mu] = -P_\mu, \qquad [D, K_\mu] = K_\mu, \qquad [D, M_{\mu\nu}] = 0, \\ &[K_\mu, P_\nu] = 2(\eta_{\mu\nu}D + M_{\mu\nu}). \end{aligned} \tag{21}$$

The quadratic Casimir operator:

$$C_2 = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + \frac{1}{2}\left(P_\mu K^\mu + K_\mu P^\mu\right), \tag{22}$$

generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu$ (the squared rest mass).

Any group element $g \in SO(4,2)$ (near the identity element) could be written as the exponential map

$$g = \exp(u), \quad u = \tau D + b^\mu P_\mu + c^\mu K_\mu + \omega^{\mu\nu} M_{\mu\nu}, \tag{23}$$

of the Lie-algebra element u (see [28,29]). The compactified Minkowski space $\mathbb{M} = \mathbb{S}^{N-1} \times_{\mathbb{Z}_2} \mathbb{S}^1$, can be obtained as the coset $\mathbb{M} = SO(N,2)/\mathbb{W}$, where $\mathbb{W}$ denotes the Weyl subgroup generated by $K_\mu, M_{\mu\nu}$ and D (i.e., a Poincaré subgroup $\mathbb{P} = SO(N-1,1) \circledS \mathbb{R}^N$ augmented by dilations $\mathbb{R}^+$). The Weyl group $\mathbb{W}$ is the stability subgroup (the little group in physical usage) of $x^\mu = 0$.

For $N=2$, the group $SO(2,2)$ is isomorphic to the direct product $SO(1,2) \times SO(1,2)$. It is well known that, in two dimensions, the conformal group is infinite dimensional. Actually, the splitting $SO(2,2) = SO(1,2) \times SO(1,2)$ has to do with the separation into holomorphic and anti-holomorphic self-maps of the infinitesimal conformal isometries of a complex domain, the generators of which,

$$L_n = -z^{n+1}\frac{\partial}{\partial z}, \qquad \bar{L}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}, \quad z = x^1 + ix^0,\ \bar{z} = x^1 - ix^0,\ n \in \mathbb{Z}, \tag{24}$$

close into the Witt algebra $[L_m, L_n] = (m-n)L_{m+n}$ (idem for $\bar{L}$). The conformal group in $N = 2+1$ dimensions, $SO(3,2)$, is also the symmetry group of the anti-de Sitter space in $3+1$ dimensions, $\mathrm{AdS}_4 = SO(3,2)/SO(3,1)$, a maximally symmetric Lorentzian manifold with constant negative scalar curvature (i.e., the Lorentzian analogue of four-dimensional hyperbolic space) which arises, for instance, as a vacuum solution of Einstein's General Relativity field equations with a negative (attractive) cosmological constant (corresponding to a negative vacuum energy density and positive pressure).

We shall focus on the 15-parameter conformal group in $3+1$ dimensions, $SO(4,2)$, which turns out to be locally isomorphic to the pseudo-unitary group

$$SU(2,2) = \left\{g \in \mathrm{Mat}_{4\times4}(\mathbb{C}):\ g^\dagger \Gamma g = \Gamma,\ \det(g) = 1\right\} \tag{25}$$

of complex special 4×4 matrices g leaving invariant the 4×4 hermitian form Γ of signature $(++--)$. Here $g^\dagger$ stands for adjoint (or conjugate/hermitian transpose) of g (it is also customary to denote it by g^*). Actually, the conformal Lie algebra (21) can be also realized in terms of the Lie algebra generators of the fundamental representation of $SU(2,2)$, given by the following 4×4 matrices

$$\begin{aligned} &D = \frac{\gamma^5}{2}, \qquad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\check{\sigma}^\nu - \sigma^\nu\check{\sigma}^\mu & 0 \\ 0 & \check{\sigma}^\mu\sigma^\nu - \check{\sigma}^\nu\sigma^\mu \end{pmatrix}, \\ &P^\mu = \gamma^\mu\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \qquad K^\mu = \gamma^\mu\frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \end{aligned} \tag{26}$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix},$$

denote the Dirac gamma matrices in the Weyl basis and

$$\sigma^0 \equiv I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{27}$$

are the Pauli matrices (we are writing $\check{\sigma}^\mu \equiv \sigma_\mu$). Indeed, using standard properties of gamma and Pauli matrices, one can easily check that the choice (26) fulfills the commutation relations (21).

To be more precise, $SU(2,2)$ is the four-cover of $SO(4,2)$, much in the same way as $SU(2)$ is the two-cover of $SO(3)$. This local isomorphism between the conformal group $SO(N,2)$ and the pseudo-unitary group $SU(M,M)$ only happens for $N=1$ and $N=4$ dimensions, where

$$SO(1,2) = SU(1,1)/\mathbb{Z}_2, \qquad SO(4,2) = SU(2,2)/\mathbb{Z}_4. \tag{28}$$

The λ-extension of the SMT given in Theorem 2.3 (and Theorem A.2) turns out to be closely related to the group $SU(2,2)$ (and $SU(N,N)$ in general), providing a kind of generating function for the unitary-representation functions of this group (the discrete series, to be more precise). This formula will be a useful mathematical tool for us, specially in proving admissibility and tight-frame conditions and providing reconstruction formulas. From this point of view, the conformal group $SO(N,2)$ in $N=4$ dimensions is singled out from the general N-dimensional case, at least in this article.

Before tackling the construction of conformal wavelets in eight-dimensional complex Minkowski space in Section 5, we shall briefly remind the simpler case of the CWT on the time axis $\mathbb{R}$ and its extension to the Lobachevsky plane $\mathbb{C}_+$ and the open unit disk $\mathbb{D}_1$, which are homogeneous spaces of $SO(1,2)$.

4. Wavelets for the affine group

Let us consider the affine or similitude group of translations and dilations in one dimension,

$$G = SIM(1) = \mathbb{R} \rtimes \mathbb{R}^+ = \{ g = (b, a) \mid b \in \mathbb{R},\ a \in \mathbb{R}^+ \},$$

with group law ($g'' = g'g$):

$$a'' = a'a,$$
$$b'' = b' + a'b.$$

This group will serve us as an introduction for studying the most interesting case of the conformal group $G = SO(4,2)$ as a "similitude" group of space–time, which will be considered in the next section.

The left-invariant Haar measure is:

$$d\mu(g) = \frac{1}{a^2} da \wedge db.$$

The representation

$$\big[\mathcal{U}_\lambda(a,b)\phi\big](x) = a^{-\lambda}\phi\left(\frac{x-b}{a}\right) \equiv \phi_{a,b}(x) \tag{29}$$

of G on $L^2(\mathbb{R}, dx)$ is unitary for $\lambda = \frac{1}{2} + is$. In fact, every $\mathcal{U}_\lambda$ is unitarily equivalent to $\mathcal{U}_{1/2}$ and one always works with $\lambda = \frac{1}{2}$. This representation is reducible and splits into two irreducible components: the positive $\omega > 0$ and negative $\omega < 0$ frequency subespaces. Restricting oneself to the subspace $\omega > 0$, the admissibility condition (123) assumes the form

$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

where $\hat{\psi}$ stands for the Fourier transform of ψ. Given an admissible function $\psi \in L^2(\mathbb{R}, dx)$, the machinery of wavelet analysis proceeds in the usual way.

4.1. Wavelets on the Lobachevsky plane $\mathbb{C}_+$

An extension of the representation (29) of the affine group, this time on the space $L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)$ of square integrable holomorphic functions on the upper half complex plane (or forward tube domain)

$$\mathbb{T}_1 \equiv \mathbb{C}_+ \equiv \{ w = x + iy \in \mathbb{C} \mid \Im(w) = y > 0 \}, \tag{30}$$

is given by:

$$\big[\tilde{\mathcal{U}}_\lambda(a,b)\phi\big](w) = a^{-\lambda}\phi\left(\frac{w-b}{a}\right). \tag{31}$$

This representation is unitary with respect to the scalar product:

$$\langle \phi | \phi' \rangle = \int_{\mathbb{C}^+} \overline{\phi(w)}\phi'(w)\, d\tilde{\nu}_\lambda(w,\bar{w}), \quad d\tilde{\nu}_\lambda(w,\bar{w}) = \frac{2\lambda - 1}{4\pi}\Im(w)^{2(\lambda-1)}|dw|, \tag{32}$$

for any $\phi, \phi' \in L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)$, where we use $|dw|$ as a shorthand for the Lebesgue measure $d\Re(w) \wedge d\Im(w)$. Although all representations $\tilde{\mathcal{U}}_\lambda, \lambda \geqslant 1$, are equivalent, they become inequivalent when the affine group is immersed inside the conformal group of the time axis $\mathbb{R}$, $SO(1,2) \simeq SL(2,\mathbb{R}) \simeq SU(1,1)$. Actually, this will be the case with the conformal group $SO(4,2)$ in the next section. This immersion of $SIM(1)$ inside $SL(2,\mathbb{R})$ is apparent for the Iwasawa decomposition KAN (see, for instance, [30]) when parameterizing and element $g \in SL(2,\mathbb{R})$ as:

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \sqrt{a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\cos\theta}{\sqrt{a}} & \frac{b\cos\theta}{\sqrt{a}} - \sqrt{a}\sin\theta \\ \frac{\sin\theta}{\sqrt{a}} & \sqrt{a}\cos\theta + \frac{b\sin\theta}{\sqrt{a}} \end{pmatrix},$$

where $\theta \in (-\pi, \pi]$ (see [27] for a unified group-theoretical treatment of wavelets on $\mathbb{R}$ and the circle $\mathbb{S}^1$ inside $SL(2,\mathbb{R})$).

4.2. Wavelets on the open unit disk $\mathbb{D}_1$

There is a one-to-one mapping between the Lobachevsky plane $\mathbb{C}_+$ and the open unit disk (or Cartan domain)

$$\mathbb{D}_1 = \{z \in \mathbb{C},\ |z| < 1\}, \tag{33}$$

given through the Cayley transformation:

$$z(w) = \frac{1+iw}{1-iw} \quad \leftrightarrow \quad w(z) = i\frac{1-z}{1+z}. \tag{34}$$

Note that the (Shilov) boundary $\mathbb{S}^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ of $\mathbb{D}_1$ is stereographically projected onto the boundary $\mathbb{R} = \{w \in \mathbb{C}\colon \Im(w) = 0\}$ of $\mathbb{C}_+$ by $w(e^{i\theta}) = \tan(\theta/2)$.

We can establish an isometry between $L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda)$ and the space $L^2_h(\mathbb{D}_1, d\nu_\lambda)$ of square integrable holomorphic functions on the unit disk $\mathbb{D}_1$ with integration measure

$$d\nu_\lambda(z,\bar{z}) = \frac{2\lambda-1}{\pi}(1-z\bar{z})^{2(\lambda-1)}|dz|, \quad \lambda \geqslant 1, \tag{35}$$

where $\bar{z}$ denotes complex conjugate. This isometry is given by the correspondence

$$\begin{aligned} \mathcal{S}_\lambda\colon\ \ L^2_h(\mathbb{D}_1, d\nu_\lambda) &\longrightarrow L^2_h(\mathbb{C}_+, d\tilde{\nu}_\lambda), \\ \phi &\longmapsto \mathcal{S}_\lambda\phi \equiv \tilde{\phi}, \end{aligned}$$

with

$$\tilde{\phi}(w) = 2^{2\lambda}(1-iw)^{-2\lambda}\phi\big(z(w)\big) \tag{36}$$

and $z(w)$ given by (34). In fact, taking into account that $(1-z\bar{z}) = 2^2\Im(w)|1-iw|^{-2}$ and the Jacobian determinant $|dz|/|dw| = 2^2|1-iw|^{-4}$, then

$$\begin{aligned} \langle\phi|\phi'\rangle_{L^2_h(\mathbb{D}_1)} &= \frac{2\lambda-1}{\pi}\int_{\mathbb{D}_1}\overline{\phi(z)}\phi'(z)(1-z\bar{z})^{2(\lambda-1)}\,|dz| \\ &= \frac{2\lambda-1}{4\pi}\int_{\mathbb{C}_+}\overline{2^{2\lambda}(1-iw)^{-2\lambda}\phi\big(z(w)\big)}2^{2\lambda}(1-iw)^{-2\lambda}\phi'\big(z(w)\big)\Im(w)^{2(\lambda-1)}\,|dw| \\ &= \int_{\mathbb{C}_+}\overline{\tilde{\phi}(w)}\tilde{\phi}(w)\,d\tilde{\nu}_\lambda(w,\bar{w}) = \langle\tilde{\phi}|\tilde{\phi}'\rangle_{L^2_h(\mathbb{C}_+)}. \end{aligned}$$

The constant factor $(2\lambda-1)/\pi$ of $d\nu_\lambda(z,\bar{z})$ is chosen so that the set of functions

$$\varphi_n(z) \equiv \binom{2\lambda+n-1}{n}^{1/2} z^n, \quad n = 0,1,2,\ldots, \tag{37}$$

constitutes an orthonormal basis of $L^2_h(\mathbb{D}_1, d\nu_\lambda)$, as can be easily checked by direct computation. These basis functions verify the following closure relation:

$$\sum_{n=0}^{\infty}\overline{\varphi_n(z)}\varphi_n\big(z'\big) = \big(1-\bar{z}z'\big)^{-2\lambda}, \tag{38}$$

which is nothing other than the *reproducing* (*Bergman*) *kernel* of $L^2_h(\mathbb{D}_1, d\nu_\lambda)$ (see e.g. [4] for a general discussion on reproducing kernels). We shall provide a four-dimensional analogue of (37) and (38) in Eqs. (65) and (67), respectively, and prove the orthonormality in Appendix C. The isometry $\mathcal{S}_\lambda$ given by (36) maps the orthonormal basis (37) of $L^2_h(\mathbb{D}_1, d\nu_\lambda)$ onto the orthonormal basis

$$\tilde{\varphi}_n(w) = 2^{2\lambda}(1-iw)^{-2\lambda}\varphi_n\big(z(w)\big), \quad n = 0,1,2,\ldots \tag{39}$$

of $L^2_h(\mathbb{C}_+, d\tilde{\nu})$, which verify the reproducing kernel relation

$$\sum_{n=0}^{\infty}\overline{\tilde{\varphi}_n(w)}\tilde{\varphi}_n\big(w'\big) = \left(\frac{i}{2}\big(\bar{w}-w'\big)\right)^{-2\lambda}. \tag{40}$$

Let us denote by $\mathcal{V}_\lambda \equiv \mathcal{S}_\lambda^{-1}\tilde{\mathcal{U}}_\lambda \mathcal{S}_\lambda$ the representation of the affine group on $L_h^2(\mathbb{D}_1, d\nu_\lambda)$ induced from (31) through the isometry (36). More explicitly:

$$\big[\mathcal{V}_\lambda(a,b)\phi\big](z) = \big[\mathcal{S}_\lambda^{-1}\tilde{\mathcal{U}}_\lambda(a,b)\tilde{\phi}\big](z) = a^{-\lambda}\left(\frac{1-i\frac{w(z)-b}{a}}{1-iw(z)}\right)^{-2\lambda}\phi\left(z\left(\frac{w(z)-b}{a}\right)\right).$$

This representation is, by construction, unitary on $L_h^2(\mathbb{D}_1, d\nu_\lambda)$.

5. Wavelets for the conformal group *SO*(4, 2)

The four-dimensional analogue of the extension of the time axis $\mathbb{R}$ to the time-energy half-plane $\mathbb{C}_+$ is the extension of the Minkowski space $\mathbb{R}^4$ to the (eight-dimensional) future tube domain $\mathbb{C}_+^4$ of the complex Minkowski space $\mathbb{C}^4$ (see later in this section). The four-dimensional analogue of the one-to-one mapping between the half-plane $\mathbb{C}_+$ and the disk $\mathbb{D}_1$ is now the Cayley transform (47) between $\mathbb{C}_+^4$ and the Cartan domain $\mathbb{D}_4 = U(2,2)/U(2)^2$, the Shilov boundary of which is the compactified Minkowski space $U(2)$ (the four-dimensional analogue of the boundary $U(1) = \mathbb{S}^1$ of the disk $\mathbb{D}_1$). Let us see all this mappings and constructions in more detail.

5.1. Wavelets on the forward tube domain $\mathbb{C}_+^4$

The four-dimensional analogue of the upper-half complex plane (30) is the future/forward tube domain

$$\mathbb{T}_4 = \mathbb{C}_+^4 \equiv \big\{W = X + iY = w_\mu\sigma^\mu \in \mathrm{Mat}_{2\times 2}(\mathbb{C}):\ Y > 0\big\} \tag{41}$$

of the complex Minkowski space $\mathbb{C}^4$, with $X = x_\mu\sigma^\mu$ and $Y = y_\mu\sigma^\mu$ hermitian matrices fulfilling the positivity condition $Y > 0 \Leftrightarrow y^0 = \Im(w^0) > \|\vec{y}\|$.

The domain $\mathbb{C}_+^4$ is naturally homeomorphic to the quotient $SU(2,2)/S(U(2)\times U(2))$ in the realization of the conformal group in terms of 4×4 complex (block) matrices f fulfilling

$$G = SU(2,2) = \left\{f = \begin{pmatrix} R & iS \\ -iT & Q\end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}):\ f^\dagger \Gamma f = \Gamma,\ \det(f) = 1\right\}, \tag{42}$$

with

$$\Gamma = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}$$

the time component of the Dirac 4×4 matrices γ^μ in the Weyl basis ($I = \sigma^0$ is the 2×2 identity matrix and $f^\dagger = f^*$ stands again for adjoint or conjugate/hermitian transpose of f). In general, Γ is a 4×4 hermitian form of signature $(++--)$. The inverse element of f is then given by:

$$f^{-1} = \gamma^0 f^\dagger \gamma^0 = \begin{pmatrix} Q^\dagger & -iS^\dagger \\ iT^\dagger & R^\dagger\end{pmatrix}.$$

The particular identification of $\mathbb{C}_+^4$ with the coset $SU(2,2)/S(U(2)^2)$ is given through:

$$W = W(f) = (S + iR)(Q + iT)^{-1} = (Q + iT)^{-1}(S + iR). \tag{43}$$

The left translation $f' \to ff'$ of G on itself induces a natural left-action of G on $\mathbb{C}_+^4$ given by:

$$W = W(f') \to W' = W(ff') = (RW + S)(TW + Q)^{-1}. \tag{44}$$

Let us make use of the standard identification $x_\mu \leftrightarrow X = x_\mu\sigma^\mu$ between the Minkowski space $\mathbb{R}^4$ and the space of 2×2 hermitian matrices X, with σ^μ the Pauli matrices (27), and $x^2 = x_\mu x^\mu = \det(X)$ the Minkowski squared-norm. Setting $W = x_\mu\sigma^\mu$, the transformations (18) can be formally recovered from (44) as follows:

(i) Standard Lorentz transformations, $x'^\mu = \Lambda^\mu_\nu(\omega)x^\nu$, correspond to $T = S = 0$ and $R = Q^{-1\dagger} \in SL(2,\mathbb{C})$, where we are making use of the homomorphism (spinor map) between $SO^+(3,1)$ and $SL(2,\mathbb{C})$ and writing $W' = RWR^\dagger$, $R \in SL(2,\mathbb{C})$ instead of $x'^\mu = \Lambda^\mu_\nu x^\nu$.
(ii) Dilations correspond to $T = S = 0$ and $R = Q^{-1} = a^{1/2}I$.
(iii) Space–time translations correspond to $R = Q = I$ and $S = b_\mu\sigma^\mu$, $T = 0$.
(iv) Special conformal transformations correspond to $R = Q = I$ and $T = c_\mu\sigma^\mu$, $S = 0$ by noting that $\det(I + TW) = 1 + 2cx + c^2x^2$.

We shall give the next proposition without proof. Instead, we address the reader to its counterpart (Proposition 5.2) in the next subsection for an equivalent proof.

Proposition 5.1. *The representation of G on square-integrable holomorphic functions $\varphi(W)$ given by*

$$[\tilde{\mathcal{U}}_\lambda(f)\varphi](W) = \det(R^\dagger - T^\dagger W)^{-\lambda}\varphi\big((Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}\big) \tag{45}$$

is unitary with respect to the integration measure

$$d\tilde{\nu}_\lambda(W, W^\dagger) \equiv \frac{c_\lambda}{2^4}\det\left(\frac{i}{2}(W^\dagger - W)\right)^{\lambda-4}|dW| = \frac{c_\lambda}{2^4}\Im(w)^{2(\lambda-4)}|dW|, \tag{46}$$

where $\lambda \in \mathbb{N}, \lambda > 3$ (the "scale or conformal dimension") is a parameter labelling non-equivalent representations, $c_\lambda \equiv (\lambda-1)(\lambda-2)^2(\lambda-3)/\pi^4$ and we are using $|dW| = \bigwedge_{\mu=0}^{3} d\Re(w_\mu)d\Im(w_\mu)$ as a shorthand for the Lebesgue measure on $\mathbb{C}^4_+$.

We identify the factor $\mathcal{M}(f, W)^{1/2} = \det(R^\dagger - T^\dagger W)^{-\lambda}$ in (45) as a multiplier or Radon–Nikodym derivative (remember the general definition in (121)). It generalizes the factor $a^{-\lambda}$ in (31) by extending (global) standard dilations $R = Q^{-1} = a^{1/2}I$, $T = S = 0$ to (local/point-dependent) "generalized dilations" with $T = c_\mu\sigma^\mu$. The representation (45) is a special (spin-less or scalar) case of the discrete series representations of $SU(2,2)$, which are characterized by λ and two spin labels s_1 and s_2. Decomposing the discrete series representations of $SU(2,2)$ into irreducible representations of the inhomogeneous Lorentz group leads to a continuous (Poincaré) mass spectrum [17].

5.2. Wavelets on the Cartan domain $\mathbb{D}_4$

Instead of working in the forward tube domain $\mathbb{C}^4_+$, we shall choose for convenience a different eight-dimensional space $\mathbb{D}_4$ generalizing the (two-dimensional) open unit disk $\mathbb{D}_1$ in (33), where we shall take advantage of the full power of the λ-extension of the SMT given by the formula (10). Both spaces, $\mathbb{C}^4_+$ and $\mathbb{D}_4$, are related by a Cayley-type transformation, which induces an isomorphism between the corresponding Hilbert spaces of square-integrable holomorphic functions on both manifolds (see later on Section 5.3).

5.2.1. Cayley transform and $\mathbb{D}_4$ as a coset of $SU(2,2)$

The four-dimensional analogue of the map (34) form the Lobachevsky plane $\mathbb{C}_+$ onto the unit disk $\mathbb{D}_1$ is now the Cayley transformation (and its inverse):

$$\begin{aligned} &W \to Z(W) = (I - iW)^{-1}(I + iW) = (I + iW)(I - iW)^{-1}, \\ &Z \to W(Z) = i(I - Z)(I + Z)^{-1} = i(I + Z)^{-1}(I - Z), \end{aligned} \tag{47}$$

that maps (one-to-one) the forward tube domain $\mathbb{C}^4_+$ onto the Cartan complex domain defined by the positive-definite matrix condition:

$$\mathbb{D}_4 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}):\ I - ZZ^\dagger > 0\}. \tag{48}$$

Note that the (Shilov) boundary

$$\check{\mathbb{D}}_4 = U(2) = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}):\ ZZ^\dagger = Z^\dagger Z = I\} = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1$$

of $\mathbb{D}_4$ is a compactification of the real Minkowski space

$$\mathbb{M}_4 = \{W \in \mathrm{Mat}_{2\times 2}(\mathbb{C}):\ W^\dagger = W\},$$

i.e., the boundary of $\mathbb{C}^4_+$ (see e.g. [18]). The restriction of the Cayley map $Z \to W(Z)$ to $Z \in U(2)$ is precisely the stereographic projection of $U(2)$ onto $\mathbb{M}_4$.

The Cartan domain $\mathbb{D}_4$ is naturally homeomorphic to the quotient $SU(2,2)/S(U(2)^2)$ in the new realization of:

$$G = SU(2,2) = \left\{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}):\ g^\dagger\gamma^5 g = \gamma^5,\ \det(g) = 1\right\}, \tag{49}$$

with

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

the fifth Dirac 4×4 gamma matrix in the Weyl basis ($I = \sigma^0$ denotes again the 2×2 identity matrix). The inverse element of g is now:

$$g^{-1} = \gamma^5 g^\dagger \gamma^5 = \begin{pmatrix} A^\dagger & -C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}.$$

Both realizations (42) and (49) are related by the map

$$f \to g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Upsilon^{-1} f \Upsilon = \frac{1}{2}\begin{pmatrix} R + iS - iT + Q & -R + iS + iT + Q \\ -R - iS - iT + Q & R - iS + iT + Q \end{pmatrix}, \tag{50}$$

with

$$\Upsilon \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$

The particular identification of $\mathbb{D}_4$ with the coset $SU(2,2)/S(U(2)^2)$ is given by (see later on Eq. (59) for more details):

$$Z(g) = BD^{-1}, \qquad Z^\dagger(g) = CA^{-1}. \tag{51}$$

Actually, making explicit the matrix restrictions $g^\dagger \gamma^5 g = \gamma^5$ in (49):

$$g^{-1}g = I_{4\times4} \quad \Leftrightarrow \quad \begin{cases} D^\dagger D - B^\dagger B = I, \\ A^\dagger A - C^\dagger C = I, \\ A^\dagger B - C^\dagger D = 0 \end{cases} \tag{52}$$

and

$$gg^{-1} = I_{4\times4} \quad \Leftrightarrow \quad \begin{cases} DD^\dagger - CC^\dagger = I, \\ AA^\dagger - BB^\dagger = I, \\ AC^\dagger - BD^\dagger = 0, \end{cases} \tag{53}$$

the positive matrix condition in (48) now reads

$$I - ZZ^\dagger = I - A^{-1\dagger}C^\dagger CA^{-1} = \left(AA^\dagger\right)^{-1} > 0, \tag{54}$$

where we have used the second condition in (52). Moreover, using the identification (51) and the first condition in (52), we can see that

$$\det\left(ZZ^\dagger\right) = \det\left(B^\dagger B\right)\det\left(I + B^\dagger B\right)^{-1} < 1. \tag{55}$$

This determinant restriction can also be proved as a direct consequence of the positive-definite matrix condition $I - ZZ^\dagger > 0$. In fact, the characteristic polynomial

$$\begin{aligned} \det\left((1-\rho)I - ZZ^\dagger\right) &= 1 - \operatorname{tr}\left(\rho I + ZZ^\dagger\right) + \det\left(\rho I + ZZ^\dagger\right) \\ &= \rho^2 - \left(2 - \operatorname{tr}\left(ZZ^\dagger\right)\right)\rho + \det\left(I - ZZ^\dagger\right) \end{aligned}$$

yields the eigenvalues

$$\rho_\pm = \frac{2 - \operatorname{tr}(ZZ^\dagger) \pm \sqrt{\Delta}}{2}, \qquad \Delta \equiv \left(2 - \operatorname{tr}\left(ZZ^\dagger\right)\right)^2 - 4\det\left(I - ZZ^\dagger\right). \tag{56}$$

Since $I - ZZ^\dagger$ is hermitian and positive definite, its eigenvalues $\rho_\pm$ are real and positive. The condition $\rho_- > 0$ implies that:

$$2 - \operatorname{tr}\left(ZZ^\dagger\right) > 0 \quad \Rightarrow \quad \operatorname{tr}\left(ZZ^\dagger\right) < 2, \tag{57}$$

and the fact that $\Delta \geqslant 0$ gives:

$$0 \leqslant \Delta = \operatorname{tr}\left(ZZ^\dagger\right)^2 - 4\det\left(ZZ^\dagger\right) \quad \Rightarrow \quad \det\left(ZZ^\dagger\right) \leqslant \frac{1}{4}\operatorname{tr}\left(ZZ^\dagger\right)^2 < 1, \tag{58}$$

where we have used (57) in the last inequality. From (57), we can regard $\mathbb{D}_4$ as an open subset of the eight-dimensional ball with radius $\sqrt{2}$. All those bounds for $Z \in \mathbb{D}_4$ will be useful for proving convergence conditions later on Section 6. See also Appendix C for a suitable parametrization of Z when computing scalar products.

5.2.2. *Haar measure, unitary representation and reproducing kernel*

Any element $g \in G$ admits a Iwasawa decomposition of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\ Z^\dagger\Delta_1 & \Delta_2 \end{pmatrix}\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{59}$$

with

$$\begin{aligned} \Delta_1 &= \left(AA^\dagger\right)^{1/2} = \left(I - ZZ^\dagger\right)^{-1/2}, \qquad U_1 = \Delta_1^{-1}A, \\ \Delta_2 &= \left(DD^\dagger\right)^{1/2} = \left(I - Z^\dagger Z\right)^{-1/2}, \qquad U_2 = \Delta_2^{-1}D. \end{aligned}$$

This decomposition is adapted to the quotient $\mathbb{D}_4 = G/H$ of $G = SU(2,2)$ by the maximal compact subgroup $H = S(U(2)^2)$; that is, $U_1, U_2 \in U(2)$ with $\det(U_1U_2) = 1$. In order to release $U_{1,2}$ from the last determinant condition, we shall work from now on with $G = U(2,2)$ and $H = U(2)^2$ instead. Likewise, a parametrization of any $U \in U(2)$, adapted to the quotient $\mathbb{S}^2 = U(2)/U(1)^2$, is (the Hopf fibration)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & -z\delta \\ \bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\beta_1} & 0 \\ 0 & e^{i\beta_2} \end{pmatrix}, \tag{60}$$

where $z = b/d \in \overline{\mathbb{C}} \simeq \mathbb{S}^2$ (the one-point compactification of $\mathbb{C}$ by inverse stereographic projection), $\delta = (1 + z\bar{z})^{-1/2}$ and $e^{i\beta_1} = a/|a|, e^{i\beta_2} = d/|d|$.

The left-invariant Haar measure (the exterior product of left-invariant one-forms $g^{-1}\,dg$) of G proves to be:

$$\begin{aligned} d\mu(g) &= d\mu(g)|_{G/H}\, d\mu(g)|_H, \\ d\mu(g)|_{G/H} &= \det\big(I - ZZ^\dagger\big)^{-4}|dZ|, \\ d\mu(g)|_H &= d\nu(U_1)\, d\nu(U_2), \end{aligned} \tag{61}$$

where we are denoting by $d\nu(U)$ the Haar measure on $U(2)$, which (using (60)) can be in turn decomposed as:

$$\begin{aligned} d\nu(U) &\equiv d\nu(U)|_{U(2)/U(1)^2}\, d\nu(U)|_{U(1)^2}, \\ d\nu(U)|_{U(2)/U(1)^2} &= d\nu(U)|_{\mathbb{S}^2} \equiv ds(U) = (1 + z\bar{z})^{-2}|dz|, \\ d\nu(U)|_{U(1)^2} &\equiv d\beta_1\, d\beta_2, \end{aligned} \tag{62}$$

where $|dz|$ and $|dZ|$ denote the Lebesgue measures in $\mathbb{C}$ and $\mathbb{C}^4$, respectively.

Let us consider the space of holomorphic functions $\phi(Z)$ on $\mathbb{D}_4$.

Proposition 5.2. *For any group element $g \in G$, the following (left-)action*

$$\begin{aligned} \phi_g(Z) &\equiv \big[\mathcal{U}_\lambda(g)\phi\big](Z) = \det\big(D^\dagger - B^\dagger Z\big)^{-\lambda}\phi\big(Z'\big), \\ Z' &= g^{-1}Z = \big(A^\dagger Z - C^\dagger\big)\big(D^\dagger - B^\dagger Z\big)^{-1} \end{aligned} \tag{63}$$

defines a unitary irreducible square integrable representation of G on $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ under the invariant scalar product

$$\begin{aligned} \langle\phi|\phi'\rangle &= \int_{\mathbb{D}_4} \overline{\phi(Z)}\phi'(Z)\, d\nu_\lambda\big(Z, Z^\dagger\big), \\ d\nu_\lambda\big(Z, Z^\dagger\big) &\equiv c_\lambda \det\big(I - ZZ^\dagger\big)^{\lambda-4}|dZ|, \end{aligned} \tag{64}$$

for any $\lambda \in \mathbb{N}$, $\lambda \geqslant 4$ (the "scale or conformal dimension"), where $c_\lambda = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)$ is chosen so that the unit function, $\phi(Z) = 1$, $\forall Z \in \mathbb{D}_4$, is normalized, i.e. $\langle\phi|\phi\rangle = 1$.

Proof. One can easily check by elementary algebra that $\mathcal{U}_\lambda(g)\mathcal{U}_\lambda(g') = \mathcal{U}_\lambda(gg')$. In order to prove unitarity, i.e. $\langle\phi_g|\phi_g\rangle = \langle\phi|\phi\rangle$ for every $g \in G$, we shall make use of (52) and (53). In fact:

$$\det\big(I - Z'Z'^\dagger\big) = \big|\det\big(D^\dagger - B^\dagger Z\big)\big|^{-2}\det\big(I - ZZ^\dagger\big),$$

and the Jacobian determinant

$$|dZ| = \big|dZ'\big|\big|\det\big(D^\dagger - B^\dagger Z\big)\big|^8,$$

give the isometry relation $\|\phi_g\|^2 = \|\phi\|^2$. Now taking $g' = g^{-1}$ implies the unitarity of $\mathcal{U}_\lambda$. For the computation of c_λ and other orthonormality properties see Appendix C. □

In the next section, we shall provide an isomorphism between $L^2_h(\mathbb{D}_4, d\nu)$ and $L^2_h(\mathbb{C}^4_+, d\tilde{\nu})$, where we enjoy more physical intuition.

In order to prove admissibility conditions in Section 5.2.3, it will be convenient to give an orthonormal basis of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$.

Proposition 5.3. *The set of homogeneous polynomials of degree* $2j+2m$:

$$\varphi^{j,m}_{q_1,q_2}(Z) \equiv \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\det(Z)^m \mathcal{D}^j_{q_1,q_2}(Z),$$
$$m\in\mathbb{N},\ j\in\mathbb{N}/2,\ q_1,q_2=-j,-j+1,\dots,j-1,j, \tag{65}$$

constitutes an orthonormal basis of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$, *that is*:

$$\langle\varphi^{j,m}_{q_1,q_2}|\varphi^{j',m'}_{q'_1,q'_2}\rangle = \delta_{j,j'}\delta_{m,m'}\delta_{q_1,q'_1}\delta_{q_2,q'_2}. \tag{66}$$

Note that the number of linearly independent polynomials $\prod_{i,j=1}^2 z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n=\sum_{i,j=1}^2 n_{ij}$ is $(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (65) with degree of homogeneity $n=2m+2j$. This proves that the set of polynomials (65) is a basis for analytic functions $\phi\in L^2_h(\mathbb{D}_4, d\nu_\lambda)$. Moreover, this basis turns out to be orthonormal. We address the interested reader to Appendix C for a proof. We prefer to omit it here in order to make the presentation more dynamic.

Note also the close resemblance between the definition (65) and the left-hand side of the equality (10) in the λ-extended SMT 2.3. In fact, taking $tX=Z^\dagger Z'$ in (10) and using the properties (6) and (7) of Wigner's $\mathcal{D}$-matrices, we can prove the following closure relation for the basis functions (65):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j}\overline{\varphi^{j,m}_{q',q}(Z)}\varphi^{j,m}_{q',q}(Z') = \det(I-Z^\dagger Z')^{-\lambda}, \tag{67}$$

which is nothing other than the *reproducing* (*Bergman*) *kernel* in $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ (see e.g. [4] for a general discussion on reproducing kernels). Note that, although the scalar product (64) is only valid for $\lambda\geqslant 4$, the expression (67) is formally valid for $\lambda\geqslant 2$, since we are just using in it the requirements of the λ-extended SMT 2.3. The case $\lambda=2$ is related to the Szegö kernel (see e.g. [18]).

5.2.3. Admissibility condition, tight frame and reconstruction formula

Theorem 5.4. *The representation* (63) *is square integrable, the constant unit function* $\psi(Z)=\varphi^{0,0}_{0,0}(Z)=1$, $\forall Z\in\mathbb{D}_4$ *being an admissible vector* (*fiducial state or mother wavelet*), *i.e.*:

$$c_\psi = \int_G \left|\langle\mathcal{U}_\lambda(g)\psi|\psi\rangle\right|^2 d\mu(g) < \infty \tag{68}$$

and the set of coherent states (*or wavelets*) $F=\{\psi_g=\mathcal{U}_\lambda(g)\psi,\ g\in G\}$ *constituting a continuous tight frame in* $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ *satisfying the resolution of the identity*:

$$\mathcal{A} = \int_G |\psi_g\rangle\langle\psi_g|\, d\mu(g) = c_\psi\mathcal{I}. \tag{69}$$

Proof. Using the extended SMT 2.3 for $tX=D^{-1}CZ$, we can expand

$$\begin{aligned}\psi_g(Z) &= \det(D^\dagger - B^\dagger Z)^{-\lambda} = \det(D^\dagger)^{-\lambda}\det\big(I-(BD^{-1})^\dagger Z\big)^{-\lambda}\\ &= \det(D^\dagger)^{-\lambda}\sum_{j=0}^{\infty}\frac{2j+1}{\lambda-1}\sum_{n=0}^{\infty}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}\\ &\quad\times\det\big((BD^{-1})^\dagger Z\big)^n\sum_{q=-j}^{j}\mathcal{D}^j_{qq}\big((BD^{-1})^\dagger Z\big).\end{aligned} \tag{70}$$

Now, taking into account that $\det((BD^{-1})^\dagger Z)^n = \det((BD^{-1})^\dagger)^n\det(Z)^n$ and the property (6) for

$$\mathcal{D}^j_{qq}\big((BD^{-1})^\dagger Z\big) = \sum_{q'=-j}^{j}\mathcal{D}^j_{qq'}\big((BD^{-1})^\dagger\big)\mathcal{D}^j_{q'q}(Z), \tag{71}$$

we recognize the orthonormal basis functions (65) in the expansion (70), so that we can write the coherent states (wavelets) as:

$$\psi_g(Z) = \sum_{j\in\mathbb{N}/2}\sum_{n=0}^{\infty}\sum_{q,q'=-j}^{j}\hat\psi^{j,n}_{q',q}(g)\varphi^{j,n}_{q'q}(Z) \tag{72}$$

with "Fourier" coefficients

$$\hat{\psi}^{j,n}_{q',q}(g) = \det\big(D^\dagger\big)^{-\lambda}\sqrt{\frac{2j+1}{\lambda-1}\binom{n+\lambda-2}{\lambda-2}\binom{n+2j+\lambda-1}{\lambda-2}}\det\big(\big(BD^{-1}\big)^\dagger\big)^n \times \sum_{q=-j}^{j}\mathcal{D}^j_{qq'}\big(\big(BD^{-1}\big)^\dagger\big) = \overline{\det(D)^{-\lambda}\varphi^{j,n}_{q',q}\big(BD^{-1}\big)}. \tag{73}$$

Using the orthogonality properties (66) of the basis functions (65), we can easily compute

$$\big|\langle\mathcal{U}_\lambda(g)\psi|\psi\rangle\big|^2 = \big|\langle\psi_g|\varphi^{0,0}_{0,0}\rangle\big|^2 = \big|\hat{\psi}^{0,0}_{0,0}(g)\big|^2 = \det\big(DD^\dagger\big)^{-\lambda} = \det\big(I-\tilde{Z}\tilde{Z}^\dagger\big)^\lambda, \tag{74}$$

where we have defined $\tilde{Z} \equiv BD^{-1}$ and used the first condition in (52). Using the Haar measure (61), the admissibility condition (68) gives:

$$c_\psi = \int_{G/H} d\mu(g)|_{G/H}\det\big(I-\tilde{Z}\tilde{Z}^\dagger\big)^\lambda \int_H d\nu(U_1)\,d\nu(U_2) = c_\lambda^{-1}\bigg(\frac{(2\pi)^3}{2}\bigg)^2 < \infty, \tag{75}$$

where we have identified $d\mu(g)|_{G/H}\det(I-\tilde{Z}\tilde{Z}^\dagger)^\lambda = c_\lambda^{-1}d\nu_\lambda(\tilde{Z},\tilde{Z}^\dagger)$ and taken into account that $\int_{\mathbb{D}_4} d\nu_\lambda(Z,Z^\dagger) = 1$ and

$$\nu\big(U(2)\big) = \int_{U(2)} d\nu(U) = \int \frac{|dz|}{(1+z\bar{z})^2}\,d\beta_1\,d\beta_2 = \frac{(2\pi)^3}{2}\int_0^\infty \frac{dx}{(1+x)^2} = \frac{(2\pi)^3}{2} \tag{76}$$

(2π times the area of the 3-sphere $\mathbb{S}^3 = SU(2)$ of unit radius).

Now it remains to prove that the resolution operator (69) is a multiple of the identity $\mathcal{I}$ in $L^2_h(\mathbb{D}_4, d\nu_\lambda)$. For this purpose, we shall compute its matrix elements:

$$\begin{aligned}\big\langle\varphi^{j,m}_{q_1,q_2}\big|\mathcal{A}\big|\varphi^{j',m'}_{q'_1,q'_2}\big\rangle &= \int_G \big\langle\varphi^{j,m}_{q_1,q_2}\big|\psi_g\big\rangle\big\langle\psi_g\big|\varphi^{j',m'}_{q'_1,q'_2}\big\rangle\,d\mu(g) = \int_G \hat{\psi}^{j,m}_{q_1,q_2}(g)\overline{\hat{\psi}^{j',m'}_{q'_1,q'_2}(g)}\,d\mu(g)\\ &= \nu\big(U(2)\big)^2\int_{G/H} d\mu(g)|_{G/H}\det\big(I-\tilde{Z}\tilde{Z}^\dagger\big)^\lambda\overline{\varphi^{j,m}_{q_1,q_2}(\tilde{Z})}\varphi^{j',m'}_{q'_1,q'_2}(\tilde{Z})\\ &= c_\psi\,\delta_{j,j'}\delta_{m,m'}\delta_{q_1,q'_1}\delta_{q_2,q'_2},\end{aligned} \tag{77}$$

where we have used (73), the orthogonality properties (66) of the basis functions (65) and the fact that $G/H = \mathbb{D}_4$. □

The reconstruction formula (128) here adopts the following form:

$$\phi(Z) = \int_G \Phi_\psi(g)\psi_g(Z)\,d\mu(g), \tag{78}$$

with wavelet coefficients

$$\Phi_\psi(g) = \frac{1}{c_\psi}\langle\psi_g|\phi\rangle = \frac{1}{c_\psi}\int_{\mathbb{D}_4}\det\big(D-Z^\dagger B\big)^{-\lambda}\phi(Z)\,d\nu_\lambda\big(Z,Z^\dagger\big). \tag{79}$$

Expanding ϕ in the basis (65)

$$\phi(Z) = \sum_{j\in\mathbb{N}/2}\sum_{n=0}^{\infty}\sum_{q,q'=-j}^{j}\hat{\phi}^{j,n}_{q,q'}\varphi^{j,n}_{q,q'}(Z),$$

and using (72) together with the orthogonality properties (66), we can write the wavelet coefficients (79) in terms of the Fourier coefficients $\hat{\phi}^{j,n}_{q',q}$ as:

$$\Phi_\psi(g) = \frac{1}{c_\psi}\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j}\overline{\hat{\psi}^{j,m}_{q,q'}(g)}\hat{\phi}^{j,m}_{q,q'} = \frac{1}{c_\psi}\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j}\det(D)^{-\lambda}\varphi^{j,m}_{q,q'}\big(BD^{-1}\big)\hat{\phi}^{j,m}_{q,q'}.$$

5.3. Isomorphism between $L_h^2(\mathbb{D}_4, d\nu)$ and $L_h^2(\mathbb{C}_+^4, d\tilde{\nu})$

For completeness, we shall give an isometry between $L_h^2(\mathbb{D}_4, d\nu)$ and $L_h^2(\mathbb{C}_+^4, d\tilde{\nu})$ which allows us to translate mathematical properties and constructions from one space into the other.

Proposition 5.5. *The correspondence*

$$\begin{aligned}\mathcal{S}_\lambda: \quad L_h^2(\mathbb{D}_4, d\nu_\lambda) &\longrightarrow L_h^2\big(\mathbb{C}_+^4, d\tilde{\nu}_\lambda\big),\\ \phi &\longmapsto \mathcal{S}_\lambda\phi \equiv \tilde{\phi},\end{aligned}$$

with

$$\tilde{\phi}(W) = 2^{2\lambda}\det(I - iW)^{-\lambda}\phi\big(Z(W)\big) \tag{80}$$

and $Z(W)$ given by the Cayley transformation (47)*, is an isometry. Actually*

$$\langle\phi|\phi'\rangle_{L_h^2(\mathbb{D}_4, d\nu_\lambda)} = \langle\mathcal{S}_\lambda\phi|\mathcal{S}_\lambda\phi'\rangle_{L_h^2(\mathbb{C}_+^4, d\tilde{\nu}_\lambda)}. \tag{81}$$

Moreover, $\mathcal{S}_\lambda$ is an intertwiner (*equivariant map*) *of the representations* (63) *and* (45)*, that is*:

$$\mathcal{U}_\lambda = \mathcal{S}_\lambda^{-1}\tilde{\mathcal{U}}_\lambda\mathcal{S}_\lambda. \tag{82}$$

Proof. The left-hand side of Eq. (81) is explicitly written as:

$$\langle\phi|\phi'\rangle_{L_h^2(\mathbb{D}_4, d\nu_\lambda)} = \int_{\mathbb{D}_4}\overline{\phi(Z)}\phi'(Z)c_\lambda\det\big(I - ZZ^\dagger\big)^{\lambda-4}|dZ|.$$

Taking into account that

$$\det\big(I - ZZ^\dagger\big) = \det\big(2i\big(W^\dagger - W\big)\big)\big|\det(I - iW)\big|^{-2}$$

and the Jacobian determinant

$$|dZ| = 2^{12}\big|\det(I - iW)\big|^{-8}|dW|,$$

then

$$\begin{aligned}d\nu_\lambda\big(Z, Z^\dagger\big) &= c_\lambda\det\big(I - ZZ^\dagger\big)^{\lambda-4}|dZ|\\ &= 2^{4\lambda-4}\big|\det(I - iW)\big|^{-2\lambda}c_\lambda\det\left(\frac{i}{2}\big(W^\dagger - W\big)\right)^{\lambda-4}|dW|\\ &= 2^{4\lambda}\big|\det(I - iW)\big|^{-2\lambda}d\tilde{\nu}_\lambda\big(W, W^\dagger\big),\end{aligned}$$

which results in:

$$\int_{\mathbb{D}_4}\overline{\phi(Z)}\phi(Z)\,d\nu_\lambda\big(Z, Z^\dagger\big) = \int_{\mathbb{C}_+^4}\overline{\tilde{\phi}(W)}\tilde{\phi}(W)\,d\tilde{\nu}_\lambda\big(W, W^\dagger\big),$$

thus proving (81).

The intertwining relation (82) can be explicitly written as:

$$\begin{aligned}[\mathcal{U}_\lambda\phi](Z) &= \det\big(D^\dagger - B^\dagger Z\big)^{-\lambda}\phi\big(\big(A^\dagger Z - C^\dagger\big)\big(D^\dagger - B^\dagger Z\big)^{-1}\big),\\ \big[\mathcal{S}_\lambda^{-1}\tilde{\mathcal{U}}_\lambda\tilde{\phi}\big](Z) &= \det(I - iW)^\lambda\det\big(R^\dagger - T^\dagger W\big)^{-\lambda}\det\big(I - iW'\big)^{-\lambda}\phi\big(Z\big(W'\big)\big),\end{aligned} \tag{83}$$

where $W' = (Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}$. On the one hand, we have that the argument of ϕ is:

$$\begin{aligned}Z\big(W'\big) &= \big(I + iW'\big)\big(I - iW'\big)^{-1}\\ &= \big(\big(R^\dagger - T^\dagger W\big) + i\big(Q^\dagger W - S^\dagger\big)\big)\big(\big(R^\dagger - T^\dagger W\big) - i\big(Q^\dagger W - S^\dagger\big)\big)^{-1}\\ &= \big(\big(R^\dagger - iS^\dagger\big) + i\big(Q^\dagger + iT^\dagger\big)W\big)\big(\big(R^\dagger + iS^\dagger\big) - i\big(Q^\dagger - iT^\dagger\big)W\big)^{-1}.\end{aligned}$$

Taking now into account the map (50) we have:

$$\begin{aligned}Z(W') &= \big((A^\dagger - C^\dagger) + i(A^\dagger + C^\dagger)W\big)\big((D^\dagger - B^\dagger) - i(D^\dagger + B^\dagger)W\big)^{-1}\\ &= \big(A^\dagger(I+iW) - C^\dagger(I-iW)\big)\big(D^\dagger(I-iW) - B^\dagger(I+iW)\big)^{-1}\\ &= \big(A^\dagger Z - C^\dagger\big)\big(D^\dagger - B^\dagger Z\big)^{-1},\end{aligned}$$

as desired. On the other hand, we have that

$$\begin{aligned}\big(I - iW'\big)\big(R^\dagger - T^\dagger W\big) &= \big(R^\dagger - T^\dagger W\big) - i\big(Q^\dagger W - S^\dagger\big) = \big(R^\dagger + iS^\dagger\big) - i\big(Q^\dagger - iT^\dagger\big)W\\ &= \big(D^\dagger - B^\dagger\big) - i\big(D^\dagger + B^\dagger\big)W = D^\dagger(I-iW) - B^\dagger(I+iW) = \big(D^\dagger - B^\dagger Z\big)(I-iW)\end{aligned}$$

which implies

$$\det(I-iW)^\lambda \det\big(R^\dagger - T^\dagger W\big)^{-\lambda}\det\big(I - iW'\big)^{-\lambda} = \det\big(D^\dagger - B^\dagger Z\big)^{-\lambda}. \tag{84}$$

That is, the equality of multipliers in (83). □

As a direct consequence of Proposition 5.5, the set of functions defined by

$$\tilde\varphi^{j,m}_{q_1,q_2}(W) \equiv 2^{2\lambda}\det(I-iW)^{-\lambda}\varphi^{j,m}_{q_1,q_2}\big(Z(W)\big), \tag{85}$$

with $\varphi^{j,m}_{q_1,q_2}$ defined in (65), constitutes an orthonormal basis of $L^2_h(\mathbb{C}^4_+, d\tilde\nu_\lambda)$ and the closure relation

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q,q'=-j}^{j}\overline{\tilde\varphi^{j,m}_{q',q}(W)}\tilde\varphi^{j,m}_{q',q}(W') = \det\bigg(\frac{i}{2}\big(W^\dagger - W'\big)\bigg)^{-\lambda}, \tag{86}$$

gives the reproducing (Bergman) kernel in $L^2_h(\mathbb{C}^4_+, d\tilde\nu_\lambda)$.

The isometry (80) also allows us to translate the results of Theorem 5.4 form $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ into $L^2_h(\mathbb{C}^4_+, d\tilde\nu_\lambda)$. Indeed, from (80) we conclude that the function $\tilde\psi \in L^2_h(\mathbb{C}^4_+, d\tilde\nu_\lambda)$ given by:

$$\tilde\psi(W) = 2^{2\lambda}\det(I-iW)^{-\lambda} \tag{87}$$

is admissible. The construction of a tight frame and a reconstruction formula form this mother wavelet parallels (69) and (78), respectively.

5.4. Symmetry properties of the proposed conformal wavelets

When working with wavelets on the sphere [8–10] it is customary to take *axisymmetric* (or zonal) wavelets, that is, admissible vectors ψ which are invariant under rotations around the (namely) z-axis, although more general implementations including directional spherical wavelets are also possible (see e.g. [31]). Let us discuss the symmetry properties of our proposed admissible wavelets (87). Applying a general $SU(2,2)$-transformation (45) to (87) gives:

$$\begin{aligned}&\big[\tilde{\mathcal{U}}_\lambda(f)\tilde\psi\big](W) = 2^{2\lambda}\det\big(R^\dagger - T^\dagger W\big)^{-\lambda}\det\big(I - iW'\big)^{-\lambda},\\ &W' = \big(Q^\dagger W - S^\dagger\big)\big(R^\dagger - T^\dagger W\big)^{-1}.\end{aligned} \tag{88}$$

Using the identity (84) we have:

$$\big[\tilde{\mathcal{U}}_\lambda(f)\tilde\psi\big](W) = \det\big(D^\dagger - B^\dagger Z\big)^{-\lambda}\tilde\psi(W), \tag{89}$$

which leaves invariant $\tilde\psi$ (up to a global phase) if:

$$B = 0 \quad\Rightarrow\quad C = 0 \quad\Rightarrow\quad S = -T, \qquad Q = R, \tag{90}$$

where we have used (52), (53) and (50). Thus, the elements $f \in SU(2,2)$ of the form

$$f = \begin{pmatrix} R & iS \\ iS & R\end{pmatrix} \tag{91}$$

leave invariant (87). The constraints $f^\dagger\gamma^0 f = \gamma^0$ imply:

$$S^\dagger S + R^\dagger R = I, \qquad S^\dagger R = R^\dagger S. \tag{92}$$

For $S = 0$, R is unitary. For $R = I$, S is hermitian with $S^2 = 0$. The last condition is satisfied for translations $S = b_\mu\sigma^\mu$ along null (light-like) vectors $b^2 = b_\mu b^\mu = \det(S) = 0$. This leaves us a seven-dimensional subgroup of $SU(2,2)$, isomorphic to $S(U(2)\times U(2))$, as the isotropy subgroup of the admissible vector (87). Any other basis state (85) could be used as a fiducial state to construct oriented wavelets.

In Fig. 1 we provide a visualization of this wavelet (modulus and argument) for the particular case of $W = w\sigma^0$ (temporal part), $w \equiv x + iy$, for which $\tilde\psi(W) = 2^{2\lambda}(1 - iw)^{-2\lambda}$ reduces to $\tilde\varphi_0(w)$ in (39). We take $\lambda = 1$ for simplicity.

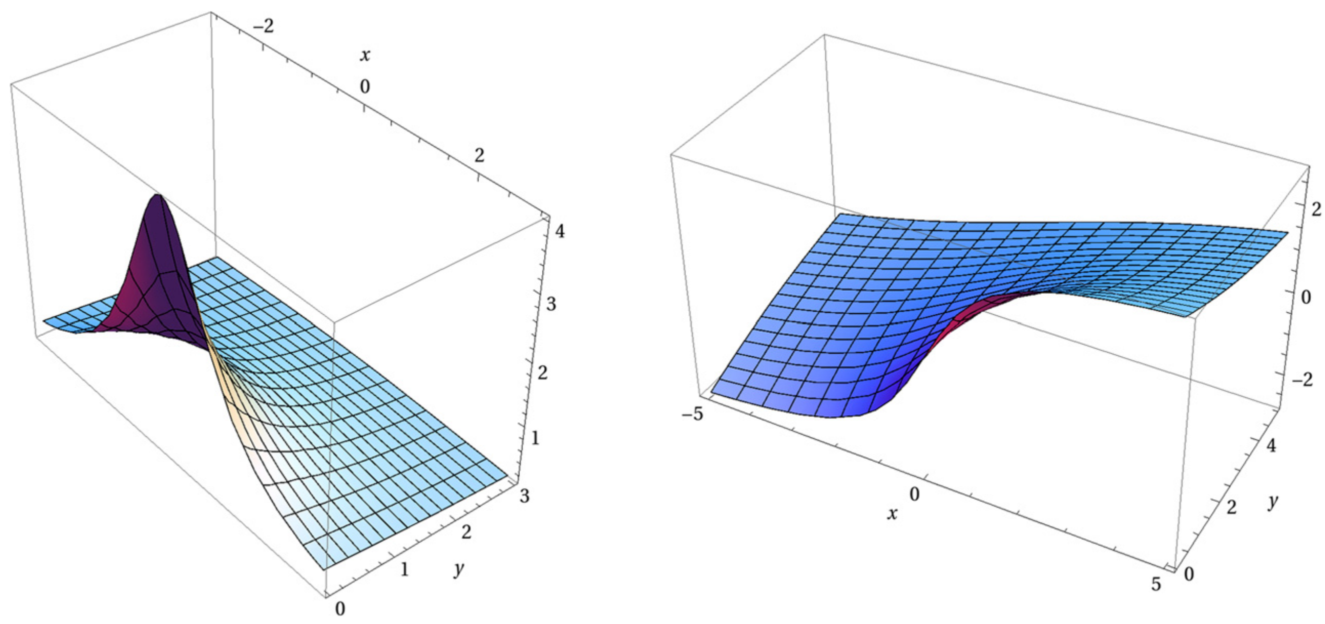

Fig. 1. Modulus and argument of $\tilde{\psi}(W)$ for $\lambda = 1$, $W = (x + iy)\sigma^0$.

5.5. The Euclidean limit

We have seen that the Shilov boundary of $\mathbb{D}_4$ is the compactified Minkowski space $U(2) = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1$ (the four-dimensional analogue of the boundary $U(1) = \mathbb{S}^1$ of the disk $\mathbb{D}_1$). One expects the wavelet transform on $\mathbb{S}^N$ to behave locally (at short scales or large values of the radius ρ) like the usual (flat) wavelet transform on $\mathbb{R}^N$. Indeed, in [27], one of the authors and collaborators discussed the Euclidean limit (infinite radius) for wavelets on $\mathbb{S}^1$. The procedure parallels that of Ref. [9] for wavelets on $\mathbb{S}^2$. In these references, the Euclidean limit is formulated as a contraction at the level of group representations. Let us restrict ourselves, for the sake of simplicity, to the conformal group $SO(1,2)$ in $1+0$ (temporal) dimensions. The realistic $1+3$-dimensional case $SO(4,2)$, although technically more complicated, follows similar guidelines and will be left for future work.

Let us denote simply by $P = P_0$ and $K = K_0$ the temporal components of P_μ and K_μ (the generators of space–time translations and accelerations). The Lie algebra commutators of $SO(1,2)$ are [remember the general N-dimensional case (21)]:

$$[D,P] = -P, \qquad [D,K] = K, \qquad [K,P] = 2D. \tag{93}$$

A contraction $\mathcal{G}'$ of the Lie algebra $\mathcal{G} = so(1,2)$ along $\mathrm{sim}(1)$ (generated by P and D) can be constructed through the one-parameter family of invertible linear mappings $\pi_\rho : \mathbb{R}^3 \to \mathbb{R}^3$, $\rho \in [1,\infty)$ defined by:

$$\pi_\rho(D) = D, \qquad \pi_\rho(P) = P, \qquad \pi_\rho(K) = \rho^{-1}K, \tag{94}$$

such that the Lie bracket of $\mathcal{G}'$ is:

$$[X,Y]' = \lim_{\rho\to\infty} \pi_\rho^{-1}[\pi_\rho X, \pi_\rho Y], \tag{95}$$

with $[\cdot,\cdot]$ the Lie bracket (93) of $\mathcal{G}$. The resulting $\mathcal{G}'$ commutators are:

$$[D,P]' = -P, \qquad [D,K]' = K, \qquad [K,P]' = 0. \tag{96}$$

The contraction process is lifted to the corresponding Lie groups $G' = \mathbb{R}^2 \ltimes \mathbb{R}^+$ and $G = SO(1,2)$ by considering the exponential mapping e^{π_ρ}. The idea is that the representation of G contract to the usual wavelet representation of the affine group $SIM(1)$ in the following sense:

Definition 5.6. Let G' be a contraction of G, defined by the contraction map $\Pi_\rho : G' \to G$, and let $\mathcal{U}'$ be a representation of G' in a Hilbert space $\mathcal{H}'$. Let $\{\mathcal{U}_\rho\}$, $\rho \in [1,\infty)$ be a one-parameter family of representations of G on a Hilbert space $\mathcal{H}_\rho$, and $\iota_\rho : \mathcal{H}_\rho \to \mathcal{D}_\rho$ a linear injective map from $\mathcal{H}_\rho$ onto a dense subspace $\mathcal{D}_\rho \subset \mathcal{H}'$. Then we shall say that $\mathcal{U}'$ is a contraction of the family $\{\mathcal{U}_\rho\}$ if there exists a dense subspace $\mathcal{D}' \subset \mathcal{H}'$ such that, for all $\phi \in \mathcal{D}'$ and $g' \in G'$, one has:

- For every ρ large enough, $\phi \in \mathcal{D}_\rho$ and $\mathcal{U}_\rho(\Pi_\rho(g'))\iota_\rho^{-1}\phi \in \iota_\rho^{-1}\mathcal{D}_\rho$.
- $\lim_{\rho\to\infty} \|\iota_\rho \mathcal{U}_\rho(\Pi_\rho(g'))\iota_\rho^{-1}\phi - \mathcal{U}'(g')\phi\|_{\mathcal{H}'} = 0$, $\forall g' \in G'$.

More precisely, one can prove that:

Theorem 5.7. *The representation* $[\mathcal{U}'(b,a)\phi](x) = \frac{1}{\sqrt{a}}\phi(\frac{x-b}{a})$ *of the affine group SIM*(1) *is a contraction of the one-parameter family* $\mathcal{U}_\rho$ *of representations of SO*(1, 2) *on* $\mathcal{H}_\rho$ *as* $\rho \to \infty$. *That is*:

$$\lim_{\rho\to\infty} \left\| \iota_\rho \mathcal{U}_\rho\big(\Pi_\rho^{\sigma'}(b,a)\big)\iota_\rho^{-1}\phi - \mathcal{U}'(b,a)\phi \right\|_{\mathcal{H}'} = 0, \quad \forall (b,a) \in \mathbb{R} \ltimes \mathbb{R}^+, \tag{97}$$

where $\Pi_\rho^{\sigma'} : SIM(1) \to SO(1,2)/\mathbb{R}$ *is the restricted contraction map, with* $\sigma' : G'/\mathbb{R} \to G'$ *a given section.*

This construction can be straightforwardly extended to $G = SO(4,2)$, the contraction G' being the so-called G_{15} group of Ref. [32]. A thorough discussion of the Euclidean limit of the conformal wavelets constructed in this paper falls beyond the scope of this article and will be left for future work [33]. Here we just wanted to give a flavor of it.

6. Convergence remarks

Schwinger's Theorem 2.1 and its extension 2.3 have been stated in the sense of generating functions in terms of formal power series in some indeterminates. From this point of view, we have disregarded convergence issues. However, infinite series expansions like for instance (12) would require in particular that $|t|^2|\det(X)| < 1$. We shall prove that such convergence requirements, together with additional restrictions coming from the basic Theorem 2.1, are automatically fulfilled inside the complex domain $\mathbb{D}_4$ for $tX = \tilde{Z}^\dagger Z$, with $\tilde{Z} = BD^{-1}$ in the expansion (70). Let us state these convergence requisites.

Proposition 6.1. *A sufficient condition for the convergence of the expansions* (4) *and* (10) *for* $t = 1$ *is that*:

$$|x_{11}| < 1, \qquad |x_{22}| < 1, \qquad |x_{12}x_{21}| < 1, \qquad \left|\det(X)\right| < 1. \tag{98}$$

Proof. Looking at the explicit expression of Wigner's $\mathcal{D}$-matrices (3)

$$\mathcal{D}^j_{q,q}(X) = \sum_{k=\max(0,2q)}^{j+q} \binom{j+q}{k}\binom{j-q}{k-2q} x_{11}^k (x_{12}x_{21})^{j+q-k} x_{22}^{k-2q} \tag{99}$$

we conclude that it is enough to have: $|x_{11}| < 1$, $|x_{22}| < 1$ and $|x_{12}x_{21}| < 1$, for the convergence of (4) for $t = 1$, because their exponents run up to infinity independent of each other. Moreover, if we require convergence in the expansions (10) and (12) for $t = 1$, then $|\det(X)| < 1$ is needed too. □

We shall see that Z and $\tilde{Z}$ fulfill (98), but before we shall prove that

Proposition 6.2. *For any matrix* $Z \in \mathbb{D}_4$ *we have that the squared norm of their rows is lesser than* 1, *that is*:

$$|z_{11}|^2 + |z_{12}|^2 < 1, \qquad |z_{21}|^2 + |z_{22}|^2 < 1. \tag{100}$$

Proof. The positivity condition (54) says that

$$\det\big(I - ZZ^\dagger\big) > 0 \quad \Leftrightarrow \quad |z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22}|^2 < \big(1 - |z_{11}|^2 - |z_{12}|^2\big)\big(1 - |z_{21}|^2 - |z_{22}|^2\big). \tag{101}$$

Hence, the last two factors must be either positive or negative. Supposing that both factors were negative would contradict $\mathrm{tr}(ZZ^\dagger) < 2$ in (57). Therefore, we conclude that both factors are positive. □

Let us remind that, since Z and $\tilde{Z}$ belong to $\mathbb{D}_4$, they must satisfy $|\det(Z)| < 1$ and $|\det(\tilde{Z})| < 1$, as we saw in (55) and (58). Now we are in condition to prove that:

Proposition 6.3. *The matrix* $X = \tilde{Z}^\dagger Z$ *verifies the convergence conditions* (98) *for every* $Z, \tilde{Z} \in \mathbb{D}_4$ *and, therefore, the expansion* (70) *is well defined for* $\tilde{Z} = BD^{-1}$.

Proof. The conditions (100) imply in particular that $|z_{11}| < 1$, $|z_{12}| < 1$, $|z_{21}| < 1$, $|z_{22}| < 1$. Using this fact, the triangle inequality and taking into account that $Z, \tilde{Z} \in \mathbb{D}$ verify (100) and the determinant restriction (58), we arrive to:

$$\begin{aligned}
&|x_{11}| = |\tilde{z}_{11}z_{11} + \tilde{z}_{12}z_{21}| \leqslant |\tilde{z}_{11}z_{11}| + |\tilde{z}_{12}z_{21}| < |\tilde{z}_{11}| + |\tilde{z}_{12}| < |\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 < 1,\\
&|x_{22}| = |\tilde{z}_{21}z_{12} + \tilde{z}_{22}z_{22}| \leqslant |\tilde{z}_{21}z_{12}| + |\tilde{z}_{22}z_{22}| < |\tilde{z}_{21}| + |\tilde{z}_{22}| < |\tilde{z}_{21}|^2 + |\tilde{z}_{22}|^2 < 1,\\
&|x_{12}| = |\tilde{z}_{11}z_{12} + \tilde{z}_{12}z_{22}| \leqslant |\tilde{z}_{11}z_{12}| + |\tilde{z}_{12}z_{22}| < |\tilde{z}_{11}| + |\tilde{z}_{12}| < |\tilde{z}_{11}|^2 + |\tilde{z}_{12}|^2 < 1,\\
&|x_{21}| = |\tilde{z}_{21}z_{11} + \tilde{z}_{22}z_{21}| \leqslant |\tilde{z}_{21}z_{11}| + |\tilde{z}_{22}z_{21}| < |\tilde{z}_{21}| + |\tilde{z}_{22}| < |\tilde{z}_{21}|^2 + |\tilde{z}_{22}|^2 < 1,\\
&\left|\det(X)\right| = \left|\det\big(\tilde{Z}^\dagger Z\big)\right| = \left|\det\big(\tilde{Z}^\dagger\big)\right|\left|\det(Z)\right| = \det\big(\tilde{Z}^\dagger\tilde{Z}\big)^{1/2}\det\big(ZZ^\dagger\big)^{1/2} < 1,
\end{aligned} \tag{102}$$

which proves the convergence conditions (98). □

7. Conclusions and outlook

We have constructed the CWT on the Cartan domain $\mathbb{D}_4 = U(2,2)/U(2)^2$ of the conformal group $SO(4,2) = SU(2,2)/\mathbb{Z}_4$ in 3+1 dimensions. The manifold $\mathbb{D}_4$ can be mapped one-to-one onto the future tube domain $\mathbb{C}^4_+$ of the complex Minkowski space through a Cayley transformation, where we enjoy more physical intuition. This construction paves the way towards a new analysis tool of fields in complex Minkowski space–time with continuum mass spectrum in terms of conformal wavelets. It is traditional in Relativistic Particle Physics to analyze fields or signals (for instance, elementary particles) in Fourier (energy–momentum) space. However, like in music where there are no infinitely lasting sounds, particles are created and destroyed in nuclear reactions. A wavelet transform based on the conformal group provides a way to analyze wave packets localized in both: space and time. Important developments in this direction have also been done in [14–16] for electromagnetic (massless) signals.

In the way, we have stated and proved a λ-extension (10) of the Schwinger's formula (4). This extension turns out to be a useful mathematical tool for us, specially as a generating function for the unitary-representation functions of $SU(2,2)$, the derivation of the reproducing (Bergman) kernel of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ and the proof of admissibility and tight frame conditions. The generalization of this theorem to matrices X of size $N \geqslant 2$ follows similar guidelines and the particular details are discussed in Appendix A, using the general $SU(N)$ solid harmonics $\mathcal{D}^p_{\alpha\alpha}(X)$ of Louck [25]. This result could be of help in studying the discrete series (infinite-dimensional) representations of the non-compact pseudo-unitary groups $SU(N,N)$.

The next step should be the discretization problem. Refs. [34–36] give us the general guidelines to construct discrete (wavelet) frames on the sphere and the hyperboloid and [37] on the Poincaré group. The conformal group is much more involved, though in principle the same scheme applies.

Looking for further potential applications of the conformal wavelets constructed in this article, we think that they could be of use in analyzing renormalizability problems in relativistic quantum field theory. When describing space and time as a continuum, certain statistical and quantum mechanical constructions are ill defined. In order to define them properly, the continuum limit has to be taken carefully starting from a discrete approach. There is a collection of techniques used to take a continuum limit, usually referred as “renormalization rules”, which determine the relationship between parameters in the theory at large and small scales. Renormalization rules fail to define a finite quantum theory of Einstein's General Relativity, one of the main breakthroughs in Theoretical Physics. The replacement of classical (commutative) space–time by a quantum (non-commutative) space–time promises to restore finiteness to quantum gravity at high energies and small (Planck) scales, where geometry becomes also *quantum* (non-commutative) [38]. Conformal wavelets could also be here of fundamental importance as an analysis tool.

Acknowledgments

Work partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219). M.C. thanks the “Universidad Politécnica de Cartagena” and C.A.R.M. for the award “Intensificación de la Actividad Investigadora”. “Este trabajo es resultado de la ayuda concedida por la Fundación Séneca, en el marco del PCTRM 2007–2010, con financiación del INFO y del FEDER de hasta un 80%”. We all thank the anonymous referees for their comments which helped to improve the paper.

Appendix A. Extended MacMahon–Schwinger's Master Theorem for matrices of size $N \geqslant 2$

We have shown the utility of Theorem 2.3 in dealing with unitary representations of $SU(2,2)$, in particular, in proving the admissibility condition 5.4. We would like to have a generalization of Theorem 2.3 for matrices of arbitrary size N, since it would be a valuable tool as a generating function for the unitary-representation functions of $SU(N,N)$.

The first step is to generalize Wigner $\mathcal{D}$-matrices. This generalization has been done in the literature (see [25] and references therein) by the so-called $SU(N)$ solid harmonics $\mathcal{D}^p_{\alpha\beta}(X)$ defined as:

$$\mathcal{D}^p_{\alpha\beta}(X) \equiv \sqrt{\alpha!\beta!} \sum_{A \in \mathbb{M}^p_N(\alpha,\beta)} \frac{X^A}{A!}, \tag{103}$$

where the following space saving notations are employed: A is a $N \times N$ matrix in the non-negative integers a_{ij}; $A! \equiv \prod_{i,j=1}^N a_{ij}!$; $X^A \equiv \prod_{i,j=1}^N x_{ij}^{a_{ij}}$; $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a sequence of N non-negative integers that sum to p (i.e., a composition of N into p non-negative parts), shortly $\alpha \vdash p$; $\alpha! \equiv \prod_{i=1}^N \alpha_i!$; $\mathbb{M}^p_N(\alpha,\beta)$ denotes the set of all matrices A such that the entries in row i sum to α_i and those in column j sum to β_j, with $\alpha \vdash p$ and $\beta \vdash p$. Hence, $\mathcal{D}^p_{\alpha\beta}(X)$ are homogeneous polynomials of degree p in the indeterminates x_{ij}.

The particular identification with Wigner's $\tilde{\mathcal{D}}$-matrices for X of size $N = 2$ is given by $\tilde{\mathcal{D}}^j_{q_1,q_2}(X) = \mathcal{D}^p_{\alpha\beta}(X)$ with $p = 2j$, $\alpha, \beta \vdash 2j$, $\alpha = (j+q_1, j-q_1)$, $\beta = (j+q_2, j-q_2)$. Matrices $A \in \mathbb{M}^p_2(\alpha,\beta)$ can be then indexed by an integer k

$$A^{(k)} \equiv \begin{pmatrix} k & j+q_1-k \\ j+q_2-k & k-q_1-q_2 \end{pmatrix} \tag{104}$$

with $\max(0, q_1+q_2) \leqslant k \leqslant \min(j+q_1, j+q_2)$.

The multiplication property (6) and the transpositional symmetry (7) for Wigner matrices are still valid for $SU(N)$-solid harmonics as:

$$\sum_{\sigma \vdash p} \mathcal{D}^p_{\alpha\sigma}(X)\mathcal{D}^p_{\sigma\beta}(Y) = \mathcal{D}^p_{\alpha\beta}(XY) \tag{105}$$

and

$$\mathcal{D}^p_{\alpha\beta}(Y) = \mathcal{D}^p_{\beta\alpha}\left(Y^T\right) \tag{106}$$

(see [25] for a combinatorial proof).

Moreover, for general $N \times N$ matrices X, the determinant $\det(I - X)$ can be expanded in terms of sums of all principal q-th minors of X as

$$\det(I - X) = \sum_{q=0}^{N} (-1)^{N+q} \sum_{\alpha \Vdash q} \partial_x^\alpha \det(X),$$

where the N-dimensional multi-index $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a partition of q with $\alpha_i \in \{0, 1\}$, a fact that we now symbolize as $\alpha \Vdash q$; $x \equiv (x_{11}, x_{22}, \ldots, x_{NN})$ and $\partial_x^\alpha = \prod_{i=1}^N \partial_{x_{ii}}^{\alpha_i}$. Let us define the sum of all principal $(N-q)$-th minors of X by

$$T_q(X) \equiv \sum_{\alpha \Vdash N-q} \partial_x^\alpha \det(X).$$

They are homogeneous polynomials of degree $q = 0, 1, \ldots, N$ in the indeterminates x_{ij}. For example: $T_0(X) = 1, T_1(X) = \mathrm{tr}(X), \ldots, T_N(X) = \det(X)$. Thus, $\det(I - X)$ can be written in terms of these homogeneous polynomials as:

$$\det(I - X) = \sum_{q=0}^{N} (-1)^q T_q(X). \tag{107}$$

Other possibility could be to use Waring's formulas [25].

To arrive at the λ-extended MacMahon–Schwinger's Master Theorem (MSMT) for $N \times N$ matrices, we shall now proceed step by step from $\lambda = 2$ to general λ. Before, let us explicitly write down the generalization of Theorem 2.1 to matrices of general size N.

Theorem A.1 *(MSMT). The identity*

$$\sum_{p=0}^{\infty} t^p \sum_{\alpha \vdash p} \mathcal{D}^p_{\alpha\alpha}(X) = \det(I - tX)^{-1} \tag{108}$$

holds for any $N \times N$ matrix X.

The action of the operator D_1 on both sides of the Basic MacMahon–Schwinger's formula (108) now gives:

$$\sum_{p=0}^{\infty} (p+1) \sum_{\alpha \vdash p} \mathcal{D}^p_{\alpha\alpha}(tX) = \frac{1 - \sum_{q=2}^N (-1)^q (q-1) T_q(tX)}{\det(I - tX)^2} = \frac{1 - \sum_{q=2}^N \widehat{T}_q(tX)}{\det(I - tX)^2}, \tag{109}$$

where we have defined $\widehat{T}_q(X) \equiv (-1)^q (q-1) T_q(X)$. We can bring the numerator of the right-hand side of (109) back to the left-hand side by using the expansion:

$$\frac{1}{1 - \sum_{q=2}^N \widehat{T}_q(tX)} = \sum_{p=0}^{\infty} \left(\sum_{q=2}^{N} \widehat{T}_q(tX) \right)^p = \sum_{\gamma=0}^{\infty} \binom{\sum_{j=2}^N \gamma_j}{\gamma} \widehat{T}(tX)^\gamma, \tag{110}$$

where we have used the following shorthand for

$$\widehat{T}(X)^\gamma \equiv \widehat{T}_2(X)^{\gamma_2} \widehat{T}_3(X)^{\gamma_3} \cdots \widehat{T}_N(X)^{\gamma_N}, \qquad \binom{\sum_{j=2}^N \gamma_j}{\gamma} \equiv \frac{(\sum_{j=2}^N \gamma_j)!}{\gamma_2! \ldots \gamma_N!}. \tag{111}$$

Note that $\widehat{T}(X)^\gamma$ are homogeneous polynomials of degree $\sum_{j=2}^N j\gamma_j$ in x_{ij}. Inserting the expansion (110) in (109) we conclude that

$$\sum_{p=0}^{\infty}(p+1)\sum_{\gamma=0}^{\infty}\left\{\binom{\sum_{j=2}^N \gamma_j}{\gamma}\right\}\widehat{T}(tX)^\gamma\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=\frac{1}{\det(I-tX)^2}. \tag{112}$$

This is the generalization of (13) for general N. Let us proceed by applying D_2 on both sides of the identity (112):

$$\sum_{p=0}^{\infty}(p+1)\sum_{\gamma=0}^{\infty}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\left(p+2+\sum_{j=2}^N j\gamma_j\right)\widehat{T}(tX)^\gamma\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=2\frac{1-\sum_{q=2}^N\widehat{T}_q(tX)}{\det(I-tX)^3}.$$

Using again (110), we have

$$\sum_{p=0}^{\infty}\frac{p+1}{2}\sum_{\gamma=0}^{\infty}\sum_{\gamma'=0}^{\infty}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\binom{\sum_{j=2}^N \gamma'_j}{\gamma'}\left(p+2+\sum_{j=2}^N j\gamma_j\right)\widehat{T}(tX)^{\gamma+\gamma'}\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)$$
$$=\frac{1}{\det(I-tX)^3}. \tag{113}$$

Rearranging series as in (15) and making the change of $(N-1)$-dimensional multi-index: $\sigma\equiv\gamma+\gamma'$, we obtain

$$\sum_{p=0}^{\infty}\frac{p+1}{2}\sum_{\sigma=0}^{\infty}\left\{\sum_{\gamma=0}^{\sigma}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\binom{\sum_{j=2}^N(\sigma_j-\gamma_j)}{\sigma-\gamma}\left(p+2+\sum_{j=2}^N j\gamma_j\right)\right\}$$
$$\times\widehat{T}(tX)^\sigma\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=\frac{1}{\det(I-tX)^3}. \tag{114}$$

Applying now D_3 on both sides of (113) results:

$$\sum_{p=0}^{\infty}\frac{p+1}{2}\sum_{\gamma=0}^{\infty}\sum_{\gamma'=0}^{\infty}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\binom{\sum_{j=2}^N \gamma'_j}{\gamma'}\left(p+2+\sum_{j=2}^N j\gamma_j\right)\left(p+3+\sum_{j=2}^N j(\gamma_j+\gamma'_j)\right)$$
$$\times\widehat{T}(tX)^{\gamma+\gamma'}\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=3\frac{1-\sum_{q=2}^N\widehat{T}_q(tX)}{\det(I-tX)^4}$$

and using again (110) we get:

$$\sum_{p=0}^{\infty}\frac{p+1}{3!}\sum_{\gamma=0}^{\infty}\sum_{\gamma'=0}^{\infty}\sum_{\gamma''=0}^{\infty}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\binom{\sum_{j=2}^N \gamma'_j}{\gamma'}\binom{\sum_{j=2}^N \gamma''_j}{\gamma''}\left(p+2+\sum_{j=2}^N j\gamma_j\right)$$
$$\times\left(p+3+\sum_{j=2}^N j(\gamma_j+\gamma'_j)\right)\widehat{T}(tX)^{\gamma+\gamma'+\gamma''}\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=\frac{1}{\det(I-tX)^4}. \tag{115}$$

Rearranging series and making the change $\sigma\equiv\gamma+\gamma'+\gamma''$, we can recast the last expression as:

$$\sum_{p=0}^{\infty}\frac{p+1}{3}\sum_{\sigma=0}^{\infty}\left\{\frac{1}{2}\sum_{\gamma=0}^{\sigma}\sum_{\gamma'=0}^{\sigma-\gamma}\binom{\sum_{j=2}^N \gamma_j}{\gamma}\binom{\sum_{j=2}^N \gamma'_j}{\gamma'}\binom{\sum_{j=2}^N(\sigma_j-\gamma_j-\gamma'_j)}{\sigma-\gamma-\gamma'}\right.$$
$$\left.\times\left(p+2+\sum_{j=2}^N j\gamma_j\right)\left(p+3+\sum_{j=2}^N j(\gamma_j+\gamma'_j)\right)\right\}\widehat{T}(tX)^\sigma\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=\frac{1}{\det(I-tX)^4}.$$

If we repeat the process $(\lambda-4)$ more times, then we arrive at the following identity:

$$\sum_{p=0}^{\infty}\frac{p+1}{(\lambda-1)!}\sum_{\gamma=0}^{\infty}\sum_{\gamma'=0}^{\infty}\cdots\sum_{\gamma^{(\lambda-2)}=0}^{\infty}\binom{\sum_{j=2}^N \gamma_j^{(\lambda-2)}}{\gamma^{(\lambda-2)}}\prod_{k=0}^{\lambda-3}\binom{\sum_{j=2}^N \gamma_j^{(k)}}{\gamma^{(k)}}$$
$$\times\left(p+k+2+\sum_{j=2}^N j\sum_{i=0}^{k}\gamma_j^{(i)}\right)\widehat{T}(tX)^{\gamma+\gamma'+\cdots+\gamma^{(\lambda-2)}}\sum_{\alpha\vdash p}\mathcal{D}^p_{\alpha\alpha}(tX)=\frac{1}{\det(I-tX)^\lambda}.$$

Making once more the change $\sigma = \gamma + \gamma' + \cdots + \gamma^{(\lambda-2)}$, we can write:

$$\sum_{p=0}^{\infty} \frac{p+1}{\lambda-1} \sum_{\sigma=0}^{\infty} C^{\lambda}_{p,\sigma} \widehat{T}(tX)^{\sigma} \sum_{\alpha\vdash p} \mathcal{D}^{p}_{\alpha\alpha}(tX) = \frac{1}{\det(I-tX)^{\lambda}},$$

where we have defined the following coefficients:

$$C^{\lambda}_{p,\sigma} \equiv \frac{1}{(\lambda-2)!} \sum_{\gamma=0}^{\sigma} \sum_{\gamma'=0}^{\sigma-\gamma} \cdots \sum_{\gamma^{(\lambda-3)}=0}^{\sigma-\gamma-\cdots-\gamma^{(\lambda-4)}} \binom{\sum_{j=2}^{N}(\sigma_j - \sum_{i=0}^{\lambda-3} \gamma_j^{(i)})}{\sigma - \sum_{i=0}^{\lambda-3} \gamma^{(i)}} \times \prod_{k=0}^{\lambda-3} \binom{\sum_{j=2}^{N} \gamma_j^{(k)}}{\gamma^{(k)}} \left(p+k+2+\sum_{j=2}^{N} j \sum_{i=0}^{k} \gamma_j^{(i)}\right). \tag{116}$$

In order to account for the particular coefficients $C^{2}_{p,\sigma}$ and $C^{3}_{p,\sigma}$, given inside curly brackets in (112) and (114), we must understand in (116) that: (1) summations on $\gamma^{(k)}$ with $k<0$ are absent, (2) empty or nullary sums are zero, and (3) empty or nullary products are 1, as customary.

Summarizing, we can enunciate the following:

Theorem A.2 *(λ-extended MSMT). For every $\lambda \in \mathbb{N}$, $\lambda \geqslant 2$ and every $N \times N$ matrix X, the following identity holds*:

$$\sum_{p=0}^{\infty} \frac{p+1}{\lambda-1} \sum_{\sigma=0}^{\infty} t^{p+\sum_{j=2}^{N} j\sigma_j} C^{\lambda}_{p,\sigma} \widehat{T}(X)^{\sigma} \sum_{\alpha\vdash p} \mathcal{D}^{p}_{\alpha\alpha}(X) = \det(I-tX)^{-\lambda}, \tag{117}$$

with $C^{\lambda}_{p,\sigma}$ given by (116) *and $\widehat{T}(X)^{\sigma}$ by* (111).

The expression (117) generalizes (10) for matrices X of arbitrary size N. In fact, for $N=2$, the coefficient (116) reduces to:

$$C^{\lambda}_{p,\sigma_2} = \binom{\lambda-2+\sigma_2}{\lambda-2} \binom{\lambda-1+p+\sigma_2}{\lambda-2}, \tag{118}$$

which agrees with (10). We have also been able to find simplifications of $C^{\lambda}_{p,\sigma}$ in the following cases (we take the binomials in the generalized sense $\binom{n}{m} = n(n-1)\cdots(n-m+1)/m!$ to account for fractional n):

(i) For $2 \leqslant \lambda \leqslant 5$, the coefficients (116) are given by:

$$C^{\lambda}_{p,\sigma} = \binom{\sum_{j=2}^{N} \sigma_j}{\sigma} \binom{\lambda-2+\sum_{k=2}^{N} \sigma_k}{\lambda-2} \times \left\{ \binom{\lambda-1+p+\frac{1}{2}\sum_{k=2}^{N} k\sigma_k}{\lambda-2} + \binom{\lambda-2+p+\frac{1}{2}\sum_{k=2}^{N} k\sigma_k}{\lambda-4} \frac{1}{4!} \sum_{k=3}^{N} (k-2)k\sigma_k \right\}.$$

(ii) For $N=3$ and $\lambda \geqslant 2$, the coefficients (116) can be obtained trough the expression:

$$C^{\lambda}_{p,\sigma} = \binom{\sigma_2+\sigma_3}{\sigma} \binom{\lambda-2+\sigma_2+\sigma_3}{\lambda-2} \sum_{i=1}^{\frac{\lambda-\xi}{2}} \binom{\lambda-i+p+\sigma_2+\frac{3}{2}\sigma_3}{\lambda-2i} \prod_{j=1}^{i-1} \frac{\sigma_3-2(j-1)}{8j}, \tag{119}$$

where we have defined $\xi \equiv \mathrm{Odd}(\lambda)$, that is, $\xi=0$ when λ is even and $\xi=1$ when odd. See that (119) reduces to (118) for $\sigma_3=0$.

Appendix B. Continuous wavelet transform on a manifold: a brief

The usual CWT on the real line $\mathbb{R}$ is derived from the natural unitary representation of the affine group $G = SIM(1)$ in the space of finite energy signals $L^2(\mathbb{R}, dx)$ (see Section 4 for a reminder). The same scheme applies to the CWT on a general manifold $\mathbb{X}$, subject to the transitive action, $x \to gx$, $g \in G$, $x \in \mathbb{X}$, of some group of transformations G which contains dilations and motions on $\mathbb{X}$. If the measure $d\nu(x)$ in $\mathbb{X}$ is G-invariant (i.e. $d\nu(gx) = d\nu(x)$), then the natural left action of G on $L^2(\mathbb{X}, d\nu)$ given by:

$$[\mathcal{U}(g)\phi](x) = \phi(g^{-1}x), \quad g \in G,\ \phi \in L^2(\mathbb{X}, d\nu), \tag{120}$$

defines a unitary representation, that is:

$$\langle \mathcal{U}(g)\varphi \big| \mathcal{U}(g)\phi \rangle = \langle \varphi|\phi\rangle \equiv \int_{\mathbb{X}} \overline{\varphi(x)}\phi(x)\,d\nu(x).$$

When $d\nu$ is not strictly invariant (i.e. $d\nu(gx) = \mathcal{M}(g,x)\,d\nu(x)$), we have to introduce a *multiplier* (Radon–Nikodym derivative)

$$\big[\mathcal{U}(g)\phi\big](x) = \mathcal{M}(g,x)^{1/2}\phi\big(g^{-1}x\big), \quad g \in G,\ \phi \in L^2(\mathbb{X}, d\nu), \tag{121}$$

in order to keep unitarity. The fact that $\mathcal{U}(g)\mathcal{U}(g') = \mathcal{U}(gg')$ (i.e. $\mathcal{U}$ is a representation of G) implies cohomology conditions for multipliers, that is:

$$\mathcal{M}\big(gg', x\big) = \mathcal{M}(g,x)\mathcal{M}\big(g', g^{-1}x\big). \tag{122}$$

Consider now the space $L^2(G, d\mu)$ of square-integrable complex functions Ψ on G, where $d\mu(g) = d\mu(g'g)$, $\forall g' \in G$, stands for the left-invariant Haar measure, which defines the scalar product

$$(\Psi|\Phi) = \int_G \overline{\Psi(g)}\Phi(g)\,d\mu(g).$$

A non-zero function $\psi \in L^2(\mathbb{X}, d\nu)$ is called *admissible* (or a *fiducial* vector) if $\Psi(g) \equiv \langle \mathcal{U}(g)\psi|\psi\rangle \in L^2(G, d\mu)$, that is, if

$$c_\psi = \int_G \overline{\Psi(g)}\Psi(g)\,d\mu(g) = \int_G \big|\langle \mathcal{U}(g)\psi \big| \psi\rangle\big|^2\,d\mu(g) < \infty. \tag{123}$$

A unitary representation for which admissible vector exists is called *square integrable*. For a square integrable representation, besides Eq. (123) the following property holds (see [1]):

$$\int_G \big|\langle \mathcal{U}(g)\psi \big| \phi\rangle\big|^2\,d\mu(g) < \infty, \quad \forall \phi \in L^2(\mathbb{X}, d\nu). \tag{124}$$

Let us assume that the representation $\mathcal{U}$ is *irreducible*, and that there exists a function ψ admissible, then a system of coherent states (CS) of $L^2(\mathbb{X}, d\nu)$ associated to (or indexed by) G is defined as the set of functions in the orbit of ψ under G

$$\psi_g \equiv \mathcal{U}(g)\psi, \quad g \in G.$$

There are representations without admissible vectors, since the integration with respect to some subgroup diverges. In this case, or even for convenience when admissible vectors exist, we can restrict ourselves to a suitable homogeneous space $Q = G/H$, for some closed subgroup H. Then, the non-zero function ψ is said to be admissible mod(H, σ) (with $\sigma : Q \to G$ a given section) and the representation $\mathcal{U}$ square integrable mod(H, σ), if the condition

$$\int_Q \big|\langle \mathcal{U}\big(\sigma(q)\big)\psi \big| \phi\rangle\big|^2\,d\check{\mu}(q) < \infty, \quad \forall \phi \in L^2(\mathbb{X}, d\nu) \tag{125}$$

holds, where $d\check{\mu}$ is a measure on Q "projected" from the left-invariant measure $d\mu$ on the whole G (see [39] for more details on this projection procedure). Note that this more general definition of square integrability includes the previous one for the trivial subgroup $H = \{e\}$ and σ the identity function. The notions of square integrability and admissibility mod(H, σ) were introduced in [40] (see also [4]).

The coherent states indexed by Q are defined as $\psi_{\sigma(q)} = U(\sigma(q))\psi$, $q \in Q$, and they form an overcomplete set in $L^2(\mathbb{X}, d\nu)$.

The condition (125) could also be written as an "expectation value"

$$0 < \int_Q \big|\langle U\big(\sigma(q)\big)\psi \big| \phi\rangle\big|^2\,d\check{\mu}(q) = \langle \phi|\mathcal{A}_\sigma|\phi\rangle < \infty, \quad \forall \phi \in L^2(\mathbb{X}, d\nu), \tag{126}$$

where $\mathcal{A}_\sigma = \int_Q |\psi_{\sigma(q)}\rangle\langle\psi_{\sigma(q)}|\,d\check{\mu}(q)$ is a positive, bounded, invertible operator. If the operator $\mathcal{A}_\sigma^{-1}$ is also bounded, then the set $F_\sigma = \{|\psi_{\sigma(q)}\rangle,\ q \in Q\}$ is called a *frame*, and a *tight frame* if $\mathcal{A}_\sigma$ is a positive multiple of the identity, $\mathcal{A}_\sigma = cI$, $c > 0$.

To avoid domain problems in the following, let us assume that ψ generates a frame (i.e., that $\mathcal{A}_\sigma^{-1}$ is bounded). The *Coherent State map* is defined as the linear map

$$\begin{aligned} \mathcal{T}_\psi:\quad & L^2(\mathbb{X}, d\nu) \longrightarrow L^2(Q, d\check{\mu}), \\ & \phi \longmapsto \mathcal{T}_\psi\phi \equiv \Phi_\psi, \end{aligned} \tag{127}$$

with $\Phi_\psi(q) = \frac{\langle\psi_{\sigma(q)}|\phi\rangle}{\sqrt{c_\psi}}$. Its range $L^2_\psi(Q, d\check{\mu}) \equiv \mathcal{T}_\psi(L^2(\mathbb{X}, d\nu))$ is complete with respect to the scalar product $(\Phi|\Phi')_\psi \equiv (\Phi|\mathcal{T}_\psi\mathcal{A}_\sigma^{-1}\mathcal{T}_\psi^{-1}\Phi')_Q$ and $\mathcal{T}_\psi$ is unitary from $L^2(\mathbb{X}, d\nu)$ onto $L^2_\psi(Q, d\check{\mu})$. Thus, the inverse map $\mathcal{T}_\psi^{-1}$ yields the *reconstruction formula*

$$\phi = \mathcal{T}_\psi^{-1}\Phi_\psi = \int_Q \Phi_\psi(q)\mathcal{A}_\sigma^{-1}\psi_{\sigma(q)}\, d\check{\mu}(q), \quad \Phi_\psi \in L^2_\psi(Q, d\check{\mu}), \tag{128}$$

which expands ϕ in terms of coherent states (wavelets) $\mathcal{A}_\sigma^{-1}\psi_{\sigma(q)}$ with wavelet coefficients $\Phi_\psi(q) = [\mathcal{T}_\psi\phi](q)$. These formulas acquire a simpler form when $\mathcal{A}_\sigma$ is a multiple of the identity, as it is precisely the case considered in this article.

Appendix C. Orthonormality of homogeneous polynomials

In order to prove the orthonormality relations (66), we shall adopt the following decomposition for a matrix $Z \in \mathbb{D}_4$

$$Z = U_1 \Xi U_2^\dagger,$$

where $U_{1,2} \in U(2)/U(1)^2$ (as in (60) with $\beta_1 = \beta_2 = 0$) and $\Xi = \operatorname{diag}(\xi_1, \xi_2\}, \xi_{1,2} \in \mathbb{D}_1$. This parametrization ensures that $Z \in \mathbb{D}_4$; in fact

$$I - ZZ^\dagger = U_1\big(I - \Xi\Xi^\dagger\big)U_1^\dagger > 0 \tag{129}$$

since the eigenvalues are $1 - |\xi_{1,2}|^2 > 0$.

Let us perform this change of variables in the invariant measure (64) of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$. On the one hand, the Lebesgue measure on $\mathbb{C}^4$ can be written as:

$$|dZ| = J|d\xi_1||d\xi_2|\, ds(U_1)\, ds(U_2),$$

with $ds(U_{1,2})$ defined in (62) and $J = \frac{1}{2}(|\xi_1|^2 - |\xi_2|^2)^2$ is the Jacobian determinant. The Lebesgue measures $|d\xi_{1,2}|$ and $|dz_{1,2}|$ will be written in polar coordinates $\xi_k = \rho_k e^{i\theta_k}$ and $z_k = r_k e^{i\alpha_k}$, $k = 1, 2$. On the other hand, the weight factor in (64) adopts the form

$$\det\big(I - ZZ^\dagger\big)^{\lambda-4} = \big(\big(1 - \rho_1^2\big)\big(1 - \rho_2^2\big)\big)^{\lambda-4} \equiv \Omega(\rho),$$

so that the invariant measure of $L^2_h(\mathbb{D}_4, d\nu_\lambda)$ reads:

$$\begin{aligned} d\nu_\lambda\big(Z, Z^\dagger\big) &= c_\lambda J(\rho)\Omega(\rho)|d\xi_1||d\xi_2|\, ds(U_1)\, ds(U_2) \\ &= c_\lambda J(\rho)\Omega(\rho)\rho_1\, d\rho_1\, d\theta_1\, \rho_2\, d\rho_2\, d\theta_2\big(1 + r_1^2\big)^{-2} r_1\, dr_1 d\alpha_1\big(1 + r_2^2\big)^{-2} r_2\, dr_2\, d\alpha_2, \end{aligned} \tag{130}$$

with $0 \leqslant \rho_{1,2} < 1$, $0 \leqslant r_{1,2} < \infty$, $0 \leqslant \theta_{1,2} < 2\pi$, $0 \leqslant \alpha_{1,2} < 2\pi$. Let us call

$$\mathcal{N}_{j,m} \equiv \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}$$

the normalization constants of the basis functions (65). We want to evaluate:

$$\big\langle\varphi^{j,m}_{q_1,q_2}\big|\varphi^{j',m'}_{q'_1,q'_2}\big\rangle = \mathcal{N}_{j,m}\mathcal{N}_{j',m'}\int_{\mathbb{D}_4} d\nu_\lambda\big(Z, Z^\dagger\big)\overline{\det(Z)^m\mathcal{D}^j_{q_1,q_2}(Z)}\,\det(Z)^{m'}\mathcal{D}^{j'}_{q'_1,q'_2}(Z). \tag{131}$$

Using determinant properties, Wigner's $\mathcal{D}$-matrix properties (6) and (7), and the fact that $\det(U_{1,2}) = 1$ and Ξ is diagonal, the previous expression can be restated as:

$$\begin{aligned} \frac{\big\langle\varphi^{j,m}_{q_1,q_2}, \varphi^{j',m'}_{q'_1,q'_2}\big\rangle}{\mathcal{N}_{j,m}\mathcal{N}_{j',m'}} &= \sum_{q=-j}^{j}\sum_{q'=-j'}^{j'} c_\lambda \int_{\mathbb{D}_1^2} J\Omega\, |d\xi_1|\,|d\xi_2|\mathcal{D}^j_{q,q}(\overline{\Xi})\mathcal{D}^{j'}_{q',q'}(\Xi)\det(\overline{\Xi})^m\det(\Xi)^{m'} \\ &\quad\times \int_{\mathbb{S}^2} ds(U_1)\mathcal{D}^j_{q_1,q}(\overline{U_1})\mathcal{D}^{j'}_{q'_1,q'}(U_1)\int_{\mathbb{S}^2} ds(U_2)\mathcal{D}^j_{q_2,q}(U_2)\mathcal{D}^{j'}_{q'_2,q'}(\overline{U_2}). \end{aligned} \tag{132}$$

Let us start evaluating the first integral. For the diagonal matrix Ξ we have that $\mathcal{D}^j_{q_1,q_2}(\Xi) = \delta_{q_1,q_2}\xi_1^{j+q_1}\xi_2^{j-q_1}$, so that

$$\begin{aligned} &\mathcal{D}^j_{q,q}(\overline{\Xi})\mathcal{D}^{j'}_{q',q'}(\Xi)\det(\overline{\Xi})^m\det(\Xi)^{m'} \\ &\quad= \overline{\xi_1}^{j+q}\overline{\xi_2}^{j-q}\xi_1^{j'+q'}\xi_2^{j'-q'}\overline{\xi_1}^m\overline{\xi_2}^m\xi_1^{m'}\xi_2^{m'} \\ &\quad= \rho_1^{j+j'+q+q'+m+m'}\rho_2^{j+j'-q-q'+m+m'}e^{i(j'-j+q'-q+m'-m)\theta_1}e^{i(j'-j+q-q'+m'-m)\theta_2}. \end{aligned} \tag{133}$$

Integrating out angular variables gives the restrictions

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{D}^{j}_{q,q}(\overline{\Xi})\mathcal{D}^{j'}_{q',q'}(\Xi)\det(\overline{\Xi})^m\det(\Xi)^{m'}\,d\theta_1\,d\theta_2 = 4\pi^2\delta_{q,q'}\delta_{j+m,j'+m'}\rho_1^{2(j+q+m)}\rho_2^{2(j-q+m)}.$$

Integrating the radial part:

$$4\pi^2 c_\lambda \int_0^1\int_0^1 J(\rho)\Omega(\rho)\rho_1^{2(j+q+m)}\rho_2^{2(j-q+m)}\rho_1\,d\rho_1\rho_2\,d\rho_2 = \frac{(j+m)^2+(j+m+2q^2+1)\lambda-5q^2-1}{\pi^2(\lambda-1)\binom{j+m+q+\lambda-1}{\lambda-1}\binom{j+m-q+\lambda-1}{\lambda-1}} \equiv \mathcal{R}^q_{j+m}$$

and putting all together in (132) we have:

$$\frac{\langle \varphi^{j,m}_{q_1,q_2}, \varphi^{j',m'}_{q'_1,q'_2}\rangle}{\mathcal{N}_{j,m}\mathcal{N}_{j',m'}} = \delta_{j+m,j'+m'}\sum_{q=-\min\{j,j'\}}^{\min\{j,j'\}}\mathcal{R}^q_{j+m} \times \int_{\mathbb{S}^2} ds(U_1)\mathcal{D}^j_{q_1,q}(\overline{U_1})\mathcal{D}^{j'}_{q'_1,q}(U_1)\int_{\mathbb{S}^2} ds(U_2)\mathcal{D}^j_{q_2,q}(U_2)\mathcal{D}^{j'}_{q'_2,q}(\overline{U_2}). \tag{134}$$

The last two integrals are easily computable. Actually they are a particular case of the orthogonality properties of Wigner's $\mathcal{D}$-matrices. More explicitly:

$$\int_{\mathbb{S}^2} ds(U)\mathcal{D}^j_{q_1,q_2}(\overline{U})\mathcal{D}^{j'}_{q'_1,q_2}(U) = \int_0^\infty\int_0^{2\pi}\frac{r\,dr\,d\alpha}{(1+r^2)^2}\mathcal{D}^j_{q_1,q_2}(\overline{U})\mathcal{D}^{j'}_{q'_1,q_2}(U) = \delta_{j,j'}\delta_{q_1,q'_1}\frac{\pi}{2j+1}.$$

Going back to (134) it results:

$$\langle \varphi^{j,m}_{q_1,q_2}|\varphi^{j',m'}_{q'_1,q'_2}\rangle = \delta_{j,j'}\delta_{m,m'}\delta_{q_1,q'_1}\delta_{q_2,q'_2}\left(\frac{\mathcal{N}_{j,m}}{2j+1}\right)^2\sum_{q=-j}^{j}\pi^2\mathcal{R}^q_{j+m}.$$

Finally, taking into account the combinatorial identity:

$$\sum_{q=-j}^{j}(\lambda-1)\pi^2\mathcal{R}^q_{j+m} = \frac{2j+1}{\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}$$

and the explicit expression of the normalization constants $\mathcal{N}_{j,m}$, we arrive at the orthonormality relations (66).

References

[1] A. Grossmann, J. Morlet, T. Paul, Transforms associated to square integrable group representations I. General results, J. Math. Phys. 26 (1985) 2473–2479.
[2] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press, 1999.
[3] G. Kaiser, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.
[4] S.T. Ali, J.-P. Antoine, J.P. Gazeau, Coherent States, Wavelets and Their Generalizations, Springer, 2000.
[5] H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Springer Lecture Notes in Math., vol. 1863, Springer-Verlag, Heidelberg, 2005.
[6] J.-P. Antoine, D. Rosca, P. Vandergheynst, Wavelet transform on manifolds: old and new approaches, Appl. Comput. Harmon. Anal. 28 (2010) 189–202.
[7] H. Führ, Painless Gabor expansions on homogeneous manifolds, Appl. Comput. Harmon. Anal. 26 (2009) 200–211.
[8] M. Holschneider, Continuous wavelet transforms on the sphere, J. Math. Phys. 37 (1996) 4156–4165.
[9] J.-P. Antoine, P. Vandergheynst, Wavelets on the 2-sphere: a group-theoretical approach, Appl. Comput. Harmon. Anal. 7 (1999) 262–291.
[10] J.-P. Antoine, P. Vandergheynst, Wavelets on the *n*-sphere and related manifolds, J. Math. Phys. 39 (1998) 3987–4008.
[11] I. Bogdanova, P. Vandergheynst, J.P. Gazeau, Continuous wavelet transform on the hyperboloid, Appl. Comput. Harmon. Anal. 23 (2007) 285–306.
[12] E. Cunningham, The principle of relativity in electrodynamics and an extension thereof, Proc. Lond. Math. Soc. 8 (1910) 77–98.
[13] H. Bateman, The transformation of the electrodynamical equations, Proc. Lond. Math. Soc. (3) 8 (1910) 223–264.
[14] G. Kaiser, Space–time-scale analysis of electromagnetic waves, in: Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, Victoria, 1992, pp. 209–212.
[15] G. Kaiser, Wavelet electrodynamics II: atomic composition of electromagnetic waves, Appl. Comput. Harmon. Anal. 1 (1994) 246–260.
[16] G. Kaiser, Physical wavelets and their sources: real physics in complex spacetime, J. Phys. A: Math. Gen. 36 (2003) R291–R338.
[17] W. Rühl, Field representations of the conformal group with continuous mass spectrum, Comm. Math. Phys. 30 (1973) 287–302.
[18] R. Coquereaux, A. Jadczyk, Conformal theories, curved phase spaces, relativistic wavelets and the geometry of complex domains, Rev. Math. Phys. 2 (1990) 1–44.
[19] J. Schwinger, On angular momentum, U.S. Atomic Energy Commission Report NYO-3071, 1952, unpublished. Reprinted in: L.C. Biedenharn, H. van Dam (Eds.), Quantum Theory of Angular Momentum, Academic Press, New York, 1965, pp. 229–279.
[20] P.A. MacMahon, Combinatory Analysis, vol. I, Cambridge Univ. Press, 1915.
[21] S. Garoufalidis, T.T.Q. Lê, D. Zeilberger, The quantum MacMahon Master Theorem, Proc. Natl. Acad. Sci. USA 38 (2006) 13928–13931.
[22] D. Foata, D. Zeilberger, Laguerre polynomials, weighted derangements, and positivity, SIAM J. Discrete Math. 1 (1988) 425–433.

[23] J.D. Louck, MacMahon's Master Theorem, double tableau polynomials, and representations of groups, Adv. in Appl. Math. 17 (1996) 143–168.
[24] W.Y.C. Chen, J.D. Louck, The combinatorics of a class of representation functions, Adv. Math. 140 (1998) 207–236.
[25] J.D. Louck, Unitary Symmetry and Combinatorics, World Scientific, Singapore, 2008.
[26] L.C. Biedenharn, J.D. Louck, Angular Momentum in Quantum Physics, Addison–Wesley, Reading, MA, 1981;
L.C. Biedenharn, J.D. Louck, The Racah–Wigner Algebra in Quantum Theory, Addison–Wesley, New York, MA, 1981.
[27] M. Calixto, J. Guerrero, Wavelet transform on the circle and the real line: a group-theoretical treatment, Appl. Comput. Harmon. Anal. 21 (2006) 204–229.
[28] A.O. Barut, J.R. Zenitt, A. Lauferts, The exponential map for the conformal group $SO(2,4)$, J. Phys. (Math. Gen.) A 27 (1994) 5239–5250.
[29] A.O. Barut, J.R. Zenitt, A. Lauferts, The exponential map for the unitary group $SU(2,2)$, J. Phys. (Math. Gen.) A 27 (1994) 6799–6805.
[30] A.O. Barut, R. Raczka, Theory of Group Representations and Applications, second ed., World Scientific Publishing Co., Singapore, 1986.
[31] J.-P. Antoine, L. Demanet, L. Jacques, P. Vandergheynst, Wavelets on the sphere: Implementation and approximations, Appl. Comput. Harmon. Anal. 13 (2002) 177–200.
[32] V. Aldaya, J.A. de Azcárraga, Group manifold analysis of the structure of relativistic quantum dynamics, Ann. Phys. 165 (1985) 484–504.
[33] M. Calixto, E. Pérez-Romero, Euclidean and non-relativistic limits of conformal wavelets, under investigation.
[34] M. Calixto, J. Guerrero, J.C. Sánchez-Monreal, Sampling theorem and discrete Fourier transform on the Riemann sphere, J. Fourier Anal. Appl. 14 (2008) 538–567.
[35] M. Calixto, J. Guerrero, J.C. Sánchez-Monreal, Sampling theorem and discrete Fourier transform on the hyperboloid, J. Fourier Anal. Appl. (2010), doi:10.1007/s00041-010-9142-5.
[36] I. Bogdanova, P. Vandergheynst, J.-P. Antoine, L. Jacques, M. Morvidone, Stereographic wavelet frames on the sphere, Appl. Comput. Harmon. Anal. 19 (2005) 223–252.
[37] J.-P. Antoine, A.L. Hohouéto, Discrete frames of Poincaré coherent states in $1+3$ dimensions, J. Fourier Anal. Appl. 9 (2003) 141–173.
[38] J. Madore, An Introduction to Noncommutative Differential Geometry and Its Physical Applications, second ed., London Math. Soc. Lecture Note Ser., vol. 257, Cambridge Univ. Press, 1999.
[39] J. Guerrero, V. Aldaya, Invariant measures on polarized submanifolds in group quantization, J. Math. Phys. 41 (2000) 6747–6765.
[40] S.T. Ali, J.-P. Antoine, J.P. Gazeau, Square integrability of group representations on homogeneous spaces I. Reproducing triples and frames, Ann. Inst. H. Poincaré 55 (1991) 829–856.

Capítulo 3

Teoría Cuántica Invariante Conforme y Reciprocidad de Born

International Journal of Geometric Methods in Modern Physics
Vol. 8, No. 3 (2011) 587–619

DOI: 10.1142/S0219887811005282

CONFORMAL SPINNING QUANTUM PARTICLES IN COMPLEX MINKOWSKI SPACE AS CONSTRAINED NONLINEAR SIGMA MODELS IN $U(2,2)$ AND BORN'S RECIPROCITY

M. CALIXTO*,‡ and E. PÉREZ-ROMERO†

*Departamento de Matemática Aplicada, Facultad de Ciencias
Universidad de Granada Campus de Fuentenueva
18071 Granada, Spain

†,‡Instituto de Astrofísica de Andalucía (IAA-CSIC)
Apartado Postal 3004, 18080 Granada, Spain
‡calixto@ugr.es

Received 30 June 2010
Accepted 11 November 2010

We revise the use of eight-dimensional conformal, complex (Cartan) domains as a base for the construction of conformally invariant quantum (field) theory, either as phase or configuration spaces. We follow a gauge-invariant Lagrangian approach (of nonlinear sigma-model type) and use a generalized Dirac method for the quantization of constrained systems, which resembles in some aspects the standard approach to quantizing coadjoint orbits of a group G. Physical wave functions, Haar measures, orthonormal basis and reproducing (Bergman) kernels are explicitly calculated in and holomorphic picture in these Cartan domains for both scalar and spinning quantum particles. Similarities and differences with other results in the literature are also discussed and an extension of Schwinger's Master theorem is commented in connection with closure relations. An adaptation of the Born's Reciprocity Principle (BRP) to the conformal relativity, the replacement of space-time by the eight-dimensional conformal domain at short distances and the existence of a maximal acceleration are also put forward.

Keywords: Coherent states; reproducing kernels; Cartan domain; conformal relativity; nonlinear sigma models; constrained quantization; Born reciprocity.

Mathematics Subject Classification: 81S10, 81T40, 22E46, 32Q15, 30C20

PACS: 03.65.Fd, 03.65.Pm, 02.20.Qs, 02.40.Tt

1. Introduction

Complex manifolds and, in particular, Cartan classical domains have been studied for many years by mathematicians and theoretical physicists (see e.g. [1] and references therein for a review). In this article we are interested in the Lie ball

$$\mathbb{D} = SO(4,2)/(SO(4) \times SO(2)) = SU(2,2)/S(U(2) \times U(2)),$$

which can be mapped one-to-one onto the eight-dimensional forward/future tube domain

$$\mathbb{T} = \{x^\mu + iy^\mu \in \mathbb{C}^{1,3},\ y^0 > \|\vec{y}\|\}$$

of the complex Minkowski space $\mathbb{C}^{1,3}$ through a Cayley transformation (see next section for more details). Both manifolds can be considered as the phase space of massive conformal particles and there is a renewed interest in its quantization (see e.g. [2] and references therein for a survey). The presentation followed in the literature is of geometric (twistor [3, 4] and Konstant–Kirillov–Souriau [5, 6] descriptions) and representation-theoretic [7, 8] nature. Here we shall adopt a (sigma-model-type) Lagrangian approach to the subject and we shall use a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the particular approach to quantizing coadjoint orbits of a group G developed many years ago in [9] (see also [10, 11] for interesting examples in $G = SU(3)$).

We share with many authors (namely, [1, 2, 12–22]) the belief that the use of complex Minkowski eight-dimensional space as a base for the construction of quantum (field) theory is not only useful from the technical point of view but can be of great physical importance. Actually, as suggested in [14], the conformal domain $\mathbb{D}$ could be considered as the replacement of the space-time at short distances (at the "microscale"). This interpretation is based on Born's Reciprocity Principle (BRP) [15, 16], originally intended to merging quantum theory and general relativity. The reciprocity symmetry between coordinates x_μ and momenta p_μ states that the laws of nature are (or should be) invariant under the transformations

$$(x_\mu, p_\mu) \to (\pm p_\mu, \mp x_\mu). \tag{1}$$

The word "reciprocity" is used in analogy with the lattice theory of crystals, where some physical phenomena (like the theory of diffraction) are sometimes better described in the p-space by means of the reciprocal (Bravais) lattice. The argument here is that Born's reciprocity implies that there must be a reciprocally conjugate relativity principle according to which the rate of change of momentum (force) should be bounded by a universal constant b, much in the same way the usual relativity principle implies a bound of the rate of change of position (velocity) by the speed of light c. As a consequence of the BRP, there must exist a minimum (namely, Planck) length $\ell_{\min} = \sqrt{\hbar c/b}$.

This symmetry led Born to conjecture that the basic underlying physical space is the eight-dimensional $\{x_\mu,\ p_\mu\}$ and to replace the Poincaré invariant line element $d\tau^2 = dx_\mu dx^\mu$ by the Finslerian-like metric (see [17, 18] for an extension to Born–Clifford phase spaces)

$$d\tilde{\tau}^2 = dx_\mu dx^\mu + \frac{\ell_{\min}^4}{\hbar^2} dp_\mu dp^\mu. \tag{2}$$

From the BRP point of view, local (versus extended) field theories like Klein–Gordon's represent the "point-particle limit" $\ell_{\min} \to 0$, for which the reciprocal symmetry is broken. Also, the Minkowski space-time is interpreted either as a local ($\ell_{\min} \to 0$) version or as a high-energy-momentum-transfer limit ($b \to \infty$) of this eight-dimensional phase-space domain. Moreover, putting $dp_\mu/d\tau = md^2x_\mu/d\tau^2 = ma_\mu$, with $m = b\ell_{\min}/c^2$ a (namely, Planck) mass and a_μ the proper acceleration (with $a^2 \leq 0$, space-like), one can write the previous extended line element as

$$d\tilde{\tau} = d\tau\sqrt{1 - \frac{|a^2|}{a^2_{\max}}}, \tag{3}$$

which naturally leads to a *maximal* (*proper*) *acceleration* $a_{\max} = c^2/\ell_{\min}$. The existence and physical consequences of a maximal acceleration was already derived by Caianiello [19]. Many papers have been published in the last years (see e.g. [20] and references therein), each one introducing the maximal acceleration starting from different motivations and from different theoretical schemes. Among the large list of physical applications of Caianiello's model we would like to point out the one in cosmology which avoids an initial singularity while preserving inflation. Also, a maximal-acceleration relativity principle leads to a variable fine structure "constant" α [20], according to which α could have been extremely small (zero) in the early Universe and then all matter in the Universe could have emerged via the Fulling–Davies–Unruh–Hawking effect (vacuum radiation due to the acceleration with respect to the vacuum frame of reference) [23–26].

There has been group-theoretical revisions of the BRP-like [21, 22] replacing the Poincaré by the Canonical (or Quaplectic) group of reciprocal relativity, which enjoys a richer structure than Poincaré. In this article we pursue a different reformulation of BRP as a natural symmetry inside the conformal group $SO(4,2)$ and the replacement of space-time by the eight-dimensional conformal domain $\mathbb{D}$ or $\mathbb{T}$ at short distances. We believe that new interesting physical phenomena remain to be unravelled inside this framework. Actually, in a coming paper [27] (see also [28] for a previous related work), we shall discuss a group-theoretical revision of the Unruh effect [25] as a spontaneous breakdown of the conformal symmetry and the consequences of a maximal acceleration. Also, a wavelet transform on the tube domain $\mathbb{T}$, based on the conformal group, could provide a way to analyze wave packets localized in both: space and time. Important developments in this direction have been done in [29, 30] for electromagnetic (massless) signals and [31] for fields with continuous mass spectrum.

In this article we shall study the geometrical and quantum mechanical underlying framework. We shall follow a gauge-invariant (singular) Lagrangian approach of nonlinear sigma-model type and we shall use a generalized Dirac method for the quantization of constrained systems.

The paper is organized as follows. In Sec. 2 we briefly review the conformal group $SO(4,2) \simeq SU(2,2)$, its Lie algebra generators and commutators, and provide different coordinate systems for the conformal domains $\mathbb{D}$ and $\mathbb{T}$; in this section we also introduce the concept of BRP in a conformally invariant setting. Section 3 is devoted to the Lagrangian formulation of conformally invariant nonlinear sigma-models on the conformal domains (either as configuration or phase spaces) and the study of their gauge invariance. The quantization of these models (for the case of Lagrangians linear in velocities) is accomplished in Sec. 4 by using a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the particular approach to quantizing coadjoint orbits of G. Physical wave functions, Haar measures, orthonormal basis and reproducing (Bergman) kernels are explicitly calculated in an holomorphic picture in the Cartan domain $\mathbb{D}$, for both scalar and spinning quantum particles in Subsecs. 4.1 and 4.2, respectively. Similarities and differences with other results in the literature are also discussed and an extension of the Schwinger Master theorem is commented in connection with closure relations. In Sec. 5 we translate (through an equivariant map) all the constructions above to the tube domain $\mathbb{T}$, where we enjoy more physical intuition. We comment on Kähler structures and generalized Born-like line elements and the existence of a maximal acceleration for conformal (quantum) particles. The last Sec. 6 is devoted to comments and outlook where we point out an interesting connection between BRP and CPT symmetry inside the conformal group and discuss on the appearance of a maximal acceleration in this scheme.

2. The Conformal Symmetry in 1 + 3D: Coordinate Systems and Generators

The conformal group $SO(4,2)$ is comprised of Poincaré (space-time translations $b^\mu \in \mathbb{R}^{1,3}$ and Lorentz $\Lambda^\mu_\nu \in SO(3,1)$) transformations augmented by dilations ($\rho = e^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, SCT, $a^\mu \in \mathbb{R}^{1,3}$) which, in Minkowski space-time, have the following realization:

$$\begin{aligned} x'^\mu &= x^\mu + b^\mu, \quad & x'^\mu &= \Lambda^\mu_\nu(\omega)x^\nu, \\ x'^\mu &= \rho x^\mu, & x'^\mu &= \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2x^2}, \end{aligned} \tag{4}$$

respectively. The interpretation of SCT as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by (see e.g. [32–34]), although alternative meanings have also been proposed. One is related to the Weyl's idea of different lengths in different points of space-time [35]: "the rule for measuring distances changes at different positions". Other is Kastrup's interpretation of SCT as geometrical gauge transformations of the Minkowski space [36] (for this point see later on Eq. (40)).

The generators of the transformations (4) are easily deduced:

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial x^\mu}, \quad & M_{\mu\nu} &= x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \\ D &= x^\mu \frac{\partial}{\partial x^\mu}, & K_\mu &= -2x_\mu x^\nu \frac{\partial}{\partial x^\nu} + x^2 \frac{\partial}{\partial x^\mu} \end{aligned} \tag{5}$$

and they close into the conformal Lie algebra

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \\
[P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho, \quad [P_\mu, P_\nu] = 0, \\
[K_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho, \quad [K_\mu, K_\nu] = 0, \\
[D, P_\mu] &= -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_{\mu\nu}] = 0, \\
[K_\mu, P_\nu] &= 2(\eta_{\mu\nu} D + M_{\mu\nu}).
\end{aligned} \tag{6}$$

We shall argue later that P_μ and K_μ are conjugated variables (they cannot be simultaneously measured) and that D can be taken to be the generator of (proper) time translations (i.e. the Hamiltonian). A BRP-like symmetry manifests here in the form:

$$P_\mu \to K_\mu, \quad K_\mu \to P_\mu, \quad D \to -D, \tag{7}$$

which leaves the commutation relations (6) unaltered. This symmetry can also be seen in the quadratic Casimir operator:

$$\begin{aligned}
C_2 &= D^2 - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} + \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu) \\
&= D^2 - \frac{1}{2} M_{\mu\nu} M^{\mu\nu} + P_\mu K^\mu + 4D,
\end{aligned} \tag{8}$$

which generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu$, just as $d\tilde{\tau}$ in (2) generalizes the Poincaré invariant line element $d\tau$. We shall provide a conformal invariant line element similar to $d\tilde{\tau}$ later in Sec. 5.

Any group element $g \in SO(4,2)$ (near the identity element 1) could be written as the exponential map

$$g = \exp(u), \quad u = \tau D + b^\mu P_\mu + a^\mu K_\mu + \omega^{\mu\nu} M_{\mu\nu}, \tag{9}$$

of the Lie-algebra element u (see [37, 38]). The compactified Minkowski space $\mathbb{M} = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1 \simeq U(2)$ can be obtained as the coset $\mathbb{M} = SO(4,2)/\mathbb{W}$, where $\mathbb{W}$ denotes the Weyl subgroup generated by $K_\mu, M_{\mu\nu}$ and D (i.e. a Poincaré subgroup $\mathbb{P} = SO(3,1) \circledS \mathbb{R}^4$ augmented by dilations $\mathbb{R}^+$). The Weyl group $\mathbb{W}$ is the stability subgroup (the little group in physical usage) of $x^\mu = 0$.

There is another interesting realization of the conformal Lie algebra (6) in terms of gamma matrices in, for instance, the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix},$$

where $\check{\sigma}^\mu \equiv \sigma_\mu$ (we are using the convention $\eta = \mathrm{diag}(1,-1,-1,-1)$ for the Minkowski metric) and σ^μ are the Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed, the choice

$$
\begin{aligned}
&D = \frac{\gamma^5}{2}, \quad M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4}\begin{pmatrix} \sigma^\mu \check{\sigma}^\nu - \sigma^\nu \check{\sigma}^\mu & 0 \\ 0 & \check{\sigma}^\mu \sigma^\nu - \check{\sigma}^\nu \sigma^\mu \end{pmatrix}, \\
&P^\mu = \gamma^\mu \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \quad K^\mu = \gamma^\mu \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^\mu & 0 \end{pmatrix}
\end{aligned} \tag{10}
$$

fulfills the commutation relations (6). These are the Lie algebra generators of the fundamental representation of the four cover of $SO(4,2)$:

$$
SU(2,2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times 4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}, \tag{11}
$$

with Γ a 4×4 hermitian form of signature $(++--)$. In particular, taking $\Gamma = \gamma^5$, the 2×2 complex matrices A, B, C, D in (11) satisfy the following restrictions:

$$
g^{-1} g = I_{4\times 4} \Leftrightarrow \begin{cases} D^\dagger D - B^\dagger B = \sigma^0 \\ A^\dagger A - C^\dagger C = \sigma^0 \\ A^\dagger B - C^\dagger D = 0, \end{cases} \tag{12}
$$

together with those of $g g^{-1} = I_{4\times 4}$. In this article we shall work with $G = U(2,2)$ instead of $SO(4,2)$ and we shall use a set of complex coordinates to parametrize G. This parametrization will be adapted to the non-compact complex Grassmannian $\mathbb{D} = G/H$ of the maximal compact subgroup $H = U(2)^2$. It can be obtained through a block-orthonormalization process with metric $\Gamma = \gamma^5$ of the matrix columns of:

$$
\begin{pmatrix} \sigma^0 & 0 \\ Z^\dagger & \sigma^0 \end{pmatrix} \to g = \begin{pmatrix} \sigma^0 & Z \\ Z^\dagger & \sigma^0 \end{pmatrix} \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \begin{cases} \Delta_1 = (\sigma^0 - Z Z^\dagger)^{-\frac{1}{2}} \\ \Delta_2 = (\sigma^0 - Z^\dagger Z)^{-\frac{1}{2}} \end{cases}.
$$

Actually, we can identify

$$
\begin{aligned}
&Z = Z(g) = B D^{-1}, \quad Z^\dagger = Z^\dagger(g) = C A^{-1}, \\
&\Delta_1 = (A A^\dagger)^{\frac{1}{2}}, \quad \Delta_2 = (D D^\dagger)^{\frac{1}{2}}.
\end{aligned} \tag{13}
$$

From (12), we obtain the positive-matrix conditions $AA^\dagger > 0$ and $DD^\dagger > 0$, which are equivalent to:

$$
\sigma^0 - Z Z^\dagger > 0, \quad \sigma^0 - Z^\dagger Z > 0. \tag{14}
$$

Moreover, from the top condition of (12), we arrive at the determinant restriction:

$$
\det(Z Z^\dagger) = \det(B^\dagger B) \det(\sigma^0 + B^\dagger B)^{-1} < 1, \tag{15}
$$

which, together with $\det(\sigma^0 - Z Z^\dagger) = 1 - \mathrm{tr}(Z Z^\dagger) + \det(Z Z^\dagger) > 0$, implies that $\mathrm{tr}(Z Z^\dagger) < 2$. Thus, we can identify the symmetric complex Cartan domain

$$
\mathbb{D} = G/H = \{ Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \sigma^0 - Z Z^\dagger > 0 \} \tag{16}
$$

with an open subset of the eight-dimensional ball with radius $\sqrt{2}$. Moreover, the compactified Minkowski space $\mathbb{M}$ is the Shilov boundary $U(2) = \{Z \in \text{Mat}_{2\times 2}(\mathbb{C}) : Z^\dagger Z = ZZ^\dagger = \sigma^0\}$ of $\mathbb{D}$.

There is a one-to-one mapping from $\mathbb{D}$ onto the future tube domain

$$\mathbb{T} = \{W = X + iY \in \text{Mat}_{2\times 2}(\mathbb{C}) : Y > 0\}, \tag{17}$$

of the complex Minkowski space $\mathbb{C}^{1,3}$, with $X = x_\mu \sigma^\mu$ and $Y = y_\mu \sigma^\mu$ hermitian matrices and $Y > 0 \Leftrightarrow y^0 > \|\vec{y}\|$. This map is given by the Cayley transformation and its inverse:

$$\begin{aligned} Z \to W(Z) &= i(\sigma^0 - Z)(\sigma^0 + Z)^{-1}, \\ W \to Z(W) &= (\sigma^0 - iW)^{-1}(\sigma^0 + iW). \end{aligned} \tag{18}$$

This is the $(3 + 1)$-dimensional analogue of the usual map form the unit disk onto the upper half-plane in two-dimensions. Actually, the forward tube domain $\mathbb{T}$ is naturally homeomorphic to the quotient G/H in a new realization of G in terms of matrices f which preserve $\Gamma = \gamma^0$, instead of $\Gamma = \gamma^5$; that is, $f^\dagger \gamma^0 f = \gamma^0$. Both realizations of G are related by the map

$$g \to f = \Upsilon g \Upsilon^{-1}, \quad \Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma^0 & -\sigma^0 \\ \sigma^0 & \sigma^0 \end{pmatrix}. \tag{19}$$

We shall come again to this "forward tube domain" realization later on in Sec. 5.

Let us proceed by giving a complete local parametrization of G adapted to the fibration $H \to G \to \mathbb{D}$. Any element $g \in G$ (in the present patch, containing the identity element) admits the Iwasawa decomposition

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta_1 & Z\Delta_2 \\ Z^\dagger \Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{20}$$

where the last factor

$$U_1 = \Delta_1^{-1} A, \quad U_2 = \Delta_2^{-1} D$$

belongs to H; i.e. $U_1, U_2 \in U(2)$. Likewise, a parametrization of any $U \in U(2)$ (in a patch containing the identity), adapted to the quotient $\mathbb{S}^2 = U(2)/U(1)^2$, is (the Hopf fibration)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta & z\delta \\ -\bar{z}\delta & \delta \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \tag{21}$$

where $z = b/d \in \overline{\mathbb{C}} \simeq \mathbb{S}^2$ (the one-point compactification of $\mathbb{C}$ by inverse stereographic projection), $\delta = (1 + z\bar{z})^{-\frac{1}{2}}$ and $e^{i\alpha} = a/|a|, e^{i\beta} = d/|d|$.

Sometimes it will be more convenient for us to use the following compact notation for the 16 coordinates of $U(2,2)$:

$$\begin{Bmatrix} \alpha_1 & z_1 & Z_{11} & Z_{12} \\ -\bar{z}_1 & \beta_1 & Z_{21} & Z_{22} \\ \bar{Z}_{11} & \bar{Z}_{21} & \alpha_2 & z_2 \\ \bar{Z}_{12} & \bar{Z}_{22} & -\bar{z}_2 & \beta_2 \end{Bmatrix} = \begin{Bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{Bmatrix} = \{x_\beta^\alpha(g)\}. \tag{22}$$

The set of coordinates $\{x_\beta^\alpha\}$ is adapted to the new Lie algebra basis of step operator matrices $(X_\alpha^\beta)_\mu^\nu \equiv \delta_\alpha^\nu \delta_\mu^\beta$ fulfilling the commutation relations:

$$\left[X_{\alpha_1}^{\beta_1}, X_{\alpha_2}^{\beta_2}\right] = \delta_{\alpha_1}^{\beta_2} X_{\alpha_2}^{\beta_1} - \delta_{\alpha_2}^{\beta_1} X_{\alpha_1}^{\beta_2}, \tag{23}$$

and the usual orthogonality properties:

$$\mathrm{tr}\left(X_\alpha^\beta X_\gamma^\rho\right) = \delta_\alpha^\rho \delta_\gamma^\beta.$$

The Cartan (maximal Abelian) subalgebra $u(1)^4 \subset \mathcal{G}$ is made of diagonal operators $\{X_\alpha^\alpha, \alpha = 1, \ldots, 4\}$.

Another realization of the conformal Lie algebra that will be useful for us is the one given in terms of left- and right-invariant vector fields, as generators of right- and left-translations of G,

$$[\mathcal{U}_g^R \psi](g') = \psi(g'g), \quad [\mathcal{U}_g^L \psi](g') = \psi(g^{-1}g'), \tag{24}$$

on complex functions $\psi : G \to \mathbb{C}$, respectively. Denoting by

$$\theta^L = -ig^{-1}dg = \theta_\beta^\alpha X_\alpha^\beta = \theta_{\beta\nu}^{\alpha\mu} dx_\mu^\nu X_\alpha^\beta \tag{25}$$

the left-invariant Maurer–Cartan one-form, the left-invariant vector fields L_α^β are defined by duality $\theta_\beta^\alpha(L_\rho^\sigma) = \delta_\rho^\alpha \delta_\beta^\sigma$. The same applies to right-invariant one-forms $\theta^R = -idgg^{-1}$ in relation with right-invariant vector fields R_α^β. They can also be computed through the group law $g'' = g'g$ as:

$$L_\alpha^\beta(g) \equiv \left.\frac{\partial x_\nu^\mu(gg')}{\partial x_\beta^\alpha(g')}\right|_{g'=1} \frac{\partial}{\partial x_\mu^\nu(g)}, \quad R_\alpha^\beta(g) \equiv \left.\frac{\partial x_\nu^\mu(g'g)}{\partial x_\beta^\alpha(g')}\right|_{g'=1} \frac{\partial}{\partial x_\mu^\nu(g)}. \tag{26}$$

The quadratic Casimir operator (8) now adopts the compact form:

$$C_2 = L_\alpha^\beta L_\beta^\alpha = R_\alpha^\beta R_\beta^\alpha.$$

Both sets of vector fields will be essential in our quantization procedure, the first ones (L) as generators of gauge transformations and the second ones (R) as the symmetry operators of our theory.

3. Nonlinear Sigma Models on G

The actual Lagrangian for quantum mechanical geodesic free motion on G, as a configuration space, is given by:

$$\mathcal{L}_G(g, \dot{g}) = \frac{1}{2}\mathrm{tr}(\vartheta^L)^2 = \frac{1}{2}\vartheta^\alpha_\beta\vartheta^\beta_\alpha = \frac{1}{2}g^{\nu\sigma}_{\mu\rho}(x)\dot{x}^\mu_\nu\dot{x}^\rho_\sigma, \tag{27}$$

where we are denoting by

$$\vartheta^L = -ig^{-1}\dot{g} = \vartheta^\alpha_\beta X^\beta_\alpha = \vartheta^{\alpha\nu}_{\beta\mu}\dot{x}^\mu_\nu X^\beta_\alpha$$

the restriction of (25) to trajectories $g = g(t)$ and writing the natural metric on G, $g^{\nu\sigma}_{\mu\rho} = \vartheta^{\alpha\nu}_{\beta\mu}\vartheta^{\beta\sigma}_{\alpha\rho}$, in terms of vielbeins ϑ^α_β. The equations of motion derived from (27) are: $\dot{\vartheta}^L = 0$, which can be converted into the standard form of geodesic motion

$$\ddot{x}^a + \Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c = 0$$

by introducing the Levi-Civita connection Γ^a_{bc} [here we used an alternative indexation $a = (\alpha\beta) = 1, \ldots, 16$, to simplify expressions]. The phase space of this theory is the cotangent bundle T^*G, which can be identified with the product of G and its Lie algebra $\mathcal{G}$ in a suitable way.

It can be shown that the Lagrangian (27) is G-invariant under both: left- and right-rigid transformations, $g(t) \to g'g(t)$ and $g(t) \to g(t)g'$, respectively; that is, $\mathcal{L}_G$ is chiral. This chirality is partially broken when we reduce the dynamics from G to certain cosets G/G^0, with G^0 the isotropy subgroup of a given Lie algebra element of the form

$$X_0 = \sum_{\alpha=1}^{4}\lambda_\alpha X^\alpha_\alpha \tag{28}$$

(with λ_α some real constants) under the adjoint action $X_0 \to gX_0g^{-1}$ of G on its Lie algebra $\mathcal{G}$. Actually, the new Lagrangian on G/G^0 can be written as a "partial trace":

$$\begin{aligned}\mathcal{L}_{G/G^0}(g, \dot{g}) &= \frac{1}{2}\mathrm{tr}_{G/G^0}(\vartheta^L)^2 \equiv \frac{1}{2}\mathrm{tr}([X_0, \vartheta^L])^2 \\ &= \frac{1}{2}\sum_{\alpha,\beta=1}^{N}(\lambda_\alpha - \lambda_\beta)^2\vartheta^\alpha_\beta\vartheta^\beta_\alpha.\end{aligned} \tag{29}$$

For example, choosing $X_0 = \frac{\lambda}{2}\gamma^5 = \lambda D$ (the dilation) we have $G^0 = H = U(2)^2$ (the maximal compact subgroup) and G/G^0 the eight-dimensional domain $\mathbb{D}$. For $\lambda_\alpha \neq \lambda_\beta, \forall\, \alpha, \beta = 1, \ldots, 4$, the isotropy subgroup of X_0 is the maximal Abelian subgroup $G^0 = U(1)^4$ and $G/G^0 = \mathbb{F}$ is a 12-dimensional "pseudo-flag" (non-compact) manifold. It is obvious that $\mathcal{L}_{G/G^0}$ is still invariant under general rigid left-transformations $g(t) \to g'g(t)$. However, this Lagrangian is now singular or, equivalently:

Proposition 3.1. *The Lagrangian* (29) *is gauge invariant under local right-transformations*

$$g(t) \to g(t)g_0(t), \quad \forall\, g_0(t) \in G^0. \tag{30}$$

Proof. We have that:

$$\vartheta^L = -ig^{-1}\dot{g} \to \vartheta'^L = -ig_0^{-1}g^{-1}(\dot{g}g_0 + g\dot{g}_0) = g_0^{-1}\vartheta^L g_0 - ig_0^{-1}\dot{g}_0$$

and

$$[X_0, \vartheta'^L] = g_0^{-1}[X_0, \vartheta^L]g_0,$$

since G^0 is the isotropy subgroup of X_0, which means $[X_0, g_0] = 0 = [X_0, \dot{g}_0]$. The cyclic property of the trace completes the proof. □

We have considered so far G/G^0 as a configuration space. In this article, we shall be rather interested in G/G^0 as a phase space. For example, we shall consider $\mathbb{D}$ [or the tube domain (17) of the complex Minkowski space $\mathbb{C}^{1,3}$] as a (complex) phase space of four-position x^μ and four-momenta y^ν, in itself. This situation will require a new singular Lagrangian of the form:

$$\mathcal{L}(g, \dot{g}) = \mathrm{tr}(X_0\vartheta^L) = \sum_{\alpha=1}^{4} \lambda_\alpha \vartheta^\alpha_\alpha. \tag{31}$$

Again, this Lagrangian is left-G-invariant under rigid transformations. The difference now is that it is linear in velocities $\dot{x}$. Moreover, we shall prove that:

Proposition 3.2. *The Lagrangian* (31) *is gauge* (*semi-*)*invariant under local right-transformations*

$$g(t) \to g(t)g_0(t), \quad \forall\, g_0(t) \in G^0 \tag{32}$$

up to a total time derivative, i.e.

$$\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}, \quad \Delta\mathcal{L} = -i\mathrm{tr}(X_0 g_0^{-1}\dot{g}_0) = \frac{d\tau}{dt}, \quad \tau = \sum_{\alpha=1}^{4} \lambda_\alpha x^\alpha_\alpha. \tag{33}$$

Proof. We shall just consider the two important cases for us:

(1) $\lambda_\alpha \neq \lambda_\beta, \forall\, \alpha \neq \beta \Rightarrow G^0 = U(1)^4, G/G^0 = \mathbb{F}$.

(2) $X_0 = \lambda D = \frac{\lambda}{2}\gamma^5 \Rightarrow G^0 = H = U(2)^2, G/G^0 = \mathbb{D}$.

For the first case, any $g_0 \in G^0$ can be written as $g_0 = \exp(ix^\alpha_\alpha X^\alpha_\alpha)$ and $\dot{g}_0 = ig_0\dot{x}^\alpha_\alpha X^\alpha_\alpha$ because G^0 is Abelian; therefore

$$\Delta\mathcal{L} = -i\mathrm{tr}(X_0 g_0^{-1}\dot{g}_0) = \sum_{\beta=1}^{4} \lambda_\beta \dot{x}^\alpha_\alpha \mathrm{tr}(X^\beta_\beta X^\alpha_\alpha) = \sum_{\alpha=1}^{4} \lambda_\alpha \dot{x}^\alpha_\alpha.$$

For the second case, $g_0 = \exp(i\varphi I + i\tau' D + i\omega^{\mu\nu} M_{\mu\nu}) = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in H$. Disregarding the trivial global phase φ, it is clear that for dilations $g_0 = d_0 = e^{i\tau' D}$ we have $\dot{d}_0 = i\dot{\tau}' D e^{i\tau' D}$ and

$$-i\mathrm{tr}(X_0 d_0^{-1}\dot{d}_0) = \lambda\dot{\tau}\mathrm{tr}(D^2) = \lambda\dot{\tau}' = \dot{\tau},$$

where $\tau \equiv \lambda\tau'$. For Lorentz transformations $g_0 = m_0 = \exp(i\omega^{\mu\nu} M_{\mu\nu})$ we have $\Delta\mathcal{L} = 0$ since $\mathrm{tr}(DM_{\mu\nu}) = 0$, which is a direct consequence of the orthogonality properties of the Pauli matrices $\mathrm{tr}(\sigma^\mu\sigma^\nu) = 2\delta^{\mu\nu}$. □

Remark 3.3. We can always fix the gauge to $\tau(t) = t$. In the case $X_0 = \lambda D$, this implies that the dilation operator D will play the role of the Hamiltonian of the quantum theory. The replacement of time translations by dilations as dynamical equations of motion has been considered in [39] and in [40] when quantizing field theories on space-like Lorentz-invariant hypersurfaces $x^2 = x^\mu x_\mu = \tau^2 = \text{constant}$. In other words, if one wishes to proceed from one surface at $x^2 = \tau_1^2$ to another at $x^2 = \tau_2^2$, this is done by scale transformations; that is, D is the evolution operator in a proper time τ.

4. Quantum Mechanics in the Phase Space G/G^0

We shall see that the constants λ_α label the (lowest weight) irreducible representations of G on which the Hilbert space of our theory is constructed. There are several ways of seeing that the values of λ_α are quantized. One way is through the path integral method. To examine this explicitly, consider the transition amplitude from an initial point g_1 at $t = t_1$ to a final point g_2 at $t = t_2$. For each path $g(t)$ connecting g_1 and g_2, there are many gauge equivalent paths

$$g'(t) = g(t)g_0(t), \quad g_0(t) \in G^0, \quad g_0(t_1) = g_0(t_2) = 1$$

that must contribute to the sum of the path integral with the same amplitude, that is:

$$e^{i\int_{t_1}^{t_2} dt\mathcal{L}(g,\dot{g})} = e^{i\int_{t_1}^{t_2} dt\mathcal{L}(g',\dot{g}')} = e^{i\int_{t_1}^{t_2} dt\mathcal{L}(g,\dot{g})} e^{i\int_{t_1}^{t_2} dt\Delta\mathcal{L}(g,\dot{g})} \Rightarrow e^{i\int_{t_1}^{t_2} dt\Delta\mathcal{L}(g,\dot{g})} = 1.$$

Using (33), the last expression can be written as $\exp(i(\tau(t_2) - \tau(t_1)) = 1$ which, together with the fact that

$$g_0(t_{1,2}) = e^{i\sum_\alpha x_\alpha^\alpha(t_{1,2})} = 1 \Leftrightarrow x_\alpha^\alpha(t_{1,2}) = 2\pi n_{1,2}^\alpha, \quad n_{1,2}^\alpha \in \mathbb{Z},$$

means that λ_α must be an integer number. Considering coverings of G, one can relax the integer to a half-integer condition, as happens with $SU(2)$ in relation with $SO(3)$.

Other alternative way to the path-integral description of realizing the integrality of λ_α is through the following operator (representation-theoretic) description. At the quantum level, finite-right gauge transformations like (32) induce constraints on "physical" wave functions $\psi(g)$ as:

$$\psi(gg_0) = \mathcal{U}_0^\lambda(g_0)\psi(g), \quad g_0 \in G^0 \tag{34}$$

where we are allowing ψ to transform non-trivially according to a representation $\mathcal{U}_0^\lambda$ of G^0 of index λ. This could be seen as a generalization of the original Dirac approach to the quantization of constrained systems (where $\mathcal{U}_0^\lambda$ is taken to be trivial) which allows new inequivalent quantizations labeled by λ_α (see e.g. [41–44] for several approaches to the subject). The finite constraint condition (34) can be written in infinitesimal form as

$$L^\alpha_\alpha \psi = \lambda_\alpha \psi, \quad \alpha = 1, \ldots, 4, \tag{35}$$

where we have used the fact that left-invariant vector fields (26) are generators of finite right-transformations. In the parametrization $\{x^\alpha_\beta\}$, the left-invariant vector fields L^β_α fulfill the same commutation relations as the step operator matrices (23). Therefore, when acting on physical/constrained states (35), they satisfy creation and annihilation harmonic-oscillator-like commutation relations:

$$[L^\beta_\alpha, L^\alpha_\beta] = (\lambda_\beta - \lambda_\alpha) \quad (\text{no sum on}\, \alpha, \beta).$$

We shall work in a holomorphic picture, which means that constrained wave functions (35) will be further restricted by holomorphicity conditions:

$$L^\beta_\alpha \psi = 0, \quad \forall\, \alpha > \beta = 1, 2, 3. \tag{36}$$

In fact, looking at (26), for $g \in G$ near the identity we have $L^\beta_\alpha(g) \sim \partial/\partial x^\alpha_\beta$ so that $L^\beta_\alpha \psi = 0$ means, roughly speaking, that $\psi(g)$ does not depend on the variables $x^\alpha_\beta, \alpha > \beta = 1, 2, 3$ in (22), that is, ψ is holomorphic. The complementary option $L^\beta_\alpha \psi = 0, \forall\, \beta > \alpha = 1, 2, 3$ then leads to anti-holomorphic functions. Those readers familiar with Geometric Quantization [5, 45] will identify the constraint Eqs. (35) and (36) as *polarization* conditions (see also [46] for a Group Approach to Quantization scheme and [47] for the extension of first-order polarizations to higher-order polarizations), intended to reduce the left-representation $\mathcal{U}^L$ (24) of G, on complex wave functions ψ, to G/G^0. Also, the constraints (35) and (36) are exactly the defining relations of a lowest-weight representation.

4.1. *Conformal scalar quantum particles*

Firstly we shall consider the (spinless) case $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 \equiv -\lambda/2$, that is, $X_0 = \frac{\lambda}{2}\gamma^5 = \lambda D$, and we shall call λ the *conformal scale or mass dimension*. In this case the gauge group is the maximal compact subgroup $G^0 = H = U(2)^2$ and the phase space is the eight-dimensional domain $\mathbb{D} = G/G^0$.

4.1.1. *Constraint conditions and physical wave functions*

The constraint conditions (35) can now be enlarged to

$$D^L \psi = -\frac{1}{2}(L^1_1 + L^2_2 - L^3_3 - L^4_4)\psi = \lambda\psi, \quad M^L_{\mu\nu}\psi = 0, \tag{37}$$

which renders translation (P_μ) and acceleration (K_ν) generators into conjugated variables. In fact, the last commutator of (6), on constrained (physical) wave functions (37), gives:

$$[K^L_\mu, P^L_\nu]\psi = 2\lambda\eta_{\mu\nu}\psi, \tag{38}$$

which states that K_μ and P_μ cannot be simultaneously measured, the conformal dimension λ playing here the role of the Planck constant $\hbar$. Note that K_μ and P_μ are conjugated but not *canonically* conjugated as such. We address the reader to Refs. [48, 49] for other definitions of quantum observables associated with positions in space-time, namely

$$X_\mu = M_{\nu\mu} \cdot \frac{P^\nu}{P^2} + D \cdot \frac{P_\mu}{P^2} \tag{39}$$

(dot means symmetrization), fulfilling canonical commutation relations $[X_\mu, P_\nu] = \eta_{\mu\nu}$ inside the conformal (enveloping) algebra (6).

A further restriction

$$K^L_\mu \psi = 0 \tag{40}$$

selects the holomorphic ("position") representation. Indeed, let us prove that:

Theorem 4.1. *The general solution to* (37) *and* (40) *can be factorized as*:

$$\psi_\lambda(g) = \mathcal{W}_\lambda(g)\phi(Z), \tag{41}$$

where the "*ground state*"

$$\begin{aligned} \mathcal{W}_\lambda(g) &= \det(D)^{-\lambda} = \det(\sigma^0 - Z^\dagger Z)^{\lambda/2} \det(U_2)^{-\lambda} \\ &= (1 - \operatorname{tr}(Z^\dagger Z) + \det(Z^\dagger Z))^{\lambda/2} \det(U_2)^{-\lambda} \end{aligned} \tag{42}$$

is a particular solution of (37), (40) *and* ϕ *is the general solution for the trivial representation* $\lambda = 0$ *of* $G^0 = H$ (*actually, an arbitrary, analytic holomorphic function of* Z), *for the decomposition* (20) *of an element* $g \in G$.

Proof. A generic proof (also valid for other symmetry groups) that the general solution of (37), (40) admits a factorization of the form (41) can be found in the Proposition 3.3 of [50]. Here we shall just prove that (41) is a solution of (37), (40). Indeed, by applying a finite right-translation (24) on $\mathcal{W}_\lambda(g)$:

$$\begin{aligned} [\mathcal{U}^R_{g'}\mathcal{W}_\lambda](g) &= \mathcal{W}_\lambda(gg') = \det(D'')^{-\lambda} = \det(CB' + DD')^{-\lambda} \\ &= \det(D')^{-\lambda}\det(CZ' + D)^{-\lambda}, \end{aligned} \tag{43}$$

we see that $\mathcal{W}_\lambda(gg')$ is not affected by translations by $Z'^\dagger = Z^\dagger(g') = C'A'^{-1}$. Infinitesimally, it means that $K^L_\mu \mathcal{W}_\lambda(g) = 0$, according to the lower-triangular choice of the generator K_μ in (10). For Lorentz transformations we have $B' = 0 = C'$ and $\det(A') = 1 = \det(D')$ and therefore $\mathcal{W}_\lambda(gg') = \mathcal{W}_\lambda(g)$, that is $M^L_{\mu\nu}\mathcal{W}_\lambda(g) = 0$. For dilations we have $B' = 0 = C'$ and $A' = e^{i\tau/2}\sigma^0 = D'^\dagger$, which gives

$\mathcal{W}_\lambda(gg') = e^{i\lambda\tau}\mathcal{W}_\lambda(g)$ or $D^L\mathcal{W}_\lambda(g) = \lambda\mathcal{W}_\lambda(g)$ for small τ. It remains to prove that $\phi(Z)$ is the general solution of (37), (40) for $\lambda = 0$. From (13) we have

$$Z'' = Z(gg') = B''D''^{-1} = (AB' + BD')(CB' + DD')^{-1}, \tag{44}$$

which is not affected by C' and gives $Z'' = Z$ for dilations and Lorentz transformations ($B' = 0$). □

Remark 4.2. In the last theorem, we are implicitly restricting ourselves to gauge transformations $g' \in S(U(2)^2)$, which means $\det(g') = \det(U_1U_2) = 1$. If we allow for transformations $g' \in U(2)^2$ with $\det(g') \neq 1$ (like $e^{i\alpha}I$) and we want them to leave physical wave functions strictly invariant $\psi(gg') = \psi(g)$ (i.e. we restrict ourselves to representations with $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$), we must choose a symmetrical form for the ground state

$$\begin{aligned}\mathcal{W}_\lambda(g) &= \det(A^\dagger)^{-\lambda/2}\det(D)^{-\lambda/2}\\ &= \det(\sigma^0 - Z^\dagger Z)^{\lambda/2}\det(U_1^\dagger)^{-\lambda/2}\det(U_2)^{-\lambda/2},\end{aligned} \tag{45}$$

which reduces to (42) for $\det(U_1U_2) = 1$.

Moreover, instead of (40), we could have chosen the complementary constraint $P^L_\mu\psi = 0$ which would have led us to an anti-holomorphic ("acceleration") representation $\psi_\lambda(g) = \check{\mathcal{W}}_\lambda(g)\phi(Z^\dagger)$ with the new ground state

$$\begin{aligned}\check{\mathcal{W}}_\lambda(g) &= \det(A)^{-\lambda/2}\det(D^\dagger)^{-\lambda/2}\\ &= \det(\sigma^0 - Z^\dagger Z)^{\lambda/2}\det(U_1)^{-\lambda/2}\det(U_2^\dagger)^{-\lambda/2},\end{aligned} \tag{46}$$

which, for $g \in SU(2,2)$, reduces to:

$$\check{\mathcal{W}}_\lambda(g) = \det(A)^{-\lambda} = \det(\sigma^0 - ZZ^\dagger)^{\lambda/2}\det(U_1)^{-\lambda} = \overline{\mathcal{W}_\lambda(g)}.$$

Therefore, the BRP-like symmetry $K^L_\mu \leftrightarrow P^L_\mu$, $D^L \to -D^L$ in (7) manifest here as a charge conjugation and time-reversal (CT) operations. See later on Sec. 6 for more details on a "BRP-CPT connection" proposal inside the conformal group.

4.1.2. *Irreducible representation, Haar measure and Bergman kernel*

The finite left-action of G on physical wave functions (41),

$$\begin{aligned}[\mathcal{U}^L_{g'}\psi_\lambda](g) &= \psi_\lambda(g'^{-1}g) = \det(D(g'^{-1}g))^{-\lambda}\phi(Z')\\ &= \mathcal{W}_\lambda(g)\det(D'^\dagger - B'^\dagger Z)^{-\lambda}\phi(Z'),\\ Z' &\equiv Z(g'^{-1}g) = (A'^\dagger Z - C'^\dagger)(D'^\dagger - B'^\dagger Z)^{-1},\end{aligned} \tag{47}$$

provides a unitary irreducible representation of G under the invariant scalar product

$$\langle\psi_\lambda\,|\,\psi'_\lambda\rangle = \int_G d\mu^L(g)\overline{\psi_\lambda(g)}\psi'_\lambda(g) \tag{48}$$

given through the left-invariant Haar measure [the exterior product of left-invariant one-form (25)] which can be decomposed as:

$$\begin{aligned} d\mu^L(g) &= c \bigwedge_{\alpha,\beta=1}^{4} \vartheta^\alpha_\beta = c \det(\vartheta^{\alpha\mu}_{\beta\nu}) \bigwedge_{\mu,\nu=1}^{4} dx^\nu_\mu \\ &= c\, d\mu(g)^L\big|_{G/H}\, d\mu^L(g)\big|_H\,, \\ d\mu^L(g)\big|_{G/H} &= \det(\sigma^0 - ZZ^\dagger)^{-4}|dZ|, \\ d\mu(g)|_H &= dv(U_1)dv(U_2), \end{aligned} \tag{49}$$

where we are denoting by $dv(U)$ the Haar measure on $U(2)$, which can be in turn decomposed as:

$$\begin{aligned} dv(U) &\equiv dv(U)|_{U(2)/U(1)^2}\, dv(U)|_{U(1)^2}\,, \\ dv(U)|_{U(2)/U(1)^2} &= dv(U)|_{\mathbb{S}^2} \equiv ds(U) = (1+z\bar{z})^{-2}|dz|, \\ dv(U)|_{U(1)^2} &\equiv d\alpha d\beta. \end{aligned} \tag{50}$$

We have used the Iwasawa decomposition of an element g given in (20), (21) and denoted by $|dz|$ and $|dZ|$ the Lebesgue measures in $\mathbb{C}$ and $\mathbb{C}^4$, respectively. The normalization constant

$$c = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)\left(\frac{(2\pi)^3}{2}\right)^{-2} \tag{51}$$

is fixed so that the ground state (42) is normalized, i.e. $\langle \mathcal{W}_\lambda \mid \mathcal{W}_\lambda\rangle = 1$ (see Appendix C of [31] for orthogonality properties), the factor $(2\pi)^3/2$ actually being the volume $v(U(2))$. The scalar product (48) is finite as long as $\lambda \geq 4$.

The infinitesimal generators of (47) are the right-invariant vector fields $R^\alpha_\beta(g)$ in (26) and constitute the operators (observables) of our quantum theory. For example, from the general expression (47), we can compute the finite left-action of dilations $g' = e^{i\tau D}$ ($B' = 0 = C'$ and $A' = e^{-i\tau/2}\sigma^0 = D'^\dagger$) on physical wave functions,

$$\psi_\lambda(g'g) = e^{i\lambda\tau}\mathcal{W}_\lambda(g)\phi(e^{i\tau}Z),$$

or infinitesimally:

$$\begin{aligned} D^R\psi_\lambda(g) &= -\frac{1}{2}(R^1_1 + R^2_2 - R^3_3 - R^4_4)\psi_\lambda(g) \\ &= \mathcal{W}_\lambda(g)\left(\lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}}\right)\phi(Z) \equiv \mathcal{W}_\lambda(g)D_\lambda\phi(Z), \end{aligned} \tag{52}$$

where we have defined the restriction of the dilation operator on holomorphic functions as:

$$D_\lambda \equiv \lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}}, \tag{53}$$

for future use. As we justified in Remark 3.3, the dilation generator D^R plays the role of the Hamiltonian operator of this theory

$$\hat{\mathcal{H}} = -i\frac{\partial}{\partial \tau} = D^R. \tag{54}$$

The conformal or mass dimension λ can be then interpreted as the zero point (vacuum) energy and the corresponding eigenfunctions are homogeneous polynomials $\phi_n(Z)$ of a certain degree (eigenvalue) n, according to Euler's theorem. We shall come back to this question later in Theorem 4.3.

Let us introduce bracket notation and write:

$$\mathcal{W}_\lambda(g) \equiv \langle g \,|\, \lambda, 0\rangle = \langle \lambda, 0|\mathcal{U}^L_{g^{-1}}|\lambda, 0\rangle, \quad \psi_\lambda(g) \equiv \langle g \,|\, \psi_\lambda\rangle. \tag{55}$$

Here we are implicitly making use of the Coherent–States machinery (see e.g. [51, 52]). Actually, we are denoting by $|g\rangle \equiv \mathcal{U}^L_g|\lambda, 0\rangle$ the set of vectors in the orbit of the ground ("fiducial") state $|\lambda, 0\rangle$ (the lowest-weight vector) under the left-action of the group G (this set is called a family of *covariant coherent states* in the literature [51, 52]). We can easily calculate the coherent state overlap:

$$\begin{aligned}\langle g' \mid g\rangle &= \langle \lambda, 0|\,\mathcal{U}^L_{g'^{-1}g}|\lambda, 0\rangle = \mathcal{W}_\lambda(g^{-1}g') = \det(D(g^{-1}g'))^{-\lambda} = \det(D^\dagger D' - B^\dagger B')^{-\lambda}\\ &= \det(D^\dagger)^{-\lambda}\det(\sigma^0 - (BD^{-1})^\dagger B' D'^{-1})^{-\lambda}\det(D')^{-\lambda}\\ &= \overline{\mathcal{W}_\lambda(g)}\det(\sigma^0 - Z^\dagger Z')^{-\lambda}\mathcal{W}_\lambda(g').\end{aligned} \tag{56}$$

The set of coherent states $\{|g\rangle,\ g \in G\}$ constitutes a tight frame (see [31] for a proof in the context of Conformal Wavelets) with resolution of unity:

$$1 = \int_G d\mu(g)\,|g\rangle\,\langle g|\,.$$

Actually, the coherent state overlap (56) is a reproducing kernel satisfying the integral equation of a projector operator

$$\langle g \mid g''\rangle = \int_G d\mu^L(g')\,\langle g \mid g'\rangle\,\langle g' \mid g''\rangle$$

and the propagator equation

$$\psi_\lambda(g') = \int_G d\mu^L(g)\,\langle g' \mid g\rangle\,\psi_\lambda(g).$$

Since the ground state $\mathcal{W}_\lambda$ is a fixed common factor of all the wave functions (41), we could factor it out and define the restricted left-action

$$[\mathcal{U}^\lambda_{g'}\phi](Z) \equiv \mathcal{W}_\lambda^{-1}(g)[\mathcal{U}^L_{g'}\psi_\lambda](g) = \det(D'^\dagger - B'^\dagger Z)^{-\lambda}\phi(Z') \equiv \phi'(Z) \tag{57}$$

of G on the arbitrary (holomorphic) part ϕ of ψ_λ, instead of (47). In standard (induced) representation theory, the factor $\det(D'^\dagger - B'^\dagger Z)^{-\lambda}$ is called a "multiplier" (Radon–Nicodym derivative) and fulfills cocycle properties. For the representation (57) of G on holomorphic functions $\phi(Z)$ to be unitary, the left-G-invariant Haar measure (49) has to be accordingly modified as:

$$d\mu_\lambda(Z, Z^\dagger) \equiv c_\lambda |\mathcal{W}_\lambda(g)|^2 d\mu^L(g)|_{G/H} = c_\lambda \det(\sigma^0 - ZZ^\dagger)^{\lambda-4}|dZ|, \tag{58}$$

where $d\mu^L(g)|_{G/H}$ in (49) is the projection of the left-G-invariant Haar measure $d\mu^L(g)$ onto G/H. Roughly speaking, we are integrating out the coordinates of H and redefining the normalization constant c in (51) as $c_\lambda = c/v(U(2)) = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)$ so that the unit constant function $\phi(Z) = 1$ (the ground state) is normalized (see [31] for orthogonality properties). As before, we could also introduce a modified bracket notation $\phi(Z) \equiv (Z \mid \phi)$ and a new set $\{|Z), Z \in \mathbb{D}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda(\mathbb{D}) = L^2(\mathbb{D}, d\mu_\lambda)$ of analytic square-integrable holomorphic functions ϕ on $\mathbb{D}$. The new coherent state overlap $(Z \mid Z')$ is nothing but the so-called reproducing Bergman's kernel $K_\lambda(Z, Z')$. It is related to (56) by:

$$K_\lambda(Z', Z) = (Z' \mid Z) = \frac{\langle g' \mid g\rangle}{\mathcal{W}_\lambda(g')\overline{\mathcal{W}_\lambda(g)}} = \det(\sigma^0 - Z^\dagger Z')^{-\lambda}. \tag{59}$$

We notice that, unlike $|g\rangle$, the coherent state $|Z)$ is not normalized. In fact,

$$\mathcal{K}_\lambda(Z, Z^\dagger) \equiv \ln{(Z \mid Z)} \tag{60}$$

is nothing but the Kähler potential, which defines $\mathbb{D}$ as a Kähler manifold with local complex coordinates $Z = z_\mu \sigma^\mu$, an hermitian Riemannian metric g and a corresponding closed two-form ω

$$ds^2 = g^{\mu\nu} dz_\mu \odot d\bar{z}_\nu, \quad \omega = -ig^{\mu\nu} dz_\mu \wedge d\bar{z}_\nu, \quad g^{\mu\nu} \equiv \frac{\partial^2 \mathcal{K}_\lambda}{\partial z_\mu \partial \bar{z}_\nu}, \tag{61}$$

where $\odot$ denotes symmetrization. We shall come back to the Riemannian structure of $\mathbb{D}$ and $\mathbb{T}$ and the connection with the BRP later on in Sec. 5.

4.1.3. *Schwinger's theorem, orthonormal basis and closure relations*

As already commented after Eq. (52), we are interested in calculating an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D})$ made of Hamiltonian eigenfunctions $\varphi_J(Z) \equiv (Z \mid \lambda, J)$, where J denotes a set of indices. This orthonormal basis would provide us with a new resolution of the identity

$$1 = \sum_J |\lambda, J)\,(\lambda, J|\,.$$

Actually, we shall identify $\varphi_J(Z)$ by looking at the expansion of the Bergman's kernel

$$K_\lambda(Z', Z) = (Z' \mid Z) = \sum_J (Z' \mid \lambda, J)\,(\lambda, J \mid Z) = \sum_J \varphi_J(Z')\overline{\varphi_J(Z)}.$$

Thus, the Bergman's kernel plays here the role of a generating function. To be more precise:

Theorem 4.3. *The infinite set of polynomials*

$$\varphi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\det(Z)^m \mathcal{D}^j_{q_1,q_2}(Z), \tag{62}$$

with

$$\mathcal{D}^j_{q_1,q_2}(Z) = \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}} \sum_{p=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)} \binom{j+q_2}{p}\binom{j-q_2}{p-q_1-q_2} \times z_{11}^p z_{12}^{j+q_1-p} z_{21}^{j+q_2-p} z_{22}^{p-q_1-q_2} \tag{63}$$

the standard Wigner's $\mathcal{D}$-matrices (j *is a non-negative half-integer*), *verifies the following closure relation* (*the reproducing Bergman kernel*):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} \overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(\sigma^0 - Z^\dagger Z')^\lambda} \tag{64}$$

and constitute an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D})$.

This theorem has been proven in [31]. It turns out to be rooted in a extension of the Schwinger's formula:

Theorem 4.4 (Schwinger's Master theorem). *The identity*

$$\sum_{j\in\mathbb{N}/2} t^{2j}\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X) = \frac{1}{\det(\sigma^0 - tX)} \tag{65}$$

holds for any 2×2 matrix X, with t an arbitrary parameter.

The above-mentioned extension of the Theorem 4.4 can be stated as:

Theorem 4.5 (λ-Extended Schwinger's Master theorem). *For every $\lambda \in \mathbb{N}$, $\lambda \geq 2$ and every 2×2 complex matrix X the following identity holds*:

$$\sum_{j\in\mathbb{N}/2}\frac{2j+1}{\lambda-1}\sum_{m=0}^{\infty} t^{2j+2m}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}\det(X)^m\sum_{q=-j}^{j}\mathcal{D}^j_{qq}(X) = \det(\sigma^0 - tX)^{-\lambda}. \tag{66}$$

We address the interested reader to [31] for a complete proof.

Scketch of proof of Theorem 4.3. Assuming the validity of (66) and replacing $tX = Z^\dagger Z'$ in it, we have:

$$\sum_{j\in\mathbb{N}/2} \frac{2j+1}{\lambda-1} \sum_{m=0}^{\infty} \binom{m+\lambda-2}{\lambda-2} \binom{m+2j+\lambda-1}{\lambda-2} \det(Z^\dagger Z')^m \sum_{q=-j}^{j} \mathcal{D}^j_{qq}(Z^\dagger Z')$$
$$= \frac{1}{\det(\sigma^0 - Z^\dagger Z')^\lambda}. \tag{67}$$

Using determinant and Wigner's $\mathcal{D}$-matrix rules

$$\det(Z^\dagger Z')^n \sum_{q=-j}^{j} \mathcal{D}^j_{qq}(Z^\dagger Z') = \det(Z^\dagger)^n \det(Z')^n \sum_{q_2,q_1=-j}^{j} \overline{\mathcal{D}^j_{q_1 q_2}(Z)} \mathcal{D}^j_{q_1 q_2}(Z'),$$

and the definition of the functions (62), we see that (67) reproduces (64). On the other hand, the number of linearly independent polynomials $\prod_{i,j=1}^{2} z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^{2} n_{ij}$ is $(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (62) with degree of homogeneity $n = 2m + 2j$. This proves that the set of polynomials (62) is a basis for analytic functions $\phi \in \mathcal{H}_\lambda(\mathbb{D}_4)$. Moreover, this basis turns out to be orthonormal under the projected integration measure (58). We address the interested reader to the Appendix C of [31] for a proof. □

Remark 4.6. The set (62) constitutes a basis of Hamiltonian eigenfunctions with energy eigenvalues E^λ_n (the homogeneity degree) given by:

$$\hat{\mathcal{H}}_\lambda \varphi^{j,m}_{q_1,q_2} = E^\lambda_n \varphi^{j,m}_{q_1,q_2}, \quad E^\lambda_n = \lambda + n, \quad n = 2j + 2m, \tag{68}$$

with $\hat{\mathcal{H}}_\lambda = D_\lambda$ defined in (53). Each energy level E^λ_n is then $(n+1)(n+2)(n+3)/6$ times degenerated. The spectrum is equi-spaced and bounded from below, with $E^\lambda_0 = \lambda$ playing the role of a zero-point energy. At this stage it is interesting to compare our Hamiltonian choice with others in the literature like [53] studying a $SU(2,2)$-harmonic oscillator on the phase space $\mathbb{D}$. In this case the quantum Hamiltonian is chosen to be the Toeplitz operator corresponding to the square of the distance with respect to the $SU(2,2)$-invariant Kähler metric (61) on the phase space $\mathbb{D}$.

4.2. *Conformal spinning quantum particles*

Let us use the following notation for

$$X_0 = \sum_{\alpha=1}^{4} \lambda_\alpha X^\alpha_\alpha = \lambda D + s_1 \Sigma^{(3)}_1 + s_2 \Sigma^{(3)}_2 + \kappa I, \tag{69}$$

where

$$\Sigma^{(3)}_1 = X^1_1 - X^2_2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^{(3)}_2 = X^3_3 - X^4_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

stand for the third spin components and I the 4×4 identity matrix. The identification (69) implies that the spin labels of the representation of the subgroup $SU(2)^2$ are $s_1 \equiv (\lambda_1 - \lambda_2)/2$ and $s_2 \equiv (\lambda_3 - \lambda_4)/2$. The conformal dimension is $\lambda = (\lambda_3 + \lambda_4 - \lambda_1 - \lambda_2)/2$ and $\kappa = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/4$ is the (trace) $U(1)$ quantum number. We shall choose, without lost of generality, $\kappa = 0$, which means that λ remains integer (as in the spinless case) and that we are restricting ourselves to representations of $SU(2,2) \subset U(2,2)$.

Theorem 4.7. *The general solution to* (35) *and* (36) *can be factorized as*:

$$\psi_\lambda^{s_1,s_2}(g) = \mathcal{W}_\lambda^{s_1,s_2}(g)\phi(Z, z_1, z_2), \tag{70}$$

where the ground state

$$\begin{aligned}\mathcal{W}_\lambda^{s_1,s_2}(g) &= \det(A^\dagger)^{-\lambda_s/2}\det(D)^{-\lambda_s/2}\mathcal{D}^{s_1}_{s_1,s_1}(U_1^\dagger)\mathcal{D}^{s_2}_{-s_2,-s_2}(U_2)\\ &= \det(A^\dagger)^{-\lambda_s/2}\det(D)^{-\lambda_s/2}\bar{a}_1^{2s_1}d_2^{2s_2}\\ &= \det(\Delta_1 U_1^\dagger)^{-\lambda_s/2}\det(\Delta_2 U_2)^{-\lambda_s/2}(\delta_1 e^{-i\alpha_1})^{2s_1}(\delta_2 e^{i\beta_2})^{2s_2}\\ &= \det(\sigma^0 - Z^\dagger Z)^{\lambda_s/2}(1 + \bar{z}_1 z_1)^{-s_1}(1 + \bar{z}_2 z_2)^{-s_2}\\ &\quad \times e^{-i\alpha_1(2s_1 - \lambda_s/2)}e^{i\beta_1\lambda_s/2}e^{-i\alpha_2\lambda_s/2}e^{i\beta_2(2s_2 - \lambda_s/2)},\end{aligned} \tag{71}$$

with $\lambda_s \equiv \lambda - s_1 - s_2$, is a particular solution of (35), (36) *and ϕ is the general solution for the trivial representation $\lambda_\alpha = 0$ of $G^0 = U(1)^4$* (*actually, an arbitrary, analytic holomorphic function of Z, z_1, z_2*), *for the decomposition* (20), (21) *of an element $g \in G$.*

Proof. On the one hand, from (43) we conclude that the factors $\det(D)^{-\lambda}$ and $\mathcal{D}^{s_2}_{-s_2,-s_2}(U_2)$, with $U_2 = (DD^\dagger)^{-1/2}D$ fulfill the holomorphicity conditions (36) for $(\beta, \alpha) = (1,3)$, $(2,3)$, $(1,4)$, $(2,4)$. Moreover, $U_1^\dagger = A^\dagger(AA^\dagger)^{-1/2}$ and we have that

$$A''^\dagger = A(gg')^\dagger = A'^\dagger A^\dagger + C'^\dagger B^\dagger = A'^\dagger(A^\dagger + (C'A'^{-1})^\dagger B^\dagger) = A'^\dagger(A^\dagger + Z'B^\dagger)$$

is not affected by $Z'^\dagger = C'A'^{-1}$ either, according to the definition (13). On the other hand, for $g' \in H$ we have that

$$\bar{a}'' = \bar{a}(gg') = \bar{a}\bar{a}' + \bar{b}\bar{c}' = \bar{a}'(\bar{a} - z'\bar{b})$$

$$d'' = d(gg') = cb' + dd' = d'(d + z'c)$$

are not affected by $\bar{z}' = -c'/a' = \bar{b}'/\bar{d}'$, according to the definition (21). This proves that the ground state (71) fulfills the holomorphicity conditions (36) for $(\beta, \alpha) = (1,2)$, $(3,4)$. It remains to prove the gauge conditions (35) or their finite counterpart (34) for $g_0 \in G^0 = U(1)^4$. Finite right (gauge) dilations $g_0 = e^{i\tau D}$ leave $\mathcal{W}_\lambda^{s_1,s_2}(gg_0) = e^{i\lambda\tau}\mathcal{W}_\lambda^{s_1,s_2}(g)$ invariant up to the phase $\mathcal{U}_0^\lambda(g_0) = e^{i\lambda\tau}$ (a character of G^0), where we have used that $\det(\cdot)$ and $\mathcal{D}^s(\cdot)$ are homogeneous of degree 2 and $2s$, respectively. Infinitesimally, it means that $D^L\psi_\lambda^{s_1,s_2} = \lambda\psi_\lambda^{s_1,s_2}$. For $g_0^{(1,2)} = e^{i\alpha\Sigma^3_{1,2}}$ the ground state transforms as expected:

$$\mathcal{W}_\lambda^{s_1,s_2}(gg_0^{(1,2)}) = \mathcal{U}_0^\lambda(g_0^{(1,2)})\mathcal{W}_\lambda^{s_1,s_2}(g), \quad \mathcal{U}_0^\lambda(g_0^{(1,2)}) = e^{2is_{1,2}\alpha}.$$

Infinitesimally, it means that

$$\Sigma_{1,2}^{L(3)}\mathcal{W}_\lambda^{s_1,s_2} = 2s_{1,2}\mathcal{W}_\lambda^{s_1,s_2}, \quad \begin{cases}\Sigma_1^{L(3)} \equiv L_1^1 - L_2^2,\\ \Sigma_2^{L(3)} \equiv L_3^3 - L_4^4.\end{cases} \tag{72}$$

Moreover, one can easily check that $\mathcal{W}_\lambda^{s_1,s_2}(gg_0) = \mathcal{W}_\lambda^{s_1,s_2}(g)$ for diagonal $U(1)$ transformations $g_0 = e^{i\theta}I$, that is, $\kappa = 0$. Finally, using similar arguments to those employed in (44), we can assert that $z'_{1,2} = z_{1,2}(gg_0) = z_{1,2}$, $\forall\, g_0 \in G^0$, which ends up proving the gauge conditions (34). □

Remark 4.8. Instead of (36), we could have chosen the complementary constraint $L_\alpha^\beta\psi = 0$, $\forall\,\alpha < \beta$ which would have led us to a anti-holomorphic representation.

As in Eq. (47), we can compute the finite left-action of G on physical wave functions (70). In particular, for the case of dilations $g' = e^{i\tau' D}$ (i.e. $B' = 0 = C'$ and $A' = e^{-i\tau'/2}\sigma^0 = D'^\dagger$) we have:

$$\psi_\lambda^{s_1,s_2}(g'^{-1}g) = e^{i\lambda\tau'}\mathcal{W}_\lambda^{s_1,s_2}(g)\phi(e^{i\tau'}Z, z_1, z_2),$$

or infinitesimally:

$$D^R\psi_\lambda^{s_1,s_2}(g) = \mathcal{W}_\lambda^{s_1,s_2}(g)\left(\lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}}\right)\phi(Z, z_1, z_2). \tag{73}$$

Comparing this expression with (52), we realize that the spin coordinates z_1, z_2 do not contribute to the degree of homogeneity of ϕ under dilations, as they correspond to "internal" (versus space-time-momentum) degrees of freedom.

As in the previous subsection, we can introduce a modified bracket notation $\phi(Z, z_1, z_2) \equiv (Z, z_1, z_2 \mid \phi)$ and a set $\{|Z, z_1, z_2), Z \in \mathbb{D}, z_1, z_2 \in \mathbb{C}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda^{s_1,s_2}(\mathbb{F})$ of analytic measurable holomorphic functions ϕ on the 12-dimensional pseudo-flag manifold $\mathbb{F} = U(2,2)/U(1)^4$, locally $\mathbb{D} \times \overline{\mathbb{C}}^2$, with integration measure

$$d\mu_\lambda^{s_1,s_2}(Z, z_1, z_2; Z^\dagger, \bar{z}_1, \bar{z}_2) \equiv d\mu_{\lambda_s}(Z, Z^\dagger)\frac{2s_1+1}{\pi}ds(U_1)\frac{2s_2+1}{\pi}ds(U_2), \tag{74}$$

where $d\mu_{\lambda_s}(Z, Z^\dagger)$ and $ds(U)$ are defined in (58) and (50), respectively. Note that the square-integrability condition $\lambda \geq 4$ in $\mathcal{H}_\lambda(\mathbb{D})$ becomes $\lambda_s \geq 4$ in $\mathcal{H}_\lambda^{s_1,s_2}(\mathbb{F})$. The constant factor $(2s_1+1)/\pi$ is introduced so that the following set of functions is normalized.

Theorem 4.9. *The infinite set of polynomials*

$$\check{\varphi}^{m,m_1,m_2}_{j,q_1,q_2}(Z,z_1,z_2) \equiv (-1)^{m_1+s_1}\varphi^{j,m}_{q_1,q_2}(Z)\frac{\mathcal{D}^{s_1}_{s_1,-m_1}(U_1^\dagger)\mathcal{D}^{s_2}_{m_2,-s_2}(U_2)}{\mathcal{D}^{s_1}_{s_1,s_1}(U_1^\dagger)\mathcal{D}^{s_2}_{-s_2,-s_2}(U_2)}$$

$$= \varphi^{j,m}_{q_1,q_2}(Z)\sqrt{\binom{2s_1}{m_1+s_1}\binom{2s_2}{m_2+s_2}}\, z_1^{m_1+s_1} z_2^{m_2+s_2}, \qquad (75)$$

(*with* $\varphi^{j,m}_{q_1,q_2}$ *in* (62) *replacing* $\lambda \to \lambda_s$) *provides an orthonormal basis of* $\mathcal{H}^{s_1,s_2}_\lambda(\mathbb{F})$. *The closure relation*:

$$\sum_{j\in\mathbb{N}/2}^{\infty}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\sum_{m_1=-s_1}^{s_1}\sum_{m_2=-s_2}^{s_2}\check{\varphi}^{m,m_1,m_2}_{j,q_1,q_2}(Z',z_1',z_2')\overline{\check{\varphi}^{m,m_1,m_2}_{j,q_1,q_2}(Z,z_1,z_2)}$$

$$= (Z',z_1',z_2' \mid Z,z_1,z_2) \qquad (76)$$

gives the reproducing Bergman's kernel for spinning particles:

$$K^{s_1,s_2}_\lambda(Z',z_1',z_2';Z,z_1,z_2) = (Z',z_1',z_2' \mid Z,z_1,z_2)$$

$$= \det(\sigma^0 - Z^\dagger Z')^{-\lambda_s}(1+\bar{z}_1 z_1')^{2s_1}(1+\bar{z}_2 z_2')^{2s_2}. \qquad (77)$$

Proof. Assuming the orthonormality of (62) (see Appendix C of [31]), and realizing that

$$\int_{\mathbb{C}}\sqrt{\binom{2s}{m}}\bar{z}^m\sqrt{\binom{2s}{m'}}z^{m'}\frac{2s+1}{\pi}ds(U) = \delta_{m,m'}, \quad m,m' = 0,\dots,2s,$$

we prove the orthonormality of the functions (75). Moreover, the number of linearly independent polynomials $\prod_{i,j=1}^2 z_{ij}^{n_{ij}}\prod_{i=1}^2 z_i^{n_i}$ with $0 \le n_i \le 2s_i$ and fixed $n = \sum_{i,j=1}^2 n_{ij}$ is $(2s_1+1)(2s_2+1)(n+1)(n+2)(n+3)/6$, which coincides with the number of linearly independent polynomials (75) with degree of homogeneity $n = 2m+2j$ in the coordinates Z. This proves that the set of polynomials (75) is a basis for analytic functions $\mathcal{H}^{s_1,s_2}_\lambda(\mathbb{F})$.

It just remains to prove the closure relation (76). This proof reduces to that of Theorem 4.3 when noting the binomial identity $\sum_{m=0}^{2s}\binom{2s}{m}(\bar{z}z')^m = (1+\bar{z}z')^{2s}$ or the Wigner $\mathcal{D}$-matrix property

$$\sum_{n=-s}^{s}\mathcal{D}^s_{sn}(U)\mathcal{D}^s_{ns}(U') = \mathcal{D}^s_{ss}(UU'). \qquad \square$$

Remark 4.10. At this point it is interesting to compare our construction with others in the literature like [8], where the proposed basis functions

$$\Phi^{m,m_1,m_2}_{j,q_1,q_2}(A,D,Z) = \mathcal{D}^{j_1}_{j_1,m_1}(A^T)\mathcal{D}^{j_2}_{m_2,j_2}(D)\varphi^{j,m}_{q_1,q_2}(Z) \qquad (78)$$

do not form an orthogonal set unless a coupling between orbital angular momentum j with spin j_1, j_2 by means of Clebsch–Gordan coefficients is made:

$$\tilde{\Phi}^{m,p_1,p_2}_{j,j_1,j_2} = \sum_{m_1,m_2,q_1,q_2} C(j,q_1;s_1,m_1-s_1\,|\,j_1,p_1)C(j,q_2;s_2,m_2-s_2\,|\,j_2,p_2)\Phi^{m,m_1,m_2}_{j,q_1,q_2}.$$

Moreover, the fact that $U_1 = (AA^\dagger)^{-1/2}A$ and $U_2 = (DD^\dagger)^{-1/2}D$ introduces a new contribution of $\mathcal{D}^{j_1}(A)$ and $\mathcal{D}^{j_2}(D)$ to the integration measure $d\mu^{j_1,j_2}_\lambda$, with respect to $\mathcal{D}^{j_1}(U_1)$ and $\mathcal{D}^{j_2}(U_2)$, such that the square-integrability condition becomes $\lambda \geq 4 + 2j_1 + 2j_2$.

The Hamiltonian of our spinning particle is $\hat{\mathcal{H}} = -i\frac{\partial}{\partial\tau}$ with τ given by (33). Its expression in terms of right-invariant vector fields R^α_β is then

$$\hat{\mathcal{H}} = \sum_{\alpha=1}^{4} \frac{1}{\lambda_\alpha} R^\alpha_\alpha = \rho_0 D^R + \rho_1 \Sigma_1^{R(3)} + \rho_2 \Sigma_2^{R(3)} + \rho_3 I, \tag{79}$$

with

$$\rho_0 = \frac{4\lambda(\lambda^2 - s_1^2 - s_2^2)}{(\lambda^2 - 4s_1^2)(\lambda^2 - 4s_2^2)}, \quad \rho_1 = \frac{-4s_1}{\lambda^2 - 4s_1^2},$$

$$\rho_2 = \frac{-4s_2}{\lambda^2 - 4s_2^2}, \quad \rho_3 = \frac{4\lambda(s_2^2 - s_1^2)}{(\lambda^2 - 4s_1^2)(\lambda^2 - 4s_2^2)},$$

and $\Sigma^{R(3)}_{1,2}$ the right-invariant version of (72). In order to compare with the spinless case, we can always renormalize

$$\hat{\mathcal{H}} \to \hat{\mathcal{H}}/\rho_0 = \hat{\mathcal{H}}' = D^R + \varrho_1 \Sigma_1^{R(3)} + \varrho_2 \Sigma_2^{R(3)} + \varrho_3 I, \tag{80}$$

with $\varrho_\alpha = \rho_\alpha/\rho_0$. We can interpret $\varrho_{1,2}$ as constant "magnetic fields" (oriented along the "z" direction) coupled to the spin degrees of freedom $\Sigma^{R(3)}_{1,2}$. The set (75) constitutes a basis of eigenfunctions of the Hamiltonian (80) with eigenvalues (energy levels) given by:

$$E^{\lambda,m_1,m_2}_{n,q_1,q_2} = \lambda + \varrho_3 + n + \varrho_1(m_1 + q_1) + \varrho_2(m_2 + q_2), \quad n = 2j + 2m. \tag{81}$$

Comparing this energy eigenvalues with the energy spectrum (68) of the spinless Hamiltonian $\hat{\mathcal{H}} = D^R$, we realize that the zero-point energy has been shifted from λ to $\lambda + \varrho_3 - s_1\varrho_1 - s_2\varrho_2$. Like in the (anomalous) *Zeeman effect*, the introduction of spin leads to a splitting of a spinless spectral line E^λ_n into $(2s_1 + 1)(2s_2 + 1)$ components in the presence of a "static magnetic field" $\varrho_{1,2}$.

5. Relation with the Tube Domain Realization

In this section we shall translate some expressions obtained from the complex Cartan domain (16) into the forward tube domain (17), where we enjoy more (Minkowskian) intuition. We shall restrict ourselves to the scalar case, since it is representative of the more general case.

5.1. *Tube domain as a homogeneous space of $SU(2,2)$*

As we have already said, the forward tube domain $\mathbb{T}$ is naturally homeomorphic to the quotient G/H in the realization of G in terms of matrices

$$f = \begin{pmatrix} R & iS \\ -iT & Q \end{pmatrix} \tag{82}$$

which preserve $\Gamma = \gamma^0$, instead of $\Gamma = \gamma^5$; that is, $f^\dagger \gamma^0 f = \gamma^0$. Both realizations of G are related by the map (19), which can be explicitly written as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \Upsilon^{-1} f \Upsilon = \frac{1}{2} \begin{pmatrix} R + iS - iT + Q & -R + iS + iT + Q \\ -R - iS - iT + Q & R - iS + iT + Q \end{pmatrix}. \tag{83}$$

The identification of $\mathbb{T}$ with the quotient G/H is given through

$$W(f) = i(R - iS)(Q + iT)^{-1}. \tag{84}$$

Hence, the left translation $f' \to ff'$ of G on itself induces a left-action of G on $\mathbb{T}$ given by:

$$W = W(f') \to W' = W(ff') = (RW + S)(TW + Q)^{-1}. \tag{85}$$

Setting $W = x_\mu \sigma^\mu$, and making use of the standard homomorphism (spinor map) between $SL(2,\mathbb{C})$ and $SO^+(3,1)$ given by: $W' = RWR^\dagger \leftrightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, R \in SL(2,\mathbb{C}), \Lambda^\mu_\nu \in SO^+(3,1)$, the transformations (4) can be recovered from (85) as follows:

(i) Standard Lorentz transformations, $x'^\mu = \Lambda^\mu_\nu(\omega) x^\nu$, correspond to $T = S = 0$ and $R = Q^{-1\dagger} \in SL(2,\mathbb{C})$.
(ii) Dilations correspond to $T = S = 0$ and $R = Q^{-1} = \rho^{1/2} I$.
(iii) Space-time translations equal $R = Q = \sigma^0$ and $S = b_\mu \sigma^\mu,\ T = 0$.
(iv) Special conformal transformations correspond to $R = Q = \sigma^0$ and $T = a_\mu \sigma^\mu,\ S = 0$ by noting that $\det(\sigma^0 + TW) = 1 + 2ax + a^2 x^2$.

5.2. *Irreducible representations, Haar measure and Bergman kernel*

Let us see the expression of the wave functions (41) in the tube domain $\mathbb{T}$. Performing the change of variables (83) in (41) we get

$$\psi_\lambda(f) = \det(Q + iT)^{-\lambda} 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \phi(Z(W)) \equiv \Omega_\lambda(f) \tilde{\phi}(W), \tag{86}$$

where we have defined a new ground state Ω_λ and a new function $\tilde{\phi}$ as:

$$\Omega_\lambda(f) \equiv \det(Q + iT)^{-\lambda}, \quad \tilde{\phi}(W) \equiv 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \phi(Z(W)). \tag{87}$$

In the same manner, the coherent-state overlap (56) can be cast as

$$\langle f' \mid f\rangle = \det(Q^\dagger - iT^\dagger)^{-\lambda} \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda} \det(Q' + iT')^{-\lambda}. \tag{88}$$

Since the ground state Ω_λ is a fixed common factor of all the wave functions (86), we can factor it out (as we did in (57) with $\mathcal{W}_\lambda$) and define the restricted action

$$\begin{aligned}[\tilde{\mathcal{U}}^\lambda_{f'}\tilde{\phi}](W) &\equiv \Omega_\lambda^{-1}(f)[\mathcal{U}^L_{f'}\psi_\lambda](f) \\ &= \det(R'^\dagger - T'^\dagger W)^{-\lambda}\tilde{\phi}((Q'^\dagger W - S'^\dagger)(R'^\dagger - T'^\dagger W)^{-1}) \equiv \tilde{\phi}'(W)\end{aligned} \tag{89}$$

of G on the arbitrary (holomorphic) part $\tilde{\phi}$ of ψ_λ. The Radon–Nicodym derivative is now $\det(R'^\dagger - T'^\dagger W)^{-\lambda}$. The representation (89) of G on holomorphic functions $\tilde{\phi}(W)$ is unitary with respect to the re-scaled integration measure

$$d\tilde{\mu}_\lambda(W, W^\dagger) \equiv |\Omega_\lambda(f)|^2 d\tilde{\mu}^L(f)|_{G/H} = \frac{c_\lambda}{2^4}\det\left(\frac{i}{2}(W^\dagger - W)\right)^{\lambda-4}|dW|, \tag{90}$$

where we are using $|dW|$ as a shorthand for the Lebesgue measure $\bigwedge_{i,j=1}^2 d\Re w_{ij} d\Im w_{ij}$ on $\mathbb{T}$. To arrive at (90), firstly, we have performed the Cayley transformation (18) in the projected integration measure:

$$\begin{aligned} d\mu^L(g)\big|_{G/H} &= c_\lambda \det(\sigma^0 - ZZ^\dagger)^{-4}|dZ| \to \\ d\tilde{\mu}^L(f)\big|_{G/H} &= \frac{c_\lambda}{2^4}\det\left(\frac{i}{2}(W^\dagger - W)\right)^{-4}|dW|, \end{aligned} \tag{91}$$

taking into account that $\det(\sigma^0 - ZZ^\dagger) = \det(2i(W^\dagger - W))|\det(\sigma^0 - iW)|^{-2}$ and the Jacobian determinant $|dZ|/|dW| = 2^{12}|\det(\sigma^0 - iW)|^{-8}$, and secondly, we have written

$$|\Omega_\lambda(f)|^2 = \det(Q^\dagger - iT^\dagger)^{-\lambda}\det(Q + iT)^{-\lambda} = \det\left(\frac{i}{2}(W^\dagger - W)\right)^{\lambda}$$

by making use of (84) and its hermitian conjugate.

As in (59), we could also introduce a modified bracket notation $\tilde{\phi}(W) \equiv (W \,|\, \tilde{\phi})$ and a new set $\{|W), W \in \mathbb{T}\}$ of coherent states in the Hilbert space $\mathcal{H}_\lambda(\mathbb{T})$ of analytic measurable holomorphic functions φ on $\mathbb{T}$. The new coherent state overlap $(W \mid W')$ is the new Bergman's kernel $\tilde{K}_\lambda(W', W)$. It is related to (88) by:

$$\tilde{K}_\lambda(W', W) = (W' \mid W) = \frac{\langle f' \mid f\rangle}{\Omega_\lambda(f')\overline{\Omega}_\lambda(f)} = \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda}. \tag{92}$$

We again notice that, unlike $|f\rangle$, the coherent state $|W)$ is not normalized. Now, the Kähler potential is $\ln(W \mid W)$, which defines $\mathbb{T}$ as a Kähler manifold too.

The identification (87) actually provides an isometry between the spaces of analytic holomorphic functions $\mathcal{H}_\lambda(\mathbb{D})$ and $\mathcal{H}_\lambda(\mathbb{T})$. Let us formally state it.

Proposition 5.1. *The correspondence*

$$\mathcal{S}_\lambda : \mathcal{H}_\lambda(\mathbb{D}) \to \mathcal{H}_\lambda(\mathbb{T})$$
$$\phi \mapsto \mathcal{S}_\lambda \phi \equiv \tilde{\phi},$$

with

$$\tilde{\phi}(W) = 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \phi(Z(W)) \tag{93}$$

and $Z(W)$ given by the Cayley transformation (18), *is an isometry, that is*:

$$\langle \phi \,|\, \phi' \rangle_{\mathcal{H}_\lambda(\mathbb{D})} = \langle \mathcal{S}_\lambda \phi \,|\, \mathcal{S}_\lambda \phi' \rangle_{\mathcal{H}_\lambda(\mathbb{T})}. \tag{94}$$

Moreover, $\mathcal{S}_\lambda$ is an intertwiner (*equivariant map*) *of the representations* (47) *and* (89), *that is*:

$$\mathcal{U}_\lambda = \mathcal{S}_\lambda^{-1} \tilde{\mathcal{U}}_\lambda \mathcal{S}_\lambda. \tag{95}$$

Proof. The isometry property is proven by construction from (87). The intertwining relation (95) can be explicitly written as:

$$[\mathcal{U}_\lambda \phi](Z) = \det(D^\dagger - B^\dagger Z)^{-\lambda} \phi((A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1})$$
$$[\mathcal{S}_\lambda^{-1} \tilde{\mathcal{U}}_\lambda \tilde{\phi}](Z) = \det(\sigma^0 - iW)^{\lambda} \det(R^\dagger - T^\dagger W)^{-\lambda} \det(\sigma^0 - iW')^{-\lambda} \phi(Z(W')), \tag{96}$$

where $W' = (Q^\dagger W - S^\dagger)(R^\dagger - T^\dagger W)^{-1}$. On the one hand, we have that the argument of ϕ is:

$$\begin{aligned} Z(W') &= (\sigma^0 + iW')(\sigma^0 - iW')^{-1} \\ &= ((R^\dagger - T^\dagger W) + i(Q^\dagger W - S^\dagger))((R^\dagger - T^\dagger W) - i(Q^\dagger W - S^\dagger))^{-1} \\ &= ((R^\dagger - iS^\dagger) + i(Q^\dagger + iT^\dagger)W)((R^\dagger + iS^\dagger) - i(Q^\dagger - iT^\dagger)W)^{-1}. \end{aligned}$$

Taking now into account the map (83) we have:

$$\begin{aligned} Z(W') &= ((A^\dagger - C^\dagger) + i(A^\dagger + C^\dagger)W)((D^\dagger - B^\dagger) - i(D^\dagger + B^\dagger)W)^{-1} \\ &= (A^\dagger(\sigma^0 + iW) - C^\dagger(\sigma^0 - iW))(D^\dagger(\sigma^0 - iW) - B^\dagger(\sigma^0 + iW))^{-1} \\ &= (A^\dagger Z - C^\dagger)(D^\dagger - B^\dagger Z)^{-1}, \end{aligned}$$

as desired. On the other hand, we have that

$$\begin{aligned} (\sigma^0 - iW')(R^\dagger - T^\dagger W) &= (R^\dagger - T^\dagger W) - i(Q^\dagger W - S^\dagger) \\ &= (R^\dagger + iS^\dagger) - i(Q^\dagger - iT^\dagger)W = (D^\dagger - B^\dagger) - i(D^\dagger + B^\dagger)W \\ &= D^\dagger(\sigma^0 - iW) - B^\dagger(\sigma^0 + iW) = (D^\dagger - B^\dagger Z)(\sigma^0 - iW) \end{aligned}$$

which implies

$$\det(\sigma^0 - iW)^{\lambda} \det(R^\dagger - T^\dagger W)^{-\lambda} \det(\sigma^0 - iW')^{-\lambda} = \det(D^\dagger - B^\dagger Z)^{-\lambda}.$$

That is, the equality of multipliers in (96). □

As a direct consequence of Proposition 5.1, the set of functions defined by

$$\tilde{\varphi}^{j,m}_{q_1,q_2}(W) \equiv 2^{2\lambda} \det(\sigma^0 - iW)^{-\lambda} \varphi^{j,m}_{q_1,q_2}(Z(W)), \tag{97}$$

with $\varphi^{j,m}_{q_1,q_2}$ defined in (62), constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{T})$ and the closure relation

$$\sum_{j\in\mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q,q'=-j}^{j} \overline{\tilde{\varphi}^{j,m}_{q',q}(W)} \tilde{\varphi}^{j,n}_{q',q}(W') = \det\left(\frac{i}{2}(W^\dagger - W')\right)^{-\lambda}, \tag{98}$$

renders again the reproducing Bergman kernel (92).

5.3. *Kähler structures, Born's reciprocity and maximal acceleration*

As we said for the Cartan domain $\mathbb{D}$ in (60) and (61), the Kähler potential

$$\tilde{\mathcal{K}}_\lambda(W, W^\dagger) \equiv \ln(W \mid W) = -\ln|\Omega_\lambda(f)|^2 = -\lambda \ln(\Im(w))^2 = -\lambda \ln y^2 \tag{99}$$

defines $\mathbb{T}$ as a Kähler manifold with local complex coordinates $W = w_\mu \sigma^\mu$, $w_\mu = x_\mu + iy_\mu$, an hermitian Riemannian metric

$$g^{\mu\nu} \equiv \frac{\partial^2 \tilde{\mathcal{K}}_\lambda}{\partial w_\mu \partial \bar{w}_\nu} = -\frac{\lambda}{2y^2}\left(\eta^{\mu\nu} - 2\frac{y_\mu y_\nu}{y^2}\right) \tag{100}$$

and a corresponding closed two-forms ω

$$\omega = -ig^{\mu\nu} dw_\mu \wedge d\bar{w}_\nu. \tag{101}$$

The line element

$$ds^2 = g^{\mu\nu} dw_\mu d\bar{w}_\nu = -\frac{\lambda}{2y^2}\left(\eta^{\mu\nu} - 2\frac{y^\mu y^\nu}{y^2}\right)(dx_\mu dx_\nu + dy_\mu dy_\nu) \tag{102}$$

turns out to be positive and provides a conformal counterpart of the Born's line element (2). The two-forms (101) defines the Poisson bracket:

$$\{a, b\} \equiv ig_{\mu\nu}\left(\frac{\partial a}{\partial w_\mu}\frac{\partial b}{\partial \bar{w}_\nu} - \frac{\partial b}{\partial w_\mu}\frac{\partial a}{\partial \bar{w}_\nu}\right) \tag{103}$$

for the inverse metric

$$g_{\mu\nu} = -\frac{2}{\lambda}(\eta^{\mu\nu} y^2 - 2y_\mu y_\nu), \tag{104}$$

so that $g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$. In particular, we have that:

$$\{x_\mu, y_\nu\} = -\frac{1}{2} g_{\mu\nu},$$

which differs from $\{x_\mu, y_\nu\} = \eta_{\mu\nu}$; that is, x_μ and y_ν are not "canonical" coordinates. However, we can define a proper conjugate four-momentum $p_\mu \equiv \lambda y_\mu / y^2$ which gives the desired (canonical) Poisson bracket

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu}, \tag{105}$$

as can be checked by direct computation. The line element (102) then becomes:

$$ds^2 = -\frac{1}{2\lambda}(\eta^{\mu\nu}p^2 - 2p^\mu p^\nu)\left(dx_\mu dx_\nu + \frac{\lambda^2}{p^4}dp_\mu dp_\nu\right). \tag{106}$$

Note the close resemblance between the coordinates, $dx^\mu(K^\nu) = -2x^\mu x^\nu + x^2\eta^{\mu\nu}$ of the vector field K^ν in (5) and the metric coefficients $(-2p^\mu p^\nu + p^2\eta^{\mu\nu})$ in (106) under the interchange $x_\mu \leftrightarrow p_\mu$. The line element (106) of the (curved) manifold $\mathbb{T}$ is the conformal counterpart of the Born's line element (2) in the (flat) complex Minkowski space $\mathbb{C}^{1,3}$, both of them considered as phase spaces of relativistic (conformal) particles. Concerning the extension of BRP to the case of curved space-times, see also [54] for the construction of a reciprocal general relativity theory as a local gauge theory of the quaplectic group of [21, 22].

Remember that one could deduce the existence of a maximal acceleration from the positivity of the Born's line element (3). The existence of a maximal acceleration inside the conformal group does not seem to be apparent from (106), although there are other arguments supporting the existence of a bound $a_{\max}$ for proper accelerations. One of them was given long time ago in [55], where the authors analyzed the physical interpretation of the singularities, $1 + 2ax + a^2x^2 = 0$, of the conformal transformations to a uniformly accelerating frame [last transformation in (4)]. When applying the transformation to an extended object of size ℓ, an upper-limit to the proper acceleration, $a_{\max} \simeq c^2/\ell$, is shown to be necessary in order that the tenets of special relativity not be violated (see [55] for more details).

In a coming paper [27], we shall provide an alternative proof of the existence of a maximal acceleration inside the conformal group. It is related to the Unruh effect (vacuum radiation in uniformly accelerated frames) and turns out to be a consequence of the finiteness of the radiated energy (black body spectrum). Contrary to other approaches to the Unruh effect, a bound for the proper acceleration does not necessarily imply a bound for the temperature.

6. Comments and Outlook

We have revised the use of complex Minkowski eight-dimensional space (more precisely, the domains $\mathbb{D}$ and $\mathbb{T}$) as a base for the construction of conformal-invariant quantum (field) theory, either as a phase space or a configuration space [the last case related to Lagrangians of type (29)]. We have followed a gauge-invariant Lagrangian approach (of nonlinear sigma-model type) and we have used a generalized Dirac method for the quantization of constrained systems, which resembles in some aspects the particular approach to quantizing coadjoint orbits of a group G developed in, for instance, [9].

One could think of these eight-dimensional domains as the replacement of space-time at short distances or high momentum transfers, as it is implicit in the original BRP [15, 16], the standard relativity theory being then the limit $\ell_{\min} \to 0$. Group-theoretical revisions of the BRP, replacing the Poincaré by the Canonical (or Quaplectic) group of reciprocal relativity, have been proposed in [21, 22]. In this article we put a (conformal) BRP-like forward, as a natural symmetry inside the conformal group $SO(4,2)$ and the replacement of space-time by the eight-dimensional conformal domain $\mathbb{D}$ or $\mathbb{T}$ at short distances. Actually, we feel tempted to establish a connection between *holomorphicity* $\leftrightarrow$ *chirality* and BRP $\leftrightarrow$ CPT symmetry inside the conformal group. Indeed, the definition of P_μ and K_μ in (10) is linked to the right- and left-handed projectors $(1+\gamma^5)/2$ and $(1-\gamma^5)/2$, respectively. According to the (conformal) BRP-like symmetry (7), conformal physics is symmetric under the interchange $P_\mu \leftrightarrow K_\mu$, as long as we perform a proper-time reversal $D \to -D$. On the other hand, $P_\mu \leftrightarrow K_\mu$ entails a swapping of chirality $(1+\gamma^5)/2 \leftrightarrow (1-\gamma^5)/2$, a complex conjugation $\psi_\lambda(g) \leftrightarrow \check{\psi}_\lambda(g) = \overline{\psi_\lambda(g)}$ (remember the discussion in Remark 4.2) and a parity inversion $\sigma_\mu \leftrightarrow \check{\sigma}_\mu = \sigma^\mu$. Nevertheless, at this stage, a BRP $\leftrightarrow$ CPT connection inside the conformal group is just conjectural and it is still premature to draw any physical conclusions based on it. It is not either the main objective of this paper.

In this article we have considered a particular class of representations (discrete series) of the conformal group, although other possibilities could also be tackled. For example, we could consider the new (vector and pseudo-vector) combinations

$$\tilde{P}_\mu \equiv \frac{1}{2}(P_\mu + K_\mu), \quad \tilde{K}_\mu \equiv \frac{1}{2}(P_\mu - K_\mu),$$

with new commutation relations:

$$[\tilde{P}_\mu, \tilde{K}_\nu] = \eta_{\mu\nu} D, \quad [\tilde{P}_\mu, \tilde{P}_\nu] = M_{\mu\nu}, \quad [\tilde{K}_\mu, \tilde{K}_\nu] = -M_{\mu\nu}. \tag{107}$$

Unlike in formulas (37) and (40), the fact that now $[D, \tilde{K}_\mu] = -\tilde{P}_\mu$ precludes the imposition of $D^L, M^L_{\mu\nu}$ and $\tilde{K}^L_\mu$ as a compatible set of constraints on wave functions. Instead, we could impose

$$M^L_{\mu\nu}\psi = 0, \quad \tilde{K}^L_\mu \psi = 0$$

together with the Casimir (8) constraint $C^L_2\psi = m^2_{00}\psi$, which leads to

$$((D^L)^2 + (\tilde{P}^L)^2)\psi = m^2_{00}\psi.$$

This equation could be seen as a *generalized* Klein–Gordon equation ($P^2\psi = m^2_0\psi$), with D replacing P_0 as the (proper) time generator and m_{00} replacing the Poincaré-invariant mass m_0, as a "conformally-invariant mass" (see e.g. [56] for the formulation of other conformally-invariant massive field equations of motion in generalized Minkowski space). This means that Cauchy hypersurfaces have dimension four. In other words, the Poincaré time is a dynamical variable, on an equal footing with position, the usual Poincaré Hamiltonian P_0 suffering Heisenberg indeterminacy relations too. Instead of the proper time (dilation) generator D, one could also consider the new combination $\tilde{P}_0 = (P_0 + K_0)/2$ as the new Hamiltonian of our theory (see [57] for this choice).

In a non-commutative geometry setting [58], the non-vanishing commutators (107), or those of the position operators X_μ in (39) giving spin generators [48, 49], can be seen as a sign of the granularity (non-commutativity) of space-time in conformal-invariant theories, along with the existence of a minimal length or, equivalently, a maximal acceleration.

The appearance of a maximal acceleration inside the conformal group will be manifest in analyzing the Unruh effect from a group-theoretical perspective [27]. In a previous paper [28], vacuum radiation in uniformly accelerated frames was related to a spontaneous breakdown of the conformal symmetry. In fact, in conformally-invariant quantum field theory, one can find degenerated pseudo-vacua (which turn out to be coherent states of conformal zero-modes) which are stable (invariant) under Poincaré transformations but are excited under accelerations and lead to a black-body spectrum. The same spontaneous-symmetry-breaking mechanism applies to general $U(N, M)$-invariant quantum field theories, where an interesting connection between "curvature and statistics" has emerged [59]. We hope this is just one of many interesting physical phenomena that remain to be unravelled inside conformal-invariant quantum field theory.

Acknowledgments

Work partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219). M. C. thanks the "Universidad Politécnica de Cartagena" and C.A.R.M. for the award "Intensificación de la Actividad Investigadora". "Este trabajo es resultado de la ayuda concedida por la Fundación Séneca, en el marco del PCTRM 2007-2010, con financiacin del INFO y del FEDER de hasta un 80%". We all thank V. Aldaya for stimulating discussions.

References

[1] R. Coquereaux and A. Jadczyk, Conformal theories, curved phase spaces, relativistic wavelets and the geometry of complex domains, *Rev. Math. Phys.* **2** (1990) 1–44.
[2] G. Jakimowicz and A. Odzijewicz, Quantum complex Minkowski space, *J. Geom. Phys.* **56** (2006) 1576–1599.
[3] R. Penrose, The twistor programme, *Rep. Math. Phys.* **12** (1977) 65–76.
[4] R. Penrose and W. Rindler, Spinor and twistor methods in space-time geometry, spinors and space-time, Cambridge University Press, 1986 (Vol. 2).
[5] J. M. Souriau, *Structure Des Systemes Dynamiques* (Dunod, Paris, 1970).
[6] A. A. Kirillov, *Elements of the Theory of Representations* (Springer, Berlin, 1976).
[7] W. Rühl, Distributions on Minkowski space and their connection with analytic representations of the conformal group, *Comm. Math. Phys.* **27** (1972) 53–86.
[8] W. Rühl, Field representations of the conformal group with continuous mass spectrum, *Comm. Math. Phys.* **30** (1973) 287–302.

[9] A. P. Balachandran, G. Marmo, B-S. Skagerstan and A. Stern, Gauge theories and fiber bundles: Applications to particle dynamics, Lecture Notes in Physics, Vol. 188 (Springer-Verlag, Berlin, 1983).

[10] G. Alexanian, A. P. Balachandran, G. Immirzi and B. Ydri, Fuzzy $\mathbb{C}P^2$, *J. Geom. Phys.* **42** (2002) arXiv: hep-th/0103023v2.

[11] V. Gerdt, R. Horan, A. Khvedelidze, M. Lavelle, D. McMullan and Yu. Palii, On the Hamiltonian reduction of geodesic motion on SU(3) to SU(3)/SU(2), *J. Math. Phys.* **47** (2006) 1129, arXiv: hep-th/0511245v1.

[12] G. Kaiser, Quantized fields in complex spacetime, *Ann. Phys.* **173** (1987) 338–354.

[13] G. Kaiser, Quantum physics, relativity, and complex spacetime, in *Towards a New Synthesis* (North-Holland, Amsterdam, 1990), arXiv: 0910.0352v2 [math-ph].

[14] A. Jadczyk, Born's reciprocity in the conformal domain, in *Spinors, Twistors, Clifford Algebras and Quanfum Deformations*, eds. Z. Oziewicz *et al.* (Kluwer Academic Publ., 1993), pp. 129–140.

[15] M. Born, A suggestion for unifying quantum theory and relativity, *Proc. R. Soc.* **A165** (1938) 291–302.

[16] M. Born, Reciprocity theory of elementary particles, *Rev. Mod. Phys.* **21** (1949) 463–473.

[17] C. Castro and M. Pavsic, Clifford algebra of spacetime and the conformal group, *Int. J. Theor. Phys.* **42** (2003) 1693–1705.

[18] C. Castro, The extended relativity theory in Born–Clifford phase spaces with a lower and upper length scales and Clifford group geometric unification, foundations of physics **35** (2005) 971–1041.

[19] E. R. Caianiello, Is there a maximal acceleration? *Lett. Nuovo Cimento* **32** (1981) 65–70.

[20] C. Castro, On the variable fine structure constant, strings and maximal-acceleration phase space relativity, *Int. J. Mod. Phys.* **A18** (2003) 5445–5473.

[21] S. G. Low, Canonically relativistic quantum mechanics: Representations of the unitary semidirect Heisenberg group $U(1,3) \otimes_s H(1,3)$, *J. Math. Phys.* **38** (1997) 2197–2209.

[22] S. G. Low, Representations of the canonical group (the semidirect product of the unitary and Weyl–Heisenberg groups), acting as a dynamical group on noncommutative extended phase space, *J. Phys.* **A35** (Math.&Gen.) (2002) 5711–5729.

[23] S. A. Fulling, Nonuniqueness of canonical field quantization in Riemannian spacetime, *Phys. Rev.* **D7** (1973) 2850–2862.

[24] P. C. W. Davies, Scalar production in Schwarzschild and Rindler metrics, *J. Phys.* **A8** (1975) 609–616.

[25] W. G. Unruh, Notes on black-hole evaporation, *Phys. Rev.* **D14** (1976) 870.

[26] S. W. Hawking, Black-hole explosions?, *Nature* **248** (1974) 30.

[27] M. Calixto, E. Pérez-Romero and V. Aldaya, to appear in Group-Theoretical Revision of the Unruh Effect and Maximal Acceleration.

[28] V. Aldaya, M. Calixto and J. M. Cerveró, Vacuum radiation and symmetry breaking in conformally invariant quantum field theory, *Comm. Math. Phys.* **200** (1999) 325–354.

[29] G. Kaiser, Space-time-scale analysis of electromagnetic waves, *Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, Victoria* (1992) 209–212.

[30] G. Kaiser, Wavelet electrodynamics II: Atomic composition of electromagnetic waves, *Appl. Comput. Harmon. Anal.* **1** (1994) 246–260.

[31] M. Calixto and E. Pérez-Romero, Extended MacMahon–Schwinger's master theorem and conformal wavelets in complex Minkowski space, in *Applied and Computational Harmonic Analysis* (2010), DOI: 10.1016/j.acha.2010.11.004, arXiv: 1002.3498v2.

[32] E. L. Hill, On accelerated coordinate systems in classical and relativistic mechanics, *Phys. Rev.* **67** (1945) 358–363.

[33] T. Fulton, F. Rohrlich and L. Witten, Physical consequences to a coordinate transformation to a uniformly accelerating frame, *Nuovo Cimento* **26** (1962) 652–671.

[34] L. J. Boya and J. M. Cerveró, Contact transformations and conformal group. I. relativistic theory, *Int. J. Theor. Phys.* **12** (1975) 47–54.

[35] H. Weyl, *Space, Time and Matter* (Dover, NY, 1st edn., 1922).

[36] H. A. Kastrup, Gauge properties of the Minkowski space, *Phys. Rev.* **150** (1966) 1183.

[37] A. O. Barut, J. R. Zenitt and A. Lauferts, The exponential map for the conformal group SO(2, 4), *J. Phys.* **A27** (*Math. Gen.*) (1994) 5239–5250.

[38] A. O. Barut, J. R. Zenitt and A. Lauferts, The exponential map for the unitary group SU(2, 2), *J. Phys.* **A27** (*Math. Gen.*) (1994) 6799–6805.

[39] V. Aldaya and J. A. de Azcárraga, Group manifold analysis of the structure of relativistic quantum dynamics, *Ann. Phys.* **165** (1985) 484–504.

[40] S. Fubini, A. J. Hanson and R. Jackiw, New approach to field theory, *Phys. Rev.* **D7** (1973) 1732–1760.

[41] V. Aldaya, J. Navarro-Salas and A. Ramirez, Algebraic quantization on a group and nonabelian constraints, *Comm. Math. Phys.* **121** (1989) 541.

[42] D. MacMullan and I. Tsutsui, On the emergence of gauge structures and generalized spin when quantizing on a coset space, *Ann. Phys.* **237** (1995) 269–321.

[43] V. Aldaya, M. Calixto and J. Guerrero, Algebraic quantization, good operators and fractional quantum numbers, *Comm. Math. Phys.* **178** (1996) 399.

[44] N. P. Landsman and N. Linden, The geometry of inequivalent quantizations, *Nucl. Phys.* **B365** (1991) 121.

[45] N. Woodhouse, *Geometric Quantization* (Oxford University Press, 1980).

[46] V. Aldaya and J. A. de Azcárraga, Quantization as a consequence of the symmetry group: An approach to geometric quantization, *J. Math. Phys.* **23** (1982) 1297.

[47] V. Aldaya, J. Guerrero and G. Marmo, Higher-order differential operators on a Lie group and quantization, *Int. J. Mod. Phys.* **A12** (1997) 3.

[48] M.-T. Jaekel and S. Reynaud, Space-time localisation with quantum fields, *Phys. Lett.* **A220** (1996) 10–16.

[49] M.-T. Jaekel and S. Reynaud, Conformal symmetry and quantum relativity, *Found. Phys.* **28** (1998) 439–456.

[50] M. Calixto and J. Guerrero, Wavelet transform on the circle and the real line: A group-theoretical treatment, *Appl. Comput. Harmon. Anal.* **21** (2006) 204–229.

[51] A. Perelomov, *Generalized Coherent States and Their Aplications* (Springer-Verlag, 1986).

[52] J. R. Klauder and B.-S. Skagerstam, *Coherent States*: *Applications in Physics and Mathematical Physics* (World Scientific, 1985).

[53] W. Mulak, Quantum $SU(2,2)$-harmonic oscillator, *Rep. Math. Phys.* **34** (1994) 143–149.

[54] C. Castro, Born's reciprocal general relativity theory and complex non-Abelian gravity as gauge theory of the quaplectic group: A novel path to quantum gravity, *Int. J. Mod. Phys.* **A23** (2008) 1487–1506.

[55] W. R. Wood, G. Papini and Y. Q. Cai, Conformal transformations and maximal acceleration, *Il Nuovo Cimento* **104** (1989) 653–663.

[56] A. O. Barut and R. B. Haugen, Theory of the conformally invariant mass, *Ann. Phys.* **71** (1972) 519–541.
[57] G. Mack, All unitary ray representations of the conformal group SU(2, 2) with positive energy, *Comm. Math. Phys.* **55** (1977) 1–28.
[58] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, 2nd edn., London Mathematical Society Lecture Note Series, Vol. 257 (Cambridge University Press, 1999).
[59] M. Calixto and V. Aldaya, Thermal vacuum radiation in spontaneously broken second-quantized theories on curved phase spaces of constant curvature, *Int. J. Geom. Meth. Mod. Phys.* **6** (2009) 513–531.

Capítulo 4

Estados Coherentes de Partículas Aceleradas y Radiación

TB, HW, US, JPhysA/410207, 3/12/2011

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **45** (2012) 015202 (16pp) — UNCORRECTED PROOF

Coherent states of accelerated relativistic quantum particles, vacuum radiation and the spontaneous breakdown of the conformal *SU*(2,2) symmetry

M Calixto[1,2], E Pérez-Romero[1] and V Aldaya[2]

[1] Departamento de Matemática Aplicada, Universidad de Granada, Facultad de Ciencias, Campus de Fuentenueva, 18071 Granada, Spain
[2] Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado Postal 3004, 18080 Granada, Spain

E-mail: calixto@ugr.es

Received 13 October 2011, in final form 20 November 2011
Published DD MMM 2011
Online at stacks.iop.org/JPhysA/45/015202

Abstract
We give a quantum mechanical description of accelerated relativistic particles in the framework of coherent states (CSs) of the (3+1)-dimensional conformal group $SU(2,2)$, with the role of accelerations and 'kinematical redshift' played by special conformal transformations (SCTs) and with the role of (proper) time translations played by dilations. The accelerated ground state $\tilde{\varphi}_0$ of first quantization is a CS of the conformal group. We compute the distribution function giving the occupation number of each energy level in $\tilde{\varphi}_0$ and, with it, the partition function $\mathcal{Z}$, mean energy $\mathcal{E}$ and entropy $\mathcal{S}$, which resemble that of an 'Einstein solid'. An effective temperature $\mathcal{T}$ can be assigned to this 'accelerated ensemble' through the thermodynamic expression $\mathrm{d}\mathcal{E}/\mathrm{d}\mathcal{S}$, which leads to a (nonlinear) relation between acceleration and temperature different from Unruh's (linear) formula. Then we construct the corresponding conformal-$SU(2,2)$-invariant second-quantized theory and its spontaneous breakdown when selecting Poincaré-invariant degenerated θ-vacua (namely, CSs of conformal zero modes). SCTs (accelerations) destabilize the Poincaré vacuum and make it to radiate.

PACS numbers: 02.20.Qs, 03.65.Fd, 03.65.Pm, 11.30.Qc, 67.40.Db
Mathematics Subject Classification: 81R30, 81R05, 42B05, 30H20, 42C15

(Some figures may appear in colour only in the online journal) **Q1**

1. Introduction

The quantum analysis of accelerated frames of reference has been studied mainly in connection with quantum field theory (QFT) in curved spacetime. For example, the case of the quantization of a Klein–Gordon field in Rindler coordinates [1, 2] entails a global mutilation of flat spacetime, with the appearance of event horizons, and leads to a quantization inequivalent to the standard Minkowski quantization. Physically one says that, whereas the Poincaré-invariant (Minkowskian) vacuum $|0\rangle$ in QFT looks the same to any inertial observer (i.e. it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

$$T = \frac{\hbar a}{2\pi c k_B} \tag{1}$$

in passing to a uniformly accelerated frame (a denotes the acceleration, c the speed of light and k_B the Boltzmann constant). This is called the Fulling–Davies–Unruh effect [1, 3, 4], which shares some features with the (black hole) Hawking [5] effect. Its explanation relies heavily upon Bogoliubov transformations, which find a natural explanation in the framework of coherent states (CSs) [6–8] and squeezed states [9].

In this paper, we also approach the quantum analysis of accelerated frames from a CS perspective but the scheme is rather different, although it shares some features with the standard approach commented before. The situation will be similar in some respects to quantum many-body condensed matter systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. We shall enlarge the Poincaré symmetry $\mathcal{P}$ to account for uniform accelerations and then spontaneously break it down back to Poincaré by selecting appropriate 'non-empty vacua'[3] stable under $\mathcal{P}$. Then the action of broken symmetry transformations (accelerations) will destabilize/excite the vacuum and make it to *radiate*. The candidate for an enlargement of $\mathcal{P}$ will be the conformal group in (3+1) dimensions $SO(4, 2)$ incorporating dilations and special conformal transformations (SCTs)

$$x^\mu \to x'^\mu = \frac{x^\mu + \mathrm{a}^\mu x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2 x^2}, \tag{2}$$

which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration (in $c = 1$ units) $a = 2\mathrm{a}$ (see e.g. [10–12] and later on equation (8)). **Q2**
From the conformal symmetry point of view, Poincaré-invariant vacua are regarded as a CS of *conformal zero modes*, which are undetectable ('dark') by inertial observers but unstable under SCTs.

A previous preliminary attempt to analyze quantum accelerated frames from a conformal group perspective was made in [13] (see also [14]), where a quite involved 'second quantization formalism on a group G' was developed and applied to the (finite part of the) conformal group in (1+1) dimensions, $SO(2, 2) \simeq SO(2, 1) \times SO(2, 1)$, which consists of two copies of the pseudo-orthogonal group $SO(2, 1)$ (left- and right-moving modes, respectively). Here we shall use more conventional methods of quantization and work in realistic (3+1) dimensions, using the (more involved) conformal group $SO(4, 2) \simeq SU(2, 2)/\mathbb{Z}_4$. New consequences of this group-theoretical approach are obtained here, regarding a similitude between the accelerated ground state and the 'Einstein solid', the computation of entropies and a deviation from Unruh's formula (1).

We would like to mention that (near-horizon *two*-dimensional) conformal symmetry has also played a fundamental role in the microscopic description of the Hawking effect. In fact, there is strong evidence that conformal field theories provide a universal (independent of the details of the particular quantum gravity model) description of low-energy black hole entropy, which is only fixed by symmetry arguments (see e.g. [15, 16]). Here, the Virasoro algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black

[3] Actually, quantum vacua are not really empty to every observer, as the quantum vacuum is filled with zero-point fluctuations of quantum fields.

hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states. However, in (3+1) dimensions, conformal invariance is necessarily global (finite-(15)-dimensional). In this paper, we shall study zero-order effects that gravity has on quantum theory (uniform accelerations). To account for higher-order effects (like non-constant accelerations) in a group-theoretical framework, we should firstly promote the 3+1 conformal symmetry $SO(4,2)$ to a higher-(infinite)-dimensional symmetry. This is not a trivial task, although some steps have been done by the authors in this direction (see e.g. [14, 17–20]).

This paper is organized as follows. In section 2, we discuss the group-theoretical backdrop (conformal transformations, infinitesimal generators and commutation relations) and justify the interpretation of SCTs as transitions to relativistic uniform accelerated frames of reference and 'kinematical redshift'. In section 3, we construct the Hilbert space and an orthonormal basis for our conformal particle in (3+1) dimensions, based on an holomorphic square-integrable irreducible representation of the conformal group on the eight-dimensional phase space $\mathbb{D}_4 = SO(4,2)/SO(4)\times SO(2)$ inside the complex Minkowski space $\mathbb{C}^4$. In section 4, we define conformal CSs, highlight the Poincaré invariance of the ground state (admissible/fiducial vector), construct the accelerated ground state as a CS of the conformal group and calculate the distribution function, mean energy, partition function and entropy of this accelerated ground state, seen as a statistical ensemble. This leads us to interpret the accelerated ground state as an Einstein solid, to obtain a deviation from Unruh's formula (1) and to discuss on the existence of a maximal acceleration. In section 5, we deal with the second-quantized (many-body) theory, where Poincaré-invariant (degenerated) pseudo-vacua are CSs of conformal zero modes. Selecting one of this Poincaré-invariant pseudo-vacua spontaneously breaks the conformal invariance and leads to vacuum radiation. Section 6 is left for conclusions and outlook.

2. The conformal group and its generators

The conformal group in (3+1) dimensions, $SO(4,2)$, is composed of Poincaré $\mathcal{P} = SO(3,1)\circledS\mathbb{R}^4$ (a semidirect product of spacetime translations $\mathrm{b}^\mu \in \mathbb{R}^4$ times Lorentz $\Lambda^\mu_\nu \in SO(3,1)$) transformations augmented by dilations ($\mathrm{e}^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (SCTs, $\mathrm{a}^\mu \in \mathbb{R}^4$) which, in Minkowski spacetime, have the following realization:

$$\begin{aligned} x'^\mu &= x^\mu + \mathrm{b}^\mu, & x'^\mu &= \Lambda^\mu_\nu(\omega)x^\nu, \\ x'^\mu &= \mathrm{e}^\tau x^\mu, & x'^\mu &= \frac{x^\mu + \mathrm{a}^\mu x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2x^2}, \end{aligned} \tag{3}$$

respectively. The infinitesimal generators (vector fields) of transformations (3) are easily deduced,

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial x^\mu}, & M_{\mu\nu} &= x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}, \\ D &= x^\mu\frac{\partial}{\partial x^\mu}, & K_\mu &= -2x_\mu x^\nu\frac{\partial}{\partial x^\nu} + x^2\frac{\partial}{\partial x^\mu}, \end{aligned} \tag{4}$$

and they close into the conformal Lie algebra:

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}, \\ &[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \quad [P_\mu, P_\nu] = 0, \\ &[K_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho, \quad [K_\mu, K_\nu] = 0, \\ &[D, P_\mu] = -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [D, M_{\mu\nu}] = 0, \\ &[K_\mu, P_\nu] = 2(\eta_{\mu\nu}D + M_{\mu\nu}). \end{aligned} \tag{5}$$

The conformal quadratic Casimir operator

$$C_2 = D^2 - \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + \frac{1}{2}\left(P_\mu K^\mu + K_\mu P^\mu\right) \tag{6}$$

generalizes the Poincaré Casimir $P^2 = P_\mu P^\mu$ which, for scalar fields ϕ, leads to the Klein–Gordon equation $P^2\phi = m_0^2\phi$, with m_0^2 the squared rest mass. The fact that $[D, P^2] = -2P^2$ implies that conformal fields must either be massless or have a continuous mass spectrum (see e.g. the classical references [22, 26]). Actually, just like the Poincaré-invariant mass m_0 comprises a continuum of 'Galilean' masses m, a conformally invariant mass m_{00} can be defined by the Casimir (6), which comprises a continuum of Poincaré masses m_0. The eigenvalue equation $C_2\phi = m_{00}^2\phi$ can be seen as a *generalized* Klein–Gordon equation, where D replaces P_0 as the (proper) time evolution generator (see [21] for more information) and m_{00} replaces m_0 (see [22] for the formulation of other conformally invariant massive field equations of motion in generalized Minkowski space).

In this paper, we shall deal with discrete series representations of the conformal group having a continuous mass spectrum and the corresponding wavefunctions having support on the whole four-dimensional Minkowski spacetime, with the dilation parameter τ playing the role of a proper time. We shall report on this model of conformal quantum particles later on in section 3. The reader can also consult our recent reference [21] for the underlying gauge-invariant Lagrangian approach (of nonlinear sigma-model type) behind our quantum model of conformal particles, which is built upon a generalized Dirac method for the quantization of constrained systems which resembles in some aspects the standard approach to quantizing coadjoint orbits of a group G (see e.g. classical references [23–25]).

2.1. Special conformal transformations as transitions to uniform relativistic accelerated frames and kinematical redshift

The interpretation of SCTs (2) as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by the authors of [10–12]. More precisely, denoting by $u^\mu = \frac{dx^\mu}{d\tau}$ and $a^\mu = \frac{du^\mu}{d\tau}$ the 4-velocity and 4-acceleration of a point particle, respectively, the relativistic motion with constant acceleration is characterized by the usual condition [27] $a_\mu a^\mu = -g^2$, where g is the magnitude of the acceleration in the instantaneous rest system. Then, from $u_\mu u^\mu = 1$ (in $c = 1$ units), we can derive the differential equation to be satisfied for all systems with constant relative acceleration[4]:

$$\frac{da^\mu}{d\tau} = g^2 u^\mu. \tag{7}$$

In 1945, Hill [10] proved that the kinematical invariance group of (7) is precisely the conformal group $SO(4,2)$ (see also [11, 12]). Here we shall provide a simple explanation of this fact. For simplicity, let us take an SCT along the 'z' axis, $\mathrm{a}^\mu = (0, 0, 0, \mathrm{a})$, and the temporal path $x^\mu = (t, 0, 0, 0)$. Then the transformation (2) reads

$$t' = \frac{t}{1 - \mathrm{a}^2t^2}, \qquad z' = \frac{\mathrm{a}t^2}{1 - \mathrm{a}^2t^2}. \tag{8}$$

[4] As a curiosity, this formula turns out to be equivalent to the vanishing of the von Laue 4-vector $F^\mu = \frac{2}{3}e^2(\frac{da^\mu}{d\tau} + a_\nu a^\nu u^\mu)$ of an accelerated point charge; that is, a compensation between the Schott term $\frac{2}{3}e^2\frac{da^\mu}{d\tau}$ and the Abraham–Lorentz–Dirac radiation reaction force $\frac{2}{3}e^2 a_\nu a^\nu u^\mu$ (minus the rate at which energy and momentum are carried away from the charge by radiation).

Writing z' in terms of t' gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

$$z' = \frac{1}{g}(\sqrt{1+g^2t'^2} - 1) \tag{9}$$

with $g = 2\mathrm{a}$. In the same year, Hill [28] also noticed a very interesting relation coming from the time ($\mu = 0$) component of SCTs (2) generated by K_0. Taking now $\mathrm{a}^\mu = (\mathrm{a}^0, 0, 0, 0)$ and denoting $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}x^0}$ and $\vec{v}' = \frac{\mathrm{d}\vec{x}'}{\mathrm{d}x'^0}$ the velocities in both reference frames, equation (2) leads to the velocity formula

$$\vec{v}' = \vec{v} - 2\mathrm{a}^0\vec{x} + O((\mathrm{a}^0)^2), \tag{10}$$

which, to first order of approximation, resembles Hubble's law of redshift when identifying $H_0 = -2\mathrm{a}^0$ (the Hubble constant). Indeed, the added term is a simple radial velocity with magnitude proportional to the distance $\vec{x}$ from the observer. Note that the previous derivation is purely kinematical and does not appeal to the relativistic theory of gravitation. The physical implications of this formula (and we think of SCTs in general) have been overlooked for nearly 65 years. It could lead to an ambiguity in current interpretations of stellar redshifts. Recently, Wulfman [29] has proposed several experiments, based on an analysis of the anomalous frequency shifts uncovered in the Pioneer 10 and 11 spacecraft studies, pursuing to determine the value of the group parameter a^0 and thereby removing the possible ambiguity in Hubble's formula.

To conclude this section, let us also say that at least two alternative meanings of SCTs have also been proposed [30, 31]. One is related to Weyl's idea of different lengths in different points of spacetime [30]: 'the rule for measuring distances changes at different positions'. The other is Kastrup's interpretation of SCTs as geometrical gauge transformations of the Minkowski space [31].

3. A model of conformal quantum particles

In this section we report on a model for quantum particles with conformal symmetry. The reader can find more details in [21], where we formulate a gauge-invariant nonlinear sigma model on the conformal group and quantize it according to a generalized Dirac method for constrained systems.

3.1. The compactified Minkowski space and the isomorphism $SO(4,2) = SU(2,2)/\mathbb{Z}_4$

In [21] it is shown how the Minkowski space arises as the support of constrained wavefunctions on the conformal group. Actually, the compactified Minkowski space $\mathbb{M}_4 = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1$ naturally lives inside the conformal group $SO(4,2)$ as the coset $\mathbb{M}_4 = SO(4,2)/\mathcal{W}$, where $\mathcal{W}$ denotes the Weyl subgroup generated by K_μ, $M_{\mu\nu}$ and D (i.e. a Poincaré subgroup $\mathcal{P}$ augmented by the dilations $\mathbb{R}^+$). The Weyl group $\mathcal{W}$ is the stability subgroup (the little group in physical usage) of $x^\mu = 0$. The conformal group acts transitively on $\mathbb{M}_4$ and is free from singularities. Instead of $SO(4,2)$, we shall work by convenience with its four covering group:

$$SU(2,2) = \left\{ g = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \mathrm{Mat}_{4\times4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma,\ \det(g) = 1 \right\}, \tag{11}$$

where Γ denotes a Hermitian form of signature $(++--)$. The conformal Lie algebra (5) can also be realized in terms of 4×4 gamma matrices in, for instance, the Weyl basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \tag{12}$$

where $\check{\sigma}^{\mu} \equiv \sigma_{\mu}$ (we are using the convention $\eta = \mathrm{diag}(1, -1, -1, -1)$) and σ^{μ} are the standard Pauli matrices. Indeed, the choice

$$\begin{aligned} &D = \frac{\gamma^5}{2}, \qquad M^{\mu\nu} = \frac{[\gamma^{\mu}, \gamma^{\nu}]}{4} = \frac{1}{4}\begin{pmatrix} \sigma^{\mu}\check{\sigma}^{\nu} - \sigma^{\nu}\check{\sigma}^{\mu} & 0 \\ 0 & \check{\sigma}^{\mu}\sigma^{\nu} - \check{\sigma}^{\nu}\sigma^{\mu} \end{pmatrix}, \\ &P^{\mu} = \gamma^{\mu}\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^{\mu} \\ 0 & 0 \end{pmatrix}, \qquad K^{\mu} = \gamma^{\mu}\frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^{\mu} & 0 \end{pmatrix} \end{aligned} \tag{13}$$

fulfills the commutation relations (5). These are the Lie algebra generators of the fundamental representation of $SU(2,2)$. The group $SU(2,2)$ acts transitively on the compactified Minkowski space $\mathbb{M}_4$, which can be identified with the set of Hermitian 2×2 matrices $X = x_{\mu}\sigma^{\mu}$, as follows:

$$X \to X' = (AX + B)(CX + D)^{-1}. \tag{14}$$

With this identification, the transformations (3) can be recovered from (14) as follows.

(i) Standard Lorentz transformations, $x'^{\mu} = \Lambda^{\mu}_{\nu}(\omega)x^{\nu}$, correspond to $B = C = 0$ and $A = D^{-1\dagger} \in SL(2, \mathbb{C})$, where we are making use of the homomorphism (spinor map) between $SO^{+}(3,1)$ and $SL(2, \mathbb{C})$ and writing $X' = AXA^{\dagger}, A \in SL(2, \mathbb{C})$ instead of $x'^{\mu} = \Lambda^{\mu}_{\nu}x^{\nu}$.

(ii) Dilations correspond to $B = C = 0$ and $A = D^{-1} = \mathrm{e}^{\tau/2}I$.

(iii) Spacetime translations are $A = D = I, C = 0$ and $B = \mathrm{b}_{\mu}\sigma^{\mu}$.

(iv) SCTs correspond to $A = D = I$ and $C = \mathrm{a}_{\mu}\sigma^{\mu}, B = 0$ by noting that $\det(CX + I) = 1 + 2\mathrm{a}x + \mathrm{a}^2x^2$:

$$X' = X(CX + I)^{-1} \leftrightarrow x'^{\mu} = \frac{x^{\mu} + \mathrm{a}^{\mu}x^2}{1 + 2\mathrm{a}x + \mathrm{a}^2x^2}.$$

3.2. Unirreps of the conformal group: discrete series

We shall consider the complex extension of the compactified Minkowski space $\mathbb{M}_4 = U(2)$ to the eight-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^{\dagger} > 0\}, \tag{15}$$

of which $\mathbb{M}_4 = \{Z \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : I - ZZ^{\dagger} = 0\}$ is the Shilov boundary. It can be proved (see e.g. [21, 32]) that the action

$$[U_{\lambda}(g)\phi](Z) = |CZ + D|^{-\lambda}\phi(Z'), \qquad Z' = (AZ + B)(CZ + D)^{-1} \tag{16}$$

constitutes a unitary irreducible representation of $SU(2,2)$ on the Hilbert space $\mathcal{H}_{\lambda}(\mathbb{D}_4)$ of square-integrable holomorphic functions ϕ with invariant integration measure

$$\mathrm{d}\mu_{\lambda}(Z, Z^{\dagger}) = \pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3)\det(I - ZZ^{\dagger})^{\lambda-4}|\mathrm{d}Z|,$$

where the label $\lambda \in \mathbb{Z}, \lambda \geqslant 4$ is the conformal, scale or mass dimension ($|\mathrm{d}Z|$ denotes the Lebesgue measure in $\mathbb{C}^4$). The factor $\pi^{-4}(\lambda - 1)(\lambda - 2)^2(\lambda - 3)$ in $\mathrm{d}\mu_{\lambda}(Z, Z^{\dagger})$ is chosen so that the constant function $\phi(Z) = 1$ has unit norm. Besides the conformal dimension λ, the discrete series representations of $SU(2,2)$ have two extra spin labels $s_1, s_2 \in \mathbb{N}/2$ associated with the (stability) subgroup $SU(2) \times SU(2)$. Here we shall restrict ourselves to scalar fields ($s_1 = s_2 = 0$) for the sake of simplicity (see e.g. [21] for the spinning unirreps of $SU(2,2)$). The reduction of this representation into unitary irreducible representations of the Poincaré subgroup indicates that we are dealing with fields with a continuous mass spectrum extending from zero to infinity [33].

3.3. The Hilbert space of our conformal particle

It has been proved in [32] that the infinite set of homogeneous polynomials

$$\varphi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\det(Z)^m\mathcal{D}^j_{q_1,q_2}(Z), \quad (17)$$

with

$$\mathcal{D}^j_{q_1,q_2}(Z) = \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}}\sum_{p=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)}\binom{j+q_2}{p}\binom{j-q_2}{p-q_1-q_2} \times z_{11}^p z_{12}^{j+q_1-p} z_{21}^{j+q_2-p} z_{22}^{p-q_1-q_2} \quad (18)$$

the standard Wigner's $\mathcal{D}$-matrices ($j \in \mathbb{N}/2$), verifies the following closure relation (the reproducing Bergman kernel or λ-extended MacMahon–Schwinger's master formula):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I-Z^\dagger Z')^\lambda} \quad (19)$$

and constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D}_4)$ (the sum on j accounts for all non-negative half-integer numbers). The identity (19) will be useful for us in the following.

3.4. Hamiltonian and energy spectrum

In [21] we have argued that the dilation operator D plays the role of the Hamiltonian of our conformal quantum theory. Actually, the replacement of time translations by dilations as kinematical equations of motion has already been considered in the literature (see e.g. [34, 35]), when quantizing field theories on space-like Lorentz-invariant hypersurfaces $x^2 = x^\mu x_\mu = \tau^2 =$ constant. In other words, if one wishes to proceed from one surface at $x^2 = \tau_1^2$ to another at $x^2 = \tau_2^2$, this is done by scale transformations; that is, $D = \frac{\partial}{\partial\tau}$ is the evolution operator in a proper time τ. We must say that other possibilities exist for choosing a conformal Hamiltonian, namely the combination $\tilde{P}_0 = (P_0 + K_0)/2$, which has been used in [26].

From the general expression (16), we can compute the finite left action of dilations ($B = 0 = C$ and $A = \mathrm{e}^{\tau/2}\sigma^0 = D^{-1} \Rightarrow g = \mathrm{e}^{\tau/2}\mathrm{diag}(1, 1, -1, -1)$) on wavefunctions:

$$[U_\lambda(g)\phi](Z) = \mathrm{e}^{\lambda\tau}\phi(\mathrm{e}^\tau Z). \quad (20)$$

The infinitesimal generator of this transformation is the Hamiltonian operator:

$$H = \lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}} = \lambda + z_\mu\frac{\partial}{\partial z_\mu}, \quad (21)$$

where we have set $Z = z_\mu\sigma^\mu$ in the last equality. This Hamiltonian has the form of that of a four-dimensional (relativistic) harmonic oscillator in the Bargmann representation. The set of functions (17) constitutes a basis of Hamiltonian eigenfunctions (homogeneous polynomials) with energy eigenvalues E^λ_n (the homogeneity degree) given by

$$H\varphi^{j,m}_{q_1,q_2} = E^\lambda_n\varphi^{j,m}_{q_1,q_2}, \qquad E^\lambda_n = \lambda + n, \qquad n = 2j + 2m. \quad (22)$$

Actually, each energy level E^λ_n is $(n + 1)(n + 2)(n + 3)/6$ times degenerated (just like a four-dimensional harmonic oscillator). This degeneracy coincides with the number of linearly independent polynomials $\prod_{i,j=1}^{2} Z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^{2} n_{ij}$. This also proves that the set of polynomials (17) is a basis for analytic functions $\phi \in \mathcal{H}_\lambda(\mathbb{D}_4)$. The spectrum is equispaced and bounded from below, with ground state $\varphi^{0,0}_{0,0} = 1$ and zero-point energy $E^\lambda_0 = \lambda$ (the conformal, scale or mass dimension).

4. Coherent states of accelerated relativistic particles, distribution functions and mean values

In the following two subsections, we shall compute the distribution function and mean values for CSs of accelerated relativistic quantum particles based on the unirrep of the conformal group previously mentioned.

4.1. Conformal CS and the accelerated ground state

Among the infinite set $\{\varphi^{j,m}_{q_1,q_2}(Z)\}$ of homogeneous polynomials (17), we shall choose the ground state $\varphi^{0,0}_{0,0}(Z) = 1$ (of zero degree/energy) as an admissible vector (see [32] for a proof of admissibility). The set of CSs in the orbit of $\varphi^{0,0}_{0,0}$ under the action (16) is

$$\tilde{\varphi}^{0,0}_{0,0}(Z) = \left[U_\lambda(g)\varphi^{0,0}_{0,0}\right](Z) = \det(CZ + D)^{-\lambda}. \tag{23}$$

Note that Poincaré transformations (zero acceleration $C = 0$ and $\det(D) = 1$) leave the ground state invariant, that is, $\varphi^{0,0}_{0,0}$ looks the same to every inertial observer. We shall call $\tilde{\varphi}^{0,0}_{0,0}$ the 'accelerated' ground state. For arbitrary accelerations, $C = \mathrm{a}_\mu\sigma^\mu \neq 0$, we can decompose $\tilde{\varphi}^{0,0}_{0,0}$ using the Bergman kernel expansion (19) as

$$\begin{aligned}\tilde{\varphi}^{0,0}_{0,0}(Z) &= \det(D)^{-\lambda}\det(D^{-1}CZ + I)^{-\lambda}\\ &= \det(D)^{-\lambda}\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\varphi^{j,m}_{q_2,q_1}(-\mathcal{C})\varphi^{j,m}_{q_1,q_2}(Z),\end{aligned} \tag{24}$$

where $\mathcal{C} \equiv D^{-1}C$ is a 'rescaled acceleration matrix'. From (24), we interpret the coefficient $\varphi^{j,m}_{q_2,q_1}(-\mathcal{C})$ as the probability amplitude of finding the accelerated ground state in the excited level $\varphi^{j,m}_{q_1,q_2}$ of energy $E^\lambda_n = \lambda + 2j + 2m = \lambda + n$ (up to a global normalizing factor $\det(D)^{-\lambda}$). In the second-quantized (many-particles) theory, the squared modulus $|\varphi^{j,m}_{q_2,q_1}(-\mathcal{C})|^2$ gives us the occupation number of the corresponding state (see section 5).

4.2. The accelerated ground state as a statistical ensemble: 'the Einstein solid'

For canonical ensembles, the (discrete) energy levels E_n of a quantum system in contact with a thermal bath at temperature T are 'populated' according to the Boltzmann distribution function $f_n(T) \sim \mathrm{e}^{-E_n/k_BT}$. For other external reservoirs or interactions (like, for instance, electric and magnetic fields acting on a charged particle) one could also compute (in principle) the distribution function giving the population of each energy level. Actually, if one were able to unitarily implement the external interaction in the original quantum system, then one could deduce the distribution function for the population of each energy level from first quantum mechanical principles. This is precisely what we have done with uniform accelerations of Poincaré-invariant relativistic quantum particles, where the unitary transformation (24) gives the population of each energy level E^λ_n in the accelerated ground state $\tilde{\varphi}^{0,0}_{0,0}$. Let us consider then the CS (23) itself as a statistical ('accelerated') ensemble. Using (19) we can explicitly compute the partition function as

$$\mathcal{Z}(\mathcal{C}) = \sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\left|\varphi^{j,m}_{q_1,q_2}(\mathcal{C})\right|^2 = \frac{1}{\det(I - \mathcal{C}^\dagger\mathcal{C})^\lambda} = \frac{1}{(1 - \mathrm{tr}(\mathcal{C}^\dagger\mathcal{C}) + \det(\mathcal{C}^\dagger\mathcal{C}))^\lambda}. \tag{25}$$

Using this result, the fact that $\varphi^{j,m}_{q_1,q_2}(\mathcal{C})$ are homogeneous polynomials of degree $2j + 2m$ in $\mathcal{C}$ (recall equation (22), with the Hamiltonian operator given by (21)) and that $\mathrm{tr}(\mathcal{C}^\dagger\mathcal{C})$

and $\det(\mathcal{C}^\dagger\mathcal{C})$ are homogeneous polynomials of degrees 1 and 2 in $\mathcal{C}$, respectively, the (dimensionless) mean energy in the accelerated ground state (23) can be calculated as

$$\mathcal{E}(\mathcal{C}) = \frac{\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j} E_n^\lambda|\varphi_{q_1,q_2}^{j,m}(\mathcal{C})|^2}{\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}|\varphi_{q_1,q_2}^{j,m}(\mathcal{C})|^2} = \lambda\frac{1-\det(\mathcal{C}^\dagger\mathcal{C})}{\det(I-\mathcal{C}^\dagger\mathcal{C})}$$
$$= \lambda + \frac{-\mathrm{tr}(\mathcal{C}^\dagger\mathcal{C})}{\det(I-\mathcal{C}^\dagger\mathcal{C})} = \mathcal{E}_0 + \mathcal{E}_B(\mathcal{C}), \tag{26}$$

where we have detached the zero-point ('dark' energy) contribution $\mathcal{E}_0 = \lambda$ from the rest ('bright' energy) $\mathcal{E}_B(\mathcal{C})$ for convenience. For the particular case of an acceleration α along the 'z' axis, $\mathcal{C} = \alpha\sigma^3$, expressions (25) and (26) acquire the simpler form:

$$\mathcal{Z}(\alpha) = (1-\alpha^2)^{-2\lambda}, \qquad \mathcal{E}(\alpha) = \lambda + 2\lambda\frac{\alpha^2}{1-\alpha^2}. \tag{27}$$

Note that the mean energy $\mathcal{E}(\alpha)$ is of Planckian type for the identification:

$$\alpha^2(T) \equiv \mathrm{e}^{-\frac{\varepsilon}{k_B T}}, \tag{28}$$

where we have introduced ε (the quantum of energy of our four-dimensional harmonic oscillator). At this stage, the identification (28) is an ad hoc assignment but, eventually, we shall justify it from first thermodynamical principles (see the following subsection).

Note also that, for the identification (28), the partition function $\mathcal{Z}(\alpha)$ matches that of an Einstein solid with 2λ degrees of freedom and Einstein temperature $T_E = \varepsilon/k_B$ (see e.g. [36]). We remind the reader that an Einstein solid consists of N independent (non-coupled) three-dimensional harmonic oscillators in a lattice (i.e. $\phi = 3N$ degrees of freedom). Let us pursue this curious analogy a bit further. The total number of ways to distribute n quanta of energy among ϕ one-dimensional harmonic oscillators is given in general by the binomial coefficient $W_\phi(n) = \binom{n+\phi-1}{\phi-1}$. For example, for $\phi = 4$ we recover the degeneracy $W_4(n) = (n+1)(n+2)(n+3)/6$ of each energy level E_n^λ of our four-dimensional 'conformal oscillator' given after (22). Let us see how $W_\phi(n)$, for $\phi = 2\lambda$, arises from the distribution function $|\varphi_{q_1,q_2}^{j,m}(\mathcal{C})|^2$. Indeed, for $\mathcal{C} = \alpha\sigma^3$, $|\varphi_{q_1,q_2}^{j,m}(\mathcal{C})|^2$ can be cast as

$$\left|\varphi_{q_1,q_2}^{j,m}(\alpha)\right|^2 = \frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}(\alpha^2)^{2m}|\mathcal{D}_{q_2,q_1}^{j}(\alpha\sigma^3)|^2$$
$$= \frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}\alpha^{4j+4m}\delta_{q_1,q_2}. \tag{29}$$

Fixing $n = 2j+2m$, the (unnormalized) probability of finding $\tilde\varphi_{0,0}^{0,0}$ in the energy level E_n^λ is

$$f_n^\lambda(\alpha) \equiv \sum_{j=[0,1/2]}^{n/2}\sum_{q=-j}^{j}|\varphi_{q,q}^{j,\frac{n}{2}-j}(\alpha)|^2$$
$$= \sum_{j=[0,1/2]}^{n/2}\frac{(2j+1)^2}{\lambda-1}\binom{\frac{n}{2}-j+\lambda-2}{\lambda-2}\binom{\frac{n}{2}+j+\lambda-1}{\lambda-2}\alpha^{2n}$$
$$= \binom{n+2\lambda-1}{2\lambda-1}\alpha^{2n} = W_{2\lambda}(n)\alpha^{2n}, \tag{30}$$

where $[0, 1/2]$ is 0 for n even and $1/2$ for n odd (in this summation, the j steps are of unity). Here, $W_{2\lambda}(n)$ plays the role of an 'effective' degeneracy and α^2 a Boltzmann-like factor. In fact, the partition function in (27) can be obtained again as

$$\mathcal{Z}(\alpha) = \sum_{n=0}^{\infty} f_n^\lambda(\alpha) = \sum_{n=0}^{\infty} W_{2\lambda}(n)\alpha^{2n} = \left(\sum_{n=0}^{\infty}\alpha^{2n}\right)^{2\lambda} = (1-\alpha^2)^{-2\lambda}, \tag{31}$$

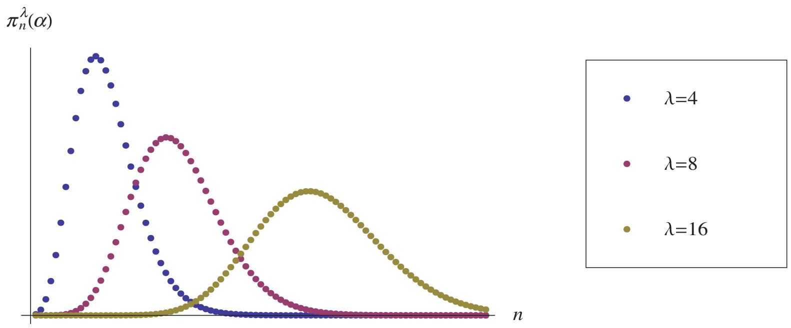

Figure 1. Probability $\pi_n^\lambda(\alpha)$ for fixed $\alpha = 0.8$ and different values of λ.

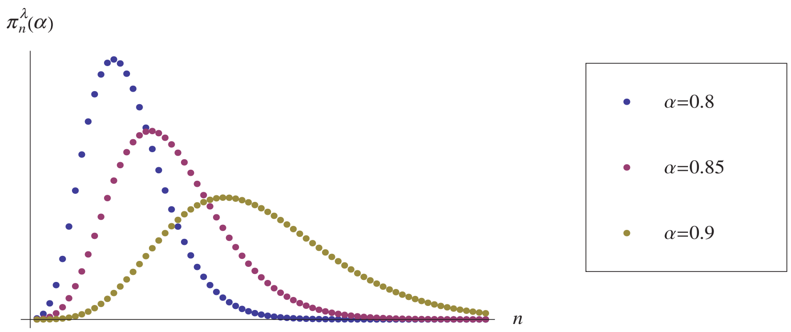

Figure 2. Probability $\pi_n^\lambda(\alpha)$ for fixed $\lambda = 4$ and different values of α.

where we have identified the Maclaurin series expansion of $(1-\alpha^2)^{-2\lambda}$ and the geometric series sum $z(\alpha) \equiv \sum_{n=0}^\infty \alpha^{2n} = 1/(1-\alpha^2)$ with ratio α^2. The fact that $\mathcal{Z}(\alpha) = (z(\alpha))^{2\lambda}$ (the product of 2λ partition functions $z(\alpha)$) reinforces the analogy between our accelerated ground state and the Einstein solid with 2λ degrees of freedom (see the following section for the computation of the entropy).

Note that the distribution function $\pi_n^\lambda(\alpha) \equiv f_n^\lambda(\alpha)/\mathcal{Z}(\alpha)$ has a maximum for a given $n = n_0(\alpha, \lambda)$, with $n_0(\alpha, \lambda)$ increasing in λ (see figure 1) and in α (see figure 2).

Furthermore, inside each energy level E_n^λ, the allowed angular momenta $j = [0, 1/2], \ldots, n/2$ appear with different (unnormalized) probabilities:

$$f_{n,j}^\lambda(\alpha) \equiv \frac{(2j+1)^2}{\lambda-1}\binom{\frac{n}{2}-j+\lambda-2}{\lambda-2}\binom{\frac{n}{2}+j+\lambda-1}{\lambda-2}\alpha^{2n}. \tag{32}$$

Actually, the distribution function $\pi_n^\lambda(j) \equiv f_{n,j}^\lambda(\alpha)/f_n^\lambda(\alpha)$, which is independent of α, has a maximum for a given $j = j_0(n, \lambda)$, with $j_0(n, \lambda)$ an increasing sequence of n and decreasing on λ (see figure 3).

4.3. Entropy, temperature and 'maximal acceleration'

Note that deriving the partition function $\mathcal{Z}(\alpha)$ and mean energy $\mathcal{E}(\alpha)$ from the distribution function (29), (30) does not involve any thermal (but just pure quantum mechanical) input. In

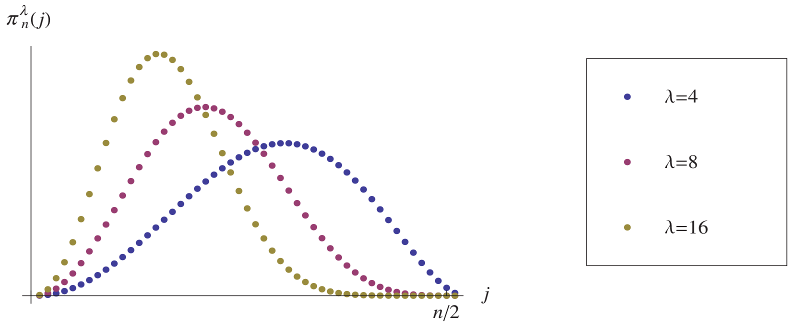

Figure 3. Probability $\pi_n^\lambda(j)$ for different values of λ.

the same way, we can also compute the entropy as a logarithmic measure of the density of states. In fact, denoting by $p_n(\alpha) = \alpha^{2n}/\mathcal{Z}(\alpha)$ the probability of finding our 'Einstein solid' in the energy level n with degeneracy $W_\lambda(n)$, the entropy can be calculated as

$$\begin{aligned}
\mathcal{S}(\alpha) &= -\sum_{n=0}^{\infty} W_{2\lambda}(n) p_n(\alpha) \ln p_n(\alpha) \\
&= -\sum_{n=0}^{\infty} \binom{2\lambda+n-1}{n} (1-\alpha^2)^{2\lambda}\alpha^{2n} \ln((1-\alpha^2)^{2\lambda}\alpha^{2n}) \\
&= -(1-\alpha^2)^{2\lambda}\left(\sum_{n=0}^{\infty}\binom{2\lambda+n-1}{n}\alpha^{2n}\ln((1-\alpha^2)^{2\lambda}) + \sum_{n=0}^{\infty}\binom{2\lambda+n-1}{n}\alpha^{2n}\ln(\alpha^{2n})\right) \\
&= -(1-\alpha^2)^{2\lambda}\left(2\lambda\ln(1-\alpha^2)\sum_{n=0}^{\infty}\binom{2\lambda+n-1}{n}\alpha^{2n} + 2\ln(\alpha)\sum_{n=1}^{\infty}\binom{2\lambda+n-1}{n} n\alpha^{2n}\right) \\
&= -2\lambda\left(\frac{\alpha^2\ln(\alpha^2)}{1-\alpha^2} + \ln(1-\alpha^2)\right),
\end{aligned} \tag{33}$$

where we have identified the partition function $\mathcal{Z}(\alpha)$ and its derivative $\alpha^2 \frac{\mathrm{d}}{\mathrm{d}(\alpha^2)}\mathcal{Z}(\alpha)$ in the last two summations. Again, there is not any thermal input up to now. If we wanted to assign an 'effective' temperature $\mathcal{T}$ to our 'accelerated ensemble', we could use the universal thermodynamic expression (derivative of the energy with respect to the entropy):

$$\mathcal{T} = \frac{\mathrm{d}\mathcal{E}(\alpha)}{\mathrm{d}\mathcal{S}(\alpha)} = -\frac{1}{\ln(\alpha^2)}, \tag{34}$$

given in units of the Einstein temperature $T_E = \varepsilon/k_B$ (i.e. $\mathcal{T} = T/T_E$)[5]. Equality (34) can be inverted to formula (28), giving the announced derivation of the assignment (28) from first thermodynamic principles. One could still check *consistency* (if desired) with other classical formulas relating mean energy and entropy to the partition function, namely,

$$\mathcal{E}(\alpha) = -\frac{\mathrm{d}\ln\mathcal{Z}(\alpha)}{\mathrm{d}\beta}, \qquad \mathcal{S}(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\mathcal{T}}(\mathcal{T}\ln\mathcal{Z}(\alpha)), \qquad \beta \equiv 1/\mathcal{T}. \tag{35}$$

[5] The semisimple character of the group $SU(2,2)$ allows us to express all kinematic magnitudes by pure numbers. From a 'Galilean' viewpoint, we could say that in conformal kinematics there is a characteristic length, a characteristic time and a characteristic speed which may be used as natural units, and then lengths, times and speeds are dimensionless (see [37, 38] for a thorough study on kinematic groups and dimensional analysis).

A hurried analysis of the relation $\alpha^2 = \mathrm{e}^{-1/\mathcal{T}}$ would lead us to think of the existence of a 'maximal acceleration' $\alpha^2 = 1$ (in dimensionless units). Actually, in the process toward the calculation of thermodynamical quantities, we have made use of a rescaling of the original acceleration $C = \mathrm{a}_\mu \sigma^\mu$, in the expression (24), to $\mathcal{C} = D^{-1}C = \alpha_\mu \sigma^\mu$. We can find the relation between a_μ and α_μ as follows. Taking into account that φ^{00}_{00} is normalized and the representation (16) is unitary (see appendix C and proposition 5.2 of [32]), we know that the accelerated ground state (24) is also normalized. This means that the normalizing global factor $\det(D)^{-\lambda}$ in (24) is related to the partition function $\mathcal{Z}(\mathcal{C})$ in (25) by

$$\det(DD^\dagger)^{-\lambda} = 1/\mathcal{Z}(\mathcal{C}) = \det(I - \mathcal{C}^\dagger\mathcal{C})^\lambda \Rightarrow \det(DD^\dagger) = \frac{1}{\det(I - \mathcal{C}^\dagger\mathcal{C})}. \tag{36}$$

Therefore, for $C = \mathrm{a}\sigma^3$ and $\mathcal{C} = \alpha\sigma^3$, the relation $\mathcal{C}^\dagger\mathcal{C} = C^\dagger(DD^\dagger)^{-1}C$ reads

$$\alpha^2 = \frac{\mathrm{a}^2}{1+\mathrm{a}^2} \Rightarrow \mathrm{a}^2 = \frac{\alpha^2}{1-\alpha^2}. \tag{37}$$

With this identification, the mean energy

$$\mathcal{E} = \lambda + 2\lambda\frac{\alpha^2}{1-\alpha^2} = \lambda + 2\lambda \mathrm{a}^2 \tag{38}$$

turns out to be a *quadratic* function of the acceleration 'a'. The dependence of 'a' with the effective temperature $\mathcal{T}$ is then

$$\mathrm{a} = \sqrt{\frac{\mathrm{e}^{-1/\mathcal{T}}}{1-\mathrm{e}^{-1/\mathcal{T}}}} = \sqrt{\mathcal{T}} + O\left(\frac{1}{\sqrt{\mathcal{T}}}\right) \quad \text{for} \quad \mathcal{T} \gg 1. \tag{39}$$

This behavior departs from Unruh's formula (1) even in the limit of high temperatures. However, at high temperatures, it is in accordance with the equipartition theorem for an Einstein solid with 2λ degrees of freedom since the energy (38) becomes $E(T) \equiv \varepsilon\mathcal{E} = E_0 + 2\lambda k_B T + O(1/\sqrt{T})$, with $E_0 = \varepsilon\lambda$ and $T = T_E\mathcal{T}$.

We have seen that the fact that α is bounded is just due to a rescaling of 'a', so that there is not a maximal acceleration in our model as such. Nevertheless, we would like to comment on other arguments in the literature supporting the existence of a bound $a_{\max}$ for proper accelerations. One was given some time ago in [39] in connection with conformal kinematics; there the authors analyzed the physical interpretation of the singularities, $1+2\mathrm{a}x+\mathrm{a}^2x^2 = 0$, of the SCT (2). When applying the transformation to an extended object of size ℓ, an upper limit to the proper acceleration, $a_{\max} \simeq 1/\ell$ (in $c = 1$ units), is shown to be necessary in order to the tenets of special relativity not to be violated (see [39] for more details). Before, Caianiello [40] derived the existence and physical consequences of a maximal acceleration connected with Born's reciprocity principle (BRP) [41, 42]. Indeed, one can deduce the existence of a maximal acceleration from the positivity of Born's line element

$$\mathrm{d}\tilde{\tau}^2 = \mathrm{d}x_\mu\,\mathrm{d}x^\mu + \frac{\ell^4_{\min}}{\hbar^2}\,\mathrm{d}p_\mu\,\mathrm{d}p^\mu = \mathrm{d}\tau\sqrt{1 - \frac{|a^2|}{a^2_{\max}}}, \tag{40}$$

where $\mathrm{d}\tau^2 \equiv \mathrm{d}x_\mu \mathrm{d}x^\mu$ and $\mathrm{d}p_\mu/\mathrm{d}\tau \equiv m\mathrm{d}^2x_\mu/\mathrm{d}\tau^2 = ma_\mu$, as usual. An adaptation of the BRP to the conformal relativity has been put forward by some of us in [21], where a conformal analogue of the line element (40) in the phase space $\mathbb{D}_4$ has been considered. However, the existence of a maximal acceleration inside the conformal group does not seem to be apparent neither from this conformal adaptation of the BRP.

In the past few years, many papers have been published (see e.g. [43] and references therein), each one introducing the maximal acceleration starting from different motivations and from different theoretical schemes. Among the large list of physical applications of

Caianiello's model we would like to point out the one in cosmology which avoids an initial singularity while preserving inflation. Also, a maximal-acceleration relativity principle leads to a variable fine structure 'constant' [43], according to which it could have been extremely small (zero) in the early Universe and then all matter in the Universe could have emerged via the Unruh effect. Moreover, in a non-commutative geometry setting [44], the non-vanishing commutators among the four components of $\tilde{P}_\mu = (P_\mu + K_\mu)/2$ can be seen as a sign of the granularity (non-commutativity) of spacetime in conformal-invariant theories, along with the existence of a minimal length $\ell_{\min}$ or, equivalently, a maximal acceleration $a_{\max} = 1/\ell_{\min}$ (in $c = 1$ units).

5. Second-quantized theory, conformal zero modes and Poincaré θ-vacua

We have discussed the effect of relativistic accelerations in first quantization. However, the proper setting to analyze radiation effects is in the second-quantized theory. Let us denote (for space-saving notation) by $n = \{j, m, q_1, q_2\}$ the multi-index of the one-particle basis wavefunctions φ_n in (17) and by $\hat{a}_n$ (resp. $\hat{a}_n^\dagger$) operators annihilating (resp. creating) a particle in the state $|n\rangle$. An orthonormal basis for the Hilbert space of the second-quantized theory is constructed by taking the orbit through the *conformal vacuum* $|0\rangle$ of the creation operators $\hat{a}_n^\dagger$:

$$|q(n_1), \ldots, q(n_p)\rangle \equiv \frac{(\hat{a}_{n_1}^\dagger)^{q(n_1)} \ldots (\hat{a}_{n_p}^\dagger)^{q(n_p)}}{(q(n_1)! \ldots q(n_p)!)^{1/2}} |0\rangle, \tag{41}$$

where $q(n) \in \mathbb{N}$ denotes the occupation number of the state n with energy $2j + 2m$.

The fact that the ground state of the first quantization, φ_0, is invariant under Poincaré transformations (remember the discussion after (23)) implies that the annihilation operator $\hat{a}_0$ of zero-('dark')-energy modes commutes with all Poincaré generators. It also commutes with all annihilation operators and creation operators of particles with positive ('bright') energy,

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0, \qquad n \neq 0. \tag{42}$$

Therefore, by Schur's lemma, $\hat{a}_0$ must behave as a multiple of the identity when conformal symmetry is broken/restricted to Poincaré symmetry. This means that we can choose Poincaré-invariant vacua $|\theta\rangle$ as being eigenstates of $\hat{a}_0$, namely,

$$\hat{a}_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = \mathrm{e}^{\theta \hat{a}_0^\dagger - \bar{\theta}\hat{a}_0}|0\rangle, \tag{43}$$

which implies that Poincaré 'θ-vacua' $|\theta\rangle$ are (canonical) *CSs of conformal zero modes*. Unlike the conformal vacuum $|0\rangle$, which is invariant under the whole conformal group, Poincaré θ-vacua $|\theta\rangle$ are not stable under SCTs (accelerations). In fact, the second-quantized version of (24), for an acceleration $\mathcal{C} = \alpha\sigma^3$ along the third axis, is given by the transformation of annihilation (resp. creation) operators:

$$\tilde{\hat{a}}_0 = \sum_{n=0}^{\infty} \varphi_n(\alpha)\hat{a}_n. \tag{44}$$

We shall assume that $\sum_n |\varphi_n(\alpha)|^2 = 1$ (normalized probabilities) so that this transformation preserves the original commutation relations $[\tilde{\hat{a}}_0, \tilde{\hat{a}}_0^\dagger] = 1$. Therefore, accelerated Poincaré θ-vacua are

$$|\tilde{\theta}\rangle = \mathrm{e}^{\theta\tilde{\hat{a}}_0^\dagger - \bar{\theta}\tilde{\hat{a}}_0}|0\rangle = \mathrm{e}^{\theta\sum_{n=1}^{\infty}\overline{\varphi_n(\alpha)}\tilde{\hat{a}}_n^\dagger}|\theta\rangle. \tag{45}$$

We can think of conformal zero modes as 'virtual particles' without 'bright' energy and undetectable by inertial observers. However, from an accelerated frame, they become 'visible' to a Poincaré observer. The average number of particles with energy E_n in the accelerated vacuum (45) is then given by

$$N_n(\alpha) = \langle\tilde{\theta}|\hat{a}_n^\dagger \hat{a}_n|\tilde{\theta}\rangle = |\theta|^2 |\varphi_n(\alpha)|^2, \tag{46}$$

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$ and $|\varphi_n(\alpha)|^2$ is the occupation number of the energy level E_n of the accelerated vacuum $|\tilde{\theta}\rangle$. The situation resembles that in many condensed-matter systems (like Bose–Einstein condensates, superconductors, etc), where one also finds non-empty, coherent ground states. In the same way, the probability $P_n(q, \alpha)$ of observing q particles with energy E_n in $|\tilde{\theta}\rangle$ can be calculated as

$$P_n(q, \alpha) = |\langle q(n)|\tilde{\theta}\rangle|^2 = \frac{\mathrm{e}^{-|\theta|^2}}{q!} |\theta|^{2q} |\varphi_n(\alpha)|^{2q} = \frac{\mathrm{e}^{-|\theta|^2}}{q!} N_n^q(\alpha). \tag{47}$$

Therefore, the relative probability of observing a state with total energy E in the excited vacuum $|\tilde{\theta}\rangle$ is

$$P(E) = \sum_{\substack{q_0,\ldots,q_k: \\ \sum_{n=0}^k E_n q_n = E}} \prod_{n=0}^{k} P_n(q_n, \alpha). \tag{48}$$

For the case studied in this paper, this distribution function can be factorized as $P(E) = \Omega(E)\mathrm{e}^{-E/\mathcal{T}}$, where $\Omega(E)$ is a relative weight proportional to the number of states with energy E and the factor $\mathrm{e}^{-E/\mathcal{T}}$ fits this weight properly to a temperature $\mathcal{T}$.

One can also compute the total mean energy

$$E(\alpha) = \langle\tilde{\theta}| \sum_{n=1}^{\infty} E_n \hat{a}_n^\dagger \hat{a}_n |\tilde{\theta}\rangle = |\theta|^2 \sum_{n=1} |\varphi_n(\alpha)|^2 E_n = |\theta|^2 \mathcal{E}(\alpha), \tag{49}$$

which, as expected, is the product of $\mathcal{E}(\alpha)$ in (27) times the average number of particles $|\theta|^2$ in $|\theta\rangle$. The free parameter $|\theta|^2$ is also linked to a vacuum ('dark') energy $E_0 = |\theta|^2 \mathcal{E}_0 = |\theta|^2 \lambda$ just like, for example, the 'cosmological constant'. Like other non-zero vacuum expectation values, zero-point energy leads to observable consequences such as, for instance, the Casimir effect and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the Universe (see e.g. [45] for a nice review). Actually, dark energy is the most popular way to explain recent observations that the Universe appears to be expanding at an accelerating rate.

6. Comments and outlook

As already commented in the introduction, conformal field theories also seem to provide a universal description of low-energy black hole thermodynamics, which is only fixed by symmetry arguments (see [15, 16] and references therein). Actually, Unruh's temperature (1) coincides with Hawking's temperature

$$T = \frac{\hbar c^3}{8\pi M k_B G} = \frac{2\pi G M \hbar}{\Sigma c k_B} \tag{50}$$

($\Sigma = 4\pi r_g^2 = 8\pi G^2 M^2/c^4$ stands for the surface of the event horizon) when the acceleration is that of a free-falling observer on the surface Σ, i.e. $a = c^4/(4GM) = GM/r_g^2$. Here, the Virasoro algebra proves to be a physically important subalgebra of the gauge algebra of surface deformations that leave the horizon fixed for an arbitrary black hole. Thus, the fields on the surface must transform according to irreducible representations of the Virasoro algebra, which is the general symmetry principle that governs the density of microscopic states. Therefore, in the Hawking effect, the calculation of thermodynamical quantities, linked to the statistical mechanical problem of counting microscopic states, is reduced to the study of the representation theory of the conformal group.

Although our approach to the quantum analysis of accelerated frames shares with the previous description of black hole thermodynamics the existence of an underlying conformal invariance, we should not confuse both schemes. Conformal invariance in the Hawking effect manifests itself as an infinite-dimensional gauge algebra of (two-dimensional) surface deformations. However, the infinite-dimensional character of conformal symmetry seems to be an exclusive patrimony of two-dimensional physics, and conformal invariance in (3+1) dimensions is finite-(15)-dimensional, thus accounting for transitions to uniformly accelerated frames only. To account for higher-order effects of gravity on QFT from a group-theoretical point of view, one should consider more general diffeomorphism (Lie) algebras. Higher-dimensional analogies of the infinite two-dimensional conformal symmetry have been proposed by us in [17, 18, 14, 19, 20]. We think that these infinite $\mathcal{W}$-like symmetries can play some fundamental role in quantum gravity models, as a gauge guiding principle.

To conclude, we would also like to mention that the same spontaneous $SU(2,2)$-symmetry breaking mechanism explained in this paper applies to general $SU(N,M)$-invariant quantum theories, where an interesting connection between 'curvature and statistics' has emerged [46, 47]. We hope that many more interesting physical phenomena remain to be unraveled inside conformal-invariant quantum (field) theory. As stated long time ago by Hill [28], a 'more complete analysis of the physical interpretation of the full conformal group of transformations will be required before all of its implications can be appreciated'.

Acknowledgments

This work is partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219, FQM1951).

References Q3

[1] Davies P C W 1975 *J. Phys. A: Math. Gen.* **8** 609–16
[2] Birrell N D and Davies P C W 1994 *Quantum Fields in Curved Space* (*Cambridge Monographs on Mathematical Physics*) Q4
[3] Fulling S A 1973 *Phys. Rev.* D **7** 2850–62
[4] Unruh W G 1976 *Phys. Rev.* D **14** 870
[5] Hawking S W 1974 *Nature* **248** 30
[6] Perelomov A 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
[7] Klauder J R and Skagerstam B-S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[8] Ali S T, Antoine J-P and Gazeau J P 2000 *Coherent States, Wavelets and their Generalizations* (Berlin: Springer)
[9] Bishop R F and Vourdas A 1986 *J. Phys. A: Math. Gen.* **19** 2525–36
[10] Hill E L 1945 *Phys. Rev.* **67** 358–63
[11] Boya L J and Cerveró J M 1975 *Int. J. Theor. Phys.* **12** 47–54
[12] Fulton T, Rohrlich F and Witten L 1962 *Nuovo Cimento* **26** 652–71
[13] Aldaya V, Calixto M and Cerveró J M 1999 *Commun. Math. Phys.* **200** 325–54
[14] Calixto M 2001 *Class. Quantum Grav.* **18** 3857–84
[15] Carlip S 1999 *Class. Quantum Grav.* **16** 3327–48
[16] Agullo I, Navarro-Salas J, Olmo G J and Parker L 2010 *Phys. Rev. Lett.* **105** 211305
[17] Calixto M 2000 *J. Phys. A: Math. Gen.* **33** L69–75
[18] Calixto M 2000 *Mod. Phys. Lett.* A **15** 939–44
[19] Calixto M 2006 *J. Geom. Phys.* **56** 143–74
[20] Aldaya V and Jaramillo J L 2003 *Int. J. Mod. Phys.* A **18** 5795

[21] Calixto M and Pérez-Romero E 2011 *Int. J. Geom. Methods Mod. Phys.* **8** 587–619 (arXiv:1006.5958)
[22] Barut A O and Haugen R B 1972 *Ann. Phys.* **71** 519–41
[23] Kostant B 1970 *Quantization and Unitary Representations (Lecture Notes in Mathematics* vol 170*)* (Berlin: Springer)
[24] Kirillov A A 1976 *Elements of the Theory of Representations* (Berlin: Springer)
[25] Balachandran A P, Marmo G, Skagerstan B-S and Stern A 1983 *Gauge Theories and Fiber Bundles (Lecture Notes in Physics* vol 188*)* (Berlin: Springer)
[26] Mack G 1977 *Commun. Math. Phys.* **55** 1–28
[27] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: Freeman)
[28] Hill E L 1945 *Phys. Rev.* **68** 232
[29] Wulfman C E 2010 *Dynamical Symmetry* (Singapore: World Scientific)
[30] Weyl H 1922 *Space, Time and Matter* 1st edn (New York: Dover)
[31] Kastrup H A 1966 *Phys. Rev.* **150** 1183
[32] Calixto M and Pérez-Romero E 2006 *Appl. Comput. Harmon. Anal.* **21** 204–29 (arXiv:1002.3498)
[33] Rühl W 1973 *Commun. Math. Phys.* **30** 287–302
[34] Aldaya V and de Azcárraga J A 1985 *Ann. Phys.* **165** 484–504
[35] Fubini S, Hanson A J and Jackiw R 1973 *Phys. Rev.* D **7** 1732–60
[36] Morse P M 1969 *Thermal Physics* (New York: Benjamin)
[37] Cariñena J, del Olmo M A and Santander M 1981 *J. Phys. A: Math. Gen.* **14** 1–14
[38] Cariñena J, del Olmo M A and Santander M 1985 *J. Phys. A: Math. Gen.* **18** 1855–72
[39] Wood W R, Papini G and Cai Y Q 1989 *Nuovo Cimento* **104** 653–63
[40] Caianiello E R 1981 *Lett. Nuovo Cimento* **32** 65–70
[41] Born M 1938 *Proc. R. Soc.* A **165** 291–302
[42] Born M 1949 *Rev. Mod. Phys.* **21** 463–73
[43] Castro C 2003 *Int. J. Mod. Phys.* A **18** 5445–73
[44] Madore J 1999 *An Introduction to Noncommutative Differential Geometry and its Physical Applications* 2nd edn *(London Mathematical Society Lecture Note Series* vol 257*)* (Cambridge: Cambridge University Press)
[45] Volovik G E 2006 *Int. J. Mod. Phys.* D **15** 1987–2010
[46] Calixto M and Aldaya V 2006 *J. Phys. A: Math. Gen.* **39** L539–45
[47] Calixto M and Aldaya V 2009 *Int. J. Geom. Methods Mod. Phys.* **6** 513–31

Capítulo 5

Revisión Grupo-Teórica del Efecto Unruh

Group-Theoretical Revision of the Unruh Effect

M. Calixto[1,2], E. Pérez-Romero[2] and V. Aldaya[2]

[1] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Paseo Alfonso XIII 56, 30203 Cartagena, Spain

[2] Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado Postal 3004, 18080 Granada, Spain

E-mail: Manuel.Calixto@upct.es

Abstract. We revise the Unruh effect (vacuum radiation in uniformly relativistic accelerated frames) in a group-theoretical setting by constructing a conformal SO(4,2)-invariant quantum field theory and its spontaneous breakdown when selecting Poincaré invariant degenerated vacua (namely, coherent states of conformal zero modes). Special conformal transformations (accelerations) destabilize the Poincaré vacuum and make it to radiate.

1. Introduction

The Fulling-Davies-Unruh effect [1, 2, 3] has to do with *vacuum radiation* in a non-inertial reference frame and shares some features with the (black-hole) Hawking [4] effect. In simple words, whereas the Poincaré invariant vacuum $|0\rangle$ in QFT looks the same to any inertial observer (i.e., it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

$$T = \frac{\hbar a}{2\pi v k_B} \tag{1}$$

in passing to a uniformly accelerated frame (a denotes the acceleration, v the speed of light and k_B the Boltzmann constant).

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved (viz, existence of horizons). This is the case of the natural quantization in Rindler coordinates [2, 5], which leads to a quantization inequivalent to the normal Minkowski quantization (see next Section), or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [1].

In the reference [6], it was showed that the reason for the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations (that is, the Unruh effect) is more profound and related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory. From this point of view, a Poincaré invariant vacuum will be regarded as a coherent state of *conformal zero modes*, which are undetectable ("dark") by inertial observers but unstable under special conformal transformations

$$x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2 x^2}, \tag{2}$$

which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration $a = 2c$ (see e.g. Ref. [7, 8, 9] and later on Eq. (14)). In the reference [6] a quite involved "second quantization formalism on a group G" was developed, which was applied to the conformal group in (1+1) dimensions, $SO(2,2) \simeq SO(2,1) \times SO(2,1)$, which consists of two copies of the pseudo-orthogonal group $SO(2,1)$ (left- and right-moving modes, respectively). Here we shall use more conventional methods of quantization and we shall work in realistic (3+1) dimensions, using the (more involved) conformal group $SO(4,2) \simeq SU(2,2)/\mathbb{Z}_4$.

The point of view exposed in this paper is consistent with the idea that quantum vacua are not really empty to every observer. Actually, the quantum vacuum is filled with zero-point fluctuations of quantum fields. The situation is similar to quantum many-body condensed mater systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. Moreover, we know that zero-point energy, like other non-zero vacuum expectation values, leads to observable consequences as, for instance, the Casimir effect, and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the universe (see e.g. [10] for a nice review). Indeed, dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelerating rate.

The organization of the paper is as follows. In Section 2 we briefly review the standard explanation for the Unruh effect, which has to do with space-time mutilation and Bogolyubov transformations. In Section 3 we construct a conformal-invariant quantum theory in 3+1 dimensions and in Section 4 we discuss its (spontaneous) breakdown to a Poincaré-invariant quantum theory by selecting Poincaré-invariant pseudo-vacua which are coherent states of conformal zero modes. We compute the mean energy of the accelerated Poincaré-invariant pseudo-vacua. This is part of a work in preparation [11], where the reader will find more details and additional results.

2. Vacuum radiation as a consequence of space-time mutilation

The existence of event horizons in passing to accelerated frames of reference leads to unitarily inequivalent representations of the quantum field canonical commutation relations and to a (ill-)definition of particles depending on the state of motion of the observer.

2.1. Field decompositions and vacua

To use an explicit example, let us consider a real scalar massless field $\phi(x)$, satisfying the Klein-Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) = 0 \,. \tag{3}$$

Let us denote by a_k, a_k^* the Fourier coefficients of the decomposition of ϕ into positive and negative frequency modes:

$$\phi(x) = \int dk(a_k f_k(x) + a_k^* f_k^*(x)). \tag{4}$$

The Fourier coefficients a_k, a_k^* are promoted to annihilation and creation operators $\hat{a}_k, \hat{a}_k^*$ of particles in the quantum field theory. The Minkowski vacuum $|0\rangle_M$ is defined as the state nullified by all annihilation operators

$$\hat{a}_k|0\rangle_M = 0, \ \forall k. \tag{5}$$

2.2. Rindler coordinate transformations

Let us consider now the Rindler coordinate transformation (see e.g. [5]):

$$t = a^{-1}e^{az'}\sinh(at'), \ \ z = a^{-1}e^{az'}\cosh(at'). \tag{6}$$

Journal of Physics: Conference Series **284** (2011) 012014 doi:10.1088/1742-6596/284/1/012014

The worldline $z' = 0$ has constant acceleration a (in natural unities). This transformation entails a mutilation of Minkowski spacetime into patches or charts with event horizons.

The new coordinate system provides a new decomposition of ϕ into Rindler positive and negative frequency modes:

$$\phi(x') = \int dq(a'_q f'_q(x') + a'^*_q f'^*_q(x')). \tag{7}$$

The Rindler vacuum $|0\rangle_R$ is defined as the state nullified by all Rindler annihilation operators:

$$\hat{a}'_q|0\rangle_R = 0\,\forall q. \tag{8}$$

Let us see that the Minkowki vacuum $|0\rangle_M$ and the Rindler vacuum $|0\rangle_R$ are not identical. In fact, the Minkowski vacuum $|0\rangle_M$ has a nontrivial content of Rindler particles.

2.3. Bogolyubov transformations

The Fourier components a'_q, a'^*_q of the field ϕ in the new (accelerated) reference frame are expressed in terms of both a_k, a^*_k through a Bogolyubov transformation:

$$\begin{array}{c} a'_q = \int dk\,(\alpha_{qk} a_k + \beta_{qk} a^*_k)\,, \\ \alpha_{qk} = \langle f'_q | f_k \rangle, \quad \beta_{qk} = \langle f'_q | f^*_k \rangle. \end{array} \tag{9}$$

The vacuum states $|0\rangle_M$ and $|0\rangle_R$, defined by the conditions (5) and (8), are not identical if the coefficients β_{qk} in (9) are not zero. In this case the Minkowski vacuum has a non-zero average number of Rindler particles given by:

$$N_R = \langle 0|\hat{N}_R|0\rangle_M = \langle 0| \int dq \hat{a}'^\dagger_q \hat{a}'_q |0\rangle_M = \int dk dq |\beta_{qk}|^2 \tag{10}$$

That is, in the second quantized theory, the vacuum states $|0\rangle_M$ and $|0\rangle_R$ are not identical if the coefficients β_{qk} are not zero. Both quantizations are inequivalent.

3. Vacuum radiation as a spontaneous breakdown of de conformal symmetry

In this section we shall offer an alternative explanation for the Unruh effect based on symmetry grounds. Actually, in Quantum Field Theory, the vacuum state is expected to be stable under some underlying group of symmetry transformations G (namely, the Poincaré group). Then the action of some spontaneously broken symmetry transformations can destabilize/excitate the vacuum and make it to *radiate*. We shall see that this is precisely the case of the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations. Here, the Poincaré invariant vacuum looks the same to every inertial observer but converts itself into a thermal bath of radiation with temperature (1) in passing to a uniformly accelerated frame. In fact, in the reference [6], it was shown that the reason for this radiation is related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory.

3.1. The conformal group and the compactified Minkowski space

The conformal group in 3+1 dimensions, $SO(4,2)$, is composed by Poincaré (spacetime translations $b^\mu \in \mathbb{R}^4$ and Lorentz $\Lambda^\mu_\nu(\in SO(3,1))$ transformations augmented by dilations ($e^\tau \in \mathbb{R}_+$) and relativistic uniform accelerations (special conformal transformations, $c^\mu \in \mathbb{R}^4$) which, in Minkowski spacetime, have the following realization:

$$x'^\mu = x^\mu + b^\mu, \;\; x'^\mu = \Lambda^\mu_\nu(\omega) x^\nu, \tag{11}$$

$$x'^\mu = e^\tau x^\mu, \;\; x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2x^2}, \tag{12}$$

respectively. The interpretation of special conformal transformations

$$x'^{\mu} = \frac{x^{\mu} + c^{\mu}x^2}{1 + 2cx + c^2x^2} \tag{13}$$

as transitions from inertial reference frames to systems of relativistic, uniformly accelerated observers was identified many years ago by [7, 8, 9], although alternative meanings have also been proposed [12, 13]. For simplicity, let use take an acceleration along the "z" axis: $c^{\mu} = (0, 0, 0, \alpha)$, and the temporal path $x^{\mu} = (t, 0, 0, 0)$. Then the transformation (13) reads:

$$t' = \frac{t}{1 - \alpha^2 t^2}\ , \qquad z' = \frac{\alpha t^2}{1 - \alpha^2 t^2}. \tag{14}$$

Writing z' in terms of t' gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

$$z' = \frac{1}{a}(\sqrt{1 + a^2 t'^2} - 1)$$

with $a = 2\alpha$.

The infinitesimal generators (vector fields) of the transformations (12) are easily deduced:

$$\begin{array}{rcl} P_{\mu} &=& \frac{\partial}{\partial x^{\mu}},\ \ M_{\mu\nu} = x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}}, \\ D &=& x^{\mu}\frac{\partial}{\partial x^{\mu}},\ \ K_{\mu} = -2x_{\mu}x^{\nu}\frac{\partial}{\partial x^{\nu}} + x^2\frac{\partial}{\partial x^{\mu}} \end{array} \tag{15}$$

and they close into the conformal Lie algebra

$$\begin{array}{rcl} [M_{\mu\nu}, M_{\rho\sigma}] &=& \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}, \\ [P_{\mu}, M_{\rho\sigma}] &=& \eta_{\mu\rho}P_{\sigma} - \eta_{\mu\sigma}P_{\rho},\ \ [P_{\mu}, P_{\nu}] = 0, \\ [K_{\mu}, M_{\rho\sigma}] &=& \eta_{\mu\rho}K_{\sigma} - \eta_{\mu\sigma}K_{\rho},\ \ [K_{\mu}, K_{\nu}] = 0, \\ [D, P_{\mu}] &=& -P_{\mu},\ \ [D, K_{\mu}] = K_{\mu},\ \ [D, M_{\mu\nu}] = 0, \\ [K_{\mu}, P_{\nu}] &=& 2(\eta_{\mu\nu}D + M_{\mu\nu}). \end{array} \tag{16}$$

The quadratic Casimir operator has the following expression:

$$C_2 = -2D^2 + M_{\mu\nu}M^{\mu\nu} - P_{\mu}K^{\mu} - K_{\mu}P^{\mu} = -2D^2 + M_{\mu\nu}M^{\mu\nu} - 2P_{\mu}K^{\mu} - 8D. \tag{17}$$

Any group element $g \in SO(4, 2)$ (near the identity element 1) could be written as the exponential map

$$g = \exp(u),\ \ u = \tau D + b^{\mu}P_{\mu} + a^{\mu}K_{\mu} + \omega^{\mu\nu}M_{\mu\nu}, \tag{18}$$

of the Lie-algebra element u.

One would be tempted to blame the singular character of the special conformal transformation (13,14) to be responsible for the radiation effect, in much the same way the (singular) Rindler transformations (6) are supposedly responsible for the Unruh effect (i.e., existence of horizons). However, one could always work with the compactified Minkowski space $\mathbb{M} = \mathbb{S}^3 \times_{\mathbb{Z}_2} \mathbb{S}^1 \simeq U(2)$, which can be obtained as a coset $\mathbb{M} = SO(4, 2)/\mathbb{W}$, where $\mathbb{W}$ denotes the Weyl subgroup generated by $K_{\mu}, M_{\mu\nu}$ and D (i.e., a Poincaré subgroup $\mathbb{P} = SO(3, 1)\circledS\mathbb{R}^4$ augmented by the dilations $\mathbb{R}^+$). The Weyl group $\mathbb{W}$ is the stability subgroup (the little group in physical usage) of $x^{\mu} = 0$. Now the conformal group acts transitively on $\mathbb{M}$ and free from singularities. So, where would the radiation come from now?. Let us see that the reason for vacuum radiation in relativistic uniformly accelerated frames is more profound an related to the spontaneous breakdown of the conformal symmetry.

3.2. SU(2,2) as the four-cover of SO(4,2)

Instead of $SO(4,2)$, we shall work by convenience with its four cover:

$$SU(2,2) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_{4\times4}(\mathbb{C}) : g^\dagger \Gamma g = \Gamma, \det(g) = 1 \right\}, \tag{19}$$

where Γ denotes a hermitian form of signature $(++,--)$.

The conformal Lie algebra (16) can also be realized in terms of gamma matrices in, for instance, the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \check{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \tag{20}$$

where $\check{\sigma}^\mu \equiv \sigma_\mu$ (we are using the convention $\eta = \mathrm{diag}(1,-1,-1,-1)$) and σ^μ are the standard Pauli matrices

$$\sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{21}$$

Indeed, the choice

$$\begin{aligned} D &= \frac{\gamma^5}{2}, \; M^{\mu\nu} = \frac{[\gamma^\mu, \gamma^\nu]}{4} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\check{\sigma}^\nu - \sigma^\nu\check{\sigma}^\mu & 0 \\ 0 & \check{\sigma}^\mu\sigma^\nu - \check{\sigma}^\nu\sigma^\mu \end{pmatrix}, \\ P^\mu &= \gamma^\mu \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & \sigma^\mu \\ 0 & 0 \end{pmatrix}, \; K^\mu = \gamma^\mu \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ \check{\sigma}^\mu & 0 \end{pmatrix} \end{aligned} \tag{22}$$

fulfils the commutation relations (16). These are the Lie algebra generators of the fundamental representation of $SU(2,2)$.

The group $SU(2,2)$ acts transitively on the compactified Minkowski space $\mathbb{M}_4 = U(2)$, with (matrix) coordinates X, as

$$X \to X' = (AX+B)(CX+D)^{-1}. \tag{23}$$

Setting $X = x_\mu\sigma^\mu$ (with σ^μ Pauli matrices) the transformations (12) can be recovered from (23) as follows:

i) Standard Lorentz transformations, $x'^\mu = \Lambda^\mu_\nu(\omega)x^\nu$, correspond to $B = C = 0$ and $A = D^{-1\dagger} \in SL(2,\mathbb{C})$.
ii) Dilations correspond to $B = C = 0$ and $A = D^{-1} = a^{1/2}\sigma^0$
iii) Spacetime translations equal $A = D = \sigma^0$, $C = 0$ and $B = b_\mu\sigma^\mu$.
iv) Special conformal transformations correspond to $A = D = \sigma^0$ and $C = c_\mu\sigma^\mu, B = 0$ by noting that $\det(CX+I) = 1 + 2cx + c^2x^2$:

$$X' = X(CX+I)^{-1} \leftrightarrow x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2x^2}$$

3.3. Unirreps of de conformal group: discrete series

We shall consider the complex extension of $\mathbb{M}_4 = U(2)$ to the 8-dimensional conformal (phase) space:

$$\mathbb{D}_4 = U(2,2)/U(2)^2 = \{Z \in \mathrm{Mat}_{2\times2}(\mathbb{C}) : I - ZZ^\dagger > 0\}$$

and the Unirrep

$$[U_\lambda(g)\phi](Z) = |CZ+D|^{-\lambda}\phi(Z'),\ Z' = (AZ+B)(CZ+D)^{-1} \tag{24}$$

on the space $\mathcal{H}_\lambda(\mathbb{D}_4)$ of square-integrable holomorphic functions ϕ with invariant integration measure

$$d\mu_\lambda(Z,Z^\dagger) = \pi^{-4}(\lambda-1)(\lambda-2)^2(\lambda-3)\det(I-ZZ^\dagger)^{\lambda-4}|dZ|,$$

where the label $\lambda \in \mathbb{Z}, \lambda \geq 4$ is the conformal, scale or mass dimension ($|dZ|$ denotes the Lebesgue measure in $\mathbb{C}^4$).

3.4. The Hilbert space of our conformal particle

It has been proved in [14] that the infinite set of homogeneous polynomials

$$\varphi^{j,m}_{q_1,q_2}(Z) = \sqrt{\frac{2j+1}{\lambda-1}\binom{m+\lambda-2}{\lambda-2}\binom{m+2j+\lambda-1}{\lambda-2}}\det(Z)^m\mathcal{D}^j_{q_1,q_2}(Z), \tag{25}$$

with

$$\mathcal{D}^j_{q_1,q_2}(Z) = \sqrt{\frac{(j+q_1)!(j-q_1)!}{(j+q_2)!(j-q_2)!}}\sum_{p=\max(0,q_1+q_2)}^{\min(j+q_1,j+q_2)}\binom{j+q_2}{p}\binom{j-q_2}{p-q_1-q_2}z_{11}^p z_{12}^{j+q_1-p}z_{21}^{j+q_2-p}z_{22}^{p-q_1-q_2} \tag{26}$$

the standard Wigner's $\mathcal{D}$-matrices ($j \in \mathbb{N}/2$), verifies the following closure relation (the reproducing Bergman kernel):

$$\sum_{j\in\mathbb{N}/2}\sum_{m=0}^{\infty}\sum_{q_1,q_2=-j}^{j}\overline{\varphi^{j,m}_{q_1,q_2}(Z)}\varphi^{j,m}_{q_1,q_2}(Z') = \frac{1}{\det(I-Z^\dagger Z')^\lambda} \tag{27}$$

and constitutes an orthonormal basis of $\mathcal{H}_\lambda(\mathbb{D}_4)$, (the sum on j accounts for all non-negative half-integer numbers)

3.5. Hamiltonian and energy spectrum

In [15] we have argued that the dilation operator D plays the role of the Hamiltonian of our quantum theory. Actually, the replacement of time translations by dilations as dynamical equations of motion has already been considered in [16] and in [17], when quantizing field theories on space-like Lorentz-invariant hypersurfaces $x^2 = x^\mu x_\mu = \tau^2$ =constant. In other words, if one wishes to proceed from one surface at $x^2 = \tau_1^2$ to another at $x^2 = \tau_2^2$, this is done by scale transformations; that is, D is the evolution operator in a proper time τ.

From the general expression (24), we can compute the finite left-action of dilations $g = e^{\tau D}$ ($B = 0 = C$ and $A = e^{-\tau/2}\sigma^0 = D^{-1}$) on wave functions,

$$[U_\lambda(g)\phi](Z) = e^{\lambda\tau}\phi(e^\tau Z). \tag{28}$$

The infinitesimal generator of this transformation is the Hamiltonian operator:

$$H = \lambda + \sum_{i,j=1}^{2} Z_{ij}\frac{\partial}{\partial Z_{ij}} = \lambda + z_\mu\frac{\partial}{\partial z_\mu}. \tag{29}$$

The set of functions (25) constitutes a basis of Hamiltonian eigenfunctions (homogeneous polynomials) with energy eigenvalues E^λ_n (the homogeneity degree) given by:

$$H\varphi^{j,m}_{q_1,q_2} = E^\lambda_n\varphi^{j,m}_{q_1,q_2},\ \ E^\lambda_n = \lambda+n,\ \ n = 2j+2m. \tag{30}$$

Journal of Physics: Conference Series **284** (2011) 012014 doi:10.1088/1742-6596/284/1/012014

Actually, each energy level E_n^λ is $(n+1)(n+2)(n+3)/6$ times degenerated. This degeneracy coincides with the number of linearly independent polynomials $\prod_{i,j=1}^2 Z_{ij}^{n_{ij}}$ of fixed degree of homogeneity $n = \sum_{i,j=1}^2 n_{ij}$. This proves that the set of polynomials (25) is a basis for analytic functions $\phi \in \mathcal{H}_\lambda(\mathbb{M})$. The spectrum is equi-spaced and bounded from below, with ground state $\varphi_{0,0}^{0,0} = 1$ and zero-point energy $E_0^\lambda = \lambda$.

3.6. The ground state is Poincaré-stable and polarized by accelerations

Under a general conformal transformation, the excited ground state is:

$$\tilde{\varphi}_{0,0}^{0,0}(Z) = [U_\lambda(g)\varphi_{0,0}^{0,0}](Z) = \det(CZ+D)^{-\lambda}. \tag{31}$$

Therefore, for Poincaré transformations we have $C = 0$ and $\det(D) = 1$, which means that the ground state $\varphi_{0,0}^{0,0}$ looks the same to every inertial observer (it is stable for zero acceleration $C = 0$, i.e., it is Poincaré invariant). We shall restrict from now on to pure accelerations: $D = A = I, B = 0$ and $C = c_\mu \sigma^\mu$. For arbitrary accelerations, $C \neq 0$, we can decompose the accelerated ground state $\tilde{\varphi}_{0,0}^{0,0}$ using the Bergman kernel expansion (27) as:

$$\tilde{\varphi}_{0,0}^{0,0}(Z) = \sum_{j\in\mathbb{N}/2} \sum_{m=0}^{\infty} \sum_{q_1,q_2=-j}^{j} \varphi_{q_2,q_1}^{j,m}(-C)\varphi_{q_1,q_2}^{j,m}(Z). \tag{32}$$

From here, we interpret the coefficient $\varphi_{q_2,q_1}^{j,m}(-C)$ as the probability amplitude of finding the accelerated ground state in the excited level $\varphi_{q_1,q_2}^{j,m}$ of energy $E_n^\lambda = \lambda + 2j + 2m = \lambda + n$

3.7. Mean Energy of the accelerated ground state

The energy expectation value in the accelerated ground state (32) can be calculated as:

$$\mathcal{E}(C) = \frac{\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi_{q_1,q_2}^{j,n}(C)|^2(\lambda+2j+2n)}{\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi_{q_1,q_2}^{j,n}(C)|^2}. \tag{33}$$

Using (27) and its derivatives in (30), with the Hamiltonian operator given by (29), we obtain:

$$\begin{aligned} \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi_{q_1,q_2}^{j,n}(C)|^2 &= \frac{1}{\det(I - C^\dagger C)^\lambda} \\ \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q_1,q_2=-j}^{j} |\varphi_{q_1,q_2}^{j,n}(C)|^2(\lambda+2j+2n) &= \lambda\frac{1-\det(C^\dagger C)}{\det(I - C^\dagger C)^{\lambda+1}}, \end{aligned} \tag{34}$$

so that the mean energy is:

$$\mathcal{E}(C) = \lambda\frac{1-\det(C^\dagger C)}{\det(I - C^\dagger C)}. \tag{35}$$

For the particular case of an acceleration along the "z" axis, $C = c_\mu\sigma^\mu = c\sigma^3$, we have:

$$\mathcal{E}(c) = \lambda\frac{1-c^4}{1-2c^2+c^4} = \lambda\frac{1+c^2}{1-c^2} = \lambda + 2\lambda\frac{c^2}{1-c^2}. \tag{36}$$

The mean energy (36) is of Planckian type for

$$c^2(T) \equiv e^{-\frac{h\nu}{k_B T}}, \tag{37}$$

where we have introduced dimensions, with $h\nu$ the quantum of energy for our harmonic oscillator. The last (*ad hoc*) assignment (37) can be in fact obtained from first thermodynamical principles (see [11] for the actual computation of the partition function, entropy and temperature). Here we shall just point out that the formula (37) gives

$$T = -T_E/\ln(c^2),\ T_E \equiv h\nu/k_B, \tag{38}$$

which differs from the Unruh formula (1). It also implies the existence of a *maximal acceleration* $a_{\max}$, so that the dimension-full acceleration is $a = a_{\max}c$ (see [18] for a discussion on the physical consequences of a maximal acceleration and [15] and [11] for more details).

4. Second-Quantized Theory, Conformal Zero Modes and θ-Vacua

We have discussed the effect of relativistic accelerations in first (one particle) quantization. However, the proper setting to analyze radiation effects is in the second quantized theory. Let us denote (for space-saving notation) by $n = \{j, m, q_1, q_2\}$ the multi-index of the basis functions (25). The Fourier coefficients a_n (and a_n^*) of the expansion in energy modes of a state

$$\phi = \sum_n a_n \varphi_n, \tag{39}$$

are promoted to annihilation $\hat{a}_n$ (and creation $\hat{a}_n^\dagger$) operators in second quantization. The fact that the ground state of first quantization, φ_0, is invariant under Poincaré transformations implies that $\hat{a}_0$ commutes with all Poincaré generators. It also commutes with all annihilation operators and creation operators

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0,\ n > 0 \tag{40}$$

of particles with non-zero energy. Therefore, by Schur's Lemma, $\hat{a}_0$ must behave as a multiple of the identity in the broken theory, which means that Poincaré "θ-vacua" fulfil

$$\hat{a}_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = e^{\theta\hat{a}_0 - \bar{\theta}\hat{a}_0^\dagger}|0\rangle. \tag{41}$$

That is, Poincaré "θ-vacua" are *coherent states of conformal zero modes* (see [19] and [20] for a thorough exposition on coherent states).

The second-quantized version of (32) for an acceleration $C = c\sigma^3$ along the third axis is:

$$\hat{a}'_0 = \sum_{n=0}^{\infty} \varphi_n(c)\hat{a}_n, \tag{42}$$

so that accelerated Poincaré θ-vacua become:

$$|\theta'\rangle = e^{\theta\hat{a}'_0 - \bar{\theta}\hat{a}'^\dagger_0}|0\rangle. \tag{43}$$

The average number of particles with energy E_n in the accelerated vacuum (43) is then given by

$$N(c) = \langle\theta'|\hat{a}_n^\dagger\hat{a}_n|\theta'\rangle = |\theta|^2|\varphi_n(c)|^2, \tag{44}$$

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$, and $|\varphi_n(c)|^2$ is the probability of finding a particle in the energy state E_n of the accelerated vacuum $|\theta'\rangle$. One can also compute the mean energy per mode and see that it reproduces the Black-Body spectrum for (37). See more details in the coming paper [11].

Acknowledgements
Work partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219, FQM1951). M. Calixto thanks the "Universidad Politécnica de Cartagena" and C.A.R.M. for the award "Intensificación de la Actividad Investigadora 2009-2010". "Este trabajo es resultado de la ayuda concedida por la Fundación Séneca, en el marco del PCTRM 2007-2010, con financiación del INFO y del FEDER de hasta un 80%".

References

[1] S.A. Fulling, Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time, Phys. Rev. **D7** (1973) 2850-2862
[2] P.C.W. Davies, Scalar production in Schwarzschild and Rindler metrics, J. Phys. **A8** (1975) 609-616
[3] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. **D14** (1976) 870
[4] S.W. Hawking, Black-hole explosions?, Nature **248** (1974) 30
[5] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge Monographs on Mathematical Physics (1994).
[6] V. Aldaya, M. Calixto and J.M. Cerveró, Commun. Math. Phys. **200** (1999) 325-354
[7] E. L. Hill, On accelerated coordinate systems in classical and relativistic mechanics, Phys. Rev. **67** (1945) 358-363.
[8] L. J. Boya and J. M. Cerveró, Contact Transformations and Conformal Group. I. Relativistic Theory, International Journal of Theoretical Physics **12** (1975) 47-54
[9] T. Fulton, F. Rohrlich and L. Witten, Physical consequences to a coordinate transformation to a uniformly accelerationg frame, Nuovo Cimento **26** (1962) 652-671
[10] G.E. Volovik: Vacuum energy: myths and reality, Int. J. Mod. Phys. **D15** (2006) 1987-2010
[11] M. Calixto, E. Pérez-Romero and V. Aldaya, The Unruh Effect and the Spontaneous Breakdown of the Conformal Symmetry, in preparation.
[12] H. Weyl, "Space, Time and Matter". Dover. NY. First Edition 1922.
[13] H.A. Kastrup, Gauge properties of the Minkowski space, Phys. Rev. **150**, 1183 (1966).
[14] M. Calixto and E. Pérez-Romero, Extended MacMahon-Schwinger's Master Theorem and Conformal Wavelets in Complex Minkowski Space, Applied and Computational Harmonic Analysis, (to appear), arXiv:1002.3498
[15] M. Calixto and E. Pérez-Romero, Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in U(2,2) and Born's Reciprocity , arXiv:1006.5958, 33pp.
[16] V. Aldaya and J.A. de Azcárraga: Group manifold analysis of the structure of relativistic quantum dynamics, Annals of Physics **165** (1985) 484-504
[17] S. Fubini, A.J. Hanson and R. Jackiw: New approach to field theory, Phys. Rev. **D7** (1973) 1732-1760
[18] E.R. Caianiello, Is there a maximal aceleration?, Lettere al Nuovo Cimento **32** (1981) 65-70
[19] A. Perelomov: Generalized Coherent States and Their Apllications, Springer-verlag (1986)
[20] J.R. Klauder and Bo-Sture Skagerstam: Coherent States: Applications in Physics and Mathematical Physics, World Scientific (1985)

Capítulo 6

ϑ-Vacíos de Poincaré como Estados Coherentes de Modos Cero

Unruh Effect Revisited: Poincaré θ-Vacua as Coherent States of Conformal Zero Modes

M. Calixto[1,2], E. Pérez-Romero[2] and V. Aldaya[2]
[1] Departamento de Matemática Aplicada, Universidad de Granada, Facultad de Ciencias, Campus de Fuentenueva, 18071 Granada, Spain
[2] Instituto de Astrofísica de Andalucía (IAA-CSIC), Apartado Postal 3004, 18080 Granada, Spain

E-mail: calixto@ugr.es

Abstract. We report on a group-theoretical revision of the Unruh effect based on the conformal group $SO(4,2)$, which has been developed by the authors and collaborators. Special Conformal Transformations (SCT) are interpreted as transitions to relativistic uniformly accelerated frames. Poincaré invariant θ-vacua (which turn out to be coherent states of conformal zero modes) are destabilized by SCT and radiate as a black body.

1. Introduction

The Fulling-Davies-Unruh effect [1, 2, 3] has to do with *vacuum radiation* in a non-inertial reference frame and shares some features with the (black-hole) Hawking [4] effect. In simple words, whereas the Poincaré invariant vacuum $|0\rangle$ in QFT looks the same to any inertial observer (i.e., it is stable under Poincaré transformations), it converts into a thermal bath of radiation with temperature

$$T = \frac{\hbar a}{2\pi v k_B} \tag{1}$$

in passing to a uniformly accelerated frame (a denotes the acceleration, v the speed of light and k_B the Boltzmann constant).

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved (viz, existence of horizons). This is the case of the natural quantization in Rindler coordinates [2, 5], which leads to a quantization inequivalent to the normal Minkowski quantization (see next Section), or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [1].

In the reference [6], it was showed that the reason for the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations (that is, the Unruh effect) is more profound and related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory. From this point of view, a Poincaré invariant vacuum will be regarded as a coherent state of *conformal zero modes*, which are undetectable ("dark") by inertial observers but unstable under Special Conformal Transformations (SCT)

$$x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2cx + c^2 x^2}. \tag{2}$$

which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers with acceleration $a = 2c$ (see e.g. Ref. [7, 8, 9]). Indeed, let use take for simplicity an acceleration along the "z" axis, $c^\mu = (0, 0, 0, \alpha)$, and the temporal path $x^\mu = (t, 0, 0, 0)$. Then the transformation (2) reads:

$$t' = \frac{t}{1 - \alpha^2 t^2}\,, \qquad z' = \frac{\alpha t^2}{1 - \alpha^2 t^2}. \tag{3}$$

Writing z' in terms of t' gives the usual formula for the relativistic uniform accelerated (hyperbolic) motion:

$$z' = \frac{1}{a}(\sqrt{1 + a^2 t'^2} - 1) \tag{4}$$

with $a = 2\alpha$. Let us say that at least two alternative meanings of SCT have also been proposed [10, 11]. One is related to the Weyl's idea of different lengths in different points of space time [10]: "the rule for measuring distances changes at different positions". Other is Kastrup's interpretation of SCT as geometrical gauge transformations of the Minkowski space [11].

The point of view exposed here is consistent with the idea that quantum vacua are not really empty to every observer. Actually, the quantum vacuum is filled with zero-point fluctuations of quantum fields. The situation is similar to quantum many-body condensed mater systems describing, for example, superfluidity and superconductivity, where the ground state mimics the quantum vacuum in many respects and quasi-particles (particle-like excitations above the ground state) play the role of matter. Moreover, we know that zero-point energy, like other non-zero vacuum expectation values, leads to observable consequences as, for instance, the Casimir effect, and influences the behavior of the Universe at cosmological scales, where the vacuum (dark) energy is expected to contribute to the cosmological constant, which affects the expansion of the universe (see e.g. [12] for a nice review). Indeed, dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelerating rate.

In the papers [13] and [14] we construct a conformal-$SO(4, 2)$-invariant quantum theory in 3+1 dimensions, giving the Hilbert space and an orthonormal basis for our conformal particle. The construction is based on an holomorphic square-integrable irreducible representation of the conformal group on the eight-dimensional phase space $\mathbb{D}_4 = SO(4, 2)/SO(4) \times SO(2)$ inside the complex Minkowski space $\mathbb{C}^4$. In [15], the Poincaré invariance of the ground state is highlighted and the mean energy, partition function and entropy of the accelerated ground state (seen as a statistical ensemble) is explicitly calculated. This leads us to interpret the accelerated ground state as an Einstein Solid, to obtain a deviation from the Unruh's formula (1) and to discuss the existence of a maximal acceleration. Here we shall just deal with the second quantized (many-body) theory, where Poincaré-invariant (degenerated) pseudo-vacua are coherent states of conformal zero modes. Selecting one of this Poincaré-invariant pseudo-vacua spontaneously breaks the conformal invariance and leads to vacuum radiation.

2. Vacuum radiation as a spontaneous breakdown of the conformal symmetry

Let us offer an alternative explanation for the Unruh effect based on symmetry grounds. Actually, in Quantum Field Theory, the vacuum state is expected to be stable under some underlying group of symmetry transformations G (namely, the Poincaré group). Then the action of some spontaneously broken symmetry transformations can destabilize/excitate the vacuum and make it to *radiate*. We argue that this is precisely the case of the Planckian radiation of the Poincaré invariant vacuum under uniform accelerations. Here, the Poincaré invariant vacuum looks the same to every inertial observer but converts itself into a thermal bath of radiation in passing to a uniformly accelerated frame. In fact, we point out that the reason for this radiation is related to the spontaneous breakdown of the conformal symmetry in *quantum* field theory (see [6] and [15] for more details).

Let us denote by φ_n the energy E_n eigenstates. The Fourier coefficients a_n (and a_n^*) of the expansion in energy modes of a state

$$\phi = \sum_n a_n \varphi_n, \tag{5}$$

are promoted to annihilation $\hat{a}_n$ (and creation $\hat{a}_n^\dagger$) operators in second quantization. The fact that the ground state of first quantization, φ_0, is invariant under Poincaré transformations (see [6] and [15]) implies that the annihilation operator of zero-("dark")-energy modes $\hat{a}_0$ commutes with all Poincaré generators. It also commutes with all annihilation operators and creation operators

$$[\hat{a}_0, \hat{a}_n^\dagger] = 0,\ n > 0 \tag{6}$$

of particles with non-zero ("bright") energy. Therefore, by Schur's Lemma, $\hat{a}_0$ must behave as a multiple of the identity in the broken theory, which means that Poincaré "θ-vacua" fulfill

$$\hat{a}_0|\theta\rangle = \theta|\theta\rangle \Rightarrow |\theta\rangle = e^{\theta\hat{a}_0 - \bar{\theta}\hat{a}_0^\dagger}|0\rangle. \tag{7}$$

That is, Poincaré "θ-vacua" are *coherent states of conformal zero modes* (see [16] and [17] for a thorough exposition on coherent states).

In the first-quantized theory, the unitary transformation $\mathcal{U}(c)$ that implements the acceleration (2) does not leave invariant the ground state φ_0 (see [15]) . In fact, a decomposition of the "accelerated ground state" in energy modes is obtained:

$$\varphi_0' = \mathcal{U}(c)\varphi_0 = \sum_{n=0}^{\infty} \phi_n(c)\varphi_n, \tag{8}$$

where $\phi_n(c)$ denotes the probability amplitude of finding φ_0' in φ_n for a given acceleration c. In the second-quantized theory, this acceleration leads to a transformation of annihilation operators

$$\hat{a}_0' = \sum_{n=0}^{\infty} \phi_n(c)\hat{a}_n, \tag{9}$$

so that accelerated Poincaré θ-vacua become:

$$|\theta'\rangle = e^{\theta\hat{a}_0' - \bar{\theta}\hat{a}_0'^\dagger}|0\rangle. \tag{10}$$

The average number of particles with energy E_n in the accelerated vacuum (10) is then given by

$$N_n(c) = \langle\theta'|\hat{a}_n^\dagger\hat{a}_n|\theta'\rangle = |\theta|^2|\phi_n(c)|^2, \tag{11}$$

where $|\theta|^2$ is the total average number of particles in $|\theta\rangle$, and $|\phi_n(c)|^2$ is the probability of finding a particle in the energy state E_n of the accelerated vacuum $|\theta'\rangle$.

In the same way, the probability $P_n(q, c)$ of observing q particles with energy E_n in $|\theta'\rangle$ can be calculated as:

$$P_n(q, c) = |\langle q(n)|\theta'\rangle|^2 = \frac{e^{-|\theta|^2}}{q!}|\theta|^{2q}\,|\phi_n(c)|^{2q} = \frac{e^{-|\theta|^2}}{q!}N_n^q(c). \tag{12}$$

Therefore, the relative probability of observing a state with total energy E in the excited vacuum $|\theta'\rangle$ is:

$$P(E) = \sum_{\substack{q_0,\ldots,q_k:\\ \sum_{n=0}^{k} E_n q_n = E}} \prod_{n=0}^{k} P_n(q_n, c)\,. \tag{13}$$

For the case studied here, this distribution function can be factorized as $P(E) = \Omega(E)e^{-\tau E}$, where $\Omega(E)$ is a relative weight proportional to the number of states with energy E and the factor $e^{-\tau E}$ fits this weight properly to a temperature $T = k_B/\tau$.

One can also compute the mean energy

$$\mathcal{E}(c) = \langle\theta'|\sum_{n=1}^{\infty} E_n \hat{a}_n^\dagger \hat{a}_n|\theta'\rangle = |\theta|^2 \sum_{n=1} |\phi_n(c)|^2 E_n. \tag{14}$$

and see that it is indeed Planckian (see [6] and [15]) .

Acknowledgements

Work partially supported by the Fundación Séneca (08814/PI/08), Spanish MICINN (FIS2008-06078-C03-01) and Junta de Andalucía (FQM219, FQM1951). M. Calixto thanks the "Universidad Politécnica de Cartagena" and C.A.R.M. for the award "Intensificación de la Actividad Investigadora 2009-2010". "Este trabajo es resultado de la ayuda concedida por la Fundación Séneca, en el marco del PCTRM 2007-2010, con financiación del INFO y del FEDER de hasta un 80%".

References

[1] S.A. Fulling, Non-uniqueness of Canonical Field Quantization in Riemannian Space-Time, Phys. Rev. **D7** (1973) 2850-2862
[2] P.C.W. Davies, Scalar production in Schwarzschild and Rindler metrics, J. Phys. **A8** (1975) 609-616
[3] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. **D14** (1976) 870
[4] S.W. Hawking, Black-hole explosions?, Nature **248** (1974) 30
[5] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge Monographs on Mathematical Physics (1994).
[6] V. Aldaya, M. Calixto and J.M. Cerveró, Vacuum Radiation and Symmetry Breaking in Conformally Invariant Quantum Field Theory, Commun. Math. Phys. **200** (1999) 325-354
[7] E. L. Hill, On accelerated coordinate systems in classical and relativistic mechanics, Phys. Rev. **67** (1945) 358-363.
[8] L. J. Boya and J. M. Cerveró, Contact Transformations and Conformal Group. I. Relativistic Theory, International Journal of Theoretical Physics **12** (1975) 47-54
[9] T. Fulton, F. Rohrlich and L. Witten, Physical consequences to a coordinate transformation to a uniformly accelerationg frame, Nuovo Cimento **26** (1962) 652-671
[10] H. Weyl, "Space, Time and Matter". Dover. NY. First Edition 1922.
[11] H.A. Kastrup, Gauge properties of the Minkowski space, Phys. Rev. **150**, 1183 (1966).
[12] G.E. Volovik: Vacuum energy: myths and reality, Int. J. Mod. Phys. **D15** (2006) 1987-2010
[13] M. Calixto and E. Pérez-Romero, Extended MacMahon-Schwinger's Master Theorem and Conformal Wavelets in Complex Minkowski Space, Applied and Computational Harmonic Analysis (2010), DOI: 10.1016/j.acha.2010.11.004, arXiv:1002.3498, 40 pages
[14] M. Calixto and E. Pérez-Romero, Conformal Spinning Quantum Particles in Complex Minkowski Space as Constrained Nonlinear Sigma Models in U(2,2) and Born's Reciprocity, International Journal of Geometric Methods in Modern Physics **8**, N° 3, 33 pages (2011), to appear, arXiv:1006.5958
[15] M. Calixto, E. Pérez-Romero and V. Aldaya, Unruh Effect and the Spontaneous Breakdown of the Conformal Symmetry, arXiv:1012.4758
[16] A. Perelomov: Generalized Coherent States and Their Apllications, Springer-verlag (1986)
[17] J.R. Klauder and Bo-Sture Skagerstam: Coherent States: Applications in Physics and Mathematical Physics, World Scientific (1985)

Parte III

BIBLIOGRAFÍA

Bibliografía

[1] S.A. Fulling, Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time, Phys. Rev. **D7** (1973) 2850-2862

[2] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. **D14** (1976) 870

[3] P.C.W. Davies, Scalar production in Schwarzschild and Rindler metrics, J. Phys. **A8** (1975) 609-616

[4] S.W. Hawking, Black-hole explosions?, Nature **248** (1974) 30

[5] A. Perelomov, Generalized Coherent States and Their Aplications, Springer-Verlag (1986)

[6] S.T. Ali, J-P. Antoine and J.P. Gazeau, Coherent States, Wavelets and Their Generalizations, Springer (2000)

[7] J.R. Klauder and Bo-Sture Skagerstam, Coherent States: Applications in Physics and Mathematical Physics, World Scientific (1985)

[8] A. Grossmann, J. Morlet and T. Paul, Transforms associated to square integrable group representations I. General results, J. Math. Phys. **26** (1985) 2473-2479.

[9] E. Cunningham, The Principle of Relativity in Electrodynamics and an Extension Thereof, Proc. London Math. Soc. **8** (1910) 77-98.

[10] H. Bateman, The Transformation of the Electrodynamical Equations, Proc. London Math. Soc. **8** (1910) 223-264.

[11] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press 1999

[12] G. Kaiser, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994

[13] H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Springer Lecture Notes in Mathematics, vol. 1863, Springer-Verlag, Heidelberg, 2005.

[14] J-P. Antoine, D. Rosca, P. Vandergheynst, Wavelet transform on manifolds: Old and new approaches, Appl. Comput. Harmon. Anal. **28** (2010) 189-202

[15] H. Führ, Painless Gabor expansions on homogeneous manifolds, Appl. Comput. Harmon. Anal. **26** (2009) 200-211

[16] M. Holschneider, Continuous Wavelet Transforms on the sphere, J. Math. Phys. **37** (1996) 4156-4165 .

[17] J-P. Antoine and P. Vandergheynst, Wavelets on the 2-sphere: a group-theoretical approach, Appl. Comput. Harmon. Anal. **7** (1999) 262-291.

[18] J-P. Antoine and P. Vandergheynst, Wavelets on the n-sphere and related manifolds, J. Math. Phys. **39** (1998) 3987-4008.

[19] I. Bogdanova, P. Vandergheynst and J.P. Gazeau, Continuous wavelet transform on the hyperboloid, Appl. Comput. Harmon. Anal. **23** (2007) 285-306

[20] G. Kaiser, Space-time-scale Analysis of Electromagnetic Waves, Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis, Victoria, (1992) 209-212.

[21] G. Kaiser, Wavelet Electrodynamics II: Atomic Composition of Electromagnetic Waves, Appl. Comput. Harmon. Anal. **1** (1994) 246-260.

[22] G. Kaiser, Physical wavelets and their sources: real physics in complex spacetime, J. Phys. A: Math. Gen. **36** (2003) R291-R338

[23] W. Rühl, Field Representations of the Conformal Group with Continuous Mass Spectrum, Commun. Math. Phys. **30** (1973) 287-302.

[24] R. Coquereaux and A. Jadczyk, Conformal theories, curved phase spaces, relativistic wavelets and the geometry of complex domains, Rev. Math. Phys. **2** (1990) 1-44

[25] J. Schwinger, On Angular Momentum, U.S. Atomic Energy Commission Report NYO-3071, 1952. Unpublished. Reprinted in Quantum Theory of Angular Momentum (L. C. Biedenharn and H. van Dam, Eds.), pp. 229-279, Academic Press, New York, 1965.

[26] P.A. MacMahon, Combinatory Analysis, Vol. I, Cambridge Univ. Press 1915.

[27] S. Garoufalidis, T. T.Q. Lê and D. Zeilberger, The Quantum MacMahon Master Theorem, Proc. Natl. Acad. of Sci. U.S.A. **38** (2006) 13928-13931.

[28] D. Foata and D. Zeilberger, Laguerre Polynomials, Weighted Derangements, and Positivity, Siam J. Disc. Math. **1** (1988) 425-433

[29] J.D. Louck, MacMahon's Master Theorem, Double Tableau Polynomials, and Representations of Groups, Advances in Applied Mathematics, **17** (1996) 143-168.

[30] W.Y.C. Chen and J.D. Louck, The Combinatorics of a Class of Representation Functions, Advances in Mathematics **140** (1998) 207-236.

[31] J.D. Louck, Unitary Symmetry and Combinatorics, World Scientific Singapore 2008.

[32] L.C. Biedenharn, J.D. Louck, Angular Momentum in Quantum Physics, Addison-Wesley, Reading, MA, 1981;
L.C. Biedenharn, J.D. Louck, The Racah-Wigner Algebra in Quantum Theory, Addison-Wesley, New York, MA 1981

[33] G. Jakimowicz and A. Odzijewicz, Quantum complex Minkowski space, J. Geom. Phys. **56** (2006) 1576-1599

[34] R. Penrose, The twistor programme, Rep. Math. Phys. **12** (1977) 65-76.

[35] Penrose, R., and W. Rindler, Spinors and Space-Time, Volume 2: Spinor and Twistor Methods in Space-Time Geometry, Cambridge University Press, 1986

[36] J.M. Souriau: Structure des systemes dynamiques, Dunod Paris (1970)

[37] A.A. Kirillov, Elements of the Theory of Representations, Springer, Berlin, (1976).

[38] W. Rühl, Distributions on Minkowski space and their connection with analytic representations of the conformal group, Commun. Math. Phys. **27** (1972) 53-86.

[39] A.P. Balachandran, G. Marmo, B-S. Skagerstan and A. Stern, Gauge Theories and Fiber Bundles: Applications to Particle Dynamics, Lecture Notes in Physics **188** (Springer-Verlag, Berlin, 1983)

[40] G. Alexanian, A.P. Balachandran, G. Immirzi and B. Ydri, Fuzzy $\mathbb{C}P^2$, J. Geom. Phys. **42** (2002) 28-53, arXiv:hep-th/0103023

[41] V. Gerdt, R. Horan, A. Khvedelidze, M. Lavelle, D. McMullan and Yu. Palii, On the Hamiltonian reduction of geodesic motion on SU(3) to SU(3)/SU(2), J. Math. Phys. **47** (2006) 112902, arXiv:hep-th/0511245

[42] G. Kaiser, Quantized Fields in Complex Spacetime, Ann. Phys. **173** (1987) 338-354

[43] G. Kaiser, Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis, North-Holland, Amsterdam (1990) arXiv:0910.0352v2 [math-ph]

[44] A. Jadczyk, Born's reciprocity in the conformal domain, in "Spinors, Twistors, Clifford Algebras and Quanfum Deformations", Eds. Z. Oziewicz et al., Kluwer Academic Publ., 1993, pp. 129-140.

[45] M. Born, A suggestion for unifying quantum theory and relativity, Proc. R. Soc. **A165** (1938) 291-302

[46] M. Born, Reciprocity theory of elementary particles, Rev. Mod. Phys. **21** (1949) 463-473

[47] C. Castro and M. Pavsic, Clifford Algebra of Spacetime and the Conformal Group, Int. J. Theor. Phys. **42** (2003) 1693-1705

[48] C. Castro, The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and Clifford Group Geometric Unification, Foundations of Physics **35** (2005) 971-1041

[49] E.R. Caianiello, Is there a maximal aceleration?, Lettere al Nuovo Cimento **32** (1981) 65-70

[50] C. Castro, On the variable fine structure constant, strings and maximal-acceleration phase space relativity, International Journal of Modern Physics **A18** (2003) 5445-5473

[51] S.G. Low, Canonically relativistic quantum mechanics: representations of the unitary semidirect Heisenberg group $U(1,3) \otimes_s H(1,3)$, J. Math. Phys. **38** (1997) 2197-209

[52] S.G. Low, Representations of the canonical group, (the semidirect product of the unitary and Weyl-Heisenberg groups), acting as a dynamical group on noncommutative extended phase space, J. Phys. **A35** (Math.&Gen.) (2002) 5711-5729

[53] V. Aldaya, M. Calixto and J.M. Cerveró, Vacuum radiation and symmetry breaking in conformally invariant quantum field theory, Commun. Math. Phys. **200** (1999) 325-354

[54] M. Calixto, Higher-U(2,2)-spin fields and higher-dimensional W-gravities: quantum AdS space and radiation phenomena, Class. Quantum Grav. **18** (2001) 3857-3884

[55] S. Carlip, Entropy from Conformal Field Theory at Killing Horizons, Class. Quant. Grav. **16** (1999) 3327-3348

[56] I. Agullo, J. Navarro-Salas, G. J. Olmo, L. Parker, Hawking radiation by Kerr black holes and conformal symmetry, Phys. Rev. Lett. **105** (2010) 211305

[57] M. Calixto, Structure constants for new infinite-dimensional Lie algebras of $U(N_+, N_-)$ tensor operators and applications, J. Phys. **A33** (2000) L69-L75

[58] M. Calixto, Promoting finite to infinite symmetries: the 3+1 analogue of the Virasoro algebra and higher-spin fields, Mod. Phys. Lett. **A15** (2000) 939-944

[59] M. Calixto, Generalized W_∞ higher-spin algebras and symbolic calculus on flag manifolds, J. Geom. Phys. **56** (2006) 143-174

[60] V. Aldaya, J.L. Jaramillo, Extended diffeomorphism algebras in (quantum) gravitational physics, Int. J. Mod. Phys. **A18** (2003) 5795

[61] A.O. Barut and R.B. Haugen, Theory of the Conformally Invariant Mass, Ann. Phys. **71** (1972) 519-541

[62] G. Mack, All Unitary Ray Representations of the Conformal Group SU(2,2) with Positive Energy, Commun. Math. Phys. **55** (1977) 1-28

[63] E. L. Hill, On accelerated coordinate systems in classical and relativistic mechanics, Phys. Rev. **67** (1945) 358-363.

[64] L. J. Boya and J. M. Cerveró, Contact Transformations and Conformal Group. I. Relativistic Theory, International Journal of Theoretical Physics **12** (1975) 47-54

[65] T. Fulton, F. Rohrlich and L. Witten, Physical consequences to a coordinate transformation to a uniformly accelerationg frame, Nuovo Cimento **26** (1962) 652-671

[66] E. L. Hill, On a Type of Kinematical "Red-Shift", Phys. Rev. **68**, 232 (1945)

[67] C. E. Wulfman, Dynamical Symmetry, (World Scientific Publ. Co., Singapore, 2010)

[68] M. Calixto, J. Guerrero and J.C. Sánchez-Monreal, Sampling Theorem and Discrete Fourier Transform on the Riemann Sphere, Journal of Fourier Analysis and Applications **14** (2008) 538-567.

[69] M. Calixto, J. Guerrero and J.C. Sánchez-Monreal, Sampling Theorem and Discrete Fourier Transform on the Hyperboloid, Journal of Fourier Analysis and Applications **17** (2011) 240-264

[70] I. Bogdanova, P. Vandergheynst, J-P. Antoine, L. Jacques and M. Morvidone, Stereographic wavelet frames on the sphere, Appl. Comput. Harmon. Anal. **19** (2005) 223-252

[71] J-P. Antoine and A.L. Hohouéto, Discrete frames of Poincaré coherent states in 1+3 dimensions, Journal of Fourier Analysis and Applications **9** (2003) 141-173

[72] J. Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications, 2nd ed., London Mathematical Society Lecture Note Series, **257** Cambridge Univ. Press. 1999

[73] M. Calixto and J. Guerrero, Wavelet Transform on the Circle and the Real Line: a Group-Theoretical Treatment, Appl. Comput. Harmon. Anal. **21** (2006) 204-229.

[74] A. O. Barut, J. R. Zenitt and A Lauferts, The exponential map for the conformal group SO(2,4), J. Phys. **A27** (Math. Gen.) (1994) 5239-5250.

[75] A. O. Barut, J. R. Zenitt and A Lauferts, The exponential map for the unitary group SU(2,2), J. Phys. **A27** (Math. Gen.) (1994) 6799-6805.

[76] A.O. Barut and R. Raczka: Theory of Group Representations and Applications, 2^{nd} ed. World Scientific Publishing Co., Singapore (1986)

[77] J-P. Antoine, L. Demanet, L. Jacques, and P. Vandergheynst, Wavelets on the sphere: Implementation and approximations. Appl. Comput. Harmon. Anal. **13** (2002) 177-200.

[78] V. Aldaya and J.A. de Azcárraga, Group manifold analysis of the structure of relativistic quantum dynamics, Ann. Phys. **165** (1985) 484-504

[79] J. Guerrero and V. Aldaya, Invariant measures on polarized submanifolds in group quantization, J. Math. Phys. **41** (2000) 6747-6765.

[80] S.T. Ali, J-P. Antoine and J.P. Gazeau, Square integrability of group representations on homogeneous spaces I. Reproducing triples and frames, Annales de l' Institut H. Poincaré **55** (1991) 829-856

[81] H. Weyl, "Space, Time and Matter". Dover. NY. First Edition 1922.

[82] H.A. Kastrup, Gauge properties of the Minkowski space, Phys. Rev. **150**, 1183 (1966).

[83] S. Fubini, A.J. Hanson and R. Jackiw, New approach to field theory, Phys. Rev. **D7** (1973) 1732-1760

[84] V. Aldaya, J. Navarro-Salas and A. Ramirez, Algebraic quantization on a group and nonabelian constraints, Commun. Math. Phys. **121**, 541 (1989)

[85] D. MacMullan and I. Tsutsui, On the emergence of gauge structures and generalized spin when quantizing on a coset space, Ann. Phys. **237** (1995) 269-321

[86] V. Aldaya, M. Calixto and J. Guerrero, Algebraic Quantization, Good Operators and Fractional Quantum Numbers, Commun. Math. Phys. **178**, 399 (1996)

[87] N.P. Landsman and N. Linden, The geometry of inequivalent quantizations, Nucl. Phys. **B365**, 121 (1991)

[88] N. Woodhouse, Geometric Quantization, Oxford University Press (1980)

[89] V. Aldaya and J.A. de Azcárraga, Quantization as a consequence of the symmetry group: an approach to geometric quantization, J. Math. Phys. **23** (1982) 1297

[90] V. Aldaya, J. Guerrero and G. Marmo, Higher-order differential operators on a Lie group and quantization, Int. J. Mod. Phys. **A12** (1997) 3

[91] Marc-Thierry Jaekel and Serge Reynaud, Space-time localisation with quantum fields, Phys. Lett. **A220** (1996) 10-16

[92] Marc-Thierry Jaekel and Serge Reynaud, Conformal Symmetry and Quantum Relativity, Found. Phys. **28** (1998) 439-456

[93] W. Mulak, Quantum $SU(2,2)$-harmonic oscillator, Rep. Math. Phys. **34** (1994) 143-149

[94] C. Castro, Born's Reciprocal General Relativity Theory and Complex Non-Abelian Gravity as Gauge Theory of the Quaplectic Group: a Novel Path to Quantum Gravity, Int. J. Mod. Phys. **A23** (2008) 1487-1506

[95] W.R. Wood, G. Papini and Y.Q. Cai, Conformal transformations and maximal acceleration, Il Nuovo Cimento **104** (1989) 653-663

[96] M. Calixto and V. Aldaya, Thermal Vacuum Radiation in Spontaneously Broken Second-Quantized Theories on Curved Phase Spaces of Constant Curvature, Int. J. Geom. Meth. Mod. Phys. **6** (2009) 513-531

[97] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, Cambridge Monographs on Mathematical Physics (1994).

[98] R. F. Bishop and A. Vourdas, Generalised coherent states and Bogoliubov transformations, J. Phys. A: Math. Gen. **19** (1986) 2525-2536.

[99] B. Kostant, Quantization and unitary representations, Lecture Notes in Math. **170**, Springer-Verlag, Berlin-Heidelberg-New York, 1970

[100] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation, Freeman, San Francisco (1973)

[101] Philip M. Morse, Thermal Physics, W.A. Benjamin 1969

[102] J. Cariñena, M. A. del Olmo and M. Santander, Kinematic groups and dimensional analysis, J. Phys. A: Math. Gen. **14** (1981) 1-14

[103] J. Cariñena, M. A. del Olmo and M. Santander, A new look at dimensional analysis from a group theoretical viewpoint, J. Phys. A: Math. Gen. **18** (1985) 1855-1872

[104] G.E. Volovik, Vacuum energy: myths and reality, Int. J. Mod. Phys. **D15** (2006) 1987-2010

[105] M. Calixto and V. Aldaya, Curvature, zero modes and quantum statistics, J. Phys. **A39** (2006) L539-L545