

PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' PROPORTIONAL REASONING

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We present the findings of a study on prospective elementary teachers' proportional reasoning. After describing some of the teachers' performance in solving multiplicative structure problems that involve ratios and relations of direct proportionality between quantities, we were able to establish classifications of their answers according to various categories of proportional reasoning.

Keywords: Direct proportionality; Multiplicative structure problems; Proportional reasoning; Prospective elementary school teachers

Razonamiento proporcional de futuros maestros de educación primaria

Presentamos los resultados de un estudio sobre el razonamiento proporcional de futuros maestros de educación primaria. Describimos las actuaciones manifestadas por un grupo de estudiantes de magisterio de la Universidad de Granada al resolver problemas de estructura multiplicativa que involucran razones y relaciones de proporcionalidad directa entre cantidades. Encontramos que sus respuestas se clasifican en distintas categorías de razonamiento proporcional.

Términos clave: Futuros maestros de educación primaria; Problemas de estructura multiplicativa; Proporcionalidad directa; Razonamiento proporcional

Mathematics educators involved in pre-service elementary school teacher education programs often express their dissatisfaction about the mathematical knowledge of the future teachers and the future teachers' attitude towards mathematics. Prospective teachers' mathematical knowledge is frequently described as insufficient (Ball, 1990; Graeber, Tirosh, & Glover, 1989; Simon, 1993; Wheeler, 1983). Research on rational numbers has shown that future teachers' mathematical knowledge is procedural and barely connected (Lamon, 2007). According to Durmus (2005), the inadequacy of teachers' pedagogical and mathematical knowledge is one of the main causes of students' difficulties, particularly regarding rational numbers. This difficult situation represents a challenge for ed-

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ucators who believe that future teachers' mathematical knowledge is crucial to their educational growth. Conscious of the relevance of analyzing prospective teachers' mathematical knowledge, we focus here on proportionality, a significant part of the curriculum, both because of its use in solving everyday problems and because of its connections with other mathematical topics. From an international perspective, the National Council of Teacher of Mathematics (NCTM, 2000) states that every student in grades 6-8 should work flexibly with percentages to solve problems and understand how to use ratios and proportions to represent quantitative relationships. The NCTM also claims that "the need to understand and be able to use mathematics in everyday life and in the workplace has never been greater and will continue to increase" (p. 3). Proportional reasoning is a type of reasoning that students are very likely to use in everyday situations, such as maps, scale models, medicine dosages related to the weight of the patient, comparison-shopping, economic and sociological predictions based on the relationship of two quantities, diluting solutions in a chemistry lab, making recipes, etc. The research literature suggests that proportional reasoning represents a milestone in students' cognitive development and claims that it is fundamental to students' later development of concepts related to functions, graphing, algebraic equations, and measurement (Ben-Chaim, Keret, & Ilany, 2007; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Vergnaud, 1988).

Spanish elementary school students must learn concepts related to the meaning of fraction, ratio, proportion, magnitude, units of measure, specific ratios such as π , similar triangles, and percentages, at least in an intuitive way, as well as problem solving that involves such topics (Ministerio de Educación y Ciencia, 2006). Because we believe that future teachers must understand these mathematical concepts, we have decided to study future elementary teachers' mathematical knowledge related to proportional reasoning. In this study, we will focus on the task resolution process that future teachers apply in order to solve problems of ratio and proportion. We believe that this type of analysis provides information about the mathematical knowledge of prospective teachers and can thus serve as a resource to guide instruction processes in education faculties.

PROPORTIONAL REASONING

Previous research shows that there are several definitions of proportional reasoning. We will present those that help to ground the different types of problems used in this study. Karplus et al. (1983) define proportional reasoning as reasoning in a system of two variables related by a linear function. Behr, Harel, Post, and Lesh (1992) consider proportional reasoning as a form of mathematical reasoning that involves a sense of covariance and multiple comparisons, and whose main characteristic is the recognition of structural similarity and invariance in a simple mathematical system. Covariation is the simultaneous change of two vari-

ables that is caused by the existence of a relationship between them. Freudenthal (1983) defines covariance as the relationship between external magnitudes or the functional relation and invariance as the equivalence between the ratio of two quantities in a variable in one magnitude and the ratio of the corresponding quantities of another variable in another magnitude—in Freudenthal's terms, preservation of ratio or equivalence of internal ratios. Lamon (2007) suggests that proportional reasoning consists of being able to provide arguments that support statements about the structural relationship between four quantities (a, b, c, d) in a context that simultaneously implies covariance of quantities and invariance of the ratio or products. This could represent the ability to distinguish a multiplicative relation between two quantities, as well as the ability to extend the same relation to other pairs of quantities. Norton (2005) claims that the term proportional reasoning is used to describe concepts and thinking required to understand rates, ratio, and proportionality.

Several Findings Regarding Students' Proportional Reasoning

Because the current understanding of teachers' proportional reasoning derives in large part from the literature based on student reasoning, we review several of the main findings on students' proportional reasoning. However, we also include some results from studies in the context of pre- or in-service teacher education.

A significant amount of research has analyzed ratio and proportion, as well as students' errors and strategies in attempting to solve problems in this area. Lamon (2007) suggests that children may begin reasoning with ratios by composing two quantities to create a new unit, which in turn can be operated upon by iterating, partitioning, or splitting. According to several studies, many secondary and elementary students focus on a single quantity when reasoning about a proportional problem, while others may link the quantities but do not recognize the multiplicative relationship between them and thus make absolute rather than relative comparisons. This is recognized as additive reasoning (Harel, Behr, Lesh, & Post, 1994; Hart, 1984; Noelting, 1980a, 1980b). Several studies have focused on proportional reasoning strategies, such as the use of unit rates or scale factors in which students may form a ratio as a multiplicative comparison of two quantities by making a relative comparison of how many times one quantity is greater than another (Ben-Chaim, Fay, Fitzgerald, Benedetto, & Miller, 1998; Cramer & Post, 1993; Tourniaire & Pulos, 1985).

Several researchers have found that students often engage in more sophisticated reasoning when not using the rule of three algorithm and that this algorithm can interfere with students' thinking (Karplus et al., 1983; Lamon, 2007). A large body of research focuses on factors that influence the difficulty of proportion problems. Among these, we find the context (Tourniaire & Pulos, 1985), the location of the missing element in a proportion in relation to the other three numbers, whether a problem concerns discrete or continuous quantities, the presence

of integral ratios as opposed to non-integral ratios, the type of problem, and the presence of particular verbal expressions (Lamon, 2007).

In contrast to the large body of literature on students' proportional reasoning, there are comparatively few studies designed to study teachers' understanding of the topic. We share the assertion by Lobato, Orrill, Druken, and Jacobson (2011) that "very little is known about teachers' specialized content knowledge and knowledge of content and students for the domain of proportional reasoning" (p. 4). The few studies that we have found suggest that many teachers of the elementary and middle grades and many prospective teachers lack a deep understanding of proportional reasoning and rely too heavily on rote procedures such as the cross-multiplication algorithm (Lobato et al., 2011). Through analysis of particular mathematical tasks, some research has shown that in-service and prospective teachers participating in training can learn to use more sophisticated strategies of proportional reasoning (Ben-Chaim et al., 2007). Some research report that many of the same misconceptions identified in students are prevalent among elementary and middle school teachers (Cramer & Post, 1993; Lobato et al., 2011; Simon & Blume, 1994).

Proportional Reasoning Categorization

In this section, we summarize some of the indicators frequently used to describe proportional reasoning. Karplus et al. (1983) use a ratio comparison problem based on the problem "lemonade puzzle" and classify the students' performance into four large categories:

Category I (Incomplete, Illogical). This category includes the performance of students who do not provide any explanations, use the data in an illogical manner, or use quantitative operations in an inappropriate way.

Category Q (Qualitative). This category contains the performance of students who justify their answers using the four terms given and compare them using qualitative expressions such as more, less, or equivalent.

Category A (Additive). This category includes the performance of students who obtain the answer by applying subtraction to the four quantities given.

Category P (Proportional). This category includes students who use proportional relations among all of the data to obtain the answer, even if there are arithmetic errors. This category is divided into three subcategories: (a) between, if they use the external or functional relation; (b) within, if they use the internal relation, and (c) unclassifiable, if they use another type of comparison.

In analyzing 6th grade children's solutions to problems of ratio and proportion, Lamon (1993) uses the following mathematical dimensions: (a) relative or absolute thinking, (b) representation type (verbal, pictorial, tabular), (c) structure of the quantity (single unit or composed unit), and (d) strategy sophistication

(wrong strategy, pre-proportional reasoning, qualitative proportional reasoning, quantitative proportional reasoning).

Lamon (1993) identifies two types of approach according to sophistication of the strategy: constructive and non-constructive. For each strategy, she proposes three levels related to more or less proportional reasoning. The levels that correspond to non-constructive strategies are avoidance, visual or additive, and construction of patterns. The levels corresponding to constructive strategies are pre-proportional reasoning, qualitative proportional reasoning, and quantitative proportional reasoning. Allain (2000) provides a scale to measure the students' performances in 10 problems of proportionality. This scale assigns each subject a number from 1 to 4 according to measures predetermined by the researcher. The aim of the scale was to provide a broad description of the proportional reasoning used by the students who took part in the study.

METHOD

This is an exploratory study. Its aim is to analyze the task resolution process that future teachers apply when solving problems of ratio and proportion. The participants in this study were 76 3rd-year future teachers of elementary education specializing in special needs education at the University of Granada (Spain). The data were collected during the 2007-2008 academic year. We used Allain's (2000) Proportional Reasoning Assessment Instrument. This paper and pencil test consists of 10 problems with different difficulty levels. The original problems were translated into Spanish and adapted to the context. In Table 1, we show the type and formulation of the problems included in the original instrument.

Table 1

Problems Included in the Proportional Reasoning Instrument

I	Problem type	Problem
1	Numerical comparison	Rose bought 3 pieces of gum for 12 cents, and Anna bought 5 pieces of gum for 20 cents. Who bought the cheaper gum, or were they equal?
2	Missing value	David needs exactly 8 cups of water to make 14 small cups of coffee. How many small cups of coffee can he make with 12 cups of water?
3	Associated sets	There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?

Table 1
Problems Included in the Proportional Reasoning Instrument

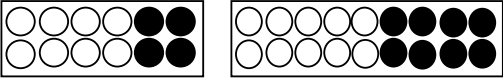
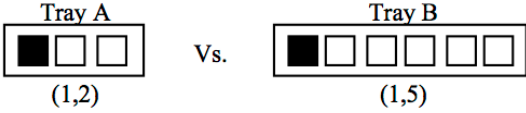

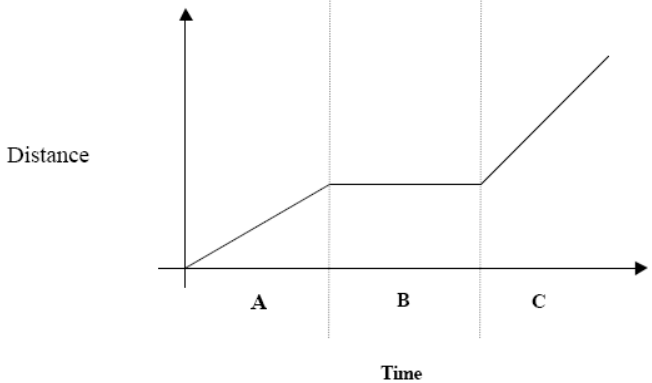
I	Problem type	Problem
4	Comparison part-part whole	<p>There are two egg cartons. The shaded circles represent brown eggs and the unshaded circles represent white eggs. The blue carton contains 8 white eggs and 4 brown eggs. The red carton contains 10 white eggs and 8 brown eggs. Which carton contains more brown eggs relative to white eggs? Explain your thinking.</p> 
5	Mixture	<p>You and your friend are going to make orange juice for a party. You will be given three different situations. In each situation, you will be presented with the contents of two trays. Each tray contains various amounts of orange juice and water. The shaded box represents the orange juice, and the unshaded box represents the water. The goal for each situation is to determine which drink will have the strongest orange taste or whether the two drinks will taste the same. Each mixture will be expressed as an ordered pair —e.g. (1,3)— with the first term corresponding to the number of glasses of orange juice and the second term to the number of glasses of water. Show any calculations and explain your thinking.</p>
6		
7		

Table 1
Problems Included in the Proportional Reasoning Instrument

I	Problem type	Problem												
8	Comparison, graphical interpretation	<p>Teresa took a bike ride this weekend. Below is a graph of her journey. The variable Distance represents the Teresa's distance from her starting point, and the variable Time represents the amount of time that has passed since she began her journey. The graph is divided into three equal intervals: A, B, and C. What information can you deduce from the graph about how fast she was traveling during each interval?</p> 												
9	Stretcher	<p>Two trees were measured five years ago. Tree A was 8 feet high, and tree B was 10 feet high. Today, tree A is 14 feet high, and tree B is 16 feet high. Over the last five years, which tree's height increased the most relative to its initial height? Show any calculations that lead you to your answer.</p>												
10a	Missing value (flag's height)	<p>You are shown a flag that measures 3 feet in length and 2 feet in height. It uses 6 square feet of cloth. If you wanted to make the second flag 3 feet longer while maintaining the same ratio of length to height, how much cloth would you need? Show your work.</p>												
10b	Non-linear stretcher (flag's area)	<table border="1" data-bbox="719 1630 1185 1724"> <thead> <tr> <th></th> <th>Length</th> <th>Height</th> <th>Surface area</th> </tr> </thead> <tbody> <tr> <td>Flag 1</td> <td>3 ft</td> <td>2 ft</td> <td>6 ft²</td> </tr> <tr> <td>Flag 2</td> <td>6 ft</td> <td></td> <td></td> </tr> </tbody> </table>		Length	Height	Surface area	Flag 1	3 ft	2 ft	6 ft ²	Flag 2	6 ft		
	Length	Height	Surface area											
Flag 1	3 ft	2 ft	6 ft ²											
Flag 2	6 ft													

Note: I = item.

In this paper we focus on future teachers' responses to items 4 and 10.

Description of Items 4 and 10

Item 4 is a numerical comparison problem. This type of problem provides two complete rates. A numerical answer is not required, but the rates must be compared. According to Allain (2000), Problem 4 is of the type “part-part-whole”. The original problems were changed replacing the egg cartons with two boxes of apples. One box contains a dozen apples and the other contains a dozen and a half apples. In addition, each box contains both red and green apples. The students must decide which box contains more red apples relative to green apples.

From an additive perspective, the students could use subtraction to determine the difference between each part (elements from each ratio). This means that the small box would contain $8 - 4 = 4$ green apples and the large box would contain $10 - 8 = 2$ green apples. Bearing this difference in mind, students may reach the following conclusion: The large box has a smaller difference between the two types of apples; consequently, the large box has more red apples relative to green apples.

The proportional reasoning involved in resolving the task can be shown in more or less sophisticated ways. This problem can also provide an example of pre-proportional reasoning, since it is possible to solve Item 4 with the building up strategy (Hart, 1984; Lamon, 2007). Using this strategy, a student establishes a ratio and extends it to a second ratio by addition. For Lamon, however, this is a primitive strategy by which students reason up to some desired quantity by using pattern recognition and replication. It does not take into consideration the constant ratio between two magnitudes. Students frequently represent this strategy in a table (see Figure 1).

Green apple	2	4	6	8	10	12	14
Red apple	1	2	3	4	5	6	7

Figure 1. Tabular representation of the building up strategy

Recognizing the multiplicative relation that links the quantities of the types of apple in each box is an indicator of proportional reasoning in solving Item 4. This means that the small box contains double the quantity of red apples as of green apples. Students can thus conclude that the large box should contain only five red apples. Since there are in fact eight, there are more red apples relative to green apples in the large box. Using the same strategy, students can recognize the scalar relation between the total number of apples and the quantity of red apples in each box.

Item 10 can be classified as a stretcher. According Allain (2000), this problem is similar to the problem posed by Lamon (1993) in her research. In this problem, the scaling up of the given quantities is non-linear. The context of this problem is constructing flags. The students are given information about the dimensions of Flag 1 and are told that Flag 2 must be three feet longer. They must determine how much cloth is needed to construct Flag 2, while maintaining the

same ratio of length to height as Flag 1. We present the information both as a written description and as a table because the information shown in a table contributes to recognizing scalar and functional relations between the quantities.

Data Analysis

Initially, we obtained the frequencies of (a) types of answers (incorrect or correct), (b) types of relation strategies, (c) errors that participants made when dealing with the given tasks, and (d) kinds of procedural knowledge applied to resolve the situations. These aspects are the observation criteria. To categorize the performance of the individuals who took the test, we constructed four different categories divided into four hierarchical levels identified by the numbers 1, 2, 3, and 4. To select the indicators that form part of each category, we drew on the studies by Karplus et al. (1983), Lamon (1993, 2007), and Allain (2000) but used primarily the participants' results, considering: (a) indicators that describe proportional reasoning, (b) indicators that describe understanding of the proportional concept, (c) types of specific strategies that can be considered correct and incorrect, and (d) error types that show how the participants are dealing with the given tasks and types of procedural knowledge applied to resolve the situations. The categories are presented in a sequence, where Category 1 denotes performance that correspond to the absence of proportional reasoning and Category 4 those that correspond to a high level of proportional reasoning. Table 2 shows the categories and indicators used in our investigation. A more detailed description of each category can be found in Valverde (2008).

Table 2

Categories and Indicators of Proportional Reasoning

Category	Indicators
1	<ul style="list-style-type: none"> a. Actions that have no interaction with the problem. b. Not solving the problem. c. No answer.
2	<ul style="list-style-type: none"> a. Giving answers without justification or giving wrong answers due to application of the wrong strategy. b. Applying absolute reasoning. c. Applying incorrect strategies.

Table 2
Categories and Indicators of Proportional Reasoning

Category	Indicators
3	<ul style="list-style-type: none"> a. Giving a wrong answer due to an error in applying some of the strategies considered at this level, a procedural error, or inadequate interpretation of the data. b. Applying relative reasoning. c. Applying the normalizing strategy. d. Applying the unit rate strategy. e. Showing no understanding of the structural properties of a proportion.
4	<ul style="list-style-type: none"> a. Obtaining the right answer. b. Applying the “ratio equivalent supposition” strategy. c. Applying the “factor of change” strategy. d. Recognizing the constant ratio of proportionality between two quantities in a situation of direct proportionality. e. Showing understanding of the structural properties of a proportion.

We assigned the foregoing categories to students’ responses according to the following criteria: (a) Category 1 was assigned to answers in which students demonstrated no interaction with the problem, (b) Category 2 was assigned if the students demonstrated one of indicators in that category, and (c) Categories 3 and 4 were assigned to the category from which the student fulfilled the most indicators. If the answer showed the same number of indicators from each category, we classified the answers according to two possibilities: First, if the student demonstrated any indicator from Category 3 and the first indicator from Category 4, the answer was assigned to Category 3. Second, answers with any other type of equality in number of indicators were assigned to Category 4.

RESULTS AND DISCUSSION

In this section, we present the results obtained in the analyzing Items 4, 10a, and 10b. We present the results for these problems only because we observed the prominence of proportional reasoning categories in each. These categories also permit us to identify errors or the application of inadequate strategies frequently used in this area that have been recorded in previous studies.

We focus on describing the general performance of the students with respect to: (a) way of linking quantities (between or within strategy), (b) procedural applied knowledge, and (c) errors and incorrect strategies. We also present the frequency of the categories assigned in solving these items. We then present examples of the students' answers, justifying the category we have assigned to them.

Overall Results from Solutions to Problem 4

The task resolutions were analyzed by considering each apple box as a system (Noelting, 1980a, 1980b). That is, if the students established a relation between the green apples and the red apples in each box, they would be applying the strategy within. If they established a relation between red apples and green apples from the two boxes, they would be implementing the strategy between.

The overall results for Problem 4 show a preference relating quantities in the same system, the within strategy: 51 participants chose this type of relation, 29 established a part-whole relation, and 22 applied a part-part relation to each box. The unit rate strategy was applied by six students and the normalizing strategy by four. We should stress that 13 participants applied the additive strategy. After raising the ratio between the number of green and red apples in each box, the students subtracted the quantities and gave their answer based on the results.

The most frequent procedures were those used in the fraction comparison (21 participants)—procedures such as conversion to decimal notation, homogenization of denominators, and cross-multiplication algorithm. Correspondingly, we found that the majority of the errors were related to misapplication of these procedures. Of a total of 76 answers, seven were assigned to Category 1, 28 to Category 2, 35 to Category 3, and six to Category 4.

In what follows, we present the work performed by some students in solving Problem 4 in order to exemplify Categories 2, 3, and 4. We have omitted Category 1, questions that remained unanswered, due to its lack of interest. We present first the student's solution and then we justify why that answer was assigned to that category.

Student 8 (Category 2)

Student 8 presented the following answer.

Small Box = 8 green, 4 red

Large Box = 10 green, 8 red

¹ Several researchers (Freudenthal, 1983; Karplus et al., 1983; Noelting, 1980a, 1980b; Vergnaud, 1988) use the idea of within and between differently in identifying multiplicative relationships inherent in proportional situations. This issue constitutes a persistent issue source of confusion (Fernández, 2009; Lamon, 2007). Our study uses the distinction proposed by Lamon (2007, p. 634), who suggests that the confusion is easily eliminated by using the terminology "within or between systems" or "within or between measure spaces".

The large box contains more red apples than green apples because there are exactly four apples left in the small box if you put the red apples on top of the green ones the red apples. In the large box, there are two green apples left over if you put the red apples on top of the green. Thus, the large box contains more red apples.

The student answers correctly, demonstrating the indicator in Category 4. However, the answer is based on additive reasoning. The student has an incomplete vision of the situation. Moreover, this type of thinking demonstrates the presence of indicator b from Category 2. Due to the absence of multiplicative reasoning, we assign this answer to Category 2.

Student 9 (Category 3)

Student 9 presented the following answer.

At first sight, the answer seems to be Box L, but when it is expressed in operations:

$$\text{Small box} \quad 4 \div 8 = 0.5$$

$$\text{Large box} \quad 4 \div 10 = 0.8$$

I divide the number of red apples in each box into the number of green apples in order to obtain the index that shows the proportion of one group in relation to the largest group (green apples) in each box.

Consequently, the large box contains 0.8 red apples for each green apple, which is a larger proportion than in the small box, which contains 0.5 red apples per green apple.

The student answers correctly, fulfilling the indicator in Category 4. The answer is based on the division of quantities. To make a comparison, the student finds the value of each ratio. This is the unit rate strategy. The subject demonstrates relational thinking, since he considers the multiplicative relation between quantities. However, the student does not use the scalar relation between the quantities of each system, since the relationship in the first box is double. The answer thus satisfies indicators b, d, and e of Category 3.

Student 38 (Category 4)

Student 9 presented the following answer.

$$12 \Rightarrow 4$$

$$18 \Rightarrow x$$

$$x = \frac{18 \cdot 4}{12} = \frac{72}{12} = 6$$

A box with 18 apples should contain 6 red apples. Since the large box has 8 apples, it contains more red apples in relation to the green apples.

The student responds correctly. He uses the “ratio equivalent supposition” strategy, thus implicitly recognizing the invariance of the ratio between the total number of apples and of red apples. He reasons correctly using the rule of three, because he determines the numerical condition to make the ratios equivalent. The answer thus satisfies indicators a, b, and c of Category 4.

Overall Results from Solutions to Problem 10, Item a

To analyze the solutions to Problem 10, we divided the study into two parts. The first corresponds to the height of Flag 2 (Item 10a) and the second to the area of Flag 2 (Item 10b). We do this because the first item can be considered a typical missing value task whose quantities remain a linear relation, whereas the quantities in the second item are not related linearly.

The item 10a solutions were analyzed considering the dimensions of each flag as a system (Noelting, 1980a, 1980b). Further, we determined that students who established a relation between the length and the width of each flag would be applying the within strategy, whereas students establishing a relation between the measures of the length of the two flags would be implementing the between strategy.

In Item 10a, the students are asked to find the width of the flag. In this case, we observe that the students relate the quantities by applying both strategies, the within strategy in 29 cases, and the between strategy in 34 cases.

Students who recognized the multiplicative factor applied the between strategy, which is associated with the relationship between the measures of length in each system. We observe that 33 students applied the factor of change strategy, answering that the constant between the lengths is 2, or double. We believe that this strategy occurred so frequently because the exchange factor was an easily recognizable integer.

The procedure used by 20 participants to determine the width of the flag was the rule of three

$$\begin{array}{l} 3 \rightarrow 2 \\ 6 \rightarrow x \end{array}$$

demonstrating that these subjects did not recognize the constant of proportionality between the dimensions of the flag. Nor did they recognize the multiplicative relation between the measures of the lengths. We believe that the application of these techniques corresponds to a school tradition that encourages the students to justify the processes of problem solving using algorithms.

Of the 19 incorrect answers, six correspond to an individual who applied the additive strategy or constant difference, since the difference between the width and the length of the first flag is one unit. These answers stated that the width of

the second flag was 5, one unit less than its length. We also note that 10 participants made mistakes in interpreting the problem. They interpreted the phrase “maintaining the same ratio of length to height” as meaning that the new flag should have the same width or that, in order to maintain the same ratio, the width should not vary. Of a total of 76 answers, two were assigned to Category 1, nine to Category 2, 32 to Category 3, and 33 to Category 4.

In what follows, we present the work done by some students in Task 10a as examples of Categories 2, 3, and 4. As before, we omit Category 1 due to its lack of interest. In the right column, we justify why we have assigned the student’s answer to each category.

Student 25 (Category 3)

Student 25 presented the following answer.

$$3 \text{ length} \rightarrow 2 \text{ height}$$

$$6 \text{ length} \rightarrow x \text{ height}$$

$$6 \times 2 = 12, 12 \div 3 = 4$$

	Length	Height	Surface area
Flag 1	3 ft	2 ft	6 ft ²
Flag 2	6 ft	4 ft	24 ft ²

The individual establishes a within relation for the dimensions of each flag. He also applies a rule of three to determine the width of the second flag. This response exemplifies the predominant use of techniques and the lack of relational reasoning on structural similarity in a proportion. The response thus fulfills indicator a of Category 4 and indicators b and e of Category 3.

Student 9 (Category 4)

Student 9 presented the following answer.

We find the ratio of length to width from the first flag and apply it to the second flag in order to determine the width we need.

$$3 \div 2 = 1.5$$

$$6 \div x = 1.5 \Rightarrow x = \frac{6}{1.5} = 4$$

	Length	Height	Surface area
Flag 1	3 ft	2 ft	6 ft ²
Flag 2	6 ft	4 ft	24 ft ²

The student establishes a within relation, which provides the correct answer. The student recognizes that the ratio between the dimensions in each flag must be constant. The answer thus shows evidence of the recognition of the invariance of ratios within each system. It satisfies indicators a, c, d, and e of Category 4.

Overall Results from Solutions to Problem 10, Item a

Part 10b of the problem discussed above asked students to determine the area of the second flag. According to Lamon (1993), this problem corresponds to the stretcher non-linear type, since the relation between the areas is not modeled by the linear function $f(x) = mx, m \neq 0$. This problem highlights the presence of the phenomenon called illusion of linearity (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). As others researchers explain (Modestou & Gatsis, 2007; Van Dooren, De Bock, & Verschaffel, 2006), the roots of this trend include different explanatory factors, among which we find: (a) elements related to linearity/proportionality, such as its intuitive nature, simplicity, and presence in everyday life; (b) elements related to students' experiences in the formal school system; and (c) the basic linguistic structure for problems involving proportionality, which includes four quantities (a , b , c , and d), of which, in most cases, three are known and one unknown, plus an implication that the same multiplicative relationship links a to b and c to d , among others. Modestou and Gatsis (2007) propose that the illusion of linearity is an epistemological obstacle, hence its continuing presence, even in older students such as this group of future teachers.

The students who did not commit the illusion of linearity error responded correctly, applying the formula for the area of a rectangle. Nevertheless, it is significant that no student argued that the ratio between the areas is equal to the square of the ratio of the sides. Due to the nature of the item, however, we do not expect to find reasoning with this level of complexity.

As to the proportional reasoning categories considered in our study, we determine that 24 of the 76 student responses to Item 10b fit Category 2. We do not consider this item useful for observing the proportional reasoning indicators considered in Categories 3 or 4; however we recognize that this item has the potential to analyze whether the incorrect strategy of the illusion of linearity persists in the students' conceptions of the problem. The 76 answers were assigned to the four categories as follows: three to Category 1, 24 to Category 2, 49 to Category 1, and none to Category 4.

The following is a sample answer that fits Category 2 for this item.

Student 15 (Category 2)

Student 15 presented the following answer.

If the length doubles, you will need double m^2 of the fabric.

	Length	Height	Surface area
Flag 1	3 ft	2 ft	6 ft ²
Flag 2	6 ft	4 ft	12 ft ²

The student's answer for Item b is incorrect. It reflects the application of a common reasoning error in the study of proportionality known as illusion of linearity. The student recognizes the linear relation between the lengths of the flags but

then applies the same multiplicative relation to the areas. This answer thus satisfies indicators a, and c of Category 2.

CONCLUSION

According to the categories assigned in the analysis of the two problems, we can see that Category 3 is the most common. This situation demonstrates the prevalence of pre-proportional reasoning in the performances of this group of future teachers. The analysis of the procedural knowledge applied by the participants shows a strong influence of procedures and algorithms related to other meanings of fractions unrelated to the subconstruct of ratio.

At the same time, we observe that the participants exhibit the same errors and primitive ideas (additive reasoning or the illusion of linearity) as children or secondary students who participated in other studies that used the same tasks (Van Dooren et al., 2005). This situation confirms that even older students do not yet understand the contrast between the additive and multiplicative relations.

Although they applied the correct strategies and procedures in resolving the tasks, the students clearly do not recognize the scalar or the functional relation between quantities. Nor do they provide any mathematical arguments to establish the relation of order between the two ratios; they merely establish the relation of division of antecedent and consequent to find the value of the ratio. In general, a series of procedural steps prevails instead of recognition of the structural properties of a proportion.

This situation prompts us to think about the importance of developing the different subconstructs of rational numbers in a connected way. Such development might include highlighting the situations or events that give meaning to each subconstruct and providing elementary school teachers with learning experiences that enable them to reconstruct a more detailed concept of ratio and its properties.

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