Universidad de Granada

E. T. S. de Ingenierías Informática y de Telecomunicación

Departamento Ciencias de la Computación e I. A.

Managing Vagueness in Ontologies

PhD Dissertation

Fernando Bobillo Ortega

Editor: Editorial de la Universidad de Granada
Autor: Fernando Bobillo Ortega
D.L.: GR.2896-2008
ISBN: 978-84-691-8334-2

Universidad de Granada

Managing Vagueness in Ontologies

Fernando Bobillo Ortega

SUPERVISOR

Miguel Delgado Calvo-Flores

Granada, October, 2008

La memoria “Managing Vagueness in Ontologies”, que presenta D. Fernando Bobillo Ortega para optar al grado de Doctor en Informática, ha sido realizada en el departamento de Ciencias de la Computación e Inteligencia Artificial de la Universidad de Granada bajo la dirección del Profesor Dr. D. Miguel Delgado Calvo-Flores, Catedrático de Universidad.

Granada, Octubre de 2008.

Miguel Delgado Calvo-Flores   Fernando Bobillo Ortega

A Esperanza, lo último que se pierde.

"Fuzzy logic is not fuzzy"

(Lotfi A. Zadeh)

Contents

List of Figures

List of Tables

Abstract

The use of ontologies as appropriate formalisms for the representation of the knowledge of many different domains of application has received a lot of attention recently. Nevertheless, classical ontologies are not suitable to represent imprecise, vague and uncertain knowledge, which is inherent to several real-world domains. As a solution, fuzzy ontologies have been proposed as a combination of ontologies with techniques from fuzzy set theory and fuzzy logic.

This dissertation presents several contributions to the field of fuzzy ontologies. Our chosen formalism is a fuzzy extension of the very expressive fuzzy Description Logic $\mathcal{SROIQ}(\mathbf{D})$, the basis of the language OWL 2. After a definition of the logic, their main properties are investigated. A reasoning algorithm is provided, based on a reduction to a classical ontology which allows to reuse current languages and reasoners. Two semantics, based on two different families of fuzzy operators (Zadeh and Gödel), are considered, and the properties of the reduction (correctness, modularity, and complexity) are studied in detail. Several optimizations have also been investigated. A possibilistic extension enabling an additional representation of uncertain pieces of knowledge is also outlined. Finally, the reasoning algorithm is implemented in a prototype called DeLorean, the first reasoner that supports fuzzy extensions of the standard language for ontology representation OWL and its recent extension OWL 2.

Agradecimientos

La presentación de esta tesis doctoral no es únicamente fruto de mi trabajo, sino que ciertamente no habría sido posible sin la ayuda de muchas personas que, de alguna u otra manera, han contribuido a que esto sea posible, y ante las que me encuentro muy agradecido.

En primer lugar me gustaría agradecer a mi director Miguel Delgado no solamente por su labor de dirección de esta tesis, sino también por su tutela ya desde mi primera beca de iniciación a la investigación. Los consejos que he recibido como consecuencia de su supervisión y amplia experiencia han sido fundamentales para mi formación.

También me gustaría mostrar mi agradecimiento a los miembros del Departamento de Ciencias de la Computación e Inteligencia Artificial, donde he pasado unos años inolvidables. De entre todos ellos, me gustaría destacar especialmente a algunas personas. A Juan Gómez-Romero, por el trabajo conjunto, sus certeras correcciones, charlas y muchas cosas compartidas, no todas científicas. También a Amparo Vila, Daniel Sánchez y Nicolás Marín por sus siempre útiles sugerencias ante mis siempre temibles seminarios. A María Dolores Ruiz-Jiménez y Pedro Magaña por el apoyo logístico cuando estuve en el extranjero pero sobre todo por su amistad y su ayuda. Y a todos los demás sufridos usuarios del despacho 16.

Desde luego, esta tesis tampoco hubiera sido posible sin la ayuda de todo tipo que me ha brindado mi familia, especialmente los más directos: mi madre Esperanza y mi hermano José Manuel. Lo vuelvo a repetir, sin ellos esta tesis no habría sido posible.

Me gustaría agradecer a los amigos que no han sido aún mencionados por darme la estabilidad emocional que necesita todo investigador, muy especialmente a Celia y a Jose (Luis Polo) por estar siempre ahí. A Elvira, por su apoyo durante la mayor parte del tiempo que he trabajado en esta tesis. A Noe, por estar a mi lado en estos momentos en que toca terminarla.

Desde el punto de vista económico, esta tesis ha sido financiada por el *Ministerio de Educación y Ciencia* a través de una *Beca de Formación de Profesorado Universitario* (FPU), dos *Ayudas para Estancias Breves* de investigación y varios proyectos de investigación que han permitido la asistencia a congresos para la presentación de algunos de los resultados de la tesis.

También me gustaría dedicar unas palabras de agradecimiento en inglés o en sus lenguas maternas a varias personas que lo merecen.

En primer lloc voldria agrair en Francesc Esteva per haver estat el meu tutor d'una *Beca de Iniciación a la Investigación para Estudiantes de Último Curso* en l'*Institut d'Investigaçió en Intel·ligència Artificial* del C*onsejo Superior de Investigaciones Científicas* (IIIA-CSIC) a Bellaterra (Barcelona), de setembre a desembre del 2004. Aquesta experiència va confirmar el meu desig de dedicar-me a l'apassionant món de la recerca. Moltíssimes gràcies a tota la gent de l'institut per fer-me sentir com a casa, i molt especialment a en Lluís Godo per les discussions científiques que hem mantingut amb l'Umberto Straccia.

Vorrei anche ringraziare Umberto Straccia per avermi lasciato lavorare con lui senza conoscermi, all' *Instituto di Scienza e Tecnologie dell'Informazione "Alessandro Faedo"* del *Consiglio Nazionale delle Ricerche* (ISTI-CNR), in a soggiorno di ricerca a Pisa (Italia), da settembre a dicembre 2006. E ancora di più, per avere accettato di lavorare con me dopo avermi conosciuto, da settembre a dicembre 2007, sempre all' ISTI-CNR. La mia permanenza è stata veramente bella ed ho imparato tantissime cose importanti.

Finally, I would like to thank Janusz Kacprzyk and Andrea G. B. Tettamanzi for writing two reports about this thesis, which are necessary for me to apply for the *European Doctorate* mention.

Part I

Introduction

CHAPTER 1

Introduction

1.1 Antecedents

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. An ontology is defined as an explicit and formal specification of a shared conceptualization [107], which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies have a lot of advantages, such as making possible to add semantics to data, making knowledge maintenance, information integration as well as the reuse of components easier. For example, ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning.

The current standard language for ontology creation is the Web Ontology Language (OWL [327]), which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full. However, since its first development, several limitations on expressivity of OWL have been identified, and consequently several extensions to the language have been proposed. Among them, the most significant is OWL 2 [75] which is its most likely immediate successor.

Description Logics (DLs for short) [13] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to be very useful as ontology languages. For instance, an ontology in OWL Lite, OWL DL or OWL 2 is equivalent to an ontology in $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ or $\mathcal{SROIQ}(\mathbf{D})$ respectively [139].

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise, vague and uncertain knowledge, which is inherent to several real-world domains. Fuzzy and possibilistic logics have proved to be suitable formalisms to handle imprecise/vague and uncertain knowledge respectively. Consequently, several fuzzy extensions of DLs can be found in the literature [181], yielding fuzzy ontologies. Several definition of fuzzy ontology have been proposed, and they have proved to be useful in several applications, such as information retrieval or the Semantic Web.

The appearance of fuzzy ontologies brings about that crisp standard languages are not suitable and can no longer be used, so new fuzzy languages must be developed. Hence, the large number of resources available for crisp ontologies are no longer appropriate and need to be adapted to the new framework, requiring an important effort. This issue affects especially reasoning engines. Previous experiences with crisp DLs have shown that there exists a significant gap between the design of a decision procedure and the achievement of a practical implementation [276], since expressive DLs has a very high worst-case complexity. Therefore, optimization of fuzzy DL reasoners will be presumably very hard and costly.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory, the representation of them using crisp DLs has not received such attention. Furthermore, the expressivity of the logics considered in this context can be enriched. For instance, the seminal of work on reduction of fuzzy DLs to crisp DLs was restricted to $\mathcal{ALCH}$ [302].

Either way, the expressivity of fuzzy DLs can be enriched in several ways, since current fuzzy DLs have several limitations:

- The nominal constructor is not fuzzified in the logics which support it.
- Although fuzzy GCIs and RIAs have been proposed, reasoning with them is not usually allowed.
- Current reasoning algorithms do not support full $\mathcal{SROIQ}(\mathbf{D})$.
- There are no fuzzy DL reasoners supporting fuzzy extensions of the languages OWL and OWL 2.
- There is a lack of a general formalism joining the management of imprecise and uncertain knowledge in DLs.

On a different topic, it is common in fuzzy logic to group the fuzzy operators in families, each of them containing a t-norm, a t-conorm, a negation and an implication function. There are three main families of fuzzy operators: Łukasiewicz, Gödel and Product [121]. The operators than L. Zadeh initially considered when he introduced fuzzy set theory (Gödel conjunction and disjunction, Łukasiewicz negation and Kleene-Dienes implication) are also of great importance in the literature, and we refer to them as Zadeh family.

It is well known that different families of fuzzy operators lead to fuzzy DLs with different properties. Most of the existing works in fuzzy DLs consider Zadeh family. Some few works consider Łukasiewicz or Product families, but Gödel family has not received such attention. In our opinion, the logical properties of Gödel family make interesting its study. For example, as well as Zadeh family, Gödel family includes an idempotent t-norm (minimum) so the conjunction is independent of the granularity of the fuzzy ontology, which is interesting in some applications. This is not the case in Łukasiewicz or Product families. But an important difference with respect to Zadeh family is that the implication of Gödel family has better logical properties than the implication of Zadeh family. For example, using the latter implication, concepts and roles do not fully subsume themselves.

1.2 Objectives

The general aim of this dissertation is to promote the achievement of fuzzy ontologies, in order to able to represent and reason with imprecise and vague pieces of knowledge.

In accordance to this, the concretes objectives of this thesis are the following:

- To review and analyze the state of the art in fuzzy ontologies and related areas such as fuzzy DLs.
  - To compare different definition of fuzzy ontologies and to analyze their limitations.
  - To identify limitations in the expressivity of current proposals for fuzzy DLs.
- To propose a new definition of fuzzy DL.
  - To increase the expressivity with respect to the related work.
  - To include some way of representing not only imprecise and vague, but also uncertain knowledge.
- To provide a crisp representation for such an expressive fuzzy DL.
  - To support the expressivity of fuzzy OWL 2.
  - To support a semantics given by different families of fuzzy operators (in particular, Gödel logic).
  - To design some optimization techniques reducing the size of the representation.
- To implement a small prototype demonstrating the feasibility of our approach.

1.3 Thesis Structure

This dissertation is structured in five clearly defined parts, each of them being composed of one or more chapters.

After this introduction, Part II starts with some necessary preliminaries. Chapter 2 reviews some basic notions on fuzzy set theory, fuzzy logic, and possibilistic logic. Chapter 3 is dedicated to classical ontologies and DLs, as the most successful formalism for ontology representation. Special attention is dedicated to the DL $\mathcal{SROIQ}(\mathbf{D})$ and the language OWL 2. Chapter 4 includes the state of the art in fuzzy extensions of ontologies and DLs, providing a more detailed contextualization of our work.

Part III presents our contributions at a theoretical level. Chapter 5 defines our fuzzy extension of the DL $\mathcal{SROIQ}(\mathbf{D})$, highlights the enhancements in the expressivity, and studies its logical properties. Chapter 6 restricts to Zadeh family and describes a reasoning preserving procedure to reduce this fuzzy DL to a classical one, in such a way that existing DL reasoners can be applied to the resulting KB. Then, Chapter 7 presents a similar result but for Gödel family. In both cases, some interesting optimizations of the reduction are presented.

Part IV covers the practical developments of this thesis. Chapter 8 presents the design and implementation of our prototype fuzzy DL reasoner, called DeLorean, which implements the reduction algorithms and the optimizations described in the precedent part. A preliminary evaluation of the procedure is also performed.

Finally, Part V concludes with some conclusiones and future work. In Chapter 9 we summarize the contributions of this thesis to the field of fuzzy ontologies. The results are analyzed in accordance with the objectives established in this introductory chapter. Finally, some ideas for future research are pointed out.

Part II

Preliminaries

CHAPTER 2

Fuzzy Set Theory and Fuzzy Logic

This chapter reviews some basic notions on fuzzy set theory and fuzzy logic, which will be necessary to read the rest of this document. For a more detailed overview, the interested reader may give look at [92, 121, 157].

Fuzzy set theory and fuzzy logic were both proposed by L. Zadeh [343][1]. While in classical sets theory elements either belong to a set or not, in fuzzy set theory elements can belong to some degree.

One of the most important features of fuzzy logic is its ability to perform *approximate reasoning* [344], a mode of reasoning with linguistic rather than numerical variables, and hence with vague information. This involves inference with premises and consequences containing fuzzy propositions, which is closest to human reasoning than classical two-valued logic.

Although Zadeh's ideas were initially met with some skepticism, fuzzy sets and fuzzy logic are well recognized as appropriate formalisms to manage imprecise and vague knowledge, and have found numerous applications in fields such as control systems of industrial processes, decision support systems, expert systems, image processing, pattern recognition, information retrieval, data mining, classification . . .

Section 2.1 starts with the definition of a fuzzy set. Section 2.2 studies the connection of fuzzy sets with classical sets by means of α-cuts. Next,

[1] However, the first use of the term "fuzzy logic" is in [105]

Section 2.3 includes some basic operations with fuzzy sets. Then, Section 2.4 considers fuzzy relations. At that point the reader will be ready to approach fuzzy logic as a whole, which is done in Section 2.5. Section 2.6 is devoted to fuzzy numbers. Finally, Section 2.7 concludes this chapter with some notions on possibilistic logic, a related formalism.

2.1 Fuzzy Sets: A Generalization of Classical Sets

The classical notion of set is deeply related to the fulfilment of a given property which is satisfied by all the members of the set. We may think of a property as a function defined over a set of objects U (which is referred as the *referential* set or *domain of discourse*) relating each of these objects to a element of the set $\{0, 1\}$. A particular element belongs to the set if the function assigns 1 to it; otherwise (if the function assigns 0 to it), the element does not belong to the set. These sets are called *crisp* or classical.

According to this, any property P determines a set S_P which is composed by the following elements:

$$S_P = \{u \in U : P(u) = 1\}$$

In the same way, any subset $S \subseteq U$ induces a property P_S which is determined by the following expression:

$$P_S(u) = 1 \text{ if and only if } u \in S$$

Fuzzy set theory, originally proposed by L. A. Zadeh in 1965 [343], generalizes this classical notion of set, having into account that the properties which define a set are defined over the referential U, but now using as an image the real interval $[0, 1]$. Any property satisfying these characteristic is said to be a *fuzzy property*, and the set that it determines is given by the following expression:

$$S_P = \{< u, \alpha >: P(x) = \alpha, u \in U, \alpha \in [0, 1]\}$$

Definition 1 ***Fuzzy set.*** *Let U be a referential set. A fuzzy subset A of U is every set of the form $A = \{(u,\alpha), u \in U, \alpha \in [0,1])\}$, that is, every set formed by the objects from U, having associated each of them some membership degree, defined in the interval $[0,1]$, to A.*

Consequently, a fuzzy set A defined over the domain of discourse U is univocally characterized using a *membership function* $\mu_A(u)$, or simply $A(u)$, which assigns any $u \in U$ to a value in the interval of real numbers between 0 and 1, representing the membership degree of the element u to A. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which u can be considered as an element of A. For example, in Figure 2.1 the elements in $[a,b]$ fully belong to A, whereas the elements in (b,c) partially belong to A.

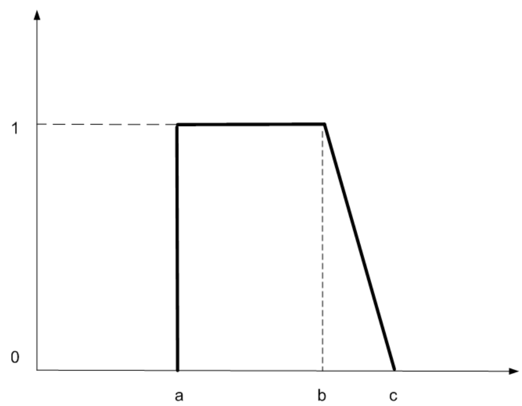

Figure 2.1: Membership function of a fuzzy set A

If the domain of discourse U is discrete ($U = \{u_1, u_2, ..., u_n\}$), the fuzzy set is usually expressed using the following notation:

$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + ... + \mu_A(u_n)/u_n$$

When U is continuous, the fuzzy set is denoted by:

$$A = \int_{u \in U} \mu_A(u)/u$$

The set of all fuzzy subsets which can be defined over a domain of discourse U is called $\tilde{\wp}(U)$. Classical sets are a special case of fuzzy sets and hence $\wp(U) \subseteq \tilde{\wp}(U)$.

2.2 Level cuts: Connection with Classical Sets

Definition 2 ***α-cut.*** *For each $\alpha \in [0,1]$ and each fuzzy set A, the α-cut of A is defined as the set of all elements of the domain of universe which have a membership degree to A which is greater or equal than α, that is:*

$$A_{\geq\alpha} = u \in U : \mu_A(u) \geq \alpha$$

The different α-cuts of a fuzzy set have an inclusion relation between them which is determined by the following property:

$$(\alpha > \beta) \Rightarrow (A_{\geq\alpha} \subseteq A_{\geq\beta})$$

Definition 3 ***Strict α-cut.*** *Analogously, for each $\alpha \in [0,1]$ and each fuzzy set A, the strict α-cut of A is defined as the set of all elements of the domain of universe which have a membership degree to A which is strictly greater than α, that is:*

$$A_{>\alpha} = u \in U : \mu_A(u) > \alpha$$

Obviously, strict α-cuts are contained in α-cuts:

$$A_{>\alpha} \subseteq A_{\geq\alpha}$$

Among the crisp sets which can be defined from a fuzzy set, there are two of special significance: the support and the core.

Definition 4 ***Support.*** *The support of a fuzzy set A defined over a domain of discourse U is the set of elements of U which have a membership degree strictly greater than 0, that is:*

$$supp(A) = \{u \in U : \mu_A(u) > 0\}$$

Definition 5 ***Core.*** *The core (or kernel) of a fuzzy set A defined over a domain of discourse U is the set of elements of U which have a membership degree equal to* 1*, that is:*

$$core(A) = \{u \in U : \mu_A(u) = 1\}$$

It is easy to see that the support of a fuzzy set A corresponds to the strict 0-cut, and that the core corresponds to the 1-cut.

Finally, Zadeh's Resolution's Identity [345] shows that a fuzzy set A can be univocally represented from its decomposition in α-cuts in the following way:

Theorem 1 ***Resolution's Identity.*** $\mu_A(u) = \sup_{\alpha \in [0,1]} \alpha \cdot A_{\geq \alpha}(u)$

More precisely, a fuzzy set A can be univocally represented from the set of all its relevant α-cuts.

Definition 6 ***Set of levels.*** *The set of values* $\alpha \in [0,1]$ *such that there exists at least one element of the domain of universe belonging to A with degree* α*, it is called the set of levels of A. Formally:*

$$\Lambda(A) = \{\alpha : \mu_A(u) = \alpha \text{ for some } u \in U\}$$

Then, it holds that:

$$\mu_A(u) = \bigcup_{\alpha \in \Lambda(A)} \alpha \cdot A_{\geq \alpha}(u)$$

2.3 Operations with Fuzzy Sets

The decomposition of fuzzy sets using α-cuts introduces a relation between classical sets and fuzzy sets which allows the operations which are performed over classical sets to be extended in such a way that they are applied over fuzzy sets. The most relevant operations over sets are union, intersection and complement. The generalization of these operations should be performed in such a way that they maintain the same behaviour when applied to classical sets.

Intersection

The intersection of two fuzzy sets is defined in the following way:

$$\mu_{A\cap B}(x) = \mu_A(x) \otimes \mu_B(x)$$

where $\otimes$ denotes a t-norm function.

Definition 7 ***T-norm.*** *A triangular norm or t-norm is a function* $\otimes : [0,1] \times [0,1] \rightarrow [0,1]$ *verifying the following properties:*

1. $\alpha \otimes 1 = \alpha$, $\forall \alpha \in [0,1]$ *(Boundary)*
2. $\beta \leq \gamma \Rightarrow \alpha \otimes \beta \leq \alpha \otimes \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Monotonicity)*
3. $\alpha \otimes \beta = \beta \otimes \alpha$, $\forall \alpha, \beta \in [0,1]$ *(Commutativity)*
4. $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Associativity)*

From this definition, it follows that every t-norm satisfies some interesting properties:

- $\alpha, \beta \geq \alpha \otimes \beta$.
- $\alpha \otimes 0 = 0$.

For example, minimum is a t-norm, and it has traditionally been the most used function in the intersection of fuzzy sets. Table 2.1 shows some examples of t-norm functions. Linear t-norm is usually called Łukasiewicz t-norm because it corresponds to Łukasiewicz logic, although it was never explicitly used by Łukasiewicz. For the same reason, minimum t-norm is also called Gödel t-norm. For a definition of some families of fuzzy operators (or fuzzy logics), see Section 2.5.

Table 2.1: Popular examples of t-norms

Name	Definition
Linear	$\max\{\alpha + \beta - 1, 0\}$
Minimum	$\min\{\alpha, \beta\}$
Product	$\alpha \cdot \beta$

A t-norm is called *idempotent* iff $\alpha \otimes \alpha = \alpha$, or *subidempotent* iff $\alpha \otimes \alpha < \alpha$, $\forall \alpha \in [0,1]$. Minimum is the only idempotent t-norm.

From a historical point of view, the concept of t-norm was proposed in 1942 by K. Menger in the frame of statistical metrics [201]. Only in the eighties, it was suggested to use t-norms to model the conjunction of fuzzy sets [6].

Definition 8 ***Residuum of a t-norm.*** *Given a left-continuous t-norm $\otimes$, there is a unique operation $\Rightarrow$, called the residuum of $\otimes$, verifying the residuation property:*

$$\alpha \otimes \beta \leq \gamma \text{ iff } \alpha \leq \beta \Rightarrow \gamma, \forall \alpha, \beta, \gamma \in [0,1]$$

and this operation is defined as:

$$\alpha \Rightarrow \beta = sup_{\gamma \in [0,1]} \{\alpha \otimes \gamma \leq \beta\}$$

Given a left-continuous t-norm $\otimes$ and its residuum $\Rightarrow$, the following identities are true:

- $\min\{\alpha, \beta\} = \alpha \otimes (\alpha \Rightarrow \beta)$
- $\max\{\alpha, \beta\} = min\{(\alpha \Rightarrow \beta) \Rightarrow \beta, (\beta \Rightarrow \alpha) \Rightarrow \alpha\}$

Union

The intersection of two fuzzy sets is defined as follows:

$$\mu_{A \cup B}(x) = \mu_A(x) \oplus \mu_B(x)$$

where $\oplus$ denotes a t-conorm function.

Definition 9 ***T-conorm.*** *A t-conorm (also known as s-norm) is a function $\oplus : [0,1] \times [0,1] \rightarrow [0,1]$ verifying the following properties:*

1. $\alpha \oplus 0 = \alpha$, $\forall \alpha \in [0,1]$ *(Boundary)*
2. $\beta \leq \gamma \Rightarrow \alpha \oplus \beta \leq \alpha \oplus \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Monotonicity)*
3. $\alpha \oplus \beta = \beta \oplus \alpha$, $\forall \alpha, \beta \in [0,1]$ *(Commutativity)*
4. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Associativity)*

From this definition, it follows that every t-conorm satisfies some interesting properties:

- $\alpha, \beta \leq \alpha \oplus \beta$.
- $\alpha \oplus 1 = 1$.

For example, maximum is a t-conorm, and it has traditionally been the most used function in the union of fuzzy sets. Table 2.2 shows some examples of t-conorm functions.

Similarly as in the case of t-norms, bounded sum, maximum, and probabilistic sum are also called Łukasiewicz, Gödel and Product t-conorms respectively, due to their relation to these logics.

Table 2.2: Popular examples of t-conorms

Name	Definition
Bounded sum	$\min\{1, \alpha + \beta\}$
Maximum	$\max\{\alpha, \beta\}$
Probabilistic sum	$\alpha + \beta - \alpha \cdot \beta$

A t-conorm is called *idempotent* iff $\alpha \oplus \alpha = \alpha$, or *superidempotent* iff $\alpha \oplus \alpha > \alpha$, $\forall \alpha \in [0, 1]$. Maximum is the only idempotent t-conorm.

Complement

The complement of a fuzzy set is defined as:

$$\mu_{\bar{A}}(x) = \ominus \mu_A(x)$$

where $\ominus$ represents a negation function [315].

Definition 10 ***Negation function.*** *A negation (or complement) function $\ominus : [0, 1] \rightarrow [0, 1]$ is a function satisfying the following properties:*

1. *$\ominus 0 = 1$, $\ominus 1 = 0$ (Boundary)*
2. *$\alpha \leq \beta \Rightarrow \ominus \alpha \geq \ominus \beta$, $\forall \alpha, \beta \in [0, 1]$ (Monotonicity)*

The following properties are usually interesting from a practical point of view:

1. $\ominus$ is a continous function (Continuity)
2. $\ominus(\ominus\alpha) = \alpha$, $\forall\alpha \in [0,1]$ (Involution)

Table 2.3 shows some examples of negation functions. The standard (or Łukasiewicz) negation is continuous and involutive, while the Gödel negation is non continuous and non involutive.

Table 2.3: Popular examples of negation functions

Name	Definition
Standard	$1-\alpha$
Gödel	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha > 0 \end{cases}$
Threshold	$\begin{cases} 1 & \text{if } \alpha \leq threshold \\ 0 & \text{if } \alpha > threshold \end{cases}$

Set Inclusion

Definition 11 ***Inclusion of fuzzy sets.*** *A fuzzy set A is included in (or, alternatively, a subset of / contained in / less or equal than) a fuzzy set B if its membership function takes no higher values over all elements of the domain of discourse:*

$$A \subseteq B \Leftrightarrow \forall u \in U, \mu_A(u) \leq \mu_B(u)$$

This definition of fuzzy set inclusion is usually referred to Zadeh's inclusion of fuzzy sets [343]. Note that according to this definition, set inclusion is a yes-no question. In order to overcome this limitation, several alternative definitions have been proposed (for instance, Definition 22).

Fuzzy Modifiers

Definition 12 ***Fuzzy modifier.*** *A fuzzy modifier (or hedge) mod is an operator which transforms a fuzzy set into another, that is, an application* $f_{mod} : [0,1] \rightarrow [0,1]$.

Fuzzy modifiers can be used to define the formal semantics for a linguistic term such as *very*, *few*, *enough*, *more or less*, *around*, *about* ...

Table 2.4 collects some examples of functions which have been used to define fuzzy modifiers.

Table 2.4: Popular examples of modifiers

Name	Definition
Triangular	$tri(x;t_1,t_2,t_3) = \begin{cases} (x-t_1)/(t_2-t_1) & \text{if } x \in [t_1,t_2] \\ (t_3-x)/(t_3-t_2) & \text{if } x \in [t_2,t_3] \\ 0 & \text{if } x \in [0,t_1] \cup [t_3,1] \end{cases}$
Lineal	$lin(x;l) = \begin{cases} (l_2/l_1)x & x \in [0,l_1] \\ 1-(x-1)(1-l_2)/(1-l_1) & x \in [l_1,1] \end{cases}$
Exponential	$exp(x;e) = x^e$

2.4 Fuzzy Relations

L. A. Zadeh also generalized the notion of relation to the fuzzy case, making use of the new notion of fuzzy set [343].

Definition 13 ***Fuzzy relation.*** *A fuzzy (binary) relation R is a fuzzy subset of the cartesian product of two domains of discourse U and V:*

$$R : U \times V \to [0,1]$$

More generally, an n-ary fuzzy relation is a fuzzy subset R of the product space $U_1 \times U_2 \times \cdots \times U_n$:

$$R : U_1 \times U_2 \times \cdots \times U_n \to [0,1]$$

Definition 14 ***Composition of fuzzy relations.*** *Let $R_1,\ldots,R_n$ be fuzzy relations. The composition of them is an operation defined as follows:*

$$R_1 \circ \cdots \circ R_n = \sup_{u_1 \in U_1,\ldots,u_{n+1} \in U_{n+1}} R_1(u_1,u_2) \otimes \cdots \otimes R_n(u_n,u_{n+1})$$

In the rest of this document we will consider fuzzy relations such that $U_1 = U_2 = \cdots = U_n$.

Some Properties of Fuzzy Relations

Given a fuzzy relation R, it is interesting to know if the following properties are verified.

Definition 15 ***Transitivity.*** *R is transitive iff*

$$\forall u_1, u_2 \in U, R(u_1, u_2) \geq \sup_{u_i \in U} R(u_1, u_i) \otimes R(u_i, u_2)$$

Definition 16 ***Reflexivity.*** *R is reflexive iff*

$$\forall u \in U, R(u, u) = 1$$

Definition 17 ***Irreflexivity.*** *R is irreflexive iff*

$$\forall u \in U, R(u, u) = 0$$

Definition 18 ***Symmetry.*** *R is symmetric iff*

$$\forall u_1, u_2 \in U, R(u_1, u_2) = R(u_2, u_1)$$

Definition 19 ***Asymmetry.*** *R is asymmetric iff*

$$\forall u_1, u_2 \in U, R(u_1, u_2) > 0 \textit{ implies } R(u_2, u_1) = 0$$

Definition 20 ***Disjointness of fuzzy relations.*** *We say that two fuzzy relation R_1, R_2 are disjoint iff*

$$\forall u_1, u_2 \in U, R(u_1, u_2) = 0 \textit{ or } R(u_2, u_1) = 0$$

Implication Functions

A specially interesting case of fuzzy relations is that of fuzzy implications.

Definition 21 ***Fuzzy implication.*** *A fuzzy implication is a function* $\Rightarrow: [0,1] \times [0,1] \to [0,1]$ *verifying the following properties:*

1. $\alpha \leq \beta$ *implies* $\alpha \Rightarrow \gamma \geq \beta \Rightarrow \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Antitonicity)*
2. $\beta \leq \gamma$ *implies* $\alpha \Rightarrow \beta \leq \alpha \Rightarrow \gamma$, $\forall \alpha, \beta, \gamma \in [0,1]$ *(Monotonicity)*
3. $0 \Rightarrow \alpha = 1$, $\forall \alpha \in [0,1]$ *(Boundary)*

4. $\alpha \Rightarrow 1 = 1, \forall \alpha \in [0,1]$ *(Boundary)*
5. $1 \Rightarrow 0 = 0$ *(Boundary)*

Recall that according to Definition 11, set inclusion is a yes-no question. In order to overcome this limitation, different alternatives have been proposed to define the set inclusion. For instance, the following definition uses a fuzzy implication function $\Rightarrow$:

Definition 22 ***Degree of inclusion of fuzzy sets.*** *The degree of inclusion $Inc(A,B)$ of a fuzzy set A in a fuzzy set B is defined as:*

$$Inc(A,B) = \inf_{u \in U} A(u) \Rightarrow B(u)$$

There are several ways of obtaining implication functions satisfying the requisites of Definition 21. According to how these functions are obtained, it is common to consider two types of fuzzy implications:

- **R-implications** (Residuated implications), which correspond to the residuum of a left continuous t-norm:

$$\alpha \Rightarrow_R \beta = sup_{\gamma \in [0,1]} \{\alpha \otimes \gamma \leq \beta\}$$

- **S-implications** (Strong implications), which extend the proposition $a \to b = \neg a \vee b$ to the fuzzy case, and are obtained from a t-conorm and a negation function as follows:

$$\alpha \Rightarrow_S \beta = (\ominus \alpha) \oplus \beta$$

R-implications verify some interesting properties:

- $\alpha \Rightarrow \beta = 1$ if and only if $\alpha \leq \beta$ (Ordering)
- A negation function can be defined as $\ominus a = a \Rightarrow 0$.
- If a proposition ϕ is true to degree α and $\phi \Rightarrow \varphi$ is true to degree β, then φ is true to degree $\alpha \otimes \beta$, where $\otimes$ is the corresponding left continuous t-norm (*Modus ponens*).

On the other hand, S-implications verify the following property:

- $\alpha \Rightarrow \beta = (\ominus \beta) \Rightarrow (\ominus \alpha)$ for some negation function $\ominus$ (Contraposition)

Table 2.5 shows some examples of implication functions.

Table 2.5: Popular examples of implication functions

Name	Type	Definition
Goguen	R-implication	$\begin{cases} 1 & \alpha \leq \beta \\ \beta/\alpha, & \alpha > \beta \end{cases}$
Gödel	R-implication	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{if } \alpha > \beta \end{cases}$
Łukasiewicz	R-implication, S-implication	$\min\{1, 1-\alpha+\beta\}$
Kleene-Dienes (KD)	S-implication	$\max\{1-\alpha, \beta\}$
Reichenbach	S-implication	$1-\alpha+\alpha\cdot\beta$

2.5 Fuzzy Logic (in a Narrow Sense)

The distinguishing characteristic of fuzzy logic is that everything is (or can be) a matter of degree. However, a common source of confusion is that the term fuzzy logic is used in two different senses:

- A *narrow sense*, in which fuzzy logic is a logical system which is a generalization of many-valued logic. Fuzzy logic in a narrow sense is usually referred to as *mathematical fuzzy logic*. However, even in its narrow sense the agenda of fuzzy logic is very different from the agendas of multi-valued logical systems.
- A *wide* (or *broad*) *sense* where fuzzy logic is the logic of classes with unsharp boundaries, including fuzzy set theory, possibility theory, calculus of fuzzy if-then rules, fuzzy arithmetic, calculus of fuzzy quantifiers and related concepts and calculi. Fuzzy logic in a narrow sense is a branch of fuzzy logic in a wide sense.

Today, fuzzy logic is mostly used in its wide sense. However, this section is devoted to the formal background of fuzzy logic in a narrow sense, that is, the study of many-valued logical systems aiming at a formalization of approximate reasoning. Fuzzy logics are truth-functional: the degree of truth of a formula can be computed from the degrees of truth of its constituents.

We will focus on the so-called t-norm based fuzzy logics, which correspond to $[0,1]$-valued calculi defined by a left-continuous t-norm, giving semantics to the strong conjunction connective, and its residuum. The residuum is an R-implication and specifies a negation as $\ominus\alpha = \alpha \Rightarrow 0$. Then, a t-conorm is defined by using duality with the t-norm, as $\alpha \oplus \beta = \ominus((\ominus\alpha) \otimes (\ominus\beta))$.

The three main families of fuzzy operators are Łukasiewicz [179] (denoted Ł), Gödel [104] (denoted G) and Product [105] (denoted Π). These three cases are important since each continuous t-norm is definable as an ordinal sum of copies of Łukasiewicz, Gödel and Product t-norms (Mostert-Shields theorem [203]). As a matter of fact, Łukasiewicz and Gödel infinitely-valued logics, were defined much before fuzzy logic was born. Jan Łukasiewicz proposed in 1930 the so-called Łukasiewicz logic [179], while in 1932 Kurt Gödel proposed the fuzzy operators of the so-called Gödel logic [104].

An S-implication $\Rightarrow$ also makes possible to specify a family of fuzzy operators. $\Rightarrow$ determines a negation and a t-conorm since $\alpha \Rightarrow \beta = \ominus\alpha \oplus \beta$, and a t-norm can be defined from them by using duality as $\alpha \otimes \beta = \ominus((\ominus\alpha) \oplus (\ominus\beta))$. For example, under Kleene-Dienes implication we get what we call here Zadeh family (the fuzzy operators originally considered by L. A. Zadeh [343], and partially inspired by Kleene's many-valued logics [156]).

Table 2.6 groups these main families of fuzzy operators: Zadeh, Łukasiewicz, Gödel and Product.

Table 2.6: Popular families of fuzzy operators.

Family	**t-norm**	**t-conorm**	**negation**	**implication**
Zadeh	$\min\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$1 - \alpha$	$\max\{1 - \alpha, \beta\}$
Łukasiewicz	$\max\{\alpha + \beta - 1, 0\}$	$\min\{\alpha + \beta, 1\}$	$1 - \alpha$	$\min\{1, 1 - \alpha + \beta\}$
Gödel	$\min\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$	$\begin{cases} 1 & \alpha \leq \beta \\ \beta, & \alpha > \beta \end{cases}$
Product	$\alpha \cdot \beta$	$\alpha + \beta - \alpha \cdot \beta$	$\begin{cases} 1, & \alpha = 0 \\ 0, & \alpha > 0 \end{cases}$	$\begin{cases} 1 & \alpha \leq \beta \\ \beta/\alpha, & \alpha > \beta \end{cases}$

Let $\otimes$, $\oplus$, $\ominus$ and $\Rightarrow$ denote the fuzzy operators of Łukasiewicz family (t-norm, t-conorm, negation and implication, respectively) and let $\wedge$, $\vee$, $\neg$ and $\rightarrow$ denote the fuzzy operators of Zadeh family. Interestingly, using the Łukasiewicz family it is possible to represent the operators of Zadeh family:

$$\begin{aligned}\neg\alpha &= \ominus\alpha\\ \alpha\wedge\beta &= \alpha\otimes(\alpha\Rightarrow\beta)\\ \alpha\vee\beta &= \neg((\neg\alpha)\wedge(\neg\beta))\\ \alpha\rightarrow\beta &= (\neg\alpha)\vee\beta\end{aligned}$$

There are many other important fuzzy logics, and we will only cite here a few outstanding examples:

- P. Hájek introduced the Basic Fuzzy logic [121] (BL) as a common fragment of Łukasiewicz, Gödel and Product logics.
- F. Esteva and L. Godo proposed the Monoidal t-norm-based logic [99] (MTL), a weaker logic than BL corresponding to the logic of left-continuous t-norms.
- Rational Pavelka logic (RPL) [235], originally proposed by J. Pavelka, extends Łukasiewicz logic with truth constants (rational numbers in $[0,1]$). Several expansions with truth-constants of other fuzzy logics have also been proposed (see [92] for a collection of references).
- ŁΠ combines operators from Łukasiewicz and Product logics [100], allowing operators of Gödel logic to be defined.

The large number of different logics proposed raises the question of what the best fuzzy logic is. This question has been discussed in [216], but with no definite answer.

2.6 Fuzzy Numbers

In the same way as precise numerical quantities are represented by real numbers, fuzzy numbers are useful to represent imprecise or vague quantities. Although several definitions of fuzzy number have been proposed in the literature, we will consider the following one [96].

Definition 23 ***Fuzzy number.*** *A fuzzy number F is a fuzzy subset of the real line $F:\mathbb{R}\rightarrow[0,1]$ which is normal and convex:*

1. $\exists x\in\mathbb{R}:\mu_F(x)=1$ *(Normality)*
2. $\mu_F(\lambda\cdot x_1+(1-\lambda)\cdot x_2)\geq\min\{\mu_F(x_1),\mu_F(x_2)\}$, $x_1,x_2\in[0,1]$ *(Convexity)*

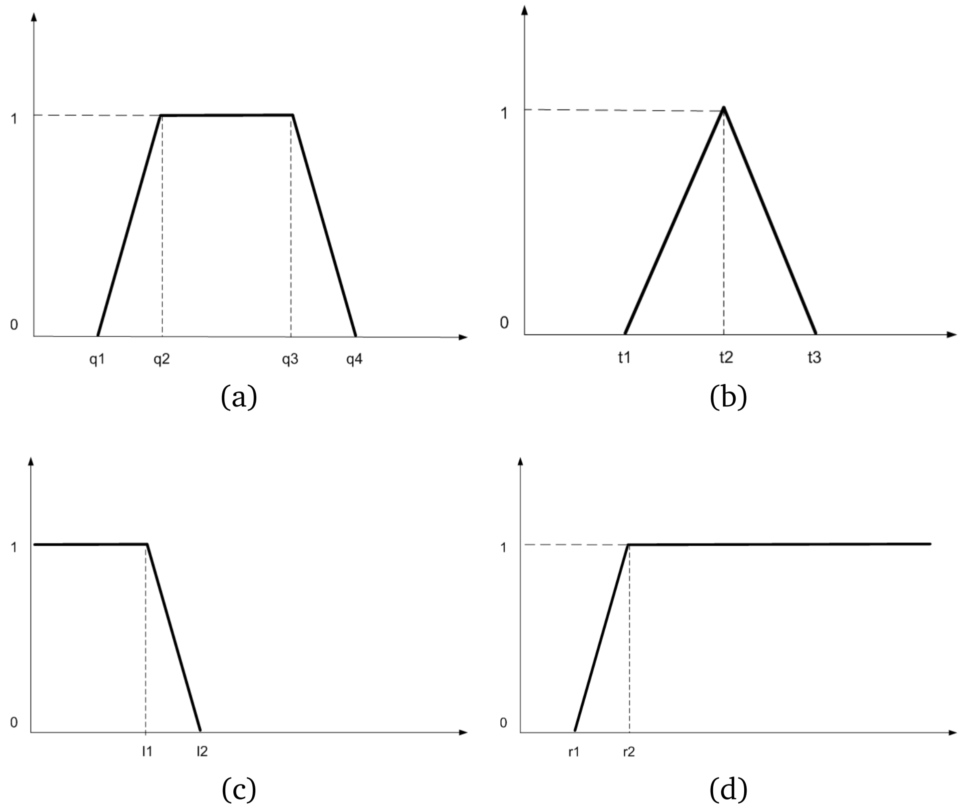

Figure 2.2: Popular examples of fuzzy numbers: (a) Trapezoidal; (b) Triangular; (c) Left-shoulder function; (d) Right-shoulder function

Some interesting cases of fuzzy numbers are triangular and trapezoidal fuzzy numbers (although they are actually intervals), as well as left and right shoulder functions (see Figure 2.2).

Definition 24 ***Trapezoidal fuzzy number.*** *A trapezoidal fuzzy number* $Q = trap(q_1, q_2, q_3, q_4)$ *is a fuzzy number which is determined by a quadruple of real numbers* (q_1, q_2, q_3, q_4) *such that:*

1. $q_1 \leq q_2 \leq q_3 \leq q_4$
2. $\mu_Q(u) = (u - q_1)/(q_2 - q_1), \forall u \in [q_1, q_2]$
3. $\mu_Q(u) = 1, \forall u \in [q_2, q_3]$
4. $\mu_Q(u) = (q_4 - u)/(q_4 - q_3), \forall u \in [q_3, q_4]$

5. $\mu_Q(u) = 0, \forall u \in [k_1, q_1] \cup [q_4, k_2]$

Definition 25 ***Triangular fuzzy number.*** *A triangular fuzzy number* $T = tri(t_1, t_2, t_3)$ *is a fuzzy number which is determined by a triple of real numbers* (t_1, t_2, t_3) *such that:*

1. $t_1 \leq t_2 \leq t_3$
2. $\mu_T(u) = (u - t_1)/(t_2 - t_1), \forall u \in [t_1, t_2]$
3. $\mu_T(u) = (t_3 - u)/(t_3 - t_2), \forall u \in [t_2, t_3]$
4. $\mu_T(u) = 0, \forall u \in [k_1, t_1] \cup [t_3, k_2]$

Definition 26 ***L-function.*** *A left-shoulder function or L-function* $L = L(l_1, l_2)$ *is a fuzzy number which is determined by a pair of real numbers* (l_1, l_2) *such that:*

1. $l_1 \leq l_2$
2. $\mu_L(u) = 1, \forall u \in [k_1, l_1]$
3. $\mu_L(u) = (l_2 - u)/(l_2 - l_1), \forall u \in [l_1, l_2]$
4. $\mu_L(u) = 0, \forall u \in [l_2, k_2]$

Definition 27 ***R-function.*** *A right-shoulder function or R-function* $R = R(r_1, r_2)$ *is a fuzzy number which is determined by a pair of real numbers* (r_1, r_2) *such that:*

1. $r_1 \leq r_2$
2. $\mu_R(u) = 0, \forall u \in [k_1, r_1]$
3. $\mu_R(u) = (r_2 - u)/(r_2 - r_1), \forall u \in [r_1, r_2]$
4. $\mu_R(u) = 1, \forall u \in [r_2, k_2]$

Notice that a trapezoidal fuzzy number defined over an interval $[k_1, k_2]$ (instead of over the real line) can be used to represent the other previous fuzzy numbers:

- A triangular fuzzy number $tri(t_1, t_2, t_3)$ can be represented using a trapezoidal fuzzy number $trap(t_1, t_2, t_2, t_3)$.

- A L-function $L(l_1, l_2)$ can be represented using a trapezoidal fuzzy number $trap(k_1, k_1, l_1, l_2)$.
- A R-function $R(r_1, r_2)$ can be represented using a trapezoidal fuzzy number $trap(l_1, l_2, k_2, k_2)$.
- A crisp number x can be represented using a trapezoidal fuzzy number $trap(x, x, x, x)$.

2.7 Possibilistic Logic

Possibilistic logic is a weighted logic which aims to enable reasoning with uncertain knowledge using possibility and necessity measures. It was introduced by L. Zadeh in 1978 [342] , although previously (1961) G. L. S. Shackle introduced a degree of impossibility as the degree of necessity of the opposite event [264]. This section contains the basic notions that will be used in this document; for a larger introduction the reader is referred to [93].

Vagueness and Uncertainty

Fuzzy and possibilistic logics have proved to be suitable formalisms to handle imprecise/vague and uncertain knowledge respectively. Imprecision/vagueness and uncertainty are very different conceptually. The former case covers all those approaches where statements are true to some degree, rather than being just either true or false. For example, “it is hot today”. In the latter case fall all those approaches in which statements are true or false in any world, but there is no knowledge about which world is the right one. For instance, “it will rain tomorrow”. In this case, it is only possible to say that statements are true or false to some degree of certainty (for example, to some probability or possibility).

Fuzzy and possibilistic logics are orthogonal, the former handling degrees of truth and the latter handling degrees of certainty. Despite that they are very different, there has been a confusion in the literature of Artificial Intelligence between their roles, meaning and properties. A good clarification paper pointing out the differences between them is [97][2].

An example proposed in [97] to emphasize the differences is that of a bottle. In terms of classical logic, a bottle is viewed as full or empty. Taking into account the amount of liquid in the bottle, one can say that the bottle is "half-full"[3]. In this sentence, "full" is a fuzzy predicate and the degree of truth of the fuzzy proposition "the bottle is full" represents how much liquid there is in the bottle. A very different case consists of expressing our ignorance about the state of the bottle, whether it is either full or empty, assuming that only one of the two situations is possible. Saying that the probability that the bottle is full is $\frac{1}{2}$ does not mean that the bottle is half full.

Actually, this misunderstanding has an old origin. Even in the three-valued propositional logic of Łukasiewicz [178], proposed to solve the problem of contingent futures by omitting the principle of contradiction in logic, the third truth-value is often wrongly understood as a lack of knowledge about truth, instead of an intermediate value between true and false.

Another well-known formalism for the management of uncertainty knowledge is *probabilistic logic* [214]. However, they represent different facets of uncertainty. In probability theory, the probability of an event is the sum of the probabilities of all the worlds satisfying the event. On the other hand, the possibility of an event is the maximum of the possibilities of all the worlds satisfying the event, thus taking the most optimistic world. Possibilistic logic is based on the idea of *ordering* elements along an ordinal scale, rather than *counting* as it happens with probabilistic logic. Moreover, reasoning in possibilistic logic usually requires less computational effort.

Note that while in fuzzy logic statements are truth-functional, uncertain statements cannot be a function of the uncertainties of their constituents [95]. In particular, in possibility theory only the disjunction in the possibility of an event and only the conjunction in the necessity of an event are truth-functional.

[2] However, the distinction was noticed many time before [82].
[3] or "half-empty", depending on how optimistic you are.

Possibility Distributions

According to L. Zadeh, a variable is a more general notion than the notion of propositional variable in logic, and is used to represent some piece of information. The ill-known value of these variables can be associated with possibility distributions or distributions mapping the domain of the variable to the interval $[0,1]$. Let x be a variable defined over a universe of discourse U. A possibility distribution π_x associated to x is a function $\pi\colon U \to [0,1]$ expressing to what extent it is plausible or possible that x takes a particular value. This value, although unknown, is unique. $\pi_x(u)$ represents the possibility degree that x takes value u $(x = u)$ for some $u \in U$.

If $\pi_x(u) = 0$, it means that it is absolutely impossible that x takes value u. The larger $\pi_x(u)$, the larger u is considered to be a more possible (or in fact, less impossible) value of x. In the extreme case, if $\pi_x(u) = 1$, there is nothing preventing u to be considered as a possible value for x, although there may exist other values u' such that $\pi_x(u') = 1$.

Since in the real world knowledge is often expressed linguistically, Zadeh uses fuzzy sets to represent incomplete pieces of information about the value of a variable. Let A be a fuzzy subset of U. A is used to represent an incomplete piece of information about the value of x. The membership degree attached to a value represents to what extent if is possible that this value is indeed the value of the variable.

If the only available knowledge about x is that "x is A", then A is interpreted as a possibility distribution representing the levels of plausibility of the possible values of x. The possibility that $x = u$ is given by the compatibility of u with the concept expressed by A (i.e., the membership degree of u to A):

$$\pi_x(u) = \mu_A(u), \forall u \in U$$

In general, π_x ranges on $[0,1]$, but the range of a possibility distribution can be any linearly ordered scale bounded by a bottom and a top element. Note that "x is A" does not mean that possibility distribution π_x is the same as the membership function μ_A. The equality $\pi_x = \mu_A$ means that given that the only available knowledge is "x is A", the degree of possibility that $x = u$ is evaluated by the degree of membership $\mu_A(u)$.

The fuzzy set A is viewed as a fuzzy restriction which serves as an elastic constraint on the value of x. For example, the sentence "John is tall" is considered as a piece of incomplete evidence, and is supposed to be the only available information about x. This is completely different to a situation where the value of x is known (e.g., $x = 180$ cm) and "tall" is used as a linguistic substitute to this value.

Possibility and Necessity Measures

The extent to which the information "x is A", represented by the possibility distribution $\pi_x = \mu_A$, is consistent with a statement like "the value of x is in subset P", for a crisp set $P \subseteq U$, is estimated by means of the possibility measure Π.

Definition 28 ***Possibility measure.*** *A possibility measure is a function* $\Pi\colon U \to [0,1]$ *satisfying:*

1. $\Pi(U) = 1$,
2. $\Pi(\emptyset) = 0$,
3. $\Pi(p \vee q) = \max\{\Pi(p), \Pi(q)\}$,

The degree of possibility is computed by the equation:

$$\Pi(P) = sup_{u \in P} \pi_x(u)$$

$\Pi(P)$ estimates the consistency of the statement $x \in P$ with what we know about the possible values of x. $\Pi(P) = 0$ means $x \in P$ is impossible knowing that "x is A". The value of $\Pi(P)$ corresponds to the elements of P having the greatest possibility degree according to π_x.

A necessity measure N can be defined as the dual measure of the possibility, since when the negation of P is impossible, then P is certain. Since they are dual, the measures verify:

$$N(P) = 1 - \Pi(\bar{P})$$
$$\Pi(P) = 1 - N(\bar{P})$$

Definition 29 ***Necessity measure.*** *A necessity measure is a function $N : U \rightarrow [0,1]$ satisfying:*

1. $N(U) = 1$,
2. $N(\emptyset) = 0$,
3. $N(p \wedge q) = \min\{N(p), N(q)\}$,

The degree of necessity is computed by the equation:

$$N(P) = inf_{u \notin P} 1 - \pi_x(u)$$

Some interesting properties of these measures are the following:

- The larger the possibility $\Pi(P)$, the smaller the necessity of the contrary event $N(\bar{P})$,
- The larger the necessity $N(P)$, the smaller the possibility of the contrary event $\Pi(\bar{P})$,
- $\Pi(P) = 1 \Leftrightarrow N(\bar{P}) = 0$. If an event is fully possible, then its contrary event cannot be necessary.
- $N(P) = 1 \Leftrightarrow \Pi(\bar{P}) = 0$. If an event is fully necessary, then its contrary event is fully impossible.
- $N(P) = 1 \Rightarrow \Pi(P) = 1$. If an event is fully necessary, then it must be possible. However, the reciprocal is not true.

CHAPTER 3

Ontologies and Description Logics

This chapter presents some basic notions on ontologies and Description Logics. Firstly, Section 3.1 discusses what ontologies are, providing an introductory historical note, a short route around the existing definitions, a description of their elements, a summary of the advantages of the use and a revision of some key applications. Then, Section 3.2 focuses on description logics, as one of the most successful logical formalisms behind ontologies. We also begin with a historical note before analyzing how they structure the knowledge. Then, we explore some important examples of logics and different approaches to reasoning with them. A particular description logic, $\mathcal{SROIQ}(\mathbf{D})$, is formally defined in Section 3.3. Finally, Section 3.4 explores the use of description logics as ontology languages, describing in detail the language OWL 2.

3.1 Ontologies

Introduction

The development of a Knowledge-Based System requires the development of a knowledge base. Unfortunately, knowledge acquisition is a difficult and complex process due to many reasons. Therefore, the possibility of reusing components from different Knowledge-Bases Systems and sharing the representations of an application domain that different agents have, seems to be very promising. Furthermore, Knowledge-Based Systems (as any other piece of software) are usually developed using their own vocabulary and assumptions about the modeled world. This leads to communication problems due to a lack of mutual understanding.

In 1991, the ARPA Knowledge Sharing Effort proposed to build Intelligent Systems by assembling reusable components [212], which meant a revolution in the way these systems were built. This way, system developers would not need to start from scratch, but they would be able to reuse the general knowledge and to concentrate on the specific knowledge of their system.

In this scenario, ontologies are developed in order to make knowledge interchange and reuse easier, as well as to interoperate with existing systems. Ontologies turn also to be very useful in such applications where it is crucial to have an explicit representation of the assumptions that we humans handle in an easy and natural way, but that are hard to represent using classical tools from Knowledge Engineering. And, most importantly, they promise to help to achieve an effective communication among humans and machines thanks to a mutual understanding.

Ontologies are becoming more and more popular nowadays and since their apparition, ontology engineering has received an important attention and has been the object of a lot of study in the field of Artificial Intelligence (see for example [279]).

Definition

The term ontology has been taken from philosophy, from the subfield that is concerned with the study of being or existence [50]. An ontology is defined there as the study of part-of relationships and entity dependencies. For example, the observation that the world is made up of objects that can be grouped into abstract classes based on shared properties us a typical ontological commitment [8]. The name comes from the juxtaposition of the Greek terms οντος and λογος, which denote *being* and *study*, respectively.

In Artificial Intelligence, the term a very different meaning. A lot of definitions have been proposed, but we shall cite here only some of the most relevant:

- Although ontologies were already used in the 1980's, probably the first definition was proposed in 1991 by R. Neches et al. (1991), for whom *"an ontology defines the basic terms and relations comprising the vocabulary of a topic area as well as the rules for combining terms and relations to define extensions to the vocabulary"* [212].

- The most widely cited definition in literature is probably that provided by T. Gruber (1993), who defined an ontology as *"an explicit specification of a conceptualization"* [107].

- N. Guarino et al. (1995) defined an ontology as *"a logical theory which gives an explicit, partial account of a conceptualization"* [110].

- Later on (1997) W. N. Borst slightly altered Gruber's definition, stating that an ontology is *"a formal specification of a shared conceptualization"* [45].

- R. Studer, V. R. Benjamins y D. Fensel interpreted this definition in the following way [308]:

  - "Conceptualization *refers to an abstract model of some phenomenon in the world by having identified the relevant concepts of that phenomenon*.

  - Explicit *means that the type of concepts used, and the constraints on their use are explicitly defined*.

  - Formal *refers to the fact that the ontology should be machine-readable*.

  - Shared *reflects the notion that an ontology captures consensual knowledge, that is, it is not private of some individual, but accepted by a group"*.

In summary, the literature provides with a wide range of definitions of the term ontology, each of them offering different and complementary points of view (see [69] for a comparative view of them).

But one should have into account that many philosophical theories concerning the construction of ontologies can provide useful guidelines for computer scientists [256]. Actually, Aristotle was responsible of the definition of a predecesor of a universal ontology [9]. Some authors even consider computational ontology as a kind of applied philosophy [278].

Components

An ontology offers some type of formalism to represent the following elements:

- *Classes* or *concepts*. They are collections of objects of the domain, representing the basic ideas in the world which is being formalized. Typically an ontology can be viewed as a taxonomic hierarchy where some concepts are explained by using others.
- *Individuals*, representing particular elements or instances of a given class.
- *Relations*, which represent interactions among individuals of the domain and determine the ontology structure. Relations are usually binary, but in general they can have n arguments. Formally, an n-ary relation R is defined as a subset of the Cartesian product of n classes $C_1, \ldots C_n$:

  $$R \subseteq C_1 \times C_2 \times \cdots \times C_n$$

  There can also be relations among concepts. Typically, concepts are structured in a concept hierarchy by means of a *taxonomic* relation, but there can be non-taxonomic relations (for example, meronimy relations of the form partOf).
- *Functions*, which constitute a special case of relations where the n-th argument has always the same value for the $n-1$ previous arguments. Formally, a function F is defined as:

  $$F : C_1 \times C_2 \times \cdots \times C_{n-1} \to C_n$$

- *Attributes*, which describe properties of the individuals of a class. They are usually called *datatypes* or *concrete domains*.
- *Axioms*, formally expressing expressing conditions to be verified by the elements of the ontology in order to guarantee its correctness, and allowing to infer new knowledge, not explicitly represented in the ontology.

Advantages

The use of ontologies offers a lot of benefits, such as the following ones:

- They offer a way to represent and share knowledge by using a *common vocabulary*. Since ontologies are, by definition, agreed among different parts, they are closer to become an standard than other knowledge models. Hence, they provide a format for exchanging knowledge and a specific communication vocabulary, thus promoting interoperability.
- They promote *knowledge reuse* and *information integration*, simplifying the development process of knowledge bases, taking advantage of previous experiences with the knowledge and making possible an automatic validation of the acquired knowledge.
- They make possible to maintain a separation between *declarative and procedural knowledge*, making assumptions about the domain explicit, promoting the modularity of the knowledge base.
- Ontologies allow knowledge to be used by *intelligent applications*, due to its formal basis and the fact that they can be automatically accessed by agents.

Applications

Ontologies have been used to build the knowledge base of expert systems and, in general, knowledge-based systems due to the advantages already mentioned, specially their ability to promote reuse and modularity. Ontologies have been used in different domains of application such as education, computer software, government, business services, life sciences, communications, media, healthcare providers, financial services, etc. [64].

Ontologies are very important in multi-agent systems, enabling automatic communication among agents by supplying knowledge sharing [89]. They have also been applied to information retrieval, since they allow to switch from a keyword-based to a much more effective content-based information retrieval [117]. Their formal nature makes them useful for word disambiguation in natural language processing [106].

But perhaps the best-known application is the *Semantic Web*, which was envisioned in 2001 by T. Berners-Lee et al. as a solution to the limitation of the current web of being only understood by humans. The Semantic Web is an extension of the current one, in which information is given well-defined meaning, better enabling computers and people to work in cooperation [24]. Semantic Web (sometimes also called *Web 3.0*) makes the automation of tasks easier. Some examples of Semantic Web use cases are grouped in [331].

In order for web resources to be directly accessed for automatic processes, they have to be annotated with metadata describing the content of the documents. Ontologies are a core element in the layered architecture of the Semantic Web as a source of shared and precisely defined terms which can be used as metadata for other resources.

3.2 Description Logics

Introduction

Description Logics (DLs) are a family of logics for representing structured knowledge (see [13] for a thorough reference).

From a historical point of view, DLs were born to provide a semantics for semantic works and frames, two classical models of structured knowledge representation. In order to overcome their lack of semantics, it was observed that they could be given a semantics by using *First Order Logic* (FOL) [123]. But, since in most of the cases frames and semantic networks do not need the full expressivity of FOL, DLs were born as a subset of FOL limiting the expressivity in exchange of a lower complexity in the reasoning. It should be noted, however, that there are some DLs which are not subsets of FOL (for instance, [63]).

Initially, they were called *terminological systems*, emphasizing the fact that they are used to define the basic terminology of an application domain. Then, the label *description logics* prevailed, reflecting the fact that the domain is described using complex concept *descriptions* and that they have a *logic*-based semantics.

The different constructors supported in a DL lead to different fragments of FOL, with different computational complexities. The different logics in the family are related through the following principle: the more expressive the language, the most difficult the reasoning [46].

The history of DLs can be summarized in the following phases:

- *Phase 0* (1965–1980). Other models for the representation of structured knowledge (such as semantic networks and frames) were used for the representation of structured knowledge, until their lack of a formal semantics was reported (see e.g. [123]).

- *Phase 1* (1980–1990) included incomplete systems based on structural algorithms. The first DL system took place in this place and was KL-ONE [47].

- *Phase 2* (1990–1995) included the development of tableau algorithms and the study of the complexity of the logics, their relation with modal logics [259] and optimization techniques.

- *Phase 3* (1995–2000) included the development of tableau algorithms for expressive DLs and the implementation of optimized reasoners, such as RACER [111] or FACT [136]).

- *Phase 4* (2000–...). This phase includes mature and commercial implementation of the reasoners (such as RACERPRO, PELLET [277] or FACT++ [135, 316]) and a big range of applications and tools. Less expressive but tractable DLs (which allow to manage very large KBs) have also grown attention.

Knowledge Bases

A DL consist of a Knowledge Base (KB) and some reasoning capabilities. This subsection covers the KB, and the next one is devoted to the reasoning services.

The vocabulary of a DL contains three types of elements: *individuals*, *concepts* and *roles*. Concepts denote unary sets of individuals, and roles denote binary relations between individuals.

A KB comprises different types of knowledge, grouped in different componentes:

- Extensive knowledge, or particular knowledge about some specific situation. This type of knowledge is contingent (it depends on several circumstances and is subject to changes) and is stored in an *Assertional Box* (or simply ABox).
- Intensive knowledge, or general knowledge about the application domain, which is usually unchangeable. This type of knowledge is structured in a *Terminological Box* (or TBox) with information about the concepts, and a *Role Box* (or RBox) with information about the roles.

DLs have a formal semantics based on the notion of interpretation, where the interpretation domain can be arbitrarily chosen (and can be infinite). Differently from other models such as databases, DLs are based on the *Open World Assumption*. A very nice example to illustrate the differences between these two formalisms may be found in [16].

Example 1 *According to Greek mythology, Oedipus killed his father, married his mother Iokaste, and had children with her, among them Polyneikes. Finally, also Polyneikes had children, among them Thersandros, which is known to not be a patricide.*

Suppose that one wants to know whether Iokaste has a child that (i) *is a patricide and,* (ii) *has a child that is not a patricide.*

Under Close World Assumption, we do not know that Polyneikes is a not a patricide, so the answer to the question is no. On the other hand, under Open World Assumption, either if Polyneikes is a patricide or not, the answer is yes. In the former case, the child is Polyneikes, while in the second case the child is Oedipus. □

Some Interesting Description Logics

In this subsection we will recall some of the most important DLs without getting too much into details of the particular syntax and semantics (Section 3.3 will formally define the DL $\mathcal{SROIQ}(\mathbf{D})$. Each DL is denoted by using a string of capital letters which identify the constructors of the logic, and therefore its complexity. We will firstly present the constructors of the logic $\mathcal{AL}$ (*$\mathcal{A}$ttributive $\mathcal{L}$anguage*) and then we will describe some extensions.

The concept constructors of $\mathcal{AL}$ are the following:

- Top and bottom concepts.
- Atomic concepts.
- Negation of atomic concepts.
- Concept intersection.
- Universal quantifications.
- Existential restrictions, but restricted to the top concept.

$\mathcal{AL}$ can be extended with several concept or role constructors, as shown in Table 3.1.

Label	Constructor
$\mathcal{U}$	concept **U**nion
$\mathcal{E}$	unrestricted **E**xistential restriction
$\mathcal{C}$	**C**omplement, concept negation
$\mathcal{R}_+$	transitive **R**oles
$\mathcal{H}$	role **H**ierarchies
$\mathcal{O}$	n**O**minals
$\mathcal{I}$	**I**nverse roles
$\mathcal{F}$	**F**unctional roles
$\mathcal{N}$	**N**umber (unqualified cardinality) restrictions
$(\mathbf{D})$	**D**atatypes
$\mathcal{R}$	additional **R**ole constructors
$(\circ)$	role composition
$\mathcal{Q}$	**Q**ualified number restrictions

Table 3.1: Some common constructors in Description Logics

Some examples of interesting DLs are the following:

- $\mathcal{ALC}$ is considered as an standard DL. It extends $\mathcal{AL}$ with concept negation (not being restricted to atomic concepts). It is equivalent to the modal logic **K** [259] and to the DL $\mathcal{ALUE}$ (since once that concept negation has been added to the language, it is possible to simulate the constructors $\mathcal{U}$ and $\mathcal{E}$ [261]).
- $\mathcal{FL}^-$ is a subset of $\mathcal{AL}$ without atomic concept negation [46].
- $\mathcal{EL}$ is a subset of $\mathcal{ALE}$ without bottom concept, atomic concept negation and universal quantifications [160].
- $\mathcal{EL}++$ extends $\mathcal{EL}$ with the bottom concept, nominals, concrete domains and GCIs [12].
- DL-Lite is a subset of $\mathcal{ALIF}$ specially designed to be efficient in the management of large numbers of individuals [62]. An n-ary extension is called DLR-Lite [61].
- $\mathcal{S}$ is an abbreviation for the logic $\mathcal{ALCR}_+$ [257], due to its equivalence with the modal logic **S4** [259].
- $\mathcal{SHIF}(\mathbf{D})$ is the subjacent logic to the OWL Lite language [138].
- $\mathcal{SHOIN}(\mathbf{D})$ is the subjacent logic to the OWL DL language [140].
- $\mathcal{SROIQ}(\mathbf{D})$ is the subjacent logic to the OWL 2 language [137].

Table 3.2 shows the complexity of reasoning in some of the most important DLs (see [225] for some background on complexity theory, and [354] for the complexity of other DLs). A recent result is that $\mathcal{SROIQ}(\mathbf{D})$ is exponentially harder than $\mathcal{SHOIN}(\mathbf{D})$, due to complex RIAs, and in particular due to their ability to chain a fixed exponential number of roles [153].

Logic	**Complexity class**
$\mathcal{ALC}$ with simple GCIs	PSPACE
$\mathcal{SHIF}(\mathbf{D})$	EXPTIME
$\mathcal{SHOIN}(\mathbf{D})$	NEXPTIME
$\mathcal{SROIQ}(\mathbf{D})$	N2EXPTIME

Table 3.2: Complexity of reasoning in some important DLs

Reasoning in Description Logics

A DL not only stores axioms and assertions, but also offers some reasoning services. Most of the common reasoning tasks are the following:

- *KB satisfiability* or *consistency*: Checks if there exists a logical model satisfying all the axioms in the KB.
- *Concept satisfiability*: Checks if a concept does not necessarily denotes the empty set.
- *Entailment*: Checks if a given fact is a logical consequence which can be derived from the axioms in the KB. If this fact is a concept assertion, the reasoning task is also called *instance checking*.
- *Concept subsumption*: Checks if a concept C can be considered more general than a concept D, that is, if D is a subset of (or subsumes) C.
- *Classification*: Computes a concept hierarchy based on the relation of concept subsumption. Essentially, classification checks subsumption for every possible pair of concepts in the KB.

However, if a DL is closed under negation, all the basic reasoning tasks concerning concepts are reducible to KB satisfiability [258], so it is usually the only task considered.

Example 2 *The following reasoning tasks can be reduced to KB satisfiability:*

- Concept satisfiability. *C is satisfiable w.r.t. a KB $\mathcal{K}$ iff $\mathcal{K} \cup \{x : C\}$ is satisfiable, where x is a new individual, i.e., an individual which does not appear in $\mathcal{K}$.*
- Entailment*: A concept assertion $a : C$ is entailed by a KB $\mathcal{K}$ (denoted $\mathcal{K} \models a : C$) iff $\mathcal{K} \cup \{a : \neg C\}$ is unsatisfiable.*

  A role assertion $(a, b) : R$ is entailed by a KB $\mathcal{K}$ (denoted $\mathcal{K} \models (a, b) : R$) iff $\mathcal{K} \cup \{b : C\} \models \{a : \exists R.C\}$, for a new concept C [298].

  If the DL contains negative role assertions, it is also true that $\mathcal{K} \models (a, b) : R$ iff $\mathcal{K} \cup \{(a, b) : \neg R\}$ is unsatisfiable.
- Concept subsumption*: D subsumes C (denoted $C \sqsubseteq D$) w.r.t. a KB $\mathcal{K}$ iff $C \sqcap \neg D$ is unsatisfiable.*

- Classification*: Classification is computed performing a subsumption test for every possible pair of concepts in the KB, so it can also be reduced to KB satisfiability.* □

There are several approaches to reason with DLs, the most popular ones being the following:

- *Tableau algorithms*. Tableau algorithms solve the problems of concept and KB satisfiability. Due to the Open World Assumption, they try to build an abstraction of a model for the KB, that is, a logical interpretation satisfying each of its axioms. The first tableaus algorithm was proposed for $\mathcal{ALC}$ [261]. Tableau algorithms for the DLs $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$ may be found in [141] and [137] respectively.

  More precisely, tableau algorithms try to create a completion graph for the KB, where the nodes represent individuals and the arcs between nodes represent binary relations between individuals. Each node is labeled with a set of concepts that the individual satisfies in this particular model. Starting from the initial KB, the graph is built by repeatedly applying a serie of expansion rules, which transform complex concept descriptions into simpler ones (in such a way that the semantics of the constructors is preserved) until no additional rules can be applied or a *clash* (an obvious contradiction in the label) occurs. If there are no clashes, the concept or the KB is satisfiable and the algorithm has found a model for the KB. If there does not exist a clash-free completion graph, then the KB is unsatisfiable.

  Although these algorithms have a high computational complexity in the worst case, they behave very well in practice, mainly to the existence of a lot of optimization techniques [142, 265, 276, 317].

- *Resolution-based algorithms*. Resolution algorithms translate KBs into FOL, given that in most of the cases DLs are subsets of First Order Logic. This allows existing reasoning algorithms and implementations of them to be reused. This approach is useful from a theoretical point of view, providing an upper bound for the complexity. In some cases, the algorithms are also worst-case optimal.Nevertheless, the practical feasibility of these algorithms is more doubtful since they do not take advantage of the fact that DLs have a lower complexity.

- *Structural algorithms*. These algorithms compute concept subsumption. They normalize the concepts to be tested and compare the syntactic structure of the normalized concept descriptions. These kind of algorithms are sound, but only complete for not very expressive DLs (sub-Boolean DLs).

- *Automata-based algorithms*. These algorithms solve the problem of concept satisfiability for DLs with the tree model property. The main idea is to translate a KB into a tree automata in such a way that they accept the same models, and them to apply an emptiness test over the automaton model.

The majority of the existing DL reasoners implement tableau algorithms. We will describe right now some of the most popular DL reasoners. A comparison of the expressivity of the supported logics may be found in Table 3.3, together with some information about the use of DIG interface.

- RACER (*Renamed ABox and Concept Expression Reasoner*) [111][1]. RACER is a tableau reasoner which supports $\mathcal{SHIQ}$ with some datatypes. The improvements of the initial versions have lead to RACERPRO, implemented in Lisp and fully supporting OWL DL datatypes. RACERPRO is a commercial product, although there are some free licences for educational and research purposes.

- PELLET [277][2]. Probably the most popular among the DL reasoners, from a historical point of view it was the first reasoner fully supporting OWL DL (including nominals and datatypes). Nowadays, is supports $\mathcal{SROIQ}(\mathbf{D})$ and hence OWL 2 (with the exception of n-ary datatypes). It is implemented in Java, has multiple interfaces to access it (including its own API) and is freely available under GNU licence. There is also an active user mailing list.

[1] http://www.racer-systems.com/
[2] http://pellet.owldl.com

- FaCT++ [135, 316][3] is the successor of FaCT(*FAst Classification of Terminologies*) [136]), using a different architecture and a more efficient implementation in C++. It is available under a GNU licence. From a historical point of view it was the first reasoner fully supporting OWL 2 and $\mathcal{SROIQ}(\mathbf{D})$.

- KAON2 [206][4] is a reasoner for $\mathcal{SHIQ}$ and other ontology languages. It does not support nominals, datatypes nor cardinality restrictions involving large integer numbers. KAON2 does not implement a tableau algorithm, but the reasoning algorithm is based on resolution,Its use is free for non commercial purposes.

- HermiT [265][5]. HermiT implements a hypertableau reasoning algorithm which is much less nondeterministic than current tableau algorithms. Some preliminary experimentation shows that HermiT outperforms other DL reasoners. Currently it supports $\mathcal{SHIQ}$, although the extension to $\mathcal{SHOIQ}$ is on its way.

Most of them implement the DIG interface; a common interface to access DL reasoners independently of the particular input language of each of them [23].

Reasoner	**DL supported**	**DIG supported**
Racer, RacerPro	$\mathcal{SHIQ}(\mathbf{D})$	Yes
Pellet	$\mathcal{SROIQ}(\mathbf{D})$	Yes
FaCT++	$\mathcal{SROIQ}(\mathbf{D})$	Yes
Kaon2	$\mathcal{SHIQ}$	Yes
HermiT	$\mathcal{SHIQ}$	No

Table 3.3: DL reasoners, supported logics and DIG interface

[3] http://owl.man.ac.uk/factplusplus/
[4] http://kaon2.semanticweb.org
[5] http://www.comlab.ox.ac.uk/people/boris.motik/HermiT/

3.3 The Description Logic $\mathcal{SROIQ}(\mathbf{D})$

This section overviews $\mathcal{SROIQ}(\mathbf{D})$ [137], the DL which will be mainly treated throughout this document.

Syntax

Definition 30 ***Concrete domain*** *A* concrete domain *is a pair* $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$*, where* $\Delta_{\mathbf{D}}$ *is a concrete interpretation domain and* $\Phi_{\mathbf{D}}$ *is a set of concrete predicates* $\mathbf{d}$ *with a predefined arity* n *and an interpretation* $\mathbf{d_D} \subseteq \Delta_{\mathbf{D}}^n$ *[14, 182].*

Concrete domains or *datatypes* are used to represent integer, real numbers, string, dates, etc.

Definition 31 ***Alphabet of*** $\mathcal{SROIQ}(\mathbf{D})$**.** $\mathcal{SROIQ}(\mathbf{D})$ *assumes three alphabets of symbols, for individuals, roles and concepts.*

- Abstract individuals *are denoted* a, b.
- Concrete individuals *are denoted* v.
- *The* abstract roles *(denoted R) of the language can be built inductively according to the following syntax rule:*

$$\begin{array}{rlll} R \rightarrow & R_A & \mid & \textit{(atomic role)} \\ & R^- & \mid & \textit{(inverse role)} \\ & U & & \textit{(universal role)} \end{array}$$

  Concrete roles *are denoted* T *and cannot be complex.*

- *Let* n, m *be natural numbers with* $n \geq 0, m > 0$ *and let* $\sharp X$ *denote the cardinality of the set* X*. The* concepts *(denoted C or D) of the language can be built inductively from atomic concepts (A), top concept* $\top$*, bottom concept* $\bot$*, named individuals (*o_i*), abstract roles (R), concrete roles (*T*), simple roles (S, which will be defined below) and concrete predicates* $\mathbf{d}$ *as follows:*

$$
\begin{array}{lrll}
C, D \rightarrow & A & \mid & \textit{(atomic concept)} \\
& \top & \mid & \textit{(top concept)} \\
& \bot & \mid & \textit{(bottom concept)} \\
& C \sqcap D & \mid & \textit{(concept conjunction)} \\
& C \sqcup D & \mid & \textit{(concept disjunction)} \\
& \neg C & \mid & \textit{(concept negation)} \\
& \forall R.C & \mid & \textit{(universal quantification)} \\
& \exists R.C & \mid & \textit{(existential quantification)} \\
& \forall T.\mathbf{d} & \mid & \textit{(concrete universal quantification)} \\
& \exists T.\mathbf{d} & \mid & \textit{(concrete existential quantification)} \\
& \{o_1, \ldots, o_m\} & \mid & \textit{(nominals)} \\
& (\geq n\, S.C) & \mid & \textit{(at-least qualified number restriction)} \\
& (\leq n\, S.C) & \mid & \textit{(at-most qualified number restriction)} \\
& (\geq n\, T.\mathbf{d}) & \mid & \textit{(concrete at-least qualified number restriction)} \\
& (\leq n\, T.\mathbf{d}) & \mid & \textit{(concrete at-most qualified number restriction)} \\
& \exists S.\mathtt{Self} & & \textit{(local reflexivity)}
\end{array}
$$

Example 3

- *Man and Woman are atomic concepts.*
- *hasChild and likes are atomic roles.*
- *Man* $\sqcap \geq 2$*hasChild.Woman is a complex concept representing a father with at least two daughters.*
- $\exists$*likes.Self represents a narcissist, a person who loves himself.* □

Expression of the form $(\geq n\ S), (\leq n\ S)$ are called unqualified number restrictions. $(= n\ S.C)$ is an abbreviation for $(\geq n\ S.C) \sqcap (\leq n\ S.C)$, and $(= n\ S)$ is an abbreviation for $(\geq n\ S) \sqcap (\leq n\ S)$. The case for concrete number restrictions is similar.

Definition 32 ***Knowledge Base.*** *A Knowledge Base (KB) comprises two parts: the intensional knowledge, i.e., general knowledge about the application domain (a Terminological Box or* TBox $\mathcal{T}$*, and a Role Box or* RBox $\mathcal{R}$*), and the extensional knowledge, i.e., particular knowledge about some specific situation (an Assertional Box or* ABox $\mathcal{A}$ *with statements about individuals).*

- *An* ABox *consists of a finite set of* assertions *about individuals:*
  - Concept assertions $a : C$*, meaning that individual* a *is an instance of* C.
  - Role assertions $(a, b):R$*, meaning that* (a, b) *is an instance of* R*.*
  - Negated role assertions $(a, b) : \neg R$*, meaning that* (a, b) *is not an instance of* R*.*
  - Concrete role assertions $(a, v):T$.
  - Negated concrete role assertions $(a, v):\neg T$.
  - Inequality assertions $a \neq b$.
  - Equality assertions $a = b$.
- *A* TBox *consists of a finite set of* general concept inclusion *(*GCI*) axioms* $C \sqsubseteq D$ *(*C *is more specific than* D*). A GCI is also called a* terminological *axiom. We also say that* C_2 *is a superclass of* C_1*, and that* C_1 *is a subclass of* C_2*.*

  A concept equivalence $C \equiv D$ *(*C *and* D *are equivalent) is a shorthand for the pair of axioms* $C \sqsubseteq D$ *and* $D \sqsubseteq C$*.*
- *Let* w *be a role chain (a finite string of roles not including the universal role* U*). An* RBox *consists of a finite set of role axioms:*
  - Role inclusion axioms (RIAs) $w \sqsubseteq R$ *(role chain* w *is more specific than* R*) or* $T_1 \sqsubseteq T_2$ *(concrete role* T_1 *subsumes concrete role* T_2*). In RIAs of the form* $R_1 \sqsubseteq R_2$ *we also say that* R_2 *is a super-role of* R_1*, and that* R_1 *is a sub-role of* R_2*.*
  - Transitive *role axioms* $\mathtt{trans}(R)$.
  - Disjoint *role axioms* $\mathtt{dis}(S_1, S_2)$ *or* $\mathtt{dis}(T_1, T_2)$.
  - Reflexive *role axioms* $\mathtt{ref}(R)$.
  - Irreflexive *role axioms* $\mathtt{irr}(S)$.
  - Symmetric *role axioms* $\mathtt{sym}(R)$.
  - Asymmetric *role axioms* $\mathtt{asy}(S)$.

  A role equivalence $R \equiv R'$ *(*R *and* R' *are equivalent) is a shorthand for the pair of axioms* $R \sqsubseteq R'$ *and* $R' \sqsubseteq R$*.*

Table 3.4: Syntax of the Axioms of the Fuzzy Description Logic $\mathcal{SROIQ}(\mathbf{D})$

ABox	
Concept assertion	$a : C$
Role assertion	$(a,b) : R, (a,b) : \neg R, (a,v) : T, (a,v) : \neg T$
Inequality assertion	$a \neq b$
Equality assertion	$a = b$
TBox	
Fuzzy GCI	$C \sqsubseteq D$
RBox	
Fuzzy RIA	$R_1 R_2 \dots R_n \sqsubseteq R, T_1 \sqsubseteq T_2$
Transitive role axiom	$\mathtt{trans}(R)$
Disjoint role axiom	$\mathtt{dis}(S_1, S_2), \mathtt{dis}(T_1, T_2)$
Reflexive role axiom	$\mathtt{ref}(R)$
Irreflexive role axiom	$\mathtt{irr}(S)$
Symmetric role axiom	$\mathtt{sym}(R)$
Asymmetric role axiom	$\mathtt{asy}(S)$

Example 4

- *The concept assertion paul: Man states that the individual Paul belongs to the class of men.*
- *The role assertion (paul, john): ¬hasChild states that John is not the child of Paul.*
- *The GCI Man $\sqsubseteq$ Human states that all men are human.*
- *The RIA owns hasPart $\sqsubseteq$ owns states the fact if somebody owns something, he also owns its components.* □

The adjective "general" in GCIs emphasizes that these axioms involve generic concepts without any restrictions on them, in contrast to some works which impose some limitations. For example, a common restriction is to assume acyclic T-Box where GCIs are of the form $A \sqsubseteq D$ and such that no atomic concept appears in more than one axiom, since KBs with such TBoxes can be transformed into an equivalent KB with an empty TBox by using an *unfolding* algorithm [210].

We note in passing that asymmetric role axioms were erroneously called antisymmetric called in the original paper [137]. As a matter fact, their semantics actually correspond to the one of asymmetric role axioms as pointed out in [275].

Now we are ready the define the notion of simple roles.

Definition 33 ***Simple role.*** Simple *roles are inductively defined as follows:*

- *R_A is simple if does not occur on the right side of a RIA.*
- *R^- is simple if R is.*
- *If R occurs on the right side of a RIA, R is simple if, for each $w \sqsubseteq R$, $w = S$ for a simple role S.*

Note that concrete roles are always simple and non-complex.

Now we will introduce some definitions which will be useful to impose some limitations in the language.

Definition 34 ***Strict partial order.*** *A strict partial order $\prec$ on a set A is an irreflexive and transitive relation on A.*

Definition 35 ***Regular order.*** *A strict partial order $\prec$ on the set of roles is called a regular order if it also satisfies $R_1 \prec R_2 \Longleftrightarrow R_2^- \prec R_1$, for all roles R_1 and R_2.*

Definition 36 **$\prec$-regularity.** *A RIA $w \sqsubseteq R$ is* $\prec$-regular *if $R = R_A$ and:*

- *$w = RR$, or*
- *$w = R^-$, or*
- *$w = S_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or*
- *$w = RS_1 \ldots S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or*
- *$w = S_1 \ldots S_n R$ and $S_i \prec R$ for all $i = 1, \ldots, n$.*

In order to guarantee the decidability of the logic, we need to assume some restrictions in the use of roles:

- Some concept constructors require simple roles: non-concrete qualified number restrictions and local reflexivity.

- Some role axioms also require simple roles: disjoint, irreflexive and asymmetric role axioms.
- Role axioms cannot contain the universal role U.
- Given a regular order $\prec$, every RIA should be $\prec$-regular.

Semantics

Definition 37 ***Interpretation.*** *An interpretation $\mathcal{I}$ with respect to a concrete domain $\mathbf{D}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and an interpretation function $\cdot^{\mathcal{I}}$ mapping:*

- *Every abstract individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.*
- *Every concrete individual v onto an element of $v_{\mathbf{D}}$ of $\Delta_{\mathbf{D}}$.*
- *Every atomic concept A onto a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.*
- *Every abstract atomic role R_A onto a relation $R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.*
- *Every concrete role T onto a relation $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$.*
- *Every n-ary concrete predicate $\mathbf{d}$ onto the interpretation $d_{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}^n$.*

The interpretation is extended to complex concepts and roles by the inductive definitions in Table 3.5.

Unique Name Assumption (UNA) is not imposed, i.e., two individual names (or nominals) might refer to the same individual.

Let $\circ$ be the standard composition of relations. An interpretation $\mathcal{I}$ satisfies (is a model of):

- $a{:}C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $(a,b){:}R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$,
- $(a,b){:}\neg R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$,
- $(a,v){:}T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \in T^{\mathcal{I}}$,
- $(a,v){:}\neg T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \notin T^{\mathcal{I}}$,
- $a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,

Table 3.5: Semantics of the Concepts and Roles in $\mathcal{SROIQ}(\mathbf{D})$

Constructor	**Semantics**
$(A)^\mathcal{I}$	$A^\mathcal{I} \subseteq \Delta^\mathcal{I}$
$(\top)^\mathcal{I}$	$\Delta^\mathcal{I}$
$(\bot)^\mathcal{I}$	$\emptyset$
$(C \sqcap D)^\mathcal{I}$	$C^\mathcal{I} \cap D^\mathcal{I}$
$(C \sqcup D)^\mathcal{I}$	$C^\mathcal{I} \cup D^\mathcal{I}$
$(\neg C)^\mathcal{I}$	$\Delta^\mathcal{I} \setminus C^\mathcal{I}$
$(\forall R.C)^\mathcal{I}$	$\{x \mid \forall y, (x,y) \notin R^\mathcal{I}$ or $y \in C^\mathcal{I}\}$
$(\exists R.C)^\mathcal{I}$	$\{x \mid \exists y, (x,y) \in R^\mathcal{I}$ and $y \in C^\mathcal{I}\}$
$(\forall T.\mathbf{d})^\mathcal{I}$	$\{x \mid \forall v, (x,v) \notin T^\mathcal{I}$ or $v \in \mathbf{d_D}\}$
$(\exists T.\mathbf{d})^\mathcal{I}$	$\{x \mid \exists v, (x,v) \in T^\mathcal{I}$ and $v \in \mathbf{d_D}\}$
$(\{o_1, \ldots, o_m\})^\mathcal{I}$	$\{o_1^\mathcal{I}, \ldots, o_m^\mathcal{I}\}$
$(\geq n\, S.C)^\mathcal{I}$	$\{x \mid \sharp\{y : (x,y) \in S^\mathcal{I}$ and $y \in C^\mathcal{I}\} \geq n\}$
$(\leq n\, S.C)^\mathcal{I}$	$\{x \mid \sharp\{y : (x,y) \in S^\mathcal{I}$ and $y \in C^\mathcal{I}\} \leq n\}$
$(\geq n\, T.\mathbf{d})^\mathcal{I}$	$\{x \mid \sharp\{v : (x,v) \in T^\mathcal{I}$ and $v \in \mathbf{d_D}\} \geq n\}$
$(\leq n\, T.\mathbf{d})^\mathcal{I}$	$\{x \mid \sharp\{v : (x,v) \in T^\mathcal{I}$ and $v \in \mathbf{d_D}\} \leq n\}$
$(\exists S.\mathtt{Self})^\mathcal{I}$	$\{x : (x,x) \in S^\mathcal{I}\}$
$(R_A)^\mathcal{I}$	$R_A^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$
$(R^-)^\mathcal{I}$	$\{(y,x) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (x,y) \in R^\mathcal{I}\}$
$(U)^\mathcal{I}$	$\Delta^\mathcal{I} \times \Delta^\mathcal{I}$

- $C \sqsubseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$,
- $R_1 \ldots R_m \sqsubseteq R$ iff $R_1^\mathcal{I} \circ \cdots \circ R_m^\mathcal{I} \subseteq R^\mathcal{I}$
- $T_1 \sqsubseteq T_2$ iff $T_1^\mathcal{I} \subseteq T_2^\mathcal{I}$
- $\mathtt{trans}(R)$ iff $(x,y) \in R^\mathcal{I}$ and $(y,z) \in R^\mathcal{I}$ imply $(x,z) \in R^\mathcal{I}$, $\forall x,y,z \in \Delta^\mathcal{I}$,
- $\mathtt{dis}(S_1, S_2)$ iff $S_1^\mathcal{I} \cap S_2^\mathcal{I} = \emptyset$,
- $\mathtt{dis}(T_1, T_2)$ iff $T_1^\mathcal{I} \cap T_2^\mathcal{I} = \emptyset$,
- $\mathtt{ref}(R)$ iff $(x,x) \in R^\mathcal{I}, \forall x \in \Delta^\mathcal{I}$,
- $\mathtt{irr}(S)$ iff $(x,x) \notin S^\mathcal{I}, \forall x \in \Delta^\mathcal{I}$,
- $\mathtt{sym}(R)$ iff $(x,y) \in R^\mathcal{I}$ implies $(y,x) \in R^\mathcal{I}, \forall x \in \Delta^\mathcal{I}$,
- $\mathtt{asy}(S)$ iff $(x,y) \in S^\mathcal{I}$ implies $(y,x) \notin S^\mathcal{I}, \forall x \in \Delta^\mathcal{I}$,
- a KB $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.

Note that GCIs can be used to express some features of OWL which are just syntactic sugar:

- *Disjointness of concepts*. The fact that $C_1 \ldots C_n$ are disjoint can be expressed as $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$.
- *Domain* of a concept. The fact that concept C is the domain of a role R can be expressed as $\top \sqsubseteq \forall R^{-}.C$ or $\exists R.\top \sqsubseteq C$.
- *Range* of a concept. The fact that concept C is the range of a role R can be expressed as $\top \sqsubseteq \forall R.C$.
- *Functionality* of a role. A role R is functional iff $R^{\mathcal{I}}(a,b)$ and $R^{\mathcal{I}}(a,c)$ imply $b = c$. This can be expressed as $\top \sqsubseteq (\leq 1\, R.\top)$.

3.4 Ontology Languages

DLs have proved to be very useful as ontology languages [15]. The current standard language for ontology representation is the Web Ontology Language (OWL) [327].

OWL

OWL is a W3C (World Wide Web Consortium) recommendation since February 2004. Essentially, OWL combines RDF [329] and RDF Schema [330] with some DLs [140], while trying to maintain backwards compatibility with previous ontology languages (such as SHOE, DAML, OIL or DAML+OIL [124]). OWL comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full.

- *OWL Lite* is the less expressive level and, accordingly, reasoning with it has the lowest computational complexity (ExpTime). It is equivalent to the DL $\mathcal{SHIF}(\mathbf{D})$.
- *OWL DL* is a balanced tradeoff between expressiveness and complexity (NExpTime), with reasoning still being decidable. OWL DL corresponds to the DL $\mathcal{SHOIN}(\mathbf{D})$, and allows all the constructors of OWL, but there are some restrictions in their usage. Every OWL Lite is a valid OWL DL ontology.

- *OWL Full* is the most expressive level, but reasoning with it becomes undecidable. OWL Full allows any constructor of OWL with no restrictions. Every OWL DL is a valid OWL Full ontology. OWL Full subsumes RDF, so every RDF document is a valid OWL Full document.

In fact, ontology entailment for OWL Lite and OWL Lite can be reduced to KB satisfiability in $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$ respectively [139].

However, since its first development, several extensions to OWL have been proposed [286].

OWL 2

Among these extensions, the most significant one is *OWL 2* [75, 233] which is its most likely immediate successor. In November 2005, the participants of the first workshop "OWL: Experiences and Directions" decided to start working on a extension of OWL. The result of a working group was OWL 1.1, is a W3C Member Submission since December 2006. Recently, the name has changed to OWL 2, and future versions of the language will be presented under this name.

Essentially, OWL 2 is based on the DL $\mathcal{SROIQ}(\mathbf{D})$ (which provides qualified cardinality restrictions and some new property axioms), but also including extended datatype support (customized datatypes), some syntactic sugar, changes in the meta-modelling (punning) and extended annotations. This increment in the expressivity makes reasoning harder (N2ExpTime) [153].

There are several syntaxes for OWL 2:

- *Functional-style* syntax [205] is the most common one, and intends to be easier for specification and for reasoning tools, and that replaces OWL abstract syntax [328] because of some problems with it [204].
- *XML* syntax is defined by an XML Schema and easier to implement [77].
- *RDF/XML* syntax allows the serialization of ontologies in RDF [76].
- *Manchester* syntax aims to be easier for non-logicians (by giving a user-friendly syntax based on the DL syntax but avoiding mathematical symbols) [133, 134].

- *Sydney* syntax is closer to natural language, using a subset of the English language [73].

An OWL 2 ontology contains descriptions of *classes* (or concepts in DL terminology), *properties* (roles in DL terminology) and *individuals*. There are two types of properties: *object properties* (abstract roles) and *datatype properties* (concrete roles). Table 3.6 includes the clases and properties constructors of OWL 2, together with their correspondences in $\mathcal{SHOIN}(\mathbf{D})$.

There are two additional types of properties which do not have a counterpart in the DL, namely *annotation properties* (`owl:AnnotationProperty`) and *ontology properties* (`owl:OntologyProperty`), but they just include some meta-properties of the ontology.

An OWL 2 document consists of optional ontology headers plus any number of axioms: facts about individuals, class axioms and property axioms, which according to the DL terminology correspond to the ABox, TBox and RBox, respectively. Ontology headers are used for meta-information, ontology import and relationships. Table 3.7 shows the OWL 2 axioms and their equivalences in $\mathcal{SHOIN}(\mathbf{D})$.

Relation Among OWL 2 and Other Languages

OWL DL and OWL Lite are two subsets of OWL 2. Table 3.8 shows the relation among OWL 2, OWL DL and OWL Lite classes, while Table 3.9 shows the relation among OWL 2, OWL DL and OWL Lite axioms.

OWL DL is a subset of OWL 2[6] which does not allow:

- Concept constructors:
  - ObjectMinCardinality (n, S, C)
  - ObjectMaxCardinality (n, S, C)
  - ObjectExactCardinality (n, S, C)
  - DataMinCardinality $(n, T, \mathbf{d})$
  - DataMaxCardinality $(n, T, \mathbf{d})$
  - DataExactCardinality $(n, T, \mathbf{d})$

[6] Note, however, that the functional-style syntax of OWL 2 is not valid in OWL.

Table 3.6: Class and property constructors in OWL 2

OWL 2 abstract syntax	DL syntax
Class (A)	A
Class (owl:Thing)	$\top$
Class (owl:Nothing)	$\bot$
ObjectIntersectionOf (C, D)	$C \sqcap D$
ObjectUnionOf (C, D)	$C \sqcup D$
ObjectComplementOf (C)	$\neg C$
ObjectAllValuesFrom (R, C)	$\forall R.C$
ObjectSomeValuesFrom (R, C)	$\exists R.C$
ObjectHasValue (R, o)	$\exists R.\{o\}$
DataAllValuesFrom ($T, \mathbf{d}$)	$\forall T.\mathbf{d}$
DataSomeValuesFrom ($T, \mathbf{d}$)	$\exists T.\mathbf{d}$
DataHasValue (T, v)	$\exists T.\{v\}$
ObjectOneOf ($o_1, \ldots, o_m$)	$\{o_1, \ldots, o_m\}$
ObjectMinCardinality (n, S, C)	$(\geq n\, S.C)$
ObjectMaxCardinality (n, S, C)	$(\leq n\, S.C)$
ObjectExactCardinality (n, S, C)	$(\geq n\, S.C) \sqcap (\leq n\, S.C)$
ObjectMinCardinality (n, S)	$(\geq n\, S.\top)$
ObjectMaxCardinality (n, S)	$(\leq n\, S.\top)$
ObjectExactCardinality (n, S)	$(\geq n\, S.\top) \sqcap (\leq n\, S.\top)$
DataMinCardinality ($n, T, \mathbf{d}$)	$(\geq n\, T.\mathbf{d})$
DataMaxCardinality ($n, T, \mathbf{d}$)	$(\leq n\, T.\mathbf{d})$
DataExactCardinality ($n, T, \mathbf{d}$)	$(\geq n\, T.\mathbf{d}) \sqcap (\leq n\, T.\mathbf{d})$
DataMinCardinality (n, T)	$(\geq n\, T.\top)$
DataMaxCardinality (n, T)	$(\leq n\, T.\top)$
DataExactCardinality (n, T)	$(\geq n\, T.\top) \sqcap (\leq n\, T.\top)$
ObjectExistsSelf (S)	$\exists S.\mathtt{Self}$
ObjectProperty (R_A)	R_A
TopObjectProperty	U
BottomObjectProperty	$\neg U$
DatatypeProperty (T)	T
TopDataProperty	U_D
BottomDataProperty	$\neg U_D$

Table 3.7: Axioms in OWL 2

OWL 2 abstract syntax	**DL syntax**
ClassAssertion (a, C)	$a : C$
ObjectPropertyAssertion $(R, a, b))$	$(a, b) : R$
NegativeObjectPropertyAssertion (R, a, b)	$(a, b) : \neg R$
DataPropertyAssertion $(T, a, v))$	$(a, v) : T$
NegativeDataPropertyAssertion (T, a, v)	$(a, v) : \neg T$
SameIndividual $(a_1, \ldots, a_m)$	$a_i = a_j, 1 \leq i < j \leq m$
DifferentIndividuals $(a_1, \ldots, a_m)$	$a_i \neq a_j, 1 \leq i < j \leq m$
SubClassOf (C_1, C_2)	$C_1 \sqsubseteq C_2$
EquivalentClasses $(C_1, \ldots, C_m)$	$C_1 \equiv \cdots \equiv C_m$
DisjointClasses $(C_1, \ldots, C_m)$	$C_1 \sqcap \cdots \sqcap C_m \sqsubseteq \bot, 1 \leq i < j \leq m$
DisjointUnion $(C, C_1, \ldots, C_m)$	$C \equiv C_1 \sqcup \cdots \sqcup C_m$, $C_i \sqcap C_j \sqsubseteq \bot, 1 \leq i < j \leq m$
SubObjectPropertyOf (subObjectPropertyChain $(R_1, \ldots, R_m)$ R)	$R_1 \ldots R_m \sqsubseteq R$
SubObjectPropertyOf $(R_1 R_2)$	$R_1 \sqsubseteq R_2$
SubDataPropertyOf (T_1, T_2)	$T_1 \sqsubseteq T_2$
EquivalentObjectProperties $(R_1, \ldots, R_m)$	$R_1 \equiv \cdots \equiv R_m$
EquivalentDataProperties $(T_1, \ldots, T_m)$	$T_1 \equiv \cdots \equiv T_m$
ObjectPropertyDomain (R, C)	$\exists R.\top \sqsubseteq C$
ObjectPropertyRange (R, C)	$\top \sqsubseteq \forall R.C$
DataPropertyDomain (T, C)	$\exists T.\top \sqsubseteq C$
DataPropertyRange $(T, \mathbf{d})$	$\top \sqsubseteq \forall T.\mathbf{d}$
InverseObjectProperties (R_1, R_2)	$R_1 \equiv R_2^-$
FunctionalObjectProperty (S)	$\top \sqsubseteq (\leq 1\, S.\top)$
FunctionalDataProperty (T)	$\top \sqsubseteq (\leq 1\, T.\top)$
InverseFunctionalObjectProperty (S)	$\top \sqsubseteq (\leq 1\, S^-.\top)$
TransitiveObjectProperty (R)	$\mathtt{trans}(R)$
DisjointObjectProperties (S_1, S_2)	$\mathtt{dis}(S_1, S_2)$
DisjointDataProperties (T_1, T_2)	$\mathtt{dis}(T_1, T_2)$
ReflexiveObjectProperty (R)	$\mathtt{ref}(R)$
IrreflexiveObjectProperty (S)	$\mathtt{irr}(S)$
SymmetricObjectProperty (R)	$\mathtt{sym}(R)$
AsymmetricObjectProperty (S)	$\mathtt{asy}(S)$

  - ObjectExistsSelf (S)
- Role constructors:
  - TopObjectProperty
  - BottomObjectProperty
  - TopDataProperty
  - BottomDataProperty
- Axioms:
  - NegativeObjectPropertyAssertion (R, a, b)
  - NegativeDataPropertyAssertion (T, a, v)
  - DisjointUnion $(C, C_1, \ldots, C_m)$
  - SubObjectPropertyOf (subObjectPropertyChain $(R_1, \ldots, R_m)$ R)
  - DisjointObjectProperties (S_1, S_2)
  - DisjointDataProperties (T_1, T_2)
  - ReflexiveObjectProperty (R)
  - IrreflexiveObjectProperty (S)
  - AsymmetricObjectProperty (S)

OWL Lite is a subset of OWL DL which does not allow:

- Concept constructors:
  - ObjectHasValue (R, o)
  - DataHasValue (T, v)
  - ObjectOneOf $(o_1, \ldots, o_m)$
  - ObjectMinCardinality (n, S)
  - ObjectMaxCardinality (n, S)
  - ObjectExactCardinality (n, S)
  - DataMinCardinality (n, T)
  - DataMaxCardinality (n, T)
  - DataExactCardinality (n, T)

Table 3.8: Relation among OWL 2, OWL DL and OWL Lite classes

Constructor	**OWL 2**	**OWL DL**	**OWL Lite**
Class (A)	✓	✓	✓
Class (owl:Thing)	✓	✓	✓
Class (owl:Nothing)	✓	✓	✓
ObjectIntersectionOf (C, D)	✓	✓	✓
ObjectUnionOf (C, D)	✓	✓	✓
ObjectComplementOf (C)	✓	✓	✓
ObjectAllValuesFrom (R, C)	✓	✓	✓
ObjectSomeValuesFrom (R, C)	✓	✓	
ObjectHasValue (R, o)	✓	✓	
DataAllValuesFrom ($T, \mathbf{d}$)	✓	✓	✓
DataSomeValuesFrom ($T, \mathbf{d}$)	✓	✓	✓
DataHasValue (T, v)	✓	✓	
ObjectOneOf ($o_1, \ldots, o_m$)	✓	✓	
ObjectMinCardinality (n, S)	✓	✓	
ObjectMaxCardinality (n, S)	✓	✓	
ObjectExactCardinality (n, S)	✓	✓	
DataMinCardinality (n, T)	✓	✓	
DataMaxCardinality (n, T)	✓	✓	
DataExactCardinality (n, T)	✓	✓	
ObjectMinCardinality (n, S, C)	✓		
ObjectMaxCardinality (n, S, C)	✓		
ObjectExactCardinality (n, S, C)	✓		
DataMinCardinality ($n, T, \mathbf{d}$)	✓		
DataMaxCardinality ($n, T, \mathbf{d}$)	✓		
DataExactCardinality ($n, T, \mathbf{d}$)	✓		
ObjectExistsSelf (S)	✓		
ObjectProperty (R_A)	✓	✓	✓
TopObjectProperty	✓		
BottomObjectProperty	✓		
DatatypeProperty (T)	✓	✓	✓
TopDataProperty	✓		
BottomDataProperty	✓		

Table 3.9: Relation among OWL 2, OWL DL and OWL Lite axioms

Axiom	OWL 2	OWL DL	OWL Lite
ClassAssertion (a, C)	✓	✓	✓
ObjectPropertyAssertion $(R, a, b))$	✓	✓	✓
NegativeObjectPropertyAssertion (R, a, b)	✓		
DataPropertyAssertion $(T, a, v))$	✓	✓	✓
NegativeDataPropertyAssertion (T, a, v)	✓		
SameIndividual $(a_1, \ldots, a_m)$	✓	✓	✓
DifferentIndividuals $(a_1, \ldots, a_m)$	✓	✓	✓
SubClassOf (C_1, C_2)	✓	✓	✓
EquivalentClasses $(C_1, \ldots, C_m)$	✓	✓	✓
DisjointClasses $(C_1, \ldots, C_m)$	✓	✓	✓
DisjointUnion $(C, C_1, \ldots, C_m)$	✓		
SubObjectPropertyOf (subObjectPropertyChain $(R_1 \ldots R_m)$ R)	✓		
SubObjectPropertyOf $(R_1 R_2)$	✓	✓	✓
SubDataPropertyOf (T_1, T_2)	✓	✓	✓
EquivalentObjectProperties $(R_1, \ldots, R_m)$	✓	✓	✓
EquivalentDataProperties $(T_1, \ldots, T_m)$	✓	✓	✓
ObjectPropertyDomain (R, C)	✓	✓	✓
ObjectPropertyRange (R, C)	✓	✓	✓
DataPropertyDomain (T, C)	✓	✓	✓
DataPropertyRange $(T, \mathbf{d})$	✓	✓	✓
InverseObjectProperties (R_1, R_2)	✓	✓	✓
FunctionalObjectProperty (S)	✓	✓	✓
FunctionalDataProperty (T)	✓	✓	✓
InverseFunctionalObjectProperty (S)	✓	✓	✓
TransitiveObjectProperty (R)	✓	✓	✓
DisjointObjectProperties (S_1, S_2)	✓		
DisjointDataProperties (T_1, T_2)	✓		
ReflexiveObjectProperty (R)	✓		
IrreflexiveObjectProperty (S)	✓		
SymmetricObjectProperty (R)	✓	✓	✓
AsymmetricObjectProperty (S)	✓		

CHAPTER 4

Fuzzy Ontologies and Fuzzy Description Logics

This chapter contains the state of the art in fuzzy extensions of ontologies and Description Logics. Section 4.1 is dedicated to fuzzy ontologies, and then Section 4.2 focuses on fuzzy Description Logics, as the most developed formalism to work with fuzzy ontologies.

4.1 Fuzzy Ontologies

This section is devoted to the field of fuzzy ontologies, including a motivation of their birth, a revision of the proposed definitions and some pointers to the most relevant applications that have been presented in the literature.

Motivation

With the sake of concrete illustration of the limitations of classical ontologies, let us consider an example in the application domain of accommodation. Our example is based on a possibilistic ontology originally proposed in [172]. Another example of application domain is medicine, since there exist some medical categories that cannot be defined in an Aristotelian sense [309].

Firstly, the concepts which are used to describe different types of lodging are inherently vague. For example, it is usually assumed that a *guesthouse* is a *"cheap, small and more hospitable hotel"*. But the notions of *cheap*, *small* and *hospitable* are clearly not well defined, having unsharp boundaries. Hence, the nature of the concepts does not make appropriate a crisp definition.

As a consequence, it may be difficult to say that a particular establishment fully belongs to one of these concepts, and that does not belong to any of the others. It is more reasonable to think that it belongs to several concepts up to different degrees.

Similarly, instead of having a concept hierarchy, we would have a fuzzy concept hierarchy, where guesthouse can be considered a subconcept of hotel to some degree. These degrees are different for the different subconcepts of hotel, since some of them can be considered hotels preferably than others.

Furthermore, a complex concept can have attributes taking vague values. For example, a guesthouse can have an attribute *hasPrice* being a linguistic variable taking values as *cheap* or *expensive* defined by means of fuzzy numbers.

Definition

There have been proposed several definitions of fuzzy ontology in the literature, although we think that neither of them is general enough. A list of the most relevant definitions is included here, where the notation has been homogenized and, in some cases, summarized. It is worth to note, however, that some of the authors do not claim to be proposing a universal definition of the term.

- A fuzzy ontology is a quintuple (I, C, R, F, A) where I is a set of individuals, C is a set of fuzzy concepts, R is a set of fuzzy n-ary relations (including fuzzy taxonomic relations), F is a set of fuzzy n-ary concrete relations, and A is the set of axioms [52, 53].
- A fuzzy ontology is a quadruple (C, R, F, A) where C is a set of fuzzy concepts, R is a set of fuzzy n-ary relations (including fuzzy taxonomic relations), F is a set of fuzzy n-ary concrete relations, and A is the set

of axioms [55, 58, 60]. Since an ontology does not contain individuals, a fuzzy Knowledge Base is also defined as a pair (O, I) where O is a fuzzy ontology, and I is a set of individuals.

- A fuzzy ontology is a quadruple (C, R, H, A), where C is a set of concepts (possibly with fuzzy attributes), R is a set of fuzzy relations, H is a concept hierarchy, and A is a set of axioms [254].

- A fuzzy ontology is a quintuple (C, R, H, P, A), where C is a set of fuzzy concepts, R is set of fuzzy relations between individuals, H is a concept hierarchy, P is a set of non-taxonomic fuzzy relations between concepts, and A is a set of fuzzy axioms [255]. According to this, an ontology does not contain individuals. They define the notion of fuzzy knowledge Base as a pair (O, I) where O is a fuzzy ontology, and I is a set of individuals.

- A fuzzy ontology is a quadruple (C, R, P, A), where C is a set of concepts, R is a set of binary relations between two individuals, P is a set of binary relations between two concepts, and A is a set of fuzzy axioms[1] [346].

- A fuzzy ontology is a quadruple (C, R, l, μ_F), where C is a finite set of nodes (concepts), $R \subseteq C \times C$ is a set of fuzzy edges (relations) that are assigned by a continuous fuzzy value and a label, l is a mapping from edges to a set L of strings called labels, and μ_F is the set of membership functions $\mu_F : R \to [0, 1]$ [162].

- A fuzzy ontology is a quintuple (C, F, H, R, E) where C is set of fuzzy concepts, F is the set of attributes of a concept, H is a concept hierarchy, R is a set of fuzzy relations, and E is the set of fuzzy events of a concept [353].

- A fuzzy ontology is a quadruple (C, R, F, U), where C is a set of concepts, R is a set of fuzzy abstract relations, F is a set of fuzzy concrete relations, and U is the universe of discourse. [1, 3].

[1] To be more precise, this is the definition of a *fuzzy domain ontology*. The authors also introduce definitions for an *extended fuzzy ontology* and a *basic fuzzy ontology*.

- A fuzzy ontology is a quadruple (C, F^C, R, A), where C is a set of concepts, F^C is a collection of sets of fuzzy attributes (one set for each concept), R is a set of relations (including taxonomic relations), and A is a set of axioms [242].
- A fuzzy ontology is a triple (C, I, R), where, C is a set of concepts, I is a set of individuals, and R is a set of binary relations between some elements of C and I, including two special types of fuzzy relations [108, 170].
- A fuzzy ontology is a quadruple (C, R, P, I), where C is a set of fuzzy concepts, R is a set of fuzzy binary relations, P is a set of fuzzy properties of concepts, and I is a set of individuals [11, 51].
- A fuzzy ontology is a pair (C, R), where C is a set of concepts, and R is a set of fuzzy relations between concepts [87].
- A fuzzy ontology is a triple (S, C, f), where C is a set of concepts, S is a set of analytic sentences describing the meaning of the symbols in C, and $f : C \to [0, 1]$ [219].
- A fuzzy ontology is an ontology extended with fuzzy values which are assigned through the functions $r : (C \cup I) \times R \to \texttt{RelationValues}) \times [0, 1]$, and $c : C \cup I \to [0, 1]$, where C is a set of concepts, R is a set of relations, and I is a set of individuals [54].
- A fuzzy ontology is an ontology extended with fuzzy values which are assigned through the functions $c : I \to [0, 1]$ and $r : \texttt{RelationValues} \to [0, 1]$, where I is a set of individuals [57].
- A fuzzy ontology is an extended domain ontology with fuzzy concepts and fuzzy relations [163].
- A fuzzy ontology is a set of Fuzzy OWL axioms [288].

Discussion

A first group of the existing definitions try to formalize the notion of fuzzy ontologies by means of an enumeration of the elements of the ontology which are extended in order to support vague knowledge representation [52, 53, 55, 58, 60, 254, 255]. Despite the merit of these definitions, this approach has some problems. Typically, different domains will need to represent vagueness and imprecision at different levels. Furthermore, future languages will offer new possibilities to be extended which are unknown nowadays, but these definition do not allow other fuzzy elements than the explicitly mentioned. As a consequence, the scalability and reusability of the definition are compromised. For example, a fuzzy role hierarchy falls out of these definitions since *taxonomic relations between roles* are not mentioned.

Most of the approaches are even more restrictive and are application-dependent, since they propose the minimal extensions which are sufficient to cover a particular application [1, 3, 54, 57, 87, 108, 162, 163, 170, 219, 242, 346, 353]. For example, [1, 3, 87] do not fuzzify concepts, whereas [219] does not fuzzify relations.

Moreover, some definitions are tied to a particular formalism. For example [288] is attached to the fuzzy language FUZZY OWL based on a fuzzy extension of the DL $\mathcal{SHOIN}$. But even if it became a standard language for fuzzy ontologies representation, there are still some interesting features which cannot be represented in it, such as fuzzy nominals or fuzzy concept or role hierarchies.

We understand fuzzy ontologies in a more general sense. In our opinion, a fuzzy ontology is simply *an ontology which uses fuzzy logic to provide a natural representation of imprecise and vague knowledge, and eases reasoning over it*. We emphasize the fact that the representation should be natural, because nowadays it is possible to represent some kind of fuzziness in crisp ontologies by means of ad-hoc solutions.

Typically, concepts, (abstract and concrete) relations and axioms can be fuzzified. But fuzziness can also be in fuzzy concrete domains. Or in external formalisms such as a fuzzy rule layer. Our informal definition is general enough to allow the ontologists to decide in which levels they want to introduce the fuzziness, makes possible to have a crisp ontology with fuzziness being dealt with using a external formalism, and, what seems more important to us, will not be compromised by the apparition of future ontology languages.

It is also worth to recall that since fuzzy logic is a generalization of classical logic, classical ontologies are special case of fuzzy ontologies, so this formalism is by nature backwards compatible with current crisp ontologies.

Finally, we note in passing that, although in the Artificial Intelligence literature the term approximate reasoning is directly related to fuzzy logic, in the ontology literature this term has nothing to do with fuzziness [248].

Applications

Without the aim of being exhaustive, we will mention in this subsection some examples of applications of fuzzy ontologies.

One of the most important applications is Semantic Web [71, 213, 232, 242] and, more generally, the Internet [253]. The World Wide Web Consortium (W3C) recently set up an incubator group on Uncertainty Reasoning for the Web, where *"uncertain is intended to encompass a variety of aspects of imperfect knowledge, including incompleteness, inconclusiveness, vagueness, ambiguity, and others"* [42].

Another important application is information retrieval [87, 21, 56, 58, 59, 60, 102, 103, 148, 217, 218, 236, 263, 332]. More specifically, medical information retrieval [228, 227, 229, 230, 231], personalized multimedia information retrieval [207, 208, 226, 333, 81] and query enrichment [19, 49, 158, 215, 336, 337] have been the subject of considerable research.

Among the other examples that appear in the literature we may cite multilingual ontologies [74], ontology mapping [215], ontology integration [3], ontology dynamics [57], context management [341], image interpretation [143], document content representation [48], data mining [98], text mining [1, 2, 4, 85, 86, 88][2], development of computing with words based systems [247], temporal knowledge representation [209], economy (a fuzzy Balanced Scorecard [36]), design process management [353], transportation systems [346], Chinese news summarization [163], semantic help-desk Support [241] or educational computer games [108, 170].

[2] [88] uses rough fuzzy ontologies.

The construction of fuzzy ontologies has for instance been considered in [1, 3, 7, 109, 163, 183, 242, 336, 337]. Regarding fuzzy ontology editors, [54, 55] presents an extension of the ontology editor KAON.

Fuzzy extensions of Semantic Web languages such as RDF [185, 196, 197, 321], OWL [101, 288] (a fuzzy probabilistic version has also been outlined [312]), OWL 2 [281], RuleML [78], SWRL [223, 335], or RIF [351] have also been proposed.

Conceptual Fuzzy Sets (CFSs) were proposed in order to overcome the fact that ordinary fuzzy sets do not dealt with the context-dependent meaning representation concerning language ambiguity. Conceptual Fuzzy Sets (CFSs) explain the meaning of concepts by chaining other related concepts, where the activation values of the other concepts correspond to the membership values [310]. By fusing CFSs with ontologies, a region of words commonly relevant to the original keyword is formed. In a series of works, T. Takagi et al. propose the combination of ontologies and Conceptual Fuzzy Sets (see e.g., [311]).

Finally, some crisp ontologies have been used for the representation of different types of fuzzy [25] and, more generally, uncertain knowledge, understanding uncertain in a general sense [42] (see also the discussion about this ontology in [66]).

4.2 Fuzzy Description Logics

This section exhaustively compiles the state of the art in fuzzy DLs, extending and updating in a very significant way the existing survey papers [79, 181, 297]. Due to the very important number of works, we have classified them and the structure of this section is as follows. The first subsection quickly illustrates the path followed in the field, from the oldest seminal works to more recent proposals dealing with expressive logics. The next subsection is devoted to some specific elements which are of interest in the fuzzy DL that we propose in this document (namely GCIs, families of fuzzy operators, fuzzy modifiers, fuzzy concrete domains, cut concepts and roles, number restrictions, reductions to crisp DLs, and fuzzy DL reasoners). We compare our contributions with the related work. Finally, the third subsection is a hotchpotch which covers the rest of the work in fuzzy DLs and some related fields, such as possibilistic DLs.

The Path Followed in Fuzzy DLs

First steps. The work of J. Yen [340] (1991) is the first effort in this area. The author defined a fuzzy extension of $\mathcal{FL}^-$ called $\mathcal{FTSL}^-$. He only considered a TBox. He extended conjunction and universal quantification concepts to the fuzzy case, considering Zadeh family. He also considered some fuzzy modifiers (`Not`, `Very` and `Slightly`) applied to atomic concepts, and fuzzy concrete domains. The extension of number restrictions is proposed, although they do not belong to $\mathcal{FL}^-$. Regarding the reasoning, he proposes an structural algorithm for concept subsumption (where this reasoning task is considered as a yes/no question).

In 1994, R. M. da Silva et al. [268] described a system for annotating variables of first order predicate calculus using a fuzzy DL.

In 1998, C. Tresp and R. Molitor [314] proposed $\mathcal{ALC}_{\mathcal{F}_\mathcal{M}}$, a multivalued extension of $\mathcal{ALC}$ under Zadeh family. They also allow the use of some fuzzy modifiers (manipulators, a special family of triangular membership functions). Concept negation is not allowed as part of the language, but it has to be simulated by using a fuzzy modifiers. Consequently, they allow the use of different types of negation functions (as long as they are manipulators). This work provides a tableau algorithm for fuzzy concept subsumption which generates an equation system to be solved by using a linear programming algorithm [249]. The algorithm is sound and complete, although not really efficient for large KBs.

The most important works in the first years of fuzzy DLs are authored by U. Straccia. In 1997, he (together with other authors) extended $\mathcal{ALC}$ with the possibility of representing fuzzy assertions of the form $\langle \tau, \alpha \rangle$ stating that the fuzzy assertion τ is true with (at least) degree α [200]. However, this work does not describe any reasoning algorithm. In 1998 he presented an extended fuzzy extension of $\mathcal{ALC}$, as well as a reasoning algorithm for KB satisfiability [292]. He also reduced other problems (entailment and computation of the Best Truth-Value Bound) to this reasoning task.

These works assume a simple TBox, as originally proposed in the crisp case by B. Nebel [211]. A simple TBox is assumed to be acyclic, to include only axioms of the form $A \sqsubseteq C$, and to be such that no atomic concept appear in the left hand side of more than one terminological axiom. In this case, it is possible to apply an unfolding algorithm to remove the TBox.

In 2001, in the most referenced paper in the field, he extended his previous works with fuzzy assertions of the form $\langle \tau \leq \alpha \rangle$ and a reduction of concept subsumption to KB satisfiability [299].

Expressive logics. After this initial phase of research in the area, more recent works have considered expressive (and very expressive) fuzzy DLs. In 2005, U. Straccia proposed a fuzzy extension of $\mathcal{SHOIN}(\mathbf{D})$ [291, 300], the logic subjacent to OWL DL, including fuzzy modifiers, fuzzy concrete domains, general families of fuzzy operators and fuzzy axioms (fuzzy GCIs and RIAs). Despite of the very high expressivity of the logic, reasoning is not dealt with.

Recently, S. Calegari and D. Ciucci extended this work with role modifiers, some kind of threshold concepts (for instance, $C_{\geq \alpha}$ is the fuzzy set of elements belonging to the fuzzy set C with a degree greater or equal than α) and proposing an alternative semantics for number restrictions forcing them to be crisp concepts [52, 53]. Unfortunately, they did not provide any reasoning algorithm either.

G. Stoilos et al. provided tableau algorithms to decide KB satisfiability for $\mathcal{SI}$ [287, 285], $\mathcal{SHIN}$ [289, 285] and $\mathcal{SHOIN}$ [288]. In the latter work they do not provide a fuzzy semantics for fuzzy nominals, but their most important contribution is the definition of FUZZY OWL, a fuzzy extension of OWL based on their fuzzy extension of $\mathcal{SHOIN}$. However, the language still lacks fuzzy nominals, fuzzy GCIs and fuzzy concrete domains, so in our opinion it should not be considered a full extension of OWL. We extended this logic with fuzzy nominals and fuzzy GCIs, and provided a reasoning algorithm based on a reduction to classical $\mathcal{SHOIN}$ [26, 27].

A extension of the reasoning algorithm for fuzzy $\mathcal{SHIF}$ which aims to support some special kinds of relations (fuzzy concept relations and fuzzy concept base relations, which come from the experience in building fuzzy ontologies) is outlined in [109]. Let C, D be fuzzy concepts, $C_1, C_2, \ldots, C_m$ be subconcepts of C and $D_1, D_2, \ldots, D_n$ be subconcepts of D. If $\mu_R(a, b)$ is always the same for all $a \in C$ and $b \in D$, R is a *fuzzy concept relation* (FCR) $\langle C, D, \mu_R(C, D)\rangle$. If there are FCRs $\langle C_i, D_j, \mu_R(C_i, D_j)\rangle$ for all pairs (C_i, D_j) where $1 \leq i \leq m$ and $1 \leq j \leq n$, there is a *fuzzy concept base relation* (FCBR) R on C and D. A FCBR can be represented by a set of triples $\{\langle C_i, D_j, \mu_R(C_i, D_j)\rangle | (C_i, C), (D_j, D) \in R_{kf}\}$.

G. Stoilos et al. defined fuzzy $\mathcal{SHOIQ}$ [284], proposing a more general semantics for qualified number restrictions and studying the logical properties, but without any reasoning algorithm. Later on, some of the authors proposed a fuzzy extension of $\mathcal{SROIQ}$ [281]. However, the reasoning algorithm only considers a fragment of it ($\mathcal{SR}^{-}\mathcal{OIN}$).

All these works restrict themselves to the Zadeh family. Regarding other families of fuzzy operators, a reasoning algorithm for fuzzy $\mathcal{SHIF}$ under Łukasiewicz family has been recently presented [298].

In this document, we extend the reduction of fuzzy $\mathcal{SROIQ}(\mathbf{D})$ into a crisp DL (under Zadeh and Gödel semantics). Part of the work has been published in [32, 33, 29].

Comparing our Contributions

General Concept Inclusions. As we have already mentioned, the first works on fuzzy DLs assumed simple TBoxes. The first work which allowed the use of GCIs used a crisp representation for the fuzzy DL $\mathcal{ALCH}$ [302]. The semantics for GCIs was based on Zadeh's inclusion of fuzzy sets, thus making concept subsumption a yes/no question.

G. Stoilos et al. (for the logic $\mathcal{ALC}$) [290] and Y. Li et al. (for the logic $\mathcal{SHI}$) [187] developed in parallel specific reasoning algorithms for DLs with GCIs. Roughly speaking, these works introduced a disjunction for every possible degree of truth appearing in the KB. This is based on the observation that, under Zadeh family, if $C \sqsubseteq D$, for each individual a and each degree α

it must hold that $C(a) \leq \alpha$ or $D(a) \geq \alpha$. Later on, Y. Li et al. applied the result to $\mathcal{SHIN}$ [189] and $\mathcal{ALCN}$ [168].

Later on, a more promising solution from a practical point of view has been presented for fuzzy GCIs, which hold to some degree. The solution, based on a combination of tableau rules and Mixed Integer Linear Programming (MILP) optimization problems [249], can be used for Zadeh and Łukasiewicz logics [304, 305], since the constructors are MILP representable. By combining tableau rules with MIQCP, a version of Product logic with Łukasiewicz negation [38] can be supported. For example, considering the semantics given by Zadeh's inclusion of fuzzy sets, for each individual a we create a new variable $x \in [0,1]$, add the restrictions $C(a) \leq x$ and $D(a) \geq x$, and finally we maximize the value of x.

A very recent work considered a fixpoint-based decision procedure for $\mathcal{ALC}$ with GCIs [154, 155].

D. Dubois et al. proposed several levels of inclusion for a GCI, based on the core and support of fuzzy sets [94]. This way, $C \sqsubseteq D$ is defined as:

- $core(C) \subseteq supp(D)$,
- $core(C) \subseteq core(D)$ or $supp(C) \subseteq supp(D)$,
- $core(C) \subseteq core(D)$ and $supp(C) \subseteq supp(D)$,
- $supp(C) \subseteq core(D)$

These definitions also force the inclusion to be either true or false; moreover, since in general $support(C) \nsubseteq core(C)$, using the latter semantics for GCIs implies that a concept does not fully subsume itself.

U. Straccia proposed to use implication functions in the semantics of fuzzy GCIs. The big advantage is that now concept subsumption hold to a certain degree in $[0,1]$ [291]. F. Bobillo et al. proposed the use of KD implication in the semantics of GCIs [26, 27], providing the first reasoning algorithm with fuzzy GCIs. Some recent works consider Łukasiewicz implication [304, 305], Gödel implication [29, 32, 33] and Goguen (or product) implication [38].

V. Haarslev et al. presented a reasoning algorithm for KBs with fuzzy GCIs in the context of a more general framework, but it only works with S-implications [113, 115].

As an example of the importance of GCIs, A. Bahri et al. defined some similarity relation between concepts (`MoreGeneral`, `LessGeneral`, `Equivalent`, `Disjoint`, `Overlap`), which are reduced to GCIs [17, 18].

Finally, although U. Straccia proposed to use implication functions also in the semantics of fuzzy RIAs [291, 300], making role subsumption hold to a certain degree; [32, 33, 298] are the only works that currently support reasoning with them.

Families of fuzzy operators. Most of the research have been restricted to the Zadeh family of fuzzy operators.

A more general logic, the Łukasiewicz family, has also been investigated. A first category of works consider reasoning algorithms based on a mixture of tableau rules and MILP optimization problems. [304, 305] consider the logic $\mathcal{ALCF}(\mathbf{D})$ with GCIs, firstly proposing a key extension of the usual blocking conditions for expressive DLs. This work has been extended in several directions: [298] moves to $\mathcal{SHIF}$, while [40] considers $\mathcal{ALCQ}$, proposing a novel semantics for qualified number restrictions. On a different matter, H. Habiballa considered a fuzzy extension of $\mathcal{ALC}$ extended with role negation, top role and bottom role, presenting a novel reasoning algorithm based on resolution [118].

The use of other fuzzy operators has been more limited. In [32, 33] we combined Zadeh family with the use of Gödel implication in the semantics of GCIs and RIAs. [37] gives a step further and considers $\mathcal{SROIQ}$ under Gödel family. In Chapter 7, we finally consider fuzzy $\mathcal{SROIQ}(\mathbf{D})$. G. Stoilos et al. also considered Gödel family, but for the non expressive logics $\mathcal{EL}^{+}$ [283] and $\mathcal{EL}^{++}$ [195].

Product logic has been studied, but replacing Gödel negation with Łukasiewicz negation [38]. In this case, the reasoning algorithm is based on a mixture of tableau rules and Mixed Integer Quadratically Constrained Programming (MIQCP).

P. Hájek studied the use of a more general family of fuzzy operators [120, 122]. In particular, he proposed a fuzzy extension of $\mathcal{ALC}$ under arbitrary continuous t-norms and defined a reasoning algorithm for concept satisfiability and validity based on a reduction to propositional logic (more precisely, to BL). Later on, he extended the results to Rational Pavelka logic, making possible to introduce degrees as upper and lower bound of the fuzzy axioms [119].

He also introduced the notion of *witnessed model* [122, 120]. Consider a formula $\varphi(x; y_1, \ldots, y_n)$. A model $\mathcal{I}$ is witnessed iff it verifies: $\sup \varphi(x; y_1, \ldots, y_n) = (\varphi(v; y_1, \ldots, y_n))^{\mathcal{I}}$ and $\inf \varphi(x; y_1, \ldots, y_n) = (\varphi(v; y_1, \ldots, y_n))^{\mathcal{I}}$ for some individual v. Since fuzzy DLs are used in knowledge representation, non-witnessed models are not interesting; what is interesting for us are those role fillers which can be represented by specifying some particular individual of the domain. Quoting P. Hájek, *"for the aims of Description Logic non-witnessed models appear to be pathological"*. In particular, he shows that in Łukasiewicz logic, if a concept is satisfiable there always exists a witnessed model satisfying it; while in Gödel and Product logics there are concepts which have an infinite model but no witnessed model (for example, the concept $(\neg\forall R.A) \sqcap (\neg\exists R.\neg A)$). He also pointed out that, under a finite set of degrees of truth including 0 and 1, in Gödel logic all models are witnessed.

V. Haarslev et al. proposeD a general framework to represent uncertainty in DLs [112, 113, 114, 115, 116], where uncertainty is understood in a general sense. This way, fuzzy, probabilistic and possibilistic extensions are only particular cases of their general approach. The different degrees corresponding to the different formalism are represented using a lattice. Currently, they are able to support $\mathcal{ALC}$.

Fuzzy modifiers. In his seminal work, J. Yen allowed the application of some fuzzy modifiers (`Not`, `Very` and `Slightly`) to atomic concepts [340]. C. Tresp and R. Molitor also allowed the use of manipulators, a special case of triangular membership functions, as fuzzy modifiers [314].

S. Hölldobler et al. have widely worked in this field. Firstly they proposed the use of exponential modifiers. Their reasoning algorithm is an extension of [299], although the reduction of the BTVB problem to KB satisfiability is not valid anymore. Initially, they only allowed modifiers to be applied to atomic concepts (logic $\mathcal{ALC}_{\mathcal{FH}}$) [127], then they extended the work to complex concepts [128, 129]. As a minor comment, S. Singh et al. slightly changed the semantics of the modifiers in the context of an information retrieval problem application [274]. A later work considers linear modifiers which can be applied to concepts and (atomic) roles (logic $\mathcal{ALC}_{\mathcal{FLH}}$) [126].

Later on, the authors reported a problem of the previous works: the fact that modifiers are not associative. For example, the concept `VeryMol(C)` (where `Mol` is a shorthand for `MoreOrLess`) can be interpreted as `VeryMol(C)` (with `VeryMol` being a modifier) or `Very(Mol(C))` (with `Very` and `Mol` being two different modifiers). This is solved in [90] (logic $\mathcal{ALC}_{\mathcal{FL}}$), although they do not allow role modifiers any more.

U. Straccia allowed concept modifiers which are MILP representable to be used [295, 296]. FUZZYDL reasoner [39] implements this theoretical works, being the only current implementation allowing concept modifiers. In particular, it allows the use of modifiers defined in terms of linear hedges and triangular functions. In our logic, we support these concept modifiers, but in addition, we allow the application of linear modifiers over fuzzy roles.

Finally, S. Calegari et al. also suggested the use of role modifiers, but unfortunately they did not detail which membership function to use nor how to reason within their proposal [52, 53].

Fuzzy concrete domains. U. Straccia introduced the concept of fuzzy concrete domains, which enable one to express that an individual has a datatype which is true with a certain degree. A fuzzy datatype is represented using a membership function associated to the fuzzy set. [295, 296] provide a fuzzy extension of $\mathcal{ALC}(\mathbf{D})$ where fuzzy concrete domains are defined using trapezoidal, triangular, left-shoulder or right shoulder functions. Reasoning is achieved using a combination of tableau rules and MILP optimization problems. A subsequent extension to $\mathcal{SHOIN}(\mathbf{D})$ has also been proposed, although reasoning is not supported yet [291, 300].

In this work we allow trapezoidal numbers, which extend the previous cases.

H. Wang et al. extended to the fuzzy case [334] an existing approach for the classical case, which allows the use of customized datatypes in the OWL language (the extended language is called OWL-Eu [220]).

S. Schockaert argued that, instead of using general fuzzy DLs, crisp DLs with fuzzy concrete domains were preferable, because most of the times it is not very clear how to compute the degrees of the truth of the fuzzy axioms in the fuzzy KB, managing vagueness at a query processing level [262]. However, we know from the classical case that the concrete and the abstract interpretation domains (the interpretation domains of the KB and the fuzzy concrete domains) should be disjoint [14], so this approach does not allow vagueness to be introduced in abstract individuals.

The remaining approaches use a combination of crisp DLs and concrete domains managing fuzziness.

M. d'Aquin et al. considered $\mathcal{ALC}$ with fuzzy concrete domains (specifically, left-shoulder and right-should functions) [80]. They considered several types of concept subsumption and showed that reasoning may be performed by using a classical $\mathcal{ALC}$ reasoner together with a fuzzy datatype reasoner. However, their approach is not efficient to compute the GLB of a GCI. Another point to be kept in mind is that they lose too much useful information since they use the representation for fuzzy sets proposed in [94], which only considers the support and the kernel of a fuzzy set.

C. Barranco et al. considered fuzzy datatypes such as fuzzy numbers, fuzzy sets defined over a scalar domain, or collections of fuzzy atomic concepts among others, and proposed to represent them using a Fuzzy Object Relation Database Management System (FORDBMS) [20].

G. Nagypál used crisp ontologies with datatypes representing vague temporal knowledge [209].

Fuzzy DLs with cut concepts and roles. Cut concepts and roles as used in our logic is not original. Y. Li et al. proposed a family of the so-called extended fuzzy DLs, which use α-cuts as atomic concept and roles [190, 174]. For example, it is possible to express using $\forall R_{0.7}.C_{0.5}$ the set of individuals which are related with degree 0.7 using role R with some individual which belongs to concept C with degree at least 0.5.

A extension of $\mathcal{ALCN}$ is defined in [191]. Subsequent works propose different extension of DLs and tableau reasoning algorithms for KB satisfiability for $\mathcal{ALCH}$ [147, 184], $\mathcal{ALCN}$ [167, 177], $\mathcal{ALCQ}$ [173] and $\mathcal{S}$ [145]. They also consider the reduction to reasoning in crisp DLs for $\mathcal{ALCH}$ [147] and $\mathcal{ALCQ}$ [188, 192]. However, they usually assume restricted TBoxes, such as in [152, 151].

In this work, we will allow elements of the form $[C \geq \alpha], [C \leq \beta], [R \geq \alpha]$, and $[R \leq \beta]$.

Number restrictions. As early as 1991, J. Yen proposed a (somewhat strange) extension of number restrictions to the fuzzy case [340]. A more natural semantics was proposed in [291, 300], which was extended in [281]. An alternative semantics for number restrictions was also presented in [17, 18], although it is not an extension of the classical case. Reasoning with qualified number restrictions (restricted to $\mathcal{ALCIQ}$) is deal with in [282].

A very interesting alternative is the application of fuzzy quantification. Fuzzy quantifiers are linguistic labels representing imprecise quantities or percentages. D. Sanchez and A. Tettamanzi proposed a fuzzy extension of $\mathcal{ALCQ}$, where quantification is extended allowing the use of not only usual quantifiers $(\forall, \exists)$, but also fuzzy quantifiers defined using piecewise-linear membership functions [251].

The evaluation of the fuzzy quantifiers is computed using the method GD [84], which is based on a nonconvex definition of fuzzy cardinality [83]. A reasoning algorithm for fuzzy concept satisfiability was proposed in [252]. For a given concept, the algorithm computes the interval [LUB, GLB], so the BTVB problem is directly solved. [250] slightly modified their approach by defining the semantics of the quantifiers by using fuzzy concrete domains. The authors also showed than giving a fuzzy semantics to universal quantification and giving a crisp semantics to GCIs is inconsistent and lead to paradoxes [43].

The extension with fuzzy quantifiers greatly increases the expressive power of the language, but makes reasoning particularly hard. As a solution, [91] proposes to use non exact evolutionary algorithms to solve concept satisfiability, and applies the result to a real-world problem with promising results.

The semantics of qualified number restrictions is still open to discussion, as pointed out in [40], where a new semantics (which we use in this document) is also introduced.

Reductions to crisp DLs. The first effort in this direction is due to U. Straccia, who showed a reasoning preserving procedure to reduce fuzzy $\mathcal{ALCH}$ to its crisp version [302]. Additionally, it was the first time that fuzzy DLs with fuzzy GCIs were supported.

A similar work from him considers fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice [294], therefore supporting quantitative reasoning (by using the interval $[0,1]$) and qualitative reasoning (by relying on a set $\{\texttt{false}, \texttt{likelyfalse}, \texttt{unknown}, \texttt{likelytrue}, \texttt{true}\}$.

We extended the former work of U. Straccia to fuzzy $\mathcal{SHOIN}$ and allowed fuzzy GCIs with a semantics given by Kleene-Dienes implication [26, 27].

An empirical evaluation of the reduction for fuzzy $\mathcal{SHIN}$ (with non-fuzzy GCIs) has been reported in [67]. The authors consider ontologies with different complexities and study the scalability of the reduction (depending on the number of individuals) for fuzzy query answering. In general, the resulting ontologies were only feasible using an optimized reduction and a small numbers of degrees of truth, and fuzzy query answering is feasible in real time in those cases where query answering with respect to the original crisp ontology is.

G. Stoilos et al. extended this work and considered the reduction of a extension of fuzzy $\mathcal{SHOIN}$ with additional role axioms: general RIAs, reflexive, asymmetric and role disjointness axioms [281]. It is not a reduction of fuzzy $\mathcal{SROIQ}$ (not even $\mathcal{SROIN}$) because they do not show how to reduce:

- the universal role,
- qualified number restrictions,
- local reflexivity concepts in expressions of the form $\rho(\exists S.\texttt{Self}, \lhd\gamma)$,
- negative role assertions.

Another limitation of the work is that GCIs and RIAs are forced to be either true or false. They also define a fuzzy version of OWL 2, but their logic still does not include fuzzy concrete domains, fuzzy nominals, fuzzy GCIs nor fuzzy RIAs, so in our opinion the extension is not really complete.

We extended this work providing a crisp representation of full $\mathcal{SROIQ}$ with fuzzy GCIs and RIAs (using a semantics given by Gödel implication) [32, 33]. A crisp representation of full $\mathcal{SROIQ}(\mathbf{D})$ under Zadeh family is given in Chapter 6, while the use of Gödel family is described in Chapter 7.

Ad-hoc solutions to represent fuzzy DLs using OWL have also been presented [52, 53, 108, 280, 320, 319]. However, it is not possible to reason directly with the resulting crisp ontology, e.g. unsatisfiability of the fuzzy KB is not equivalent to unsatisfiability of the crisp KB.

A different approach is due to Y. Li et al., who introduced a family of fuzzy DLs using α-cuts as atomic concept and roles [190, 174]. The approach is slightly different because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce an $\mathcal{ALCQ}$ ABox [188, 192] and an $\mathcal{ALCH}$ concept [147] to their crisp versions. But unfortunately, both of these works assume an empty TBox.

D. Dubois et al. combined possibilistic and fuzzy logics in the context of DLs (more concretely, in $\mathcal{ALCIN}(\circ)$) [94]. Interestingly, they proposed to represent every fuzzy set using two crisp sets (its support and its core) and commented the possibility of extending their work by using more crisp sets (intermediate α-cuts), in order to have a more refined representation. Although for some applications this representation may be enough, there is a loss of information which do not come about in other approaches.

The previous work has restricted to Zadeh family, with the exception of a reduction of $\mathcal{ALCHIO}$ under Łukasiewicz family [41] (which is more general than Zadeh family). There are two important remarks which are worth to be kept it mind. Firstly, under this family is necessary to assume a fixed set of allowed degrees of truth. Secondly, from a practical point of view, the size of the resulting KB is much more complex in this case, so the practical feasibility of this approach has to be empirically verified.

Finally, we mention in passing that P. Vojtáš also sketched a reduction of fuzzy $\mathcal{EL}$ to crisp $\mathcal{EL}$ with concrete domains [325].

Reasoners. There exist several implementations of reasoners for fuzzy DLs.

- The oldest one is FUZZYDL [39], which is publicly available[3], and under constant update. It is perhaps the most worked reasoner, supporting some unique features. FUZZYDL extends fuzzy $\mathcal{SHIF}(\mathbf{D})$ with concept modifiers (using linear hedges and triangular functions), explicit definitions of fuzzy concepts (by means of triangular, trapezoidal, left-shoulder and right-shoulder functions), concrete features or *datatypes*, which can have a value with is an integer, a real or a string, among other novel constructors: weighted concepts, weighted sum concepts and threshold concepts. From a reasoning point of view, it is able to compute several queries, ranging from typical reasoning tasks such as the BDB, concept satisfiability and subsumption problems, to variable optimization and defuzzifications. FUZZYDL also supports both Zadeh and Łukasiewicz families of fuzzy operators, being the only one supporting it. Reasoning is based on a mixture of a tableau and a MILP optimization problem. Another interesting feature is that the degrees of the fuzzy axioms may not only be numerical constants, but also variables, thus being able to deal with unknown degrees of truth.

- DLMEDIA [306, 307][4] is an ontology-based multimedia information retrieval system combining logic-based retrieval with multimedia feature-based similarity retrieval. An ontology layer may be used to define the application domain, using DLR-Lite with fuzzy concrete domains expressing similarity relations between keywords.

- FIRE [272, 280] implements the tableau algorithm for fuzzy $\mathcal{SHIN}$ described in [289, 290]. It restricts itself to the Zadeh family. It is also publicly available[5], An interesting feature is its graphical interface, although users need to deal directly with the syntax of the language for the representation of the fuzzy KB. Moreover, it can serialize ontologies in fuzzy $\mathcal{SHIF}$ into RDF triples, and is integrated with classical RDF

[3] http://faure.isti.cnr.it/~straccia/software/fuzzyDL/fuzzyDL.html
[4] http://faure.isti.cnr.it/~straccia/software/DL-Media/DL-Media.html
[5] http://www.image.ece.ntua.gr/~nsimou/FiRE/

storing systems, which provide persistent storing and querying over large-scale fuzzy information [273].

- GURDL [114] supports an extension of $\mathcal{ALC}$ with an abstract and more general notion of uncertainty. The reasoning algorithm is also based on a mixture of tableau rules and the resolution of a set of inequations. Moreover, it implements some interesting techniques of optimization: lexical normalization, concept simplification, partitions based on individual connectivity and caching. The applicability of some techniques used in the crisp case is also studied.

- GERDS [118][6] implements a resolution algorithm for fuzzy $\mathcal{ALC}$ with role negation, top role and bottom role under Łukasiewicz family.

- KAON2[7] implements the reduction of fuzzy DLs to crisp DLs proposed in [302], as mentioned in [5]. An empirical study of the scalability of reasoning with fuzzy ontologies using this reasoner has been performed in [67].

- ONTOSEARCH2 [313, 224] [8] is the first scalable query engine for fuzzy ontologies. It implements an instance retrieval algorithm from a KB in fuzzy DL-Lite [221], allowing queries to be defined using a fuzzy extension of SPARQL [237].

- YADLR [159][9] is a recent implementation of a resolution-based algorithm for Łukasiewicz logic, which also allows dealing with unknown degrees of truth in the fuzzy assertions of the KB.

- Our proposal is DeLorean [30, 32, 33], which reduces reasoning in fuzzy $\mathcal{SROIQ}(\mathbf{D})$ under Zadeh or Gödel family to reasoning in crisp $\mathcal{SROIQ}(\mathbf{D})$, taking into account several interesting optimizations. As a consequence, it allows the reuse of classical languages and resources (editors, tools, reasoners . . .). For a full description, see Chapter 8.

[6] http://www.volny.cz/habiballa/files/gerds.zip
[7] http://kaon2.semanticweb.org/
[8] http://dipper.csd.abdn.ac.uk/OntoSearch/
[9] http://sourceforge.net/projects/yadlr/

Other Related Work

Complexity. In general, fuzzy DLs (under Zadeh family) belong to the same complexity classes as in the crisp case. This is probably the reason why complexity analysis has not received a lot of attention in the literature.

U. Straccia proved that the reasoning algorithm for KB satisfiability in fuzzy $\mathcal{ALC}$ with restricted TBoxes under Zadeh family is PSPACE [299]. Some extensions of this logic have also shown to be in PSPACE [127, 167, 252, 294, 303]. Recently, U. Keller and H. Heymans have shown that several reasoning tasks in $\mathcal{ALC}$ with GCIs are EXPTIME [155].

P. Bonatti y A. Tettamanzi analyzed the complexity of reasoning (concept satisfiability, KB satisfiability and concept subsumption) in the former logic extended with GCIs, obtaining some interesting results [44].

Tractable logics. In the last years, less expressive but tractable fuzzy DLs have started to receive attention.

U. Straccia proposed a fuzzy extension of DL-Lite and a novel reasoning algorithm to solve the problem of the fuzzy retrieval of a yes/no query [293]. Reusing previous reasoning algorithms would be very inefficient, because it would require to solve the BTVB for every individual of the KB, and then to rank the individuals according to this value.

Some related works considered a complementary perspective: crisp DL-Lite but allowing fuzzy predicates in the queries [246, 301]. Methods for a efficient handling of fuzzy retrieval are proposed, and implemented in the DLMEDIA system [306, 307].

J. Pan et al. proposed two query languages which increment the expressivity of the language proposed in [293]. In particular, they allow the representation of more general queries and the specification of different thresholds in them. They also introduced some reasoning algorithms, an implementation and a preliminar evaluation [221, 222].

P. Vojtáš proposed a fuzzy extension of $\mathcal{EL}$ [323, 324, 325]. His extension assumes that roles are crisp and makes possible to express aggregation functions with the aim of combining user preferences to obtain a global result. He did not allow concept negation either, assuming that it can be represented using atomic concepts. A short consideration about the combination of fuzzy $\mathcal{EL}$ and Bayesian $\mathcal{EL}$ can be found in [322, 326].

G. Stoilos et al. proposed a fuzzy extension of $\mathcal{EL}^+$ and efficient algorithms to compute fuzzy subsumption and hence to classify fuzzy ontologies under Gödel family [283]. In a subsequent work they considered a fuzzy extension of $\mathcal{EL}^{++}$ [195].

Qualitative reasoning. U. Straccia is the author of several works which allow qualitative reasoning in fuzzy DLs, that is, degrees of truth are not numbers, but they are taking from a certainty lattice. Complex concepts are defined using the lattice operators. For example, concept conjunction and disjunction are defined using meet and join operators, respectively. This is a more general approach than fuzzy logic. [294, 303] define the logic $\mathcal{L}$-$\mathcal{ALC}$ and provided an algorithm for KB satisfiability. If the lattice verifies some reasonable conditions (*safe lattices*), the algorithm is decidable. If the lattice verifies some additional conditions (*ps-safe lattices*), the complexity is PSPACE. However, the reduction of the BTBV problem to KB satisfiability deserves special attention.

D. Dinh-Khac et al. extended the expressivity of the logic by adding concept modifiers, given rise to the logic $\mathcal{ALC}_{\mathcal{FL}}$ [90]. In particular, they used a linear symmetric hedge algebra.

Another work which considers lattices is [347], which defines fuzzy interpretations based on boolean lattices. In order to obtain a fuzzy DL verifying laws of contradiction and excluded middle, Boolean lattice based membership degrees are mapped to Zadeh membership degrees. In this work the degree of truth of a fuzzy assertion is equal to the proportion of observers who think that the (crisp) assertion is true.

Applications. In addition to the purely theoretical works, fuzzy DLs have been applied to some particular application domains, such as medicine [202, 260], oncology [80], matchmaking and electronic commerce [5, 243, 244, 245, 246], information retrieval [274], multimedia information representation and retrieval [166, 199, 271, 269, 270, 306, 307], semantic web portals [350], semantic search engines [165], context representation [35] (an extension of the `cider` model [28, 34]), planning [70, 266], ontology mapping [339, 68], fuzzy control [39], modeling [144, 171, 349, 348], human models [273] or appraisal of Chinese spirits [109].

Miscellaneous. A *type-2* fuzzy DL has also been presented, allowing the definition of a lower and an upper bound for the membership degree of an individual to a fuzzy concept and a pair of individuals to a fuzzy role [165]. Such an interval can also be added to the GCIs. This approach generalizes type-1 fuzzy DLs. An hybrid model allowing type-1 and type-2 fuzzy sets has also been proposed [166].

An extension of $\mathcal{SI}$ with *vague sets*, allowing degrees of truth and degrees of falsity, has been presented in [194].

Another interesting extension of fuzzy DLs allows the use of *comparison expressions* [146, 149, 150, 175], making possible to represent, for instance, that "a is taller than b" without knowing explicitly to what extent the individuals a and b belong to the concept TALL.

Distributed fuzzy DLs have been introduced in [186, 176, 352], permitting to use e-connections to relate individuals belonging to different interpretation domains, corresponding to different fuzzy KBs [161].

Fuzzy DLs have also been *criticized* because the lack of an explicit examination of the underlying epistemological issues about the source of the numerical values causes several problems [267].

Beyond fuzzy DLs. *Possibilistic logic* has been used to extend DLs in order to deal with uncertain knowledge. Some works reduce reasoning to the classical case, taking advantage of existing DL reasoners [130, 239, 240]. Tableaux algorithms have also been proposed [72, 164, 238]. Information retrieval has been investigated as an application of possibilistic DLs [169], whereas trustworthiness in the Semantic Web has been considered a potential application [65]. Possibilistic extensions of fuzzy DLs have also been presented [31, 94], as we will see at the end of this chapter.

In addition to fuzzy and possibilistic logic, DLs have also been extended with other formalisms, such as probabilistic logic, neutrosophic logic, tetra-valued logic (for paraconsistent reasoning) and modal supervaluation logics.

There are also some extensions of DL programs, rule languages and other Semantic Web languages. For more information on some of these extensions we refer the reader to [297].

There have been some attempts to handle both *uncertainty and vagueness* in DLs. There exist several fuzzy and probabilistic / possibilistic extensions of DLs in the literature. These extensions are appropriate to handle either vagueness or uncertainty, but handling both of them has not received such attention. An exception is [94], where every fuzzy set is represented using two crisp sets (its support and its core) and then axioms are extended with necessity degrees. Although for some applications this representation may be enough (and the own authors suggest to consider more α-cuts), there is a loss of information.

Our approach combines possibilistic and fuzzy logics by building a possibilistic layer on top of a fuzzy DL [31], solving the mentioned loss of information.

Another related work combines fuzzy vagueness and probabilistic uncertainty but in the field of DL programs [180]. A fuzzy probabilistic version of OWL language has also been sketched [312].

Part III

Theoretical developments

CHAPTER 5

The Fuzzy Description Logic $\mathcal{SROIQ}(\mathbf{D})$

In this chapter we define the fuzzy DL $\mathcal{SROIQ}(\mathbf{D})$, which is a fuzzy extension of the logic $\mathcal{SROIQ}(\mathbf{D})$ where concepts denote fuzzy sets of individuals, roles denote fuzzy binary relations, and axioms are extended to the fuzzy case in such a way that some of them hold to a degree in $[0, 1]$. Then, we sketch how to build a layer to deal with uncertain knowledge on top of a fuzzy KB, by annotating the axioms with possibility and necessity degrees.

This chapter is organized as follows. Section 5.1 provides a formal definition of the syntax and semantics of the logic. Then, Section 5.2 studies its main logical properties. Finally, Section 5.3 shows how to build a possibilistic layer on top of the fuzzy DL.

5.1 Definition

Syntax

Definition 38 ***Fuzzy concrete domain*** *[295]. A* fuzzy concrete domain $\mathbf{D}$ *is a pair* $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$*, where:*

- $\Delta_{\mathbf{D}}$ *is a concrete interpretation domain.*

- $\Phi_{\mathbf{D}}$ *is a set of fuzzy concrete predicates* $\mathbf{d}$ *with an arity* n *and an interpretation* $\mathbf{d}_{\mathbf{D}} : \Delta_{\mathbf{D}}^n \to [0,1]$*, which is an* n*-ary fuzzy relation over* $\Delta_{\mathbf{D}}$.

For simplicity we assume arity 1.

Definition 39 ***Alphabet of fuzzy*** $\mathcal{SROIQ}(\mathbf{D})$**.** *Similarly as in their crisp counterpart, fuzzy* $\mathcal{SROIQ}(\mathbf{D})$ *assumes three alphabets of symbols, for concepts, roles and individuals.*

- *Let* n, m *be natural numbers (*$n \geq 0, m > 0$*) and* $\alpha_i \in (0,1]$*. The* concepts *(denoted* C *or* D*) of the language can be built inductively from atomic concepts (*A*), top concept* $\top$*, bottom concept* $\bot$*, named individuals (*o_i*), abstract roles (*R*), concrete roles (*T*), simple roles (*S*, which will be defined below) and fuzzy concrete predicates (*$\mathbf{d}$*) as:*

$$
\begin{array}{llrll}
C, D \to & & A & \mid & \textit{(atomic concept)} \\
& & \top & \mid & \textit{(top concept)} \\
& & \bot & \mid & \textit{(bottom concept)} \\
& & C \sqcap D & \mid & \textit{(concept conjunction)} \\
& & C \sqcup D & \mid & \textit{(concept disjunction)} \\
& & \neg C & \mid & \textit{(concept negation)} \\
& & \forall R.C & \mid & \textit{(universal quantification)} \\
& & \exists R.C & \mid & \textit{(existential quantification)} \\
& & \forall T.\mathbf{d} & \mid & \textit{(concrete universal quantification)} \\
& & \exists T.\mathbf{d} & \mid & \textit{(concrete existential quantification)} \\
& & \{\alpha_1/o_1, \dots, \alpha_m/o_m\} & \mid & \textit{(fuzzy nominals)} \\
& & (\geq m\, S.C) & \mid & \textit{(at-least qualified number restriction)} \\
& & (\leq n\, S.C) & \mid & \textit{(at-most qualified number restriction)} \\
& & (\geq m\, T.\mathbf{d}) & \mid & \textit{(concrete at-least qualified number restriction)} \\
& & (\leq n\, T.\mathbf{d}) & \mid & \textit{(concrete at-most qualified number restriction)} \\
& & \exists S.\mathtt{Self} & \mid & \textit{(local reflexivity)} \\
& & mod(C) & \mid & \textit{(modified concept)} \\
& & [C \geq \alpha] & \mid & \textit{(cut concept)} \\
& & [C \leq \beta] & & \textit{(cut concept)}
\end{array}
$$

- *The* abstract roles *(denoted R) of the language can be built inductively according to the following syntax rule:*

$$
\begin{array}{llll}
R \rightarrow & R_A & | & \textit{(atomic role)} \\
& R^- & | & \textit{(inverse role)} \\
& U & | & \textit{(universal role)} \\
& mod(R) & | & \textit{(modified role)} \\
& [R \geq \alpha] & | & \textit{(cut role)}
\end{array}
$$

  Concrete roles *are denoted T and cannot be complex*[1].

- Abstract individuals *are denoted $a, b \in \Delta^{\mathcal{I}}$, and* concrete individuals *are denoted $v \in \Delta_{\mathbf{D}}$.*

In the case of concepts, the only difference with the crisp case are fuzzy nominals [26], modified and cut concepts. Regarding roles, the only difference with the crisp case are modified and cut roles.

Note that negative cut roles of the form $[R \leq \beta]$ are somewhat equivalent to negated roles. Since negated roles cannot be arbitrarily used in crisp $\mathcal{SROIQ}$, we will not allow negative cut roles.

Example 5

- $\{1/\textsf{germany}, 1/\textsf{austria}, 0.67/\textsf{switzerland}\}$ *represents the concept of German-speaking country, with Germany and Austria fully belonging to it, but Switzerland belonging only with degree* 0.67.
- $\textsf{very}(\textsf{Tall})$ *represents the fuzzy set of individuals which are very tall.*
- $[\textsf{isFriendOf} \geq 0.8]$ *represents the pairs of individuals which are friends at least to degree* 0.8. □

In the rest of the paper we will assume $\bowtie \in \{\geq, <, \leq, >\}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$ and $\gamma \in [0, 1]$.

Definition 40 ***Symmetric and negation of an operator***. *Let $\bowtie \in \{\geq, <, \leq, >\}$. The symmetric $\bowtie^-$ and the negation $\neg \bowtie$ of an operator are defined as:*

[1] Extending the logic with concrete cut roles is immediate, but it is omitted here for simplicity of the presentation.

$\bowtie$	$\bowtie^-$	$\neg\bowtie$
$\geq$	$\leq$	$<$
$>$	$<$	$\leq$
$\leq$	$\geq$	$>$
$<$	$>$	$\geq$

Definition 41 ***Fuzzy Knowledge Base.*** *A fuzzy Knowledge Base comprises a fuzzy ABox $\mathcal{A}$, a fuzzy TBox $\mathcal{T}$ and a fuzzy RBox $\mathcal{R}$.*

- *A* fuzzy ABox *consists of a finite set of* fuzzy assertions *of one of the following types:*
  - *A fuzzy concept assertion, or constraint on the truth value of a concept assertion, $\langle a : C \geq \alpha\rangle$, $\langle a : C > \beta\rangle$, $\langle a : C \leq \beta\rangle$ or $\langle a : C < \alpha\rangle$.*
  - *A fuzzy role assertion, or constraint on the truth value of a role assertion, $\langle \Psi \geq \alpha\rangle$, $\langle \Psi > \beta\rangle$, $\langle \Psi \leq \beta\rangle$ or $\langle \Psi < \alpha\rangle$, where Ψ is of the form $(a, b) : R$, $(a, b) : \neg R$, $(a, v) : T$ or $(a, v) : \neg T$.*
  - *An inequality assertion $\langle a \neq b\rangle$.*
  - *An equality assertion $\langle a = b\rangle$.*
- *A* fuzzy TBox *consists of* fuzzy GCIs*, which constrain the truth value of a GCI i.e., they are expressions of the form $\langle C \sqsubseteq D \geq \alpha\rangle$ or $\langle C \sqsubseteq D > \beta\rangle$.*
- *A* fuzzy RBox *consists of a finite set of* role axioms *of the following types:*
  - Fuzzy RIAs $\langle w \sqsubseteq R \geq \alpha\rangle$, $\langle w \sqsubseteq R > \beta\rangle$, *where $w = R_1R_2 \ldots R_m$ is a role chain, $\langle T_1 \sqsubseteq T_2 \geq \alpha\rangle$, or $\langle T_1 \sqsubseteq T_2 > \beta\rangle$.*
  - Transitive *role axioms $\mathtt{trans}(R)$.*
  - Disjoint *role axioms $\mathtt{dis}(S_1, S_2)$, $\mathtt{dis}(T_1, T_2)$.*
  - Reflexive *role axioms $\mathtt{ref}(R)$.*
  - Irreflexive *role axioms $\mathtt{irr}(S)$.*
  - Symmetric *role axioms $\mathtt{sym}(R)$.*
  - Asymmetric *role axioms $\mathtt{asy}(S)$.*

The types of axioms in a KB are summarized in Table 5.1.

Note that all the role axioms except RIAs are syntactically equivalent to the crisp case.

Table 5.1: Syntax of the Axioms of the Fuzzy Description Logic $\mathcal{SROIQ}(\mathbf{D})$

Fuzzy ABox	
Concept assertion	$\langle a:C \geq \alpha \rangle, \langle a:C > \beta \rangle, \langle a:C \leq \beta \rangle, \langle a:C < \alpha \rangle$
Role assertion	$\langle (a,b):R \geq \alpha \rangle$, $\langle (a,b):R > \beta \rangle$, $\langle (a,b):R \leq \beta \rangle$, $\langle (a,b):R < \alpha \rangle$, $\langle (a,b):\neg R \geq \alpha \rangle$, $\langle (a,b):\neg R > \beta \rangle$, $\langle (a,b):\neg R \leq \beta \rangle$, $\langle (a,b):\neg R < \alpha \rangle$, $\langle (a,v):T \geq \alpha \rangle$, $\langle (a,v):T > \beta \rangle$, $\langle (a,v):T \leq \beta \rangle$, $\langle (a,v):T < \alpha \rangle$
Inequality assertion	$\langle a \neq b \rangle$
Equality assertion	$\langle a = b \rangle$
Fuzzy TBox	
General concept inclusion	$\langle C \sqsubseteq D \geq \alpha \rangle, \langle C \sqsubseteq D > \beta \rangle$
Fuzzy RBox	
Role inclusion axiom	$\langle R_1 R_2 \ldots R_m \sqsubseteq R \geq \alpha \rangle, \langle R_1 R_2 \ldots R_m \sqsubseteq R > \beta \rangle$, $\langle T_1 \sqsubseteq T_2 \geq \alpha \rangle, \langle T_1 \sqsubseteq T_2 > \beta \rangle$
Transitive role axiom	$\texttt{trans}(R)$
Disjoint role axiom	$\texttt{dis}(S_1, S_2), \texttt{dis}(T_1, T_2)$
Reflexive role axiom	$\texttt{ref}(R)$
Irreflexive role axiom	$\texttt{irr}(S)$
Symmetric role axiom	$\texttt{sym}(R)$
Asymmetric role axiom	$\texttt{asy}(S)$

Example 6

- *The fuzzy concept assertion* $\langle \mathsf{paul}: \mathsf{Tall} \geq 0.5 \rangle$ *states that Paul is tall with at least degree* 0.5.
- *The fuzzy RIA* $\langle \mathsf{isFriendOf}\ \mathsf{isFriendOf} \sqsubseteq \mathsf{isFriendOf} \geq 0.75 \rangle$ *states that the friends of my friends can also be considered my friends with at least degree* 0.75. □

We are ready now to formally define simple roles.

Definition 42 ***Simple role.*** *Simple roles are defined as in the crisp case:*

- R_A *is simple if it does not occur on the right side of a RIA.*
- R^- *is simple if* R *is.*
- *If* R *occurs on the right side of a RIA,* R *is simple if, for each* $\langle w \sqsubseteq R \rhd \gamma \rangle$, $w = S$ *for a simple role* S.

Note that concrete roles are always simple.

Definition 43 ***Positive fuzzy axiom.*** *A fuzzy axiom is* positive *(denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geq \alpha \rangle$ or $\langle \tau > \beta \rangle$.*

Definition 44 ***Negative fuzzy axiom.*** *A fuzzy axiom is* negative *(denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leq \beta \rangle$ or $\langle \tau < \alpha \rangle$.*

Notice that negative fuzzy GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs respectively, which are not part of crisp $\mathcal{SROIQ}(\mathbf{D})$.

$\langle \tau = \alpha \rangle$ is equivalent to the pair of axioms $\langle \tau \geq \alpha \rangle$ and $\langle \tau \leq \alpha \rangle$ [127].

Of course, if nothing is specified we assume that a fuzzy axiom is true with degree 1, so we can use the abbreviation:

$$\tau = \langle \tau \geq 1 \rangle$$

Now we will introduce some definitions which will be useful to impose some limitations in the expressivity of the language.

Definition 45 ***Strict partial order.*** *A strict partial order $\prec$ on a set S is an irreflexive and transitive relation on S.*

Definition 46 ***Regular order.*** *A strict partial order $\prec$ on the set of roles is called a regular order if it also satisfies $R_1 \prec R_2 \Leftrightarrow R_2^- \prec R_1$, for all roles R_1 and R_2.*

Definition 47 **$\prec$-regularity.** *A RIA $\langle w \sqsubseteq R \rhd \gamma \rangle$ is* $\prec$-regular *if $R = R_A$ and:*

- *$w = RR$, or*
- *$w = R^-$, or*
- *$w = S_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or*
- *$w = RS_1 \dots S_n$ and $S_i \prec R$ for all $i = 1, \dots, n$, or*
- *$w = S_1 \dots S_n R$ and $S_i \prec R$ for all $i = 1, \dots, n$.*

As in the crisp case, there are some restrictions in the use of roles, in order to guarantee the decidability of the logic:

- Firstly, some concept constructors require simple roles: non-concrete qualified number restrictions and local reflexivity.

- Some role axioms also require simple roles: disjoint, irreflexive and asymmetric role axioms.
- Role axioms cannot contain the universal role U.
- Finally, given a regular order $\prec$, every RIA should be $\prec$-regular.

Semantics

Definition 48 ***Fuzzy interpretation.*** *A fuzzy interpretation $\mathcal{I}$ with respect to a fuzzy concrete domain* $\mathbf{D}$ *is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:*

- *Every* abstract individual *a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.*
- *Every* concrete individual *v onto an element $v_{\mathbf{D}}$ of $\Delta_{\mathbf{D}}$.*
- *Every* concept *C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$.*
- *Every* abstract role *R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.*
- *Every* concrete role *T onto a function $T^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \to [0,1]$.*
- *Every n-ary* concrete fuzzy predicate $\mathbf{d}$ *onto the fuzzy relation $\mathbf{d}_{\mathbf{D}} : \Delta^{n}_{\mathbf{D}} \to [0,1]$.*
- *Every* modifier *mod onto a function $f_{mod} : [0,1] \to [0,1]$.*

Given arbitraries t-norm $\otimes$, t-conorm $\oplus$, negation function $\ominus$ and implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles *as shown in Table 5.2. Our semantics of fuzzy number restrictions is appropriated to several fuzzy logics, as shown in [40].*

The fuzzy interpretation function is extended to fuzzy axioms *in Table 5.3.*

$C^{\mathcal{I}}$ denotes the membership function of the fuzzy concept C with respect to the fuzzy interpretation $\mathcal{I}$. $C^{\mathcal{I}}(x)$ gives us the degree of being the individual x an element of the fuzzy concept C under $\mathcal{I}$.

Similarly, $R^{\mathcal{I}}$ denotes the membership function of the fuzzy role R with respect to $\mathcal{I}$. $R^{\mathcal{I}}(x,y)$ gives us the degree of being (x,y) an element of the fuzzy role R under $\mathcal{I}$.

As in the classical case, we do not impose Unique Name Assumption.

Table 5.2: Semantics of the Fuzzy Concepts and Roles in Fuzzy $\mathcal{SROIQ}(\mathbf{D})$

Constructor	Semantics
$(\top)^{\mathcal{I}}(x)$	1
$(\bot)^{\mathcal{I}}(x)$	0
$(A)^{\mathcal{I}}(x)$	$A^{\mathcal{I}}(x)$
$(C \sqcap D)^{\mathcal{I}}(x)$	$C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
$(C \sqcup D)^{\mathcal{I}}(x)$	$C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$
$(\neg C)^{\mathcal{I}}(x)$	$\ominus C^{\mathcal{I}}(x)$
$(\forall R.C)^{\mathcal{I}}(x)$	$\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$
$(\exists R.C)^{\mathcal{I}}(x)$	$\sup_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
$(\forall T.\mathbf{d})^{\mathcal{I}}(x)$	$\inf_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \Rightarrow \mathbf{d}_{\mathbf{D}}(v)\}$
$(\exists T.\mathbf{d})^{\mathcal{I}}(x)$	$\sup_{v \in \Delta_{\mathbf{D}}} \{T^{\mathcal{I}}(x,v) \otimes \mathbf{d}_{\mathbf{D}}(v)\}$
$(\{\alpha_1/o_1, \ldots, \alpha_m/o_m\})^{\mathcal{I}}(x)$	$\sup_{i \mid x = o_i^{\mathcal{I}}} \alpha_i$
$(\geq m\, S.C)^{\mathcal{I}}(x)$	$\sup_{y_1,\ldots,y_m \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{m} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \bigotimes (\otimes_{j<k} \{y_j \neq y_k\})]$
$(\leq n\, S.C)^{\mathcal{I}}(x)$	$\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})]$
$(\geq m\, T.\mathbf{d})^{\mathcal{I}}(x)$	$\sup_{v_1,\ldots,v_m \in \Delta_{\mathbf{D}}} [(\otimes_{i=1}^{m} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \bigotimes (\otimes_{j<k} \{v_j \neq v_k\})]$
$(\leq n\, T.\mathbf{d})^{\mathcal{I}}(x)$	$\inf_{v_1,\ldots,v_{n+1} \in \Delta_{\mathbf{D}}} [(\otimes_{i=1}^{n+1} \{T^{\mathcal{I}}(x,v_i) \otimes \mathbf{d}_{\mathbf{D}}(v_i)\}) \Rightarrow (\oplus_{j<k} \{v_j = v_k\})]$
$(\exists S.Self)^{\mathcal{I}}(x)$	$S^{\mathcal{I}}(x,x)$
$(mod(C))^{\mathcal{I}}(x)$	$f_{mod}(C^{\mathcal{I}}(x))$
$([C \geq \alpha])^{\mathcal{I}}(x)$	1 if $C^{\mathcal{I}}(x) \geq \alpha$, 0 otherwise
$([C \leq \beta])^{\mathcal{I}}(x)$	1 if $C^{\mathcal{I}}(x) \leq \beta$, 0 otherwise
$(R_A)^{\mathcal{I}}(x,y)$	$R_A^{\mathcal{I}}(x,y)$
$(R^-)^{\mathcal{I}}(x,y)$	$R^{\mathcal{I}}(y,x)$
$(U)^{\mathcal{I}}(x,y)$	1
$(mod(R))^{\mathcal{I}}(x,y)$	$f_{mod}(R^{\mathcal{I}}(x,y))$
$([R \geq \alpha])^{\mathcal{I}}(x,y)$	1 if $R^{\mathcal{I}}(x,y) \geq \alpha$, 0 otherwise
$(T)^{\mathcal{I}}(x,v)$	$T^{\mathcal{I}}(x,v)$

Table 5.3: Semantics of the Axioms in Fuzzy $\mathcal{SROIQ}(\mathbf{D})$

Axiom	Semantics
$(a:C)^{\mathcal{I}}$	$C^{\mathcal{I}}(a^{\mathcal{I}})$
$((a,b):R)^{\mathcal{I}}$	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$
$((a,b):\neg R)^{\mathcal{I}}$	$\ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$
$((a,v):T)^{\mathcal{I}}$	$T^{\mathcal{I}}(a^{\mathcal{I}}, v_{\mathbf{D}})$
$((a,v):\neg T)^{\mathcal{I}}$	$\ominus T^{\mathcal{I}}(a^{\mathcal{I}}, v_{\mathbf{D}})$
$(C \sqsubseteq D)^{\mathcal{I}}$	$\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
$(R_1 \ldots R_m \sqsubseteq R)^{\mathcal{I}}$	$\sup_{x_1 \ldots x_{n+1} \in \Delta^{\mathcal{I}}} (R_1^{\mathcal{I}}(x_1,x_2) \otimes \cdots \otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1})$
$(T_1 \sqsubseteq T_2)^{\mathcal{I}}$	$\sup_{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}}} T_1^{\mathcal{I}}(x,v) \Rightarrow T_2^{\mathcal{I}}(x,v)$

Definition 49 ***Satisfaction of a fuzzy axiom.*** *A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):*

- $\langle a : C \bowtie \gamma \rangle$ *iff* $(a : C)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a, b) : R \bowtie \gamma \rangle$ *iff* $((a, b) : R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a, b) : \neg R \bowtie \gamma \rangle$ *iff* $((a, b) : \neg R)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a, v) : T \bowtie \gamma \rangle$ *iff* $((a, v) : T)^{\mathcal{I}} \bowtie \gamma$,
- $\langle (a, v) : \neg T \bowtie \gamma \rangle$ *iff* $((a, v) : \neg T)^{\mathcal{I}} \bowtie \gamma$,
- $\langle a \neq b \rangle$ *iff* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $\langle a = b \rangle$ *iff* $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $\langle C \sqsubseteq D \rhd \gamma \rangle$ *iff* $(C \sqsubseteq D)^{\mathcal{I}} \rhd \gamma$,
- $\langle R_1 \ldots R_m \sqsubseteq R \rhd \gamma \rangle$ *iff* $(R_1 \ldots R_m \sqsubseteq R)^{\mathcal{I}} \rhd \gamma$,
- $\langle T_1 \sqsubseteq T_2 \rhd \gamma \rangle$ *iff* $(T_1 \sqsubseteq T_2)^{\mathcal{I}} \rhd \gamma$,
- $\mathtt{trans}(R)$ *iff* $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, z) \otimes R^{\mathcal{I}}(z, y)$,
- $\mathtt{dis}(S_1, S_2)$ *iff* $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = 0$ *or* $S_2^{\mathcal{I}}(x, y) = 0$,
- $\mathtt{dis}(T_1, T_2)$ *iff* $\forall x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}}, T_1^{\mathcal{I}}(x, v) = 0$ *or* $T_2^{\mathcal{I}}(x, v) = 0$,
- $\mathtt{ref}(R)$ *iff* $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, x) = 1$,
- $\mathtt{irr}(S)$ *iff* $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x, x) = 0$,
- $\mathtt{sym}(R)$ *iff* $\forall x, y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, y) = R^{\mathcal{I}}(y, x)$,
- $\mathtt{asy}(S)$ *iff* $\forall x, y \in \Delta^{\mathcal{I}}$, *if* $S^{\mathcal{I}}(x, y) > 0$ *then* $S^{\mathcal{I}}(y, x) = 0$,
- *a fuzzy KB iff it satisfies each element in $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{R}$.*

Notice that individual assertions are considered to be crisp, since the equality and inequality of individuals has always been considered crisp in the fuzzy DL literature [291, 289].

From a historical point of view, the reduction to a crisp DL was the first reasoning algorithm supporting to reason with fuzzy GCIs and RIAs.

As in the crisp case, fuzzy GCIs can be used to express some interesting axioms.

- *Disjointness of concepts*. The fact that $C_1 \dots C_n$ are disjoint can be expressed as $\langle C_1 \sqcap \cdots \sqcap C_n \sqsubseteq \bot \geq 1 \rangle$.
- *Domain* of a concept. The fact that concept C is the domain of a role R can be expressed as $\langle \top \sqsubseteq \forall R^{-}.C \geq 1 \rangle$ or $\langle \exists R.\top \sqsubseteq C \geq 1 \rangle$.
- *Range* of a concept. The fact that concept C is the range of a role R can be expressed as $\langle \top \sqsubseteq \forall R.C \geq 1 \rangle$.
- *Functionality* of a role. The fact that a role R is functional can be expressed as $\langle \top \sqsubseteq (\leq 1\, R.\top) \geq 1 \rangle$.

In the rest of the paper we will only consider fuzzy KB satisfiability, since (as in the crisp case) most inference problems can be reduced to it [299].

Example 7 *The following reasoning tasks can be reduced to fuzzy KB satisfiability:*

- Concept satisfiability. *C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K} \cup \{\langle x : C \geq \alpha \rangle\}$ is satisfiable, where x is a new individual, i.e., an individual which does not appear in $\mathcal{K}$.*
- Entailment*: A fuzzy concept assertion $a : C \bowtie \alpha$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle a : C \bowtie \alpha \rangle$) iff $\mathcal{K} \cup \{\langle a : C \neg\!\bowtie \alpha \rangle\}$ is unsatisfiable.*

  The case for fuzzy role assertions is similar. A fuzzy role assertion $(a, b) : R \bowtie \alpha$ is entailed by a fuzzy KB $\mathcal{K}$ (denoted $\mathcal{K} \models \langle (a, b) : R \bowtie \alpha \rangle$) iff $\mathcal{K} \cup \{\langle (a, b) : R \neg\!\bowtie \alpha \rangle\}$ is unsatisfiable.
- Greatest lower bound. *The greatest lower bound of a concept or role assertion τ is defined as the $\sup\{\alpha : \mathcal{K} \models \langle \tau \geq \alpha \rangle\}$. Under Łukasiewicz, Zadeh and Gödel semantics it can be computed by performing several entailment tests, more concretely at most $log|\mathcal{N}^{\mathcal{K}}|$ tests [299].*[2].
- Concept subsumption*: Under an S-implication, D subsumes C to degree α ($C \sqsubseteq D \geq \alpha$) w.r.t. a fuzzy KB $\mathcal{K}$ iff $C \sqcap \neg D$ is not α-satisfiable.* □

In order to manage correctly infima and suprema in the reasoning, we need to define the notion of *witnessed* interpretations or models [120].

[2]In Zadeh and Łukasiewicz logics, $\mathcal{N}^{\mathcal{K}}$ is the set of degrees of truth γ in the fuzzy KB together with their complementaries $1 - \gamma$ [299], while in Gödel logic we need to assume a finite set of degrees of truth $\mathcal{N}^{\mathcal{K}}$ including 0 and 1 [120].

Definition 50 ***Witnessed interpretations.*** *A fuzzy interpretation $\mathcal{I}$ is* witnessed *iff it verifies:*

- *for all $x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\exists R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$, and*
- *for all $x \in \Delta^{\mathcal{I}}$, there is $v \in \Delta_{\mathbf{D}}$ such that $(\exists T.\mathbf{d})^{\mathcal{I}}(x) = T^{\mathcal{I}}(x,v) \otimes \mathbf{d}_{\mathbf{D}}(v)$, and*
- *for all $x \in \Delta^{\mathcal{I}}$, there is $y \in \Delta^{\mathcal{I}}$ such that $(\forall R.C)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$, and*
- *for all $x \in \Delta^{\mathcal{I}}$, there is $v \in \Delta_{\mathbf{D}}$ such that $(\forall T.\mathbf{d})^{\mathcal{I}}(x) = T^{\mathcal{I}}(x,v) \Rightarrow \mathbf{d}_{\mathbf{D}}(v)$, and*
- *there is $x \in \Delta^{\mathcal{I}}$ such that $(C \sqsubseteq D)^{\mathcal{I}} = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$, and*
- *there are $x_1, \dots, x_{m+1} \in \Delta^{\mathcal{I}}$ such that $(R_1 \dots R_m \sqsubseteq R)^{\mathcal{I}} = (R_1^{\mathcal{I}}(x_1, x_2) \bigotimes \dots \bigotimes R_n^{\mathcal{I}}(x_m, x_{m+1})) \Rightarrow R^{\mathcal{I}}(x_1, x_{m+1})$, and*
- *there are $x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}}$ such that $(T_1 \sqsubseteq T_2)^{\mathcal{I}} = T_1^{\mathcal{I}}(x,v) \Rightarrow T_2^{\mathcal{I}}(x,v)$, and*
- *if $\mathcal{I} \models \mathit{trans}(R)$, for all $x, y \in \Delta^{\mathcal{I}}$, there is $z \in \Delta^{\mathcal{I}}$ such that $\sup_{z' \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z') \otimes R^{\mathcal{I}}(z',y) = R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$.*

Fuzzy nominals. Recall that previous approaches [284, 288, 291, 300] considered a crisp semantics defined as:

$$\{o_i\}^{\mathcal{I}}(x) = \begin{cases} 1 & \text{if } x = o_i^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, [284, 288] are restricted to Zadeh logic (so it uses maximum t-conorm), whereas [291, 300] do not show how to reason with nominals.

Now we will shortly justify our decision of fuzzifying the nominal construct by showing an example.

Example 8 *Suppose we want to represent the concept of country where German is a widely spoken language. Previous approaches allow the representation of a fuzzy disjunction of nominals $C \equiv \{\mathsf{germany}\} \sqcup \{\mathsf{austria}\} \sqcup \{\mathsf{switzerland}\}$.*

Since the semantics of the nominal construct is crisp, it forces switzerland *to fully belong to the concept or not, despite of German-speaking community of Switzerland represents only about two thirds of the total population of the country. On the contrary, following our approach we are able to define* $C \equiv \{1/\textit{germany}, 1/\textit{austria}, 0.67/\textit{switzerland}\}$. □

Let us comment the semantics of the fuzzy nominals (recall that it is defined as $\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}^{\mathcal{I}}(x) = \sup_{i \mid x = o_i^{\mathcal{I}}} \alpha_i$). Since we are not imposing Unique Name Assumption, it is possible that $x = o_i^{\mathcal{I}}$ for more than one o_i. This is the reason why we need to compute the supremum over the α_i associated to these named individuals o_i. And, of course, if $\forall i \in \{1, \ldots, m\}, x \neq o_i^{\mathcal{I}}$, then $\{\alpha_1/o_1, \ldots, \alpha_m/o_m\}^{\mathcal{I}}(x) = \sup \emptyset = 0$.

Note that previous approaches consider nominals to be crisp singletons arguing that they do not represent real-life concepts [284, 288, 291, 300]. In these approaches it is possible to represent a fuzzy disjunction of crisp singletons. However, we consider fuzzy nominals as proper fuzzy sets, which do represent real-life concepts. It is easy to see that our definition generalizes the previous definition for the nominal construct, as $\{o_1\} \sqcup \cdots \sqcup \{o_m\}$ is equivalent to $\{1/o_1, \ldots, 1/o_m\}$ under maximum t-conorm.

Sometimes it is possible to represent explicitly the vagueness of a concept by defining a fuzzy concrete domain [295], for example, a trapezoidal membership function defined over the rational numbers. However, sometimes there exist concepts without a subjacent semantic representation, so it is not possible or unnatural to define a fuzzy concrete domain (for example, the concept in Example 8). In these cases, fuzzy nominals make possible to explicitly define the membership function of a fuzzy set, stating the meaning that a fuzzy concept has for the ontology developer.

Fuzzy concrete domains. Apart from our work, FUZZYDL reasoner is the only current implementation allowing fuzzy concrete domains to be used. In particular, it allows the use of trapezoidal, triangular, left shoulder and right shoulder functions. In the rest of this work we will restrict ourselves to the trapezoidal membership function (see Chapter 2 for details) defined over an interval $[k_1, k_2]$ because it can be used to represent the other membership

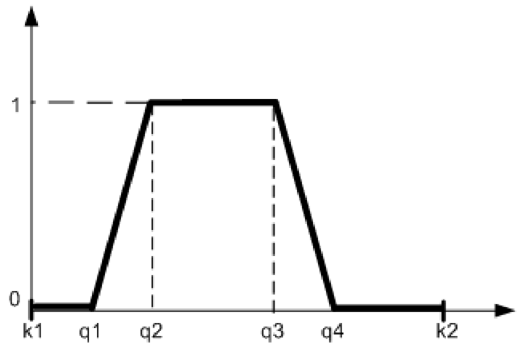

Figure 5.1: Trapezoidal membership function defined in $[k_1, k_2]$.

functions. Hence, we assume a unique fuzzy predicate $\mathbf{d} = trap_{k_1,k_2}(q_1, q_2, q_3, q_4)$ defined as in Figure 5.1.

Fuzzy modifiers. In the rest of this work we will restrict ourselves to the triangular modifier and the linear modifier, which are defined as follows:

- The semantics of a triangular modifier $mTri$ (Figure 5.2 (a)) is given by a function $f_{mTri}(x; t_1, t_2, t_3)$, where $t_1, t_2, t_3 \in [0, 1]$ and:

$$f_{mTri}(x; t_1, t_2, t_3) = \begin{cases} f_{left}(x; t_1, t_2, t_3) = t_1 + x(1 - t_1)/t_2 & x \in [0, t_2] \\ f_{right}(x; t_1, t_2, t_3) = 1 - (x - t_2)(1 - t_3)/(1 - t_2) & x \in [t_2, 1] \end{cases}$$

  Note that $f_{mTri}(0) = t_1$, $f_{mTri}(t_2) = 1$ and $f_{mTri}(1) = t_3$.

- The semantics of a linear modifier $mLin$ (Figure 5.2 (b)) is given by a function $f_{mLin}(x; l)$, with $l \in [0, 1]$, $l_1 = \frac{l}{l+1}$ and $l_2 = \frac{1}{l+1}$, defined as follows:

$$f_{mLin}(x; c) = \begin{cases} (l_2/l_1)x & x \in [0, l_1] \\ 1 - (x - 1)(1 - l_2)/(1 - l_1) & x \in [l_1, 1] \end{cases}$$

  Note that $f_{mLin}(0) = 0$, $f_{mLin}(l_1) = l_2$ and $f_{mLin}(1) = 1$.

The reason for this assumption is the fact that FUZZYDL, the only current implementation apart from our work allowing concept modifiers to be used,

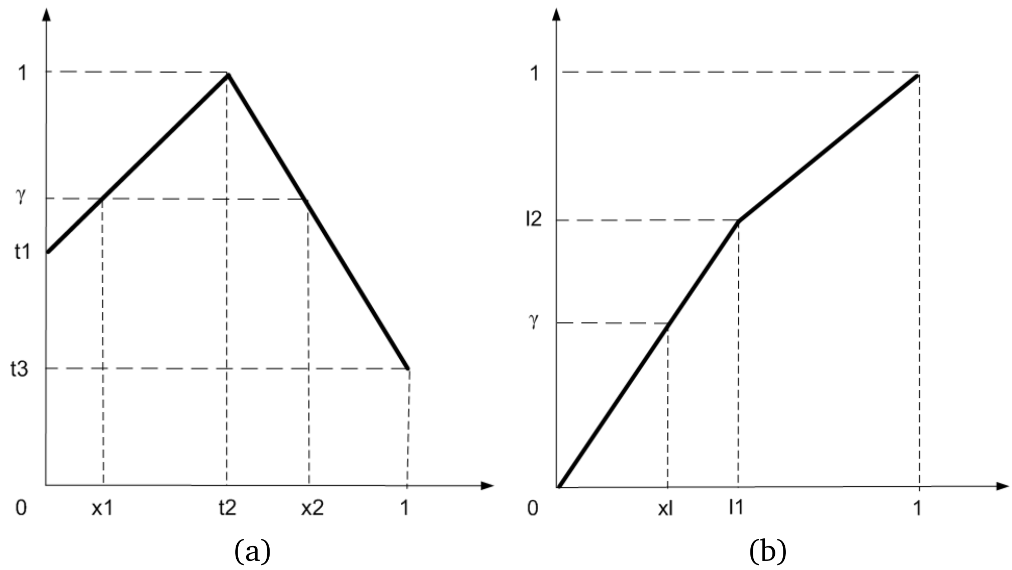

Figure 5.2: Popular examples of fuzzy modifiers (a) Triangular modifier; (b) Linear modifier.

only supports modifiers defined in terms of linear and triangular membership functions. In addition, we allow the application of linear modifiers over fuzzy roles.

5.2 Logical Properties

The following proposition shows that fuzzy $\mathcal{SROIQ}(\mathbf{D})$ is a sound extension of its crisp counterpart $\mathcal{SROIQ}(\mathbf{D})$.

Proposition 1 *Fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to* $\{0, 1\}$.

Due to the standard properties of the fuzzy operators, the following concept equivalences hold [299]:

$$\neg\top \equiv \bot,$$
$$\neg\bot \equiv \top,$$
$$C \sqcap \top \equiv C,$$
$$C \sqcup \bot \equiv C,$$
$$C \sqcap \bot \equiv \bot,$$
$$C \sqcup \top \equiv \top,$$
$$\exists R.\bot = \bot,$$
$$\forall R.\top = \top.$$

Laws of *excluded middle* and *contradiction* do not hold in general:

$$C \sqcup \neg C \not\equiv \top$$
$$C \sqcap \neg C \not\equiv \bot.$$

In the fuzzy DLs literature, the notation $f_i\mathcal{DL}$ has been proposed [288], where i is the fuzzy implication function considered. We point out, however, that this notation is not appropriate for the logic considered in [38], which combines Goguen implication with Łukasiewicz negation. The notation that we will use is $F\ \mathcal{DL}$ [298], where F is a symbol denoting a family of fuzzy operators such as Z for Zadeh, G for Gödel, Ł for Ł ukasiewicz or Π for Product.

Zadeh Family

In this subsection we will concentrate on $Z\ \mathcal{SROIQ}(\mathbf{D})$, restricting ourselves to the Zadeh family: minimum t-norm, maximum t-conorm, Łukasiewicz negation and KD implication.

The choice of the fuzzy operators implies that the following properties (which extend the crisp case) hold:

1. *Involution*: $\neg\neg C \equiv C$.
2. *De Morgan* laws: $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$ and $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$.
3. *Inter-definability of conjunction and disjunction*: $C \sqcap D \equiv \neg(\neg C \sqcap \neg D)$ and $C \sqcup D \equiv \neg(\neg C \sqcup \neg D)$.
4. *Idempotence* of conjunction and disjunction: $C \sqcap C \equiv C$ and $C \sqcup C \equiv C$.

5. *Inter-definability of quantifiers*: $\forall R.C = \neg\exists R.(\neg C)$ and $\exists R.C = \neg\forall R.(\neg C)$. Obviously, this is equivalent to $\neg\exists R.C = \forall R.(\neg C)$ and $\neg\forall R.C = \exists R.(\neg C)$.
6. *Inter-definability of number restrictions*: $(\leq n\ S.C) \equiv \neg(\geq m+1\ S.C)$ and $(\geq m\ S.C) \equiv \neg(\leq m-1\ S.C)$.

It would be possible to transform concept expressions into a semantically equivalent *Negation Normal Form* (NNF) [131], which is obtained by using the previous equivalences to push negation in front of atomic concepts, fuzzy nominals and local reflexivity concepts.

We can assume that negated role assertions of the form $\langle (a,b):\neg R \bowtie \gamma\rangle$ do not appear in the fuzzy KB (and similarly for concrete roles) due to the following equivalence:

$$\langle (a,b):\neg R \bowtie \gamma\rangle \equiv \langle (a,b):R \bowtie^{-} 1-\gamma\rangle$$

The use of KD implication in the semantics of fuzzy GCIs allows reasoning with *modus tolens* since they verify:

$$C \sqsubseteq D \equiv \neg D \sqsubseteq \neg C$$

Unfortunately, using KD in the semantics of fuzzy GCIs and RIAs brings about two counter-intuitive effects:

- Firstly, a concept does not fully subsume itself i.e. $C \sqsubseteq C$ iff $\inf_{x\in\Delta^{\mathcal{I}}} \max\{1-C^{\mathcal{I}}(x), C^{\mathcal{I}}(x)\} \geq 0.5$. The case for roles is similar: a role does not fully subsume itself since $R \sqsubseteq R$ iff $\inf_{x,y\in\Delta^{\mathcal{I}}} \max\{1-R^{\mathcal{I}}(x,y), R^{\mathcal{I}}(x,y)\} \geq 0.5$
- Secondly, crisp concept subsumption forces fuzzy concepts to be crisp i.e. $\langle C \sqsubseteq D \geq 1\rangle \Rightarrow \inf_{x\in\Delta^{\mathcal{I}}} \max\{1-C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \geq 1$ which is true iff for each element of the domain $D^{\mathcal{I}}(x) = 1$ or $1 - C^{\mathcal{I}}(x) \geq 1 \Leftrightarrow C^{\mathcal{I}}(x) = 0$. Again, the case for roles is similar.

These problems point out the need of alternative fuzzy operators. For example, using an R-implication it is always true that $a \Rightarrow b = 1$ if $a \leq b$, which would fix the first problem; while Łukasiewicz (as in [41]) or Gödel (as in the next subsection) implications fix the second one.

Similarly as in [292], $\mathcal{Z}\ \mathcal{SROIQ}(\mathbf{D})$ allows some sort of *modus ponens* over concepts and roles, even with the new semantics of fuzzy GCIs:

Proposition 2 *For* $\alpha, \beta \in [0,1]$, $\rhd \in \{\geq, >\}$ *and* $\alpha \not\rhd 1-\beta$ ($\not\geq = >, \not> = \geq$)*, the following properties are verified:*

(i) $\langle a : C \rhd \alpha \rangle$ *and* $\langle C \sqsubseteq D \rhd \beta \rangle$ *imply* $\langle a : D \rhd \beta \rangle$.

(ii) $\langle (a,b) : R \rhd \alpha \rangle$ *and* $\langle R \sqsubseteq R' \rhd \beta \rangle$ *imply* $\langle (a,b) : R' \rhd \beta \rangle$.

(iii) $\langle (a,b) : R \rhd \alpha \rangle$ *and* $\langle a : \forall R.C \rhd \beta \rangle$ *imply* $\langle b : C \rhd \beta \rangle$.

Proof.

(i) $\langle a : C \rhd \alpha \rangle$ implies $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \alpha$. $\langle C \sqsubseteq D \rhd \beta \rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \rhd \beta = \inf_{x \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)\} \rhd \beta$. Since this is true for the infimum, it is also true for $a^{\mathcal{I}}$, so $\max\{1 - C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})\} \rhd \beta$. But since $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \alpha$ and $\alpha \not\rhd 1-\beta$, it is not possible that $1 - C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \beta$. Hence, $D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \beta$, so $\langle a : D \rhd \beta \rangle$ holds.

(ii) $\langle (a,b) : R \rhd \alpha \rangle$ implies $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \alpha$. $\langle R \sqsubseteq R' \rhd \beta \rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, y) \Rightarrow R'^{\mathcal{I}}(x,y) \rhd \beta = \inf_{x,y \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(x,y), R'^{\mathcal{I}}(x,y)\} \rhd \beta$. Since this is true for the infimum, it is also true for the pair $(a^{\mathcal{I}}, b^{\mathcal{I}})$, so $\max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}), R'^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})\} \rhd \beta$. But since $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \alpha$ and $\alpha \not\rhd 1-\beta$, it is not possible that $1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \beta$. Hence, $R'^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \beta$, so $\langle a : R' \rhd \beta \rangle$ holds.

(iii) $\langle (a,b) : R \rhd \alpha \rangle$ implies $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \alpha$. $\langle a : \forall R.C \rhd \beta \rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a^{\mathcal{I}}, x) \Rightarrow C^{\mathcal{I}}(x) \rhd \beta = \inf_{x \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, x), C^{\mathcal{I}}(x)\} \rhd \beta$. Since this is true for the infimum, it is also true for $b^{\mathcal{I}}$, so $\max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}), C^{\mathcal{I}}(b^{\mathcal{I}})\} \rhd \beta$. But since $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \alpha$ and $\alpha \not\rhd 1-\beta$, it is not possible that $1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \beta$. Hence, $C^{\mathcal{I}}(b^{\mathcal{I}}) \rhd \beta$, so $\langle b : C \rhd \beta \rangle$ holds. □

Gödel Family

In this subsection, we will concentrate on $G\,\mathcal{SROIQ}$, restricting ourselves to the Gödel family: minimum t-norm, maximum t-conorm, Gödel negation and Gödel implication.

In general, Gödel logic does not have the witnessed model property, i.e., there can exist fuzzy KBs which have an infinite model, but they do not have a witnessed model. For instance, the concept $(\neg \forall R.A) \sqcap (\neg \exists R.(\neg A))$ [120]. However, due to the limited precision of computers, we will deal with a finite set of truth values, and in Gödel logic over a fixed finite subset of degrees of truth including 0 and 1, all models (finite or infinite) are witnessed [120]. We will use such a set of degrees of truth, and hence in our logic every interpretation $\mathcal{I}$ is witnessed.

The choice of the fuzzy operators implies the following properties:

1. Negation is not *involutive*: $\neg\neg C \not\equiv C$.
2. Law of *contradiction*: $C \sqcap \neg C \equiv \bot$.
3. *Idempotence* of conjunction and disjunction: $C \sqcap C \equiv C$ and $C \sqcup C \equiv C$.
4. *De Morgan* laws: $\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$ and $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$.
5. *Non inter-definability of conjunction and disjunction*: $C \sqcup D \not\equiv \neg(\neg C \sqcap \neg D)$ and $C \sqcap D \not\equiv \neg(\neg C \sqcup \neg D)$.
6. *Non inter-definability of quantifiers*: $\forall R.C \not\equiv \neg\exists R.(\neg C)$ and $\exists R.C \not\equiv \neg\forall R.(\neg C)$. Moreover, $\neg\forall R.C \not\equiv \exists R.(\neg C)$ but $\neg\exists R.C \equiv \forall R.(\neg C)$.
7. *Non inter-definability of qualified number restrictions*: $(\geq m\ S.C) \not\equiv \neg(\leq m-1\ S.C)$, but $(\leq n\ S.C) \equiv \neg(\geq n+1\ S.C)$.

Properties 1–5 follow immediately from the semantics of the fuzzy operators. Although in general quantifiers and qualified number restrictions are not inter-definable, the following proposition shows that two interesting equivalences hold.

Proposition 3 *Under G $\mathcal{SROIQ}$ the following properties hold:*

1. $\neg\exists R.C \equiv \forall R.(\neg C)$

2. $(\leq n\ S.C) \equiv \neg(\geq m+1\ S.C)$

Proof.

1. $(\neg\exists R.C)^{\mathcal{I}}(x) = \ominus(\sup_{y\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\})$. There are two possibilities.
    - $(\neg\exists R.C)^{\mathcal{I}}(x) = 1$ if the supremum is 0, that is, $\forall y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) = 0$, which is true if $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $C^{\mathcal{I}}(y) = 0$ holds. In other words, the value is 1 if there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$.

- $(\neg\exists R.C)^{\mathcal{I}}(x) = 0$ if the supremum is greater than 0, that is, $\exists y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) > 0$, which is true if $\exists y \in \Delta^{\mathcal{I}}$, with $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. In other words, the value is 0 if there exists some element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$.

Now, consider $(\forall R.(\neg C))^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\}$. Firstly, assume that there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. Then, $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $C^{\mathcal{I}}(y) = 0$ hold, which is equivalent to say that $\forall y \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(x,y) = 0$ or $(\neg C)^{\mathcal{I}}(y) = 1$ holds.

- If $R^{\mathcal{I}}(x,y) = 0$, then $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y) = 0 \Rightarrow (\neg C)^{\mathcal{I}}(y) = 1$.
- If $(\neg C)^{\mathcal{I}}(y) = 1$, then $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y) = R^{\mathcal{I}}(x,y) \Rightarrow 1 = 1$.

In any case, we end up with $\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\} = 1$.

Finally, assume that there exists some element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$. Then, $\exists y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$ holds. Hence, $\exists y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x,y) > 0$ and $(\neg C)^{\mathcal{I}}(y) = 0$ holds, and hence it satisfies $R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y) = 0$. So, $\inf_{y \in \Delta^{\mathcal{I}}} \{R^{\mathcal{I}}(x,y) \Rightarrow (\neg C)^{\mathcal{I}}(y)\} = 0$.

Summing up, in any case (either there does not exist any element y of the domain such that $R^{\mathcal{I}}(x,y) > 0$ and $C^{\mathcal{I}}(y) > 0$, or there exists such an element), $\neg\exists R.C \equiv \forall R.(\neg C)$.

2. $(\leq n\, S.C)^{\mathcal{I}}(x) = \inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})]$. Note that $\oplus_{j<k}\{y_j = y_k\}$ can be either 0 or 1, so the result of the Gödel implication is either 0 or 1 and hence $(\leq n\, S.C)$ is actually a crisp concept.

   - Let $\inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}} [(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})] = 0$. Then, there exist $y_1, \ldots, y_{n+1} \in \Delta^{\mathcal{I}}$ such that $[(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k} \{y_j = y_k\})] = 0$. This is true if there exist $n+1$ mutually different elements y_i such that $(\otimes_{i=1}^{n+1} \{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}) > 0$, that is, $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i)\}) > 0$.

- Assume that $\inf_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}[(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 1$. Then, $\forall y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}, [(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\}) \Rightarrow (\oplus_{j<k}\{y_j = y_k\})] = 1$. This is true in two cases:
  - $\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\} = 0$, so there exist some y_i such that $S^{\mathcal{I}}(x,y_i) = 0$ or $C^{\mathcal{I}}(y_i) = 0$ holds.
  - $\oplus_{j<k}\{y_j = y_k\} = 0$ holds.

  This means that there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$.

Now, consider $(\neg(\geq m{+}1\ S.C))^{\mathcal{I}}(x) = \ominus(\sup_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\})\bigotimes(\otimes_{j<k}\{y_j \neq y_k\}))$. Firstly, assume that there exist $n+1$ mutually different elements y_i such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i)\} > 0$. Then, $\sup_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\})\bigotimes(\otimes_{j<k}\{y_j \neq y_k\}) > 0$, so $\ominus(\sup_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\})\bigotimes(\otimes_{j<k}\{y_j \neq y_k\})) = 0$. Now, assume that there exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$. Then, $\sup_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\})\bigotimes(\otimes_{j<k}\{y_j \neq y_k\}) = 0$, so $\ominus(\sup_{y_1,\ldots,y_{n+1}\in\Delta^{\mathcal{I}}}(\otimes_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i)\otimes C^{\mathcal{I}}(y_i)\})\bigotimes(\otimes_{j<k}\{y_j \neq y_k\})) = 1$.

Summing up, in any case (either there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(x,y_i) > 0$ and $C^{\mathcal{I}}(y_i) > 0$, or there do exist such elements), $(\leq n\ S.C) \equiv \neg(\geq m{+}1\ S.C)$. □

In crisp DLs, the assertion $a : C$ is equivalent to the GCI $\{a\} \sqsubseteq C$. This can be extended to the fuzzy case, as the following proposition shows:

Proposition 4 *In fuzzy $\mathcal{SROIQ}$ under an R-implication, the following equivalence holds:*

$$\langle a : C \geq \alpha\rangle \equiv \langle\{\alpha/a\} \sqsubseteq C \geq 1\rangle$$

Proof. On the one hand, $\langle a : C \geq \alpha\rangle$ implies $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$. On the other hand, from $\langle\{\alpha/a\} \sqsubseteq C \geq 1\rangle$ and under an R-implication, we can deduce that, for every individual x of the domain, $(\{\alpha/a\})^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$. In particular, for $a^{\mathcal{I}}$ we have that $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq (\{\alpha/a\})^{\mathcal{I}}(a^{\mathcal{I}}) = \alpha$. □

Similarly as in [292], $G\,\mathcal{SROIQ}$ allows some sort of *modus ponens* and *chaining* of GCIs and RIAs:

Proposition 5 *For $\alpha, \beta \in [0,1]$ and $\triangleright \in \{\geq, >\}$, the following properties are verified:*

(i) $\langle a : C \triangleright \alpha \rangle$ *and* $\langle C \sqsubseteq D \triangleright \beta \rangle$ *imply* $\langle a : D \triangleright \alpha \otimes \beta \rangle$.

(ii) $\langle (a,b) : R \triangleright \alpha \rangle$ *and* $\langle R \sqsubseteq R' \triangleright \beta \rangle$ *imply* $\langle (a,b) : R' \triangleright \alpha \otimes \beta \rangle$.

(iii) $\langle C \sqsubseteq D \triangleright \alpha \rangle$ *and* $\langle D \sqsubseteq E \triangleright \beta \rangle$ *imply* $\langle C \sqsubseteq E \triangleright \alpha \otimes \beta \rangle$.

(iv) $\langle R \sqsubseteq R' \triangleright \alpha \rangle$ *and* $\langle R' \sqsubseteq R'' \triangleright \beta \rangle$ *imply* $\langle R \sqsubseteq R'' \triangleright \alpha \otimes \beta \rangle$.

Proof.

(i) $\langle a : C \triangleright \alpha \rangle$ implies $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \alpha$. $\langle C \sqsubseteq D \triangleright \beta \rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \beta$. Since this is true for the infimum, it is also true for $a^{\mathcal{I}}$, so $C^{\mathcal{I}}(a^{\mathcal{I}}) \Rightarrow D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \beta$. But from $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \alpha$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \Rightarrow D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \beta$, using *modus ponens* with Gödel implication, it follows that $D^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright \min\{\alpha, \beta\}$. Hence, $\langle a : D \triangleright \min\{\alpha, \beta\} \rangle$ holds.

(ii) $\langle (a,b) : R \triangleright \alpha \rangle$ implies $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \alpha$. $\langle R \sqsubseteq R' \triangleright \beta \rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, y) \Rightarrow R'^{\mathcal{I}}(x,y) \triangleright \beta$. Since this is true for the infimum, in particular $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \Rightarrow R'^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \beta$. Similarly as in the previous case, using *modus ponens* with Gödel implication, it follows that $R'^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \min\{\alpha, \beta\}$. Hence, $\langle (a,b) : R' \triangleright \min\{\alpha, \beta\} \rangle$ holds.

(iii) $\langle C \sqsubseteq D \triangleright \alpha \rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \alpha$, and $\langle D \sqsubseteq E \triangleright \beta \rangle$ implies $\inf_{x \in \Delta^{\mathcal{I}}} D^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$. Now, for an individual x there are three possibilities:

1. $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x) \leq E^{\mathcal{I}}(x)$. It follows that $C^{\mathcal{I}}(x) \leq E^{\mathcal{I}}(x)$ and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) = 1 \triangleright \min\{\alpha, \beta\}$.
2. $C^{\mathcal{I}}(x) > D^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x) \leq E^{\mathcal{I}}(x)$. From $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \triangleright \alpha$ it follows that $D^{\mathcal{I}}(x) \triangleright \alpha$. Since $E^{\mathcal{I}}(x) \geq D^{\mathcal{I}}(x)$, then $E^{\mathcal{I}}(x) \triangleright \alpha$. Since the result of Gödel implication is either 1 or $E^{\mathcal{I}}(x)$, $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \alpha$, and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \min\{\alpha, \beta\}$.
3. $D^{\mathcal{I}}(x) > E^{\mathcal{I}}(x)$. From $D^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$ it follows that $E^{\mathcal{I}}(x) \triangleright \beta$. Since the result of Gödel implication is either 1 or $E^{\mathcal{I}}(x)$, $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \beta$, and hence $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \triangleright \min\{\alpha, \beta\}$.

In summary, for every element x of the domain we can always conclude that $C^{\mathcal{I}}(x) \Rightarrow E^{\mathcal{I}}(x) \rhd \min\{\alpha, \beta\}$, so $\langle C \Rightarrow E \rhd \min\{\alpha, \beta\}\rangle$ holds.

(iv) $\langle R \sqsubseteq R' \rhd \alpha\rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow R'^{\mathcal{I}}(x,y) \rhd \alpha$, and $\langle R' \sqsubseteq R'' \rhd \beta\rangle$ implies $\inf_{x,y \in \Delta^{\mathcal{I}}} R'^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \beta$. Now, for an individual x, y there are three possibilities:

1. $R^{\mathcal{I}}(x,y) \leq R'^{\mathcal{I}}(x,y)$ and $R'^{\mathcal{I}}(x,y) \leq R''^{\mathcal{I}}(x,y)$. It follows that $R^{\mathcal{I}}(x,y) \leq R''^{\mathcal{I}}(x,y)$ and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) = 1 \rhd \min\{\alpha, \beta\}$.
2. $R^{\mathcal{I}}(x,y) > R'^{\mathcal{I}}(x,y)$ and $R'^{\mathcal{I}}(x,y) \leq R''^{\mathcal{I}}(x,y)$. From $R^{\mathcal{I}}(x,y) \Rightarrow R'^{\mathcal{I}}(x,y) \rhd \alpha$ it follows that $R'^{\mathcal{I}}(x,y) \rhd \alpha$. Since $R''^{\mathcal{I}}(x,y) \geq R'^{\mathcal{I}}(x,y)$, then $R''^{\mathcal{I}}(x,y) \rhd \alpha$. Since the result of Gödel implication is either 1 or $R''^{\mathcal{I}}(x,y)$, $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \alpha$, and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \min\{\alpha, \beta\}$.
3. $R'^{\mathcal{I}}(x,y) > R''^{\mathcal{I}}(x,y)$. From $R'^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \beta$ it follows that $R''^{\mathcal{I}}(x,y) \rhd \beta$. Since the result of Gödel implication is either 1 or $R''^{\mathcal{I}}(x,y)$, $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \beta$, and hence $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \min\{\alpha, \beta\}$.

Hence, in every case and for every pair of elements x, y of the domain we can conclude that $R^{\mathcal{I}}(x,y) \Rightarrow R''^{\mathcal{I}}(x,y) \rhd \min\{\alpha, \beta\}$, so $\langle R \Rightarrow R'' \rhd \min\{\alpha, \beta\}\rangle$ holds. □

Irreflexive, transitive and symmetric role axioms are syntactic sugar for any R-implication (and consequently it can be assumed that they do not appear in fuzzy KBs) due to some equivalences with fuzzy GCIs and RIAs.

Proposition 6 *In fuzzy $\mathcal{SROIQ}$ under an R-implication, the following equivalences hold:*

- $\mathtt{irr}(S) \equiv \langle \top \sqsubseteq \neg \exists S.\mathtt{Self} \geq 1\rangle$,
- $\mathtt{trans}(R) \equiv \langle RR \sqsubseteq R \geq 1\rangle$,
- $\mathtt{sym}(R) \equiv \langle R \sqsubseteq R^{-} \geq 1\rangle$.

Proof.

- On the one hand, $\texttt{irr}(S)$ implies that $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$. On the other hand, $\langle \top \sqsubseteq \neg\exists S.\texttt{Self} \geq 1 \rangle$ implies that, for every individual x of the domain, $(\top)^{\mathcal{I}}(x) \Rightarrow (\neg\exists S.\texttt{Self})^{\mathcal{I}}(x) \geq 1$. Since it is an R-implication, $(\top)^{\mathcal{I}}(x) = 1 \leq (\neg\exists S.\texttt{Self})^{\mathcal{I}}(x)$. Due to the standard properties of negation functions, $(\neg\exists S.\texttt{Self})^{\mathcal{I}}(x) \geq 1$ implies that $(\exists S.\texttt{Self})^{\mathcal{I}}(x) = 0$. Hence, $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$.

- On the one hand, $\texttt{trans}(R)$ implies that $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$. On the other hand, $\langle RR \sqsubseteq R \geq 1 \rangle$ implies that, for every pair of individuals (x,y) of the domain, $\sup_{z \in \Delta^{\mathcal{I}}}(R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)) \Rightarrow R^{\mathcal{I}}(x,y) \geq 1$. Since it is an R-implication, it follows that $R^{\mathcal{I}}(x,y) \geq \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$.

- On the one hand, $\texttt{sym}(R)$ implies that $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. On the other hand, $\langle R \sqsubseteq R^{-} \geq 1 \rangle$ implies that, for every pair of individuals (x,y) of the domain, $R^{\mathcal{I}}(x,y) \Rightarrow (R^{-})^{\mathcal{I}}(x,y) \geq 1$. Since it is an R-implication, $R^{\mathcal{I}}(x,y) \leq (R^{-})^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. But if we consider the pair (y,x), it follows than $R^{\mathcal{I}}(y,x) \leq (R^{-})^{\mathcal{I}}(y,x) = R^{\mathcal{I}}(x,y)$. Hence, $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. □

Joining Zadeh and Gödel Logics

There is no conceptual problem in assuming an arbitrary combination of the fuzzy operators of Zadeh and Gödel logics in our context (recall that our interpretations are witnessed). This has also been done in for Zadeh, Gödel and Łukasiewicz logics [39].

Another interesting possibility is to have a representation language supporting the use of two types of GCIs and RIAs $\sqsubseteq_{KD}$ y $\sqsubseteq_{G}$ (with semantics based on KD and Gödel implications respectively). This is similar to other works which allow several types of subsumption (for instance, [193]). This way, the ontology developer would be free to choose the better option for his own needs. In general, Gödel implication provides better logical properties than KD, but KD for example makes possible to reason with *modus tolens*.

5.3 A Possibilistic Extension

Imprecision/vagueness and uncertainty are very different conceptually, being possible to manage them by means of fuzzy and possibilistic logics, respectively.

In this section we will combine them by means of a possibilistic and fuzzy extension of an ontology. In particular, we build a layer to deal with uncertain knowledge on top of a fuzzy KB, by annotating the axioms with possibility and necessity degrees.

Syntax

Definition 51 ***Possibilistic fuzzy knowledge base.*** *Let* $\alpha \in (0,1]$. *A possibilistic fuzzy KB* $\mathcal{P}$ *is a fuzzy KB* $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ *where each fuzzy axiom* $\tau \in \mathcal{A}$ *is equipped with one of the following degrees:*

- *A* possibility degree $(\tau, \Pi\ \alpha)$.
- *A* necessity degree $(\tau, N\ \alpha)$.

The possibility degree expresses to what extent a formula is possible, whereas the necessity degree expresses to what extent a formula is necessarily true.

Example 9 *The following axiom means that it is possible with degree* 0.2 *that Tom is considered a high person with (at least) degree* 0.5*:*

$$(\langle \mathit{tom} \colon \mathit{High} \geq 0.5 \rangle, \Pi\ 0.2)$$

□

By default axioms are interpreted as necessarily true, i.e., if no degree is specified, τ is interpreted as $(\tau, N\ 1)$.

Semantics

Firstly, we define the notion of negation a fuzzy axiom. We will restrict ourselves to fuzzy axioms in a fuzzy ABox, that is, fuzzy concept and role assertions.

Definition 52 ***Negation of an $\mathcal{SROIQ}(\mathbf{D})$ fuzzy axiom.*** *Let τ be a fuzzy axiom in an $\mathcal{SROIQ}(\mathbf{D})$ ABox. The negation $\neg\tau$ is defined as such a way that a fuzzy interpretation $\mathcal{I}$ satisfies τ iff $\mathcal{I}$ does not satisfy $\neg\tau$.*

Proposition 7 *Let ϕ be a concept or role assertion in $\mathcal{SROIQ}(\mathbf{D})$. Then:*

$$\neg(\langle \phi \bowtie \gamma \rangle) = \langle \phi \neg \bowtie \gamma \rangle$$

Proof. Assume that $\phi = a : C$. A fuzzy interpretation $\mathcal{I}$ satisfies $\langle \phi \bowtie \gamma \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$. Then, $\mathcal{I}$ does not satisfy $C^{\mathcal{I}}(a^{\mathcal{I}}) \neg \bowtie \gamma$. Hence, $\mathcal{I}$ does not satisfy $\neg(\langle a : C \bowtie \gamma) = \neg\tau$. The cases $\phi = (a, b) : R$, $\phi = (a, b) : \neg R$, $\phi = (a, v) : T$, $\phi = (a, v) : \neg T$ are similar. □

Definition 53 ***Possibilistic interpretation.*** *Let $\mathfrak{I}$ be the set of all (possibly fuzzy) interpretations. A possibilistic interpretation is a mapping $\pi : \mathfrak{I} \rightarrow [0, 1]$ such that $\pi(\mathcal{I}) = 1$ for some $\mathcal{I} \in \mathfrak{I}$.*

The intuition here is that $\pi(\mathcal{I})$ represents the degree to which the world $\mathcal{I}$ is possible. $\mathcal{I}$ is impossible if $\pi(\mathcal{I}) = 0$ and fully possible if $\pi(\mathcal{I}) = 1$.

Definition 54 ***Possibility of an axiom.*** *The possibility of a fuzzy axiom τ in a possibilistic interpretation π is defined as:*

$$Poss(\tau) = \sup\{\pi(\mathcal{I}) \mid \mathcal{I} \in \mathfrak{I}, \mathcal{I} \models \tau\}$$

where $\sup \emptyset = 0$.

Definition 55 ***Necessity of an axiom.*** *The necessity of a fuzzy axiom τ in a possibilistic interpretation π is defined as:*

$$Nec(\tau) = 1 - Poss(\neg\tau)$$

The reason for the restriction to fuzzy axioms in the fuzzy ABox is the following. According to the latter two definitions, the possibility and the necessity of an axiom is based on the notion of entailment, which in the fuzzy DLs literature has only been defined for fuzzy concept and role assertions [299]. A similar definition could be given for the rest of axioms in $\mathcal{SROIQ}(\mathbf{D})$, but most of them seem to be counter-intuitive or unnatural. For example, let $\mathcal{N}^{\mathcal{K}}$ be the set of relevant degrees of truth of a fuzzy KB $\mathcal{K}$. We could define the negation of a transitive role axiom by introducing some new individuals preventing the role to be transitive:

$$\neg(\mathtt{trans}(R) = \langle(x_1, x_2):R \geq \alpha\rangle \cup \langle(x_2, x_3):R \geq \alpha\rangle \cup \langle(x_1, x_3):R < \alpha\rangle$$

where x_1, x_2, x_3 are new individuals in $\mathcal{K}$, and for some $\alpha \in \mathcal{N}^{\mathcal{K}}$. But note that this is true for some α and we do not know exactly for which of them (so we would need to try with all of them).

CHAPTER 6

A Crisp Representation for Fuzzy $\mathcal{SROIQ}(\mathbf{D})$ under Zadeh Family

In this chapter we show how to reduce a $\mathcal{Z}$ $\mathcal{SROIQ}(\mathbf{D})$ fuzzy KB to a crisp KB. The procedure preserves reasoning, in such a way that existing $\mathcal{SROIQ}(\mathbf{D})$ reasoners could be applied to the resulting KB. First we will describe the reduction and then we will provide an illustrating example.

The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics. Finally, every axiom in the ABox, the TBox and the RBox is represented, independently of other axioms, using these new crisp elements.

Section 6.1 studies the relevant set of degrees of truth to be considered in the reasoning. Section 6.2 describes the process of creation of new crisp concepts and roles. Then, the elements in the fuzzy DL are mapped: abstract concepts and roles (Section 6.3), concrete predicates (Section 6.4), axioms (Section 6.5) and modified concepts and roles (Section 6.7). An illustrating example is presented in Section 6.6. Section 6.8 studies some properties of the reduction. Some important optimizations of the reasoning procedure are described in Section 6.9. Finally, Section 6.10 shows how to reason with a possibilistic extension of the fuzzy DL.

6.1 Relevant Set of Degrees of Truth

Let **A** be the set of atomic concepts, **R** the set of atomic abstract roles and **T** the set of concrete roles in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$.

U. Straccia showed [302] that the set of the degrees which must be considered for any reasoning task in fuzzy $Z\ \mathcal{ALCH}$ is composed of those degrees appearing in the fuzzy KB together with their complementaries. Formally, the set of degrees is defined as

$$N^{\mathcal{K}} = X^{\mathcal{K}} \cup \{1 - \gamma \mid \gamma \in X^{\mathcal{K}}\}$$

where $X^{\mathcal{K}} = \{0, 0.5, 1\} \cup \{\gamma \mid \langle \tau \bowtie \gamma \rangle \in \mathcal{K}\}$. This also holds in $Z\ \mathcal{SROIQ}(\mathbf{D})$, but it must be noted that it is not necessarily true when other fuzzy operators are considered.

Without loss of generality, it can be assumed that $N^{\mathcal{K}} = \{\gamma_1, \ldots, \gamma_{|N^{\mathcal{K}}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leq i \leq |N^{\mathcal{K}}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|N^{\mathcal{K}}|} = 1$.

6.2 Adding New Elements

For each $\alpha, \beta \in N^{\mathcal{K}}$ with $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, for each $A \in \mathbf{A}$, two new atomic concepts $A_{\geq \alpha}, A_{> \beta}$ are introduced. $A_{\geq \alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e. the α-cut of A. $A_{> \beta}$ is defined in a similar way.

Similarly, for each $R_A \in \mathbf{R}$ and for each $T \in \mathbf{T}$ two new atomic abstract roles $R_{A \geq \alpha}, R_{A > \beta}$ and two new concrete roles $T_{\geq \alpha}, T_{> \beta}$ are introduced.

The atomic elements $A_{>1}, R_{A>1}, T_{>1}, A_{\geq 0}, R_{A \geq 0}$ and $T_{\geq 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB. For example, we do not allow fuzzy axioms of the forms $\tau \geq 0$, $\tau > 1$, $\tau \leq 1$ or $\tau < 0$.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq |N^{\mathcal{K}}| - 1, 2 \leq j \leq |N^{\mathcal{K}}| - 1$ and for each $A \in \mathbf{A}$, $T(N^{\mathcal{K}})$ is the smallest T-Box containing these two axioms:

$$A_{\geq\gamma_{i+1}} \sqsubseteq A_{>\gamma_i} \qquad A_{>\gamma_j} \sqsubseteq A_{\geq\gamma_j}$$

Similarly, for each $R_A \in \mathbf{R}$, $R_a(N^{\mathcal{K}})$ is the smallest R-Box containing:

$$R_{A\geq\gamma_{i+1}} \sqsubseteq R_{A>\gamma_i} \qquad R_{A>\gamma_i} \sqsubseteq R_{A\geq\gamma_i}$$

and for each $T \in \mathbf{T}$, $R_c(N^{\mathcal{K}})$ is the smallest R-Box containing:

$$T_{\geq\gamma_{i+1}} \sqsubseteq T_{>\gamma_i} \qquad T_{>\gamma_i} \sqsubseteq T_{\geq\gamma_i}$$

Now we will introduce some customized datatypes which will be used to represent membership trapezoidal functions. $real[a,b]$ denotes a real number defined in the interval $[a,b]$ and can be defined in OWL 2 syntax as follows:

```
<owl:DataRange rdf:about="#real[a,b]">
  <owl2:onDataRange rdf:resource="&xsd;double"/>
  <owl2:minInclusive rdf:datatype="&xsd;double">a</owl2:minInclusive>
  <owl2:maxInclusive rdf:datatype="&xsd;double">b</owl2:maxInclusive>
</owl:DataRange>
```

$real(a,b)$ denotes a real number defined in (a,b) and can be defined similarly as before, but using `owl2:minExclusive` and `owl2:maxExclusive` instead of `owl2:minInclusive` and `owl2:maxInclusive`.

Finally, $union\text{–}real[k_1, a, b, k_2]$ stands for a real number in $[k_1, a] \cup [b, k_2]$ and can be defined in the following way:

```
<owl:DataRange rdf:about="#union-real[k1,a,b,k2]">
  <owl:complementOf>
    <owl:DataRange>
      <owl2:onDataRange rdf:resource="real(a,b)"/>
    </owl:DataRange>
  </owl:complementOf>
  <owl2:minInclusive rdf:datatype="&xsd;double">k1</owl2:minInclusive>
  <owl2:maxInclusive rdf:datatype="&xsd;double">k2</owl2:maxInclusive>
</owl:DataRange>
```

Notice that these customized datatypes are just subsets of $\mathbb{R}$, that is, double numbers with a restricted set of allowed values. Hence, it is possible to use current available algorithms to reason with customized real numbers. Observe that using our own datatype representing trapezoidal membership functions is possible but we would need to implement a new reasoning procedure capable of dealing with them.

6.3 Mapping Fuzzy Concepts and Roles

Fuzzy concept and role expressions are reduced using mapping ρ, as shown in Tables 6.1 and 6.2. Concrete predicates are reduced in Section 6.4, while modified concept and roles are discussed in Section 6.7.

Given a fuzzy concept C, $\rho(C, \geq \alpha)$ is the α-cut of C, a crisp set containing all the elements which belong to C with a degree greater or equal than α. The other cases $\rho(C, \bowtie \gamma)$ are similar.

Given a fuzzy role R, $\rho(R, \geq \alpha)$ is a crisp set containing all the pair of elements which are related through R with a degree greater or equal than α. The other cases $\rho(R, \bowtie \gamma)$ and $\rho(T, \bowtie \gamma)$ are similar.

Obviously, the mapping verifies some interesting properties. Firstly, we have the following equivalences:

$$\rho(C, \bowtie \gamma) \equiv \neg \rho(C, \neg \bowtie \gamma)$$
$$\rho(R, \bowtie \gamma) \equiv \neg \rho(R, \neg \bowtie \gamma)$$

It is also interesting to remark that $\rho(A, \leq \beta) = \neg A_{>\beta}$ is different from $\rho(\neg A, \geq \alpha) = \rho(A, \leq 1 - \alpha) = \neg A_{>1-\alpha}$.

Finally, due to the restrictions in the definition of the fuzzy KB, some expressions cannot appear during the process:

- $\rho(A, \geq 0), \rho(A, > 1), \rho(A, \leq 1), \rho(A, < 0)$.
- $\rho(\top, \geq 0), \rho(\top, > 1), \rho(\top, \leq 1), \rho(\top, < 0)$.
- $\rho(\bot, \geq 0), \rho(\bot, > 1), \rho(\bot, \leq 1), \rho(\bot, < 0)$.
- $\rho(R_A, \geq 0), \rho(R_A, > 1), \rho(R_A, \leq 1), \rho(R_A, < 0)$.
- $\rho(T, \geq 0), \rho(T, > 1), \rho(T, \leq 1), \rho(T, < 0)$.
- $\rho(U, \geq 0), \rho(U, > 1), \rho(U, \leq 1), \rho(U, < 0)$.
- $\rho(R, \lhd \gamma), \rho(U, \lhd \gamma)$ and $\rho(T, \lhd \gamma)$ can only appear in a (crisp) negated role assertion.

Table 6.1: Mapping of concept expressions under Zadeh semantics.

x	y	$\rho(x,y)$
$\top$	$\triangleright\gamma$	$\top$
$\top$	$\triangleleft\gamma$	$\bot$
$\bot$	$\triangleright\gamma$	$\bot$
$\bot$	$\triangleleft\gamma$	$\top$
A	$\triangleright\gamma$	$A_{\triangleright\gamma}$
A	$\triangleleft\gamma$	$\neg A_{\neg\triangleleft\gamma}$
$\neg C$	$\bowtie\gamma$	$\rho(C, \bowtie^{-} 1-\gamma)$
$C \sqcap D$	$\triangleright\gamma$	$\rho(C,\triangleright\gamma) \sqcap \rho(D,\triangleright\gamma)$
$C \sqcap D$	$\triangleleft\gamma$	$\rho(C,\triangleleft\gamma) \sqcup \rho(D,\triangleleft\gamma)$
$C \sqcup D$	$\triangleright\gamma$	$\rho(C,\triangleright\gamma) \sqcup \rho(D,\triangleright\gamma)$
$C \sqcup D$	$\triangleleft\gamma$	$\rho(C,\triangleleft\gamma) \sqcap \rho(D,\triangleleft\gamma)$
$\exists R.C$	$\triangleright\gamma$	$\exists\rho(R,\triangleright\gamma).\rho(C,\triangleright\gamma)$
$\exists R.C$	$\triangleleft\gamma$	$\forall\rho(R,\neg\triangleleft\gamma).\rho(C,\triangleleft\gamma)$
$\exists T.\mathbf{d}$	$\triangleright\gamma$	$\exists\rho(T,\triangleright\gamma).\rho(\mathbf{d},\triangleright\gamma)$
$\exists T.\mathbf{d}$	$\triangleleft\gamma$	$\forall\rho(T,\neg\triangleleft\gamma).\rho(\mathbf{d},\triangleleft\gamma)$
$\forall R.C$	$\{\geq,>\}\gamma$	$\forall\rho(R,\{>,\geq\}1-\gamma).\rho(C,\{\geq,>\}\gamma)$
$\forall R.C$	$\triangleleft\gamma$	$\exists\rho(R,\triangleleft^{-}1-\gamma).\rho(C,\triangleleft\gamma)$
$\forall T.\mathbf{d}$	$\{\geq,>\}\gamma$	$\forall\rho(T,\{>,\geq\}1-\gamma).\rho(\mathbf{d},\{\geq,>\}\gamma)$
$\forall T.\mathbf{d}$	$\triangleleft\gamma$	$\exists\rho(T,\triangleleft^{-}1-\gamma).\rho(\mathbf{d},\triangleleft\gamma)$
$\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$	$\bowtie\gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq n\}$
$\geq m\, S.C$	$\triangleright\gamma$	$\geq m\, \rho(S,\triangleright\gamma).\rho(C,\triangleright\gamma)$
$\geq m\, S.C$	$\triangleleft\gamma$	$\leq m-1\, \rho(S,\neg\triangleleft\gamma).\rho(C,\neg\triangleleft\gamma)$
$\geq m\, T.\mathbf{d}$	$\triangleright\gamma$	$\geq m\, \rho(T,\triangleright\gamma).\rho(\mathbf{d},\triangleright\gamma)$
$\geq m\, T.\mathbf{d}$	$\triangleleft\gamma$	$\leq m-1\, \rho(T,\neg\triangleleft\gamma).\rho(\mathbf{d},\neg\triangleleft\gamma)$
$\leq n\, S.C$	$\{\geq,>\}\gamma$	$\leq n\, \rho(S,\{>,\geq\}\,1-\gamma).\rho(C,\{>,\geq\}\,1-\gamma)$
$\leq n\, S.C$	$\triangleleft\gamma$	$\geq n+1\, \rho(S,\triangleleft^{-}\,1-\gamma).\rho(C,\triangleleft^{-}\,1-\gamma)$
$\leq n\, T.\mathbf{d}$	$\{\geq,>\}\gamma$	$\leq n\, \rho(T,\{>,\geq\}\,1-\gamma).\rho(\mathbf{d},\{>,\geq\}\,1-\gamma)$
$\leq n\, T.\mathbf{d}$	$\triangleleft\gamma$	$\geq n+1\, \rho(T,\triangleleft^{-}\,1-\gamma).\rho(\mathbf{d},\triangleleft^{-}\,1-\gamma)$
$\exists S.\mathtt{Self}$	$\triangleright\gamma$	$\exists\rho(S,\triangleright\gamma).\mathtt{Self}$
$\exists S.\mathtt{Self}$	$\triangleleft\gamma$	$\neg\exists\rho(S,\neg\triangleleft\gamma).\mathtt{Self}$
$[C \geq \alpha]$	$\triangleright\gamma$	$\rho(C,\geq\alpha)$
$[C \geq \alpha]$	$\triangleleft\gamma$	$\rho(C,<\alpha)$
$[C \leq \beta]$	$\triangleright\gamma$	$\rho(C,\leq\beta)$
$[C \leq \beta]$	$\triangleleft\gamma$	$\rho(C,>\beta)$

Table 6.2: Mapping of role expressions under Zadeh semantics.

x	y	$\rho(x,y)$
R_A	$\triangleright\gamma$	$R_{A\triangleright\gamma}$
R_A	$\triangleleft\gamma$	$\neg R_{A\neg\triangleleft\gamma}$
R^-	$\bowtie\gamma$	$\rho(R,\bowtie\gamma)^-$
U	$\triangleright\gamma$	U
U	$\triangleleft\gamma$	$\neg U$
$[R \geq \alpha]$	$\triangleright\gamma$	$\rho(R,\geq \alpha)$
$[R \geq \alpha]$	$\triangleleft\gamma$	$\rho(R,< \alpha)$
T	$\triangleright\gamma$	$T_{\triangleright\gamma}$

6.4 Mapping Concrete Predicates

Concrete predicate expressions are reduced using mapping ρ, as shown in Table 6.3. We recall that we are allowing only fuzzy concrete predicates $\mathbf{d} = trap_{k_1,k_2}(q_1,q_2,q_3,q_4)$.

In the case $< \alpha$ we use a sufficiently small number $\epsilon > 0$ which simulates strict inequalities, since $a \geq b + \epsilon$ is equivalent to $a > b$.

Table 6.3: Mapping of concrete expressions.

x	y	$\rho(x,y)$
$\mathbf{d}$	$\geq \alpha$	$real[q_1 + \alpha(q_2 - q_1), q_4 - \alpha(q_4 - q_3)]$
$\mathbf{d}$	$> \beta$	$real(q_1 + \beta(q_2 - q_1), q_4 - \beta(q_4 - q_3))$
$\mathbf{d}$	$\leq \beta$	$union\text{–}real[k_1, q_1 + \beta(q_2 - q_1), q_4 - \beta(q_4 - q_3), k_2]$
$\mathbf{d}$	$< \alpha$	$union\text{–}real[k_1, q_1 + \alpha(q_2 - q_1) - \epsilon, q_4 - \alpha(q_4 - q_3) + \epsilon, k_2]$

6.5 Mapping Fuzzy Axioms

Axioms are reduced as in Table 6.4, where $\kappa(\tau)$ maps a fuzzy axiom τ in $Z\ \mathcal{SROIQ}(D)$ to a set of crisp axioms in $\mathcal{SROIQ}(D)$. We note $\kappa(\mathcal{A})$ (resp.

$\kappa(\mathcal{T})$, $\kappa(\mathcal{R})$) the union of the reductions of all the fuzzy axioms in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$)[1].

Example 10 *Let us consider some cases of the reduction.*

- *Consider an assertion* $\langle a : \forall R.C \geq \alpha \rangle$. *If it is satisfied, there exists a fuzzy interpretation* $\mathcal{I}$ *such that* $\inf_{y \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, y), C^{\mathcal{I}}(y)\} \geq \alpha$. *For an arbitrary* y, $R^{\mathcal{I}}(a^{\mathcal{I}}, y) \leq 1 - \alpha$ *or* $C^{\mathcal{I}}(y) \geq \alpha$ *must hold. Hence, if* $R^{\mathcal{I}}(a^{\mathcal{I}}, y) \leq 1 - \alpha$ *is not satisfied (i.e.,* $R^{\mathcal{I}}(a^{\mathcal{I}}, y) > 1 - \alpha$*), then we deduce that* $C^{\mathcal{I}}(y) \geq \alpha$*, which is the semantics of the crisp assertion* $a : \forall \rho(R, > 1 - \alpha).\rho(C, \geq \alpha)$.
- *Consider* $\langle a : (\geq m\, S.C) \leq \beta \rangle$. *If it is satisfied, it follows that* $\sup_{y_1,\dots,y_m \in \Delta^{\mathcal{I}}} (\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, y_i) \otimes C^{\mathcal{I}}(y_i)\}) \leq \beta$*, so there cannot exist* m *different individuals* y_i *with* $(\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, y_i) \otimes C^{\mathcal{I}}(y_i)\}) > \beta$*, and it follows that the crisp assertion* $a : (\leq m-1\; \rho(S, > \beta).\rho(C, > \beta))$ *is satisfied.* □

In the definition of the logic we have not allowed negative fuzzy GCIs and RIAs because they do not have a clear equivalence in the crisp case. However, it would be possible to include fuzzy GCIs in the logic. The interesting difference is that they can be reduced to concept assertions respectively. The reduction of these axioms has also been included in Table 6.4, where x is a new individual (not appearing in the original fuzzy KB).

Summing up, a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced to a KB $\texttt{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R}) \rangle$.

6.6 Examples

In this section, we will illustrate the procedure with a couple of examples. Firstly, we will present an example concerning interchange of medical knowledge. Then, we will consider an accommodation domain.

[1] More precisely, the reduction of transitive and symmetric role axioms should be noted as $\kappa(\tau, N^{\mathcal{K}})$. Similarly, the reduction of the fuzzy RBox should be noted as $\kappa(\mathcal{R}, N^{\mathcal{K}})$ respectively. However, for the sake of simplicity we omit $N^{\mathcal{K}}$ since it is clear from the context.

Table 6.4: Reduction of the axioms under Zadeh semantics.

τ	$\kappa(\tau)$
$\langle a : C \bowtie \gamma \rangle$	$\{a : \rho(C, \bowtie \gamma)\}$
$\langle (a,b) : R \bowtie \gamma \rangle$	$\{(a,b) : \rho(R, \bowtie \gamma)\}$
$\langle (a,v) : T \bowtie \gamma \rangle$	$\{(a,v) : \rho(T, \bowtie \gamma)\}$
$\langle a \neq b \rangle$	$\{a \neq b\}$
$\langle a = b \rangle$	$\{a = b\}$
$\langle C \sqsubseteq D \geq \alpha \rangle$	$\{\rho(C, > 1-\alpha) \sqsubseteq \rho(D, \geq \alpha)\}$
$\langle C \sqsubseteq D > \beta \rangle$	$\{\rho(C, \geq 1-\beta) \sqsubseteq \rho(D, > \beta)\}$
$\langle C \sqsubseteq D \leq \beta \rangle$	$\{x : \rho(C, \geq 1-\beta) \sqcap \rho(D, \leq \beta)\}$
$\langle C \sqsubseteq D < \alpha \rangle$	$\{x : \rho(C, > 1-\alpha) \sqcap \rho(D, < \alpha)\}$
$\langle R_1 \dots R_m \sqsubseteq R \geq \alpha \rangle$	$\{\rho(R_1, > 1-\alpha) \dots \rho(R_m, > 1-\alpha) \sqsubseteq \rho(R, \geq \alpha)\}$
$\langle R_1 \dots R_m \sqsubseteq R > \beta \rangle$	$\{\rho(R_1, \geq 1-\beta) \dots \rho(R_m, \geq 1-\beta) \sqsubseteq \rho(R, > \beta)\}$
$\langle T_1 \sqsubseteq T_2 \geq \alpha \rangle$	$\{\rho(T_1, > 1-\alpha) \sqsubseteq \rho(T_2, \geq \alpha)\}$
$\langle T_1 \sqsubseteq T_2 > \beta \rangle$	$\{\rho(T_1, \geq 1-\beta) \sqsubseteq \rho(T_2, > \beta)\}$
$\mathtt{trans}(R)$	$\bigcup_{\gamma \in N^{\mathcal{K}} \setminus \{0\}} \{\mathtt{trans}(\rho(R, \geq \gamma))\} \bigcup_{\gamma \in N^{\mathcal{K}} \setminus \{1\}} \{\mathtt{trans}(\rho(R, > \gamma))\}$
$\mathtt{dis}(S_1, S_2)$	$\{\mathtt{dis}(\rho(S_1, > 0), \rho(S_2, > 0))\}$
$\mathtt{dis}(T_1, T_2)$	$\{\mathtt{dis}(\rho(T_1, > 0), \rho(T_2, > 0))\}$
$\mathtt{ref}(R)$	$\{\mathtt{ref}(\rho(R, \geq 1))\}$
$\mathtt{irr}(S)$	$\{\mathtt{irr}(\rho(S, > 0))\}$
$\mathtt{sym}(R)$	$\bigcup_{\gamma \in N^{\mathcal{K}} \setminus \{0\}} \{\mathtt{sym}(\rho(R, \geq \gamma))\} \bigcup_{\gamma \in N^{\mathcal{K}} \setminus \{1\}} \{\mathtt{sym}(\rho(R, > \gamma))\}$
$\mathtt{asy}(S)$	$\{\mathtt{asy}(\rho(S, > 0)\}$

Example 11 *A known issue in health-care support is that consensus in the used vocabulary is required to achieve understanding among different physicians and systems. Medical taxonomies are an effort in this direction, as they provide a well-defined catalogue of codes to label diseases univocally. Two examples are ICD*[2] *(for general medicine) and DSM-IV [10] (for mental disorders), which identify prototypical clinical medical profiles with a name and a code. Medical taxonomies have been developed to be essentially crisp, so they can be transcribed almost directly to OWL.*

However, vagueness could be introduced at different levels of the taxonomy so that richer semantics would be represented:

- *In order to associate diagnostic codes to patient electronic records, fuzzy assertions would be be useful, allowing the knowledge base to contain*

[2]http://www.who.int/classifications/icd/en/

statements such as "Patient001's Serotonin Level is quite low" or "Patient001's disease is likely to be an Obsessive-Compulsive Disorder".

- *In the current version of DSM-IV, "Substance-Induced Anxiety Disorder" is defined as a subclass of "Substance-Related Disorder". A fuzzy GCI could express that a "Substance-Induced Anxiety Disorder can be partially considered a Substance-Related Disorder", as well as an "Anxiety Disorder".*

Assume a fuzzy KB $\mathcal{K}$ representing this knowledge:

$$\mathcal{K} = \{$$
$$\langle \mathit{patient001} : \exists \mathit{hasSerotoninLevel}.\mathit{HighLevel} \leq 0.25 \rangle,$$
$$\langle \mathit{patient001} : \exists \mathit{hasDisease}.\mathit{ObsessiveCompulsiveDisorder} \geq 0.75 \rangle,$$
$$\langle \mathit{SubstanceInducedAnxietyDisorder} \sqsubseteq \mathit{AnxietyDisorder} \geq 0.75 \rangle$$
$$\}$$

Firstly, we compute the number of degrees of truth to be considered: $X^{\mathcal{K}} = N^{\mathcal{K}} = \{0, 0.25, 0.5, 1, 0.75\}$.

Next, we create some new elements and some axioms preserving their semantics. The new axioms in $R_a(N^{\mathcal{K}})$, due to the new atomic roles, are:

$$R_a(N^{\mathcal{K}}) = \{$$
$$\mathit{hasSerotoninLevel}_{\geq 1} \sqsubseteq \mathit{hasSerotoninLevel}_{>0.75},$$
$$\mathit{hasSerotoninLevel}_{>0.75} \sqsubseteq \mathit{hasSerotoninLevel}_{\geq 0.75},$$
$$\mathit{hasSerotoninLevel}_{\geq 0.75} \sqsubseteq \mathit{hasSerotoninLevel}_{>0.5},$$
$$\mathit{hasSerotoninLevel}_{>0.5} \sqsubseteq \mathit{hasSerotoninLevel}_{\geq 0.5},$$
$$\mathit{hasSerotoninLevel}_{\geq 0.5} \sqsubseteq \mathit{hasSerotoninLevel}_{>0.25},$$
$$\mathit{hasSerotoninLevel}_{>0.25} \sqsubseteq \mathit{hasSerotoninLevel}_{\geq 0.25},$$
$$\mathit{hasSerotoninLevel}_{\geq 0.25} \sqsubseteq \mathit{hasSerotoninLevel}_{>0},$$
$$\mathit{hasDisease}_{\geq 1} \sqsubseteq \mathit{hasDisease}_{>0.75},$$
$$\mathit{hasDisease}_{>0.75} \sqsubseteq \mathit{hasDisease}_{\geq 0.75},$$
$$\mathit{hasDisease}_{\geq 0.75} \sqsubseteq \mathit{hasDisease}_{>0.5},$$
$$\mathit{hasDisease}_{>0.5} \sqsubseteq \mathit{hasDisease}_{\geq 0.5},$$
$$\mathit{hasDisease}_{\geq 0.5} \sqsubseteq \mathit{hasDisease}_{>0.25},$$
$$\mathit{hasDisease}_{>0.25} \sqsubseteq \mathit{hasDisease}_{\geq 0.25},$$
$$\mathit{hasDisease}_{\geq 0.25} \sqsubseteq \mathit{hasDisease}_{>0}$$
$$\}$$

The new axioms in $T(N^{\mathcal{K}})$, due to the new atomic concepts, are:

$T(N^{\mathcal{K}}) = \{$

$HighLevel_{\geq 1} \sqsubseteq HighLevel_{>0.75}$,
$HighLevel_{>0.75} \sqsubseteq HighLevel_{\geq 0.75}$,
$HighLevel_{\geq 0.75} \sqsubseteq HighLevel_{>0.5}$,
$HighLevel_{>0.5} \sqsubseteq HighLevel_{\geq 0.5}$,
$HighLevel_{\geq 0.5} \sqsubseteq HighLevel_{>0.25}$,
$HighLevel_{>0.25} \sqsubseteq HighLevel_{\geq 0.25}$,
$HighLevel_{\geq 0.25} \sqsubseteq HighLevel_{>0}$,
$AnxietyDisorder_{\geq 1} \sqsubseteq AnxietyDisorder_{>0.75}$,
$AnxietyDisorder_{>0.75} \sqsubseteq AnxietyDisorder_{\geq 0.75}$,
$AnxietyDisorder_{\geq 0.75} \sqsubseteq AnxietyDisorder_{>0.5}$,
$AnxietyDisorder_{>0.5} \sqsubseteq AnxietyDisorder_{\geq 0.5}$,
$AnxietyDisorder_{\geq 0.5} \sqsubseteq AnxietyDisorder_{>0.25}$,
$AnxietyDisorder_{>0.25} \sqsubseteq AnxietyDisorder_{\geq 0.25}$,
$AnxietyDisorder_{\geq 0.25} \sqsubseteq AnxietyDisorder_{>0}$,
$ObsessiveCompulsiveDisorder_{\geq 1} \sqsubseteq ObsessiveCompulsiveDisorder_{>0.75}$,
$ObsessiveCompulsiveDisorder_{>0.75} \sqsubseteq ObsessiveCompulsiveDisorder_{\geq 0.75}$,
$ObsessiveCompulsiveDisorder_{\geq 0.75} \sqsubseteq ObsessiveCompulsiveDisorder_{>0.5}$,
$ObsessiveCompulsiveDisorder_{>0.5} \sqsubseteq ObsessiveCompulsiveDisorder_{\geq 0.5}$,
$ObsessiveCompulsiveDisorder_{\geq 0.5} \sqsubseteq ObsessiveCompulsiveDisorder_{>0.25}$,
$ObsessiveCompulsiveDisorder_{>0.25} \sqsubseteq ObsessiveCompulsiveDisorder_{\geq 0.25}$,
$ObsessiveCompulsiveDisorder_{\geq 0.25} \sqsubseteq ObsessiveCompulsiveDisorder_{>0}$,
$SubstanceInducedAnxietyDisorder_{\geq 1} \sqsubseteq SubstanceInducedAnxietyDisorder_{>0.75}$,
$SubstanceInducedAnxietyDisorder_{>0.75} \sqsubseteq SubstanceInducedAnxietyDisorder_{\geq 0.75}$,
$SubstanceInducedAnxietyDisorder_{\geq 0.75} \sqsubseteq SubstanceInducedAnxietyDisorder_{>0.5}$,
$SubstanceInducedAnxietyDisorder_{>0.5} \sqsubseteq SubstanceInducedAnxietyDisorder_{\geq 0.5}$,
$SubstanceInducedAnxietyDisorder_{\geq 0.5} \sqsubseteq SubstanceInducedAnxietyDisorder_{>0.25}$,
$SubstanceInducedAnxietyDisorder_{>0.25} \sqsubseteq SubstanceInducedAnxietyDisorder_{\geq 0.25}$,
$SubstanceInducedAnxietyDisorder_{\geq 0.25} \sqsubseteq SubstanceInducedAnxietyDisorder_{>0}$

$\}$

Finally, we add the reduction of every axiom in the fuzzy KB:

- $\kappa(\langle \mathit{patient001} : \exists \mathit{hasSerotoninLevel.HighLevel} \leq 0.25\rangle) =$ $\mathit{patient001} : \forall \mathit{hasSerotoninLevel}_{>0.25}.\mathit{HighLevel}_{\leq 0.25}$
- $\kappa(\langle \mathit{patient001} : \exists \mathit{hasDisease.ObsessiveCompulsiveDisorder} \geq 0.75\rangle) =$ $\mathit{patient001} : \exists \mathit{hasDisease}_{\geq 0.75}.\mathit{ObsessiveCompulsiveDisorder}_{\geq 0.75}$
- $\kappa(\langle \mathit{SubstanceInducedAnxietyDisorder} \sqsubseteq \mathit{AnxietyDisorder}\rangle \geq 0.75) =$ $\mathit{SubstanceInducedAnxietyDisorder}_{>0.25} \sqsubseteq \mathit{AnxietyDisorder}_{\geq 0.75}$ □

Example 12 *Consider a travel agency, offering accommodation to the audience of some language courses, aiming to represent the following knowledge:*

- h_1 *is a hotel located at a German speaking country, being this axiom true with at least degree* 0.75*,*
- *the price of* h_1 *can be defined using a trapezoidal function* $trap_{0,150}(x; 0, 50, 100, 150)$ *with at least degree* 0.5*,*
- h_2 *is not a hotel.*
- h_1 *and* h_2 *are close with a degree not greater than* 0.5*.*

In order to represent the previous knowledge, $\mathcal{K}$ *is defined as follows:*

$$
\begin{aligned}
\mathcal{K} = \{ & \\
& \langle \mathit{h1} : \mathit{Hotel} \sqcap \exists \mathit{isIn}.\{(\mathit{germany}, 1), (\mathit{austria}, 1), (\mathit{switzerland}, 0.67)\} \geq 0.75\rangle, \\
& \langle \mathit{h1} : \exists \mathit{hasPrice}.trap_{0,150}(x; 0, 50, 100, 150) \geq 0.5\rangle, \\
& \langle \mathit{h2} : \neg \mathit{Hotel} \geq 1\rangle, \\
& \langle (\mathit{h1}, \mathit{h2}) : \mathit{isCloseTo} \leq 0.5\rangle\} \\
\} &
\end{aligned}
$$

Then, we compute the number of truth values which have to be considered: $X^{\mathcal{K}} = \{0, 0.5, 1, 0.75\}$*, so* $N^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$.

In the next step, we create some new elements and some axioms preserving their semantics.

The new axioms due to the new concepts are:

$$T(N^{\mathcal{K}}) = \{ \\ \mathit{Hotel}_{\geq 1} \sqsubseteq \mathit{Hotel}_{>0.75}, \\ \mathit{Hotel}_{>0.75} \sqsubseteq \mathit{Hotel}_{\geq 0.75}, \\ \mathit{Hotel}_{\geq 0.75} \sqsubseteq \mathit{Hotel}_{>0.5}, \\ \mathit{Hotel}_{>0.5} \sqsubseteq \mathit{Hotel}_{\geq 0.5}, \\ \mathit{Hotel}_{\geq 0.5} \sqsubseteq \mathit{Hotel}_{>0.25}, \\ \mathit{Hotel}_{>0.25} \sqsubseteq \mathit{Hotel}_{\geq 0.25}, \\ \mathit{Hotel}_{\geq 0.25} \sqsubseteq \mathit{Hotel}_{>0} \\ \}$$

The new axioms generated for the abstract roles are:

$$R_a(N^{\mathcal{K}}) = \{ \\ \mathit{isIn}_{\geq 1} \sqsubseteq \mathit{isIn}_{>0.75}, \\ \mathit{isIn}_{>0.75} \sqsubseteq \mathit{isIn}_{\geq 0.75}, \\ \mathit{isIn}_{\geq 0.75} \sqsubseteq \mathit{isIn}_{>0.5}, \\ \mathit{isIn}_{>0.5} \sqsubseteq \mathit{isIn}_{\geq 0.5}, \\ \mathit{isIn}_{\geq 0.5} \sqsubseteq \mathit{isIn}_{>0.25}, \\ \mathit{isIn}_{>0.25} \sqsubseteq \mathit{isIn}_{\geq 0.25}, \\ \mathit{isIn}_{\geq 0.25} \sqsubseteq \mathit{isIn}_{>0}, \\ \mathit{isCloseTo}_{\geq 1} \sqsubseteq \mathit{isCloseTo}_{>0.75}, \\ \mathit{isCloseTo}_{>0.75} \sqsubseteq \mathit{isCloseTo}_{\geq 0.75}, \\ \mathit{isCloseTo}_{\geq 0.75} \sqsubseteq \mathit{isCloseTo}_{>0.5}, \\ \mathit{isCloseTo}_{>0.5} \sqsubseteq \mathit{isCloseTo}_{\geq 0.5}, \\ \mathit{isCloseTo}_{\geq 0.5} \sqsubseteq \mathit{isCloseTo}_{>0.25}, \\ \mathit{isCloseTo}_{>0.25} \sqsubseteq \mathit{isCloseTo}_{\geq 0.25}, \\ \mathit{isCloseTo}_{\geq 0.25} \sqsubseteq \mathit{isCloseTo}_{>0} \\ \}$$

Similarly, for the concrete role hasPrice *we have:*

$$
\begin{aligned}
R_c(N^{\mathcal{K}}) = \quad \{ & \\
& hasPrice_{\geq 1} \sqsubseteq hasPrice_{>0.75}, \\
& hasPrice_{>0.75} \sqsubseteq hasPrice_{\geq 0.75}, \\
& hasPrice_{\geq 0.75} \sqsubseteq hasPrice_{>0.5}, \\
& hasPrice_{>0.5} \sqsubseteq hasPrice_{\geq 0.5}, \\
& hasPrice_{\geq 0.5} \sqsubseteq hasPrice_{>0.25}, \\
& hasPrice_{>0.25} \sqsubseteq hasPrice_{\geq 0.25}, \\
& hasPrice_{\geq 0.25} \sqsubseteq hasPrice_{>0} \\
\} &
\end{aligned}
$$

Finally, we add the reduction of every axiom in the fuzzy ABox of $\mathcal{K}$*:*

- $\kappa(\langle h1\colon Hotel \sqcap \exists isIn.\{(germany, 1), (austria, 1), (switzerland, 0.67)\} \geq 0.75\rangle) = h1\colon Hotel_{\geq 0.75} \sqcap \exists isIn_{\geq 0.75}.\{germany, austria\}$.
- $\kappa(\langle h1\colon \exists hasPrice.trap_{0,150}(x; 0, 50, 100, 150) \geq 0.5\rangle) = h1\colon \exists hasPrice_{\geq 0.5}.real[25, 125]$.
- $\kappa(\langle h2\colon \neg Hotel \geq 1\rangle) = h2\colon \rho(\neg Hotel, \geq 1) = h2\colon \neg Hotel_{>0}$.
- $\kappa(\langle (h1, h2)\colon isCloseTo \leq 0.5\rangle) = (h1, h2)\colon \neg isCloseTo_{>0.5}$.

We also need to include the following definition of $real[25, 125]$*:*

```
<owl:DataRange rdf:about="#real[25,125]">
  <owl2:onDataRange rdf:resource="&xsd;double"/>
  <owl2:minInclusive rdf:datatype="&xsd;double">25</owl2:minInclusive>
  <owl2:maxInclusive rdf:datatype="&xsd;double">125</owl2:maxInclusive>
</owl:DataRange>
```

□

6.7 Mapping Modified Fuzzy Concepts and Roles

In this section we will show how to extend our reduction in order to allow concept and role modifiers in the language. We will restrict ourselves to the triangular modifier and the linear modifier (see Chapter 2).

The first thing to be kept in mind is that it is no longer enough to consider the degrees in $N^{\mathcal{K}}$. Consider a fuzzy KB with one assertion $\langle a : mod(C) \geq \gamma\rangle$, where the modifier mod is defined as in Figure 5.2 (a). We can deduce that $\langle a : C \geq t_1 \rangle$ and $\langle a : C \leq t_2 \rangle$, so we should also consider the degree t_1, t_2 in $N^{\mathcal{K}}$. But this is not enough, since we might have a concept of the form $mod(mod(mod(\ldots mod(C))\ldots))$.

Our solution to this problem is to restrict the modifiers to those such that its membership function verifies the following property:

$$\forall \gamma \in N^{\mathcal{K}}, f_{mod}(\gamma) \in N^{\mathcal{K}}$$

Let $x_1 \in [0, b]$ and $x_2 \in [b, 1]$ be those numbers such that $f_{left}(x_1; t_1, t_2, t_3) = \gamma$ and $f_{right}(x_2; t_1, t_2, t_3) = \gamma$ respectively, for a triangular modifier $mTri$. Note that x_1 does not exist if $\gamma < t_1$, and that x_2 does not exist if $\gamma > t_2$. The reduction of modified concepts depend on the values of the parameters of the modifier, as Table 6.5 shows.

In the case of role modifiers, we only allow linear modifiers, because triangular modifiers would need to use role conjunction, role disjunction and expressions of the form $\rho(R, \triangleleft\gamma)$ outside the ABox, which are not part of crisp $\mathcal{SROIQ}(\mathbf{D})$.

On the other hand, linear modifiers are reduced as:

$$\rho(mLin(C), \bowtie \gamma) = \rho(C, \bowtie x_l)$$
$$\rho(mLin(R), \bowtie \gamma) = \rho(R, \bowtie x_l)$$

with x_l being that number such that $f_{mLin}(x_l; l)(lin) = \gamma$.

Example 13 *Assume a fuzzy KB such that* $N^{\mathcal{K}} = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. *Let us consider the reduction of the axiom* a: $\mathsf{around}(C) \geq 0.8$, *where* around *is a triangular modifier defined as follows:* $f_{\mathsf{around}}(x; 0.6, 0.4, 0.4)$.

Firstly, we verify that indeed $\forall \gamma \in N^{\mathcal{K}}, f_{\mathsf{around}}(\gamma) \in N^{\mathcal{K}}$:

- $f_{\mathsf{around}}(0) = 0.6 \in N^{\mathcal{K}}$,
- $f_{\mathsf{around}}(0.1) = 0.7 \in N^{\mathcal{K}}$,
- $f_{\mathsf{around}}(0.2) = 0.8 \in N^{\mathcal{K}}$,
- $f_{\mathsf{around}}(0.3) = 0.9 \in N^{\mathcal{K}}$,
- $f_{\mathsf{around}}(0.4) = 1 \in N^{\mathcal{K}}$,

Table 6.5: Reduction of modified concepts.

Reduction of $\rho(mTri(C), \geq \gamma)$
if $(\alpha > t_1)$ and $(\alpha > t_3)$ then $\rho(C, \geq x_1) \sqcap \rho(C, \leq x_2)$
if $(\alpha > t_1)$ and $(\alpha \leq t_3)$ then $\rho(C, \geq x_1)$
if $(\alpha \leq t_1)$ and $(\alpha > t_3)$ then $\rho(C, \leq x_2)$
if $(\alpha \leq t_1)$ and $(\alpha \leq t_3)$ then $\top$
Reduction of $\rho(mTri(C), > \gamma)$
if $(\beta \geq t_1)$ and $(\beta \geq t_3)$ then $\rho(C, > x_1) \sqcap \rho(C, < x_2)$
if $(\beta \geq t_1)$ and $(\beta < t_3)$ then $\rho(C, > x_1)$
if $(\beta < t_1)$ and $(\beta \geq t_3)$ then $\rho(C, < x_2)$
if $(\beta < t_1)$ and $(\beta < t_3)$ then $\top$
Reduction of $\rho(mTri(C), \leq \gamma)$
if $(\beta \geq t_1)$ and $(\beta \geq t_3)$ then $\rho(C, \leq x_1) \sqcup \rho(C, \geq x_2)$
if $(\beta \geq t_1)$ and $(\beta < t_3)$ then $\rho(C, \leq x_1)$
if $(\beta < t_1)$ and $(\beta \geq t_3)$ then $\rho(C, \geq x_2)$
if $(\beta < t_1)$ and $(\beta < t_3)$ then $\bot$
Reduction of $\rho(mTri(C), < \gamma)$
if $(\alpha > t_1)$ and $(\alpha > t_3)$ then $\rho(C, \geq x_1) \sqcup \rho(C, \leq x_2)$
if $(\alpha > t_1)$ and $(\alpha \leq t_3)$ then $\rho(C, < x_1)$
if $(\alpha \leq t_1)$ and $(\alpha > t_3)$ then $\rho(C, > x_2)$
if $(\alpha \leq t_1)$ and $(\alpha \leq t_3)$ then $\bot$

- $f_{around}(0.5) = 0.9 \in N^{\mathcal{K}}$,
- $f_{around}(0.6) = 0.8 \in N^{\mathcal{K}}$,
- $f_{around}(0.7) = 0.7 \in N^{\mathcal{K}}$,
- $f_{around}(0.8) = 0.6 \in N^{\mathcal{K}}$,
- $f_{around}(0.9) = 0.5 \in N^{\mathcal{K}}$,
- $f_{around}(1) = 0.4 \in N^{\mathcal{K}}$,

Now, x_1, x_2 are those points such that the modifier takes the value 0.8, so $x_1 = 0.2$ and $x_2 = 0.6$. Hence, the reduction of the axiom is $\kappa(\langle a : around(C) \geq 0.8\rangle) = a : \rho(C, \geq x_1) \sqcap \rho(C, \leq x_2) = a : \rho(C, \geq 0.2) \sqcap \rho(C, \leq 0.6)$.

6.8 Properties of the Reduction

Firstly, it is worth to note that the reduction preserves simplicity of the roles and regularity of the RIAs.

Correctness of the Reduction

The following theorem shows that the reduction preserves reasoning.

Theorem 2 *A Z $\mathcal{SROIQ}(\mathbf{D})$ fuzzy KB $\mathcal{K}$ is satisfiable iff* `crisp`$(\mathcal{K})$ *is satisfiable.*

Proof. We will show the proof for the only-if direction. From $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ with respect to a fuzzy concrete domain $\mathbf{D} = \langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Phi_{\mathbf{D}}$ only contains the fuzzy concrete predicate $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$, satisfying every axiom in $\mathcal{K}$. Now, it is possible to build a (crisp) interpretation $\mathcal{I}_C = \{\Delta^{\mathcal{I}_C}, \cdot^{\mathcal{I}_C}\}$ with respect to a crisp concrete domain $\mathbf{D_C} = \langle \Delta_{\mathbf{D_C}}, \Phi_{\mathbf{D_C}} \rangle$ as:

- $\Delta^{\mathcal{I}_C} = \Delta^{\mathcal{I}}$.
- $\Delta_{\mathbf{D_C}} = \Delta_{\mathbf{D}}$.
- $x^{\mathcal{I}_C} = x^{\mathcal{I}}$, for all $x \in \Delta^{\mathcal{I}}$.
- $v_{\mathbf{D_C}} = v_{\mathbf{D}}$, for all $v \in \Delta_{\mathbf{D}}$.
- $A_{\geq \alpha}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq \alpha\}$, for each $A \in \mathbf{A}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $A_{> \beta}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) > \beta\}$, for each $A \in \mathbf{A}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $R_{A \geq \alpha}^{\mathcal{I}_C} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) \geq \alpha\}$, for each $R_A \in \mathbf{R}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $R_{A > \beta}^{\mathcal{I}_C} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) > \beta\}$, for each $R_A \in \mathbf{R}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $T_{\geq \alpha}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) \geq \alpha\}$, for each $T \in \mathbf{T}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $T_{> \beta}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) > \beta\}$, for each $T \in \mathbf{T}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $\Phi_{\mathbf{D_C}}$ will contain some concrete predicates of the form $real[a, b]$, $real(a, b)$ and $union\text{–}real[k_1, a, b, k_2]$, with $a, b, k_1, k_2 \in \mathbb{R}$.

Now, we will show that $\mathcal{I}_C$ satisfies every axiom in `crisp`$(\mathcal{K})$. For every axiom $\tau \in \mathcal{K}$, there are several cases:

1. τ is an inequality assertion. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_C$, $a^{\mathcal{I}_C} \neq b^{\mathcal{I}_C}$, so $\mathcal{I}_C \models \langle a \neq b \rangle \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a \neq b \rangle)$. The case of equality assertions is similar.

2. τ is a role assertion. Assume that $\mathcal{I} \models \langle (a,b) : R \bowtie \gamma \rangle$. We show, by induction on the structure of roles, that $\mathcal{I}_C \models \kappa(\langle (a,b) : R \bowtie \gamma \rangle)$.

   - *Atomic role*. Assume that $\mathcal{I} \models \langle (a,b) : R_A \rhd \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in R_{A \rhd \gamma}^{\mathcal{I}_C}$. By definition of ρ, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R_A, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R_A, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : R_A \rhd \gamma \rangle)$.

     Now assume that $\mathcal{I} \models \langle (a,b) : R_A \lhd \gamma \rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \lhd \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \notin (R_{A \neg\rhd \gamma})^{\mathcal{I}_C}$ and hence $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\neg R_{A \neg\rhd \gamma})^{\mathcal{I}_C}$. By definition of ρ, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R_A, \lhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R_A, \lhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : R_A \lhd \gamma \rangle)$.

   - *Concrete roles*. This case is similar to that of atomic roles.

   - *Inverse role*. Assume that $\mathcal{I} \models \langle (a,b) : R^- \bowtie \gamma \rangle$. Then, $R^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}_C}$. Consequently, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R, \bowtie \gamma)^{\mathcal{I}_C})^- \Leftrightarrow \mathcal{I}_C \models (a,b) \in \rho(R, \bowtie \gamma)^- \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : R^- \bowtie \gamma \rangle)$.

   - *Universal role*. Assume that $\mathcal{I} \models \langle (a,b) : U \rhd \gamma \rangle$. Then, $U^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1 \geq \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \Delta^{\mathcal{I}_C} \times \Delta^{\mathcal{I}_C}$ and consequently $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in U^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(U, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(U, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : U \rhd \gamma \rangle)$. The case $\mathcal{I} \models \langle (a,b) : U \lhd \gamma \rangle$ is similar.

   - *Modified role*. Assume that $\mathcal{I} \models \langle (a,b) : mLin(R) \rhd \gamma \rangle$ for a linear modifier $mLin$ such that $f_{mLin}(x; l)$. Then, it follows that $f_{mLin}(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}); l) \rhd \gamma$. Let $x_l \in [0,1]$ be such that $f_{mLin}(x_l; l) = \gamma$. Then, it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \rhd x_l$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}}) \in \rho(R, \rhd x_l)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R, \rhd x_l) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : mLin(R) \rhd \gamma \rangle)$.

     The case $\mathcal{I} \models \langle (a,b) : mLin(R) \lhd \gamma \rangle$ is similar, but now it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \lhd x_l$ so we end up with $\mathcal{I}_C \models (a,b) : \rho(R, \lhd x_l) \Leftrightarrow$

$\mathcal{I}_{\mathcal{C}} \models \kappa(\langle (a,b) : mLin(R) \lhd \gamma \rangle)$.

- *Cut role*. Assume that $\mathcal{I} \models \langle (a,b) : [R \geq \alpha] \rhd \gamma \rangle$. Then, it follows that $([R \geq \alpha])^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$, which is the case if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(R, \geq \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R, \geq \alpha) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho([R \geq \alpha], \rhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa((a,b) : [R \geq \alpha] \rhd \gamma)$.

  Now assume that $\mathcal{I} \models \langle (a,b) : [R \geq \alpha] \lhd \gamma \rangle$. Then, it follows that $([R \geq \alpha])^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 0$, which is the case if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) < \alpha$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(R, < \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho(R, < \alpha) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models (a,b) : \rho([R \geq \alpha], \lhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa((a,b) : [R \geq \alpha] \lhd \gamma)$.

3. τ is a concept assertion. Assume that $\mathcal{I} \models \langle a : C \bowtie \gamma \rangle$. We show, by induction on the structure of concepts and roles, that $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \bowtie \gamma \rangle)$.

   - *Atomic concept*. Assume that $\mathcal{I} \models \langle a : A \rhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_C} : A^{\mathcal{I}_C}_{\rhd \gamma}$. Consequently, $a^{\mathcal{I}_C} \in (\rho(A, \rhd \gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \rhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \rhd \gamma \rangle)$. Now assume that $\mathcal{I} \models \langle a : A \lhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_C} \notin A^{\mathcal{I}_C}_{\neg \rhd \gamma} \Leftrightarrow a^{\mathcal{I}_C} \in \neg A^{\mathcal{I}_C}_{\neg \rhd \gamma} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(A, \lhd \gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(A, \lhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : A \lhd \gamma \rangle)$.
   - *Top concept*. Assume that $\mathcal{I} \models \langle a : \top \rhd \gamma \rangle$. Then, $\top^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_{\mathcal{C}}$, it follows that $a^{\mathcal{I}_C} \in \Delta^{\mathcal{I}_C} = \top$. Consequently, $a^{\mathcal{I}_C} \in (\rho(\top, \rhd \gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(\top, \rhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \top \rhd \gamma \rangle)$. The case $\mathcal{I} \models \langle a : \top \lhd \gamma \rangle$ is not possible. If $\mathcal{I} \models \langle a : \top \leq \beta \rangle$ we have that $1 \leq \beta$, which is a contradiction with the restriction $\beta \in [0,1)$. If $\mathcal{I} \models \langle a : \top < \alpha \rangle$ we have that $1 < \alpha$, which is a contradiction with the restriction $\alpha \in (0,1]$.
   - *Bottom concept*. This case is similar to the previous one.
   - *Concept negation*. Assume that $\mathcal{I} \models \langle a : \neg C \bowtie \gamma \rangle$. Then, $1 - C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie^{-} 1 - \gamma$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \bowtie^{-} 1 - \gamma)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a \in \rho(C, \bowtie^{-} 1 - \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \neg C \bowtie \gamma \rangle)$.

- *Concept conjunction*. Assume that $\mathcal{I} \models \langle a : C \sqcap D \rhd \gamma\rangle$. Then, $\min\{C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})\} \rhd \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$ and $D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \rhd\gamma)^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \in \rho(D, \rhd\gamma)^{\mathcal{I}_C}$. Consequently, $a^{\mathcal{I}_C} \in (\rho(C, \rhd\gamma) \sqcap \rho(D, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(C \sqcap D, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(C \sqcap D, \rhd\gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcap D \rhd \gamma\rangle)$.

  In the case $\mathcal{I} \models \langle a : C \sqcap D \lhd \gamma\rangle$, it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$ or $D^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \lhd\gamma)^{\mathcal{I}_C}$ or $a^{\mathcal{I}_C} \in \rho(D, \lhd\gamma)^{\mathcal{I}_C}$. In this case, we end up with $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : C \sqcap D \lhd \gamma\rangle)$.

- *Concept disjunction*. It can be easily obtained using inter-definability of conjunction and disjunction.

- *Universal quantification*. Assume that $\mathcal{I} \models \langle a : \forall R.C \geq \alpha\rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b), C^{\mathcal{I}}(b)\} \geq \alpha$. Since this is true for the infimum, an arbitrary individual $b \in \Delta^{\mathcal{I}}$ must satisfy $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leq 1 - \alpha$ or $C^{\mathcal{I}}(b) \geq \alpha$. By induction hypothesis, $(a^{\mathcal{I}_C}, b) \in \rho(R, \leq 1 - \alpha)^{\mathcal{I}_C}$ or $b \in \rho(C, \geq \alpha)^{\mathcal{I}_C}$ for an arbitrary individual $b \in \Delta^{\mathcal{I}_C}$, which is equivalent to $a^{\mathcal{I}_C} \in (\forall \rho(R, > 1 - \alpha).\rho(C, \geq \alpha))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(\forall R.C \geq \alpha))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : (\rho(\forall R.C \geq \alpha) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \forall R.C, \geq \alpha\rangle)$. The case $> \beta$ is quite straightforward.

  Now, assume that $\mathcal{I} \models \langle a : \forall R.C \leq \beta\rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}} \max\{1 - R^{\mathcal{I}}(a^{\mathcal{I}}, b), C^{\mathcal{I}}(b)\} \leq \beta$. P. Hájek showed for Łukasiewicz logic (which is a more general case of Zadeh logic), that if there is a model, then there is also a witnessed model [120] so we can assume that, if this is true for the infimum, there exists an individual b satisfying $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq 1 - \beta$ and $C^{\mathcal{I}}(b) \leq \beta$. By induction hypothesis, $(a^{\mathcal{I}_C}, b) \in (\rho(R, \geq 1 - \beta))^{\mathcal{I}_C}$ and $b \in (\rho(C, \leq \beta))^{\mathcal{I}_C}$ for some individual $b \in \Delta^{\mathcal{I}_C}$. In this case, we end up with $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \forall R.C, \leq \beta\rangle)$. The case $< \alpha$ is quite straightforward.

- *Existential quantification*. Use inter-definability of quantifiers.

- *Fuzzy nominals*. Assume that $\mathcal{I} \models \langle a : \{\alpha_1/o_1, \dots, \alpha_n/o_n\} \rhd \gamma\rangle$. Let $o_{i1}, \dots, o_{ik}$ be such that $\alpha_{ij} \rhd \gamma$. Then, $\sup\{\alpha_{i_1}, \dots, \alpha_{i_k}\} \rhd \gamma$, with $a^{\mathcal{I}} \in \{o_{i_1}, \dots, o_{i_k}\}^{\mathcal{I}}$. By construction of $\mathcal{I}_{\mathcal{C}}$, it holds that $a^{\mathcal{I}_C} \in \{o_{i_1}, \dots, o_{i_k}\}^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in \rho(\{\alpha_1/o_1, \dots, \alpha_n/o_n\}^{\mathcal{I}_C}, \rhd\gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a :$

$\rho(\{\alpha_1/o_1,\ldots,\alpha_n/o_n\}, \triangleright\gamma\rangle) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \{\alpha_1/o_1,\ldots,\alpha_n/o_n\} \triangleright \gamma\rangle)$. The case $\triangleleft\gamma$ is quite straightforward.

- *At-least qualified number restriction*. Assume that $\mathcal{I} \models \langle a : (\geq m\, S.C) \geq \alpha\rangle$. Then, $\sup_{b_1,\ldots,b_m \in \Delta^\mathcal{I}} [(\min_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \bigotimes (\otimes_{j<k}\{b_j \neq b_k\})] \geq \alpha$. Note that $(\otimes_{j<k}\{b_j \neq b_k\})$ can be either 0 or 1. If it is 0, then we have that $\sup_{b_1,\ldots,b_m \in \Delta^\mathcal{I}} [(\min_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \bigotimes 0] = 0 \geq \alpha$, which is not possible because by definition $\alpha \in (0,1]$. Hence, $(\otimes_{j<k}\{b_j \neq b_k\}) = 1$ and consequently $\sup_{b_1,\ldots,b_m \in \Delta^\mathcal{I}} [(\min_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \bigotimes 1]$. This is equivalent to $\sup_{b_1,\ldots,b_m \in \Delta^\mathcal{I}} (\min_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}) \geq \alpha$. This implies that there exist m different $b_i \in^{\mathcal{I}_C}$ such that $\min_{i=1}^m \{S^\mathcal{I}(a^\mathcal{I}, b_i) \otimes C^\mathcal{I}(b_i)\}$ and hence $S^\mathcal{I}(a^\mathcal{I}, b_i) \geq \alpha$ and $C^\mathcal{I}(b_i) \geq \alpha$, for $1 \leq i \leq m$. By induction hypothesis, $(a^{\mathcal{I}_C}, b_i) \in (\rho(S, \geq \alpha))^{\mathcal{I}_C}$ and $b_i \in (\rho(C, \geq \alpha))^{\mathcal{I}_C}$, for $1 \leq i \leq m$. Consequently, $a^{\mathcal{I}_C} \in (\geq m\, \rho(S, \geq \alpha).\rho(C, \geq \alpha))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in \rho(\geq m\, S.C, \geq \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(\geq m\, S.C, \geq \alpha) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : (\geq m\, S.C) \geq \alpha\rangle)$. The case $> \beta$ is quite similar.

  Now assume that $\mathcal{I} \models \langle a : (\geq m\, S.C) \leq \beta\rangle$. In this case, it follows that $\sup_{b_1,\ldots,b_m \in \Delta^\mathcal{I}} (\min_{i=1}^m \{S^\mathcal{I}(a^{\mathcal{I}_C}, b_i) \otimes C^\mathcal{I}(b_i)\}) \leq \beta$. Consequently, there cannot exist m different individuals b_i with $(\min_{i=1}^m \{S^\mathcal{I}(a^{\mathcal{I}_C}, b_i) \otimes C^\mathcal{I}(b_i)\}) > \beta$, so we end up with $\mathcal{I}_\mathcal{C} \models \kappa(\langle a : (\geq m\, S.C) \leq \beta\rangle)$. The case $< \alpha$ is quite similar.

- *At-most qualified number restriction*. It can be easily obtained using inter-definability of qualified number restrictions.

- *Local reflexivity*. Assume that $\mathcal{I} \models \langle a : \exists S.Self \triangleright \gamma\rangle$. Then, $S^\mathcal{I}(a^\mathcal{I}, a^\mathcal{I}) \triangleright \gamma$. By induction hypothesis, $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(S, \triangleright\gamma)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_\mathcal{C} \models (a,a) : \rho(S, \triangleright\gamma) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \exists S.Self \triangleright \gamma\rangle)$. Now assume that $\mathcal{I} \models \langle a : \exists S.Self \triangleleft \gamma\rangle$. Then, $S^\mathcal{I}(a^\mathcal{I}, a^\mathcal{I}) \triangleleft \gamma$. By induction hypothesis, $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(S, \triangleleft\gamma)^{\mathcal{I}_C}$. Hence, it follows that $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \notin (\rho(S, \neg \triangleleft \gamma))^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \neg(\rho(S, \neg \triangleleft \gamma))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(\exists S.Self, \triangleleft\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(\exists S.Self, \triangleleft\gamma) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : \exists S.Self, \triangleleft\gamma\rangle)$.

- *Modified concept*. Firstly, let us consider the case of a triangular modifier $mTri$ such that $f_{mTri}(x;t_1,t_2,t_3)$. Assume that $\langle \mathcal{I} \models a : mTri(C) \geq \alpha \rangle$. Then, it follows that $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}};t_1,t_2,t_3) \geq \alpha$. Let $x_1 \in [0,t_2]$ and $x_2 \in [t_2,1]$ be those numbers such that $f_{left}(x_1;t_1,t_2,t_3) = \alpha$ and $f_{right}(x_2;t_1,t_2,t_3) = \alpha$. There are several options now, depending on the value of α with respect to t_1 and t_3.

  a) If $(\alpha > t_1)$ and $(\alpha > t_3)$, then $C^{\mathcal{I}}(a^{\mathcal{I}})$ is lower bounded by x_1 (since $f_{left}(x_1;t_1,t_2,t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}};t_1,t_2,t_3) \geq \alpha$) and upper bounded by x_2 (since $f_{right}(x_2;t_1,t_2,t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}};t_1,t_2,t_3) \geq \alpha)$. That is, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq x_1$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq x_2$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \geq x_1)^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \in \rho(C, \leq x_2)^{\mathcal{I}_C}$. It follows that $a^{\mathcal{I}_C} \in \rho(C, \geq x_1)^{\mathcal{I}_C} \sqcap \rho(C, \leq x_2)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C, \geq x_1) \sqcap \rho(C, \leq x_2) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

  b) If $(\alpha > t_1)$ and $(\alpha \leq t_3)$, then $C^{\mathcal{I}}(a^{\mathcal{I}})$ is lower bounded by x_1 as in the previous case, but x_2 does not introduce an upper bounded now: as noted in Section 6.7, $f_{mTri}(1) = t_3$, and since $\alpha \leq t_3$ and f_{right} is a strictly decreasing function, the possible upper bound for $C^{\mathcal{I}}(a^{\mathcal{I}})$ would be greater than 1, but we already know that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in [0,1]$. That is, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq x_1$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \geq x_1)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C, \geq x_1) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

  c) The case $(\alpha \leq t_1)$ and $(\alpha > t_3)$ is similar, but now $C^{\mathcal{I}}(a^{\mathcal{I}})$ is upper bounded by x_2 and not lower bounded. Now, $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq x_2$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \leq x_2)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C, \leq x_2) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

  d) Finally, in the case $(\alpha \leq t_1)$ and $(\alpha \leq t_3)$ there are no bounds, so we only now that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in [0,1]$ and hence we only know that $\top^{\mathcal{I}}(a^{\mathcal{I}})$. By induction hypothesis, $a^{\mathcal{I}_C} \in \top^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \top \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

The other cases $\langle \mathcal{I} \models a : mTri(C) \bowtie \gamma \rangle$ are similar.

Now, let us consider the case of a triangular modifier $mLin$ such that $f_{mLin}(x;l)$. Assume that $\langle \mathcal{I} \models a : mLin(C) \triangleright \gamma \rangle$. Then, it follows that $f_{mLin}(C^{\mathcal{I}}(a^{\mathcal{I}};l) \triangleright \gamma$. Let $x_l \in [0,1]$ be such that $f_{mLin}(x_l;l) = \gamma$. Then, it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleright x_l$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \triangleright x_l)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \triangleright x_l) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mLin(C) \triangleright \gamma \rangle)$.

The case $\langle \mathcal{I} \models a : mLin(C) \triangleleft \gamma \rangle$ is similar, but now it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \triangleleft x_l$ so we end up with $\mathcal{I}_\mathcal{C} \models a : \rho(C, \triangleleft x_l) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mLin(C) \triangleleft \gamma \rangle)$.

- *Cut concept*. Assume that $\langle \mathcal{I} \models a : [C \geq \alpha] \triangleright \gamma \rangle$. Then, it follows that $([C \geq \alpha])^{\mathcal{I}}(a^{\mathcal{I}}) = 1$, which is the case if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \geq \alpha)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \geq \alpha) \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho([C \geq \alpha], \triangleright \gamma)$. The case $\mathcal{I} \models a : [C \leq \alpha] \leq \gamma \rangle$ is similar.

  Now assume that $\langle \mathcal{I} \models a : [C \geq \beta] \triangleleft \gamma \rangle$. Then, it follows that $([C \geq \beta])^{\mathcal{I}}(a^{\mathcal{I}}) = 0$, which is the case if $C^{\mathcal{I}}(a^{\mathcal{I}}) < \beta$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, < \beta)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, < \beta) \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho([C \geq \beta], \triangleleft \gamma)$. The case $\mathcal{I} \models a : [C \leq \beta] \geq \gamma \rangle$ is similar.

- *Concrete concept constructs*. Concrete concept constructs are similar to their abstract versions. The only point which deserves a special comment is the existence of some expressions of the form $v_i : d_{\mathbf{D}} \bowtie \gamma$, with $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$. Assume that $v_i : d_{\mathbf{D}} > \beta$. In order to guarantee that the trapezoidal function takes a value x_{v_i} which is greater or equal than β, we have that $x_{v_i} > a + \beta(b-a)$ and $x_{v_i} < d - \beta(d-c)$, which is equivalent to say that $v_i \in real(a + \beta(b-a), d - \beta(d-c))$. The case $\geq \alpha$ is similar, but using a customized datatype of the form $real[x, y]$ instead of $real(x, y)$. Now, assume that $v_i : d_{\mathbf{D}} \leq \beta$. In this case we have that either $x_{v_i} \leq k_1, a + \beta(b-a)$ or $x_{v_i} \geq d - \beta(d-c), k_2]$, which is equivalent to say that $v_i \in union\text{–}real[k_1, a + \beta(b-a), d - \beta(d-c), k_2]$. The case $< \alpha$ is similar, using ϵ to get the strict inequality.

4. τ is a fuzzy GCI. Assume that $\mathcal{I} \models \langle a : C \sqsubseteq D \geq \alpha \rangle$. Then, $\inf_{a \in \Delta^{\mathcal{I}}} \max\{1 - C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\} \geq \alpha$. As this is true for the infimum, every individual a of the domain must satisfy that condition, which is equivalent to satisfy $\langle C^{\mathcal{I}}(a) \leq 1 - \alpha \rangle$ or $\langle D^{\mathcal{I}}(a) \geq \alpha \rangle$. By induction, it follows that $\forall a \in \Delta^{\mathcal{I}_C}, a \in (\rho(C, \leq 1 - \alpha))^{\mathcal{I}_C}$ or $a \in (\rho(D, \geq \alpha))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models \rho(C, > 1 - \alpha) \sqsubseteq \rho(D, \geq \alpha) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle C \sqsubseteq D \geq \alpha \rangle)$. Similar arguments can be used for $\langle C \sqsubseteq D > \beta \rangle$.

5. τ is a fuzzy RIA. Assume that $\mathcal{I} \models \langle R_1 \ldots R_m \sqsubseteq R \geq \alpha \rangle$. The case is similar to the previous one, with the difference that there appears a minimum i.e., $\min\{R_1^{\mathcal{I}}(b_1, b_2), \ldots, R_n^{\mathcal{I}}(b_n, b_{n+1})\} \Rightarrow \{R^{\mathcal{I}}(b_1, b_{n+1}) \geq \alpha$. As a consequence, the left side of the crisp RIAs will contain $\rho(R_1, > 1 - \alpha) \ldots \rho(R_m, > 1 - \alpha)$ in the left side, instead of $\rho(C, > 1 - \alpha)$. The case for $> \beta$ is similar.

6. τ is a transitive role axiom. Assume that $\mathcal{I} \models \texttt{trans}(R)$. Then, $\forall x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, y) \geq \sup_{z \in \Delta^{\mathcal{I}}} \min\{R^{\mathcal{I}}(x, z), R^{\mathcal{I}}(z, y)\}$. It is easy to see that this implies that $\kappa(\tau)$ holds i.e. $R_{\geq \alpha}, R_{> \beta}$ are transitive, for each $\alpha, \beta \in N^{\mathcal{K}}$. For example, assume that $R^{\mathcal{I}}(x, z) \geq \gamma$ and $R^{\mathcal{I}}(z, y) \geq \gamma$ for some individual z. By transitivity, $R^{\mathcal{I}}(x, y) \geq \gamma$ holds. By induction, $(x^{\mathcal{I}_C}, z^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$, $(z^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$ and $(x^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$. Since $\forall x, y, z \in \Delta^{\mathcal{I}}, (x^{\mathcal{I}_C}, z^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$ and $(z^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$ imply $(x^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R_{\geq \gamma}^{\mathcal{I}_C}$, then $R_{\geq \gamma}$ is transitive.

7. τ is a role disjoint axiom. Assume that $\mathcal{I} \models \texttt{dis}(S_1, S_2)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = 0$ or $S_2^{\mathcal{I}}(x, y) = 0$. By induction, $\forall x, y \in \Delta^{\mathcal{I}_C}, (x, y) \in (\rho(S_1, \leq 0))^{\mathcal{I}_C}$ or $(x, y) \in (\rho(S_2, \leq 0))^{\mathcal{I}_C} \Leftrightarrow \forall x, y \in \Delta^{\mathcal{I}_C}, (x, y) \notin (\rho(S_1, > 0))^{\mathcal{I}_C}$ or $(x, y) \notin (\rho(S_2, > 0))^{\mathcal{I}_C} \Leftrightarrow (\rho(S_1, > 0))^{\mathcal{I}_C} \cap (\rho(S_2, > 0))^{\mathcal{I}_C} = \emptyset \Leftrightarrow \mathcal{I}_C \models (dis(\rho(S_1, > 0), \rho(S_2, > 0))) \Leftrightarrow \mathcal{I}_C \models \kappa(\texttt{dis}(S_1, S_2))$. The case $\tau = \texttt{dis}(T_1, T_2)$ is similar.

8. τ is a reflexive role axiom. Assume that $\mathcal{I} \models \texttt{ref}(R)$. Then, $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x, x) = 1$. By induction, $\forall x \in \Delta^{\mathcal{I}_C}, (x, x) \in (\rho(R, \geq 1))^{\mathcal{I}_C} \Leftrightarrow \forall x \in \Delta^{\mathcal{I}_C}, \mathcal{I}_C \models (x, x) : \rho(R, \geq 1) \Leftrightarrow \mathcal{I}_C \models \kappa(\texttt{ref}(R))$.

9. τ is an irreflexive role axiom. Assume that $\mathcal{I} \models \texttt{irr}(S)$. Then, $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x, x) = 0$. By induction, $\forall x \in \Delta^{\mathcal{I}_C}, (x, x) \in (\rho(S, \leq 0))^{\mathcal{I}_C} \Leftrightarrow$

$\forall x \in \Delta^{\mathcal{I}_C}, (x,x) \notin (\rho(S,>0))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models \texttt{irr}(\rho(S,>0)) \Leftrightarrow \mathcal{I}_C \models \kappa(\texttt{irr}(S))$.

10. τ is a symmetry role axiom. Assume that $\mathcal{I} \models \texttt{sym}(R)$. Then, $\forall x,y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$. It is easy to see that this implies that $\kappa(\tau)$ holds i.e. $R_{\geq\alpha}, R_{>\beta}$ are symmetric, for each $\alpha,\beta \in N^{\mathcal{K}}$. For example, assume that $R^{\mathcal{I}}(x,y) \geq \gamma$. By symmetry, $R^{\mathcal{I}}(y,x) \geq \gamma$ holds. By induction, $(x^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R^{\mathcal{I}_C}_{\geq\gamma}$ and $(y^{\mathcal{I}_C}, x^{\mathcal{I}_C}) \in R^{\mathcal{I}_C}_{\geq\gamma}$. Since $\forall x,y \in \Delta^{\mathcal{I}}, (x^{\mathcal{I}_C}, y^{\mathcal{I}_C}) \in R^{\mathcal{I}_C}_{\geq\gamma}$ implies $(y^{\mathcal{I}_C}, x^{\mathcal{I}_C}) \in R^{\mathcal{I}_C}_{\geq\gamma}$, then $R_{\geq\gamma}$ is symmetric.
11. τ is an asymmetry role axiom. Assume that $\mathcal{I} \models \texttt{asy}(S)$. Then, $\forall x,y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$. By induction, $\forall x,y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S,>0))^{\mathcal{I}_C}$ then $(y,x) \in (\rho(S,\leq 0))^{\mathcal{I}_C} \Leftrightarrow \forall x,y \in \Delta^{\mathcal{I}_C}$, if $(x,y) \in (\rho(S,>0))^{\mathcal{I}_C}$ then $(y,x) \notin (\rho(S,>0))^{\mathcal{I}_C}$. Consequently, $\mathcal{I}_C \models \kappa(\texttt{asy}(\rho(S,>0)))$.

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If $crisp(\mathcal{K})$ is satisfiable, it is not possible (due to the axioms in $T(N^{\mathcal{K}})$) to have an individual a such that $a^{\mathcal{I}_C} \in (A_{\rhd\gamma_1})^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \notin (A_{\rhd\gamma_2})^{\mathcal{I}_C}$ with $\gamma_2 < \gamma_1$, so for every individual a we can compute the maximum value α such that $a : A_{\geq\alpha}$ holds, or the maximum value β such that $a : A_{>\beta}$ holds, and use these values in the construction of the fuzzy interpretation. The case for roles in $R_a(N^{\mathcal{K}})$ and $R_c(N^{\mathcal{K}})$ is similar. □

Modularity

An interesting property of the procedure is that, under certain conditions given by Theorem 3, the reduction of an ontology can be reused when adding new axioms, and only the reduction of the new axioms has to be included. From an implementation point of view, this property makes possible to compute the reduction off-line and update it incrementally. We note that the reduction of the new axioms may include the definitions of new customized datatypes in case concrete concepts appear in them.

Theorem 3 *Let $\mathcal{K}$ be a Z $\mathcal{SROIQ}(\mathbf{D})$ fuzzy knowledge base involving a set of fuzzy atomic roles* $\mathbf{A}$, *a set of a set of atomic roles* $\mathbf{R_a}$ *and a set of concrete* *roles* $\mathbf{R_c}$*, let* $N^{\mathcal{K}}$ *be the set of relevant degrees to be considered and let* τ *be a* $Z\ \mathcal{SROIQ}(\mathbf{D})$ *axiom such that:*

1. *for every atomic concept* A *which appears in* τ*,* $A \in \mathbf{A}$*,*
2. *for every atomic role* R_A *which appears in* τ*,* $R_A \in \mathbf{R_a}$*,*
3. *for every concrete role* T *which appears in* τ*,* $T \in \mathbf{R_c}$*,*
4. *if* γ *appears in* τ*, then* $\gamma \in N^{\mathcal{K}}$*.*

Then, the reduction of the union of the KB and the axiom is equivalent to the union of the reduction of $\mathcal{K}$ *and the reduction of* τ*:*

$$crisp(\mathcal{K} \cup \tau) = crisp(\mathcal{K}) \cup \kappa(\tau)$$

Proof. Trivial from the following observations:

- Every axiom is reduced to a combination of new crisp elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- τ does not introduce atomic concepts, atomic abstract roles, concrete roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others.

□

The theorem assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often, which are reasonable assumptions because ontologies do not usually change once that their development has finished, and because it has been shown that the set of the degrees which must be considered for any reasoning task is $N^{\mathcal{K}}$ [302].

Consequently, even in case of an entailment test, it makes sense to use a degree in $N^{\mathcal{K}}$. Regarding the computation of any greatest lower bound, we recall that it has been shown that, in the worst case, it requires to compute $log|N^{\mathcal{K}}|$ satisfiability tests [299], which is another argument to fix the set of allowed degrees.

This property is very useful when it is necessary to add a new axiom to an ontology in order to perform some reasoning task.

Example 14 *Suppose that we want to classify a fuzzy ontology $\mathcal{O}$ i.e., to compute a fuzzy hierarchy for the concepts. Then, for every pair of concepts $C, D \in \mathcal{O}$ we have to compute the degree of subsumption of $C \sqsubseteq D$ w.r.t. $\mathcal{O}$. Recall that we can reduce this to a (un)satisfiability test. Using this property, while $C \sqsubseteq D$ has to be reduced for every pair of concepts (generating one new axiom every time), $\mathcal{O}$ only has to be reduced the first time.* □

Now we will consider what to do if the conditions of Theorem 3 are not satisfied.

- If τ introduces a new atomic concept, $T(N^{fK})$ needs to be recomputed.
- If τ introduces a new atomic abstract role, $R_a(N^{fK})$ needs to be recomputed.
- If τ introduces a new concrete role, $R_c(N^{fK})$ needs to be recomputed.
- If τ introduces a new degree of truth, $X^{\mathcal{K}}$ changes. As a consequence, $N^{\mathcal{K}}$ may change. If $N^{\mathcal{K}}$ changes, we need to recompute:
  - $T(N^{fK})$,
  - $R_a(N^{fK})$,
  - $R_c(N^{fK})$,
  - The reduction of every transitive role axiom in $\mathcal{K}$.
  - The reduction of every symmetric role axiom in $\mathcal{K}$.

Complexity

Definition 56 ***Depth of a concept expression.*** *The depth of a concept expression is inductively defined as follows:*

- $depth(A) = depth(\top) = depth(\bot) = depth(\{\alpha_1/o_1, \dots \alpha_m/o_m\}) = depth(\exists S.\,Self) = 1$,
- $depth(\exists R.C) = depth(\forall R.C) = depth(\neg C) = depth(\geq m\ S.C) = depth(\leq n\ S.C) = depth(\ mod(C)) = depth(C \geq \alpha]) = depth([C \leq \beta]) = depth(C)$,
- $depth(C \sqcap D) = depth(C \sqcup D) = max\{depth(C), depth(D)\}$,

It is easy to see that:

- Every fuzzy concept expression of depth k generates a crisp concept expression of depth k.
- Most of the axioms of the fuzzy KB generate one crisp axiom, but some of them (transitive and symmetric role axioms) generate several $(2 \cdot (|N^{\mathcal{K}}| - 1))$ crisp axioms.

Recall that in order to preserve the semantics of the new elements, we are also introducing some new crisp axioms.

Now, the size of the resulting KB is $O(|\mathcal{K}|^2)$. The ABox is actually linear while the TBox and the RBox are both quadratic:

- $|N^{\mathcal{K}}|$ is linearly bounded by $|\mathcal{K}|$: $|N^{\mathcal{K}}| \leq |\mathcal{A}| + |\mathcal{T}| + |\mathcal{R}|$,
- $|\kappa(\mathcal{A})| = |\mathcal{A}|$,
- $|T(N^{\mathcal{K}})| = (2 \cdot (|N^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{A}|$,
- $|\kappa(\mathcal{T})| = |\mathcal{T}|$,
- $|R_a(N^{\mathcal{K}})| = (2 \cdot (|N^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{R}|$,
- $|R_c(N^{\mathcal{K}})| = (2 \cdot (|N^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{T}|$,
- $|\kappa(\mathcal{R})| \leq 2 \cdot (|N^{\mathcal{K}}| - 1) \cdot |\mathcal{R}|$.

The resulting KB is quadratic because it depends on the number of relevant degrees $|N^{\mathcal{K}}|$. An immediate solution to obtain a KB which is linear in complexity is to fix the number of degrees which can appear in the knowledge base. From a practical point of view, in most of the applications it is sufficient to consider a small number of degrees.

Example 15 *Let* $N^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$, *i.e.,* $\alpha \in \{0.25, 0.5, 0.75, 1\}$ *and* $\beta \in \{0, 0.25, 0.5, 0.75\}$. *Now, the size of the resulting KB is linear since now:*

- $|T(N^{\mathcal{K}})| = 7 \cdot |\mathbf{A}|$,
- $|R_a(N^{\mathcal{K}})| = 7 \cdot |\mathbf{R}|$,
- $|R_c(N^{\mathcal{K}})| = 7 \cdot |\mathbf{T}|$,
- $|\kappa(\mathcal{R})| \leq 8 \cdot |\mathcal{R}|$. □

6.9 Optimizations

Reducing the Number of New Elements

Previous works [302, 26] use two more atomic concepts $A_{\leq\beta}, A_{<\alpha}$ and the following additional axioms (for $2 \leq k \leq |N^{\mathcal{K}}|$):

$$\begin{array}{rclcrcl} A_{<\gamma_k} & \sqsubseteq & A_{\leq\gamma_k}, & & A_{\leq\gamma_i} & \sqsubseteq & A_{<\gamma_{i+1}} \\ A_{\geq\gamma_k} \sqcap A_{<\gamma_k} & \sqsubseteq & \bot, & & A_{>\gamma_i} \sqcap A_{\leq\gamma_i} & \sqsubseteq & \bot \\ \top & \sqsubseteq & A_{\geq\gamma_k} \sqcup A_{<\gamma_k}, & & \top & \sqsubseteq & A_{>\gamma_i} \sqcup A_{\leq\gamma_i} \end{array}$$

In contrast to this, we use $\neg A_{>\gamma_k}$ rather than $A_{\leq\gamma_k}$ and $\neg A_{\geq\gamma_k}$ instead of $A_{<\gamma_k}$. The six axioms above follow immediately from the semantics of the crisp concepts as Proposition 8 shows:

Proposition 8 *If $A_{\geq\gamma_{i+1}} \sqsubseteq A_{>\gamma_i}$ and $A_{>\gamma_k} \sqsubseteq A_{\geq\gamma_k}$ hold, then the followings axioms are verified:*

$$\begin{array}{ll} (1)\ \neg A_{\geq\gamma_k} \sqsubseteq \neg A_{>\gamma_k} & (2)\ \neg A_{>\gamma_i} \sqsubseteq \neg A_{\geq\gamma_{i+1}} \\ (3)\ A_{\geq\gamma_k} \sqcap \neg A_{\geq\gamma_k} \sqsubseteq \bot & (4)\ A_{>\gamma_i} \sqcap \neg A_{>\gamma_i} \sqsubseteq \bot \\ (5)\ \top \sqsubseteq A_{\geq\gamma_k} \sqcup \neg A_{\geq\gamma_k} & (6)\ \top \sqsubseteq A_{>\gamma_i} \sqcup \neg A_{>\gamma_i} \end{array}$$

Proof.

- (1) and (2) derive from the fact that in crisp DLs $A \sqsubseteq B \equiv \neg B \sqsubseteq \neg A$.
- (3) and (4) come from the law of contradiction $A \sqcap \neg A \sqsubseteq \bot$.
- (5) and (6) derive from the law of excluded middle $\top \sqsubseteq A \sqcup \neg A$. □

As a minor comment, those works also introduce unnecessarily a couple of elements $A_{\geq 0}$ and $R_{\geq 0}$, as well as the axioms $A_{>0} \sqsubseteq A_{\geq 0}$, $R_{>0} \sqsubseteq R_{\geq 0}$ [302, 26].

In the case of roles, this optimization is essential in order to represent some role constructors of $\mathcal{SROIQ}(\mathbf{D})$ (negated role assertions and self reflexivity concepts). Actually, it is not possible to use a role of the form $R_{A\leq\gamma_k}$ rather than $\neg R_{A>\gamma_k}$ and $R_{A<\gamma_k}$ instead of $\neg R_{A\geq\gamma_k}$. The reason is that the logic does not make possible to express the corresponding version of the axioms

(3), (4), (5) and (6), which would be necessary to guarantee the correctness of the reduction, because the role conjunction and the bottom role are not allowed, and the universal role cannot appear in RIAs. On the contrary, we have represented expressions of the form $\rho(R_A, \lhd\gamma)$ using $\neg R_{A\neg\lhd\gamma}$.

Without this optimization, the size of $T(N^{\mathcal{K}})$ is:

$$|T(N^{\mathcal{K}})| = 8 \cdot (|N^{\mathcal{K}}| - 1) \cdot |\mathbf{A}|$$

Optimizing Irreflexive Role Axioms

We note that [281] proposes the following reduction for irreflexive role axioms:

$$\kappa(irr(R)) = \bigcup_{\gamma \in \mathcal{N}^{\mathcal{K}} \setminus \{0\}} irr(\rho(R, \geq \gamma)) \bigcup_{\gamma \in \mathcal{N}^{\mathcal{K}}} irr(\rho(R, > \gamma))$$

However, this reduction could be optimized to $\kappa(irr(R)) = irr(\rho(R, > 0))$. Proposition 9 shows that the other axioms follow immediately.

Proposition 9 *If $R_1 \sqsubseteq R_2$ and $irr(R_2)$, then it holds that $irr(R_1)$.*

Proof. Assume that $(x, y) \in R_1^{\mathcal{I}}$. Since $R_1 \sqsubseteq R_2$ is satisfied, then $(x, y) \in R_2^{\mathcal{I}}$ for some interpretation $\mathcal{I}$. Since $irr(R_2)$, then it holds that $(y, x) \notin R_2^{\mathcal{I}}$. But the role inclusion also implies that $(y, x) \notin R_1^{\mathcal{I}}$. For every pair of individuals, we have shown that $(x, y) \in R_1^{\mathcal{I}}$ implies $(y, x) \notin R_1^{\mathcal{I}}$. Hence, $irr(R_1)$ holds.

□

Allowing Crisp Concepts and Roles

It is easy to see that the complexity of the crisp representation is caused by fuzzy atomic concepts and roles. Fortunately, in real applications not all concepts and roles will be fuzzy. Therefore, an interesting optimization is enabling to specify that an atomic concept (resp. an atomic abstract role, a concrete role) is crisp.

For instance, suppose that A is a fuzzy atomic concept. Then, we need $|N^{\mathcal{K}}| - 1$ concepts of the form $A_{\geq\alpha}$ and another $|N^{\mathcal{K}}| - 1$ concepts of the form $A_{>\beta}$ to represent it, as well as $2(|N^{\mathcal{K}}|-1)-1$ axioms to preserve their semantics. On the other hand, if A is declared to be crisp, we just need one crisp concept A_{crisp} to represent it and no new axioms.

The case for atomic abstract roles R_a (resp. concrete roles T) is similar, only one crisp element $R_{A\,crisp}$ (resp. T_{crisp}) is needed.

Handling these crisp elements is very easy, because we only need to consider the following extension of ρ for those elements asserted to be interpreted as crisp:

x	y	$\rho(x,y)$
A	$\rhd\gamma$	A_{crisp}
A	$\lhd\gamma$	$\neg A_{crisp}$
R_A	$\rhd\gamma$	$R_{A\,crisp}$
R_A	$\lhd\gamma$	$\neg R_{A\,crisp}$
T	$\rhd\gamma$	T_{crisp}
T	$\lhd\gamma$	$\neg T_{crisp}$

Let $\mathbf{F_c}$, $\mathbf{F_{r_a}}$, $\mathbf{F_{r_c}}$ be the set of fuzzy concepts, abstract roles and concrete roles, respectively (that is, we exclude the crisp ones). Now the complexity is reduced:

- $|T(N^{\mathcal{K}})| = (2\cdot(|N^{\mathcal{K}}|-1)-1)\cdot|\mathbf{F_c}|$,
- $|R_a(N^{\mathcal{K}})| = (2\cdot(|N^{\mathcal{K}}|-1)-1)\cdot|\mathbf{F_{r_a}}|$,
- $|R_c(N^{\mathcal{K}})| = (2\cdot(|N^{\mathcal{K}}|-1)-1)\cdot|\mathbf{F_{r_c}}|$.

Of course, this optimization requires some manual intervention: the ontology developer needs to identify which elements can be interpreted as crisp.

Reasoning Ignoring Superfluous Elements

Our reduction is optimized to promote reusing under the conditions shown in Theorem 3. However, before performing a satisfiability test, some axioms do not need to be considered. These axioms cannot be removed from the crisp KB, because they may be necessary when new axioms are added to it, but they are superfluous for computing the satisfiability of the current crisp KB.

Definition 57 ***Superfluous element for reasoning in*** $crisp(\mathcal{K})$*. Let* $crisp(\mathcal{K})$ $= \langle \kappa(\mathcal{A}), T(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R}) \rangle$.

- *An* atomic concept A *is superfluous for reasoning in* $crisp(\mathcal{K})$ *if* $A \in T(N^{\mathcal{K}})$ *but* $A \notin crisp(\mathcal{K}) \setminus T(N^{\mathcal{K}})$ *(i.e., it does not appear in the other parts of* $crisp(\mathcal{K})$*).*

- *An* atomic abstract role R_A *is superfluous for reasoning in* $crisp(\mathcal{K})$ *if* $R_A \in R_a(N^{\mathcal{K}})$ *but* $R_A \notin crisp(\mathcal{K}) \setminus R_a(N^{\mathcal{K}})$ *(i.e., it does not appear in the other parts of* $crisp(\mathcal{K})$*).*

- *A* concrete role T *is superfluous for reasoning in* $crisp(\mathcal{K})$ *if* $T \in R_c(N^{\mathcal{K}})$ *but* $T \notin crisp(\mathcal{K}) \setminus R_c(N^{\mathcal{K}})$ *(i.e., it does not appear in the other parts of* $crisp(\mathcal{K})$*).*

The intuition here is that superfluous concepts and roles cannot cause a contradiction. But please note that if additional axioms are added to $\mathcal{K}$, $crisp(\mathcal{K})$ will be different and superfluous concept and roles may not be superfluous any more.

Let $1 \leq i \leq |N^{\mathcal{K}}| - 1, 2 \leq j \leq |N^{\mathcal{K}}| - 1$ and let $\sqsubseteq^*$ be the transitive closure of $\sqsubseteq$. Before reasoning with $crisp(\mathcal{K})$ we replace some parts of it with other such that they do not contain any superfluous concept or role. More precisely, we proceed as follows:

1. Replace $T(N^{\mathcal{K}})$ with $T'(N^{\mathcal{K}})$, which is defined as the smallest terminology containing, for each $A_1, A_2 \in T(N^{\mathcal{K}})$ such that A_1, A_2 are not superfluous and $A_1 \sqsubseteq^* A_2$, the axiom $A_1 \sqsubseteq A_2$.

2. Replace $R_a(N^{\mathcal{K}})$ with $R'_a(N^{\mathcal{K}})$, which is defined as the smallest terminology containing, for each $R_1, R_2 \in R_a(N^{\mathcal{K}})$ such that R_1, R_2 are not superfluous and $R_1 \sqsubseteq^* R_2$, the axiom $R_1 \sqsubseteq R_2$.

3. Replace $R_c(N^{\mathcal{K}})$ with $R'_c(N^{\mathcal{K}})$, which is defined as the smallest terminology containing, for each $T_1, T_2 \in R_c(N^{\mathcal{K}})$ such that T_1, T_2 are not superfluous and $T_1 \sqsubseteq^* T_2$, the axiom $T_1 \sqsubseteq T_2$.

Note that $T'(N^{\mathcal{K}})$, $R'_a(N^{\mathcal{K}})$ and $R'_c(N^{\mathcal{K}})$ are the versions of $T(N^{\mathcal{K}})$, $R_a(N^{\mathcal{K}})$ and $R_c(N^{\mathcal{K}})$ respectively, but including only non superfluous concepts, abstract roles and concrete roles.

A very interesting case is when there do not exist two atomic non superfluous concepts A_1, A_2 in $T(N^{\mathcal{K}})$ such that $A_1 \sqsubseteq^* A_2$, so $T'(N^{\mathcal{K}})$ is empty. A very common situation is that, for a non superfluous concept A_1 there is no A_2 in $T(N^{\mathcal{K}})$ such that it is not superfluous and $A_1 \sqsubseteq^* A_2$. The case for roles is similar.

Proposition 10 $\langle \kappa(\mathcal{A}), T(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R})\rangle$ *is satisfiable iff* $crisp(\mathcal{K}) = \langle \kappa(\mathcal{A}), T'(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R'(N^{\mathcal{K}}) \cup R'_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R})\rangle$ *is satisfiable.*

Proof. Consider a superfluous atomic concept A. By definition, $A \in T(N^{\mathcal{K}})$ and $A \notin crisp(\mathcal{K}) \setminus T(N^{\mathcal{K}})$. $A \notin crisp(\mathcal{K}) \setminus T(N^{\mathcal{K}})$ implies that it cannot cause a contradiction with $crisp(\mathcal{K}) \setminus T(N^{\mathcal{K}})$, so it does not affect the satisfiability. Moreover, from the structure of $T(N^{\mathcal{K}})$, it follows that it cannot cause a contradiction in $T(N^{\mathcal{K}})$, so it cannot cause it with $crisp(\mathcal{K})$. But $T'(N^{\mathcal{K}})$ is a modification of $T(N^{\mathcal{K}})$ including only non superfluous concepts, so the satisfiability of the full fuzzy KB is preserved. The case of roles is similar. □

Finally, a couple of examples will illustrate how the optimization works.

Example 16 *Consider Example 12 again. The procedure has created atomic concepts* $Hotel_{\geq 1}$, $Hotel_{>0.75}$, $Hotel_{\geq 0.75}$, $Hotel_{>0.5}$, $Hotel_{\geq 0.5}$, $Hotel_{>0.25}$, $Hotel_{\geq 0.25}$ *and* $Hotel_{>0}$. *However, if we consider* $crisp(\mathcal{K}) \setminus T^{\mathcal{K}}$, *it only contains* $Hotel_{\geq 0.75}$ *and* $Hotel_{>0}$. *Hence, the other concepts are superfluous. Since it follows that* $Hotel_{\geq 0.75} \sqsubseteq^* Hotel_{>0}$, *we may replace* $T(N^{\mathcal{K}})$ *with* $T'(N^{\mathcal{K}}) = \{Hotel_{\geq 0.75} \sqsubseteq Hotel_{>0}\}$. □

Example 17 *Consider Example 11 again. If we consider* $crisp(\mathcal{K}) \setminus T^{\mathcal{K}}$, *it only contains atomic concepts* $HighLevel_{\leq 0.25}$, $ObsessiveCompulsiveDisorder_{\geq 0.75}$, $SubstanceInducedAnxietyDisorder_{>0.25}$, $AnxietyDisorder_{\geq 0.75}$. *The other concepts are hence superfluous, so we may replace* $T(N^{\mathcal{K}})$ *with* $T'(N^{\mathcal{K}}) = \emptyset$. *Note that the set is empty, because there are no atomic non superfluous concepts such that they are subclasses or superclasses of any of the non superfluous concepts.*

In a similar way, the KB without $R_a^{\mathcal{K}}$ *only contains* $hasSerotoninLevel_{>0.25}$ *and* $hasDisease_{\geq 0.75}$ *only. Hence, the other roles are superfluous, so we may replace* $R_a(N^{\mathcal{K}})$ *with* $R'_a(N^{\mathcal{K}}) = \emptyset$.

But please note that if additional axioms are added to $\mathcal{K}$, $\mathtt{crisp}(\mathcal{K})$ will be different and previous superfluous concept and roles may not be superfluous any more. For example, if we want to check if $\mathcal{K} \cup \langle \mathsf{disorder} : \mathsf{AnxietyDisorder} \geq 0.5\rangle$ is satisfiable, then the concept $\mathsf{AnxietyDisorder}_{\geq 0.5}$ is no longer superfluous. Now, $T'(N^{\mathcal{K}}) = \{\mathsf{AnxietyDisorder}_{\geq 0.75} \sqsubseteq \mathsf{AnxietyDisorder}_{\geq 0.5}\}$.

□

6.10 Reasoning with a Possibilistic Extension

B. Hollunder showed that reasoning within a possibilistic DL can be reduced to reasoning within a classical DL [130]. He used a result which applies to every fragment of first-order logic such that classical entailment is decidable for it. Let KB be a possibilistic knowledge base, p be a first-order formula, $\alpha \in (0,1]$ and:

$$KB^{\alpha} = \{p \mid (p, N\alpha') \in KB, \alpha' \geq \alpha\}$$
$$KB_{\alpha} = \{p \mid (p, N\alpha') \in KB, \alpha' > \alpha\}$$

Then:

1. $KB \models (p, N\alpha)$ iff $KB_{\alpha} \models p$, and
2. $KB \models (p, \Pi\alpha)$ iff
   a) $KB^{0} \models p$ or,
   b) there is some $(q, \Pi\beta) \in KB$ such that $\beta \leq \alpha$ and $KB^{1-\beta} \cup \{q\} \models p$.

We will reduce here our possibilistic fuzzy DL to a possibilistic DL. As already seen, a fuzzy KB $\mathcal{K}$ can be reduced to a crisp KB $crisp(\mathcal{K})$ and every axiom $\tau \in \mathcal{K}$ is reduced to $\kappa(\tau)$, which can be an axiom or a set of axioms.

Adding degrees of certainty to an axiom $\tau \in \mathcal{K}$ is equivalent to adding degrees of certainty to their reductions $\kappa(\tau) \in crisp(\mathcal{K})$, as long as we also consider the axioms preserving the semantics of the whole process $T(N^{\mathcal{K}}) \cup R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}})$ (which are assumed to be necessarily true and do not have any degree of certainty associated). Hence:

- For every axiom $(\tau, \Pi\gamma) \in p\mathcal{K}$, $Poss(\tau) \geq \gamma$ iff $Poss(\kappa(\tau)) \geq \gamma$.
- For every axiom $(\tau, N\gamma) \in p\mathcal{K}$, $Nec(\tau) \geq \gamma$ iff $Nec(\kappa(\tau)) \geq \gamma$.

The following theorem shows that the reductions preserves reasoning.

Theorem 4 *A possibilistic fuzzy ontology $\mathcal{P}$ defined over $\mathcal{K}$ is satisfiable iff the possibilistic ontology $\mathcal{P}$ defined over* $\texttt{crisp}(\mathcal{K})$ *is satisfiable.*

Proof. The proof is trivial since entailment is reduced to KB satisfiability and that testing the satisfiability of $\mathcal{K}$ is equivalent to testing the satisfiability of $\texttt{crisp}(\mathcal{K})$. Given a fuzzy interpretation $\mathcal{I}$ such that $\mathcal{I} \models \tau$ for every axiom $\tau \in \mathcal{K}$, we can build a crisp interpretation $\mathcal{I}_{\mathcal{C}}$ such that $\mathcal{I}_{\mathcal{C}} \models \kappa(\tau)$ (and viceversa). Hence, $\sup\{\pi(\mathcal{I}) \mid \mathcal{I} \in \mathfrak{I}, \mathcal{I} \models \tau\} = \sup\{\pi(\mathcal{I}_c) \mid \mathcal{I}_c \in \mathfrak{I}, \mathcal{I}_c \models \kappa(\tau)\}$ □

Example 18 *Consider again Example 9. The axiom* $(\langle \mathit{tom}\colon \mathit{High} \geq 0.5\rangle, \Pi\ 0.2)$ *is reduced to:*

$$(\langle \mathit{tom}\colon \mathit{High}_{\geq 0.5}\rangle, \Pi\ 0.2)$$

This new axiom means that it is possible with degree 0.2 *that Tom belongs to the crisp set* $\mathit{High}_{\geq 0.5}$.

The final crisp KB would also need some additional axioms (consequence of the reduction of the fuzzy KB):

$$
\begin{aligned}
T(N^{\mathcal{K}}) = \{ & \\
& \mathit{High}_{\geq 0.5} \sqsubseteq \mathit{High}_{>0}, \\
& \mathit{High}_{>0.5} \sqsubseteq \mathit{High}_{\geq 0.5}, \\
& \mathit{High}_{\geq 1} \sqsubseteq \mathit{High}_{>0.5} \\
\} &
\end{aligned}
$$

□

In the previous sections of this chapter, we have reduced a fuzzy KB to a crisp KB. Then, reasoning is performed by computing a consistency test on the crisp KB. The case is more difficult now, because we need to perform several entailment tests.

It is also worth to keep in mind that, although it is possible to reason using a crisp reasoner, how to represent a possibilistic ontology using a classical ontology remains an open issue.

Figure 6.1 illustrates the differences between the reduction of fuzzy ontologies and the reduction of possibilistic fuzzy ontologies.

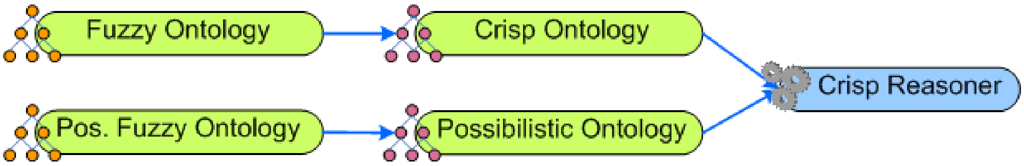

Figure 6.1: Reasoning with fuzzy ontologies and possibilistic fuzzy ontologies.

CHAPTER 7

A Crisp Representation for Fuzzy $\mathcal{SROIQ}(\mathbf{D})$ under Gödel Family

In this chapter we show how to reduce a G $\mathcal{SROIQ}(\mathbf{D})$ fuzzy KB to a crisp KB. The procedure preserves reasoning, so existing $\mathcal{SROIQ}(\mathbf{D})$ reasoners could be applied to the resulting KB. First we will describe the reduction and then we will provide an illustrating example.

Once again, the basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics. Finally, every axiom in the ABox, the TBox and the RBox is represented, independently of other axioms, using these new crisp elements.

Section 7.1 studies the relevant set of degrees of truth to be considered in the reasoning. Section 7.2 describes the process of creation of new crisp concepts and roles. Abstract concepts and roles are mapped in Section 7.3, whereas axioms are reduced in Section 7.4. The cases of concrete concepts and roles, and modified concept and roles are not addressed in this chapter, since they are dealt with exactly as in Zadeh family, and hence we refer the reader to Chapter 6. An illustrating example of the reduction is presented in Section 7.5. Section 7.6 studies some properties of the reduction. Finally, some important optimizations are described in 7.7. Reasoning with the possibilistic layer is not repeated here either, since it can be dealt with exactly as in Zadeh family (Section 6.10).

7.1 Relevant Set of Degrees of Truth

In the previous chapter we have seen that in Zadeh logic, it is enough to consider the set of degrees of truth $\mathcal{N}^{\mathcal{K}}$. Interestingly, in Gödel logic it is enough to consider a fixed set of degrees truth $\mathcal{N}^{\mathcal{K}}$ including 0 and 1, since the fuzzy operators of this family do not introduce new degrees of truth. We define $\mathcal{TV} = \{0,1\} \cup \{\gamma \mid \langle \tau \bowtie \gamma \rangle \in \mathcal{K}\}$. For every $\gamma_1, \gamma_2 \in \mathcal{TV}$:

- The value of $\ominus\gamma_1$ is either 0 or 1.
- The value of $\gamma_1 \otimes \gamma_2$ and $\gamma_1 \oplus \gamma_2$ is either γ_1 or γ_2.
- The value of $\gamma_1 \Rightarrow \gamma_2$ is either 1 or γ_2.

And, by definition, 0, 1, γ_1 and γ_2 belong to $\mathcal{N}^{\mathcal{K}}$.

Given a fuzzy KB $\mathcal{K}$, we will assume a fixed set of degrees of truth $\mathcal{N}^{\mathcal{K}}$ such that it includes at last every degree in $\mathcal{K}$ plus 0 and 1. For convenience we also define $(\mathcal{N}^{\mathcal{K}})^+ = \mathcal{N}^{\mathcal{K}} \setminus \{0\}$.

7.2 Adding New Elements

Now we proceed exactly as for Zadeh logic. Let $\mathbf{A}$ be the set of atomic concepts, $\mathbf{R}$ the set of atomic abstract roles and $\mathbf{T}$ the set of concrete roles in a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$. Let $1 \le i \le |\mathcal{N}^{\mathcal{K}}| - 1, 2 \le j \le |\mathcal{N}^{\mathcal{K}}| - 1$. For each $\alpha, \beta \in \mathcal{N}^{\mathcal{K}}$ with $\alpha \in (0,1]$ and $\beta \in [0,1)$, we introduce:

- For each $A \in \mathbf{A}$,
  - We create two new atomic concepts $A_{\ge\alpha}, A_{>\beta}$.
  - We add to $T(\mathcal{N}^{\mathcal{K}})$ the set of axioms:

$$A_{\ge\gamma_{i+1}} \sqsubseteq A_{>\gamma_i} \qquad A_{>\gamma_j} \sqsubseteq A_{\ge\gamma_j}$$

- For each $R_A \in \mathbf{R}$.

– Two new atomic abstract roles $R_{A \geq \alpha}, R_{A > \beta}$.
– We add to $R_a(\mathcal{N}^{\mathcal{K}})$ the set of axioms:

$$R_{A \geq \gamma_{i+1}} \sqsubseteq R_{A > \gamma_i} \qquad R_{A > \gamma_i} \sqsubseteq R_{A \geq \gamma_i}$$

- For each $T \in \mathbf{T}$,
  – Two concrete roles $T_{\geq \alpha}, T_{> \beta}$.
  – We add to $R_c(\mathcal{N}^{\mathcal{K}})$ the set of axioms:

$$T_{\geq \gamma_{i+1}} \sqsubseteq T_{> \gamma_i} \qquad T_{> \gamma_i} \sqsubseteq T_{\geq \gamma_i}$$

7.3 Mapping Fuzzy Concepts and Roles

Fuzzy concept and role expressions are reduced using mapping ρ, as shown in Tables 7.1 and 7.2 respectively. Concrete predicates are reduced exactly as in Section 6.7, while modified concept and roles are reduced exactly as in Section 6.9.

Once again, note that $\rho(A, \leq \beta) = \neg A_{> \beta}$ is different from $\rho(\neg A, \geq \alpha) = \rho(A, \leq 0) = \neg A_{> 0}$ and that, due to the restrictions in the definition of the fuzzy KB, there are some expressions that cannot appear during in the reduction:

- $\rho(A, \geq 0), \rho(A, > 1), \rho(A, \leq 1), \rho(A, < 0)$.
- $\rho(\top, \geq 0), \rho(\top, > 1), \rho(\top, \leq 1), \rho(\top, < 0)$.
- $\rho(\bot, \geq 0), \rho(\bot, > 1), \rho(\bot, \leq 1), \rho(\bot, < 0)$.
- $\rho(R_A, \geq 0), \rho(R_A, > 1), \rho(R_A, \leq 1), \rho(R_A, < 0)$.
- $\rho(T, \geq 0), \rho(T, > 1), \rho(T, \leq 1), \rho(T, < 0)$.
- $\rho(U, \geq 0), \rho(U, > 1), \rho(U, \leq 1), \rho(U, < 0)$.
- $\rho(R, \lhd \gamma), \rho(U, \lhd \gamma)$ and $\rho(T, \lhd \gamma)$ can only appear in a (crisp) negated role assertion.

Table 7.1: Mapping of concept expressions under Gödel semantics.

x	y	$\rho(x,y)$
$\top$	$\rhd\gamma$	$\top$
$\top$	$\lhd\gamma$	$\bot$
$\bot$	$\rhd\gamma$	$\bot$
$\bot$	$\lhd\gamma$	$\top$
A	$\rhd\gamma$	$A_{\rhd\gamma}$
A	$\lhd\gamma$	$\neg A_{\neg\lhd\gamma}$
$\neg C$	$\rhd\gamma$	$\rho(C,\leq 0)$
$\neg C$	$\lhd\gamma$	$\rho(C,> 0)$
$C \sqcap D$	$\rhd\gamma$	$\rho(C,\rhd\gamma) \sqcap \rho(D,\rhd\gamma)$
$C \sqcap D$	$\lhd\gamma$	$\rho(C,\lhd\gamma) \sqcup \rho(D,\lhd\gamma)$
$C \sqcup D$	$\rhd\gamma$	$\rho(C,\rhd\gamma) \sqcup \rho(D,\rhd\gamma)$
$C \sqcup D$	$\lhd\gamma$	$\rho(C,\lhd\gamma) \sqcap \rho(D,\lhd\gamma)$
$\exists R.C$	$\rhd\gamma$	$\exists\rho(R,\rhd\gamma).\rho(C,\rhd\gamma)$
$\exists R.C$	$\lhd\gamma$	$\forall\rho(R,\neg\lhd\gamma).\rho(C,\lhd\gamma)$
$\exists T.\mathbf{d}$	$\rhd\gamma$	$\exists\rho(T,\rhd\gamma).\rho(\mathbf{d},\rhd\gamma)$
$\exists T.\mathbf{d}$	$\lhd\gamma$	$\forall\rho(T,\neg\lhd\gamma).\rho(\mathbf{d},\lhd\gamma)$
$\forall R.C$	$\geq\alpha$	$\sqcap_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\alpha}(\forall\rho(R,\geq\gamma).\rho(C,\geq\gamma)) \sqcap_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma<\alpha}(\forall\rho(R,>\gamma).\rho(C,>\gamma))$
$\forall R.C$	$>\beta$	$\sqcap_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\beta}(\forall\rho(R,\geq\gamma).\rho(C,\geq\gamma)) \sqcap_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma\leq\beta}(\forall\rho(R,>\gamma).\rho(C,>\gamma))$
$\forall R.C$	$\leq\beta$	$\sqcup_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma\leq\beta}(\exists\rho(R,>\gamma).\rho(C,\leq\gamma))$
$\forall R.C$	$<\alpha$	$\sqcup_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\alpha}(\exists\rho(R,\geq\gamma).\rho(C,<\gamma))$
$\forall T.\mathbf{d}$	$\geq\alpha$	$\sqcap_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\alpha}(\forall\rho(T,\geq\gamma).\rho(\mathbf{d},\geq\gamma)) \sqcap_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma<\alpha}(\forall\rho(T,>\gamma).\rho(\mathbf{d},>\gamma))$
$\forall T.\mathbf{d}$	$>\beta$	$\sqcap_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\beta}(\forall\rho(T,\geq\gamma).\rho(\mathbf{d},\geq\gamma)) \sqcap_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma\leq\beta}(\forall\rho(T,>\gamma).\rho(\mathbf{d},>\gamma))$
$\forall T.\mathbf{d}$	$\leq\beta$	$\sqcup_{\gamma\in\mathcal{N}^{\mathcal{K}} \mid \gamma\leq\beta}(\exists\rho(T,>\gamma).\rho(\mathbf{d},\leq\gamma))$
$\forall T.\mathbf{d}$	$<\alpha$	$\sqcup_{\gamma\in(\mathcal{N}^{\mathcal{K}})^+ \mid \gamma\leq\alpha}(\exists\rho(T,\geq\gamma).\rho(\mathbf{d},<\gamma))$
$\{\alpha_1/o_1,\ldots,\alpha_m/o_m\}$	$\bowtie\gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1\leq i\leq m\}$
$\geq m\, S.C$	$\rhd\gamma$	$\geq m\; \rho(S,\rhd\gamma).\rho(C,\rhd\gamma)$
$\geq m\, S.C$	$\lhd\gamma$	$\leq m-1\; \rho(S,\neg\lhd\gamma).\rho(C,\neg\lhd\gamma)$
$\geq m\; T.\mathbf{d}$	$\rhd\gamma$	$\geq m\; \rho(T,\rhd\gamma).\rho(\mathbf{d},\rhd\gamma)$
$\geq m\; T.\mathbf{d}$	$\lhd\gamma$	$\leq m-1\; \rho(T,\neg\lhd\gamma).\rho(\mathbf{d},\neg\lhd\gamma)$
$\leq n\, S.C$	$\rhd\gamma$	$\leq n\; \rho(S,>0).\rho(C,S,>0)$
$\leq n\, S.C$	$\lhd\gamma$	$\geq n+1\; \rho(S,>0).\rho(C,S,>0)$
$\leq n\; T.\mathbf{d}$	$\rhd\gamma$	$\leq n\; \rho(T,>0).\rho(\mathbf{d},>0)$
$\leq n\; T.\mathbf{d}$	$\lhd\gamma$	$\geq n+1\; \rho(T,>0).\rho(\mathbf{d},>0)$
$\exists S.Self$	$\rhd\gamma$	$\exists\rho(S,\rhd\gamma).Self$
$\exists S.Self$	$\lhd\gamma$	$\neg\exists\rho(S,\neg\lhd\gamma).Self$
$[C\geq\alpha]$	$\rhd\gamma$	$\rho(C,\geq\alpha)$
$[C\geq\alpha]$	$\lhd\gamma$	$\rho(C,<\alpha)$
$[C\leq\beta]$	$\rhd\gamma$	$\rho(C,\leq\beta)$
$[C\leq\beta]$	$\lhd\gamma$	$\rho(C,>\beta)$

Table 7.2: Mapping of role expressions under Gödel semantics.

x	y	$\rho(x,y)$
R_A	$\triangleright\gamma$	$R_{A\triangleright\gamma}$
R_A	$\triangleleft\gamma$	$\neg R_{A\neg\triangleleft\gamma}$
R^-	$\bowtie\gamma$	$\rho(R,\bowtie\gamma)^-$
U	$\triangleright\gamma$	U
U	$\triangleleft\gamma$	$\neg U$
$[R\geq\alpha]$	$\triangleright\gamma$	$\rho(R,\geq\alpha)$
$[R\geq\alpha]$	$\triangleleft\gamma$	$\rho(R,<\alpha)$
$\neg R$	$\triangleright\gamma$	$\rho(R,\leq 0)$
$\neg R$	$\triangleleft\gamma$	$\rho(R,>0)$
T	$\triangleright\gamma$	$T_{\triangleright\gamma}$

7.4 Mapping Fuzzy Axioms

Axioms are reduced as in Table 7.3, where $\kappa(\tau)$ maps a fuzzy axiom τ in $G\,\mathcal{SROIQ}(\mathbf{D})$ to a set of crisp axioms in $\mathcal{SROIQ}(\mathbf{D})$.

Note that in fuzzy GCIs and RIAs of the form $\langle\tau\triangleright\gamma\rangle$, the number of new crisp axioms depends on the value of γ: the higher γ, the more axioms are generated.

Observe that $\kappa(\langle C\sqsubseteq D\geq 1\rangle)$ is equivalent to the reduction of a GCI, under a semantics based on Zadeh's inclusion of fuzzy sets, proposed in [302], although this work introduces two unnecessary axioms $C_{\geq 0}\sqsubseteq D_{\geq 0}$ and $C_{>1}\sqsubseteq D_{>1}$.

Recall that in Gödel family we may assume that irreflexive, transitive and symmetric role axioms do not appear.

Let us illustrate how the reduction of an axiom works by showing an example.

Example 19 *Consider the GCI $\langle C\sqsubseteq D\geq\alpha\rangle$. If it is satisfied, $\inf_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\geq\alpha$. As this is true for the infimum, an arbitrary $x\in\Delta^{\mathcal{I}}$ must satisfy $C^{\mathcal{I}}(x)\Rightarrow D^{\mathcal{I}}(x)\geq\alpha$. From the semantics of Gödel implication, this is true if $C^{\mathcal{I}}(x)\leq D^{\mathcal{I}}(x)$ or $D^{\mathcal{I}}(x)\geq\alpha$. Hence, for each $\gamma\in(\mathcal{N}^{\mathcal{K}})^+$ such that $\gamma\leq\alpha$, $C^{\mathcal{I}}(x)\geq\gamma$ implies $D^{\mathcal{I}}(x)\geq\gamma$ (which is expressed as $\rho(C,\geq\gamma)\sqsubseteq\rho(D,\geq\gamma)$)*

and for each $\gamma \in \mathcal{N}^{\mathcal{K}} \mid \gamma < \alpha$, $C^{\mathcal{I}}(x) > \alpha$ *implies* $D^{\mathcal{I}}(x) > \alpha$ *(which is expressed as* $\rho(C, > \alpha) \sqsubseteq \rho(D, > \alpha)$*).* □

Table 7.3: Reduction of the axioms under Gödel semantics.

τ	$\kappa(\tau)$
$\langle a : C \bowtie \gamma \rangle$	$\{a : \rho(C, \bowtie \gamma)\}$
$\langle (a,b) : R \bowtie \gamma \rangle$	$\{(a,b) : \rho(R, \bowtie \gamma)\}$
$\langle (a,b) : \neg R \bowtie \gamma \rangle$	$\{(a,b) : \rho(\neg R, \bowtie \gamma)\}$
$\langle a \neq b \rangle$	$\{a \neq b\}$
$\langle a = b \rangle$	$\{a = b\}$
$\langle C \sqsubseteq D \geq \alpha \rangle$	$\bigcup_{\gamma \in (\mathcal{N}^{\mathcal{K}})^+ \mid \gamma \leq \alpha} \{\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)\}$ $\bigcup_{\gamma \in \mathcal{N}^{\mathcal{K}} \mid \gamma < \alpha} \{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\}$
$\langle C \sqsubseteq D > \beta \rangle$	$\kappa(C \sqsubseteq D \geq \beta) \cup \{\rho(C, > \beta) \sqsubseteq \rho(D, > \beta)\}$
$\langle R_1 \dots R_m \sqsubseteq R \geq \alpha \rangle$	$\bigcup_{\gamma \in (\mathcal{N}^{\mathcal{K}})^+ \mid \gamma \leq \alpha} \{\rho(R_1, \geq \gamma) \dots \rho(R_m, \geq \gamma) \sqsubseteq \rho(R, \geq \gamma)\}$ $\bigcup_{\gamma \in \mathcal{N}^{\mathcal{K}} \mid \gamma < \alpha} \{\rho(R_1, > \gamma) \dots \rho(R_m, > \gamma) \sqsubseteq \rho(R, > \gamma)\}$
$\langle R_1 \dots R_m \sqsubseteq R > \beta \rangle$	$\kappa(\langle R_1 \dots R_m \sqsubseteq R \geq \beta \rangle) \cup \{\rho(R_1, > \beta) \dots \rho(R_m, > \beta) \sqsubseteq \rho(R, > \beta)\}$
$\langle T_1 \sqsubseteq T_2 \geq \alpha \rangle$	$\bigcup_{\gamma \in (\mathcal{N}^{\mathcal{K}})^+ \mid \gamma \leq \alpha} \{\rho(T_1, \geq \gamma) \sqsubseteq \rho(T_2, \geq \gamma)\}$ $\bigcup_{\gamma \in \mathcal{N}^{\mathcal{K}} \mid \gamma < \alpha} \{\rho(T_1, > \gamma) \sqsubseteq \rho(T_2, > \gamma)\}$
$\langle T_1 \sqsubseteq T_2 > \beta \rangle$	$\kappa(\langle T_1 \sqsubseteq T_2 \geq \beta \rangle) \cup \{\rho(T_1, > \beta) \sqsubseteq \rho(T_2, > \beta)\}$
$\mathtt{dis}(S_1, S_2)$	$\{\mathtt{dis}(\rho(S_1, > 0), \rho(S_2, > 0))\}$
$\mathtt{dis}(T_1, T_2)$	$\{\mathtt{dis}(\rho(T_1, > 0), \rho(T_2, > 0))\}$
$\mathtt{ref}(R)$	$\{\mathtt{ref}(\rho(R, \geq 1))\}$
$\mathtt{asy}(S)$	$\{\mathtt{asy}(\rho(S, > 0)\}$

Summing up, a fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced to a KB $\langle \kappa(\mathcal{A}), T(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R}) \rangle$.

7.5 Example

Now we will illustrate the whole procedure with an example.

Example 20 *Consider a fuzzy KB* $\mathcal{K}$ *in a musical domain where we define* $\mathcal{TV} = \{0, 0.3, 0.5, 0.7, 1\}$ *and that represents the following knowledge:*

- *Radiohead is very likely not among Fernando's favourite bands:*

$$\langle (\mathit{fernando}, \mathit{radiohead}) : \neg \mathit{hasFavouriteBand} \geq 0.7 \rangle$$

- *Fernando has at most two favourite bands playing flamenco:*

$$\langle \mathit{fernando}\colon\ \leq 2\ \mathit{hasFavouriteBand}.\mathit{FlamencoBand} \geq 0.7\rangle$$

- *Every Radiohead's record is not a live record (we introduce a degree to reflect the fact that there exist several non-official live records):*

$$\langle \mathit{radiohead}\colon \forall \mathit{hasRecord}.(\neg \mathit{LiveRecord}) \geq 0.7\rangle$$

Next, we create some new elements and some axioms preserving their semantics. The new axioms due to the new concepts are:

$$
\begin{aligned}
T(\mathcal{N}^{\mathcal{K}}) = \quad \{ & \\
& \mathit{LiveRecord}_{\geq 1} \sqsubseteq \mathit{LiveRecord}_{>0.75}, \\
& \mathit{LiveRecord}_{>0.75} \sqsubseteq \mathit{LiveRecord}_{\geq 0.75}, \\
& \mathit{LiveRecord}_{\geq 0.75} \sqsubseteq \mathit{LiveRecord}_{>0.5}, \\
& \mathit{LiveRecord}_{>0.5} \sqsubseteq \mathit{LiveRecord}_{\geq 0.5}, \\
& \mathit{LiveRecord}_{\geq 0.5} \sqsubseteq \mathit{LiveRecord}_{>0.25}, \\
& \mathit{LiveRecord}_{>0.25} \sqsubseteq \mathit{LiveRecord}_{\geq 0.25}, \\
& \mathit{LiveRecord}_{\geq 0.25} \sqsubseteq \mathit{LiveRecord}_{>0}, \\
& \mathit{FlamencoBand}_{\geq 1} \sqsubseteq \mathit{FlamencoBand}_{>0.75}, \\
& \mathit{FlamencoBand}_{>0.75} \sqsubseteq \mathit{FlamencoBand}_{\geq 0.75}, \\
& \mathit{FlamencoBand}_{\geq 0.75} \sqsubseteq \mathit{FlamencoBand}_{>0.5}, \\
& \mathit{FlamencoBand}_{>0.5} \sqsubseteq \mathit{FlamencoBand}_{\geq 0.5}, \\
& \mathit{FlamencoBand}_{\geq 0.5} \sqsubseteq \mathit{FlamencoBand}_{>0.25}, \\
& \mathit{FlamencoBand}_{>0.25} \sqsubseteq \mathit{FlamencoBand}_{\geq 0.25}, \\
& \mathit{FlamencoBand}_{\geq 0.25} \sqsubseteq \mathit{FlamencoBand}_{>0} \\
\} &
\end{aligned}
$$

The new axioms generated for the roles are:

$$
\begin{aligned}
R_a(\mathcal{N}^{\mathcal{K}}) = \{ & \\
& \mathit{hasFavouriteBand}_{\geq 1} \sqsubseteq \mathit{hasFavouriteBand}_{>0.75}, \\
& \mathit{hasFavouriteBand}_{>0.75} \sqsubseteq \mathit{hasFavouriteBand}_{\geq 0.75}, \\
& \mathit{hasFavouriteBand}_{\geq 0.75} \sqsubseteq \mathit{hasFavouriteBand}_{>0.5}, \\
& \mathit{hasFavouriteBand}_{>0.5} \sqsubseteq \mathit{hasFavouriteBand}_{\geq 0.5}, \\
& \mathit{hasFavouriteBand}_{\geq 0.5} \sqsubseteq \mathit{hasFavouriteBand}_{>0.25}, \\
& \mathit{hasFavouriteBand}_{>0.25} \sqsubseteq \mathit{hasFavouriteBand}_{\geq 0.25}, \\
& \mathit{hasFavouriteBand}_{\geq 0.25} \sqsubseteq \mathit{hasFavouriteBand}_{>0}, \\
& \mathit{hasRecord}_{\geq 1} \sqsubseteq \mathit{hasRecord}_{>0.75}, \\
& \mathit{hasRecord}_{>0.75} \sqsubseteq \mathit{hasRecord}_{\geq 0.75}, \\
& \mathit{hasRecord}_{\geq 0.75} \sqsubseteq \mathit{hasRecord}_{>0.5}, \\
& \mathit{hasRecord}_{>0.5} \sqsubseteq \mathit{hasRecord}_{\geq 0.5}, \\
& \mathit{hasRecord}_{\geq 0.5} \sqsubseteq \mathit{hasRecord}_{>0.25}, \\
& \mathit{hasRecord}_{>0.25} \sqsubseteq \mathit{hasRecord}_{\geq 0.25}, \\
& \mathit{hasRecord}_{\geq 0.25} \sqsubseteq \mathit{hasRecord}_{>0} \\
\} &
\end{aligned}
$$

Finally, we map the three axioms in the ABox:

- $\kappa(\langle(\mathit{fernando}, \mathit{radiohead}) \colon \neg \mathit{hasFavouriteBand} \geq 0.7\rangle) =$
  $(\mathit{fernando}, \mathit{radiohead}) \colon \rho(\neg \mathit{hasFavouriteBand}, \geq 0.7) =$
  $(\mathit{fernando}, \mathit{radiohead}) \colon \rho(\mathit{hasFavouriteBand}, \leq 0) =$
  $(\mathit{fernando}, \mathit{radiohead}) \colon \neg \rho(\mathit{hasFavouriteBand}_{>0}$.

- $\kappa(\langle \mathit{fernando} \colon \leq 2\ \mathit{hasFavouriteBand}.\mathit{FlamencoBand} \geq 0.7\rangle) =$
  $\mathit{fernando} \colon \leq 2\ \mathit{hasFavouriteBand}_{>0}.\mathit{FlamencoBand}_{>0}$.

- $\kappa(\langle \mathit{radiohead} \colon \forall \mathit{hasRecord}.(\neg \mathit{LiveRecord}) \geq 0.7\rangle) =$

$$
\begin{aligned}
\mathit{radiohead}: \quad & (\forall \mathit{hasRecord}_{>0}.\rho(\neg \mathit{LiveRecord}, > 0)) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.3}.\rho(\neg \mathit{LiveRecord}, \geq 0.3)) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.3}.\rho(\neg \mathit{LiveRecord}, > 0.3)) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.5}.\rho(\neg \mathit{LiveRecord}, \geq 0.5)) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.5}.\rho(\neg \mathit{LiveRecord}, > 0.5)) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.7}.\rho(\neg \mathit{LiveRecord}, \geq 0.7)) = \\
\mathit{radiohead}: \quad & (\forall \mathit{hasRecord}_{>0}.\mathit{LiveRecord}_{\leq 0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.3}.\mathit{LiveRecord}_{\leq 0}) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.3}.\mathit{LiveRecord}_{\leq 0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.5}.\mathit{LiveRecord}_{\leq 0}) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.5}.\mathit{LiveRecord}_{\leq 0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.7}.\mathit{LiveRecord}_{\leq 0}) = \\
\mathit{radiohead}: \quad & (\forall \mathit{hasRecord}_{>0}.\neg \mathit{LiveRecord}_{>0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.3}.\neg \mathit{LiveRecord}_{>0}) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.3}.\neg \mathit{LiveRecord}_{>0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.5}.\neg \mathit{LiveRecord}_{>0}) \sqcap \\
& (\forall \mathit{hasRecord}_{>0.5}.\neg \mathit{LiveRecord}_{>0}) \sqcap \\
& (\forall \mathit{hasRecord}_{\geq 0.7}.\neg \mathit{LiveRecord}_{>0})
\end{aligned}
$$

Observe that the reduction of the latter axiom can be simplified to

$$\mathit{radiohead}: \forall \mathit{hasRecord}_{>0}.\neg \mathit{LiveRecord}_{>0}$$

but in general, this is not possible and the reduction of a fuzzy universal quantification is a conjunction of universal quantifications. □

7.6 Properties of the Reduction

Firstly, the reduction preserves simplicity of the roles and regularity of the RIAs.

Decidability of the Logic and Correctness of the Reduction

The following theorem shows the logic is decidable under Gödel semantics and that the reductions preserves reasoning.

Theorem 5 *The satisfiability problem in G $\mathcal{SROIQ}(\mathbf{D})$ is decidable. Furthermore, a G $\mathcal{SROIQ}(\mathbf{D})$ fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is satisfiable iff its crisp representation* $\texttt{crisp}(\mathcal{K}) = \langle \kappa(\mathcal{A}), T(N^{\mathcal{K}}) \cup \kappa(\mathcal{T}), R_a(N^{\mathcal{K}}) \cup R_c(N^{\mathcal{K}}) \cup \kappa(\mathcal{R}) \rangle$ *is satisfiable.*

Proof. We will show the proof for the only-if direction. From $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\}$ with respect to a fuzzy concrete domain $\mathbf{D} = \langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Phi_{\mathbf{D}}$ only contains the fuzzy concrete predicate $\mathbf{d} = trap_{k_1,k_2}(q_1, q_2, q_3, q_4)$, satisfying every axiom in $\mathcal{K}$. Now, it is possible to build a (crisp) interpretation $\mathcal{I}_C = \{\Delta^{\mathcal{I}_C}, \cdot^{\mathcal{I}_C}\}$ with respect to a crisp concrete domain $\mathbf{D_C} = \langle \Delta_{\mathbf{D_C}}, \Phi_{\mathbf{D_C}} \rangle$ as:

- $\Delta^{\mathcal{I}_C} = \Delta^{\mathcal{I}}$.
- $\Delta_{\mathbf{D_C}} = \Delta_{\mathbf{D}}$.
- $x^{\mathcal{I}_C} = x^{\mathcal{I}}$, for all $x \in \Delta^{\mathcal{I}}$.
- $v_{\mathbf{D_C}} = v_{\mathbf{D}}$, for all $v \in \Delta_{\mathbf{D}}$.
- $A_{\geq\alpha}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) \geq \alpha\}$, for each $A \in \mathbf{A}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $A_{>\beta}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}} \mid A^{\mathcal{I}}(x) > \beta\}$, for each $A \in \mathbf{A}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $R_{A\geq\alpha}^{\mathcal{I}_C} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) \geq \alpha\}$, for each $R_A \in \mathbf{R}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $R_{A>\beta}^{\mathcal{I}_C} = \{x, y \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid R_A^{\mathcal{I}}(x, y) > \beta\}$, for each $R_A \in \mathbf{R}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $T_{\geq\alpha}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) \geq \alpha\}$, for each $T \in \mathbf{T}$, $\alpha \in N^{\mathcal{K}} \setminus \{0\}$.
- $T_{>\beta}^{\mathcal{I}_C} = \{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathbf{D}} \mid T^{\mathcal{I}}(x, v) > \beta\}$, for each $T \in \mathbf{T}$, $\beta \in N^{\mathcal{K}} \setminus \{1\}$.
- $\Phi_{\mathbf{D_C}}$ will contain some concrete predicates of the form $real[a, b]$, $real(a, b)$ and $union\text{–}real[k_1, a, b, k_2]$, with $a, b, k_1, k_2 \in \mathbb{R}$.

Now, we will show that $\mathcal{I}_C$ satisfies every axiom in $\texttt{crisp}(\mathcal{K})$. For every axiom $\tau \in \mathcal{K}$, there are several cases:

1. τ is an inequality assertion. Assume that $\mathcal{I} \models \langle a \neq b \rangle$. Then, $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. By definition of $\mathcal{I}_C$, $a^{\mathcal{I}_C} \neq b^{\mathcal{I}_C}$, so $\mathcal{I}_C \models \langle a \neq b \rangle \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a \neq b \rangle)$. The case of equality assertions is similar.
2. τ is a role assertion. Assume that $\mathcal{I} \models \langle (a, b) : R \bowtie \gamma \rangle$. We show, by induction on the structure of roles, that $\mathcal{I}_C \models \kappa(\langle (a, b) : R \bowtie \gamma \rangle)$.

- *Atomic role*. Assume that $\mathcal{I} \models \langle(a,b) : R_A \triangleright \gamma\rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in R_{A \triangleright \gamma}^{\mathcal{I}_C}$. By definition of ρ, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R_A, \triangleright\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R_A, \triangleright\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : R_A \triangleright \gamma\rangle)$.

  Now assume that $\mathcal{I} \models \langle(a,b) : R_A \triangleleft \gamma\rangle$. Then, $R_A^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleleft \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \notin (R_{A \neg\triangleright\gamma})^{\mathcal{I}_C}$ and hence $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\neg R_{A\neg\triangleright\gamma})^{\mathcal{I}_C}$. By definition of ρ, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R_A, \triangleleft\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R_A, \triangleleft\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : R_A \triangleleft \gamma\rangle)$.

- *Concrete roles*. This case is similar to that of atomic roles.

- *Negated role*. Assume that $\mathcal{I} \models \langle(a,b) : \neg R \geq \alpha\rangle$. Then, $\ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$. Since the result of Gödel negation is either 0 or 1, and given that $\alpha > 0$, it follows that $\ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$ and hence $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 0$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \notin \rho(R, > 0)^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(R, \leq 0)^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(\neg R, \geq \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(\neg R, \geq \alpha) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : \neg R \geq \alpha\rangle)$. The other cases are similar.

- *Inverse role*. Assume that $\mathcal{I} \models \langle(a,b) : R^- \bowtie \gamma\rangle$. Then, $R^{\mathcal{I}}(b^{\mathcal{I}}, a^{\mathcal{I}}) \bowtie \gamma$. By induction hypothesis, $(b^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(R, \bowtie \gamma)^{\mathcal{I}_C}$. Consequently, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(R, \bowtie \gamma)^{\mathcal{I}_C})^- \Leftrightarrow \mathcal{I}_C \models (a,b) \in \rho(R, \bowtie \gamma)^- \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : R^- \bowtie \gamma\rangle)$.

- *Universal role*. Assume that $\mathcal{I} \models \langle(a,b) : U \triangleright \gamma\rangle$. Then, $U^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1 \geq \gamma$. By definition of $\mathcal{I}_C$, it follows that $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \Delta^{\mathcal{I}_C} \times \Delta^{\mathcal{I}_C}$ and consequently $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in U^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in (\rho(U, \triangleright\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(U, \triangleright\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : U \triangleright \gamma\rangle)$. The case $\mathcal{I} \models \langle(a,b) : U \triangleleft \gamma\rangle$ is similar.

- *Modified role*. Assume that $\mathcal{I} \models \langle(a,b) : mLin(R) \triangleright \gamma\rangle$ for a linear modifier $mLin$ such that $f_{mLin}(x;l)$. Then, it follows that $f_{mLin}(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}); l) \triangleright \gamma$. Let $x_l \in [0,1]$ be such that $f_{mLin}(x_l; l) = \gamma$. Then, it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \triangleright x_l$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}}) \in \rho(R, \triangleright x_l)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R, \triangleright x_l) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle(a,b) : mLin(R) \triangleright \gamma\rangle)$.

The case $\mathcal{I} \models \langle (a,b) : mLin(R) \lhd \gamma \rangle$ is similar, but now it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \lhd x_l$ so we end up with $\mathcal{I}_C \models (a,b) : \rho(R, \lhd x_l) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle (a,b) : mLin(R) \lhd \gamma \rangle)$.

- *Cut role*. Assume that $\mathcal{I} \models \langle (a,b) : [R \geq \alpha] \rhd \gamma \rangle$. Then, it follows that $([R \geq \alpha])^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 1$, which is the case if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(R, \geq \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R, \geq \alpha) \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho([R \geq \alpha], \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa((a,b) : [R \geq \alpha] \rhd \gamma)$.

  Now assume that $\mathcal{I} \models \langle (a,b) : [R \geq \alpha] \lhd \gamma \rangle$. Then, it follows that $([R \geq \alpha])^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = 0$, which is the case if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) < \alpha$. By induction hypothesis, $(a^{\mathcal{I}_C}, b^{\mathcal{I}_C}) \in \rho(R, < \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho(R, < \alpha) \Leftrightarrow \mathcal{I}_C \models (a,b) : \rho([R \geq \alpha], \lhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa((a,b) : [R \geq \alpha] \lhd \gamma)$.

3. τ is a concept assertion. Assume that $\mathcal{I} \models \langle a : C \bowtie \gamma \rangle$. We show, by induction on the structure of concepts and roles, that $\mathcal{I}_C \models \kappa(\langle a : C \bowtie \gamma \rangle)$.

   - *Atomic concept*. Assume that $\mathcal{I} \models \langle a : A \rhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_C$, it follows that $a^{\mathcal{I}_C} : A^{\mathcal{I}_C}_{\rhd\gamma}$. Consequently, $a^{\mathcal{I}_C} \in (\rho(A, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(A, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : A \rhd \gamma \rangle)$. Now assume that $\mathcal{I} \models \langle a : A \lhd \gamma \rangle$. Then, $A^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By definition of $\mathcal{I}_C$, it follows that $a^{\mathcal{I}_C} \notin A^{\mathcal{I}_C}_{\neg\rhd\gamma} \Leftrightarrow a^{\mathcal{I}_C} \in \neg A^{\mathcal{I}_C}_{\neg\rhd\gamma} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(A, \lhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(A, \lhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : A \lhd \gamma \rangle)$.
   - *Top concept*. Assume that $\mathcal{I} \models \langle a : \top \rhd \gamma \rangle$. Then, $\top^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By definition of $\mathcal{I}_C$, it follows that $a^{\mathcal{I}_C} \in \Delta^{\mathcal{I}_C} = \top$. Consequently, $a^{\mathcal{I}_C} \in (\rho(\top, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(\top, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : \top \rhd \gamma \rangle)$. The case $\mathcal{I} \models \langle a : \top \lhd \gamma \rangle$ is not possible. If $\mathcal{I} \models \langle a : \top \leq \beta \rangle$ we have that $1 \leq \beta$, which is a contradiction with the restriction $\beta \in [0,1)$. If $\mathcal{I} \models \langle a : \top < \alpha \rangle$ we have that $1 < \alpha$, which is a contradiction with the restriction $\alpha \in (0,1]$.
   - *Bottom concept*. This case is similar to the previous one.
   - *Concept negation*. Assume that $\mathcal{I} \models \langle a : \neg C \geq \alpha \rangle$. Then, $\ominus C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$. Since $\alpha > 0$, it follows that $\ominus C^{\mathcal{I}}(a^{\mathcal{I}}) = 1$ and hence $C^{\mathcal{I}}(a^{\mathcal{I}}) =$

0. By induction hypothesis, $a^{\mathcal{I}_C} \notin \rho(C, > 0)^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in \rho(C, \leq 0)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C, \leq 0) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : \neg C \geq \alpha \rangle)$. The case $> \beta$ is similar.

In the case $\mathcal{I} \models \langle a : \neg C \leq \beta \rangle$ it follows that $\ominus C^{\mathcal{I}}(a^{\mathcal{I}}) \leq \beta$. Since $\beta < 1$, it follows that $\ominus C^{\mathcal{I}}(a^{\mathcal{I}}) = 0$ and hence $C^{\mathcal{I}}(a^{\mathcal{I}}) > 0$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, > 0)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C, > 0) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : \neg C \leq \beta \rangle)$. The case $< \alpha$ is similar.

- *Concept conjunction*. Assume that $\mathcal{I} \models \langle a : C \sqcap D \rhd \gamma \rangle$. Then, $\min\{C^{\mathcal{I}}(a^{\mathcal{I}}), D^{\mathcal{I}}(a^{\mathcal{I}})\} \rhd \gamma$, so it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$ and $D^{\mathcal{I}}(a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \rhd\gamma)^{\mathcal{I}_C}$ and $a^{\mathcal{I}_C} \in \rho(D, \rhd\gamma)^{\mathcal{I}_C}$. Consequently, $a^{\mathcal{I}_C} \in (\rho(C, \rhd\gamma) \sqcap \rho(D, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(C \sqcap D, \rhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(C \sqcap D, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : C \sqcap D \rhd \gamma \rangle)$.

  In the case $\mathcal{I} \models \langle a : C \sqcap D \lhd \gamma \rangle$, it follows that $C^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$ or $D^{\mathcal{I}}(a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \lhd\gamma)^{\mathcal{I}_C}$ or $a^{\mathcal{I}_C} \in \rho(D, \lhd\gamma)^{\mathcal{I}_C}$. In this case, we end up with $\mathcal{I}_C \models \kappa(\langle a : C \sqcap D \lhd \gamma \rangle)$.

- *Concept disjunction*. This case is similar to concept conjunction.

- *Universal quantification*. Assume that $\mathcal{I} \models \langle a : \forall R.C \geq \alpha \rangle$. Then, $\inf_{b \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow C^{\mathcal{I}}(b)\} \geq \alpha$. Since this is true for the infimum, an arbitrary individual $b \in \Delta^{\mathcal{I}}$ must satisfy that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \Rightarrow C^{\mathcal{I}}(b) \geq \alpha$ and hence one of the following conditions holds *(i)* $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leq C^{\mathcal{I}}(b)$ (which makes the Gödel implication equal to $1 \geq \alpha$), or *(ii)* $C^{\mathcal{I}}(b) \geq \alpha$ (which makes the Gödel implication take a value $\geq \alpha$). The former condition is equivalent to $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq \gamma$ implies $C^{\mathcal{I}}(b) \geq \gamma$ for every $\gamma \in N^{\mathcal{K}}$ [1].The latter condition enables us to restrict to those $\gamma \in$ such that $\gamma \leq \alpha$. By induction hypothesis, it follows that $(a^{\mathcal{I}_C}, b) \in (\rho(R, \rhd\gamma))^{\mathcal{I}_C}$ implies $b^{\mathcal{I}_C} \in (\rho(C, \rhd\gamma))^{\mathcal{I}_C}$ or $b^{\mathcal{I}_C} \in (\rho(C, \geq \alpha))^{\mathcal{I}_C}$ for an arbitrary $b \in \Delta^{\mathcal{I}_C}$. Consequently, $\mathcal{I}_C \models a\colon \sqcap_{\gamma \in N^{\mathcal{K}} \setminus \{0\} \mid \gamma \leq \alpha}(\forall \rho(R, \geq \gamma).\rho(C, \geq$

[1] It is easy to see that (i) implies this condition. To see the equivalence, consider γ' such that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) = \gamma'$ and assume that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq \gamma$ implies $C^{\mathcal{I}}(b) \geq \gamma$. Since $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \geq \gamma'$ is true, $C^{\mathcal{I}}(b) \geq \gamma'$ holds. Now, it follows that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) \leq C^{\mathcal{I}}(b)$, because if $R^{\mathcal{I}}(a^{\mathcal{I}}, b) > C^{\mathcal{I}}(b)$, then $R^{\mathcal{I}}(a^{\mathcal{I}}, b) > C^{\mathcal{I}}(b) \geq \gamma'$, which is in contradiction with the assumption that $R^{\mathcal{I}}(a^{\mathcal{I}}, b) = \gamma'$.

$\gamma))\sqcap_{\gamma\in N^{\mathcal{K}} \mid \gamma<\alpha}(\forall\rho(R,>\gamma).\ \rho(C,>\gamma)) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a\colon \forall R.C \geq \alpha\rangle)$. The case for $>\beta$ is quite similar.

Now assume that $\mathcal{I} \models \langle a : \forall R.C \leq \beta\rangle$. Then, $\inf_{b\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(a^{\mathcal{I}},b) \Rightarrow C^{\mathcal{I}}(b)\} \leq \beta$. Due to the witnessed model property, there is an individual $b \in \Delta^{\mathcal{I}}$ satisfying that $R^{\mathcal{I}}(a^{\mathcal{I}},b) \Rightarrow C^{\mathcal{I}}(b)\} \leq \beta$. Since $\beta < 1$, it follows that *(i)* $R^{\mathcal{I}}(a^{\mathcal{I}},b) > C^{\mathcal{I}}(b)$, and *(ii)* $C^{\mathcal{I}}(b) \leq \beta$. In this case we end up with $\mathcal{I}_{\mathcal{C}} \models a\colon \sqcup_{\gamma\in\mathcal{N} \mid \gamma\leq\beta}(\exists\rho(R,>\gamma).\rho(C,\leq\gamma))$ and hence $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a\colon \forall R.C \leq \beta\rangle)$. The case for $<\alpha$ is quite similar.

- *Existential quantification*. Assume that $\mathcal{I} \models \langle a : \exists R.C \rhd \gamma\rangle$. Then, $\sup_{b\in\Delta^{\mathcal{I}}}\min\{R^{\mathcal{I}}(a^{\mathcal{I}},b), C^{\mathcal{I}}(b)\} \rhd \gamma$. P. Hájek showed for Gödel logic under a finite set of degrees of truth, that if there is a model, then there is also a witnessed model [120]. Hence, if this is true for the supremum, then there exists an individual b satisfying $\min\{R^{\mathcal{I}}(a^{\mathcal{I}},b), C^{\mathcal{I}}(b)\} \rhd\gamma$, so $R^{\mathcal{I}}(a^{\mathcal{I}},b) \rhd \gamma$ and $C^{\mathcal{I}}(b) \rhd \gamma$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}},b) \in (\rho(R,\rhd\gamma))^{\mathcal{I}_{\mathcal{C}}}$ and $b \in (\rho(C,\rhd\gamma))^{\mathcal{I}_{\mathcal{C}}}$ for some individual $b \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, which is equivalent to say that $a^{\mathcal{I}_{\mathcal{C}}} \in (\exists\rho(R,\rhd\gamma).\rho(C,\rhd\gamma))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(\exists R.C \rhd \gamma))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(\exists R.C \rhd \gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \exists R.C, \rhd\gamma\rangle)$.

  Now, assume that $\mathcal{I} \models \langle a : \exists R.C \leq \beta\rangle$. Then, $\sup_{b\in\Delta^{\mathcal{I}}}\min\{R^{\mathcal{I}}(a^{\mathcal{I}}, b), C^{\mathcal{I}}(b)\} \leq \beta$. Since this is true for the supremum, an arbitrary individual $b \in \Delta^{\mathcal{I}}$ must satisfy $R^{\mathcal{I}}(a^{\mathcal{I}},b) \leq \beta$ or $C^{\mathcal{I}}(b) \leq \beta$. By induction hypothesis, $(a^{\mathcal{I}_{\mathcal{C}}},b) \in (\rho(R,\leq \beta))^{\mathcal{I}_{\mathcal{C}}}$ or $b \in (\rho(C,\leq\beta))^{\mathcal{I}_{\mathcal{C}}}$ for some individual $b \in \Delta^{\mathcal{I}_{\mathcal{C}}}$, which is equivalent to $a^{\mathcal{I}_{\mathcal{C}}} \in (\forall\rho(R,>\beta).\rho(C,\leq\beta))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow a^{\mathcal{I}_{\mathcal{C}}} \in (\forall\rho(R,\neg\leq\beta).\rho(C,\leq \beta))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow a^{\mathcal{I}_{\mathcal{C}}} \in (\rho(\exists R.C \leq \beta))^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(\exists R.C \leq \beta) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \exists R.C, \leq \beta\rangle)$. The case $<\alpha$ is similar.

- *Fuzzy nominals*. Assume that $\mathcal{I} \models \langle a : \{\alpha_1/o_1,\ldots,\alpha_n/o_n\} \rhd \gamma\rangle$. Let $o_{i1},\ldots,o_{ik}$ be such that $\alpha_{ij} \rhd \gamma$. Then, $\sup\{\alpha_{i_1},\ldots,\alpha_{i_k}\} \rhd \gamma$, with $a^{\mathcal{I}} \in \{o_{i_1},\ldots,o_{i_k}\}^{\mathcal{I}}$. By construction of $\mathcal{I}_{\mathcal{C}}$, it holds that $a^{\mathcal{I}_{\mathcal{C}}} \in \{o_{i_1},\ldots,o_{i_k}\}^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow a^{\mathcal{I}_{\mathcal{C}}} \in \rho(\{\alpha_1/o_1,\ldots,\alpha_n/o_n\}^{\mathcal{I}_{\mathcal{C}}}, \rhd\gamma) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(\{\alpha_1/o_1,\ldots,\alpha_n/o_n\}, \rhd\gamma))) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : \{\alpha_1/o_1,\ldots,\alpha_n/o_n\} \rhd \gamma\rangle)$. The case $\lhd\gamma$ is quite straightforward.

- *At-least qualified number restriction*. Assume that $\mathcal{I} \models \langle a : (\geq m\, S.C) \geq \alpha \rangle$. Then, $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes (\otimes_{j<k}\{b_j \neq b_k\})] \geq \alpha$. Note that $(\otimes_{j<k}\{b_j \neq b_k\})$ can be either 0 or 1. If it is 0, then we have that $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes 0] = 0 \geq \alpha$, which is not possible because by definition $\alpha \in (0,1]$. Hence, $(\otimes_{j<k}\{b_j \neq b_k\}) = 1$ and so $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \bigotimes 1]$. This is equivalent to $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} (\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \geq \alpha$. This implies that there exist m different $b_i \in^{\mathcal{I}_C}$ such that $\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}$ and hence $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \geq \alpha$ and $C^{\mathcal{I}}(b_i) \geq \alpha$, for $1 \leq i \leq m$. By induction hypothesis, $(a^{\mathcal{I}_C}, b_i) \in (\rho(S, \geq \alpha))^{\mathcal{I}_C}$ and $b_i \in (\rho(C, \geq \alpha))^{\mathcal{I}_C}$, for $1 \leq i \leq m$. Consequently, $a^{\mathcal{I}_C} \in (\geq m\, \rho(S, \geq \alpha).\rho(C, \geq \alpha))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in \rho(\geq m\, S.C, \geq \alpha)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(\geq m\, S.C, \geq \alpha) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : (\geq m\, S.C) \geq \alpha \rangle)$. The case $> \beta$ is quite similar.

  Now assume that $\mathcal{I} \models \langle a : (\geq m\, S.C) \leq \beta \rangle$. In this case, it follows that $\sup_{b_1,\ldots,b_m \in \Delta^{\mathcal{I}}} (\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}_C}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \leq \beta$. Consequently, there cannot exist m different individuals b_i with $(\min_{i=1}^{m} \{S^{\mathcal{I}}(a^{\mathcal{I}_C}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) > \beta$, so we end up with $\mathcal{I}_{\mathcal{C}} \models \kappa(\langle a : (\geq m\, S.C) \leq \beta \rangle)$. The case $< \alpha$ is quite similar.

- *At-most qualified number restriction*. Assume that $\mathcal{I} \models \langle a : (\leq n\, S.C) \geq \alpha \rangle$. Then, $\inf_{b_1,\ldots,b_{n+1} \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \geq \alpha$. Note that $(\oplus_{j<k}\{b_j = b_k\})$ can be either 0 or 1, so the result of the Gödel implication is either 0 or 1 and hence $(\leq n\, S.C)$ is actually a crisp concept. Since $\alpha > 0$, it follows that $\inf_{b_1,\ldots,b_{n+1} \in \Delta^{\mathcal{I}}} [(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \geq \alpha = 1$. Then, $\forall b_1, \ldots, b_{n+1} \in \Delta^{\mathcal{I}}, [(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] = 1$. This is true in two cases: *(i)* $(\min_{i=1}^{n+1} \{S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) = 0$, so there exist some b_i such that $S^{\mathcal{I}}(x, b_i) = 0$ or $C^{\mathcal{I}}(b_i) = 0$ hold, or *(ii)* $\oplus_{j<k}\{b_j = b_k\} = 0$ holds. This means that there do not exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) > 0$ and $C^{\mathcal{I}}(b_i) > 0$. By induction hypothesis, there do not exist $n+1$ mutually different individuals

$b_i \in \Delta^{\mathcal{I}_C}$ such that $S^{\mathcal{I}_C}(a^{\mathcal{I}_C}, b_i) > 0$ and $C^{\mathcal{I}_C}(b_i) > 0$. Hence, $a^{\mathcal{I}_C} \in (\leq n\ \rho(S, > 0).\rho(C, > 0))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : (\leq n\ \rho(S, > 0).\rho(C, > 0)) \Leftrightarrow \mathcal{I}_C \models a : \rho(\leq n\ S.C, \geq \alpha) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : (\leq n\ S.C) \geq \alpha \rangle)$. The case $> \beta$ is quite similar.

Let us assume now the case $\mathcal{I} \models \langle a : (\leq n\ S.C) \leq \beta \rangle$. Then, $\inf_{b_1,\ldots,b_{n+1} \in \Delta^{\mathcal{I}}}[(\min_{i=1}^{n+1}\{\, S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) \otimes C^{\mathcal{I}}(b_i)\}) \Rightarrow (\oplus_{j<k}\{b_j = b_k\})] \leq \beta$. Thanks to the witnessed model property, it follows that there exist $n+1$ mutually different individuals such that $S^{\mathcal{I}}(a^{\mathcal{I}}, b_i) > 0$ and $C^{\mathcal{I}}(b_i) > 0$. By induction hypothesis, there exist $n+1$ mutually different individuals $b_i \in \Delta^{\mathcal{I}_C}$ such that $S^{\mathcal{I}_C}(a^{\mathcal{I}_C}, b_i) > 0$ and $C^{\mathcal{I}_C}(b_i) > 0$. Hence, $a^{\mathcal{I}_C} \in (\geq n+1\ \rho(S, > 0).\ \rho(C, > 0))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (\geq n+1\ \rho(S, > 0).\rho(C, > 0)) \Leftrightarrow \mathcal{I}_C \models a : \rho(\leq n\ S.C, \leq \beta) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : (\leq n\ S.C) \leq \beta \rangle)$. The case $< \alpha$ is similar.

- *Local reflexivity*. Assume that $\mathcal{I} \models \langle a : \exists S.Self \rhd \gamma \rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \rhd \gamma$. By induction hypothesis, $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(S, \rhd\gamma)^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models (a, a) : \rho(S, \rhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : \exists S.Self \rhd \gamma \rangle)$. Now assume that $\mathcal{I} \models \langle a : \exists S.Self \lhd \gamma \rangle$. Then, $S^{\mathcal{I}}(a^{\mathcal{I}}, a^{\mathcal{I}}) \lhd \gamma$. By induction hypothesis, $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \rho(S, \lhd\gamma)^{\mathcal{I}_C}$. Hence, it follows that $(a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \notin (\rho(S, \neg \lhd \gamma))^{\mathcal{I}_C} \Leftrightarrow (a^{\mathcal{I}_C}, a^{\mathcal{I}_C}) \in \neg(\rho(S, \neg \lhd \gamma))^{\mathcal{I}_C} \Leftrightarrow a^{\mathcal{I}_C} \in (\rho(\exists S.Self, \lhd\gamma))^{\mathcal{I}_C} \Leftrightarrow \mathcal{I}_C \models a : \rho(\exists S.Self,\ \lhd\gamma) \Leftrightarrow \mathcal{I}_C \models \kappa(\langle a : \exists S.Self, \lhd\gamma \rangle)$.
- *Modified concept*. Firstly, let us consider the case of a triangular modifier $mTri$ such that $f_{mTri}(x; t_1, t_2, t_3)$. Assume that $\langle \mathcal{I} \models a : mTri(C) \geq \alpha \rangle$. Then, it follows that $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geq \alpha$. Let $x_1 \in [0, t_2]$ and $x_2 \in [t_2, 1]$ be those numbers such that $f_{left}(x_1; t_1, t_2, t_3) = \alpha$ and $f_{right}(x_2; t_1, t_2, t_3) = \alpha$. There are several options now, depending on the value of α with respect to t_1 and t_3.
  a) If $(\alpha > t_1)$ and $(\alpha > t_3)$, then $C^{\mathcal{I}}(a^{\mathcal{I}})$ is lower bounded by x_1 (since $f_{left}(x_1; t_1, t_2, t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geq \alpha$) and upper bounded by x_2 (since $f_{right}(x_2; t_1, t_2, t_3) = \alpha$ and $f_{mTri}(C^{\mathcal{I}}(a^{\mathcal{I}}; t_1, t_2, t_3) \geq \alpha)$. That is, $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq x_1$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq x_2$. By induction hypothesis, $a^{\mathcal{I}_C} \in \rho(C, \geq x_1)^{\mathcal{I}_C}$

and $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \leq x_2)^{\mathcal{I}_\mathcal{C}}$. It follows that $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \geq x_1)^{\mathcal{I}_\mathcal{C}} \sqcap \rho(C, \leq x_2)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \geq x_1) \sqcap \rho(C, \leq x_2) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

b) If $(\alpha > t_1)$ and $(\alpha \leq t_3)$, then $C^\mathcal{I}(a^\mathcal{I})$ is lower bounded by x_1 as in the previous case, but x_2 does not introduce an upper bounded now: as noted in Section 6.7, $f_{mTri}(1) = t_3$, and since $\alpha \leq t_3$ and f_{right} is a strictly decreasing function, the possible upper bound for $C^\mathcal{I}(a^\mathcal{I})$ would be greater than 1, but we already know that $C^\mathcal{I}(a^\mathcal{I}) \in [0, 1]$. That is, $C^\mathcal{I}(a^\mathcal{I}) \geq x_1$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \geq x_1)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \geq x_1) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

c) The case $(\alpha \leq t_1)$ and $(\alpha > t_3)$ is similar, but now $C^\mathcal{I}(a^\mathcal{I})$ is upper bounded by x_2 and not lower bounded. Now, $C^\mathcal{I}(a^\mathcal{I}) \leq x_2$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \leq x_2)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \leq x_2) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

d) Finally, in the case $(\alpha \leq t_1)$ and $(\alpha \leq t_3)$ there are no bounds, so we only now that $C^\mathcal{I}(a^\mathcal{I}) \in [0, 1]$ and hence we only know that $\top^\mathcal{I}(a^\mathcal{I})$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \top^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \top \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mTri(C) \geq \alpha \rangle)$.

The other cases $\langle \mathcal{I} \models a : mTri(C) \bowtie \gamma \rangle$ are similar.

Now, let us consider the case of a triangular modifier $mLin$ such that $f_{mLin}(x; l)$. Assume that $\langle \mathcal{I} \models a : mLin(C) \rhd \gamma \rangle$. Then, it follows that $f_{mLin}(C^\mathcal{I}(a^\mathcal{I}; l) \rhd \gamma$. Let $x_l \in [0, 1]$ be such that $f_{mLin}(x_l; l) = \gamma$. Then, it follows that $C^\mathcal{I}(a^\mathcal{I}) \rhd x_l$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \rhd x_l)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \rhd x_l) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mLin(C) \rhd \gamma \rangle)$.

The case $\langle \mathcal{I} \models a : mLin(C) \lhd \gamma \rangle$ is similar, but now it follows that $C^\mathcal{I}(a^\mathcal{I}) \lhd x_l$ so we end up with $\mathcal{I}_\mathcal{C} \models a : \rho(C, \lhd x_l) \Leftrightarrow \mathcal{I}_\mathcal{C} \models \kappa(\langle a : mLin(C) \lhd \gamma \rangle)$.

- *Cut concept*. Assume that $\langle \mathcal{I} \models a : [C \geq \alpha] \rhd \gamma \rangle$. Then, it follows that $([C \geq \alpha])^\mathcal{I}(a^\mathcal{I}) = 1$, which is the case if $C^\mathcal{I}(a^\mathcal{I}) \geq \alpha$. By induction hypothesis, $a^{\mathcal{I}_\mathcal{C}} \in \rho(C, \geq \alpha)^{\mathcal{I}_\mathcal{C}} \Leftrightarrow \mathcal{I}_\mathcal{C} \models a : \rho(C, \geq$

$\alpha) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho([C \geq \alpha], \rhd\gamma)$. The case $\mathcal{I} \models a : [C \leq \alpha] \leq \gamma\rangle$ is similar.

Now assume that $\langle \mathcal{I} \models a : [C \geq \beta] \lhd \gamma\rangle$. Then, it follows that $([C \geq \beta])^{\mathcal{I}}(a^{\mathcal{I}}) = 0$, which is the case if $C^{\mathcal{I}}(a^{\mathcal{I}}) < \beta$. By induction hypothesis, $a^{\mathcal{I}_{\mathcal{C}}} \in \rho(C, < \beta)^{\mathcal{I}_{\mathcal{C}}} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho(C, < \beta) \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models a : \rho([C \geq \beta], \lhd\gamma)$. The case $\mathcal{I} \models a : [C \leq \beta] \geq \gamma\rangle$ is similar.

- *Concrete concept constructs*. Concrete concept constructs are similar to their abstract versions. The only point which deserves a special comment is the existence of some expressions of the form $v_i : d_{\mathbf{D}} \bowtie \gamma$, with $\mathbf{d} = trap_{k_1,k_2}(x; a, b, c, d)$. Assume that $v_i : d_{\mathbf{D}} > \beta$. In order to guarantee that the trapezoidal function takes a value x_{v_i} which is greater or equal than β, we have that $x_{v_i} > a + \beta(b - a)$ and $x_{v_i} < d - \beta(d - c)$, which is equivalent to say that $v_i \in real(a + \beta(b - a), d - \beta(d - c))$. The case $\geq \alpha$ is similar, but using a customized datatype of the form $real[x, y]$ instead of $real(x, y)$. Now, assume that $v_i : d_{\mathbf{D}} \leq \beta$. In this case we have that either $x_{v_i} \leq k_1, a + \beta(b - a)$ or $x_{v_i} \geq d - \beta(d - c), k_2]$, which is equivalent to say that $v_i \in union{-}real[k_1, a + \beta(b - a), d - \beta(d - c), k_2]$. The case $< \alpha$ is similar, using ϵ to get the strict inequality.

4. τ is a fuzzy GCI. Assume that $\mathcal{I} \models \langle C \sqsubseteq D \geq \alpha\rangle$. Then, $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$. Hence, for an arbitrary individual $x \in \Delta^{\mathcal{I}}$ it follows that $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq \alpha$ and hence one of the following conditions holds *(i)* $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$ (which makes the Gödel implication equal to $1 \geq \alpha$), or *(ii)* $D^{\mathcal{I}}(x) \geq \alpha$ (which makes the Gödel implication take a value $\geq \alpha$). Note that the former condition is equivalent to: $C^{\mathcal{I}}(x) \rhd \gamma$ implies $D^{\mathcal{I}}(x) \rhd \gamma$ for every $\gamma \in N^{\mathcal{K}}$. The latter condition enables us to restrict to those $\gamma \in \mathcal{TV}$ such that $\gamma \leq \alpha$. By induction, it follows that $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(C, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}$ implies $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(D, \geq \gamma))^{\mathcal{I}_{\mathcal{C}}}$ or $x^{\mathcal{I}_{\mathcal{C}}} \in (\rho(D, \geq \alpha))^{\mathcal{I}_{\mathcal{C}}}$, for an arbitrary $x \in \Delta^{\mathcal{I}_{\mathcal{C}}}$. Consequently, it follows that $\mathcal{I}_{\mathcal{C}} \models \bigcup_{\gamma \in N^{\mathcal{K}} \setminus \{0\} \mid \gamma \leq \alpha} \{\rho(C, \geq \gamma) \sqsubseteq \rho(D, \geq \gamma)\} \bigcup_{\gamma \in N^{\mathcal{K}} \mid \gamma < \alpha} \{\rho(C, > \gamma) \sqsubseteq \rho(D, > \gamma)\} \Leftrightarrow \mathcal{I}_{\mathcal{C}} \models \kappa(\langle C \sqsubseteq D \geq \alpha\rangle)$. The case for $> \beta$ is quite similar.

5. τ is a fuzzy RIA. Assume that $\mathcal{I} \models \langle R_1 \dots R_m \sqsubseteq R \rhd \gamma \rangle$. The case is similar to the previous one, with the difference that there appears a minimum i.e., $\min\{R_1^{\mathcal{I}}(y_1, y_2), \dots, R_n^{\mathcal{I}}(y_n, y_{n+1})\} \leq \{R^{\mathcal{I}}(y_1, y_{n+1})$. As a consequence, the left side of the crisp RIAs will contain $\rho(R_1, \rhd\gamma) \dots \rho(R_m, \rhd\gamma)$ in the left side, instead of $\rho(C, \rhd\gamma)$.

6. τ is a role disjoint axiom. Assume that $\mathcal{I} \models \mathtt{dis}(S_1, S_2)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x, y) = 0$ or $S_2^{\mathcal{I}}(x, y) = 0$. By induction, $\forall x, y \in \Delta^{\mathcal{I}_c}$, $(x, y) \in (\rho(S_1, \leq 0))^{\mathcal{I}_c}$ or $(x, y) \in (\rho(S_2, \leq 0))^{\mathcal{I}_c} \Leftrightarrow \forall x, y \in \Delta^{\mathcal{I}_c}, (x, y) \notin (\rho(S_1, > 0))^{\mathcal{I}_c}$ or $(x, y) \notin (\rho(S_2, > 0))^{\mathcal{I}_c} \Leftrightarrow (\rho(S_1, > 0))^{\mathcal{I}_c} \cap (\rho(S_2, > 0))^{\mathcal{I}_c} = \emptyset \Leftrightarrow \mathcal{I}_C \models (dis(\rho(S_1, > 0), \rho(S_2, > 0))) \Leftrightarrow \mathcal{I}_C \models \kappa(\mathtt{dis}(S_1, S_2))$. The case $\tau = \mathtt{dis}(T_1, T_2)$ is similar.

7. τ is a reflexive role axiom. Assume that $\mathcal{I} \models \mathtt{ref}(R)$. Then, $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,\ x) = 1$. By induction, $\forall x \in \Delta^{\mathcal{I}_c}, (x, x) \in (\rho(R, \geq 1))^{\mathcal{I}_c} \Leftrightarrow \forall x \in \Delta^{\mathcal{I}_c}, \mathcal{I}_C \models (x, x) : \rho(R, \geq 1) \Leftrightarrow \mathcal{I}_C \models \kappa(\mathtt{ref}(R))$.

8. τ is an asymmetry role axiom. Assume that $\mathcal{I} \models \mathtt{asy}(S)$. Then, $\forall x, y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x, y) > 0$ then $S^{\mathcal{I}}(y, x) = 0$. By induction, $\forall x, y \in \Delta^{\mathcal{I}_c}$, if $(x, y) \in (\rho(S, > 0))^{\mathcal{I}_c}$ then $(y, x) \in (\rho(S, \leq 0))^{\mathcal{I}_c} \Leftrightarrow \forall x, y \in \Delta^{\mathcal{I}_c}$, if $(x, y) \in (\rho(S, > 0))^{\mathcal{I}_c}$ then $(y, x) \notin (\rho(S, > 0))^{\mathcal{I}_c}$. Consequently, $\mathcal{I}_C \models \kappa(\mathtt{asy}(\rho(S, > 0)))$.

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If $crisp(\mathcal{K})$ is satisfiable, it is not possible (due to the axioms in $T(N^{\mathcal{K}})$) to have an individual a such that $a^{\mathcal{I}_c} \in (A_{\rhd\gamma_1})^{\mathcal{I}_c}$ and $a^{\mathcal{I}_c} \notin (A_{\rhd\gamma_2})^{\mathcal{I}_c}$ with $\gamma_2 < \gamma_1$, so for every individual a we can compute the maximum value α such that $a : A_{\geq\alpha}$ holds, or the maximum value β such that $a : A_{>\beta}$ holds, and use these values in the construction of the fuzzy interpretation. The case for roles in $R_a(N^{\mathcal{K}})$ and $R_c(N^{\mathcal{K}})$ is similar. □

Modularity

As it is the case for Zadeh logic, the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included.

Theorem 6 *Let $\mathcal{K}$ be a G $\mathcal{SROIQ}(\mathbf{D})$ fuzzy knowledge base involving a set of fuzzy atomic roles $\mathbf{A}$, a set of a set of atomic roles $\mathbf{R_a}$ and a set of concrete roles $\mathbf{R_c}$, let $\mathcal{N}^{\mathcal{K}}$ be the set of relevant degrees to be considered (including 0 and 1) and let τ be a G $\mathcal{SROIQ}(\mathbf{D})$ axiom such that:*

1. *for every atomic concept A which appears in τ, $A \in \mathbf{A}$,*
2. *for every atomic role R_A which appears in τ, $R_A \in \mathbf{R}$,*
3. *for every concrete role T which appears in τ, $T \in \mathbf{R_c}$,*
4. *if γ appears in τ, then $\gamma \in \mathcal{N}^{\mathcal{K}}$.*

Then, the reduction of the union of the KB and the axiom is equivalent to the union of the reduction of $\mathcal{K}$ and the reduction of τ:

$$crisp(\mathcal{K} \cup \tau) = crisp(\mathcal{K}) \cup \kappa(\tau)$$

Proof. Trivial from the following observations:

- Every axiom is reduced to a combination of new crisp elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- τ does not introduce atomic concepts, atomic abstract roles, concrete roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others. □

If the conditions of Theorem 6 are not satisfied, we proceed similarly as with Zadeh logic:

- If τ introduces a new atomic concept, $T(N^{fK})$ needs to be recomputed.
- If τ introduces a new atomic abstract role, $R_a(N^{fK})$ needs to be recomputed.
- If τ introduces a new concrete role, $R_c(N^{fK})$ needs to be recomputed.
- If τ introduces a new degree of truth, $X^{\mathcal{K}}$ changes. As a consequence, $N^{\mathcal{K}}$ may change. If $N^{\mathcal{K}}$ changes, we need to recompute:
  - $T(N^{fK})$,
  - $R_a(N^{fK})$,

- $R_c(N^{fK})$,
- The reduction of every universal quantification concept in $\mathcal{K}$.
- The reduction of every fuzzy GCI in $\mathcal{K}$.
- The reduction of every fuzzy RIA in $\mathcal{K}$.

Complexity

Definition 58 ***Depth of a universal quantification concept.*** *The depth of a universal quantification concept is inductively defined as follows:*

- $depth(A) = depth(\top) = depth(\bot) = depth(\{\alpha_1/o_1, \dots \alpha_m/o_m\}) = depth(\exists S.Self) = 1$,
- $depth(\exists R.C) = depth(\neg C) = depth(\geq m\, S.C) = depth(\leq n\, S.C) = depth(mod(C)) = depth(C \geq \alpha]) = depth([C \leq \beta]) = depth(C)$,
- $depth(C \sqcap D) = depth(C \sqcup D) = max\{depth(C), depth(D)\}$,
- $depth(\forall R.C) = 1 + depth(C)$,

It is easy to see that:

- Every fuzzy concept expression of depth k generates a crisp concept expression of depth k except universal restrictions.
- The reduction of a universal quantification concept of depth k generates a crisp concept expression of depth $2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) \cdot k$.
- Most of the axioms of the fuzzy KB generate one crisp axiom, but some of them (fuzzy GCIs and fuzzy RIAs) generate several crisp axioms.

$|\mathcal{N}^{\mathcal{K}}|$ is bounded by $|\mathcal{K}| + 2$. In this case, the size of the resulting KB is $\mathcal{O}(|\mathcal{N}^{\mathcal{K}}|^k)$, where k is the maximal depth of the universal quantification concepts appearing in KB:

- $|\kappa(\mathcal{A})| = |\mathcal{A}|$,
- $|T(\mathcal{N}^{\mathcal{K}})| = (2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{A}|$,
- $|\kappa(\mathcal{T})| \leq 2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) \cdot |\mathcal{T}|$,
- $|R_a(\mathcal{N}^{\mathcal{K}})| = (2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{R}|$,
- $|R_c(\mathcal{N}^{\mathcal{K}})| = (2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) - 1) \cdot |\mathbf{T}|$,

- $|\kappa(\mathcal{R})| \leq 2 \cdot (|\mathcal{N}^{\mathcal{K}}| - 1) \cdot |\mathcal{R}|$.

We recall that under Zadeh semantics, the size of the resulting KB is quadratic (or linear if we fix the number of degrees of truth). The increment of spatial complexity is due to the use of Gödel implication in universal quantifications. In this case it is not possible to infer the exact degrees of truth, but we need to guess them, building disjunctions or conjunctions over all possible combinations of the degrees of truth.

However, in most of the cases universal quantifications of the form $(\forall R.C)$ can be approximated by using cut concepts and roles, replacing them by $(\forall [R \geq \alpha_1].[C \geq \alpha_2])$, meaning that every individual which is related through role R with degree (at least) α_1 must belong to C with (at least) degree α_2. Now the reduction is:

$$\rho(\forall [R \geq \alpha_1].[C \geq \alpha_2], \rhd \gamma) = \forall \rho(R, \geq \alpha_1).\rho(C, \geq \alpha_2)$$
$$\rho(\forall [R \geq \alpha_1].[C \geq \alpha_2], \lhd \gamma) = \exists \rho(R, \geq \alpha_1).\rho(C, < \alpha_2)$$

Whenever this approximation is possible, the resulting KB is linear ($\mathcal{O}(|\mathcal{N}^{\mathcal{K}}|)$ (recall that we are assuming a fixed finite set of degrees of truth $|\mathcal{N}^{\mathcal{K}}|$)).

7.7 Optimizations

All the optimization that we have described for Zadeh logic (Chapter 6.9) can also be applied in Gödel logic:

- Reducing the number of new elements.
- Optimizing irreflexive role axioms[2].
- Allowing crisp concepts and roles.
- Reasoning ignoring superfluous elements.

In addition, we can optimize the reductions of some fuzzy GCIs.

[2]For the sake of clarity, we have assumed that irreflexive axioms do not appear under Gödel logic and that they and are represented using fuzzy RIAs, but however the optimized representations is preferable.

Optimizing Fuzzy GCI Reductions

Firstly, $\langle C \sqsubseteq \top \triangleright \gamma\rangle$ and $\langle \bot \sqsubseteq D \triangleright \gamma\rangle$ are tautologies for every fuzzy implication function, so their reductions are unnecessary in the resulting KB.

Furthermore, in some particular (but important) cases, the reduction of a fuzzy GCI can be optimized, by applying the following proposition:

Proposition 11 *If an ontology contains the axioms $C_1 \sqsubseteq C_2$, $C_1 \sqsubseteq C_3$ and $C_2 \sqsubseteq C_3$, then $C_1 \sqsubseteq C_3$ is unnecessary.*

Proof. From $C_1 \sqsubseteq C_2$ it follows that $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for some interpretation $\mathcal{I}$, and from $C_2 \sqsubseteq C_3$ it follows that $C_2^{\mathcal{I}} \subseteq C_3^{\mathcal{I}}$. It follows that $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}} \subseteq C_3^{\mathcal{I}}$ and hence $C_1 \sqsubseteq C_3$ holds. □

There are several cases in which this proposition can be applied:

- $\kappa(\langle \top \sqsubseteq D \triangleright \gamma\rangle) = \{\top \sqsubseteq \rho(D, \triangleright\gamma)\}$. Note that this kind of axiom appears in domain, range, and functional role axioms.

  Example 21 *Let us consider the range axiom $\langle \top \sqsubseteq \forall \mathit{madeFromFruit}.(\mathit{NonSweetFruit} \sqcup \mathit{SweetFruit}) \geq 1\rangle$ with $\mathcal{N}^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$. The original reduction is:*

$$
\begin{aligned}
&\{\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{>0}.(\mathit{NonSweetFruit}_{\geq 1} \sqcup \mathit{SweetFruit}_{\geq 1}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{\geq 0.25}.(\mathit{NonSweetFruit}_{>0.75} \sqcup \mathit{SweetFruit}_{>0.75}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{>0.25}.(\mathit{NonSweetFruit}_{\geq 0.75} \sqcup \mathit{SweetFruit}_{\geq 0.75}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{\geq 0.5}.(\mathit{NonSweetFruit}_{>0.5} \sqcup \mathit{SweetFruit}_{>0.5}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{>0.5}.(\mathit{NonSweetFruit}_{\geq 0.5} \sqcup \mathit{SweetFruit}_{\geq 0.5}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{\geq 0.75}.(\mathit{NonSweetFruit}_{>0.25} \sqcup \mathit{SweetFruit}_{>0.25}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{>0.75}.(\mathit{NonSweetFruit}_{\geq 0.25} \sqcup \mathit{SweetFruit}_{\geq 0.25}),\\
&\quad \top \sqsubseteq \forall \mathit{madeFromFruit}_{\geq 1}.(\mathit{NonSweetFruit}_{>0} \sqcup \mathit{SweetFruit}_{>0}),\\
&\}
\end{aligned}
$$

  but it can be simplified to:

$$\top \sqsubseteq \forall \mathit{madeFromFruit}_{>0}.(\mathit{NonSweetFruit}_{\geq 1} \sqcup \mathit{SweetFruit}_{\geq 1})$$

  since the other unnecessary axioms trivially hold. □

- $\kappa(\langle C \sqsubseteq \bot \rhd \gamma \rangle) = \{\rho(C, > 0) \sqsubseteq \bot\}$. This axiom appears when two concepts are disjoint.

  Example 22 *Assuming $\mathcal{N}^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$, the reduction of the axiom $\langle \mathit{ReadMeat} \sqcap \mathit{NonReadMeat} \sqsubseteq \bot \geq 1 \rangle$ is:*

$$
\begin{array}{l}
\{ \\
\quad \mathit{ReadMeat}_{>0} \sqcap \mathit{NonReadMeat}_{>0} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{\geq 0.25} \sqcap \mathit{NonReadMeat}_{\geq 0.25} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{>0.25} \sqcap \mathit{NonReadMeat}_{>0.25} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{\geq 0.5} \sqcap \mathit{NonReadMeat}_{\geq 0.5} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{>0.5} \sqcap \mathit{NonReadMeat}_{>0.5} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{\geq 0.75} \sqcap \mathit{NonReadMeat}_{\geq 0.75} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{>0.75} \sqcap \mathit{NonReadMeat}_{>0.75} \sqsubseteq \bot, \\
\quad \mathit{ReadMeat}_{\geq 1} \sqcap \mathit{NonReadMeat}_{\geq 1} \sqsubseteq \bot, \\
\}
\end{array}
$$

  but it can be optimized to:

$$\mathit{ReadMeat}_{>0} \sqcap \mathit{NonReadMeat}_{>0} \sqsubseteq \bot$$

  since the other unnecessary axioms trivially hold. □

- The optimization is also useful in concept definitions involving the nominal constructor or a crisp concept, as Examples 23 and 24 show:

  Example 23 *Assuming $N^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$, the reduction of the fuzzy GCI $\langle C \sqsubseteq \{1/o_1, 0.5/o_2\} \rangle$ is:*

$$
\begin{array}{l}
\{ \\
\quad C_{>0} \sqsubseteq \{o_1, o_2\}, \\
\quad C_{\geq 0.5} \sqsubseteq \{o_1, o_2\}, \\
\quad C_{>0.5} \sqsubseteq \{o_1\}, \\
\quad C_{\geq 1} \sqsubseteq \{o_1\} \\
\}
\end{array}
$$

  but it can be optimized to:

$$\{$$
$$C_{>0} \sqsubseteq \{o_1, o_2\},$$
$$C_{>0.5} \sqsubseteq \{o_1\},$$
$$\}$$

since the two unnecessary axioms trivially hold:

$$\{C_{\geq 0.5} \sqsubseteq C_{>0}, C_{>0} \sqsubseteq \{o_1, o_2\}\} \models C_{\geq 0.5} \sqsubseteq \{o_1, o_2\}$$
$$\{C_{\geq 1} \sqsubseteq C_{>0.5}, C_{>0.5} \sqsubseteq \{o_1\}\} \models C_{\geq 1} \sqsubseteq \{o_1\}$$

□

Example 24 *Assuming $N^{\mathcal{K}} = \{0, 0.25, 0.5, 0.75, 1\}$ and that* Port *is a crisp concept, the reduction of the fuzzy GCI $\langle \mathsf{Port} \sqsubseteq \mathsf{RedWine} \geq 1 \rangle$ is as follows:*

$$\kappa(\langle \mathsf{Port} \sqsubseteq \mathsf{RedWine} \geq 1 \rangle) = \{$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{>0},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{\geq 0.25},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{>0.25},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{\geq 0.5},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{>0.5},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{\geq 0.75},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{>0.75},$$
$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{\geq 1}$$
$$\}$$

but it can be optimized to:

$$\mathsf{Port} \sqsubseteq \mathsf{RedWine}_{>0}$$

since the other unnecessary axioms trivially hold. □

One could think that this optimization makes reasoning harder, because the reasoning algorithm needs to deduce the axiom $C_1 \sqsubseteq C_3$, but due to the existence of terminological optimizations such as *lazy unfolding*, these axioms are computed only when necessary [317].

Part IV

Implementations

CHAPTER 8

DeLorean Fuzzy Description Logic Reasoner

This chapter describes our implementation of the reduction algorithms and optimizations already described in this document, as well as a preliminary evaluation. Our prototype implementation is called DeLorean (DEscription LOgic REasoner with vAgueNess) and is the first reasoner that supports fuzzy extensions of the languages OWL and OWL 2. Given a fuzzy ontology in our fuzzy extension of OWL or OWL 2, DeLorean computes its crisp representation in OWL or OWL 2, respectively.

Relying on OWL is important since it is the current standard language for ontology representation. Furthermore, supporting a fuzzy extension of OWL 2 (or OWL 2) is very interesting since *"the broad acceptance of the forthcoming OWL 1.1 ontology language will largely depend on the availability of editors, reasoners and numerous other tools that support the use of OWL 1.1 from a high-level/application perspective"* [132].

In a strict sense, DeLorean is not a reasoner but a *translator* from a fuzzy ontology language to a classical ontology language (the standard language OWL or OWL 2, depending on the expressivity of the original ontology). Then, a classical DL reasoner is employed to reason with the resulting ontology. But due to this ability of combining the reduction procedure with the crisp reasoning, we will refer to DeLorean as a reasoner.

Section 8.1 describes the main features and the architecture of the reasoner. The syntax of the fuzzy language and the functions of the API are analyzed in Sections 8.2 and 8.3 respectively. Finally, a concrete use case is presented in Section 8.4, together with a preliminary evaluation of the usefulness of the optimizations and the performance of the reasoner.

8.1 Main Features and Architecture

In 2007 we developed a first version based on Jena API. This first version was developed in Java programming language, and using the parser generator JavaCC to read the inputs, and DIG 1.1 interface to communicate with crisp DL reasoners.

JavaCC[1] (Java Compiler Compiler) is an open-source parser generator for the Java programming language. Given a formal specification in EBNF (Extended Backus-Naur Form, [338]) notation of a grammar, it produces as output the Java source code of the parser.

Jena [198][2] is probably the most used Semantic Web API. It is an open-source framework which includes an RDF API, a SPARQL query engine and an OWL API. Jena does not allow directly to reason with OWL ontologies, but it can use an external reasoner by means of the DIG interface.

DIG (Description logic Implementation Group) is a common interface to access DL reasoners [23], avoiding the need to know the particularities of the representation languages of all of them. Currently, the latest version is 1.1 and it supports $\mathcal{SHOIQ}$. Version 2.0[3] is still under development and it is expected to support $\mathcal{SROIQ}$ [318].

The use of Java makes DeLorean platform independent. Moreover, DeLorean can take advantage of any crisp reasoner as long as it supports DIG interface. However, DIG interface does not yet support full $\mathcal{SROIQ}$, so the logic supported by this first version of DeLorean was restricted to $Z\ \mathcal{SHOIN}$ (OWL DL). From a historical point of view, this version was the first reasoner that supported a fuzzy extension of the OWL DL language [32].

[1] https://javacc.dev.java.net
[2] http://jena.sourceforge.net
[3] http://dig.cs.manchester.ac.uk/

With the aim of augmenting the expressivity of the logic, we have implemented a newer version which uses OWL API 2 instead of Jena API.

OWL API 2[4] is an open-source API for manipulating OWL 2 ontologies [132], which extends the previous OWL API (or WonderWeb API) [22] that only supported OWL ontologies. Applications can use some integrated DL reasoners such as Pellet [277] and FaCT++ [316], but the API also supports integration with DIG-compliant DL reasoners. Currently, the API does not support the use of the universal role.

In the current version, DeLorean supports the fuzzy DLs $Z\ \mathcal{SROIQ}(\mathbf{D})$ and $G\ \mathcal{SROIQ}(\mathbf{D})$, which correspond to fuzzy versions of the DL $\mathcal{SROIQ}(\mathbf{D})$ (equivalent to OWL 2) under a semantics given by Zadeh and Gödel families of fuzzy operators, respectively. The only limitation is that the universal role cannot be used, since OWL API 2 does not currently allow it. DeLorean is the first reasoner that supports a fuzzy extension of OWL 2.

Since DIG interface does not currently allow the full expressivity of OWL 2, our solution was to integrate directly DeLorean with a concrete crisp ontology reasoner: Pellet, which can be directly used from the current version of the OWL API 2. This way, the user is free to choose to use either a generic crisp reasoner (restricting the expressivity to $\mathcal{SHOIQ}$) or Pellet (with no expressivity limitations).

Figure 8.1 illustrates the architecture of the system:

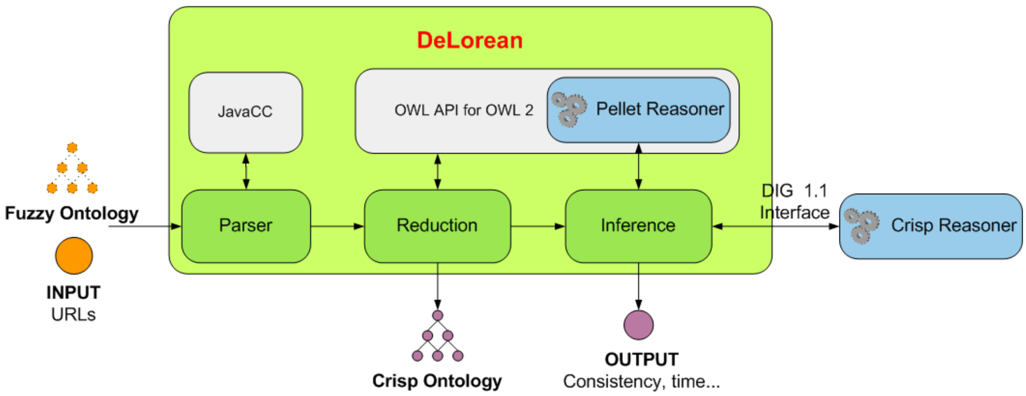

Figure 8.1: Architecture of DeLorean reasoner.

[4] http://owlapi.sourceforge.net

- The *Parser* reads a fuzzy ontology contained in an input physical URI and translates it into an internal representation of a fuzzy KB. The point here is that we can use any language to encode the fuzzy ontology, as long as the *Parser* can understand the representation and the reduction is properly implemented. Moreover, we could have several parsers, each of them being responsible of the translation of a different fuzzy ontology language.
- The *Reduction* module implements the reduction procedures described in Chapters 6 and 7, building an OWL API 2 model with an equivalent crisp ontology, which can be exported to an OWL 2 file. The implementation also takes into account all the optimizations already discussed along this document. Given an input fuzzy KB and an output physical URI, the pseudo-code of the reduction is:

```
// Get degrees in the KB and their complementaries
TreeSet x = kb.getDegrees();

// Create axioms for the new alpha-cuts and alpha-roles
createNewAxioms(x);

// Reduce every axiom in the ABox
reduceABox();

// Reduce every axiom in the TBox
reduceTBox(x);

// Reduce every axiom in the RBox
reduceRBox(x);

// Create output OWL file
write(ontology, physicalURI);

// Perform a consistency test
boolean consistent = isOntologyConsistent(ontology);
```

- The *Inference* module tests the consistency of the ontology, using either Pellet or any crisp reasoner through the DIG interface. Interestingly, crisp reasoning does not take into account superfluous elements as discussed in Section 6.9.

- A simple *User interface* manages inputs and outputs (see Figure 8.2 for a screen shot). The inputs of the system are the path of the fuzzy ontology, the path of the output crisp ontology and the reasoner. The reasoner can be Pellet, or the URL of any DIG-complaint DL reasoner. The outputs of the system are the result of the consistency test, the reduction time and the reasoning time. Of course, some error messages are shown if necessary.

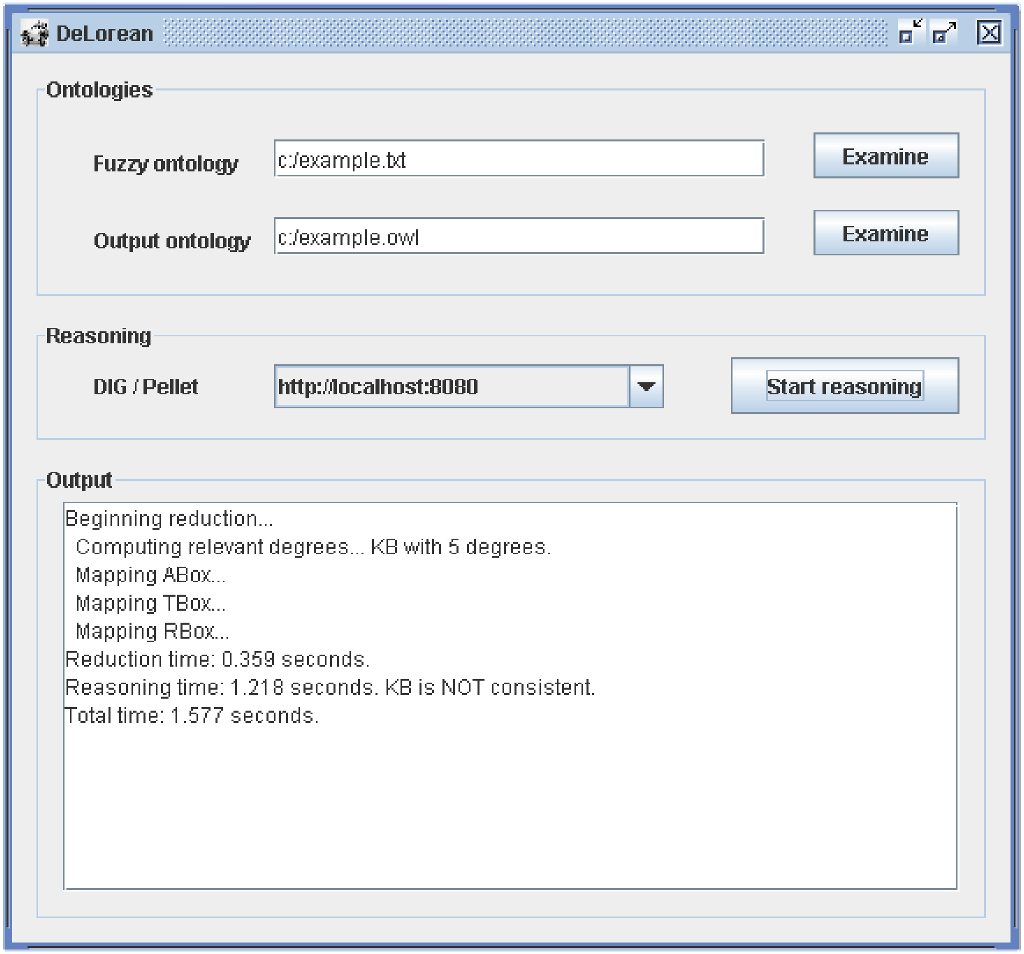

Figure 8.2: User interface of DeLorean reasoner.

8.2 Syntax of the Fuzzy Language

In this section we shall describe the syntax of the fuzzy language supported by DeLorean. We will begin by introducing the notation that we will use:

- C denotes a concept.
- a denotes an individual.
- R denotes a role.
- S denotes a simple role.
- M denotes a strictly positive natural number.
- N denotes a natural number.
- $\bowtie$ denotes one of the following: $>=$ (greater or equal), $>$ (greater), $<=$ (less or equal) or $<$ (less).
- $\triangleright$ denotes one of the following: $>=$ (greater or equal) or $>$ (greater).
- γ denotes a rational number in [0,1].

Input files are text files with a fuzzy KB. A fuzzy KB consists of a set of axioms (one per line) with the syntax shown in Table 8.1. Fuzzy concepts can be built by using complex expressions with the syntax shown in Table 8.1.

It can be seen that the syntax of axioms and fuzzy concepts draws inspiration from the Knowledge Representation System Specification [234].

There are some restrictions in the degree that can be used. If $\bowtie$ is $>=$ or $<$, then γ has to be in $(0,1]$, while if $\bowtie$ is $>$ or $<=$, then γ has to be in $[0,1)$. Of course, if $\bowtie \gamma$ are omitted, ≥ 1 is assumed.

Furthermore, in fuzzy nominal concepts, γ_i should be in $(0,1]$ and, in case it is not specified, 1 is assumed.

In order to make the representation of fuzzy KBs easier, DeLorean also allows the possibility of importing OWL 2 ontologies. These (crisp) ontologies are saved as a text file which the user can edit and extend, for example adding membership degrees to the fuzzy axioms or specifying a particular fuzzy operator (Zadeh or Gödel family) for some complex concept.

Table 8.1: Axioms and fuzzy concepts in DELOREAN

Fuzzy axioms	
(instance a C $[\bowtie \gamma]$)	Concept assertion
(related a_1 a_2 R $[\bowtie \gamma]$)	Role assertion
(not-related a_1 a_2 R $[\bowtie \gamma]$)	Negated role assertion
(different-as a_1 a_2)	Inequality assertion
(same-as a_1 a_2)	Equality assertion
(cris-concept C	Crisp concept axiom
(implies-concept C_1 C_2 $[\triangleright \gamma]$)	GCI
(equivalent-concepts C_1 C_2)	Concept equivalence
(disjoint-concepts $C_1 \dots C_n$)	Disjoint concept axiom
(domain R C)	Domain axiom
(range R C)	Range axiom
(cris-role R	Crisp role axiom
(implies-role $R_1 \dots R_m$ R $[\triangleright \gamma]$)	RIA
(equivalent-roles R_1 R_2)	Equivalence role axiom
(inverse-role R_1 R_2)	Inverse role axiom
(disjoint-roles $S_1 \dots S_n$)	Disjoint role axiom
(transitive R)	Transitive role axiom
(symmetric R)	Symmetric role axiom
(asymmetric S)	Asymmetric role axiom
(reflexive R)	Reflexive role axiom
(irreflexive S)	Irreflexive role axiom
Fuzzy concepts	
A	Atomic concept
top	Top concept
bottom	Bottom concept
(and $C_1 \dots C_n$)	Concept conjunction
(or $C_1 \dots C_n$)	Concept disjunction
(not C)	Concept negation
(some R C)	Existential quantification
(all R C)	Universal quantification
(one-of I_1 $[\gamma_1]$ I_2 $[\gamma_2]$ $\dots I_n$ $[\gamma_n]$)	Fuzzy nominal
(at-least M S)	At-least unqualified number restriction
(at-most N S)	At-most unqualified number restriction
(exactly N S)	Exact unqualified number restriction
(at-least M S C)	At-least qualified number restriction
(at-most N S C)	At-most qualified number restriction
(exactly M S C)	Exact qualified number restriction
(self S)	Local reflexivity

Example 25 *This is a fuzzy ontology about wines (see Section 8.4):*

```
(crisp-concept AlsatianWine)
(crisp-concept AmericanWine)
(...)
(crisp-concept Wine)
(crisp-concept WineDescriptor)
(crisp-concept WineGrape)
(crisp-concept Winery)
(crisp-role madeFromFruit)
(crisp-role madeFromGrape)
(crisp-role madeIntoWine)
(crisp-role producesWine)
(instance Chicken LightMeatFowl >= 0.5)
(instance Steak NonSpicyRedMeat >= 0.8)
(instance RoastBeef NonSpicyRedMeat >= 0.85)
(...)
(related AnjouRegion LoireRegion locatedIn >= 0.95)
(related LongridgeMerlot Moderate hasFlavor >= 0.4)
(...)
(different-as Sweet Dry)
(different-as OffDry Sweet)
(different-as OffDry Dry)
(g-implies-concept RedBurgundy (at-most 1 madeFromGrape) >= 0.5)
(g-implies-concept RedBurgundy (some madeFromGrape (one-of PinotNoirGrape)) >= 0.5)
(g-implies-concept SweetRiesling (all hasFlavor (one-of Moderate Strong)) >= 0.5)
(g-implies-concept SweetRiesling (some hasBody (one-of Full)) >= 0.5)
(g-implies-concept SweetRiesling DessertWine >= 0.5)
(...)
(g-implies-concept (and NonRedMeat RedMeat) *bottom* >= 1.0)
(g-implies-concept (some adjacentRegion *top*) Region >= 1.0)
(g-implies-concept *top* (all adjacentRegion Region) >= 1.0)
(g-implies-concept (some hasWineDescriptor *top*) Wine >= 1.0)
(g-implies-concept *top* (all hasWineDescriptor (or WineTaste WineColor)) >= 1.0)
(g-implies-concept *top* (at-most 1 hasSugar) >= 1.0)
(g-implies-concept *top* (all hasSugar (one-of Sweet OffDry Dry)) >= 1.0)
(...)
(equivalent-concepts RedBurgundy (and Burgundy RedWine))
(equivalent-concepts WineDescriptor (or WineTaste WineColor))
(...)
(transitive locatedIn)
(inverse madeFromGrape madeIntoWine)
(inverse hasMaker producesWine)
(g-implies-role hasColor hasWineDescriptor >= 1.0)
(g-implies-role hasFlavor hasWineDescriptor >= 1.0)
(g-implies-role hasBody hasWineDescriptor >= 1.0)
(g-implies-role hasSugar hasWineDescriptor >= 1.0)
(g-implies-role madeFromGrape madeFromFruit >= 1.0)
```

8.3 DeLorean API

As we have seen, DELOREAN can be used as a stand-alone application, but it can also be used as an API from other Java applications. In this section we explain how to use the latter feature.

Firstly, we present a sample template code which can be adapted according to the user needs:

```
// Import packages
import edu.es.ugr.arai.delorean.*;
import edu.es.ugr.arai.delorean.parser.*;

public class UsingDeLorean
{

    public static void main(String args[])
    {
        // Create KB
        KnowledgeBase kb = new KnowledgeBase();

        // Add axioms to the KB
        // ...

        // Prepare reduction
        Reduction r = new OptimizedReduction(kb, reasonerURL,
                                nameSpace, jTextArea, outputPath);

        // Start reduction
        r.run();
    }
}
```

Note that KnowledgeBase class represents a fuzzy ontology and that the constructor of the OptimizedReduction class receives five arguments:

1. kb: a fuzzy knowledge base, instance of the KnowledgeBase class.
2. reasonerURL: URL of the DIG reasoner. For example, a string of the form "http://localhost:8080". In order to use the integrated Pellet reasoner, the value should be "Pellet".

3. nameSpace: name space of the ontology. For example, a possible namespace is the string "http://arai.ugr.es/testont.owl".
4. printableArea: an instance of the JTextArea class which is used to print output messages. If it takes the value "null", then no messages will be shown.
5. outputFilePath: path of the output file. For example, in Windows systems, a string of the form "c:/example.owl".

The remaining part of this section explains how to populate the fuzzy KB with axioms.

Adding axioms to the fuzzy KB. The easiest way to do this is by loading an input file with the syntax specified in Section 8.2. Assume that inputFile is a string with the path of the input file. For example, in Windows systems, something of the form "c:/example.txt". The source code for this is the following:

```
// Load input file
Parser parser = new Parser(new FileInputStream(inputFile));

// Create fuzzy KB from file
KnowledgeBase kb = parser.Start();
```

An alternative is to use the API to add each of the axioms. Table 8.2 describes the representation that the KnowledgeBase class uses for the different elements of the fuzzy KB. Having into account this representation, the user can use the following methods to add new axioms:

- addAsymmetricRole(Role role).
- addConceptAssertion(Individual a, Concept c, int inequality, double degree).
- addConceptEquivalence(Concept c1, Concept c2).
- addCrispConcept(String conceptName).
- addCrispRole(String roleName).
- addDisjointConcepts(ArrayList disjointConcepts).

- addDisjointRoles(ArrayList disjointRoles).
- addEquality(String ind1, String ind2).
- addGoedelGCI(Concept c1, Concept c2, int inequality, double degree).
- addGoedelNegatedRoleAssertion(Individual a, Role role, Individual b, int inequality, double degree).
- addGoedelRIA(ArrayList roleC, Role roleP, int inequality, double degree).
- addIndividual(String indName, Individual ind).
- addInequality(String ind1, String ind2).
- addInverseRole(String roleName, String inverseRoleName).
- addIrreflexiveRole(Role role).
- addKDGCI(Concept c1, Concept c2, int inequality, double degree).
- addKDRIA(ArrayList roleC, Role roleP, int inequality, double degree).
- addNegatedRoleAssertion(Individual a, Role role, Individual b, int inequality, double degree).
- addReflexiveRole(Role role).
- addRoleAssertion(Individual a, Role role, Individual b, int inequality, double degree).
- addRoleDomain(Role role, Concept conc).
- addRoleEquivalence(Role role1, Role role2).
- addRoleRange(Role role, Concept conc).
- addSymmetricRole(Role role).
- addTransitiveRole(Role role).
- addZadehGCI(Concept c1, Concept c2).
- addZadehRIA(ArrayList roleC, Role roleP).

The names of the methods are self-descriptive, and there is only a remark. If the semantics of the axioms depends on the choice of the fuzzy operators, there are different methods to create them (one for each of them):

Table 8.2: Variables of the KnowledgeBase class

Variable	Meaning
ArrayList<ConceptAssertion> assertions	Concept assertions
ArrayList<Role> asymmetricRoles	Asymmetric role axioms
ArrayList<ConceptEquivalence> conceptEquivalences	Concept equivalence axioms
Hashtable concepts	Alphabet of concepts
HashSet<java.lang.String> crispConcepts	Crisp concepts
HashSet<java.lang.String> crispRoles	Crisp roles
Hashtable disjointRoles	Disjoint role axioms
ArrayList<ConceptInclusion> gcis	GCIs
Hashtable individuals	Alphabet of individuals
Hashtable inverseRoles	Inverse roles
ArrayList<Role> irreflexiveRoles	Irreflexive role axioms
ArrayList<Role> reflexiveRoles	Reflexive role axioms
ArrayList<RoleInclusion> rias	RIAs
ArrayList<RoleEquivalence> roleEquivalences	Role equivalence axioms
Hashtable roles	Alphabet of roles
ArrayList<Role> symmetricRoles	Symmetric role axioms
ArrayList<Role> transitiveRoles	Transitive role axioms

- In *fuzzy GCIs*, addKDGCI adds a fuzzy GCI with a semantics given by Kleene-Dienes implication, addGoedelGCI uses Gödel implication and addZadehGCI uses Zadeh's inclusion of fuzzy sets.
- In *fuzzy RIAs*, addKDRIA adds a fuzzy RIA with a semantics given by Kleene-Dienes implication, addGoedelRIA uses Gödel implication and addZadehRIA uses Zadeh's inclusion of fuzzy sets.
- In *negated role assertions*, addGoedelNegatedRoleAssertion uses Gödel negation in the semantics, and addNegatedRoleAssertion uses standard negation.

Now, it only remains to explain how to build inequalities, degrees, individuals, roles and concepts.

Adding inequalities. Inequalities can be built by using some public constants of the class Inequality, namely:

- Inequality.GREATEREQUAL for $\geq$.
- Inequality.GREATER for $>$.
- Inequality.LESS for $<$.
- Inequality.LESSEQUAL for $\leq$.

Adding degrees of truth. Degrees are double numbers, with the special feature that they are required to be in $[0, 1]$.

Adding individuals. Individuals are created using the method getIndividual of the class KnowledgeBase, which returns an individual with a specified name creating it in case there does not already exist an individual with such a name:

```
Individual ind = kb.getIndividual(individualName);
```

Adding roles. Roles are directly built using the constructor of the class Role:

```
Role aRole = new Role (String roleName);
```

Adding concepts. Concepts can be obtained by using the different constructors of the Concepts class:

- Concept(String name): Constructor for atomic concepts.
- Concept(String name, int type): Constructor for top and bottom concepts.
- Concept(int type, Concept c1, Concept c2, String name): Constructor for union and intersection concepts.
- Concept(int type, Role role, Concept c): Constructor for existential and universal quantifications.
- Concept(int type, Role role, TrapezoidalNumber t): Constructor for concrete existential and universal quantifications.
- Concept(int type, int card, Role role): Constructor for at-most and at-least unqualified number restrictions.

- Concept(int type, String name, ArrayList nom): Constructor for fuzzy nominal concepts.
- Concept(int type, int card, Role role, Concept c): Constructor for at-most and at-least qualified number restrictions.
- Concept(int type, int card, Role role, TrapezoidalNumber t): Constructor for concrete at-most and at-least qualified number restrictions.
- Concept(int type, Role role): Constructor for local reflexivity concepts.

The type of a concept can be one of the public constants defined in the Concept class, namely:

- Concept.ATOMIC
- Concept.TOP
- Concept.BOTTOM
- Concept.COMPLEMENT
- Concept.GOEDEL_COMPLEMENT
- Concept.AND
- Concept.OR
- Concept.SOME
- Concept.ALL
- Concept.GOEDEL_ALL
- Concept.NOMINAL
- Concept.ATLEAST
- Concept.ATMOST
- Concept.GOEDEL_ATMOST
- Concept.QATLEAST
- Concept.QATMOST
- Concept.GOEDEL_QATMOST
- Concept.SELF

- Concept.CONCRETE_SOME
- Concept.CONCRETE_ALL
- Concept.CONCRETE_GOEDEL_ALL
- Concept.CONCRETE_ATLEAST
- Concept.CONCRETE_ATMOST
- Concept.CONCRETE_GOEDEL_ATMOST
- Concept.CONCRETE_QATLEAST
- Concept.CONCRETE_QATMOST
- Concept.CONCRETE_GOEDEL_QATMOST

In fuzzy nominals, nom is an ArrayList of objects of the class Nominal, which are created using the constructor:

```
Nominal nom = new Nominal (Individual i, double n);
```

Finally, trapezoidal membership functions are built by using the constructor of the TrapezoidalNumber class:

```
TrapezoidalNumber trap = new TrapezoidalNumber(double a,
                         double b, double c, double d);
```

8.4 Use case: A fuzzy Wine ontology

This section considers a concrete use case, a fuzzy extension of the well-known Wine ontology[5], a highly expressive ontology (in $\mathcal{SHOIN}(\mathbf{D})$). Some metrics of the ontology are shown in the first column of Table 8.3. In the context of an empirical evaluation of the reductions of fuzzy DLs to crisp DLs, P. Cimiano et al. wrote that *"the Wine ontology showed to be completely intractable both with the optimized and unoptimized reduction even using only 3 degrees"* [67]. They only considered there what we have called here "optimization of the number of new elements and axioms". We will show that the rest of the optimizations, specially the (natural) assumption that there

[5] http://www.w3.org/TR/2003/CR-owl-guide-20030818/wine.rdf

are some crisp elements, reduce significantly the number of axioms, even if tractability of the reasoning is to be verified.

A fuzzy extension of the ontology. We have defined a fuzzy version of the Wine ontology by adding a degree to the axioms. Given a variable set of degrees $\mathcal{N}^{\mathcal{K}}$, the degrees of the truth for fuzzy assertions is randomly chosen in $\mathcal{N}^{\mathcal{K}}$. In the case of fuzzy GCIs and RIAs, the degree is always 1 in special GCIs (that is, in concept equivalences and disjointness, and in domain, range and functional role axioms) or if there is a crisp element in the left side; otherwise, the degree is 0.5.

Moreover, in most of the times fuzzy assertions are of the form $\langle \tau \triangleright \beta \rangle$ with $\beta \neq 1$. Clearly, this favors the use of elements of the forms $C_{\triangleright \beta}$ and $R_{\triangleright \gamma}$, reducing the number of superfluous concepts. As a consequence, we are in the worst case from the point of view of the size of the resulting crisp ontology. Nonetheless, in practice we will be often able to say that an individual fully belongs to a fuzzy concept, or that two individuals are fully related by means of a fuzzy role.

An excerpt of the fuzzy Wine ontology has been included in Example 25.

Crisp concepts and roles. A careful analysis of the fuzzy KB brings about that most of the concepts and the roles should indeed be interpreted as crisp.

For example, most of the subclasses of the class Wine refer to the geographical origin of the wines. For instance, Alsatian wine is a wine which has been produced in the French region of Alsace:

$$\mathsf{AlsatianWine} \equiv \mathsf{Wine} \sqcap \exists \mathsf{locatedAt}.\{\mathsf{alsaceRegion}\}$$

Although there are geographical areas which do not have a well defined meaning, all of the geographical places used in the Wine ontology have a clear boundary and thus can be interpreted as crisp. Some examples of imprecise geographical areas are Scandinavia or Lapland (as shown in [125], a crisp relation partOf cannot represent the partial overlap between Lapland and the countries Finland, Sweden, Norway, and Russia).

Another important number of subclasses of Wine refer to the type of grape used, which is also a crisp concept. For instance, Riesling is a wine which has been produced from Riesling grapes:

$$\text{Riesling} \equiv \text{Wine} \sqcap \exists \text{madeFromGrape}.\{\text{RieslingGrape}\} \sqcap \geq 1 \text{ madeFromGrape}.\top$$

The result of our study has identified 50 fuzzy concepts in the Wine ontology. The source of the vagueness is summarized in several categories[6]:

- Color of the wine: WineColor, RedWine, RoseWine, WhiteWine, RedBordeaux, RedBurgundy, RedTableWine, WhiteBordeaux, WhiteBurgundy, WhiteLoire, WhiteTableWine.
- Sweetness of the wine: WineSugar, SweetWine, SweetRiesling, WhiteNonSweetWine, DryWine, DryRedWine, DryRiesling, DryWhiteWine.
- Body of the wine: WineBody, FullBodiedWine.
- Flavor of the wine: WineFlavor, WineTaste.
- Age of the harvest: LateHarvest, EarlyHarvest.
- Spiciness of the food: NonSpicyRedMeat, NonSpicyRedMeatCourse, SpicyRedMeat, PastaWithSpicyRedSauce, PastaWithSpicyRedSauceCourse, PastaWithNonSpicyRedSauce, PastaWithNonSpicyRedSauceCourse, SpicyRedMeatCourse.
- Sweetness of the food: SweetFruit, SweetFruitCourse, SweetDessert, SweetDessertCourse, NonSweetFruit, NonSweetFruitCourse.
- Type of the meat: RedMeat, NonRedMeat, RedMeatCourse, NonRedMeatCourse. These concept are fuzzy because, according to the age of the animal, pork and lamb are classified as red (old animals) or white (young animals) meat.
- Heaviness of the cream: PastaWithHeavyCreamSauce, PastaWithLightCreamSauce. In this case the terms “heavy” and “light” depend on the fat percentage, and thus can be a matter of degree.

[6]Clearly, these categories are not disjoint and some concepts may belong to more than one, meaning that they are fuzzy for several reasons. For example, DryRedWine is a fuzzy concept because both “dry” and “red” are vague terms.

- Desserts: Dessert, CheeseNutsDessert, DessertCourse, CheeseNutsDessertCourse, DessertWine. We make these concepts fuzzy because the question whether something is a dessert or not does not have a clear answer.

As already discussed, the color, the sweetness, the body and the flavor of a wine are fuzzy. As a consequence, we can identify 5 fuzzy roles: hasColor, hasSugar, hasBody, hasFlavor, and hasWineDescriptor, where the role hasWineDescriptor is a super-role of the other four roles.

Measuring the importance of the optimizations. We have focused our experimentation in a variant of $\mathcal{Z}\ \mathcal{SHOIN}$ (so we have omitted the concrete role yearValue) using Gödel implication in the semantics of fuzzy GCIs and RIAs. This combination is reasonable in order to overcome the already mentioned counter-intuitive effects of Kleene-Dienes implication.

We have performed several reductions of the fuzzy ontology. Figure 8.3 shows an excerpt of the crisp representation of the fuzzy Wine ontology with 5 degrees of truth. Note for example that the reduction of the fuzzy concept WineColor introduces 8 new crisp concepts.

Table 8.3 show the metrics of the crisp ontologies obtained after applying different optimizations. The meaning of the columns of the table is the following:

1. Column “Original” shows some metrics of the original ontology.
2. “None” considers the reduction obtained without applying any optimization.
3. “(NEW)” considers the reduction obtained after optimizing the number of new elements and axioms.
4. “(GCI)” considers the reduction obtained after optimizing GCI reductions.
5. “(C/S)” considers the reduction obtained after allowing crisp concepts and roles and ignoring superfluous elements.
6. Finally, “All” applies all the previous optimizations.

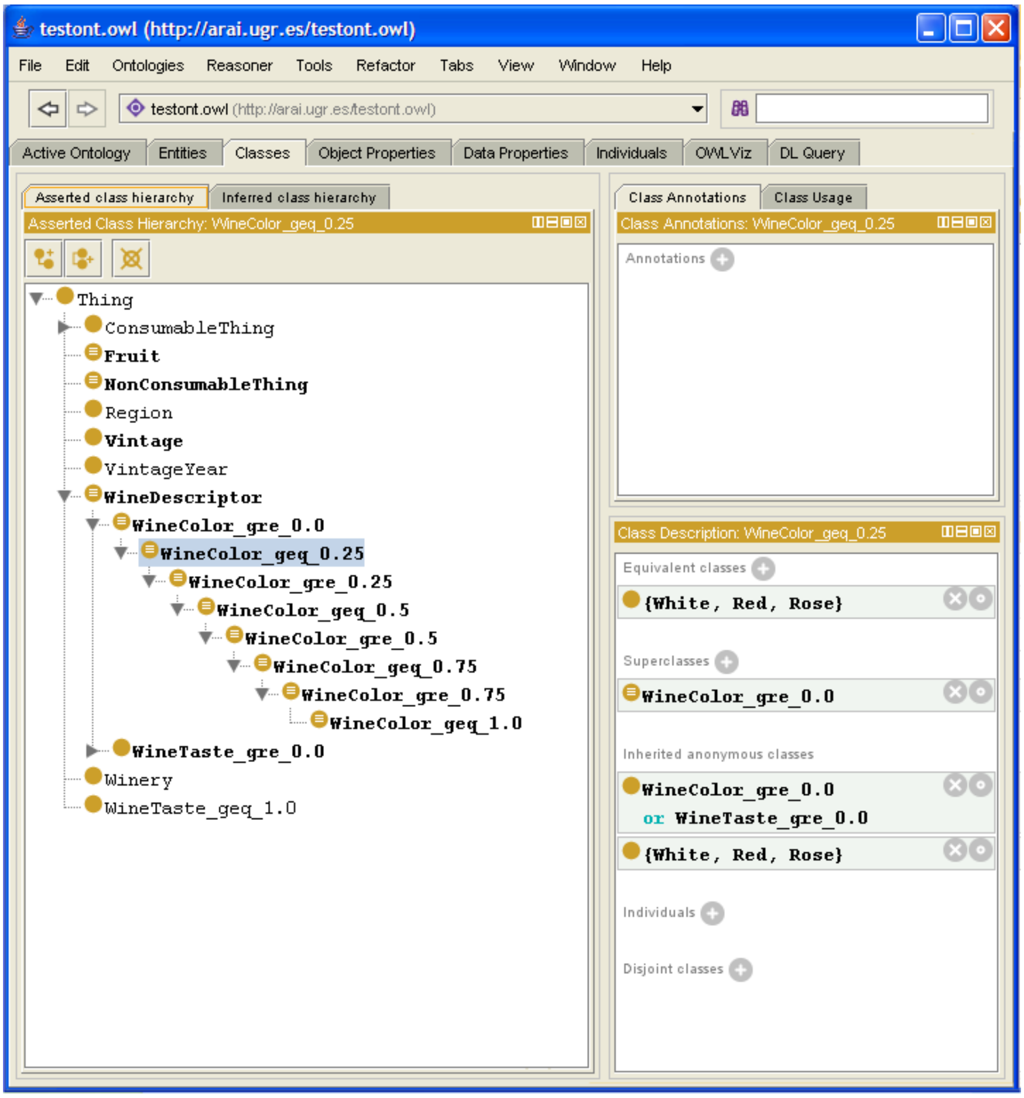

Figure 8.3: An excerpt of the crisp representation of the fuzzy Wine ontology.

Table 8.3: Metrics of the Wine ontology and its fuzzy versions with 5 degrees

	Original	**None**	**(NEW)**	**(GCI)**	**(C/S)**	**All**
Individuals	206	206	206	206	206	206
Named concepts	136	2176	486	2176	800	191
Abstract roles	16	128	128	128	51	20
Concrete roles	1	0	0	0	0	0
Concept assertions	194	194	194	194	194	194
Role assertions	246	246	246	246	246	246
Inequality assertions	3	3	3	3	3	3
Equality assertions	0	0	0	0	0	0
New GCIs	0	4352	952	4352	1686	324
Subclass axioms	275	1288	1288	931	390	390
Concept equivalences	87	696	696	696	318	318
Disjoint concepts	19	152	152	19	152	19
Domain role axioms	13	104	104	97	104	97
Range role axioms	10	80	80	10	80	10
Functional role axioms	6	48	48	6	48	6
New RIAs	0	136	119	136	34	34
Sub-role axioms	5	40	40	40	33	33
Role equivalences	0	0	0	0	0	0
Inverse role axioms	2	16	16	16	2	2
Transitive role axioms	1	8	8	8	1	1

We have put together the optimizations of crisp and superfluous elements because in this ontology handling superfluous concepts is not always useful, due to the existence of a lot of concept definitions, as we will see in the next example.

Example 26 *Consider the fuzzy concept* $\mathsf{NonRedMeat}$*.*

Firstly, this concept appears as part of a fuzzy assertion stating that pork is a non read meat:

$$\sigma(\langle \mathsf{Pork} : \mathsf{NonRedMeat} \triangleright \alpha_1 \rangle) = \mathsf{Pork} : \mathsf{NonRedMeat}_{\triangleright \alpha_1}$$

Secondly, non read meat is declared to be disjoint from read meat:

$$\kappa(\langle \mathsf{RedMeat} \sqcap \mathsf{NonRedMeat} \sqsubseteq \bot \geq 1 \rangle) = \mathsf{RedMeat}_{>0} \sqcap \mathsf{NonRedMeat}_{>0} \sqsubseteq \bot$$

Thirdly, non read meat is a kind of meat:

$$\kappa(\langle \mathsf{NonRedMeat} \sqsubseteq \mathsf{Meat} \geq \alpha_2 \rangle) = \mathsf{NonRedMeat}_{>0} \sqsubseteq \mathsf{Meat}$$

If these were the only occurrences of NonRedMeat*, then the reduction would create only two non-superfluous crisp concepts, namely* $\mathsf{NonRedMeat}_{>0}$ *and* $\mathsf{NonRedMeat}_{\triangleright\alpha_1}$*, and in order to preserve the semantics of them we would need to add just one axiom during the reduction:*

$$\mathsf{NonRedMeat}_{\triangleright\alpha_1} \sqsubseteq \mathsf{NonRedMeat}_{>0}$$

However, this is not true because NonRedMeat *appears in the definition of the fuzzy concept* NonRedMeatCourse*. In fact,* $\kappa(\mathsf{NonRedMeatCourse} \equiv \mathsf{MealCourse} \sqcap \forall \mathsf{hasFood.NonRedMeat})$ *introduces non-superfluous crisp concepts for the rest of the degrees in* $N^{\mathcal{K}}$*. Consequently, for each* $1 \leq i \leq |N^{\mathcal{K}}| - 1, 2 \leq j \leq |N^{\mathcal{K}}| - 1$*, the reduction adds to* $T(N^{\mathcal{K}})$ *the following axioms:*

$$\mathsf{NonRedMeat}_{\geq\gamma_{i+1}} \sqsubseteq \mathsf{NonRedMeat}_{>\gamma_i}$$

$$\mathsf{NonRedMeat}_{>\gamma_j} \sqsubseteq \mathsf{NonRedMeat}_{\geq\gamma_j}$$

□

Note that the size of the ABox is always the same, because every axiom in the fuzzy ABox generates exactly one axiom in the reduced ontology.

The number of new GCIs and RIAs added to preserve the semantics of the new elements is much smaller in the optimized versions. In particular, we reduce from 4352 to 324 GCIs (7.44%) and from 136 to 34 RIAs (25%). This shows the importance of reducing the number of new crisp elements and their corresponding axioms, as well as of defining crisp concepts and roles and (to a lesser extent) handling superfluous concepts.

Optimizing GCI reductions turns out to be very useful in reducing the number of disjoint concepts, domain, range and functional role axioms: 152 to 19 (12.5 %), 104 to 97 (93.27 %), 80 to 10 (12.5 %), and 48 to 6 (12.5 %), respectively.

In the case of domain role axioms the optimization is not very high because, in order to apply the optimizations, we need to use an inverse role, which is defined only for one of the roles that are involved in this kind of axiom.

Every fuzzy GCI or RIA generates several axioms in the reduced ontology. Combining the optimization of GCI reductions with the definition of crisp concepts and roles reduces the number of new axioms, from 1288 to 390 subclass axioms (30.28 %), from 696 to 318 concept equivalences (45.69 %) and from 40 to 33 sub-role axioms (82.5 %).

Finally, the number of inverse and transitive role axioms is reduced in the optimized version because fuzzy roles interpreted as crisp introduce one inverse or transitive axiom, instead of several ones. This allows a reduction from 16 to 2 axioms, and from 8 to 1, respectively, which corresponds to the 12.5 % in both cases.

Table 8.4 shows the influence of the number of degrees on the size of the resulting crisp ontology, as well as on the reduction time (which is shown in seconds). The reduction time is small enough to make possible to recompute the reduction of an ontology in real-time when necessary, thus allowing superfluous concepts and roles in the reduction to be avoided.

Table 8.4: Influence of the number of degrees in the reduction.

	Crisp	3	5	7	9	11	21
Number of axioms	811	1166	1674	2182	2690	3198	5738
Reduction time	-	0.343	0.453	0.64	0.782	0.859	1.75

Part V

Conclusions

CHAPTER 9

Conclusions and Future Work

This chapter summarizes the contributions of this thesis to the field of fuzzy ontologies, analyzing the results in accordance with the initial objectives.

This dissertation means an important step towards the achievement of fuzzy ontologies, capable of representing and handling imprecise, vague and uncertain knowledge. Throughout the document, we have presented examples in different application domains, such as medicine, accommodation, music, or enology.

Objective 1. Chapter 4 fulfills the first of our objectives, which was to perform a critical review of the state of the art in fuzzy ontologies and fuzzy DLs. We have observed that none of the existing definitions of fuzzy ontology fits well with our more general view, and we have provided an alternative definition. We have also identified some limitations on the expressivity of current fuzzy DLs. Some of them have been overcome in the next chapter.

Objective 2. Our definition of the fuzzy DL $\mathcal{SROIQ}(\mathbf{D})$ in Chapter 5 fulfills the second of our objectives, which was to obtain a fuzzy DL increasing the expressivity with respect to the related work and representing uncertain knowledge.

Among other features, our logic includes fuzzy nominals, making possible to give extensive definitions to fuzzy sets, cut concept and roles, fuzzy

GCIs and RIAs, modified concepts and roles, and fuzzy concrete domains. We accept all the fuzzy modifiers and fuzzy concrete domains which have previously been implemented in some fuzzy DL reasoner. Historically, we provided the first reasoning algorithm that supported fuzzy GCIs and RIAs, as well as modified roles.

The properties of the logic have been studied, paying special attention to the cases of Zadeh and Gödel families of fuzzy operators. They have some different features, and each of them has some advantages and some drawbacks, so different applications may require one or the other. One of the most interesting properties of Zadeh family is that the use of Kleene-Dienes implication in the semantics of fuzzy GCIs and RIAs brings about two counter-intuitive effects. On the other hand, Gödel family forces at-most qualified number restrictions to be crisp concepts, just to cite an example. These families can also be combined, and their fuzzy operators can be used together.

We have also added a possibilistic layer on top of the fuzzy DL allowing uncertain knowledge to be represented.

Objective 3. Chapters 6 and 7 fulfill the third objective: the representation of fuzzy ontologies using crisp ontologies, in such a way that crisp ontology languages, reasoners and other related tools can be reused. An immediate practical application of fuzzy ontologies is feasible, because of its tight relation with already existing languages and tools which have proved their validity.

Our proposal has several advantages:

- There is no need to agree on a new standard fuzzy language, but every developer could use its own language expressing a fuzzy DL, as long as he implements the reduction to the standard language.
- We can continue using standard languages with a lot of resources available, avoiding the need (and cost) of adapting them to the new fuzzy language. Although it would be desirable to assist the user in tasks such as fuzzy ontology editing, reducing the fuzzy ontology into a crisp one or fuzzy querying, once the reduction is performed, we may use the resources available for the crisp language.

- We may continue using existing crisp reasoners. Even if we do not claim that reasoning will be more efficient, this approach offers a work-around to support early reasoning in future fuzzy languages. Nowadays, the unique fuzzy DL reasoner fully supporting fuzzy extensions of the languages OWL and OWL 2 is based on this approach.

- It can help to proof the decidability of reasoning tasks in fuzzy DLs for which no other reasoning algorithm is known. For instance, we have proved that fuzzy $\mathcal{SROIQ}(\mathbf{D})$ under Gödel family is decidable.

- It is possible to reason with more expressive fuzzy DLs. Current specific reasoning algorithms to reason with fuzzy DLs are restricted to $\mathcal{SHOIN}$ [288] or $\mathcal{SHIF}(\mathbf{D})$ [298] under Zadeh family, or $\mathcal{ALC}$ [120] under Gödel family, but we have considered instead $\mathcal{SROIQ}(\mathbf{D})$.

We have studied the complexity of the resulting crisp KBs, which is quadratic in the case of Zadeh family, or linear if we fix the set of degrees of truth. In the case of Gödel family, the complexity is polinomial, or linear if we approximate universal quantification concepts by using cut concepts and roles.

We have shown that, under some reasonable conditions, the reduction of a fuzzy KB can be reused when additional axioms are added to it. These requirements are to use the ontology vocabulary and to restrict the set of possible degrees of truth. This is not a very restrictive assumption, since in practical applications it is usual to work with a small number of them.

In the case of Gödel family, using a fixed set of degrees of truth including 0 and 1 is mandatory in order to guarantee the witnessed model property. We recall that in the case of Łukasiewicz family (not addressed in this dissertation) it is also mandatory the use of a fixed set of degrees [41].

Restricting the degrees of truth turned also to be essential in order to compute efficiently greatest lower bounds, since they need to perform several entailment tests and the number depends on the size of set of degrees of truth.

We have introduced some interesting optimization techniques which allow a reduction of the size of the resulting KB, namely a reduction of the number of new crisp atomic elements and their corresponding axioms, the use of crisp concepts and roles, reasoning ignoring superfluous elements, and an optimization of the reductions of irreflexive role axioms and, in the case of Gödel family, of some cases of fuzzy GCIs.

We have also studied how to reason with a possibilistic fuzzy DL using only a crisp reasoner, although in this case a crisp representation has not been found.

Objective 4. Chapter 8 fulfills the fourth objective, an implementation of a small prototype demonstrating the feasibility of our approach, which is called DeLorean. It is the more expressive fuzzy DL reasoner that we are aware of, since it supports fuzzy OWL 2. A preliminary evaluation has confirmed that the designed optimizations help to reduce significantly the size of the resulting ontology. Moreover, the reduction time is small enough to enable recomputing the reduction of an ontology in real-time.

Future work. The main direction for future work is to perform a more detailed benchmark of the reasoner, taking into account not only the size of the resulting ontology but also the reasoning time. On the one hand, we will compare fuzzy reasoning with DeLorean and crisp reasoning. On the other hand, we will compare DeLorean with other fuzzy DL reasoners.

In the former case, we will consider a real-world fuzzy ontology. As far as we know, there does not exist any significant fuzzy KB; the only one that we are aware of is a fuzzy extension of LUBM [222], but it is also a non expressive ontology (in fuzzy DL Lite). Wine ontology was designed as an exercise of use of all the features of OWL, and reasoning with it is unnecessary difficult. We hope that fuzzy versions of more realistic ontologies would be tractable under our approach.

The latter option is complicated because different reasoners support different features and expressivities, and have different input formats. In that regard, it would be interesting to design a common fuzzy DIG interface. In the case of Gödel family, we plan to design and implement a tableau algorithm for some fuzzy DL. Otherwise, we will not be able to compare the two approaches.

It would also be of interest to consider alternative reasoning tasks, such as the computation of the greatest lower bound and answering conjunctive fuzzy queries, a reasoning task which has not investigated in expressive fuzzy DLs.

In future versions of DeLorean, we will adapt it to DIG 2.0 as soon as it is available. We would also like to rely on the hypertableau reasoner HermiT, which seems to outperform other DL reasoners [265] but, unfortunately, it does not yet support DIG interface. In the meanwhile of the definition of a fuzzy common language for the different reasoners, we will develop a parser for each of them (or, more precisely, the fragment of them that our reasoner is able to support). We will also adapt the user interface to support alternative reasoning tasks.

From a theoretical point of view, we would like to increase the expressivity of the logic and to consider alternative fuzzy operators. In the former case, we would like to consider some kind of fuzzy quantifiers [250]. In the latter case, we will study Łukasiewicz logic and try to extend the current results which are restricted to $\mathcal{ALCHOI}$ [41].

Bibliography

References

[1] Muhammad Abulaish and Lipika Dey. “A Fuzzy Ontology Generation Framework for Handling Uncertainties and Nonuniformity in Domain Knowledge Description”. In: *Proceedings of the 2007 International Conference on Computing: Theory and Applications (ICCTA 2007)*. Kolkata, India: IEEE Computer Society, 2007. Pages 287–293.

[2] Muhammad Abulaish and Lipika Dey. “Biological Relation Extraction and Query Answering from MEDLINE Abstracts Using Ontology-Based Text Mining”. In: *Data Knowledge and Engineering* 61.2 (2007). Pages 228–262.

[3] Muhammad Abulaish and Lipika Dey. “Interoperability among Distributed Overlapping Ontologies - A Fuzzy Ontology Framework”. In: *Proceedings of the 2006 IEEE / WIC / ACM International Conference on Web Intelligence (WI 2006)*. Hong Kong, China: IEEE Computer Society, 2006. Pages 397–403.

[4] Muhammad Abulaish and Lipika Dey. “Knowledge Enhancement Through Ontology-Guided Text Mining”. In: *Proceedings of the First International Conference on Pattern Recognition and Machine Intelligence (PReMI 2005)*. Edited by Sankar K. Pal, Sanghamitra Bandyopadhyay, and Sambhunath Biswas. Volume 3776. Lecture Notes in Computer Science. Kolkata, India: Springer-Verlag, 2005. Pages 601–604.

[5] Sudhir Agarwal and Pascal Hitzler. “sMart - A Semantic Matchmaking Portal for Electronic Markets”. In: *Proceedings of the 7th International IEEE Conference on E-Commerce Technology (CEC’05)*. Munich, Germany: IEEE Computer Society, 2005.

[6] Claudi Alsina, Enric Trillas, and Llorenç Valverde. “On Some Logical Connectives for Fuzzy Sets Theory”. In: *Journal of Mathematical Analysis and Applications* 93.1 (1983). Pages 15–26.

[7] Rafal A. Angryk, Jacob Dolan, and Frederick E. Petry. “Development of Ontologies by the Lowest Common Abstraction of Terms Using Fuzzy Hypernym Chains”. In: *Soft Computing in Ontologies and Semantic Web*. Volume 204. Studies in Fuzziness and Soft Computing. Springer-Verlag, 2005. Pages 123–148.

[8] Grigoris Antoniou and Frank van Harmelen. *A Semantic Web Primer*. MIT Press, 2004. ISBN: 0-262-01210-3.

[9] Aristotle. “Categoriae and de interpretatione”. In: *The Works of Aristotle*. Edited by William David Ross and John Alexander Smith. Volume 1: Logic. Oxford University Press, 1928.

[10] American Psychiatric Association. *Diagnostic and Statistical Manual of Mental Disorders: DSM-IV-TR*. 1994.

[11] Ching man Au Yeung and Ho fung Leung. “Ontology with Likeliness and Typicality of Objects in Concepts”. In: *Proceedings of the 25th International Conference on Conceptual Modeling (ER 2006)*. Edited by David W. Embley, Antoni Olivé, and Sudha Ram. Volume 4215. Lecture Notes in Computer Science. Tucson, AZ, USA: Springer-Verlag, 2006. Pages 98–111.

[12] Franz Baader, Sebastian Brandt, and Carsten Lutz. “Pushing the $\mathcal{EL}$ Envelope”. In: *Proceedings of the 19th International Joint Conference on Artificial Intelligence (IJCAI 2005)*. Edinburgh, Scotland, UK 2005. Pages 364–369.

[13] Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter F. Patel-Schneider. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, 2003.

[14] Franz Baader and Philipp Hanschke. "A Scheme for Integrating Concrete Domains into Concept Languages". In: *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991)*. Edited by John Mylopoulos and Ray Reiter. Sydney, Australia 1991. Pages 452–457.

[15] Franz Baader, Ian Horrocks, and Ulrike Sattler. "Description Logics as Ontology Languages for the Semantic Web". In: *Mechanizing Mathematical Reasoning: Essays in Honor of Jörg H. Siekmann on the Occasion of His 60th Birthday*. Edited by Dieter Hutter and Werner Stephan. Volume 2605. Lecture Notes in Artificial Intelligence. Springer-Verlag, 2005. Pages 228–248.

[16] Franz Baader and Werner Nutt. "Basic Description Logics". In: *The Description Logic Handbook: Theory, Implementation, and Applications*. Edited by Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, and Peter F. Patel-Schneider. Cambridge University Press, 2003.

[17] Afef Bahri, Lotfi Bejaoui, and Salem Chakhar. "On the Similarity Relation within Fuzzy Ontologies Components". In: *Proceedings of the 2nd International Symposium on Computational Intelligence and Intelligent Informatics (ISCIII'05)*. Hammamet, Tunisia 2005.

[18] Afef Bahri, Rafik Bouaziz, and Faiez Gargouri. "Dealing with Similarity Relations in Fuzzy Ontologies". In: *Proceedings of the 16th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*. London, United Kingdom 2007. Pages 1836–1841.

[19] Stefania Bandini, Silvia Calegari, and Paolo Radaelli. "Towards Fuzzy Ontology Handling Vagueness of Natural Languages". In: *Proceedings of the 1st International Conference on Rough Sets and Knowledge Technology (RSKT 2006)*. Edited by Guoyin Wang, James F. Peters, Andrzej

Skowron, and Yiyu Yao. Volume 4062. Lecture Notes in Computer Science. Chongquing, China: Springer-Verlag, 2006. Pages 693–700.

[20] Carlos D. Barranco, Jesus R. Campaña, Juan Miguel Medina, and Olga Pons. “On Storing Ontologies including Fuzzy Datatypes in Relational Databases”. In: *Proceedings of the 16th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*. London, United Kingdom 2007. Pages 1709–1714.

[21] Mustapha Baziz, Mohand Boughanem, Gabriella Pasi, and Henri Prade. “A Fuzzy Logic Approach to Information Retrieval Using an Ontology-based Representation”. In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 363–377.

[22] Sean Bechhofer, Phillip Lord, and Raphael Volz. “Cooking the Semantic Web with the OWL API”. In: *Proceedings of the 2nd International Semantic Web Conference (ISWC 2003)*. Sanibel Island, FL, USA 2003.

[23] Sean Bechhofer, Ralf Möller, and Peter Crowther. “The DIG Description Logic Interface: DIG/1.1”. In: *Proceedings of the 16th International Workshop on Description Logics (DL 2003)*. Rome, Italy 2003.

[24] Tim Berners-Lee, James Hendler, and Ora Lassila. “The Semantic Web”. In: *Scientific American* 284.5 (2001). Pages 34–43.

[25] Ignacio Blanco, Carmen Martínez-Cruz, Nicolás Marín, and María Amparo Vila. “About the Use of Ontologies for Fuzzy Knowledge Representation”. In: *Proceedings of the Joint 4th EUSFLAT & 11th LFA Conference (EUSFLAT-LFA'2005)*. Barcelona, Spain 2005. Pages 106–111.

[26] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “A Crisp Representation for Fuzzy $\mathcal{SHOIN}$ with Fuzzy Nominals and General Concept Inclusions”. In: *Proceedings of the 2nd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2006)*. Edited by Paulo C. G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, Francis Fung, and Michael Pool. Volume 218. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[27] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “A Crisp Representation for Fuzzy $\mathcal{SHOIN}$ with Fuzzy Nominals and General Concept Inclusions”. In: *Uncertainty Reasoning for the Semantic Web I*. Edited by Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Ken Laskey, Thomas Lukasiewicz, Matthias Nickles, and Michael Pool. Volume 5327. Lecture Notes in Computer Science. Springer Verlag, 2008. Pages 174–188.

[28] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “A Design Pattern for Representing Relevance in OWL Ontologies”. In: *Proceedings of the 6th International Semantic Web Conference (ISWC 2007)*. Edited by Karl Aberer, Key-Sun Choi, Natasha Fridman Noy, Dean Allemang, Kyung-Il Lee, Lyndon J. B. Nixon, Jennifer Golbeck, Peter Mika, Diana Maynard, Riichiro Mizoguchi, Guus Schreiber, and Philippe Cudré-Mauroux. Volume 4825. Lecture Notes in Computer Science. Busan, South Korea: Springer Verlag, 2007. Pages 72–85.

[29] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “Crisp Representations and Reasoning for Fuzzy Ontologies”. In: *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* (To appear).

[30] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “DeLorean: A Reasoner for Fuzzy OWL 1.1”. In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[31] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. “Extending Fuzzy Description Logics with a Possibilistic Layer”. In: *Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Francis Fung, Thomas

Lukasiewicz, Trevor Martin, Matthias Nickles, Yun Peng, Michael Pool, Pavel Smrž, and Peter Vojtáš. Volume 327. Busan, South Korea: CEUR Workshop Proceedings, 2007.

[32] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. "Optimizing the Crisp Representation of the Fuzzy Description Logic $\mathcal{SROIQ}$". In: *Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Francis Fung, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Yun Peng, Michael Pool, Pavel Smrž, and Peter Vojtáš. Volume 327. Busan, South Korea: CEUR Workshop Proceedings, 2007.

[33] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. "Optimizing the Crisp Representation of the Fuzzy Description Logic $\mathcal{SROIQ}$". In: *Uncertainty Reasoning for the Semantic Web I*. Edited by Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Ken Laskey, Thomas Lukasiewicz, Matthias Nickles, and Michael Pool. Volume 5327. Lecture Notes in Computer Science. Springer Verlag, 2008. Pages 189–206.

[34] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. "Representation of Context-dependant Knowledge in Ontologies: A Model and an Application". In: *Expert Systems with Applications* 35.4 (2008). Pages 1899–1908.

[35] Fernando Bobillo, Miguel Delgado, and Juan Gómez-Romero. "Representation of Context-dependant Relevance Relations with Fuzzy Ontologies". In: *Proceedings of the 1st Workshop on Advancing Reasoning on the Web: Scalability and Commonsense (ARea2008)*. Edited by Frank van Harmelen, Andreas Herzig, Pascal Hitzler, Zuoquan Lin, Ruzica Piskac, and Guilin Qi. Volume 350. Tenerife, Spain: CEUR Workshop Proceedings, 2008.

[36] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, and Enrique López. "A Semantic Fuzzy Expert System for a Fuzzy Bal-

anced Scorecard”. In: *Expert Systems with Applications* 36.1 (2009). Pages 423–433.

[37] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, and Umberto Straccia. “Fuzzy Description Logics under Gödel Semantics”. In: *International Journal of Approximate Reasoning* (To appear).

[38] Fernando Bobillo and Umberto Straccia. “A Fuzzy Description Logic with Product T-norm”. In: *Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*. London, UK: IEEE Press, 2007. Pages 652–657.

[39] Fernando Bobillo and Umberto Straccia. “fuzzyDL: An Expressive Fuzzy Description Logic Reasoner”. In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 923–930.

[40] Fernando Bobillo and Umberto Straccia. “On Qualified Cardinality Restrictions in Fuzzy Description Logics under Łukasiewicz Semantics”. In: *Proceedings of the 12th International Conference of Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2008)*. Edited by Luis Magdalena, Manuel Ojeda-Aciego, and José Luis Verdegay. Torremolinos, Málaga, Spain 2008. Pages 1008–1015.

[41] Fernando Bobillo and Umberto Straccia. “Towards a Crisp Representation of Fuzzy Description Logics under Łukasiewicz Semantics”. In: *Proceedings of the 17th International Symposium on Methodologies for Intelligent Systems (ISMIS 2008)*. Edited by Aijun An, Stan Matwin, Zbigniew W. Raś, and Dominik Ślęzak. Volume 4994. Lecture Notes in Computer Science. Toronto, Ontario, Canada: Springer Verlag, 2008. Pages 309–318.

[42] Fernando Bobillo et al. *Uncertainty Reasoning for the World Wide Web (URW3) Incubator Group Report*. Technical report. http://www.w3.org/2005/Incubator/urw3/XGR-urw3. W3C Incubator Group Final Report, 2008.

[43] Piero A. Bonatti, Daniel Sánchez, and Andrea G. B. Tettamanzi. *Quantification in Fuzzy Description Logics*. Technical report. Dipartimento di Tecnologie dell'Informazione, Università degli Studi di Milano, Italy, 2004.

[44] Piero A. Bonatti and Andrea G. B. Tettamanzi. "Some Complexity Results on Fuzzy Description Logics". In: *Proceedings of the 5th International Workshop on Fuzzy Logic and Applications (WILF 2003)*. Edited by Vito Di Gesù, Francesco Masulli, and Alfredo Petrosino. Volume 2955. Lecture Notes in Computer Science. Naples, Italia: Springer-Verlag, 2003. Pages 19–24.

[45] Willem Nico Borst. "Construction of Engineering Ontologies for Knowledge Sharing and Reuse". PhD thesis. Centre for Telematica and Information Technology, Universiteit Twente, Enschede, The Netherlands, 1997.

[46] Ronald J. Brachman and Hector J. Levesque. "The Tractability of Subsumption in Frame-Based Description Languages". In: *Proceedings of the 4th National Conference on Artificial Intelligence (AAAI 1984)*. Austin, Texas, USA 1984. Pages 34–37.

[47] Ronald J. Brachman and James G. Schmolze. "An Overview of the KL-ONE Knowledge Representation System". In: *Cognitive Science* 9.2 (1985). Pages 171–216.

[48] Jean Sébastien Brunner and Thibaud Latour. "Referencing Text Documents in Multidimensional Concept Spaces for Technological and Scientific Watch. A Conceptual Overview of Text Models in the Context of a Collaborative Scientific Watch System". In: *Proceedings of the Workshop on Terminology, Ontology and Knowledge Representation*. Lyon (France) 2004. Pages 11–18.

[49] Henrik Bulskov, Rasmus Knappe, and Troels Andreasen. "On Measuring Similarity for Conceptual Querying". In: *Proceedings of the 5th International Conference on Flexible Query Answering Systems (FQAS 2002)*. Edited by Troels Andreasen, Amihai Motro, Henning Chris-

tiansen, and Henrik Legind Larsen. Volume 2522. Lecture Notes in Artificial Intelligence. Copenhagen, Denmark: Springer-Verlag, 2002. Pages 100–111.

[50] Mario Bunge. *Ontology I: The Furniture of the World*. Volume 3. Treatise on Basic Philosophy. Reidel, 1977.

[51] Yi Cai and Ho fung Leung. "A Formal Model of Fuzzy Ontology with Property Hierarchy and Object Membership". In: *Proceedings of the 27th International Conference on Conceptual Modeling (ER 2008)*. Volume 5231. Lecture Notes in Computer Science. Barcelona, Spain: Springer-Verlag, 2008. Pages 69–82.

[52] Silvia Calegari and Davide Ciucci. "Fuzzy Ontology and Fuzzy-OWL in the KAON Project". In: *Proceedings of the 16th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*. London, United Kingdom 2007. Pages 1415–1420.

[53] Silvia Calegari and Davide Ciucci. "Fuzzy Ontology, Fuzzy Description Logics and Fuzzy-OWL". In: *Proceedings of the International Workshop on Fuzzy Logic and Applications (WILF 2007)*. Volume 4578. Lecture Notes in Computer Science. Portofino Vetta - Ruta di Camogli, Italy: Springer-Verlag, 2007.

[54] Silvia Calegari and Davide Ciucci. "Integrating Fuzzy Logic in Ontologies". In: *Proceedings of the 8hth International Conference on Enterprise Information Systems: Databases and Information Systems Integration (ICEIS 2006)*. Edited by Yannis Manolopoulos, Joaquim Filipe, Panos Constantopoulos, and José Cordeiro. Paphos, Cyprus 2006. Pages 66–73.

[55] Silvia Calegari and Davide Ciucci. "Towards a Fuzzy Ontology Definition and a Fuzzy-Extension of an Ontology Editor". In: *Enterprise Information Systems 8th International Conference, ICEIS 2006 (Selected Papers)*. Edited by Yannis Manolopoulos, Joaquim Filipe, Panos Constantopoulos, and José Cordeiro. Volume 3. Lecture Notes in Business Information Processing. Springer, 2008. Pages 147–158.

[56] Silvia Calegari and Fabio Farina. "Fuzzy Ontologies and Scale-free Networks Analysis". In: *International Journal of Computer Science and Applications* IV.II (2007). Pages 125–144.

[57] Silvia Calegari and Marco Loregian. "Using Dynamic Fuzzy Ontologies to Understand Creative Environments". In: *Proceedings of the 7th International Conference on Flexible Query Answering Systems (FQAS 2006)*. Edited by Henrik Legind Larsen, Gabriella Pasi, Daniel Ortiz Arroyo, Troels Andreasen, and Henning Christiansen. Volume 4027. Lecture Notes in Computer Science. Milan, Italy: Springer-Verlag, 2006. Pages 404–415.

[58] Silvia Calegari and Elie Sanchez. "A Fuzzy Ontology-Approach to improve Semantic Information Retrieval". In: *Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Francis Fung, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Yun Peng, Michael Pool, Pavel Smrž, and Peter Vojtáš. Volume 327. Busan, South Korea: CEUR Workshop Proceedings, 2007.

[59] Silvia Calegari and Elie Sanchez. "A Query Refinement to Concept-based Information Retrieval". In: *Proceedings of the 12th International Conference of Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2008)*. Edited by Luis Magdalena, Manuel Ojeda-Aciego, and José Luis Verdegay. Torremolinos, Málaga, Spain 2008. Pages 1068–1075.

[60] Silvia Calegari and Elie Sanchez. "Object-Fuzzy Concept Network: An Enrichment of Ontologies in Semantic Information Retrieval". In: *Journal of the American Society for Information Science and Technology* (To appear).

[61] Diego Calvanese, Giuseppe De Giacomo, Domenico Lembo, Maurizio Lenzerini, and Riccardo Rosati. "Data Complexity of Query Answering in Description Logics". In: *Proceedings of the 10th International Conference of Knowledge Representation and Reasoning (KR 2006)*. Edited

by Patrick Doherty, John Mylopoulos, and Christopher A. Welty. Lake District, UK 2006. Pages 260–270.

[62] Diego Calvanese, Giuseppe De Giacomo, Domenico Lembo, Maurizio Lenzerini, and Riccardo Rosati. “DL-Lite: Tractable Description Logics for Ontologies”. In: *Proceedings of the 20th National Conference on Artificial Intelligence (AAAI 2005)*. Edited by Manuela M. Veloso and Subbarao Kambhampati. Pittsburgh, Pennsylvania, USA: AAAI Press, 2005. Pages 602–607.

[63] Diego Calvanese, Giuseppe De Giacomo, and Maurizio Lenzerini. “On the Decidability of Query Containment under Constraints”. In: *Proceedings of the Seventeenth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS’98)*. Seattle, Washington, United States 1998. Pages 149–158.

[64] Jorge Cardoso. “The Semantic Web Vision: Where Are We?”. In: *IEEE Intelligent Systems* 22.5 (2007). Pages 84–88.

[65] Paolo Ceravolo, Ermesto Damiani, and Cristiano Fugazza. “Trustworthiness-related Uncertainty of Semantic Web-style Metadata: A Possibilistic Approach”. In: *Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d’Amato, Nicola Fanizzi, Francis Fung, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Yun Peng, Michael Pool, Pavel Smrž, and Peter Vojtáš. Volume 327. Busan, South Korea: CEUR Workshop Proceedings, 2007.

[66] Paolo Ceravolo, Ernesto Damiani, and Marcello Leida. “Which Role for an Ontology of Uncertainty?”. In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d’Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[67] Philipp Cimiano, Peter Haase, Qiu Ji, Theofilos Mailis, Giorgos B. Stamou, Giorgos Stoilos, Thanh Tran, and Vassilis Tzouvaras. "Reasoning with Large A-Boxes in Fuzzy Description Logics using DL Reasoners: An Experimental Valuation". In: *Proceedings of the 1st Workshop on Advancing Reasoning on the Web: Scalability and Commonsense (ARea2008)*. Edited by Frank van Harmelen, Andreas Herzig, Pascal Hitzler, Zuoquan Lin, Ruzica Piskac, and Guilin Qi. Volume 350. Tenerife, Spain: CEUR Workshop Proceedings, 2008.

[68] Resolution of Conflicts Among Ontology Mappings: A Fuzzy Approach. "Alfio Ferrara and Davide Lorusso and Giorgos Stamou and Giorgos Stoilos and Vassilis Tzouvaras and Tassos Venetis". In: *Proceedings of the 3rd International Workshop on Ontology Matching (OM 2008)*. Edited by Pavel Shvaiko, Jérôme Euzenat, Fausto Giunchiglia, and Heiner Stuckenschmidt. Karlsruhe, Germany 2008.

[69] Óscar Corcho, Mariano Fernández-López, and Asunción Gómez-Pérez. "Methodologies, Tools and Languages for Building Ontologies. Where is their Meeting Point?". In: *Data & Knowledge Engineering* 46 (2003). Pages 41–64.

[70] Célia da Costa Pereira and Andrea G. B. Tettamanzi. "Possibilistic Planning Using Description Logics: A First Step". In: *Proceedings of the 6th International Workshop on Fuzzy Logic and Applications (WILF 2005)*. Edited by Isabelle Bloch, Alfredo Petrosino, and Andrea G. B. Tettamanzi. Volume 3849. Lecture Notes in Computer Science. Crema, Italy: Springer-Verlag, 2006. Pages 81–88.

[71] Paulo C. G. Costa, Kathryn B. Laskey, and Thomas Lukasiewicz. "Uncertainty Representation and Reasoning in the Semantic Web". In: *Semantic Web Engineering in the Knowledge Society*. Edited by Jorge Cardoso and Miltiadis D. Lytras. 2008.

[72] Olivier Couchariere, Marie-Jeanne Lesot, and Bernadette Bouchon-Meunier. "Consistency Checking for Extended Description Logics". In: *Proceedings of the 21st International Workshop on Description Logics*

(DL 2008). Volume 353. Dresden, Germany: CEUR Workshop Proceedings, 2008.

[73] Anne Cregan, Rolf Schwitter, and Thomas Meyer. “Sydney OWL Syntax - Towards a Controlled Natural Language Syntax for OWL 1.1”. In: *Proceedings of the 3rd International Workshop on OWL: Experiences and Directions (OWLED 2007)*. Volume 258. Innsbruck, Austria: CEUR Workshop Proceedings, 2007.

[74] Valerie V. Cross and Clare Voss. “Fuzzy Ontologies for Multilingual Document Exploitation”. In: *Proceedings of the 18th International Conference of the North American Fuzzy Information Processing Society (NAFIPS’99)*. New York City, New York, USA 1999. Pages 392–397.

[75] Bernardo Cuenca-Grau, Ian Horrocks, Boris Motik, Bijan Parsia, Peter Patel-Schneider, and Ulrike Sattler. “OWL 2: The next step for OWL”. In: *Journal of Web Semantics* (To appear).

[76] Bernardo Cuenca-Grau and Boris Motik. *OWL 1.1 Web Ontology Language: Mapping to RDF Graphs*. [Online] Available: http://www.w3.org/TR/owl11-mapping-to-rdf/. 2008.

[77] Bernardo Cuenca-Grau, Boris Motik, and Peter F. Patel-Schneider. *OWL 1.1 Web Ontology Language: XML Syntax*. [Online] Available: http://www.webont.com/owl/1.1/xml_syntax.html. 2007.

[78] Carlos Viegas Damásio, Jeff Z. Pan, Giorgos Stoilos, and Umberto Straccia. “Representing Uncertainty Rules in RuleML”. In: *Fundamenta Informaticae* 82.3 (2008). Pages 1–24.

[79] Mathieu d’Aquin, Jean Lieber, and Amedeo Napoli. “Étude de quelques logiques de descriptions floues et de formalismes apparentés (in French)”. In: *Rencontres francophones sur la Logique Floue et ses Applications (LFA 2004)*. Edited by Elie Sanchez. Nantes, France: Cépadues-Editions, 2004. Pages 255–262.

[80] Mathieu d'Aquin, Jean Lieber, and Amedeo Napoli. "Towards a Semantic Portal for Oncology using a Description Logic with Fuzzy Concrete Domains". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 379–393.

[81] Stamatia Dasiopoulou, Carsten Saathoff, Phivos Mylonas, Yannis Avrithis, Yiannis Kompatsiaris, Steffen Staab, and Michael G. Strinztis. "Introducing Context and Reasoning in Visual Content Analysis: An Ontology-based Framework". In: *Semantic Multimedia and Ontologies: Theory and Applications*. Edited by Yiannis Kompatsiaris and Paola Hobson. 2008. Pages 99–122.

[82] Bruno De Finetti. "La logique de la probabilité (in French)". In: *Actes du Congres International de Philosophie Scientifique*. Volume 4. Paris, France 1935.

[83] Miguel Delgado, María José Martín-Bautista, Daniel Sánchez, and María Amparo Vila. "A Probabilistic Definition of a Nonconvex Fuzzy Cardinality". In: *Fuzzy Sets and Systems* 126.2 (2002). Pages 41–54.

[84] Miguel Delgado, Daniel Sánchez, and María Amparo Vila. "Fuzzy Cardinality Based Evaluation of Quantified Sentences". In: *International Journal of Approximate Reasoning* 23.1 (2000). Pages 23–66.

[85] Lipika Dey and Muhammad Abulaish. "A Fuzzy Ontology-Based Text Mining System for Knowledge Acquisition, Ontology Enhancement and Query Answering from Bio-Medical Texts". In: *Computational Intelligence in Bioinformatics*. Edited by Yi Pan Gary B. Fogel David W. Corne. Wiley Inc., 2007.

[86] Lipika Dey and Muhammad Abulaish. "Enhancing a Biological Concept Ontology to Fuzzy Relational Ontology with Relations Mined from Text". In: *Proceedings of the 5th International Conference on Rough Sets and Current Trends in Computing (RSCTC 2006)*. Edited by Salvatore Greco, Yutaka Hata, Shoji Hirano, Masahiro Inuiguchi, Sadaaki Miyamoto, Hung Son Nguyen, and Roman Slowin-

ski. Volume 4259. Lecture Notes in Computer Science. Kobe, Japan: Springer-Verlag, 2006. Pages 527–536.

[87] Lipika Dey and Muhammad Abulaish. "Fuzzy Ontologies for Handling Uncertainties and Inconsistencies in Domain Knowledge Description". In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 1366–1373.

[88] Lipika Dey, Muhammad Abulaish, Rohit Goyal, and Kumar Shubham. "A Rough-Fuzzy Ontology Generation Framework and Its Application to Bio-medical Text Processing". In: *Proceedings of the 5th Atlantic Web Intelligence Conference (AWIC 2007)*. Edited by Katarzyna M. Wegrzyn-Wolska and Piotr S. Szczepaniak. Volume 43. Advances in Soft Computing. Fontainebleau, France: Springer-Verlag, 2007. Pages 74–79.

[89] Jonathan Dileo, Timothy Jacobs, and Scott Deloach. "Integrating Ontologies into Multi-Agent Systems Engineering". In: *Proceedings of 4th International Bi-Conference Workshop on Agent Oriented Information Systems (AOIS 2002)*. Edited by Paolo Giorgini, Yves Lespérance, Gerd Wagner, and Eric S. K. Yu. Volume 59. Bologna, Italy: CEUR Workshop Proceedings, 2002.

[90] Dzung Dinh-Khac, Dinh-Khang Tran, and Steffen Hölldobler. "The Fuzzy Linguistic Description Logic $\mathcal{ALC}_{\mathcal{FL}}$". In: *Proceedings of the 11th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU 2006)*. Paris, France 2006. Pages 2096–2103.

[91] Mauro Dragoni and Andrea G. B. Tettamanzi. "Evolutionary Algorithms for Reasoning in Fuzzy Description Logics with Fuzzy Quantifiers". In: *Proceedings of Genetic and Evolutionary Computation Conference (GECCO 2007)*. Edited by Hod Lipson. London, UK: ACM, 2007. Pages 1967–1974.

[92] Didier Dubois, Francesc Esteva, Lluís Godo, and Henri Prade. "Fuzzy-set Based Logics – An History-oriented Presentation of their Main Developments". In: *The Many Valued and Non-monotonic Turn in Logic*. Edited by Dov M. Gabbay and John Woods. Volume 8. Handbook of the History of Logic. Elsevier, 2007. Pages 325–449.

[93] Didier Dubois, Jérôme Lang, and Henri Prade. "Possibilistic Logic". In: *Handbook of Logic in Artificial Intelligence and Logic Programming, Volume 3: Nonmonotonic Reasoning and Uncertain Reasoning*. Edited by Dov M. Gabbay, Christopher J. Hogger, and J. A. Robinson. Clarendon Press, 1994. Pages 439–513.

[94] Didier Dubois, Jértme Mengin, and Henri Prade. "Possibilistic Uncertainty and Fuzzy Features in Description Logic. A Preliminary Discussion". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 101–113.

[95] Didier Dubois and Henri Prade. "Can we Enforce Full Compositionality in Uncertainty Calculi?". In: *Proceedings of the 12th National Conference on Artificial Intelligence (AAAI 1994)*. Seattle, WA, USA 1994. Pages 149–154.

[96] Didier Dubois and Henri Prade. "Operations on Fuzzy Numbers". In: *International Journal of Systems Sciences* 9 (1978). Pages 613–626.

[97] Didier Dubois and Henri Prade. "Possibility Theory, Probability Theory and Multiple-valued Logics: A Clarification". In: *Annals of Mathematics and Artificial Intelligence* 32.1–4 (2001). Pages 35–66.

[98] Eduardo L. G. Escovar, Cristiane A. Yaguinuma, and Mauro Biajiz. "Using Fuzzy Ontologies to Extend Semantically Similar Data Mining". In: *Proceedings of the XXI Simpósio Brasileiro de Banco de Dados (SBBD 2006)*. Florianópolis, SC, Brasil 2006. Pages 16–30.

[99] Francesc Esteva and Lluís Godo. "Monoidal T-norm Based Logic: Towards a Logic for Left-continuous T-norms". In: *Fuzzy Sets and Systems* 124.3 (2001). Pages 271–288.

[100] Francesc Esteva, Lluís Godo, and Franco Montagna. "The ŁΠ and ŁΠ$\frac{1}{2}$ Logics: Two Complete Fuzzy Systems Joining Łukasiewicz and Product Logics". In: *Archive for Mathematical Logic* 40.1 (2001). Pages 39–67.

[101] Mingxia Gao and Chunnian Liu. "Extending OWL by Fuzzy Description Logic". In: *Proceedings of the 17th IEEE International Conference on Tools with Artificial Intelligence (ICTAI 2005)*. Hong Kong, China: IEEE Computer Society, 2005. Pages 562–567.

[102] Pablo J. Garcés, José A. Olivas, and Francisco P. Romero. "Concept-Matching IR Systems Versus Word-Matching Information Retrieval Systems: Considering Fuzzy Interrelations for Indexing Web Pages". In: *Journal of the American Society for Information Science and Technology* 57.4 (2006). Pages 564–576. ISSN: 1532-2882.

[103] Pablo J. Garcés, José A. Olivas, and Francisco P. Romero. "FIS-CRM: A Representation Model based on Fuzzy Interrelations for Internet Search". In: *Proceedings of the International Conference on Artificial Intelligence (ICAI 2002)*. Volume I. Las Vegas, NE, USA 2002. Pages 219–224.

[104] Kurt Gödel. "Zum intuitionistischen Aussagenkalkül (in German)". In: *Anzieger Akademie der Wissenschaften in Wien*. Volume 69. 1932. Pages 65–66.

[105] Joseph Amadee Goguen. "The Logic of Inexact Concepts". In: *Synthese* 19.3–4 (1969). Pages 325–373.

[106] Asunción Gómez-Pérez and María Socorro Bernardos-Galindo. "Ontologies and Natural Language Processing". In: *Reflections on Language Use in the Academic Context*. 2005. Pages 121–161.

[107] Thomas R. Gruber. "A Translation Approach to Portable Ontologies". In: *Knowledge Acquisition* 5.2 (1993). Pages 199–220.

[108] Hua-Mao Gu, Xun Wang, Yun Ling, and Jin-Qin Shi. "Building a Fuzzy Ontology of Edutainment Using OWL". In: *Proceedings of the 7th International Conference on Computational Science (ICCS 2007), Part III*. Edited by Yong Shi, Geert Dick van Albada, Jack Dongarra, and Peter M. A. Sloot. Volume 4489. Lecture Notes in Computer Science. Beijing, China: Springer-Verlag, 2007. Pages 591–594.

[109] Huamao Gu, Hexin Lv, Ji Gao, and Jinqin Shi. "Towards a General Fuzzy Ontology and its Construction". In: *Proceedings of the International Conference on Intelligent Systems and Knowledge Engineering (ISKE 2007)*. Advances in Intelligent Systems Research. 2007.

[110] Nicola Guarino and Pierdaniele Giaretta. "Ontologies and Knowledge Bases: Towards a Terminological Clarification". In: *Towards Very Large Knowledge Bases, Knowledge Building and Knowledge Sharing*. Edited by Nicolaas J. I. Mars. IOS Press, 1995. Pages 25–32.

[111] Volker Haarslev and Ralf Möller. "Racer System Description". In: *Proceedings of International Joint Conference on Automated Reasoning (IJCAR-01)*. Edited by R. Goré, A. Leitsch, and T. Nipkow. Volume 2083. Lecture Notes in Computer Science. Siena, Italy: Springer-Verlag, 2001. Pages 701–705.

[112] Volker Haarslev, Hsueh-Ieng Pai, and Nematollaah Shiri. "A Generic Framework for Description Logics with Uncertainty". In: *Proceedings of the 1st ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2005)*. Edited by Paulo Cesar G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, and Michael Pool. Volume 173. Galway, Ireland: CEUR Workshop Proceedings, 2005. Pages 67–76.

[113] Volker Haarslev, Hsueh-Ieng Pai, and Nematollaah Shiri. "Completion Rules for Uncertainty Reasoning with the Description Logic $\mathcal{ALC}$". In: *Proceedings of the Canadian Semantic Web Working Symposium (CSWWS 2006)*. Volume 4. Semantic Web and Beyond: Computing for Human Experience. Quèbec City, Quèbec, Canada: Springer Verlag, 2006. Pages 205–225.

[114] Volker Haarslev, Hsueh-Ieng Pai, and Nematollaah Shiri. “Optimizing Tableau Reasoning in ALC Extended with Uncertainty”. In: *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*. Edited by Diego Calvanese, Enrico Franconi, Volker Haarslev, Domenico Lembo, Boris Motik, Sergio Tessaris, and Anni-Yasmin Turhan. Volume 250. Brixen-Bressanone, Italy: CEUR Workshop Proceedings, 2007. Pages 307–314.

[115] Volker Haarslev, Hsueh-Ieng Pai, and Nematollaah Shiri. “Uncertainty Reasoning for Ontologies with General TBoxes in Description Logic”. In: *Uncertainty Reasoning for the Semantic Web I*. Edited by Paulo C. G. da Costa, Claudia d’Amato, Nicola Fanizzi, Kathryn B. Laskey, Ken Laskey, Thomas Lukasiewicz, Matthias Nickles, and Michael Pool. Volume 5327. Lecture Notes in Computer Science. Springer Verlag, 2008.

[116] Volker Haarslev, Hsueh-Ieng Pai, and Nematollaah Shiri. “Uncertainty Reasoning in Description Logics: A Generic Approach”. In: *Proceedings of the 19th International FLAIRS Conference*. Melbourne Beach, FL, USA: AAAI Press, 2006. Pages 818–823.

[117] Hele-Mai Haav and Tanel-Lauri Lubi. “A Survey of Concept-based Information Retrieval Tools on the Web”. In: *Proceedings of the 5th East-European Conference on Advances in Databases and Information Systems (ADBIS 2001)*. Edited by Albertas Caplinskas and Johann Eder. Volume 2. Vilnius, Lithuania: Technika, 2001. Pages 29–41.

[118] Hashim Habiballa. “Resolution Strategies for Fuzzy Description Logic”. In: *Proceedings of the 5th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2007)*. Edited by Martin Štěpnička, Vilém Novák, and Ulrich Bodenhofer. Volume 2. Ostrava, Czech Republic 2007. Pages 27–36.

[119] Petr Hájek. “Computational Complexity of T-norm Based Propositional Fuzzy Logics with Rational Truth Constants”. In: *Fuzzy Sets and Systems* 157.5 (2006). Pages 677–682.

[120] Petr Hájek. “Making Fuzzy Description Logic More General”. In: *Fuzzy Sets and Systems* 154.1 (2005). Pages 1–15.

[121] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[122] Petr Hájek. “What Does Mathematical Fuzzy Logic Offer to Description Logic?”. In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 91–100.

[123] Patrick J. Hayes. “The Logic for Frames”. In: *Frame Conception and Text Understanding*. Edited by D. Metzing. Walter de Gruyter and Co., 1979. Pages 46–61.

[124] James Hendler and Deborah L. McGuinness. “The Semantic Web and its Languages”. In: *IEEE Intelligent Systems* 15.6 (2000). Pages 67–73.

[125] Markus Holi and Eero Hyvönen. “Modeling Uncertainty in Semantic Web Taxonomies”. In: *Soft Computing in Ontologies and Semantic Web*. Volume 204. Studies in Fuzziness and Soft Computing. Springer-Verlag, 2005. Pages 31–46.

[126] Steffen Hölldobler, Hoang-Nga Nguyen, and Dinh-Khang Tran. “The Fuzzy Description Logic $\mathcal{ALC}_{\mathcal{FLH}}$”. In: *Proceedings of the International Workshop on Description Logics (DL 2005)*. Edinburgh, Scotland, UK 2005.

[127] Steffen Hölldobler, Hans-Peter Störr, and Dinh-Khang Tran. “A Fuzzy Description Logic with Hedges as Concept Modifiers”. In: *Proceedings of the Joint Third International Conference on Intelligent Technologies and Third Vietnam-Japan Symposium on Fuzzy Systems and Applications (InTech/VJFuzzy’2002)*. Edited by Nguyen Hoang Phuong, Hung T. Nguyen, Nguyen Cat Ho, and Pratit Santiprabho. Hanoi, Vietnam: Science and Technics Publishing House, 2002. Pages 25–34.

[128] Steffen Hölldobler, Hans-Peter Störr, and Dinh-Khang Tran. “The Fuzzy Description Logic $\mathcal{ALC}_{\mathcal{FH}}$ with Hedge Algebras as Concept Modifiers”. In: *Journal of Advanced Computational Intelligence* 7.3 (2003). Pages 294–305.

[129] Steffen Hölldobler, Hans-Peter Störr, Dinh-Khang Tran, and Hoang-Nga Nguyen. “The Subsumption Problem of the Fuzzy Description Logic $\mathcal{ALC}_{\mathcal{FH}}$”. In: *Proceedings of the 10th International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU 2004)*. Perugia, Italy: Editrice Università La Sapienza, Italy, 2004. Pages 243–250.

[130] Bernhard Hollunder. “An Alternative Proof Method for Possibilistic Logic and its Application to Terminological Logics.”. In: *Proceedings of the Tenth Annual Conference on Uncertainty in Artificial Intelligence (UAI 1994)*. Edited by Ramon López de Màntaras and David Poole. Seattle, Washington, USA: Morgan Kaufmann, 1994. Pages 327–335.

[131] Bernhard Hollunder, Werner Nutt, and Manfred Schmidt-Schauß. “Subsumption Algorithms for Concept Description Languages”. In: *Proceedings of the 9th European Conference of Artificial Intelligence (ECAI 1990)*. Stockholm, Sweden 1990. Pages 348–353.

[132] Matthew Horridge, Sean Bechhofer, and Olaf Noppens. “Igniting the OWL 1.1 Touch Paper: The OWL API”. In: *Proceedings of the 3rd International Workshop on OWL: Experiences and Directions (OWLED 2007)*. Volume 258. Innsbruck, Austria: CEUR Workshop Proceedings, 2007.

[133] Matthew Horridge, Nick Drummond, John Goodwin, Alan Rector, Robert Stevens, and Hai H Wang. “The Manchester OWL Syntax”. In: *Proceedings of the 2nd International Workshop on OWL: Experience and Directions (OWLED 2006)*. Edited by Bernardo Cuenca-Grau, Pascal Hitzler, Conor Shankey, and Evan Wallace. Volume 216. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[134] Matthew Horridge and Peter F. Patel-Schneider. “Manchester Syntax for OWL 1.1”. In: *Proceedings of the 4th International Workshop on OWL: Experience and Directions (OWLED 2008 DC)*. Washington, DC, USA 2008.

[135] Matthew Horridge, Dmitry Tsarkov, and Timothy Redmond. "Supporting Early Adoption of OWL 1.1 with Protege-OWL and FaCT++". In: *Proceedings of the 2nd International Workshop on OWL: Experience and Directions (OWLED 2006)*. Edited by Bernardo Cuenca-Grau, Pascal Hitzler, Conor Shankey, and Evan Wallace. Volume 216. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[136] Ian Horrocks. "Using an Expressive Description Logic: FaCT or Fiction?". In: *Proceedings of the Sixth International Conference on the Principles of Knowledge Representation and Reasoning (KR 1998)*. Edited by A. G. Cohn, L. Schubert, and S. C. Shapiro. Trento, Italy: Morgan Kaufmann Publishers, 1998. Pages 636–647.

[137] Ian Horrocks, Oliver Kutz, and Ulrike Sattler. "The Even More Irresistible $\mathcal{SROIQ}$". In: *Proceedings of the 10th International Conference of Knowledge Representation and Reasoning (KR 2006)*. Edited by Patrick Doherty, John Mylopoulos, and Christopher A. Welty. Lake District, UK 2006. Pages 452–457.

[138] Ian Horrocks and Peter F. Patel-Schneider. "A Description Logic with Transitive and Inverse Roles and Role hierarchies". In: *Journal of Logic and Computation* 9.3 (1999). Pages 385–410.

[139] Ian Horrocks and Peter F. Patel-Schneider. "Reducing OWL Entailment to Description Logic Satisfiability". In: *Journal of Web Semantics* 1.4 (2004). Pages 345–357. ISSN: 1570-8268.

[140] Ian Horrocks, Peter F. Patel-Schneider, and Frank van Harmelen. "From $\mathcal{SHIQ}$ and RDF to OWL: The Making of a Web Ontology Language". In: *Journal of Web Semantics* 1.1 (2003). Pages 7–26.

[141] Ian Horrocks and Ulrike Sattler. "A Tableaux Decision Procedure for $\mathcal{SHOIQ}$". In: *Journal of Automated Reasoning* 39.3 (2007). Pages 249–276.

[142] Ian Horrocks, Ulrike Sattler, and Stephan Tobies. "Practical Reasoning for Very Expressive Description Logics". In: *Logic Journal of the IGPL* 8.3 (2000). Pages 239–264.

[143] Céline Hudelot, Jamal Atif, and Isabelle Bloch. "Fuzzy Spatial Relation Ontology for Image Interpretation". In: *Fuzzy Sets and Systems* 159.15 (2008). Pages 1929–1951.

[144] Yun Cheng Jiang, Yong Tang, and Ju Wang. "Fuzzy ER Modeling with Description Logics". In: *Journal of Software* 17.1 (2006). Pages 20–30.

[145] Da-Zhou Kang, Bao-Wen Xu, Jian-Jiang Lu, and Yan-Hui Li. "Reasoning within Extended Fuzzy Description Logic Supporting Terminological Axiom Restrictions". In: *Journal of Software* 18.7 (2007). Pages 1563–1572.

[146] Dazhou Kang and Jianjiang Lu. "A Family of Fuzzy Description Logics with Comparison Expressions". In: *Proceedings of Computation and Logic in the Real World (CIE 2007)*. Siena, Italy 2007.

[147] Dazhou Kang, Jianjiang Lu, Baowen Xu, Yanhui Lv, and Yanxiang He. "Two Reasoning Methods for Extended Fuzzy $\mathcal{ALCH}$". In: *Proceedings of OTM Conferences 2005, Part II*. Edited by Robert Meersman, Zahir Tari, Mohand-Said Hacid, John Mylopoulos, Barbara Pernici, Özalp Babaoglu, Hans-Arno Jacobsen, Joseph P. Loyall, Michael Kifer, and Stefano Spaccapietra. Volume 3761. Lecture Notes in Computer Science. Agia Napa, Cyprus: Springer-Verlag, 2005. Pages 1588–1595.

[148] Dazhou Kang, Baowen Xu, Jianjiang Lu, Keyue Li, William C.Chu, and Hanwu Chen. "Approximate Information Retrieval for Heterogeneity Ontologies". In: *Proceedings of the 2005 International Conference on Cyberworlds (CW'05)*. IEEE Computer Society, 2005. Pages 539–544.

[149] Dazhou Kang, Baowen Xu, Jianjiang Lu, and Yanhui Lv. "A Fuzzy Extension of Description Logic $\mathcal{ALC}$ with Comparison Expressions". In: *Proceedings of the 10th International Conference on Knowledge-Based Intelligent Information and Engineering Systems (KES 2006), Part I*. Edited by Bogdan Gabrys, Robert J. Howlett, and Lakhmi C. Jain. Volume 4251. Lecture Notes in Computer Science. Bournemouth, UK: Springer-Verlag, 2006. Pages 647–654.

[150] Dazhou Kang, Baowen Xu, Jianjiang Lu, and Yanhui Lv. "Reasoning for a Fuzzy Description Logic with Comparison Expressions". In: *Proceedings of the 2006 International Workshop on Description Logics (DL 2006)*. Lake District, UK 2006. Pages 111–118.

[151] Dazhou Kang, Baowen Xu, Jianjiang Lu, and Yanhui Lv. "Reasoning within the Extended Fuzzy Description Logics with Restricted TBoxes". In: *Proceedings of Second Conference on Computability in Europe (CiE 2006)*. Swansea, UK 2006.

[152] Dazhou Kang, Baowen Xu, and Yanhui Lv. "Reasoning within the Extended Fuzzy Description Logics with Restricted TBoxes". In: *Proceedings of the International Conference on Semantic Computing (ICSC 2007)*. Irvine, California, USA: IEEE Computer Society, 2007.

[153] Yevgeny Kazakov. "$\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ are Harder than $\mathcal{SHOIQ}$". In: *Proceedings of the 21st International Workshop on Description Logics (DL 2008)*. Volume 353. Dresden, Germany: CEUR Workshop Proceedings, 2008.

[154] Uwe Keller and Stijn Heymans. "Deciding Fuzzy Description Logics by Type Elimination". In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[155] Uwe Keller and Stijn Heymans. "Fuzzy Description Logic Reasoning using a Fixpoint Algorithm". In: *Proceedings of the 6th International Symposium on Logical Foundations of Computer Science (LFCS 2009)*. Deerfield Beach, FL, USA 2009.

[156] Stephen Cole Kleene. *Introduction to Metamathematics*. North Holland, 1952.

[157] George J. Klir and Bo Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice-Hall, 1995.

[158] Rasmus Knappe, Henrik Bulskov, and Troels Andreasen. "Perspectives on Ontology-based Querying". In: *International Journal of Intelligent Systems* 22.7 (2007). Pages 739–761.

[159] Stasinos Konstantopoulos and Georgios Apostolikas. "Fuzzy-DL Reasoning over Unknown Fuzzy Degrees". In: *Proceedings of the 3rd International Workshop on Semantic Web and Web Semantics (SWWS 07), Part II*. Edited by Robert Meersman, Zahir Tari, and Pilar Herrero. Volume 4806. Lecture Notes in Computer Science. Vilamoura, Portugal: Springer-Verlag, 2007. Pages 1312–1318.

[160] Ralf Küsters and Ralf Molitor. "Approximating Most Specific Concepts in Description Logics with Existential Restrictions". In: *AI Communications* 15.1 (2002). Pages 47–59. ISSN: 0921-7126.

[161] Oliver Kutz, Carsten Lutz, Frank Wolter, and Michael Zakharyaschev. "E-connections of Abstract Description Systems". In: *Artificial Intelligence* 156.1 (2004). Pages 1–73.

[162] Toby H. W. Lam. "Fuzzy Ontology Map - A Fuzzy Extension of the Hard-Constraint Ontology". In: *Proceedings of the 2006 IEEE / WIC / ACM International Conference on Web Intelligence (WI 2006)*. Hong Kong, China: IEEE Computer Society, 2006. Pages 506–509.

[163] Chang-Shing Lee, Zhi-Wei Jian, and Lin-Kai Huang. "A Fuzzy Ontology and Its Application to News Summarization". In: *IEEE Transactions on Systems, Man and Cybernetics, Part B* 35.5 (2005). Pages 859–880.

[164] Marie-Jeanne Lesot, Olivier Couchariere, Bernadette Bouchon-Meunier, and Jean-Luc Rogier. "Inconsistency Degree Computation for Possibilistic Description Logic: An Extension of the Tableau Algorithm". In: *Proceedings of the Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS 2008)*. New York, NY, USA: IEEE Press, 2008.

[165] Ruixuan Li, Xiaolin Sun, Zhengding Lu, Kunmei Wen, and Yuhua Li. “Towards a Type-2 Fuzzy Description Logic for Semantic Search Engine”. In: *Proceedings of the Joint 9th Asia-Pacific Web Conference (APWeb 2007) and the 8th International Conference on Web-Age Information Management (WAIM 2007)*. Edited by Guozhu, Xuemin Lin, Wei Wang, Yun Yang, and Jeffrey Xu Yu. Volume 4505. Lecture Notes in Computer Science. Huang Shan, China: Springer-Verlag, 2007. Pages 805–812.

[166] Ruixuan Li, Kunmei Wen, Zhengding Lu, Xiaolin Sun, and Zhigang Wang. “An Improved Semantic Search Model Based on Hybrid Fuzzy Description Logic”. In: *Proceedings of the Japan-China Joint Workshop on Frontier of Computer Science and Technology (FCST 2006)*. Fukushima, Japan: IEEE Computer Society, 2006. Pages 139–146.

[167] Yan-Hui Li, Bao-Wen Xu, Jian-Jiang Lu, and Da-Zhou Kang. “On Computational Complexity of the Extended Fuzzy Description Logic with Numerical Restriction”. In: *Journal of Software* 17.5 (2006). Pages 968–975.

[168] Yan-Hui Li, Bao-Wen Xu, Jian-Jiang Lu, and Da-Zhou Kang. “Reasoning with General Terminological Axioms in Fuzzy Description Logic F$\mathcal{ALCN}$”. In: *Journal of Software* 19.3 (2008). Pages 594–604.

[169] Churn-Jung Liau and Y. Y. Yao. “Information Retrieval by Possibilistic Reasoning”. In: *Proceedings of the 12th International Conference on Database and Expert Systems Applications (DEXA 2001)*. Edited by Heinrich C. Mayr, Jirí Lazanský, Gerald Quirchmayr, and Pavel Vogel. Volume 2113. Lecture Notes in Computer Science. Munich, Germany: Springer-Verlag, 2001. Pages 52–61.

[170] Yun Ling, Hua-Mao Gu, Xun Wang, and Jin-Qin Shi. “A Fuzzy Ontology and Its Application to Computer Games”. In: *Proceedings of the 4th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2007)*. Volume 4. 2007. Pages 442–446.

[171] Ou Liu, Qijian Tian, and Jian Ma. "A Fuzzy Description Logic Approach to Model Management in R&D Project Selection". In: *Proceedings of the 8th Pacific Asia Conference on Information Systems (PACIS 2004)*. Shanghai, China 2006.

[172] Yannick Loiseau, Henri Prade, and Mohand Boughanem. "Qualitative Pattern Matching with Linguistic Terms". In: *AI Communications* 17.1 (2004). Pages 25–34. ISSN: 0921-7126.

[173] Jian-Jiang Lu, Yan-Hui Li, Ya-Fei Zhang, Bo Zhou, and Da-Zhou Kang. "Reasoning Complexity for Extended Fuzzy Description Logic with Qualifying Number Restriction". In: *Journal of Southeast University* 23.2 (2007). Pages 236–240.

[174] Jian-Jiang Lu, Bao-Wen Xu, Yan-Hui Li, and Da-Zhou Kang. "A Family of Extended Fuzzy Description Logics". In: *International Journal of Business Intelligence and Data Mining* 1.4 (2006). Pages 384–400.

[175] Jianjiang Lu, Dazhou Kang, Yafei Zhang, Yanhui Lv, and Bo Zhou. "A Family of Fuzzy Description Logics with Comparison Expressions". In: *Proceedings of the 3rd International Conference on Rough Sets and Knowledge Technology (RSKT 2008)*. Edited by Guoyin Wang, Tian rui Li, Jerzy W. Grzymala-Busse, Duoqian Miao, Andrzej Skowron, and Yiyu Yao. Volume 5009. Lecture Notes in Computer Science. Chengdu, China: Springer-Verlag, 2008. Pages 395–402.

[176] Jianjiang Lu, Yanhui Lv, Bo Zhou, Dazhou Kang, and Yafei Zhang. "Distributed Reasoning with Fuzzy Description Logics". In: *Proceedings of the 7th International Conference on Computational Science (ICCS 2007), Part I*. Edited by Yong Shi, G. Dick van Albada, Jack Dongarra, and Peter M. A. Sloot. Volume 4487. Lecture Notes in Computer Science. Beijing, China: Springer-Verlag, 2007. Pages 196–203.

[177] Jianjiang Lu, Baowen Xu, Yanhui Lv, Dazhou Kang, and Peng Wang. "Extended Fuzzy $\mathcal{ALCN}$ and its Tableau Algorithm". In: *Proceedings of the Second International Conference on Fuzzy Systems and Knowledge Discovery (FSKD 2005)*. Edited by Lipo Wang and Yaochu Jin.

Volume 3613. Lecture Notes in Computer Science. Changsha, China: Springer-Verlag, 2005. Pages 232–242.

[178] Jan Łukasiewicz. "O logice trojwartosciowej (in Polish, On three-valued logic)". In: *Ruch filozoficzny* 5 (1920). Pages 170–171.

[179] Jan Łukasiewicz. "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls (in German)". In: *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*. Volume 23. 1930. Chapter iii, pages 51–77.

[180] Thomas Lukasiewicz and Umberto Straccia. "Description Logic Programs under Probabilistic Uncertainty and Fuzzy Vagueness". In: *Proceedings of the 9th European Conferences on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2007)*. Edited by Khaled Mellouli. Volume 4724. Lecture Notes in Computer Science. Hammamet, Tunisia: Springer-Verlag, 2007. Pages 187–198.

[181] Thomas Lukasiewicz and Umberto Straccia. "Managing Uncertainty and Vagueness in Description Logics for the Semantic Web". In: *Journal of Web Semantics* 6.4 (2008). Pages 51–68.

[182] Carsten Lutz. "Description Logics with Concrete Domains - A Survey". In: volume 4. Advances in Modal Logics. King's College Publications, 2003.

[183] Yanhui Lv. "A Fuzzy Ontology Generation Framework from Fuzzy Relational Database". In: *International Journal on Semantic Web and Information Systems* 4.2 (2008).

[184] Yanhui Lv, Jianjiang Lu, Baowen Xu, Dazhou Kang, and Jixiang Jiang. "A Fuzzy Extension of Description Logic $\mathcal{ALCH}$". In: *Proceedings of the 4th Mexican International Conference on Artificial Intelligence (MICAI 2005)*. Volume 3789. Lecture Notes in Computer Science. Monterrey, Mexico: Springer-Verlag, 2005. Pages 152–161.

[185] Yanhui Lv, Zongmin M. Ma, and Li Yan. “Fuzzy RDF: A Data Model to Represent Fuzzy Metadata”. In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 1439–1445.

[186] Yanhui Lv, Baowen Xu, Jianjiang Lu, and Dazhou Kang. “A Distributed and Fuzzy Extension of Description Logics”. In: *Proceedings of the 10th International Conference on Knowledge-Based Intelligent Information and Engineering Systems (KES 2006), Part I*. Edited by Bogdan Gabrys, Robert J. Howlett, and Lakhmi C. Jain. Volume 4251. Lecture Notes in Computer Science. Bournemouth, UK: Springer-Verlag, 2006. Pages 655–662.

[187] Yanhui Lv, Baowen Xu, Jianjiang Lu, and Dazhou Kang. “Discrete Tableau Algorithms for $f\mathcal{SHI}$”. In: *Proceedings of the 2006 International Workshop on Description Logics (DL 2006)*. Lake District, UK 2006. Pages 39–50.

[188] Yanhui Lv, Baowen Xu, Jianjiang Lu, and Dazhou Kang. “Reasoning Technique for Extended Fuzzy $\mathcal{ALCQ}$”. In: *Proceedings of the International Conference on Computational Science and Its Applications (ICCSA 2006), Part II*. Edited by Marina L. Gavrilova, Osvaldo Gervasi, Vipin Kumar, Chih Jeng Kenneth Tan, David Taniar, Antonio Laganà, Youngsong Mun, and Hyunseung Choo. Volume 3981. Lecture Notes in Computer Science. Glasgow, Scotland, UK: Springer-Verlag, 2006. Pages 1179–1188.

[189] Yanhui Lv, Baowen Xu, Jianjiang Lu, and Dazhou Kang. “Reasoning with Fuzzy Ontologies”. In: *Proceedings of the 2nd International Workshop on Contexts and Ontologies: Theory, Practice and Applications (C&O-2006)*. Edited by Pavel Shvaiko, Jérôme Euzenat, Alain Léger, Deborah L. McGuinness, and Holger Wache. Volume 210. Riva del Garda, Italy: CEUR Workshop Proceedings, 2006.

[190] Yanhui Lv, Baowen Xu, Jianjiang Lu, Dazhou Kang, and Peng Wang. “A Family of Extended Fuzzy Description Logics”. In: *Proceedings of the 29th Annual International Computer Software and Applications Con-*

ference (COMPSAC 2005). Volume 1. Edinburgh, Scotland, UK: IEEE Computer Society, 2005. Pages 221–226.

[191] Yanhui Lv, Baowen Xu, Jianjiang Lu, Dazhou Kang, and Peng Wang. "Extended Fuzzy Description Logic $\mathcal{ALCN}$". In: *Proceedings of the 9th International Conference on Knowledge-Based Intelligent Information and Engineering Systems, Part III (KES 2005)*. Edited by Rajiv Khosla, Robert J. Howlett, and Lakhmi C. Jain. Volume 3684. Lecture Notes in Computer Science. Melbourne, Australia: Springer-Verlag, 2005. Pages 896–902.

[192] Yanhui Lv, Baowen Xu, Jianjiang Lu, Dazhou Kang, and Jie Xu. "Reasoning Technique for Extended Fuzzy Description Logics". In: *Proceedings of the 17th IEEE International Conference on Tools with Artificial Intelligence (ICTAI 2005)*. Hong Kong, China: IEEE Computer Society, 2005. Pages 665–666.

[193] Yue Ma, Pascal Hitzler, and Zuoquan Lin. "Algorithms for Paraconsistent Reasoning with OWL". In: *Proceedings of the 4th European Semantic Web Conference (ESWC 2007)*. Edited by Enrico Franconi, Michael Kifer, and Wolfgang May. Innsbruck, Austria 2007. Pages 399–413.

[194] Zongmin M. Ma, Jingwei Cheng, Hailong Wang, and Li Yan. "A Vague Description Logic $\mathcal{SI}$". In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 1345–1349.

[195] Theofilos Mailis, Giorgos Stoilos, and Giorgos Stamou. "Tractable Reasoning Based on the Fuzzy $\mathcal{EL}++$ Algorithm". In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[196] Mauro Mazzieri. "A Fuzzy RDF Semantics to Represent Trust Metadata". In: *Proceedings of the 1st Workshop on Semantic Web Applications and Perspectives (SWAP 2004)*. Ancona, Italy 2004.

[197] Mauro Mazzieri and Aldo Franco Dragoni. "A Fuzzy Semantics for Semantic Web Languages". In: *Proceedings of the 1st ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2005)*. Edited by Paulo Cesar G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, and Michael Pool. Volume 173. Galway, Ireland: CEUR Workshop Proceedings, 2005.

[198] Brian McBride. "Jena: A Semantic Web Toolkit". In: *IEEE Internet Computing* 6.6 (2002). Pages 55–59.

[199] Carlo Meghini, Fabrizio Sebastiani, and Umberto Straccia. "A Model of Multimedia Information Retrieval". In: *Journal of the ACM* 48.5 (2001). Pages 909–970.

[200] Carlo Meghini, Fabrizio Sebastiani, and Umberto Straccia. "Reasoning about the Form and Content for Multimedia Objects (Extended Abstract)". In: *Proceedings of AAAI 1997 Spring Symposium on Intelligent integration and Use of Text, Image, Video and Audio*. Stanford, California, USA 1997. Pages 89–94.

[201] Karl Menger. "Statistical Metrics". In: *Proceedings of the National Academy of Sciences* 28.12 (1942). Pages 535–537.

[202] Ralf Molitor and Christopher B. Tresp. "Extending Description Logics to Vague Knowledge in Medicine". In: *Fuzzy Systems in Medicine*. Edited by P. Szczepaniak, P. J. G. Lisboa, and S. Tsumoto. Volume 41. Studies in Fuzziness and Soft Computing. Springer Verlag, 2000. Pages 617–635.

[203] Paul S. Mostert and Allen L. Shields. "On the Structure of Semigroups on a Compact Manifold with Boundary". In: *Annals of Mathematics* 65.1 (1957). Pages 117–143.

[204] Boris Motik and Ian Horrocks. "Problems with OWL Syntax". In: *Proceedings of the 2nd International Workshop on OWL: Experience and Directions (OWLED 2006)*. Edited by Bernardo Cuenca-Grau, Pascal Hitzler, Conor Shankey, and Evan Wallace. Volume 216. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[205] Boris Motik, Peter F. Patel-Schneider, and Ian Horrocks. *OWL 1.1 Web Ontology Language: Structural Specification and Functional-Style Syntax*. [Online] Available: http://www.w3.org/TR/owl11-syntax/. 2008.

[206] Boris Motik and Ulrike Sattler. "A Comparison of Reasoning Techniques for Querying Large Description Logic ABoxes". In: *Proceedings of the 13th International Conference on Logic for Programming Artificial Intelligence and Reasoning (LPAR 2006)*. Edited by Miki Hermann and Andrei Voronkov. Volume 4246. Lecture Notes in Computer Science. Phnom Penh, Cambodia: Springer-Verlag, 2006. Pages 351–384.

[207] Phivos Mylonas, Thanos Athanasiadis, Manolis Wallace, Yannis Avrithis, and Stefanos Kollias. "Semantic Representation of Multimedia Content - Knowledge Representation and Semantic Indexing". In: *Multimedia Tools and Applications* 39.3 (2008). Pages 293–327.

[208] Phivos Mylonas, David Vallet, Pablo Castells, Miriam Fernández, and Yannis Avrithis. "Personalized Information Retrieval Based on Context and Ontological Knowledge". In: *Knowledge Engineering Review* 23.1 (2008). Pages 73–100.

[209] Gábor Nagypál and Boris Motik. "A Fuzzy Model for Representing Uncertain, Subjective and Vague Temporal Knowledge in Ontologies". In: *Proceedings of the 2nd International Conference on Ontologies, DataBases and Applications of SEmantics (ODBASE 2003)*. Edited by Robert Meersman, Zahir Tari, and Douglas C. Schmidt. Volume 2888. Lecture Notes in Computer Science. Catania, Italy: Springer-Verlag, 2003. Pages 906–923.

[210] Bernhard Nebel. “Computational Complexity of Terminological Reasoning in BACK”. In: *Artificial Intelligence* 34.3 (1988). Pages 371–383.

[211] Bernhard Nebel. “Terminological Reasoning is Inherently Intractable”. In: *Artificial Intelligence* 43.2 (1990). Pages 235–249.

[212] Robert Neches, Richard E. Fikes, Timothy Finin, Thomas R. Gruber, Ramesh Patil, Ted E. Senator, and William R. Swartout. “Enabling Technology for Knowledge Sharing”. In: *AI Magazine* 12.3 (1991). Pages 36–56.

[213] Masoud Nikravesh, Tomohiro Takagi, Martine De Cock, and Sergio Guadarrama. “The Automatic Construction of Fuzzy Ontologies for Search in the WWW and the Use of Fuzzy Spiders for the Detection of Objects in Images”. In: *2004 EECS Research Summary*. 2004. Pages 135–139.

[214] Nils J. Nilsson. “Probabilistic Logic”. In: *Artificial Intelligence* 28.1 (1986). Pages 71–87.

[215] Vít Novácek and Pavel Smrz. “Empirical Merging of Ontologies - A Proposal of Universal Uncertainty Representation Framework”. In: *Proceedings of the 3rd European Semantic Web Conference (ESWC 2006)*. Edited by York Sure and John Domingue. Volume 4011. Lecture Notes in Computer Science. Budva, Montenegro: Springer-Verlag, 2006. Pages 65–79.

[216] Vilém Novák. “Which Logic is the Real Fuzzy Logic?”. In: *Fuzzy Sets and Systems* 157.5 (2006). Pages 635–641.

[217] José A. Olivas, Pablo J. Garcés, and Francisco P. Romero. “An Application of the FIS-CRM Model to the FISS Metasearcher: Using Fuzzy Synonymy and Fuzzy Generality for Representing Concepts in Documents”. In: *International Journal of Approximate Reasoning* 34.2–3 (2003). Pages 201–219.

[218] José A. Olivas, Pablo J. Garcés, and Francisco P. Romero. "FISS: Application of Fuzzy Technologies to an Internet Metasearcher". In: *Proceedings of the 21st International Conference of the North American Fuzzy Information Processing Society (NAFIPS 2002)*. New Orleans, LA, USA 2002. Pages 140–145.

[219] Dmitry E. Pal'chunov. "Logical Definition of Object Domain Ontology". In: *Proceedings of the 9th Asian Logic Conference*. Novosibirsk, Russia: World Scientific, 2005.

[220] Jeff Z. Pan and Ian Horrocks. "OWL-Eu: Adding Customised Datatypes into OWL". In: *Journal of Web Semantics* 4.1 (2006). Pages 29–39.

[221] Jeff Z. Pan, Giorgos Stamou, Giorgos Stoilos, and Edward Thomas. "Expressive Querying over Fuzzy DL-Lite Ontologies". In: *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*. Edited by Diego Calvanese, Enrico Franconi, Volker Haarslev, Domenico Lembo, Boris Motik, Sergio Tessaris, and Anni-Yasmin Turhan. Volume 250. Brixen-Bressanone, Italy: CEUR Workshop Proceedings, 2007. Pages 427–434.

[222] Jeff Z. Pan, Giorgos Stamou, Giorgos Stoilos, Edward Thomas, and Stuart Taylor. "Scalable Querying Service over Fuzzy Ontologies". In: *Proceedings of the 17th International World Wide Web Conference (WWW2008)*. Beijing, China 2008.

[223] Jeff Z. Pan, Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, and Ian Horrocks. "f-SWRL: A Fuzzy Extension of SWRL". In: *Journal of Data Semantics* VI (Special issue on Emergent Semantics 2006). Pages 28–46.

[224] Jeff Z. Pan, Edward Thomas, and Derek Sleeman. "ONTOSEARCH2: Searching and Querying Web Ontologies". In: *Proceeedings of the IADIS International Conference WWW/Internet 2006*. Murcia, Spain 2006.

[225] Christos H. Papadimitriou. *Computational Complexity*. Addison Wesley, 1994.

[226] Georgios Th. Papadopoulos, Phivos Mylonas, Vasileios Mezaris, Yannis Avrithis, and Ioannis Kompatsiaris. “Knowledge-Assisted Image Analysis Based on Context and Spatial Optimization”. In: *International Journal on Semantic Web & Information Systems* 2.3 (2006). Pages 17–36.

[227] David Parry. “A Fuzzy Ontology for Medical Document Retrieval”. In: *Proceedings of the 2nd Australasian Workshop on Data Mining and Web Intelligence (DMWI2004)*. Edited by Martin Purvis. Volume 32. Dunedin, New Zealand: Australian Computer Society, Inc., 2004. Pages 121–126.

[228] David Parry. “Evaluation of a Fuzzy Ontology-Based Medical Information System”. In: *International Journal of Health Information Systems and Informatics* 1.1 (2006). Pages 40–49.

[229] David Parry. “Fuzzification of a Standard Ontology to Encourage Reuse”. In: *Proceedings of the 2004 IEEE International Conference on Information Reuse and Integration (IEEE IRI-2004)*. Las Vegas, NE, USA 2004. Pages 582–587.

[230] David Parry. “Fuzzy Ontologies for Information Retrieval on the WWW”. In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 21–48.

[231] David Parry. “Fuzzy Ontology for Medical Document Retrieval”. In: *Proceedings of the 11th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU 2006)*. Paris, France 2006. Pages 2276–2283.

[232] David Parry. “Tell me the Important Stuff - Fuzzy Ontologies and Personal Assessments for Interaction with the Mobile Semantic Web”. In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems*

(FUZZ-IEEE 2008). Hong Kong, China: IEEE Computer Society, 2008. Pages 1295–1300.

[233] Bijan Parsia and Peter F. Patel-Schneider. *OWL 2 Web Ontology Language: Primer*. [Online] Available: `http://www.w3.org/TR/owl2-primer/`. 2008.

[234] Peter F. Patel-Schneider and Bill Swartout. *Description-Logic Knowledge Representation System Specification from the KRSS Group of the ARPA Knowledge Sharing Effort*. Technical report. DARPA Knowledge Representation System Specification (KRSS) Group of the Knowledge Sharing Initiative, 1993.

[235] Jan Pavelka. "On Fuzzy Logic I, II, III". In: *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik* 25 (1979). Pages 45–52, 119–134, 447–464.

[236] Rachel Pereira, Ivan Ricarte, and Fernando Gomide. "Fuzzy Relational Ontological Model in Information Search Systems". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 395–412.

[237] Jorge Pérez, Marcelo Arenas, and Claudio Gutiérrez. "The Semantics and Complexity of SPARQL". In: *Proceedings of the 5th International Semantic Web Conference (ISWC 2006)*. Volume 4273. Lecture Notes in Computer Science. Athens, GA, USA: Springer-Verlag, 2006. Pages 30–43.

[238] Guilin Qi and Jeff Z. Pan. "Tableau Algorithm for Possibilistic Description Logic $\mathcal{ALC}$". In: *Proceedings of the 2nd International Conference on Web Reasoning and Rule Systems (RR 2008)*. Edited by Diego Calvanese and Georg Lausen. Volume 5341. Lecture Notes in Computer Science. Karlsruhe, Germany: Springer-Verlag, 2008. Pages 238–239.

[239] Guilin Qi, Jeff Z. Pan, and Qiu Ji. "A Possibilistic Extension of Description Logics". In: *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*. Edited by Diego Calvanese, Enrico

Franconi, Volker Haarslev, Domenico Lembo, Boris Motik, Sergio Tessaris, and Anni-Yasmin Turhan. Volume 250. Brixen-Bressanone, Italy: CEUR Workshop Proceedings, 2007. Pages 435–442.

[240] Guilin Qi, Jeff Z. Pan, and Qiu Ji. "A Possibilistic Extension of Description Logics". In: *Proceedings of the 9th European Conferences on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2007)*. Edited by Khaled Mellouli. Volume 4724. Lecture Notes in Computer Science. Hammamet, Tunisia: Springer-Verlag, 2007. Pages 828–839.

[241] Thanh Tho Quan, Siu Cheung Hui, and Alvis Cheuk M. Fong. "Automatic Fuzzy Ontology Generation for Semantic Help-Desk Support". In: *IEEE Transactions on Industrial Informatics* 2.3 (2006). Pages 155–164.

[242] Thanh Tho Quan, Siu Cheung Hui, Alvis Cheuk M. Fong, and Tru Hoang Cao. "Automatic Fuzzy Ontology Generation for Semantic Web". In: *IEEE Transactions on Knowledge and Data Engineering* 18.6 (2006). Pages 842–856.

[243] Azzurra Ragone, Umberto Straccia, Fernando Bobillo, Tommaso Di Noia, and Eugenio Di Sciascio. "Fuzzy Bilateral Matchmaking in e-Marketplaces". In: *Proceedings of the 12th International Conference on Knowledge-Based and Intelligent Information & Engineering Systems (KES 2008), Part III*. Volume 5179. Lecture Notes in Computer Science. Springer Verlag, 2008. Pages 293–301.

[244] Azzurra Ragone, Umberto Straccia, Fernando Bobillo, Tommaso Di Noia, and Eugenio Di Sciascio. "Fuzzy Description Logics for Bilateral Matchmaking in e-Marketplaces". In: *Proceedings of the 21st International Workshop on Description Logics (DL 2008)*. Volume 353. Dresden, Germany: CEUR Workshop Proceedings, 2008.

[245] Azzurra Ragone, Umberto Straccia, Fernando Bobillo, Tommaso Di Noia, Eugenio Di Sciascio, and Francesco M. Donini. "A Fuzzy Description Logic Approach to Bilateral Matchmaking in Electronic Mar-

ketplaces". In: *Proceedings of the 16th Italian Symposium on Advanced Database Systems (SEBD 2008)*. Mondello, Palermo, Italy 2008.

[246] Azzurra Ragone, Umberto Straccia, Tommaso Di Noia, Eugenio Di Sciascio, and Francesco M. Donini. "Vague Knowledge Bases for Matchmaking in P2P E-Marketplaces". In: *Proceedings of the 4th European Semantic Web Conference (ESWC 2007)*. Edited by Enrico Franconi, Michael Kifer, and Wolfgang May. Innsbruck, Austria 2007. Pages 414–428.

[247] Marek Reformat and Cuong Ly. "Ontological Approach to Development of Computing with Words Based Systems". In: *International Journal of Approximate Reasoning* (To appear).

[248] Sebastian Rudolph, Tuvshintur Tserendorj, and Pascal Hitzler. "What Is Approximate Reasoning?". In: *Proceedings of the 2nd International Conference on Web Reasoning and Rule Systems (RR 2008)*. Edited by Diego Calvanese and Georg Lausen. Volume 5341. Lecture Notes in Computer Science. Karlsruhe, Germany: Springer-Verlag, 2008. Pages 150–164.

[249] Harvey M. Salkin and Kamlesh Mathur. *Foundations of Integer Programming*. North-Holland, 1989.

[250] Daniel Sánchez and Andrea G. B. Tettamanzi. "Fuzzy Quantification in Fuzzy Description Logics". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 135–159.

[251] Daniel Sánchez and Andrea G. B. Tettamanzi. "Generalizing Quantification in Fuzzy Description Logics". In: *Proceedings of the 8th Fuzzy Days in Dortmund*. Edited by Bernd Reusch. Volume 33. Advances in Soft Computing. Dortmund, Germany: Springer-Verlag, 2005.

[252] Daniel Sánchez and Andrea G. B. Tettamanzi. "Reasoning and Quantification in Fuzzy Description Logics". In: *Proceedings of the 6th International Workshop on Fuzzy Logic and Applications (WILF 2005)*. Edited by Isabelle Bloch, Alfredo Petrosino, and Andrea G. B. Tet-

tamanzi. Volume 3849. Lecture Notes in Computer Science. Crema, Italy: Springer-Verlag, 2006. Pages 81–88.

[253] Elie Sanchez, editor. *Fuzzy Logic and the Semantic Web*. Volume 1. Capturing Intelligence. Elsevier Science, 2006.

[254] Elie Sanchez and Takahiro Yamanoi. "Fuzzy Logic and the Semantic Web". In: *Proceedings of the 10th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2004)*. Perugia, Italy: Editrice Università La Sapienza, Italy, 2004. Pages 1033–1038.

[255] Elie Sanchez and Takahiro Yamanoi. "Fuzzy Ontologies for the Semantic Web". In: *Proceedings of the 7th International Conference on Flexible Query Answering Systems (FQAS 2006)*. Edited by Henrik Legind Larsen, Gabriella Pasi, Daniel Ortiz Arroyo, Troels Andreasen, and Henning Christiansen. Volume 4027. Lecture Notes in Computer Science. Milan, Italy: Springer-Verlag, 2006. Pages 691–699.

[256] Thomas Sandholm. *The Philosophy of the Grid: Ontology Theory - From Aristotle to Self-Managed IT Resources*. Technical report TRITA-NA-0532. Stockholm, Sweden: Royal Institute of Technology, 2005.

[257] Ulrike Sattler. "A Concept Language Extended with Different Kinds of Transitive Roles". In: *Proceedings of the 20th Annual German Conference on Artificial Intelligence (KI-96)*. Edited by Günther Görz and Steffen Hölldobler. Volume 1137. Lecture Notes in Computer Science. Dresden, Germany: Springer-Verlag, 1996. Pages 333–345.

[258] Andrea Schaerf. "Reasoning with Individuals in Concept Languages". In: *Data & Knowledge Engineering* 13.2 (1994). Pages 141–176.

[259] Klaus Schild. "A Correspondence Theory for Terminological Logics: Preliminary Report". In: *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991)*. Sidney, Australia 1991. Pages 466–471.

[260] Stefan Schlobach, Michel C. A. Klein, and Linda Peelen. "Description Logics with Approximate Definitions - Precise Modeling of Vague Concepts". In: *Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI 2007)*. Edited by Manuela M. Veloso. Hyderabad, India 2007. Pages 557–562.

[261] Manfred Schmidt-Schauß and Gert Smolka. "Attributive Concept Descriptions with Complements". In: *Artificial Intelligence* 48.1 (1991). Pages 1–26.

[262] Steven Schockaert, Patricia Victor, Geert-Jan Houben, Chris Cornelis, Martine De Cock, and Etienne Kerre. "Reflections on Modelling Vagueness in Description Logics". In: *Proceedings of the 2nd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2006)*. Edited by Paulo C. G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, Francis Fung, and Michael Pool. Volume 218. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[263] Ebru Sezer, Adnan Yazıcı, and Ünal Yarımağan. "Semantic Information Retrieval on the Web". In: *Proceedings of the 4th International Conference on Advances in Information Systems (ADVIS 2006)*. Edited by Tatyana M. Yakhno and Erich J. Neuhold. Volume 4243. Lecture Notes in Computer Science. Izmir, Turkey: Springer-Verlag, 2006. Pages 158–167.

[264] George L. S. Shackle. *Decision, Order and Time in Human Affairs*. Cambridge University Press, 1961.

[265] Rob Shearer, Boris Motik, and Ian Horrocks. "HermiT: A Highly-Efficient OWL Reasoner". In: *Proceedings of the 5th International Workshop on OWL: Experiences and Directions (OWLED 2008)*. Karlsruhe, Germany 2008.

[266] Lian Shi, Jigui Sun, Shuai Lu, and Minghao Yin. "Flexible Planning Using Fuzzy Description Logics: Theory and Application". In: *Applied Soft Computing* 9.1 (2009). Pages 142–148.

[267] Miguel Ángel Sicilia. "On Some Problems of Decision–Making Under Uncertainty in the Semantic Web". In: *Soft Computing in Ontologies and Semantic Web*. Volume 204. Studies in Fuzziness and Soft Computing. Springer-Verlag, 2005. Pages 231–246.

[268] Rita Maria da Silva, Antonio Eduardo C. Pereira, and Marcio Andrade Netto. "A System of Knowledge Representation Based in Formulae of Predicate Calculus Whose Variables Are Annotated by Expressions of a "Fuzzy" Terminological Logic". In: *Proceedings of the 5th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU-94)*. Edited by Bernadette Bouchon-Meunier, Ronald R. Yager, and Lotfi A. Zadeh. Volume 945. Lecture Notes in Computer Science. Paris, France: Springer-Verlag, 1994. Pages 409–417.

[269] Nikolaos Simou, Thanos Athanasiadis, and Stefanos Kollias. "An Architecture for Multimedia Analysis and Retrieval Based on Fuzzy Description Logics". In: *Proceedings of the 2nd K-Space PhD Students Workshop*. Volume 379. Paris, France: CEUR Workshop Proceedings, 2008.

[270] Nikolaos Simou, Thanos Athanasiadis, Giorgos Stoilos, and Stefanos Kollias. "Image Indexing and Retrieval using Expressive Fuzzy Description Logics". In: *Signal, Image and Video Processing* (To appear).

[271] Nikolaos Simou, Thanos Athanasiadis, Vassilis Tzouvaras, and Stefanos Kollias. "Multimedia Reasoning with f-$\mathcal{SHIN}$". In: *Proceedings of the 2nd International Workshop on Semantic Media Adaptation and Personalization (SMAP 2007)*. London, UK 2007.

[272] Nikolaos Simou and Stefanos Kollias. "FiRE: A Fuzzy Reasoning Engine for Imprecise Knowledge". In: *Proceedings of the 1st K-Space PhD Students Workshop*. Berlin, Germany 2007.

[273] Nikolaos Simou, Giorgos Stoilos, and Giorgos Stamou. "Storing and Querying Fuzzy Knowledge in the Semantic Web". In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Se-*

mantic Web (URSW 2008). Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[274] Shailendra Singh, Lipika Dey, and Muhammad Abulaish. "A Framework for Extending Fuzzy Description Logic to Ontology Based Document Processing". In: *Proceedings of the 2nd International Atlantic Web Intelligence Conference (AWIC 2004)*. Edited by Jesús Favela, Ernestina Menasalvas Ruiz, and Edgar Chávez. Volume 3034. Lecture Notes in Computer Science. Cancun, Mexico: Springer-Verlag, 2004. Pages 95–104.

[275] Evren Sirin. *Discussion on OWL Mailing List*. [Online] Available: http://lists.w3.org/Archives/Public/public-owl-dev/2007JanMar/0195.html. 2005.

[276] Evren Sirin, Bernardo Cuenca-Grau, and Bijan Parsia. "From Wine to Water: Optimizing Description Logic Reasoning for Nominals". In: *Proceedings of the 10th International Conference of Knowledge Representation and Reasoning (KR-2006)*. Lake District, UK 2006. Pages 90–99.

[277] Evren Sirin, Bijan Parsia, Bernardo Cuenca-Grau, Aditya Kalyanpur, and Yarden Katz. "Pellet: A Practical OWL-DL Reasoner". In: *Journal of Web Semantics* 5.2 (2007). Pages 51–53.

[278] John F. Sowa. *Conceptual Structures: Information Processing in Mind and Machine*. Systems Programming. Addison Wesley, 1984.

[279] Steffen Staab and Rudi Studer, editors. *Handbook on Ontologies*. International Handbooks on Information Systems. Springer, 2004. ISBN: 3-540-40834-7.

[280] Giorgos Stoilos, Nikolaos Simou, Giorgos Stamou, and Stefanos Kollias. "Uncertainty and the Semantic Web". In: *IEEE Intelligent Systems* 21.5 (2006). Pages 84–87.

[281] Giorgos Stoilos and Giorgos Stamou. "Extending Fuzzy Description Logics for the Semantic Web". In: *Proceedings of the 3rd International Workshop on OWL: Experiences and Directions (OWLED 2007)*. Volume 258. Innsbruck, Austria: CEUR Workshop Proceedings, 2007.

[282] Giorgos Stoilos, Giorgos Stamou, and Stefanos Kollias. "Reasoning with Qualified Cardinality Restrictions in Fuzzy Description Logics". In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 637–644.

[283] Giorgos Stoilos, Giorgos Stamou, and Jeff Z. Pan. "Classifying Fuzzy Subsumption in Fuzzy-$\mathcal{EL}+$". In: *Proceedings of the 21st International Workshop on Description Logics (DL 2008)*. Volume 353. Dresden, Germany: CEUR Workshop Proceedings, 2008.

[284] Giorgos Stoilos, Giorgos Stamou, and Jeff Z. Pan. "Handling Imprecise Knowledge with Fuzzy Description Logic". In: *Proceedings of the 2006 International Workshop on Description Logics (DL 2006)*. Lake District, UK 2006. Pages 119–126.

[285] Giorgos Stoilos, Giorgos Stamou, Jeff Z. Pan, Vassilis Tzouvaras, and Ian Horrocks. "Reasoning with Very Expressive Fuzzy Description Logics". In: *Journal of Artificial Intelligence Research* 30.8 (2007). Pages 273–320.

[286] Giorgos Stoilos, Giorgos Stamou, Rob Searer, Ian Horrocks, Jeff Z. Pan, Alex Polleres, Holger Lausen, Mustafa Jarrar, Carlos F. Enguix, Holger Wache, and York Sure. *Analysis of Requirements for Further Language Extensions*. Technical report D2.5.4. Knowledge Web Technical Report, 2007.

[287] Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, Jeff Z. Pan, and Ian Horrocks. "A Fuzzy Description Logic for Multimedia Knowledge Representation". In: *Proceedings of the International Workshop on Multimedia and the Semantic Web*. Heraklion, Greece 2005. Pages 12–19.

[288] Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, Jeff Z. Pan, and Ian Horrocks. "Fuzzy OWL: Uncertainty and the Semantic Web". In: *Proceedings of the 1st International Workshop on OWL: Experience and Directions (OWLED 2005)*. Volume 188. Galway, Ireland: CEUR Workshop Proceedings, 2005.

[289] Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, Jeff Z. Pan, and Ian Horrocks. "The Fuzzy Description Logic $f\text{-}\mathcal{SHIN}$". In: *Proceedings of the 1st ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2005)*. Edited by Paulo Cesar G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, and Michael Pool. Volume 173. Galway, Ireland: CEUR Workshop Proceedings, 2005. Pages 67–76.

[290] Giorgos Stoilos, Umberto Straccia, Giorgos Stamou, and Jeff Z. Pan. "General Concept Inclusions in Fuzzy Description Logics". In: *Proceedings of the 17th European Conference on Artificial Intelligence (ECAI 2006)*. Riva del Garda, Italy 2006. Pages 457–461.

[291] Umberto Straccia. "A Fuzzy Description Logic for the Semantic Web". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 73–90.

[292] Umberto Straccia. "A Fuzzy Description Logic". In: *Proceedings of the 15th National Conference on Artificial Intelligence (AAAI 1998)*. Madison, Wisconsin, USA 1998. Pages 594–599.

[293] Umberto Straccia. "Answering Vague Queries in Fuzzy DL-Lite". In: *Proceedings of the 11th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU 2006)*. Paris, France 2006. Pages 2238–2245.

[294] Umberto Straccia. "Description Logics over Lattices". In: *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 14.1 (2006). Pages 1–16.

[295] Umberto Straccia. "Description Logics with Fuzzy Concrete Domains". In: *Proceedings of the 21st Conference on Uncertainty in Ar-*

tificial Intelligence (UAI 2005). Edited by Fahiem Bachus and Tommi Jaakkola. Edinburgh, Scotland, UK: AUAI Press, 2005.

[296] Umberto Straccia. “Fuzzy $\mathcal{ALC}$ with Fuzzy Concrete Domains”. In: *Proceedings of the International Workshop on Description Logics (DL 2005)*. Edinburgh, Scotland, UK 2005.

[297] Umberto Straccia. “Managing Uncertainty and Vagueness in Description Logics, Logic Programs and Description Logic Programs”. In: *Proceedings of the 4th International Summer School Reasoning Web 2008*. Edited by Cristina Baroglio, Piero A. Bonatti, Jan Maluszynski, Massimo Marchiori, Axel Polleres, and Sebastian Schaffert. Volume 5224. Lecture Notes in Computer Science. Venice, Italy: Springer-Verlag, 2008. Pages 54–103.

[298] Umberto Straccia. *Reasoning in Ł-$\mathcal{SHIF}$: an Expressive Fuzzy Description Logic under Łukasiewicz Semantics*. Technical report ISTI-018/2007. ISTI-CNR, 2007.

[299] Umberto Straccia. “Reasoning within Fuzzy Description Logics”. In: *Journal of Artificial Intelligence Research* 14 (2001). Pages 137–166.

[300] Umberto Straccia. “Towards a Fuzzy Description Logic for the Semantic Web (Preliminary Report)”. In: *Proceedings of the Second European Semantic Web Conference (ESWC 2005)*. Edited by Asunción Gómez-Pérez and Jérôme Euzenat. Volume 3532. Lecture Notes in Computer Science. Crete, Greece: Springer-Verlag, 2005. Pages 167–181.

[301] Umberto Straccia. “Towards Top-k Query Answering in Description Logics: the case of DL-Lite”. In: *Proceedings of the 10th European Conference on Logics in Artificial Intelligence (JELIA 2006)*. Edited by Michael Fisher, Wiebe van der Hoek, Boris Konev, and Alexei Lisitsa. Volume 4160. Lecture Notes in Computer Science. Liverpool, UK: Springer Verlag, 2006. Pages 439–451.

[302] Umberto Straccia. "Transforming Fuzzy Description Logics into Classical Description Logics". In: *Proceedings of the 9th European Conference on Logics in Artificial Intelligence (JELIA 2004)*. Edited by José Júlio Alferes and João Alexandre Leite. Volume 3229. Lecture Notes in Computer Science. Lisbon, Portugal: Springer-Verlag, 2004. Pages 385–399.

[303] Umberto Straccia. "Uncertainty in Description Logics: a Lattice-based Approach". In: *Proceedings of the 10th International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU 2004)*. Perugia, Italy: Editrice Università La Sapienza, Italy, 2004. Pages 251–258.

[304] Umberto Straccia and Fernando Bobillo. "Mixed Integer Programming, General Concept Inclusions and Fuzzy Description Logics". In: *Proceedings of the 5th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2007)*. Edited by Martin Štěpnička, Vilém Novák, and Ulrich Bodenhofer. Volume 2. Ostrava, Czech Republic 2007. Pages 213–220.

[305] Umberto Straccia and Fernando Bobillo. "Mixed Integer Programming, General Concept Inclusions and Fuzzy Description Logics". In: *Mathware & Soft Computing* 14.3 (2007). Pages 247–259.

[306] Umberto Straccia and Giulio Visco. "DL-Media: an Ontology Mediated Multimedia Information Retrieval System". In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d'Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[307] Umberto Straccia and Giulio Visco. "DLMedia: an Ontology Mediated Multimedia Information Retrieval System". In: *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*. Edited by Diego Calvanese, Enrico Franconi, Volker Haarslev, Domenico

Lembo, Boris Motik, Sergio Tessaris, and Anni-Yasmin Turhan. Volume 250. Brixen-Bressanone, Italy: CEUR Workshop Proceedings, 2007. Pages 467–474.

[308] Rudi Studer, V. Richard Benjamins, and Dieter Fensel. "Knowledge Engineering: Principles and Methods". In: *Data & Knowledge Engineering* 25.1-2 (1998). Pages 161–197.

[309] György Surján. "Non Aristotelian Categories in Medicine". In: *Studies in Health Technology and Informatics* 124 (2006). Pages 735–740.

[310] Tomohiro Takagi, Atsushi Imura, Hirohide Ushida, and Toru Yamaguchi. "Conceptual Fuzzy Sets as a Meaning Representation and their Inductive Construction". In: *International Journal of Intelligent Systems* 10.11 (1995). Pages 929–945.

[311] Tomohiro Takagi and Kazushi Kawase. "A Trial for Data Retrieval Using Conceptual Fuzzy Sets". In: *IEEE Transactions on Fuzzy Sistems* 9.4 (2001). Pages 497–505.

[312] Christopher Thomas and Amit Sheth. "On the Expressiveness of the Languages for the Semantic Web - Making a Case for 'A Little More'". In: *Fuzzy Logic and the Semantic Web*. Edited by Elie Sanchez. Volume 1. Capturing Intelligence. Elsevier Science, 2006. Pages 3–20.

[313] Edward Thomas, Jeff Z. Pan, and Derek Sleeman. "ONTOSEARCH2: Searching Ontologies Semantically". In: *Proceedings of the 3rd International Workshop on OWL: Experiences and Directions (OWLED 2007)*. Volume 258. Innsbruck, Austria: CEUR Workshop Proceedings, 2007.

[314] Christopher B. Tresp and Ralf Molitor. "A Description Logic for Vague Knowledge". In: *Proceedings of the 7th European Conference of Artificial Intelligence (ECAI 1998)*. Brighton, England 1998.

[315] Enric Trillas. "On Negation Functions in Fuzzy Set Theory". In: *Advances of Fuzzy Logic*. Edited by Senén Barro, Sobrino José Antonio Vila Sobrino, and Alberto Bugarín. English version of the Spanish original of 1979. Universidade de Santiago de Compostela, 1998. Pages 31–40.

[316] Dmitry Tsarkov and Ian Horrocks. “FaCT++ Description Logic Reasoner: System Description”. In: *Proceedings of the 3rd International Joint Conference on Automated Reasoning (IJCAR 2006)*. Seattle, Washington, USA 2006.

[317] Dmitry Tsarkov, Ian Horrocks, and Peter F. Patel-Schneider. “Optimizing Terminological Reasoning for Expressive Description Logics”. In: *Journal of Automated Reasoning* 39.3 (2007). Pages 277–316.

[318] Anni-Yasmin Turhan, Sean Bechhofer, Alissa Kaplunova, Thorsten Liebig, Marko Luther, Ralf Möller, Olaf Noppens, Peter Patel-Schneider, Boontawee Suntisrivaraporn, and Timo Weithöner. “DIG 2.0 - Towards a Flexible Interface for Description Logic Reasoners”. In: *Proceedings of the 2nd International Workshop on OWL: Experience and Directions (OWLED 2006)*. Edited by Bernardo Cuenca-Grau, Pascal Hitzler, Conor Shankey, and Evan Wallace. Volume 216. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[319] Miroslav Vacura, Vojtěch Svátek, and Pavel Smrž. “A Pattern-based Framework for Representation of Uncertainty in Ontologies”. In: *Proceedings of the 11th International Conference on Text, Speech, and Dialogue (TSD 2008)*. Brno, Czech Republic 2008. Pages 1–8.

[320] Miroslav Vacura, Vojtěch Svátek, Pavel Smrž, and Nikos Simou. “A Pattern-based Framework for Representation of Uncertainty in Ontologies”. In: *Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d’Amato, Nicola Fanizzi, Francis Fung, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Yun Peng, Michael Pool, Pavel Smrž, and Peter Vojtáš. Volume 327. Busan, South Korea: CEUR Workshop Proceedings, 2007.

[321] Veronika Vaneková, Ján Bella, Peter Gurský, and Tomáš Horváth. “Fuzzy RDF in the Semantic Web: Deduction and Induction”. In: *Proceedings of the 6th Workshop on Data Analysis (WDA 2005)*. Abaujszanto, Hungary 2005. Pages 16–29.

[322] Peter Vojtáš. “A Description Logic Modeling Querying Web and Learning Imperfect User Preferences”. In: *Proceedings of the 2nd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2006)*. Edited by Paulo C. G. da Costa, Kathryn B. Laskey, Kenneth J. Laskey, Francis Fung, and Michael Pool. Volume 218. Athens, GA, USA: CEUR Workshop Proceedings, 2006.

[323] Peter Vojtáš. “A Description Logic with Combination of User Preference Concepts, Crisp Roles and Existential Restrictions”. In: *Proceedings of the 16th European-Japanese Conference on Information Modelling and Knowledge Bases (EJC 2006)*. Trojanovice, Czech Republic 2006. Pages 176–183.

[324] Peter Vojtáš. “A Fuzzy $\mathcal{EL}$ Description Logic with Crisp Roles and Fuzzy Aggregation for Web Consulting”. In: *Proceedings of the 11th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems (IPMU 2006)*. Paris, France 2006. Pages 1834–1841.

[325] Peter Vojtáš. “$\mathcal{EL}$ Description Logic with Aggregation of User Preference Concepts”. In: *Information modelling and Knowledge Bases*. Edited by Marie Duží, Hannu Jaakkola, Yasushi Kiyoki, and Hannu Kangassalo. Volume XVIII. IOS Press, 2007. Pages 154–165.

[326] Peter Vojtáš and Alan Eckhardt. “Towards Ontology Language Handling Imperfection”. In: *1st Workshop on Intelligent and Knowledge oriented Technologies (WIKT 2006)*. Edited by Michal Laclavík, Ivana Budinská, and Ladislav Hluchý. Bratislava, Slovakia 2006. Pages 124–125.

[327] W3C. *OWL Web Ontology Language Overview*. Edited by Deborah L. McGuinness and Frank van Harmelen. [Online] Available: http://www.w3.org/TR/owl-features/. 2004.

[328] W3C. *OWL Web Ontology Language Semantics and Abstract Syntax*. Edited by Peter F. Patel-Schneider, Patrick Hayes, and Ian Horrocks.

[Online] Available: http://www.w3.org/TR/owl-semantics/. 2004.

[329] W3C. *RDF Primer*. Edited by Frank Manola and Eric Miller. [Online] Available: http://www.w3.org/TR/rdf-primer/. 2004.

[330] W3C. *RDF Vocabulary Description Language 1.0: RDF Schema*. Edited by Dan Brickley and R.V. Guha. [Online] Available: http://www.w3.org/TR/rdf-schema/. 2004.

[331] W3C. *Semantic Web Case Studies and Use Cases*. [Online] Available: http://www.w3.org/2001/sw/sweo/public/UseCases/. 2008.

[332] Manolis Wallace and Yannis Avrithis. "Fuzzy Relational Knowledge Representation and Context in the Service of Semantic Information Retrieval". In: *Proceedings of the 2004 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2004)*. Volume 1. Budapest, Hungary: IEEE Press, 2004. Pages 1397–1402.

[333] Manolis Wallace, Phivos Mylonas, Giorgos Akrivas, Yannis Avrithis, and Stefanos Kollias. "Automatic Thematic Categorization of Multimedia Documents using Ontological Information and Fuzzy Algebra". In: *Soft Computing in Ontologies and Semantic Web*. Volume 204. Studies in Fuzziness and Soft Computing. Springer-Verlag, 2005. Pages 247–274.

[334] Hailong Wang, Zongmin M. Ma, Li Yan, and Jingwei Cheng. "A Fuzzy Description Logic with Fuzzy Data Type Group". In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 1534–1541.

[335] Xing Wang, Zongmin M. Ma, Li Yan, and Xiangfu Meng. "Vague SWRL: A Fuzzy Extension of SWRL". In: *Proceedings of the 2nd International Conference on Web Reasoning and Rule Systems (RR 2008)*. Edited by Diego Calvanese and Georg Lausen. Volume 5341. Lecture Notes in Computer Science. Karlsruhe, Germany: Springer-Verlag, 2008. Pages 232–233.

[336] Dwi H. Widyantoro and John Yen. “A Fuzzy Ontology-based Abstract Search Engine and Its User Studies”. In: *Proceedings of the 10th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2001)*. Volume 3. Melbourne, Australia 2001. Pages 1291–1294.

[337] Dwi H. Widyantoro and John Yen. “Using Fuzzy Ontology for Query Refinement in a Personalized Abstract Search Engine”. In: *Proceedings of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference (IFSA-NAFIPS 2001)*. Volume 1. Vancouver, British Columbia, Canada 2001. Pages 610–615.

[338] Niklaus Wirth. “What Can We Do About the Unnecessary Diversity of Notation for Syntactic Definitions?”. In: *Communications of the ACM* 20.11 (1977). Pages 822–823.

[339] Baowen Xu, Dazhou Kang, Jianjiang Lu, Yanhui Lv, and Jixiang Jiang. “Mapping Fuzzy Concepts Between Fuzzy Ontologies”. In: *Proceedings of the 9th International Conference on Knowledge-Based Intelligent Information and Engineering Systems, Part IV (KES 2005)*. Edited by Rajiv Khosla, Robert J. Howlett, and Lakhmi C. Jain. Volume 3684. Lecture Notes in Computer Science. Melbourne, Australia: Springer-Verlag, 2005. Pages 199–205.

[340] John Yen. “Generalizing Term Subsumption Languages to Fuzzy Logic”. In: *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991)*. Sidney, Australia 1991. Pages 472–477.

[341] Chingman Au Yeung and Hofung Leung. “Formalizing Typicality of Objects and Context-sensitivity in Ontologies”. In: *Proceedings of the 5th International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2006)*. Hakodate, Japan 2006. Pages 946–948.

[342] Lofti A. Zadeh. “Fuzzy Sets as a Basis for a Theory of Possibility”. In: *Fuzzy Sets and Systems* 1 (1978). Pages 3–28.

[343] Lofti A. Zadeh. “Fuzzy Sets”. In: *Information and Control* 8 (1965). Pages 338–353.

[344] Lofti A. Zadeh. “Outline of a New Approach to the Analysis of Complex Systems and Decision Processes”. In: *IEEE Transactions on Systems, Man and Cybernetics* 3.1 (1973). Pages 28–44.

[345] Lofti A. Zadeh. “Similarity Relations and Fuzzy Orderings”. In: *Information Sciences* 3.2 (1971). Pages 177–200.

[346] Jun Zhai, Lixin Shen, Zhou Zhou, and Yan Liang. “Fuzzy Ontology Model for Knowledge Management”. In: *Proceedings of the 2007 International Conference Proceedings on Intelligent Systems and Knowledge Engineering (ISKE 2007)*. Edited by Tianrui Li, Yang Xu, and Da Ruan. Volume 3. Advances in Intelligent Systems Research. Chengdu, China: Atlantic Press, 2007.

[347] Changli Zhang, Jian Wu, and Zhengguo Hu. “A Boolean Lattice Based Fuzzy Description Logic in Web Computing”. In: *Proceedings of the 20th International Workshop on Description Logics (DL 2007)*. Edited by Diego Calvanese, Enrico Franconi, Volker Haarslev, Domenico Lembo, Boris Motik, Sergio Tessaris, and Anni-Yasmin Turhan. Volume 250. Brixen-Bressanone, Italy: CEUR Workshop Proceedings, 2007. Pages 499–506.

[348] Fu Zhang, Zongmin M. Ma, Yanhui Lv, and Xing Wang. “Formal Semantics-Preserving Translation from Fuzzy ER Model to Fuzzy OWL DL Ontology”. In: *Proceedings of the 2008 IEEE/WIC/ACM International Conferences on Web Intelligence and Intelligent Agent Technology (WI-IAT 2008)*. Sydney, Australia 2008.

[349] Fu Zhang, Zongmin M. Ma, and Li Yan. “Representation and Reasoning of Fuzzy ER Model with Description Logic”. In: *Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008)*. Hong Kong, China: IEEE Computer Society, 2008. Pages 1358–1365.

[350] Lei Zhang, Yong Yu, Jian Zhou, ChenXi Lin, and Yin Yang. “An Enhanced Model for Searching in Semantic Portals”. In: *Proceedings*

of the 14th international conference on World Wide Web (WWW-05). Chiba, Japan 2005. Pages 453–462.

[351] Judy Zhao and Harold Boley. “Uncertainty Treatment in the Rule Interchange Format: From Encoding to Extension”. In: *Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008)*. Edited by Fernando Bobillo, Paulo C. G. da Costa, Claudia d’Amato, Nicola Fanizzi, Kathryn B. Laskey, Kenneth J. Laskey, Thomas Lukasiewicz, Trevor Martin, Matthias Nickles, Michael Pool, and Pavel Smrž. Karlsruhe, Germany: CEUR Workshop Proceedings, 2008.

[352] Bo Zhou, Jianjiang Lu, Yanhui Lv, Yafei Zhang, and Dazhou Kang. “Reasoning Problems on Distributed Fuzzy Ontologies”. In: *Proceedings of the 3rd International Conference on Rough Sets and Knowledge Technology (RSKT 2008)*. Edited by Guoyin Wang, Tian rui Li, Jerzy W. Grzymala-Busse, Duoqian Miao, Andrzej Skowron, and Yiyu Yao. Volume 5009. Lecture Notes in Computer Science. Chengdu, China: Springer-Verlag, 2008. Pages 588–595.

[353] Liang Zhou, Lei Zhang, Jun Chen, Qiang Xie, Qiulin Ding, and Zheng Xing Sun. “The Application of Fuzzy Ontology in Design Management”. In: *Proceedings of the International Conference on Artificial Intelligence (ICAI 2006)*. Edited by Hamid R. Arabnia. Volume 1. Las Vegas, Nevada, USA 2006. Pages 278–282.

[354] Evgeny Zolin. *Complexity of reasoning in Description Logics*. [Online] Available: `http://www.cs.man.ac.uk/~ezolin/dl/`.

Resumen

El uso de ontologías como un formalismo apropiado para la representación del conocimiento perteneciente a diferentes dominios de aplicación ha sido objeto de mucha atención durante los últimos años. Sin embargo, las ontologías clásicas no resultan apropriadas para representar conocimiento afectado por imprecisión, vaguedad e incertidumbre; aspectos que son inherentes a numerosos dominios del mundo real. Como solución se han propuesto las ontologías difusas, que combinan las ontologías con técnicas pertenecientes a la teoría de conjuntos difusos y a la lógica difusa.

Esta tesis presenta numerosas contribuciones al área de las ontologías difusas. Como formalismo se ha escogido una extensión difusa de la muy expresiva lógica de descripciones difusa $\mathcal{SROIQ}(\mathbf{D})$, en la que se basa el lenguaje OWL 2. Una vez introducida la definición de la lógica, se han investigado sus principales propiedades. Se ha proporcionado un algoritmo de razonamiento basado en una reducción a una ontología clásica, que permite la reutilización de lenguajes y razonadores existentes en la actualidad. Se han considerado dos semánticas, basadas en dos familias de operadores difusos diferentes (Zadeh y Gödel), se han estudiado con detalle las propiedades de la reducción (corrección, modularidad, complejidad) y se han propuesto varias optimizaciones. También se ha introducido una extensión posibilística que permite la representación adicional de conocimiento afectado por incertidumbre. Finalmente, el algoritmo de razonamiento ha sido implementado en un prototipo llamado DeLorean, que constituye el primer razonador que soporta extensiones difusas del lenguaje estándar para la representación de ontologías OWL, así como de su reciente extensión OWL 2.

Introducción y Motivación

En los últimos años, el uso de ontologías como formalismos para la representación del conocimiento perteneciente a numerosos dominios de aplicación ha aumentado de un modo significativo. Una ontología se define como una especificación explícita y formal de una conceptualización compartida [107], lo que significa que las ontologías representan los conceptos y las relaciones de un dominio, fomentado la interrelación con otros modelos, así como el procesamiento automático. Las ontologías presentan numerosas ventajas, como permitir que se añada semántica a los datos, haciendo más fácil el mantenimiento del conocimiento, la integración de información y la reutilización de componentes. Por ejemplo, las ontologías se han utilizado satisfactoriamente como parte de sistemas expertos y multiagente, además de ser un elemento fundamental en la Web Semántica, que propone extender la web actual dando a la información un significado definido de forma precisa.

El actual lenguaje estándar para la creación de ontologías es OWL (Web Ontology Language [327]), que incluye tres sublanguajes con una capacidad expresiva creciente: OWL Lite, OWL DL y OWL Full. Sin embargo, desde su nacimiento se han identificado diversas limitaciones en la expresividad de OWL y, como consecuencia, se han propuesto numerosas extensiones al lenguaje. Entre ellas, la más significativa es OWL 2 [75] que muy probablemente se convertirá en su sucesor.

Las lógicas de descripciones son una familia de lógicas utilizadas para la representación de conocimiento estructurado [13]. Cada lógica se denota mediante una cadena de letras mayúsculas que identifican los constructores admitidos en la lógica y por consiguiente su complejidad. Las lógicas de descripciones han demostrado son muy útiles como lenguajes para ontologías. Por ejemplo, una ontología en OWL Lite, OWL DL o OWL 2 es equivalente a una ontología en $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ o $\mathcal{SROIQ}(\mathbf{D})$ respectivamente [139].

Las lógicas difusa y posibilística han mostrado ser formalismos apropiados para manejar conocimiento impreciso/vago e incierto, respectivamente.

La lógica difusa es una extensión de la lógica clásica donde los elementos no están limitados a pertenecer o no pertenecer a un determinado conjunto, sino que puede hacerlo con un cierto grado. Por otra parte, la lógica posibilística permite asignar grados de posibilidad y necesidad a los axiomas, permitiendo manejar incertidumbre sobre ellos.

Sin embargo, ha sido ampliamente discutido el hecho de que las ontologías clásicas no son apropiadas para manejar conocimiento impreciso, vago e incierto, aspectos que son inherentes a muchos dominios del mundo real. Como medio para la obtención de ontologías difusas, pueden encontrarse en la literatura varias extensiones difusas de las lógicas de descripciones [181]. Las ontologías difusas han demostrado su utilidad en diferentes aplicaciones como en recuperación de información o en la Web Semántica.

La aparición de las ontologías difusas origina que los lenguajes clásicos para su representación dejen de ser apropiados, debiendo desarrollarse nuevos lenguajes difusos. Por tanto, el gran número de recursos disponibles actualmente para las ontologías dejarían también de ser apropiados y requerirían ser adaptados al nuevo marco de trabajo, lo que supondría un importante esfuerzo.

Esto afecta especialmente a los motores de inferencia. La experiencia previa con lógicas de descripciones clásicas ha mostrado que existe un salto importante entre el diseño de un algoritmo de razonamiento y la obtención de una implementación que funcione bien en la práctica [276], puesto que las lógicas de descripciones expresivas presentan una complejidad muy alta en el peor caso. Por ello, la optimización de los razonadores para lógicas de descripciones difusas será presumiblemente muy complicada y costosa.

Aunque se ha realizado un cantidad considerable de trabajo extendiendo lógicas de descripciones con la teoría de conjuntos difusos, la representación de ellas utilizando lógicas de descripciones clásicas no ha recibido tanta atención. Además, la expresividad de las lógicas consideradas en este contexto puede enriquecerse. Por ejemplo, el trabajo seminal en reducción de lógicas de descripciones difusas se restringe a $\mathcal{ALCH}$ [302].

Aumentar la expresividad de las lógicas de descripciones difusas es posible puesto que las propuestas actuales poseen varias limitaciones:

- El constructor nominal no se extiende al caso difuso.
- Aunque se han propuesto axiomas difusos de inclusión de conceptos y relaciones , razonar con ellos no siempre se permite.
- Los algoritmos de razonamientos actuales no soportan completamente la lógica $\mathcal{SROIQ}(\mathbf{D})$.
- No hay razonadores implementados que soporten extensiones difusas de los lenguajes OWL y OWL 2.
- No existe un formalismo general que permita una gestión unificada del conocimiento impreciso e incierto en el ámbito de las lógicas de descripciones.

Por otra parte, es común en la lógica difusa agrupar los operadores difusos en familias, cada una de las cuales contiene una t-norma, una t-conorma, una negación y una función de implicación. Hay tres familias principales de operadores difusos: Łukasiewicz, Gödel y del Producto [121]. Los operadores que L. Zadeh consideró cuando introdujo la lógica difusa (es decir, la conjunción y la disyunción de Gödel, la negación de Łukasiewicz y la implicación de Kleene-Dienes) son también de gran importancia en la literatura, y nos referiremos a ellos como la familia de Zadeh.

Es un hecho bien conocido que diferentes familias de operadores difusos conducen a lógicas de descripciones difusas con diferentes propiedades. La mayor parte de los trabajos existentes en el campo de las lógicas de descripciones difusas consideran la familia de Zadeh. Algunos otros trabajos consideran las familias de Łukasiewicz o del Producto, pero la familia de Gödel no ha recibido tanta atención.

En nuestra opinión, las propiedades lógicas de la familia de Gödel hacen interesante su studio. Por ejemplo, al igual que la familia de Zadeh, la familia de Gödel incluye una t-norma idempotente (el mínimo), por lo que la conjunción es independiente de la granularidad de la ontología difusa, lo que es interesante en algunas aplicaciones. Esto no sucede en las familias de Łukasiewicz o del Producto. Sin embargo, existe una importante diferencia con respecto a la familia de Zadeh, y es que la implicación de la familia de Gödel presenta mejores propiedades lógicas. Por ejemplo, utilizando la implicación de la familia de Zadeh, los conceptos y relaciones no están completamente incluidos en sí mismos.

Objetivos

El objetivo general de esta tesis es la obtención de ontologías difusas, capaces de representar y razonar con conocimiento afectado por imprecisión y vaguedad. De acuerdo con esto, los objetivos concretos de la tesis son los siguientes:

1. Revisar y analizar la literatura previa sobre ontologías difusas y sus áreas relacionadas, como las lógicas de descripciones difusas.
    - Comparar diferentes definiciones de ontología difusa y analizar sus limitaciones.
    - Identificar las limitaciones en la expresividad de las propuestas actuales para lógicas de descripciones difusas.
2. Proponer una nueva definición de lógica de descripciones difusa.
    - Aumentar la expresividad con respecto al trabajo previo.
    - Incluir algún modo de representar no solamente conocimiento impreciso y vago, sino también conocimiento afectado por incertidumbre.
3. Proporcionar una representación no difusa para una lógica de descripciones difusa tan expresiva.
    - Soportar la expresividad equivalente al lenguaje OWL 2 (difuso).
    - Permitir una semántica dada por varias familias de operadores difusos.
    - Diseñar técnicas de optimización que permitan reducir el tamaño de la representación.
4. Implementar un pequeño prototipo que demuestre la viabilidad de nuestra propuesta.

Contenidos

Esta tesis se estructura en cinco partes bien diferenciadas, cada una de las cuales se compone de uno o más capítulos.

La Parte I contiene el Capítulo 1, que incluye una introducción donde, partiendo de los antecedentes en el área, se motiva nuestro trabajo, se establecen los objetivos de la tesis y se describe el contenido del documento.

A continuación, la Parte II recuerda algunos preliminares necesarios para la lectura del texto. El Capítulo 2 revisa algunas nociones básicas sobre la teoría de conjuntos difusos, la lógica difusa y la lógica posibilística. El Capítulo 3 se dedica a las ontologías y lógicas de descripciones, como el principal formalismo utilizado para la representación de ontologías. Se dedica especial atención a la lógica de descripciones $\mathcal{SROIQ}(\mathbf{D})$ y al lenguaje OWL 2. El Capítulo 4 incluye con una revisión de la literatura sobre extensiones difusas de ontologías y lógicas de descripciones, proporcionando una contextualización más detallada de nuestro trabajo.

La Parte III presenta nuestras contribuciones teóricas. El Capítulo 5 define nuestra extensión difusa de la lógica de descripciones $\mathcal{SROIQ}(\mathbf{D})$, subrayando el aumento en la expresividad obtenido, y estudia sus propiedades lógicas. El Capítulo 6 se restringe a la familia de Zadeh y describe un procedimiento para representar una ontología difusa utilizado una ontología clásica, de modo que se pueden utilizar los razonadores existentes para razonar sobre la ontología obtenida. El Capítulo 7 presenta un resultado similar para la familia de Gödel. En ambos casos, se incluyen optimizaciones del procedimiento de reducción interesantes desde un punto de vista práctico.

La Parte IV se ocupa de nuestras aportaciones prácticas. El Capítulo 8 presenta el diseño e implementación de nuestro prototipo, un razonador para lógicas de descripciones difusas denominado DeLorean, que implementa el algoritmo de reducción y las optimizaciones descritas en la parte anterior. También se lleva a cabo una evaluación preliminar del procedimiento.

Finalmente, la Part V concluye la memoria. El Capítulo 9 incluye las principales conclusiones, resume nuestras contribuciones relacionándolas con los objetivos de la tesis y apunta algunas ideas para el trabajo futuro.

Aportaciones y Consecución de los Objetivos

Nuestro trabajo supone un importante avance hacia la obtención de ontologías difusas, capaces de representar y tratar conocimiento impreciso, vago e incierto. A lo largo de este documento, se han presentado ejemplos en diferentes dominios como medicina, alojamiento, música o enología, lo que refleja la naturaleza imprecisa de la mayor parte de los campos de aplicación del mundo real. En esta sección se resumen las principales aportaciones de la tesis al área de las ontologías difusas, estableciendo una relación entre los resultados y el procedimiento de consecución de los objetivos inicialmente establecidos. No nos centraremos por tanto en las capítulos exclusivamente dedicados a la introducción, los preliminares y conclusiones.

Capítulo 4

Este capítulo satisface el primero de nuestros objetivos, que era llevar a cabo una revisión crítica de la literatura relativa a las extensiones difusas de ontologías y lógicas de descripciones. En primer lugar, se ha realizado un estudio detallado acerca de las propuestas previamente existentes sobre ontologías difusas. Tras haber observado que ninguna de las definiciones existentes de ontología difusa se adecúa a nuestra visión más general, hemos propuesto una definición alternativa más adecuada.

A continuación, el estudio se ha centrado en las extensiones difusas de las lógicas de descripciones. Esto permite identificar una serie de limitaciones en la expresividad de las lógicas de descripciones difusas existentes que se superarán a lo largo de la tesis. De esta manera, se obtiene una contextualización más detallada de nuestro trabajo, al permitir diferenciar las aportaciones propias de las de otros autores.

Capítulo 5

En este capítulo se introduce una definición de la lógica de descripciones difusa $\mathcal{SROIQ}(\mathbf{D})$ que cumple el segundo de nuestros objetivos, que era la obtención de una lógica de descripciones difusa que aumente la expresividad con respecto al trabajo previo.

El aumento de la expresividad de la lógica se ha logrado así:

- Se ha definido una extensión difusa del constructor nominal, de forma que se permiten definiciones extensivas de conjuntos difusos (esto es, definiciones mediante una definición de los elementos que lo componen junto a su grado de pertenencia).
- Se utiliza una semántica para las restricciones cualificadas de cardinalidad que modifica otras propuestas previas y que presenta algunas propiedades lógicas deseables desde el punto de vista práctico.
- Se permite el uso de conceptos y relaciones basados en α-cortes positivos y negativos. Es decir, es posible definir el conjunto (crisp) formado por aquellos elementos que pertenecen a un conjunto difuso con grado mayor o igual que α, o menor o igual que α. Análogamente, se puede definir la relación (crisp) formada por aquellos pares de elementos relacionados con grado mayor o igual que α.
- Se permite la aplicación de modificadores difusos sobre conceptos y relaciones. El razonamiento con relaciones modificadas no era posible con anterioridad a nuestra propuesta.
- Se permite el razonamiento con axiomas difusos de inclusión de conceptos y relaciones, que hacen posible representar que un concepto está parcialmente incluido en otro, o que una relación está parcialmente incluida en otra.

La definición es independiente de la elección de la familia de operadores difusos, por lo que posee una gran generalidad. Se han estudiado las propiedades de la lógica obtenida, prestando especial atención a los casos especiales de las familias de operadores difusos de Zadeh y Gödel. Cada familia posee ventajas e inconvenientes, y la elección de una u otra depende de la aplicación concreta.

Adicionalmente, se permite la representación de conocimiento afectado por incertidumbre, cumpliendo así el segundo de los subobjetivos. Para ello, se establece una una capa posibilística sobre la lógica de descripciones difusa, de forma que es posible añadir grados de posibilidad y necesidad a los axiomas de una ontología difusa.

Capítulo 6

Este capítulo y el siguiente satisfacen el tercer objetivo: la representación de ontologías difusas utilizando ontologías clásicas. De esta forma, es posible reutilizar los lenguajes, razonadores y otras herramientas actualmente disponibles para ellas. En esta primera parte del bloque de capítulos que cumplen el tercer objetivo nos centramos en los operadores difusos de la familia de Zadeh.

La idea de la reducción se basa en la introducción de una serie de nuevos conceptos y relaciones, que se utilizan para representar los α-cortes de los conceptos y relaciones difusos de la ontología difusa. A continuación, los axiomas de la ontología se representan utilizando una ontología clásica, y los conceptos y relaciones difusos se representan mediante estos nuevos elementos. Esta transformación utiliza una aplicación κ que transforma axiomas de la ontología difusa en axiomas de la ontología crisp, y otra aplicación ρ que calcula el α-corte (posiblemente estricto) de un concepto o una relación.

Este procedimiento es posible porque el número de α-cortes relevantes puede conocerse de antemano, debido a la semántica de los operadores de la familia de Zadeh. Además, es necesario introducir nuevos axiomas para mantener la semántica entre los diferentes α-cortes introducidos.

Las propiedades de la reducción (corrección, modularidad, complejidad) se analizan en detalle, incluyendo demostraciones de las proposiciones y de los teoremas formulados.

También se han introducido algunas técnicas de optimización interesantes que permiten reducir el tamaño de la ontología resultante. Estas técnicas son la reducción del número de nuevos α-cortes introducidos (así como de sus correspondientes axiomas), el uso de conceptos y relaciones no difusos en el lenguaje, el razonamiento ignorando elementos superfluos y una optimización de la reducción de las relaciones irreflexivas.

Para finalizar, se ha estudiado cómo razonar con nuestra extensión posibilística de una ontología difusa. En este caso es posible utilizar únicamente un razonador clásico, si bien no se ha encontrado una representación clásica para una ontología de este tipo.

Capítulo 7

Junto al anterior, en este capítulo se lleva a cabo la satisfacacción del tercero de los objetivos: la representación de ontologías difusas utilizando ontologías clásicas. Pero en este caso, la familia de operadores difusos considerada es la de Gödel.

La idea del procedimiento es similar, por lo que en este capítulo se muestran únicamente aquellos aspectos que difieren al caso anterior. La definición de κ y de ρ es diferente ahora, lo que causa ciertas diferencias en las propiedades de la reducción, especialmente en cuanto a la complejidad de la ontología resultante. Además, es necesario asumir un conjunto de grados de verdad para poder trabajar correctamente con los modelos de la lógica.

Adicionalmente, se estudian las optimizaciones de algunos casos especiales de axiomas difusos de inclusión de conceptos, como son los axiomas de dominio, rango y funcionalidad de una relación, o los axiomas que establecen que dos conceptos son disjuntos. En general, se define una condición para la optimización que puede aplicarse en otros casos (como por ejemplo cuando aparecen nominales difusos o conceptos crisp en la definición de un concepto difuso).

Capítulo 8

Finalmente, este capítulo cumple el cuarto objetivo mediante la implementación de un pequeño prototipo que demuestra la viabilidad de nuestra propuesta. El prototipo se denomina DELOREAN y supone el razonador para lógicas de descripciones más expresivo en la actualidad, puesto que soporta versiones difusas del lenguaje estándar OWL y de su extensión OWL 2.

A partir de una ontología difusa en OWL difuso o en OWL 2 difuso, DELOREAN calcula una representación no difusa, es decir, una ontología equivalente en OWL u OWL 2, respectivamente. Terminar utilizando OWL es importante porque es el lenguaje estándar actual para la representación de ontologías. OWL 2 se perfila como el próximo lenguaje estándar, por lo que la posibilidad de terminar utilizándolo es también muy interesante.

Para evaluar hasta qué punto nuestra propuesta es factible, se ha realizado una evaluación preliminar que ha confirmado que las optimizaciones propuestas permiten una reducción muy significativa del tamaño de la ontología no difusa resultante, y que el tiempo que se consume en obtener la ontología equivalente es suficientemente pequeño como para permitir su cálculo en tiempo real.

Conclusiones

La aplicación práctica inmediata de las ontologías difusas es factible, debido a su estrecha relación con los lenguajes y herramientas actualmente existentes y que han demostrado su validez. Nuestra propuesta presenta varias ventajas:

- No hay necesidad de consensuar un nuevo lenguaje estándar difuso, sino que cada desarrollador puede utilizar su propio lenguaje para expresar una lógica de descripciones difusa, siempre que implemente la reducción a un lenguaje estándar.
- Es posible continuar utilizando lenguajes estándar con muchos recursos disponibles, evitando la necesidad (y el coste) de adaptarlos a un nuevo lenguaje difuso. Aunque sería deseable asistir al usuario en tareas como la edición de ontologías difusas, la reducción a ontologías clásicas o el razonamiento difuso, una vez que la reducción se lleva a cabo, es posible reutilizar los recursos existentes para el lenguaje crisp.
- Es posible continuar utilizando razonadores para lógicas de descripciones existentes. Si bien el razonamiento no necesariamente será más eficiente, se ofrece un método para soportar fácilmente el razonamiento en futuros lenguajes difusos. Actualmente, el único razonador para lógicas de descripciones difusas que soporta completamente extensiones difusas de los lenguajes OWL y OWL 2 se basa en esta idea.
- Permite ayudar a demostrar la decidibilidad del razonamiento en lógicas de descripciones difusas para los cuales no se conoce otro algoritmo de razonamiento. Por ejemplo, hemos demostrado que $\mathcal{SROIQ}(\mathbf{D})$ difusa con la semántica de la familia de Gödel es decidible.
- Es posible razonar con lógicas de descripciones difusas más expresivas. Los algoritmos específicos actuales para razonar con lógicas de descrip-

ciones difusas se restringen a $\mathcal{SHOIN}$ [288] o $\mathcal{SHIF}(\mathbf{D})$ [298] para la familia de Zadeh, o $\mathcal{ALC}$ [120] para la familia de Gödel, mientras que por el contrario nosotros hemos considerado $\mathcal{SROIQ}(\mathbf{D})$.

Nuestra definición de la lógica de descripciones difusa $\mathcal{SROIQ}(\mathbf{D})$ permite el uso de todos los modificadores y dominios de concretos difusos que soporta algún razonador para lógicas de descripciones difusas. Desde un punto de vista histórico, supone el primer algoritmo de razonamiento para una lógica con axiomas difusos de inclusión de conceptos y relaciones , así como con modificadores de relaciones .

La posibilidad de utilizar las familias de operadores difusos de Zadeh y Gödel aumenta la flexibilidad del lenguaje. Cada una de estas familias presenta algunas características especiales, algunas ventajas y algunos inconvenientes, por lo que diferentes aplicaciones pueden requerir el uso de una u otra. Una de las propiedades más relevantes de la familia de Zadeh es que el uso de la implicación de Kleene-Dienes en la semántica de los axiomas difusos de inclusión de conceptos y relaciones implica dos efectos contraintuitivos. Por otra parte, la familia de Gödel obliga a que las restricciones cualificadas de cardinalidad mínima sean conceptos no difusos, por citar un ejemplo concreto. Estas familias también pueden combinarse, de modo que sus operadores difusos pueden usarse simultáneamente.

La complejidad de las ontologías clásicas obtenidas como resultado de nuestro procedimiento es cuadrática en el caso de la familia de Zadeh, o lineal si fijamos el conjunto de posibles grados de verdad. En el caso de la familia de Gödel, la complejidad es polinomial, o lineal si aproximamos la cuantificación universal utilizando conceptos y relaciones basados en α-cortes.

Bajo ciertas condiciones razonables, la reducción de una ontología difusa puede reutilizarse cuando se añaden axiomas adicionales. Estos requisitos son el uso del vocabulario de la ontología y de un conjunto fijo de posibles grados de verdad. Esto último se trata de una suposición razonable, ya que en las aplicaciones prácticas es habitual trabajar con un número pequeño de grados.

En el caso de la familia de Gödel, utilizar un conjunto fijo de grados de verdad que incluya 0 y 1 es necesario para garantizar el cumplimiento de la propiedad de modelos *witnessed*. Es conveniente recordar que en el caso de la familia de Łukasiewicz (que no se ha considerado en esta tesis) también es necesario utilizar un conjunto fijo de grados de verdad para obtener una representación no difusa [41].

Fijar el conjunto de grados de verdad también resulta ser esencial para calcular de un modo eficiente el mayor límite inferior posible para un axioma difuso, puesto que esta tarea requiere el cálculo de varios tests lógicos, el número de los cuales depende logarítmicamente del número de posibles grados de verdad.

DeLorean es el razonador para lógicas de descripciones difusas más expresivo que conocemos, puesto que soporta una versión difusa de OWL 2. Una evaluación preliminar ha confirmado que las optimizaciones diseñadas ayudan a reducir de un modo significativo el tamaño de la ontología obtenida como resultado de la reducción. Además, el tiempo necesario para la reducción es suficientemente pequeño como para permitir volver a calcular la reducción de una ontología en tiempo real.

Trabajo Futuro

La principal línea de investigación en el futuro será la realización de una evaluación más detallada del razonador, teniendo en cuanta no solamente el tamaño de la ontología resultante, sino también el tiempo de razonamiento. Por una parte, se comparará el razonamiento difuso en DeLorean con el razonamiento en el caso clásico. Por otro lado, se comparará DeLorean con otros razonadores para lógicas de descripciones difusas.

En el primer caso, consideraremos una ontología difusa perteneciente al mundo real. Hasta donde nosotros conocemos, no existe ninguna base de conocimiento difusa significativa. Si bien existe una versión difusa de la ontología LUBM [222], se trata de una ontología muy poco expresiva (en DL-Lite difuso). La ontología Wine que hemos utilizado nosotros fue designada como un ejercicio que permitiera el uso de todas las posibilidades de OWL, lo que hace que el razonamiento en ella sea innecesariamente complicado. Se espera que el razonamiento con ontologías difusas más realísticas sea factible bajo nuestra aproximación.

La segunda línea de investigación presenta ciertas dificultades porque diferentes razonadores soportan diferentes características y expresividad en sus lenguajes, además de requerir diferentes formatos para la especificación de las ontologías difusas de entrada. En este sentido, sería interesante diseñar una versión de la interfaz DIG común a las ontologías difusas. En el caso de la lógica de Gödel, también se planea el diseño y la implementación de un algoritmo tableau para alguna lógica de descripciones. De otro modo, no sería posible la comparación con nuestra propuesta.

También resultaría de interés el considerar tareas de razonamiento alternativas, como el cálculo del mayor límite inferior posible para un axioma difuso, o la resolución de respuesta a consultas conjuntivas difusas, una tarea que no ha sido investigada para lógicas de descripciones difusas expresivas.

En futuras versiones de DeLorean, lo adaptaremos a la interfaz DIG 2.0 tan pronto como esté disponible. También nos gustaría utilizar como razonador crisp HermiT, basado en la técnica del hipertableau y que parece mejorar sustancialmente a los demás razonadores para lógicas de descripciones [265], si bien desafortunadamente todavía no soporta la interfaz DIG. Mientras tiene lugar la definición de un lenguaje común para las ontologías difusas, desarrollaremos diferentes módulos para aceptar como entrada los lenguajes de cada uno de ellos (o, de modo más preciso, el fragmento de ellos que seamos capaces de soportar). También adaptaremos la interfaz de usuario para soportar tareas de razonamiento alternativas.

Desde un punto de vista teórico, nos gustaría aumentar la expresividad de la lógica y considerar operadores difusos alternativos. En el primer caso, nos gustaría estudiar algunos tipos de cuantificadores difusos [250]. En el segundo caso, estudiaremos la familia de Łukasiewicz, intentando extender los resultados actuales que se restringen a la lógica $\mathcal{ALCHOI}$ [41].