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Adjoint and Frobenius Pairs of Functors, Equivalences and the Picard Group for Corings

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by

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Introduction

The aim of this work is:

1. to study adjoint and Frobenius pairs of functors between comodule categories over corings;

2. to extend the comatrix coring defined in [35, 17] to a quasi-finite comodule over a coring and to study equivalences between of comodule categories over corings. We think that our generalized comatrix coring gives more concrete characterizations of these equivalences;

3. to extend the Picard group from algebras [9, 10] and coalgebras over fields [91] to corings and extend some of their properties;

4. and to apply the obtained results to induction functors over different base rings and to corings coming from entwined structures and graded ring theory, in order to give more concrete and new results. For example, we give a characterization of adjoint and Frobenius pairs of functors between categories of graded modules over G-sets.

We conclude that the notion of a coring unifies several important algebraic structures of current interest and is a powerful tool which allows to obtain more concrete and simple results than the already existing ones.

Corings were introduced by M. Sweedler in [88] as a generalization of coalgebras over commutative rings to the case of non-commutative rings, to give a formulation of a predual of the Jacobson-Bourbaki's theorem for intermediate extensions of division ring extensions. Thus, a coring over an associative ring with unit A is a comonoid in the monoidal category of all A-bimodules. Further study of corings are given by several authors, for example, see [51, 52]. It is well known that algebras and coalgebras over fields (see [87, 1, 31]), or more generally over rings (see [19]), are special cases of corings.

Recently, corings were intensively studied. The main motivation of these studies is an observation by Takeuchi, namely that an entwining structure (resp. an entwined module) can be viewed as a suitable coring (resp. as a comodule over a suitable coring) (see Theorem 1.5.7). T. Brzeziński has given in [15] some new examples and general properties of corings. Among them, a study of Frobenius corings is developed, extending previous results on entwining structures [14] and relative Hopf modules [23].

A pair of functors (F, G) is said to be a *Frobenius pair* [26], if G is at the same time a left and right adjoint to F. That is a standard name which we use instead of Morita's original "*strongly adjoint pairs*" [71]. The functors F and G are known as *Frobenius functors* [23]. The study of Frobenius functors was motivated by a paper of K. Morita, where he proved [71, Theorem 5.1] that given a ring extension $i : A \to B$, the induction functor $- \otimes_A B : \mathcal{M}_A \to \mathcal{M}_B$ is a Frobenius functor if and only if the morphism i is Frobenius in the sense of [59] (see also [72]): ${}_AB$ is finitely generated projective and ${}_BB_A \simeq \mathrm{Hom}_A({}_AB, {}_AA)$ as (B, A)-bimodules. There is another interesting result relating Forbenius extensions of rings and corings [56, Proposition 4.3]: A ring extension $i : A \to B$ is a Frobenius extension if and only if the bimodule ${}_AB_A$ is endowed by a structure of A-coring such that the comultiplication is a B-bimodule map.

The dual result of Morita's [71, Theorem 5.1] for coalgebras over fields was proved in [26, Theorem 3.5] and it states that the corestriction functor $(-)_\varphi : \mathcal{M}^C \to \mathcal{M}^D$ associated to a morphism of coalgebras $\varphi : C \to D$ is Frobenius if and only if C_D is quasi-finite and injective and there exists an isomorphism of bicomodules $\mathrm{h}_D({}_CC_D, D) \simeq {}_DC_C$ (here, $\mathrm{h}_D(C, -)$ denotes the "cohom" functor). Since corings generalize both rings and coalgebras over fields, one may expect that [71, Theorem 5.1] and [26, Theorem 3.5] are specializations of a general statement on homomorphisms of corings. In Chapter 2, we find such a result (Theorem 2.3.1) and introduce the notion of a (right) Frobenius extension of corings (see Definition 2.3.2). To prove such a result, we study adjoint pairs and Frobenius pairs of functors between categories of comodules over rather general corings. Precedents for categories of modules and categories of comodules over coalgebras over fields are contained in [71], [90], [26] and [20]. More recently, *Frobenius corings* (i.e., corings for which the functor forgetting the coaction is Frobenius), have intensively been studied in [15, 16, 17, 19, 20].

We think that our general approach produces results of independent interest, beyond the aforementioned extension to the coring setting of [71, Theorem 5.1] and [26, Theorem 3.5] and contributes to the understanding of the behavior of the cotensor product functor for corings. In fact, our general results, although they are sometimes rather technical, have other applications and will probably find more. For instance, we have used them to prove new results on equivalences of comodule categories over corings that are expounded in Chapter 3. Moreover, when applied to corings stemming from different algebraic theories of current interest, they boil down to new (more concrete) results. As an illustration, we consider entwined modules over an entwining structure in Section 2.4 and graded modules over G-sets in Section 2.5.

Comatrix corings were introduced in [35] in order to give the structure of corings whose category of right comodules has a finitely generated projective generator, and the structure of cosemisimple corings which were defined and studied in [37]. In [17] the authors have given an equivalent definition of them and studied when are cosplit, coseparable, and Frobenius.

In Section 3.1, we extend comatrix coring to the case of quasi-finite comodules and we study when is cosplit, coseparable, and Frobenius. Suppose that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ and suppose that $-\square_{\mathfrak{C}}X$ is a left adjoint to $-\square_{\mathfrak{D}}\Lambda$. If $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat and ${}_{\mathfrak{C}}X$, ${}_{\mathfrak{D}}\Lambda$ are coflat, or A and B are von Neumann or if $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable, then the B-bimodule $\Lambda\square_{\mathfrak{C}}X$ is endowed by a structure of B-coring (Theorem 3.1.1). The coproduct is

$$
\begin{tikzcd}
\Delta : \Lambda\square_{\mathfrak{C}}X \arrow[r, "\simeq"] & \Lambda\square_{\mathfrak{C}}(\mathfrak{C}\square_{\mathfrak{C}}X) \arrow[r, "\Lambda\square_{\mathfrak{C}}(\eta_{\mathfrak{C}}\square_{\mathfrak{C}}X)"] & \Lambda\square_{\mathfrak{C}}((X\square_{\mathfrak{D}}\Lambda)\square_{\mathfrak{C}}X) \arrow[r, "\simeq"] & (\Lambda\square_{\mathfrak{C}}X)\square_{\mathfrak{D}}(\Lambda\square_{\mathfrak{C}}X) \arrow[d, hook] \\
& & & (\Lambda\square_{\mathfrak{C}}X)\otimes_B(\Lambda\square_{\mathfrak{C}}X),
\end{tikzcd}
$$

where η is the counit of this adjunction, and the counit is $\epsilon = \epsilon_{\mathfrak{D}} \circ \omega : \Lambda\square_{\mathfrak{C}}X \longrightarrow B$.

The most important property of this coring is the following: Let $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ be a bicomodule, quasi-finite as a right $\mathfrak{C}$-comodule, such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. Set $X = \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. If

(a) $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat, the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is exact and ${}_{\mathfrak{D}}\Lambda$ is coflat, or

(b) A and B are von Neumann regular rings and the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is exact or

(c) $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable corings,

then the canonical isomorphism $\delta_\Lambda : \mathrm{e}_{\mathfrak{C}}(\Lambda) \longrightarrow \Lambda\square_{\mathfrak{C}}X$ is an isomorphism of B-corings (see Proposition 3.1.4). We think that, under these conditions, this coring isomorphism gives a concrete description of the coendomorphism coring even in the case of coalgebras over fields. Indeed, let C be a coalgebra over a field k and $\Lambda \in \mathcal{M}^C$ a quasi-finite and injective comodule. Then $\mathrm{e}_C(\Lambda) \simeq \Lambda\square_C\mathrm{h}_C(\Lambda, C)$ as coalgebras (the case (b)) (see Corollary 3.1.5). We also think that the last isomorphism gives a more concrete characterization of equivalence between categories of comodules over coalgebras over a field, than that of Takeuchi [90]. Of course, this characterization of the coendomorphism coring is also useful in the characterization of equivalence between categories of comodules over corings. Indeed, in Section 3.2, we study equivalences between categories of comodules over rather general corings. We generalize and improve (using results we give in Chapter 1) the main results concerning equivalences between categories of comodules given in [90], [4] and [19]. We also give new characterizations of equivalences between categories of comodules over coseparable corings or corings with a duality. We apply our results to the particular case of the adjoint pair of functors associated to a morphism of corings over different base rings. Finally, when applied to corings associated to entwining structures and that associated to a G-graded algebra and a right G-set, we obtain new results concerning entwined modules and graded modules. We think that our result, Theorem 3.4.4, is more simple than Del Río's [81, Theorem 2.3].

In [82] the Picard group of an Azumaya algebra is defined and the connections between this group and the algebra automorphisms are studied. H. Bass generalized in [9, 10] the Picard group and these connections for arbitrary algebras. Further study of the Picard group of algebras is given in [39]. Morita theory for rings with local units was developed in [2] or [6]. In [11] the authors introduced the Picard group of a ring with local units and gave the versions of the corresponding connections for this ring, in order to study the Picard group of the category $R-gr$, where R is a G-graded ring. In [91], the authors defined the Picard group of a coalgebra over a field and studied the corresponding connections for it.

The purpose of Chapter 4 is to introduce and study the right Picard group of corings. The motivation is the fact that there is an isomorphism of groups between the Picard group of the category $\mathcal{M}^{\mathfrak{C}}$, for a certain coring $\mathfrak{C}$ and the right Picard group of $\mathfrak{C}$, which is defined as the group of the isomorphism classes of right invertible $\mathfrak{C}$-bicomodules (=$T(\mathfrak{C})$ for some k-autoequivalence of $\mathcal{M}^{\mathfrak{C}}$, T) with the composition law induced by the cotensor product (see Proposition 4.1.3). It is clear that the Picard group of corings generalizes that of algebras and coalgebras over fields. We recall two theorems, the first is [10, Proposition II (5.2)(3)] (see also [30, Theorems 55.9, 55.11]) and the second is the particular case of [91, Theorem 2.7] where $R = k$.

Theorem 0.0.1. *For a k-algebra A, there is an exact sequence*

$$1 \longrightarrow \mathrm{Inn}_k(A) \longrightarrow \mathrm{Aut}_k(A) \longrightarrow \mathrm{Pic}_k(A)\ ,$$

where $\mathrm{Inn}_k(A) := \{a \mapsto bab^{-1} \mid b \text{ invertible in } A\}$, *the* group of inner automorphisms *of A.*

Theorem 0.0.2. *For a coalgebra C over a field k, there is an exact sequence*

$$1 \longrightarrow \mathrm{Inn}_k(C) \longrightarrow \mathrm{Aut}_k(C) \longrightarrow \mathrm{Pic}_k(C)\ ,$$

where $\mathrm{Inn}_k(C)$ *(the* group of inner automorphisms *of C) is the set of $\varphi \in \mathrm{Aut}_k(C)$ such that there is $p \in C^*$ invertible with $\varphi(c) = \sum p(c_{(1)})c_{(2)}p^{-1}(c_{(3)})$, for every $c \in C$.*

Our Theorem 4.1.6 generalizes both Theorem 0.0.1 and Theorem 0.0.2. We extend the Aut-Pic property from algebras [12] and coalgebras over fields [29] to corings, namely, a coring $\mathfrak{C}$, which is flat on both sides over its base ring A, has the right Aut-Pic property, if the morphism $\mathrm{Aut}_k(\mathfrak{C}) \to \mathrm{Pic}_k^r(\mathfrak{C})$ is an epimorphism. We also extend a result which is useful to show that a given coring $\mathfrak{C}$ has the right Aut-Pic property (Proposition 2.7). Of course, all of the examples of algebras and coalgebras having the Aut-Pic property that are given in [12, 29] are corings having this property. In this chapter, we give some new examples of corings having the Aut-Pic property. We also simplify the computation of the right Picard group of several interesting corings (see Proposition 4.2.6). Finally, in section 4, we give the corresponding exact sequences for the category of entwined modules over an entwining structure, the category of Doi-Koppinen-Hopf modules over a Doi-Koppinen structure and the category of graded modules by a G-set, where G is a group. Our result, regarding the Picard group of the category $gr-(A, X, G)$, is more natural than the Beattie-Del Río result given in [11, §2].

Now, we move to a more detailed outline of the contents of this work.

Chapter 1. The aim of this chapter is to collect some results that will be useful in the next chapters. Several of them are new. In Section 1, we state certain results of category and ring theory. Sometimes, we also give some proofs in order to give the reader clear tools that are necessary to understand the rest of this work. Section 2 is devoted to the study of the category of comodules over a coring. In Section 3, we study the cotensor product over corings. Section 4 deals with the cohom functor and the coendomorphism coring. We give complete and rigorous proofs of the most results in Section 2 and of all the stated results in Sections 3 and 4. Finally, in Section 5, we recall the definitions and some results regarding induction functors, entwining structures and graded rings.

For an introduction to the theory of abelian and Grothendieck categories, we refer to the very famous and monumental paper by A. Grothendieck (known as the "Tohoku paper") [47]. For a complete treatment, we refer to [40, 38, 69, 79, 64, 85] and the very recent book by M. Kashiwara and P. Schapira [60]. For the foundation of the theory of categories, using Grothendieck's universes, see [7]. For a detailed discussion of corings, we refer to [19]. We refer to [19, 24] for a study of entwining structures and to [76, 73, 77] for graded ring theory.

Chapter 2. Section 2.1 deals with adjoint and Frobenius pairs on categories of comodules. Some refinements of results from [43] and [19, §23] on the representation as cotensor product functors of certain functors between comodule categories are needed and are, thus, included in Section 2.1. From our general discussion on adjoint pairs of cotensor product functors, we will derive our main general result on Frobenius pairs between comodule categories (Theorem 2.1.19) that extends the known characterizations in the setting of modules over rings and of comodules over coalgebras. In the first case, the key property to derive the result on modules from Theorem 2.1.19 is the separability of the trivial corings (see Remark 2.1.21). In the case of coalgebras, the fundamental additional property is the duality between finite left and right comodules. We already consider a much more general situation in Section 2.2, where we introduce the class of so called *corings having a duality* for which we prove characterizations of Frobenius pairs that are similar to the coalgebra case.

Section 2.3 is one of the principal motivations of this study. After the technical development of sections 2.1 and 2.2, our main results follow without difficulty. We prove, in particular, that the induction functor $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ associated to a homomorphism $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ of corings $\mathfrak{C}$ and $\mathfrak{D}$ flat over their respective base rings A and B is Frobenius if and only if the $(\mathfrak{C}, \mathfrak{D})$-bicomodule $\mathfrak{C}\otimes_A B$ is quasi-finite and injector as a right $\mathfrak{D}$-comodule and there exists an isomorphism of $(\mathfrak{D}, \mathfrak{C})$-bicomodules $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C}\otimes_A B, \mathfrak{D}) \simeq B\otimes_A \mathfrak{C}$ (Theorem 2.3.1). We show as well how this theorem unifies previous results for ring homomorphisms [71, Theorem 5.1], coalgebra maps [26, Theorem 3.5] and Frobenius corings [19, 27.10, 28.8].

In Section 2.4, we specialize one of the general results on corings to entwining structures.

In Section 2.5, we particularize our results in the previous sections to the coring associated to a G-graded algebra and a G-set, where G ia a group. Then, we obtain a series of new results for graded modules by G-sets.

Chapter 3. In Section 3.1, we generalize the comatrix coring introduced in [35] and [17]. We also generalize some of its interesting properties given in [17]. Section 3.2 is devoted to the study of equivalences of comodule categories over corings. Our results given in [95] and Section 3.1 will allow us to generalize and improve the main results in both [4] and [19] (see Propositions 3.2.2, 3.2.6, Theorems 3.2.8, 3.2.9) and also to give new results concerning equivalences of comodule categories over coseparable corings (see Propositions 3.2.2, 3.2.6, Theorems 3.2.8, 3.2.10) and over corings over QF rings (Theorem 3.2.16). Obviously, our last theorem generalizes [90, Theorem 3.5] and [4, Corollary 7.6]. In Section 3.3, we deal with the application of some of our results given in Section 3.2 to the induction functor. In Section 2.4, we apply some of our results given in the previous sections to the corings associated to entwining structures, in particular, those associated to a G-graded algebra and a right G-set, where G is a group.

Chapter 4. The aim of Section 4.1 is to introduce and study the Picard group of corings. Section 4.2 is devoted to define the Aut-Pic property for corings and give some new examples of corings having this property. In the final section 4.3, we apply our result, Theorem 4.1.6, to the corings coming from entwining structures, specially from graded ring theory.

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Some general notations

(i) Throughout this work, unless otherwise stated, k denotes an associative and commutative ring with unit, A, A', A'', A_1, A_2, B, B_1 and B_2 denote associative and unitary algebras over k and $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{D}$, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ denote corings over A, A', A'', A_1, A_2, B, B_1 and B_2, respectively.

(ii) **Set** will denote the category of sets, **Ab** will denote the category of abelian groups, and $\mathcal{M}_A$, ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_B$ will denote the categories of right A-modules, left A-modules and (A, B)-modules, respectively.

(iii) The notation $\otimes$ will stand for the tensor product over k.

(iv) We denote the dual algebra of an algebra A by A°. We denote the dual category of a category $\mathbf{C}$ by $\mathbf{C}^\circ$.

Chapter 1

Generalities

1.1 Some category and ring theory

We refer to the Grothendieck's paper [47] for the definitions of the basic notions of abelian categories. We recall only the Grothendieck's following axioms for a category $\mathbf{C}$:

AB 1) Every morphism in $\mathbf{C}$ has a kernel and cokernel.

AB 2) For every morphism $u : A \to B$ in $\mathbf{C}$, the induced morphism $\overline{u} : \mathrm{Coim}(u) \to \mathrm{Im}(u)$ defined to be the unique morphism making commutative the diagram

$$\begin{tikzcd} \mathrm{Ker}(u) \arrow[r] & A \arrow[r, "u"] \arrow[d] & B \arrow[r] & \mathrm{Coker}(u) \\ & \mathrm{Coim}(u) \arrow[r, "\overline{u}"] & \mathrm{Im}(u) \arrow[u] & \end{tikzcd} ,$$

is an isomorphism.

AB 3) The direct sum of every family of objects $(A_i)_{i\in I}$ of $\mathbf{C}$ exists.

AB 4) The axiom AB 3) holds, and the direct sum of a family of monomorphisms is a monomorphism.

AB 5) The axiom AB 3) holds, and if $(A_i)_{i\in I}$ is a directed family of subobjects of an object $A \in \mathbf{C}$, and B is an arbitrary subobject of A, then

$$(\sum_i A_i) \cap B = \sum_i (A_i \cap B).$$

AB 6) The axiom AB 3) holds, and for every object $A \in \mathbf{C}$ and every family $(B^j)_{j\in J}$, where $B^j = (B^j_i)_{i\in I_j}$ is a directed family of subobjects of A for every $j \in J$,

$$\bigcap_{j\in J} \Big(\sum_{i\in I_j} B^j_i \Big) = \sum_{(i_j)\in \prod I_j} \Big(\bigcap_{j\in J} B^j_{i_j} \Big).$$

(This axiom includes implicitly the existence of the greatest lower bound of every family of subobjects of A.)

Obviously the axioms AB 1) and AB 2) coincide with their duals. We define also the dual axioms AB 3*), AB 4*), AB 5*), and AB 6*).

A category is called *additive* if it is preadditive, has a zero object and finite products (or coproducts). A category is called *abelian* if it is additive and satisfies AB 1) and AB 2).

k-categories, k-functors. A category $\mathbf{C}$ is said to be k-*category* if for every $C, C' \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}}(C, C')$ is a k-module, and the composition is k-bilinear. In particular, for every $C \in \mathbf{C}$, $\mathrm{End}_{\mathbf{C}}(C)$ is a k-algebra. For a k-algebra A, $\mathcal{M}_A$ is a k-category.

A functor between k-categories is said to be k-*functor* or k-*linear functor* if it is k-linear on the k-modules of morphisms.

Of course, if $k = \mathbb{Z}$, we find the very well known notions of a preadditive category and additive functor.

A k-category is called a k-*abelian category* if it is an abelian category.

Let $\mathbf{C}$ be a k-category. Following Mac Lane ([64, p. 194]), a *biproduct diagram* for the objects $A, B \in \mathbf{C}$ is a diagram

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} C \underset{i_2}{\overset{p_2}{\rightleftarrows}} B \tag{1.1}$$

such that the morphisms p_1, p_2, i_1, i_2 satisfy the conditions

$$p_1 i_1 = 1_A, \quad p_2 i_2 = 1_B, \quad i_1 p_1 + i_2 p_2 = 1_C. \tag{1.2}$$

In particular, p_1, p_2 are epimorphisms, i_1, i_2 are monomorphisms, $p_2 i_1 = 0$ and $p_1 i_2 = 0$.

If such biproduct exists, it is both a product and coproduct.

Proposition 1.1.1. *Let $\mathbf{C}$ and $\mathbf{D}$ be two k-categories such that all two objects have a biproduct diagram. Then a functor $T : \mathbf{C} \to \mathbf{D}$ is k-linear if and only if the following hold*

(a) T carries every biproduct diagram in $\mathbf{C}$ to a biproduct diagram in $\mathbf{D}$,

(b) $T(h1_A) = h1_{T(A)}$ for every $A \in \mathbf{C}$ and $h \in k$.

Proof. First, by Grothendieck's [64, Proposition VIII.4], T is additive if and only if the condition (a) holds. On the other hand, for every morphism $f : A \to B$ in $\mathbf{C}$ and $h \in k$, $hf = (h1_A)f$. Hence the claimed result follows. □

A functor $T : \mathbf{C} \to \mathbf{D}$ between abelian categories is said to be *middle-exact* if for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence $T(A) \to T(B) \to T(C)$ is exact. Obviously every left (right) exact functor is middle-exact.

Proposition 1.1.2. *Every middle-exact functor is additive.*

Proof. Let $A, B \in \mathbf{C}$. Let p_1, p_2, i_1, i_2 be morphisms satisfying (1.1). It is easy to verify that (1.2) is equivalent to

(a) $p_1 i_1 = 1_A$ and $p_2 i_2 = 1_B$, and

(b) the sequences $0 \longrightarrow A \xrightarrow{i_1} C \xrightarrow{p_2} B \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{i_2} C \xrightarrow{p_1} A \longrightarrow 0$ are exact.

Finally, the use of Proposition 1.1.1 achieves the proof. $\square$

Abelian subcategory. Let $\mathbf{C}$ be an abelian category. A nonempty full subcategory of $\mathbf{C}$, $\mathbf{B}$, is called *an abelian subcategory* if $\mathbf{B}$ is abelian and the injection functor $\mathbf{B} \to \mathbf{C}$ is exact. The proof of the following lemma is straightforward and easy.

Lemma 1.1.3. *Let* $\mathbf{C}$ *be an abelian category, and* $\mathbf{B}$ *be a nonempty full subcategory of* $\mathbf{C}$*. Then*

(1) $\mathbf{B}$ *is additive if and only if* $0 \in \mathbf{B}$ *and for every* $B_1, B_2 \in \mathbf{B}$*,* $B_1 \oplus B_2 \in \mathbf{B}$*.*

(2) $\mathbf{B}$ *is an abelian subcategory of* $\mathbf{C}$ *if and only if* $\mathbf{B}$ *is additive, and for every morphism* $f : B \to B'$ *in* $\mathbf{B}$*,* $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathbf{B}$ ([38, Theorem 3.41]).

Diagram categories. We propose here a detailed exposition of diagram categories. This paragraph is essentially due to Grothendieck, see [47].

A *diagram scheme* is a triple $S = (I, \Phi, d)$ consisting of two sets I and Φ and a map $d : \Phi \to I \times I$. The elements of I are called the *vertices*, the elements of Φ are called the *arrows* of the scheme. If φ is an arrow of the diagram, $d(\varphi) = (i, j)$ is called its *direction*, characterized by the origin and the extremity of the arrow: i and j. In this situation we write $\varphi : i \to j$.

A *composite arrow* φ of origin i and extremity j is defined to be a nonempty finite sequence of arrows $\varphi = (\varphi_1, \ldots, \varphi_n)$ such that the origin of φ_1 is i and the extremity of φ_n is j. We say that n is the *length* of φ.

Let $\mathbf{C}$ be a category and S a diagram scheme. A *diagram in* $\mathbf{C}$ *of scheme* S is a map D which assigns to every $i \in I$ an object $D(i) \in \mathbf{C}$ and assigns to every arrow $\varphi : i \to j$, a morphism $D(\varphi) : D(i) \to D(j)$ in $\mathbf{C}$. A morphism υ from a diagram D to another D' is a family of morphisms $\upsilon_i : D(i) \to D'(i)$ such that for every arrow $\varphi : i \to j$, the diagram in $\mathbf{C}$

$$\begin{tikzcd} D(i) \arrow[r, "\upsilon_i"] \arrow[d, "D(\varphi)"'] & D'(i) \arrow[d, "D'(\varphi)"] \\ D(j) \arrow[r, "\upsilon_j"] & D'(j) \end{tikzcd}$$

is commutative. Let $\upsilon : D \to D'$ and $\upsilon' : D' \to D''$ be morphisms of diagrams. We define two morphisms of diagrams $\upsilon'\upsilon : D \to D''$ and $1_D : D \to D$ by $(\upsilon'\upsilon)_i = \upsilon'_i \upsilon_i$ and $(1_D)_i = 1_{D(i)}$ for every $i \in I$.

The diagrams in $\mathbf{C}$ of scheme S with their morphisms is a category which we denote by $\mathbf{C}^S$.

A diagram scheme $S = (I, \Phi, d)$ and a diagram $D \in \mathbf{C}^S$ are called *finite* (resp. *discrete*) if I is a finite set (resp. $\Phi = \emptyset$).

Now let D be a diagram $\mathbf{C}$ of scheme S and $\varphi = (\varphi_1, \dots, \varphi_n)$ be a composite arrow of origin i and extremity j. Define the morphism $D(\varphi) = D(\varphi_n) \dots D(\varphi_1) : D(i) \to D(j)$ in $\mathbf{C}$. We say that D is a *commutative diagram* if $D(\varphi) = D(\varphi')$ for every composite arrows φ and φ' which have the same origin and the same extremity.

Lemma 1.1.4. [The Cube Lemma]
Consider the following diagram in a fixed category.

$$\begin{tikzcd}
& A \arrow[rr, "f_1"] \arrow[dl, "f_3"'] \arrow[dd, "h_1"] & & B \arrow[dl, "f_2"] \arrow[dd, "h_2"] \\
C \arrow[rr, crossing over, "f_4"] \arrow[dd, "h_3"] & & D & \\
& E \arrow[rr, "g_1"] \arrow[dl, "g_3"] & & F \arrow[dl, "g_2"] \\
G \arrow[rr, "g_4"] & & H \arrow[from=uu, crossing over, "h_4"] &
\end{tikzcd}.$$

If h_4 is a monomorphism, and all squares, save possibly the top one, are commutative, then the cube is also commutative.

Proof. We have

$$h_4 f_2 f_1 = g_2 h_2 f_1 = g_2 g_1 h_1 = g_4 g_3 h_1 = g_4 h_3 f_3 = h_4 f_4 f_3.$$

Since h_4 is a monomorphism, we obtain $f_2 f_1 = f_4 f_3$. $\square$

Let $S = (I, \Phi, d)$ be a diagram scheme. We define the *dual scheme* of S to be $S^\circ = (I, \Phi, d^\circ)$ where $d^\circ : \Phi \to I \times I$ is the map defined by $d^\circ(\varphi) = (j, i)$ for every $\varphi \in \Phi$ such that $d(\varphi) = (i, j)$. Consider the contravariant functor

$$(-)^\circ : \mathbf{C}^S \to (\mathbf{C}^\circ)^{S^\circ} \tag{1.3}$$

which assigns to a diagram $D \in \mathbf{C}^S$ the diagram $D^\circ \in (\mathbf{C}^\circ)^{S^\circ}$ defined by $D^\circ(i) = D(i)$ and $D^\circ(\varphi^\circ) = D(\varphi)^\circ$ for every $i \in I$ and $\varphi \in \Phi$. It also assigns to a morphism of diagrams $\upsilon : D \to D'$ the morphism of diagrams $\upsilon^\circ : D'^\circ \to D^\circ$ such that $(\upsilon^\circ)_i = (\upsilon_i)^\circ$ for every $i \in I$. This functor leads to an isomorphism of categories

$$(\mathbf{C}^S)^\circ \simeq (\mathbf{C}^\circ)^{S^\circ}. \tag{1.4}$$

An object A of a category $\mathbf{C}$ is called an *initial object* if $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is a singleton for every $B \in \mathbf{C}$. An initial object is determined up to an isomorphism. Dually we define a final object. An object is called a *zero object* if it is at the same time an initial and final object. In a preadditive category, an object is an initial object if and only it is a final object, if and only if it is a zero object.

Now we will see that the category $\mathbf{C}^S$ inherits interesting properties from the category $\mathbf{C}$.

Lemma 1.1.5. *Let $\mathbf{C}$ be a category and $S = (I, \Phi, d)$ a diagram scheme. Then*

(i) If $\mathbf{C}$ is a k-category then the same hold for $\mathbf{C}^S$.

(ii) If $\mathbf{C}$ has an initial (resp. a final, resp. a zero) object then the same hold for $\mathbf{C}^S$.

(iii) If $\mathbf{C}$ has products (finite products) (resp. direct sums (finite direct sums)) then the same hold for $\mathbf{C}^S$.

(iv) If $\mathbf{C}$ has kernels (resp. cokernels) then the same hold for $\mathbf{C}^S$.

(v) Let $\mathbf{C}$ be a k-abelian category. Then so is $\mathbf{C}^S$. A subobject of a diagram $D \in \mathbf{C}^S$ is a family of objects $(D'(i))_{i\in I}$ such that $D'(i)$ is a subobject of $D(i)$ for every $i \in I$, and for every arrow $\varphi : i \to j$ of S, $D(\varphi).D'(i) \subset D'(j)$ as subobjects of $D(j)$. The morphism $D'(\varphi)$ is then defined to be the unique morphism making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & D'(i) \arrow[r] \arrow[d, "D'(\varphi)"'] & D(i) \arrow[d, "D(\varphi)"] \\
0 \arrow[r] & D'(j) \arrow[r] & D(j).
\end{tikzcd}$$

If moreover $\mathbf{C}$ satisfy one of the statements AB 4), AB 5), AB 6), or one of their dual statements AB 4), AB 5*), AB 6*), then the same hold for $\mathbf{C}^S$.*

Proof. (i) Let $\mathbf{C}$ be a k-category and $\upsilon, \omega : D \to D'$ two morphisms of diagrams in $\mathbf{C}^S$. Set

$$(\upsilon + \omega)_i = \upsilon_i + \omega_i, \ (h\upsilon)_i = h\upsilon_i$$

for all $i \in I$ and $h \in k$. It is easy to verify that $\mathrm{Hom}_{\mathbf{C}^S}(D, D')$ is a k-module with the morphism of diagrams $0_i = 0 : D(i) \to D'(i)$, $i \in I$, as neutral element for the addition, and the composition is k-bilinear.

(ii) Let C be an initial object in $\mathbf{C}$. Let $D \in \mathbf{C}^S$ be the diagram defined by: $D(i) = C$ for every $i \in I$, and $D(\varphi)$ be the unique morphism $C \to C$ in $\mathbf{C}$ for every $\varphi \in \Phi$. Obviously D is an initial object in $\mathbf{C}^S$. Analogously we prove the second statement. The third one is obvious from the first and the second ones.

(iii) Suppose that $\mathbf{C}$ has products. Let $(D_\lambda)_{\lambda\in\Lambda}$ be a nonempty family of diagrams in $\mathbf{C}^S$. For every $i \in I$ let $(\prod_{\lambda\in\Lambda} D_\lambda(i), (p_{\lambda,i}))$ be a product of $D_\lambda(i)$, $\lambda \in \Lambda$, where

$$p_{\lambda,i} : \prod_{\mu\in\Lambda} D_\mu(i) \to D_\lambda(i),$$

$\lambda \in \Lambda$, are the canonical projections. Let $\varphi : i \to j$ be an arrow of S. Consider the morphism

$$\prod_{\lambda\in\Lambda} D_\lambda(\varphi) : \prod_\lambda D_\lambda(i) \to \prod_\lambda D_\lambda(j)$$

in $\mathbf{C}$, i.e. the unique morphism making commutative the diagram

$$\begin{tikzcd} \prod_\mu D_\mu(i) \arrow[r, "p_{\lambda,i}"] \arrow[d, "\prod_\mu D_\mu(\varphi)"'] & D_\lambda(i) \arrow[d, "D_\lambda(\varphi)"] \\ \prod_\mu D_\mu(j) \arrow[r, "p_{\lambda,j}"] & D_\lambda(j) \end{tikzcd}$$

for every $\lambda \in \Lambda$. Then we obtain the diagram defined by

$$(\prod_{\lambda\in\Lambda} D_\lambda)(i) = \prod_{\lambda\in\Lambda} D_\lambda(i),\ (\prod_{\lambda\in\Lambda} D_\lambda)(\varphi) = \prod_{\lambda\in\Lambda} D_\lambda(\varphi),$$

and for every $\lambda \in \Lambda$, $p_\lambda = (p_{\lambda,i})_{i\in I} : \prod_\mu D_\mu \to D_\lambda$ is a morphism of diagrams.

Finally, let $\upsilon_\lambda : D \to D_\lambda$, $\lambda \in \Lambda$, be a family of morphisms of diagrams. We should prove that there is a unique morphism of diagrams $\upsilon : D \to \prod_\mu D_\mu$ making commutative the diagram

$$\begin{tikzcd} D \arrow[r, "\upsilon_\lambda"] \arrow[dr, "\upsilon"] & D_\lambda \\ & \prod_\mu D_\mu \arrow[u, "p_\lambda"'] \end{tikzcd}$$

for every $\lambda \in \Lambda$.

For this let $i \in I$. Then there is a unique morphism $\upsilon_i : D(i) \to \prod_\mu D_\mu(i)$ in $\mathbf{C}$ making commutative the diagram

$$\begin{tikzcd} D(i) \arrow[r, "(\upsilon_\lambda)_i"] \arrow[dr, "\upsilon_i"] & D_\lambda(i) \\ & \prod_\mu D_\mu(i) \arrow[u, "p_{\lambda,i}"'] \end{tikzcd}$$

for every $\lambda \in \Lambda$. It remains to verify that υ is a morphism of diagrams. Let $\varphi : i \to j$ be an arrow of S and $\lambda \in \Lambda$. Consider the diagram

$$\begin{tikzcd} D(i) \arrow[rr, "D(\varphi)"] \arrow[dr, "(\upsilon_\lambda)_i"] \arrow[dd, "\upsilon_i"'] & & D(j) \arrow[dr, "(\upsilon_\lambda)_j"] \arrow[dd, "\upsilon_j"] & \\ & D_\lambda(i) \arrow[rr, "D_\lambda(\varphi)"] & & D_\lambda(j) \\ \prod_\mu D_\mu(i) \arrow[ur, "p_{\lambda,i}"] \arrow[rr, "\prod_\mu D_\mu(\varphi)"] & & \prod_\mu D_\mu(j) \arrow[ur, "p_{\lambda,j}"'] & \end{tikzcd} .$$

It follows that

$$p_{\lambda,j}\upsilon_j D(\varphi) = p_{\lambda,j}(\prod_{\mu} D_\mu(\varphi))\upsilon_i$$

for every $\lambda \in \Lambda$. Hence $\upsilon_j D(\varphi) = (\prod_\mu D_\mu(\varphi))\upsilon_i$.

Analogously we prove the second statement.

Now suppose that $\mathbf{C}$ has direct sums. Let $(D_\lambda)_{\lambda\in\Lambda}$ be a nonempty family of diagrams in $\mathbf{C}^S$. For every $i \in I$ let $(\oplus_{\lambda\in\Lambda} D_\lambda(i), (u_{\lambda,i}))$ be a direct sum of $D_\lambda(i)$, $\lambda \in \Lambda$, where

$$u_{\lambda,i} : D_\lambda(i) \to \oplus_{\mu\in\Lambda} D_\mu(i),$$

$\lambda \in \Lambda$, are the canonical injections. Define as above the diagram in $\mathbf{C}^S$

$$(\oplus_{\lambda\in\Lambda} D_\lambda)(i) = \oplus_{\lambda\in\Lambda} D_\lambda(i),\ (\oplus_{\lambda\in\Lambda} D_\lambda)(\varphi) = \oplus_{\lambda\in\Lambda} D_\lambda(\varphi).$$

Moreover, we have that $u_\lambda = (u_{\lambda,i})_i : D_\lambda \to \oplus_{\mu\in\Lambda} D_\mu$ is a morphism in $\mathbf{C}^S$ for every $\lambda \in \Lambda$.

Let $(-)^\circ : \mathbf{C}^S \to (\mathbf{C}^\circ)^{S^\circ}$ be the contravariant functor (1.3). We have

$$(\oplus_{\lambda\in\Lambda} D_\lambda)^\circ(i) = \oplus_{\lambda\in\Lambda} D_\lambda(i) = \oplus_{\lambda\in\Lambda} D^\circ_\lambda(i) = (\prod_{\lambda\in\Lambda} D^\circ_\lambda)(i)$$

and

$$(\oplus_{\lambda\in\Lambda} D_\lambda)^\circ(\varphi^\circ) = \big[(\oplus_{\lambda\in\Lambda} D_\lambda)(\varphi)\big]^\circ = \big(\oplus_{\lambda\in\Lambda} D_\lambda(\varphi)\big)^\circ = \prod_{\lambda\in\Lambda} D^\circ_\lambda(\varphi^\circ) = (\prod_{\lambda\in\Lambda} D^\circ_\lambda)(\varphi^\circ),$$

for every $i \in I$ and $\varphi \in \Phi$. Hence

$$(\oplus_{\lambda\in\Lambda} D_\lambda)^\circ = \prod_{\lambda\in\Lambda} D^\circ_\lambda. \tag{1.5}$$

Finally, $(\oplus_{\lambda\in\Lambda} D_\lambda, (u_\lambda))$ is then a direct sum of the family $(D_\lambda)_\lambda$ in $\mathbf{C}^S$.

Analogously we prove the last statement.

(iv) Suppose that $\mathbf{C}$ is additive and has kernels. Let $\upsilon : D \to D'$ be a morphism in $\mathbf{C}^S$. Set $K(i) = \mathrm{Ker}(\upsilon_i)$ for all $i \in I$. Let $\varphi : i \to j$ be an arrow of S. Then there is a unique morphism $K(\varphi) : K(i) \to K(j)$ in $\mathbf{C}$ making commutative the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K(i) \arrow[r, "\ker(\upsilon_i)"] \arrow[d, "K(\varphi)"] & D(i) \arrow[r, "\upsilon_i"] \arrow[d, "D(\varphi)"] & D'(i) \arrow[d, "D'(\varphi)"] \\
0 \arrow[r] & K(j) \arrow[r, "\ker(\upsilon_i)"] & D(j) \arrow[r, "\upsilon_j"] & D'(j).
\end{tikzcd} \tag{1.6}$$

We obtain that $K \in \mathbf{C}^S$ and $(\ker(\upsilon_i))_{i\in I} : K \to D$ is a morphism in $\mathbf{C}^S$.

Now, let $\xi : D'' \to D$ be a morphism in $\mathbf{C}^S$ such that $\upsilon\xi = 0$. Then, for all $i \in I$, there is a unique morphism $\omega_i : D''(i) \to k(i)$ in $\mathbf{C}$ making commutative the diagram

$$\begin{tikzcd}
 & & D''(i) \arrow[dll, "\omega_i"'] \arrow[dr, "\xi_i"] & & \\
0 \arrow[r] & K(i) \arrow[rr, "\ker(\upsilon_i)"] & & D(i) \arrow[r, "\upsilon_i"] & D'(i).
\end{tikzcd}$$

To complete the proof of the first statement it is enough to verify that $\omega = (\omega_i)_i : D'' \to K$ is a morphism in $\mathbf{C}^S$. For this, let $\varphi : i \to j$ be an arrow of S and consider the diagram

$$\begin{tikzcd}
D''(i) \arrow[rr, "D''(\varphi)"] \arrow[dr, "\xi_i"] \arrow[dd, "\omega_i"'] & & D''(j) \arrow[dr, "\xi_j"] \arrow[dd, "\omega_j"] & \\
& D(i) \arrow[rr, "D(\varphi)"] & & D(j) \\
K(i) \arrow[rr, "K(\varphi)"] \arrow[ur, "\ker(\upsilon_i)"] & & K(j) \arrow[ur, "\ker(\upsilon_j)"'] &
\end{tikzcd}.$$

Since $\ker(\upsilon_j)$ is a monomorphism in $\mathbf{C}$, the rectangle is commutative. Hence K is the kernel of υ in $\mathbf{C}^S$.

Suppose that $\mathbf{C}$ is additive and has cokernels. Let $\upsilon : D \to D'$ be a morphism in $\mathbf{C}^S$. Set $K'(i) = \mathrm{Coker}(\upsilon_i)$ for all $i \in I$. Let $\varphi : i \to j$ be an arrow of S. Then there is a unique morphism $K'(\varphi) : K'(i) \to K'(j)$ in $\mathbf{C}$ making commutative the diagram with exact rows

$$\begin{tikzcd}
D(i) \arrow[r, "\upsilon_i"] \arrow[d, "D(\varphi)"] & D'(i) \arrow[r, "\mathrm{coker}(\upsilon_i)"] \arrow[d, "D'(\varphi)"] & K'(i) \arrow[r] \arrow[d, "K'(\varphi)"] & 0 \\
D(j) \arrow[r, "\upsilon_j"] & D(j) \arrow[r, "\mathrm{coker}(\upsilon_j)"] & K'(j) \arrow[r] & 0.
\end{tikzcd}$$

Then $(K'^{\circ}, (\mathrm{coker}(\upsilon_i)^{\circ})_i)$ is the kernel of $\upsilon^{\circ} : D'^{\circ} \to D^{\circ}$ in $(\mathbf{C}^{\circ})^{S^{\circ}}$. Hence $(K', (\mathrm{coker}(\upsilon_i))_i)$ is the cokernel of $\upsilon : D \to D'$ in $\mathbf{C}^S$.

Finally, (v) is obvious from the statements (i) to (iv), their proofs, and the fact that a morphism $\upsilon : D \to D'$ in $\mathbf{C}^S$ is an isomorphism if and only if $\upsilon_i : D(i) \to D'(i)$ is an isomorphism in $\mathbf{C}$ for every $i \in I$. $\square$

Remark 1.1.6. Let $\mathbf{C}$ be a category and $S = (I, \Phi, d)$ a diagram scheme.

(a) Suppose that $\mathbf{C}$ has products. Let $(D_\lambda)_{\lambda \in \Lambda}$ be a nonempty family of diagrams in $\mathbf{C}^S$. For a composite arrow φ, $(\prod_\lambda D_\lambda)(\varphi) = \prod_\lambda D_\lambda(\varphi)$.

(b) Suppose that $\mathbf{C}$ is additive and has kernels. Let $\upsilon : D \to D'$ be a morphism in $\mathbf{C}^S$. Let K be the diagram in $\mathbf{C}^S$ defined as in the proof of Lemma 1.1.5 (iv). Let φ be a composite arrow of S of origin i and extremity j. Then $K(\varphi) : K(i) \to K(j)$ is the unique morphism making commutative the diagram (1.6).

Now we consider very interesting full subcategories of $\mathbf{C}^S$. Let S be a diagram scheme and $\mathbf{C}$ an arbitrary category.

(i) Let R be a set of couples (φ, φ') where φ and φ' are two composite arrows having the same origin and the same extremity, and of composite arrows φ whose origin and extremity coincide. Consider $\mathbf{C}^{(S,R)}$ the full subcategory of $\mathbf{C}^S$ consisting of the diagrams $D \in \mathbf{C}^S$ such that $D(\varphi) = D(\varphi')$ for every $(\varphi, \varphi') \in R$ and $D(\varphi) = 1_{D(i)}$ for every $\varphi \in R$ of origin and extremity i.

(ii) Now assume that $\mathbf{C}$ is a k-category. Let $(i, j) \in I \times I$. Let R_{ij} be a set of formal linear combinations of composite arrows of origin i and extremity j with coefficients in k, and if $i = j$ we add some formal linear combinations of composite arrows of origin and extremity i, and a fixed element e_i, with coefficients in k. Let D be a diagram in $\mathbf{C}^S$. We define for every $L \in R_{ij}$, the morphism $D(L) : D(i) \to D(j)$ by replacing in the expression of L, a composite arrow φ by $D(\varphi)$, and e_i by $1_{D(i)}$. Set $R = \cup_{(i,j)\in I\times I} R_{ij}$.

A diagram D in $\mathbf{C}^S$ is called *R-commutative* if $D(L) = 0$ for every $L \in R$. The couple $(S, R) = \Sigma$ is called a *diagram scheme with commutation relations*. The equalities $L = 0$, $L \in R$, are called the *commutation relations*. Consider $\mathbf{C}^\Sigma$ the full subcategory of $\mathbf{C}^S$ consisting of the R-commutative diagrams.

Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a composite arrow of S of origin i and extremity j. Define the composite arrow $\varphi^\circ = (\varphi_n^\circ, \dots, \varphi_1^\circ)$ of S° of origin j and extremity i. Let $(-)^\circ : \mathbf{C}^S \to (\mathbf{C}^\circ)^{S^\circ}$ be the contravariant functor (1.3). Then $D^\circ(\varphi^\circ) = D(\varphi)^\circ$. Let $(i, j) \in I \times I$ and $L \in R_{ij}$. Let L° be the linear combination obtained by replacing in the expression of L each composite arrow φ by φ°, and if $i = j$, e_i by a fixed element e_i°. Let R_{ji}° be the set of L° where $L \in R_{ij}$. Set $R^\circ = \cup_{(i,j)\in I\times I} R_{ji}^\circ$. If $L \in R_{ij}$, then $D^\circ(L^\circ) = D(L)^\circ$. Therefore, D is R-commutative if and only if D° is R°-commutative. Hence we obtain an isomorphism of categories

$$(\mathbf{C}^\Sigma)^\circ \simeq (\mathbf{C}^\circ)^{\Sigma^\circ}. \tag{1.7}$$

Proposition 1.1.7. *Let $\mathbf{C}$ a k-category and $(S, R) = \Sigma$ a diagram scheme with commutation relations. Then*

(1) $\mathbf{C}^\Sigma$ is a k-category and the injection functor $\mathbf{C}^\Sigma \to \mathbf{C}^S$ is k-linear. Moreover, if $\mathbf{C}$ has a zero object, then the zero object of $\mathbf{C}^S$ belongs in $\mathbf{C}^\Sigma$.

(2) Suppose that $\mathbf{C}$ has products (resp. finite products). Let $(D_\lambda)_{\lambda\in\Lambda}$ be a nonempty family (resp. finite family) of diagrams in $\mathbf{C}^\Sigma$. Then $\prod_{\lambda\in\Lambda} D_\lambda$ belongs in $\mathbf{C}^\Sigma$. The same hold for direct sums.

(3) Suppose that $\mathbf{C}$ has kernels. Let $\upsilon : D \to D'$ be a morphism in $\mathbf{C}^\Sigma$ and K its kernel in $\mathbf{C}^S$. Then K belongs in $\mathbf{C}^\Sigma$. The same hold for cokernels.

(4) If $\mathbf{C}$ is an abelian category, then $\mathbf{C}^\Sigma$ is an abelian subcategory of $\mathbf{C}^S$.

Proof. (1) Obvious.

(2) Let $(i, j) \in I \times I$ and $L = \sum_{l=1}^m h_l \varphi^l$, where φ^l is a composite arrow of origin i and

extremity j and $h_l \in k$ for every l. Then

$$\begin{aligned}
(\prod_\lambda D_\lambda)(L) &= \sum_{l=1}^{m} h_l (\prod_\lambda D_\lambda)(\varphi^l) \\
&= \sum_{l=1}^{m} h_l \prod_\lambda D_\lambda(\varphi^l) \\
&= \prod_\lambda (\sum_{l=1}^{m} h_l D_\lambda(\varphi^l)) \\
&= \prod_\lambda D_\lambda(L).
\end{aligned}$$

Hence, if $L \in R_{ij}$ then $(\prod_\lambda D_\lambda)(L) = 0$.

On the other hand,

$$\begin{aligned}
(\prod_\lambda D_\lambda)(e_i) &= 1_{(\prod_\lambda D_\lambda)(i)} \\
&= 1_{\prod_\lambda D_\lambda(i)} \\
&= \prod_\lambda 1_{D_\lambda(i)} \\
&= \prod_\lambda D_\lambda(e_i).
\end{aligned}$$

Let $h \in k$. Then

$$\begin{aligned}
(\prod_\lambda D_\lambda)(L + he_i) &= (\prod_\lambda D_\lambda)(L) + h(\prod_\lambda D_\lambda)(e_i) \\
&= \prod_\lambda D_\lambda(L) + h \prod_\lambda D_\lambda(e_i) \\
&= \prod_\lambda D_\lambda(L) + hD_\lambda(e_i) \\
&= \prod_\lambda D_\lambda(L + he_i).
\end{aligned}$$

Hence, if $L + he_i \in R_{ii}$ then $(\prod_\lambda D_\lambda)(L + he_i) = 0$.

Analogously we prove the second statements.

The last statement follows from the first one and the isomorphism of categories (1.7).

(3) Let $(i, j) \in I \times I$ and $L = \sum_{l=1}^{m} h_l \varphi^l$, where φ^l is a composite arrow of origin i and extremity j and $h_l \in k$ for every l. Then

$$D(L) \ker(\upsilon_i) = \ker(\upsilon_j) K(L).$$

Hence, if $L \in R_{ij}$ then $K(L) = 0$.

Let $h \in k$. Then
$$D(L + he_i)\ker(\upsilon_i) = \ker(\upsilon_j)K(L + he_i).$$
Hence, if $L + he_i \in R_{ii}$ then $K(L + he_i) = 0$.

The last statement follows from the first one and the isomorphism of categories (1.7).

(4) Obvious using Lemma 1.1.3 and the statements (1), (2) and (3). □

Now let $F : \mathbf{C} \to \mathbf{C}'$ be a functor and $S = (I, \Phi, d)$ a diagram scheme. Define the functor $F^S : \mathbf{C}^S \to \mathbf{C}'^S$ such that
$$F^S(D)(i) = F(D(i)),\ F^S(D)(\varphi) = F(D(\varphi))$$
for every $D \in \mathbf{C}^S$, $i \in I$, $\varphi \in \Phi$, and
$$F^S(\upsilon)_i = F(\upsilon_i) : F(D(i)) \to F(D'(i))$$
for every morphism $\upsilon : D \to D'$ in $\mathbf{C}^S$ and $i \in I$.

If $F' : \mathbf{C} \to \mathbf{C}'$ another functor and $\eta : F \to F'$ is a natural transformation, then $\eta^S : F^S \to F'^S$ defined by
$$(\eta^S_D)_i = \eta_{D(i)} : F(D(i)) \to F'(D(i)),$$
for every $D \in \mathbf{C}^S$ and $i \in I$, is a natural transformation.

Lemma 1.1.8. *(a) $F \mapsto F^S, \eta \mapsto \eta^S$ acts as a functor.*

(b) Let $G : \mathbf{C}' \to \mathbf{C}''$ be another functor. Then $(GF)^S = G^S F^S$.

(c) If $\mathbf{C}$ and $\mathbf{C}'$ are abelian categories and F is left (resp. right) exact, then F^S is so.

Proof. Easy verifications. □

Now assume that $\mathbf{C}$ and $\mathbf{C}'$ are k-categories and let $\Sigma = (S, R)$ be a diagram scheme with commutation relations and $F : \mathbf{C} \to \mathbf{C}'$ a k-linear functor. The functor F^S is k-linear and induces a k-linear functor $F^\Sigma : \mathbf{C}^\Sigma \to \mathbf{C}'^\Sigma$. The functors F^S and F^Σ are called *canonical prolongations of F to diagrams.* Sometimes and if there is no risk of confusion we denote the functors F^S and F^Σ by F.

If $F' : \mathbf{C} \to \mathbf{C}'$ another k-linear functor and $\eta : F \to F'$ is a natural transformation, then η^S induces a natural transformation $\eta^\Sigma : F^\Sigma \to F'^\Sigma$.

Proposition 1.1.9. *(a) $F \mapsto F^\Sigma, \eta \mapsto \eta^\Sigma$ acts as a functor.*

(b) Let $G : \mathbf{C}' \to \mathbf{C}''$ be another k-linear functor. Then $(GF)^\Sigma = G^\Sigma F^\Sigma$.

(c) If $\mathbf{C}$ and $\mathbf{C}'$ are abelian categories and F is left (resp. right) exact, then F^Σ is so.

Proof. Obvious from the last lemma. □

Examples 1.1.10. Let $\mathbf{C}$ be an arbitrary category.

(1) Assume that $\mathbf{C}$ is a k-category. Let I be a singleton and $\Phi = \emptyset$. The commutation relations are of the form $he = 0$. Let $h_1, \ldots, h_n$ be of elements of k. Consider the commutation relations $h_l e = 0$, $l = 1, \ldots, n$. Then $\mathbf{C}^\Sigma$ is the full subcategory of $\mathbf{C}$ consisting of the objects vanished by h_l for all l.

(2) Now we take I is a set which has exactly two elements i and j, and $\Phi = \{\varphi\}$ is a singleton with φ of origin i and extremity j. Then $\mathbf{C}^S$ is the category of morphisms $u : A \to B$ between objects of $\mathbf{C}$. If we add the commutation relation $h\varphi = 0$ where $h \in k$, we restricts ourselves to A and B. u is vanished by h. If $h = 1$, then $\mathbf{C}^\Sigma \simeq \mathbf{C} \times \mathbf{C}$.

(3) **Functor categories.** Let $\mathbf{I}$ be a small category. Let $S = (I, \Phi, d)$ be the diagram scheme such that I and Φ are respectively the sets of objects and morphisms of the category $\mathbf{I}$. The directions of the arrows is defined by a natural way. Let R be the set of $(gf, (f, g))$ where f and g are morphisms in $\mathbf{I}$ such that (f, g) is a composite arrow of length is 2, and the identity morphisms. Then the category of functors $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$ is the category $\mathbf{C}^{(S,R)}$.

   Assume moreover that $\mathbf{I}$ and $\mathbf{C}$ are k-categories. $\mathbf{Hom}_k(\mathbf{I}, \mathbf{C})$ denotes the full subcategory of $\mathbf{Fun}(\mathbf{I}, \mathbf{C})$ consisting of the k-functors. If we add the commutation relations $(f + g) - f - g = 0$ and $(hf) - h.f = 0$, where f and g are morphisms in $\mathbf{I}$ and $h \in k$, we obtain that $\mathbf{Hom}_k(\mathbf{I}, \mathbf{C})$ can be viewed as a category $\mathbf{C}^\Sigma$.

(4) **Complexes with values in C.** $I = \mathbb{Z}$ (the set of integers), $\Phi = \{d_n \mid n \in \mathbb{Z}\}$. For every n, d_n of origin n and extremity $n+1$. The commutation relations are $d_{n+1} d_n = 0$. In order to consider complexes with positive degrees we take $I = \mathbb{N}$ (the set of positive integers).

Faithful functors. Let $T : \mathbf{C} \to \mathbf{D}$ be a functor. T is called *faithful* (*full*, *fully faithful*) if the maps

$$T : \mathrm{Hom}_{\mathbf{C}}(C, C') \to \mathrm{Hom}_{\mathbf{D}}(T(C), T(C'))$$

are injective (surjective, bijective), for every $C, C' \in \mathbf{C}$.

T is called *conservative* if it reflects isomorphisms (for every morphism in $\mathbf{C}$, $f : A \to A'$, $T(f)$ is an isomorphism implies f is also an isomorphism). Obviously a fully faithful functor is conservative.

Proposition 1.1.11. [Freyd] ([69, Theorem II.7.1])
Let $T : \mathbf{C} \to \mathbf{D}$ be a faithful functor. Then

(1) T reflects monomorphisms, epimorphisms, and commutative diagrams.

(2) If $\mathbf{C}$ and $\mathbf{D}$ are preadditive categories with zero and T is additive, then T reflects zero objects.

(3) If $\mathbf{C}$ and $\mathbf{D}$ are abelian categories and T is zero preserving, then reflects exact sequences.

Proposition 1.1.12. *Let $T : \mathbf{C} \to \mathbf{D}$ be a functor between abelian categories. Consider the below statements*

(1) T is faithful.

(2) T is conservative.

(3) $T(C) \simeq 0 \Longrightarrow C \simeq 0$ for every $C \in \mathbf{C}$.

Then:

(i) (1) implies (2).

(ii) If T is additive, then (2) implies (3).

(iii) If T is exact (and then additive by Proposition 1.1.2), then (1), (2) and (3) are equivalent.

Proof. First (1) $\Longrightarrow$ (2) Follows from Proposition 1.1.11 (1).

(ii) Assume that T is additive and (2) holds. Let $C \in \mathbf{C}$ such that $T(C) \simeq 0$. Then $T(0) = 0 : T(C) \to T(C)$ is an isomorphism. Thus, $0 : C \to C$ is also an isomorphism. Hence $\mathrm{Hom}_{\mathbf{C}}(C, C)$ is a singleton.

(iii) Assume that T is exact. It is enough to show (3) $\Longrightarrow$ (1). For this, let f be a morphism in $\mathbf{C}$ such that $T(f) = 0$, that is $\mathrm{Im}(T(f)) = 0$. We have $T(\mathrm{Im}(f)) = \mathrm{Im}(T(f))$. Hence $\mathrm{Im}(f) = 0$, i.e. $f = 0$. □

Remark 1.1.13. We refer to [60, Exercises 8.26] for an example which shows that (2) does not imply (1) in general, and to [60, Exercises 8.27] for an example which shows that (3) does not imply (2) (and then does not imply (1)) in general.

Completely faithful modules. Let R be a ring and U_R and ${}_RM$ modules. The *annihilator in M of U* is

$$\mathrm{Ann}_M(U) = \{m \in M \mid u \otimes m = 0 \text{ in } U \otimes_R M \text{ for all } u \in U\}.$$

We say that U_R is *${}_RM$-faithful* if $\mathrm{Ann}_M(U) = 0$. Observe that U_R is ${}_RR$-faithful if and only if it is faithful.

Lemma 1.1.14. *Let $f : M \to M'$ be an epimorphism in $\mathcal{M}_R$ and $g : N \to N'$ an epimorphism in ${}_R\mathcal{M}$. Then $h = f \otimes_R g : M \otimes_R N \to M' \otimes_R N'$ is an epimorphism and $\mathrm{Ker}(h)$ is the set K of $\sum m_i \otimes n_i$ such that $m_i \in \mathrm{Ker}(f)$ or $n_i \in \mathrm{Ker}(g)$ for all i.*

Proof. First h is surjective since it is a composition of two surjective maps. It is obvious that $K \subset \mathrm{Ker}(h)$. Let Q be the quotient group $(M \otimes_R N)/K$ and $p : M \otimes_R N \to Q$ is the canonical epimorphism. Then h induces a morphism $h^* : Q \to M' \otimes_R N'$ such that $h = h^* p$.

Let $m' \in M'$ and $n' \in N'$ and suppose that $m' = f(m_1) = f(m_2)$ and $n' = g(n_1) = g(n_2)$ with $m_1, m_2 \in M$ and $n_1, n_2 \in N$. Then $m = m_2 - m_1 \in \mathrm{Ker}(f)$ and $n = n_2 - n_1 \in \mathrm{Ker}(g)$. Therefore,

$$m_2 \otimes n_2 - m_1 \otimes n_1 = m \otimes n_1 + m_1 \otimes n + m \otimes n \in K.$$

Hence, $p(m_1 \otimes n_1) = p(m_2 \otimes n_2)$.

Now, we can define a map $j : M' \times N' \to Q$ as follows. For $m' \in M'$ and $n' \in N'$, $j(m' \otimes n') = p(m \otimes n)$ where $m \in M$ and $n \in N$ such that $m' = f(m)$ and $n' = g(n)$. It is easy to verify that j is R-balanced. By the definition of the tensor product of modules (see [5, §19]), there is a unique abelian group morphism $j^* : M' \otimes_R N' \to Q$ such that $j^*(m \otimes n) = j(m, n)$ for every $m \in M, n \in N$.

Finally, we have that $j^* h^* p = j^* h = p$. Then $j^* h^* = 1_Q$ and h^* is injective. Hence $\mathrm{Ker}(h) = K$. □

In the following lemma we collect some basic properties of the annihilator that we need to prove Lemma 1.1.16. We refer to [5, Exercises 19.18, 19.20] for more properties.

Lemma 1.1.15. *Let R be a ring and U_R and ${}_RM$ modules.*

(a) $\mathrm{Ann}_M(U)$ *is the (unique) smallest submodule K of M such that U is M/K-faithful.*

(b) Let $(U_i)_{i \in I}$ be a family of right R-modules. Then

$$\mathrm{Ann}_M(\oplus_{i \in I} U_i) = \cap_{i \in I} \mathrm{Ann}_M(U_i).$$

(c) If U_R generates V_R, then $\mathrm{Ann}_M(U) \leq \mathrm{Ann}_M(V)$.

(d) U is M-faithful if and only if for every morphism $f : N \to M$ in ${}_R\mathcal{M}$,

$$U \otimes_R f = 0 \Longrightarrow f = 0.$$

Proof. (a) is straightforward using Lemma 1.1.14.

(b) is clear using the canonical isomorphism

$$(\oplus_{i \in I} U_i) \otimes_R M \to \oplus_{i \in I}(U_i \otimes_R M),\ (u_i)_i \otimes m \mapsto (u_i \otimes m)_i.$$

(c) Assume that U_R generates V_R. Then there is an epimorphism $U^{(I)} \to V$ in $\mathcal{M}_R$, where I is a nonempty set. Hence, $\mathrm{Ann}_M(U) = \mathrm{Ann}_M(U^{(I)}) \leq \mathrm{Ann}_M(V)$ (by (b)).

(d) ($\Rightarrow$) Obvious. ($\Leftarrow$) Let $m \in \mathrm{Ann}_M(U)$. To obtain $m = 0$, it is enough to take $N = Rm$ and $f : Rm \to M$ the canonical injection. □

Following [5, Exercise 19.19], a module W_R is said to be *completely faithful* if it is ${}_RM$-faithful for every left R-module M.

Lemma 1.1.16. *Let R be a ring and W_R a module.*

(1) R_R is completely faithful.

(2) Every generator in $\mathcal{M}_R$ is completely faithful.

(3) W_R is completely faithful if and only if the functor $W \otimes_R - : {}_R\mathcal{M} \to \boldsymbol{Ab}$ is faithful.

Proof. (1) is obvious. (2) is obvious from Lemma 1.1.15 (c) and (1). (3) is obvious from Lemma 1.1.15 (d). □

Generators and cogenerators. Let $\mathbf{C}$ be a category, and $(U_i)_{i\in I}$ a family of objects of $\mathbf{C}$. Following Grothendieck [47, p. 134], we say that $(U_i)_{i\in I}$ is a *family of generators* of $\mathbf{C}$ if for all $A \in \mathbf{C}$ and all subobject $B \neq A$, there exist $i \in I$ and a morphism $u : U_i \to A$ that does not factor through $B \to A$. We say that U is a *generator* if the family $\{U\}$ is a family of generators.

A category $\mathbf{C}$ is called *locally small* if for every object $A \in \mathbf{C}$, the class of subobjects of A is a set. We refer to [69] for the definitions and the basic properties of pullbacks and equalizers and their duals.

Proposition 1.1.17. [Grothendieck]
Every category $\mathbf{C}$ that has a family of generators and pullbacks is locally small.

Proof. Let $(U_i)_{i\in I}$ be a family of generators of $\mathbf{C}$ and $A \in \mathbf{C}$. Set $E = \bigcup_{i\in I} \mathrm{Hom}_{\mathbf{C}}(U_i, A)$. Let $\mathsf{L}(A)$ be the class of subobjects of A and $\mathcal{P}(E)$ the set of subsets of E. Consider the correspondence $\phi : \mathsf{L}(A) \to \mathcal{P}(E)$ defined by: for a subobject B of A, $\phi(B)$ is the set of morphisms $f : U_i \to A$ that factors through $B \to A$ for some $i \in I$. Let $B, B' \in \mathsf{L}(A)$ such that $B \neq B'$ and a pullback

$$\begin{tikzcd} P \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & B' \arrow[d, "\alpha_2"] \\ B \arrow[r, "\alpha_1"'] & A \end{tikzcd}$$

associated to the canonical injections α_1 and α_2. By [69, proposition I.7.1], β_1 and β_2 are monomorphisms. We have that $P \neq B$ or $P \neq B'$ since $B \neq B'$. Assume for example that $P \neq B$. Obviously $\phi(P) \subset \phi(B)$. This inclusion is strict from the definition of a family of generators. Now suppose that $\phi(B) = \phi(B')$ and let $u \in \phi(B)$. Therefore, there is an index i such that $u : U_i \to A$ is a morphism with $u = \alpha_1\xi_1 = \alpha_2\xi_2$ for some morphisms $\xi_1 : U_i \to B$ and $\xi_2 : U_i \to B'$. By the definition of a pullback, there is a morphism $\gamma : U_i \to P$ making commutative the following diagram

$$\begin{tikzcd} U_i \arrow[dr, "\gamma"] \arrow[drr, bend left, "\xi_2"] \arrow[ddr, bend right, "\xi_1"'] & & \\ & P \arrow[r, "\beta_2"] \arrow[d, "\beta_1"] & B' \arrow[d, "\alpha_2"] \\ & B \arrow[r, "\alpha_1"'] & A. \end{tikzcd}$$

Then $u = \alpha_1\beta_1\gamma$. Thus $\phi(B) = \phi(P)$. We obtain then a contradiction. Hence the correspondence ϕ is injective. It follows that $\mathsf{L}(A)$ is a set and $\mathrm{Card}(\mathsf{L}(A)) \leq \mathrm{Card}(\mathcal{P}(E))$. $\square$

Proposition 1.1.18. [47, Proposition 1.9.1]
Let $\mathbf{C}$ be an abelian category that has arbitrary direct sums. Let $(U_i)_{i\in I}$ be a family of objects of $\mathbf{C}$. Set $U = \oplus_{i\in I}U_i$. Then the below statements are equivalent

(1) $(U_i)_{i\in I}$ is a family of generators of $\mathbf{C}$;

(2) U is a generator of $\mathbf{C}$;

(3) every $A \in \mathbf{C}$ is isomorphic to a quotient object of $U^{(J)}$ where J is a set.

Proposition 1.1.19. [79, Proposition 2.3.4]
Let $\mathbf{C}$ be an abelian category and $(U_i)_{i\in I}$ a family of objects of $\mathbf{C}$. Then the below statements are equivalent

(1) $(U_i)_{i\in I}$ is a family of generators of $\mathbf{C}$;

(2) for every $A, C \in \mathbf{C}$ and every distinct morphisms $f, g \in \mathrm{Hom}_{\mathbf{C}}(A, C)$, there are an index i_0 and a morphism $h \in \mathrm{Hom}_{\mathbf{C}}(U_{i_0}, A)$ such that $fh \neq gh$;

(3) for every $A, C \in \mathbf{C}$ and every nonzero morphism $f \in \mathrm{Hom}_{\mathbf{C}}(A, C)$, there are an index i_0 and a morphism $h \in \mathrm{Hom}_{\mathbf{C}}(U_{i_0}, A)$ such that $fh \neq 0$.

Hence, $U \in \mathbf{C}$ is a generator if and only if the covariant functor $\mathrm{Hom}_{\mathbf{C}}(U, -)$ is faithful.

From the last result and Proposition 1.1.12, we obtain

Proposition 1.1.20. [85, Proposition V.6.3]
A projective object P of an abelian category is a generator if and only if there is a nonzero morphism $P \to C$ for every nonzero object C.

Dually we define a family of cogenerators of $\mathbf{C}$.

An abelian category is called a *Grothendieck category* if it has a generator and satisfying the AB 5) condition. The following result is due to Grothendieck and it extends Baer's criterion for injective modules.

Proposition 1.1.21. [47, Lemme 1, p. 136]
Let $\mathbf{C}$ be a Grothendieck category with a generator U. Then $M \in \mathbf{C}$ is injective if and only for every subobject V of U, and every morphism $v : V \to M$, there is a morphism $U \to M$ extending v.

Notice that Proposition 1.1.21 remains true for an arbitrary family of generators.

An abelian category is said to have *enough injectives* if for every object is a subobject of an injective object. Dually we define a category having enough projectives.

Proposition 1.1.22. [47, Théorème 1.10.1]
Every Grothendieck category has enough injectives.

Proposition 1.1.23. [Grothendieck] ([79, Lemma 3.7.12])
Every abelian category which has arbitrary direct sums, a generator, and enough injectives (e.g. Grothendieck categories), has an injective cogenerator object.

Adjoint and Frobenius pairs of functors. Adjoint functors were introduced and studied in [58].

Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors, and

$$\eta : \mathrm{Hom}_{\mathbf{D}}(S(-), -) \to \mathrm{Hom}_{\mathbf{C}}(-, T(-))$$

a natural transformation of functors $\mathbf{C}^{\circ} \times \mathbf{D} \to \mathbf{Set}$. Let

$$\zeta_C := \eta_{C,S(C)}(1_{S(C)}) : C \to TS(C),$$

for every $C \in \mathbf{C}$.

The following five results are due to W. Shih [84] (see also [64]).

Lemma 1.1.24. *ζ is a natural transformation and the correspondence*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathbf{D}}(S(-), -), \mathrm{Hom}_{\mathbf{C}}(-, T(-))) \to \mathrm{Nat}(1_{\mathbf{C}}, TS), \quad \eta \mapsto \zeta,$$

is bijective with inverse map $\zeta \mapsto \eta^{\zeta}$, where

$$\eta^{\xi}_{C,D} : \mathrm{Hom}_{\mathbf{D}}(S(C), D) \to \mathrm{Hom}_{\mathbf{C}}(C, T(D)), \quad \beta \mapsto T(\beta) \circ \zeta_C,$$

for every $C \in \mathbf{C}, D \in \mathbf{D}$.

Now let

$$\eta' : \mathrm{Hom}_{\mathbf{C}}(-, T(-)) \to \mathrm{Hom}_{\mathbf{D}}(S(-), -)$$

be a natural transformation, and

$$\xi_D := \eta'_{T(D),D}(1_{T(D)}) : ST(D) \to D,$$

for every $D \in \mathbf{D}$. Dually we obtain

Lemma 1.1.25. *ξ is a natural transformation and the correspondence*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-, T(-)), \mathrm{Hom}_{\mathbf{D}}(S(-), -)) \to \mathrm{Nat}(ST, 1_{\mathbf{D}}), \quad \eta' \mapsto \xi,$$

is bijective with inverse map $\xi \mapsto \eta'^{\xi}$, where

$$\eta'^{\xi}_{C,D} : \mathrm{Hom}_{\mathbf{C}}(C, T(D)) \to \mathrm{Hom}_{\mathbf{D}}(S(C), D), \quad \alpha \mapsto \xi_D \circ S(\alpha),$$

for every $C \in \mathbf{C}, D \in \mathbf{D}$.

Now let η and η' be natural transformations as above, and ζ ad ξ the associated natural transformations.

Proposition 1.1.26. *$\eta'\eta$ is the identity transformation of the functor* $\mathrm{Hom}_{\mathbf{D}}(S(-),-)$ *if and only if the composition of natural transformations*

$$S \xrightarrow{S\zeta} STS \xrightarrow{\xi S} S$$

is the identity transformation of the functor S, i.e.

$$\xi_{S(C)}S(\zeta_C) = 1_{S(C)} \text{ for all } C \in \mathbf{C}. \tag{1.8}$$

Dually we obtain

Proposition 1.1.27. *$\eta\eta'$ is the identity transformation of the functor* $\mathrm{Hom}_{\mathbf{C}}(-,T(-))$ *if and only if the composition of natural transformations*

$$T \xrightarrow{\zeta T} TST \xrightarrow{T\xi} T$$

is the identity transformation of the functor T, i.e.

$$T(\xi_D)\zeta_{T(D)} = 1_{T(D)} \text{ for all } D \in \mathbf{D}. \tag{1.9}$$

Corollary 1.1.28. *Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors. To give a natural equivalence*

$$\eta : \mathrm{Hom}_{\mathbf{D}}(S(-),-) \to \mathrm{Hom}_{\mathbf{C}}(-,T(-)) \tag{1.10}$$

is the same as to give two natural transformations $\zeta : 1_{\mathbf{C}} \to ST$ and $\xi : ST \to 1_{\mathbf{D}}$ such that the composition of natural transformations

$$S \xrightarrow{S\zeta} STS \xrightarrow{\xi S} S$$

is the identity transformation of the functor S, and the composition of natural transformations

$$T \xrightarrow{\zeta T} TST \xrightarrow{T\xi} T$$

is the identity transformation of the functor T, i.e.

$$\xi_{S(C)}S(\zeta_C) = 1_{S(C)}, \quad T(\xi_D)\zeta_{T(D)} = 1_{T(D)} \quad \text{for all } C \in \mathbf{C}, D \in \mathbf{D}. \tag{1.11}$$

Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors. We say that T is a *right adjoint* to S (and symmetrically, S is a *left adjoint* to T), or (S,T) is a *pair of adjoint functors*, if one of the equivalent conditions of the last corollary holds. Sometimes and following [58] we write $\eta : S \dashv T$.

The natural equivalence (1.10) is called the *adjunction isomorphism.* The natural transformations $\zeta : 1_{\mathbf{C}} \to ST$ and $\xi : ST \to 1_{\mathbf{D}}$ are called the *unit* and the *counit* of the adjunction while and the equalities (1.11) are called *the adjunction equalities.*

Proposition 1.1.29. [58, Theorem 3.2]
Let $S, S' : \mathbf{C} \to \mathbf{D}$ and $T, T' : \mathbf{D} \to \mathbf{C}$ be functors and let $\eta : S \dashv T$ and $\eta' : S' \dashv T'$. Assume that $\sigma : S' \to S$ is a natural transformation. Then there is a unique natural transformation $\tau : T \to T'$ such that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(S(C), D) \arrow[r, "\eta"] \arrow[d, "\mathrm{Hom}_{\mathbf{D}}(\sigma_C, D)"'] & \mathrm{Hom}_{\mathbf{C}}(C, T(D)) \arrow[d, "\mathrm{Hom}_{\mathbf{C}}(C, \tau_D)"] \\
\mathrm{Hom}_{\mathbf{D}}(S'(C), D) \arrow[r, "\eta'"] & \mathrm{Hom}_{\mathbf{C}}(C, T'(D))
\end{tikzcd}$$

for every $C \in \mathbf{C}, D \in \mathbf{D}$.

Corollary 1.1.30. *(1) Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors and let $\eta : S \dashv T$. Then the natural transformation defined as in Prop. 1.1.29 associated to the natural transformation $1_S : S \to S$ is $1_T : T \to T$.*

(2) Let $S, S', S'' : \mathbf{C} \to \mathbf{D}$ and $T, T', T'' : \mathbf{D} \to \mathbf{C}$ be functors and let $\eta : S \dashv T$, $\eta' : S' \dashv T'$ and $\eta'' : S'' \dashv T''$. Assume that $\sigma : S' \to S$ and $\sigma' : S'' \to S'$ are natural transformations. Let $\tau : T \to T'$ and $\tau' : T' \to T''$ be the natural transformations defined as in Prop. 1.1.29 associated to σ and σ' respectively. Then the natural transformations defined as in Prop. 1.1.29 associated to $\sigma\sigma'$ is $\tau'\tau$.

(3) Let $S, S' : \mathbf{C} \to \mathbf{D}$ and $T, T' : \mathbf{D} \to \mathbf{C}$ be functors and let $\eta : S \dashv T$ and $\eta' : S' \dashv T'$. Assume that $\sigma : S' \to S$ is a natural equivalence. Then the natural transformations defined as in Prop. 1.1.29 associated to σ is so.

Now, we give the duality theorem for adjoint functors. Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors and $S^\circ : \mathbf{C}^\circ \to \mathbf{D}^\circ$ and $T^\circ : \mathbf{D}^\circ \to \mathbf{C}^\circ$ be their duals ($S^\circ(C^\circ) = S(C)^\circ$ and $S^\circ(f^\circ) = S(f)^\circ$ for every object $C \in \mathbf{C}$ and every morphism f in $\mathbf{C}$).

Proposition 1.1.31. [58, Theorem 3.4]
Assume that $\eta : S \dashv T$. Define for every $C \in \mathbf{C}$ and $D \in \mathbf{D}$ a map

$$\eta^{\#}_{D^\circ, C^\circ} : \mathrm{Hom}_{\mathbf{C}^\circ}(T^\circ(D^\circ), C^\circ) \to \mathrm{Hom}_{\mathbf{D}^\circ}(D^\circ, S^\circ(C^\circ))$$

by

$$\eta^{\#}_{D^\circ, C^\circ} = \eta^{-1}_{C,D}.$$

Then $\eta^{\#}$ is a natural equivalence, and hence $\eta^{\#} : T^\circ \dashv S^\circ$. Furthermore, $\eta^{\#\#} = \eta$.

Notice that by this duality theorem we obtain the dual result of Proposition 1.1.29.

The following result whose proof is obvious will be useful.

Proposition 1.1.32. *Let $\mathbf{C}, \mathbf{C}', \mathbf{C}''$ be categories and let*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, shift left, "S"] & \mathbf{C}' \arrow[l, shift left, "T"] \arrow[r, shift left, "S'"] & \mathbf{C}'' \arrow[l, shift left, "T'"]
\end{tikzcd}$$

be functors. If (S, T) and (S', T') are pairs of adjoint functors, then $(S'S, TT')$ is a pair of adjoint functors.

Proposition 1.1.33. *Let $S : \mathbf{C} \to \mathbf{D}$ be a functor between k-categories and $T : \mathbf{D} \to \mathbf{C}$ a right adjoint of T. If S (resp. T) is k-linear, then T (resp. S) is k-linear and the adjunction bijections*

$$\eta_{C,D} : \mathrm{Hom}_{\mathbf{D}}(S(C), D) \simeq \mathrm{Hom}_{\mathbf{C}}(C, T(D))$$

are isomorphisms of k-modules ($C \in \mathbf{C}, D \in \mathbf{D}$). Conversely, if the bijections $\eta_{C,D}$ are isomorphisms of k-modules, then S and T are k-linear.

Proof. Suppose that S is k-linear. By [69, Proposition V.1.4], T is additive and the bijections $\eta_{C,D}$ are isomorphisms of abelian groups. Moreover, for every morphism $\alpha : C \to T(D)$ in $\mathbf{C}$ and $h \in k$,

$$\begin{aligned} \eta^{-1}_{C,D}(h\alpha) &= \xi_D \circ S(h\alpha) \\ &= \xi_D \circ (hS(\alpha)) \\ &= h\xi_D \circ S(\alpha) \quad \text{(Since } \mathbf{D} \text{ is a } k\text{-category)} \\ &= h\eta^{-1}_{C,D}(\alpha). \end{aligned}$$

Hence $\eta_{C,D}$ are isomorphisms of k-modules. On the other hand, for every morphism $\beta : D \to D'$ in $\mathbf{D}$ and $h \in k$,

$$\begin{aligned} \eta^{-1}_{T(D),D'}(T(h\beta)) &= \xi_{D'} \circ ST(h\beta) \\ &= (h\beta) \circ \xi_D \quad \text{(Since } \xi \text{ is a natural transformation)} \\ &= h\beta \circ \xi_D, \end{aligned}$$

and

$$\begin{aligned} \eta^{-1}_{T(D),D'}(hT(\beta)) &= \xi_{D'} \circ S(hT(\beta)) \\ &= \xi_{D'} \circ (hST(\beta)) \\ &= h\xi_{D'} \circ ST(\beta) \\ &= h\beta \circ \xi_D \quad \text{(Since } \xi \text{ is a natural transformation).} \end{aligned}$$

Then $T(h\beta) = hT(\beta)$. Hence T is k-linear.

Finally, using Proposition 1.1.31, we prove the second statement. The converse is proved by a similar way. □

Proposition 1.1.34. *Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories, and let $S : \mathbf{C} \to \mathbf{D}$ be a functor. Let $T : \mathbf{D} \to \mathbf{C}$ be a right adjoint of S.*

(1) If $N \in \mathbf{D}$ is injective and S is exact, then $T(N) \in \mathbf{C}$ is injective ([79, Lemma 3.2.7]).

(2) If moreover $\mathbf{D}$ is a category with sufficiently many injectives (e.g. if $\mathbf{D}$ is a Grothendieck category), then S is exact if and only if T preserves injective objects ([79, Theorem 3.2.8]).

Let $S : \mathbf{C} \to \mathbf{D}$ be a functor. If $T : \mathbf{D} \to \mathbf{C}$ is both a right and a left adjoint functor to S, we say that (S, T) is a *Frobenius pair of functors*. In this case, S and T are called *Frobenius functors*.

Adjoint functors in several variables. To give a covariant functor $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is the same as to give

(a) a covariant functor $S(-, Y) : \mathbf{X} \to \mathbf{Z}$ for every $Y \in \mathbf{Y}$, and

(b) a natural transformation $S(-, y) : S(-, Y) \to S(-, Y')$ for every morphism $y : Y \to Y'$ in $\mathbf{Y}$ such that

(i) $S(-, y'y) = S(-, Y)S(-, Y)$ for every morphisms $y : Y \to Y'$ and $y' : Y' \to Y''$ in $\mathbf{Y}$, and

(ii) $S(-, 1_Y) = 1_{S(-,Y)}$ for every $Y \in \mathbf{Y}$.

Now assume that for every $Y \in \mathbf{Y}$ there are a functor $T_Y : \mathbf{Z} \to \mathbf{X}$ and a natural equivalence

$$\eta_Y : \mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, Y), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T_Y(\mathbf{Z})),$$

i.e. $\eta_Y : S(-, Y) \dashv T_Y$. From Proposition 1.1.29, for every morphism $y : Y \to Y'$ in $\mathbf{Y}$ there is a unique natural transformation $T_y : T_{Y'} \to T_Y$ making commutative the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, Y), \mathbf{Z}) \arrow[r, "\eta_Y"] & \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T_Y(\mathbf{Z})) \\
\mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, Y'), \mathbf{Z}) \arrow[r, "\eta_{Y'}"] \arrow[u, "\mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X},y),\mathbf{Z})"] & \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T_Y(\mathbf{Z})). \arrow[u, "\mathrm{Hom}_{\mathbf{X}}(\mathbf{X},(T_y)_{\mathbf{Z}})"']
\end{tikzcd}$$

Let $y' : Y' \to Y''$ be a morphism $y' : Y' \to Y''$ in $\mathbf{Y}$. By Corollary 1.1.30 (1),(2), we obtain $T_{y'y} = T_y T_{y'}$ and $T_{1_Y} = 1_{T_Y}$. We have then a functor $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ defined by

$$T(Y, Z) = T_Y(Z),\ T(Y, -) = T_Y,\ T(y, -) = T_y$$

for every $Y \in \mathbf{Y}, Z \in \mathbf{Z}$ and $y : Y \to Y'$ in $\mathbf{Y}$. Hence

$$\eta : \mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T(\mathbf{Y}, \mathbf{Z}))$$

defined by $\eta_{(X,Y,Z)} = (\eta_Y)_{(X,Z)}$ for every $X \in \mathbf{X}, Y \in \mathbf{Y}, Z \in \mathbf{Z}$, is a natural equivalence. Thus we get:

Proposition 1.1.35. *Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor. Assume that for every $Y \in \mathbf{Y}$ there are a functor $T_Y : \mathbf{Z} \to \mathbf{X}$ and a natural equivalence*

$$\eta_Y : \mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, Y), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T_Y(\mathbf{Z})),$$

i.e. $\eta_Y : S(-, Y) \dashv T_Y$. Then there is a unique functor $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ and a unique natural equivalence

$$\eta : \mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T(\mathbf{Y}, \mathbf{Z}))$$

such that for every $X \in \mathbf{X}, Y \in \mathbf{Y}, Z \in \mathbf{Z}$

$$T(Y, Z) = T_Y(Z),\ T(Y, -) = T_Y,\ \eta_{(X,Y,Z)} = (\eta_Y)_{(X,Z)}.$$

Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor and $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ a functor contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$. If there is a natural equivalence

$$\eta : \mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T(\mathbf{Y}, \mathbf{Z})),$$

we say that S is a *left adjoint of* T, and symmetrically, T is a *right adjoint of* S. Sometimes we will use the notation $\eta : S \dashv T$.

Now using Proposition 1.1.29 and Lemma 1.1.4 we get

Proposition 1.1.36. *Let $S, S' : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be covariant functors and $T, T' : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ functors contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ such that $\eta : S \dashv T$ and $\eta' : S' \dashv T'$. Let $\sigma : S' \to S$ be a natural transformation. Then there is a unique natural transformation $\tau : T \to T'$ such that the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Z}}(S(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) \arrow[r, "\eta"] \arrow[d, "\mathrm{Hom}_{\mathbf{Z}}(\sigma_{(\mathbf{X},\mathbf{Y})}, \mathbf{Z})"'] & \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T(\mathbf{Y}, \mathbf{Z})) \arrow[d, "\mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, \tau_{(\mathbf{Y},\mathbf{Z})})"] \\
\mathrm{Hom}_{\mathbf{Z}}(S'(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) \arrow[r, "\eta'"] & \mathrm{Hom}_{\mathbf{X}}(\mathbf{X}, T'(\mathbf{Y}, \mathbf{Z})).
\end{tikzcd}$$

Note that Corollary 1.1.30 is also true for adjoint functors in two variables.

Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor. Consider the functor $S^{\#} : \mathbf{Y} \times \mathbf{X}^{\circ} \to \mathbf{Z}^{\circ}$ contravariant in $\mathbf{Y}$ and covariant in $\mathbf{X}^{\circ}$ defined by

$$S^{\#}(Y, X^{\circ}) = S(X, Y)^{\circ}, \quad S^{\#}(y, x^{\circ}) = S(x, y)^{\circ},$$

for every objects $Y \in \mathbf{Y}$, $X \in \mathbf{X}$ and morphisms y in $\mathbf{Y}$ and x in $\mathbf{X}$.

Let $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ be a functor contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$. Consider the covariant functor $T^{\#} : \mathbf{Z}^{\circ} \times \mathbf{Y} \to \mathbf{X}^{\circ}$ defined by $T^{\#\#} = T$.

We have

$$\mathrm{Hom}_{\mathbf{X}^{\circ}}(T^{\#}(Z^{\circ}, Y), X^{\circ}) = \mathrm{Hom}_{\mathbf{X}}(X, T(Y, Z)),$$
$$\mathrm{Hom}_{\mathbf{Z}^{\circ}}(Z^{\circ}, S^{\#}(Y, X^{\circ})) = \mathrm{Hom}_{\mathbf{Z}}(S(X, Y), Z)$$

for every $X \in \mathbf{X}, Y \in \mathbf{Y}, Z \in \mathbf{Z}$.

It is easy to prove the following duality theorem:

Proposition 1.1.37. *Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor and $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ a functor contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ such that $\eta : S \dashv T$. Define for every $X \in \mathbf{X}, Y \in \mathbf{Y}, Z \in \mathbf{Z}$ the map*

$$\eta^{\#}_{(Z^{\circ}, Y, X^{\circ})} = \eta^{-1}_{(X,Y,Z)} : \mathrm{Hom}_{\mathbf{X}^{\circ}}(T^{\#}(Z^{\circ}, Y), X^{\circ}) \to \mathrm{Hom}_{\mathbf{Z}^{\circ}}(Z^{\circ}, S^{\#}(Y, X^{\circ})).$$

Then $\eta^{\#}$ is a natural equivalence, i.e. $\eta^{\#} : T^{\#} \dashv S^{\#}$. Moreover, $\eta^{\#\#} = \eta$.

Note that Propositions 1.1.31 and 1.1.37 give the dual of Proposition 1.1.35. Also Proposition 1.1.37 gives the dual of Proposition 1.1.36.

Finally we consider functors in more than two variables. Let $S : \mathbf{X}\times\mathbf{Y}_1\times\cdots\times\mathbf{Y}_n \to \mathbf{Z}$ be a covariant functor and $T : \mathbf{Y}_1\times\cdots\times\mathbf{Y}_n\times\mathbf{Z} \to \mathbf{X}$ a functor contravariant in $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ and covariant in $\mathbf{Z}$. Let $Y = \prod_i \mathbf{Y}_i$ be the cartesian product category. S can be viewed as a covariant functor $\mathbf{Y}\times\mathbf{Z} \to \mathbf{X}$ and T can be viewed as a functor $\mathbf{Y}\times\mathbf{Z} \to \mathbf{X}$ contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$. We return then to the case of functors in two variables.

Equivalences of categories. A functor $S : \mathbf{C} \to \mathbf{D}$ is called an equivalence, if there exists a functor $T : \mathbf{D} \to \mathbf{C}$ with natural equivalences $\eta : 1 \to TS$ and $\varepsilon : ST \to 1$. A functor $S : \mathbf{C} \to \mathbf{D}$ is called *essentially surjective* if for every $D \in \mathbf{D}$, there is $C \in C$ such that $S(C) \simeq D$.

The following result is very well known.

Theorem 1.1.38. *A functor $S : \mathbf{C} \to \mathbf{D}$ is an equivalence if and only if it is faithful, full and essentially surjective.*

A functor between k-categories is said to be a *k-equivalence* if it is k-linear and an equivalence.

The Yoneda-Grothendieck lemma and representable functors. Let $\mathbf{C}$ be a category and $X \in \mathbf{C}$. We have then a covariant functor $h_X = \mathrm{Hom}_{\mathbf{C}}(-, X) : \mathbf{C}^\circ \to \mathbf{Set}$. Let $X', Y \in \mathbf{C}$ and let $u : X \to X'$ be a morphism. Then

$$h_u = \mathrm{Hom}_{\mathbf{C}}(Y, u) : h_X \to h_{X'}$$

is a natural transformation. Hence we have a "functor"

$$h : \mathbf{C} \to \mathbf{Fun}(\mathbf{C}^\circ, \mathbf{Set}).$$

Let $F : \mathbf{C}^\circ \to \mathbf{Set}$ be a covariant functor. Define

$$\alpha : \mathrm{Nat}(h_X, F) \to F(X),\ \eta \mapsto \eta_X(1_X) \in F(X)$$

and

$$\beta : F(X) \to \mathrm{Nat}(h_X, F),\ \xi \mapsto \eta$$

where η is the natural transformation defined by

$$\eta_Y : \mathrm{Hom}_{\mathbf{C}}(Y, X) \to F(X),\ \upsilon \mapsto F(\upsilon)(\xi)$$

for every $Y \in \mathbf{C}$.

Proposition 1.1.39. [Yoneda-Grothendieck]
The maps α and β are bijections inverse to each other. (This includes that $\mathrm{Nat}(h_X, F)$ is a set.)

If we take $F = h_{X'}$ for $X' \in \mathbf{C}$, the map

$$\beta : \mathrm{Hom}_{\mathbf{C}}(X, X') \to \mathrm{Nat}(h_X, h_{X'})$$

is exactly the map $u \mapsto h_u$. Thus:

Corollary 1.1.40. *The canonical "functor" $h : \mathbf{C} \to \boldsymbol{Fun}(\mathbf{C}^\circ, \boldsymbol{Set})$ is fully faithful.*

We say that a contravariant functor $F : \mathbf{C} \to \mathbf{Set}$ is *representable* if there is $X \in \mathbf{C}$ such that $F \simeq h_X$. Since h is conservative then X is determined up to an isomorphism. Then F is representable if and only if there exist $X \in \mathbf{C}, \xi \in F(X)$ such that

$$\mathrm{Hom}_{\mathbf{C}}(Y, X) \to F(Y),\ v \mapsto F(v)(\xi)$$

is an isomorphism, for every $Y \in \mathbf{C}$. We say that the couple (X, ξ) *represents* F. Usually we omit ξ and we say only that X *represents* F. It follows from Corollary 1.1.40 that every category $\mathbf{C}$ is equivalent to the category $h(\mathbf{C})$ of all representable contravariant functors $\mathbf{C} \to \mathbf{Set}$.

For the details we refer to [48] or [49].

Using the axiom of choice and the Yoneda-Grothendieck lemma, we can easily prove the following useful result:

Proposition 1.1.41. *Let $S : \mathbf{C} \to \mathbf{D}$ be a functor. Then the following are equivalent*

(1) the functor S has a right adjoint;

(2) for every object $D \in \mathbf{D}$, the contravariant functor $C \mapsto \mathrm{Hom}_{\mathbf{D}}(S(C), D)$ is representable.

We also define for $X \in \mathbf{C}$ a covariant functor $h'_X = \mathrm{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Set}$. Dually, we define representable covariant functors.

Using also the Yoneda-Grothendieck lemma we prove the result below:

Proposition 1.1.42. [64, Theorem IV.3.1]
Let $S : \mathbf{C} \to \mathbf{D}$ be a functor. Suppose that there is an adjunction (S, T) with unit ζ and counit ξ.

(i) T is faithful if and only if ξ_D is an epimorphism for every $D \in \mathbf{D}$.

(ii) T is full if and only if ξ_D is a section for every $D \in \mathbf{D}$.

Hence, T is faithful and full if and only if ξ_D is an isomorphism for every $D \in \mathbf{D}$.

Limits (Projective limits). Let $\mathbf{C}$ be a category and $S = (I, \Phi, d)$ a diagram scheme. A family of morphisms $\{u_i : X \to D(i) \mid i \in I\}$, is called *compatible* for D if $D(\varphi)u_i = u_j$ for every arrow $\varphi : i \to j$ of S. Assume that (u_i) is compatible and let $v_i : Y \to D(i)$ be another compatible family. A *morphism* from (u_i) to (v_i) is a morphism in $\mathbf{C}$ such that

$v_i = u_i v$ for every $i \in I$. Then we obtain the category of all compatible families for D. A final object for the this category is called a *limit* for D. Hence a limit is determined up to an isomorphism. We denote it by $\{\varprojlim D \to D(i) \mid i \in I\}$.

Obviously pullbacks, products and equalizers are special cases of limits. We say that $\mathbf{C}$ is S-*complete* if every diagram in $\mathbf{C}^S$ has a limit. We say that $\mathbf{C}$ is *complete* (resp. *finitely complete*) if it is $\mathbf{C}$ is S-complete for every (resp. finite) diagram scheme S. Every finitely complete category has pullbacks, equalizers and finite products, and every complete category has products.

Dually we define colimits (inductive limits).

Let $\upsilon : D \to D'$ be a morphism in $\mathbf{C}^S$. By the definition of limit there exists a unique morphism $\varprojlim \upsilon : \varprojlim D \to \varprojlim D'$ making the following diagram commutative

$$\begin{tikzcd}
\varprojlim D \arrow[r, "\varprojlim \upsilon"] \arrow[d] & \varprojlim D' \arrow[d] \\
D(i) \arrow[r, "\upsilon_i"] & D'(i)
\end{tikzcd}$$

for every $i \in I$. Then we obtain a functor

$$\varprojlim : \mathbf{C}^S \to \mathbf{C}.$$

Now let $A \in \mathbf{C}$. Define $D^A \in \mathbf{C}^S$ by $D^A(i) = A$ and $D^A(\varphi) = 1_A$ for every $i \in I$ and $\varphi \in \Phi$. A morphism $D^A \to D$ in $\mathbf{C}^S$ can be identified with a compatible family $\{A \to D(i) \mid i \in I\}$ for D. Let $u : A \to B$ be a morphism in $\mathbf{C}$. Consider the morphism $\upsilon^u : D^A \to D^B$ in $\mathbf{C}^S$ defined by $\upsilon^u_i = u$ for every $i \in I$. We obtain then a functor $I : \mathbf{C} \to \mathbf{C}^S$. Let $X \in \mathbf{C}$ and $\xi \in \mathrm{Hom}_{\mathbf{C}^S}(D^X, D)$. Then (X, ξ) is a limit for D if and only if (X, ξ) represents the contravariant functor $h_D I = \mathrm{Hom}_{\mathbf{C}^S}(-, D)I : \mathbf{C} \to \mathbf{Set}$. Hence, by Proposition 1.1.41, $\mathbf{C}$ is S-complete if and only if the functor I has a right adjoint. In such a case, I is a left adjoint to $\varprojlim$.

Let $F : \mathbf{C} \to \mathbf{C}'$ be a functor and S a diagram scheme. We know that F induces a functor $F^S : \mathbf{C}^S \to \mathbf{C}'^S$. We say that F *preserves limits* if $(F(u_i))_i$ is the limits of $F(D)$ whenever (u_i) is the limits of D. Of course, if F preserves limits then it preserves pullbacks, equalizers and products. Dually, we say that a functor $F : \mathbf{C} \to \mathbf{C}'$ *preserves colimits* if its dual $F^\circ : \mathbf{C}^\circ \to \mathbf{C}'^\circ$ preserves limits.

Let $G : \mathbf{C} \to \mathbf{C}'$ be another functor and $\eta : F \to G$ be a natural transformation. If $\{v_i : Y \to F(D(i)) \mid i \in I\}$ is a compatible family for the diagram $F(D) \in \mathbf{C}'^S$, then the $(\eta_{D(i)} v_i)_i$ is a compatible family for $G(D)$. Assume that η is a natural equivalence and let $u_i : X \to D(i)$, $i \in I$, be a family of morphism in $\mathbf{C}$. Then $(F(u_i))_i$ is a limit for $F(D)$ if and only if $(G(u_i))_i$ is a limit for $G(D)$. Hence F preserves limits if and only if G preserves limits.

It is also easy to verify the following two facts. A morphism u in $\mathbf{C}$ is a monomorphism if and only if $h'_A(u)$ is an injective map for every $A \in \mathbf{C}$. Let $\{u_i : X \to D(i) \mid i \in I\}$ be a family of morphism in $\mathbf{C}$. Then $(u_i)_i$ is a limit for D if and only if $(h'_A(u_i))_i$ is a limit for

$h'_A(D)$ for every $A \in \mathbf{C}$. Using the fact that $h'_{A^\circ} = h_A : \mathbf{C}^\circ \to \mathbf{Set}$, we obtain the dual of these considerations.

Obviously, an object $X \in \mathbf{C}$ is a final object if and only if $h'_A(X)$ is a final object of the category **Set** for every $A \in \mathbf{C}$ (Since the final objects of **Set** are the sets with one element). Let $\mathbf{C}$ and $\mathbf{C}'$ be k-categories and $F : \mathbf{C} \to \mathbf{C}'$ a functor. We have F is k-linear if and only if $h'_X F$ is k-linear for every $X \in \mathbf{C}$ (of course the functors h'_X are k-linear). Moreover, F is k-linear if and only if its dual F° is k-linear.

A functor F is called a *monofunctor* if it preserves monomorphisms. Dually, a functor F is an *epifunctor* if its dual F° is a monofunctor, or equivalently, if it preserves epimorphisms.

From the dual of Proposition 1.1.41, a functor $G : \mathbf{C}' \to \mathbf{C}$ has a left adjoint if and only if $h'_A G$ is representable for every $A \in \mathbf{C}$.

Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor and $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ a functor contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ such that $\eta : S \dashv T$. Consider the covariant functors $S_X = S(X, -) : \mathbf{Y} \to \mathbf{Z}$ and $T_Z = T(-, Z) : \mathbf{Y}^\circ \to \mathbf{X}$. We have then a natural equivalence

$$h'_{Z^\circ}(S_X)^\circ \simeq h'_X T_Z : \mathbf{Y}^\circ \to \mathbf{Set}$$

for every $X \in \mathbf{X}, Z \in \mathbf{Z}$.

Then from the above considerations we prove the following result:

Proposition 1.1.43. *(1) Let $F : \mathbf{C} \to \mathbf{C}'$ be a functor and $G : \mathbf{C}' \to \mathbf{C}$ a right adjoint functor to F. Then*

(i) G is a monofunctor and preserves limits. G preserves also final objects.

(ii) Dually, F is an epifunctor and preserves colimits. F preserves also initial objects.

(2) Let $S : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ be a covariant functor and $T : \mathbf{Y} \times \mathbf{Z} \to \mathbf{X}$ a functor contravariant in $\mathbf{Y}$ and covariant in $\mathbf{Z}$ such that $\eta : S \dashv T$. Consider the covariant functors $S_X = S(X, -) : \mathbf{Y} \to \mathbf{Z}$ and $T_Z = T(-, Z) : \mathbf{Y}^\circ \to \mathbf{X}$. Then

(a) S_X preserves colimits for every $X \in \mathbf{X}$ if and only if T_Z preserves limits (as a covariant functor) for every $Z \in \mathbf{Z}$.

(b) Assume moreover that the categories $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are k-categories. Then S_X is k-linear for every $X \in \mathbf{X}$ if and only if T_Z is k-linear (as a covariant functor) for every $Z \in \mathbf{Z}$.

Corollary 1.1.44. *Let $\mathbf{C}$ be a category and S a diagram scheme. If $\mathbf{C}$ is S-complete, then the functor $\varprojlim : \mathbf{C}^S \to \mathbf{C}$ is a monofunctor and preserves limits.*

Proposition 1.1.45. [Freyd] (see [69, Theorem II.7.1])
Let $F : \mathbf{C} \to \mathbf{C}'$ be a faithful functor. Then

(1) If $\mathbf{C}$ is an abelian category and $\mathbf{C}'$ is additive, then F reflects limits and colimits of finite diagrams.

(2) If moreover F is full, then F reflects limits and colimits.

For many more interesting properties of limits, see [69].

Separable functors. Separable functors were introduced and studied in [75]. Further study of these functors is given in [80].

Let $T : \mathbf{C} \to \mathbf{D}$ be a functor between arbitrary categories. T is called a *separable functor* if for every $C, C' \in \mathbf{C}$, there exists a map

$$\Phi = \Phi_{C,C'} : \mathrm{Hom}_{\mathbf{D}}(T(C), T(C')) \to \mathrm{Hom}_{\mathbf{C}}(C, C')$$

such that

(SF1) For every morphism $f : C \to C'$ in $\mathbf{C}$, $\Phi(T(f)) = f$.

(SF2) For every morphisms $f : C_1 \to C_1'$ and $g : C_2 \to C_2'$ in $\mathbf{C}$, and every commutative diagram in $\mathbf{D}$

$$\begin{tikzcd} T(C_1) \arrow[r, "h"] \arrow[d, "T(f)"'] & T(C_2) \arrow[d, "T(g)"] \\ T(C_1') \arrow[r, "h'"] & T(C_2') \end{tikzcd} ,$$

the following diagram

$$\begin{tikzcd} C_1 \arrow[r, "\Phi(h)"] \arrow[d, "f"'] & C_2 \arrow[d, "g"] \\ C_1' \arrow[r, "\Phi(h')"] & C_2' \end{tikzcd}$$

is also commutative.

Obviously, from (SF1), every separable functor is faithful. Let $T : \mathbf{C} \to \mathbf{D}$ be a separable functor and $T^\circ : \mathbf{C}^\circ \to \mathbf{D}^\circ$ its dual. Then, for every objects $C, C' \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{D}^\circ}(T^\circ(C^\circ), T^\circ(C'^\circ)) = \mathrm{Hom}_{\mathbf{D}}(T(C'), T(C))$ and $\mathrm{Hom}_{\mathbf{C}^\circ}(C^\circ, C'^\circ) = \mathrm{Hom}_{\mathbf{C}}(C', C)$. Define for every objects $C, C' \in \mathbf{C}$ a map

$$\Phi^{\#}_{C^\circ, C'^\circ} : \mathrm{Hom}_{\mathbf{D}^\circ}(T^\circ(C^\circ), T^\circ(C'^\circ)) \to \mathrm{Hom}_{\mathbf{C}^\circ}(C^\circ, C'^\circ)$$

by

$$\Phi^{\#}_{C^\circ, C'^\circ} = \Phi_{C',C}.$$

We have, for every morphism $h : T(C') \to T(C)$ in $\mathbf{C}$,

$$\Phi^{\#}_{C^\circ, C'^\circ}(h^\circ) = \Phi_{C',C}(h)^\circ.$$

The functor $T^\circ : \mathbf{C}^\circ \to \mathbf{D}^\circ$ is clearly separable. We use this fact to prove the dual results of separable functors.

Examples 1.1.46. [75, Lemma 1.1]

(1) An equivalence of categories is separable.

(2) Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ be functors. Then

  (i) If F and G are separable then GF is separable.
  (ii) If GF is separable then F is separable.

Proposition 1.1.47. [75, Proposition 1.2]
Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and $C \in \mathbf{C}$. Then

(1) F reflects sections and retractions.

Now, assume that $\mathbf{C}$ and $\mathbf{D}$ are abelian categories. Then

(2) If F is a monofunctor, then F reflects quasi-simple objects (=every subject splits off).

(3) If F is a monofunctor, then F reflects injective objects. Dually, if F is an epifunctor, then F reflects projective objects.

Proposition 1.1.48. [Rafael] ([80, Theorem 1.2])
Let $S : \mathbf{C} \to \mathbf{D}$ be a functor and $T : \mathbf{D} \to \mathbf{C}$ a right adjoint to S. Let $\zeta : 1_{\mathbf{C}} \to TS$ and $\xi : ST \to 1_{\mathbf{D}}$ be the unit and the counit of the adjunction, respectively.

(1) S is separable if and only if ζ splits (i.e. there is a natural transformation $\zeta' : TS \to 1_{\mathbf{C}}$ such that $\zeta'_C \circ \zeta_C = 1_C$ for all $C \in \mathbf{C}$).

(2) Dually, T is separable if and only if ξ cosplits (i.e. there is a natural transformation $\xi' : 1_{\mathbf{D}} \to ST$ such that $\xi_D \circ \xi'_D = 1_D$ for all $D \in \mathbf{D}$).

Split exact and Z-pure sequences. A monomorphism $f : A \to B$ in a category $\mathbf{C}$ is called a *section* (or a *coretraction*) if there is a morphism $g : B \to A$ such that $gf = 1_A$. A section is an epimorphism if and only if it is an isomorphism. Dually we define a *retraction*.

Proposition 1.1.49. [60, Proposition 8.3.14]
Let $0 \longrightarrow B \xrightarrow{f} A \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in an abelian category $\mathbf{C}$. Then the following are equivalent

(1) there is $h : C \to A$ such that $gh = 1_C$ (i.e. g is a retraction);

(2) there is $k : A \to B$ such that $kf = 1_B$ (i.e. f is a section);

(3) there is $h : C \to A$ and $k : A \to B$ such that $1_A = fk + hg$;

(4) there is $h : C \to A$ and $k : A \to B$ such that $gh = 1_C$, $kf = 1_B$, $kh = 0$, and $1_A = fk + hg$;

(5) there are $\varphi = (k, g) : A \to B \oplus C$ and $\psi = f \oplus h : B \oplus C \to A$ are isomorphisms inverse to each other.

In such a case we say that the sequence is *split exact*.

Proposition 1.1.50. *Let* $0 \longrightarrow A_1 \xrightarrow{f_1} A \xrightarrow{g_2} A_2 \longrightarrow 0$ *be a short sequence in an abelian category* $\mathbf{C}$*. Then the following are equivalent*

(1) the sequence is split exact;

(2) there is a sequence $0 \longrightarrow A_2 \xrightarrow{f_2} A \xrightarrow{g_1} A_1 \longrightarrow 0$ *(necessarily split exact) such that for every* $i, j \in \{1, 2\}$,

$$g_i f_j = \delta_{i,j} 1_{A_i} \quad \text{and} \quad f_1 g_1 + f_2 g_2 = 1_A;$$

(3) there is an isomorphism $h : A_1 \oplus A_2 \to A$ *such that the following diagram is commutative*

$$\begin{tikzcd} & & A_1 \oplus A_2 \arrow[dd, "h"] \arrow[dr, "p_2"] & & \\ 0 \arrow[r] & A_1 \arrow[ur, "\iota_1"] \arrow[dr, "f_1"'] & & A_2 \arrow[r] & 0 \\ & & A \arrow[ur, "g_2"'] & & \end{tikzcd}.$$

where $\iota_i : A_i \to A_1 \oplus A_2$ *and* $p_i : A_1 \oplus A_2 \to A_i$, $i \in \{1, 2\}$, *are such that* $p_j \iota_i = \delta_{i,j}$ *and* $\iota_1 p_1 + \iota_2 p_2 = 1_{A_1 \oplus A_2}$.

Proof. The proof of [5, Proposition 5.3] works here. $\square$

Corollary 1.1.51. *Every additive functor between abelian categories preserves split exactness.*

More generally, and following [19, 40.5], we say that a sequence $B \xrightarrow{f} A \xrightarrow{g} C$ of morphisms in an abelian category is *split exact* if it is exact (i.e. $\mathrm{Ker}(g) = \mathrm{Im}(f)$) and the canonical monomorphism $\mathrm{Im}(g) \to C$ is a section.

Let $n \in \{2, \ldots\} \subset \mathbb{N}$. We say that an exact sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1}$$

in an abelian category is *split exact* if the canonical monomorphism $\mathrm{Im}(f_i) \to A_{i+1}$ is a section for every $i \in \{2, \ldots, n\}$.

Let $n \in \{2, \ldots\} \subset \mathbb{N}$ and $Z \in {}_A\mathcal{M}$. Following [19, 40.13] we say that an exact sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \tag{1.12}$$

in $\mathcal{M}_A$, is Z-*pure* if it still exact under the functor $- \otimes_A Z$. If this is the case for every $Z \in {}_A\mathcal{M}$, we say simply that the sequence (1.12) is *pure*. Of course if Z is flat, then every exact sequence in $\mathcal{M}_A$ is Z-pure.

Let $f : M \to N$ be a monomorphism in $\mathcal{M}_A$. f is called *Z-pure* and M is called a *Z-pure submodule* of N if the exact sequence $0 \longrightarrow M \xrightarrow{f} N$ is Z-pure, that is if $f \otimes_A Z$ is a monomorphism.

Lemma 1.1.52. *The exact sequence* (1.12) *is Z-pure if and only if the canonical monomorphism* $\mathrm{Im}(f_i) \to M_{i+1}$ *is Z-pure for every $i \in \{2, \dots, n\}$.*

In particular if the sequence (1.12) *is split exact then it is pure.*

Proof. We begin by factorizing the map f_i through its image for all $i \in \{2, \dots, n\}$ as follows

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] \arrow[dr, "p_2"] & M_3 \arrow[r, "f_3"] & \dots M_n \arrow[r, "f_n"] \arrow[dr, "p_n"] & M_{n+1} & \\
 & & \mathrm{Im}(f_2)\dots \arrow[u, hook, "\iota_2"] \arrow[dr] & & \mathrm{Im}(f_n) \arrow[u, hook, "\iota_n"] \arrow[dr] & \\
 & & & 0\dots & & 0.
\end{tikzcd}$$

Since ι_2 is a monomorphism, $\mathrm{Im}(f_1) = \mathrm{Ker}(f_2) = \mathrm{Ker}(p_2)$. Then the sequence $M_1 \xrightarrow{f_1} M_2 \xrightarrow{p_2} \mathrm{Im}(f_2) \longrightarrow 0$ is exact. Since $- \otimes_A Z$ is right exact, $M_1 \otimes_A Z \xrightarrow{f_1 \otimes_A Z} M_2 \otimes_A Z \xrightarrow{p_2 \otimes_A Z} \mathrm{Im}(f_2) \otimes_A Z \longrightarrow 0$ is also exact. The same thing hold for every $i \in \{3, \dots, n\}$. Then the proof follows easily. $\square$

Corollary 1.1.53. *Let $Z \in {}_A\mathcal{M}$ and $f : M \to N$ be a morphism in $\mathcal{M}_A$. Then the following are equivalent*

(1) the sequence

$$0 \longrightarrow \mathrm{Ker}(f) \otimes_A Z \longrightarrow M \otimes_A Z \xrightarrow{f \otimes_A Z} N \otimes_A Z$$

is exact;

(2) the canonical monomorphisms $\mathrm{Ker}(f) \to M$ *and* $\mathrm{Im}(f) \to N$ *are Z-pure.*

In such a case we say that f is *Z-pure*. If this the case for every $Z \in {}_A\mathcal{M}$, we say simply that the sequence f is *pure*.

Locally finitely generated and locally noetherian categories. Let $\mathbf{C}$ be a Grothendieck category. An object C of $\mathbf{C}$ is said to be *finitely generated* [85, p. 121] if the (upper continuous) lattice of all subobjects $\mathsf{L}(C)$ is compact, that is, if $C = \sum_i C_i$ for a direct family of subobjects C_i of C, then $C = C_{i_0}$ for some index i_0. Notice that a subobject B of C is finitely generated as object if and only if B is compact in the lattice $\mathsf{L}(C)$ (Since there is a canonical order isomorphism (and then a lattice isomorphism) $[0, B] \to \mathsf{L}(B)$).

An object $C \in \mathbf{C}$ is called *finitely presented* if it is finitely generated and every epimorphism $B \to C$ where B is finitely generated has a finitely generated kernel. An object $C \in \mathbf{C}$ is called *noetherian* if the lattice (of all subobjects) $\mathsf{L}(C)$ is noetherian, that is, every infinite ascending chain in it is stationary.

Lemma 1.1.54. [85, Lemma V.3.1]
Let $0 \to C' \to C \to C'' \to 0$ *be a short exact sequence in* $\mathbf{C}$. *Then*

(a) If C *is finitely generated, then so is* C''.

(b) If C' *and* C'' *are finitely generated, then so is* C.

Proposition 1.1.55. [85, Proposition V.3.2]
The following are equivalent

(1) C *is finitely generated;*

(2) for every directed family of subobjects $(D_i)_{i\in I}$ *of a certain object* D *of* $\mathbf{C}$, *and every monomorphism* $\alpha : C \to \sum_I D_i$ *in* $\mathbf{C}$, *there exists some* $i_0 \in I$ *and a morphism* $\beta : C \to D_{i_0}$ *making commutative the following diagram*

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[dr, "\beta"] & \sum D_i \\ & D_{i_0}; \arrow[u] \end{tikzcd}$$

(3) the canonical monomorphism of abelian groups

$$\varinjlim_I \mathrm{Hom}_{\mathbf{C}}(C, D_i) \to \mathrm{Hom}_{\mathbf{C}}(C, \sum D_i)$$

is an isomorphism. In such a case, we say that the functor $\mathrm{Hom}_{\mathbf{C}}(C, -)$ preserves direct unions.

Proof. (2) $\Longrightarrow$ (1) Obvious. (1) $\Longrightarrow$ (2) From Lemma 1.1.54, $\mathrm{Im}(\alpha)$ is finitely generated. We have $(D_i)_{i\in I}$ is a directed family of subobjects of $\sum D_i$. Hence $\mathrm{Im}(\alpha) \subset D_{i_0}$ for some $i_0 \in I$. Finally, (3) is a reformulation of (2). $\square$

Lemma 1.1.56. *Let* $S : \mathbf{C} \to \mathbf{D}$ *be a functor between Grothendieck categories, and* $T : \mathbf{D} \to \mathbf{C}$ *be a right adjoint functor to* S, *that is exact and preserves coproducts. Then* S *preserves finitely generated objects.*

In particular, a Frobenius functor between Grothendieck categories preserves finitely generated objects.

Proof. Let C be a finitely generated object in $\mathbf{C}$, $(D_i)_{i\in I}$ be a directed family of subobjects of a certain object $D \in \mathbf{C}$, and

$$\eta : \mathrm{Hom}_{\mathbf{C}}(-, T(-)) \to \mathrm{Hom}_{\mathbf{D}}(S(-), -)$$

be the natural equivalence.

Since $\eta_{C,-}$ is natural in $D_i \to \sum D_i$, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(C, T(D_i)) \arrow[r, "\eta_{C,D_i}"] \arrow[d] & \mathrm{Hom}_{\mathbf{D}}(S(C), D_i) \arrow[d] \\
\mathrm{Hom}_{\mathbf{C}}(C, T(\sum D_i)) \arrow[r, "\eta_{C,\sum D_i}"] & \mathrm{Hom}_{\mathbf{D}}(S(C), \sum D_i).
\end{tikzcd} \tag{1.13}$$

Since T is exact, $(T(D_i))_{i\in I}$ is a directed family of subobjects of $T(D)$. Let $\iota_i : D_i \to D$ be the canonical injection for every $i \in I$. Since T is exact and preserves coproducts, we have

$$\sum_I T(D_i) = \mathrm{Im}(T(\oplus_I \iota_i)) = T\big(\mathrm{Im}(\oplus_I \iota_i)\big) = T\Big(\sum_I D_i\Big).$$

(To obtain the last statement we can also use that T preserves inductive limits.) Now, from the commutativity of the diagram (1.13) and Proposition 1.1.55, $S(C)$ is finitely generated. □

Proposition 1.1.57. [85, Proposition V.3.4]
Let $\mathbf{C}$ be a locally finitely generated category. An object $C \in \mathbf{C}$ is finitely presented if and only if the functor $\mathrm{Hom}_{\mathbf{C}}(C, -)$ *preserves direct limits.*

Part of the following result is [85, Proposition V.4.1].

Proposition 1.1.58. *Let $C \in \mathbf{C}$. The following are equivalent*

(1) C is noetherian;

(2) every subobject of C is finitely generated;

(3) every nonempty set of subobjects of C has a maximal element.

Since the above result is stated in [85] without proof, we propose a proof of it.

Proof. (1) $\Longrightarrow$ (3) Let L be a nonempty set of subobjects of C. Suppose that L does not have a maximal element, that is, for every $A \in \mathsf{L}$, the set $\{A' \in \mathsf{L} \mid A < A'\}$ is not empty. Hence, by the axiom of choice, there is an infinite strictly ascending chain of subobjects of C.

(3) $\Longrightarrow$ (2) Let B be a subobject of C, and $B = \sum B_i$, where $(B_i)_i$ is a directed family of subobjects of B. This family has a maximal element, say B_{i_m}. Hence $B = B_{i_m}$.

(2) $\Longrightarrow$ (1) Suppose that every subobject of C is finitely generated. Let

$$C_1 \subset C_2 \subset \ldots \subset C_n \subset \ldots$$

be an ascending chain of subobjects of C. Set $B = \sum_{\mathbb{N}-\{0\}} C_n$. Since B is finitely generated, there is n such that $B = C_n$. Hence $B = C_{n+i}$ for every $i \in \mathbb{N}$. □

Remark 1.1.59. The equivalence (1) $\Leftrightarrow$ (3) of Proposition 1.1.58 is true in a more general context. Indeed, a lattice L is noetherian if and only if every nonempty subset of L has a maximal element (see [85, Proposition III.3.4]).

Proposition 1.1.60. [85, Proposition V.4.2]
Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence in $\mathbf{C}$. Then C is noetherian if and only if both C' and C'' are noetherian.

Proposition 1.1.61. *(1) Let $C = C_1 \oplus \ldots \oplus C_n \in \mathbf{C}$. Then C is finitely generated if and only each C_i $(i = 1, \ldots, n)$ is finitely generated.*

(2) If C is the sum of many finitely generated subobjects, then C is finitely generated.

(3) Every finitely generated projective object is finitely presented.

(4) Let $C = C_1 \oplus \ldots \oplus C_n \in \mathbf{C}$. Then C is noetherian if and only each C_i $(i = 1, \ldots, n)$ is finitely noetherian.

(5) If C is the sum of many noetherian subobjects, then C is noetherian.

(6) The following statements are equivalent

(a) every finitely generated object is noetherian;

(b) every finitely generated object is finitely presented.

In particular, a ring R is right noetherian if and only if every finitely generated right R-module is finitely presented.

Proof. (1) It is enough to prove it for $n = 2$. In such a case, there are morphisms $\iota_i : C_i \to C$ and $\pi_i : C \to C_i$ $(i = 1, 2)$ such that $\pi_i \iota_j = \delta_{ij}$ and $\iota_1\pi_1 + \iota_2\pi_2 = 1$. Hence we have two short exact sequences

$$0 \longrightarrow C_1 \xrightarrow{\iota_1} C \xrightarrow{\pi_2} C_2 \longrightarrow 0$$

and

$$0 \longrightarrow C_2 \xrightarrow{\iota_2} C \xrightarrow{\pi_1} C_1 \longrightarrow 0.$$

By Lemma 1.1.54, (1) follows.

(2) follows directly from (1) and Lemma 1.1.54(i). Notice that we can prove it directly using [85, Lemma III.5.2].

(3) Let C be a finitely generated projective object, and $\alpha : B \to C$ be an epimorphism, where B is finitely generated. Then we obtain the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0.$$

Since C is projective, this sequence splits and $B \simeq \mathrm{Ker}(\alpha) \oplus C$. Hence, by (1), $\mathrm{Ker}(\alpha)$ is finitely generated.

The proofs of (4) and (5) are analogous to that of (1) and (2), respectively, using Proposition 1.1.60.

(6) Follows directly from Propositions 1.1.60, 1.1.58, and Lemma 1.1.54(i). $\square$

The category $\mathbf{C}$ is *locally finitely generated* if the lattice $\mathsf{L}(C)$ is compactly generated for all $C \in \mathbf{C}$, i.e. every object of $\mathbf{C}$ is a sum of a (directed) family of finitely generated subobjects (see [85, p. 73]).

Proposition 1.1.62. *Let* $\mathbf{C}$ *be a Grothendieck category. The following are equivalent*

(1) $\mathbf{C}$ *is locally finitely generated;*

(2) $\mathbf{C}$ *has a family of finitely generated generators.*

Proof. (1) $\Longrightarrow$ (2) Let U be a generator of $\mathbf{C}$. Then there is a family of finitely generated subobjects of C, $(U_i)_{i\in I}$, such that there is an epimorphism $\bigoplus_{i\in I} U_i \to U$. Hence $(U_i)_{i\in I}$ is a family of finitely generated generators of $\mathbf{C}$.

(2) $\Longrightarrow$ (1) Let $(U_i)_{i\in I}$ be a family of finitely generated generators of $\mathbf{C}$. Set $U = \bigoplus_{i\in I} U_i$. Let $C \in \mathbf{C}$. Then there are a set J and an epimorphism $U^{(J)} \to C$. Consider the family $(U_{(i,j)})_{(i,j)\in I\times J}$ such that $U_{(i,j)} = U_i$ for every $(i,j) \in I \times J$. We have $U^{(J)} \simeq \bigoplus_{(i,j)\in I\times J} U_{(i,j)}$. It follows that there is an epimorphism $f : \bigoplus_{(i,j)\in I\times J} U_{(i,j)} \to C$. Let $f_{(i,j)} : U_{(i,j)} \to C$, $(i,j) \in I \times J$, be the morphisms defining f. Let $f_{(i,j)} = \mathrm{im}(f_{(i,j)}) \circ p_{(i,j)}$ be the canonical factorization, for every $(i,j) \in I \times J$. Then we obtain the commutative diagram

$$\begin{tikzcd}
\bigoplus_{(i,j)} U_{(i,j)} \arrow[rr, "f"] \arrow[dr, "\bigoplus_{(i,j)} p_{(i,j)}"'] & & C \\
& \bigoplus_{(i,j)} \mathrm{Im}(f_{(i,j)}) \arrow[ur] &
\end{tikzcd}.$$

Since f is an epimorphism, then the morphism $\bigoplus_{(i,j)\in I\times J} \mathrm{Im}(f_{(i,j)}) \to C$ is also an epimorphism. Finally, by Lemma 1.1.54 (a), each $\mathrm{Im}(f_{(i,j)})$ is finitely generated. $\square$

The category $\mathbf{C}$ is *locally noetherian* [85, p. 123] if it has a family of noetherian generators.

Proposition 1.1.63. *Let* $\mathbf{C}$ *be a Grothendieck category. The following are equivalent*

(1) $\mathbf{C}$ *is locally noetherian;*

(2) every object of $\mathbf{C}$ *is a sum of a (directed) family of noetherian subobjects;*

(3) $\mathbf{C}$ *is locally finitely generated and every finitely generated object of* $\mathbf{C}$ *is noetherian;*

(4) $\mathbf{C}$ *is locally finitely generated and every finitely generated object of* $\mathbf{C}$ *is finitely presented.*

Proof. First observe that if an object of $\mathbf{C}$ is a sum of a family of noetherian subobjects, then it is a sum of a directed family of noetherian subobjects (by Proposition 1.1.61 (5)).

(1) $\Longleftrightarrow$ (2) Analogous to that of Proposition 1.1.62 using Proposition 1.1.60.

(2) $\Longrightarrow$ (3) Obvious from Proposition 1.1.58.

(3) $\Longrightarrow$ (1) Obvious from Proposition 1.1.62.

(3) $\Longleftrightarrow$ (4) Obvious from Proposition 1.1.61 (6). $\square$

Small objects. Let C be an additive category and X an object in $\mathbf{C}$.

Lemma 1.1.64. [69, lemma II.16.1]
Let $f : X \to \bigoplus_{i\in I} X_i$ be a morphism in $\mathbf{C}$ and F is a nonempty finite subset of I. The following are equivalent

(a) there exists a factorization of f

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & \bigoplus_{i\in I} X_i , \\ & \bigoplus_{j\in F} X_j \arrow[ur, "u_F"'] & \end{tikzcd} \tag{1.14}$$

where u_F is the unique morphism satisfying $u_F u'_j = u_j$ for every $j \in F$, with u_i, $i \in I$, are the canonical injections of the direct sum $\bigoplus_{i\in I} X_i$ and u'_j, $j \in J$, are these of $\bigoplus_{j\in F} X_j$;

(b) $f = \sum_{j\in J} u_j p_j f$, where u_j, $j \in J$, are the canonical injections and p_j, $j \in J$, are the associated projections.

Proposition 1.1.65. [69, Proposition II.16.2]
The following are equivalent

(1) Every morphism $f : X \to \bigoplus_{i\in I} X_i$ in $\mathbf{C}$ has the factorization (1.14) *for some nonempty finite subset F of I;*

(2) the functor $h_X = \mathrm{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \boldsymbol{Ab}$ preserves direct sums.

X is called a *small object* if the conditions of the above proposition hold.

Proposition 1.1.66. [85, Exercise V.12, p. 134]
Let C be a Grothendieck category. The following are equivalent

(1) X is small;

(2) the functor $h_X = \mathrm{Hom}_{\mathbf{C}}(X, -)$ preserves denumerable direct sums;

(3) the functor $h_X = \mathrm{Hom}_{\mathbf{C}}(X, -)$ preserves denumerable direct unions;

(4) if $X = \sum X_i$ where $X_1 \subset X_2 \subset \dots$ is a denumerable ascending chain of subobjects of X, then $X = X_n$ for some n.

In [85] the terminology "$\sum$-generated object" was used for small object.

Let C be a Grothendieck category. From Proposition 1.1.66, every finitely generated of $\mathbf{C}$ is small. Following [77, p. 76], we say that $\mathbf{C}$ is a *steady category* if every small object of $\mathbf{C}$ is finitely generated. For instance, every locally noetherian category is a steady category (see [77, Exercice 2.12.22, p. 76]).

Ring action on a k-category. To give a k-category with only one object is the same as to give a k-algebra. Let A be a k-algebra and $\mathbf{C}$ be a k-category. Consider A as the category with only one object $*$. To give a k-functor $T : A \to \mathbf{C}$ is the same as to give an object $T(*) = C$ and a k-algebra morphism

$$\rho : A = \mathrm{Hom}_A(*, *) \to \mathrm{Hom}_{\mathbf{C}}(T(*), T(*)) = \mathrm{Hom}_{\mathbf{C}}(C, C).$$

The functor $T : A \to \mathbf{C}$ is called a left A-*object* in $\mathbf{C}$ and is denoted by (C, ρ). The category of left A-objects in $\mathbf{C}$ is the category of k-functors $\mathbf{Hom}_k(A, \mathbf{C})$ and it is denoted by ${}_A\mathbf{C}$. Thus, a morphism $(C, \rho) \to (C', \rho')$ in ${}_A\mathbf{C}$ is a morphism $f : C \to C'$ in $\mathbf{C}$ such that, for all $a \in A$, the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\rho(a)"'] & C' \arrow[d, "\rho'(a)"] \\ C \arrow[r, "f"] & C'. \end{tikzcd}$$

From Proposition 1.1.7 and Example 1.1.10 (3), if $\mathbf{C}$ is a k-category (resp. k-abelian category), then so is ${}_A\mathbf{C}$.

Dually, a right A-*object* is a k-functor $A^\circ \to \mathbf{C}$. The category of right A-objects in $\mathbf{C}$ is the category of k-functors $\mathbf{Hom}_k(A^\circ, \mathbf{C})$ and is denoted by $\mathbf{C}_A$. Obviously, $\mathbf{C}_A \simeq {}_{A^\circ}\mathbf{C}$.

If $\mathbf{C} = \mathcal{M}_k$, then ${}_A\mathbf{C} = {}_A\mathcal{M}$ and $\mathbf{C}_A = \mathcal{M}_A$.

Let A be k-algebra and $F : \mathbf{C} \to \mathbf{D}$ be a k-functor. If $C \in {}_A\mathbf{C}$, then $F(C) \in {}_A\mathbf{D}$. Hence, we obtain a functor ${}_AF = F : {}_A\mathbf{C} \to {}_A\mathbf{D}$. Notice that ${}_AF$ is nothing else that the functor F^Σ defined in "Diagram categories". From this proposition, if F is left (resp. right) exact, then the functor ${}_AF$ is so. If F is a contravariant functor, we obtain a functor $F : {}_A\mathbf{C} \to \mathbf{D}_A$. For a k-category $\mathbf{C}$ and a k-algebra morphism $A \to B$, we obtain a functor ${}_A(-) : {}_B\mathbf{C} \to {}_A\mathbf{C}$ (the restriction functor).

Lemma 1.1.67. *Let $F, G : \mathbf{C} \to \mathbf{D}$ be k-functors and $\eta : F \to G$ a natural transformation. Then ${}_A\eta = \eta : {}_AF \to {}_AG$ is a natural transformation. In particular, if $C \in {}_A\mathbf{C}$ then $\eta_C : F(C) \to G(C)$ is a morphism in ${}_A\mathbf{D}$.*

Proof. Obvious. (This lemma is also an immediate consequence from the definition of the natural transformation η^Σ, see "Diagram categories".) □

Now, let $T : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ be a k-linear bifunctor (i.e. k-linear in each variable) and (C, ρ) is a left A-object in $\mathbf{C}$, then $T(C, D)$ is a left A-object in $\mathbf{E}$ for all $D \in \mathbf{D}$. Moreover, it is easy to verify that, for a morphism $f : C \to C'$ in $\mathbf{C}$ and a morphism $g : D \to D'$ in $\mathbf{D}$, $T(f, g)$ is a morphism in ${}_A\mathbf{E}$. Hence, we obtain a functor ${}_A\mathbf{C} \times \mathbf{D} \to {}_A\mathbf{E}$. If T is contravariant in $\mathbf{C}$ and covariant in $\mathbf{D}$, then we obtain a bifunctor ${}_A\mathbf{C} \times \mathbf{D} \to \mathbf{E}_A$. A special case is the following: Let $\mathbf{C}$ be a k-category. The k-linear bifunctor $\mathrm{Hom}_{\mathbf{C}}(-, -) : \mathbf{C} \times \mathbf{C} \to \mathcal{M}_k$ induces a bifunctor $\mathrm{Hom}_{\mathbf{C}}(-, -) : {}_A\mathbf{C} \times \mathbf{C} \to \mathcal{M}_A$.

Proposition 1.1.68. *Let $\mathbf{C}$ be a k-category and A a k-algebra.*

(1) Assume that $\mathbf{C}$ *has direct sums and cokernels. Then*

Let $X \in {}_A\mathbf{C}$ *and* $N \in \mathcal{M}_A$. *The functor* $\mathbf{C} \to \mathcal{M}_k$ *defined by*

$$Y \mapsto \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{C}}(X, Y))$$

is representable. Set $N \otimes_A X$ *its representative.*

(ii) The functor contravariant in the first variable and covariant in the second one

$$\mathrm{Hom}_{\mathbf{C}}(-,-) : {}_A\mathbf{C} \times \mathbf{C} \to \mathcal{M}_A$$

has a left adjoint functor

$$- \otimes_A - : \mathcal{M}_A \times {}_A\mathbf{C} \to \mathbf{C}$$

which assigns to each $(N, X) \in \mathcal{M}_A \times {}_A\mathbf{C}$, $N \otimes_A X$. *In particular, the functor* $- \otimes_A -$ *is* k*-linear and preserves limits in each variable.*

(2) Assume that $\mathbf{C}$ *has products and kernels. Then*

(i) let $X \in {}_A\mathbf{C}$ *and* $M \in {}_A\mathcal{M}$. *the functor* $\mathbf{C} \to \mathcal{M}_k$ *defined by*

$$Y \mapsto \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{C}}(Y, X))$$

is representable. Set $\mathrm{Hom}_A(M, X)$ *its representative.*

(ii) There is a functor

$$\mathrm{Hom}_A(-,-) : ({}_A\mathcal{M})^\circ \times {}_A\mathbf{C} \to \mathbf{C}$$

which is k*-linear and preserves colimits in each variable, and assigns to each* $(M, X) \in ({}_A\mathcal{M})^\circ \times {}_A\mathbf{C}$, $\mathrm{Hom}_A(M, X)$, *such that the isomorphism*

$$\mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{C}}(Y, X)) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, \mathrm{Hom}_A(M, X))$$

is natural in M, Y *and* X.

Proof. (1) (i) First we consider the particular case where $N = A^{(I)}$ with I is a set. Then there is an isomorphism which is natural in $Y \in \mathbf{C}$:

$$\mathrm{Hom}_A(A^{(I)}, \mathrm{Hom}_{\mathbf{C}}(X, Y)) \simeq \mathrm{Hom}_{\mathbf{C}}(X, Y)^I \simeq \mathrm{Hom}_{\mathbf{C}}(X^{(I)}, Y).$$

For the general case, let $A^{(J)} \xrightarrow{f} A^{(I)} \longrightarrow N \longrightarrow 0$ be an exact sequence with I and J are sets. By the Yoneda-Grothendieck lemma, there is a morphism $g : X^{(J)} \to X^{(I)}$ in $\mathbf{C}$ and a unique isomorphism of k-modules $\eta_Y : \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{C}}(X, Y)) \to \mathrm{Hom}_{\mathbf{C}}(\mathrm{Coker}(g), Y)$ making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{C}}(X, Y)) \arrow[r] \arrow[d, "\eta_Y"] & \mathrm{Hom}_A(A^{(I)}, \mathrm{Hom}_{\mathbf{C}}(X, Y)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(A^{(J)}, \mathrm{Hom}_{\mathbf{C}}(X, Y)) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(\mathrm{Coker}(g), Y) \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(X^{(I)}, Y) \arrow[r, "- \circ g"] & \mathrm{Hom}_{\mathbf{C}}(X^{(J)}, Y)
\end{tikzcd}.$$

By Lemma 1.1.4, η_Y is natural in $Y \in \mathbf{C}$.

(ii) Follows immediately using Proposition 1.1.41, the dual of Proposition 1.1.35, and Propositions 1.1.43, 1.1.33.

(2) Apply (1) to $\mathbf{C}^\circ$ and A°. $\square$

Remark 1.1.69. If $\mathbf{C}$ has finite direct sums and cokernels (resp. $\mathbf{C}$ has finite direct sums and kernels) and N (resp. M) is finitely presented, then, by a slightly different proof, we obtain that $N \otimes_A X$ (resp. $\mathrm{Hom}_A(M, X)$) is well defined.

Now, we will recall a well known result of Gabriel-Popescu ([79, Theorem 3.7.9]). Let $\mathbf{C}$ be a Grothendieck category and $U \in \mathbf{C}$. Set $A = \mathrm{End}_{\mathbf{C}}(U)$. Then $U \in {}_A\mathbf{C}$. Consider the functors

$$T = \mathrm{Hom}_{\mathbf{C}}(U, -) : \mathbf{C} \to \mathcal{M}_A, \quad S = - \otimes_A U : \mathcal{M}_A \to \mathbf{C}.$$

We have that (S, T) is an adjoint pair.

Theorem 1.1.70. [Gabriel-Popescu]
Let $\mathbf{C}$ be a Grothendieck category and $U \in \mathbf{C}$. The assertions below are equivalent

(1) U is a generator of $\mathbf{C}$;

(2) T is fully faithful and S is exact.

For an alternative proof of it see that of [60, Theorem 8.5.8].

The following result is contained in [53, Proposition 3].

Theorem 1.1.71. [Harada]
Let $\mathbf{C}$ be a Grothendieck category and $U \in \mathbf{C}$. The assertions below are equivalent

(1) S is an equivalence of categories;

(2) U is a generator, projective and small in $\mathbf{C}$.

Some Hom-tensor relations. Let R and S be rings.

Lemma 1.1.72. *(1) Let $\mathbf{C}$ and $\mathbf{D}$ be abelian categories, F and G be covariant (contravariant) additive functors from $\mathbf{C}$ to $\mathbf{D}$, and let $\eta : F \to G$ be a natural transformation.*

(a) If

$$0 \to C' \to C \to C'' \to 0$$

is split exact in $\mathbf{C}$, then η_C is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if $\eta_{C'}$ and $\eta_{C''}$ are monomorphisms (resp. epimorphisms, resp. isomorphisms).

(b) *If* $C_1, \dots, C_n \in \mathbf{C}$, *then* $\eta_{\oplus_{i=1}^n C_i}$ *is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if* $\eta_{C_1}, \dots, \eta_{C_n}$ *are monomorphisms (resp. epimorphisms, resp. isomorphisms).*

In particular, if $\mathbf{C}$ *and* $\mathbf{D}$ *are the categories of left or right modules over* R *and* S, *respectively, and if* η_R *is a monomorphism (resp. epimorphism, resp. isomorphism), then so is* η_P *for every finitely generated projective* R*-module* P.

(2) *Let* F *and* G *be left exact contravariant (additive) functors from* ${}_R\mathcal{M}$ *to* $\boldsymbol{Ab}$, *and let* $\eta : F \to G$ *be a natural transformation. If* $\eta_R : F(R) \to G(R)$ *is a monomorphism (resp. isomorphism), then* $\eta_M : F(M) \to G(M)$ *is also a monomorphism (resp. isomorphism) for every finitely generated (resp. finitely presented) module* ${}_RM$.

The version of this statement for right exact covariant (additive) functors is obtained by replacing monomorphism by epimorphism.

(3) *In the situation* $({}_SL, {}_SU_R, {}_RQ)$, *there is a homomorphism natural in* L, U *and* Q,

$$\eta : \mathrm{Hom}_S(L, U) \otimes_R Q \to \mathrm{Hom}_S(L, U \otimes_R Q)$$

defined by

$$\eta(\gamma \otimes q) : l \mapsto \gamma(l) \otimes q.$$

(i) *If* ${}_SL$ *is finitely generated and* ${}_RQ$ *is flat, then* η *is a monomorphism.*

(ii) *If*

(a) ${}_SL$ *is finitely generated and projective, or*

(b) ${}_SL$ *is finitely presented and* ${}_RQ$ *is flat, or*

(c) ${}_RQ$ *is finitely generated and projective,*

then η *is an isomorphism.*

(4) *In the situation* $(L_S, {}_RU_S, Q_R)$, *there is a homomorphism natural in* L, U *and* Q,

$$\eta : Q \otimes_R \mathrm{Hom}_S(L, U) \to \mathrm{Hom}_S(L, Q \otimes_R U)$$

defined by

$$\eta(q \otimes \gamma) : l \mapsto q \otimes \gamma(l).$$

(i) *If* L_S *is finitely generated and* Q_R *is flat, then* η *is a monomorphism.*

(ii) *If*

(a) L_S *is finitely generated and projective, or*

(b) L_S *is finitely presented and* Q_R *is flat, or*

(c) Q_R *is finitely generated and projective,*

then η *is an isomorphism.*

(5) *Let* M *and* N *be left* A*-modules. If*

(a) ${}_AN$ *is finitely generated and projective, or*

(b) ${}_AN$ *is finitely presented and* ${}_AM$ *is flat, or*

(c) ${}_AM$ *is finitely generated and projective,*

then the map

$$\theta_{N,M} : N^* \otimes_A M \to \mathrm{Hom}_A(N, M)$$

defined by

$$\theta_{N,M}(\gamma \otimes m) : n \mapsto \gamma(n)m,$$

is an isomorphism which is natural in M *and* N.

(6) *Let* M *and* N *be right* A*-modules. If*

(a) N_A *is finitely generated and projective, or*

(b) N_A *is finitely presented and* M_A *is flat, or*

(c) M_A *is finitely generated and projective,*

then the map

$$\theta_{N,M} : M \otimes_A N^* \to \mathrm{Hom}_A(N, M)$$

defined by

$$\theta_{N,M}(m \otimes \gamma) : n \mapsto m\gamma(n),$$

is an isomorphism which is natural in M *and* N.

(7) *In the situation* $(L_S, {}_RU_S, {}_RQ)$*, there is a homomorphism natural in* L, U *and* Q,

$$\nu : \mathrm{Hom}_S(U, L) \otimes_R Q \to \mathrm{Hom}_S(\mathrm{Hom}_R(Q, U), L)$$

defined by

$$\nu(\gamma \otimes q) : \alpha \mapsto \gamma(\alpha(q)).$$

(i) *If* ${}_RQ$ *is finitely generated and* L_S *is injective, then* ν *is an epimorphism.*

(ii) *If*

(a) ${}_RQ$ *is finitely generated and projective, or*

(b) ${}_RQ$ *is finitely presented and* L_S *is injective,*

then ν *is an isomorphism.*

(8) *In the situation* $({}_SL, {}_SU_R, Q_R)$*, there is a homomorphism natural in* L, U *and* Q,

$$\nu : Q \otimes_R \mathrm{Hom}_S(U, L) \to \mathrm{Hom}_S(\mathrm{Hom}_R(Q, U), L)$$

defined by

$$\nu(q \otimes \gamma) : \alpha \mapsto \gamma(\alpha(q)).$$

(i) If Q_R is finitely generated and ${}_SL$ is injective, then ν is an epimorphism.

(ii) If

(a) Q_R is finitely generated and projective, or

(b) Q_R is finitely presented and ${}_SL$ is injective,

then ν is an isomorphism.

Proof. (1) (a) From Proposition 1.1.50, there are two split exact short sequences

$$0 \to C' \to C \to C'' \to 0$$

and

$$0 \to C'' \to C \to C' \to 0.$$

By Corollary 1.1.51, we obtain then, in both cases (the "covariant" case and the "contravariant" one), two diagrams with split exact rows

$$\begin{tikzcd} 0 \arrow[r] & F(C') \arrow[r] \arrow[d, "\eta_{C'}"] & F(C) \arrow[r] \arrow[d, "\eta_C"] & F(C'') \arrow[r] \arrow[d, "\eta_{C''}"] & 0 \\ 0 \arrow[r] & G(C') \arrow[r] & G(C) \arrow[r] & G(C'') \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & F(C'') \arrow[r] \arrow[d, "\eta_{C''}"] & F(C) \arrow[r] \arrow[d, "\eta_C"] & F(C') \arrow[r] \arrow[d, "\eta_{C'}"] & 0 \\ 0 \arrow[r] & G(C'') \arrow[r] & G(C) \arrow[r] & G(C') \arrow[r] & 0. \end{tikzcd}$$

The short five lemma achieves then the proof.

(b) It follows by induction using (a).

(2) Let ${}_RM$ be a finitely generated (resp. finitely presented) module. Then there is an exact sequence $R^{(I)} \to R^n \to M \to 0$, where I is a (resp. a finite) nonempty index set and $n \in \mathbb{N} - \{0\}$. Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & F(M) \arrow[r] \arrow[d, "\eta_M"] & F(R^n) \arrow[r] \arrow[d, "\eta_{R^n}"] & F(R^{(I)}) \arrow[d, "\eta_{R^{(I)}}"] \\ 0 \arrow[r] & G(M) \arrow[r] & G(R^n) \arrow[r] & G(R^{(I)}). \end{tikzcd}$$

By (1), η_{R^n} is a monomorphism (resp. η_{R^n} and $\eta_{R^{(I)}}$ are isomorphisms). Hence η_M is also a monomorphism (resp. an isomorphism).

Analogously we prove the second statement.

(3), (4), (7) and (8) are straightforward from (1) and (2). (5) is obvious from (3) while (6) is obvious from (4). $\square$

Separable bimodules. Separable bimodules are introduced by Sugano in [86]. Let ${}_AM_B$ be a bimodule. Define the evaluation map $ev : M \otimes_B {}^*M \to A$ by $m \otimes f \mapsto f(m)$ which is an A-bimodule morphism. M is *separable* or *A is M-separable over B* if ev is a retraction in ${}_A\mathcal{M}_A$.

A *ring extension* A/S is a ring morphism $\iota : S \to A$. A ring extension A/S is called a *separable extention* if

$$\mu_S : A \otimes_S A \to A,\ a \otimes b \mapsto ab,$$

is a retraction in ${}_A\mathcal{M}_A$.

Let ${}_AM_B$ be a bimodule. We have, $E = \mathrm{End}_B(M)$ (resp. $E' = \mathrm{End}_A(M)$) is a ring extension over A (resp. B) via left (resp. right) multiplication.

Proposition 1.1.73. [86, Theorem 1, Proposition 2] (see also [57, Theorem 3.1])
Let ${}_AM_B$ be a bimodule.

1) If M is separable then E/A is a separable extension.

2) If M_B is finitely generated projective and E/A is a separable extension, then M is separable.

3) If ${}_AM$ is finitely generated projective and M is separable, then E'/B is a separable extension.

4) If ${}_AM$ is a generator and E'/B is a separable extension, then M is separable.

Locally projective modules. Let R be a ring. Following Zimmermann-Huisgen [98], a right R-module M is called *locally projective* if for every diagram in $\mathcal{M}_A$ with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "i"] & M \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ & L \arrow[r, "f"] & N \arrow[r] & 0, \end{tikzcd}$$

where F is finitely generated, there exists $h : M \to L$ with $gi = fhi$. Obviously, every projective module is locally projective.

For every $M \in \mathcal{M}_R$ and $N \in {}_R\mathcal{M}$, let

$$\alpha_{M,N} : M \otimes_R N \to \mathrm{Hom}_R(M^*, N)$$

be the map defined by $\alpha_{M,N}(m \otimes n)(f) = f(m)n$, for every $m \in M, n \in N, f \in M^*$.

Following Bass, M is called *torsionless* if $\alpha_{M,R}$ is injective and *reflexive* if $\alpha_{M,R}$ is an isomorphism.

Following Garfinkel [41], M is called *universally torsionless* (UTL) if $\alpha_{M,N}$ is injective for all $N \in {}_R\mathcal{M}$ and *universally ring torsionless* (URTL) if $M \otimes_R S$ is S-torsionless for all ring extensions S/R.

Proposition 1.1.74. [41, Theorem 2.2]
Let M be a module over a commutative ring R. The following are equivalent

(1) M is UTL;

(2) M is URTL;

(3) $M \otimes_R S$ is S-torsionless for all commutative ring extensions S/R.

Proposition 1.1.75. [41, Proposition 2.4]
Let M be a right R-module. The following are equivalent

(1) M is UTL;

(2) M is URTL and flat.

Let $\beta : G \to M$ be a map in $\mathcal{M}_R$ and $n \in \mathbb{N} - \{0\}$. Let us consider the property $P(n)$: for all $x_1, \ldots, x_n \in G$, there is a map $\varphi : M \to G$ in $\mathcal{M}_R$ such that $\beta\varphi\beta(x_i) = \beta(x_i)$ for $i = 1, \ldots, n$.

By Lemma [41, Lemma 3.1], we have that, for every n, $P(1)$ and $P(n)$ are equivalent.

A submodule K of M_R is called *ideal pure* if for every left ideal I of R, $KI = K \cap MI$. Following [41, p. 124], a map $\beta : G \to M$ in $\mathcal{M}_R$ is said to be *split* if there a map $\varphi : M \to G$ in $\mathcal{M}_R$ such that $\beta = \beta\varphi\beta$, *finitely split* if it satisfies $P(1)$, and *(ideal) pure* if its image is (ideal) pure. Note that, if the image of β is finitely generated, then it is split if and only if it is finitely split.

From [41, Theorem 2.6] and [98, Theorem 2.1] we obtain the following proposition. [98, Theorem 2.1] provides more equivalent conditions to the (1) to (11) ones.

Proposition 1.1.76. *Let M be a right R-module. The following are equivalent*

(1) M is locally projective;

(2) M is UTL;

(3) M is a (ideal) pure submodule of a right UTL module P;

(4) $\alpha_{M,N}$ is injective for every cyclic left R-module N;

(5) every $m \in M$ belongs in $M.M^*(m)$ with $M^*(m) = \{f(m) \mid f \in M^*\}$;

(6) for every $m \in M$, there are $m_1, \ldots, m_n \in M$ and $f_1, \ldots, f_n \in M^*$ such that $m = \sum m_i f_i(m)$;

(7) every ideal pure map $\beta : G \to M$ finitely splits;

(8) every epimorphism $\beta : G \to M$ finitely splits;

(9) for every finitely generated submodule K of M there is a finitely generated free right module F and maps $\alpha : F \to M$ and $\gamma : M \to F$ such that $\alpha\gamma$ is the identity on K;

(10) for every finitely generated submodule K of M there are $m_1, \ldots, m_n \in M$ and $f_1, \ldots, f_n \in M^*$ such that $x = \sum m_i f_i(x)$ for $x \in K$;

(11) for every epimorphism $\beta : G \to M$ the induced map

$$\mathrm{Hom}_R(C, \beta) : \mathrm{Hom}_R(C, G) \to \mathrm{Hom}_R(C, M)$$

is an epimorphism for every finitely generated module C such that for every $m \in M$ there is an $\alpha \in \mathrm{End}_R(M)$ with $\alpha(m) = m$ and the image of α is contained in a finitely generated submodule of M.

Corollary 1.1.77. [41, p. 126]

(a) Let M_R be UTL and S/R a ring extension. Then the right S-module $M \otimes_R S$ is also UTL.

(b) Each projective module is UTL.

(c) Any finitely generated UTL module is projective.

(d) Any finitely generated pure submodule of a UTL module is a projective direct summand.

(e) Any countably generated submodule of a UTL module M is contained in a countably generated pure projective submodule.

We refer to [41, 98] for more details.

The following example has been communicated to me by Edgard Enochs.

An example of a self-injective commutative ring which is not coherent. Let $(R_i)_{i \in I}$ be any family of rings indexed by the set I and let $R = \prod_{i \in I} R_i$. Then the following are easy to prove. If I is a finitely generated ideal of R such that I is generated by $n \geq 1$ elements, then for each $i \in I$ there are ideals $I_i \in R_i$ such that I_i is generated by n elements and such that $I = \prod I_i$. Conversely, given an $n \geq 1$ and such ideal I_i for each $i \in I$ we get that $I = \prod I_i$ is a finitely generated ideal of R and moreover that I is generated by n elements.

The above can be generalized to finitely generated submodules of R^m for any $m \geq 1$.

Now using the above facts it is not hard to see that if R is coherent the following two conditions hold:

(a) each R_i is coherent,

(b) for each i and $n \geq 1$ if I_i is an ideal of R_i generated by n elements and if $R_i^n \to I_i$ is surjective and linear then there is some $m \geq 1$ such that each of these kernels is generated by m elements.

Now for an example, let $I = P$ (P the set of positive natural numbers) and let $R_n = k[[x_1, \ldots, x_n]]$ modulo the ideal generated by $x_1^2, \ldots, x_n^2$ Then each R_n is artinian and so coherent. But now look at [33, Proposition 21.5 (p. 530), Corollary 21.19]. Then we see that each R_n has a unique minimal ideal I_n. But it is not hard to see that if $R_n \to I_n$ is surjective then the kernel (which is the ideal generated by the cosets of $x_1, \ldots, x_n$) cannot be generated by fewer than n elements. So in this case, the product ring R is not coherent.

By the results in [33] mentioned above, we have that each R_n is self-injective (in his language, Gorenstein of dimension 0). Finally, from [61, Corollary 3.11B], the product of self-injective rings is also self-injective. So this is an example of a commutative ring R that is self-injective but which is not coherent.

1.2 The category of comodules

We recall from [88] the definition of a coring.

Definition 1.2.1. An A-*coring* is an A-bimodule $\mathfrak{C}$ with two A-bimodule maps $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ (coproduct or comultiplication) and $\epsilon : \mathfrak{C} \to A$ (counit) such that $(\mathfrak{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathfrak{C}) \circ \Delta$ (coassociative property) and $(\epsilon \otimes_A \mathfrak{C}) \circ \Delta = (\mathfrak{C} \otimes_A \epsilon) \circ \Delta = 1_{\mathfrak{C}}$ (counit property), that is the following diagrams are commutative

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\mathfrak{C}\otimes_A\Delta"] \\ \mathfrak{C} \otimes_A \mathfrak{C} \arrow[r, "\Delta\otimes_A\mathfrak{C}"] & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{tikzcd} \quad \begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta"] \arrow[dr, "\simeq"] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\epsilon\otimes_A\mathfrak{C}"] \\ & A \otimes_A \mathfrak{C} \end{tikzcd} \quad \begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta"] \arrow[dr, "\simeq"] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\mathfrak{C}\otimes_A\epsilon"] \\ & \mathfrak{C} \otimes_A A \end{tikzcd} .$$

As for coalgebras, we will use the Sweedler's sigma notation, for every $c \in \mathfrak{C}$, we write

$$\Delta(c) = \sum c_{(1)} \otimes_A c_{(2)}.$$

Then, as for coalgebras, the coassociative property can be expressed by

$$\sum \Delta(c_{(1)}) \otimes_A c_{(2)} = \sum c_{(1)} \otimes_A \Delta(c_{(2)}) = \sum c_{(1)} \otimes_A c_{(2)} \otimes_A c_{(3)},$$

and the counit property can be expressed by

$$\sum \epsilon(c_{(1)})c_{(2)} = c = \sum c_{(1)}\epsilon(c_{(2)}).$$

Sometimes the symbol $\sum$ will be omitted.

Examples 1.2.2. (1) If we take $A = k$ we find the notion of a k-coalgebra.

(2) $\mathfrak{C} = A$ endowed with the obvious structure maps is an A-coring. This coring is called *the trivial coring.*

(3) Let $\rho : A \to B$ be a morphism of k-algebras, and $\mathfrak{C}$ be an A-coring. $B\mathfrak{C}B = B\otimes_A\mathfrak{C}\otimes_A B$ is a B-coring with coproduct and counit:

$$\Delta_{B\mathfrak{C}B} := B \otimes_A \upsilon \otimes_A B : B \otimes_A \mathfrak{C} \otimes_A B \to B \otimes_A \mathfrak{C} \otimes_A B \otimes_A \mathfrak{C} \otimes_A B,$$

$$\epsilon_{B\mathfrak{C}B} : B \otimes_A \mathfrak{C} \otimes_A B \to B, \quad b \otimes_A c \otimes_A b' \mapsto b\rho(\epsilon(c))b',$$

where

$$\upsilon : \mathfrak{C} \to \mathfrak{C} \otimes_A B \otimes_A \mathfrak{C}, \quad c \mapsto \sum c_{(1)} \otimes_A 1_B \otimes_A c_{(2)}.$$

This coring is called *a base ring extension* of $\mathfrak{C}$.

(4) Let X be a nonempty set. $AX = A^{(X)}$ is an A-coring with coproduct and counit:

$$\Delta(x) = x \otimes_A x, \quad \epsilon(x) = 1_A,$$

for every $x \in X$. This coring is called the *grouplike coring on* X. If we take $A = k$ we find the definition of a grouplike coalgebra (cocommutative).

(5) Let M be a $B-A$-bimodule with M_A is finitely generated projective. Let $\{e_i, e_i^*\}_{i\in I}$ be a finite dual basis of M. Define $\tau : B \to M\otimes_A M^*, \quad b \mapsto \sum_{i\in I} be_i\otimes e_i^* = \sum_{i\in I} e_i\otimes e_i^* b$, and $\epsilon_{M^*\otimes_B M} : M^* \otimes_B M \to B, \quad \varphi \otimes m \mapsto \varphi(m)$.

Then we have $(A, B, {}_AM^*_B, {}_BM_A, \epsilon_{M^*\otimes_B M}, \tau)$ is a comatrix coring context, i.e. $\epsilon_{M^*\otimes_B M}$ and τ are bimodule maps, and the following diagrams are commutative

$$\begin{tikzcd} M^* \otimes_B M \otimes_A M^* \arrow[r, "\epsilon_{M^*\otimes_B M}\otimes_A M^*"] & A \otimes_A M^* \arrow[d, "\simeq"] \\ M^* \otimes_B B \arrow[u, "M^*\otimes_B B"] \arrow[r, "\simeq"] & N \end{tikzcd} \quad \begin{tikzcd} M \otimes_A M^* \otimes_B M \arrow[r, "M\otimes_A \epsilon_{M^*\otimes_B M}"] & M \otimes_A A \arrow[d, "\simeq"] \\ B \otimes_B M \arrow[u, "\tau\otimes_B M"] \arrow[r, "\simeq"] & M \end{tikzcd} .$$

By [35, Proposition 2.1] or [17, Theorem 2.4(2)], $M^* \otimes_B M$ is an A-coring with coproduct

$$\Delta_{M^*\otimes_B M} : M^* \otimes_B M \to M^* \otimes_B M \otimes_A M^* \otimes_B M, \quad \varphi \otimes m \mapsto \sum_{i\in I} \varphi \otimes e_i \otimes e_i^* \otimes m,$$

and counit $\epsilon_{M^*\otimes_B M}$. This coring is called the *comatrix coring* associated to the bimodule M. We propose a generalization of this coring in Chapter 3. In fact, we consider, under certain suitable conditions, a coring associated to a quasi-finite comodule over a coring.

(6) We take in (5) $M = A^n$ for $n \in \mathbb{N} - \{0\}$. $M^* \otimes_A M$ can be identified with the ring of all $n \times n$ matrices with entries in A, $M_n(A)$. Let $\{e_{i,j}\}_{1\leq i,j\leq n}$ be the canonical A-basis of $M_n(A)$. Then $M_n(A)$ is an A-coring with coproduct and counit defined as follows

$$\Delta(e_{i,j}) = \sum_k e_{i,k} \otimes_A e_{k,j}, \quad \epsilon(e_{i,j}) = \delta_{i,j}.$$

This coring is called the (n, n)-*matrix coring* over A and it is denoted by $M_n^c(A)$.

Following [88], for an A-coring, $\mathfrak{C}$, we define:

$$\mathfrak{C}^* = \mathrm{Hom}_{-A}(\mathfrak{C}, A), \quad {}^*\mathfrak{C} = \mathrm{Hom}_{A-}(\mathfrak{C}, A), \quad {}^*\mathfrak{C}^* = \mathrm{Hom}_{(A,A)}(\mathfrak{C}, A) = \mathfrak{C}^* \cap {}^*\mathfrak{C}.$$

The following result that gives the elementary properties of these k-modules is essentially due to Sweedler, see [88, Proposition 3.2] (see also [19, 17.8]).

Proposition 1.2.3. *(1) $\mathfrak{C}^*$ is a k-algebra with product $f *^r g = g(f \otimes_A \mathfrak{C})\Delta$, and unit ϵ. More explicitly, $f *^r g(c) = \sum g(f(c_{(1)})c_{(2)})$, for every $c \in \mathfrak{C}$. Moreover the map $i_R : A \to \mathfrak{C}^*, \quad a \mapsto \epsilon(a-)$, is an anti-morphism of k-algebras.*

(2) ${}^*\mathfrak{C}$ is a k-algebra with product $f *^l g = f(\mathfrak{C} \otimes_A g)\Delta$, and unit ϵ. More explicitly, $f *^l g(c) = \sum f(c_{(1)}g(c_{(2)}))$, for every $c \in \mathfrak{C}$. Moreover the map $i_L : A \to {}^*\mathfrak{C}, \quad a \mapsto \epsilon(-a)$, is an anti-morphism of k-algebras.

(3) ${}^*\mathfrak{C}^*$ is a k-algebra with product $f * g(c) = \sum f(c_{(1)})g(c_{(2)})$, for every $f, g \in {}^*\mathfrak{C}^*, c \in \mathfrak{C}$, and unit ϵ. Moreover the map $i : \mathrm{Z}(A) \to \mathrm{Z}({}^*\mathfrak{C}^*), \quad a \mapsto \epsilon(a-) = \epsilon(-a)$, is a morphism of k-algebras. ($\mathrm{Z}(A)$ and $\mathrm{Z}({}^*\mathfrak{C}^*)$ denote the center of A and ${}^*\mathfrak{C}^*$ respectively.)

(4) $\mathrm{Z}(\mathfrak{C}^*) \subset \mathrm{Z}({}^*\mathfrak{C}^*)$ and $\mathrm{Z}({}^*\mathfrak{C}) \subset \mathrm{Z}({}^*\mathfrak{C}^*)$.

(5) If $A = k$ ($\mathfrak{C}$ is a k-coalgebra), then $\mathfrak{C}^* = {}^*\mathfrak{C} = {}^*\mathfrak{C}^*$.

(6) If $\mathfrak{C}$ is the trivial A-coring, then $\mathfrak{C}^* \simeq A^\circ$ (the opposite algebra of A) and ${}^*\mathfrak{C} \simeq A$.

Definitions 1.2.4. A right $\mathfrak{C}$-*comodule* is a pair (M, ρ_M) consisting of a right A-module M and an A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$ (right coaction of $\mathfrak{C}$ on M) satisfying $(M \otimes_A \Delta) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M$ and $(M \otimes_A \epsilon) \circ \rho_M = 1_M$. That means that the following diagrams are commutative

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"] & M \otimes_A \mathfrak{C} \arrow[d, "M\otimes_A\Delta"] \\ M \otimes_A \mathfrak{C} \arrow[r, "\rho_M\otimes_A\mathfrak{C}"] & M \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "\simeq"] & M \otimes_A \mathfrak{C} \arrow[d, "M\otimes_A\epsilon"] \\ & M \otimes_A A \end{tikzcd} . \tag{1.15}$$

A *comodule morphism* $f : M \to N$ of right $\mathfrak{C}$-comodules (M, ρ_M) and (N, ρ_N) is a right A-linear map such that $(f \otimes_A \mathfrak{C}) \circ \rho_M = \rho_N \circ f$. That means the the following diagram is commutative

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"] & N \arrow[d, "\rho_N"] \\ M \otimes_A \mathfrak{C} \arrow[r, "f\otimes_A\mathfrak{C}"] & N \otimes_A \mathfrak{C} \end{tikzcd} . \tag{1.16}$$

The set of all such morphisms is a k-submodule of $\mathrm{Hom}_A(M, N)$, and it will be denoted by $\mathrm{Hom}_{\mathfrak{C}}(M, N)$.

Analogously we define a left *comodule* M, and a *comodule morphism* of left comodules. The left coaction of $\mathfrak{C}$ on M, $M \to \mathfrak{C} \otimes_A M$, is denoted by λ_M.

Since $\mathcal{M}_A$ is a k-category, right comodules over $\mathfrak{C}$ and their morphisms form a k-category, and it is denoted by $\mathcal{M}^{\mathfrak{C}}$. When $\mathfrak{C}$ is the trivial A-coring, $\mathcal{M}^A$ is the category of right A-modules $\mathcal{M}_A$. Analogously left comodules over $\mathfrak{C}$ and their morphisms form a k-category, and it is denoted by ${}^{\mathfrak{C}}\mathcal{M}$.

Coproducts and cokernels (and then inductive limits) in $\mathcal{M}^{\mathfrak{C}}$ exist and they coincide respectively with coproducts and cokernels in the category of right A-modules $\mathcal{M}_A$ (see Proposition 1.2.13). If ${}_A\mathfrak{C}$ is flat, then $\mathcal{M}^{\mathfrak{C}}$ is a k-abelian category (see Proposition 1.2.13). Moreover it is a Grothendieck category (see also Lemma 1.2.16).

As for coalgebras, we will use the Sweedler's sigma notation, for every $m \in M$, we write

$$\rho_M(m) = \sum m_{(0)} \otimes_A m_{(1)}.$$

Again as for coalgebras, the commutativity of the diagrams (1.15) can be expressed by

$$\sum \rho_M(m_{(0)}) \otimes_A m_{(1)} = \sum m_{(0)} \otimes_A \Delta(m_{(1)}) = \sum m_{(0)} \otimes_A m_{(1)} \otimes_A m_{(2)},$$

and

$$\sum m_{(0)}\epsilon(m_{(1)}) = m.$$

Also the commutativity of the diagram (1.16) can be expressed by

$$\sum f(m)_{(0)} \otimes_A f(m)_{(1)} = \sum f(m_{(0)}) \otimes_A m_{(1)}.$$

Sometimes the symbol $\sum$ will be omitted.

Proposition 1.2.5. *(1) Let T be a k-algebra and $M \in {}_T\mathcal{M}^{\mathfrak{C}}$. If $X \in \mathcal{M}_T$, then $X \otimes_T M$ is a right $\mathfrak{C}$-comodule. If f is a morphism in $\mathcal{M}_T$, then $f \otimes_T M$ is a morphism in $\mathcal{M}^{\mathfrak{C}}$. This yields a functor $- \otimes_T M : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$. In particular, we obtain a functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$.*

Right $\mathfrak{C}$-comodules of the type $X \otimes_A \mathfrak{C}$ where $X \in \mathcal{M}_A$, are called standard comodules.

(2) The forgetful functor (forgets the coaction) $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a left adjoint to the functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$. In particular, the functor $- \otimes_A \mathfrak{C}$ preserves projective limits.

(3) Let T be a k-algebra and $M \in {}_T\mathcal{M}^{\mathfrak{C}}$. The functor $- \otimes_T M : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$ is a left adjoint to the functor $\mathrm{Hom}_{\mathfrak{C}}(M, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_T$*. In particular, the functor $- \otimes_T M$ preserves inductive limits.*

Let I be a nonempty index set. $A^{(I)} \otimes_A \mathfrak{C} \simeq \mathfrak{C}^{(I)}$ as comodules, and for $M \in \mathcal{M}^{\mathfrak{C}}$, there is a surjective $\mathfrak{C}$-comodule $\mathfrak{C}^{(I')} \to M \otimes_A \mathfrak{C}$, where I' is a set.

(4) Let $M \in \mathcal{M}^{\mathfrak{C}}$. M is a direct summand of the right A-module $M \otimes_A \mathfrak{C}$. The coaction $\rho_M : M \to M \otimes_A \mathfrak{C}$ is a morphism of right $\mathfrak{C}$-comodules, and then M is a subcomodule of a standard comodule.

Proof. (1) Easy verifications.

(2) For every $M \in \mathcal{M}^{\mathfrak{C}}$ and $X \in \mathcal{M}_A$, the k-linear map

$$\varphi : \mathrm{Hom}_{\mathfrak{C}}(M, X \otimes_A \mathfrak{C}) \to \mathrm{Hom}_A(M, X),\ f \mapsto (X \otimes_A \epsilon)f,$$

is bijective, with inverse map $g \mapsto (g \otimes_A \mathfrak{C})\rho_M$. Moreover, φ is natural in M (then also in X).

(3) From the Hom-tensor relations, we obtain that for every $X \in \mathcal{M}_T$ and $N \in \mathcal{M}^{\mathfrak{C}}$, the k-linear map

$$\psi : \mathrm{Hom}_T(X, \mathrm{Hom}_A(M, N)) \to \mathrm{Hom}_A(X \otimes_T M, N),$$

$\psi(f)(x \otimes m) = f(x)(m)$, is bijective, with inverse map $\psi^{-1}(g)(x) = g(x \otimes -)$. Moreover, ψ is natural in both X and N. It is easy to verify that ψ induces a bijection

$$\psi' : \mathrm{Hom}_T(X, \mathrm{Hom}_{\mathfrak{C}}(M, N)) \to \mathrm{Hom}_{\mathfrak{C}}(X \otimes_T M, N).$$

Since ψ is natural in X, ψ' is also natural in X (and then in N). This achieves the proof of the first statement.

The first part of the last statement follows directly from the first statement. To show the second part, take a surjective A-linear map $f : A^{(M)} \to M$. Then, $f \otimes_A \mathfrak{C} : A^{(M)} \otimes_A \mathfrak{C} \to M \otimes_A \mathfrak{C}$ is a surjective comodule morphism, and $A^{(M)} \otimes_A \mathfrak{C} \simeq \mathfrak{C}^{(M)}$ as comodules.

(4) By the counit property, ρ_M is a section in $\mathcal{M}_A$ ($(M \otimes_A \epsilon)\rho_M = M$). In particular, ρ_M is an injective A-linear map. On the other hand, from the coassociativity property, ρ_M is a comodule morphism. □

Definitions 1.2.6. A $A' - A$-bimodule M is a $\mathfrak{C}' - \mathfrak{C}$-*bicomodule* if

(1) M is a the same time a right and a left comodule over $\mathfrak{C}$ and $\mathfrak{C}'$ respectively.

(2) ρ_M is A'-linear and λ_M is A-linear.

(3) one of the following equivalent statements holds

  (a) $\rho_M : M \to M \otimes_A \mathfrak{C}$ is a morphism of left $\mathfrak{C}'$-comodules;

  (b) $\lambda_M : M \to \mathfrak{C}' \otimes_{A'} M$ is a morphism of right $\mathfrak{C}$-comodules;

  (c) the following diagram is commutative

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\lambda_M"] & M \otimes_A \mathfrak{C} \arrow[d, "\lambda_M \otimes_A \mathfrak{C}"] \\ \mathfrak{C}' \otimes_{A'} M \arrow[r, "\mathfrak{C}' \otimes_{A'} \rho_M"] & \mathfrak{C}' \otimes_{A'} M \otimes_A \mathfrak{C}. \end{tikzcd}$$

A *morphism of bicomodules* is a morphism of right and left comodules.

$\mathfrak{C}' - \mathfrak{C}$-bicomodules with their morphisms form a k-category ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. Coproducts and cokernels (and then inductive limits) in this category exist and they coincide respectively with coproducts and cokernels in the category of right $A'-A$-modules, ${}_{A'}\mathcal{M}_A$ (see Proposition 1.2.13). If ${}_A\mathfrak{C}$ and $\mathfrak{C}'_{A'}$ are flat, ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ is a k-abelian category (see Proposition 1.2.13). Moreover it is a Grothendieck category (see also Corollary 1.2.8, Proposition 1.2.11, and Lemma 1.2.16). If $\mathfrak{C}$ and $\mathfrak{C}'$ are the trivial A-coring and A'-coring respectively, then ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ is exactly the category ${}_{A'}\mathcal{M}_A$.

Let R, S and T be k-algebras. In the situation $(M_{R-S}, {}_RN_T)$, $M \otimes_R N$ is a right $S \otimes T$-module via $(m \otimes n)(s \otimes t) = ms \otimes nt$. The situation ${}_{R-S}M$ is equivalent to that of ${}_{R\otimes S}M$ via $(r \otimes s)m = s(rm) = r(sm)$.

Lemma 1.2.7. *Let R, S and T be k-algebras. In the situation $(M_{R-S}, {}_RN_T, {}_{S-T}L)$, there is a unique morphism*

$$\alpha : (M \otimes_R N) \otimes_{S\otimes T} L \to M \otimes_{R\otimes S} (N \otimes_T L)$$

such that $\alpha((m \otimes n) \otimes l) = m \otimes (n \otimes l)$. Moreover, this morphism is an isomorphism and yields a natural isomorphism of functors.

Proof. Easy verifications. □

Corollary 1.2.8. *Let S and T be k-algebras. In the situation (M_S, N_T), if M_S and N_T are flat, then $(M \otimes N)_{S\otimes T}$ is also flat.*

Lemma 1.2.9. *In the situation $({}_AM_A, {}_AN_A, {}_BL_B, {}_BP_B)$, there is a unique morphism*

$$\beta : (M \otimes_A N) \otimes (L \otimes_B P) \to (M \otimes L) \otimes_{A\otimes B} (N \otimes P)$$

such that $\beta((m \otimes n) \otimes (l \otimes p)) = (m \otimes l) \otimes (n \otimes p)$. Moreover, this morphism is an isomorphism.

Proof. Easy verifications. □

Proposition 1.2.10. [44, Proposition 1.5]
Let $\mathfrak{C}$ be an A-coring and $\mathfrak{D}$ be a B-coring. Then $\mathfrak{C} \otimes \mathfrak{D}$ is an $A \otimes B$-coring with coproduct

$$\mathfrak{C} \otimes \mathfrak{D} \xrightarrow{\Delta_{\mathfrak{C}} \otimes \Delta_{\mathfrak{D}}} (\mathfrak{C} \otimes_A \mathfrak{C}) \otimes (\mathfrak{D} \otimes_B \mathfrak{D}) \xrightarrow{\simeq} (\mathfrak{C} \otimes \mathfrak{D}) \otimes_{A\otimes B} (\mathfrak{C} \otimes \mathfrak{D}),$$

and counit

$$\mathfrak{C} \otimes \mathfrak{D} \xrightarrow{\epsilon_{\mathfrak{C}} \otimes \epsilon_{\mathfrak{D}}} A \otimes B.$$

For example, if $\mathfrak{C}$ is the trivial A-coring and $\mathfrak{D}$ is the trivial B-coring, then the $A \otimes B$-coring $\mathfrak{C} \otimes \mathfrak{D}$ is noting else that the trivial $A \otimes B$-coring.

Every (A, B)-bimodule M yields a (B°, A°)-bimodule M°. Let $\mathfrak{C}$ be an A-coring. Then $\mathfrak{C}^\circ$ is an A°-bimodule. The *opposite coring* of $\mathfrak{C}$, that is denoted by $\mathfrak{C}^\circ$, is the A°-coring with coproduct

$$\Delta^\circ : \mathfrak{C}^\circ \xrightarrow{\Delta} \mathfrak{C} \otimes_A \mathfrak{C} \xrightarrow{\tau} \mathfrak{C}^\circ \otimes_{A^\circ} \mathfrak{C}^\circ\ ,$$

where τ is the isomorphism of k-modules defined by $\tau(c \otimes c') = c' \otimes c$, and counit $\epsilon^\circ = \epsilon : \mathfrak{C}^\circ \to A^\circ$.

As examples, the opposite coring of the trivial A-coring is the trivial A°-coring. The opposite coring of a k-coalgebra C is the opposite coalgebra of C.

The following result generalizes both the algebra case and the coalgebra case.

Proposition 1.2.11. [44, Proposition 1.8]
Let $\mathfrak{C}$ be an A-coring and $\mathfrak{D}$ be a B-coring. Then there are isomorphisms of categories

$$^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}} \simeq {}^{\mathfrak{C}\otimes\mathfrak{D}^\circ}\mathcal{M} \simeq \mathcal{M}^{\mathfrak{C}^\circ\otimes\mathfrak{D}}.$$

Lemma 1.2.12. *(1) Let $M, N, K \in \mathcal{M}^{\mathfrak{C}}$, $p : M \to N$ be a morphism in $\mathcal{M}^{\mathfrak{C}}$ which is a surjective map, and $f : N \to K$ be a map such that fp is a morphism in $\mathcal{M}^{\mathfrak{C}}$. Then f is a morphism in $\mathcal{M}^{\mathfrak{C}}$.*

(2) Let $M, N, K \in \mathcal{M}^{\mathfrak{C}}$, $f : M \to N$ be a map, and $i : N \to K$ be a morphism in $\mathcal{M}^{\mathfrak{C}}$, such that i and $i \otimes_A \mathfrak{C}$ are injective maps, and if is a morphism in $\mathcal{M}^{\mathfrak{C}}$. Then f is a morphism in $\mathcal{M}^{\mathfrak{C}}$.

Proof. The proof is easy. We prove for example (2). It is obvious that f is A-linear. We have moreover, $(i \otimes_A \mathfrak{C})(f \otimes_A \mathfrak{C})\rho_M = (if \otimes_A \mathfrak{C})\rho_M = \rho_K if = (i \otimes_A \mathfrak{C})\rho_N f$. Since $i \otimes_A \mathfrak{C}$ is injective, then $(f \otimes_A \mathfrak{C})\rho_M = \rho_N f$, that is, f is a morphism in $\mathcal{M}^{\mathfrak{C}}$. $\square$

Proposition 1.2.13. *(1) Let $\{M_i\}_I$ be a family of $^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. Put $M = \bigoplus_I M_i$ in $_{A'}\mathcal{M}_A$. Then there is a unique morphism $\rho_M : M \to M \otimes_A \mathfrak{C}$ in $_{A'}\mathcal{M}_A$ making the following diagrams commutative ($i \in I$)*

$$\begin{tikzcd} M_i \arrow[r, "\rho_{M_i}"] \arrow[d, "\iota_i"] & M_i \otimes_A \mathfrak{C} \arrow[d, "\iota_i \otimes_A \mathfrak{C}"] \\ M \arrow[r, "\rho_M"] & M \otimes_A \mathfrak{C} \end{tikzcd} .$$

(M, ρ_M) is a right $\mathfrak{C}$-comodule and $\iota_i : M_i \to M$ is a morphism of right $\mathfrak{C}$-comodules. Moreover $\{M, \{\iota_i\}_I\}$ is the coproduct of $\{M_i\}_I$ in $\mathcal{M}^{\mathfrak{C}}$.

Analogously we define $\lambda_M : M \to \mathfrak{C}' \otimes_{A'} M$ to be the unique morphism in $_{A'}\mathcal{M}_A$ making commutative the diagrams ($i \in I$)

$$\begin{tikzcd} M_i \arrow[r, "\lambda_{M_i}"] \arrow[d, "\iota_i"] & \mathfrak{C}' \otimes_{A'} M_i \arrow[d, "\mathfrak{C}' \otimes_{A'} \iota_i"] \\ M \arrow[r, "\lambda_M"] & \mathfrak{C}' \otimes_{A'} M \end{tikzcd} .$$

We have (M, ρ_M, λ_M) is a $\mathfrak{C}' - \mathfrak{C}$-bicomodule and $\iota_i : M_i \to M$ is a morphism of $\mathfrak{C}' - \mathfrak{C}$-bicomodules. Moreover $\{M, \{\iota_i\}_I\}$ is the coproduct of $\{M_i\}_I$ in $^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$.

(2) The category ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ has cokernels, which coincide with those in ${}_{A'}\mathcal{M}_A$.

(3) Let $f : M \to N$ be a morphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, and let $i : \mathrm{Ker}(f) \to M$ be its kernel in ${}_{A'}\mathcal{M}_A$. If f is $\mathfrak{C}'_{A'}$-pure and ${}_A\mathfrak{C}$-pure, and the following

$$i \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}, \quad \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} i \quad \textit{and} \quad \mathfrak{C}' \otimes_{A'} i \otimes_A \mathfrak{C}$$

are injective maps, then $\mathrm{Ker}(f)$ is the kernel of f in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. This is the case if f is $(\mathfrak{C}' \otimes_{A'} \mathfrak{C}')_{A'}$-pure, ${}_A(\mathfrak{C} \otimes_A \mathfrak{C})$-pure, and $\mathfrak{C}' \otimes_{A'} f$ is ${}_A\mathfrak{C}$-pure (e.g. if $\mathfrak{C}'_{A'}$ and ${}_A\mathfrak{C}$ are flat).

Proof. (1) The proof of the first part is well known and we left it to the reader. For the second part consider, for all $i \in I$, the diagram

$$\begin{tikzcd}
& M_i \arrow[rr, "\rho_{M_i}"] \arrow[dl, "\lambda_{M_i}"'] \arrow[dd, "\iota_i"] & & M_i \otimes_A \mathfrak{C} \arrow[dl, "\lambda_{M_i} \otimes_A \mathfrak{C}"] \arrow[dd, "\iota_i \otimes_A \mathfrak{C}"] \\
\mathfrak{C}' \otimes_{A'} M_i \arrow[rr, "\mathfrak{C}' \otimes_{A'} \rho_{M_i}"] \arrow[dd, "\mathfrak{C}' \otimes_{A'} \iota_i"'] & & \mathfrak{C}' \otimes_{A'} M_i \otimes_A \mathfrak{C} \arrow[dd, "\mathfrak{C}' \otimes_{A'} \iota_i \otimes_A \mathfrak{C}"] & \\
& M \arrow[rr, "\rho_M"] \arrow[dl, "\lambda_M"'] & & M \otimes_A \mathfrak{C} \arrow[dl, "\lambda_M \otimes_A \mathfrak{C}"] \\
\mathfrak{C}' \otimes_{A'} M \arrow[rr, "\mathfrak{C}' \otimes_{A'} \rho_M"] & & \mathfrak{C}' \otimes_{A'} M \otimes_A \mathfrak{C} &
\end{tikzcd} .$$

Then for all $i \in I$, all the squares, save possibly the bottom one, are commutative. Hence the bottom square is also commutative.

(2) The proof is well known and we left it to the reader.

(3) First put $M' = \mathrm{Ker}(f)$. From the assumptions there is a unique morphism $\rho_{M'} : M' \to M' \otimes_A \mathfrak{C}$ in ${}_{A'}\mathcal{M}_A$ making commutative in ${}_{A'}\mathcal{M}_A$ the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, "\rho_{M'}"] & M \arrow[r, "f"] \arrow[d, "\rho_M"] & N \arrow[d, "\rho_N"] \\
0 \arrow[r] & M' \otimes_A \mathfrak{C} \arrow[r, "i \otimes_A \mathfrak{C}"] & M \otimes_A \mathfrak{C} \arrow[r, "f \otimes_A \mathfrak{C}"] & N \otimes_A \mathfrak{C}
\end{tikzcd} .$$

Now consider the diagram

$$\begin{tikzcd}
M' \arrow[r, "\rho_{M'}"] \arrow[d, "\rho_{M'}"] & M' \otimes_A \mathfrak{C} \arrow[r, "i \otimes_A \mathfrak{C}"] \arrow[d, "M' \otimes_A \Delta_{\mathfrak{C}}"] & M \otimes_A \mathfrak{C} \arrow[d, "M \otimes_A \Delta_{\mathfrak{C}}"] \\
M' \otimes_A \mathfrak{C} \arrow[r, "\rho_{M'} \otimes_A \mathfrak{C}"] & M' \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \arrow[r, "i \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}"] & M \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}
\end{tikzcd} . \tag{1.17}$$

$i \otimes_A \mathfrak{C}$ is a morphism in $\mathcal{M}^{\mathfrak{C}}$ means that the second rectangle of (1.17) is commutative. Since

$$\begin{aligned}
(M \otimes_A \Delta_{\mathfrak{C}})(i \otimes_A \mathfrak{C})\rho_{M'} &= (M \otimes_A \Delta_{\mathfrak{C}})\rho_M i \\
&= (\rho_M \otimes_A \mathfrak{C})\rho_M i \\
&= (\rho_M \otimes_A \mathfrak{C})(i \otimes_A \mathfrak{C})\rho_{M'} \\
&= (\rho_M i \otimes_A \mathfrak{C})\rho_{M'} \\
&= (i \otimes_A \mathfrak{C} \otimes_A \mathfrak{C})(\rho_{M'} \otimes_A \mathfrak{C})\rho_{M'}.
\end{aligned}$$

Then the diagram (1.17) is commutative. Hence the coassociative property of $\rho_{M'}$ holds.

The counit property of $\rho_{M'}$ follows from the commutativity of the following diagram and the counit property of ρ_M

$$\begin{tikzcd}
M' \arrow[r, "\rho_{M'}"] \arrow[d, "i"] & M' \otimes_A \mathfrak{C} \arrow[r, "M' \otimes_A \epsilon_{\mathfrak{C}}"] \arrow[d, "i \otimes_A \mathfrak{C}"] & M' \otimes_A A \arrow[d, "i \otimes_A A"] \\
M \arrow[r, "\rho_M"] & M \otimes_A \mathfrak{C} \arrow[r, "M \otimes_A \epsilon_{\mathfrak{C}}"] & M \otimes_A A
\end{tikzcd} .$$

Hence $(M', \rho_{M'})$ is a right $\mathfrak{C}$-comodule.

By the same way we define $\lambda_{M'} : M' \to \mathfrak{C}' \otimes_{A'} M'$ which endow M' with a structure of left $\mathfrak{C}'$-comodule.

Now consider the diagram

$$\begin{tikzcd}
& M' \arrow[rr, "\rho_{M'}"] \arrow[dl, "\lambda_{M'}"'] \arrow[dd, "i"] & & M' \otimes_A \mathfrak{C} \arrow[dl, "\lambda_{M'} \otimes_A \mathfrak{C}"'] \arrow[dd, "i \otimes_A \mathfrak{C}"] \\
\mathfrak{C}' \otimes_{A'} M' \arrow[rr, "\mathfrak{C}' \otimes_{A'} \rho_{M'}"] \arrow[dd, "\mathfrak{C}' \otimes_{A'} i"] & & \mathfrak{C}' \otimes_{A'} M' \otimes_A \mathfrak{C} \arrow[dd, "\mathfrak{C}' \otimes_{A'} i \otimes_A \mathfrak{C}"] & \\
& M \arrow[rr, "\rho_M"] \arrow[dl, "\lambda_M"'] & & M \otimes_A \mathfrak{C} \arrow[dl, "\lambda_M \otimes_A \mathfrak{C}"'] \\
\mathfrak{C}' \otimes_{A'} M \arrow[rr, "\mathfrak{C}' \otimes_{A'} \rho_M"] & & \mathfrak{C}' \otimes_{A'} M \otimes_A \mathfrak{C} &
\end{tikzcd} .$$

It follows from Lemma 1.1.4 ($\mathfrak{C}' \otimes_{A'} i \otimes_A \mathfrak{C}$ is an injective map), that ρ_M is a morphism of left $\mathfrak{C}'$-comodules (or λ_M is a morphism of right $\mathfrak{C}$-comodules). Hence $M' \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and i is a monomorphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$.

Finally, let $\xi : X \to M$ be a monomorphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ such that $f\xi = 0$. Then there is a unique morphism $\gamma : X \to M'$ in ${}_{A'}\mathcal{M}_A$ such that $\xi = i\gamma$. By Lemma 1.2.12 ($i \otimes_A \mathfrak{C}$ and $\mathfrak{C}' \otimes_{A'} i$ are injective), γ is a morphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. Therefore (M', i) is the kernel of f in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. $\square$

Corollary 1.2.14. [52, Proposition 1.1]
Let $f : M \to N$ be a morphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, and let $i : \mathrm{Ker}(f) \to M$ be its kernel in ${}_{A'}\mathcal{M}_A$. If i is section in ${}_{A'}\mathcal{M}_A$ and $\mathrm{Im}(f)$ is a $\mathfrak{C}'_{A'}$-pure submodule and a ${}_A\mathfrak{C}$-pure submodule of N, then $\mathrm{Ker}(f)$ is the kernel of f in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$.

Corollary 1.2.15. [19, 18.14]
For an A-coring $\mathfrak{C}$, the following are equivalent

(1) ${}_A\mathfrak{C}$ is flat;

(2) the forgetful functor $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a monofunctor;

(3) every monomorphism $U \to \mathfrak{C}$ in $\mathcal{M}^{\mathfrak{C}}$ is injective (in $\mathcal{M}_A$).

Proof. (1) $\Longrightarrow$ (2) Clear. (2) $\Longrightarrow$ (3) Trivial.
(3) $\Longrightarrow$ (1) By the Flat Test Lemma [5, 19.17], it is enough to show that $\mathfrak{C}$ is A_A-flat, that is, for every right ideal I of A, the map $I \otimes_A \mathfrak{C} \to A \otimes_A \mathfrak{C}$ is injective. Since $A \otimes_A \mathfrak{C} \simeq \mathfrak{C}$ in $\mathcal{M}^{\mathfrak{C}}$ and the functor $- \otimes_A \mathfrak{C}$ preserves monomorphisms, the claimed statement holds. □

Lemma 1.2.16. *If ${}_A\mathfrak{C}$ is flat, then the subcomodules of $\mathfrak{C}^n$, $n \in \mathbb{N}$, form a generating family for the category $\mathcal{M}^{\mathfrak{C}}$.*

Proof. Let $M \in \mathcal{M}^{\mathfrak{C}}$. M is a subcomodule of a standard comodule M'. Then there is an epimorphism $f : \mathfrak{C}^{(\Lambda)} \to M'$ for some index set Λ. Let K be a subcomodule of M such that $K \neq M$, and let $m \in M - K$. There are $n \in \mathbb{N}$ and a monomorphism $i_n : \mathfrak{C}^{(n)} \to \mathfrak{C}^{(\Lambda)}$ such that $m \in \mathrm{Im}(fi_n)$. Then we obtain the following diagram

$$\begin{tikzcd} \mathfrak{C}^{(n)} \arrow[r, "i_n"] & \mathfrak{C}^{(\Lambda)} \arrow[r, "f"] & M' \\ (fi_n)^{-1}(M) \arrow[u] \arrow[rr, "g"] & & M \arrow[u, hook] \end{tikzcd}.$$

Then, by Lemma 1.2.12, g is a morphism in $\mathcal{M}^{\mathfrak{C}}$. g cannot be factorized through a morphism $(fi_n)^{-1}(M) \to K$. Hence the mentioned family is a family of generators of $\mathcal{M}^{\mathfrak{C}}$. □

Now, we end this section by recalling some corings of particular interest.

Coseparable corings. Following [52], a coring $\mathfrak{C}$ is said to be *coseparable* if the comultiplication map $\Delta_{\mathfrak{C}}$ is a section in the category ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$. Obviously the trivial coring is coseparable.

Definition 1.2.17. A comodule $M \in \mathcal{M}^{\mathfrak{C}}$ is called A-relative injective comodule or $(\mathfrak{C}, A)$-injective comodule if for every morphism in $\mathcal{M}^{\mathfrak{C}}$, $i : N \to L$, that is a section in $\mathcal{M}_A$, every morphism in $\mathcal{M}^{\mathfrak{C}}$, $f : N \to M$, there is a morphism in $\mathcal{M}^{\mathfrak{C}}$, $g : L \to M$, such that $f = gi$.

Proposition 1.2.18. [19, 18.18]

(1) Let $M \in \mathcal{M}^{\mathfrak{C}}$. The following are equivalent

(a) M is $(\mathfrak{C}, A)$-injective;

(b) every morphism in $\mathcal{M}^{\mathfrak{C}}$, $i : M \to L$, that is a section in $\mathcal{M}_A$, is also a section in $\mathcal{M}^{\mathfrak{C}}$;

(c) the coaction $\rho_M : M \to M \otimes_A \mathfrak{C}$ is a section in $\mathcal{M}^{\mathfrak{C}}$.

(2) For every $X \in \mathcal{M}_A$, $X \otimes_A \mathfrak{C}$ is $(\mathfrak{C}, A)$-injective.

(3) For every $M \in \mathcal{M}^{\mathfrak{C}}$ that is $(\mathfrak{C}, A)$-injective, and every $L \in \mathcal{M}^{\mathfrak{C}}$, the canonical sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(L, M) \xrightarrow{i} \mathrm{Hom}_A(L, M) \xrightarrow{\gamma} \mathrm{Hom}_A(L, M \otimes_A \mathfrak{C})$$

is split exact in $\mathcal{M}_B$, where $B = \mathrm{End}_{\mathfrak{C}}(L)$ and $\gamma(f) = \rho_M f - (f \otimes_A \mathfrak{C})\rho_L$.

For instance the comodule $\mathfrak{C} \in \mathcal{M}^{\mathfrak{C}}$ is $(\mathfrak{C}, A)$-injective.

Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is flat. An exact sequence in $\mathcal{M}^{\mathfrak{C}}$ is called $(\mathfrak{C}, A)$-*exact* if it is split exact in $\mathcal{M}_A$. A functor on $\mathcal{M}^{\mathfrak{C}}$ is called right (left) $(\mathfrak{C}, A)$-*exact* if it is right (left) exact on short $(\mathfrak{C}, A)$-exact sequences. For instance M is $(\mathfrak{C}, A)$-injective if and only if $\mathrm{Hom}_{\mathfrak{C}}(-, M)$ is $(\mathfrak{C}, A)$-exact.

Proposition 1.2.19. [19, 26.1]
Let $\mathfrak{C}$ be an A-coring. Then the following are equivalent

(1) $\mathfrak{C}$ is coseparable;

(2) there is an A-bimodule map $\delta : \mathfrak{C} \otimes_A \mathfrak{C} \to A$ satisfying

$$\delta\Delta = \epsilon \quad \textit{and} \quad (\mathfrak{C} \otimes_A \delta)(\Delta \otimes_A \mathfrak{C}) = (\delta \otimes_A \mathfrak{C})(\mathfrak{C} \otimes_A \Delta); \tag{1.18}$$

(3) the forgetful functor $(-)_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is separable;

(4) the forgetful functor ${}_A(-) : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is separable;

(5) the forgetful functor ${}_A(-)_A : {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}} \to {}_A\mathcal{M}_A$ is separable;

(6) $\mathfrak{C}$ is (A, A)-relative semisimple as a $\mathfrak{C}$-bicomodule, that is any monomorphism in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$ that splits as an A-bimodule map also splits in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$;

(7) $\mathfrak{C}$ is (A, A)-relative injective as a $\mathfrak{C}$-bicomodule;

(8) $\mathfrak{C}$ is $(A, \mathfrak{C})$-relative injective as a $\mathfrak{C}$-bicomodule;

(9) $\mathfrak{C}$ is $(\mathfrak{C}, A)$-relative injective as a $\mathfrak{C}$-bicomodule.

In such a case, $\mathfrak{C}$ is right and left $(\mathfrak{C}, A)$-semisimple, that is, all comodules in $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}$ are $(\mathfrak{C}, A)$-injective.

Definition 1.2.20. An A-bimodule map $\delta : \mathfrak{C} \otimes_A \mathfrak{C} \to A$ satisfying conditions (1.18) is called a *cointegral* in $\mathfrak{C}$.

Cosplit corings. A coring is called a *cosplit coring* if it satisfy the conditions of the proposition below.

Proposition 1.2.21. *Let $\mathfrak{C}$ be an A-coring. Then the following are equivalent*

(1) the functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$ is separable;

(2) the functor $\mathfrak{C} \otimes_A - : {}_A\mathcal{M} \to {}^{\mathfrak{C}}\mathcal{M}$ is separable;

(3) ϵ is a retraction in ${}_A\mathcal{M}_A$.

Proof. By Rafael's Proposition 1.1.48, $- \otimes_A \mathfrak{C}$ is separable if and only if there exists a natural transformation $\xi' : 1_{\mathcal{M}_A} \to - \otimes_A \mathfrak{C}$ such that $\xi_X \circ \xi'_X = 1_X$ for every $X \in \mathcal{M}_A$, where $\xi_X = X \otimes_A \epsilon$, $X \in \mathcal{M}_A$, is the counit of the adjunction $(U_r, - \otimes_A \mathfrak{C})$. ($U_r$ is the forgetful functor.) By Lemma 1.1.67 (or by Lemma 2.1.16), a natural transformation $\xi' : 1_{\mathcal{M}_A} \to - \otimes_A \mathfrak{C}$ is entirely determined by the data of an A-bimodule map $\xi' : A \to \mathfrak{C}$ such that $\xi'_X = X \otimes_A \xi'$ for every $X \in \mathcal{M}_A$. Hence the equivalence (1) $\Leftrightarrow$ (3) follows. (2) $\Leftrightarrow$ (3) follows by symmetry. □

Cosemisimple corings. Let $\mathbf{C}$ be an abelian category. $\mathbf{C}$ is called a *spectral category* if every short exact sequence in $\mathbf{C}$ splits, or equivalently, if every object is injective (resp. projective). As for modules, an object S of $\mathbf{C}$ is *simple* if $S \neq 0$ has exactly two object 0 and S, and it is *semisimple* if it a sum of simple subjects. 0 is, by definition, a semisimple object. A spectral category is called *discrete* if every object is semisimple.

Proposition 1.2.22. [85, Propsition 6.7]
Let $\mathbf{C}$ be a Grothendieck category. Then the following are equivalent

(1) $\mathbf{C}$ is a discrete spectral category;

(2) $\mathbf{C}$ is a locally finitely generated spectral category;

(3) every object of $\mathbf{C}$ is semisimple;

(4) $\mathbf{C}$ has a family of simple generators;

(5) $\mathbf{C}$ is equivalent to a product category $\prod_I \mathcal{M}_{K_i}$, where K_i, $i \in I$, is a family of division rings.

For more details we refer to [85, Chap.V, §6].

Proposition 1.2.23. [37, Theorem 3.1]
Let $\mathfrak{C}$ be an A-coring. Then the following are equivalent

(1) $\mathcal{M}^{\mathfrak{C}}$ is an abelian and a discrete spectral category;

(2) ${}^{\mathfrak{C}}\mathcal{M}$ *is an abelian and a discrete spectral category;*

(3) $\mathfrak{C}$ *is semisimple in* $\mathcal{M}^{\mathfrak{C}}$ *and* ${}_A\mathfrak{C}$ *is flat;*

(4) $\mathfrak{C}$ *is semisimple in* ${}^{\mathfrak{C}}\mathcal{M}$ *and* $\mathfrak{C}_A$ *is flat;*

(5) $\mathfrak{C}$ *is a semisimple right* $\mathfrak{C}^*$*-module and* $\mathfrak{C}_A$ *is projective;*

(6) $\mathfrak{C}$ *is a semisimple left* ${}^*\mathfrak{C}$*-module and* ${}_A\mathfrak{C}$ *is projective.*

If the above conditions hold, we say that $\mathfrak{C}$ is a *cosemisimple coring*.

Semiperfect corings. In [8], H. Bass defined and studied perfect and semiperfect rings. In [63], L.-P. Lin introduced and studied semiperfect coalgebras over fields. In [54, §3], M. Harada defined perfect and semiperfect Grothendieck categories and characterized them with respect to a generating set. Let $\mathbf{C}$ be a Grothendieck category. $\mathbf{C}$ is called *perfect* (resp. *semiperfect*) if every object of $\mathbf{C}$ has a projective cover (which is defined as for modules).

Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is flat. $\mathfrak{C}$ is called right *semiperfect* if $\mathcal{M}^{\mathfrak{C}}$ is a semiperfect category.

Proposition 1.2.24. [22, Theorem 3.1]
Let $\mathfrak{C}$ *be an* A*-coring such that* A *is a right artinian ring and* ${}_A\mathfrak{C}$ *is projective. Then the following are equivalent*

(1) $\mathfrak{C}$ *is right semiperfect;*

(2) every simple object of $\mathcal{M}^{\mathfrak{C}}$ *has a projective cover;*

(3) every finitely generated object of $\mathcal{M}^{\mathfrak{C}}$ *has a finitely generated projective cover;*

(4) every simple object of $\mathcal{M}^{\mathfrak{C}}$ *has a finitely generated projective cover;*

(5) the category $\mathcal{M}^{\mathfrak{C}}$ *has enough projectives;*

(6) the category $\mathcal{M}^{\mathfrak{C}}$ *has a projective generator.*

For a study of semiperfect corings over QF rings, see [36] or [22, §4].

1.3 Cotensor product over corings

First, we recall that the cotensor product over coalgebras over fields was introduced by Milnor and Moore in [68]. A detailed study of the cotensor product over corings is given in [19].

Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$. The map

$$\omega_{M,N} = \rho_M \otimes_A N - M \otimes_A \lambda_N : M \otimes_A N \to M \otimes_A \mathfrak{C} \otimes_A N$$

is a $\mathfrak{C}' - \mathfrak{C}''$-bicomodule map.

Notice that $W \otimes_{A'} \omega_{M,N} = \omega_{W\otimes_{A'} M,N}$ and $\omega_{M,N} \otimes_{A''} V = \omega_{M,N\otimes_{A''} V}$ for every $W \in \mathcal{M}_{A'}$ and $V \in {}_{A''}\mathcal{M}$.

Definition 1.3.1. The kernel of $\omega_{M,N}$ in ${}_{A'}\mathcal{M}_{A''}$ is the *cotensor product* of M and N, and it is denoted by $M\square_{\mathfrak{C}}N$. We denote the canonical injection $M\square_{\mathfrak{C}}N \to M \otimes_A N$ by $\iota_{M,N}$.

Proposition 1.3.2. *If $\omega_{M,N}$ is $\mathfrak{C}'_{A'}$- and ${}_{A''}\mathfrak{C}''$-pure, and the following*

$$\iota_{M,N} \otimes_{A''} \mathfrak{C}'' \otimes_{A''} \mathfrak{C}'', \quad \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} \iota_{M,N} \quad \textit{and} \quad \mathfrak{C}' \otimes_{A'} \iota_{M,N} \otimes_{A''} \mathfrak{C}''$$

are injective maps, then $M\square_{\mathfrak{C}}N$ is the kernel of $\omega_{M,N}$ in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''}$.

These conditions are fulfilled if $\omega_{M,N}$ is $(\mathfrak{C}' \otimes_{A'} \mathfrak{C}')_{A'}$- and ${}_{A''}(\mathfrak{C}'' \otimes_{A''} \mathfrak{C}'')$-pure, and $\mathfrak{C}' \otimes_{A'} \omega_{M,N}$ is ${}_{A''}\mathfrak{C}''$-pure (e.g. if $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ are flat).

Proof. Follows immediately from Proposition 1.2.13. □

Definition 1.3.3. Let $f : M \to M'$ be a morphism in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, and $g : N \to N'$ be a morphism in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$. There is a unique morphism in ${}_{A'}\mathcal{M}_{A''}$, $f\square_{\mathfrak{C}}g : M\square_{\mathfrak{C}}N \to M'\square_{\mathfrak{C}}N'$, making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & M\square_{\mathfrak{C}}N \arrow[r] \arrow[d, "f\square_{\mathfrak{C}}g"] & M \otimes_A N \arrow[r, "\omega_{M,N}"] \arrow[d, "f\otimes_A g"] & M \otimes_A \mathfrak{C} \otimes_A N \arrow[d, "f\otimes_A \mathfrak{C}\otimes_A g"] \\
0 \arrow[r] & M'\square_{\mathfrak{C}}N' \arrow[r] & M' \otimes_A N' \arrow[r, "\omega_{M',N'}"] & M' \otimes_A \mathfrak{C} \otimes_A N'
\end{tikzcd} .$$

$f\square_{\mathfrak{C}}g$ is called the *cotensor product* of f and g.

Proposition 1.3.4. *If for every $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$, $\omega_{M,N}$ is $\mathfrak{C}'_{A'}$- and ${}_{A''}\mathfrak{C}''$-pure, then we have a k-linear bifunctor*

$$-\square_{\mathfrak{C}}- : {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}} \times {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''} \longrightarrow {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''} . \tag{1.19}$$

In particular, if $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ are flat, then the bifunctor (1.19) *is well defined.*

Proof. Follows immediately from Proposition 1.3.2 and Lemma 1.2.12. □

Lemma 1.3.5. *(1) Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ and $W \in \mathcal{M}_{A'}$. We define $\psi_{W,M,N}$: $W \otimes_{A'} (M\square_{\mathfrak{C}}N) \to (W \otimes_{A'} M)\square_{\mathfrak{C}}N$ to be the unique morphism making commutative the following diagram*

$$\begin{tikzcd}
& W \otimes_{A'} (M\square_{\mathfrak{C}}N) \arrow[r] \arrow[d, "\psi_{W,M,N}"] & W \otimes_{A'} (M \otimes_A N) \arrow[r, "W\otimes_{A'}\omega_{M,N}"] \arrow[d, "\simeq"] & W \otimes_{A'} (M \otimes_A \mathfrak{C} \otimes_A N) \arrow[d, "\simeq"] \\
0 \arrow[r] & (W \otimes_{A'} M)\square_{\mathfrak{C}}N \arrow[r] & (W \otimes_{A'} M) \otimes_A N \arrow[r, "\omega_{W\otimes_{A'} M,N}"] & (W \otimes_{A'} M) \otimes_A \mathfrak{C} \otimes_A N
\end{tikzcd} .$$

($\psi_{W,M,N}\big(\sum w_i \otimes (m_j \otimes n_k)\big) = \sum (w_i \otimes m_j) \otimes n_k$.) Then

(a) *$\psi_{W,M,N}$ is natural in W, M and N.*

(b) *$\psi_{W,M,N}$ is an isomorphism if and only if $\omega_{M,N}$ is $W_{A'}$-pure. In particular this the case if $W_{A'}$ is flat.*

(2) We have analogous statements for $W \in {}_{A''}\mathcal{M}$.

Proof. (1) (a) For example we verify that $\psi_{W,M,N}$ is natural in N. For this, let $f : N \to N'$ be a morphism in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$, and consider the diagram

$$\begin{tikzcd}
 & W \otimes_{A'} (M\Box_{\mathfrak{C}}N) \arrow[rr, "W\otimes_{A'}(M\Box_{\mathfrak{C}} f)"] \arrow[dl, "\psi_{W,M,N}"'] \arrow[dd] & & W \otimes_{A'} (M\Box_{\mathfrak{C}}N') \arrow[dl, "\psi_{W,M,N'}"] \arrow[dd] \\
(W \otimes_{A'} M)\Box_{\mathfrak{C}}N \arrow[rr, "(W\otimes_{A'}M)\Box_{\mathfrak{C}} f"] \arrow[dd] & & (W \otimes_{A'} M)\Box_{\mathfrak{C}}N' \arrow[dd] & \\
 & W \otimes_{A'} (M \otimes_A N) \arrow[rr, "W\otimes_{A'}(M\otimes_A f)"] \arrow[dl, "\simeq"] & & W \otimes_{A'} (M \otimes_A N') \arrow[dl, "\simeq"] \\
(W \otimes_{A'} M) \otimes_A N \arrow[rr, "(W\otimes_{A'}M)\otimes_A f"] & & (W \otimes_{A'} M) \otimes_A N' &
\end{tikzcd}$$

Then the claimed statement follows from Lemma 1.1.4.

(b) Obvious.

(2) Analogous to (1). $\square$

Lemma 1.3.6. *Let A and T be k-algebras. Let $M \in \mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}_T$.*

(1) If the functor $M\Box_{\mathfrak{C}}-$ is right exact, then $\omega_{M,N}$ is a pure morphism in $\mathcal{M}_T$.

(2) If M is $(\mathfrak{C}, A)$-injective, then the sequence

$$0 \longrightarrow M\Box_{\mathfrak{C}}N \longrightarrow M \otimes_A N \xrightarrow{\omega_{M,N}} M \otimes_A \mathfrak{C} \otimes_A N$$

is split exact in $\mathcal{M}_T$. In particular it is pure in $\mathcal{M}_T$.

Analogous statements are true for $M \in {}_S\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}$, where S is a k-algebra.

Proof. (1) Let $X \in {}_T\mathcal{M}$. Then there is an exact sequence in ${}_T\mathcal{M}$, $F' \to F \to X \to 0$, where F and F' are free. We obtain the commutative diagram with exact rows

$$\begin{tikzcd}
(M\Box_{\mathfrak{C}}N) \otimes_T F' \arrow[r] \arrow[d, "\simeq"', "\psi'_{M,N,F'}"] & (M\Box_{\mathfrak{C}}N) \otimes_T F \arrow[r] \arrow[d, "\simeq"', "\psi'_{M,N,F}"] & (M\Box_{\mathfrak{C}}N) \otimes_T X \arrow[r] \arrow[d, "\psi'_{M,N,X}"] & 0 \\
M\Box_{\mathfrak{C}}(N \otimes_T F') \arrow[r] & M\Box_{\mathfrak{C}}(N \otimes_T F) \arrow[r] & M\Box_{\mathfrak{C}}(N \otimes_T X) \arrow[r] & 0.
\end{tikzcd}$$

$\psi'_{M,N,F}$ and $\psi'_{M,N,F'}$ are isomorphisms since F and F' are flat. Hence $\psi'_{M,N,X}$ is also an isomorphism. The statement follows then from Lemma 1.3.5.

(2) It enough to show that the canonical injections in $\mathcal{M}_T$, $M\Box_{\mathfrak{C}}N \to M\otimes_A N$ and $\mathrm{Im}(\omega_{M,N}) \to M\otimes_A \mathfrak{C}\otimes_A N$ are sections. Let $\lambda : M\otimes_A \mathfrak{C} \to M$ be a morphism in $\mathcal{M}^{\mathfrak{C}}$ such that $\lambda\circ\rho_M = 1_M$. Consider $\alpha = (\lambda\otimes_A N)\circ(M\otimes_A \lambda_N) : M\otimes_A N \to M\otimes_A N$ which is a morphism in $\mathcal{M}_T$.

First we have $\mathrm{Im}(\alpha) \subset M\Box_{\mathfrak{C}}N$ since

$$\begin{aligned}
(\rho_M\otimes_A N)\circ\alpha &= (\rho_M\circ\lambda\otimes_A N)\circ(M\otimes_A\lambda_N)\\
&= (\lambda\otimes_A\mathfrak{C}\otimes_A N)\circ(M\otimes_A\Delta_{\mathfrak{C}}\otimes_A N)\circ(M\otimes_A\lambda_N)\\
&\ (\text{Since }\lambda\text{ is a morphism in }\mathcal{M}^{\mathfrak{C}}\text{ i.e. }\rho_M\circ\lambda = (\lambda\otimes_A\mathfrak{C})\circ(M\otimes_A\Delta_{\mathfrak{C}}))\\
&= (\lambda\otimes_A\mathfrak{C}\otimes_A N)\circ\Big(M\otimes_A\big((\Delta_{\mathfrak{C}}\otimes_A N)\lambda_N\big)\Big),
\end{aligned}$$

and

$$\begin{aligned}
(M\otimes_A\lambda_N)\circ\alpha &= (\lambda\otimes_A\mathfrak{C}\otimes_A N)\circ(M\otimes_A\mathfrak{C}\otimes_A\lambda_N)\circ(M\otimes_A\lambda_N)\\
&= (\lambda\otimes_A\mathfrak{C}\otimes_A N)\circ\Big(M\otimes_A\big((\Delta_{\mathfrak{C}}\otimes_A N)\lambda_N\big)\Big)\\
&\ (\text{Since }\lambda_N\text{ is a morphism in }{}^{\mathfrak{C}}\mathcal{M}\text{ i.e. }(\mathfrak{C}\otimes_A\lambda_N)\circ\lambda_N = (\Delta_{\mathfrak{C}}\otimes_A N)\circ\lambda_N).
\end{aligned}$$

On the other hand, for every $x\in M\Box_{\mathfrak{C}}N$, $\alpha(x) = (\lambda\otimes_A N)\circ(\rho_M\otimes_A N)(x) = x$. Hence $M\Box_{\mathfrak{C}}N \to M\otimes_A N$ is a section.

Now consider the canonical diagram

$$\begin{tikzcd}
M\otimes_A N \arrow[r, "\omega_{M,N}"] \arrow[d, "\pi"] & M\otimes_A\mathfrak{C}\otimes_A N\ , \\
(M\otimes_A N)/(M\Box_{\mathfrak{C}}N) \arrow[r, "\bar{\omega}_{M,N}", "\simeq"'] & \mathrm{Im}(\omega_{M,N}) \arrow[u, hook, "\iota"']
\end{tikzcd}$$

and the map $\gamma = \bar{\omega}_{M,N}\circ\pi\circ(\lambda\otimes_A N) : M\otimes_A\mathfrak{C}\otimes_A N \to \mathrm{Im}(\omega_{M,N})$. We have $(\lambda\otimes_A N)\circ\omega_{M,N} = 1_{M\otimes_A N} - \alpha$. Then $\pi\circ(\lambda\otimes_A N)\circ\omega_{M,N} = \pi$. Hence $\gamma\circ\iota = 1_{\mathrm{Im}(\omega_{M,N})}$.

The last assertion is proved by a similar method. $\square$

Corollary 1.3.7. *Let $M\in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N\in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$. If $\mathfrak{C}$ is a coseparable A-coring, then $M\Box_{\mathfrak{C}}N$ is the kernel of $\omega_{M,N}$ in ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''}$.*

Proof. Follows immediately from Propositions 1.3.2, 1.2.19, and 1.3.6. $\square$

Corollary 1.3.8. *If $\mathfrak{C}$ is a coseparable A-coring, then the bifunctor* (1.19) *is well defined. In the special case when $\mathfrak{C}$ is the trivial A-coring, $-\Box_{\mathfrak{C}}- = -\otimes_A -$.*

Proof. Follows immediately from Propositions 1.3.4, 1.2.19, and 1.3.6. $\square$

Proposition 1.3.9. *For every $M\in\mathcal{M}^{\mathfrak{C}}$, the functor $M\Box_{\mathfrak{C}}-$ preserves direct limits.*

Proof. Let I be a direct set, $\mathbf{I}$ be the associated category, and let (N_i, u_{ij}) be a directed system in ${}^{\mathfrak{C}}\mathcal{M}$ indexed by I. Let (N, u_i) be a direct limit in ${}_A\mathcal{M}$ of this direct system. It is clear that N is a left $\mathfrak{C}$-comodule via the unique morphism λ_N making commutative the following diagram

$$\begin{tikzcd} N_i \arrow[r, "u_i"] \arrow[d, "\lambda_{N_i}"] & N \arrow[d, "\lambda_N"] \\ \mathfrak{C} \otimes_A N_i \arrow[r, "\mathfrak{C}\otimes_A u_i"] & \mathfrak{C} \otimes_A N. \end{tikzcd}$$

We have moreover u_i is a morphism in ${}^{\mathfrak{C}}\mathcal{M}$. Let $v_i : N_i \to L$, $i \in I$, be a compatible family with (N_i, u_{ij}) in ${}^{\mathfrak{C}}\mathcal{M}$. Let $v : N \to L$ be the unique morphism in ${}_A\mathcal{M}$ making commutative the diagram

$$\begin{tikzcd} N_i \arrow[r, "u_i"] \arrow[dr, "v_i"'] & N \arrow[d, "v"] \\ & L. \end{tikzcd}$$

Therefore, $(\mathfrak{C} \otimes_A v)\lambda_N u_i = (\mathfrak{C} \otimes_A vu_i)\lambda_{N_i} = (\mathfrak{C} \otimes_A v_i)\lambda_{N_i}$ and $\lambda_L v u_i = \lambda_L v_i = (\mathfrak{C} \otimes_A v_i)\lambda_{N_i}$. Hence $(\mathfrak{C} \otimes_A v)\lambda_N u_i = \lambda_L v u_i$ for every $i \in I$, and by the definition of direct limit, $(\mathfrak{C} \otimes_A v)\lambda_N = \lambda_L v$, i.e. v is a morphism in ${}^{\mathfrak{C}}\mathcal{M}$. It follows that (N, u_i) is the direct limit of (N, u_i) in ${}^{\mathfrak{C}}\mathcal{M}$.

We have moreover, for every $i \in I$, a commutative diagram in $\mathcal{M}_k$ with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M\Box_{\mathfrak{C}}N_i \arrow[r] \arrow[d, "M\Box_{\mathfrak{C}}u_i"] & M \otimes_A N_i \arrow[r, "\omega_{M,N_i}"] \arrow[d, "M\otimes_A u_i"] & M \otimes_A \mathfrak{C} \otimes_A N_i \arrow[d, "M\otimes_A\mathfrak{C}\otimes_A u_i"] \\ 0 \arrow[r] & M\Box_{\mathfrak{C}}N \arrow[r] & M \otimes_A N \arrow[r, "\omega_{M,N}"] & M \otimes_A \mathfrak{C} \otimes_A N. \end{tikzcd}$$

The first rows of the the above diagrams define an exact sequence in $\mathbf{Fun}(\mathbf{I}, \mathcal{M}_k)$. Since direct limits are exact in $\mathcal{M}_k$, $(M\Box_{\mathfrak{C}}N, M\Box_{\mathfrak{C}}u_i)$ is the direct limit of the direct system $(M\Box_{\mathfrak{C}}N_i, M\Box_{\mathfrak{C}}u_{ij})$ in $\mathcal{M}_k$. $\square$

It is well known that the cotensor functor, for coalgebras over fields, is left exact (see [90]). But for corings, the cotensor functor is not left exact nor right exact in general. For a counterexample, see [50, Counterexample 2.5, p. 338] which is an example of a $\mathbb{Z}$-coalgebra C such that the cotensor product $\Box_C$ is not associative. We will give in the next result sufficient conditions to have left exactness of the cotensor functor.

Proposition 1.3.10. *Let ${}_A\mathfrak{C}$ be flat and $M \in {}^{\mathfrak{C}}\mathcal{M}$. Let*

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$

be an exact sequence in $\mathcal{M}^{\mathfrak{C}}$. Then the resulting sequence

$$0 \longrightarrow N_1\Box_{\mathfrak{C}}M \xrightarrow{f\Box_{\mathfrak{C}}M} N_2\Box_{\mathfrak{C}}M \xrightarrow{g\Box_{\mathfrak{C}}M} N_3\Box_{\mathfrak{C}}M$$

is exact if $f \otimes_A M$ and $f \otimes_A \mathfrak{C} \otimes_A M$ are injective maps.

In particular, the functor $-\Box_{\mathfrak{C}} M : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_k$ is left $(\mathfrak{C}, A)$-exact, and it is left exact if ${}_A M$ is flat.

Proof. Consider the following commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \Box_{\mathfrak{C}} M \arrow[r, "f\Box_{\mathfrak{C}} M"] \arrow[d] & N_2 \Box_{\mathfrak{C}} M \arrow[r, "g\Box_{\mathfrak{C}} M"] \arrow[d] & N_3 \Box_{\mathfrak{C}} M \arrow[d] & \\
0 \arrow[r] & N_1 \otimes_A M \arrow[r, "f\otimes_A M"] \arrow[d, "\omega_{N_1,M}"] & N_2 \otimes_A M \arrow[r, "g\otimes_A M"] \arrow[d, "\omega_{N_2,M}"] & N_3 \otimes_A M \arrow[r] \arrow[d, "\omega_{N_3,M}"] & 0 \\
0 \arrow[r] & N_1 \otimes_A \mathfrak{C} \otimes_A M \arrow[r, "f\otimes_A \mathfrak{C} \otimes_A M"] & N_2 \otimes_A \mathfrak{C} \otimes_A M \arrow[r, "g\otimes_A \mathfrak{C} \otimes_A M"] & N_3 \otimes_A \mathfrak{C} \otimes_A M \arrow[r] & 0.
\end{tikzcd}$$

By assumptions, the second and the third rows are exact. From Snake Lemma ([25, Lemmas III.3.2, III.3.3]) the top row is also exact. $\Box$

Remark 1.3.11. As a corollary of the above result we state: Let ${}_A\mathfrak{C}$ is flat and $M \in {}^{\mathfrak{C}}\mathcal{M}$. Let

$$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$$

be an exact sequence in $\mathcal{M}^{\mathfrak{C}}$ which is ${}_A M$- and ${}_A(\mathfrak{C} \otimes_A M)$-pure. Then the resulting sequence

$$0 \longrightarrow N_1 \Box_{\mathfrak{C}} M \xrightarrow{f\Box_{\mathfrak{C}} M} N_2 \Box_{\mathfrak{C}} M \xrightarrow{g\Box_{\mathfrak{C}} M} N_3 \Box_{\mathfrak{C}} M$$

is exact.

Lemma 1.3.12. *Let $M \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ and $V \in {}_A\mathcal{M}$. Then there is a canonical isomorphism*

$$M \Box_{\mathfrak{C}} (\mathfrak{C} \otimes_A V) \simeq M \otimes_A V.$$

If moreover $\omega_{M, \mathfrak{C}\otimes_A V}$ is $(\mathfrak{D} \otimes_B \mathfrak{D})_B$-pure for every $V \in {}_A\mathcal{M}$, then

$$M \Box_{\mathfrak{C}} (\mathfrak{C} \otimes_A -) \simeq M \otimes_A - : {}_A\mathcal{M} \to {}^{\mathfrak{D}}\mathcal{M}.$$

Proof. First consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \Box_{\mathfrak{C}} (\mathfrak{C} \otimes_A V) \arrow[r, "\iota"] & M \otimes_A \mathfrak{C} \otimes_A V \arrow[r, "\omega_{M,\mathfrak{C}\otimes_A V}"] \arrow[dl, shift left, "M\otimes_A \epsilon_{\mathfrak{C}} \otimes_A V"] & M \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A V \, , \\
 & M \otimes_A V \arrow[u, dashed, "\psi_M"] \arrow[ur, shift left, "\rho_M \otimes_A V"] & &
\end{tikzcd}$$

By the universal property of kernel, there is a unique map ψ_M such that $\iota\psi_M = \rho_M \otimes_A V$. Moreover we have $(M \otimes_A \epsilon_{\mathfrak{C}} \otimes_A V)(\rho_M \otimes_A V) = M \otimes_A V$ and

$$\begin{aligned}
(\rho_M \otimes_A V)(M \otimes_A \epsilon_{\mathfrak{C}} \otimes_A V)\iota &= (M \otimes_A \mathfrak{C} \otimes_A \epsilon_{\mathfrak{C}} \otimes_A V)(\rho_M \otimes_A \mathfrak{C} \otimes_A V)\iota \\
&= (M \otimes_A \mathfrak{C} \otimes_A \epsilon_{\mathfrak{C}} \otimes_A N)(M \otimes_A \Delta_{\mathfrak{C}} \otimes_A V)\iota \\
&= (M \otimes_A \mathfrak{C} \otimes_A V)\iota \\
&= \iota.
\end{aligned}$$

Then ψ_M is bijective with inverse map $(M \otimes_A \epsilon_{\mathfrak{C}} \otimes_A V)\iota$.

Now let $f : V \to V'$ be a morphism in ${}_A\mathcal{M}$. Consider the diagram

$$\begin{tikzcd}
 & M \otimes_A V \arrow[rr, "\simeq"] \arrow[dl, "M\otimes_A f"'] \arrow[ddrr, "\rho_M\otimes_A V"] & & M\square_{\mathfrak{C}}(\mathfrak{C} \otimes_A V) . \arrow[dl, "M\square_{\mathfrak{C}}(\mathfrak{C}\otimes_A f)"'] \arrow[dd] \\
M \otimes_A V' \arrow[rr, "\simeq"] \arrow[ddrr, "\rho_M\otimes_A V'"] & & M\square_{\mathfrak{C}}(\mathfrak{C} \otimes_A V') \arrow[dd] & \\
 & & & M \otimes_A \mathfrak{C} \otimes_A V \arrow[dl, "M\otimes_A\mathfrak{C}\otimes_A f"] \\
 & & M \otimes_A \mathfrak{C} \otimes_A V' &
\end{tikzcd}$$

All sides, except perhaps the top one, are commutative, then the top side is also commutative. ◻

Definition 1.3.13. Let $\mathfrak{C}_A$ be flat. A comodule $M \in \mathcal{M}^{\mathfrak{C}}$ is called coflat (resp. faithfully coflat) if the functor $M\square_{\mathfrak{C}}- : {}^{\mathfrak{C}}\mathcal{M} \to \mathcal{M}_k$ is exact (resp. faithfully exact).

Corollary 1.3.14. *(1) Let $M \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. If $\mathrm{Im}(\omega_{M,\mathfrak{C}})$ is a ${}_A\mathfrak{C}$-pure submodule of $M \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}$, then*

$$M\square_{\mathfrak{C}}\mathfrak{C} \simeq M$$

as $(\mathfrak{D}, \mathfrak{C})$-bicomodules.

(2) Let $M \in \mathcal{M}^{\mathfrak{C}}$. If $\mathfrak{C}_A$ is flat and M is coflat in $\mathcal{M}^{\mathfrak{C}}$ then M_A is flat.

Proof. (1) Follows from Proposition 1.3.2, Lemmas 1.3.12, 1.3.6 ($\mathfrak{C}$ is $(\mathfrak{C}, A)$-injective), and Lemma 1.2.12. (2) Obvious from Lemma 1.3.12. ◻

Proposition 1.3.15. *Suppose that ${}_A\mathfrak{C}$ is flat and for every $X \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, $Y \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ and $Z \in {}^{\mathfrak{C}'''}\mathcal{M}^{\mathfrak{C}'}$, we have $\omega_{X,Y}$ is $\mathfrak{C}'_{A'}$- and ${}_{A''}\mathfrak{C}''$-pure and $\omega_{Z,X}$ is $\mathfrak{C}'''_{A'''}$-pure.*

Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ and $L \in {}^{\mathfrak{C}'''}\mathcal{M}^{\mathfrak{C}'}$. If the following hold

(a) $\omega_{M,N}$ is $L_{A'}$-, $(L \otimes_{A'} \mathfrak{C}')_{A'}$-pure,

(b) $\omega_{L,M}$ is ${}_AN$-, ${}_A(\mathfrak{C} \otimes_A N)$-pure,

(c) $\omega_{L\otimes_{A'}M,N}$ *is* ${}_{A''}\mathfrak{C}''$*-,* $\mathfrak{C}'''_{A'''}$*-pure,*

(d) $\omega_{L,M\square_{\mathfrak{C}}N}$ *is* ${}_{A''}\mathfrak{C}''$*-,* $\mathfrak{C}'''_{A'''}$*-pure,*

then there is a canonical isomorphism of $\mathfrak{C}''' - \mathfrak{C}'$*-bicomodules*

$$L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \simeq (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N.$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \arrow[r] \arrow[d, dashed, "\psi"] & L\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[r, "\omega_{L,M\square_{\mathfrak{C}}N}"] \arrow[d, "\psi_{L,M,N}"] & L\otimes_{A'}\mathfrak{C}'\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[d, "\psi_{L\otimes_{A'}\mathfrak{C}',M,N}"] \\
0 \arrow[r] & (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N \arrow[r] & (L\otimes_{A'}M)\square_{\mathfrak{C}}N \arrow[r, "\omega_{L,M}\square_{\mathfrak{C}}N"] & (L\otimes_{A'}\mathfrak{C}'\otimes_{A'}M)\square_{\mathfrak{C}}N.
\end{tikzcd}$$

The top row is exact from the definition of the cotensor product and the second one is exact from Remark 1.3.11 (see Condition (b)). Then there is a unique map $\psi : L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \to (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N$ making the above diagram commutative. From Lemma 1.3.5 (see Condition (a)), $\psi_{L,M,N}$ and $\psi_{L\otimes_{A'}\mathfrak{C}',M,N}$ are isomorphisms. Hence ψ is also an isomorphism. From Lemmas 1.2.12 and 1.3.5 (see Condition (c)), $\psi_{L,M,N}$ is an isomorphism of $\mathfrak{C}''' - \mathfrak{C}'$-bicomodules. Again from Lemma 1.2.12 (see Condition (d)), we obtain that ψ is an isomorphism of $\mathfrak{C}''' - \mathfrak{C}'$-bicomodules. $\square$

Proposition 1.3.16. *All the assumed conditions in Proposition 1.3.15 are fulfilled if at least one of the following hold*

(1) ${}_{A}\mathfrak{C}$*,* $\mathfrak{C}'_{A'}$*,* ${}_{A''}\mathfrak{C}''$*,* $\mathfrak{C}'''_{A'''}$*,* $L_{A'}$ *and* ${}_{A}N$ *are flat;*

(2) ${}_{A}\mathfrak{C}$*,* $\mathfrak{C}'_{A'}$*,* ${}_{A''}\mathfrak{C}''$*,* $\mathfrak{C}'''_{A'''}$ *are flat,* N *is coflat in* ${}^{\mathfrak{C}}\mathcal{M}$*;*

(3) ${}_{A}\mathfrak{C}$*,* $\mathfrak{C}'_{A'}$*,* ${}_{A''}\mathfrak{C}''$*,* $\mathfrak{C}'''_{A'''}$ *are flat,* L *is coflat in* $\mathcal{M}^{\mathfrak{C}'}$*;*

(4) ${}_{A}\mathfrak{C}$*,* $\mathfrak{C}'_{A'}$*,* ${}_{A''}\mathfrak{C}''$*,* $\mathfrak{C}'''_{A'''}$ *are flat,* N *is* $(\mathfrak{C}, A)$*-injective (e.g. if* $\mathfrak{C}$ *is coseparable), and* L *is* $(\mathfrak{C}', A')$*-injective (e.g. if* $\mathfrak{C}'$ *is coseparable);*

(5) If ${}_{A}\mathfrak{C}$ *is flat and* $\mathfrak{C}$ *and* $\mathfrak{C}'$ *are coseparable.*

Proof. Immediate from the aforementioned results. $\square$

Proposition 1.3.17. *Let* $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$*,* $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ *and* $L \in {}^{\mathfrak{C}'''}\mathcal{M}^{\mathfrak{C}'}$*. Then the canonical isomorphism of* $\mathfrak{C}''' - \mathfrak{C}'$*-bicomodules*

$$L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \simeq (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N$$

is natural in L*,* M *and* N *provided that it is well defined.*

Proof. For example let $f : L \to L'$ be a morphism in ${}^{\mathfrak{C}''}\mathcal{M}^{\mathfrak{C}'}$. Consider the diagram

$$\begin{tikzcd}
 & L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \arrow[rr, "\simeq"] \arrow[dl, "f\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N)"'] \arrow[dd] & & (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N \arrow[dl, "(f\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N"] \arrow[dd] \\
L'\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \arrow[rr, "\simeq"] \arrow[dd] & & (L'\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N \arrow[dd] & \\
 & L\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[rr, "\simeq"] \arrow[dl, "f\otimes_{A'}(M\square_{\mathfrak{C}}N)"'] & & (L\otimes_{A'}M)\square_{\mathfrak{C}}N \arrow[dl, "(f\otimes_{A'}M)\square_{\mathfrak{C}}N"] \\
L'\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[rr, "\simeq"] & & (L'\otimes_{A'}M)\square_{\mathfrak{C}}N &
\end{tikzcd}$$

By Lemmas 1.1.4 and 1.3.5, the top square is commutative. $\square$

1.4 Cohom functors and coendomorphism corings

Definition 1.4.1. A bicomodule $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is said to be *quasi-finite* as a right $\mathfrak{D}$-comodule or $(A, \mathfrak{D})$-*quasi-finite*, if the functor $-\otimes_A N : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{D}}$ has a left adjoint $\mathrm{h}_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}_A$, and we call it the *cohom functor* associated to N.

Example 1.4.2. If $\mathfrak{D}$ is the trivial B-coring, then a $(\mathfrak{C}, B)$-bicomodule N is (A, B)-quasi-finite if and only if ${}_AN$ is finitely generated and projective. The cohom functor is $-\otimes_B {}^*N : \mathcal{M}_B \to \mathcal{M}_A$, where ${}^*N = \mathrm{Hom}_A(N, A)$. In Chapter 2 we will give a generalization of this fact for bigraded modules.

Remark 1.4.3. Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. It is clear that, if ${}_B\mathfrak{D}$ is flat and N is $(A, \mathfrak{D})$-quasi-finite, then ${}_AN$ is flat.

Now, let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a $(A, \mathfrak{D})$-quasi-finite comodule, and let

$$\Phi_{X,Y} : \mathrm{Hom}_A(\mathrm{h}_{\mathfrak{D}}(N, X), Y) \to \mathrm{Hom}_{\mathfrak{D}}(X, Y \otimes_A N), \tag{1.20}$$

where $Y \in \mathcal{M}_A$ and $X \in \mathcal{M}^{\mathfrak{D}}$, be the natural isomorphism giving the adjunction $(-\otimes_A N, \mathrm{h}_{\mathfrak{D}}(N, -))$. Let $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to \mathrm{h}_{\mathfrak{D}}(N, -) \otimes_A N$ be the unit of this adjunction. Then $\Phi_{X,Y}(f) = (f \otimes_A N)\theta_X$ for every $f \in \mathrm{Hom}_A(\mathrm{h}_{\mathfrak{D}}(N, X), Y)$.

Let $X \in \mathcal{M}^{\mathfrak{D}}$. Then there is a unique A-linear map

$$\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} : \mathrm{h}_{\mathfrak{D}}(N, X) \to \mathrm{h}_{\mathfrak{D}}(N, X) \otimes_A \mathfrak{C}$$

such that

$$(\mathrm{h}_{\mathfrak{D}}(N, X) \otimes_A \lambda_N)\theta_X = (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X.$$

Proposition 1.4.4. *We have that* $(\mathrm{h}_{\mathfrak{D}}(N,X), \rho_{\mathrm{h}_{\mathfrak{D}}(N,X)})$ *is a right* $\mathfrak{C}$*-comodule, and we obtain a functor*

$$\mathrm{h}_{\mathfrak{D}}(N,-) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}.$$

Proof. We have

$$\begin{aligned}(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \mathfrak{C} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X &= (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \mathfrak{C} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \lambda_N)\theta_X \\ &= (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \lambda_N)\theta_X,\end{aligned}$$

and

$$\begin{aligned}(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \Delta_{\mathfrak{C}} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X &= (\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \Delta_{\mathfrak{C}} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \lambda_N)\theta_X \\ &= \big(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A (\mathfrak{C} \otimes_A \lambda_N)\lambda_N\big)\theta_X \\ &= (\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \mathfrak{C} \otimes_A \lambda_N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X \\ &= (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \lambda_N)\theta_X.\end{aligned}$$

Hence $(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \mathfrak{C})\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} = (\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \Delta_{\mathfrak{C}})\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)}$.

On the other hand,

$$\begin{aligned}(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \epsilon_{\mathfrak{C}} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X &= (\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \epsilon_{\mathfrak{C}} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \lambda_N)\theta_X \\ &= \theta_X.\end{aligned}$$

Hence $(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \epsilon_{\mathfrak{C}})\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} = \mathrm{h}_{\mathfrak{D}}(N,X)$.

Now let $f : X \to X'$ be a morphism in $\mathcal{M}^{\mathfrak{D}}$. Now, we will verify that $\mathrm{h}_{\mathfrak{D}}(N,f)$ is $\mathfrak{C}$-colinear. First, since θ is a natural transformation, $\theta_{X'}f = (\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A N)\theta_X$. Then

$$\begin{aligned}(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X')} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A N)\theta_X &= (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X')} \otimes_A N)\theta_{X'}f \\ &= (\mathrm{h}_{\mathfrak{D}}(N,X') \otimes_A \lambda_N)\theta_{X'}f \\ &= (\mathrm{h}_{\mathfrak{D}}(N,X') \otimes_A \lambda_N)(\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A N)\theta_X \\ &= (\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \lambda_N)\theta_X.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \mathfrak{C} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X &= (\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \mathfrak{C} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \lambda_N)\theta_X \\ &= (\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \lambda_N)\theta_X.\end{aligned}$$

Hence $\rho_{\mathrm{h}_{\mathfrak{D}}(N,X')}\mathrm{h}_{\mathfrak{D}}(N,f) = (\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \mathfrak{C})\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)}$. □

Lemma 1.4.5. *Let* $X \in \mathcal{M}^{\mathfrak{D}}$*,* $Y \in \mathcal{M}^{\mathfrak{C}}$*, and* $f \in \mathrm{Hom}_A(\mathrm{h}_{\mathfrak{D}}(N,X), Y)$*. Then* f *is* $\mathfrak{C}$*-colinear if and only if* $\mathrm{Im}((f \otimes_A N)\theta_X) \subset Y\square_{\mathfrak{C}}N$.

Proof. We have,

$$\begin{aligned}(f \otimes_A \mathfrak{C} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X &= (f \otimes_A \mathfrak{C} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N,f) \otimes_A \lambda_N)\theta_X \\ &= (Y \otimes_A \lambda_N)(f \otimes_A N)\theta_X.\end{aligned}$$

Hence,

$$\begin{aligned} f \text{ is } \mathfrak{C}\text{-colinear} &\iff (\rho_Y \otimes_A N)(f \otimes_A N)\theta_X = (f \otimes_A \mathfrak{C} \otimes_A N)(\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A N)\theta_X \\ &\iff (\rho_Y \otimes_A N)(f \otimes_A N)\theta_X = (Y \otimes_A \lambda_N)(f \otimes_A N)\theta_X \\ &\iff \mathrm{Im}((f \otimes_A N)\theta_X) \subset Y \square_{\mathfrak{C}} N. \end{aligned}$$

□

Proposition 1.4.6. *Assume that* $\omega_{Y,N} := \rho_Y \otimes_A N - Y \otimes_A \lambda_N$ *is* ${}_B(\mathfrak{D} \otimes_B \mathfrak{D})$*-pure for every right* $\mathfrak{C}$*-comodule* Y *(e.g.,* ${}_B\mathfrak{D}$ *is flat or* $\mathfrak{C}$ *is coseparable).*

(1) If N *is* $(A, \mathfrak{D})$*-quasi-finite then* $-\square_{\mathfrak{C}} N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *has a left adjoint, which is* $\mathrm{h}_{\mathfrak{D}}(N, -)$*.*

(2) If $-\square_{\mathfrak{C}} N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *has a left adjoint, then* N *is* $(A, \mathfrak{D})$*-quasi-finite.*

Proof. (1) From Lemmas 1.4.5 and 1.2.12, the isomorphism (1.20) induces an isomorphism $\Phi_{X,Y} : \mathrm{Hom}_{\mathfrak{C}}(\mathrm{h}_{\mathfrak{D}}(N, X), Y) \to \mathrm{Hom}_{\mathfrak{D}}(X, Y \square_{\mathfrak{C}} N)$. Hence (1) follows.

(2) From Lemma 1.3.12, $(- \otimes_A \mathfrak{C})\square_{\mathfrak{C}} N \simeq - \otimes_A N : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{D}}$. We know that $- \otimes_A \mathfrak{C}$ is a right adjoint to the forgetful functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$. Hence $- \otimes_A N$ has a left adjoint (by Proposition 1.1.32). □

Definition 1.4.7. A bicomodule $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is said to be *injector* as a right $\mathfrak{D}$-comodule or $(A, \mathfrak{D})$-*injector* if the functor $- \otimes_A N : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{D}}$ preserves injective objects.

By Proposition 1.1.34, if ${}_A\mathfrak{C}$ is flat, then $\mathfrak{C}$ is $(A, \mathfrak{C})$-injector.

Proposition 1.4.8. *Let* $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *be a bicomodule with* ${}_A\mathfrak{C}$ *and* ${}_B\mathfrak{D}$ *are flat. Suppose that* N *is* $(A, \mathfrak{D})$*-quasi-finite. Then the following are equivalent*

(1) N *is* $(A, \mathfrak{D})$*-injector;*

(2) the cohom functor $h_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}_A$ *is exact;*

(3) the cohom functor $h_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$ *is exact;*

(4) the functor $-\square_{\mathfrak{C}} N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *preserves injective objects.*

Proof. Follows directly from Propositions 1.1.34, 1.4.6. □

Now, let $N \in {}_A\mathcal{M}^{\mathfrak{D}}$ be $(A, \mathfrak{D})$-quasi-finite. Set $\mathrm{e}_{\mathfrak{D}}(N) = \mathrm{h}_{\mathfrak{D}}(N, N)$. From Subsection 2.1, $\mathrm{e}_{\mathfrak{D}}(N)$ is an A-bimodule. From Lemma 2.1.8, θ_N is a morphism in ${}_A\mathcal{M}^{\mathfrak{D}}$. Consider the map in $\mathcal{M}^{\mathfrak{D}}$

$$N \xrightarrow{\theta_N} \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N \xrightarrow{\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N} \mathrm{e}_{\mathfrak{D}}(N) \otimes_A \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N.$$

Then there is a unique right A-linear map

$$\Delta_e : \mathrm{e}_{\mathfrak{D}}(N) \to \mathrm{e}_{\mathfrak{D}}(N) \otimes_A \mathrm{e}_{\mathfrak{D}}(N)$$

such that

$$\Phi_{N,\mathrm{e}_{\mathfrak{D}}(N)\otimes_A \mathrm{e}_{\mathfrak{D}}(N)}(\Delta_e) = (\Delta_e \otimes_A N)\theta_N = (\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N)\theta_N.$$

Let $\rho^N : A \to \mathrm{End}_{\mathfrak{D}}(N)$ and $\rho : A \to \mathrm{End}_{\mathfrak{C}}(\mathrm{e}_{\mathfrak{D}}(N))$ be the maps defining the left A-action on N and $\mathrm{e}_{\mathfrak{D}}(N)$, respectively.

We want to verify that Δ_e is left A-linear, that is, for every $a \in A$, the diagram in $\mathcal{M}_A$

$$\begin{tikzcd}
\mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "\Delta_e"] \arrow[d, "\rho(a)"'] & \mathrm{e}_{\mathfrak{D}}(N) \otimes_A \mathrm{e}_{\mathfrak{D}}(N) \arrow[d, "\rho(a)\otimes_A \mathrm{e}_{\mathfrak{D}}(N)"] \\
\mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "\Delta_e"] & \mathrm{e}_{\mathfrak{D}}(N) \otimes_A \mathrm{e}_{\mathfrak{D}}(N)
\end{tikzcd}$$

commutes.

Let $a \in A$. We have the commutativity of the following diagram

$$\begin{tikzcd}
N \arrow[r, "\theta_N"] \arrow[d, "\rho^N(a)"'] & \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N \arrow[d, "\rho(a)\otimes_A N"] \\
N \arrow[r, "\theta_N"] & \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N.
\end{tikzcd} \tag{1.21}$$

Then

$$\begin{aligned}
(\rho(a) \otimes_A \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N)(\Delta_e \otimes_A N)\theta_N &= (\rho(a) \otimes_A \theta_N)\theta_N \\
&= (\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N)(\rho(a) \otimes_A N)\theta_N \\
&= (\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N)\theta_N\rho^N(a),
\end{aligned}$$

and

$$\begin{aligned}
(\Delta_e \otimes_A N)(\rho(a) \otimes_A N)\theta_N &= (\Delta_e \otimes_A N)\theta_N\rho^N(a) \\
&= (\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N)\theta_N\rho^N(a).
\end{aligned}$$

Hence Δ_e is left A-linear.

There is also a unique right A-linear map

$$\epsilon_e : \mathrm{e}_{\mathfrak{D}}(N) \to A$$

such that

$$\Phi_{N,A}(\epsilon_e) = (\epsilon_e \otimes_A N)\theta_N = N.$$

Now we want to verify that ϵ_e is left A-linear, that is, for every $a \in A$, the diagram in $\mathcal{M}_A$

$$\begin{tikzcd}
\mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "\epsilon_e"] \arrow[d, "\rho(a)"'] & A \arrow[d, "a-"] \\
\mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "\epsilon_e"] & A
\end{tikzcd}$$

commutes.

Indeed, for every $a \in A$, we have

$$\begin{aligned}(\epsilon_e \otimes_A N)(\rho(a) \otimes_A N)\theta_N &= (\epsilon_e \otimes_A N)\theta_N \rho^N(a)\\ &= \rho^N(a),\end{aligned}$$

and $(a- \otimes_A N)(\epsilon_e \otimes_A N)\theta_N = \rho^N(a)$.

Proposition 1.4.9. *Let $N \in {}_A\mathcal{M}^{\mathfrak{D}}$ be $(A, \mathfrak{D})$-quasi-finite. Set $\mathrm{e}_{\mathfrak{D}}(N) = \mathrm{h}_{\mathfrak{D}}(N, N)$. Then $(\mathrm{e}_{\mathfrak{D}}(N), \Delta_e, \epsilon_e)$ is an A-coring, where Δ_e and ϵ_e defined above. Moreover, N is a $(\mathrm{e}_{\mathfrak{D}}(N), \mathfrak{D})$-bicomodule by $\theta_N : N \to \mathrm{e}_{\mathfrak{D}}(N) \otimes_A N$, and there is an algebra anti-isomorphism*

$$\mathrm{e}_{\mathfrak{D}}(N)^* = \mathrm{Hom}_A(\mathrm{e}_{\mathfrak{D}}(N), A) \xrightarrow{\Phi_{N,A}} \mathrm{End}_{\mathfrak{D}}(N) .$$

Proof. First set $\mathfrak{C} = \mathrm{e}_{\mathfrak{D}}(N)$. The coassociativity of Δ_e follows from

$$\begin{aligned}(\mathfrak{C} \otimes_A \Delta_e \otimes_A N)(\Delta_e \otimes_A N)\theta_N &= (\mathfrak{C} \otimes_A \Delta_e \otimes_A N)(\mathfrak{C} \otimes_A \theta_N)\theta_N\\ &= (\mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \theta_N)(\mathfrak{C} \otimes_A \theta_N)\theta_N,\end{aligned}$$

and

$$\begin{aligned}(\Delta_e \otimes_A \mathfrak{C} \otimes_A N)(\Delta_e \otimes_A N)\theta_N &= (\Delta_e \otimes_A \mathfrak{C} \otimes_A N)(\mathfrak{C} \otimes_A \theta_N)\theta_N\\ &= (\Delta_e \otimes_A \theta_N)\theta_N\\ &= (\mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \theta_N)(\mathfrak{C} \otimes_A \theta_N)\theta_N.\end{aligned}$$

The counit property follows from

$$\begin{aligned}(\epsilon_e \otimes_A \mathfrak{C} \otimes_A N)(\Delta_e \otimes_A N)\theta_N &= (\epsilon_e \otimes_A \mathfrak{C} \otimes_A N)(\mathfrak{C} \otimes_A \theta_N)\theta_N\\ &= (A \otimes_A \theta_N)(\epsilon_e \otimes_A N)\theta_N\\ &= (A \otimes_A \theta_N)N\\ &= \theta_N,\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{C} \otimes_A \epsilon_e \otimes_A N)(\Delta_e \otimes_A N)\theta_N &= (\mathfrak{C} \otimes_A \epsilon_e \otimes_A N)(\mathfrak{C} \otimes_A \theta_N)\theta_N\\ &= \big(\mathfrak{C} \otimes_A (\epsilon_e \otimes_A N)\theta_N\big)\theta_N\\ &= \theta_N.\end{aligned}$$

Finally, we know that $\theta_N : N \to \mathfrak{C} \otimes_A N$ is a morphism in ${}_A\mathcal{M}^{\mathfrak{D}}$. Then, by the definitions of Δ_e and ϵ_e, N is a $(\mathfrak{C}, \mathfrak{D})$-bicomodule.

Let $f, g \in \mathfrak{C}^*$. Since θ_N is left A-linear, $(f \otimes_A \mathfrak{C} \otimes_A N)(\mathfrak{C} \otimes_A \theta_N) = \theta_N(f \otimes_A N)$. Hence

$$\begin{aligned}\Phi_{N,A}(f *^r g) &= (g \otimes_A N)(f \otimes_A \mathfrak{C} \otimes_A N)(\Delta_e \otimes_A N)\theta_N\\ &= (g \otimes_A N)(f \otimes_A \mathfrak{C} \otimes_A N)(\mathfrak{C} \otimes_A \theta_N)\theta_N\\ &= (g \otimes_A N)\theta_N(f \otimes_A N)\theta_N\\ &= \Phi_{N,A}(g)\Phi_{N,A}(f).\end{aligned}$$

□

Now, we recall from [88] the definition of a morphism of corings.

Definition 1.4.10. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be two A-corings. An A-bilinear map $f : \mathfrak{C} \to \mathfrak{C}'$ is called a *morphism of corings* if the diagrams

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "f"] \arrow[d, "\Delta_{\mathfrak{C}}"'] & \mathfrak{C}' \arrow[d, "\Delta_{\mathfrak{C}'}"] \\ \mathfrak{C} \otimes_A \mathfrak{C} \arrow[r, "f \otimes_A f"] & \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C} \arrow[r, "f"] \arrow[dr, "\epsilon_{\mathfrak{C}}"'] & \mathfrak{C}' \arrow[d, "\epsilon_{\mathfrak{C}'}"] \\ & A \end{tikzcd}$$

are commutative.

Proposition 1.4.11. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be $(A, \mathfrak{D})$-quasi-finite. Let $f : \mathrm{e}_{\mathfrak{D}}(N) \to \mathfrak{C}$ be the unique morphism in $\mathcal{M}_A$ such that*

$$(f \otimes_A N)\theta_N = \lambda_N.$$

Then f is a morphism of A-corings.

Proof. First we must verify that f is left A-linear, that is, for every $a \in A$, the diagram in $\mathcal{M}_A$

$$\begin{tikzcd} \mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "f"] \arrow[d, "\rho(a)"'] & \mathfrak{C} \arrow[d, "a-"] \\ \mathrm{e}_{\mathfrak{D}}(N) \arrow[r, "f"] & \mathfrak{C} \end{tikzcd}$$

commutes, where $\rho : A \to \mathrm{End}_{\mathfrak{C}}(\mathrm{e}_{\mathfrak{D}}(N))$ is the map defining the left A-action on $\mathrm{e}_{\mathfrak{D}}(N)$. We have for every $a \in A$,

$$\begin{array}{rcl} (f \otimes_A N)(\rho(a) \otimes_A N)\theta_N & = & (f \otimes_A N)\theta_N \rho^N(a) \text{ (see Diagram 1.21)} \\ & = & \lambda_N \rho^N(a), \end{array}$$

and $(a- \otimes_A N)(f \otimes_A N)\theta_N = (a- \otimes_A N)\lambda_N$. Hence, since λ_N is left A-linear, f is also left A-linear.

Moreover,

$$\begin{array}{rcl} (f \otimes_A f \otimes_A N)(\Delta_e \otimes_A N)\theta_N & = & (f \otimes_A f \otimes_A N)(\mathrm{e}_{\mathfrak{D}}(N) \otimes_A \theta_N)\theta_N \\ & = & (f \otimes_A \lambda_N)\theta_N \\ & = & (\mathfrak{C} \otimes_A \lambda_N)\lambda_N, \end{array}$$

and

$$\begin{array}{rcl} (\Delta_{\mathfrak{C}} \otimes_A N)(f \otimes_A N)\theta_N & = & (\Delta_{\mathfrak{C}} \otimes_A N)\lambda_N \\ & = & (\mathfrak{C} \otimes_A \lambda_N)\lambda_N. \end{array}$$

Hence, $(f \otimes_A f)\Delta_e = \Delta_{\mathfrak{C}} f$.

Finally, we have

$$\begin{aligned}(\epsilon_{\mathfrak{C}} \otimes_A N)(f \otimes_A N)\theta_N &= (\epsilon_{\mathfrak{C}} \otimes_A N)\lambda_N \\ &= N,\end{aligned}$$

and $(\epsilon_e \otimes_A N)\theta_N = N$. Hence, $\epsilon_{\mathfrak{C}} f = \epsilon_e$. □

Remark 1.4.12. By Lemma 1.4.5, the map f defined in Proposition 1.4.11 is right $\mathfrak{C}$-colinear. Hence, if $\omega_{Y,N} := \rho_Y \otimes_A N - Y \otimes_A \lambda_N$ is ${}_B(\mathfrak{D} \otimes_B \mathfrak{D})$-pure for every right $\mathfrak{C}$-comodule Y (e.g., ${}_B\mathfrak{D}$ is flat or $\mathfrak{C}$ is coseparable), then $f = \chi_{\mathfrak{C}} \circ \mathrm{h}_{\mathfrak{D}}(N, \lambda_N)$, where $\chi : \mathrm{h}_{\mathfrak{D}}(N, -\Box_{\mathfrak{C}} N) \to 1_{\mathcal{M}^{\mathfrak{C}}}$ is the counit of the adjunction $(\mathrm{h}_{\mathfrak{D}}(N, -), -\Box_{\mathfrak{C}} N)$. Hence, Proposition 1.4.11 and this remark recover [35, Proposition 5.2].

1.5 Induction functors, the category of entwining modules, the category of graded modules

1.5.1 Induction functors

We start this subsection by recalling from [43] and [19] the following

Definition 1.5.1. A coring homomorphism from the coring $(\mathfrak{C} : A)$ to the coring $(\mathfrak{D} : B)$ is a pair (φ, ρ), where $\rho : A \to B$ is a homomorphism of k-algebras and $\varphi : \mathfrak{C} \to \mathfrak{D}$ is a homomorphism of A-bimodules such that the diagrams

$$\begin{tikzcd} \mathfrak{C} \arrow[rr, "\Delta_{\mathfrak{C}}"] \arrow[dd, "\varphi"'] & & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[dr, "\varphi \otimes_A \varphi"] & \\ & & & \mathfrak{D} \otimes_A \mathfrak{D} \arrow[dl, "\omega_{\mathfrak{D},\mathfrak{D}}"] \\ \mathfrak{D} \arrow[rr, "\Delta_{\mathfrak{D}}"] & & \mathfrak{D} \otimes_B \mathfrak{D} & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C} \arrow[r, "\epsilon_{\mathfrak{C}}"] \arrow[d, "\varphi"'] & A \arrow[d, "\rho"] \\ \mathfrak{D} \arrow[r, "\epsilon_{\mathfrak{D}}"] & B \end{tikzcd}$$

are commutative, where $\omega_{\mathfrak{D},\mathfrak{D}} : \mathfrak{D} \otimes_A \mathfrak{D} \to \mathfrak{D} \otimes_B \mathfrak{D}$ is the canonical map induced by $\rho : A \to B$. Equivalently, the map

$$\varphi' : B \otimes_A \mathfrak{C} \otimes_A B \xrightarrow{B \otimes_A \varphi \otimes_A B} B \otimes_A \mathfrak{D} \otimes_A B \xrightarrow{\omega} B \otimes_B \mathfrak{D} \otimes_B B \xrightarrow{\simeq} \mathfrak{D}$$

$(\varphi'(b \otimes_A c \otimes_A b') = b\varphi(c)b')$ is a morphism of B-corings, where ω is the canonical map and $B \otimes_A \mathfrak{C} \otimes_A B$ is the base ring extension of $\mathfrak{C}$ (see Example 1.2.2 (3)).

Example 1.5.2. Let $\rho : A \to B$ be a morphism of k-algebras and $\mathfrak{C}$ an A-coring. Consider the map $\varphi : \mathfrak{C} \to B \otimes_A \mathfrak{C} \otimes_A B,\ c \mapsto 1_B \otimes c \otimes 1_B$. Then $(\varphi, \rho) : (\mathfrak{C} : A) \to (B \otimes_A \mathfrak{C} \otimes_A B : B)$ is a morphism of corings.

Let $(\varphi, \rho) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ be a coring morphism. Now we introduce the *induction functor*

$$- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}.$$

It assigns to a comodule $M \in \mathcal{M}^{\mathfrak{C}}$, $M \otimes_A B$ which is a right $\mathfrak{D}$-comodule by the coaction

$$\rho_{M \otimes_A B}(m \otimes_A b) = \sum m_{(0)} \otimes_A 1_B \otimes_B \varphi(m_{(1)})b,$$

where $\rho_M(m) = \sum m_{(0)} \otimes_A m_{(1)}$, and assigns to a morphism f in $\mathcal{M}^{\mathfrak{C}}$, the map $f \otimes_A B$.

Definition 1.5.3. We say that (φ, ρ) is a *pure morphism of corings* [19, 24.8], if $\omega_{Y, B \otimes_A \mathfrak{C}}$ is ${}_A(\mathfrak{C} \otimes_A \mathfrak{C})$-pure for every right $\mathfrak{D}$-comodule Y.

Let $(\varphi, \rho) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ be a pure morphism of corings. We also define the *coinduction functor*

$$-\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}},$$

where the left coaction on the left B-module $B \otimes_A \mathfrak{C}$ is given by:

$$\lambda_{B \otimes_A \mathfrak{C}} : B \otimes_A \mathfrak{C} \to \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \simeq \mathfrak{D} \otimes_A \mathfrak{C}, \quad b \otimes_A c \mapsto \sum b\varphi(c_{(1)}) \otimes_A c_{(2)},$$

where $\Delta_{\mathfrak{C}}(c) = \sum c_{(1)} \otimes_A c_{(2)}$.

The following result is given in [43, Proposition 5.4] or [19, 24.11], but there is a missing condition, in both results, that guaranties that the coinduction functor is well defined.

Proposition 1.5.4. *If (φ, ρ) is a pure morphism of corings, then we have an adjoint pair of functors*

$$(- \otimes_A B, -\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})).$$

From the proof of [19, 24.11], the counit of the adjunction $(- \otimes_A B, -\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}))$ is given by

$$\psi_N : (N\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})) \otimes_A B \to N, \ \sum_i n_i \otimes_A c_i \otimes_A b \mapsto \sum_i n_i \rho(\epsilon_{\mathfrak{C}}(c_i))b$$

$(N \in \mathcal{M}^{\mathfrak{D}})$. We obtain a commutative diagram

$$\begin{tikzcd}
(\mathfrak{D}\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})) \otimes_A B \arrow[rr, "\psi_{\mathfrak{D}}"] \arrow[dr, "\simeq"'] & & \mathfrak{D} \\
& B \otimes_A \mathfrak{C} \otimes_A B \arrow[ur, "\varphi'"'] &
\end{tikzcd} \tag{1.22}$$

,

where φ' is the map defined in Definition 1.5.1.

Proposition 1.5.5. [19, 23.9]
Let $\rho : A \to B$ be a morphism of k-algebras and $\mathfrak{C}$ an A-coring such that ${}_A\mathfrak{C}$ is flat. The coendomorphism coring of the $(B, \mathfrak{C})$-quasi-finite comodule $B \otimes_A \mathfrak{C}$ is isomorphic as coring to the B-coring $B \otimes_A \mathfrak{C} \otimes_A B$.

Proof. Let $\varphi : \mathfrak{C} \to B \otimes_A \mathfrak{C} \otimes_A B$ be the map defined by $\varphi(c) = 1_B \otimes c \otimes 1_B$. By Example 1.5.2, $(\varphi, \rho) : (\mathfrak{C} : A) \to (B \otimes_A \mathfrak{C} \otimes_A B : B)$ is a morphism of coring. In this case φ' is the identity map. From the commutativity of the diagram (1.22), $\psi_{B\otimes_A \mathfrak{C} \otimes_A B}$ is an isomorphism. Hence, by Remark 1.4.12, there is an isomorphism of corings $f : \mathrm{e}_{\mathfrak{C}}(B \otimes_A \mathfrak{C}) \to B \otimes_A \mathfrak{C} \otimes_A B$. $\square$

1.5.2 The category of entwined modules

Entwining structures were introduced in [18] with the aim of preserving in noncommutative geometry symmetry properties of principal bundles. Entwined modules over entwining structures were introduced in [13] as a generalization of Doi-Koppinen Hopf modules and, in particular, graded modules by G-sets, Hopf modules and Yetter-Drinfeld modules.

Concerning the subject of this subsection and for the rest of this work, we adopt the notations of [24].

We recall from [19] that a *right-right entwining structure* over k is a triple (A, C, ψ), where A is a k-algebra, C is a k-coalgebra, and $\psi : C \otimes A \to A \otimes C$ is a k-linear map, such that

(ES1) $\psi \circ (1_C \otimes m) = (m \otimes 1_C) \circ (1_A \otimes \psi) \circ (\psi \otimes 1_A)$, or equivalently, for all $a, b \in A$, $c \in C$, $\sum (ab)_\psi \otimes c^\psi = \sum a_\psi b_\Psi \otimes c^{\psi\Psi}$,

(ES2) $(1_A \otimes \Delta) \circ \psi = (\psi \otimes 1_C) \circ (1_C \otimes \psi) \circ (\Delta \otimes 1_A)$, or equivalently, for all $a \in A$, $c \in C$, $\sum a_\psi \otimes \Delta(c^\psi) = \sum a_{\psi\Psi} \otimes c_{(1)}^{\Psi} \otimes c_{(2)}^{\psi}$,

(ES3) $\psi \circ (1_C \otimes \eta) = \eta \otimes 1_C$, or equivalently, for all $c \in C$, $\sum (1_A)_\psi \otimes c^\psi = 1_A \otimes c$,

(ES4) $(1_A \otimes \epsilon) \circ \psi = \epsilon \otimes 1_A$, or equivalently, for all $a \in A$, $c \in C$, $\sum a_\psi \epsilon(c^\psi) = a\epsilon(c)$.

where m and η are respectively the multiplication and the unit maps of A, and $\psi(c \otimes a) = a_\psi \otimes c^\psi = a_\Psi \otimes c^\Psi$.

We say that A and C are *entwined* by ψ while ψ is called the *entwining map*.

The conditions (ES1) to (ES4) mean that the following diagram is commutative

$$\begin{tikzcd}
& C\otimes A\otimes A \arrow[r, "C\otimes m"] \arrow[ddl, "\psi\otimes A"'] & C\otimes A \arrow[r, "\Delta\otimes A"] \arrow[dd, "\psi"] \arrow[dr, "\epsilon\otimes A"] & C\otimes C\otimes A \arrow[ddr, "C\otimes\psi"] & \\
& & & & \\
A\otimes C\otimes C \ \textbf{(ES1)} \arrow[ddr, "A\otimes\psi"'] & C \arrow[uur, "C\otimes\eta"] \arrow[r, phantom, "\textbf{(ES3)}"] \arrow[ddr, "\eta\otimes C"'] & {} & A \arrow[l, phantom, "\textbf{(ES4)}"'] & \textbf{(ES2)}\ C\otimes A\otimes C \arrow[ddl, "\psi\otimes C"] \\
& & & & \\
& A\otimes A\otimes C \arrow[r, "m\otimes C"'] & A\otimes C \arrow[r, "A\otimes\Delta"'] \arrow[uur, "A\otimes\epsilon"'] & A\otimes C\otimes C. &
\end{tikzcd}$$

A *morphism* $(A, C, \psi) \to (A', C', \psi')$ is a pair (α, γ) with $\alpha : A \to A'$ is a morphism of algebras, and $\gamma : C \to C'$ is a morphism of coalgebras such that the following diagram is commutative

$$\begin{tikzcd}
C \otimes A \arrow[r, "\psi"] \arrow[d, "\gamma\otimes\alpha"'] & A \otimes C \arrow[d, "\alpha\otimes\gamma"] \\
C' \otimes A' \arrow[r, "\psi'"] & A' \otimes C',
\end{tikzcd}$$

or equivalently, for all $a \in A$, $c \in C$,

$$\sum \alpha(a_\psi) \otimes \gamma(c^\psi) = \sum \alpha(a)_{\psi'} \otimes \gamma(c)^{\psi'}.$$

We denote this category by $\mathbb{E}^\bullet_\bullet(k)$.

We can also define (τ and τ' are the twist maps)

- Left-right entwining structures over k: triples (A, C, ψ), where $\psi : A \otimes C \to A \otimes C$ such that $(A^\circ, C, \psi \circ \tau) \in \mathbb{E}^\bullet_\bullet(k)$. We denote this category by ${}_\bullet\mathbb{E}^\bullet(k)$.
- Right-left entwining structures over k: triples (A, C, ψ), where $\psi : C \otimes A \to C \otimes A$ such that $(A, C^\circ, \tau \circ \psi) \in \mathbb{E}^\bullet_\bullet(k)$. We denote this category by ${}^\bullet\mathbb{E}_\bullet(k)$.
- Left-left entwining structures over k: triples (A, C, ψ), where $\psi : A \otimes C \to C \otimes A$ such that $(A^\circ, C^\circ, \tau' \circ \psi \circ \tau) \in \mathbb{E}^\bullet_\bullet(k)$. We denote this category by ${}^\bullet_\bullet\mathbb{E}(k)$.

Proposition 1.5.6. *The categories $\mathbb{E}^\bullet_\bullet(k)$, ${}_\bullet\mathbb{E}^\bullet(k)$, ${}^\bullet\mathbb{E}_\bullet(k)$, and ${}^\bullet_\bullet\mathbb{E}(k)$ are isomorphic.*

Again we recall from [19], that a *right-right entwined modules* over a right-right entwining structure (A, C, ψ) is a k-module M which is a right A-module with multiplication ψ_M, and a right C-comodule with comultiplication ρ_M such that the following diagram is commutative

$$\begin{tikzcd}
M \otimes A \arrow[r, "\rho_M\otimes A"] \arrow[dd, "\psi_M"'] & M \otimes C \otimes A \arrow[dr, "M\otimes\psi"] & \\
& & M \otimes A \otimes C \arrow[dl, "\psi_M\otimes C"] \\
M \arrow[r, "\rho_M"] & M \otimes C, &
\end{tikzcd}$$

or equivalently, for all $m \in M$ and $a \in A$,

$$\rho_M(ma) = \sum m_{(0)} a_\psi \otimes m_{(1)}^\psi.$$

A *morphism* between entwined modules is a morphism of right A-modules and right C-comodules at the same time. We denote this category by $\mathcal{M}(\psi)_A^C$.

The following result shows that entwining structures and entwined modules are very related to corings and comodules over corings respectively.

Theorem 1.5.7. [Takeuchi]

(a) Let A be an algebra, C be a k-coalgebra, and let $\psi : C \otimes A \to A \otimes C$ be a k-linear map. Obviously $A \otimes C$ has a structure of left A-module by $b(a \otimes c) = ba \otimes c$ for $a, b \in A$ and $c \in C$. Define the right A-module action on $A \otimes C$,

$$\psi^r_{A\otimes C} : A \otimes C \otimes A \xrightarrow{A\otimes\psi} A \otimes A \otimes C \xrightarrow{m\otimes C} A \otimes C\ ,$$

that is, $(a \otimes c)b = a\psi(c \otimes b)$ for $a, b \in A$ and $c \in C$. Define also

$$\Delta : A \otimes C \xrightarrow{A\otimes\Delta_C} A \otimes C \otimes C \simeq (A \otimes C) \otimes_A (A \otimes C)$$

(for every $a \in A, c \in C$, $\Delta(a \otimes c) = \sum(a \otimes c_{(1)}) \otimes_A (1_A \otimes c_{(2)})$, where $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$, and

$$\epsilon : A \otimes C \xrightarrow{A\otimes\epsilon_C} A \otimes k \simeq A$$

($\epsilon(a \otimes c) = a\epsilon_C(c)$). Thus, $(A \otimes C, \Delta, \epsilon)$ is an A-coring if and only if (A, C, ψ) is a right-right entwining structure.

(b) Let (A, C, ψ) be a right-right entwining structure, M be a right A-module with the action $\psi^r_M : M \otimes A \to M$, and let $\rho_M : M \to M \otimes C$ be a k-linear map. Define

$$\rho'_M : M \xrightarrow{\rho_M} M \otimes C \simeq M \otimes_A (A \otimes C)$$

(for every $m \in M$, $\rho'_M(m) = \sum m_{(0)} \otimes_A (1_A \otimes m_{(1)})$, where $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$). Then, (M, ρ'_M) is a right $A \otimes C$-comodule if and only if (M, ψ^r_M, ρ_M) is a right-right entwined module over (A, C, ψ). Hence, there is an isomorphism of categories

$$\mathcal{M}^{A\otimes C} \simeq \mathcal{M}^C_A(\psi).$$

Proof. (Sketch) (a) It is easy to verify that

- For all $a, b \in A, c \in C$, $(1 \otimes c)(ab) = ((1 \otimes c)a)b \iff$ (ES1).
- For all $c \in C$, $(1 \otimes c)1 = 1 \otimes c \iff$ (ES3).
- For all $a \in A, c \in C$, $\Delta((1 \otimes c)a) = \Delta(1 \otimes c)a \iff$ (ES2).
- For all $a \in A, c \in C$, $\epsilon((1 \otimes c)a) = \epsilon(1 \otimes c)a \iff$ (ES4).

Moreover, the coassociativity of Δ and the counit property of ϵ follow from that of Δ_C and ϵ_C. Hence (a) follows.

(b) It is easy to verify that

- ρ'_M is A-linear if and only if for all $m \in M$ and $a \in A$, $\rho_M(ma) = \sum m_{(0)}a_\psi \otimes m^\psi_{(1)}$.

- ρ'_M is coassociative if and only if ρ_M is coassociative.
- The counit property of ϵ holds if and only if that of ϵ_C holds. Hence (b) follows.

□

The last theorem has a left-handed version:

Theorem 1.5.8. *(a) Let A be an algebra, C be a k-coalgebra, and let $\varphi : A\otimes C \to C\otimes A$ be a k-linear map. Obviously $C\otimes A$ has a structure of right A-module by $(c\otimes a)b = c\otimes ab$ for $a, b \in A$ and $c \in C$. Define the left A-module action on $C \otimes A$,*

$$\psi^l_{C\otimes A} : A \otimes C \otimes A \xrightarrow{\varphi\otimes A} C \otimes A \otimes A \xrightarrow{C\otimes m} C \otimes A\,,$$

that is, $b(c \otimes a) = \varphi(b \otimes c)a$ for $a, b \in A$ and $c \in C$. Define also

$$\Delta : C \otimes A \xrightarrow{\Delta_C\otimes A} C \otimes C \otimes A \simeq (C \otimes A) \otimes_A (C \otimes A)$$

(for every $a \in A, c \in C$, $\Delta(c \otimes a) = \sum(c_{(1)} \otimes 1_A) \otimes_A (c_{(2)} \otimes a)$, where $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$, and

$$\epsilon : C \otimes A \xrightarrow{\epsilon_C\otimes A} k \otimes A \simeq A$$

($\epsilon(c \otimes a) = \epsilon_C(c)a$). Thus, $(C \otimes A, \Delta, \epsilon)$ is an A-coring if and only if (A, C, φ) is a left-left entwining structure.

(b) Let (A, C, ψ) be a left-left entwining structure, M be a left A-module with the action $\psi^l_M : A \otimes M \to M$, and let $\lambda_M : M \to C \otimes M$ be a k-linear map. Define

$$\lambda'_M : M \xrightarrow{\lambda_M} C \otimes M \simeq (C \otimes A) \otimes_A M$$

(for every $m \in M$, $\lambda'_M(m) = \sum(m_{(-1)} \otimes_A 1_A) \otimes m_{(0)}$, where $\rho_M(m) = \sum m_{(-1)} \otimes m_{(0)}$). Then, (M, λ'_M) is a left $C\otimes A$-comodule if and only if (M, ψ^l_M, λ_M) is a left-left entwined module over (A, C, φ). Hence, there is an isomorphism of categories

$$^{C\otimes A}\mathcal{M} \simeq {}^C_A\mathcal{M}(\varphi).$$

Let $(A, C, \psi) \in \mathbb{E}^{\bullet}_{\bullet}(k)$ be a right-right entwining structure and $(B, D, \varphi) \in {}^{\bullet}_{\bullet}\mathbb{E}(k)$ a left-left entwining structure. A *two-sided entwined module* [24, pp. 68–69] is a k-module M that have a structure of a left-left (B, D, φ)-module and a right-right (A, C, ψ)-module such that the following extra conditions hold:

(1) M is a (B, A)-bimodule;

(2) M is a (D, C)-bicomodule;

(3) the right A-action is left D-colinear, that is,

$$\lambda_M(ma) = m_{(-1)} \otimes m_{(0)}a, \text{ for all } m \in M, a \in A;$$

(4) the left B-action is right C-colinear, that is,

$$\rho_M(bm) = bm_{(0)} \otimes m_{(1)}, \text{ for all } m \in M, b \in B.$$

We denote the category of two-sided entwined modules by ${}^D_B\mathcal{M}(\varphi, \psi)^C_A$.

A right-right *Doi-Koppinen structure or simply DK structure* over k [24] is a triple (H, A, C), where H is a bialgebra, A is a right H-comodule algebra, and C is a right H-module coalgebra. A *morphism of DK structures* is a triple $(\hbar, \alpha, \gamma) : (H, A, C) \to (H', A', C')$, where $\hbar : H \to H'$, $\alpha : A \to A'$, and $\gamma : C \to C'$ are respectively a bialgebra morphism, an algebra morphism, and a coalgebra morphism such that

$$\rho_{A'}(\alpha(a)) = \alpha(a_{(0)}) \otimes \hbar(a_{(1)})$$

and

$$\gamma(ch) = \gamma(c)\hbar(h),$$

for all $a \in A, c \in C, h \in H$. This yields a category which we denote by $\mathbb{DK}^\bullet_\bullet(k)$.

The category of right *Doi-Koppinen-Hopf modules* over the right-right DK structure (H, A, C) is exactly the category of $\mathcal{M}(\psi)^C_A$, and it is denoted by $\mathcal{M}(H)^C_A$.

If H has a twisted antipode $\overline{S}$ (see Subsection 1.5.3), then ψ is bijective and $(A, C, \psi^{-1}) \in {}^\bullet_\bullet\mathbb{E}(k)$. In this case, ${}^C_A\mathcal{M}(\psi^{-1})$ will denoted by ${}^C_A\mathcal{M}(H)$. The objects of this category are left A-modules and left C-comodules such that

$$\rho_C(am) = m_{(-1)}\overline{S}(a_{(1)}) \otimes a_{(0)}m_{(0)}. \tag{1.23}$$

Analogous considerations are true for left-right, right-left, and left-left Doi-Koppinen structures.

1.5.3 The category of graded modules

For the rest of this work we adopt the notations of [73] and [81].

In this subsection G stand for a group, and A for a G-graded k-algebra (i.e. $A = \bigoplus_{g \in G} A_g$, where A_g, $g \in G$, are k-submodules of A such that $A_g A_\sigma \subset A_{g\sigma}$ for all $g, \sigma \in G$). We denote the unit of G by e and the unit of A by 1.

Let X be a right G-set (i.e. there is a map $X \times G \to X$, $(x, g) \mapsto xg$ such that $(xg)\sigma = x(g\sigma)$ and $xe = x$ for all $x \in X, g, \sigma \in G$). Of course, X is a left G-set via $g.x = xg^{-1}$ for all $g \in G, x \in X$.

We consider the category of X-graded right A-modules $gr - (A, X, G)$. This category is introduced and studied in [73].

A right A-module M is called a *X-graded right A-module* if $M = \bigoplus_{x\in X} M_x$, where M_x, $x \in X$, is k-submodules of M such that $M_x A_g \subset M_{xg}$ for all $g \in G, x \in X$.

Let M and N be two X-graded right A-modules. A morphism of right A-modules $f : M \to N$ is called a *morphism of X-graded right A-modules* if $f(M_x) \subset N_x$ for all $x \in X$. Notice that $\mathrm{Hom}_{gr-(A,X,G)}(M, N)$ is a k-submodule of $\mathrm{Hom}_A(M, N)$.

Since $\mathcal{M}_A$ is a k-category, X-graded right A-modules and their morphisms define the k-category of X-graded right A-modules. We denote it by $gr - (A, X, G)$.

Analogously, if X be a left G-set, then we define the k-category of X-graded left A-modules, and we denote it by $(G, X, A) - gr$. For every $M \in (G, X, A) - gr$, we denote the x-th component of M by ${}_xM$, for all $x \in X$.

If $X = G$ with the obvious action, then $gr - (A, G, G)$ is the category of graded right A-modules, and it is denoted by $gr - A$. If X is a singleton, then $gr - (A, X, G) = \mathcal{M}_A$.

Let $M \in gr - A$ ($M \in A - gr$ and X be a left G-set) and $x \in X$. The x-th *suspension* of M is, by definition, the X-graded right (left) A-module $M(x)$ ($(x)M$) that is equal to M as right (left) A-module endowed with the grading $M(x)_y = \bigoplus\{M_g \mid g \in G, xg = y\}$ $\big({}_y(x)M = \bigoplus\{{}_gM \mid g \in G, gx = y\}\big)$, $y \in X$.

By [73, Theorem 2.8], $gr - (A, X, G)$ is a Grothendieck category with $\{A(x) \mid x \in X\}$ as a family of projective generators. We can drive that $gr - (A, X, G)$ is a Grothendieck category from the isomorphism of categories (1.32) and Theorem 1.5.7 (2).

Now, let G and G' be two groups, X a right G-set, X' a right G'-set, A a G-graded algebra, and A' a G'-graded algebra.

Definition 1.5.9. Let $M \in gr-(A, X, G)$ and $N \in (G, X, A)-gr$. $M\widehat{\otimes}_A N$ is defined as the k-submodule of $M \otimes_A N$ generated by the elements $m \otimes n$ where $x \in X, m \in M_x, n \in {}_xN$.

Definition 1.5.10. Let Z be a (G, G')-set (i.e. Z a right G-set and a right G'-set such that $(g.z)g' = g.(zg')$ for all $g \in G, g' \in G', z \in Z$). A (A, A')-bimodule M is called a *Z-graded (A, A')-bimodule* if it has a Z-grading $M = \bigoplus_{z\in Z} M_z$ such that $A_g M_z A'_{g'} \subset M_{gzg'}$ for all $g \in G, g' \in G', z \in Z$.

Obviously, $X \times X'$ is a (G, G')-set via

$$g.(x, x')g' = (g.x, x'g'),\ g \in G, g' \in G', x \in X, x' \in X'.$$

Let $P = \bigoplus_{(x,x')\in X\times X'} P_{(x,x')}$ be an $X \times X'$-graded (A, A')-bimodule.

We have that, for every $x \in X$, ${}_xP$ is a submodule of $P_{A'}$ and ${}_xP = \bigoplus_{x'\in X'} P_{(x,x')} \in gr-(A', X', G')$ with the X'-grading $({}_xP)_{x'} = P_{(x,x')}$, $x' \in X'$. Furthermore, $P = \bigoplus_{x'\in X'} P_{x'} \in gr - (A', X', G')$, and P is the coproduct of the family $\{{}_xP, x \in X\}$ in $gr - (A', X', G')$.

Analogously, for every $x' \in X'$, $P_{x'} = \bigoplus_{x\in X} P_{(x,x')} \in (G, X, A) - gr$ with the X-grading ${}_x(P_{x'}) = P_{(x,x')}$, $x \in X$, $P = \bigoplus_{x\in X} {}_xP \in (G, X, A) - gr$, and P is the coproduct of the family $\{P_{x'}, x' \in X'\}$ in $(G, X, A) - gr$.

Now we will introduce two functors associated to P.

(1) **The functor** $-\widehat{\otimes}_A P : gr-(A,X,G) \to gr-(A',X',G')$.

Let $M \in gr-(A,X,G)$. $M\widehat{\otimes}_A P$ is a submodule of $(M \otimes_A P)_{A'}$. Then $M\widehat{\otimes}_A P \in gr-(A',X',G')$ with the X'-grading

$$(M\widehat{\otimes}_A P)_{x'} = M\widehat{\otimes}_A P_{x'},\ x' \in X'.$$

For a morphism $f : M \to N$ in $gr-(A,X,G)$, we define $f\widehat{\otimes}_A P : m \otimes p \mapsto f(m) \otimes p$.

(2) **The functor** $\mathrm{H}(P_{A'}, -) : gr-(A',X',G') \to gr-(A,X,G)$.

Let $M \in gr-(A',X',G')$. We have that $\mathrm{Hom}_{A'}(P,M')$ is a right A-module via $(fa)(p) = f(ap),\ f \in \mathrm{Hom}_{A'}(P,M'), a \in A, p \in P$. Consider the A-submodule of $\mathrm{Hom}_{A'}(P,M')$

$$\mathrm{H}(P_{A'}, M') = \{g \in \mathrm{Hom}_{gr-(A',X',G')}(P,M') \mid f({}_xP) = 0 \text{ for almost all } x \in X\}.$$

$\mathrm{H}(P_{A'}, M')$ can be viewed as a X-graded right A-module via the grading

$$(\mathrm{H}(P_{A'}, M'))_x = \big\{g \in \mathrm{Hom}_{gr-(A',X',G')}(P,M') \mid f({}_yP) = 0 \text{ for all } y \in X - \{x\}\big\},\ x \in X.$$

Let $f : M' \to N'$ be a morphism in $gr-(A',X',G')$. $\mathrm{H}(P_{A'}, f) : \mathrm{H}(P_{A'}, M') \to \mathrm{H}(P_{A'}, N')$ is defined by $\mathrm{H}(P_{A'}, f)(\varphi) = f \circ \varphi,\ \varphi \in \mathrm{H}(P_{A'}, M')$.

Proposition 1.5.11. [65, Proposition 1.2]
For every $X \times X'$-graded (A,A')-bimodule P, we have an adjunction

$$(-\widehat{\otimes}_A P, \mathrm{H}(P_{A'}, -)).$$

The unit and the counit of this adjunction are given respectively by

$$\eta_M : M \to \mathrm{H}(P_{A'}, M\widehat{\otimes}_A P),$$

$\eta_M(m)(p) = \sum_{x\in X} m_x \otimes_A {}_xp$ ($m = \sum_{x\in X} m_x \in M, p = \sum_{x\in X} {}_xp \in P$), and

$$\varepsilon_N : \mathrm{H}(P_{A'}, N)\widehat{\otimes}_A P \to N,$$

$\varepsilon_N(f \otimes_A P) = f(p)$ ($f \in \mathrm{H}(P_{A'}, N)_x, p \in {}_xP, x \in X$).

Now, set, for $a \in A$, $\lambda_a = a- : A \to A$. We have, for every $a \in A_g, x \in X$, $\lambda_a : A(x) \to A(g.x)$ is a morphism in $gr-(A,X,G)$.

Definition 1.5.12. Let $F : gr-(A,X,G) \to gr-(A',X',G')$ be a (covariant) functor. The $X \times X'$*-graded* (A,A')*-bimodule associated to* F is the X'-graded right A'-module $P = \bigoplus_{x\in X} F(A(x))$ endowed with the following left A-module structure

$$ap = \big(F(\lambda_a)\big)(p),\ g \in G, a \in A_g, p \in F(A(x)),$$

and the $X \times X'$-grading

$$P_{(x,x')} = F(A(x))_{x'},\ x \in X, x' \in X'.$$

We will denote the $X \times X$-graded (A, A)-bimodule associated to the identity functor $1_{gr-(A,X,G)}$ by $\widehat{A}$.

For more details we refer to [65] and [81].

Now, we will recall the definition and some well known results of bialgebras.

Proposition 1.5.13. *Let H be a k-algebra with the multiplication map m_H and the unit map η_H, and a k-coalgebra. The following are equivalent*

(1) m_H and η_H are morphisms of coalgebras;

(2) Δ_H and ϵ_H are morphisms of algebras;

(3) for all $g, h \in H$,

$$\begin{aligned} \Delta(gh) &= g_{(1)}h_{(1)} \otimes g_{(2)}h_{(2)} && (1.24)\\ \epsilon(gh) &= \epsilon(g)\epsilon(h) && (1.25)\\ \Delta(1_H) &= 1_H \otimes 1_H && (1.26)\\ \epsilon(1_H) &= 1_k. && (1.27) \end{aligned}$$

In this case, we say that H is a bialgebra. *A map between bialgebras over the same ring k is called a* morphism of bialgebras *if it is both a morphism of algebras and a morphism of coalgebras.*

Proof. It is a direct consequence from the following remarks
m_H is a morphism of coalgebras if and only if (1.24) and (1.25) hold,
η_H is a morphism of coalgebras if and only if (1.26) and (1.27) hold,
Δ_H is a morphism of algebras if and only if (1.24) and (1.26) hold,
ϵ_H is a morphism of algebras if and only if (1.25) and (1.27) hold. □

Definition 1.5.14. Let H be a bialgebra. We say that H is a *Hopf algebra* if 1_H is invertible in the convolution algebra $\mathrm{Hom}_k(H, H)$ (the multiplication is defined by $f * g = m_H(f \otimes g)\Delta_H$ for all $g, h \in \mathrm{Hom}_k(H, H)$, and $\eta_H\epsilon_H$ as the unit map), that is, there is $S \in \mathrm{Hom}_k(H, H)$ such that

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \eta_H(\epsilon_H(h)), \text{ for all } h \in H. \quad (1.28)$$

S is called the *antipode* of H.

Let H and K be two Hopf algebras and $f : H \to K$ be a morphism of bialgebras. It is very-known that f preserves the antipode, i.e.

$$S_K f = f S_H.$$

In this case, we say that f is a *morphism of Hopf algebras.*

Example 1.5.15. Let G be a group. Then kG is a k-algebra and a k-coalgebra (see Example 1.2.2 (4)). Moreover kG is a Hopf algebra with the antipode $S(g) = g^{-1}$ for all $g \in G$.

To differentiate the opposite algebra from that of coalgebra, we will denote the opposite coalgebra of a k-coalgebra C by C^{co}.

If H is a bialgebra, H°, H^{co} and $H^{\circ co}$ are also bialgebras. If H has an antipode S, then S is also an antipode of $H^{\circ co}$. Then, if H° has an antipode $\overline{S}$, that is, $\overline{S}$ satisfy

$$h_{(2)}\overline{S}(h_{(1)}) = \overline{S}(h_{(2)})h_{(1)} = \eta_H(\epsilon_H(h)), \text{ for all } h \in H, \tag{1.29}$$

then $\overline{S}$ is also an antipode of H^{co}. $\overline{S}$ is called a *twisted antipode.*

Proposition 1.5.16. [24, Proposition 2]
Let H be a Hopf algebra. Then $S : H \to H^{\circ co}$ is a morphism of bialgebras. If S is bijective, then S^{-1} is a twisted antipode. If H is commutative or cocommutative, then $S^2 = 1_H$. In particular, $S = \overline{S}$.

Lemma 1.5.17. *Let H be a bialgebra and M and N two right H-modules. Then $M \otimes N$ is also a right H-module via*

$$(m \otimes n)h = mh_{(1)} \otimes nh_{(2)}$$

for all $m \in M, n \in N, h \in H$.

Proof. Let $\psi_M : M \otimes H \to M$ and $\psi_N : N \otimes H \to N$ be the right H-actions on M and N, respectively. Consider the k-linear map

$$\psi : \ M \otimes N \otimes H \xrightarrow{M\otimes N\otimes \Delta_H} M \otimes N \otimes H \otimes H \xrightarrow{\simeq} M \otimes H \otimes N \otimes H \xrightarrow{\psi_M \otimes \psi_N} M \otimes N.$$

Set $(m \otimes n)h = \psi(m \otimes n \otimes h) = mh_{(1)} \otimes nh_{(2)}$ for all $m \in M, n \in N, h \in H$.

Now, let $m \in M, n \in N, g, h \in H$. By (1.24), $(m \otimes n)(gh) = ((m \otimes n)g)h$. Finally, by (1.26), $(m \otimes n)1_H = m \otimes n$. □

Definition 1.5.18. Let H be a bialgebra. A k-module C which is a k-coalgebra and a right H-module is called a right *H-module coalgebra* if the comultiplication and the counit maps are right H-linear, that is,

$$\Delta_C(ch) = c_{(1)}h_{(1)} \otimes c_{(2)}h_{(2)} \text{ and } \epsilon_C(ch) = \epsilon_C(c)\epsilon_H(h) \tag{1.30}$$

for all $h \in H, c \in C$.

Analogously, we define left module coalgebras. Moreover, if C is a right H-module coalgebra, then C^{co} is a left $H^{\circ co}$-module coalgebra.

Example 1.5.19. Let G be a group and X a right G-set. Then the k-coalgebra kX is a right kG-module coalgebra. Indeed, $kX \otimes kG$ is a free k-module with a basis $\{x \otimes g \mid x \in X, g \in G\}$. Define $\psi_{kX} : kX \otimes kG \to kX$ such that $\psi_{kX}(x \otimes g) = xg$ for all $x \in X, g \in G$. It is obvious that (kX, ψ_{kX}) is a right kG-module. From Example 1.2.2 (4), kX is also a k-coalgebra. It is easy to verify that kX is a right kG-module coalgebra.

Let A be a k-algebra and X a nonempty set. Let $G = \{e\}$ be the trivial group. Then A is trivially a G-graded algebra and X is trivially a right G-set. Moreover, $M \in gr-(A, X, G)$ if and only if M is a right A-module and $M = \bigoplus_{x \in X} M_x$ with each M_x is an A-submodule of M. In the following example we will consider the particular case where $G = \{e\}$ and $A = k$.

Example 1.5.20. Let $C = kX$ be a grouplike coalgebra with X is a nonempty set. Let M be X-graded k-module, that is, $M = \bigoplus_{x \in X} M_x$ such that each M_x is a k-submodule of M. Then M is a right C-comodule via

$$\rho_M(m) = m \otimes x$$

for all $x \in X, m \in M_x$. If $M \in \mathcal{M}^C$, then M is an X-graded k-module via the grading

$$M_x = \{m \in M \mid \rho_M(m) = m \otimes x\},\ x \in X.$$

This defines an isomorphism from $\mathcal{M}^C$ to the category of X-graded k-modules $gr-(k, X, \{e\})$.

Lemma 1.5.21. *Let H be a bialgebra and M and N are right H-comodules. Then $M \otimes N$ is also a right H-comodule via*

$$\rho_{M\otimes N}(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}$$

for all $m \in M, n \in N$.

Proof. Let $\rho_M : M \to M \otimes H$ and $\rho_N : N \to N \otimes H$ be the right H-coactions on M and N, respectively, and η the multiplication map of H. Consider the k-linear map

$$\rho :\ M \otimes N \xrightarrow{\rho_M \otimes \rho_N} M \otimes H \otimes N \otimes H \xrightarrow{\simeq} M \otimes N \otimes H \otimes H \xrightarrow{M\otimes N\otimes\eta} M \otimes N \otimes H.$$

Then $\rho(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}$ for all $m \in M, n \in N$.

Finally, for all $m \in M, n \in N$,

$$\begin{aligned}(M \otimes N \otimes \Delta)\rho(m \otimes n) &= m_{(0)} \otimes n_{(0)} \otimes \Delta(m_{(1)}n_{(1)}) \\ &= m_{(0)} \otimes n_{(0)} \otimes m_{(1)(1)}n_{(1)(1)} \otimes m_{(1)(2)}n_{(1)(2)} \text{ (by (1.24))} \\ &= m_{(0)(0)} \otimes n_{(0)(0)} \otimes m_{(0)(1)}n_{(0)(1)} \otimes m_{(1)}n_{(1)} \\ &= (\rho \otimes H)\rho(m \otimes n),\end{aligned}$$

and

$$\begin{aligned}(M \otimes N \otimes \epsilon)\rho(m \otimes n) &= m_{(0)} \otimes n_{(0)}\epsilon(m_{(1)}n_{(1)}) \\ &= m_{(0)} \otimes n_{(0)}\epsilon(m_{(1)})\epsilon(n_{(1)}) \text{ (by (1.25))} \\ &= m \otimes n.\end{aligned}$$

□

Definition 1.5.22. Let H be a bialgebra. Since the map $\eta_H : k \to H$ is a morphism of coalgebras, then k is a H-coalgebra with the coaction $k \xrightarrow{\simeq} k \otimes k \xrightarrow{k\otimes\eta_H} k \otimes H$.

A k-module A which is a k-algebra and a right H-comodule is called a right *H-comodule algebra* if the multiplication and the unit maps are right H-colinear, that is,

$$\rho_A(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)} \text{ and } \rho_A(1_A) = 1_A \otimes 1_H \tag{1.31}$$

for all $a, b \in A$.

Analogously, we define left comodule algebras. Moreover, if A is a right H-comodule algebra, then A° is a left $H^{\circ co}$-comodule algebra.

Example 1.5.23. Let G be a group, and $H = kG$ the associated Hopf algebra. To give a kG-comodule algebra is the same as to give a G-graded k-algebra.

Let G be a group, X a right G-set, and A a G-graded k-algebra. We know that (kG, A, C) is a DK structure with kG a Hopf algebra. Then, (A, kX, ψ) is an entwining structure where $\psi : kX \otimes A \to A \otimes kX$ is the map defined by $\psi(x \otimes a_g) = a_g \otimes xg$ for all $x \in X, g \in G, a_g \in A_g$. Moreover, we have

$$gr - (A, X, G) \simeq \mathcal{M}(kG)_A^{kX}. \tag{1.32}$$

From Theorem 1.5.7, we have an A-coring, $A \otimes kX$. The comultiplication and the counit maps of the coring $A \otimes kX$ are defined by:

$$\Delta_{A\otimes kX}(a \otimes x) = (a \otimes x) \otimes_A (1_A \otimes x), \quad \epsilon_{A\otimes kX}(a \otimes x) = a \quad (a \in A, x \in X). \tag{1.33}$$

We will verify in Section 2.5 that this coring is coseparable.

Chapter 2

Adjoint and Frobenius Pairs of Functors for Corings

2.1 Frobenius functors between categories of comodules

Let T be a k-algebra and $M \in {}_T\mathcal{M}_A$. Let $\rho : T \to \mathrm{End}_A(M)$ be the morphism of k-algebras given by the left T-module structure of the bimodule ${}_TM_A$. Now, suppose moreover that $M \in \mathcal{M}^{\mathfrak{C}}$. Then $\mathrm{End}_{\mathfrak{C}}(M)$ is a subalgebra of $\mathrm{End}_A(M)$. We have that $\rho(T) \subset \mathrm{End}_{\mathfrak{C}}(M)$ if and only if ρ_M is T-linear. Indeed, let $m \in M$ and $t \in T$, and let $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$. We have, $(\rho(t) \otimes_A \mathfrak{C})\rho_M(m) = \sum tm_{(0)} \otimes m_{(1)} = t\rho_M(m)$, and $\rho_M\rho(t)(m) = \rho_M(tm)$. Hence the left T-module structure of a $T-\mathfrak{C}$-bicomodule M can be described as a morphism of k-algebras $\rho : T \to \mathrm{End}_{\mathfrak{C}}(M)$, that is, as a T-object (see Section 1.1). Moreover, the category ${}^T\mathcal{M}^{\mathfrak{C}}$ is noting else than the category of left T-objects in $\mathcal{M}^{\mathfrak{C}}$, ${}_T\mathcal{M}^{\mathfrak{C}}$.

Given a k-linear functor $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ and $M \in {}_T\mathcal{M}^{\mathfrak{C}}$. We know that $F(M) \in {}_T\mathcal{M}^{\mathfrak{D}}$ via $(F(M), \{F(\rho(t)) \mid t \in T\})$. We have then two k-linear bifunctors

$$- \otimes_T F(-),\ F(- \otimes_T -) : \mathcal{M}_T \times {}_T\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}.$$

Let $\Upsilon_{T,M}$ be the unique isomorphism of $\mathfrak{D}$-comodules making the following diagram commutative

$$\begin{tikzcd} T \otimes_T F(M) \arrow[rr, "\Upsilon_{T,M}"] \arrow[dr, "\simeq"'] & & F(T \otimes_T M) \arrow[dl, "\simeq"] \\ & F(M) & \end{tikzcd} \tag{2.1}$$

for every $M \in {}_T\mathcal{M}^{\mathfrak{C}}$. It is easy to verify that $\Upsilon_{T,M}$ is natural in T.

Theorem 2.1.1. [Mitchell] ([79, Theorem 3.6.5])
Let $\mathbf{C}$ be a category satisfying AB 5) and $\mathbf{C}'$ be a full subcategory whose objects form a set of

projective generators of $\mathbf{C}$*. Let* $T, S : \mathbf{C} \to \mathbf{D}$ *be additive covariant functors where* $\mathbf{D}$ *is an abelian category which has direct sums. Suppose that* T *preserves inductive limits. Let* T' *and* S' *be the restrictions of* T *and* S *to* $\mathbf{C}'$*, respectively. Then, every natural transformation* $\eta : T' \to S'$ *can be uniquely extended to a natural transformation* $\overline{\eta} : T \to S$*.*

Corollary 2.1.2. [Mitchell] ([79, Corollary 3.6.6])
In the set up of Theorem 2.1.1, if S *preserves inductive limits and* η *is a natural equivalence, then so is* $\overline{\eta}$*.*

Remark 2.1.3. Theorem 2.1.1 holds also if we only suppose that the target category $\mathbf{D}$ is preadditive and has direct sums, or if the category $\mathbf{D}$ is preadditive and the functor S preserves direct sums. Note also that Corollary 2.1.2 also holds if we suppose only that the category $\mathbf{D}$ is preadditive.

Now, by Remark 2.1.3, there exists a unique natural transformation

$$\Upsilon_{-,M} : - \otimes_T F(M) \to F(- \otimes_T M)$$

extending the natural transformation $\Upsilon_{T,M}$. (The natural transformation $\Upsilon_{-,M}$ exists even if the category $\mathcal{M}^{\mathfrak{D}}$ is not abelian.)

Remark 2.1.4. Let $\mathbf{C}$ and $\mathbf{D}$ be k-categories having direct sums and cokernels, $F : \mathbf{C} \to \mathbf{D}$ a k-linear functor, and T a k-algebra. We have two k-linear bifunctors

$$- \otimes_T F(-),\ F(- \otimes_T -) : \mathcal{M}_T \times {}_T\mathbf{C} \to \mathbf{D}.$$

Let $M = (M, \rho) \in {}_T\mathbf{C}$. By the definition of tensor product (see Proposition 1.1.68 and its proof), $T \otimes_T F(M) = F(M)$ and $T \otimes_T M = M$. Define

$$\Upsilon_{T,M} = 1_{F(M)} : T \otimes_T F(M) \to F(T \otimes_T M).$$

That $\Upsilon_{T,M}$ is natural in T follows from the fact that for a morphism $f : T \to T$ in $\mathcal{M}_T$, $f \otimes_T M = \rho(f(1)) : M \to M$. By Remark 2.1.3, there exists a unique natural transformation

$$\Upsilon_{-,M} : - \otimes_T F(M) \to F(- \otimes_T M) : \mathcal{M}_T \to \mathbf{D}$$

extending the natural transformation $\Upsilon_{T,M}$.

Let $g : M \to M'$ be a morphism in ${}_T\mathbf{C}$. Consider the two natural transformation

$$F(- \otimes_T g)\Upsilon_{-,M},\ \Upsilon_{-,M}(- \otimes_T F(g)) : - \otimes_T F(M) \to F(- \otimes_T M').$$

Since $T \otimes_T g = g$, $F(T \otimes_T g)\Upsilon_{T,M} = \Upsilon_{T,M}(T \otimes_T F(g))$. Then, by Remark 2.1.3, these natural transformations coincide. Hence $\Upsilon_{T,M}$ is natural in M. Again, by Remark 2.1.3, if moreover F preserves inductive limits, then Υ is a natural equivalence of bifunctors. Now we give two interesting applications:

(i) Take $\mathbf{C} = \mathcal{M}_T$ and $M = T$ and assume that $F : \mathcal{M}_T \to \mathbf{D}$ is k-linear and preserves inductive limits. Then we obtain a natural equivalence

$$F \simeq F(- \otimes_T T) \simeq - \otimes_T F(T).$$

This result generalizes [69, Exercise 5, p. 157] which generalizes a result due independently to Watts [92] and Eilenberg [32].

(ii) Take $\mathbf{C} = \mathcal{M}_R$ with R a k-algebra, $F = - \otimes_R N : \mathcal{M}_R \to \mathbf{D}$ with $N \in {}_R\mathbf{D}$ and let $X \in \mathcal{M}_T$ and $M \in {}_T\mathcal{M}_R$. Then we obtain a natural equivalence of bifunctors:

$$\Upsilon_{X,M} : X \otimes_T (M \otimes_R N) \to (X \otimes_T M) \otimes_R N.$$

Using Remark 2.1.3 this isomorphism is also natural in N. This result generalizes [69, Exercise 4, p. 157].

The following lemma is a generalization of [3, Folgerung III.4.3].

Lemma 2.1.5. *(i) If F preserves direct sums then $\Upsilon_{X,M}$ is an isomorphism for every projective right T-module X.*

(ii) If F preserves direct limits then $\Upsilon_{X,M}$ is an isomorphism for every flat right T-module X.

(iii) If F preserves inductive limits then $\Upsilon_{X,M}$ is an isomorphism for every right T-module X.

Proof. Set $\eta = \Upsilon_{-,M}$, $S = - \otimes_T F(M)$, and $T = F(- \otimes_T M)$. First observe that S preserves inductive limits.

(i) Let $(X_i)_{i\in I}$ be a family of right T-modules. It is clear that $\eta_{\oplus_{i\in I} X_i} = \oplus_{i\in I}\eta_{X_i}$. Then $\eta_{\oplus_{i\in I} X_i}$ is an isomorphism if and only if $\oplus_{i\in I}\eta_{X_i}$ is so for every $i \in I$. Hence (i) follows since a projective right module is a direct summand of a free module.

(ii) Let X be a flat right T-module. By Lazard's structural theorem [62, Théorème 1.2], every flat module is a direct limit of finitely generated free modules. Then $X = \varinjlim_I X_i$ for a directed set I and a direct system of finitely generated free modules (X_i, u_{ij}). Then, $S(X) = \varinjlim_I S(X_i)$ and $T(X) = \varinjlim_I T(X_i)$. Hence, $\eta_{X_i} : S(X_i) \to T(X_i)$, $i \in I$, is a morphism from $(S(X_i), S(u_{ij}))$ to $(T(X_i), T(u_{ij}))$, and $\varinjlim_I \eta_{X_i} = \eta_X$. Finally, from (i), η_X is an isomorphism.

(iii) Let X be a right T-module. Consider a free presentation in $\mathcal{M}_T$ of X

$$T^{(J)} \to T^{(I)} \to X \to 0,$$

where I and J are sets. This yields the commutative diagram with exact rows (F is right exact)

$$\begin{tikzcd} T^{(J)} \otimes_T F(M) \arrow[r] \arrow[d, "\eta_{T^{(J)}}"] & T^{(I)} \otimes_T F(M) \arrow[r] \arrow[d, "\eta_{T^{(I)}}"] & X \otimes_T F(M) \arrow[r] \arrow[d, "\eta_X"] & 0 \\ F(T^{(J)} \otimes_T M) \arrow[r] & F(T^{(I)} \otimes_T M) \arrow[r] & F(X \otimes_T M) \arrow[r] & 0. \end{tikzcd}$$

From (i), $\eta_{T^{(I)}}$ and $\eta_{T^{(J)}}$ are isomorphisms. Hence so is η_X. $\square$

Clearly we have

Proposition 2.1.6. Υ *is a natural transformation of bifunctors. Moreover,* $\Upsilon_{X,-}$ *is a natural isomorphism for all projective right T-module X. If F preserves direct limits, then* $\Upsilon_{X,-}$ *is a natural isomorphism for all flat right T-module X. If F preserves inductive limits, then Υ is a natural equivalence.*

Example 2.1.7. Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. Suppose that ${}_B\mathfrak{D}$ is flat or $\mathfrak{C}$ is coseparable. Consider the functor $F = -\square_{\mathfrak{C}}N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ and the natural transformation associated to F, Υ. By Mitchell's Theorem, $\Upsilon_{W,M}$ is exactly the canonical map (see Proposition 1.3.5)

$$W \otimes_T (M\square_{\mathfrak{C}}N) \to (W \otimes_T M)\square_{\mathfrak{C}}N,$$

for every $W \in \mathcal{M}_T, M \in {}_T\mathcal{M}^{\mathfrak{C}}$.

From Lemma 1.1.67 we obtain

Lemma 2.1.8. *Let $F, G : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be k-linear functors, and let $\eta : F \to G$ be a natural transformation. For each $M \in {}_T\mathcal{M}^{\mathfrak{C}}$, $\eta_M : F(M) \to G(M)$ is a morphism of $(T, \mathfrak{D})$-bicomodules.*

Using Remark 2.1.3, we prove easily the two following lemmas. The condition "F (and G) preserve(s) direct sums" is superfluous. Moreover, these lemmas remain true for k-categories which have direct sums and cokernels (see Remark 2.1.4).

Lemma 2.1.9. [43, Lemma 3.2]
Let $F, G : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be k-linear functors and $\eta : F \to G$ a natural transformation. For $X \in \mathcal{M}_T$ and $M \in {}_T\mathcal{M}^{\mathfrak{C}}$, the following diagram is commutative

$$\begin{tikzcd} X \otimes_T F(M) \arrow[r, "X\otimes_T\eta_M"] \arrow[d, "\Upsilon^F_{X,M}"] & X \otimes_T G(M) \arrow[d, "\Upsilon^G_{X,M}"] \\ F(X \otimes_T M) \arrow[r, "\eta_{X\otimes_T M}"] & G(X \otimes_T M). \end{tikzcd}$$

Lemma 2.1.10. [43, Lemma 3.3]
Let S, T be k-algebras, and Let $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be a k-linear functor. In the situation $(X_S, {}_SY_T, {}_TM_{\mathfrak{C}})$,

$$\Upsilon_{X,Y\otimes_T M} \circ (X \otimes_S \Upsilon_{Y,M}) = \Upsilon_{X\otimes_S Y,M}.$$

Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ be a bicomodule. We say that a k-linear functor $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is *M-compatible* if $\Upsilon_{\mathfrak{C}',M}$ and $\Upsilon_{\mathfrak{C}'\otimes_{A'}\mathfrak{C}',M}$ are isomorphisms. For example, F is M-compatible for every bicomodule M either if F preserves inductive limits, or ${\mathfrak{C}'}_{A'}$ is flat and F preserves direct limits, or ${\mathfrak{C}'}_{A'}$ is projective and F preserves coproducts (since, by [25, Proposition II.5.3], $(\mathfrak{C}' \otimes_{A'} \mathfrak{C}')_{A'}$ is projective).

The following lemma will be useful in the proof of the next theorem.

Lemma 2.1.11. *If $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, and $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is a M-compatible k-linear functor, then for all $X \in \mathcal{M}_{A'}$,*

$$\Upsilon_{X\otimes_{A'}\mathfrak{C}',M} \circ (X \otimes_{A'} \lambda_{F(M)}) = F(X \otimes_{A'} \lambda_M) \circ \Upsilon_{X,M}.$$

Proof. Let us consider the diagram

$$\begin{tikzcd}
X \otimes_{A'} F(M) \arrow[rr, "X\otimes_{A'}\lambda_{F(M)}"] \arrow[dr, "X\otimes_{A'}F(\lambda_M)"] \arrow[dd, "\Upsilon_{X,M}"'] & & X \otimes_{A'} \mathfrak{C}' \otimes_{A'} F(M) \arrow[dl, "X\otimes_{A'}\Upsilon_{\mathfrak{C}',M}"] \arrow[dd, "\Upsilon_{X\otimes_{A'}\mathfrak{C}',M}"] \\
& X \otimes_{A'} F(\mathfrak{C}' \otimes_{A'} M) \arrow[dr, "\Upsilon_{X,\mathfrak{C}'\otimes_{A'}M}"] & \\
F(X \otimes_{A'} M) \arrow[rr, "F(X\otimes_{A'}\lambda_M)"'] & & F(X \otimes_{A'} \mathfrak{C}' \otimes_{A'} M)
\end{tikzcd}$$

The commutativity of the top triangle follows from the definition of $\lambda_{F(M)}$, while the right triangle commutes by Lemma 2.1.10 (we take $S = T = A'$, and $Y = \mathfrak{C}'$), and the left triangle is commutative since $\Upsilon_{X,-}$ is natural. Therefore, the commutativity of the rectangle holds. $\square$

Now suppose that F is M-compatible. We define

$$\lambda_{F(M)} := \Upsilon^{-1}_{\mathfrak{C}',M} \circ F(\lambda_M) : F(M) \to \mathfrak{C}' \otimes_{A'} F(M).$$

We have that $\lambda_{F(M)}$ is a morphism of $A' - \mathfrak{D}$-bicomodules.

Proposition 2.1.12. [43, Proposition 3.4]
Let F be M-compatible functor. $F(M)$ endowed with $\rho_{F(M)}$ and $\lambda_{F(M)}$, is a $\mathfrak{C}' - \mathfrak{D}$-bicomodule. Moreover, if $F, G : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ are M-compatible, and $\eta : F \to G$ is a natural transformation, then the map $\eta_M : F(M) \to G(M)$ is a morphism of $\mathfrak{C}' - \mathfrak{D}$-bicomodules.

Proof. To prove the first statement, consider the diagram

$$\begin{tikzcd}
 & F(M) \arrow[rr, "\lambda_{F(M)}"] \arrow[dl, "\lambda_{F(M)}"'] \arrow[dd, equal] & & \mathfrak{C}' \otimes_{A'} F(M) \arrow[dl, "\Delta_{\mathfrak{C}'} \otimes_{A'} F(M)"] \arrow[dd, "\Upsilon_{\mathfrak{C}',M}"] \\
\mathfrak{C}' \otimes_{A'} F(M) \arrow[rr, "\mathfrak{C}' \otimes_{A'} \lambda_{F(M)}"] \arrow[dd, "\Upsilon_{\mathfrak{C}',M}"] & & \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} F(M) \arrow[dd, "\Upsilon_{\mathfrak{C}' \otimes_{A'} \mathfrak{C}',M}"] & \\
 & F(M) \arrow[rr, "F(\lambda_M)"] \arrow[dl, "F(\lambda_M)"'] & & F(\mathfrak{C}' \otimes_{A'} M) \arrow[dl, "F(\Delta_{\mathfrak{C}'} \otimes_{A'} M)"] \\
F(\mathfrak{C}' \otimes_{A'} M) \arrow[rr, "F(\mathfrak{C}' \otimes_{A'} \lambda_M)"] & & F(\mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} M) &
\end{tikzcd}$$

Since $\Upsilon_{-,M}$ is a natural transformation, and from the definition of $\lambda_{F(M)}$, the coassociative property of λ_M, and Lemma 2.1.11, all squares, save possibly the top one, are commutative. Hence, by Lemma 1.1.4 ($\Upsilon_{\mathfrak{C}' \otimes_{A'} \mathfrak{C}',M}$ is injective), the top square is also commutative.

To show the counity property, consider the diagram

$$\begin{tikzcd}
\mathfrak{C}' \otimes_{A'} F(M) \arrow[rr, "\Upsilon_{\mathfrak{C}',M}"] \arrow[dd, "\epsilon_{\mathfrak{C}'} \otimes_{A'} F(M)"'] & & F(\mathfrak{C}' \otimes_{A'} M) \arrow[dd, "F(\epsilon_{\mathfrak{C}'} \otimes_{A'} M)"] \\
 & F(M) \arrow[ul, "\lambda_{F(M)}"] \arrow[ur, "F(\lambda_M)"'] \arrow[dl, "\simeq"] \arrow[dr, "\simeq"] & \\
A' \otimes_{A'} F(M) \arrow[rr, "\Upsilon_{A',M}"] & & F(A' \otimes_{A'} M).
\end{tikzcd}$$

From the definition of $\Upsilon_{A',M}$ and $\lambda_{F(M)}$, the counity property of the left $\mathfrak{C}'$-comodule M, and the fact that $\Upsilon_{-,M}$ is natural, all the triangles, save possibly the left one, and the rectangle are commutative. Since $\Upsilon_{A',M}$ is injective, then the left triangle is also commutative.

Now we will verify the last statement. For this consider the diagram

$$\begin{tikzcd}
F(M) \arrow[rr, "\eta_M"] \arrow[dr, "F(\lambda_M)"] \arrow[dd, "\lambda_{F(M)}"] & & G(M) \arrow[dr, "G(\lambda_M)"] \arrow[dd, "\lambda_{G(M)}"] & \\
 & F(\mathfrak{C}' \otimes_{A'} M) \arrow[rr, "\eta_{\mathfrak{C}' \otimes_{A'} M}"] & & G(\mathfrak{C}' \otimes_{A'} M) \\
\mathfrak{C}' \otimes_{A'} F(M) \arrow[ur, "\Upsilon^F_{\mathfrak{C}',M}"] \arrow[rr, "\mathfrak{C}' \otimes_{A'} \eta_M"] & & \mathfrak{C}' \otimes_{A'} G(M) \arrow[ur, "\Upsilon^G_{\mathfrak{C}',M}"] &
\end{tikzcd}$$
.

The commutativity of the triangles follows by the definition of $\lambda_{F(M)}$ and $\lambda_{G(M)}$. The upper parallelogram is commutative since η is natural while the bottom one commutes from Lemma 2.1.9(2). Since $\Upsilon^G_{\mathfrak{C}',M}$ is an injective map, the triangle is also commutative. $\square$

From now on, every result for comodules over coseparable corings applies in particular for modules over rings (since the trivial A-coring is coseparable).

Now we propose a generalization of [90, Proposition 2.1], [19, 23.1(1)], and [43, Theorem 3.5].

Theorem 2.1.13. *Let $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be a k-linear functor, such that*

(I) ${}_B\mathfrak{D}$ is flat and F preserves the kernel of $\rho_N \otimes_A \mathfrak{C} - N \otimes_A \Delta_{\mathfrak{C}}$ for every $N \in \mathcal{M}^{\mathfrak{C}}$, or

(II) $\mathfrak{C}$ is a coseparable A-coring and the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$ are abelian.

Assume that at least one of the following statements holds

1. *$\mathfrak{C}_A$ is projective, F preserves coproducts, and $\Upsilon_{N,\mathfrak{C}}$, $\Upsilon_{N\otimes_A\mathfrak{C},\mathfrak{C}}$ are isomorphisms for all $N \in \mathcal{M}^{\mathfrak{C}}$ (e.g. , if A is semisimple and F preserves coproducts), or*

2. *$\mathfrak{C}_A$ is flat, F preserves direct limits, and $\Upsilon_{N,\mathfrak{C}}$, $\Upsilon_{N\otimes_A\mathfrak{C},\mathfrak{C}}$ are isomorphisms for all $N \in \mathcal{M}^{\mathfrak{C}}$ (e.g. , if A is a von Neumann regular ring and F preserves direct limits), or*

3. *F preserves inductive limits (e.g., if F has a right adjoint).*

Then F is naturally equivalent to $-\square_{\mathfrak{C}}F(\mathfrak{C})$.

Proof. In each case, we have F is $\mathfrak{C}$-compatible where $\mathfrak{C} \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$. Therefore, by Proposition 2.1.12, $F(\mathfrak{C})$ can be viewed as a $\mathfrak{C}-\mathfrak{D}$-bicomodule. From Lemma 2.1.11, and since $\Upsilon_{-,\mathfrak{C}}$ is a natural transformation, we have, for every $N \in \mathcal{M}^{\mathfrak{C}}$, the commutativity of the following diagram with exact rows in $\mathcal{M}^{\mathfrak{D}}$

$$\begin{tikzcd}
0 \arrow[r] & N\square_{\mathfrak{C}}F(\mathfrak{C}) \arrow[r] & N \otimes_A F(\mathfrak{C}) \arrow[r, "\rho_N\otimes_A F(\mathfrak{C}) - N\otimes_A \lambda_{F(\mathfrak{C})}"] \arrow[d, "\Upsilon_{N,\mathfrak{C}}"', "\simeq"] & N \otimes_A \mathfrak{C} \otimes_A F(\mathfrak{C}) \arrow[d, "\Upsilon_{N\otimes_A\mathfrak{C},\mathfrak{C}}"', "\simeq"] \\
0 \arrow[r] & F(N) \arrow[r, "F(\rho_N)"] & F(N \otimes_A \mathfrak{C}) \arrow[r, "F(\rho_N\otimes_A\mathfrak{C} - N\otimes_A\Delta_{\mathfrak{C}})"] & F(N \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}).
\end{tikzcd}$$

The exactness of the bottom sequence is assumed in the case (I). For the case (II), it follows by factorizing the map $\omega_{N,\mathfrak{C}} = \rho_N \otimes_A \mathfrak{C} - N \otimes_A \Delta_{\mathfrak{C}}$ through its image, and using the facts that the sequence $0 \longrightarrow N \xrightarrow{\rho_N} N \otimes_A \mathfrak{C} \xrightarrow{\omega_{N,\mathfrak{C}}} N \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}$ is split exact in $\mathcal{M}^{\mathfrak{C}}$, and that additive functors between abelian categories preserve split exactness. By the universal property of the kernel, there exists a unique isomorphism $\eta_N : N\square_{\mathfrak{C}}F(\mathfrak{C}) \to F(N)$ in $\mathcal{M}^{\mathfrak{D}}$ making commutative the above diagram.

Now, we will verify that η is natural. For this let $f : N \to N'$ be a morphism in $\mathcal{M}^{\mathfrak{C}}$, and consider the following diagram

$$\begin{tikzcd}
 & N\square_{\mathfrak{C}}F(\mathfrak{C}) \arrow[rr, "\eta_N"] \arrow[dl, "f\square_{\mathfrak{C}}F(\mathfrak{C})"'] \arrow[dd] & & F(N) \arrow[dl, "F(f)"] \arrow[dd, "F(\rho_N)"] \\
N'\square_{\mathfrak{C}}F(\mathfrak{C}) \arrow[rr, "\eta_{N'}"] \arrow[dd] & & F(N') \arrow[dd, "F(\rho_{N'})"] & \\
 & N \otimes_A F(\mathfrak{C}) \arrow[rr, "\Upsilon_{N,\mathfrak{C}}"] \arrow[dl, "f\otimes_A F(\mathfrak{C})"'] & & F(N \otimes_A \mathfrak{C}) \arrow[dl, "F(f\otimes_A \mathfrak{C})"] \\
N' \otimes_A F(\mathfrak{C}) \arrow[rr, "\Upsilon_{N',\mathfrak{C}}"] & & F(N' \otimes_A \mathfrak{C}) &
\end{tikzcd} .$$

From Lemma 1.1.4, the top square is commutative as desired. Hence $F \simeq -\square_{\mathfrak{C}}F(\mathfrak{C})$. $\square$

As an immediate consequence of the last theorem we have the following generalization of Eilenberg-Watts Theorem [85, Proposition VI.10.1].

Corollary 2.1.14. *Let $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ be a k-linear functor.*

(1) If ${}_B\mathfrak{D}$ is flat and A is a semisimple ring (resp. a von Neumann regular ring), then the following statements are equivalent

(a) F is left exact and preserves coproducts (resp. left exact and preserves direct limits);

(b) $F \simeq -\square_{\mathfrak{C}}M$ for some bicomodule $M \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$.

(2) If ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat, then the following statements are equivalent

(a) F is exact and preserves inductive limits;

(b) $F \simeq -\square_{\mathfrak{C}}M$ for some bicomodule $M \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ which is coflat in ${}^{\mathfrak{C}}\mathcal{M}$.

(3) If $\mathfrak{C}$ is a coseparable A-coring and the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$ are abelian, then the following statements are equivalent

(a) F preserves inductive limits;

(b) F preserves cokernels and $F \simeq -\square_{\mathfrak{C}}M$ for some bicomodule $M \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$.

(4) If $\mathfrak{C} = A$ and the category $\mathcal{M}^{\mathfrak{D}}$ is abelian, then the following statements are equivalent

(a) F has a right adjoint;

(b) F preserves inductive limits;

(c) $F \simeq - \otimes_A M$ *for some bicomodule* $M \in {}_A\mathcal{M}^{\mathfrak{D}}$.

Proof. Follows directly from Theorem 2.1.13 and Proposition 1.3.9. □

The particular case of the following statement when the cohom is exact generalizes [4, Corollary 3.12].

Corollary 2.1.15. *Let* $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *be a bicomodule, quasi-finite as a right* $\mathfrak{D}$*-comodule, such that* ${}_A\mathfrak{C}$ *and* ${}_B\mathfrak{D}$ *are flat. If the cohom functor* $\mathrm{h}_{\mathfrak{D}}(N,-)$ *is exact or if* $\mathfrak{D}$ *is a coseparable* B*-coring, then we have*

$$\mathrm{h}_{\mathfrak{D}}(N,-) \simeq -\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(N,\mathfrak{D}) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}.$$

Proof. The functor $\mathrm{h}_{\mathfrak{D}}(N,-)$ is k-linear and preserves inductive limits, since it is a left adjoint to the k-linear functor $-\square_{\mathfrak{C}}N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ (see Propositions 1.4.6, 1.1.33). Hence Theorem 2.1.13 achieves the proof. □

Now we will use the following generalization of [90, Lemma 2.2].

Lemma 2.1.16. *Let* Λ*,* Λ' *be bicomodules in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*. Suppose that*

(a) $\mathrm{Im}(\omega_{\mathfrak{D},\Lambda})$ *is a* $\mathfrak{D}_B$*-pure submodule of* $\mathfrak{D} \otimes_B \mathfrak{D} \otimes_B \Lambda$*, and* $\mathrm{Im}(\omega_{\mathfrak{D},\Lambda'})$ *is a* $\mathfrak{D}_B$*-pure submodule of* $\mathfrak{D} \otimes_B \mathfrak{D} \otimes_B \Lambda'$*,*

(b) $\omega_{X,\Lambda}$ *and* $\omega_{X,\Lambda'}$ *are* ${}_A(\mathfrak{C} \otimes_A \mathfrak{C})$*-pure for every* $X \in \mathcal{M}^{\mathfrak{D}}$ *(e.g.* ${}_A\mathfrak{C}$ *is flat or* $\mathfrak{D}$ *is coseparable).*

Let $G = -\square_{\mathfrak{D}}\Lambda, G' = -\square_{\mathfrak{D}}\Lambda' : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$*. Then*

$$\mathrm{Nat}(G, G') \simeq \mathrm{Hom}_{(\mathfrak{D},\mathfrak{C})}(\Lambda, \Lambda').$$

Proof. We want to show that the correspondence

$$\mathrm{Hom}_{(\mathfrak{D},\mathfrak{C})}(\Lambda, \Lambda') \to \mathrm{Nat}(G, G'), \quad \widetilde{\eta} \mapsto -\square_{\mathfrak{D}}\widetilde{\eta} \tag{2.2}$$

is bijective.

Let $\eta : G \to G'$ be a natural transformation. By Lemma 2.1.8, $\eta_{\mathfrak{D}}$ is left B-linear.

Let $M \in \mathcal{M}_B$ and $m \in M$. Define $f_m : \mathfrak{D} \to M \otimes_B \mathfrak{D}$, $f_m(d) = m \otimes d$, $m \in M, d \in \mathfrak{D}$. f_m is clearly right $\mathfrak{D}$-colinear. Then, by Lemma 1.3.12, we obtain the commutative diagram

$$\begin{tikzcd}
G(\mathfrak{D}) \simeq \Lambda \arrow[r, "\eta_{\mathfrak{D}}"] \arrow[d, "G(f_m)"'] & \Lambda' \simeq G'(\mathfrak{D}) \arrow[d, "G'(f_m)"] \\
G(M \otimes_B \mathfrak{D}) \simeq M \otimes_B \Lambda \arrow[r, "\eta_{M \otimes_B \mathfrak{D}}"] & M \otimes_B \Lambda' \simeq G'(M \otimes_B \mathfrak{D}),
\end{tikzcd}$$

and $G(f_m)(\lambda) = m \otimes \lambda$ for all $\lambda \in \Lambda$. Hence $\eta_{M \otimes_B \mathfrak{D}} = M \otimes_B \eta_{\mathfrak{D}}$.

Now, let $M \in \mathcal{M}^{\mathfrak{D}}$. Again from Lemma 1.3.12 we have the commutative diagram

$$\begin{tikzcd}
M\Box_{\mathfrak{D}}\Lambda \arrow[r, "\eta_M"] \arrow[d, "\rho_M\Box_{\mathfrak{D}}\Lambda"'] & M\Box_{\mathfrak{D}}\Lambda' \arrow[d, "\rho_M\Box_{\mathfrak{D}}\Lambda'"] \\
(M\otimes_B \mathfrak{D})\Box_{\mathfrak{D}}\Lambda \arrow[r, "\eta_{M\otimes_B\mathfrak{D}}"] \arrow[d, "\simeq"'] & (M\otimes_B \mathfrak{D})\Box_{\mathfrak{D}}\Lambda' \arrow[d, "\simeq"] \\
M\otimes_B \Lambda \arrow[r, "M\otimes_B\eta_{\mathfrak{D}}"] & M\otimes_B \Lambda'
\end{tikzcd} \tag{2.3}$$

and the compositions of the vertical maps are the inclusion maps. From the commutativity of the diagram (2.3), we get the following commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "\eta_{\mathfrak{D}}"] \arrow[d, "\lambda_\Lambda"'] & \Lambda' \arrow[d, "\lambda_{\Lambda'}"] \\
\mathfrak{D}\otimes_B \Lambda \arrow[r, "\mathfrak{D}\otimes_B\eta_{\mathfrak{D}}"] & \mathfrak{D}\otimes_B \Lambda'
\end{tikzcd} ,$$

i.e. $\eta_{\mathfrak{D}}$ is left $\mathfrak{D}$-colinear. Therefore, $\eta_M = M\Box_{\mathfrak{D}}\eta_{\mathfrak{D}}$. Hence the correspondence (2.2) is bijective. Finally, it is clearly an isomorphism of abelian groups. $\Box$

The following proposition generalizes [71, Theorem 2.1] from bimodules over rings to bicomodules over corings.

Proposition 2.1.17. *Suppose that ${}_A\mathfrak{C}$, $\mathfrak{C}_A$, ${}_B\mathfrak{D}$ and $\mathfrak{D}_B$ are flat. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. Consider the following properties:*

(1) $-\Box_{\mathfrak{C}}X$ is left adjoint to $-\Box_{\mathfrak{D}}\Lambda$;

(2) Λ is quasi-finite as a right $\mathfrak{C}$-comodule and $-\Box_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$;

(3) Λ is quasi-finite as a right $\mathfrak{C}$-comodule and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$;

(4) there exist bicolinear maps

$$\psi : \mathfrak{C} \to X\Box_{\mathfrak{D}}\Lambda \text{ and } \omega : \Lambda\Box_{\mathfrak{C}}X \to \mathfrak{D}$$

in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{D}}$ respectively, such that

$$(\omega\Box_{\mathfrak{D}}\Lambda) \circ (\Lambda\Box_{\mathfrak{C}}\psi) = \Lambda \text{ and } (X\Box_{\mathfrak{D}}\omega) \circ (\psi\Box_{\mathfrak{C}}X) = X; \tag{2.4}$$

(5) $\Lambda\Box_{\mathfrak{C}}-$ is left adjoint to $X\Box_{\mathfrak{D}}-$.

Then

(1) and (2) are equivalent, and they imply (3).

(3) implies (2) if $\mathfrak{C}$ is a coseparable A-coring. If ${}_AX$ and ${}_B\Lambda$ are flat, and $\omega_{X,\Lambda} = \rho_X \otimes_B \Lambda - X \otimes_A \rho_\Lambda$ is pure as an A-linear map and $\omega_{\Lambda,X} = \rho_\Lambda \otimes_A X - \Lambda \otimes_B \rho_X$ is pure

as a B-linear map (e.g. if ${}_{\mathfrak{C}}X$ and ${}_{\mathfrak{D}}\Lambda$ are coflat (see Lemma 1.3.6) or A and B are von Neumann regular rings), or if $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable, then (4) implies (1). The converse is true if ${}_{\mathfrak{C}}X$ and ${}_{\mathfrak{D}}\Lambda$ are coflat, or if A and B are von Neumann regular rings, or if $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable. Finally, if $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable, or if X and Λ are coflat on both sides, or if A, B are von Neumann regular rings, then (1), (4) and (5) are equivalent.

Proof. The equivalence between (1) and (2) follows from Proposition 1.4.6. That (2) implies (3) is a consequence of Proposition 2.1.12. If $\mathfrak{C}$ is coseparable and we assume (3) then, by Corollary 2.1.15, $\mathrm{h}_{\mathfrak{C}}(\Lambda, -) \simeq -\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \simeq -\square_{\mathfrak{C}}X$. That (1) implies (4) follows from Lemma 2.1.16 by evaluating the unit and the counit of the adjunction at $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Conversely, if we put $F = -\square_{\mathfrak{C}}X$ and $G = -\square_{\mathfrak{D}}\Lambda$, we have $GF \simeq -\square_{\mathfrak{C}}(X\square_{\mathfrak{D}}\Lambda)$ and $FG \simeq -\square_{\mathfrak{D}}(\Lambda\square_{\mathfrak{C}}X)$ by Proposition 1.3.16. Define natural transformations

$$\eta : 1_{\mathcal{M}^{\mathfrak{C}}} \xrightarrow{\simeq} -\square_{\mathfrak{C}}\mathfrak{C} \xrightarrow{-\square_{\mathfrak{C}}\psi} GF$$

and

$$\varepsilon : FG \xrightarrow{-\square_{\mathfrak{D}}\omega} -\square_{\mathfrak{D}}\mathfrak{D} \xrightarrow{\simeq} 1_{\mathcal{M}^{\mathfrak{D}}},$$

which become the unit and the counit of an adjunction by (3.1). This gives the equivalence between (1) and (4). The equivalence between (4) and (5) follows by symmetry. □

Proposition 2.1.18. *Suppose that ${}_{A}\mathfrak{C}$ and ${}_{B}\mathfrak{D}$ are flat. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. The following statements are equivalent*

(i) $-\square_{\mathfrak{C}}X$ *is left adjoint to* $-\square_{\mathfrak{D}}\Lambda$*, and* $-\square_{\mathfrak{C}}X$ *is left exact (or* ${}_{A}X$ *is flat or* ${}_{\mathfrak{C}}X$ *is coflat);*

(ii) Λ *is quasi-finite as a right* $\mathfrak{C}$*-comodule,* $-\square_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$*, and* $-\square_{\mathfrak{C}}X$ *is left exact (or* ${}_{A}X$ *is flat or* ${}_{\mathfrak{C}}X$ *is coflat);*

(iii) Λ *is quasi-finite and injector as a right* $\mathfrak{C}$*-comodule and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$.

Proof. First, observe that if ${}_{\mathfrak{C}}X$ is coflat, then ${}_{A}X$ is flat (see Corollary 1.3.14 (2)), and that if ${}_{A}X$ is flat, then the functor $-\square_{\mathfrak{C}}X$ is left exact. Thus, in view of Proposition 2.1.17, it suffices if we prove that the version of (*ii*) with $-\square_{\mathfrak{C}}X$ left exact implies (*iii*), and this last implies the version of (*ii*) with ${}_{\mathfrak{C}}X$ coflat. Assume that $-\square_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ with $-\square_{\mathfrak{C}}X$ left exact. By Proposition 2.1.12, $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. Being a left adjoint, $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is right exact and, henceforth, exact. By Proposition 1.4.8, $\Lambda_{\mathfrak{C}}$ is an injector and we have proved (*iii*). Conversely, if $\Lambda_{\mathfrak{C}}$ is a quasi-finite injector and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ as bicomodules, then $-\square_{\mathfrak{C}}X \simeq -\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ and, by Proposition 1.4.8, we get that $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is an exact functor. By Corollary 2.1.15, $\mathrm{h}_{\mathfrak{C}}(\Lambda, -) \simeq -\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \simeq -\square_{\mathfrak{C}}X$, and ${}_{\mathfrak{C}}X$ is coflat. □

From the foregoing propositions, it is easy to deduce our characterization of Frobenius functors between categories of comodules over corings.

Theorem 2.1.19. *Suppose that ${}_{A}\mathfrak{C}$ and ${}_{B}\mathfrak{D}$ are flat. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. The following statements are equivalent* *(i)* $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ *is a Frobenius pair;*

(ii) $-\Box_{\mathfrak{C}}X$ *is a Frobenius functor, and* $\mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D}) \simeq \Lambda$ *as bicomodules;*

(iii) there is a Frobenius pair (F, G) *for* $\mathcal{M}^{\mathfrak{C}}$ *and* $\mathcal{M}^{\mathfrak{D}}$ *such that* $F(\mathfrak{C}) \simeq \Lambda$ *and* $G(\mathfrak{D}) \simeq X$ *as bicomodules;*

(iv) $\Lambda_{\mathfrak{C}}, X_{\mathfrak{D}}$ *are quasi-finite injectors, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *as bicomodules;*

(v) $\Lambda_{\mathfrak{C}}, X_{\mathfrak{D}}$ *are quasi-finite, and* $-\Box_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ *and* $-\Box_{\mathfrak{D}}\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, -)$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ This is obvious, after Theorem 2.1.13 and Proposition 2.1.12.

$(i) \Leftrightarrow (iv)$ Follows from Proposition 2.1.18.

$(iv) \Leftrightarrow (v)$ If $X_{\mathfrak{D}}$ and $\Lambda_{\mathfrak{C}}$ are quasi-finite, then ${}_AX$ and ${}_B\Lambda$ are flat. Now, apply Proposition 2.1.18. $\square$

From Proposition 2.1.17 and Proposition 2.1.18 (or Theorem 2.1.19) we get the following

Theorem 2.1.20. *Let* $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. *Suppose that* ${}_A\mathfrak{C}$, $\mathfrak{C}_A$, ${}_B\mathfrak{D}$ *and* $\mathfrak{D}_B$ *are flat. The following statements are equivalent*

1. $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ *is a Frobenius pair, with* $X_{\mathfrak{D}}$ *and* $\Lambda_{\mathfrak{C}}$ *coflat;*
2. $(\Lambda\Box_{\mathfrak{C}}-, X\Box_{\mathfrak{D}}-)$ *is a Frobenius pair, with* ${}_{\mathfrak{C}}X$ *and* ${}_{\mathfrak{D}}\Lambda$ *coflat;*
3. X *and* Λ *are coflat quasi-finite injectors on both sides, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

If moreover $\mathfrak{C}$ *and* $\mathfrak{D}$ *are coseparable (resp.* A *and* B *are von Neumann regular rings), then the following statements are equivalent*

1. $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ *is a Frobenius pair;*
2. $(\Lambda\Box_{\mathfrak{C}}-, X\Box_{\mathfrak{D}}-)$ *is a Frobenius pair;*
3. X *and* Λ *are quasi-finite (resp. quasi-finite injectors) on both sides, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

Remark 2.1.21. In the case of rings (i.e., $\mathfrak{C} = A$ and $\mathfrak{D} = B$, Theorem 2.1.19 and the second part of Theorem 2.1.20 give [26, Theorem 2.1]. To see this, observe that ${}_AX_B$ is quasi-finite as a right B-module if and only if $- \otimes_A X : \mathcal{M}_A \to \mathcal{M}_B$ has a left adjoint, that is, if and only if ${}_AX$ is finitely generated and projective. In such a case, the left adjoint is $- \otimes_B \mathrm{Hom}_A(X, A) : \mathcal{M}_B \to \mathcal{M}_A$. Of course, $-\Box_A X = - \otimes_A X$.

The dual characterization in the framework of coalgebras over fields [26, Theorem 3.3] will be deduced in Section 2.2.

2.2 Frobenius functors between corings with a duality

We will look to Frobenius functors for corings closer to coalgebras over fields, in the sense that the categories of comodules share a fundamental duality.

First we recall a finiteness theorem:

Lemma 2.2.1. [19, 19.12(1)]
Assume that ${}_A\mathfrak{C}$ is locally projective and let $M \in \mathcal{M}^{\mathfrak{C}}$. Then every nonempty finite subset of M is a subset of a subcomodule of M which is finitely generated as a right A-module. In particular, minimal ${}^*\mathfrak{C}$-submodules are finitely generated as right A-modules.

Proposition 2.2.2. *Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is flat.*

(1) A comodule $M \in \mathcal{M}^{\mathfrak{C}}$ is finitely generated if and only if M_A is finitely generated.

(2) A comodule $M \in \mathcal{M}^{\mathfrak{C}}$ is finitely presented if M_A is finitely presented. The converse is true whenever $\mathcal{M}^{\mathfrak{C}}$ is locally finitely generated.

(3) If ${}_A\mathfrak{C}$ is locally projective, then $\mathcal{M}^{\mathfrak{C}}$ is locally finitely generated.

Proof. (1) From the definition of a finitely generated object, every comodule that is finitely generated as right A-module is a finitely generated comodule. The forgetful functor $U : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ has a left adjoint $-\otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$ which is exact and preserves coproducts. Thus, from Lemma 1.1.56, U preserves finitely generated objects.

(2) If $M \in \mathcal{M}^{\mathfrak{C}}$ is such that M_A is finitely presented, then for every exact sequence $0 \to K \to L \to M \to 0$ in $\mathcal{M}^{\mathfrak{C}}$ with L finitely generated, we get an exact sequence $0 \to K_A \to L_A \to M_A \to 0$ with M_A finitely presented. Thus, K_A is finitely generated and, by (1), $K \in \mathcal{M}^{\mathfrak{C}}$ is finitely generated. This proves that M is a finitely presented comodule. The converse follows directly from Lemma 1.1.57 and the fact that the forgetful functor preserves inductive limits.

(3) is a consequence of (1) and Lemma 2.2.1. $\square$

Proposition 2.2.3. *Let $\mathfrak{C}$ be an A-coring.*

(1) If ${}_A\mathfrak{C}$ is flat, A is a right noetherian algebra, and $\mathfrak{C}_A$ is finitely generated module, then the category $\mathcal{M}^{\mathfrak{C}}$ is locally finitely noetherian.

(2) If ${}_A\mathfrak{C}$ is flat and A is a right noetherian algebra, then every finitely generated object of $\mathcal{M}^{\mathfrak{C}}$ is noetherian.

(3) If ${}_A\mathfrak{C}$ is flat and A is a right noetherian algebra, then the category $\mathcal{M}^{\mathfrak{C}}$ is locally finitely noetherian.

Proof. (1) Immediate from Lemma 1.2.16 and the fact that every $M \in \mathcal{M}^{\mathfrak{C}}$ such that M_A is noetherian, is noetherian in $\mathcal{M}^{\mathfrak{C}}$ (by Proposition 1.1.58).

(2) Let $M \in \mathcal{M}^{\mathfrak{C}}$ be a finitely generated comodule. From Proposition 2.2.2, M_A is finitely generated. Then it is noetherian. Hence $M_{\mathfrak{C}}$ is noetherian.

(3) Obvious from Proposition 2.2.2 and (2). $\square$

The notation $\mathbf{C}_f$ stands for the full subcategory of a Grothendieck category $\mathbf{C}$ whose objects are the finitely generated ones. The special case of the following lemma where $\mathbf{C}$ is the category of right modules over a ring R, recovers [85, Exercise 4, p. 109].

Lemma 2.2.4. *Let $\mathbf{C}$ be a locally finitely generated category.*

(1) The category $\mathbf{C}_f$ is additive.

(2) The category $\mathbf{C}_f$ has cokernels, and every monomorphism in $\mathbf{C}_f$ is a monomorphism in $\mathbf{C}$.

(3) The following statements are equivalent:

(a) The category $\mathbf{C}_f$ has kernels;

(b) $\mathbf{C}$ is locally noetherian;

(c) $\mathbf{C}_f$ is abelian;

(d) $\mathbf{C}_f$ is an abelian subcategory of $\mathbf{C}$.

Proof. (1) Obvious from Lemma 1.1.3 and Proposition 1.1.61(1).

(2) That $\mathbf{C}_f$ has cokernels is straightforward from Lemma 1.1.54 (i). Now, let $f : M \to N$ be a monomorphism in $\mathbf{C}_f$ and $\xi : X \to M$ be a morphism in $\mathbf{C}$ such that $f\xi = 0$. Suppose that $X = \bigcup_{i\in I} X_i$, where $X_i \in \mathbf{C}_f$, and $\iota_i : X_i \to X$, $i \in I$, the canonical injections. Then $f\xi\iota_i = 0$, and $\xi\iota_i = 0$, for every i, and by the definition of the inductive limit, $\xi = 0$.

(3) $(b) \Rightarrow (a)$ Straightforward from Proposition 1.1.58.

$(d) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are trivial.

$(a) \Rightarrow (b)$ Let $M \in \mathbf{C}_f$, and K be a subobject of M. Let $\iota : L \to M$ be the kernel of the canonical morphism $f : M \to M/K$ in $\mathbf{C}_f$. Suppose that $K = \bigcup_{i\in I} K_i$, where $K_i \in \mathbf{C}_f$, for every $i \in I$. By the universal property of kernel, there exist a unique morphism $\alpha : L \to K$, and a unique morphism $\beta_i : K_i \to L$, for every $i \in I$, making commutative the diagrams

$$\begin{tikzcd}
& K_i \arrow[dl, "\beta_i"'] \arrow[d] & \\
L \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & M \arrow[r, "f"] & M/K \\
& K \arrow[u] &
\end{tikzcd}.$$

By (2), ι is a monomorphism in $\mathbf{C}$, then for every $K_i \subset K_j$, the diagram

$$\begin{tikzcd}
K_i \arrow[r, "\beta_i"] \arrow[d] & L \\
K_j \arrow[ur, "\beta_j"'] &
\end{tikzcd}$$

commutes. Therefore we have the commutative diagram

$$\begin{tikzcd} & K \arrow[dl, "\varinjlim_I \beta_i"'] \arrow[d] \\ L \arrow[r, "\iota"] & M. \end{tikzcd}$$

Then $K \simeq L$, and hence $K \in \mathbf{C}_f$. Finally, by Proposition 1.1.58, M is noetherian in $\mathbf{C}$.

$(b) \Rightarrow (d)$ Straightforward from Lemma 1.1.3. □

The following generalization of [26, Proposition 3.1] will allow us to give an alternative proof to the equivalence "(1) ⇔ (4)" of Theorem 2.2.14 (see Remark 2.2.15 (1)).

Proposition 2.2.5. *Let $\mathbf{C}$ and $\mathbf{D}$ be two locally noetherian categories. Then*

(1) If $F : \mathbf{C} \to \mathbf{D}$ is a Frobenius functor, then its restriction $F_f : \mathbf{C}_f \to \mathbf{D}_f$ is a Frobenius functor.

(2) If $H : \mathbf{C}_f \to \mathbf{D}_f$ is a Frobenius functor, then H can be uniquely extended to a Frobenius functor $\overline{H} : \mathbf{C} \to \mathbf{D}$.

(3) The assignment $F \mapsto F_f$ defines a bijective correspondence (up to natural isomorphisms) between Frobenius functors from $\mathbf{C}$ to $\mathbf{D}$ and Frobenius functors from $\mathbf{C}_f$ to $\mathbf{D}_f$.

(4) In particular, if $\mathbf{C} = \mathcal{M}^{\mathfrak{C}}$ and $\mathbf{D} = \mathcal{M}^{\mathfrak{D}}$ are locally noetherian such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat, then $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is a Frobenius functor if and only if it preserves direct limits and comodules which are finitely generated as right A-modules, and the restriction functor $F_f : \mathcal{M}^{\mathfrak{C}}_f \to \mathcal{M}^{\mathfrak{D}}_f$ is a Frobenius functor.

Proof. The proofs of [26, Proposition 3.1 and Remark 3.2] remain valid for our situation, but with some minor modifications: to prove that $\overline{H}$ is well-defined, we use Lemma 2.2.4. In the proof of the statements (1), (2) and (3) we use the Grothendieck AB 5) condition. □

In order to generalize [89, Proposition A.2.1] and its proof, we need the following lemma.

Lemma 2.2.6. *(1) Let $\mathbf{C}$ be a locally noetherian category, let $\mathbf{D}$ be an arbitrary Grothendieck category, $F : \mathbf{C} \to \mathbf{D}$ be an arbitrary functor which preserves direct limits, and $F_f : \mathbf{C}_f \to \mathbf{D}$ be its restriction to $\mathbf{C}_f$. Then F is exact (faithfully exact, resp. left, right exact) if and only if F_f is exact (faithfully exact, resp. left, right exact).*

In particular, an object M in $\mathbf{C}_f$ is projective (resp. projective generator) if and only if it is projective (resp. projective generator) in $\mathbf{C}$.

(2) Let $\mathbf{C}$ be a locally noetherian category. For every object M of $\mathbf{C}$, the following conditions are equivalent

(a) M is injective (resp. an injective cogenerator);

(b) *the contravariant functor* $\mathrm{Hom}_{\mathbf{C}}(-,M):\mathbf{C}\to\mathbf{Ab}$ *is exact (resp. faithfully exact);*

(c) *the contravariant functor* $\mathrm{Hom}_{\mathbf{C}}(-,M)_f : \mathbf{C}_f \to \mathbf{Ab}$ *is exact (resp. faithfully exact).*

In particular, an object M in $\mathbf{C}_f$ is injective (resp. injective cogenerator) if and only if it is injective (resp. injective cogenerator) in $\mathbf{C}$.

Proof. (1) The "only if" part is straightforward from the fact that the injection functor $\mathbf{C}_f \to \mathbf{C}$ is faithfully exact.

For the "if" part, suppose that F_f is left exact. Let $f: M\to N$ be a morphism in $\mathbf{C}$. Put $M=\bigcup_{i\in I} M_i$ and $N=\bigcup_{j\in J} N_j$, as direct union of directed families of finitely generated subobjects. For $(i,j)\in I\times J$, let $M_{i,j}=M_i\cap f^{-1}(N_j)$, and $f_{i,j}:M_{i,j}\to N_j$ be the restriction of f to $M_{i,j}$. We have $f=\varinjlim_{I\times J} f_{i,j}$ and then $F(f)=\varinjlim_{I\times J} F_f(f_{i,j})$. Hence

$\ker F(f) = \ker \varinjlim_{I\times J} F_f(f_{i,j}) = \varinjlim_{I\times J} \ker F_f(f_{i,j}) = \varinjlim_{I\times J} F_f(\ker f_{i,j}) = \varinjlim_{I\times J} F(\ker f_{i,j})$ (by Lemma 2.2.4) $= F(\varinjlim_{I\times J} \ker f_{i,j}) = F(\ker f)$.

Finally F is left exact. Analogously, it can be proved that F_f is right exact implies that F is also right exact. Now, suppose that F_f is faithfully exact. We have already proved that F is exact. It remains to prove that F is faithful. For this, let $0\neq M=\bigcup_{i\in I} M_i$ be an object of $\mathbf{C}$, where M_i is finitely generated for every $i\in I$. We have

$$F(M)=\varinjlim_{I} F_f(M_i)\simeq \sum_i F_f(M_i)$$

(since F is exact). Since $M\neq 0$, there exists some $i_0\in I$ such that $M_{i_0}\neq 0$. By Proposition 1.1.12, $F_f(M_{i_0})\neq 0$, hence $F(M)\neq 0$. Also by Proposition 1.1.12, F is faithful.

(2) $(a)\Leftrightarrow(b)$ Obvious.

$(b)\Rightarrow(c)$ Analogous to that of the "only if" part of (1).

$(c)\Rightarrow(a)$ That M is injective is a consequence of Propositions 1.1.21, 1.1.58. Now, suppose moreover that $\mathrm{Hom}_{\mathbf{C}}(-,M)_f$ is faithful. Let L be a nonzero object of $\mathbf{C}$, and K be a nonzero finitely generated subobject of L. By Proposition 1.1.12, there exists a nonzero morphism $K\to M$. Since M is injective, there exists a nonzero morphism $L\to M$ making commutative the following diagram

$$\begin{tikzcd} & M & \\ 0 \arrow[r] & K \arrow[u] \arrow[r] & L. \arrow[ul] \end{tikzcd}$$

From the dual of Proposition 1.1.20, it follows that M is a cogenerator. ☐

Lemma 2.2.7. [19, 19.19 (1)]
Let M be a right $\mathfrak{C}$-module. If

(a) M_A is finitely generated and projective, or

(b) $\mathfrak{C}_A$ is flat and M_A is finitely presented, or

(c) $\mathfrak{C}_A$ is finitely generated and projective,

then the dual module $M^* = \mathrm{Hom}_A(M, A)$ is a left $\mathfrak{C}$-module via the structure map

$$\lambda_{M^*} : M^* \to \mathrm{Hom}_A(M, \mathfrak{C}) \simeq \mathfrak{C} \otimes_A M^*, \quad f \mapsto (f \otimes_A \mathfrak{C}) \circ \rho_M.$$

(See Lemma 1.1.72(6).)

If $\mathfrak{C}_A$ is flat and $M \in \mathcal{M}^{\mathfrak{C}}$ is finitely presented as a right A-module, then, by Lemma 2.2.7, the dual A-module M^* has a left $\mathfrak{C}$-comodule structure. Now, if ${}_AM^*$ turns out to be finitely presented and ${}_A\mathfrak{C}$ is flat, then ${}^*(M^*) = \mathrm{Hom}_A(M^*, A)$ is a right $\mathfrak{C}$-comodule and the canonical map $\sigma_M : M \to {}^*(M^*)$ is a homomorphism in $\mathcal{M}^{\mathfrak{C}}$. This construction leads to a duality

$$(-)^* : \mathcal{M}_0^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{C}}\mathcal{M}_0 : {}^*(-)$$

between the full subcategories $\mathcal{M}_0^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}_0$ of $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}$ whose objects are the comodules which are finitely generated and projective over A on the corresponding side (this holds even without flatness assumptions of $\mathfrak{C}$). Call it the *basic duality* (details may be found in [21]). Of course, in the case that A is semisimple (e.g. for coalgebras over fields) these categories are that of finitely generated comodules, and this basic duality plays a remarkable role in the study of several notions in the coalgebra setting (e.g. Morita equivalence [90], semiperfect coalgebras [63], Morita duality [45], [46], or Frobenius functors [26]). It would be interesting to know, in the coring setting, to what extent the basic duality can be extended to the subcategories $\mathcal{M}_f^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}_f$, since, as we will try to show in this section, this allows to obtain better results. Of course, this is the underlying idea when the ground ring A is assumed to be quasi-Frobenius (see [36]).

Consider contravariant functors between Grothendieck categories $H : \mathbf{A} \leftrightarrows \mathbf{A}' : H'$, together with natural transformations $\tau : 1_{\mathbf{A}} \to H' \circ H$ and $\tau' : 1_{\mathbf{A}'} \to H \circ H'$, satisfying the condition $H(\tau_X) \circ \tau'_{H(X)} = 1_{H(X)}$ and $H'(\tau'_{X'}) \circ \tau_{H'(X')} = 1_{H'(X')}$ for $X \in \mathbf{A}$ and $X' \in \mathbf{A}'$. Following [28], this situation is called a *right adjoint pair.*

Proposition 2.2.8. *Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are flat. Assume that $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}$ are locally noetherian categories. If ${}_AM^*$ and ${}^*N_A$ are finitely generated modules for every $M \in \mathcal{M}_f^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}_f$, then the basic duality extends to a right adjoint pair $(-)^* : \mathcal{M}_f^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{C}}\mathcal{M}_f : {}^*(-)$.*

Proof. If $M \in \mathcal{M}_f^{\mathfrak{C}}$ then, since $\mathcal{M}^{\mathfrak{C}}$ is locally noetherian, $M_{\mathfrak{C}}$ is finitely presented. By Proposition 2.2.2, M_A is finitely presented and the left $\mathfrak{C}$-comodule M^* makes sense. Now, the assumption ${}_AM^*$ finitely generated implies, by Lemma 2.2.2, that $M^* \in {}^{\mathfrak{C}}\mathcal{M}_f$. We have then the functor $(-)^* : \mathcal{M}_f^{\mathfrak{C}} \to {}^{\mathfrak{C}}\mathcal{M}_f$. The functor ${}^*(-)$ is analogously defined, and the rest of the proof consists of straightforward verifications. □

Example 2.2.9. The hypotheses are fulfilled if ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are locally projective and A is left and right noetherian. But there are situations in which no finiteness condition need to be required to A: this is the case, for instance, of cosemisimple corings (see Proposition 1.2.23). In particular, if an arbitrary ring A contains a division ring B, then, by Proposition 1.2.23, the canonical coring $A \otimes_B A$ satisfies all hypotheses in Proposition 2.2.8.

Definition 2.2.10. Let $\mathfrak{C}$ be a coring over A satisfying the assumptions of Proposition 2.2.8. We will say that $\mathfrak{C}$ *has a duality* if the basic duality extends to a duality

$$(-)^* : \mathcal{M}_f^{\mathfrak{C}} \leftrightarrows {}^{\mathfrak{C}}\mathcal{M}_f : {}^*(-).$$

We have the following examples of a coring which has a duality:

(i) $\mathfrak{C}$ is a coring over a QF ring A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are flat (and hence projective);

(ii) $\mathfrak{C}$ is a cosemisimple coring, where $\mathcal{M}_f^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}_f$ are equal to $\mathcal{M}_0^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}_0$, respectively;

(iii) $\mathfrak{C}$ is a coring over A such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are flat and semisimple, $\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{C}}\mathcal{M}$ are locally noetherian categories, and the dual of every simple right (resp. left) A-module in the decomposition of $\mathfrak{C}_A$ (resp. ${}_A\mathfrak{C}$) as a direct sum of simple A-modules is finitely generated and A-reflexive (in fact, every right (resp. left) $\mathfrak{C}$-comodule M becomes a submodule of the semisimple right (resp. left) A-module $M \otimes_A \mathfrak{C}$, and hence M_A (resp. ${}_AM$) is also semisimple).

Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is flat, and $M \in \mathcal{M}^{\mathfrak{C}}$. A comodule $Q \in \mathcal{M}^{\mathfrak{C}}$ is said to be *M-injective* if for every subcomodule M' of M, and every morphism in $\mathcal{M}^{\mathfrak{C}}$, $f : M' \to Q$, there is $g : M \to Q$ such that the diagram

$$\begin{tikzcd}
 & & Q \\
0 \arrow[r] & M' \arrow[r] \arrow[ur, "f"] & M \arrow[u, dashed, "g"']
\end{tikzcd}$$

commutes. Obviously, $Q \in \mathcal{M}^{\mathfrak{C}}$ is injective if and only if it is M-injective for all $M \in \mathcal{M}^{\mathfrak{C}}$.

The following is a generalization of [89, Proposition A.2.1] and $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$ of [31, Theorem 2.4.17].

Proposition 2.2.11. *Suppose that the coring $\mathfrak{C}$ has a duality. Let $Q \in \mathcal{M}^{\mathfrak{C}}$ such that Q_A is flat. The following are equivalent*

(1) Q is coflat (resp. faithfully coflat);

(2) Q is M-injective for all finitely generated right $\mathfrak{C}$-comodule M (resp. the last statement hold and moreover $0 \neq M \in \mathcal{M}^{\mathfrak{C}}$ is finitely generated implies $\mathrm{Hom}_{\mathfrak{C}}(M, Q) \neq 0$;

(3) M is injective (resp. an injective cogenerator).

Proof. Let $Q \in \mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}_f$. We have the following commutative diagram (in $\mathcal{M}_k$)

$$\begin{tikzcd}
0 \arrow[r] & Q\square_{\mathfrak{C}}N \arrow[r] & Q \otimes_A N \arrow[r, "\rho_Q\otimes_A N - Q\otimes_A\lambda_N"] \arrow[d] & Q \otimes_A \mathfrak{C} \otimes_A N \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_{\mathfrak{C}}(N^*, Q) \arrow[r] & \mathrm{Hom}_A(N^*, Q) \arrow[r, "f\mapsto\rho_Q f-(f\otimes_A\mathfrak{C})\rho_{N^*}"] & \mathrm{Hom}_A(N^*, Q \otimes_A \mathfrak{C}),
\end{tikzcd}$$

where the vertical maps are the canonical maps. By the universal property of kernel, there is a unique morphism $\eta_{Q,N} : Q\square_{\mathfrak{C}}N \to \mathrm{Hom}_{\mathfrak{C}}(N^*, Q)$ making commutative the above diagram. By Lemma 1.1.4, η is a natural transformation of bifunctors. For example we will verify that η is natural in N. For this let $f : N_1 \to N_2$ be a morphism in ${}^{\mathfrak{C}}\mathcal{M}_f$. It is enough to consider the diagram

$$\begin{tikzcd}
& Q\square_{\mathfrak{C}}N_1 \arrow[rr, "\eta_{Q,N_1}"] \arrow[dl, "Q\square_{\mathfrak{C}}f"'] \arrow[dd] & & \mathrm{Hom}_{\mathfrak{C}}(N_1^*, Q) \arrow[dl, "\mathrm{Hom}_{\mathfrak{C}}(f^*,Q)"] \arrow[dd] \\
Q\square_{\mathfrak{C}}N_2 \arrow[rr, "\eta_{Q,N_2}"] \arrow[dd] & & \mathrm{Hom}_{\mathfrak{C}}(N_2^*, Q) \arrow[dd] & \\
& Q \otimes_A N_1 \arrow[rr] \arrow[dl, "Q\otimes_A f"'] & & \mathrm{Hom}_A(N_1^*, Q) \arrow[dl, "\mathrm{Hom}_A(f^*,Q)"] \\
Q \otimes_A N_2 \arrow[rr] & & \mathrm{Hom}_A(N_2^*, Q). &
\end{tikzcd}$$

If Q_A is flat then $\eta_{Q,N}$ is an isomorphism for every $N \in {}^{\mathfrak{C}}\mathcal{M}_f$. We have

$$Q\square_{\mathfrak{C}}- \simeq \mathrm{Hom}_{\mathfrak{C}}(-, Q)_f \circ (-)^* : {}^{\mathfrak{C}}\mathcal{M}_f \to \mathcal{M}_k.$$

Then, by Lemma 2.2.6, $Q_{\mathfrak{C}}$ is coflat (resp. faithfully coflat) iff $Q\square_{\mathfrak{C}}- : {}^{\mathfrak{C}}\mathcal{M}_f \to \mathcal{M}_k$ is exact (resp. faithfully exact) iff $\mathrm{Hom}_{\mathfrak{C}}(-, Q)_f : \mathcal{M}_f^{\mathfrak{C}} \to \mathcal{M}_k$ is exact (resp. faithfully exact) iff $Q_{\mathfrak{C}}$ is injective (resp. an injective cogenerator). $\square$

The particular case of the second part of the following result for coalgebras over a commutative ring is given in [4].

Corollary 2.2.12. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule. Suppose that A is a QF ring.*

(a) If N is an injector as a right $\mathfrak{D}$-comodule then N is injective in $\mathcal{M}^{\mathfrak{D}}$.

(b) If $\mathfrak{D}$ has a duality, N_B is flat and N is injective in $\mathcal{M}^{\mathfrak{D}}$, then N is an injector as a right $\mathfrak{D}$-comodule.

Proof. (a) Since A is a QF ring, then A_A is injective. Hence $N_{\mathfrak{D}} \simeq (A \otimes_A N)_{\mathfrak{D}}$ is injective.

(b) Let X_A be an injective module. Since A is a QF ring, X_A is projective. We have then the natural isomorphism

$$(X \otimes_A N)\square_{\mathfrak{D}}- \simeq X \otimes_A (N\square_{\mathfrak{D}}-) : {}^{\mathfrak{D}}\mathcal{M} \to \mathcal{M}_k.$$

By Proposition 2.2.11, $N_{\mathfrak{D}}$ is coflat, and then $X \otimes_A N$ is coflat. Now, since $X \otimes_A N$ is a flat right B-module, and by Proposition 2.2.11, $X \otimes_A N$ is injective in $\mathcal{M}^{\mathfrak{D}}$. □

The last two results allow to improve our general statements in Section 2.1 for corings having a duality.

Proposition 2.2.13. *Suppose that $\mathfrak{C}$ and $\mathfrak{D}$ have a duality. Consider the following statements*

1. $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ *is an adjoint pair of functors, with ${}_AX$ and Λ_A flat;*

2. Λ *is quasi-finite injective as a right $\mathfrak{C}$-comodule, with ${}_AX$ and Λ_A flat and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$.*

We have (1) implies (2), and the converse is true if in particular B is a QF ring.

Proof. (1) ⇒ (2) From Proposition 2.2.11, $\mathfrak{D}$ is injective in $\mathcal{M}^{\mathfrak{D}}$. Since the functor $-\square_{\mathfrak{C}}X$ is exact, $\Lambda \simeq \mathfrak{D}\square_{\mathfrak{D}}\Lambda$ is injective in $\mathcal{M}^{\mathfrak{C}}$.

(2) ⇒ (1) Assume that B is a QF ring. From Corollary 2.2.12, Λ is quasi-finite injector as a right $\mathfrak{C}$-comodule, and Proposition 2.1.18 achieves the proof. □

We are now in a position to state and prove our main result of this section.

Theorem 2.2.14. *Suppose that $\mathfrak{C}$ and $\mathfrak{D}$ have a duality. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. The following statements are equivalent*

1. $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ *is a Frobenius pair, with X_B and Λ_A flat;*

2. $(\Lambda\square_{\mathfrak{C}}-, X\square_{\mathfrak{D}}-)$ *is a Frobenius pair, with ${}_AX$ and ${}_B\Lambda$ flat;*

3. X *and Λ are quasi-finite injector on both sides, and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ in ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.*

   In particular, if A and B are QF rings, then the above statements are equivalent to

4. X *and Λ are quasi-finite injective on both sides, and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ in ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.*

Finally, suppose that $\mathfrak{C}$ and $\mathfrak{D}$ are cosemisimple corings. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. The following statements are equivalent

1. $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ *is a Frobenius pair;*

2. $(\Lambda\square_{\mathfrak{C}}-, X\square_{\mathfrak{D}}-)$ *is a Frobenius pair;*

3. X *and* Λ *are quasi-finite on both sides, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

Proof. We start by proving the first part. In view of Theorem 2.1.20 and Theorem 2.1.19 it suffices to show that if $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ is a Frobenius pair, the condition "$X_{\mathfrak{D}}$ and $\Lambda_{\mathfrak{C}}$ are coflat" is equivalent to "X_B and Λ_A are flat". Indeed, the first implication is obvious, for the converse, assume that X_B and Λ_A are flat. By Proposition 2.2.13, X and Λ are injective in $\mathcal{M}^{\mathfrak{D}}$ and $\mathcal{M}^{\mathfrak{C}}$ respectively, and they are coflat by Proposition 2.2.11. The particular case is straightforward from Corollary 2.2.12 and the above equivalences.

Now we will show the second part. We know that every comodule category over a cosemisimple coring is a spectral category (see Proposition 1.2.23). From Proposition 2.2.11, the bicomodules ${}_{\mathfrak{C}}X_{\mathfrak{D}}$ and ${}_{\mathfrak{D}}\Lambda_{\mathfrak{C}}$ are coflat and injector on both sides (we can see this directly by using Corollary 1.1.51). Finally, Theorem 2.1.20 achieves the proof. □

Remarks 2.2.15. (1) The equivalence "(1) ⇔ (4)" of the last theorem is a generalization of [26, Theorem 3.3]. The proof of [26, Theorem 3.3] gives an alternative proof of "(1) ⇔ (4)" of Theorem 2.2.14, using Proposition 2.2.5.

(2) The adjunction of Proposition 2.1.17 and Proposition 2.2.13 generalizes the coalgebra version of Morita's theorem [20, Theorem 4.2].

Example 2.2.16. Let A be a k-algebra. Put $\mathfrak{C} = A$ and $\mathfrak{D} = k$. The bicomodule $A \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is quasi-finite as a right $\mathfrak{D}$-comodule. A is an injector as a right $\mathfrak{D}$-comodule if and only if the k-module A is flat. If we take $A = k = \mathbb{Z}$, the bicomodule A is quasi-finite and injector as a right $\mathfrak{D}$-comodule but it is not injective in $\mathcal{M}^{\mathfrak{D}}$. Hence, the assertion "$(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ is an adjoint pair of functors" does not imply in general the assertion "Λ is quasi-finite injective as a right $\mathfrak{C}$-comodule and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$", and the following statements are not equivalent in general:

(1) $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ is a Frobenius pair;

(2) X and Λ are quasi-finite injective on both sides, and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ in ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

On the other hand, there exists a commutative self-injective ring which is not coherent (see Section 1.1). By a theorem of S.U. Chase (see for example [5, Theorem 19.20]), there exists then a k-algebra A which is injective, but not flat as k-module. Hence, the bicomodule $A \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is quasi-finite and injective as a right $\mathfrak{D}$-comodule, but not an injector as a right $\mathfrak{D}$-comodule.

2.3 Applications to induction functors

We will characterize when the induction functor $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ defined in Subsection 1.5.1 is a Frobenius functor.

Theorem 2.3.1. *Let* $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ *be a homomorphism of corings such that* ${}_A\mathfrak{C}$ *and* ${}_B\mathfrak{D}$ *are flat. The following statements are equivalent* *(a)* $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *is a Frobenius functor;*

(b) the $\mathfrak{C}-\mathfrak{D}$*-bicomodule* $\mathfrak{C}\otimes_A B$ *is quasi-finite and injector as a right* $\mathfrak{D}$*-comodule and there exists an isomorphism of* $\mathfrak{D}-\mathfrak{C}$*-bicomodules* $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C}\otimes_A B, \mathfrak{D}) \simeq B\otimes_A \mathfrak{C}$.

Moreover, if $\mathfrak{C}$ *and* $\mathfrak{D}$ *are coseparable, then the condition "injector" in (b) can be deleted.*

Proof. First observe that $-\otimes_A B$ is a Frobenius functor if and only if $(-\otimes_A B, -\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))$ is a Frobenius pair (by Proposition 1.5.4). Let $M \in \mathcal{M}^{\mathfrak{C}}$. The map $\rho_M \otimes_A B : M\otimes_A B \to M\otimes_A \mathfrak{C}\otimes_A B$ is a morphism of $\mathfrak{D}$-comodules. As in the proof of Lemma 1.3.12, and using Proposition 1.3.2 (${}_B\mathfrak{D}$ is flat), we have a commutative diagram in $\mathcal{M}^{\mathfrak{D}}$ with exact row

$$\begin{tikzcd}
0 \arrow[r] & M\square_{\mathfrak{C}}(\mathfrak{C}\otimes_A B) \arrow[r, "\iota"] & M\otimes_A \mathfrak{C}\otimes_A B \arrow[r, "\omega_{M,\mathfrak{C}\otimes_A B}"] & M\otimes_A \mathfrak{C}\otimes_A \mathfrak{C}\otimes_A B\,, \\
& M\otimes_A B \arrow[u, "\psi_M"] \arrow[ur, "\rho_M\otimes_A B"'] & &
\end{tikzcd}$$

where ψ_M is the isomorphism defined by the universal property of kernel. Now, we will verify that ψ is a natural isomorphism. For this, let $f : M \to M'$ be a morphism in $\mathcal{M}^{\mathfrak{C}}$. Consider the diagram

$$\begin{tikzcd}
& M\otimes_A B \arrow[rr, "\psi_M"] \arrow[dl, "f\otimes_A B"] \arrow[ddrr, "\rho_M\otimes_A B"] & & M\square_{\mathfrak{C}}(\mathfrak{C}\otimes_A B) \arrow[dl, "f\square_{\mathfrak{C}}(\mathfrak{C}\otimes_A B)"] \arrow[dd] \\
M'\otimes_A B \arrow[rr, "\psi_{M'}"] \arrow[ddrr, "\rho_{M'}\otimes_A B"] & & M'\square_{\mathfrak{C}}(\mathfrak{C}\otimes_A B) \arrow[dd] & \\
& & & M\otimes_A \mathfrak{C}\otimes_A B \arrow[dl, "f\otimes_A\mathfrak{C}\otimes_A B"] \\
& & M'\otimes_A \mathfrak{C}\otimes_A B. &
\end{tikzcd}$$

All sides, except perhaps the top one, are commutative, then the top side is so. The equivalence between (a) and (b) is then obvious from Propositions 2.1.18, 2.1.17. $\square$

Let us consider the particular case where $\mathfrak{C} = A$ and $\mathfrak{D} = B$ are the trivial corings (which are separable). Then Theorem 2.3.1 gives functorial Morita's characterization of Frobenius ring extensions given in [71, Theorem 5.1]: In the case $\mathfrak{C} = A$, $\mathfrak{D} = B$ we have that (by Example 1.4.2), $A\otimes_A B \simeq B$ is quasi-finite as a right B-comodule if and only if ${}_AB$ is finitely generated an projective, and, in this case, $\mathrm{h}_B(B,-) \simeq -\otimes_B \mathrm{Hom}_A({}_AB, A)$.

Theorem 2.3.3 generalizes the characterization of Frobenius extension of coalgebras over fields [26, Theorem 3.5]. It is then reasonable to give the following definition.

Definition 2.3.2. Let $(\varphi,\rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings. It is said to be a *right Frobenius* morphism of corings if $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is a Frobenius functor.

Theorem 2.3.3. *Suppose that A and B are QF rings. Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that the modules ${}_A\mathfrak{C}$, $\mathfrak{C}_A$ and ${}_B\mathfrak{D}$ are projective. Then the following statements are equivalent*

(a) $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *is a Frobenius functor;*

(b) the $\mathfrak{C}$*–*$\mathfrak{D}$*-bicomodule* $\mathfrak{C}\otimes_A B$ *is quasi-finite as a right* $\mathfrak{D}$*-comodule,* $(\mathfrak{C}\otimes_A B)_{\mathfrak{D}}$ *is injective and there exists an isomorphism of* $\mathfrak{D} - \mathfrak{C}$*-bicomodules* $\mathrm{h}_{\mathfrak{D}}(\mathfrak{C} \otimes_A B, \mathfrak{D}) \simeq B \otimes_A \mathfrak{C}$.

Proof. Obvious from Theorem 2.3.1 and Corollary 2.2.12. □

Lemma 2.3.4. *Let $\mathfrak{C}$ be an A-coring. If $- \otimes_A \mathfrak{C}$ is a left adjoint functor to the forgetful functor, then ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are finitely generated and projective.*

Proof. Suppose that the forgetful functor $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a Frobenius functor. Then the functor $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}_A$ is also a Frobenius functor (since it is a composition of two Frobenius functors) and ${}_A\mathfrak{C}$ is finitely generated projective. On the other hand, since $- \otimes_A \mathfrak{C} : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{C}}$ is a left adjoint to $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$. Then $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}, -)$ is a Frobenius functor. Therefore, $\mathfrak{C}$ is finitely generated projective in $\mathcal{M}^{\mathfrak{C}}$, and hence in $\mathcal{M}_A$. □

Lemma 2.3.5. [15, Lemma 4.3]
Let $\mathfrak{C}$ be an A-coring and let R the opposite algebra of $\mathfrak{C}^*$. If ${}_A\mathfrak{C}$ is finitely generated and projective, then the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}_R$ are isomorphic to each other. Indeed, if $M \in \mathcal{M}^{\mathfrak{C}}$ then M is a right R by $m.r = \sum m_{(0)}r(m_{(1)})$. This yields an isomorphism of categories $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$.

Lemma 2.3.6. *Let R be the opposite algebra of ${}^*\mathfrak{C}$.*

(1) $\mathfrak{C} \in {}^{\mathfrak{C}}\mathcal{M}^A$ *is quasi-finite (resp. quasi-finite and injector) as a right A-comodule if and only if ${}_A\mathfrak{C}$ is finitely generated projective (resp. ${}_A\mathfrak{C}$ is finitely generated projective and ${}_AR$ is flat). Let $\mathrm{h}_A(\mathfrak{C}, -) = - \otimes_A R : \mathcal{M}^A \to \mathcal{M}^{\mathfrak{C}}$ be the cohom functor.*

(2) If ${}_A\mathfrak{C}$ is finitely generated projective and ${}_AR$ is flat, then

$$ {}_A\mathrm{h}_A(\mathfrak{C}, A)_{\mathfrak{C}} \simeq {}_AR_{\mathfrak{C}}, $$

where the right $\mathfrak{C}$-comodule structure of R is defined as in Lemma 2.3.5

Proof. (1) Straightforward.

(2) From Lemma 2.3.5, the forgetful functor $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is the composition of functors $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R \to \mathcal{M}_A$. By Proposition 1.4.6, $\mathrm{h}_A(\mathfrak{C}, -)$ is a left adjoint to $-\Box_{\mathfrak{C}}\mathfrak{C} : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ which is isomorphic to the forgetful functor $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$. Then $\mathrm{h}_A(\mathfrak{C}, -)$ is isomorphic to the composition of functors

$$ \mathcal{M}_A \xrightarrow{-\otimes_A R} \mathcal{M}_R \longrightarrow \mathcal{M}^{\mathfrak{C}}. $$

In particular, ${}_A\mathrm{h}_A(\mathfrak{C}, A)_{\mathfrak{C}} \simeq {}_A(A \otimes_A R)_{\mathfrak{C}} \simeq {}_AR_{\mathfrak{C}}$. □

Corollary 2.3.7. [19, 27.10]
Let $\mathfrak{C}$ be an A-coring and let R be the opposite algebra of ${}^*\mathfrak{C}$. Then the following statements are equivalent

(a) The forgetful functor $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a Frobenius functor;

(b) ${}_A\mathfrak{C}$ is finitely generated projective and $\mathfrak{C} \simeq R$ as (A, R)-bimodules, such that $\mathfrak{C}$ is a right R-module by $c.r = c_{(1)}.r(c_{(2)})$, for every $c \in \mathfrak{C}$ and $r \in R$.

Proof. Straightforward from Theorem 2.3.1 and Lemma 2.3.6. □

The following proposition gives sufficient conditions to have that a morphism of corings is right Frobenius if and only if it is left Frobenius. Note that it says in particular that the notion of Frobenius homomorphism of coalgebras over fields (by (b)) or of rings (by (d)) is independent on the side. Of course, the latter is well known.

Proposition 2.3.8. *Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat. Assume that at least one of the following holds*

(a) $\mathfrak{C}$ and $\mathfrak{D}$ have a duality, and ${}_AB$ and B_A are flat;

(b) A and B are von Neumann regular rings;

(c) $B \otimes_A \mathfrak{C}$ is coflat in ${}^{\mathfrak{D}}\mathcal{M}$ and $\mathfrak{C} \otimes_A B$ is coflat in $\mathcal{M}^{\mathfrak{D}}$ and ${}_AB$ and B_A are flat;

(d) $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable corings.

Then the following statements are equivalent

1. *$- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is a Frobenius functor;*
2. *$B \otimes_A - : {}^{\mathfrak{C}}\mathcal{M} \to {}^{\mathfrak{D}}\mathcal{M}$ is a Frobenius functor.*

Proof. Obvious from Theorem 2.1.20 and Theorem 2.2.14. □

Let us finally show how to derive from our results a remarkable characterization of the so called *Frobenius corings.*

Corollary 2.3.9. [19, 27.8]
The following statements are equivalent

(a) the forgetful functor $\mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is a Frobenius functor;

(b) the forgetful functor ${}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is a Frobenius functor;

(c) there exist an (A, A)-bimodule map $\eta : A \to \mathfrak{C}$ and a $(\mathfrak{C}, \mathfrak{C})$-bicomodule map $\pi : \mathfrak{C} \otimes_A \mathfrak{C} \to \mathfrak{C}$ such that $\pi(\mathfrak{C} \otimes_A \eta) = \mathfrak{C} = \pi(\eta \otimes_A \mathfrak{C})$.

Proof. The proof of "(1) ⇔ (4)" in Proposition 2.1.17 for $X = \mathfrak{C} \in {}^A\mathcal{M}^{\mathfrak{C}}$ and $\Lambda = \mathfrak{C} \in {}^{\mathfrak{C}}\mathcal{M}^A$ remains valid for our situation. Finally, notice that the condition (4) in this case is exactly the condition (c). □

In such a case we say that $\mathfrak{C}$ is a *Frobenius coring.*

2.4 Applications to entwined modules

In this section we particularize some of our results in Section 2.3 to the category of entwined modules. We start with some useful results.

Proposition 2.4.1. *Consider a right-right entwining structure $(A, C, \psi) \in \mathbb{E}^{\bullet}_{\bullet}(k)$ and a left-left entwining structure $(B, D, \varphi) \in {}^{\bullet}_{\bullet}\mathbb{E}(k)$. The category of two-sided entwined modules ${}^{D}_{B}\mathcal{M}(\varphi, \psi)^{C}_{A}$ (defined in Subsection 1.5.2) is isomorphic to the category of bicomodules ${}^{D\otimes B}\mathcal{M}^{A\otimes C}$ over the associated corings.*

Proof. Let $\mathfrak{C} = A \otimes C$ and $\mathfrak{D} = D \otimes B$ the associated A-coring and B-coring respectively (see Theorems 1.5.7 and 1.5.8). Let $M \in {}^{D}_{B}\mathcal{M}(\varphi, \psi)^{C}_{A}$ with $\rho_M : M \to M \otimes C$ and $\lambda_M : M \to D \otimes M$ as coaction maps and $\psi^r_M : M \otimes A \to M$ and $\psi^l_M : B \otimes M \to M$ as action maps. Let $(M, \rho'_M) \in \mathcal{M}^{\mathfrak{C}}$ and $(M, \lambda'_M) \in {}^{\mathfrak{D}}\mathcal{M}$ the associated right $\mathfrak{C}$-comodule and left $\mathfrak{D}$-comodule respectively (see Theorems 1.5.7 and 1.5.8).

First, we have, ρ'_M is left B-linear if and only if ψ^l_M is right C-colinear, and λ'_M is right A-linear if and only if ψ^r_M is left D-colinear.

Now, consider the diagram

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "\lambda_M"'] \arrow[dr, phantom, "\mathbf{(1)}"] & M \otimes C \arrow[r, "\simeq"] \arrow[d, "\lambda_M \otimes C"] \arrow[dr, phantom, "\mathbf{(2)}"] & M \otimes_A (A \otimes C) \arrow[d, "\lambda_M \otimes_A (A\otimes C)"] \\
D \otimes M \arrow[r, "D\otimes \rho_M"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathbf{(3)}"] & D \otimes M \otimes C \arrow[r, "\simeq"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\mathbf{(4)}"] & (D \otimes M) \otimes_A (A \otimes C) \arrow[d, "\simeq"] \\
(D \otimes B) \otimes_B M \arrow[r, "(D\otimes B)\otimes_B \rho_M"] & (D \otimes B) \otimes_B (M \otimes C) \arrow[r, "\simeq"] & (D \otimes B) \otimes_B M \otimes_A (A \otimes C).
\end{tikzcd}$$

The diagrams (2), (3) and (4) are commutative. Hence, the greatest rectangle is commutative if and only if the rectangle (1) is commutative. Finally, from Theorems 1.5.7 and 1.5.8, ${}^{D}_{B}\mathcal{M}(\varphi, \psi)^{C}_{A} \simeq {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. □

Proposition 2.4.2. [24, Proposition 34]
If (A, C, ψ) belongs to $\mathbb{E}^{\bullet}_{\bullet}(k)$ and is such that ψ is an isomorphism, then ψ is an isomorphism of corings.

Lemma 2.4.3. *Let $\varphi : \mathfrak{C} \to \mathfrak{C}'$ be an isomorphism of A-corings and $\mathfrak{D}$ a B-coring. Then there is a canonical isomorphism of categories*

$$F : {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}} \to {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{D}}.$$

Let $M \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. Set $F(M) = M$ as $(A, \mathfrak{D})$-bicomodules. The left $\mathfrak{C}'$-coaction of $F(M)$ is

$$M \xrightarrow{\lambda_M} \mathfrak{C} \otimes_A M \xrightarrow{\varphi \otimes_A M} \mathfrak{C}' \otimes_A M.$$

Proof. Straightforward. □

Corollary 2.4.4. *If (A, C, ψ) and (A', C', ψ') belong to $\mathbb{E}^{\bullet}_{\bullet}(k)$ and are such that ψ is an isomorphism. Then*

(a) We have

$$^{A\otimes C}\mathcal{M}^{A'\otimes C'} \simeq {}^{C}_{A}\mathcal{M}(\psi^{-1}, \psi')^{C'}_{A'}.$$

(b) If the coalgebra C is flat as a k-module, then the modules ${}_A(A \otimes C)$ and $(A \otimes C)_A$ are flat.

Proof. Follows immediately from Propositions 2.4.2, 2.4.1, and Lemma 2.4.3. □

Now, let $(\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')$ be a morphism in $\mathbb{E}^{\bullet}_{\bullet}(k)$. We know that $(\alpha \otimes \gamma, \alpha) : A \otimes C \to A' \otimes C'$ is a morphism of corings. From [24, Lemma 8], we have a functor $F : \mathcal{M}(\psi)^C_A \to \mathcal{M}(\psi')^{C'}_{A'}$ defined as follows: Let $M \in \mathcal{M}(\psi)^C_A$. $F(M) = M \otimes_A A' \in \mathcal{M}_{A'}$. The right C'-coaction is defined by

$$\rho_{M\otimes_A A'}(m \otimes a') = (m_{(0)} \otimes a'_{\psi'}) \otimes \gamma(m_{(1)})^{\psi'}, \tag{2.5}$$

$m \in M, a' \in A'$. For every morphism $f : M \to N$ in $\mathcal{M}(\psi)^C_A$, $F(f) = f \otimes A'$.

Proposition 2.4.5. *We have the commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}^{A\otimes C} & \xrightarrow{-\otimes_A A'} & \mathcal{M}^{A'\otimes C'} \\
\big\downarrow \simeq & & \big\downarrow \simeq \\
\mathcal{M}(\psi)^C_A & \xrightarrow[F=-\otimes_A A']{} & \mathcal{M}(\psi')^{C'}_{A'},
\end{array}$$

where $- \otimes_A A' : \mathcal{M}^{A\otimes C} \to \mathcal{M}^{A'\otimes C'}$ is the induction functor.

Proof. Let $M \in \mathcal{M}(\psi)^C_A$. Then

$$\begin{aligned}
\rho'_{M\otimes_A A'}(m \otimes_A a') &= m_{(0)} \otimes_A 1_{A'} \otimes_{A'} (\alpha \otimes \gamma)(1_A \otimes m_{(1)})a' \\
&= m_{(0)} \otimes_A 1_{A'} \otimes_{A'} \big(1_{A'} \otimes \gamma(m_{(1)})\big)a' \\
&= m_{(0)} \otimes_A 1_{A'} \otimes_{A'} \big(a'_{\psi'} \otimes \gamma(m_{(1)})^{\psi'}\big) \\
&= m_{(0)} \otimes_A 1_{A'} \otimes_{A'} a'_{\psi'}\big(1_{A'} \otimes \gamma(m_{(1)})^{\psi'}\big) \\
&= m_{(0)} \otimes_A a'_{\psi'} \otimes_{A'} \big(1_{A'} \otimes \gamma(m_{(1)})^{\psi'}\big).
\end{aligned}$$

□

We obtain the following result concerning the category of entwined modules.

Theorem 2.4.6. *Let $(\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')$ be a morphism in $\mathbb{E}^{\bullet}_{\bullet}(k)$, such that ${}_kC$ and ${}_kD$ are flat.*

1. *The following statements are equivalent*

(a) *The functor* $F = - \otimes_A A' : \mathcal{M}(\psi)_A^C \to \mathcal{M}(\psi')_{A'}^{C'}$ *defined above is a Frobenius functor;*

(b) *the* $A \otimes C - A' \otimes C'$*-bicomodule* $(A \otimes C) \otimes_A A'$ *is quasi-finite injector as a right* $A' \otimes C'$*-comodule and there exists an isomorphism of* $A' \otimes C' - A \otimes C$*-bicomodules* $\mathrm{h}_{A' \otimes C'}((A \otimes C) \otimes_A A', A' \otimes C') \simeq A' \otimes_A (A \otimes C)$.

Moreover, if $A \otimes C$ *and* $A' \otimes C'$ *are coseparable corings, then the condition "injector" in (b) can be deleted.*

2. *If* A *and* A' *are QF rings and the module* $(A' \otimes C')_{A'}$ *is projective, then the following are equivalent*

(a) *The functor* $F = - \otimes_A A' : \mathcal{M}(\psi)_A^C \to \mathcal{M}(\psi')_{A'}^{C'}$ *defined above is a Frobenius functor;*

(b) *the* $A \otimes C - A' \otimes C'$*-bicomodule* $(A \otimes C) \otimes_A A'$ *is quasi-finite and injective as a right* $A' \otimes C'$*-comodule and there exists an isomorphism of* $A' \otimes C' - A \otimes C$*-bicomodules* $\mathrm{h}_{A' \otimes C'}((A \otimes C) \otimes_A A', A' \otimes C') \simeq A' \otimes_A (A \otimes C)$.

Proof. Follows from Theorem 2.3.1 and Theorem 2.3.3. □

Remark 2.4.7. Let a right-right entwining structure $(A, C, \psi) \in \mathbb{E}^{\bullet}_{\bullet}(k)$. The coseparability of the coring $A \otimes C$ is characterized in [24, Theorem 38 (1)] (see also Proposition 1.2.19).

2.5 Applications to graded ring theory

In this section we apply our results in the previous sections to the category of graded modules by a G-set. Let G be a group, A a G-graded k-algebra, and X a left G-set. We begin by giving some useful lemmas.

2.5.1 Some useful lemmas

Let $C = kX$ and $C' = kX'$ be two grouplike coalgebras, where X and X' are arbitrary nonempty sets. We know (see Example 1.5.20) that the category $\mathcal{M}^C$ is isomorphic to the category of X-graded modules. Moreover we have the following:

Lemma 2.5.1. *For a k-module M which is both a X-graded and a X'-graded module, the following are equivalent*

(a) $M \in {}^{C'}\mathcal{M}^C$;

(b) *for every* $m \in M, x \in X, x' \in X'$, $\quad {}_{x'}(m_x) \in M_x$;

(c) *for every* $m \in M, x \in X, x' \in X'$, $\quad ({}_{x'}m)_x \in {}_{x'}M$;

(d) *for every* $m \in M, x \in X, x' \in X'$, $\quad {}_{x'}(m_x) = ({}_{x'}m)_x$.

Proof. Let $M = \bigoplus_{x\in X} M_x = \bigoplus_{x'\in X'} {}_{x'}M$. At first observe that the condition (a) is equivalent to the fact that the diagram

$$\begin{tikzcd}
M \arrow[rr, "\rho_M"] \arrow[d, "\lambda_M"'] & & M \otimes kX \arrow[d, "\lambda_M \otimes kX"] \\
kX' \otimes M \arrow[r, "kX' \otimes \rho_M"'] & kX' \otimes M \otimes kX \simeq M^{(X' \times X)} &
\end{tikzcd}$$

is commutative.

$(a) \Rightarrow (d)$ Let $m \in M$. We have, $\rho_M(m) = \sum_{x\in X} m_x \otimes x$ and $\lambda_M(m) = \sum_{x'\in X'} x' \otimes {}_{x'}m$. From the commutativity of the above diagram,

$$\sum_{x\in X}\sum_{x'\in X'} x' \otimes {}_{x'}(m_x) \otimes x = \sum_{x'\in X'}\sum_{x\in X} x' \otimes ({}_{x'}m)_x \otimes x.$$

Hence, ${}_{x'}(m_x) = ({}_{x'}m)_x$ for all $x \in X$, and $x' \in X'$.

$(d) \Rightarrow (b)$ Trivial.

$(b) \Rightarrow (a)$ Let $m \in M$. We have,

$$(\lambda_M \otimes kX)\rho_M(m) = \sum_{x\in X}\sum_{x'\in X'} x' \otimes {}_{x'}(m_x) \otimes x,$$

and

$$(kX' \otimes \rho_M)\lambda_M(m) = (kX' \otimes \rho_M)\big(\sum_{x\in X}\sum_{x'\in X'} x' \otimes {}_{x'}(m_x)\big) = \sum_{x\in X}\sum_{x'\in X'} x' \otimes {}_{x'}(m_x) \otimes x.$$

Hence (a) follows.

$(a) \Leftrightarrow (d) \Leftrightarrow (c)$ Follows by symmetry. □

Now let G and G' be two groups, A a G-graded k-algebra, A' a G'-graded k-algebra, X a right G-set, and X' a left G'-set. Let kG and kG' be the canonical Hopf algebras.

Let $\psi : kX \otimes A \to A \otimes kX$, be the map defined by $x \otimes a_g \mapsto a_g \otimes xg$, and $\psi' : A' \otimes kX' \to kX' \otimes A'$, be the map defined by $a'_{g'} \otimes x' \mapsto g'x' \otimes a'_{g'}$.

From Subsection 1.5.3, we have $(kG, A, kX) \in \mathbb{DK}^{\bullet}_{\bullet}(k)$, $(kG', A', kX') \in {}^{\bullet}_{\bullet}\mathbb{DK}(k)$, $(A, kX, \psi) \in \mathbb{E}^{\bullet}_{\bullet}(k)$, $(A', kX', \psi') \in {}^{\bullet}_{\bullet}\mathbb{E}(k)$, and $\mathcal{M}(kG)^{kX}_A \simeq gr - (A, X, G)$, ${}^{kX'}_{A'}\mathcal{M}(kG') \simeq (G', X', A') - gr$.

Lemma 2.5.2. *(1) Let M be a k-module having the structure of an X-graded right A-module and an X'-graded left A'-module. The following are equivalent*

(a) $M \in {}^{kX'}_{A'}\mathcal{M}(\psi', \psi)^{kX}_A$;

(b) the following conditions hold

(i) M is a (A', A)-bimodule,

(ii) *for every* $m \in M, x \in X, x' \in X'$, ${}_{x'}(m_x) \in M_x$ *(or* $({}_{x'}m)_x \in {}_{x'}M$, *or* ${}_{x'}(m_x) = ({}_{x'}m)_x$*),*

(iii) *for every* $x \in X$, M_x *is a submodule of* ${}_{A'}M$*,*

(iv) *for every* $x' \in X'$, ${}_{x'}M$ *is a submodule of* M_A*.*

(2) $M \in {}^{kX'}_{A'}\mathcal{M}(\psi', \psi)^{kX}_{A}$ *if and only if* M *is an* $X' \times X$*-graded* (A', A)*-bimodule (see Subsection 1.5.3).*

Proof. (1) $(a) \Leftrightarrow (b)$ We will use the definition of an object in the category of two-sided entwined modules ${}^{kX'}_{A'}\mathcal{M}(\psi', \psi)^{kX}_{A}$. By Lemma 2.5.1, the condition "$M \in {}^{kX'}\mathcal{M}^{kX}$" is equivalent to the condition (ii). We have moreover that the left A'-action on M is kX-colinear if and only if for every $x \in X, m \in M_x$, $\rho_M(a'm) = (a'm) \otimes x$, if and only if for every $x \in X, m \in M_x$, $a'm \in M_x$, if and only if (iii) holds. By symmetry, the condition "the right A-action on M is kX'-colinear" is equivalent to the condition (iv).

(2) The "if" part is clear. For the "only if" part, put $M = \bigoplus_{(x',x) \in X' \times X} M_{(x',x)}$, where $M_{(x',x)} = {}_{x'}(M_x) = ({}_{x'}M)_x$. Let $a' \in A'_{g'}, m \in M$. Since $a'.{}_{x'}(m_x) \in M_x$ and $a'.{}_{x'}(m_x) \in {}_{g'x'}(M)$, $a'.{}_{x'}(m_x) \in {}_{a'x'}(M_x)$. Therefore, $A'_{g'}.{}_{x'}(M_x) \subset {}_{a'x'}(M_x)$. By symmetry we obtain $A'_{g'}M_{(x',x)}A_g \subset M_{g'(x',x)g} = M_{(g'x',xg)}$ $(g \in G, g' \in G', x \in X, x' \in X')$. □

2.5.2 Adjoint pairs and Frobenius pairs of functors between categories of graded modules over G-sets

Throughout this subsection, G and G' will be two groups, A a G-graded k-algebra, A' a G'-graded k-algebra, X a right G-set, and X' a right G'-set. Let kG and kG' be the canonical Hopf algebras. Let $\psi : kX \otimes A \to A \otimes kX$ be the map defined by $x \otimes a_g \mapsto a_g \otimes xg$. Analogously we define the map $\psi' : kX' \otimes A' \to A' \otimes kX'$. The comultiplication and the counit maps of the coring $A \otimes kX$ are defined by:

$$\Delta(a \otimes x) = (a \otimes x) \otimes_A (1_A \otimes x), \quad \epsilon(a \otimes x) = a \quad (a \in A, x \in X).$$

Proposition 2.5.3. *The corings* $A \otimes kX$ *and* $A' \otimes kX'$ *are coseparable.*

Proof. First, [24, Proposition 101] states that the forgetful functor

$$\mathcal{M}(kG)^{kX}_{A} \to \mathcal{M}_A$$

is separable. By Proposition 1.2.19, $A \otimes kX$ is a coseparable coring.

The following direct proof in the setting of corings is shown to me by T. Brzeziński. The map $\delta : (A \otimes kX) \otimes_A (A \otimes kX) \to A$ defined by $\delta(a \otimes x \otimes y) = a\delta_{x,y}$ (Kronecker's delta) for $a \in A, x, y \in X$, is a cointegral in the coring $A \otimes kX$ (see Definition 1.2.20). Indeed, δ

is clearly left A-linear. On the other hand, for all $a \in A, x, y \in X, g \in G, b_g \in A_g$,

$$\begin{aligned}
\delta\big((a \otimes x) \otimes_A (1_A \otimes y)b_g\big) &= \delta\big((a \otimes x) \otimes_A \psi(y \otimes b_g)\big) \\
&= \delta\big((a \otimes x) \otimes_A (b_g \otimes yg)\big) \\
&= \delta\big(a\psi(x \otimes b_g) \otimes_A (1_A \otimes yg)\big) \\
&= \delta\big((ab_g \otimes xg) \otimes_A (1_A \otimes yg)\big) \\
&= ab_g\delta_{xg,yg} \\
&= ab_g\delta_{x,y} \\
&= \delta\big((a \otimes x) \otimes_A (1_A \otimes y)\big)b_g.
\end{aligned}$$

Then δ is right A-linear. Moreover, it is clear that $\delta \circ \Delta = \epsilon$. Finally, set $\mathfrak{C} = A \otimes kX$. Then, we have for all $a \in A, x, y \in X$,

$$\begin{aligned}
(\mathfrak{C} \otimes_A \delta) \circ (\Delta \otimes_A \mathfrak{C})\big((a \otimes x) \otimes_A (1_A \otimes y)\big) &= (\mathfrak{C} \otimes_A \delta)\big((a \otimes x) \otimes_A (1_A \otimes x) \otimes_A (1_A \otimes y)\big) \\
&= (a \otimes x)\delta_{x,y},
\end{aligned}$$

and

$$\begin{aligned}
(\delta \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A \Delta)\big((a \otimes x) \otimes_A (1_A \otimes y)\big) &= (\delta \otimes_A \mathfrak{C})\big((a \otimes x) \otimes_A (1_A \otimes y) \otimes_A (1_A \otimes y)\big) \\
&= a\delta_{x,y}(1_A \otimes y) \\
&= a\delta_{x,y} \otimes y.
\end{aligned}$$

Hence, $(\mathfrak{C} \otimes_A \delta) \circ (\Delta \otimes_A \mathfrak{C}) = (\delta \otimes_A \mathfrak{C}) \circ (\mathfrak{C} \otimes_A \Delta)$. By Proposition 1.2.19, $A \otimes kX$ is a coseparable coring. $\square$

Lemma 2.5.4. *(1) ψ is bijective, $(A, kX, \psi^{-1}) \in {}^{\bullet}_{\bullet}\mathbb{E}(k)$, and*

$$ {}_A^{kX}\mathcal{M}(kG) := {}_A^{kX}\mathcal{M}(\psi^{-1}) \simeq (G, X, A) - gr, $$

where the structure of left G-set on X is given by $g.x = xg^{-1}$ $(g \in G, x \in X)$.

(2) Every object of the category ${}^{A'\otimes kX'}\mathcal{M}^{A\otimes kX} \simeq {}_{A'}^{kX'}\mathcal{M}((\psi')^{-1}, \psi)_A^{kX}$ can be identified to an $X' \times X$-graded (A', A)-bimodule.

Proof. (1) From Proposition 1.5.16, and since the antipode of $H = kG$ is $S(g) = g^{-1}$, for all $g \in G$, then $S \circ S = 1_H$ and $\overline{S} = S^{-1} = S$ is a twisted antipode of H. Hence ψ is bijective and $(A, kX, \psi^{-1}) \in {}^{\bullet}_{\bullet}\mathbb{E}(k)$ (see Subsection 1.5.2), and ${}_A^{kX}\mathcal{M}(kG) := {}_A^{kX}\mathcal{M}(\psi^{-1}) \simeq (G, X, A) - gr$, since $\psi^{-1} : A \otimes kX \to kX \otimes A$, $a_g \otimes x \mapsto g.x \otimes a_g$, where $g.x = xg^{-1}$.

(2) It follows from (1) and Lemma 2.5.2. $\square$

Lemma 2.5.5. *(1) Let $M \in gr - (A, X, G)$, and $N \in (G, X, A) - gr$. We know that $M \in \mathcal{M}^{A\otimes kX}$ by the coaction: $\rho_M : M \to M \otimes_A (A \otimes kX)$, where $\rho_M(m_x) = m_x \otimes_A (1_A \otimes x)$, and $N \in {}^{A\otimes kX}\mathcal{M}$ by the coaction: $\lambda_N : N \to (A \otimes kX) \otimes_A N$, where $\lambda_N({}_xn) = (1_A \otimes x) \otimes_A {}_xn$. We have $M\Box_{(A\otimes kX)}N = M\widehat{\otimes}_A N$, where $M\widehat{\otimes}_A N$ is the additive subgroup of $M \otimes_A N$ generated by the elements $m \otimes_A n$ where $x \in X, m \in M_x$, and $n \in {}_xN$ (see Subsection 1.5.3).*

(2) Let P be an $X \times X'$-graded (A, A')-bimodule. We have the commutative diagram:

$$\begin{tikzcd} gr-(A,X,G) \arrow[r, "-\widehat{\otimes}_A P"] \arrow[d, "\simeq"] & gr-(A',X',G') \arrow[d, "\simeq"] \\ \mathcal{M}^{A\otimes kX} \arrow[r, "-\square_{(A\otimes kX)}P"] & \mathcal{M}^{A'\otimes kX'}, \end{tikzcd}$$

where $-\widehat{\otimes}_A P : gr-(A,X,G) \to gr-(A',X',G')$ is the functor defined in Subsection 1.5.3.

Proof. (1) At first we will show that the right A-module $A \otimes kX$ is free. More precisely that the family $\{1_A \otimes x \mid x \in X\}$ is a right basis of it. It is clear that $a_g \otimes x = (1_A \otimes xg^{-1})a_g$ for $a_g \in A_g$ and $x \in X$. Now suppose that $\sum_i (1_A \otimes x_i)a_i = 0$, where $x_i \in X, a_i \in A$. Then $\sum_i \psi(x_i \otimes a_i) = 0$. Since ψ is bijective, we obtain $\sum_i x_i \otimes a_i = 0$. Therefore $a_i = 0$ for all i. Hence the above mentioned family is a basis of the right A-module $A \otimes kX$. (There is a shorter and indirect proof of this fact by using that ψ is an isomorphism of A-bimodules, and $\{x \otimes 1_A | x \in X\}$ is a basis of the right A-module $kX \otimes A$.)

Finally, suppose that $\sum_i m_{x_i} \otimes_A {}_{y_i}n \in M\square_{(A\otimes kX)}N$. Then

$$\sum_i m_{x_i} \otimes_A 1_A \otimes x_i \otimes_A {}_{y_i}n = \sum_i m_{x_i} \otimes_A 1_A \otimes y_i \otimes_A {}_{y_i}n.$$

Hence (the right A-module $A \otimes kX$ is free), $x_i = y_i$ for all i, and $M\square_{(A\otimes kX)}N \subset M\widehat{\otimes}_A N$. The other inclusion is obvious.

(2) It suffices to show that the map $M\widehat{\otimes}_A P \to M\widehat{\otimes}_A P \otimes_{A'} (A' \otimes kX')$ defined by

$$\sum_{x\in F} m_x \otimes_A {}_x p \longmapsto \sum_{x'\in X'} \left(\sum_{x \in F} m_x \otimes_A {}_x p \right)_{x'} \otimes_{A'} (1_{A'} \otimes x'),$$

where F is a finite subset of X, makes commutative the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M\widehat{\otimes}_A P \arrow[r, "i"] \arrow[d] & M \otimes_A P \arrow[d] \\ 0 \arrow[r] & M\widehat{\otimes}_A P \otimes_{A'} (A' \otimes kX') \arrow[r, "i\otimes_{A'}(A'\otimes kX')"] & M \otimes_A P \otimes_{A'} (A' \otimes kX'). \end{tikzcd}$$

That is clear since $(\sum_{x\in F} m_x \otimes_A {}_x p)_{x'} = \sum_{x\in F} m_x \otimes_A ({}_x p)_{x'} = \sum_{x\in F} m_x \otimes_A p_{(x,x')}$, where $p = \sum_{x\in F} {}_x p$. $\square$

Let $\widehat{A} = A \otimes kX$ be the $X \times X$-graded (A, A)-bimodule associated to the $(A \otimes kX) - (A \otimes kX)$-bicomodule $A \otimes kX$. It is clear that $-\widehat{\otimes}_A \widehat{A} = -\square_{(A\otimes kX)}(A \otimes kX) \simeq 1_{gr-(A,G,X)}$ and then $\widehat{A}$ is isomorphic as a bigraded bimodule to Del Río's "$\widehat{A}$" (see Subsection 1.5.3). The gradings are $\widehat{A}_x = A \otimes kx$, and ${}_x\widehat{A} = \{\sum_i a_i \otimes x_i \mid x_i g^{-1} = x, \forall i, \forall g \in G : (a_i)_g \neq 0\}$ $(x \in X)$.

Proposition 2.5.6. [65, Proposition 1.3, Corollary 1.4]

(1) The following statements are equivalent for a k-linear functor $F : gr-(A, X, G) \to gr-(A', X', G')$.

(a) F has a right adjoint;

(b) F is right exact and preserves coproducts;

(c) $F \simeq -\widehat{\otimes}_A P$ for some $X \times X'$-graded (A, A')-bimodule P.

(2) A k-linear functor $G : gr-(A', X', G') \to gr-(A, X, G)$ has a left adjoint if and only if $G \simeq \mathrm{H}(P_{A'}, -)$ for some $X \times X'$-graded (A, A')-bimodule P.

Proof. (1) $(a) \Rightarrow (b)$ Clear. $(b) \Rightarrow (c)$ It follows from Theorem 2.1.13 and Lemma 2.5.5. $(c) \Rightarrow (a)$ Obvious from the above mentioned result.

(2) It follows from (1) and Proposition 1.5.11. □

Lemma 2.5.7. *Let P be an $X \times X'$-graded (A, A')-bimodule.*

(1) $\mathrm{H}(P_{A'}, -)$ is right exact and preserves direct limits if and only if ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$.

(2) Suppose that ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$.

(a) For every k-algebra T,

$$\Upsilon_{Z,M} : Z \otimes_T \mathrm{H}(P_{A'}, M) \xrightarrow{\simeq} \mathrm{H}(P_{A'}, Z \otimes_T M)$$

defined by

$$\Upsilon_{Z,M}(z \otimes_T f)(p) = \sum_{x \in X} z \otimes_T f_x({}_xp)$$

$(z \in Z, f = \sum_{x \in X} f_x \in \mathrm{H}(P_{A'}, M), p = \sum_{x \in X} {}_xp \in P)$, is the natural isomorphism associated to the functor (see Section 2.1)

$$\mathrm{H}(P_{A'}, -) : gr-(A', X', G') \to gr-(A, X, G).$$

(b) Moreover we have the natural isomorphism

$$\eta_N : N \widehat{\otimes}_{A'} \mathrm{H}(P_{A'}, \widehat{A}') \xrightarrow{\simeq} \mathrm{H}(P_{A'}, N)$$

defined by

$$\eta_N(n_{x'} \otimes_{A'} {}_{x'}f)(p) = \sum_{x \in X} n_{x'} \delta\big((1_{A'} \otimes x') \otimes_{A'} f_{(x',x)}({}_xp)\big)$$

($n_{x'} \in N_{x'}, {}_{x'}f \in {}_{x'}\mathrm{H}(P_{A'}, \widehat{A'}), p = \sum_{x\in X} {}_xp \in P$), where δ is the cointegral in the coring $A' \otimes kX'$ defined in the beginning of this subsection. The left grading on $\mathrm{H}(P_{A'}, \widehat{A'})$ is given by

$$_{x'}\mathrm{H}(P_{A'}, \widehat{A'}) = \left\{ f \in \mathrm{H}(P_{A'}, \widehat{A'}) \mid \Delta_{(A'\otimes kX')}\big(f(p)\big) = (1_{A'}\otimes x')\otimes_{A'} \sum_{x\in X} f_x(xp), \forall p \in P \right\}$$

($x' \in X'$).

Proof. (1) We have

$$\mathrm{H}(P_{A'}, -) \simeq \bigoplus\nolimits_{x\in X} \mathrm{Hom}_{gr-(A',X',G')}({}_xP, -) : gr - (A', X', G') \longrightarrow \mathbf{Ab}.$$

Hence, $\mathrm{H}(P_{A'}, -)$ is right exact and preserves direct limits if and only if $\mathrm{Hom}_{gr-(A',X',G')}({}_xP, -)$ is right exact and preserves direct limits for every $x \in X$ if and only if ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$ (by Proposition 1.1.57 and Lemma 1.1.61 (3)).

(2) (a) Let T be a k-algebra. Let M be an $X_0 \times X'$-graded $T - A'$-bimodule, where X_0 is a singleton, Z be a right T-module, and $x \in X$. We have a sequence of T-submodules

$$\mathrm{H}(P_{A'}, M)_x \leq \mathrm{H}(P_{A'}, M) \leq \mathrm{Hom}_{gr-(A',X',G')}(P, M) \leq {}_T\mathrm{Hom}_{A'}(P_{A'}, M),$$

and the induced structure of left T-module on $\mathrm{H}(P_{A'}, M)$ is the same structure (see Section 2.1) of left T-module associated to the functor $\mathrm{H}(P_{A'}, -)$ on it. Moreover, we have the isomorphism of left T-modules $\mathrm{H}(P_{A'}, M)_x \simeq \mathrm{Hom}_{gr-(A',X',G')}({}_xP, M)$. From Lemma 1.1.72 (4), for every $x \in X$, there is an isomorphism

$$\eta_x : Z \otimes_T \mathrm{Hom}_{A'}({}_xP, M) \xrightarrow{\simeq} \mathrm{Hom}_{A'}({}_xP, Z \otimes_T M) \tag{2.6}$$

defined by $\eta_x(z \otimes_T \gamma_x) : {}_xp \mapsto z \otimes_T \gamma_x({}_xp)$ ($z \in Z, \gamma_x \in \mathrm{Hom}_{A'}({}_xP, M)$). For every $x \in X$, η_x induces an isomorphism

$$\eta'_x : Z \otimes_T \mathrm{Hom}_{gr-(A',X',G')}({}_xP, M) \xrightarrow{\simeq} \mathrm{Hom}_{gr-(A',X',G')}({}_xP, Z \otimes_T M)\,. \tag{2.7}$$

Now let us consider the isomorphism

$$\eta'_{Z,M} := \bigoplus\nolimits_{x\in X} \eta'_x : Z \otimes_T \mathrm{H}(P_{A'}, M) \xrightarrow{\simeq} \mathrm{H}(P_{A'}, Z \otimes_T M)\,.$$

We have $\eta'_{Z,M}(z \otimes_T f)(p) = \sum_{x\in X} z \otimes_T f_x({}_xp)$ ($z \in Z, f = \sum_{x\in X} f_x \in \mathrm{H}(P_{A'}, M), p = \sum_{x\in X} {}_xp \in P$). We can verify easily that $\eta'_{Z,M}$ is a morphism of right A-modules. Hence it is a morphism in $gr - (A, X, G)$. It is clear that $\eta'_{Z,M}$ is natural in Z, and $\eta'_{T,M}$ makes commutative the diagram (2.1) (see Section 2.1). Finally, by Mitchell's Theorem 2.1.1, $\eta'_{Z,M} = \Upsilon_{Z,M}$.

(b) To prove the first statement it suffices to use the proof of Theorem 2.1.13, the property (a), and the fact that every comodule over an A-coseparable coring is A-relative injective comodule (see Proposition 1.2.19).

Finally, we will prove the last statement. We know that $\lambda_{\mathrm{H}(P_{A'},\widehat{A'})}$ is defined to be the unique A'-linear map making commutative the following diagram

$$\begin{tikzcd} \mathrm{H}(P_{A'},\widehat{A'}) \arrow[rr, "\lambda_{\mathrm{H}(P_{A'},\widehat{A'})}"] \arrow[dr, "\mathrm{H}(P_{A'},\Delta_{(A'\otimes kX')})"'] & & \widehat{A'}\otimes_{A'}\mathrm{H}(P_{A'},\widehat{A'}) \arrow[dl, "\Upsilon_{\widehat{A'},\widehat{A'}}"] \\ & \mathrm{H}(P_{A'},\widehat{A'}\otimes_{A'}\widehat{A'}) & \end{tikzcd}$$

On the other hand, for each $x'\in X'$, ${}_{x'}\mathrm{H}(P_{A'},\widehat{A'})=\{f\in \mathrm{H}(P_{A'},\widehat{A'}) \mid \lambda_{\mathrm{H}(P_{A'},\widehat{A'})}(f)=(1_{A'}\otimes x')\otimes_{A'} f\}$. Since $\Upsilon_{\widehat{A'},\widehat{A'}}$ is an isomorphism, ${}_{x'}\mathrm{H}(P_{A'},\widehat{A'})=\{f\in \mathrm{H}(P_{A'},\widehat{A'}) \mid \Delta_{(A'\otimes kX')}(f(p))=(1_{A'}\otimes x')\otimes_{A'}\sum_{x\in X} f_x({}_xp) \text{ for all } p\in P\}$. □

Remark 2.5.8. Let Σ be a finitely generated projective right A-module, and M be a right A-module. It easy to verify that the map $\alpha_{\Sigma,M}:\mathrm{Hom}_A(\Sigma,M)\to M\otimes_A\Sigma^*$ defined by $\alpha_{\Sigma,M}(\varphi)=\sum_i\varphi(e_i)\otimes_A e_i^*$, where $\{e_i,e_i^*\}_i$ is a dual basis of Σ, is an isomorphism, with $\alpha_{\Sigma,M}^{-1}(m\otimes_A f)(u)=mf(u)$ $(m\in M, f\in\Sigma^*, u\in\Sigma)$.
Now we assume that the condition of Lemma 2.5.7 (2) holds. It is easy to verify that for every $x\in X$, $\eta_x=\alpha^{-1}_{{}_xP,Z\otimes_T M}\circ(Z\otimes_T\alpha_{{}_xP,M})$, where η_x is the isomorphism (2.6). Therefore $\eta_x^{-1}=(Z\otimes_T\alpha^{-1}_{{}_xP,M})\circ\alpha_{{}_xP,Z\otimes_T M}$. Hence $\Upsilon^{-1}_{Z,M}(g)=\sum_{x\in X}\sum_{i\in I_x}(Z\otimes_T\alpha^{-1}_{{}_xP,M})(g_x(e_{x,i})\otimes_{A'}e^*_{x,i})$, where $\{e_{x,i},e^*_{x,i}\}_{i\in I_x}$ is a dual basis of ${}_xP$ $(x\in X)$, and $g=\sum_{x\in X}g_x\in\mathrm{H}(P_{A'},Z\otimes_T M)$. Finally, for each $N\in gr-(A',X',G')$,

$$\eta_N^{-1}:\mathrm{H}(P_{A'},N)\xrightarrow{\simeq} N\widehat{\otimes}_{A'}\mathrm{H}(P_{A'},\widehat{A'})$$

is $\eta_N^{-1}=\Upsilon^{-1}_{N,\widehat{A'}}\circ\mathrm{H}(P_{A'},\rho_N)$, and then

$$\eta_N^{-1}(\varphi)=\sum_{x\in X}\sum_{i\in I_x}\sum_{x'\in X'}(\varphi_x(e_{x,i}))_{x'}\otimes_{A'}\psi'(x'\otimes e^*_{x,i}(-)),$$

where $\varphi=\sum_{x\in X}\varphi_x\in\mathrm{H}(P_{A'},N)$.

In particular, if $X'=G'=\{e'\}$, then

$$\eta_N^{-1}:\mathrm{H}(P_{A'},N)\xrightarrow{\simeq} N\otimes_{A'}\mathrm{H}(P_{A'},A')$$

is defined by

$$\eta_N^{-1}(\varphi)=\sum_{x\in X}\sum_{i\in I_x}\varphi(e_{x,i})\otimes_{A'}e^*_{x,i}\qquad(\varphi\in\mathrm{H}(P_{A'},N)).$$

Lemma 2.5.9. *(1) Let $N\in{}_{A'}\mathcal{M}^{A\otimes kX}$. Then N is quasi-finite as a right $A\otimes kX$-comodule if and only if N_x is finitely generated projective in ${}_{A'}\mathcal{M}$, for every $x\in X$. In this case, the cohom functor $\mathrm{h}_{A\otimes kX}(N,-)$ is the composite*

$$\mathcal{M}^{A\otimes kX}\xrightarrow{\simeq} gr-(A,X,G)\xrightarrow{-\widehat{\otimes}_A P}\mathcal{M}_{A'},$$

where P is the $X\times X'_0$-graded $A-A'$-bimodule $\mathrm{H}({}_{A'}N,A')$ with X'_0 is a singleton.

(2) Now suppose that $N \in {}^{A'\otimes kX'}\mathcal{M}^{A\otimes kX}$, and N_x is finitely generated projective in ${}_{A'}\mathcal{M}$, for every $x \in X$. Let $\{e_{x,i}, e^*_{x,i}\}_{i\in I_x}$ be a dual basis of N_x $(x \in X)$. Let P be the bigraded bimodule defined as above, $\theta : 1_{\mathcal{M}^{A\otimes kX}} \to -\widehat{\otimes}_A P \otimes_{A'} N$ be the unit of the adjunction $(-\widehat{\otimes}_A P, -\otimes_{A'} N)$, and let $M \in gr-(A, X, G)$.

We can endow P with a structure of $X \times X'$-graded $A-A'$-bimodule such that

$$\mathrm{h}_{A\otimes kX}(N, -) \simeq -\widehat{\otimes}_A P : gr-(A, X, G) \to gr-(A', X', G').$$

The $A' \otimes kX'$-comodule structure on P,

$$\rho_P : P \to P \otimes_{A'} (A' \otimes kX'),$$

is defined to be the unique A'-linear map satisfying the condition:

$$\sum_{i\in I_x} \rho_P(ae^*_{x,i}) \otimes_{A'} e_{x,i} = \sum_{i\in I_x}\sum_{x'\in X'} ae^*_{x,i} \otimes_{A'} (1_{A'} \otimes x') \otimes_{A'} (e_{x,i})_{x'}, \tag{2.8}$$

for every $a \in A, x \in X$.

Proof. (1) It follows from Lemma 2.5.5 (2), that the functor $F := -\widehat{\otimes}_{A'} N : \mathcal{M}_{A'} = gr-(A', X'_0, G') \to gr-(A, X, G)$, where X'_0 is a singleton, is the composite

$$\mathcal{M}_{A'} = gr-(A', X'_0, G') \xrightarrow{-\otimes_{A'}N} \mathcal{M}^{A\otimes kX} \xrightarrow{\simeq} gr-(A, X, G)\ .$$

Then, N is quasi-finite as a right $A \otimes kX$-comodule if and only if F has a left adjoint, if and only if (by Corollary 2.5.6 (2)) there exists an $X \times X'_0$-graded $A-A'$-bimodule P such that $F \simeq \mathrm{H}(P_{A'}, -)$, if and only if (by Lemma 2.5.7, Lemma 2.1.8, Theorem 2.1.13) there exists an $X \times X'_0$-graded $A-A'$-bimodule P such that ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$, and $N \simeq \mathrm{H}(P_{A'}, A')$ in ${}^{A'}\mathcal{M}^{A\otimes kX}$.

Now let us consider the $X \times X'_0$-graded $A-A'$-bimodule $P := \mathrm{H}({}_{A'}N, A')$, and the $X' \times X$-graded (A', A)-bimodule $M := \mathrm{H}(P_{A'}, A')$, where X'_0 is a singleton. We have

$$M = \{f \in P^* | f({}_xP) = 0 \text{ for almost all } x \in X\}$$

$$M_x = \left\{f \in P^* | f({}_yP) = 0 \text{ for all } y \in X - \{x\}\right\} \quad (x \in X),$$

where $P^* = \mathrm{Hom}_{A'}(P_{A'}, A'_{A'})$. The structure of $A'-A$-bimodule on P^* is given by $(fa)(p) = f(ap), (a'f)(p) = a'f(p)$ $(f \in P^*, a \in A, a' \in A', p \in P)$. We have

$$M \leq (P^*)_A, \quad M_x \leq M \leq {}_{A'}(P^*), \quad \text{and} \quad M_x \simeq ({}_xP)^* \text{ in } {}_{A'}\mathcal{M} \quad (x \in X).$$

Analogously,

$$P = \{f \in {}^*N | f(N_x) = 0 \text{ for almost all } x \in X\}$$

$${}_xP = \{f \in {}^*N | f(N_y) = 0 \text{ for all } y \in X - \{x\}\} \quad (x \in X),$$

where ${}^*N = \mathrm{Hom}_{A'}({}_{A'}N, {}_{A'}A')$. The structure of $A - A'$-bimodule on ${}^*N$ is given by $(af)(n) = f(na), (fa')(n) = a'f(n)$ $(f \in {}^*N, a \in A, a' \in A', n \in N)$. We have

$$P \leq {}_A({}^*N), \quad {}_xP \leq P \leq ({}^*N)_{A'}, \quad \text{and} \quad {}_xP \simeq {}^*(N_x) \text{ in } \mathcal{M}_{A'} \quad (x \in X).$$

Hence, N_x is finitely generated projective in ${}_{A'}\mathcal{M}$, for every $x \in X$ implies that ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$, for every $x \in X$, which implies that M_x is finitely generated projective in ${}_{A'}\mathcal{M}$, for every $x \in X$.

Finally, let us consider for every $x \in X$ the isomorphism of left A'-modules

$$H_x : N_x \xrightarrow{\simeq} ({}^*(N_x))^* \xrightarrow{\simeq} ({}_xP)^* \xrightarrow{\simeq} M_x \,.$$

We have $H_x(n_x)(\gamma) = \gamma(n_x)$ if $\gamma \in {}_xP$, and $H_x(n_x)(\gamma) = 0$ if $\gamma \in {}_yP$ and $y \in X - \{x\}$. Set

$$H = \bigoplus_{x\in X} H_x : N \xrightarrow{\simeq} M \,.$$

It can be proved easily that H is a morphism of right A-modules. Hence it is an isomorphism of graded bimodules.

(2) We have $\{e^*_{x,i}, \sigma_x(e_{x,i})\}_{i\in I_x}$ is a dual basis of ${}^*(N_x) \simeq {}_xP$, where σ_x is the evaluation map $(x \in X)$. From the proof of Lemma 2.5.6 and Remark 2.5.8, the unit of the adjunction $(-\widehat{\otimes}_A P, -\otimes_{A'} N)$ is $\theta_M : M \to M\widehat{\otimes}_A P\otimes_{A'} N$, $\theta_M(m) = \sum_{x\in X}\sum_{i\in I_x} m_x \otimes_A e^*_{x,i} \otimes_{A'} e_{x,i}$ $(m \in M)$. The coaction on $\mathrm{h}_{A\otimes kX}(N, M) = M\widehat{\otimes}_A P$: $\rho_{M\widehat{\otimes}_A P} : M\widehat{\otimes}_A P \to M\widehat{\otimes}_A P \otimes_{A'} (A' \otimes kX')$ is the unique A'-linear map satisfying the commutativity of the diagram

$$\begin{tikzcd}
M \arrow[r, "\theta_M"] \arrow[d, "\theta_M"'] & M\widehat{\otimes}_A P \otimes_{A'} N \arrow[d, "\rho_{M\widehat{\otimes}_A P}\otimes_{A'} N"] \\
M\widehat{\otimes}_A P \otimes_{A'} N \arrow[r, "M\widehat{\otimes}_A P\otimes_{A'}\lambda_N"] & M\widehat{\otimes}_A P \otimes_{A'} (A' \otimes kX') \otimes_{A'} N.
\end{tikzcd}$$

(See Section 1.4.) Since $(A \otimes kX)\widehat{\otimes}_A P \simeq P$ as $(A \otimes kX) - A'$-bicomodules, P can be endowed with a structure of $(A\otimes kX)-(A'\otimes kX')$-bicomodule such that $(A\times kX)\widehat{\otimes}_A P \simeq P$ as $(A \otimes kX) - (A' \otimes kX')$-bicomodules. Moreover, the right coaction on P, $\rho_P : P \to P \otimes_{A'} (A' \otimes kX')$, is the unique A'-linear map satisfying the commutativity of the diagram

$$\begin{tikzcd}
A \otimes kX \arrow[r, "\theta_{A\otimes kX}"] \arrow[d, "\theta_{A\otimes kX}"'] & (A \otimes kX)\widehat{\otimes}_A P \otimes_{A'} N \arrow[r, "\simeq"] \arrow[d, "\rho_{((A\otimes kX)\widehat{\otimes}_A P)}\otimes_{A'} N"] & P \otimes_{A'} N \arrow[d, "\rho_P\otimes_{A'} N"] \\
(A \otimes kX)\widehat{\otimes}_A P \otimes_{A'} N \arrow[r, "(A\otimes kX)\widehat{\otimes}_A P\otimes_{A'}\lambda_N"] & (A \otimes kX)\widehat{\otimes}_A P \otimes_{A'} (A' \otimes kX') \otimes_{A'} N \arrow[r, "\simeq"] & P \otimes_{A'} (A' \otimes kX') \otimes_{A'} N.
\end{tikzcd}$$

The commutativity of the above diagram is equivalent to the condition (2.8). Finally, By Theorem 2.1.13, $\mathrm{h}_{A\otimes kX}(N, -) \simeq -\widehat{\otimes}_A((A \otimes kX)\widehat{\otimes}_A P) \simeq -\widehat{\otimes}_A P$. $\square$

Now we are in a position to state and prove the main results of this section which characterize adjoint pairs and Frobenius pairs of functors between categories of graded modules over G-sets.

Theorem 2.5.10. *Let M be an $X \times X'$-graded (A, A')-bimodule and N an $X' \times X$-graded (A', A)-bimodule. Then the following are equivalent*

(1) $(-\widehat{\otimes}_A M, -\widehat{\otimes}_{A'} N)$ is an adjoint pair;

(2) $(N\widehat{\otimes}_A -, M\widehat{\otimes}_{A'} -)$ is an adjoint pair;

(3) N_x is finitely generated projective in ${}_{A'}\mathcal{M}$, for every $x \in X$, and $\mathrm{H}({}_{A'}N, A') \simeq M$ as $X \times X'$-graded $A - A'$-bimodules;

(4) there exist bigraded maps

$$\psi : \widehat{A} \to M\widehat{\otimes}_{A'} N \text{ and } \omega : N\widehat{\otimes}_A M \to \widehat{A}',$$

such that

$$(\omega\widehat{\otimes}_{A'} N) \circ (N\widehat{\otimes}_A \psi) = \Lambda \text{ and } (M\widehat{\otimes}_{A'} \omega) \circ (\psi\widehat{\otimes}_A M) = M. \tag{2.9}$$

In particular, $(-\widehat{\otimes}_A M, -\widehat{\otimes}_{A'} N)$ is a Frobenius pair if and only if $(M\widehat{\otimes}_{A'} -, N\widehat{\otimes}_A -)$ is a Frobenius pair.

Proof. We know that the corings $A \otimes kX$ and $A' \otimes kX'$ are coseparable. Proposition 2.1.17 then achieves the proof. □

Theorem 2.5.11. *Let G and G' be two groups, A a G-graded k-algebra, A' a G'-graded k-algebra, X a right G-set, and X' a right G'-set. For a pair of k-linear functors $F : gr - (A, X, G) \to gr - (A', X', G')$ and $G : gr - (A', X', G') \to gr - (A, X, G)$, the following statements are equivalent*

(a) (F, G) is a Frobenius pair;

(b) there exist an $X \times X'$-graded (A, A')-bimodule M, and an $X' \times X$-graded (A', A)-bimodule N, with the following properties

(1) ${}_xM$ and N_x is finitely generated projective in $\mathcal{M}_{A'}$ and ${}_{A'}\mathcal{M}$ respectively, for every $x \in X$, and $M_{x'}$ and ${}_{x'}N$ is finitely generated projective in ${}_A\mathcal{M}$ and $\mathcal{M}_A$ respectively, for every $x' \in X'$,

(2) $\mathrm{H}({}_AM, A) \simeq N$ as $X' \times X$-graded (A', A)-bimodules, and $\mathrm{H}({}_{A'}N, A') \simeq M$ as $X \times X'$-graded (A, A')-bimodules,

(3) $F \simeq -\widehat{\otimes}_A M$ and $G \simeq -\widehat{\otimes}_{A'} N$.

Proof. Straightforward from Theorem 2.1.20. □

2.5.3 When is the induction functor T^* Frobenius?

Finally, let $f : G \to G'$ be a morphism of groups, X a right G-set, X' a right G'-set, $\varphi : X \to X'$ a map such that $\varphi(xg) = \varphi(x)f(g)$ for every $g \in G$, $x \in X$. Let A be a G-graded k-algebra, A' a G'-graded k-algebra, and $\alpha : A \to A'$ a morphism of algebras such that $\alpha(A_g) \subset A'_{f(g)}$ for every $g \in G$.

At first we recall some functors from [65] and [67, Theorem 2.27].

Let $T_* : gr-(A', X', G') \to gr-(A, X, G)$ be the functor defined by

$$T_*(M) = \bigoplus_{x\in X} M^x_{\varphi(x)},$$

for $M \in gr-(A', X', G')$, where $M^x_{\varphi(x)} = M_{\varphi(x)}$ for every $x \in X$.

Set $m^x = m$ for $m \in M_{\varphi(x)}$ and $x \in X$. The right A-action on $T_*(M)$ is defined by

$$m^x.a_g = (m\alpha(a_g))^{xg}$$

for all $g \in G, x \in X, a_g \in A_g, m \in M_{\varphi(x)}$.

Moreover, we have for every $x \in X$,

$$(T_*(M))_x \simeq \mathrm{Hom}_{gr-(A',X',G')}(A'(\varphi(x)), M).$$

Hence,

$$T_*(M) \simeq \bigoplus_{x\in X} \mathrm{Hom}_{gr-(A',X',G')}(A'(\varphi(x)), M).$$

The functor T_* is a covariant exact functor and has a left adjoint T^*. We will recall the definition of this functor later.

Now, we define another functor $\widetilde{T} : gr-(A, X, G) \to gr-(A', X', G')$. For every $L \in gr-(A, X, G)$,

$$\widetilde{T}(L) = \bigoplus_{x'\in X'} (\widetilde{T}(L))_{x'},$$

where

$$(\widetilde{T}(L))_{x'} = \mathrm{Hom}_{gr-(A,X,G)}(T_*(A'(x')), L).$$

Let $g' \in g', a'_{g'} \in A'_{g'}, x' \in X', \xi \in (\widetilde{T}(L))_{x'}$. Set

$$\xi.a'_{g'} = \xi \circ T_*(\lambda_{a'_{g'}}),$$

where $\lambda_{a'_{g'}} : A'(x'g') \to A'(x')$ is the left multiplication by $a'_{g'}$ on A'.

The functor $\widetilde{T}$ is a right adjoint of T_*. Then $\widetilde{T}$ is left exact.

For more details we refer to [65].

Theorem 2.5.12. [67, Theorem 2.27]
The functors T^* and $\widetilde{T}$ are isomorphic if and only if the following conditions hold:

(1) for every $x' \in X'$, $T_*(A'(x'))$ is finitely generated and projective in $\mathcal{M}_A$;

(2) for every $x \in X$, there exists an isomorphism in $gr-(A', X', G')$

$$\theta_x : A'(\varphi(x)) \xrightarrow{\sim} \widetilde{T}(A(x))$$

such that

$$\big((\theta_{x_1}(1))(a')\big).a = (\theta_{x_2}(1))(a'.\alpha(a))$$

for every $x_1, x_2, y \in X, a \in A(x_2)_{x_1}, a' \in A'(\varphi(x_1))_{\varphi(y)}$.

We have, $\gamma : kX \to kX'$ such that $\gamma(x) = \varphi(x)$ for each $x \in X$, is a morphism of coalgebras, and $(\alpha, \gamma) : (A, kX, \psi) \to (A', kX', \psi')$ is a morphism in $\mathbb{E}^{\bullet}_{\bullet}(k)$.

Let $-\otimes_A A' : gr-(A, X, G) \to gr-(A', X', G')$ be the functor making commutative the following diagram

$$\begin{tikzcd}
gr-(A,X,G) \arrow[r, "-\otimes_A A'"] \arrow[d, "\simeq"] & gr-(A',X',G') \arrow[d, "\simeq"] \\
\mathcal{M}(kG)_A^{kX} \arrow[r, "-\otimes_A A'"] & \mathcal{M}(kG')_{A'}^{kX'}.
\end{tikzcd}$$

Let $M \in gr-(A, X, G)$. We have $M \otimes_A A'$ is a right A'-module, and

$$\rho(m_x \otimes_A a'_{g'}) = (m_x \otimes_A a'_{g'}) \otimes \varphi(x)g'.$$

Therefore, $(M \otimes_A A')_{x'}$ is the k-module of $M \otimes_A A'$ spanned by the elements of the form $m_x \otimes_A a'_{g'}$ where $x \in X$, $g' \in G'$, $\varphi(x)g' = x'$, $m_x \in M_x$, $a'_{g'} \in A'_{g'}$, for every $x' \in X'$.

Therefore, the functor $-\otimes_A A' : gr-(A, X, G) \to gr-(A', X', G')$ is exactly the induction functor T^* defined in [65] (or [67, p. 531]). Hence T^* is a Frobenius functor if and only if the induction functor $-\otimes_A A' : \mathcal{M}^{A\otimes kX} \to \mathcal{M}^{A'\otimes kX'}$ is a Frobenius functor (see Section 2.4).

Moreover, we have the commutativity of the following diagram

$$\begin{tikzcd}
(G,X,A)-gr \arrow[r, "(T^*)'=A'\otimes_A -"] \arrow[d, "\simeq"] & (G',X',A')-gr \arrow[d, "\simeq"] \\
{}_A^{kX}\mathcal{M}(\psi^{-1}) \arrow[r, "A'\otimes_A -"] \arrow[d, "\simeq"] & {}_{A'}^{kX'}\mathcal{M}((\psi')^{-1}) \arrow[d, "\simeq"] \\
{}^{kX\otimes A}\mathcal{M} \arrow[r, "A'\otimes_A -"] \arrow[d, "\simeq"] & {}^{kX'\otimes A'}\mathcal{M} \arrow[d, "\simeq"] \\
{}^{A\otimes kX}\mathcal{M} \arrow[r, "A'\otimes_A -"] & {}^{A'\otimes kX'}\mathcal{M}.
\end{tikzcd}$$

The following consequence of Theorem 2.4.6 and Theorem 2.5.12 give two different characterizations when the induction functor T^* is a Frobenius functor.

Theorem 2.5.13. *The following statements are equivalent*

(a) the functor $T^* : gr-(A, X, G) \to gr-(A', X', G')$ is a Frobenius functor;

(b) $(T^*(\widehat{A}))_{x'}$ is finitely generated projective in ${}_A\mathcal{M}$, for every $x' \in X'$, and there exists an isomorphism of $X' \times X$ graded (A', A)-bimodules

$$\mathrm{H}({}_A T^*(\widehat{A}), A) \simeq (T^*)'(\widehat{A}).$$

As an immediate consequence of Proposition 2.3.8, we obtain

Proposition 2.5.14. *The following are equivalent*

(a) the functor $T^* : gr-(A, X, G) \to gr-(A', X', G')$ is a Frobenius functor;

(b) the functor $(T^*)' : (G, X, A)-gr \to (G', X', A')-gr$ is a Frobenius functor.

Chapter 3

Comatrix Coring Generalized and Equivalences of Categories of Comodules

3.1 Comatrix coring generalized

In this section we generalize the concept of a comatrix coring defined in [17] and [35] to the case of a quasi-finite comodule, and generalize some of its properties.

The conjunction of [95, Proposition 2.7] and the following result generalizes [17, Theorem 2.4].

Theorem 3.1.1. *Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. Assume that at least one of the following conditions holds*

(1) (a) ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat,

(b) ${}_{\mathfrak{C}}X$ and ${}_{\mathfrak{D}}\Lambda$ are coflat; or

(2) A and B are von Neumann regular rings; or

(3) ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$ are flat, and $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable corings.

If there exist bicolinear maps

$$\psi : \mathfrak{C} \to X\Box_{\mathfrak{D}}\Lambda \text{ and } \omega : \Lambda\Box_{\mathfrak{C}}X \to \mathfrak{D}$$

in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{D}}$ respectively, such that the diagrams

$$\begin{tikzcd}
\Lambda \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \Lambda\Box_{\mathfrak{C}}\mathfrak{C} \arrow[d, "\Lambda\Box_{\mathfrak{C}}\psi"] \\
\mathfrak{D}\Box_{\mathfrak{D}}\Lambda & \Lambda\Box_{\mathfrak{C}}X\Box_{\mathfrak{D}}\Lambda \arrow[l, "\omega\Box_{\mathfrak{D}}\Lambda"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathfrak{C}\Box_{\mathfrak{C}}X \arrow[d, "\psi\Box_{\mathfrak{C}}X"] \\
X\Box_{\mathfrak{D}}\mathfrak{D} & X\Box_{\mathfrak{D}}\Lambda\Box_{\mathfrak{C}}X \arrow[l, "X\Box_{\mathfrak{D}}\omega"]
\end{tikzcd} \tag{3.1}$$

commute, then $\Lambda\Box_{\mathfrak{C}}X$ *is a* B*-coring with coproduct*

$$\begin{tikzcd}
\Delta : \Lambda\Box_{\mathfrak{C}}X \arrow[r, "\simeq"] & \Lambda\Box_{\mathfrak{C}}(\mathfrak{C}\Box_{\mathfrak{C}}X) \arrow[r, "\Lambda\Box_{\mathfrak{C}}(\psi\Box_{\mathfrak{C}}X)"] & \Lambda\Box_{\mathfrak{C}}((X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}X) \arrow[r, "\simeq"] & (\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \arrow[d, hook] \\
 & & & (\Lambda\Box_{\mathfrak{C}}X)\otimes_B(\Lambda\Box_{\mathfrak{C}}X),
\end{tikzcd}$$

and counit $\epsilon = \epsilon_{\mathfrak{D}} \circ \omega : \Lambda\Box_{\mathfrak{C}}X \longrightarrow B$.

Proof. By the bicolinearity of ψ, we have the commutativity of the two diagrams

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[d, "\psi"'] \arrow[ddrr, "(\psi\Box_{\mathfrak{C}}\psi)\circ\Delta_{\mathfrak{C}}"] & \mathfrak{C}\Box_{\mathfrak{C}}\mathfrak{C} \arrow[r, "\psi\Box_{\mathfrak{C}}\mathfrak{C}"] & (X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}\mathfrak{C} \arrow[dd, "(X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}\psi"] \\
(X\Box_{\mathfrak{D}}\Lambda) \arrow[d, "\simeq"'] & & \\
(X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}\mathfrak{C} \arrow[rr, "(X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}\psi"'] & & (X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda)
\end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[d, "\psi"'] \arrow[ddrr, "(\psi\Box_{\mathfrak{C}}\psi)\circ\Delta_{\mathfrak{C}}"] & \mathfrak{C}\Box_{\mathfrak{C}}\mathfrak{C} \arrow[r, "\mathfrak{C}\Box_{\mathfrak{C}}\psi"] & \mathfrak{C}\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda) \arrow[dd, "\psi\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda)"] \\
(X\Box_{\mathfrak{D}}\Lambda) \arrow[d, "\simeq"'] & & \\
\mathfrak{C}\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda) \arrow[rr, "\psi\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda)"'] & & (X\Box_{\mathfrak{D}}\Lambda)\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda).
\end{tikzcd} \tag{3.3}$$

Therefore, we have the commutativity of the diagram

$$\begin{tikzcd}
\Lambda\Box_{\mathfrak{C}}X \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X \arrow[r, "\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X"] & (\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \arrow[d, "\simeq"] \\
\Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X \arrow[d, "\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X"'] & & (\Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \arrow[d, "(\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X)"] \\
(\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \arrow[dr, "\simeq"'] & & (\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \\
 & (\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X) \arrow[ur, "(\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X)"'] &
\end{tikzcd}$$

.

Hence the coassociative property of Δ follows. On the other hand, if we put $i : (\Lambda\Box_{\mathfrak{C}}X)\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \hookrightarrow (\Lambda\Box_{\mathfrak{C}}X) \otimes_B (\Lambda\Box_{\mathfrak{C}}X)$ the canonical injection, we have,

$$\begin{aligned}&[(\epsilon_{\mathfrak{D}} \circ \omega \otimes_B (\Lambda\Box_{\mathfrak{C}}X)] \circ i \circ (\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X) \circ [\Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X] \\ &= [\epsilon_{\mathfrak{D}} \otimes_B (\Lambda\Box_{\mathfrak{C}}X)] \circ i' \circ (\omega\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X)) \circ (\Lambda\Box_{\mathfrak{C}}\psi\Box_{\mathfrak{C}}X) \circ [\Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X], \quad (3.4)\end{aligned}$$

where $i' : \mathfrak{D}\Box_{\mathfrak{D}}(\Lambda\Box_{\mathfrak{C}}X) \hookrightarrow \mathfrak{D} \otimes_B (\Lambda\Box_{\mathfrak{C}}X)$ is the canonical injection.

The first diagram of (3.1) is commutative means that the diagram

$$\begin{tikzcd}
\Lambda\Box_{\mathfrak{C}}\mathfrak{C} \arrow[r, "\Lambda\Box_{\mathfrak{C}}\psi"] \arrow[dr, "\simeq"'] & \Lambda\Box_{\mathfrak{C}}(X\Box_{\mathfrak{D}}\Lambda) \arrow[r, "\omega\Box_{\mathfrak{D}}\Lambda"] & \mathfrak{D}\Box_{\mathfrak{D}}\Lambda \\
& \Lambda \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

commutes. The composition of morphisms

$$\Lambda \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C} \xrightarrow{\simeq} \Lambda \xrightarrow{\simeq} \mathfrak{D}\Box_{\mathfrak{D}}\Lambda \hookrightarrow \mathfrak{D} \otimes_B \Lambda \xrightarrow{\epsilon_{\mathfrak{D}}\otimes_B\Lambda} B \otimes_B \Lambda$$

is exactly the morphism $[\lambda \mapsto 1_B \otimes_B \lambda]$. Then,

$$(3.4) = \Big[\sum_i \lambda_i \otimes_A x_i \mapsto 1_B \otimes_B (\sum_i \lambda_i \otimes_A x_i)\Big].$$

Analogously, by using the commutativity of the second diagram of (3.1), we complete the proof of the counit property. $\Box$

Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule, quasi-finite as a right $\mathfrak{D}$-comodule, such that ${}_B\mathfrak{D}$ is flat or $\mathfrak{C}$ is a coseparable A-coring. Let T be a k-algebra.
We will give an other characterization of the natural isomorphism (see Section 2.1):

$$\Upsilon_{-,-} : - \otimes_T \mathrm{h}_{\mathfrak{D}}(N, -) \to \mathrm{h}_{\mathfrak{D}}(N, - \otimes_T -)$$

associated to the cohom functor $\mathrm{h}_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$.
Let $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to h_{\mathfrak{D}}(N, -) \otimes_A N$ be the unit of the adjunction $(\mathrm{h}_{\mathfrak{D}}(N, -), - \otimes_A N)$.
Let $M \in {}^T\mathcal{M}^{\mathfrak{D}}$ and $W \in \mathcal{M}^T$. Since the functors $\mathrm{h}_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^A$ and $- \otimes_A N : \mathcal{M}^A \to \mathcal{M}^{\mathfrak{D}}$ are k-linear and preserve inductive limits, then the functor $\mathrm{h}_{\mathfrak{D}}(N, -) \otimes_A N : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{D}}$ is also k-linear and preserves inductive limits. By Lemma 2.1.9 (1), θ_M is a morphism in ${}^T\mathcal{M}^{\mathfrak{D}}$. Define

$$\Xi_{W,M} : \mathrm{h}_{\mathfrak{D}}(N, W \otimes_T M) \to W \otimes_T \mathrm{h}_{\mathfrak{D}}(N, M)$$

to be the unique morphism in $\mathcal{M}^A$ satisfying $(\Xi_{W,M} \otimes_A N)\theta_{W\otimes_T M} = W \otimes_T \theta_M$. We have

$$\begin{aligned}&\Big[\rho_{W\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M)} \otimes_A N - \big(W \otimes_T \mathrm{h}_{\mathfrak{D}}(N, M)\big) \otimes_A \lambda_N\Big](W \otimes_T \theta_M) \\ &\qquad = W \otimes_T \Big[\big(\rho_{\mathrm{h}_{\mathfrak{D}}(N,M)} \otimes_A N - \mathrm{h}_{\mathfrak{D}}(N, M) \otimes_A \lambda_N\big)\theta_M\Big] = 0. \quad (3.5)\end{aligned}$$

Then $\mathrm{Im}(W\otimes_T \theta_M) \subset (W\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M)\square_{\mathfrak{C}} N$, and by Lemma 1.4.5, $\Xi_{W,M}$ is a morphism in $\mathcal{M}^{\mathfrak{C}}$. Now we will verify that $\Xi_{-,M}$ is a natural transformation. For this, let $f : W \to W'$ be a morphism in $\mathcal{M}^T$. We will verify the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{h}_{\mathfrak{D}}(N, W\otimes_T M) \arrow[r, "\Xi_{W,M}"] \arrow[d] & W\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M) \arrow[d] \\
\mathrm{h}_{\mathfrak{D}}(N, W'\otimes_T M) \arrow[r, "\Xi_{W',M}"] & W'\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M).
\end{tikzcd}$$

Since θ is a natural transformation, the diagram

$$\begin{tikzcd}
W\otimes_T M \arrow[r, "\theta_{W\otimes_T M}"] \arrow[d] & \mathrm{h}_{\mathfrak{D}}(N, W\otimes_T M)\otimes_A N \arrow[d] \\
W'\otimes_T M \arrow[r, "\theta_{W'\otimes_T M}"] & \mathrm{h}_{\mathfrak{D}}(N, W'\otimes_T M)\otimes_A N
\end{tikzcd}$$

commutes. Then, we have

$$\begin{aligned}
\Big[\big(\Xi_{W',M}\mathrm{h}_{\mathfrak{D}}(N, f\otimes_T M)\big)\otimes_A N\Big]\theta_{W\otimes_T M} &= (\Xi_{W',M}\otimes_A N)\big(\mathrm{h}_{\mathfrak{D}}(N, f\otimes_T M)\otimes_A N\big)\theta_{W\otimes_T M} \\
&= (\Xi_{W',M}\otimes_A N)\theta_{W'\otimes_T M}(f\otimes_T M) \\
&= (W'\otimes_T \theta_M)(f\otimes_T M) \\
&= \big(f\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M)\otimes_A N\big)(W\otimes_T \theta_M) \\
&= \big(f\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M)\otimes_A N\big)(\Xi_{W,M}\otimes_A N)\theta_{W\otimes_T M} \\
&= \Big[\Big(\big(f\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M)\big)\Xi_{W,M}\Big)\otimes_A N\Big]\theta_{W\otimes_T M}.
\end{aligned}$$

By uniqueness, $\Xi_{W',M}\mathrm{h}_{\mathfrak{D}}(N, f\otimes_T M) = (f\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M))\Xi_{W,M}$. Finally, we will verify that $\Upsilon_{T,M} = \Xi_{T,M}^{-1}$, i.e. $\Xi_{T,M}$ satisfies the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{h}_{\mathfrak{D}}(N, T\otimes_T M) \arrow[rr, "\Xi_{T,M}"] \arrow[dr, "\simeq"'] & & T\otimes_T \mathrm{h}_{\mathfrak{D}}(N,M) \arrow[dl, "\simeq"] \\
& \mathrm{h}_{\mathfrak{D}}(N,M) &
\end{tikzcd}$$

(the canonical isomorphism $T\otimes_T M \xrightarrow{\simeq} M$ is an isomorphism in $\mathcal{M}^{\mathfrak{D}}$). Since θ is a natural transformation, the diagram

$$\begin{tikzcd}
T\otimes_T M \arrow[r, "\theta_{T\otimes_T M}"] \arrow[d, "\simeq"] & \mathrm{h}_{\mathfrak{D}}(N, T\otimes_T M)\otimes_A N \arrow[d, "\simeq"] \\
M \arrow[r, "\theta_M"] & \mathrm{h}_{\mathfrak{D}}(N,M)\otimes_A N
\end{tikzcd}$$

commutes. It follows from the commutativity of the last diagram and the fact that θ_M is T-linear, that the following diagram is also commutative

$$\begin{tikzcd}
T \otimes_T M \arrow[r, "\theta_{T\otimes_T M}"] \arrow[d, "\simeq"] & \mathrm{h}_{\mathfrak{D}}(N, T \otimes_T M) \otimes_A N \arrow[d, "\simeq"] \\
T \otimes_T \mathrm{h}_{\mathfrak{D}}(N, M) \otimes_A N \arrow[r, "\simeq"] & \mathrm{h}_{\mathfrak{D}}(N, M) \otimes_A N.
\end{tikzcd}$$

By Corollary 2.1.2, $\Xi_{W,M}$ is an isomorphism, and by Mitchell's Theorem 2.1.1, $\Upsilon_{W,M} = \Xi^{-1}_{W,M}$.

We obtain then the following generalization of [90, 1.6]:

Proposition 3.1.2. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule, quasi-finite as a right $\mathfrak{D}$-comodule, such that ${}_B\mathfrak{D}$ is flat or $\mathfrak{C}$ is a coseparable A-coring. Let $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to \mathrm{h}_{\mathfrak{D}}(N, -) \otimes_A N$ be the unit of the adjunction $(\mathrm{h}_{\mathfrak{D}}(N, -), - \otimes_A N)$, and let T be a k-algebra. If $\Upsilon_{-,-} : - \otimes_T \mathrm{h}_{\mathfrak{D}}(N, -) \to \mathrm{h}_{\mathfrak{D}}(N, - \otimes_T -)$ is the natural isomorphism associated to the cohom functor $\mathrm{h}_{\mathfrak{D}}(N, -) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$, then $\Upsilon_{W,M} = \Xi^{-1}_{W,M}$, where $\Xi_{W,M}$ is defined as above.*

As a consequence of the last result we get the following generalization of [90, 1.13].

Proposition 3.1.3. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule, quasi-finite as a right $\mathfrak{D}$-comodule, such that ${}_B\mathfrak{D}$ is flat. Suppose that ${}_A\mathfrak{C}$ is flat (resp. $\mathcal{M}^{\mathfrak{C}}$ is an abelian category). Let $\theta : 1_{\mathcal{M}^{\mathfrak{D}}} \to \mathrm{h}_{\mathfrak{D}}(N, -) \otimes_A N$ be the unit of the adjunction $(\mathrm{h}_{\mathfrak{D}}(N, -), - \otimes_A N)$. If the cohom functor $\mathrm{h}_{\mathfrak{D}}(N, -)$ is exact (resp. $\mathfrak{D}$ is a coseparable B-coring), then the natural isomorphism δ (see Corollary 2.1.15):*

$$\mathrm{h}_{\mathfrak{D}}(N, -) \simeq -\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(N, \mathfrak{D}) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$$

satisfy the property:

For every $M \in \mathcal{M}^{\mathfrak{D}}$, $\delta_M : \mathrm{h}_{\mathfrak{D}}(N, M) \to M\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(N, \mathfrak{D})$ is the unique right A-linear map satisfying

$$(\delta_M \otimes_A N)\theta_M = (M\square_{\mathfrak{D}}\theta_{\mathfrak{D}})\rho_M.$$

Proof. Let $M \in \mathcal{M}^{\mathfrak{D}}$, we have $(\Xi_{M,\mathfrak{D}} \otimes_A N)\theta_{M\otimes_B\mathfrak{D}} = M \otimes_B \theta_{\mathfrak{D}}$. On the other hand, since θ is a natural transformation, $(\Xi_{M,\mathfrak{D}} \otimes_A N)\theta_{M\otimes_B\mathfrak{D}}\rho_M = (\Xi_{M,\mathfrak{D}} \otimes_A N)(\mathrm{h}_{\mathfrak{D}}(N, \rho_M) \otimes_A N)\theta_M = (i\delta_M \otimes_A N)\theta_M$, where $i : M\square_{\mathfrak{D}}\mathrm{h}_{\mathfrak{D}}(N, \mathfrak{D}) \hookrightarrow M \otimes_B \mathrm{h}_{\mathfrak{D}}(N, \mathfrak{D})$ is the canonical injection. Therefore, $(i \otimes_A N)(\delta_M \otimes_A N)\theta_M = (M \otimes_B \theta_{\mathfrak{D}})\rho_M$. Since $i \otimes_A N$ is a monomorphism (${}_AN$ is flat), $(\delta_M \otimes_A N)\theta_M = (M\square_{\mathfrak{D}}\theta_{\mathfrak{D}})\rho_M$. □

Now we will give a series of properties concerning our comatrix coring.

Proposition 3.1.4. *Let $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$ be a bicomodule, quasi-finite as a right $\mathfrak{C}$-comodule, such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. Set $X = \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. If*

(a) $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat, the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is exact and ${}_{\mathfrak{D}}\Lambda$ is coflat, or

(b) A and B are von Neumann regular rings and the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda,-)$ *is exact, or*

(c) $\mathfrak{C}$ *and* $\mathfrak{D}$ *are coseparable corings,*

then we have

(1) $\delta_\Lambda : \mathrm{e}_{\mathfrak{C}}(\Lambda) \longrightarrow \Lambda \Box_{\mathfrak{C}} X$ *is an isomorphism of B-corings;*

(2) $\chi_{\mathfrak{D}} \circ \mathrm{h}_{\mathfrak{C}}(\Lambda, \lambda_\Lambda) = \omega \circ \delta_\Lambda : \mathrm{e}_{\mathfrak{C}}(\Lambda) \longrightarrow \mathfrak{D}$ *and* $\omega : \Lambda \Box_{\mathfrak{C}} X \longrightarrow \mathfrak{D}$ *is a homomorphism of B-corings, where* $\lambda_\Lambda : \Lambda \to \mathfrak{D} \Box_{\mathfrak{D}} \Lambda$ *is the left coaction of* Λ*,* χ *the counit of the adjunction* $(\mathrm{h}_{\mathfrak{C}}(\Lambda,-), -\Box_{\mathfrak{D}}\Lambda)$*, and* ω *is the unique* $\mathfrak{D}-\mathfrak{D}$*-bicolinear map such that* (3.7) *is the counit of the adjunction* $(-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}), -\Box_{\mathfrak{D}}\Lambda)$.

Proof. Let $\theta : 1_{\mathcal{M}^{\mathfrak{C}}} \to \mathrm{h}_{\mathfrak{C}}(\Lambda,-) \otimes_B \Lambda$ be the unit of the adjunction $(\mathrm{h}_{\mathfrak{C}}(\Lambda,-), - \otimes_B \Lambda)$.

(1) By Theorem 2.1.13, $\mathrm{h}_{\mathfrak{C}}(\Lambda,-) \simeq -\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$. In the first case, $\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ is coflat as a left $\mathfrak{C}$-comodule. By Proposition 2.1.12, $\theta_{\mathfrak{C}} : \mathfrak{C} \to \mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\Lambda$ is a $\mathfrak{C}$-bicolinear map.

From Proposition 3.1.3, for every $M \in \mathcal{M}^{\mathfrak{C}}$, we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\simeq\quad} & M\Box_{\mathfrak{C}}\mathfrak{C} \\ {\scriptstyle\theta_M}\downarrow & & \downarrow{\scriptstyle M\Box_{\mathfrak{C}}\theta_{\mathfrak{C}}} \\ \mathrm{h}_{\mathfrak{C}}(\Lambda, M)\Box_{\mathfrak{D}}\Lambda & \xrightarrow[\delta_M\Box_{\mathfrak{D}}\Lambda]{} & M\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\Lambda. \end{array}$$

From Lemma 2.1.16, the unit and the counit of the adjunction $(-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}), -\Box_{\mathfrak{D}}\Lambda)$ are

$$\eta : 1_{\mathcal{M}^{\mathfrak{C}}} \xrightarrow{\simeq} -\Box_{\mathfrak{C}}\mathfrak{C} \xrightarrow{-\Box_{\mathfrak{C}}\theta_{\mathfrak{C}}} -\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\Lambda \tag{3.6}$$

and

$$\varepsilon : -\Box_{\mathfrak{D}}\Lambda\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}) \xrightarrow{-\Box_{\mathfrak{D}}\omega} -\Box_{\mathfrak{D}}\mathfrak{D} \xrightarrow{\simeq} 1_{\mathcal{M}^{\mathfrak{D}}}. \tag{3.7}$$

To show that δ_Λ is a homomorphism of B-corings, it suffices to prove (see Proposition 1.4.9):

$$\begin{aligned} &(\delta_\Lambda \otimes_B \delta_\Lambda \otimes_B \Lambda) \circ (\mathrm{e}_{\mathfrak{C}}(\Lambda) \otimes_B \theta_\Lambda) \circ \theta_\Lambda \\ &\quad = (\Lambda\Box_{\mathfrak{C}}\theta_{\mathfrak{C}}\Box_{\mathfrak{C}}X \otimes_B \Lambda) \circ [\,\Lambda\Box_{\mathfrak{C}}X \otimes_B \Lambda \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X \otimes_B \Lambda\,] \circ (\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda, \end{aligned} \tag{3.8}$$

and

$$(\epsilon_{\mathfrak{D}} \otimes_B \Lambda) \circ (\omega \otimes_B \Lambda) \circ (\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda = [\,\Lambda \xrightarrow{\simeq} B \otimes_B \Lambda\,]. \tag{3.9}$$

Now we will prove (8). We have,

$$\begin{aligned}(\delta_\Lambda \otimes_B \delta_\Lambda \otimes_B \Lambda) \circ (\mathrm{e}_{\mathfrak{C}}(\Lambda) \otimes_B \theta_\Lambda) \circ \theta_\Lambda &= \Big[\delta_\Lambda \otimes_B \big((\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda\big)\Big] \circ \theta_\Lambda \\ &= \Big[(\Lambda \square_{\mathfrak{C}} X) \otimes_B \big((\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda\big)\Big] \circ (\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda \\ &= \Big[(\Lambda \square_{\mathfrak{C}} X) \otimes_B \big((\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ \rho_\Lambda\big)\Big] \circ (\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ \rho_\Lambda \\ &\quad \text{(using Proposition 3.1.3)} \\ &= (\Lambda \square_{\mathfrak{C}} X \otimes_B \Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ (\Lambda \square_{\mathfrak{C}} X \otimes_B \rho_\Lambda) \circ (\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ \rho_\Lambda \end{aligned} \tag{3.10}$$

On the other hand, by Proposition 3.1.3,

$$\begin{aligned}&(\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}} \square_{\mathfrak{C}} X \otimes_B \Lambda) \circ [\, \Lambda \square_{\mathfrak{C}} X \otimes_B \Lambda \xrightarrow{\simeq} \Lambda \square_{\mathfrak{C}} \mathfrak{C} \square_{\mathfrak{C}} X \otimes_B \Lambda \,] \circ (\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda \\ &\quad = (\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}} \square_{\mathfrak{C}} X \otimes_B \Lambda) \circ [\, \Lambda \square_{\mathfrak{C}} X \otimes_B \Lambda \xrightarrow{\simeq} \Lambda \square_{\mathfrak{C}} \mathfrak{C} \square_{\mathfrak{C}} X \otimes_B \Lambda \,] \circ (\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ \rho_\Lambda. \end{aligned} \tag{3.11}$$

Now, by Proposition 2.1.12, $\theta_{\mathfrak{C}} : \mathfrak{C} \to \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \otimes_B \Lambda$ is $\mathfrak{C}$-bicolinear. Then $\theta_{\mathfrak{C}}$ makes commutative the diagrams (3.2) and (3.3) (by replacing ψ by $\theta_{\mathfrak{C}}$). Hence (10) = (11).

Finally, we will prove (9).

$$\begin{aligned}(\epsilon_{\mathfrak{D}} \otimes_B \Lambda) \circ (\omega \otimes_B \Lambda) \circ (\delta_\Lambda \otimes_B \Lambda) \circ \theta_\Lambda &= (\epsilon_{\mathfrak{D}} \otimes_B \Lambda) \circ (\omega \otimes_B \Lambda) \circ (\Lambda \square_{\mathfrak{C}} \theta_{\mathfrak{C}}) \circ \rho_\Lambda \\ &\quad \text{(using Proposition 3.1.3)} \\ &= (\epsilon_{\mathfrak{D}} \otimes_B \Lambda) \circ \lambda_\Lambda \qquad \text{(using the first equality of (3.1))} \\ &= [\, \Lambda \xrightarrow{\simeq} B \otimes_B \Lambda \,]. \end{aligned}$$

(2) Since δ is a natural isomorphism, the diagram

$$\begin{tikzcd} \mathrm{h}_{\mathfrak{C}}(\Lambda, \Lambda) \arrow[r, "\delta_\Lambda"] \arrow[d, "\mathrm{h}_{\mathfrak{C}}(\Lambda, \lambda_\Lambda)"'] & \Lambda \square_{\mathfrak{C}} X \arrow[d, "\lambda_\Lambda \square_{\mathfrak{C}} X"] \\ \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{D} \square_{\mathfrak{D}} \Lambda) \arrow[r, "\delta_{\mathfrak{D} \square_{\mathfrak{D}} \Lambda}"'] & (\mathfrak{D} \square_{\mathfrak{D}} \Lambda) \square_{\mathfrak{C}} X \end{tikzcd}$$

is commutative. The counit of the adjunction $(\mathrm{h}_{\mathfrak{C}}(\Lambda, -), - \square_{\mathfrak{D}} \Lambda)$ is

$$\mathrm{h}_{\mathfrak{C}}(\Lambda, - \square_{\mathfrak{D}} \Lambda) \xrightarrow{\delta_{- \square_{\mathfrak{D}} \Lambda}} - \square_{\mathfrak{D}} \Lambda \square_{\mathfrak{C}} X \xrightarrow{- \square_{\mathfrak{D}} \omega} - \square_{\mathfrak{D}} \mathfrak{D} \xrightarrow{\simeq} 1_{\mathcal{M}^{\mathfrak{D}}}. \tag{3.12}$$

Then, $\chi_{\mathfrak{D}} \circ \mathrm{h}_{\mathfrak{C}}(\Lambda, \lambda_\Lambda) = [\, \mathfrak{D} \square_{\mathfrak{D}} \mathfrak{D} \xrightarrow{\simeq} \mathfrak{D} \,] \circ (\mathfrak{D} \square_{\mathfrak{D}} \omega) \circ (\lambda_\Lambda \square_{\mathfrak{C}} X) \circ \delta_\Lambda$.
Since ω is left $\mathfrak{D}$-colinear, and $\lambda_{\Lambda \square_{\mathfrak{C}} X} = \lambda_\Lambda \square_{\mathfrak{C}} X$, and from the commutativity of the following diagram

$$\begin{tikzcd} \Lambda \square_{\mathfrak{C}} X \arrow[r, "\simeq"] \arrow[dr, "\lambda_{\Lambda \square_{\mathfrak{C}} X}"'] & \mathfrak{D} \square_{\mathfrak{D}} (\Lambda \square_{\mathfrak{C}} X) \arrow[r, "\mathfrak{D} \square_{\mathfrak{D}} \omega"] \arrow[d, hook] & \mathfrak{D} \square_{\mathfrak{D}} \mathfrak{D} \arrow[r, "\simeq"] \arrow[d, hook] & \mathfrak{D} \arrow[dl, "\Delta_{\mathfrak{D}}"] \\ & \mathfrak{D} \otimes_B (\Lambda \square_{\mathfrak{C}} X) \arrow[r, "\mathfrak{D} \otimes_B \omega"'] & \mathfrak{D} \otimes_B \mathfrak{D} & \end{tikzcd},$$

$\chi_{\mathfrak{D}} \circ \mathrm{h}_{\mathfrak{C}}(\Lambda, \lambda_\Lambda) = \omega \circ \delta_\Lambda$. Finally, by Remark 1.4.12, it is a homomorphism of B-corings. Hence, ω is also a homomorphism of B-corings. $\square$

Corollary 3.1.5. *Let C be a coalgebra over a field k, and let $\Lambda \in \mathcal{M}^C$ be a quasi-finite and injective comodule. Then $\mathrm{e}_C(\Lambda) \simeq \Lambda \Box_C \mathrm{h}_C(\Lambda, C)$ as coalgebras.*

Example 3.1.6. Let ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ be flat. Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings. Suppose that $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat and ${}_{\mathfrak{D}}(B \otimes_A \mathfrak{C})$ is coflat, or $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable corings. Then, we have $(- \otimes_A B, -\Box_{\mathfrak{D}}(B \otimes_A \mathfrak{C}))$ is an adjoint pair, and $- \otimes_A B \simeq -\Box_{\mathfrak{C}}(\mathfrak{C} \otimes_A B)$ (see the proof of Theorem 2.3.1). By Proposition 1.5.5, the comatrix coring $(B \otimes_A \mathfrak{C})\Box_{\mathfrak{C}}(\mathfrak{C} \otimes_A B)$ ($\simeq \mathrm{e}_{\mathfrak{C}}(B \otimes_A \mathfrak{C})$) is isomorphic (as corings) to the coring $B\mathfrak{C}B$ (see Example 1.2.2 (3)).

Now, we need the left-handed version of Corollary 2.3.7.

Lemma 3.1.7. [19, 27.13]
Let $\mathfrak{C}$ be an A-coring and $T = \mathfrak{C}^*$. Then the following statements are equivalent

(a) $\mathfrak{C}$ is a Frobenius coring;

(b) $\mathfrak{C}_A$ is finitely generated projective and $\mathfrak{C} \simeq T$ as (A°, T)-bimodules, such that $\mathfrak{C}$ is a right T-module by $c.t = t(c_{(1)}).c_{(2)}$, for every $c \in \mathfrak{C}$ and $t \in T$.

Theorem 3.1.8. *Let ${}_A\mathfrak{C}$ be flat. Let $\Lambda \in {}_B\mathcal{M}^{\mathfrak{C}}$ be a bicomodule, quasi-finite as a right $\mathfrak{C}$-comodule. Set $X = \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \in {}^{\mathfrak{C}}\mathcal{M}_B$. If $\mathfrak{C}_A$ is flat and the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is exact, or if $\mathfrak{C}$ is a coseparable coring, then we have*

(1) The functor $- \otimes_B \Lambda : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}}$ is separable if and only if the comatrix coring $\Lambda \Box_{\mathfrak{C}} X$ is a cosplit B-coring.

(2) If the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is separable then the comatrix coring $\Lambda \Box_{\mathfrak{C}} X$ is a coseparable B-coring.

(3) If $- \otimes_B \Lambda : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}}$ is a Frobenius functor, either X_B is flat and $\Lambda_{\mathfrak{C}}$ is coflat or $\mathfrak{C}$ is coseparable, and if for all $f \in (\Lambda \Box_{\mathfrak{C}} X)^*$ and all $b \in B$, $bH(f) = H(f)b$, where $H : (\Lambda \Box_{\mathfrak{C}} X)^* \longrightarrow \Lambda \Box_{\mathfrak{C}} X$ is the isomorphism defined by (3.15) (this is the case if $\Lambda \Box_{\mathfrak{C}} X$ is a coalgebra (i.e. $B = k$)), then the comatrix coring $\Lambda \Box_{\mathfrak{C}} X$ is a Frobenius B-coring.

Proof. (1) Let $\omega : \Lambda \Box_{\mathfrak{C}} X \to B$ the unique B-bilinear map such that

$$\varepsilon : - \otimes_B \Lambda \Box_{\mathfrak{C}} \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}) \xrightarrow{-\otimes_B \omega} - \otimes_B \mathfrak{D} \xrightarrow{\simeq} 1_{\mathcal{M}_B} \tag{3.13}$$

is the counit of the adjunction $(-\Box_{\mathfrak{C}} \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C}), - \otimes_B \Lambda)$. We have, $\epsilon_{\Lambda \Box_{\mathfrak{C}} X} = \omega$. By Lemma 2.1.16 and Rafael's Proposition 1.1.48, the functor $- \otimes_B \Lambda : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}}$ is separable if and only if there exists a B-bilinear map $\omega' : B \to \Lambda \Box_{\mathfrak{C}} X$ satisfying $\omega \circ \omega' = 1_B$. Hence, (1) follows from the definition of a cosplit coring.

(2) We know that the unit of the adjunction $(-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}), -\otimes_B\Lambda)$ is

$$\eta : 1_{\mathcal{M}^{\mathfrak{C}}} \xrightarrow{\simeq} -\Box_{\mathfrak{C}}\mathfrak{C} \xrightarrow{-\Box_{\mathfrak{C}}\theta_{\mathfrak{C}}} -\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\otimes_B\Lambda. \tag{3.14}$$

By Lemma 2.1.16 and Rafael's Proposition 1.1.48, the cohom functor is separable if and only if there exists a $\mathfrak{C}-\mathfrak{C}$-bicolinear map $\psi' : \mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\otimes_B\Lambda \to \mathfrak{C}$ satisfying $\psi'\circ\theta_{\mathfrak{C}} = 1_{\mathfrak{C}}$. We have, $\Delta_{\Lambda\Box_{\mathfrak{C}}X} = (\Lambda\Box_{\mathfrak{C}}\theta_{\mathfrak{C}}\Box_{\mathfrak{C}}X)\circ[\ \Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X\ ]$.
Let $f = [\ \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\ ]\circ(\Lambda\Box_{\mathfrak{C}}\psi'\Box_{\mathfrak{C}}X)$. We have, $f\circ\Delta_{\Lambda\Box_{\mathfrak{C}}X} = 1_{\Lambda\Box_{\mathfrak{C}}X}$,

$$\begin{aligned}\Delta_{\Lambda\Box_{\mathfrak{C}}X}\circ f &= \big[[\ \Lambda\Box_{\mathfrak{C}}\mathfrak{C} \xrightarrow{\simeq} \Lambda\ ]\circ(\Lambda\Box_{\mathfrak{C}}\psi')\big]\Box_{\mathfrak{C}}\big[(\psi\Box_{\mathfrak{C}}X)\circ[\ X \xrightarrow{\simeq} \mathfrak{C}\Box_{\mathfrak{C}}X\ ]\big]\\ &= (f\otimes_B\Lambda\Box_{\mathfrak{C}}X)\circ(\Lambda\Box_{\mathfrak{C}}X\otimes_B\Delta_{\Lambda\Box_{\mathfrak{C}}X}),\end{aligned}$$

and

$$\begin{aligned}\Delta_{\Lambda\Box_{\mathfrak{C}}X}\circ f &= \big[(\Lambda\Box_{\mathfrak{C}}\psi)\circ[\ \Lambda \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}\mathfrak{C}\ ]\big]\Box_{\mathfrak{C}}\big[[\ \mathfrak{C}\Box_{\mathfrak{C}}X \xrightarrow{\simeq} X\ ]\circ(\psi'\Box_{\mathfrak{C}}X)\big]\\ &= (\Lambda\Box_{\mathfrak{C}}X\otimes_B f)\circ(\Delta_{\Lambda\Box_{\mathfrak{C}}X}\otimes_B\Lambda\Box_{\mathfrak{C}}X).\end{aligned}$$

From the definition of a coseparable coring, $\Lambda\Box_{\mathfrak{C}}X$ is a coseparable coring.

(3) We will prove it using Lemma 3.1.7. Set $T = (\Lambda\Box_{\mathfrak{C}}X)^*$, $\Delta = \Delta_{\Lambda\Box_{\mathfrak{C}}X}$ and $\epsilon = \epsilon_{\Lambda\Box_{\mathfrak{C}}X}$. At first from Theorem 2.1.20, $(X\otimes_B -, \Lambda\Box_{\mathfrak{C}}-)$ is a Frobenius pair, and then $\Lambda\Box_{\mathfrak{C}}X\otimes_B -$ is a Frobenius functor. Hence $(\Lambda\Box_{\mathfrak{C}}X)_B$ is finitely generated projective.
Let us consider the isomorphism

$$H : T \xrightarrow{\simeq} \mathrm{e}_{\mathfrak{C}}(\Lambda)^* \xrightarrow{\phi_{\Lambda,B}} \mathrm{End}_{\mathfrak{C}}(\Lambda) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{C}}(B\otimes_B\Lambda,\Lambda) \xrightarrow{\simeq} \mathrm{Hom}_B(B,\Lambda\Box_{\mathfrak{C}}X) \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\ ,$$

where $\phi_{\Lambda,B} : \mathrm{e}_{\mathfrak{C}}(\Lambda)^* \longrightarrow \mathrm{End}_{\mathfrak{C}}(\Lambda)$ is the canonical anti-isomorphism of rings defined in Proposition 1.4.9. Then, (by using the following consequence of Proposition 3.1.3; $\Delta = \big[(\delta_\Lambda\otimes_B\Lambda)\theta_\Lambda\big]\Box_{\mathfrak{C}}X)$, for $f\in T$,

$$\begin{aligned}&H(f) = \Big[\big(\phi_{\Lambda,B}(f\delta_\Lambda)[\ B\otimes_B\Lambda \xrightarrow{\simeq} \Lambda\ ]\big)\Box_{\mathfrak{C}}X\Big]\xi_B(1_B)\\ &= [\ B\otimes_B\Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\ ](f\otimes_B\Lambda\Box_{\mathfrak{C}}X)\Delta[\ B\otimes_B\Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\ ]\xi_B(1_B),\end{aligned} \tag{3.15}$$

where $\xi : 1_{\mathcal{M}_B} \to -\otimes_B\Lambda\Box_{\mathfrak{C}}X$ is the unit of the adjunction $(-\otimes_B\Lambda, -\Box_{\mathfrak{C}}X)$.

Now, let $f, f'\in T$, $\phi_{\Lambda,B}(f *^r f') = \phi_{\Lambda,B}(f'\delta_\Lambda)\phi_{\Lambda,B}(f'\delta_\Lambda)$.

$$\begin{aligned}H(f *^r f') &= \big(\phi_{\Lambda,B}(f'\delta_\Lambda)\Box_{\mathfrak{C}}X\big)\big(H(f)\big)\\ &= [\ B\otimes_B\Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\ ](f'\otimes_B\Lambda\Box_{\mathfrak{C}}X)\Delta\big(H(f)\big)\\ \\ &= H(f).f'.\end{aligned}$$

Finally, let $f \in T$, $b \in B$ and let $i_R : B \to T$ be the anti-morphism of rings defined by $i_R(b) = \epsilon(b-)$ for $B \in B$ (see Proposition 1.2.3 (1)).

$$H(b.f) = H(i_R(b) *^r f) \ = \Big[\Big(\phi_{\Lambda,B}(f\delta_\Lambda)\phi_{\Lambda,B}\big(i_R(b)\delta_\Lambda\big)[\,\Lambda \xrightarrow{\simeq} B \otimes_B \Lambda\,]\Big)\Box_{\mathfrak{C}}X\Big]\xi_B(1_B).$$

On the other hand,

$$\begin{aligned}\phi_{\Lambda,B}\big(i_R(b)\delta_\Lambda\big)\Box_{\mathfrak{C}}X &= [\,B \otimes_B \Lambda\Box_{\mathfrak{C}}X \xrightarrow{\simeq} \Lambda\Box_{\mathfrak{C}}X\,]\big(\epsilon(b-) \otimes_B \Lambda\Box_{\mathfrak{C}}X\big)\Delta \\ &= b1_{\Lambda\Box_{\mathfrak{C}}X}.\end{aligned}$$

Hence $H(b.f) = bH(f)$. □

The last result is a generalization of [17, Theorem 3.2(1), Theorem 3.5(1)].

Corollary 3.1.9. *Let ${}_BM_A$ be a (B, A)-bimodule such that M_A is finitely generated projective. Then*

(1) ${}_AM^*_B$ is a separable bimodule if and only if the comatrix coring $M^* \otimes_B M$ is a cosplit A-coring.

(2) If M is a separable bimodule, then the comatrix coring $M^* \otimes_B M$ is a coseparable A-coring.

(3) If M is a Frobenius bimodule, then the comatrix coring $M^* \otimes_B M$ is a Frobenius A-coring.

Proof. Define $\psi : B \to M\otimes_A M^*$ by $b \mapsto \sum_{i=1}^n be_i\otimes e_i^* = \sum_{i=1}^n e_i\otimes e_i^*b$, where $\{e_i, e_i^*\}_{i\in\{1,\dots,n\}}$ is a dual basis basis of M_A, and $\omega : M^* \otimes_B M \to A$ by $\varphi \otimes m \mapsto \varphi(m)$ (the evaluation map). Then $(A, B, {}_AM^*_B, {}_BM_A, \omega, \psi)$ is a comatrix coring context (see Example 1.2.2 (5)). From Proposition 2.1.17, we have the adjunction $(- \otimes_B M, - \otimes_A M^*)$, in particular, $\Lambda = M^* \in {}_A\mathcal{M}_B$ is quasi-finite as a right B-(co)module, and the cohom functor is $- \otimes_B M$.

(1) It is clear from the definition of a separable bimodule (see Section 1.1), that ${}_AM^*_B$ is a separable bimodule if and only if ω is an A-bimodule retraction.

(2) Let $S = \mathrm{End}_A(M_A)$. Define $\lambda : B \to S$ by $b \mapsto \lambda_B : M \to M, [m \mapsto bm]$, and $\phi : S \to M \otimes_A M^*$ by $s \mapsto \sum_i s(e_i) \otimes e_i^*$, where $\{e_i, e_i^*\}_{i\in\{1,\dots,n\}}$ is a dual basis of M_A. ϕ is an isomorphism of B-bimodule, with inverse map $m \otimes \varphi \mapsto [x \mapsto m\varphi(x)]$. It is easy to see that $\psi = \phi \circ \lambda$. From Sugano's Proposition 1.1.73, the bimodule M is separable if and only if λ is a split extention, i.e., λ is a B-bimodule section. However that is equivalent to ψ is a B-bimodule section.

(3) This is [17, Theorem 3.7(1)]. □

3.2 Applications to equivalences of categories of comodules

In this section we will generalize and improve the main results concerning equivalences between categories of comodules given in [90], [4] and [19]. We also give new characterizations of equivalences between categories of comodules over separable corings or corings with a duality.

We recall from [64, p. 93] that an *adjoint equivalence of categories* is an adjunction $(S, T, \eta, \varepsilon)$ in which both the unit $\eta : 1 \to TS$ and the counit $\varepsilon : ST \to 1$ are natural isomorphisms.

Proposition 3.2.1. *Let $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ be functors. Then (S, T) is a pair of inverse equivalences if and only if S and T are part of an adjoint equivalence $(S, T, \eta, \varepsilon)$.*

Proof. The proof is analogous to that of [64, Theorem IV.4.1]. The "if" part is trivial. For the "only if" part, let $\eta : 1 \to TS$ be a natural equivalence, then $\varphi_{C,D}(\alpha) = T(\alpha).\eta_C$ for $\alpha : S(C) \to D$, is a natural transformation in C and D. Since T is faithful and full, φ is a natural equivalence, and T is a right adjoint to S. Therefore the counit of this adjunction $\varepsilon : ST \to 1$ satisfies $T(\varepsilon_D).\eta_{T(D)} = 1_{T(D)}$ for every $D \in \mathbf{D}$, and consequently $T(\varepsilon_D) = (\eta_{T(D)})^{-1}$ is invertible. Since T is faithful and full, ε_D is also invertible, and ε is a natural equivalence (to show that ε is a natural equivalence we can also use Proposition 1.1.42). $\square$

The following result generalizes the case of the category of modules [10, II (2.4)].

Proposition 3.2.2. *Suppose that ${}_A\mathfrak{C}$, $\mathfrak{C}_A$, ${}_B\mathfrak{D}$ and $\mathfrak{D}_B$ are flat. Let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.*

The following statements are equivalent:

(1) $(-\Box_{\mathfrak{C}} X, -\Box_{\mathfrak{D}} \Lambda)$ is a pair of inverse equivalences;

(2) there exist bicomodule isomorphisms

$$f : X \Box_{\mathfrak{D}} \Lambda \to \mathfrak{C} \quad \text{and} \quad g : \Lambda \Box_{\mathfrak{C}} X \to \mathfrak{D}$$

in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$ and ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{D}}$ respectively, such that

(a) ${}_A X$ and ${}_B\Lambda$ are flat, and $\omega_{X,\Lambda} = \rho_X \otimes_B \Lambda - X \otimes_A \rho_\Lambda$ is pure in ${}_A\mathcal{M}$ and $\omega_{\Lambda,X} = \rho_\Lambda \otimes_A X - \Lambda \otimes_B \rho_X$ is pure in ${}_B\mathcal{M}$, or

(b) ${}_{\mathfrak{C}}X$ and ${}_{\mathfrak{D}}\Lambda$ are coflat.

In such a case the diagrams

$$\begin{tikzcd}
\Lambda\Box_{\mathfrak{C}}X\Box_{\mathfrak{D}}\Lambda \arrow[r, "\Lambda\Box_{\mathfrak{C}}f"] \arrow[d, "g\Box_{\mathfrak{D}}\Lambda"'] & \Lambda\Box_{\mathfrak{C}}\mathfrak{C} \arrow[d, "\simeq"] \\
\mathfrak{D}\Box_{\mathfrak{D}}\Lambda \arrow[r, "\simeq"] & \Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
X\Box_{\mathfrak{D}}\Lambda\Box_{\mathfrak{C}}X \arrow[r, "f\Box_{\mathfrak{C}}X"] \arrow[d, "X\Box_{\mathfrak{D}}g"'] & \mathfrak{C}\Box_{\mathfrak{C}}X \arrow[d, "\simeq"] \\
X\Box_{\mathfrak{D}}\mathfrak{D} \arrow[r, "\simeq"] & X
\end{tikzcd}
\tag{3.16}$$

commute.

If A and B are von Neumann regular rings, or if $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable corings (without $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat), the conditions (a) and (b) can be deleted.

Proof. Clear from Lemma 2.1.16 and Propositions 2.1.17, 3.2.1. □

Definition 3.2.3. A bicomodule $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is called a *cogenerator preserving* (resp. *an injector-cogenerator*) as a right $\mathfrak{D}$-comodule if the functor $- \otimes_A N : \mathcal{M}_A \to \mathcal{M}^{\mathfrak{D}}$ preserves cogenerator (resp. injective cogenerator) objects.

The version for module categories of the first part of the following lemma is given in [5, Exercise 20.8].

Lemma 3.2.4. *Let $\mathbf{C}$ and $\mathbf{D}$ be two abelian categories and $S : \mathbf{C} \to \mathbf{D}$ a functor. Let $T : \mathbf{D} \to \mathbf{C}$ be a right adjoint of S. We have the following properties:*

(1) If $N \in \mathbf{D}$ is cogenerator and $T(N) \in \mathbf{C}$ is injective, then S is exact.

(2) If $N \in \mathbf{D}$ is cogenerator and S is faithful, then $T(N) \in \mathbf{C}$ is a cogenerator.

(3) If $T(N) \in \mathbf{C}$ is a cogenerator for some $N \in \mathbf{D}$, then S is faithful.

If moreover $\mathbf{D}$ is an AB 3 category with generators and has enough injectives (e.g. if $\mathbf{D}$ is a Grothendieck category), then the following properties hold:

(4) S is faithful if and only if T preserves cogenerator objects.

(5) S is faithfully exact if and only if T preserves injective cogenerator objects.

Proof. Let $N \in \mathbf{D}$. Then we have a natural isomorphism

$$\eta_{-,N} : \mathrm{Hom}_{\mathbf{D}}(S(-), N) \to \mathrm{Hom}_{\mathbf{C}}(-, T(N)).$$

The functor $\mathrm{Hom}_{\mathbf{D}}(S(-), N)$ is the composition of covariant functors

$$\mathbf{C}^{\circ} \xrightarrow{S^{\circ}} \mathbf{D}^{\circ} \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(-,N)} \mathbf{Ab}\,,$$

where S° is the dual functor of S. Note that S° is exact (resp. faithful) if and only if S is exact (resp. faithful).

(1) By assumptions, $\mathrm{Hom}_{\mathbf{D}}(-, T(N))$ is exact and $\mathrm{Hom}_{\mathbf{C}}(-, N)$ is faithful. By Theorem 1.1.11, $\mathrm{Hom}_{\mathbf{C}}(-, N)$ reflects exact sequences. Then S° is exact. Hence S is so.

(2) Follows directly from the fact that the composition of two faithful functors is also faithful.

(3) Follows directly from the fact that if the composition of two functors GF is faithful then F is also faithful.

(4) is an immediate consequence of (2), (3) and the fact that $\mathbf{D}$ has a cogenerator object (see Proposition 1.1.23).

(5) The "only if" part is obvious from Lemma 1.1.34 and (4). The "if" part is an immediate consequence of Proposition 1.1.23 and (1) and (3). $\square$

Corollary 3.2.5. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule, quasi-finite as a right $\mathfrak{D}$-comodule, such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. N is a cogenerator preserving (resp. an injector-cogenerator) as a right $\mathfrak{D}$-comodule if and only if the cohom functor $h_{\mathfrak{D}}(N,-)$ is faithful (resp. faithfully exact).*

The first part of the following is [19, 23.10]. We think that the proof we give here is more clear.

Proposition 3.2.6. *Let ${}_A\mathfrak{C}$ be flat and let $\Lambda \in {}_B\mathcal{M}^{\mathfrak{C}}$ be quasi-finite as a right $\mathfrak{C}$-comodule. Let $\mathrm{h}_{\mathfrak{C}}(\Lambda,-)$ and $\mathrm{e}_{\mathfrak{C}}(\Lambda)$ be the cohom functor and the coendomorphism coring of Λ respectively. Suppose that $\mathrm{e}_{\mathfrak{C}}(\Lambda)$ is flat as left B-module.*

(1) If $\mathfrak{C}_A$ is flat and $\mathrm{e}_{\mathfrak{C}}(\Lambda)$ is flat as right B-module, and Λ is an injector-cogenerator as a right $\mathfrak{C}$-comodule, then the functors

$$-\square_{\mathrm{e}_{\mathfrak{C}}(\Lambda)}\Lambda : \mathcal{M}^{\mathrm{e}_{\mathfrak{C}}(\Lambda)} \to \mathcal{M}^{\mathfrak{C}}, \qquad \mathrm{h}_{\mathfrak{C}}(\Lambda,-) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathrm{e}_{\mathfrak{C}}(\Lambda)}, \tag{3.17}$$

are inverse equivalences.

(2) If $\mathfrak{C}$ and $\mathrm{e}_{\mathfrak{C}}(\Lambda)$ are coseparable corings, and Λ is a cogenerator preserving as a right $\mathfrak{C}$-comodule, then the functors of (3.17) are inverse equivalences.

Proof. We will prove at the same time the two statements. Let $\theta : 1_{\mathcal{M}^{\mathfrak{C}}} \to \mathrm{h}_{\mathfrak{C}}(\Lambda,-) \otimes_B \Lambda$ be the unit of the adjunction $(\mathrm{h}_{\mathfrak{C}}(\Lambda,-), -\otimes_B \Lambda)$. By Theorem 2.1.13, $\mathrm{h}_{\mathfrak{C}}(\Lambda,-) \simeq -\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$. In the case (1), $\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ is coflat as a left $\mathfrak{C}$-comodule. We put $\mathfrak{D} = \mathrm{e}_{\mathfrak{C}}(\Lambda)$. By Proposition 2.1.12, $\delta_\Lambda : \mathfrak{D} \to \Lambda\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ is $\mathfrak{D}$-bicolinear. Therefore, $-\square_{\mathfrak{D}}\Lambda\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}) \simeq -\square_{\mathfrak{D}}\mathfrak{D}$ is exact. Hence, in the case (1), the functor $-\square_{\mathfrak{D}}\Lambda$ is exact (since $\mathrm{h}_{\mathfrak{C}}(\Lambda,-)$ is faithful). From the proof of Proposition 3.1.4(1), the unit of the adjunction $(-\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}), -\square_{\mathfrak{D}}\Lambda)$ is

$$\eta : 1_{\mathcal{M}^{\mathfrak{C}}} \xrightarrow{\simeq} -\square_{\mathfrak{C}}\mathfrak{C} \xrightarrow{-\square_{\mathfrak{C}}\theta_{\mathfrak{C}}} -\square_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\square_{\mathfrak{D}}\Lambda.$$

From Proposition 3.1.3, δ_Λ is the unique map making the following diagram commutative

$$\begin{tikzcd}
\Lambda \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \Lambda\square_{\mathfrak{C}}\mathfrak{C} \arrow[d, "\Lambda\square_{\mathfrak{C}}\theta_{\mathfrak{C}}"] \\
\mathfrak{D}\square_{\mathfrak{D}}\Lambda \arrow[r, "\delta_\Lambda\square_{\mathfrak{D}}\Lambda"'] & \Lambda\square_{\mathfrak{C}}(\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\square_{\mathfrak{D}}\Lambda).
\end{tikzcd}$$

By Lemma 2.1.16, Poposition 2.1.17, the counit of the adjunction $(-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}),-\Box_{\mathfrak{D}}\Lambda)$ is

$$\varepsilon : -\Box_{\mathfrak{D}}\Lambda\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}) \xrightarrow{-\Box_{\mathfrak{D}}\delta_{\Lambda}^{-1}} -\Box_{\mathfrak{D}}\mathfrak{D} \xrightarrow{\simeq} 1_{\mathcal{M}^{\mathfrak{D}}},$$

and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathfrak{C}\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}) \arrow[d, "\theta_{\mathfrak{C}}\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})"] \\
\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\mathfrak{D} & \mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\Lambda\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}). \arrow[l, "\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})\Box_{\mathfrak{D}}\delta_{\Lambda}^{-1}"']
\end{tikzcd}$$

Then, $\theta_{\mathfrak{C}}\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ is an isomorphism. Since $-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ is faithful, $\theta_{\mathfrak{C}}$ is also an isomorphism. Finally, the unit and the counit of the adjunction $(-\Box_{\mathfrak{C}}\mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C}),-\Box_{\mathfrak{D}}\Lambda)$ are natural isomorphisms. □

The first part of the following is contained in [35, Theorem 3.10].

Corollary 3.2.7. *Let ${}_BM_A$ be a (B,A)-bimodule such that M_A is finitely generated projective.*

(1) The following statements are equivalent

(a) ${}_A(M^*\otimes_B M)$ is flat and $-\otimes_B M : \mathcal{M}_B \to \mathcal{M}^{M^*\otimes_B M}$ is an equivalence of categories;

(b) ${}_BM$ is faithfully flat.

(2) If $\mathcal{M}^{M^*\otimes_B M}$ is an abelian category, and $M^*\otimes_B M$ is an A-coseparable coring, then the following statements are equivalent

(a) $-\otimes_B M : \mathcal{M}_B \to \mathcal{M}^{M^*\otimes_B M}$ is an equivalence of categories;

(b) ${}_BM$ is completely faithful.

Proof. It suffices to take $\Lambda = M^* \in {}_A\mathcal{M}_B$ in Proposition 3.2.6. □

The first part of the following is contained in [19, 23.12]. We think that the proof we give here is more clear.

Proposition 3.2.8. *Suppose that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat, and let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. If $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat, then the following statements are equivalent*

(1) $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ is a pair of inverse equivalences;

(2) Λ is quasi-finite injector-cogenerator as a right $\mathfrak{C}$-comodule, $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ as corings and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda,\mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$;

(3) X is quasi-finite injector-cogenerator as a right $\mathfrak{D}$-comodule, $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ as corings and $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X,\mathfrak{D})$ in ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

If moreover $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable or cosemisimple, then (1) is equivalent to

(4) Λ *is quasi-finite cogenerator preserving as a right* $\mathfrak{C}$*-comodule,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(5) X *is quasi-finite cogenerator preserving as a right* $\mathfrak{D}$*-comodule,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*.*

Proof. At first we will prove the first part.

(1) $\Rightarrow$ (2) By Proposition 2.1.18, Λ is quasi-finite injector as a right $\mathfrak{C}$-comodule, and $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ in ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$. Therefore, $-\square_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ is faithful. By Lemma 3.2.4, Λ is quasi-finite injector-cogenerator as a right $\mathfrak{C}$-comodule. Finally, by Proposition 3.1.4, $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ as corings.

(2) $\Rightarrow$ (1) By Proposition 2.1.18, $-\square_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$. Hence (1) follows obviously from Proposition 3.2.6(1).

(1) $\Leftrightarrow$ (3) Follows by symmetry.

Now we will prove the second part. The case "cosemisimple" is obvious from the first part. It suffices to show the "coseparable" case.

(1) $\Rightarrow$ (4) Obvious from the first part.

(4) $\Rightarrow$ (1) By Proposition 2.1.17, $-\square_{\mathfrak{C}}X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, -)$. Hence (1) follows obviously from Proposition 3.2.6(2).

(1) $\Leftrightarrow$ (5) Follows by symmetry. $\square$

As an immediate consequence of Proposition 2.1.17 and Proposition 3.2.8, we get the two following theorems.

Theorem 3.2.9. *Suppose that* ${}_A\mathfrak{C}$*,* $\mathfrak{C}_A$*,* ${}_B\mathfrak{D}$ *and* $\mathfrak{D}_B$ *are flat, and let* $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*. The following statements are equivalent:*

(1) $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ *is a pair of inverse equivalences with* $X_{\mathfrak{D}}$ *and* $\Lambda_{\mathfrak{C}}$ *coflat;*

(2) $(\Lambda\square_{\mathfrak{C}}-, X\square_{\mathfrak{D}}-)$ *is a pair of inverse equivalences with* ${}_{\mathfrak{C}}X$ *and* ${}_{\mathfrak{D}}\Lambda$ *coflat;*

(3) Λ *is quasi-finite injector-cogenerator as a right* $\mathfrak{C}$*-comodule with* $X_{\mathfrak{D}}$ *and* $\Lambda_{\mathfrak{C}}$ *coflat,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(4) X *is quasi-finite injector-cogenerator as a right* $\mathfrak{D}$*-comodule with* $X_{\mathfrak{D}}$ *and* $\Lambda_{\mathfrak{C}}$ *coflat,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*;*

(5) Λ *is quasi-finite injector-cogenerator coflat on both sides,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(6) X *is quasi-finite injector-cogenerator coflat on both sides,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings, and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*.*

Theorem 3.2.10. *Suppose that* ${}_A\mathfrak{C}$*,* $\mathfrak{C}_A$*,* ${}_B\mathfrak{D}$ *and* $\mathfrak{D}_B$ *are flat, and let* $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ *and* $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*. If* $\mathfrak{C}$ *and* $\mathfrak{D}$ *are coseparable or cosemisimple (resp.* A *and* B *are von Neumann regular ring), then the following statements are equivalent:*

(1) $(-\square_{\mathfrak{C}}X, -\square_{\mathfrak{D}}\Lambda)$ *is a pair of inverse equivalences;*

(2) $(\Lambda\square_{\mathfrak{C}}-, X\square_{\mathfrak{D}}-)$ *is a pair of inverse equivalences;*

(3) Λ *is quasi-finite cogenerator preserving (resp. injector-cogenerator) as a right* $\mathfrak{C}$*-comodule,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(4) X *is quasi-finite cogenerator preserving (resp. injector-cogenerator) as a right* $\mathfrak{D}$*-comodule,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*;*

(5) Λ *is quasi-finite cogenerator preserving (resp. injector-cogenerator) on both sides,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(6) X *is quasi-finite cogenerator preserving (resp. injector-cogenerator) on both sides,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings, and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*.*

The following result, which recovers Morita's characterization of equivalence [5, Theorem 22.2], is a relevant consequence of the last theorem.

Corollary 3.2.11. *Let A and B be k-algebras and let*

$$F : \mathcal{M}_A \to \mathcal{M}_B \qquad \textit{and} \qquad G : \mathcal{M}_B \to \mathcal{M}_A$$

be k-linear functors. Then the following statements are equivalent

(1) F *and* G *are inverse equivalences;*

(2) there exists a bimodule ${}_AM_B$ *such that:*

- *(a)* ${}_AM$ *and* M_B *are progenerators (i.e., finitely generated, projective and generators),*
- *(b)* ${}_AM_B$ *is (faithfully) balanced (see [5, p. 60]),*
- *(c)* $F \simeq - \otimes_A M \qquad$ *and* $\qquad G \simeq \mathrm{Hom}_B(M, -)$*;*

(3) there exists a bimodule ${}_AM_B$ *such that:* M_B *is finitely generated projective,* ${}_AM$ *is completely faithful, the evaluation map* $M^* \otimes_A M \to B$ *is an isomorphism, and* $F \simeq - \otimes_A M \qquad$ *and* $\qquad G \simeq \mathrm{Hom}_B(M, -)$*.*

Moreover, in such a case, ${}_BM^*$ *and* M^*_A *are progenerators, and*

$$F \simeq \mathrm{Hom}_A(M^*, -) \qquad \textit{and} \qquad G \simeq - \otimes_B M^*.$$

Proof. (1) $\Rightarrow$ (2) There exists a bimodule ${}_AM_B$ such that $F \simeq - \otimes_A M$ and $G \simeq \mathrm{Hom}_B(M, -)$. Since G is an equivalence, $G \simeq - \otimes_B M^*$ (by Eilenberg-Watts Theorem or by Theorem 2.1.13). Then M_B is finitely generated and projective. From G faithful, it follows that M_B is a progenerator. By symmetry, it follows that ${}_AM$, ${}_BM^*$ and M^*_A are also progenerators.

As in the proof of Corollary 3.1.9, $(B, A, {}_BM^*_A, {}_AM_B, \omega, \psi)$ is a comatrix coring context, and $\psi = \phi\circ\lambda$, where ω , ψ, ϕ are defined as in that proof, and $\lambda : A \to \mathrm{End}_B(M_B)$ is defined by $\lambda(a) = \lambda_a$ with $\lambda_a(m) = am$. In particular we have an adjunction $(- \otimes_A M, - \otimes_B M^*)$. Using Proposition 1.1.33 and since ϕ is an isomorphism, we obtain that F is fully faithful if and only if ψ is an isomorphism, if and only if λ is an isomorphism. Then, by symmetry, ${}_AM_B$ is faithfully balanced.

(2) ⇒ (3) From Lemma 1.1.16 (2), it follows that ${}_AM$ is completely faithful. In particular it is faithful. By symmetry ${}_AM_B$ is faithfully balanced. Now, we will prove that the coring homomorphism $\epsilon_{M^*\otimes_A M} : M^* \otimes_A M \to B$ is an isomorphism. Let $N \in \mathcal{M}_B$ and let us consider the map

$$\xi_N : \mathrm{Hom}_B(M, N) \otimes_A M \xrightarrow{\nu} \mathrm{Hom}_B(\mathrm{Hom}_A(M, M), N) \xrightarrow{\simeq} \mathrm{Hom}_B(B, N) \xrightarrow{\simeq} N\ ,$$

where ν is defined by $\nu(\gamma \otimes_A m) : \alpha \mapsto \gamma(\alpha(m))$. From Lemma 1.1.72 (7) (${}_AM$ is finitely generated projective), ν and ξ_N are isomorphisms. It is easy to see that $\xi_N(\gamma\otimes_A m) = \gamma(m)$, for every γ in $\mathrm{Hom}_B(M, N)$ and every m in M. In particular ($N = B$), $\xi_B = \epsilon_{M^*\otimes_A M}$.

(3) ⇒ (1) Since M_B is finitely generated projective, $\Lambda = {}_BM^*_A$ is (B, A)-quasi-finite, and the cohom functor is $F \simeq - \otimes_A M$. F is faithful since ${}_AM$ is completely faithful. Moreover the coring homomorphism $\epsilon_{M^*\otimes_A M} : M^* \otimes_A M \to B$ is an isomorphism. The implication (3) ⇒ (1) of Theorem 3.2.10 then achieves the proof. □

Corollary 3.2.12. *Let A and B be rings. $\mathcal{M}_A$ and $\mathcal{M}_B$ are equivalent if and only if ${}_A\mathcal{M}$ and ${}_B\mathcal{M}$ are equivalent. In such a case we say that A and B are* Morita equivalent*.*

Proof. Obvious from Theorem 3.2.10. □

Now, we will study when the category $\mathcal{M}^{\mathfrak{C}}$ for some coring $\mathfrak{C}$ is equivalent to a category of modules $\mathcal{M}_B$.

Let $\Sigma \in {}_B\mathcal{M}^{\mathfrak{C}}$ such that Σ_A is finitely generated projective with a dual basis $\{e_i, e^*_i\}_i$, and let $M \in \mathcal{M}^{\mathfrak{C}}$. By [21, Proposition 1.4], the canonical isomorphism $\mathrm{Hom}_A(\Sigma, M) \to M \otimes_A \Sigma^*$ yields an isomorphism $\mathrm{Hom}_{\mathfrak{C}}(\Sigma, M) \to M\Box_{\mathfrak{C}}\Sigma^*$, where the left coaction on Σ^* is given by

$$\lambda_{\Sigma^*} : \Sigma^* \to \mathfrak{C} \otimes_A \Sigma^*,\ \lambda_{\Sigma^*}(f) = \sum_i f(e_{i(0)})e_{i(1)} \otimes_A e^*_i.$$

Moreover, we have an adjoint pair (F, G), where

$$F = - \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}},$$

and

$$G = \mathrm{Hom}_{\mathfrak{C}}(\Sigma, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_B.$$

The unit and the counit of this adjunction are given by: For $N \in \mathcal{M}_B$,

$$\upsilon_N : N \to \mathrm{Hom}_{\mathfrak{C}}(\Sigma, N \otimes_B \Sigma),\ \upsilon_N(n)(u) = n \otimes_B u,$$

or

$$\upsilon_N : N \to (N \otimes_B \Sigma)\Box_{\mathfrak{C}}\Sigma^*,\ \upsilon_N(n) = \sum_i (n \otimes_B e_i) \otimes_A e_i^*,$$

and for $M \in \mathcal{M}^{\mathfrak{C}}$:

$$\zeta_M : \mathrm{Hom}_{\mathfrak{C}}(\Sigma, M) \otimes_B \Sigma \to M,\ \zeta_M(\varphi \otimes_B u) = \varphi(u),$$

or

$$\zeta_M : (M\Box_{\mathfrak{C}}\Sigma^*) \otimes_B \Sigma \to M,\ \zeta_M\Big((\sum_j m_j \otimes_A f_j) \otimes_B u\Big) = \sum_j m_j f_j(u)$$

(see [21, Proposition 1.5]).

Let us define the map

$$\mathsf{can} : \Sigma^* \otimes_B \Sigma \to \mathfrak{C},\ \mathsf{can}(f \otimes_B u) = f(u_{(0)})u_{(1)}.$$

From [21, Lemma 3.1], we have can is a morphism of corings. It follows from this that can is a $\mathfrak{C}$-bicolinear map. Moreover, we can verify easily that for every $M \in \mathcal{M}^{\mathfrak{C}}$, the following diagram is commutative:

$$\begin{tikzcd}
(M\Box_{\mathfrak{C}}\Sigma^*) \otimes_B \Sigma \arrow[r, "\zeta_M"] \arrow[d, "\psi_M"] & M \arrow[d, "\simeq"] \\
M\Box_{\mathfrak{C}}(\Sigma^* \otimes_B \Sigma) \arrow[r, "M\Box_{\mathfrak{C}}\mathsf{can}"] & M\Box_{\mathfrak{C}}\mathfrak{C},
\end{tikzcd} \tag{3.18}$$

where ψ_M is the canonical map. We obtain that if ψ_M is isomorphism, for every $M \in \mathcal{M}^{\mathfrak{C}}$ (for example if ${}_B\Sigma$ is flat or if $\mathfrak{C}$ is coseparable), then G is fully faithful if and only if can is an isomorphism (see Proposition 1.1.42).

Let us consider the map $\upsilon : B \to \Sigma\Box_{\mathfrak{C}}\Sigma^*,\ \upsilon(b) = \sum_i be_i \otimes_A e_i^*$ $(b \in B)$. For every $N \in \mathcal{M}_B$, we have the following commutative diagram

$$\begin{tikzcd}
N \arrow[r, "\upsilon_N"] \arrow[d, "\simeq"'] & (N \otimes_B \Sigma)\Box_{\mathfrak{C}}\Sigma^* \\
N \otimes_B B \arrow[r, "N\otimes_B\upsilon"] & N \otimes_B (\Sigma\Box_{\mathfrak{C}}\Sigma^*), \arrow[u, "\psi_N"']
\end{tikzcd} \tag{3.19}$$

where ψ_N is the canonical map. Let $\phi : B \to \mathrm{End}_{\mathfrak{C}}(\Sigma)$ be the canonical morphism of k-algebras which define the left action on Σ $(\phi(b)(u) := bu)$ (see Section 2.1). We have $\upsilon :\ B \xrightarrow{\phi} \mathrm{End}_{\mathfrak{C}}(\Sigma) \xrightarrow{\simeq} \Sigma\Box_{\mathfrak{C}}\Sigma^*$. We obtain that if ψ_N is an isomorphism, for every $N \in \mathcal{M}_B$ (for example if $\Sigma_{\mathfrak{C}}$ is projective or if $\mathfrak{C}$ is coseparable), then F is fully faithful if and only if ϕ is an isomorphism (see Proposition 1.1.42).

Furthermore, if ${}_B\Sigma$ is flat or $\mathfrak{C}$ is coseparable, then we have the commutative diagram:

$$\begin{tikzcd}
\Sigma \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & B \otimes_B \Sigma \arrow[d, "\upsilon\otimes_B\Sigma"] \\
\Sigma\Box_{\mathfrak{C}}\mathfrak{C} & \Sigma\Box_{\mathfrak{C}}\Sigma^* \otimes_B \Sigma. \arrow[l, "\Sigma\Box_{\mathfrak{C}}\mathsf{can}"]
\end{tikzcd} \tag{3.20}$$

Now we are ready to state and prove the following theorem. In the particular case where $B = \mathrm{End}_{\mathfrak{C}}(\Sigma)$, the first part of Theorem is known, see [35, Theorem 3.2] and [19, 18.27].

Theorem 3.2.13. *Let $\Sigma \in {}_B\mathcal{M}^{\mathfrak{C}}$ such that Σ_A is finitely generated projective. Let $F = - \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}}$ and $G = \mathrm{Hom}_{\mathfrak{C}}(\Sigma, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_B$. The following statements are equivalent:*

(1) (F, G) is a pair of inverse equivalences with ${}_A\mathfrak{C}$ flat;

(2) ${}_B\Sigma$ is flat, $\Sigma_{\mathfrak{C}}$ is projective, and can*, and the morphism of k-algebras $\phi : B \to \mathrm{End}_{\mathfrak{C}}(\Sigma)$ defined by $\phi(b)(u) = bu$ for $b \in B, u \in \Sigma$, are isomorphisms;*

(3) ${}_A\mathfrak{C}$ is flat, $\Sigma_{\mathfrak{C}}$ is a projective generator, and the morphism of k-algebras $\phi : B \to \mathrm{End}_{\mathfrak{C}}(\Sigma)$ defined by $\phi(b)(u) = bu$ for $b \in B, u \in \Sigma$, is an isomorphism;

(4) ${}_B\Sigma$ is faithfully flat, and can *is an isomorphism.*

If moreover $\mathfrak{C}$ is coseparable, and ${}_A\mathfrak{C}$ is projective, then the following statements are equivalent:

(1) (F, G) is a pair of inverse equivalences;

(2) $\Sigma_{\mathfrak{C}}$ is a generator, and the morphism of k-algebras $\phi : B \to \mathrm{End}_{\mathfrak{C}}(\Sigma)$ defined by $\phi(b)(u) = bu$ for $b \in B, u \in \Sigma$, is an isomorphism;

(3) the morphism of k-algebras $\phi : B \to \mathrm{End}_{\mathfrak{C}}(\Sigma)$ defined by $\phi(b)(u) = bu$ for $b \in B, u \in \Sigma$, is an isomorphism, and can *is a surjective map;*

(4) ${}_B\Sigma$ is completely faithful, and can *is a bijective map.*

Proof. First we will prove the first statement. It is obvious that the condition (1) implies the other conditions.

(2) $\Rightarrow$ (1) That ${}_A\mathfrak{C}$ is flat follows from ${}_B\Sigma$ is flat and can is an isomorphism of A-bimodules. To prove that (F, G) is a pair of equivalences, it is enough to use the commutativity of the diagrams (3.18) and (3.19).

(3) $\Rightarrow$ (1) Follows from the Gabriel-Popescu Theorem 1.1.70, and the commutativity of the diagram (3.19).

(4) $\Rightarrow$ (1) From can is an isomorphism, it follows that $-\square_{\mathfrak{C}}(\Sigma^* \otimes_B \Sigma) \simeq -\square_{\mathfrak{C}}\mathfrak{C}$, and the following diagram is commutative and each of its morphisms is an isomorphism:

$$\begin{tikzcd}
N \otimes_B (\Sigma\square_{\mathfrak{C}}\Sigma^*) \otimes_B \Sigma \arrow[r, "\psi_{N\otimes_B\Sigma}"] \arrow[d, "\simeq"] & \big((N \otimes_B \Sigma)\square_{\mathfrak{C}}\Sigma^*\big) \otimes_B \Sigma \arrow[d, "\simeq"] \\
N \otimes_B \big(\Sigma\square_{\mathfrak{C}}(\Sigma^* \otimes_B \Sigma)\big) \arrow[r, "\simeq"] & (N \otimes_B \Sigma)\square_{\mathfrak{C}}(\Sigma^* \otimes_B \Sigma).
\end{tikzcd}$$

Then ψ_N is an isomorphism (since $-\otimes_B \Sigma$ is faithful). From the commutativity of the diagram (3.20), can is an isomorphism, and $-\otimes_B \Sigma$ is faithful, we have υ is an isomorphism. Finally from the commutativity of the diagram (3.19), υ_N is an isomorphism for every $N \in \mathcal{M}_B$.

Now we will prove the second statement. Obviously the condition (1) implies the condition (4).

(2) $\Rightarrow$ (1) By Proposition 2.1.17, $(\Sigma^* \otimes_B -, \Sigma\Box_{\mathfrak{C}}-)$ is an adjoint pair, and furthermore, $\Sigma_{\mathfrak{C}}$ is generator if and only if $\Sigma\Box_{\mathfrak{C}}-$ is faithful. From the commutativity of the diagram (3.20), and $\Sigma\Box_{\mathfrak{C}}-$ is faithful, it follows that can is an isomorphism. Hence (2) follows.

(3) $\Rightarrow$ (2) Since ${}_A\mathfrak{C}$ is projective, and can is surjective, can is a retraction in ${}_A\mathcal{M}$. Since $\mathfrak{C}$ is coseparable, it follows that can is a retraction in ${}^{\mathfrak{C}}\mathcal{M}$. Then for every $M \in \mathcal{M}^{\mathfrak{C}}$, $M\Box_{\mathfrak{C}}\mathsf{can}$ is a retraction in $\mathcal{M}_k$, and from the commutativity of the diagram (3.18), ζ_M is surjective for every $M \in \mathcal{M}^{\mathfrak{C}}$. By Proposition 1.1.42, $\Sigma_{\mathfrak{C}}$ is a generator.

(4) $\Rightarrow$ (3) We have that $\Sigma\Box_{\mathfrak{C}}\mathsf{can}$ is bijective. From the commutativity of the diagram (3.20), $\upsilon \otimes_B \Sigma$ is bijective. Since ${}_B\Sigma$ is completely faithful, υ and ϕ are also bijective. $\square$

Remark 3.2.14. In [21, Proposition 5.6], the authors have a similar version of our second statement. They state that if $\mathfrak{C}_A$ is projective, $B = \mathrm{End}_{\mathfrak{C}}(\Sigma)$, and $\mathfrak{C}$ is coseparable, then (F, G) defined as above is a pair of inverse equivalences.

In order to give a generalization of [90, Theorem 3.5] and [4, Corollary 7.6], we need the following result.

Lemma 3.2.15. *Let $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ be a bicomodule. Suppose that A is a QF ring.*

(a) If N is an injector-cogenerator as a right $\mathfrak{D}$-comodule, then N is an injective cogenerator in $\mathcal{M}^{\mathfrak{D}}$.

(b) If $\mathfrak{D}$ has a duality, and N is an injective cogenerator in $\mathcal{M}^{\mathfrak{D}}$ such that N_B is flat, then N is an injector-cogenerator as a right $\mathfrak{D}$-comodule.

Proof. (a) Since A is a QF ring, then A_A is an injective cogenerator. Hence $N_{\mathfrak{D}} \simeq (A \otimes_A N)_{\mathfrak{D}}$ is an injective cogenerator.

(b) Let X_A be an injective cogenerator module. Since A is a QF ring, X_A is projective. We have then the natural isomorphism

$$(X \otimes_A N)\Box_{\mathfrak{D}}- \simeq X \otimes_A (N\Box_{\mathfrak{D}}-) : {}^{\mathfrak{D}}\mathcal{M} \to \mathcal{M}_k.$$

By Proposition 2.2.11, $N_{\mathfrak{D}}$ and X_A are faithfully coflat, and then $X \otimes_A N$ is faithfully coflat. Once again by Proposition 2.2.11 ($X \otimes_A N$ is a flat right B-module), $X \otimes_A N$ is injective cogenerator in $\mathcal{M}^{\mathfrak{D}}$. $\square$

Theorem 3.2.16. *Suppose that ${}_A\mathfrak{C}$, $\mathfrak{C}_A$, ${}_B\mathfrak{D}$ and $\mathfrak{D}_B$ are flat, and let $X \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ and $\Lambda \in {}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$. If $\mathfrak{C}$ and $\mathfrak{D}$ have a duality, then the following statements are equivalent:*

(1) $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ is a pair of inverse equivalences with X_B and Λ_A are flat;

(2) $(\Lambda\Box_{\mathfrak{C}}-, X\Box_{\mathfrak{D}}-)$ *is a pair of inverse equivalences with* ${}_AX$ *and* ${}_B\Lambda$ *are flat;*

If in particular A and B are QF rings, then (1) and (2) are equivalent to

(3) Λ *is quasi-finite injective cogenerator as a right $\mathfrak{C}$-comodule with X_B and Λ_A are flat,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(4) X *is quasi-finite injective cogenerator as a right $\mathfrak{D}$-comodule with X_B and Λ_A are flat,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$*;*

(5) Λ *is quasi-finite injective cogenerator on both sides,* $\mathrm{e}_{\mathfrak{C}}(\Lambda) \simeq \mathfrak{D}$ *as corings, and* $X \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, \mathfrak{C})$ *in* ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$*;*

(6) X *is quasi-finite injective cogenerator on both sides,* $\mathrm{e}_{\mathfrak{D}}(X) \simeq \mathfrak{C}$ *as corings, and* $\Lambda \simeq \mathrm{h}_{\mathfrak{D}}(X, \mathfrak{D})$ *in* ${}^{\mathfrak{D}}\mathcal{M}^{\mathfrak{C}}$.

Proof. The equivalence between (1) and (2) follows from Theorem 3.2.9, and the fact that if $(-\Box_{\mathfrak{C}}X, -\Box_{\mathfrak{D}}\Lambda)$ is a Frobenius pair, $X_{\mathfrak{D}}$ and $\Lambda_{\mathfrak{C}}$ are coflat if and only if X_B and Λ_A are flat (see the proof of Theorem 2.2.14).

(1) $\Leftrightarrow$ (3) Obvious from Proposition 3.2.8(I), and Lemma 3.2.15.

(1) $\Leftrightarrow$ (4) The proof is analogous to that of "(1) $\Leftrightarrow$ (3)".

(5) $\Rightarrow$ (3) Trivial.

(1) $\Rightarrow$ (5) Obvious from the equivalence of (1), (2) and (3).

(1) $\Leftrightarrow$ (6) Follows by symmetry.

□

Remark 3.2.17. The second part of [4, Corollary 7.6] (and also the second part of [19, 12.14]) is true in a more general context. Let ${}_A\mathfrak{C}$ be flat. Consider the statements:

(1) $\Lambda \in {}_B\mathcal{M}^{\mathfrak{C}}$ be quasi-finite as a right $\mathfrak{C}$-comodule;

(2) $\mathrm{Hom}_{\mathfrak{C}}(M_0, \Lambda)$ is a finitely generated left B-module, for every finitely generated comodule $M_0 \in \mathcal{M}^{\mathfrak{C}}$;

(3) $\mathrm{Hom}_{\mathfrak{C}}(M_0, \Lambda)$ is a finitely generated projective left B-module, for every finitely generated comodule $M_0 \in \mathcal{M}^{\mathfrak{C}}$.

We have, (1) $\Rightarrow$ (2) holds if B is a QF ring, and the category $\mathcal{M}^{\mathfrak{C}}$ is locally finitely generated, and the converse implication holds if in particular B is a semisimple ring, and the category $\mathcal{M}^{\mathfrak{C}}$ is locally finitely generated. (1) $\Leftrightarrow$ (3) holds if $\mathfrak{C}$ is a cosemisimple coring. Moreover, for the two cases, if (1) holds, then for every comodule $M \in \mathcal{M}^{\mathfrak{C}}$,

$$\mathrm{h}_{\mathfrak{C}}(\Lambda, M) \simeq \varinjlim_{I} \mathrm{Hom}_{\mathfrak{C}}(M_i, \Lambda)^*,$$

where $(M_i)_{i\in I}$ is the family of all finitely generated subcomodules of M.

We will prove at the same time (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let $M_0 \in \mathcal{M}^{\mathfrak{C}}$ be a finitely generated comodule. We have, $\mathrm{h}_{\mathfrak{C}}(\Lambda, M_0)^* = \mathrm{Hom}_B(\mathrm{h}_{\mathfrak{C}}(\Lambda, M_0), B) \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, \Lambda)$.

Since the functor $- \otimes_B \Lambda$ is exact and preserves coproducts, the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -)$ preserves finitely generated (resp. finitely generated projective) objects. In particular, $\mathrm{h}_{\mathfrak{C}}(\Lambda, M_0)$ is a finitely generated (resp. finitely generated projective) right B-module, and $\mathrm{Hom}_{\mathfrak{C}}(M_0, \Lambda)$ so is. Therefore, $\mathrm{h}_{\mathfrak{C}}(\Lambda, M_0) \simeq \mathrm{h}_{\mathfrak{C}}(\Lambda, M_0)^{**} \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, \Lambda)^*$. Hence, $\mathrm{h}_{\mathfrak{C}}(\Lambda, M) \simeq \varinjlim_I \mathrm{h}_{\mathfrak{C}}(\Lambda, M_i) \simeq \varinjlim_I \mathrm{Hom}_{\mathfrak{C}}(M_i, \Lambda)^*$, where $(M_i)_{i \in I}$ is the family of all finitely generated subcomodules of M (since the cohom functor preserves inductive limits).

The proof of the implication $(i) \Rightarrow (ii)$ of [90, Proposition 1.3], remains valid to prove $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$.

3.3 Applications to induction functors

In this section we particularize our results in the previous sections to induction functors defined in Subsection 1.5.1 .

Theorem 3.3.1. *Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that ${}_A\mathfrak{C}$ and ${}_B\mathfrak{D}$ are flat. If $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat (resp. $\mathfrak{C}$ and $\mathfrak{D}$ are coseparable), then the following statements are equivalent*

(a) $(- \otimes_A B, -\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}))$ is a pair of inverse equivalences;

(b) the functor $- \otimes_A B$ is exact and faithful (resp. faithful), and there exists an isomorphism of B-corings $\mathfrak{D} \simeq B\mathfrak{C}B$.

Proof. From the proof of Theorem 2.3.1, $- \otimes_A B \simeq -\square_{\mathfrak{C}}(\mathfrak{C} \otimes_A B)$. The use of Proposition 3.2.8, Proposition 3.1.4, and Proposition 1.5.5 achieves the proof.

□

Corollary 3.3.2. *Let $\mathfrak{C}$ be an A-coring. Then the following statements are equivalent*

(a) The forgetful functor $U_r : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_A$ is an equivalence of categories;

(b) The forgetful functor $U_l : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is an equivalence of categories;

(c) there exists an isomorphism of A-corings $A \simeq \mathfrak{C}$.

Proof. It is enough to apply the last theorem to the particular homomorphism of corings $(\epsilon_{\mathfrak{C}}, 1_A) : \mathfrak{C} \to A$ which gives the well known adjunction $(U_r, - \otimes_A \mathfrak{C})$, and observing that the map $A\mathfrak{C}A \to \mathfrak{C}$ defined by $a \otimes c \otimes a' \mapsto aca'$, is an isomorphism of corings. □

Finally, given a homomorphism of corings, we give sufficient conditions to have that the right induction functor is an equivalence if and only if the left induction functor so is. Note that for the case of coalgebras over fields (by (b)), or of rings (well known) (by (d)), we have the left right symmetry.

Proposition 3.3.3. *Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings such that ${}_A\mathfrak{C}$, ${}_B\mathfrak{D}$, $\mathfrak{C}_A$ and $\mathfrak{D}_B$ are flat. Assume that at least one of the following holds* *(a)* $\mathfrak{C}$ *and* $\mathfrak{D}$ *have a duality, and* ${}_AB$ *and* B_A *are flat;*

(b) A *and* B *are von Neumann regular rings;*

(c) $B \otimes_A \mathfrak{C}$ *is coflat in* ${}^{\mathfrak{D}}\mathcal{M}$ *and* $\mathfrak{C} \otimes_A B$ *is coflat in* $\mathcal{M}^{\mathfrak{D}}$ *and* ${}_AB$ *and* B_A *are flat;*

(d) $\mathfrak{C}$ *and* $\mathfrak{D}$ *are coseparable corings.*

Then the following statements are equivalent

1. $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ *is an equivalence of categories;*

2. $B \otimes_A - : {}^{\mathfrak{C}}\mathcal{M} \to {}^{\mathfrak{D}}\mathcal{M}$ *is an equivalence of categories.*

Proof. Obvious from Theorem 3.2.9, Theorem 3.2.10 and Theorem 3.2.16. □

3.4 Applications to entwined modules and graded ring theory

In this section we particularize our results to corings associated to entwined structures and in particular those associated to a G-graded algebra and a right G-set, where G is group.

We obtain the following result concerning the category of entwined modules.

Theorem 3.4.1. *Let* $(\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')$ *be a morphism in* $\mathbb{E}^{\bullet}_{\bullet}(k)$*, such that* ${}_kC$ *and* ${}_kD$ *are flat. If either* $(A \otimes C)_A$ *and* $(A' \otimes C')_{A'}$ *are flat (e.g., if* ψ *and* ψ' *are isomorphisms), (resp.* $A \otimes C$ *and* $A' \otimes C'$ *are coseparable), then the following statements are equivalent*

(a) The functor $- \otimes_A A' : \mathcal{M}(\psi)^C_A \to \mathcal{M}(\psi')^{C'}_{A'}$ *defined in Section 2.4 is an equivalence;*

(b) the functor $- \otimes_A A'$ *is exact and faithful (resp. faithful), and there exists an isomorphism of* A'*-corings* $A' \otimes C' \simeq A'(A \otimes C)A'$.

Proof. Follows immediately from Theorem 3.3.1. □

Corollary 3.4.2. [81, Proposition 2.1]
Let G *and* G' *be two groups,* A *a* G*-graded* k*-algebra,* A' *a* G'*-graded* k*-algebra,* X *a right* G*-set, and* X' *a right* G'*-set. The following statements are equivalent*

(1) the categories $gr - (A, X, G)$ *and* $gr - (A', X', G')$ *are equivalent;*

(2) there are an $X \times X'$*-graded* (A, A')*-bimodule* P *and an* $X' \times X$*-graded* (A', A)*-bimodule* Q *such that*

$$P \widehat{\otimes}_{A'} Q \simeq \widehat{A} \quad \textit{and} \quad Q \widehat{\otimes}_A P \simeq \widehat{A}'.$$

Moreover, if P *and* Q *satisfy the condition (2), then* $-\widehat{\otimes}_A P$ *and* $-\widehat{\otimes}_{A'} Q$ *are inverse equivalences.*

Proof. Clear from Proposition 3.2.2 (using the fact that the corings $A \otimes kX$ and $A' \otimes kX'$ are coseparable) and Proposition 2.5.6. $\square$

The following result is similar to [81, Corollary 2.4].

Corollary 3.4.3. *Let A be a G-graded k-algebra, X a right G-set, and B a k-algebra. The following statements are equivalent*

(1) the category $gr-(A, X, G)$ is equivalent to $\mathcal{M}_B$;

(2) the category $(G, X, A) - gr$ is equivalent to ${}_B\mathcal{M}$;

(3) there exists an $X_0 \times X$-graded $B - A$-bimodule P, such that X_0 is a singleton, P is finitely generated projective in $\mathcal{M}_A$, and generator in $gr-(A, X, G)$, and the morphism of k-algebras $\phi : B \to \mathrm{End}_{gr-(A,X,G)}(P)$ defined by $\phi(b)(p) = bp$ for $b \in B, p \in P$, is an isomorphism.

Proof. Follows immediately from Theorem 3.2.13. $\square$

In [81, Theorem 2.3], A. Del Río gave a characterization when $(-\widehat{\otimes}_A P, \mathrm{H}(P_{A'}, -))$ is a pair of inverse equivalences. We think that the following result gives a simple characterization of it. Our result is also a generalization of Morita's characterization of equivalence Remark 3.2.11.

Theorem 3.4.4. *Let P be an $X \times X'$-graded (A, A')-bimodule. Then the following are equivalent*

(1) $(-\widehat{\otimes}_A P, \mathrm{H}(P_{A'}, -))$ is a pair of inverse equivalences;

(2) (a) ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$, and $P_{x'}$ is finitely generated projective in ${}_A\mathcal{M}$ for every $x' \in X'$ (resp. and P is a generator in both $gr - (A', X', G')$ and $(G, X, A) - gr$),

(b) the following bigraded bimodules maps: $\psi : \widehat{A} \to \mathrm{H}(P_{A'}, P)$ defined by $\psi(a \otimes x)(p) = a({}_xp) \in P$ ($a \in A, x \in X, p \in P$), and $\psi' : \widehat{A'} \to \mathrm{H}({}_AP, P)$ defined by $\psi'(a' \otimes x')(p) = (p_{x'})a' \in P$ ($a' \in A', x' \in X', p \in P$), are isomorphisms (resp. are surjective maps).

(3) (a) ${}_xP$ is finitely generated projective in $\mathcal{M}_{A'}$ for every $x \in X$,

(b) the evaluation map

$$\varepsilon_{\widehat{A'}} : \mathrm{H}(P_{A'}, \widehat{A'})\widehat{\otimes}_A P \to \widehat{A'},\ \varepsilon_{\widehat{A'}}(f \otimes_A P) = f(p)$$

($f \in \mathrm{H}(P_{A'}, \widehat{A'})_x, p \in {}_xP, x \in X$) is an isomorphism,

(c) the functor $-\widehat{\otimes}_A P$ is faithful.

Proof. From [65, Proposition 1.2], the unit and the counit of the adjunction $(-\widehat{\otimes}_A P, \mathrm{H}(P_{A'}, -))$ are given respectively by $\eta_M : M \to \mathrm{H}(P_{A'}, M\widehat{\otimes}_A P)$, $\eta_M(m)(p) = \sum_{x\in X} m_x \otimes_A {}_x p$ ($m = \sum_{x\in X} m_x \in M, p = \sum_{x\in X} {}_x p \in P$), and $\varepsilon_N : \mathrm{H}(P_{A'}, N)\widehat{\otimes}_A P \to N$, $\varepsilon_N(f \otimes_A P) = f(p)$ ($f \in \mathrm{H}(P_{A'}, N)_x, p \in {}_x P, x \in X$). By Lemma 2.5.7 (1), the functor $\mathrm{H}(P_{A'}, -))$ preserves inductive limits if and only if the condition (3)(a) holds. Then by Theorem 2.1.13 (the coring $A' \otimes kX'$ is coseparable), there is a natural isomorphism $\delta : \mathrm{H}(P_{A'}, -) \xrightarrow{\simeq} -\widehat{\otimes}_{A'} \mathrm{H}(P_{A'}, \widehat{A}')$. Set $Q = \mathrm{H}(P_{A'}, \widehat{A}')$.

(1) $\Leftrightarrow$ (3) It follows from Proposition 3.2.8(II) and Proposition 3.1.4(2).

(1) $\Leftrightarrow$ (2) We can suppose that the condition (2)(a) holds. We have $\eta_{\widehat{A}} : M \to \mathrm{H}(P_{A'}, \widehat{A}\widehat{\otimes}_A P)$, $\eta_{\widehat{A}}(a \otimes x)(p) = (a \otimes x) \otimes_A {}_x p = a(1_A \otimes x) \otimes_A {}_x p = a\lambda_P({}_x p) = \lambda_P(a({}_x p)) = \lambda_P(\psi(a \otimes x)(p))$, ($a \in A, x \in X, p \in P$). Since δ is a natural isomorphism, the following diagram is commutative:

$$\begin{tikzcd}
\widehat{A} \arrow[r, "\eta_{\widehat{A}}"] \arrow[dr, "\psi"'] & \mathrm{H}(P_{A'}, \widehat{A}\widehat{\otimes}_A P) \arrow[r, "\delta_{\widehat{A}\widehat{\otimes}_A P}"] & \widehat{A}\widehat{\otimes}_A P \widehat{\otimes}_{A'} Q \arrow[r, "\simeq"] & P\widehat{\otimes}_{A'} Q \\
 & \mathrm{H}(P_{A'}, P) \arrow[u, "\simeq"', "\mathrm{H}(P_{A'}, \lambda_P)"] \arrow[urr, "\delta_P"] & &
\end{tikzcd} .$$

Therefore the unit of the adjunction $(-\widehat{\otimes}_A P, -\widehat{\otimes}_{A'} Q)$ is

$$1_{gr-(A,X,G)} \xrightarrow{\simeq} -\widehat{\otimes}_A \widehat{A} \xrightarrow{-\widehat{\otimes}_A \psi_0} -\widehat{\otimes}_A P \widehat{\otimes}_{A'} Q,$$

where ψ_0 is the map $\psi_0 : \widehat{A} \xrightarrow{\psi} \mathrm{H}(P_{A'}, P) \xrightarrow{\delta_P} P\widehat{\otimes}_{A'} Q$. Hence $-\widehat{\otimes}_A P$ is fully faithful if and only if ψ is an isomorphism. By Proposition 2.1.17, $(Q\widehat{\otimes}_A -, P\widehat{\otimes}_{A'} -)$ is an adjoint pair. Moreover $-\widehat{\otimes}_{A'} Q$ is fully faithful if and only if $P\widehat{\otimes}_{A'} -$ is fully faithful, if and only if ψ' is an isomorphism (see Proposition 1.1.42).

Finally, if the maps ψ and ψ' defined in (2)(b) are surjective, then, P is a generator in $gr - (A', X', G')$ if and only if $-\widehat{\otimes}_{A'} Q$ is faithful, if and only if $P\widehat{\otimes}_{A'} -$ is faithful, if and only if ψ' is an injective map (see Proposition 1.1.42). By symmetry, P is a generator in $(G, X, A) - gr$ if and only if ψ is an injective map. $\square$

Finally, let $f : G \to G'$ be a morphism of groups, X a right G-set, X' a right G'-set, $\varphi : X \to X'$ a map such that $\varphi(xg) = \varphi(x)f(g)$ for every $g \in G$, $x \in X$. Let A be a G-graded k-algebra, A' a G'-graded k-algebra, and $\alpha : A \to A'$ a morphism of algebras such that $\alpha(A_g) \subset A'_{f(g)}$ for every $g \in G$.

We have, $\gamma : kX \to kX'$ such that $\gamma(x) = \varphi(x)$ for each $x \in X$, is a morphism of coalgebras, and $(\alpha, \gamma) : (A, kX, \psi) \to (A', kX', \psi')$ is a morphism in $\mathbb{E}^{\bullet}_{\bullet}(k)$.

Let $T^* = - \otimes_A A' : gr-(A,X,G) \to gr-(A',X',G')$ be the functor defined in Section 2.5. We know that T^* makes commutative the following diagram

$$\begin{tikzcd} gr-(A,X,G) \arrow[r, "T^*"] \arrow[d, "\simeq"] & gr-(A',X',G') \arrow[d, "\simeq"] \\ \mathcal{M}(kG)_A^{kX} \arrow[r, "-\otimes_A A'"] & \mathcal{M}(kG')_{A'}^{kX'}. \end{tikzcd}$$

Moreover, we have the commutativity of the following diagram

$$\begin{tikzcd} (G,X,A)-gr \arrow[r, "(T^*)'=A'\otimes_A -"] \arrow[d, "\simeq"] & (G',X',A')-gr \arrow[d, "\simeq"] \\ {}_A^{kX}\mathcal{M}(\psi^{-1}) \arrow[r, "A'\otimes_A -"] \arrow[d, "\simeq"] & {}_{A'}^{kX'}\mathcal{M}((\psi')^{-1}) \arrow[d, "\simeq"] \\ {}^{kX\otimes A}\mathcal{M} \arrow[r, "A'\otimes_A -"] \arrow[d, "\simeq"] & {}^{kX'\otimes A'}\mathcal{M} \arrow[d, "\simeq"] \\ {}^{A\otimes kX}\mathcal{M} \arrow[r, "A'\otimes_A -"] & {}^{A'\otimes kX'}\mathcal{M}. \end{tikzcd}$$

A. Del Río gave in [81, Example 2.6] an interesting characterization when T^* is an equivalence. Our result gives an other characterization of it.

Theorem 3.4.5. *The following statements are equivalent*

(1) the functor $T^* : gr-(A,X,G) \to gr-(A',X',G')$ is an equivalence;

(2) T^* is faithful, and the map

$$\omega : A' \otimes_A \widehat{A} \otimes_A A' \to \widehat{A'},\ a' \otimes_A (a \otimes x) \otimes_A a'' \mapsto a'\alpha(a)a'' \otimes \varphi(x)g'$$

$(a \in A, a' \in A', x \in X, g' \in G', a'' \in A'_{g'})$, is bijective.

Proof. Let $(\varphi, \rho) : \mathfrak{C} \to \mathfrak{D}$ be a homomorphism of corings. From the proof of [19, 24.11], the counit of the adjunction $(- \otimes_A B, -\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}))$ is given by

$$\psi_N : (N\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})) \otimes_A B \to N,\ \sum_i n_i \otimes_A c_i \otimes_A b \mapsto \sum_i n_i\rho(\epsilon_{\mathfrak{C}}(c_i))b$$

$(N \in \mathcal{M}^{\mathfrak{D}})$. This yields the map:

$$\omega : B \otimes_A \mathfrak{C} \otimes_A B \to \mathfrak{D},\ b' \otimes_A c \otimes_A b \mapsto \sum b'\varphi(c_{(1)})\rho(\epsilon_{\mathfrak{C}}(c_{(2)}))b, \quad (c \in \mathfrak{C}, b, b' \in B).$$

In our case, the last map is exactly that mentioned in the condition (2). Finally, our result follows from Theorem 3.2.13. $\square$

Finally we give the following consequence of Proposition 3.3.3:

Proposition 3.4.6. *The following are equivalent*

(1) the functor $T^* : gr - (A, X, G) \to gr - (A', X', G')$ *is an equivalence;*

(2) the functor $(T^*)' : (G, X, A) - gr \to (G', X', A') - gr$ *is an equivalence.*

Chapter 4

The Picard Group of Corings

In this chapter, all the considered corings are flat on both sides over their base rings.

4.1 The Picard group of corings

Corings over k-algebras and their morphisms form a category. We denote it by $\mathbf{Crg}_k$. Notice that the category of k-algebras $\mathbf{Alg}_k$, and the category of k-coalgebras $\mathbf{Coalg}_k$, are full subcategories of $\mathbf{Crg}_k$.

Lemma 4.1.1. *A morphism $(\varphi, \rho) \in \mathbf{Crg}_k$ is an isomorphism if and only if φ and ρ are bijective.*

Proof. Straightforward. $\square$

Now we recall from [10] the definition of the Picard group of a k-abelian category.

Let $\mathbf{C}$ be a k-abelian category, $\mathrm{Pic}_k(\mathbf{C})$ is the group of isomorphism classes (T) of k-equivalences $T : \mathbf{C} \to \mathbf{C}$ with the composition law $(T)(S) = (ST)$.

Definition 4.1.2. We say that a $(\mathfrak{C}, \mathfrak{D})$-bicomodule X is *right invertible* if the functor $-\square_{\mathfrak{C}} X : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is an equivalence. We say that $\mathfrak{C}$ and $\mathfrak{D}$ are *Morita-Takeuchi right equivalent* if there is a right invertible $(\mathfrak{C}, \mathfrak{D})$-bicomodule.

The isomorphism classes (X) of right invertible $\mathfrak{C}$-bicomodules with the composition law

$$(X_1)(X_2) = (X_1 \square_{\mathfrak{C}} X_2),$$

form the *right Picard group* of $\mathfrak{C}$. We denote it by $\mathrm{Pic}^r_k(\mathfrak{C})$. This law is well defined since $-\square_{\mathfrak{C}}(X_1 \square_{\mathfrak{C}} X_2) \simeq (-\square_{\mathfrak{C}} X_1)\square_{\mathfrak{C}} X_2$ (From Propositions 1.3.16, 1.3.17 (${}_{\mathfrak{C}}X_2$ is coflat)). The associativity of this law follows from the associativity of the cotensor product (${}_{\mathfrak{C}}X_1$ and ${}_{\mathfrak{C}}X_2$ are coflat). The isomorphism class $(\mathfrak{C})$ is the identity element. $(X)^{-1} = (\Lambda)$, where Λ is such that $(-\square_{\mathfrak{C}} X, -\square_{\mathfrak{C}} \Lambda)$ is a pair of inverse equivalences (see Proposition 3.2.2).

It follows from Theorems 3.2.10, 3.2.16 that if $\mathfrak{C}$ is coseparable (this case includes that of algebras), or cosemisimple, or A is von Neumann regular ring, or if $\mathfrak{C}$ is a coalgebra over a QF ring, then a $\mathfrak{C}$-bicomodule X is right invertible if and only if it is left invertible. Hence, for each case we have $\mathrm{Pic}_k^r(\mathfrak{C}) = \mathrm{Pic}_k^l(\mathfrak{C})$. We will denote this last by $\mathrm{Pic}_k(\mathfrak{C})$.

The following proposition is the motivation of the study of the Picard group of corings. The proof is straightforward using Theorem 2.1.13.

Proposition 4.1.3. *There are inverse isomorphisms of groups*

$$\mathrm{Pic}_k^r(\mathfrak{C}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Pic}_k(\mathcal{M}^{\mathfrak{C}}) \ ,$$

$\alpha((X)) = (-\Box_{\mathfrak{C}} X)$ *and* $\beta((T)) = (T(\mathfrak{C}))$.

Now we introduce two categories, $\mathfrak{Crg}_k$ and $\mathfrak{MT} - \mathfrak{Crg}_k^r$. The objects of both are the corings. The morphisms of $\mathfrak{Crg}_k$ are the isomorphisms of corings. The morphism of $\mathfrak{MT} - \mathfrak{Crg}_k^r$ are the Morita-Takeuchi right equivalences. A right *Morita-Takeuchi equivalence* $\mathfrak{C} \sim \mathfrak{D}$ is simply an isomorphism class (M) where ${}_{\mathfrak{D}}M_{\mathfrak{C}}$ is a right invertible $(\mathfrak{D}, \mathfrak{C})$-bicomodule. If (N) is a Morita-Takeuchi right equivalence $\mathfrak{D} \sim \mathfrak{C}'$, the composite is given by $(N \Box_{\mathfrak{D}} M)$.

The categories $\mathfrak{Crg}_k$ and $\mathfrak{MT} - \mathfrak{Crg}_k^r$ are groupoids (i.e., categories in which every morphism is an isomorphism). Then $\mathrm{Aut}_k(\mathfrak{C})$ and $\mathrm{Pic}_k^r(\mathfrak{C})$ are respectively the endomorphism group of the object $\mathfrak{C}$ in $\mathfrak{Crg}_k$ and in $\mathfrak{MT} - \mathfrak{Crg}_k$.

Let $f : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ and $g : (\mathfrak{C}_1 : A_1) \to (\mathfrak{D}_1 : B_1)$ be two morphisms of corings, and let ${}_{\mathfrak{C}_1}X_{\mathfrak{C}}$ be a bicomodule.

Consider the right induction functor $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ and the left induction functor $B_1 \otimes_{A_1} - : {}^{\mathfrak{C}_1}\mathcal{M} \to {}^{\mathfrak{D}_1}\mathcal{M}$.

Now, consider ${}_gX_f := B_1 \otimes_{A_1} X \otimes_A B$ which is a $(\mathfrak{D}_1, \mathfrak{D})$-bicomodule, and $X_f := X \otimes_A B$ which is a $(\mathfrak{C}_1, \mathfrak{D})$-bicomodule. It can be showed easily that ${}_1X_f \simeq X_f$ as bicomodules, and that if $\mathfrak{C} \xrightarrow{f=(\varphi,\rho)} \mathfrak{D} \xrightarrow{f'=(\varphi',\rho')} \mathfrak{C}'$ are morphisms of corings, then $(X_f)_{f'} \simeq X_{f'f}$ as bicomodules. Moreover, from Theorem 2.1.13, $X_f \simeq X \Box_{\mathfrak{C}} \mathfrak{C}_f$ as $(\mathfrak{C}_1, \mathfrak{D})$-bicomodules.

Now, assume that f is an isomorphism. Define $X'_f = X_f$ as $\mathfrak{C}_1$-comodules, with right B-action and right $\mathfrak{D}$-coaction given by

$$xb = x\rho^{-1}(b) \text{ and } \rho_{X'_f}(x) = x_{(0)} \otimes_B \varphi(x_{(1)}),$$

where $x \in X, b \in B$ and $\rho_X(x) = x_{(0)} \otimes_A x_{(1)}$. X_f and X'_f are isomorphic as $(\mathfrak{C}, \mathfrak{D})$-bicomodules. The connecting isomorphism is the canonical isomorphism of abelian groups $\alpha : X_f \to X'_f$ defined by $\alpha(x \otimes_A b) = x\rho^{-1}(b)$ whose inverse map is defined by $\beta(x) = x \otimes_A 1$. Similarly, we define ${}_fX'$.

Lemma 4.1.4. *We have a functor*

$$\Omega : \mathfrak{Crg}_k \to \mathfrak{MT} - \mathfrak{Crg}_k^r,$$

which is the identity on objects and associates with an isomorphism of corings $f : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ *the isomorphism class of the right invertible* $(\mathfrak{D}, \mathfrak{C})$*-bicomodule* ${}_f\mathfrak{C}$.

Proof. Let $f = (\varphi, \rho) : (\mathfrak{C} : A) \to (\mathfrak{D} : B)$ be an isomorphism of corings. Since $B \otimes_A \mathfrak{C} \otimes_A B \to \mathfrak{D}$, $b \otimes_A c \otimes_A b \mapsto b\varphi(c)b$ is a morphism of B-corings, it follows from Theorem 3.3.1 that ${}_f\mathfrak{C}$ is a right invertible $(\mathfrak{D}, \mathfrak{C})$-bicomodule. Now let $\mathfrak{C} \xrightarrow{f} \mathfrak{D} \xrightarrow{g} \mathfrak{C}'$ be two morphisms in $\mathfrak{Crg}_k$. Then ${}_{gf}\mathfrak{C} \simeq {}_g({}_f\mathfrak{C}) \simeq {}_g\mathfrak{D}\square_{\mathfrak{D}}{}_f\mathfrak{C}$ is an isomorphism of $(\mathfrak{C}', \mathfrak{C})$-bicomodules. $\square$

Lemma 4.1.5. [19, 18.12 (1)]
The map $\mathrm{End}_{\mathfrak{C}}(\mathfrak{C}) \to \mathfrak{C}^*$, $f \mapsto \epsilon f$, *is an algebra anti-morphism with inverse map* $h \mapsto (h \otimes_A \mathfrak{C})\Delta$.

Now we are ready to state and prove the main result of this section.

Theorem 4.1.6. *Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are flat.*

(1) We have an exact sequence

$$1 \longrightarrow \mathrm{Inn}_k^r(\mathfrak{C}) \longrightarrow \mathrm{Aut}_k(\mathfrak{C}) \xrightarrow{\Omega} \mathrm{Pic}_k^r(\mathfrak{C}) \ ,$$

where $\mathrm{Inn}_k^r(\mathfrak{C})$ is the set of $f = (\varphi, \rho) \in \mathrm{Aut}_k(\mathfrak{C})$ such that there is $p \in \mathfrak{C}^*$ invertible with

$$\varphi(c_{(1)})p(c_{(2)}) = p(c_{(1)})c_{(2)}, \tag{4.1}$$

for every $c \in \mathfrak{C}$. As for algebras and coalgebras over fields, we call $\mathrm{Inn}_k^r(\mathfrak{C})$ the right group of inner automorphisms *of $\mathfrak{C}$*

In particular, there is a monomorphism from the quotient group $\mathrm{Out}_k^r(\mathfrak{C}) := \mathrm{Aut}_k(\mathfrak{C})/\mathrm{Inn}_k^r(\mathfrak{C})$ to $\mathrm{Pic}_k^r(\mathfrak{C})$. As for algebras and coalgebras over fields, we call $\mathrm{Out}_k^r(\mathfrak{C})$ the right outer automorphism group *of $\mathfrak{C}$.*

(2) If $\mathfrak{C}$ and $\mathfrak{D}$ are Morita-Takeuchi right equivalent, then $\mathrm{Pic}_k^r(\mathfrak{C}) \simeq \mathrm{Pic}_k^r(\mathfrak{D})$.

Proof. (1) Let $f = (\varphi, \rho) \in \mathrm{Aut}_k(\mathfrak{C})$ and ${}_f\mathfrak{C} \simeq \mathfrak{C}$ as $\mathfrak{C}$-bicomodules. We have ${}_f\mathfrak{C} \simeq {}_f\mathfrak{C}'$ as $\mathfrak{C}$-bicomodules. Now, let $h : {}_f\mathfrak{C}' \to \mathfrak{C}$ be an isomorphism of $\mathfrak{C}$-bicomodules. h is a morphism of right $\mathfrak{C}$-bicomodules is equivalent to the existence of $p \in \mathfrak{C}^*$ invertible such that $h(c) = p(c_{(1)})c_{(2)}$, for every $c \in \mathfrak{C}$ (see Lemma 4.1.5). Then, $p = \epsilon \circ h$. That h is a morphism of left $\mathfrak{C}$-bicomodules means that for every $a \in A, c \in \mathfrak{C}$

$$h(ac) = \rho(a)h(c), \tag{4.2}$$

and for every $c \in \mathfrak{C}$,

$$\varphi(c_{(1)}) \otimes_A h(c_{(2)}) = h(c)_{(1)} \otimes_A h(c)_{(2)}. \tag{4.3}$$

The condition (4.3) is equivalent to

$$\varphi(c_{(1)}) \otimes_A p(c_{(2)})c_{(3)} = p(c_{(1)})c_{(2)} \otimes_A c_{(3)},$$

for every $c \in \mathfrak{C}$. Hence the condition (4.1) follows. To prove the converse, take $p \in \mathfrak{C}^*$ invertible satisfying (4.1) and set $h(c) = p(c_{(1)})c_{(2)}$, for every $c \in \mathfrak{C}$. Using the left A-linearity of φ, the condition (4.2) follows. On the other hand, the condition (4.3) follows easily.

(2) It follows immediately from that $\mathfrak{C}$ and $\mathfrak{D}$ are isomorphic to each other in the category $\mathfrak{MT}-\mathfrak{Crg}_k^r$. More explicitly, let ${}_{\mathfrak{C}}M_{\mathfrak{D}}$ be a right invertible bicomodule. The map

$$(X) \mapsto (M^{-1}\Box_{\mathfrak{C}} X \Box_{\mathfrak{C}} M)$$

is an isomorphism from $\mathrm{Pic}_k^r(\mathfrak{C})$ to $\mathrm{Pic}_k^r(\mathfrak{D})$. □

If we apply Theorem 4.1.6 in the case $\mathfrak{C} = A$, then we obtain [10, Proposition II (5.2)(3)] (see also [30, Theorems 55.9, 55.11]). The special case where $\mathfrak{C}$ is a k-coalgebra ($A = k$) with $\mathfrak{C}$ is flat as a k-module, recovers [91, Theorem 2.7].

Corollary 4.1.7. *(1) For a k-algebra A, we have an exact sequence*

$$1 \longrightarrow \mathrm{Inn}_k(A) \longrightarrow \mathrm{Aut}_k(A) \xrightarrow{\Omega} \mathrm{Pic}_k(A)\ ,$$

where $\mathrm{Inn}_k(A) := \{a \mapsto bab^{-1} \mid b \textit{ invertible in } A\}$, *the* group of inner automorphisms *of A.*

(2) For a k-coalgebra C such that ${}_kC$ is flat, we have an exact sequence

$$1 \longrightarrow \mathrm{Inn}_k(C) \longrightarrow \mathrm{Aut}_k(C) \xrightarrow{\Omega} \mathrm{Pic}_k^r(C)\ ,$$

where $\mathrm{Inn}_k(C)$ *(the* group of inner automorphisms *of C) is the set of $\varphi \in \mathrm{Aut}_k(C)$ such that there is $p \in C^*$ invertible with $\varphi(c) = p(c_{(1)})c_{(2)}p^{-1}(c_{(3)})$, for every $c \in C$.*

Proof. (1) It suffices to see that for $p \in \mathfrak{C}^*$ where $\mathfrak{C} = A$, p is invertible if and only if $p = \lambda_b : a \mapsto ba$ and b is invertible in A.

(2) Suppose that $\varphi \in \mathrm{Aut}_k(C)$ and $\varphi(c_{(1)})p(c_{(2)}) = p(c_{(1)})c_{(2)}$, for every $c \in C$. Then $\varphi(c) = \varphi(c_{(1)})\epsilon_C(c_{(2)}) = \varphi(c_{(1)})p(c_{(2)})p^{-1}(c_{(3)}) = p(c_{(1)})c_{(2)}p^{-1}(c_{(3)})$, for every $c \in C$. The converse is obvious. □

In order to consider the one-sided comodule structure of invertible bicomodules we need

Proposition 4.1.8. *Let $(M), (N) \in \mathrm{Pic}_k^r(\mathfrak{C})$. Then*

(1) $(N) \in \mathrm{Im}\Omega.(M)$ if and only if $N \simeq {}_fM$ as bicomodules for some $f \in \mathrm{Aut}_k(\mathfrak{C})$.

(2) ${}_AM_{\mathfrak{C}} \simeq {}_AN_{\mathfrak{C}}$ if and only if $N \simeq {}_fM$ as bicomodules for some $f = (\varphi, 1_A) \in \mathrm{Aut}_k(\mathfrak{C})$.

Proof. (1) Straightforward from the fact that for every $f \in \mathrm{Aut}_k(\mathfrak{C})$, there is an isomorphism of $\mathfrak{C}$-bicomodules ${}_fM \simeq {}_f\mathfrak{C}\Box_{\mathfrak{C}}M$.

(2) ($\Leftarrow$) Obvious. ($\Rightarrow$) Let $h : {}_AM_{\mathfrak{C}} \to {}_AN_{\mathfrak{C}}$ be a bicomodule isomorphism. Since M is a $(A, \mathfrak{C})$-quasi-finite comodule (see Proposition 3.2.8), it has a structure of $(\mathrm{e}_{\mathfrak{C}}(M), \mathfrak{C})$-bicomodule. Then N has a structure of $(\mathrm{e}_{\mathfrak{C}}(M), \mathfrak{C})$-bicomodule induced by h, and h is an $(\mathrm{e}_{\mathfrak{C}}(M), \mathfrak{C})$-bicomodule isomorphism. On the other hand, N is a $(A, \mathfrak{C})$-quasi-finite comodule. Let $F =: \mathrm{h}_{\mathfrak{C}}(N, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathrm{e}_{\mathfrak{C}}(M)}$ be the cohom functor, and let $\theta : 1_{\mathcal{M}^{\mathfrak{C}}} \to$

$F(-)\Box_{\mathrm{e}_{\mathfrak{C}}(M)}N$ and $\chi : F(-\Box_{\mathrm{e}_{\mathfrak{C}}(M)}N) \to 1_{\mathcal{M}^{\mathrm{e}_{\mathfrak{C}}(M)}}$ be respectively the unit and the counit of this adjunction. From Proposition 1.4.11 and Remark 1.4.12, the map $\varphi := \chi_{\mathrm{e}_{\mathfrak{C}}(M)} \circ F(\lambda_N^h) : \mathrm{e}_{\mathfrak{C}}(N) \to \mathrm{e}_{\mathfrak{C}}(M)$, where $\lambda_N^h : N \to \mathrm{e}_{\mathfrak{C}}(M)\Box_{\mathrm{e}_{\mathfrak{C}}(M)}N$ is the left comodule structure map on N, is a morphism of A-corings, and making commutative the diagram

$$\begin{tikzcd} N \arrow[r, "\theta_N"] \arrow[dr, "\lambda_N^h"'] & \mathrm{e}_{\mathfrak{C}}(N) \otimes_A N \arrow[d, "\varphi\otimes_A 1_N"] \\ & \mathrm{e}_{\mathfrak{C}}(M) \otimes_A N. \end{tikzcd}$$

Since F is a left adjoint to $-\Box_{\mathrm{e}_{\mathfrak{C}}(M)}N$ which is an equivalence, $\chi_{\mathrm{e}_{\mathfrak{C}}(M)}$ and φ are isomorphisms. Since M and N are right invertible, $\mathrm{e}_{\mathfrak{C}}(M) \simeq \mathfrak{C} \simeq \mathrm{e}_{\mathfrak{C}}(N)$ (see Proposition 3.2.8). We identify $\mathrm{e}_{\mathfrak{C}}(M)$ and $\mathrm{e}_{\mathfrak{C}}(N)$ with $\mathfrak{C}$. Then φ is a coring endomorphism of $\mathfrak{C}$. Hence $M \simeq {}_fN$ as $\mathfrak{C}$-bicomodules, where $f = (\varphi, 1_A)$. $\square$

The special case where $\mathfrak{C}$ is a k-coalgebra ($A = k$) with $\mathfrak{C}$ is flat as a k-module, recovers [91, Proposition 2.8]. Its version for algebras is given in [10, Proposition II.5.2(4)] or [30, Theorem 55.12].

Corollary 4.1.9. *Let C be a k-coalgebra such that ${}_kC$ is flat, and let $(M), (N) \in \mathrm{Pic}_k^r(C)$. Then $M_C \simeq N_C$ if and only if $(N) \in \mathrm{Im}\Omega.(M)$, that is, $N \simeq {}_fM$ as bicomodules for some $f \in \mathrm{Aut}_k(C)$.*

Proposition 4.1.10. [Bass]
Let A be a k-algebra and let $(M), (N) \in \mathrm{Pic}_k(A)$. Then $M_A \simeq N_A$ if and only if $(N) \in \mathrm{Im}\Omega.(M)$, that is, $N \simeq {}_fM$ as bicomodules for some $f \in \mathrm{Aut}_k(A)$.

4.2 The Aut-Pic Property

Following [12] and [29], we give the following

Definition 4.2.1. We say that a coring $\mathfrak{C}$ has the *right Aut-Pic property* if the group morphism Ω of Theorem 4.1.6 is surjective. In such a case, $\mathrm{Out}_k^r(\mathfrak{C}) \simeq \mathrm{Pic}_k^r(\mathfrak{C})$.

Bolla proved ([12, p. 264]) that every ring such that all right progenerators (=finitely generated projective generators) are free, has Aut-Pic. In particular, local rings (by using a theorem of Kaplansky), principal right ideal domains (by [61, Corollary 2.27]), and polynomial rings $k[X_1, \ldots, X_n]$, where k is a field (by Quillen-Suslin Theorem), have Aut-Pic. He also proved that every basic (semiperfect) ring has Aut-Pic (see [12, Proposition 3.8]). Moreover, it is well known that a semiperfect ring A is Morita equivalent to its basic ring eAe (see [5, Proposition 27.14]), and by Theorem 4.1.6, $\mathrm{Pic}_k(A) \simeq \mathrm{Out}_k(eAe)$. In [29], the authors gave several interesting examples of coalgebras over fields having Aut-Pic. For instance, they proved that every basic coalgebra has Aut-Pic. On the other hand, we know [27, Corollary 2.2] that given a coalgebra over a field C is Morita-Takeuchi equivalent to a basic coalgebra C_0, and by Theorem 4.1.6, $\mathrm{Pic}_k(C) \simeq \mathrm{Out}_k(C_0)$.

Of course all of the examples of rings and coalgebras over fields having Aut-Pic mentioned in [12] and [29] are examples of corings having Aut-Pic. In order to give other examples of corings satisfying the Aut-Pic property we need the next lemma which is a generalization of [74, Proposition 4.1].

We recall from [74] that an object M in an additive category $\mathbf{C}$ has the *invariant basis number property* (IBN for short) if $M^n \simeq M^m$ implies $n = m$. For example every nonzero finitely generated projective module over a semiperfect ring has IBN (from [5, Theorem 27.11]).

Lemma 4.2.2. *Let $\mathfrak{C}$ be an A-coring such that ${}_A\mathfrak{C}$ is flat. If $0 \neq M \in {}_B\mathcal{M}^{\mathfrak{C}}$ is $(B, \mathfrak{C})$-quasi-finite such that*

(a) B is semisimple and $\mathcal{M}^{\mathfrak{C}}$ is locally finitely generated; or

(b) B is semiperfect and there is a finitely generated projective M_0 in $\mathcal{M}^{\mathfrak{C}}$ such that $\mathrm{Hom}_{\mathfrak{C}}(M_0, M) \neq 0$. *(the last condition is fulfilled if $\mathfrak{C}$ is cosemisimple, or (by Proposition 1.2.24) if A is right artinian, ${}_A\mathfrak{C}$ is projective, and $\mathfrak{C}$ is right semiperfect.);*

then M has IBN as a $(B, \mathfrak{C})$-bicomodule.

In particular, if A is semisimple, or A is semiperfect and there is a nonzero finitely generated projective in $\mathcal{M}^{\mathfrak{C}}$, then $0 \neq \mathfrak{C}$ has IBN as an $(A, \mathfrak{C})$-bicomodule.

Proof. We prove at the same time the two statements. Let M_0 be a finitely generated subcomodule of M (resp. a finitely generated projective in $\mathcal{M}^{\mathfrak{C}}$ such that $\mathrm{Hom}_{\mathfrak{C}}(M_0, M) \neq 0$). We have $\mathrm{h}_{\mathfrak{C}}(M, M_0)^* = \mathrm{Hom}_B(\mathrm{h}_{\mathfrak{C}}(M, M_0), B) \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, M)$ as left B-modules. Since the functor $- \otimes_B M : \mathcal{M}_B \to \mathcal{M}^{\mathfrak{C}}$ is exact and preserves coproducts, the cohom functor $\mathrm{h}_{\mathfrak{C}}(\Lambda, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_B$ preserves finitely generated (resp. finitely generated projective) objects. In particular, $\mathrm{h}_{\mathfrak{C}}(M, M_0)$ is a finitely generated (resp. finitely generated projective) right B-module. Hence the left B-module $\mathrm{Hom}_{\mathfrak{C}}(M_0, M)$ is so.

Now suppose that $M^m \simeq M^n$. Since the functor $\mathrm{Hom}_{\mathfrak{C}}(M_0, -) : {}_B\mathcal{M}^{\mathfrak{C}} \to {}_B\mathcal{M}$ is k-linear, then $\mathrm{Hom}_{\mathfrak{C}}(M_0, M^m) \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, M^n)$ as left B-bimodules. Hence $\mathrm{Hom}_{\mathfrak{C}}(M_0, M)^m \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, M)^n$ as left B-bimodules. Finally, by [5, Theorem 27.11], $m = n$.

For the particular case we take $M = \mathfrak{C}$. □

Corollary 4.2.3. *Let C be a k-coalgebra such that ${}_kC$ is flat. If $0 \neq M \in \mathcal{M}^C$ is quasi-finite, such that*

(a) k is semisimple; or

(b) k is semiperfect and there is a finitely generated projective M_0 in $\mathcal{M}^C$ such that $\mathrm{Hom}_C(M_0, M) \neq 0$;

then M has IBN as a C-comodule.

In particular, if k is semisimple, or k is semiperfect and there is a nonzero finitely generated projective in $\mathcal{M}^C$, then $0 \neq C$ has IBN as an C-bicomodule.

Proposition 4.2.4. *Let $\mathfrak{C} \neq 0$ be an A-coring such that A is semisimple, or A is semiperfect and there is a nonzero finitely generated projective in $\mathcal{M}^{\mathfrak{C}}$. If every $\mathfrak{C}$-bicomodule which is $(A, \mathfrak{C})$-injector, is isomorphic to $\mathfrak{C}^{(I)}$ as $(A, \mathfrak{C})$-bicomodules for some set I, then the coring $\mathfrak{C}$ has right Aut-Pic.*

Proof. The proof is analogous to that of [29, Proposition 2.4] and the proof of Bolla ([12, p. 264]) of the fact that every ring such that all left progenerators are free has Aut-Pic.

Let M be a right invertible $\mathfrak{C}$-bicomodule. By assumptions, $M \simeq \mathfrak{C}^{(I)}$ as $(A, \mathfrak{C})$-bicomodules for some set I. Let M_0 be a finitely generated (resp. finitely generated projective) right comodule. Since M_0 is a small object, then the k-linear functor $\mathrm{Hom}_{\mathfrak{C}}(M_0, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_k$ preserves coproducts. Therefore, $\mathrm{Hom}_{\mathfrak{C}}(M_0, M) \simeq \mathrm{Hom}_{\mathfrak{C}}(M_0, \mathfrak{C})^{(I)}$ as left A-modules. Hence I is a finite set, and $M \simeq \mathfrak{C}^{(n)}$ as $(A, \mathfrak{C})$-bicomodules for some $n \geq 1$. Let N be a bicomodule such that (N) is the inverse of (M) in $\mathrm{Pic}^r_k(\mathfrak{C})$, then

$$\mathfrak{C} \simeq M \square_{\mathfrak{C}} N \simeq \mathfrak{C}^{(n)} \square_{\mathfrak{C}} N \simeq N^{(n)}$$

as $(A, \mathfrak{C})$-bicomodules. On the other hand, $N \simeq \mathfrak{C}^{(m)}$ as $(A, \mathfrak{C})$-bicomodules for some set $m \geq 1$. It follows that $\mathfrak{C} \simeq \mathfrak{C}^{(nm)}$ as $(A, \mathfrak{C})$-bicomodules. By Lemma 4.2.2, $nm = 1$ and then $M \simeq \mathfrak{C}$ as $(A, \mathfrak{C})$-bicomodules. Finally, from Proposition 4.1.8, $M \simeq {}_f\mathfrak{C}$ as bicomodules for some $f = (\varphi, 1_A) \in \mathrm{Aut}_k(\mathfrak{C})$. □

Corollary 4.2.5. *Let C be a k-coalgebra such that k is semisimple, or k is a QF ring and there is a nonzero finitely generated projective in $\mathcal{M}^C$. If every right injective C-comodule is free, then the coalgebra C has right Aut-Pic.*

The following result allows to simplify the computation of the Picard group of some interesting corings.

Proposition 4.2.6. *(1) Let ${}_B\Sigma_A$ be a bimodule such that Σ_A is finitely generated and projective and ${}_B\Sigma$ is faithfully flat, and $(\Sigma^* \otimes_B \Sigma)_A$ is flat. Then* $\mathrm{Pic}^r_k(\Sigma^* \otimes_B \Sigma) \simeq \mathrm{Pic}_k(B)$*, where $\Sigma^* \otimes_B \Sigma$ is the comatrix coring associated to Σ. If moreover B has Aut-Pic then* $\mathrm{Pic}^r_k(\Sigma^* \otimes_B \Sigma) \simeq \mathrm{Out}_k(B)$.
In particular, For a k-algebra A and $n \in \mathbb{N}$, we have $\mathrm{Pic}^r_k(M^c_n(A)) \simeq \mathrm{Pic}_k(A)$*, where $M^c_n(A)$ is the (n,n)-matrix coring over A defined in Example 1.2.2 (6). If moreover A has Aut-Pic then* $\mathrm{Pic}^r_k(M^c_n(A)) \simeq \mathrm{Out}_k(A)$.

(2) Let $\mathfrak{C}$ be an A-coring such that $\mathfrak{C}_A$ is flat, and R the opposite algebra of ${}^*\mathfrak{C}$. If ${}_A\mathfrak{C}$ is finitely generated projective (e.g. if $\mathfrak{C}$ is a Frobenius ring), then $\mathrm{Pic}^r_k(\mathfrak{C}) \simeq \mathrm{Pic}_k(R)$.

(3) If $\mathfrak{C}$ is an A-coring such that ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are flat and the category $\mathcal{M}^{\mathfrak{C}}$ has a small projective generator U, then $\mathrm{Pic}^r_k(\mathfrak{C}) \simeq \mathrm{Pic}_k(\mathrm{End}_{\mathfrak{C}}(U))$.

Proof. (1) From Corollary 3.2.7 (1), we have an equivalence of categories $-\otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}^{\Sigma^*\otimes_B\Sigma}$. Theorem 4.1.6 achieves then the proof. Now we prove the particular case. Let A be a k-algebra and $n \in \mathbb{N}$. If we take $\Sigma = A^n$, the comatrix coring $\Sigma^* \otimes_B \Sigma$ can be identified with $M_n^c(A)$.

(2) From Lemma 2.3.5, the categories $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}_R$ are isomorphic to each other. It is enough to apply Theorem 4.1.6.

(3) This follows immediately from Theorems 1.1.71 and 4.1.6. □

4.3 Application to the Picard group of $gr-(A, X, G)$

We begin by giving two useful lemmas whose proofs are straightforward.

Lemma 4.3.1. *A morphism $(\alpha, \gamma) \in \mathbb{E}^{\bullet}_{\bullet}(k)$ is an isomorphism if and only if α and γ are bijective.*

Lemma 4.3.2. *If $(\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')$ is a morphism of entwined structures, then $(\alpha \otimes \gamma, \alpha) : A \otimes C \to A \otimes C$ is a morphism of corings. Hence we have a functor*

$$F : \mathbb{E}^{\bullet}_{\bullet}(k) \to \mathbf{Crg}_k.$$

By Theorem 4.1.6, we obtain

Proposition 4.3.3. *For an entwining structure (A, C, ψ) there is an exact sequence*

$$1 \longrightarrow \operatorname{Ker}(\Omega \circ F) \longrightarrow \operatorname{Aut}_k((A, C, \psi)) \xrightarrow{\Omega\circ F} \operatorname{Pic}^r_k(A \otimes C) \simeq \operatorname{Pic}^r_k(\mathcal{M}(\psi)^C_A),$$

and $\operatorname{Ker}(\Omega \circ F)$ *is the set of* $(\alpha, \gamma) \in \operatorname{Aut}_k((A, C, \psi))$ *such that there is* $p \in (A \otimes C)^*$ *satisfying* $\sum \big(\alpha(a) \otimes \gamma(c_{(1)})\big)p(1_A \otimes c_{(2)}) = \sum p(a \otimes c_{(1)}) \otimes c_{(2)}$ *for all* $a \in A, c \in C$.

Lemma 4.3.4. *A morphism $(\hbar, \alpha, \gamma) \in \mathbb{DK}^{\bullet}_{\bullet}(k)$ is an isomorphism if and only if $\hbar$, α and γ are bijective.*

Proof. Straightforward. □

Lemma 4.3.5. [24, Proposition 17]
We have a faithful functor

$$G : \mathbb{DK}^{\bullet}_{\bullet}(k) \to \mathbb{E}^{\bullet}_{\bullet}(k)$$

defined by $G((H, A, C)) = (A, C, \psi)$ *with* $\psi : C \otimes A \to A \otimes C$ *and* $\psi(c \otimes a) = a_{(0)} \otimes ca_{(1)}$, *and* $G((\hbar, \alpha, \gamma)) = (\alpha, \gamma)$.

Hence Proposition 4.3.3 yields the following

Proposition 4.3.6. *For a DK structure (H, A, C) there is an exact sequence*

$$1 \longrightarrow \operatorname{Ker}(\Omega \circ F \circ G) \longrightarrow \operatorname{Aut}_k((H, A, C)) \xrightarrow{\Omega\circ F\circ G} \operatorname{Pic}^r_k(A \otimes C) \simeq \operatorname{Pic}^r_k(\mathcal{M}(H)^C_A)\ ,$$

and $\operatorname{Ker}(\Omega \circ F \circ G)$ *is the set of* $(\hbar, \alpha, \gamma) \in \operatorname{Aut}_k((H, A, C))$ *such that there is* $p \in (A \otimes C)^*$ *satisfying* $\sum \big(\alpha(a) \otimes \gamma(c_{(1)})\big)p(1_A \otimes c_{(2)}) = \sum p(a \otimes c_{(1)}) \otimes c_{(2)}$ *for all* $a \in A, c \in C$.

Finally, we consider the category of right modules graded by a G-set, $gr-(A,X,G)$, where G is a group and X is a right G-set. This category is equivalent to a category of modules over a ring if $X=G$ and A is strongly graded (by Dade's theorem [77, Theorem 3.1.1]), or if X is a finite set (by [73, Theorem 2.13]). But there are examples of a category of graded modules which is not equivalent to a category of modules, see [66, Remark 2.4].

Let G be a group, X a right G-set, and A be a G-graded k-algebra. We know that (kG,A,C) is a DK structure, with kG is a Hopf algebra. Then we have, (A,kX,ψ) is an entwining structure where $\psi: kX\otimes A\to A\otimes kX$ is the map defined by $\psi(x\otimes a_g)=a_g\otimes xg$ for all $x\in X, g\in G, a_g\in A_g$. $A\otimes kX$ is an A-coring with the comultiplication and the counit maps:

$$\Delta_{A\otimes kX}(a\otimes x)=(a\otimes x)\otimes_A(1_A\otimes x),\quad \epsilon_{A\otimes kX}(a\otimes x)=a\quad (a\in A, x\in X).$$

We know that this coring is coseparable (see Proposition 2.5.3). From Subsection 1.5.3, $gr-(A,X,G)\simeq \mathcal{M}(kG)_A^{kX}$. Moreover, we know that the family $\{1_A\otimes x\mid x\in X\}$ is a basis of both the left and the right A-module $A\otimes kX$ (see the proof of Lemma 2.5.5). Each $p\in(A\otimes kX)^*$ is entirely determined by the data of $p(1_A\otimes x)$, for all $x\in X$. The same thing holds for φ, where $(\varphi,\rho)\in\mathrm{Aut}_k(A\otimes kX)$.

By Theorem 4.1.6 we get

Proposition 4.3.7. *(a) We have, $p\in(A\otimes kX)^*$ is invertible if and only if there exists $q\in(A\otimes kX)^*$ such that $\sum_{h\in G} q(1_A\otimes xh^{-1})p(1_A\otimes x)_h=1_A=\sum_{h\in G}p(1_A\otimes xh^{-1})q(1_A\otimes x)_h$ for every $x\in X$.*

(b) We have an exact sequence

$$1\longrightarrow \mathrm{Ker}(\Omega)\longrightarrow \mathrm{Aut}_k(A\otimes kX)\xrightarrow{\Omega}\mathrm{Pic}_k(A\otimes kX)\simeq \mathrm{Pic}_k(gr-(A,X,G)),$$

and $(\varphi,\rho)\in\mathrm{Ker}(\Omega)$ if and only if there exists $p\in(A\otimes kX)^*$ invertible such that

(i) for all $x\in X$, $a_y^x p(1_A\otimes x)_h=0$ for all $y\in X, h\in G$ such that $yh\neq x$, where $\varphi(1_A\otimes x)=\sum_{y\in X}a_y^x\otimes y$, and

(ii) for all $x\in X, a\in A$, $p(a\otimes x)=\rho(a)p(1_A\otimes x)$.

Proof. We have, $p\in(A\otimes kX)^*$ is invertible if and only if there exists $q\in(A\otimes kX)^*$ such that

$$q\big(p(a\otimes x)(1_A\otimes x)\big)=\epsilon(a\otimes x)=a \tag{4.4}$$
$$p\big(q(a\otimes x)(1_A\otimes x)\big)=\epsilon(a\otimes x)=a, \tag{4.5}$$

for all $a \in A, x \in X$. (from Proposition 1.2.3.) On the other hand,

$$\begin{aligned}
q\big(p(a_g \otimes x)(1_A \otimes x)\big) &= q\big(p(1_A \otimes xg^{-1})(a_g \otimes x)\big) \\
&= q\big(\sum_{h\in G}(b_h a_g) \otimes x\big), \quad \text{where} \quad p(1_A \otimes xg^{-1}) = \sum_{h\in G} b_h \\
&= \sum_{h\in G} q\big(1_A \otimes x(hg)^{-1}\big)(b_h a_g), \quad \text{since} \quad b_h a_g \in A_{hg} \\
&= \sum_{h\in G} q(1_A \otimes xg^{-1}h^{-1})p(1_A \otimes xg^{-1})_h a_g,
\end{aligned}$$

for all $x \in X, g \in G, a_g \in A_g$. Hence

$$\begin{aligned}
(4.4) \iff & q\big(p(a_g \otimes x)(1_A \otimes x)\big) = a_g \quad \text{for all}\, x \in X, g \in G, a_g \in A_g \\
\iff & \sum_{h\in G} q(1_A \otimes xh^{-1})p(1_A \otimes x)_h = 1_A \quad \text{for all}\, x \in X.
\end{aligned}$$

By symmetry,

$$(4.5) \iff \sum_{h\in G} p(1_A \otimes xh^{-1})q(1_A \otimes x)_h = 1_A \quad \text{for all}\, x \in X.$$

By Theorem 4.1.6, $(\varphi, \rho) \in \mathrm{Ker}(\Omega)$ if and only if there exists $p \in (A \otimes kX)^*$ invertible with $\varphi(a \otimes x)p(1_A \otimes x) = p(a \otimes x) \otimes x$ for all $a \in A, x \in X$. On the other hand,

$$\begin{aligned}
\varphi(a \otimes x)p(1_A \otimes x) &= \rho(a)\varphi(1_A \otimes x)p(1_A \otimes x) \\
&= \rho(a)\sum_{y\in X}\sum_{h\in G}(a_y^x \otimes y)b_h, \,\text{where}\, \varphi(1_A \otimes x) = \sum_{y\in X} a_y^x \otimes y,\, p(1_A \otimes x) = \sum_{h\in G} b_h \\
&= \rho(a)\sum_{y\in X}\sum_{h\in G} a_y^x b_h \otimes yh.
\end{aligned}$$

Since $\sum_{y\in X} a_y^x = \epsilon \circ \varphi(1_A \otimes x) = \rho \circ \epsilon(1_A \otimes x) = 1_A$, (b) follows. $\square$

Now, let $f : G \to G'$ be a morphism of groups, X a right G-set, X' a right G'-set, and $\varphi : X \to X'$ a map. Let A be a G-graded k-algebra, A' a G'-graded k-algebra, and $\alpha : A \to A'$ a morphism of algebras. We have, $\hbar : kG \to kG'$ defined by $\hbar(g) = f(g)$ for each $g \in G$, is a morphism of Hopf algebras, and $\gamma : kX \to kX'$ defined by $\gamma(x) = \varphi(x)$ for each $x \in X$, is a morphism of coalgebras. It is easy to show that $(\hbar, \alpha, \gamma)$ is a morphism of DK structures if and only if

$$\varphi(xg) = \varphi(x)f(g) \quad \text{for all}\, g \in G, x \in X \tag{4.6}$$
$$\alpha(A_g) \subset A'_{f(g)} \quad \text{for all}\, g \in G. \tag{4.7}$$

We define a category as follows. The objects are the triples (G, X, A), where G is a group, X is a right G-set, and A is a graded k-algebra. The morphisms are the triples

(f, φ, α), where $f : G \to G'$ is a morphism of groups, $\varphi : X \to X'$ is a map, and $\alpha : A \to A'$ is a morphism of algebras such that (4.6) and (4.7) hold. We denote this category by $\mathbb{G}^r(k)$. There is a faithful functor

$$H : \mathbb{G}^r(k) \to \mathbb{DK}^{\bullet}_{\bullet}(k)$$

defined by $H((G, X, A)) = (kG, A, kX)$ and $H((f, \varphi, \alpha)) = (\hbar, \alpha, \gamma)$. We obtain then the next result. We think that our subgroup of $\mathrm{Pic}_k(gr - (A, X, G))$ is a natural subgroup and it is more simple than the Beattie-Del Río subgroup (see [11, §2]).

Proposition 4.3.8. *There is an exact sequence*

$$1 \longrightarrow \mathrm{Ker}(\Omega \circ F \circ G \circ H) \longrightarrow \mathrm{Aut}_k((G, X, A)) \xrightarrow{\Omega \circ F \circ G \circ H} \mathrm{Pic}_k(A \otimes kX) \simeq \mathrm{Pic}_k(gr - (A, X, G)) \,,$$

and $\mathrm{Ker}(\Omega \circ F \circ G \circ H)$ *is the set of* $(f, \varphi, \alpha) \in \mathrm{Aut}_k((G, X, A))$ *such that there is* $p \in (A \otimes kX)^*$ *invertible satisfying*

1) $p(a \otimes x) = \alpha(a)p(1_A \otimes x)$ *for all* $x \in X, a \in A$, *and*

2) $p(1_A \otimes x)_h = 0$ *for all* $x \in X, h \in G$ *such that* $\varphi(x)h \neq x$.

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